

Deflections and Free Vibrations of Circular Isotropic Plates of Thickness Varying in Accordance with a Parabola



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Abstract The paper considers the relationship between the dynamic and static parameters of circular isotropic plates under various boundary conditions. The studies of the plates were carried out under static and dynamic loading, taking into account the variability of the thickness. The authors established the relationship between the maximum deflection and the natural frequencies of the transverse vibrations of the plates, and assessed the matching of the coefficient K obtained by numerical studies with its analytical one. The curves for the frequencies of free vibrations and deflections under the static load and the change in the coefficient K depending on the thickness of the plate and boundary conditions were plotted. Studies showed that the coefficient K complies within 5% of the dependence of Professor V.I. Korobko only when the ratio of the thickness in the center to the thickness on the support $t_2/t_1 = 60/50 < 1.2$ for both support schemes. This is due to the fact that formula (16) was derived for isotropic plates with constant thickness and the distribution of mass evenly over the entire area of the plate leads to a significant error already at the stage of a small difference between the thicknesses at the support and in the center. With a thickness ratio $t_2/t_1 = 100/50 = 2$, the difference between the K coefficient and the analytical one is about 16%.

Keywords Circular plate · Boundary conditions · Frequency of free transverse vibrations · Maximum deflection

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1 Introduction

A large number of works are devoted to the calculation of solid and composite plates [1–9]. This article presents the study of the coefficient K , which express the relation between the frequencies of natural vibrations and the maximum deflections of the plates.

The determination of static and dynamic characteristics leads to determining the deflections [10–15] and frequencies of system vibrations [16–19] in solving the relevant differential equations. The functional relationship between the maximum deflection and the frequency of the fundamental mode of free transverse vibrations of elastic isotropic plates was proved by V. I. Korobko [5].

2 Materials and Methods

The differential equation of the plate transverse deflection has the form:

$$\frac{\partial^4 W}{\partial x^4} + 2 \frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} = \frac{q}{D} \quad (1)$$

With the use of biharmonic operators, the equation takes the form:

$$D \nabla^2 \nabla^2 W = \frac{q}{D}. \quad (2)$$

where $W = W(x, y)$ is the deflection function of the plate at the transverse deflection; $\nabla^2 \nabla^2 \nabla^2 \nabla^2$ —is a biharmonic operator; $D = EH^3/(12(1 - \nu^2))$ is cylindrical stiffness of the plate;

$q(x, y)$ is the law of the lateral load change.

The differential equation of plate free vibrations:

$$D \left(\frac{\partial^4 W}{\partial x^4} + 2 \frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} \right) + m \frac{\partial^2 W}{\partial t^2} = 0 \quad (3)$$

$$D \nabla^2 \nabla^2 W + m \frac{\partial^2 W}{\partial t^2} = 0. \quad (4)$$

where $W = W(x, y, t)$ is the deflection function of a freely oscillating plate; m is the mass per unit area of the plate; E, ν are respectively the modulus of elasticity of the material and the Poisson's ratio.

If the vibrations are harmonic

$$W = W(x, y) \cdot \cos(\omega t), \quad (5)$$

then Eq. (1) can be transformed to the following form:

$$D\nabla^2\nabla^2W - m\omega^2W = 0$$

or

$$D\nabla^2\nabla^2W - \beta^2W = 0,$$

where $\beta^2 = m\omega^2/D$ is the eigenvalue of the differential equation of vibrations of the plates.

Let us represent the deflection function as a product of the maximum deflection W_0 by the unit function $f(x, y)$ and substitute it in the differential equations of transverse deflection and free vibrations of the plates:

$$W(x, y) = W_0f(x, y) \tag{6}$$

$$\begin{cases} D\nabla^2\nabla^2f - m\omega^2f = 0 \\ DW_0\nabla^2\nabla^2f - q(x, y) = 0 \end{cases}$$

It should be noted that the precise solution of these differential equations is valid only in the frequent cases of plate forms and boundary conditions. Therefore, in practice, approximate methods of solution are mainly used.

If we assume that the plate is under a uniformly distributed load q , then having integrated Eq. (6) over the entire area of the region, and having performed the necessary transformations, we will get:

$$W_0 = \frac{q}{D} \frac{A}{\iint_A \nabla^2\nabla^2f dA}, \quad \omega^2 = \frac{D}{m} \frac{\iint_A \nabla^2\nabla^2f dA}{\iint_A f dA}. \tag{7}$$

The deflection function $W(x, y)$ can approximately be put down in a one-parameter form in the polar coordinate system:

$$W(x, y) = W_0f(x, y) = W_0g\left[\frac{t}{r(\phi)}\right] = W_0g(\rho) \tag{8}$$

where $r = r(\varphi)$ is the equation of the contour of the plate in the polar coordinate system, t and φ are polar coordinates, $\rho = t/r(\varphi)$ is the dimensionless polar coordinate.

This function describes a surface which level lines are similar to the region contour and are similarly located. The representation of the function of deflections in this form is justified by the fact that through it we can write down the exact solution to the problem of transverse deflection of a rigidly pinched elliptical plate under the action of a uniformly distributed load. Since just in a single case it is possible to

represent the real deflection function in the form of a one-parameter function (8), further results are of an approximate nature.

We transform the integrals in (7), taking into account the deflection function in form (8).

$$\iint_A f dA = \int_0^{2\pi} \int_0^r g(\rho) t dt d\phi. \tag{9}$$

Multiplying and dividing the right-hand side by r^2 , we get after the transformations:

$$\iint_A f dA = 2A \int_0^1 g(\rho) \rho d\rho. \tag{10}$$

Completing the transformation of the integral of the biharmonic operator according to, we finally write:

$$I = (K_f^2 \Phi_{g1} + K_f \Phi_{g2})/A = K_f (K_f \Phi_{g1} + \Phi_{g2})/A \tag{11}$$

where

$$\begin{aligned} \Phi_{g1} &= \frac{1}{2} \int_0^1 (g^{IV} \rho - 12g''' - 21g'' \rho^{-1} - 3g' \rho^{-2}) d\rho, \\ \Phi_{g2} &= \frac{\pi}{2} \int_0^1 (2g^{IV} \rho + 14g''' + 22g'' \rho^{-1} + 3g' \rho^{-2}) d\rho, \end{aligned} \tag{12}$$

The sign of the approximate equality in (11) appeared under the transformation of integrals by means of the Bunyakovsky inequality. We substitute integrals (9) and (11) into expressions (6). After the necessary transformations, we get:

$$\left\{ \begin{aligned} W_0 &\approx \frac{qA^2}{D} \frac{1}{K_f^2 \Phi_{g1} + K_f \Phi_{g2}}, \\ \omega^2 &\approx \frac{D}{2A^2 m} \frac{K_f^2 \Phi_{g1} + K_f \Phi_{g2}}{\int_0^1 g \rho d\rho} \end{aligned} \right. \tag{13}$$

Since all the values of the definite integrals occurring in the expressions (13) are constant numbers depending on the accuracy of the choice of function $g(\rho)$, they can be represented as the proportionality coefficients K_w, K_ω and B . Then

$$W_0 = K_w \frac{q}{D} \frac{A^2}{K_f^2 + BK_f}, \quad \omega^2 = K_\omega \frac{D}{m} \frac{K_f^2 + BK_f}{A^2}. \quad (14)$$

where

$$K_w = 1/\Phi_{g1}; \quad K_\omega = \frac{1}{2}\Phi_{g1} \int_0^1 g\rho d\rho; \quad B = \Phi_{g2}/\Phi_{g1}. \quad (15)$$

Strictly speaking, the signs of approximate equalities should be put in expressions (14), in view of (12) and the approximation of function $g(\rho)$.

Let us multiply the expressions (14) to each other:

$$W_0\omega^2 = K_w K_\omega \frac{q}{m} = K \frac{q}{m}. \quad (16)$$

Taking into account that the coefficients K_w and K_ω depend on the shape of the plate, the following regularity can be obtained from the expression (16): for elastic isotropic plates of identical shapes with homogeneous boundary conditions, the product of the maximum deflection W_0 from the action of the uniformly distributed load q per square of their fundamental frequency of transverse vibrations in the unloaded state, ω^2 with accuracy up to the dimensional factor q/m is a constant. Thus, it is mathematically and rigorously proved that for the whole set of plates with homogeneous boundary conditions the product $W_0 \cdot \omega^2$ will be represented by a single curve. An important feature of the formulated regularity is the fact that the product $W_0 \cdot \omega^2$, which is considered in it, does not depend on the flexural rigidity and dimensions of constructions.

The design structure is a circular isotropic plate, the thickness of which varies in accordance with a parabola (Fig. 1). The thickness of the plate on the support is 5 cm, the thickness in the span varies according to the parabola with maximum value at the middle:

$$t = 5 + k \cdot x^{\frac{1}{2}} (\text{cm}). \quad (17)$$

Numerical studies of the plates were carried out by the finite element method. The design schemes of composite plates are shown in Fig. 2. When calculating the plates, two support schemes were investigated: rigid pinching along the contour (Fig. 2a) and hinged support along the contour (Fig. 2b).

The plate with a diameter of 6 m is divided into 240 finite elements—24 elements in the annular direction and 10 finite elements in the radial direction (Fig. 1). The thickness of the plate on the support was taken constant 0.05 m; the thickness in the center was a variable parameter and varied from 0.05 m (plate of constant thickness) to 0.10 m with a step of 0.005 m. The plate was taken from steel of ordinary quality, volumetric weight 78.5 kN / m³. The modulus of elasticity is taken as $E = 2.06$

Fig. 1 Circular plate with a linearly variable thickness (a—finite element scheme; b—plate thickness)

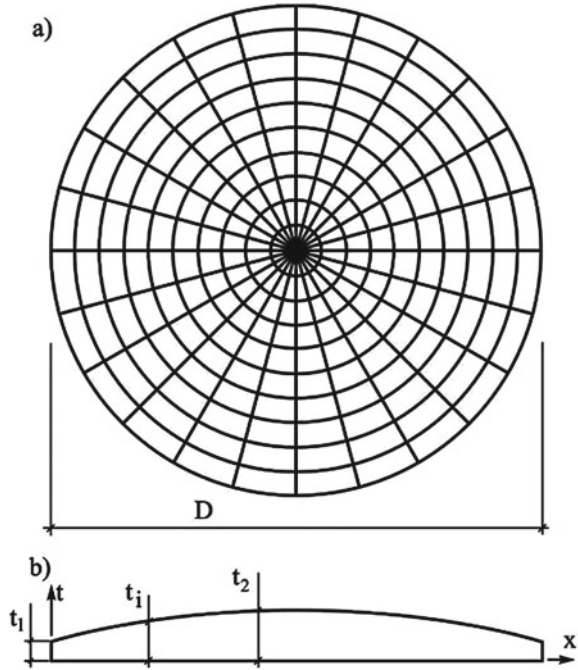
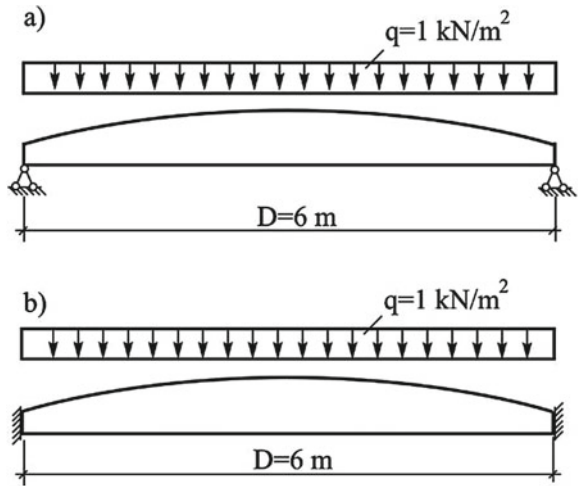


Fig. 2 Design diagrams of plates (a—with hinged support along the contour; b—with pinching along the contour)



·10⁵ MPa according to the Building Code of Russian Federation SP 16.13330.2017 “Steel structures”. All studies were carried out under the assumption of the elastic work of the material. Uniformly distributed load was assumed to be $q = 1 \text{ kN/m}^2$ (Fig. 2). The support was carried out along the contour in the contour nodes of the plates. Two support schemes were provided—hinged support and fixing along the

contour. To determine the natural frequencies of the transverse vibrations of the plates, concentrated masses from the empty weight of the plate were applied to the structural nodes in accordance with the load area of the nodes.

3 Results and Discussion

Determination of vibration and deflection frequencies was carried out using the SCAD software package [20]. The results of numerical studies of the plate are shown in Tables 1 and 2.

According to the data of Tables 1 and 2, graphs of changes in the maximum deflections and vibration frequencies in the studied plates and the proportionality coefficient K are plotted. The deviation of the actual value of the coefficient K from the theoretical one was determined by the formula:

$$\Delta = \frac{K_{theor} - K}{K_{theor}} \cdot 100\% \tag{18}$$

Based on the results of the research, curves for the frequency of natural oscillations (Fig. 3), maximum deflections (Fig. 4) and the K coefficient (Fig. 5) are plotted.

Table 1 Results of numerical studies of a circular plate, the thickness of which varies in accordance with a parabola, $D = 6$ m with hinged support

t_1 (mm)	t_2 (mm)	Circular frequency of fundamental tone, ω (s ⁻¹)	Maximum deflection, W_0 (mm)	$K = W_0\omega^2 / (q/m)$	$K = W_0\omega^2 / (q/m)$ based on analytical W_0 and ω	Deviation of K from K_{theor} . %
50	50	42.52237	21.8409	1.581	1.579	-0.11
50	55	45.19654	17.9303	1.542		2.35
50	60	47.88706	14.8697	1.507		4.58
50	65	50.56988	12.4814	1.477		6.45
50	70	53.25339	10.5797	1.451		8.10
50	75	55.93603	9.0470	1.428		9.55
50	80	58.61656	7.7981	1.408		10.84
50	85	61.29428	6.7700	1.390		12.00
50	90	63.96914	5.9159	1.373		13.04
50	95	66.64001	5.2006	1.358		13.99
50	100	69.30723	4.5969	1.345	14.84	

Table 2 Results of numerical studies of a circular plate plate, the thickness of which varies in accordance with a parabola, $d = 6$ m when pinched along the contour

t_1 (mm)	t_2 (mm)	Circular frequency of fundamental tone, ω (s^{-1})	Maximum deflection, W_0 (mm)	$K = W_0\omega^2/(q/m)$	$K = W_0w^2/(q/m)$ based on analytical W_0 and ω	Deviation of K from $K_{theor.}$ %
50	50	89.80446	5.1396	1.658	1.629	-1.81
50	55	93.04019	4.4117	1.608		1.30
50	60	96.26179	3.8158	1.562		4.09
50	65	99.4389	3.3322	1.525		6.40
50	70	102.5884	2.9315	1.492		8.40
50	75	105.7178	2.5957	1.464		10.15
50	80	108.8287	2.3116	1.438		11.70
50	85	111.9266	2.0693	1.416		13.06
50	90	115.014	1.8609	1.396		14.29
50	95	118.0914	1.6806	1.378		15.39
50	100	121.1638	1.5235	1.362	16.39	

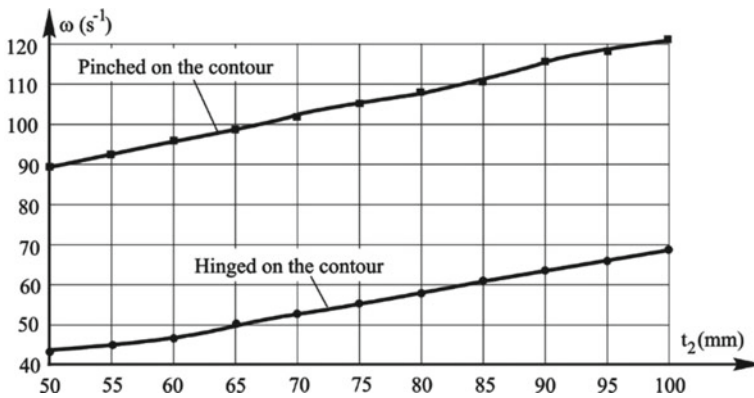


Fig. 3 Change in free vibration frequencies depending on the thickness of the plate t_2 in the center

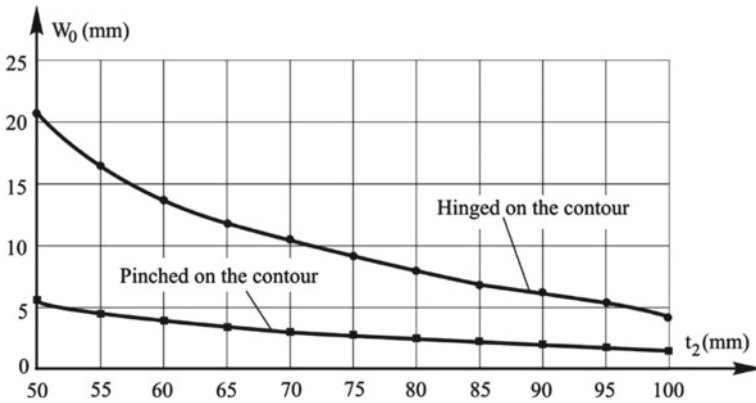


Fig. 4 Change in deflections by static load depending on the thickness of the plate t_2 in the center

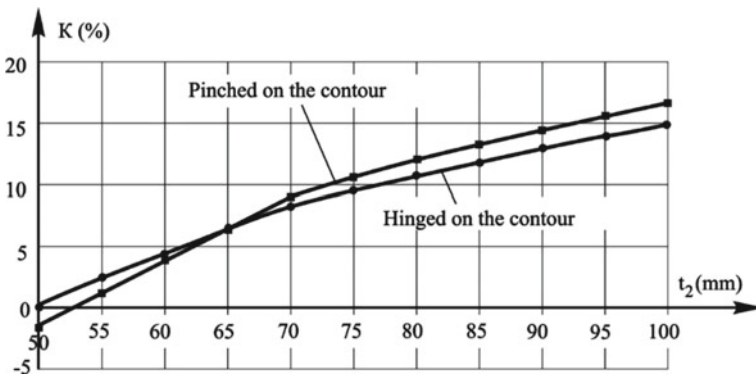


Fig. 5 Change in coefficient K depending on the thickness of the plate t_2 in the center

4 Conclusion

As a result of numerical studies, the maximum deflections and free vibration frequencies were determined for circular isotropic plates with a thickness varying in accordance with parabola with a thickening in the center. Studies showed that the coefficient K matches within 5% of the dependence of Professor V.I. Korobko only for the ratio of the thickness in the center to the thickness on the support $t_2 / t_1 = 60/50 < 1.2$ for both support schemes. This is due to the fact that formula (16) was derived for isotropic plates with constant thickness and the distribution of mass evenly over the entire area of the plate. This leads to a significant error already at the stage of a small difference between the thicknesses at the support and in the center. With a thickness ratio $t_2 / t_1 = 100/50 = 2$, the difference between the K coefficient and the analytical one is about 16%, and it should be expected that the difference will increase with increasing plate thickness in the center.

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