Chapter 14
Linear Diophantine Fuzzy Information Aggregation with Multi-criteria Decision-Making

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1 Introduction

The act of selecting steps to take after compiling relevant information and analyzing the relative merits of several potential solutions is known as decision-making. Establishing pertinent information and identifying available alternatives are the first two stages of a decision-making process that should be carried out according to a step-by-step methodology. Depending on the objectives and available options, the decision-making process may be either strategically, tactically, or operational. Since the beginning of the twentieth century, one of the most significant challenges faced by society has been confusing and inaccurate information. In several areas, such as economics, administration, psychology, mathematics, engineering, cognitive systems, and autonomous systems, data aggregation is a crucial phase in the decision-making process. Knowledge of the alternative has traditionally been conceptualized by individuals as a restricted amount or linguistic number. On the other hand, it is difficult to synthesize the information due to the substantial ambiguity involved. The multi-criteria decision-making (MCDM) approach is a frequently used intellectual activity instrument whose primary objective is to pick from a restricted number of possibilities based on the details provided by decision-makers (DMs). The MCDM approach, on the other hand, is prone to becoming ambiguous and inaccurate. This is because it integrates the complexity of human reasoning skills, making it difficult for DMs to engage in the review process in an accurate manner. In addition to addressing the issue of uncertainty, Zadeh [1] was a pioneer in developing fuzzy set theory. It is imperative that a solution be found for this issue. Atanassov [2] developed the "intuitionistic fuzzy set (IFS)."

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Yager [3–5] introduced "Pythagorean fuzzy set (PFS)" as an extended form of IFS. Yager added some generalizations to the IFS and PFS, and he developed the concept of the "q-rung orthopair fuzzy set (q-ROFS)" [6]. A constraint of the q-ROFS is that the sum of qth membership degree (MSD) power and non-membership degree (NMSD) power might be equal to or less than one. Riaz and Hashmi established the notion of the linear Diophantine fuzzy set (LDFS) [17]. After the advent of this notion, a number of academics were drawn to it and began working in this field.

Xu et al. [7–9] gave some Aos related to IFS. Wei et al. [11], Feng et al. [14], Mahmood et al. [10], Zhang et al. [12], Zhao et al. [13], Garg [15], and Rahman et al. [16] introduced many AOs for different extensions of fuzzy sets. Some work related to AOs and graph structures can be seen in [18, 19]. Extensive work related to bipolar fuzzy set is given in [20, 21]. Feng et al. [22] proposed some novel score functions related to orthopair fuzzy set. Senapati and Yager proposed Fermatean fuzzy set as the extension of IFS [23]. Smarandache proposed a novel idea of neutrosophic set [24, 25]. Farid and Riaz introduced some Einstein interacting geometric AOs for q-ROFSs [26]. Many AOs for "linear Diophantine fuzzy numbers" are given in [27, 28]. Ashraf et al. proposed some distance metric for cubic picture fuzzy set [29, 30]. Saha et al. [31, 32] introduced some hybrid AOs for different extensions of fuzzy set. Wei and Zhang [33] gave some single-valued neutrosophic Bonferroni power AOs. Riaz et al. proposed a number of AOs, including Einstein prioritized [35], interactive [36], hybrid [34], and prioritized with PDs [37]. Some extra-ordinary work related to proposed work is given in [38–41]. Ejegwa and Davvaz proposed the improved composite relation for q-ROFSs [42]. Ejegwa and Ahemen introduced some enhanced IF similarity measures [43]. Ejegwa et al. described the Pythagorean fuzzy correlation approach from a statistical standpoint [44]. Jana et al. [45] gave the notion of picture fuzzy Dombi AOs. Naeem et al. [46] presented some features related to topology in m-polarity PFSs. Peng et al. [47] proposed upgraded "single valued neutrosophic number" (SVNN) operations and established their associated AOs. Nancy and Garg [48] established AOs by employing Frank operations. Liu et al. [49] developed some AOs for SVNNs based on "Hamacher operations." Farid and Riaz [50] proposed Einstein interactive AOs for SVNNs. Zhang et al. [51] provided the AOs in the context of an "interval-valued neutrosophic set." Wu et al. [52] developed the prioritized AOs with SVNNs. Wei [53] proposed some similarity measures, Singh [54] idea of correlation coefficients, and Son [55] gave some clustering method for picture fuzzy set.

Multi-criteria decision-making (MCDM) is a method used to evaluate and select the best option among a set of alternatives based on multiple criteria. It is a powerful tool for decision-makers, as it allows for the consideration of multiple factors that may have an impact on the success of a decision. MCDM has been applied in a wide range of fields, including agriculture, where it can be used to make important decisions related to crop selection, land use, irrigation systems, and more.

One of the main advantages of MCDM in agriculture is that it takes into account the multiple and often conflicting objectives that farmers and other stakeholders may have. For example, when selecting a crop to plant, a farmer may consider factors such as expected yield, market demand, and pest resistance. Each of these factors may have different levels of importance to the farmer, and MCDM allows for the weighting of these factors to reflect this. Additionally, MCDM can be used to evaluate the trade-offs between different factors, such as the relationship between yield and water use efficiency.

Another important use of MCDM in agriculture is in land use planning. MCDM can be used to evaluate different land use options and determine the best option based on multiple criteria such as economic profitability, environmental sustainability, and social acceptability. This can be particularly useful in situations where there is a need to balance competing interests such as urbanization and agricultural production.

MCDM can also be used in irrigation systems. In this case, the farmer can evaluate different irrigation options based on criteria such as water use efficiency, cost, and impact on the environment. Additionally, MCDM can be used to evaluate the trade-offs between different irrigation options, such as the relationship between cost and water use efficiency. This can be particularly useful in areas where water is scarce, and farmers need to make decisions about how to use water resources in the most efficient and sustainable way.

Furthermore, MCDM can be used in the context of climate change, where farmers need to make decisions about crop selection, irrigation systems, and land use in the face of changing weather patterns, rising temperatures, and increased water scarcity. MCDM allows for the consideration of multiple factors such as crop resilience, water use efficiency, and environmental impact, which can help farmers make more informed decisions about how to adapt to changing conditions.

MCDM is an important tool for decision-making in agriculture. It allows for the consideration of multiple and often conflicting objectives, and it can be used to evaluate the trade-offs between different factors. This makes MCDM a valuable tool for farmers and other stakeholders in the agricultural sector, as it can help them make more informed decisions that balance economic profitability, environmental sustainability, and social acceptability. The main objectives of the manuscript are as follows:

- Some basic AOs are proposed for the aggregation of linear Diophantine fuzzy information.
- The essential properties of proposed AOs are also examined.
- Decision-making algorithm based on proposed Aos is also explained.
- Numerical example related to agriculture land selection is also given to show the practical implication of proposed algorithm.

This format is maintained for the remainder of the paper. In the second portion, we will talk about some essential LDFS concepts. The third section offers several potential AOs for LDFNs. In Sect. 4, an MCDM framework is shown for the recommended AOs. Section 5 has a test scenario with numerical information. The most important findings from the research are discussed in the sixth section.

2 Preliminary

In this part, we will go over some of the most fundamental aspects of LDFS.

Definition 1 ([17]) An LDFS R^r in X can be characterized by

$$R^r = \left\{\left(\Xi, \langle {\zeta^\tau}_{R^r}(\Xi), {\eta^\upsilon}_{R^r}(\Xi)\rangle, \langle {\mathscr{J}^\aleph}_{R^r}(\Xi), {\mathscr{C}^\gamma}_{R^r}(\Xi)\rangle\right) : \Xi \in X\right\},$$

where ${\zeta^\tau}_{R^r}(\Xi)$, ${\eta^\upsilon}_{R^r}(\Xi)$, ${\mathscr{J}^\aleph}_{R^r}(\Xi)$, ${\mathscr{C}^\gamma}_{R^r}(\Xi) \in [0, 1]$ are the MSD, the NMSD, and the corresponding reference parameters (RPs), respectively. Moreover,

$$0 \leq {\mathscr{J}^\aleph}_{R^r}(\Xi) + {\mathscr{C}^\gamma}_{R^r}(\Xi) \leq 1,$$

and

$$0 \leq {\mathscr{J}^\aleph}_{R^r}(\Xi){\zeta^\tau}_{R^r}(\Xi) + {\mathscr{C}^\gamma}_{R^r}(\Xi){\eta^\upsilon}_{R^r}(\Xi) \leq 1$$

for all $\Xi \in X$. The LDFS

$$R^r_X = \{(\Xi, \langle 1, 0\rangle, \langle 1, 0\rangle) : \Xi \in X\}$$

is recognized the "absolute LDFS" in X. The LDFS

$$R^r_\phi = \{(\Xi, \langle 0, 1\rangle, \langle 0, 1\rangle) : \Xi \in X\}$$

is recognized the "null LDFS" in X.

Modeling or categorization certain structures can be accomplished with the help of the RPs. We are able to describe a wide variety of systems by altering the fundamental significance of the RPs. Moreover, $\eta_{R^r}(\Xi)\pi_{R^r}(\Xi) = 1-({\mathscr{J}^\aleph}_{R^r}(\Xi){\zeta^\tau}_{R^r}(\Xi)+ {\mathscr{C}^\gamma}_{R^r}(\Xi){\eta^\upsilon}_{R^r}(\Xi))$ is called the "indeterminacy degree" and its corresponding RP of Ξ to R^r.

It is very evident that our suggested conception is more appropriate and advanced, and it includes a range of RPs. This procedure is applicable to a wide range of projects, including those in the fields of industry, medicine, cognitive computing, and MCDM.

Definition 2 ([17]) A "linear Diophantine fuzzy number" (LDFN) is the form of $\daleth^\varsigma = (\langle {\zeta^\tau}_{\hbar^\kappa}, {\eta^\upsilon}_{\daleth^\varsigma}\rangle, \langle {\mathscr{J}^\aleph}_{\daleth^\varsigma}, {\mathscr{C}^\gamma}_{\daleth^\varsigma}\rangle)$ having the given characteristics:

(1) $0 \leq {\zeta^\tau}_{\daleth^\varsigma}, {\eta^\upsilon}_{\daleth^\varsigma}, {\mathscr{J}^\aleph}_{\daleth^\varsigma}, {\mathscr{C}^\gamma}_{\daleth^\varsigma} \leq 1.$
(2) $0 \leq {\mathscr{J}^\aleph}_{\daleth^\varsigma} + {\mathscr{C}^\gamma}_{\daleth^\varsigma} \leq 1.$
(3) $0 \leq {\mathscr{J}^\aleph}_{\daleth^\varsigma}{\zeta^\tau}_{\daleth^\varsigma} + {\mathscr{C}^\gamma}_{\daleth^\varsigma}{\eta^\upsilon}_{\daleth^\varsigma} \leq 1.$

Definition 3 ([17]) Consider $\daleth^{\varsigma} = (\langle \zeta^{\tau}{}_{\daleth^{\varsigma}}, \eta^{\upsilon}{}_{\daleth^{\varsigma}} \rangle, \langle \mathscr{J}^{\aleph}{}_{\daleth^{\varsigma}}, \mathscr{C}^{\gamma}{}_{\daleth^{\varsigma}} \rangle)$ is the LDFN, and then the "score function" (SF) $\breve{\P}(\daleth^{\varsigma})$ is defined by $\maltese(\daleth^{\varsigma}) : LDFN(X) \rightarrow [-1, 1]$ and given by

$$\maltese(\daleth^{\varsigma}) = \frac{1}{2}[(\zeta^{\tau}{}_{\daleth^{\varsigma}} - \eta^{\upsilon}{}_{\daleth^{\varsigma}}) + (\mathscr{J}^{\aleph}{}_{\daleth^{\varsigma}} - \mathscr{C}^{\gamma}{}_{\daleth^{\varsigma}}),]$$

where $LDFN(X)$ is the collection of LDFNs on X.

Definition 4 ([17]) Consider $\daleth^{\varsigma} = (\langle \zeta^{\tau}{}_{\daleth^{\varsigma}}, \eta^{\upsilon}{}_{\daleth^{\varsigma}} \rangle, \langle \mathscr{J}^{\aleph}{}_{\daleth^{\varsigma}}, \mathscr{C}^{\gamma}{}_{\daleth^{\varsigma}} \rangle)$ is the LDFN, and then the "accuracy function" is defined by $\psi : LDFN(X) \rightarrow [0, 1]$ and given as

$$\psi(\daleth^{\varsigma}) = \frac{1}{2}\Big[\Big(\frac{\zeta^{\tau}{}_{\daleth^{\varsigma}} + \eta^{\upsilon}{}_{\daleth^{\varsigma}}}{2}\Big) + (\mathscr{J}^{\aleph}{}_{\daleth^{\varsigma}} + \mathscr{C}^{\gamma}{}_{\daleth^{\varsigma}})\Big]$$

Definition 5 ([17]) Let $\daleth^{\varsigma}{}_1 = (\langle \zeta^{\tau}{}_1, \eta^{\upsilon}{}_1 \rangle, \langle \mathscr{J}^{\aleph}{}_1, \mathscr{C}^{\gamma}{}_1 \rangle)$ be an LDFN and $\mathfrak{X} > 0$. Then:

- $\daleth^{\varsigma c}_1 = (\langle \eta^{\upsilon}{}_1, \zeta^{\tau}{}_1 \rangle, \langle \mathscr{C}^{\gamma}{}_1, \mathscr{J}^{\aleph}{}_1 \rangle)$.
- $\mathfrak{X}\daleth^{\varsigma}{}_1 = \Big(\langle 1 - (1 - \zeta^{\tau}{}_1)^{\mathfrak{X}}, \eta^{\upsilon \mathfrak{X}}_1 \rangle, \langle 1 - (1 - \mathscr{J}^{\aleph}{}_1)^{\mathfrak{X}}, \mathscr{C}^{\gamma \mathfrak{X}}_1 \rangle\Big)$.
- $\daleth^{\varsigma \mathfrak{X}}_1 = \Big(\langle \zeta^{\tau \mathfrak{X}}_1, 1 - (1 - \eta^{\upsilon}{}_1)^{\mathfrak{X}} \rangle, \langle \mathscr{J}^{\aleph \mathfrak{X}}_1, 1 - (1 - \mathscr{C}^{\gamma}{}_1)^{\mathfrak{X}} \rangle\Big)$.

Definition 6 ([17]) Let $\daleth^{\varsigma}{}_i = (\langle \zeta^{\tau}{}_i, \eta^{\upsilon}{}_i \rangle, \langle \mathscr{J}^{\aleph}{}_i, \mathscr{C}^{\gamma}{}_i \rangle)$ be two LDFNs with $i = 1, 2$. Then:

- $\daleth^{\varsigma}{}_1 \subseteq \daleth^{\varsigma}{}_2 \Leftrightarrow \zeta^{\tau}{}_1 \leq \zeta^{\tau}{}_2, \eta^{\upsilon}{}_2 \leq \eta^{\upsilon}{}_1, \mathscr{J}^{\aleph}{}_1 \leq \mathscr{J}^{\aleph}{}_2, \mathscr{C}^{\gamma}{}_2 \leq \mathscr{C}^{\gamma}{}_1$.
- $\daleth^{\varsigma}{}_1 = \daleth^{\varsigma}{}_2 \Leftrightarrow \zeta^{\tau}{}_1 = \zeta^{\tau}{}_2, \eta^{\upsilon}{}_1 = \eta^{\upsilon}{}_2, \mathscr{J}^{\aleph}{}_1 = \mathscr{J}^{\aleph}{}_2, \mathscr{C}^{\gamma}{}_1 = \mathscr{C}^{\gamma}{}_2$.
- $\daleth^{\varsigma}{}_1 \oplus \daleth^{\varsigma}{}_2 = (\langle \zeta^{\tau}{}_1 + \zeta^{\tau}{}_2 - \zeta^{\tau}{}_1\zeta^{\tau}{}_2, \eta^{\upsilon}{}_1\eta^{\upsilon}{}_2 \rangle, \langle \mathscr{J}^{\aleph}{}_1 + \mathscr{J}^{\aleph}{}_2 - \mathscr{J}^{\aleph}{}_1\mathscr{J}^{\aleph}{}_2, \mathscr{C}^{\gamma}{}_1\mathscr{C}^{\gamma}{}_2 \rangle)$.
- $\daleth^{\varsigma}{}_1 \otimes \daleth^{\varsigma}{}_2 = (\langle \zeta^{\tau}{}_1\zeta^{\tau}{}_2, \eta^{\upsilon}{}_1 + \eta^{\upsilon}{}_2 - \eta^{\upsilon}{}_1\eta^{\upsilon}{}_2 \rangle, \langle \mathscr{J}^{\aleph}{}_1\mathscr{J}^{\aleph}{}_2, \mathscr{C}^{\gamma}{}_1 + \mathscr{C}^{\gamma}{}_2 - \mathscr{C}^{\gamma}{}_1\mathscr{C}^{\gamma}{}_2 \rangle)$.

Definition 7 ([17]) Let $\daleth^{\varsigma}{}_i = (\langle \zeta^{\tau}{}_i, \eta^{\upsilon}{}_i \rangle, \langle \mathscr{J}^{\aleph}{}_i, \mathscr{C}^{\gamma}{}_i \rangle)$ be the assemblage of LDFNs with $i \in \Delta$. Then:

- $\bigcup_{i \in \Delta} \daleth^{\varsigma}{}_i = \Big(\langle \sup_{i \in \Delta} \zeta^{\tau}{}_i, \inf_{i \in \Delta} \eta^{\upsilon}{}_i \rangle, \langle \sup_{i \in \Delta} \mathscr{J}^{\aleph}{}_i, \inf_{i \in \Delta} \mathscr{C}^{\gamma}{}_i \rangle\Big)$.
- $\bigcap_{i \in \Delta} \daleth^{\varsigma}{}_i = \Big(\langle \inf_{i \in \Delta} \zeta^{\tau}{}_i, \sup_{i \in \Delta} \eta^{\upsilon}{}_i \rangle, \langle \inf_{i \in \Delta} \mathscr{J}^{\aleph}{}_i, \sup_{i \in \Delta} \mathscr{C}^{\gamma}{}_i \rangle\Big)$.

There are many AOs for the aggregation of LDFNs, namely, Einstein AOs [28], prioritized AOs [27], and fairly AOs [56].

Definition 8 ([28]) Consider $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth} \rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth} \rangle)$ the agglomeration of LDFNs and $\Re^{\Im} = (\Re^{\Im}{}_1, \Re^{\Im}{}_2, \ldots, \Re^{\Im}{}_n)^T$ be the weight vector (WV) with $\sum_{\beth=1}^{n} \Re^{\Im}{}_{\beth} = 1$. Then "linear Diophantine fuzzy Einstein weighted average

(LDFEWA) operator" is defined as

$$LDFEWA(\hbar_1^{\kappa}, \hbar_2^{\kappa}, \hbar_3^{\kappa}, \ldots, \hbar_n^{\kappa}) =$$

$$\sum_{\beth=1}^{n} \Re^{\Im}{}_{\Lambda} \daleth^{\varsigma}{}_{\beth} = \Re^{\Im}{}_{1} \cdot_{\mathcal{E}} \hbar_1^{\kappa} \oplus_{\mathcal{E}} \Re^{\Im}{}_{2} \cdot_{\mathcal{E}} \hbar_2^{\kappa} \oplus_{\mathcal{E}} \Re^{\Im}{}_{3} \cdot_{\mathcal{E}} \gamma_3 \oplus_{\mathcal{E}} \ldots \oplus_{\mathcal{E}} \Re^{\Im}{}_{n} \cdot_{\mathcal{E}} \hbar_n^{\kappa}.$$

In LDFEWA operator, we use $\Re^{\Im}$ as a WV and $\daleth^{\varsigma}{}_{\beth}$ are the LDFNs, where $\beth = 1, 2, \ldots, n$.

Theorem 1 ([28]) *Let $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth} \rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth} \rangle)$ be an agglomeration of LDFNs and $\Re^{\Im} = (\Re^{\Im}{}_{1}, \Re^{\Im}{}_{2}, \ldots, \Re^{\Im}{}_{n})^T$ be the WV with $\sum_{\beth=1}^{n} \Re^{\Im}{}_{\beth} = 1$. Then the LDFEWA operator can also be written as*

$$\begin{aligned} &LDFEWA(\hbar_1^{\kappa}, \hbar_2^{\kappa}, \ldots, \hbar_n^{\kappa}) \\ &= \Bigg(\Bigg\langle \frac{\prod_{\beth=1}^{n}(1+\zeta^{\tau}{}_{\beth})^{\Re^{\Im}{}_{\beth}} - \prod_{\beth=1}^{n}(1-\zeta^{\tau}{}_{\beth})^{\Re^{\Im}{}_{\beth}}}{\prod_{\beth=1}^{n}(1+\zeta^{\tau}{}_{\beth})^{\Re^{\Im}{}_{\beth}} + \prod_{\beth=1}^{n}(1-\zeta^{\tau}{}_{\beth})^{\Re^{\Im}{}_{\beth}}}, \\ &\frac{2\prod_{\beth=1}^{n} \eta^{\upsilon}{}_{\beth}^{\Re^{\Im}{}_{\beth}}}{\prod_{\beth=1}^{n}(2-\eta^{\upsilon}{}_{\beth})^{\Re^{\Im}{}_{\beth}} + \prod_{\beth=1}^{n}(\eta^{\upsilon}{}_{\beth})^{\Re^{\Im}{}_{\beth}}} \Bigg\rangle, \\ &\Bigg\langle \frac{\prod_{\beth=1}^{n}(1+\mathscr{J}^{\aleph}{}_{\beth})^{\Re^{\Im}{}_{\beth}} - \prod_{\beth=1}^{n}(1-\mathscr{J}^{\aleph}{}_{\beth})^{\Re^{\Im}{}_{\beth}}}{\prod_{\beth=1}^{n}(1+\mathscr{J}^{\aleph}{}_{\beth})^{\Re^{\Im}{}_{\beth}} + \prod_{\beth=1}^{n}(1-\mathscr{J}^{\aleph}{}_{\beth})^{\Re^{\Im}{}_{\beth}}}, \\ &\frac{2\prod_{\beth=1}^{n} \mathscr{C}^{\gamma}{}_{\beth}^{\Re^{\Im}{}_{\beth}}}{\prod_{\beth=1}^{n}(2-\mathscr{C}^{\gamma}{}_{\beth})^{\Re^{\Im}{}_{\beth}} + \prod_{\beth=1}^{n}(\mathscr{C}^{\gamma}{}_{\beth})^{\Re^{\Im}{}_{\beth}}} \Bigg\rangle \Bigg). \end{aligned}$$

Definition 9 ([28]) Consider $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth} \rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth} \rangle)$ is the agglomeration of LDFNs and $\Re^{\Im} = (\Re^{\Im}{}_{1}, \Re^{\Im}{}_{2}, \ldots, \Re^{\Im}{}_{n})^T$ be the WV with $\sum_{\beth=1}^{n} \Re^{\Im}{}_{\beth} = 1$. Then "linear Diophantine fuzzy Einstein weighted geometric (LDFEWG) operator" is defined as

$$LDFEWG(\hbar_1^{\kappa}, \hbar_2^{\kappa}, \hbar_3^{\kappa}, \ldots, \hbar_n^{\kappa}) =$$

$$\prod_{\beth=1}^{n} \Re^{\Im}{}_{\beth} \daleth^{\varsigma}{}_{\beth} = \Re^{\Im}{}_{1} \cdot_{\mathcal{E}} \hbar_1^{\kappa} \otimes_{\mathcal{E}} \Re^{\Im}{}_{2} \cdot_{\mathcal{E}} \hbar_2^{\kappa} \otimes_{\mathcal{E}} \Re^{\Im}{}_{3} \cdot_{\mathcal{E}} \hbar_3^{\kappa} \otimes_{\mathcal{E}} \ldots \otimes_{\mathcal{E}} \Re^{\Im}{}_{n} \cdot_{\mathcal{E}} \hbar_n^{\kappa}.$$

Theorem 2 *[[28]] Let $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth} \rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth} \rangle)$ be the agglomeration of LDFNs and $\Re^{\Im} = (\Re^{\Im}{}_{1}, \Re^{\Im}{}_{2}, \ldots, \Re^{\Im}{}_{n})^T$ be the WV with $\sum_{\beth=1}^{n} \Re^{\Im}{}_{\beth} = 1$. Then*

LDFEWG operator can also be written as

$$LDFEWG(\hbar_1^{\kappa}, \hbar_2^{\kappa}, \ldots, \hbar_n^{\kappa})$$
$$= \Bigg(\Bigg\langle \frac{2\prod_{\beth=1}^{n} {\zeta^{\tau}}_{\beth}^{\Re^{\Im}\beth}}{\prod_{\beth=1}^{n}(2-{\zeta^{\tau}}_{\beth})^{\Re^{\Im}\beth} + \prod_{\beth=1}^{n}({\zeta^{\tau}}_{\beth})^{\Re^{\Im}\beth}},$$
$$\frac{\prod_{\beth=1}^{n}(1+{\eta^{\upsilon}}_{\beth})^{\Re^{\Im}\beth} - \prod_{\beth=1}^{n}(1-{\eta^{\upsilon}}_{\beth})^{\Re^{\Im}\beth}}{\prod_{\beth=1}^{n}(1+{\eta^{\upsilon}}_{\beth})^{\Re^{\Im}\beth} + \prod_{\beth=1}^{n}(1-{\eta^{\upsilon}}_{\beth})^{\Re^{\Im}\beth}}\Bigg\rangle,$$
$$\Bigg\langle \frac{2\prod_{\beth=1}^{n} {\mathscr{J}^{\aleph}}_{\beth}^{\Re^{\Im}\beth}}{\prod_{\beth=1}^{n}(2-{\mathscr{J}^{\aleph}}_{\beth})^{\Re^{\Im}\beth} + \prod_{\beth=1}^{n}({\mathscr{J}^{\aleph}}_{\beth})^{\Re^{\Im}\beth}},$$
$$\frac{\prod_{\beth=1}^{n}(1+{\mathscr{C}^{\gamma}}_{\beth})^{\Re^{\Im}\beth} - \prod_{\beth=1}^{n}(1-{\mathscr{C}^{\gamma}}_{\beth})^{\Re^{\Im}\beth}}{\prod_{\beth=1}^{n}(1+{\mathscr{C}^{\gamma}}_{\beth})^{\Re^{\Im}\beth} + \prod_{\beth=1}^{n}(1-{\mathscr{C}^{\gamma}}_{\beth})^{\Re^{\Im}\beth}}\Bigg\rangle\Bigg)$$

Definition 10 ([27]) Assume that ${\daleth^{\varsigma}}_{\beth} = (\langle {\zeta^{\tau}}_{\beth}, {\eta^{\upsilon}}_{\beth}\rangle, \langle {\mathscr{J}^{\aleph}}_{\beth}, {\mathscr{C}^{\gamma}}_{\beth}\rangle)$ is the agglomeration of LDFNs, and LDFPWA : $\$^n \rightarrow \$$ is the mapping. If

$$\text{LDFPWA}({\daleth^{\varsigma}}_1, {\daleth^{\varsigma}}_2, \ldots {\daleth^{\varsigma}}_n) = \frac{\breve{\hbar}_1}{\sum_{\beth=1}^{n}\breve{\hbar}_{\beth}}{\daleth^{\varsigma}}_1 \oplus \frac{\breve{\hbar}_2}{\sum_{\beth=1}^{n}\breve{\hbar}_{\beth}}{\daleth^{\varsigma}}_2 \oplus \ldots, \oplus \frac{\breve{\hbar}_n}{\sum_{\beth=1}^{n}\breve{\hbar}_{\beth}}{\daleth^{\varsigma}}_n, \tag{14.1}$$

then the mapping LDFPWA is called "linear Diophantine fuzzy prioritized weighted averaging (LDFPWA) operator," where $\breve{\hbar}_{\beth} = \overline{\prod}_{k=1}^{j-1}\mathscr{H}({\daleth^{\varsigma}}_k)$ $(j = 2\ldots, n)$, $\breve{\hbar}_1 = 1$, and $\mathscr{H}({\daleth^{\varsigma}}_k)$ is the expectation score function of kth LDFN.

Theorem 3 ([27]) *Assuming that* ${\daleth^{\varsigma}}_{\beth} = (\langle {\zeta^{\tau}}_{\beth}, {\eta^{\upsilon}}_{\beth}\rangle, \langle {\mathscr{J}^{\aleph}}_{\beth}, {\mathscr{C}^{\gamma}}_{\beth}\rangle)$ *is the agglomeration of LDFNs, we can find LDFPWA by*

$$LDFPWA({\daleth^{\varsigma}}_1, {\daleth^{\varsigma}}_2, \ldots {\daleth^{\varsigma}}_n)$$
$$= \Bigg(\Bigg\langle 1 - \overline{\prod}_{\beth=1}^{n}(1-{\zeta^{\tau}}_{\beth})^{\frac{\breve{\hbar}_{\beth}}{\sum_{\beth=1}^{n}\breve{\hbar}_{\beth}}}, \overline{\prod}_{\beth=1}^{n}{\eta^{\upsilon}}_{\beth}^{\frac{\breve{\hbar}_{\beth}}{\sum_{\beth=1}^{n}\breve{\hbar}_{\beth}}}\Bigg\rangle,$$
$$\Bigg\langle 1 - \overline{\prod}_{\beth=1}^{n}(1-{\mathscr{J}^{\aleph}}_{\beth})^{\frac{\breve{\hbar}_{\beth}}{\sum_{\beth=1}^{n}\breve{\hbar}_{\beth}}}, \overline{\prod}_{\beth=1}^{n}{\mathscr{C}^{\gamma}}_{\beth}^{\frac{\breve{\hbar}_{\beth}}{\sum_{\beth=1}^{n}\breve{\hbar}_{\beth}}}\Bigg\rangle\Bigg). \tag{14.2}$$

Definition 11 ([27]) Assume that ${\daleth^{\varsigma}}_{\beth} = (\langle {\zeta^{\tau}}_{\beth}, {\eta^{\upsilon}}_{\beth}\rangle, \langle {\mathscr{J}^{\aleph}}_{\beth}, {\mathscr{C}^{\gamma}}_{\beth}\rangle)$ is the agglomeration of LDFNs and $LDFPWG : \$^n \rightarrow \$$ is the mapping. If

$$\text{LDFPWG}({\daleth^{\varsigma}}_1, {\daleth^{\varsigma}}_2, \ldots {\daleth^{\varsigma}}_n) = {\daleth^{\varsigma}}_1^{\frac{\breve{\hbar}_1}{\sum_{\beth=1}^{n}\breve{\hbar}_{\beth}}} \otimes {\daleth^{\varsigma}}_2^{\frac{\breve{\hbar}_2}{\sum_{\beth=1}^{n}\breve{\hbar}_{\beth}}} \otimes \ldots, \otimes {\daleth^{\varsigma}}_n^{\frac{\breve{\hbar}_n}{\sum_{\beth=1}^{n}\breve{\hbar}_{\beth}}}, \tag{14.3}$$

then the mapping LDFPWG is called “linear Diophantine fuzzy prioritized weighted geometric (LDFPWG) operator.”

Theorem 4 ([27]) *Assuming that* $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth}\rangle)$ *is the agglomeration of LDFNs, we can find LDFPWG by*

$$
\begin{aligned}
&LDFPWG(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{n}) \\
&= \left(\left\langle \prod_{\beth=1}^{n} \zeta^{\tau}{}_{\beth}^{\frac{\check{\hbar}_{\beth}}{\sum_{\beth=1}^{n}\check{\hbar}_{\beth}}}, 1 - \prod_{\beth=1}^{n}(1-\eta^{\upsilon}{}_{\beth})^{\frac{\check{\hbar}_{\beth}}{\sum_{\beth=1}^{n}\check{\hbar}_{\beth}}} \right\rangle, \right. \\
&\left. \left\langle \prod_{\beth=1}^{n} \mathscr{J}^{\aleph}{}_{\beth}^{\frac{\check{\hbar}_{\beth}}{\sum_{\beth=1}^{n}\check{\hbar}_{\beth}}}, 1 - \prod_{\beth=1}^{n}(1-\mathscr{C}^{\gamma}{}_{\beth})^{\frac{\check{\hbar}_{\beth}}{\sum_{\beth=1}^{n}\check{\hbar}_{\beth}}} \right\rangle\right).
\end{aligned}
\tag{14.4}
$$

Definition 12 ([56]) Let $\daleth^{\varsigma}{}_{\beth} = \left(\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth}\rangle\right)$ be the agglomeration of LDFNs and LDFFWA: $\mathscr{F}^{n} \to \mathscr{F}$ be a n dimension mapping. If

$$
\text{LDFFWA}(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{e}) = \left(\Re^{\Im}{}_{1} * \daleth^{\varsigma}{}_{1} \tilde{\oplus} \Re^{\Im}{}_{2} * \daleth^{\varsigma}{}_{2} \tilde{\oplus} \ldots, \tilde{\oplus} \Re^{\Im}{}_{e} * \daleth^{\varsigma}{}_{e}\right),
\tag{14.5}
$$

then the mapping LDFFWA is called “linear Diophantine fuzzy fairly weighted averaging (LDFFWA) operator,” and here $\Re^{\Im}{}_{i}$ is the weight vector (WV) of $\daleth^{\varsigma}{}_{i}$ with $\Re^{\Im}{}_{i} > 0$ and $\sum_{i=1}^{e} \Re^{\Im}{}_{i} = 1$.

Theorem 5 ([56]) *Let* $\daleth^{\varsigma}{}_{\beth} = \left(\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth}\rangle\right)$ *be the agglomeration of LDFNs, and we can also find LDFFWA by*

$$
LDFFWA(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots, \daleth^{\varsigma}{}_{e})
$$

$$
= \left(\begin{array}{l}
\left\langle \frac{1}{2} \frac{\prod_{i=1}^{e}(\zeta^{\tau}{}_{i})^{\Re^{\Im}{}_{i}}}{\prod_{i=1}^{e}(\zeta^{\tau}{}_{i})^{\Re^{\Im}{}_{i}} + \prod_{i=1}^{e}(\eta^{\upsilon}{}_{i})^{\Re^{\Im}{}_{i}}} \times \left(1 + \prod_{i=1}^{e}\left(2 - \zeta^{\tau}{}_{i} - \eta^{\upsilon}{}_{i}\right)^{\Re^{\Im}{}_{i}}\right), \right. \\
\left. \frac{1}{2} \frac{\prod_{i=1}^{e}(\eta^{\upsilon}{}_{i})^{\Re^{\Im}{}_{i}}}{\prod_{i=1}^{e}(\zeta^{\tau}{}_{i})^{\Re^{\Im}{}_{i}} + \prod_{i=1}^{e}(\eta^{\upsilon}{}_{i})^{\Re^{\Im}{}_{i}}} \times \left(1 + \prod_{i=1}^{e}\left(2 - \zeta^{\tau}{}_{i} - \eta^{\upsilon}{}_{i}\right)^{\Re^{\Im}{}_{i}}\right)\right\rangle, \\
\left\langle \frac{\prod_{i=1}^{e}(\mathscr{J}^{\aleph}{}_{i})^{\Re^{\Im}{}_{i}}}{\prod_{i=1}^{e}(\mathscr{J}^{\aleph}{}_{i})^{\Re^{\Im}{}_{i}} + \prod_{i=1}^{e}(\mathscr{C}^{\gamma}{}_{i})^{\Re^{\Im}{}_{i}}} \times \left(1 - \prod_{i=1}^{e}\left(1 - \mathscr{J}^{\aleph}{}_{i} - \mathscr{C}^{\gamma}{}_{i}\right)^{\Re^{\Im}{}_{i}}\right), \right. \\
\left. \frac{\prod_{i=1}^{e}(\mathscr{C}^{\gamma}{}_{i})^{\Re^{\Im}{}_{i}}}{\prod_{i=1}^{e}(\mathscr{J}^{\aleph}{}_{i})^{\Re^{\Im}{}_{i}} + \prod_{i=1}^{e}(\mathscr{C}^{\gamma}{}_{i})^{\Re^{\Im}{}_{i}}} \times \left(1 - \prod_{i=1}^{e}\left(1 - \mathscr{J}^{\aleph}{}_{i} - \mathscr{C}^{\gamma}{}_{i}\right)^{\Re^{\Im}{}_{i}}\right)\right\rangle,
\end{array}\right)
$$

where $\Re^{\Im}{}_{i}$ *is the WV of* $\daleth^{\varsigma}{}_{i}$ *with* $\Re^{\Im}{}_{i} > 0$ *and* $\sum_{i=1}^{e} \Re^{\Im}{}_{i} = 1$.

Definition 13 ([56]) Let $\daleth^{\varsigma}{}_{\beth} = \left(\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth}\rangle\right)$ be the agglomeration of LDFNs and LDFFOWA: $\mathscr{F}^{n} \to \mathscr{F}$ be a n dimension mapping. If

$$\text{LDFFOWA}(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_e)$$
$$= \left(\Re^{\Im}{}_1 * \daleth^{\varsigma}{}_{\tau(1)} \tilde{\oplus} \Re^{\Im}{}_2 * \daleth^{\varsigma}{}_{\tau(2)} \tilde{\oplus} \ldots, \tilde{\oplus} \Re^{\Im}{}_e * \daleth^{\varsigma}{}_{\tau(e)}\right), \quad (14.6)$$

then the mapping LDFFOWA is called "linear Diophantine fuzzy fairly ordered weighted averaging (LDFFOWA) operator," and here $\Re^{\Im}{}_i$ is the WV of $\daleth^{\varsigma}{}_i$ with $\Re^{\Im}{}_i > 0$ and $\sum_{i=1}^{e} \Re^{\Im}{}_i = 1$.
$\zeta^{\tau} : 1, 2, 3, \ldots\ldots, n \to 1, 2, 3, \ldots\ldots, n$ is a permutation map s.t. $\daleth^{\varsigma}{}_{\tau(i-1)} \geq \daleth^{\varsigma}{}_{\tau(i)}$.

Theorem 6 ([56]) *Let $\daleth^{\varsigma}{}_{\beth} = \left(\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth}\rangle\right)$ be the agglomeration of LDFNs, and we can also find LDFFOWA by*

$$LDFFOWA(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots, \daleth^{\varsigma}{}_e)$$

$$= \begin{pmatrix} \left\langle \frac{1}{2} \frac{\prod_{i=1}^{e}\left(\zeta^{\tau}{}_{\tau(i)}\right)^{\Re^{\Im}{}_i}}{\prod_{i=1}^{e}\left(\zeta^{\tau}{}_{\tau(i)}\right)^{\Re^{\Im}{}_{\tau(i)}} + \prod_{i=1}^{e}\left(\eta^{\upsilon}{}_{\tau(i)}\right)^{\Re^{\Im}{}_i}} \times \left(1 + \prod_{i=1}^{e}\left(2 - \zeta^{\tau}{}_{\tau(i)} - \eta^{\upsilon}{}_{\tau(i)}\right)^{\Re^{\Im}{}_i}\right), \right. \\ \left. \frac{1}{2} \frac{\prod_{i=1}^{e}\left(\eta^{\upsilon}{}_{\tau(i)}\right)^{\Re^{\Im}{}_i}}{\prod_{i=1}^{e}\left(\zeta^{\tau}{}_{\tau(i)}\right)^{\Re^{\Im}{}_i} + \prod_{i=1}^{e}\left(\eta^{\upsilon}{}_{\tau(i)}\right)^{\Re^{\Im}{}_i}} \times \left(1 + \prod_{i=1}^{e}\left(2 - \zeta^{\tau}{}_{\tau(i)} - \eta^{\upsilon}{}_{\tau(i)}\right)^{\Re^{\Im}{}_i}\right) \right\rangle, \\ \left\langle \frac{\prod_{i=1}^{e}\left(\mathscr{J}^{\aleph}{}_{\tau(i)}\right)^{\Re^{\Im}{}_i}}{\prod_{i=1}^{e}\left(\mathscr{J}^{\aleph}{}_{\tau(i)}\right)^{\Re^{\Im}{}_{\tau(i)}} + \prod_{i=1}^{e}\left(\mathscr{C}^{\gamma}{}_{\tau(i)}\right)^{\Re^{\Im}{}_i}} \times \left(1 - \prod_{i=1}^{e}\left(1 - \mathscr{J}^{\aleph}{}_{\tau(i)} - \mathscr{C}^{\gamma}{}_{\tau(i)}\right)^{\Re^{\Im}{}_i}\right), \right. \\ \left. \frac{\prod_{i=1}^{e}\left(\mathscr{C}^{\gamma}{}_{\tau(i)}\right)^{\Re^{\Im}{}_i}}{\prod_{i=1}^{e}\left(\mathscr{J}^{\aleph}{}_{\tau(i)}\right)^{\Re^{\Im}{}_i} + \prod_{i=1}^{e}\left(\mathscr{C}^{\gamma}{}_{\tau(i)}\right)^{\Re^{\Im}{}_i}} \times \left(1 - \prod_{i=1}^{e}\left(1 - \mathscr{J}^{\aleph}{}_{\tau(i)} - \mathscr{C}^{\gamma}{}_{\tau(i)}\right)^{\Re^{\Im}{}_i}\right) \right\rangle \end{pmatrix},$$

where $\Re^{\Im}{}_i$ is the WV of $\daleth^{\varsigma}{}_i$ with $\Re^{\Im}{}_i > 0$ and $\sum_{i=1}^{e} \Re^{\Im}{}_i = 1$.

Definition 14 ([28]) Consider $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth}\rangle)$ the agglomeration of LDFNs and $\Re^{\Im} = (\Re^{\Im}{}_1, \Re^{\Im}{}_2, \ldots, \Re^{\Im}{}_n)^T$ be the weight vector (WV) with $\sum_{\beth=1}^{n} \Re^{\Im}{}_{\beth} = 1$. Then "linear Diophantine fuzzy Einstein weighted average (LDFEWA) operator" is defined as

$$LDFEWA(\hbar_1^{\kappa}, \hbar_2^{\kappa}, \hbar_3^{\kappa}, \ldots, \hbar_n^{\kappa}) =$$
$$\sum_{\beth=1}^{n} \Re^{\Im}{}_{\Lambda} \daleth^{\varsigma}{}_{\beth} = \Re^{\Im}{}_1 \cdot_{\mathcal{E}} \hbar_1^{\kappa} \oplus_{\mathcal{E}} \Re^{\Im}{}_2 \cdot_{\mathcal{E}} \hbar_2^{\kappa} \oplus_{\mathcal{E}} \Re^{\Im}{}_3 \cdot_{\mathcal{E}} \gamma_3 \oplus_{\mathcal{E}} \ldots \oplus_{\mathcal{E}} \Re^{\Im}{}_n \cdot_{\mathcal{E}} \hbar_n^{\kappa}.$$

In LDFEWA operator, we use $\Re^{\Im}$ as a WV and $\daleth^{\varsigma}{}_{\beth}$ are the LDFNs, where $\beth = 1, 2, \ldots, n$.

Theorem 7 ([28]) *Let $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth}\rangle)$ be an agglomeration of LDFNs and $\Re^{\Im} = (\Re^{\Im}{}_1, \Re^{\Im}{}_2, \ldots, \Re^{\Im}{}_n)^T$ be the WV with $\sum_{\beth=1}^{n} \Re^{\Im}{}_{\beth} = 1$. Then the LDFEWA operator can also be written as*

$$LDFEWA(\hbar_1^{\kappa}, \hbar_2^{\kappa}, \ldots, \hbar_n^{\kappa}) = \left(\left\langle \frac{\prod_{\beth=1}^{n}(1+\zeta^{\tau}{}_{\beth})^{\Re^{\Im}{}_{\beth}} - \prod_{\beth=1}^{n}(1-\zeta^{\tau}{}_{\beth})^{\Re^{\Im}{}_{\beth}}}{\prod_{\beth=1}^{n}(1+\zeta^{\tau}{}_{\beth})^{\Re^{\Im}{}_{\beth}} + \prod_{\beth=1}^{n}(1-\zeta^{\tau}{}_{\beth})^{\Re^{\Im}{}_{\beth}}}, \frac{2\prod_{\beth=1}^{n} \eta^{\upsilon}{}_{\beth}^{\Re^{\Im}{}_{\beth}}}{\prod_{\beth=1}^{n}(2-\eta^{\upsilon}{}_{\beth})^{\Re^{\Im}{}_{\beth}} + \prod_{\beth=1}^{n}(\eta^{\upsilon}{}_{\beth})^{\Re^{\Im}{}_{\beth}}} \right\rangle, \left\langle \frac{\prod_{\beth=1}^{n}(1+\mathscr{J}^{\aleph}{}_{\beth})^{\Re^{\Im}{}_{\beth}} - \prod_{\beth=1}^{n}(1-\mathscr{J}^{\aleph}{}_{\beth})^{\Re^{\Im}{}_{\beth}}}{\prod_{\beth=1}^{n}(1+\mathscr{J}^{\aleph}{}_{\beth})^{\Re^{\Im}{}_{\beth}} + \prod_{\beth=1}^{n}(1-\mathscr{J}^{\aleph}{}_{\beth})^{\Re^{\Im}{}_{\beth}}}, \frac{2\prod_{\beth=1}^{n} \mathscr{C}^{\gamma}{}_{\beth}^{\Re^{\Im}{}_{\beth}}}{\prod_{\beth=1}^{n}(2-\mathscr{C}^{\gamma}{}_{\beth})^{\Re^{\Im}{}_{\beth}} + \prod_{\beth=1}^{n}(\mathscr{C}^{\gamma}{}_{\beth})^{\Re^{\Im}{}_{\beth}}} \right\rangle\right).$$

Definition 15 ([28]) Consider $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth}\rangle)$ is the agglomeration of LDFNs and $\Re^{\Im} = (\Re^{\Im}{}_1, \Re^{\Im}{}_2, \ldots, \Re^{\Im}{}_n)^T$ be the WV with $\sum_{\beth=1}^{n} \Re^{\Im}{}_{\beth} = 1$. Then "linear Diophantine fuzzy Einstein weighted geometric (LDFEWG) operator" is defined as

$$LDFEWG(\hbar_1^{\kappa}, \hbar_2^{\kappa}, \hbar_3^{\kappa}, \ldots, \hbar_n^{\kappa}) = \prod_{\beth=1}^{n} \Re^{\Im}{}_{\beth} \daleth^{\varsigma}{}_{\beth} = \Re^{\Im}{}_1 \cdot_{\mathcal{E}} \hbar_1^{\kappa} \otimes_{\mathcal{E}} \Re^{\Im}{}_2 \cdot_{\mathcal{E}} \hbar_2^{\kappa} \otimes_{\mathcal{E}} \Re^{\Im}{}_3 \cdot_{\mathcal{E}} \hbar_3^{\kappa} \otimes_{\mathcal{E}} \ldots \otimes_{\mathcal{E}} \Re^{\Im}{}_n \cdot_{\mathcal{E}} \hbar_n^{\kappa}.$$

Theorem 8 ([28]) *Let $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth}\rangle)$ be the agglomeration of LDFNs and $\Re^{\Im} = (\Re^{\Im}{}_1, \Re^{\Im}{}_2, \ldots, \Re^{\Im}{}_n)^T$ be the WV with $\sum_{\beth=1}^{n} \Re^{\Im}{}_{\beth} = 1$. Then LDFEWG operator can also be written as*

$$LDFEWG(\hbar_1^{\kappa}, \hbar_2^{\kappa}, \ldots, \hbar_n^{\kappa}) = \left(\left\langle \frac{2\prod_{\beth=1}^{n} \zeta^{\tau}{}_{\beth}^{\Re^{\Im}{}_{\beth}}}{\prod_{\beth=1}^{n}(2-\zeta^{\tau}{}_{\beth})^{\Re^{\Im}{}_{\beth}} + \prod_{\beth=1}^{n}(\zeta^{\tau}{}_{\beth})^{\Re^{\Im}{}_{\beth}}}, \frac{\prod_{\beth=1}^{n}(1+\eta^{\upsilon}{}_{\beth})^{\Re^{\Im}{}_{\beth}} - \prod_{\beth=1}^{n}(1-\eta^{\upsilon}{}_{\beth})^{\Re^{\Im}{}_{\beth}}}{\prod_{\beth=1}^{n}(1+\eta^{\upsilon}{}_{\beth})^{\Re^{\Im}{}_{\beth}} + \prod_{\beth=1}^{n}(1-\eta^{\upsilon}{}_{\beth})^{\Re^{\Im}{}_{\beth}}} \right\rangle,$$

$$\left\langle \frac{2\prod_{\beth=1}^{n} \mathscr{I}^{\aleph}{}_{\beth}^{\mathfrak{R}^{\mathfrak{I}}{}_{\beth}}}{\prod_{\beth=1}^{n}(2-\mathscr{I}^{\aleph}{}_{\beth})^{\mathfrak{R}^{\mathfrak{I}}{}_{\beth}}+\prod_{\beth=1}^{n}(\mathscr{I}^{\aleph}{}_{\beth})^{\mathfrak{R}^{\mathfrak{I}}{}_{\beth}}}, \right.$$
$$\left. \frac{\prod_{\beth=1}^{n}(1+\mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{R}^{\mathfrak{I}}{}_{\beth}}-\prod_{\beth=1}^{n}(1-\mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{R}^{\mathfrak{I}}{}_{\beth}}}{\prod_{\beth=1}^{n}(1+\mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{R}^{\mathfrak{I}}{}_{\beth}}+\prod_{\beth=1}^{n}(1-\mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{R}^{\mathfrak{I}}{}_{\beth}}} \right\rangle\Bigg).$$

AOs are used in a variety of fields to summarize and analyze large sets of data. They are commonly used in business and finance to summarize financial data, in computer science and programming to analyze log files and performance metrics, and in data science and machine learning to extract insights from large datasets.

In business and finance, AOs are used to summarize financial data such as sales and revenue. For example, a company may use an operator to calculate the total revenue for a particular product or product line. This information can then be used to make decisions about pricing, production, and marketing.

In computer science and programming, AOs are used to analyze log files and performance metrics. For example, a web developer may use an operator to calculate the average response time of a web server or the number of requests per second. This information can be used to identify performance bottlenecks and optimize the performance of the system.

In data science and machine learning, AOs are used to extract insights from large datasets. For example, a data scientist may use an operator to calculate the average of a particular variable in a dataset. This information can be used to identify patterns and trends in the data, which can inform decisions about which variables to include in a model or which groups to target in a marketing campaign.

In the field of natural language processing, AOs are used to extract insights from text data. For example, a researcher may use an operator to calculate the most common words or phrases in a dataset of text. This information can be used to identify topics or themes in the data, which can inform decisions about which algorithms to use for text classification or sentiment analysis.

In bioinformatics, AOs are used to summarize and analyze large sets of genetic data. For example, a researcher may use an operator to calculate the frequency of a particular genetic variant in a population. This information can be used to identify genetic risk factors for diseases and inform drug development.

In general, AOs are a powerful tool for extracting insights from large sets of data. They can be used to summarize data, identify patterns and trends, and inform decisions across a wide range of fields.

3 Linear Diophantine Fuzzy Aggregation Operators

In this section, we discussed “linear Diophantine fuzzy weighted average (LDFWA) operator, linear Diophantine fuzzy ordered weighted average (LDFOWA) operator, linear Diophantine fuzzy weighted geometric (LDFWG) operator and linear Diophantine fuzzy weighted ordered geometric (LDFOWG) operator."

3.1 LDFWA Operator

Definition 16 Consider $\daleth^{\varsigma}{}_{\gimel} = (\langle \zeta^{\tau}{}_{\gimel}, \eta^{\upsilon}{}_{\gimel}\rangle, \langle \mathscr{J}^{\aleph}{}_{\gimel}, \mathscr{C}^{\gamma}{}_{\gimel}\rangle)$ is the agglomeration of LDFNs, and LDFWA : $\$^n \rightarrow \$$ be the mapping.

$$\text{LDFWA}(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) = \mathfrak{P}^{\gamma}{}_1 \daleth^{\varsigma}{}_1 \oplus \mathfrak{P}^{\gamma}{}_2 \daleth^{\varsigma}{}_2 \oplus \ldots, \oplus \mathfrak{P}^{\gamma}{}_n \daleth^{\varsigma}{}_n. \quad (14.7)$$

Then LDFWA is known as LDFWA operator, where $(\mathfrak{P}^{\gamma}{}_1, \mathfrak{P}^{\gamma}{}_2, \ldots, \mathfrak{P}^{\gamma}{}_n)$ be the weight vector (WV) with the constraint $\mathfrak{P}^{\gamma}{}_{\gimel} > 0$ and $\sum_{h=1}^{n} \mathfrak{P}^{\gamma}{}_{\gimel} = 1$.

We also evaluate LDFWA operator by the following theorem.

Theorem 9 *Consider* $\daleth^{\varsigma}{}_{\gimel} = (\langle \zeta^{\tau}{}_{\gimel}, \eta^{\upsilon}{}_{\gimel}\rangle, \langle \mathscr{J}^{\aleph}{}_{\gimel}, \mathscr{C}^{\gamma}{}_{\gimel}\rangle)$ *is the agglomeration of LDFNs, and we can find LDFWA by*

$$\begin{aligned} &LDFWA(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) \\ &= \left(\left\langle 1 - \overline{\prod}_{\gimel=1}^{n} (1 - \zeta^{\tau}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \overline{\prod}_{\gimel=1}^{n} \eta^{\upsilon}{}_{\gimel}^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle, \right. \\ &\left. \left\langle 1 - \overline{\prod}_{\gimel=1}^{n} (1 - \mathscr{J}^{\aleph}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \overline{\prod}_{\gimel=1}^{n} \mathscr{C}^{\gamma}{}_{\gimel}^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle \right). \end{aligned} \quad (14.8)$$

Proof It is quite simple for the first assertion to come before Definition 17 and Theorem 13. The following instances demonstrate this point further:

$$\begin{aligned} &\text{LDFWA}(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) \\ &= \left(\mathfrak{P}^{\gamma} 1 \daleth^{\varsigma}{}_1 \oplus \mathfrak{P}^{\gamma}{}_2 \daleth^{\varsigma}{}_2 \oplus \ldots, \mathfrak{P}^{\gamma}{}_n \daleth^{\varsigma}{}_n \right) \\ &= \left(\left\langle 1 - \overline{\prod}_{\gimel=1}^{n} (1 - \zeta^{\tau}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \overline{\prod}_{\gimel=1}^{n} \eta^{\upsilon}{}_{\gimel}^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle, \right. \\ &\left. \left\langle \overline{\prod}_{\gimel=1}^{n} (1 - \mathscr{J}^{\aleph}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \overline{\prod}_{\gimel=1}^{n} \mathscr{C}^{\gamma}{}_{\gimel}^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle \right). \end{aligned}$$

In order to demonstrate the validity of this theorem, we turned to mathematics induction.

For $n = 2$

$$\mathfrak{P}^{\gamma}{}_1 ℸ^{\varsigma}{}_1 = \left(\left\langle 1-(1-\zeta^{\tau}{}_1)^{\mathfrak{P}^{\gamma}{}_1}, \eta^{\upsilon}{}_1^{\mathfrak{P}^{\gamma}{}_1}\right\rangle, \left\langle 1-(1-\mathscr{I}^{\aleph}{}_1)^{\mathfrak{P}^{\gamma}{}_1}, \mathscr{C}^{\gamma}{}_1^{\mathfrak{P}^{\gamma}{}_1}\right\rangle\right)$$

$$\mathfrak{P}^{\gamma}{}_2 ℸ^{\varsigma}{}_2 = \left(\left\langle 1-(1-\zeta^{\tau}{}_2)^{\mathfrak{P}^{\gamma}{}_1}, \eta^{\upsilon}{}_2^{\mathfrak{P}^{\gamma}{}_1}\right\rangle, \left\langle 1-(1-\mathscr{I}^{\aleph}{}_2)^{\mathfrak{P}^{\gamma}{}_1}, \mathscr{C}^{\gamma}{}_2^{\mathfrak{P}^{\gamma}{}_1}\right\rangle\right).$$

Then

$$\mathfrak{P}^{\gamma}{}_1 ℸ^{\varsigma}{}_1 \oplus \mathfrak{P}^{\gamma}{}_2 ℸ^{\varsigma}{}_2$$

$$= \left(\left\langle 1-(1-\zeta^{\tau}{}_1)^{\mathfrak{P}^{\gamma}{}_1}, \eta^{\upsilon}{}_1^{\mathfrak{P}^{\gamma}{}_1}\right\rangle, \left\langle 1-(1-\mathscr{I}^{\aleph}{}_1)^{\mathfrak{P}^{\gamma}{}_1}, \mathscr{C}^{\gamma}{}_1^{\mathfrak{P}^{\gamma}{}_1}\right\rangle\right) \oplus$$

$$\left(\left\langle 1-(1-\zeta^{\tau}{}_2)^{\mathfrak{P}^{\gamma}{}_1}, \eta^{\upsilon}{}_2^{\mathfrak{P}^{\gamma}{}_1}\right\rangle, \left\langle 1-(1-\mathscr{I}^{\aleph}{}_2)^{\mathfrak{P}^{\gamma}{}_1}, \mathscr{C}^{\gamma}{}_2^{\mathfrak{P}^{\gamma}{}_1}\right\rangle\right)$$

$$= \left(\left\langle 1-(1-\zeta^{\tau}{}_1)^{\mathfrak{P}^{\gamma}{}_1} + 1-(1-\zeta^{\tau}{}_2)^{\mathfrak{P}^{\gamma}{}_1} - \left((1-(1-\zeta^{\tau}{}_1)^{\mathfrak{P}^{\gamma}{}_1}\right)\right.\right.$$

$$\left.\left((1-(1-\zeta^{\tau}{}_2)^{\mathfrak{P}^{\gamma}{}_1}\right), \eta^{\upsilon}{}_1^{\mathfrak{P}^{\gamma}{}_1}.\eta^{\upsilon}{}_2^{\mathfrak{P}^{\gamma}{}_1}\right\rangle, \left\langle 1-(1-\mathscr{I}^{\aleph}{}_1)^{\mathfrak{P}^{\gamma}{}_1}+1-(1-\mathscr{I}^{\aleph}{}_2)^{\mathfrak{P}^{\gamma}{}_1}\right.$$

$$\left.\left. - \left((1-(1-\mathscr{I}^{\aleph}{}_1)^{\mathfrak{P}^{\gamma}{}_1}\right)\left(1-(1-\mathscr{I}^{\aleph}{}_2)^{\mathfrak{P}^{\gamma}{}_1}\right), \mathscr{C}^{\gamma}{}_1^{\mathfrak{P}^{\gamma}{}_1}.\mathscr{C}^{\gamma}{}_2^{\mathfrak{P}^{\gamma}{}_1}\right\rangle\right)$$

$$= \left(\left\langle 1-(1-\zeta^{\tau}{}_1)^{\mathfrak{P}^{\gamma}{}_1}(1-\zeta^{\tau}{}_2)^{\mathfrak{P}^{\gamma}{}_1}, \eta^{\upsilon}{}_1^{\mathfrak{P}^{\gamma}{}_1}.\eta^{\upsilon}{}_2^{\mathfrak{P}^{\gamma}{}_1}\right\rangle,\right.$$

$$\left.\left\langle 1-(1-\mathscr{I}^{\aleph}{}_1)^{\mathfrak{P}^{\gamma}{}_1}(1-\mathscr{I}^{\aleph}{}_2)^{\mathfrak{P}^{\gamma}{}_1}, \mathscr{C}^{\gamma}{}_1^{\mathfrak{P}^{\gamma}{}_1}.\mathscr{C}^{\gamma}{}_2^{\mathfrak{P}^{\gamma}{}_1}\right\rangle\right)$$

$$= \left(\left\langle 1-\overline{\prod}_{ℶ=1}^{2}(1-\zeta^{\tau}{}_ℶ)^{\mathfrak{P}^{\gamma}{}_ℶ}, \overline{\prod}_{ℶ=1}^{2}\eta^{\upsilon}{}_ℶ^{\mathfrak{P}^{\gamma}{}_ℶ}\right\rangle,\right.$$

$$\left.\left\langle 1-\overline{\prod}_{ℶ=1}^{2}(1-\mathscr{I}^{\aleph}{}_ℶ)^{\mathfrak{P}^{\gamma}{}_ℶ}, \overline{\prod}_{ℶ=1}^{2}\mathscr{C}^{\gamma}{}_ℶ^{\mathfrak{P}^{\gamma}{}_ℶ}\right\rangle\right).$$

This demonstrates that Eq. (14.10) is correct for the value of n equal to two; now assume that Eq. (14.10) is accurate for the value of n equal to k, i.e.,

$$\text{LDFWA}(ℸ^{\varsigma}{}_1, ℸ^{\varsigma}{}_2, \ldots ℸ^{\varsigma}{}_k)$$

$$= \left(\left\langle 1-\overline{\prod}_{\beth=1}^{k}(1-\zeta^{\tau}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}, \overline{\prod}_{\beth=1}^{k}\eta^{\upsilon\mathfrak{P}^{\gamma}{}_{\beth}}_{\beth}\right\rangle, \left\langle 1-\overline{\prod}_{\beth=1}^{k}(1-\mathscr{I}^{\aleph}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}, \overline{\prod}_{\beth=1}^{k}\mathscr{C}^{\gamma\mathfrak{P}^{\gamma}{}_{\beth}}_{\beth}\right\rangle\right).$$

Now that $n = k + 1$, according to the operational laws that govern LDFNs, we obtain

$$\text{LDFWA}(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{k+1}) = \text{LDFWA}(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{k}) \oplus \mathfrak{P}^{\gamma}{}_{\beth}\daleth^{\varsigma}{}_{k+1}$$

$$= \left(\left\langle 1-\overline{\prod}_{\beth=1}^{k}(1-\zeta^{\tau}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}, \overline{\prod}_{\beth=1}^{k}\eta^{\upsilon\mathfrak{P}^{\gamma}{}_{\beth}}_{\beth}\right\rangle, \left\langle 1-\overline{\prod}_{\beth=1}^{k}(1-\mathscr{I}^{\aleph}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}, \overline{\prod}_{\beth=1}^{k}\mathscr{C}^{\gamma\mathfrak{P}^{\gamma}{}_{\beth}}_{\beth}\right\rangle\right) \oplus$$

$$\left(\left\langle 1-(1-\zeta^{\tau}{}_{k+1})^{\mathfrak{P}^{\gamma}{}_{k+1}}, \eta^{\upsilon\mathfrak{P}^{\gamma}{}_{k+1}}_{k+1}\right\rangle, \left\langle 1-(1-\mathscr{I}^{\aleph}{}_{k+1})^{\mathfrak{P}^{\gamma}{}_{k+1}}, \mathscr{C}^{\gamma\mathfrak{P}^{\gamma}{}_{k+1}}_{k+1}\right\rangle\right)$$

$$= \left(\left\langle 1-\overline{\prod}_{\beth=1}^{k}(1-\zeta^{\tau}{}_{k})^{\mathfrak{P}^{\gamma}{}_{\beth}} + 1-(1-\zeta^{\tau}{}_{k+1})^{\mathfrak{P}^{\gamma}{}_{k+1}} - \left(1-\overline{\prod}_{\beth=1}^{k}(1-\zeta^{\tau}{}_{k})^{\mathfrak{P}^{\gamma}{}_{\beth}}\right)\left(1-(1-\zeta^{\tau}{}_{k+1})^{\mathfrak{P}^{\gamma}{}_{k+1}}\right),\right.\right.$$

$$\left.\overline{\prod}_{\beth=1}^{k}\eta^{\upsilon\mathfrak{P}^{\gamma}{}_{\beth}}_{k}.\eta^{\upsilon\mathfrak{P}^{\gamma}{}_{k+1}}_{k+1}\right\rangle, \left\langle 1-\overline{\prod}_{\beth=1}^{k}(1-\mathscr{I}^{\aleph}{}_{k})^{\mathfrak{P}^{\gamma}{}_{\beth}} + 1-(1-\mathscr{I}^{\aleph}{}_{k+1})^{\mathfrak{P}^{\gamma}{}_{k+1}}\right.$$

$$\left.\left. - \left(1-\overline{\prod}_{\beth=1}^{k}(1-\mathscr{I}^{\aleph}{}_{k})^{\mathfrak{P}^{\gamma}{}_{\beth}}\right)\left(1-(1-\mathscr{I}^{\aleph}{}_{k+1})^{\mathfrak{P}^{\gamma}{}_{k+1}}\right), \overline{\prod}_{\beth=1}^{k}\mathscr{C}^{\gamma\mathfrak{P}^{\gamma}{}_{\beth}}_{k}.\mathscr{C}^{\gamma\mathfrak{P}^{\gamma}{}_{k+1}}_{k+1}\right\rangle\right)$$

$$= \left(\left\langle 1-\overline{\prod}_{\beth=1}^{k}(1-\zeta^{\tau}{}_{k})^{\mathfrak{P}^{\gamma}{}_{\beth}}(1-\zeta^{\tau}{}_{k+1})^{k+1}, \overline{\prod}_{\beth=1}^{k}\eta^{\upsilon\mathfrak{P}^{\gamma}{}_{\beth}}_{k}.\eta^{\upsilon\mathfrak{P}^{\gamma}{}_{k+1}}_{k+1}\right\rangle,\right.$$

$$\left.\left\langle 1-\overline{\prod}_{\beth=1}^{k}(1-\mathscr{I}^{\aleph}{}_{k})^{\mathfrak{P}^{\gamma}{}_{\beth}}(1-\mathscr{I}^{\aleph}{}_{k+1})^{k+1}, \overline{\prod}_{\beth=1}^{k}\mathscr{C}^{\gamma\mathfrak{P}^{\gamma}{}_{\beth}}_{k}.\mathscr{C}^{\gamma\mathfrak{P}^{\gamma}{}_{k+1}}_{k+1}\right\rangle\right)$$

$$= \left(\left\langle 1-\overline{\prod}_{\beth=1}^{k+1}(1-\zeta^{\tau}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}, \overline{\prod}_{\beth=1}^{k+1}\eta^{\upsilon\mathfrak{P}^{\gamma}{}_{\beth}}_{\beth}\right\rangle, \left\langle 1-\overline{\prod}_{\beth=1}^{k+1}(1-\mathscr{I}^{\aleph}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}, \overline{\prod}_{\beth=1}^{k+1}\mathscr{C}^{\gamma\mathfrak{P}^{\gamma}{}_{\beth}}_{\beth}\right\rangle\right).$$

This shows that for $n = k + 1$,, Eq. (14.10) holds. Then,

$$\begin{aligned}
&\text{LDFWA}(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) \\
&= \Bigg(\Big\langle 1 - \overline{\prod}_{\beth=1}^{n} (1 - \zeta^{\tau}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}, \overline{\prod}_{\beth=1}^{n} \eta^{\upsilon}{}_{\beth}^{\mathfrak{P}^{\gamma}{}_{\beth}} \Big\rangle, \\
&\Big\langle 1 - \overline{\prod}_{\beth=1}^{n} (1 - \mathscr{J}^{\aleph}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}, \overline{\prod}_{\beth=1}^{n} \mathscr{C}^{\gamma}{}_{\beth}^{\mathfrak{P}^{\gamma}{}_{\beth}} \Big\rangle\Bigg).
\end{aligned}$$

The next couple of paragraphs will discuss a few of the beneficial qualities that LDFWA operator has.

Theorem 10 (Idempotency) *Assume that* $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth}\rangle)$ *is the agglomeration of LDFNs, where* $\breve{\hbar}_{\beth} = \overline{\prod}_{k=1}^{j-1} \mathscr{H}(\daleth^{\varsigma}{}_k)$ $(j = 2\ldots, n)$, $\breve{\hbar}_1 = 1$, *and* $\mathscr{H}(\daleth^{\varsigma}{}_k)$ *is the expectation SF of kth LDFN. If all* $\daleth^{\varsigma}{}_{\beth}$ *are equal, i.e.,* $\daleth^{\varsigma}{}_{\beth} = \daleth^{\varsigma}$ *for all* j*, then*

$$LDFWA(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) = \daleth^{\varsigma}.$$

Proof From Definition 17, we have

$$\begin{aligned}
\text{LDFWA}(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) &= \mathfrak{P}^{\gamma}{}_1 \daleth^{\varsigma}{}_1 \oplus \mathfrak{P}^{\gamma}{}_2 \daleth^{\varsigma}{}_2 \oplus \ldots, \oplus \mathfrak{P}^{\gamma}{}_n \daleth^{\varsigma}{}_n \\
&= \mathfrak{P}^{\gamma}{}_1 \daleth^{\varsigma} \oplus \mathfrak{P}^{\gamma}{}_2 \daleth^{\varsigma} \oplus \ldots, \oplus \mathfrak{P}^{\gamma}{}_n \daleth^{\varsigma} \\
&= (\mathfrak{P}^{\gamma}{}_1 + \mathfrak{P}^{\gamma}{}_2 + \ldots + \mathfrak{P}^{\gamma}{}_n) \daleth^{\varsigma} \\
&= \daleth^{\varsigma}.
\end{aligned}$$

Corollary 1 *If* $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth}\rangle)$, $j = (1, 2, \ldots n)$ *is the agglomeration of largest LDFNs, i.e.,* $\daleth^{\varsigma}{}_{\beth} = \langle (1, 0), (1, 0)\rangle$ *for all* j*, then*

$$LDFWA(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) = \langle (1, 0), (1, 0)\rangle.$$

Proof We can easily obtain Corollary similar to Theorem 10.

Theorem 11 (Monotonicity) *Assume that* $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth}\rangle)$ *and* $\daleth^{\varsigma *}_{\beth} = (\langle \zeta^{\tau *}_{\beth}, \eta^{\upsilon *}_{\beth}\rangle, \langle \mathscr{J}^{\aleph *}_{\beth}, \mathscr{C}^{\gamma *}_{\beth}\rangle)$ *are the agglomerations of LDFNs. If* $\zeta^{\tau *}_{\beth} \geq \zeta^{\tau}{}_{\beth}$, $\eta^{\upsilon *}_{\beth} \leq \eta^{\upsilon}{}_{\beth}$, $\mathscr{J}^{\aleph *}_{\beth} \geq \mathscr{J}^{\aleph}{}_{\beth}$, *and* $\mathscr{C}^{\gamma *}_{\beth} \leq \mathscr{C}^{\gamma}{}_{\beth}$ *for all* j*, then*

$$LDFWA(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) \leq LDFWA(\daleth^{\varsigma *}_1, \daleth^{\varsigma *}_2, \ldots \daleth^{\varsigma *}_n).$$

Proof Here, $\zeta^{\tau *}_{\beth} \geq \zeta^{\tau}{}_{\beth}$ and $\eta^{\upsilon *}_{\beth} \leq \eta^{\upsilon}{}_{\beth}$ for all j. If $\zeta^{\tau *}_{\beth} \geq \zeta^{\tau}{}_{\beth}$:

$\Leftrightarrow \zeta^{\tau *}_{\beth} \geq \zeta^{\tau}{}_{\beth} \Leftrightarrow 1 - \zeta^{\tau *}_{\beth} \leq 1 - \zeta^{\tau}{}_{\beth}$
$\Leftrightarrow (1 - \zeta^{\tau *}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \leq (1 - \zeta^{\tau}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}$
$\Leftrightarrow \overline{\prod}_{\beth=1}^{n}(1 - \zeta^{\tau *}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \leq \overline{\prod}_{\beth=1}^{n}(1 - \zeta^{\tau}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}$
$\Leftrightarrow 1 - \overline{\prod}_{\beth=1}^{n}(1 - \zeta^{\tau}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \leq 1 - \overline{\prod}_{\beth=1}^{n}(1 - \zeta^{\tau *}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}$

Again:

$\mathscr{J}^{\aleph *}_{\beth} \geq \mathscr{J}^{\aleph}{}_{\beth}$ and $\mathscr{C}^{\gamma *}_{\beth} \leq \mathscr{C}^{\gamma}{}_{\beth}$ for all j. If $\mathscr{J}^{\aleph *}_{\beth} \geq \mathscr{J}^{\aleph}{}_{\beth}$,
$\Leftrightarrow \mathscr{J}^{\aleph *}_{\beth} \geq \mathscr{J}^{\aleph}{}_{\beth} \Leftrightarrow 1 - \mathscr{J}^{\aleph *}_{\beth} \leq 1 - \mathscr{J}^{\aleph}{}_{\beth}$
$\Leftrightarrow (1 - \mathscr{J}^{\aleph *}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \leq (1 - \mathscr{J}^{\aleph}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}$
$\Leftrightarrow \overline{\prod}_{\beth=1}^{n}(1 - \mathscr{J}^{\aleph *}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \leq \overline{\prod}_{\beth=1}^{n}(1 - \mathscr{J}^{\aleph}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}$
$\Leftrightarrow 1 - \overline{\prod}_{\beth=1}^{n}(1 - \mathscr{J}^{\aleph}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \leq 1 - \overline{\prod}_{\beth=1}^{n}(1 - \mathscr{J}^{\aleph *}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}$

Now:

$\eta^{\upsilon *}_{\beth} \leq \eta^{\upsilon}{}_{\beth}$
$\Leftrightarrow (\eta^{\upsilon *}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \leq (\eta^{\upsilon}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}$
$\Leftrightarrow \overline{\prod}_{\beth=1}^{n}(\eta^{\upsilon *}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \leq \overline{\prod}_{\beth=1}^{n}(\eta^{\upsilon}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}$

And:

$\mathscr{C}^{\gamma *}_{\beth} \leq \mathscr{C}^{\gamma}{}_{\beth}$
$\Leftrightarrow (\mathscr{C}^{\gamma *}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \leq (\mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}$
$\Leftrightarrow \overline{\prod}_{\beth=1}^{n}(\mathscr{C}^{\gamma *}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \leq \overline{\prod}_{\beth=1}^{n}(\mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}$

Let

$$\overline{\daleth^{\varsigma}} = \text{LDFWA}(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{n})$$

and

$$\overline{\daleth^{\varsigma *}} = \text{LDFWA}(\daleth^{\varsigma *}_{1}, \daleth^{\varsigma *}_{2}, \ldots \daleth^{\varsigma *}_{n})$$

We get that $\overline{\daleth^{\varsigma *}} \geq \overline{\daleth^{\varsigma}}$. So,

$$\text{LDFWA}(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{n}) \leq \text{LDFWA}(\daleth^{\varsigma *}_{1}, \daleth^{\varsigma *}_{2}, \ldots \daleth^{\varsigma *}_{n}).$$

Theorem 12 *Assume that* $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth}\rangle)$ *and* $\digamma^{\gamma}{}_{\beth} = (\langle \phi_{\beth}, \varphi_{\beth}\rangle, \langle \mathscr{K}_{\beth}, \mathscr{M}_{\beth}\rangle)$ *are two families of LDFNs. If* $r > 0$ *and* $\digamma^{\gamma} = (\langle \zeta^{\tau}{}_{\digamma^{\gamma}}, \eta^{\upsilon}{}_{\digamma^{\gamma}}\rangle, \langle \mathscr{J}^{\aleph}{}_{\digamma^{\gamma}}, \mathscr{C}^{\gamma}{}_{\digamma^{\gamma}}\rangle)$ *is an LDFN, then:*

1. *LDFWA*$(\daleth^{\varsigma}{}_{1} \oplus \digamma^{\gamma}, \daleth^{\varsigma}{}_{2} \oplus \digamma^{\gamma}, \ldots \daleth^{\varsigma}{}_{n} \oplus \digamma^{\gamma}) =$ *LDFWA*$(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{n}) \oplus \digamma^{\gamma}$
2. *LDFWA*$(r\daleth^{\varsigma}{}_{1}, r\daleth^{\varsigma}{}_{2}, \ldots r\daleth^{\varsigma}{}_{n}) = r$ *LDFWA*$(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{n})$

3. $LDFWA(\daleth^{\varsigma}{}_1 \oplus \digamma^{\gamma}{}_1, \daleth^{\varsigma}{}_2 \oplus \digamma^{\gamma}{}_2, \ldots \daleth^{\varsigma}{}_n \oplus \digamma^{\gamma}{}_n) = LDFWA(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) \oplus LDFWA(\digamma^{\gamma}{}_1, \digamma^{\gamma}{}_2, \ldots \digamma^{\gamma}{}_n)$
4. $LDFWA(r\daleth^{\varsigma}{}_1 \oplus \digamma^{\gamma}, r\daleth^{\varsigma}{}_2 \oplus \digamma^{\gamma}, \ldots \oplus r\daleth^{\varsigma}{}_n \oplus \digamma^{\gamma}) = r\ LDFWA(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) \oplus \digamma^{\gamma}$

Proof Here, we just proof 1 and 3.
1. Since,

$$\daleth^{\varsigma}{}_{\gimel} \oplus \digamma^{\gamma} = \left(\left(1-(1-\zeta^{\tau}{}_{\gimel})(1-\zeta^{\tau}{}_{\digamma^{\gamma}}), \eta^{\upsilon}{}_{\gimel}\eta^{\upsilon}{}_{\digamma^{\gamma}}\right),\right.$$

$$\left.\left(1-(1-\mathscr{J}^{\aleph}{}_{\gimel})(1-\mathscr{J}^{\aleph}{}_{\digamma^{\gamma}}), \mathscr{C}^{\gamma}{}_{\gimel}\mathscr{C}^{\gamma}{}_{\digamma^{\gamma}}\right)\right).$$

By Theorem 13,

$$\mathrm{LDFWA}(\daleth^{\varsigma}{}_1 \oplus \digamma^{\gamma}, \daleth^{\varsigma}{}_2 \oplus \digamma^{\gamma}, \ldots \daleth^{\varsigma}{}_n \oplus \digamma^{\gamma})$$

$$= \left(\left\langle (1-\overline{\prod}_{\gimel=1}^{n}\left((1-\zeta^{\tau}{}_{\gimel})(1-\zeta^{\tau}{}_{\digamma^{\gamma}})\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \overline{\prod}_{\gimel=1}^{n}\left(\eta^{\upsilon}{}_{\digamma^{\gamma}}\eta^{\upsilon}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}\right\rangle,\right.$$

$$\left.\left\langle (1-\overline{\prod}_{\gimel=1}^{n}\left((1-\mathscr{J}^{\aleph}{}_{\gimel})(1-\mathscr{J}^{\aleph}{}_{\digamma^{\gamma}})\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \overline{\prod}_{\gimel=1}^{n}\left(\mathscr{C}^{\gamma}{}_{\digamma^{\gamma}}\mathscr{C}^{\gamma}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}\right\rangle\right)$$

$$= \left(\left\langle (1-\left(1-\zeta^{\tau}{}_{\digamma^{\gamma}}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}\overline{\prod}_{\gimel=1}^{n}\left(1-\zeta^{\tau}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \left(\eta^{\upsilon}{}_{\digamma^{\gamma}}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}\overline{\prod}_{\gimel=1}^{n}\left(\eta^{\upsilon}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}\right\rangle,\right.$$

$$\left.\left\langle (1-\left(1-\mathscr{J}^{\aleph}{}_{\digamma^{\gamma}}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}\overline{\prod}_{\gimel=1}^{n}\left(1-\mathscr{J}^{\aleph}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \left(\mathscr{C}^{\gamma}{}_{\digamma^{\gamma}}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}\overline{\prod}_{\gimel=1}^{n}\left(\mathscr{C}^{\gamma}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}\right\rangle\right)$$

$$= \left(\left\langle (1-\left(1-\zeta^{\tau}{}_{\digamma^{\gamma}}\right)\overline{\prod}_{\gimel=1}^{n}\left(1-\zeta^{\tau}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}},\right.\right.$$

$$\left.\left(\eta^{\upsilon}{}_{\digamma^{\gamma}}\right)\overline{\prod}_{\gimel=1}^{n}\left(\eta^{\upsilon}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}\right\rangle,$$

$$\left.\left\langle (1-\left(1-\mathscr{J}^{\aleph}{}_{\digamma^{\gamma}}\right)\overline{\prod}_{\gimel=1}^{n}\left(1-\mathscr{J}^{\aleph}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \left(\mathscr{C}^{\gamma}{}_{\digamma^{\gamma}}\right)\overline{\prod}_{\gimel=1}^{n}\left(\mathscr{C}^{\gamma}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}\right\rangle\right).$$

Now, by operational laws of LDFNs,

$$\mathrm{LDFWA}(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) \oplus \digamma^{\gamma}$$

$$= \left(\left\langle (1-\overline{\prod}_{\gimel=1}^{n}(1-\zeta^{\tau}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \overline{\prod}_{\gimel=1}^{n}\eta^{\upsilon}{}_{\gimel}^{\mathfrak{P}^{\gamma}{}_{\gimel}}\right\rangle,\right.$$

$$\left\langle (1-\overline{\prod}_{\beth=1}^{n}(1-\mathscr{J}^{\aleph}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}, \overline{\prod}_{\beth=1}^{n}\mathscr{C}^{\gamma}{}_{\beth}^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle \oplus$$

$$\left((\langle \zeta^{\tau}{}_{F^{\gamma}}, \eta^{\upsilon}{}_{F^{\gamma}} \rangle, \langle \mathscr{J}^{\aleph}{}_{F^{\gamma}}, \mathscr{C}^{\gamma}{}_{F^{\gamma}} \rangle) \right)$$

$$= \left(\left\langle (1-\left(1-\zeta^{\tau}{}_{F^{\gamma}}\right)\overline{\prod}_{\beth=1}^{n}\left(1-\zeta^{\tau}{}_{\beth}\right)^{\mathfrak{P}^{\gamma}{}_{\beth}}, \left(\eta^{\upsilon}{}_{F^{\gamma}}\right)\overline{\prod}_{\beth=1}^{n}\left(\eta^{\upsilon}{}_{\beth}\right)^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle, \right.$$

$$\left. \left\langle (1-\left(1-\mathscr{J}^{\aleph}{}_{F^{\gamma}}\right)\overline{\prod}_{\beth=1}^{n}\left(1-\mathscr{J}^{\aleph}{}_{\beth}\right)^{\mathfrak{P}^{\gamma}{}_{\beth}}, \left(\mathscr{C}^{\gamma}{}_{F^{\gamma}}\right)\overline{\prod}_{\beth=1}^{n}\left(\mathscr{C}^{\gamma}{}_{\beth}\right)^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle \right).$$

Thus,

$$\text{LDFWA}(\daleth^{\varsigma}{}_{1} \oplus F^{\gamma}, \daleth^{\varsigma}{}_{2} \oplus F^{\gamma}, \ldots \daleth^{\varsigma}{}_{n} \oplus F^{\gamma}) = \text{LDFWA}(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{n}) \oplus F^{\gamma}.$$

3. According to Theorem 13,

$$\text{q-ROFWA}(\daleth^{\varsigma}{}_{1} \oplus F^{\gamma}{}_{2}, \daleth^{\varsigma}{}_{2} \oplus F^{\gamma}{}_{2}, \ldots \daleth^{\varsigma}{}_{n} \oplus F^{\gamma}{}_{n})$$

$$= \left(\left\langle 1-\overline{\prod}_{\beth=1}^{n}\left((1-\zeta^{\tau}{}_{\beth})(1-\phi_{\beth})\right)^{\mathfrak{P}^{\gamma}{}_{\beth}}, \overline{\prod}_{\beth=1}^{n}\left(\varphi_{\beth}\eta^{\upsilon}{}_{\beth}\right)^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle, \right.$$

$$\left. \left\langle 1-\overline{\prod}_{\beth=1}^{n}\left((1-\mathscr{J}^{\aleph}{}_{\beth})(1-\mathscr{K}_{\beth})\right)^{\mathfrak{P}^{\gamma}{}_{\beth}}, \overline{\prod}_{\beth=1}^{n}\left(\mathscr{M}_{\beth}\mathscr{C}^{\gamma}{}_{\beth}\right)^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle \right)$$

$$= \left(\left\langle 1-\overline{\prod}_{\beth=1}^{n}\left(1-\phi_{\beth}\right)^{\mathfrak{P}^{\gamma}{}_{\beth}}\overline{\prod}_{\beth=1}^{n}\left(1-\zeta^{\tau}{}_{\beth}\right)^{\mathfrak{P}^{\gamma}{}_{\beth}}, \right. \right.$$

$$\left. \overline{\prod}_{\beth=1}^{n}\left(\varphi_{\beth}\right)^{\mathfrak{P}^{\gamma}{}_{\beth}}\overline{\prod}_{\beth=1}^{n}\left(\eta^{\upsilon}{}_{\beth}\right)^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle,$$

$$\left\langle 1-\overline{\prod}_{\beth=1}^{n}\left(1-\mathscr{K}_{\beth}\right)^{\mathfrak{P}^{\gamma}{}_{\beth}}\overline{\prod}_{\beth=1}^{n}\left(1-\mathscr{J}^{\aleph}{}_{\beth}\right)^{\mathfrak{P}^{\gamma}{}_{\beth}}, \right.$$

$$\left. \left. \overline{\prod}_{\beth=1}^{n}\left(\mathscr{M}_{\beth}\right)^{\mathfrak{P}^{\gamma}{}_{\beth}}\overline{\prod}_{\beth=1}^{n}\left(\mathscr{C}^{\gamma}{}_{\beth}\right)^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle \right).$$

Now,

$$\text{LDFWA}(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{n}) \oplus \text{LDFWA}(F^{\gamma}{}_{1}, F^{\gamma}{}_{2}, \ldots F^{\gamma}{}_{n})$$

$$= \left(\left\langle 1-\overline{\prod}_{\beth=1}^{n}(1-\zeta^{\tau}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}, \overline{\prod}_{\beth=1}^{n}\eta^{\upsilon}{}_{\beth}^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle, \right.$$

$$\left. \left\langle 1-\overline{\prod}_{\beth=1}^{n}(1-\mathscr{J}^{\aleph}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}, \overline{\prod}_{\beth=1}^{n}\mathscr{C}^{\gamma}{}_{\beth}^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle \right) \oplus$$

$$\left(\left\langle 1-\overline{\prod}_{\daleth=1}^{n}(1-\phi_\daleth)^{\mathfrak{P}^{\gamma}{}_\daleth}, \overline{\prod}_{\daleth=1}^{n}\varphi_\daleth^{\mathfrak{P}^{\gamma}{}_\daleth}\right\rangle,\right.$$

$$\left.\left\langle 1-\overline{\prod}_{\daleth=1}^{n}(1-\mathscr{K}_\daleth)^{\mathfrak{P}^{\gamma}{}_\daleth}, \overline{\prod}_{\daleth=1}^{n}\mathscr{M}_\daleth^{\mathfrak{P}^{\gamma}{}_\daleth}\right\rangle\right)$$

$$=\left(\left\langle 1-\overline{\prod}_{\daleth=1}^{n}\left(1-\phi_\daleth\right)^{\mathfrak{P}^{\gamma}{}_\daleth}\overline{\prod}_{\daleth=1}^{n}\left(1-\zeta^{\tau}{}_\daleth\right)^{\mathfrak{P}^{\gamma}{}_\daleth},\right.\right.$$

$$\left.\overline{\prod}_{\daleth=1}^{n}\left(\varphi_\daleth\right)^{\mathfrak{P}^{\gamma}{}_\daleth}\overline{\prod}_{\daleth=1}^{n}\left(\eta^{\upsilon}{}_\daleth\right)^{\mathfrak{P}^{\gamma}{}_\daleth}\right\rangle,$$

$$\left\langle 1-\overline{\prod}_{\daleth=1}^{n}\left(1-\mathscr{K}_\daleth\right)^{\mathfrak{P}^{\gamma}{}_\daleth}\overline{\prod}_{\daleth=1}^{n}\left(1-\mathscr{J}^{\aleph}{}_\daleth\right)^{\mathfrak{P}^{\gamma}{}_\daleth},\right.$$

$$\left.\left.\overline{\prod}_{\daleth=1}^{n}\left(\mathscr{M}_\daleth\right)^{\mathfrak{P}^{\gamma}{}_\daleth}\overline{\prod}_{\daleth=1}^{n}\left(\mathscr{C}^{\gamma}{}_\daleth\right)^{\mathfrak{P}^{\gamma}{}_\daleth}\right\rangle\right).$$

Thus,

$$\begin{aligned}&\text{LDFWA}(\daleth^{\varsigma}{}_1\oplus F^{\gamma}{}_2, \daleth^{\varsigma}{}_2\oplus F^{\gamma}{}_2, \ldots \daleth^{\varsigma}{}_n\oplus F^{\gamma}{}_n)\\ &\quad=\text{LDFWA}(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n)\oplus\text{LDFWA}(F^{\gamma}{}_1, F^{\gamma}{}_2, \ldots F^{\gamma}{}_n).\end{aligned}$$

3.2 LDFOWA Operator

Definition 17 Consider $\daleth^{\varsigma}{}_\daleth = (\langle\zeta^{\tau}{}_\daleth, \eta^{\upsilon}{}_\daleth\rangle, \langle\mathscr{J}^{\aleph}{}_\daleth, \mathscr{C}^{\gamma}{}_\daleth\rangle)$ is the agglomeration of LDFNs, and LDFOWA : $\$^n \to \$$ be the mapping.

$$\text{LDFOWA}(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) = \mathfrak{P}^{\gamma}{}_1\daleth^{\varsigma}{}_{\sigma(1)}\oplus\mathfrak{P}^{\gamma}{}_2\daleth^{\varsigma}{}_{\sigma(2)}\oplus\ldots,\oplus\mathfrak{P}^{\gamma}{}_n\daleth^{\varsigma}{}_{\sigma(n)}, \tag{14.9}$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$, such that $\daleth^{\varsigma}{}_{\sigma(r-1)} \geq \daleth^{\varsigma}{}_{\sigma(r)}$, for any r. Then LDFOWA is known as LDFOWA operator, where $(\mathfrak{P}^{\gamma}{}_1, \mathfrak{P}^{\gamma}{}_2, \ldots, \mathfrak{P}^{\gamma}{}_n)$ be the WV with the constraint $\mathfrak{P}^{\gamma}{}_\daleth > 0$ and $\sum_{h=1}^{n}\mathfrak{P}^{\gamma}{}_\daleth = 1$.

We might also think about LDFOWA by employing the theorem following.

Theorem 13 *Consider $\daleth^{\varsigma}{}_\daleth = (\langle\zeta^{\tau}{}_\daleth, \eta^{\upsilon}{}_\daleth\rangle, \langle\mathscr{J}^{\aleph}{}_\daleth, \mathscr{C}^{\gamma}{}_\daleth\rangle)$ is the agglomeration of LDFNs, and we can find LDFWA by*

$$\mathit{LDFOWA}(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n)$$

$$= \left(\left\langle 1 - \overline{\prod}_{\daleth=1}^{n} (1 - \zeta^{\tau}{}_{\sigma(j)})^{\mathfrak{P}^{\gamma}{}_{\daleth}}, \overline{\prod}_{\daleth=1}^{n} \eta^{\upsilon}{}_{\sigma(j)}^{\mathfrak{P}^{\gamma}{}_{\daleth}} \right\rangle, \right.$$

$$\left. \left\langle 1 - \overline{\prod}_{\daleth=1}^{n} (1 - \mathscr{J}^{\aleph}{}_{\sigma(j)})^{\mathfrak{P}^{\gamma}{}_{\daleth}}, \overline{\prod}_{\daleth=1}^{n} \mathscr{C}^{\gamma}{}_{\sigma(j)}^{\mathfrak{P}^{\gamma}{}_{\daleth}} \right\rangle\right). \tag{14.10}$$

Theorem 14 (Monotonicity) *Assume that* $\urcorner^{\varsigma}{}_{\daleth} = (\langle \zeta^{\tau}{}_{\daleth}, \eta^{\upsilon}{}_{\daleth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\daleth}, \mathscr{C}^{\gamma}{}_{\daleth}\rangle)$ *and* $\urcorner^{\varsigma *}_{\daleth} = (\langle \zeta^{\tau *}_{\daleth}, \eta^{\upsilon *}_{\daleth}\rangle, \langle \mathscr{J}^{\aleph *}_{\daleth}, \mathscr{C}^{\gamma *}_{\daleth}\rangle)$ *are the agglomerations of LDFNs. If* $\zeta^{\tau *}_{\daleth} \geq \zeta^{\tau}{}_{\daleth}$, $\eta^{\upsilon *}_{\daleth} \leq \eta^{\upsilon}{}_{\daleth}$, $\mathscr{J}^{\aleph *}_{\daleth} \geq \mathscr{J}^{\aleph}{}_{\daleth}$, *and* $\mathscr{C}^{\gamma *}_{\daleth} \leq \mathscr{C}^{\gamma}{}_{\daleth}$ *for all j, then*

$$LDFOWA(\urcorner^{\varsigma}{}_1, \urcorner^{\varsigma}{}_2, \ldots \urcorner^{\varsigma}{}_n) \leq LDFOWA(\urcorner^{\varsigma *}_1, \urcorner^{\varsigma *}_2, \ldots \urcorner^{\varsigma *}_n).$$

Proof This is the same as Theorem 14.

Theorem 15 *Assume that* $\urcorner^{\varsigma}{}_{\daleth} = (\langle \zeta^{\tau}{}_{\daleth}, \eta^{\upsilon}{}_{\daleth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\daleth}, \mathscr{C}^{\gamma}{}_{\daleth}\rangle)$ *and* $F^{\gamma}{}_{\daleth} = (\langle \phi_{\daleth}, \varphi_{\daleth}\rangle, \langle \mathscr{K}_{\daleth}, \mathscr{M}_{\daleth}\rangle)$ *are two families of LDFNs. If* $r > 0$ *and* $F^{\gamma} = (\langle \zeta^{\tau}{}_{F^{\gamma}}, \eta^{\upsilon}{}_{F^{\gamma}}\rangle, \langle \mathscr{J}^{\aleph}{}_{F^{\gamma}}, \mathscr{C}^{\gamma}{}_{F^{\gamma}}\rangle)$ *is an LDFN, then:*

1. $LDFOWA(\urcorner^{\varsigma}{}_1 \oplus F^{\gamma}, \urcorner^{\varsigma}{}_2 \oplus F^{\gamma}, \ldots \urcorner^{\varsigma}{}_n \oplus F^{\gamma}) = LDFOWA(\urcorner^{\varsigma}{}_1, \urcorner^{\varsigma}{}_2, \ldots \urcorner^{\varsigma}{}_n) \oplus F^{\gamma}$
2. $LDFOWA(r\urcorner^{\varsigma}{}_1, r\urcorner^{\varsigma}{}_2, \ldots r\urcorner^{\varsigma}{}_n) = r\ LDFOWA(\urcorner^{\varsigma}{}_1, \urcorner^{\varsigma}{}_2, \ldots \urcorner^{\varsigma}{}_n)$
3. $LDFOWA(\urcorner^{\varsigma}{}_1 \oplus F^{\gamma}{}_1, \urcorner^{\varsigma}{}_2 \oplus F^{\gamma}{}_2, \ldots \urcorner^{\varsigma}{}_n \oplus F^{\gamma}{}_n) = LDFOWA(\urcorner^{\varsigma}{}_1, \urcorner^{\varsigma}{}_2, \ldots \urcorner^{\varsigma}{}_n) \oplus LDFOWA(F^{\gamma}{}_1, F^{\gamma}{}_2, \ldots F^{\gamma}{}_n)$
4. $LDFOWA(r\urcorner^{\varsigma}{}_1 \oplus F^{\gamma}, r\urcorner^{\varsigma}{}_2 \oplus F^{\gamma}, \ldots \oplus r\urcorner^{\varsigma}{}_n \oplus F^{\gamma}) = rLDFOWA(\urcorner^{\varsigma}{}_1, \urcorner^{\varsigma}{}_2, \ldots \urcorner^{\varsigma}{}_n) \oplus F^{\gamma}$

3.3 LDFWG Operator

Definition 18 Consider $\urcorner^{\varsigma}{}_{\daleth} = (\langle \zeta^{\tau}{}_{\daleth}, \eta^{\upsilon}{}_{\daleth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\daleth}, \mathscr{C}^{\gamma}{}_{\daleth}\rangle)$ is the agglomeration of LDFNs and $LDFWG : \$^n \to \$$ be a mapping.

$$\text{LDFWG}(\urcorner^{\varsigma}{}_1, \urcorner^{\varsigma}{}_2, \ldots \urcorner^{\varsigma}{}_n) = \urcorner^{\varsigma}{}_1^{\mathfrak{P}^{\gamma}{}_1} \otimes \urcorner^{\varsigma}{}_2^{\mathfrak{P}^{\gamma}{}_2} \otimes \ldots, \otimes \urcorner^{\varsigma}{}_n^{\mathfrak{P}^{\gamma}{}_n}. \tag{14.11}$$

Then the mapping LDFWG is called LDFWG operator, where $(\mathfrak{P}^{\gamma}{}_1, \mathfrak{P}^{\gamma}{}_2, \ldots, \mathfrak{P}^{\gamma}{}_n)$ be the WV with the constraint $\mathfrak{P}^{\gamma}{}_i > 0$ and $\sum_{i=1}^{n} \mathfrak{P}^{\gamma}{}_i = 1$.

We may also consider LDFWG using the theorem below based on LDFNs operational law.

Theorem 16 *Assume that* $\urcorner^{\varsigma}{}_{\daleth} = (\langle \zeta^{\tau}{}_{\daleth}, \eta^{\upsilon}{}_{\daleth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\daleth}, \mathscr{C}^{\gamma}{}_{\daleth}\rangle)$ *is the agglomeration of LDFNs, and we can find LDFWG by* $LDFWG(\urcorner^{\varsigma}{}_1, \urcorner^{\varsigma}{}_2, \ldots \urcorner^{\varsigma}{}_n)$

$$= \left(\left\langle \overline{\prod}_{\daleth=1}^{n} \zeta^{\tau}{}_{\daleth}^{\mathfrak{P}^{\gamma}{}_{\daleth}}, 1 - \overline{\prod}_{\daleth=1}^{n} (1 - \eta^{\upsilon}{}_{\daleth})^{\mathfrak{P}^{\gamma}{}_{\daleth}} \right\rangle, \right.$$

$$\left\langle \overline{\prod}_{\beth=1}^{n} \mathscr{J}^{\aleph \mathfrak{P}^{\gamma}{}_{\beth}}_{\beth}, 1-\overline{\prod}_{\beth=1}^{n} (1-\mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle\Bigg). \tag{14.12}$$

Proof It is quite simple for the first assertion to come before Definition 19 and Theorem 20. The following instances demonstrate this point further:

$$\begin{aligned}
&\text{LDFWG}(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{n}) \\
&= \daleth^{\varsigma}{}_{1}^{\mathfrak{P}^{\gamma}{}_{1}} \otimes \daleth^{\varsigma}{}_{2}^{\mathfrak{P}^{\gamma}{}_{2}} \otimes \ldots, \otimes \daleth^{\varsigma}{}_{n}^{\mathfrak{P}^{\gamma}{}_{n}} \\
&= \Bigg(\left\langle \overline{\prod}_{\beth=1}^{n} \zeta^{\tau}{}_{\beth}^{\mathfrak{P}^{\gamma}{}_{\beth}}, 1-\overline{\prod}_{\beth=1}^{n} (1-\eta^{\upsilon}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle, \\
&\quad \left\langle \overline{\prod}_{\beth=1}^{n} \mathscr{J}^{\aleph}{}_{\beth}^{\mathfrak{P}^{\gamma}{}_{\beth}}, 1-\overline{\prod}_{\beth=1}^{n} (1-\mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle\Bigg).
\end{aligned}$$

In order to demonstrate the validity of this theorem, we turned to mathematics induction.

For $n = 2$

$$\daleth^{\varsigma}{}_{1}^{\mathfrak{P}^{\gamma}{}_{1}} = \left(\left\langle \zeta^{\tau}{}_{1}^{\mathfrak{P}^{\gamma}{}_{1}}, 1-(1-\eta^{\upsilon}{}_{1})^{\mathfrak{P}^{\gamma}{}_{1}} \right\rangle, \left\langle \mathscr{J}^{\aleph}{}_{1}^{\mathfrak{P}^{\gamma}{}_{1}}, 1-(1-\mathscr{C}^{\gamma}{}_{1})^{\mathfrak{P}^{\gamma}{}_{1}} \right\rangle\right)$$

$$\daleth^{\varsigma}{}_{2}^{\mathfrak{P}^{\gamma}{}_{2}} = \left(\left\langle \zeta^{\tau}{}_{2}^{\mathfrak{P}^{\gamma}{}_{1}}, 1-(1-\eta^{\upsilon}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{1}} \right\rangle, \left\langle \mathscr{J}^{\aleph}{}_{2}^{\mathfrak{P}^{\gamma}{}_{1}}, 1-(1-\mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{1}} \right\rangle\right).$$

Then

$$\begin{aligned}
&\daleth^{\varsigma}{}_{1}^{\mathfrak{P}^{\gamma}{}_{1}} \otimes \daleth^{\varsigma}{}_{2}^{\mathfrak{P}^{\gamma}{}_{2}} \\
&= \left(\left\langle \zeta^{\tau}{}_{1}^{\mathfrak{P}^{\gamma}{}_{1}}, 1-(1-\eta^{\upsilon}{}_{1})^{\mathfrak{P}^{\gamma}{}_{1}} \right\rangle, \left\langle \mathscr{J}^{\aleph}{}_{1}^{\mathfrak{P}^{\gamma}{}_{1}}, 1-(1-\mathscr{C}^{\gamma}{}_{1})^{\mathfrak{P}^{\gamma}{}_{1}} \right\rangle\right) \otimes \\
&\quad \left(\left\langle \zeta^{\tau}{}_{2}^{\mathfrak{P}^{\gamma}{}_{1}}, 1-(1-\eta^{\upsilon}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{1}} \right\rangle, \left\langle \mathscr{J}^{\aleph}{}_{2}^{\mathfrak{P}^{\gamma}{}_{1}}, 1-(1-\mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{1}} \right\rangle\right) \\
&= \Bigg(\Bigg\langle \zeta^{\tau}{}_{1}^{\mathfrak{P}^{\gamma}{}_{1}}.\zeta^{\tau}{}_{2}^{\mathfrak{P}^{\gamma}{}_{1}}, 1-(1-\eta^{\upsilon}{}_{1})^{\mathfrak{P}^{\gamma}{}_{1}}+1-(1-\eta^{\upsilon}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{1}} - \left(1-(1-\eta^{\upsilon}{}_{1})^{\mathfrak{P}^{\gamma}{}_{1}}\right) \\
&\quad \left(1-(1-\eta^{\upsilon}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{1}}\right)\Bigg\rangle, \Bigg\langle \mathscr{J}^{\aleph}{}_{1}^{\mathfrak{P}^{\gamma}{}_{1}}.\mathscr{J}^{\aleph}{}_{2}^{\mathfrak{P}^{\gamma}{}_{1}}, 1-(1-\mathscr{C}^{\gamma}{}_{1})^{\mathfrak{P}^{\gamma}{}_{1}}+1 \\
&\quad -(1-\mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{1}} - \left(1-(1-\mathscr{C}^{\gamma}{}_{1})^{\mathfrak{P}^{\gamma}{}_{1}}\right)\left(1-(1-\mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{1}}\right)\Bigg\rangle\Bigg)
\end{aligned}$$

$$= \left(\left\langle \zeta^{\tau \mathfrak{P}^{\gamma}{}_1}_{1} . \zeta^{\tau \mathfrak{P}^{\gamma}{}_1}_{2}, 1 - (1 - \eta^{\upsilon}{}_1)^{\mathfrak{P}^{\gamma}{}_1} (1 - \eta^{\upsilon}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_1} \right\rangle, \right.$$

$$\left. \left\langle \mathscr{J}^{\aleph \mathfrak{P}^{\gamma}{}_1}_{1} . \mathscr{J}^{\aleph \mathfrak{P}^{\gamma}{}_1}_{2}, 1 - (1 - \mathscr{C}^{\gamma}{}_1)^{\mathfrak{P}^{\gamma}{}_1} (1 - \mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_1} \right\rangle \right)$$

$$= \left(\left\langle \overline{\prod}_{\beth=1}^{2} \zeta^{\tau \mathfrak{P}^{\gamma}{}_{\beth}}_{\beth}, 1 - \overline{\prod}_{\beth=1}^{2} (1 - \eta^{\upsilon}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle, \right.$$

$$\left. \left\langle \overline{\prod}_{\beth=1}^{2} \mathscr{J}^{\aleph \mathfrak{P}^{\gamma}{}_{\beth}}_{\beth}, 1 - \overline{\prod}_{\beth=1}^{2} (1 - \mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle \right).$$

This shows that Eq. (14.14) is true for $n = 2$, and now assume that Eq. (14.14) holds for $n = k$, i.e.,

$$\text{LDFWG}(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_k)$$

$$= \left(\left\langle \overline{\prod}_{\beth=1}^{k} \zeta^{\tau \mathfrak{P}^{\gamma}{}_{\beth}}_{\beth}, 1 - \overline{\prod}_{\beth=1}^{k} (1 - \eta^{\upsilon}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle, \right.$$

$$\left. \left\langle \overline{\prod}_{\beth=1}^{k} \mathscr{J}^{\aleph \mathfrak{P}^{\gamma}{}_{\beth}}_{\beth}, 1 - \overline{\prod}_{\beth=1}^{k} (1 - \mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle \right).$$

Now $n = k + 1$, and by operational laws of LDFNs, we have

$$\text{LDFWG}(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_{k+1}) = \text{LDFWG}(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_k) \otimes \daleth^{\varsigma \mathfrak{P}^{\gamma}{}_{\beth}}_{k+1}$$

$$= \left(\left\langle \overline{\prod}_{\beth=1}^{k} \zeta^{\tau \mathfrak{P}^{\gamma}{}_{\beth}}_{\beth}, 1 - \overline{\prod}_{\beth=1}^{k} (1 - \eta^{\upsilon}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle, \right.$$

$$\left. \left\langle \overline{\prod}_{\beth=1}^{k} \mathscr{J}^{\aleph \mathfrak{P}^{\gamma}{}_{\beth}}_{\beth}, 1 - \overline{\prod}_{\beth=1}^{k} (1 - \mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle \right) \otimes$$

$$\left(\left\langle \zeta^{\tau \mathfrak{P}^{\gamma}{}_{k+1}}_{k+1}, 1 - (1 - \eta^{\upsilon}{}_{k+1})^{\mathfrak{P}^{\gamma}{}_{k+1}} \right\rangle, \left\langle \mathscr{J}^{\aleph \mathfrak{P}^{\gamma}{}_{k+1}}_{k+1}, 1 - (1 - \mathscr{C}^{\gamma}{}_{k+1})^{\mathfrak{P}^{\gamma}{}_{k+1}} \right\rangle \right)$$

$$= \left(\left\langle \overline{\prod}_{\beth=1}^{k} \zeta^{\tau \mathfrak{P}^{\gamma}{}_{\beth}}_{k} . \zeta^{\tau \mathfrak{P}^{\gamma}{}_{k+1}}_{k+1}, 1 - \overline{\prod}_{\beth=1}^{k} (1 - \eta^{\upsilon}{}_k)^{\mathfrak{P}^{\gamma}{}_{\beth}} (1 - \eta^{\upsilon}{}_{k+1})^{k+1} \right\rangle, \right.$$

$$\left. \left\langle \overline{\prod}_{\beth=1}^{k} \mathscr{J}^{\aleph \mathfrak{P}^{\gamma}{}_{\beth}}_{k} . \mathscr{J}^{\aleph \mathfrak{P}^{\gamma}{}_{k+1}}_{k+1}, 1 - \overline{\prod}_{\beth=1}^{k} (1 - \mathscr{C}^{\gamma}{}_k)^{\mathfrak{P}^{\gamma}{}_{\beth}} (1 - \mathscr{C}^{\gamma}{}_{k+1})^{k+1} \right\rangle \right)$$

$$= \left(\left\langle \overline{\prod}_{\beth=1}^{k+1} \zeta^{\tau}{}_{\beth}^{\mathfrak{P}^{\gamma}{}_{\beth}}, 1-\overline{\prod}_{\beth=1}^{k+1}(1-\eta^{\upsilon}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}\right\rangle,\right.$$
$$\left.\left\langle \overline{\prod}_{\beth=1}^{k+1} \mathscr{J}^{\aleph}{}_{\beth}^{\mathfrak{P}^{\gamma}{}_{\beth}}, 1-\overline{\prod}_{\beth=1}^{k+1}(1-\mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}\right\rangle\right).$$

This shows that for $n = k+1$, Eq. (14.10) holds. Then,

$$\text{LDFWG}(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n)$$
$$= \left(\left\langle \overline{\prod}_{\beth=1}^{n} \zeta^{\tau}{}_{\beth}^{\mathfrak{P}^{\gamma}{}_{\beth}}, 1-\overline{\prod}_{\beth=1}^{n}(1-\eta^{\upsilon}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}\right\rangle,\right.$$
$$\left.\left\langle \overline{\prod}_{\beth=1}^{n} \mathscr{J}^{\aleph}{}_{\beth}^{\mathfrak{P}^{\gamma}{}_{\beth}}, 1-\overline{\prod}_{\beth=1}^{n}(1-\mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}\right\rangle\right).$$

A few of LDFWG's promising properties are described below.

Theorem 17 *(Idempotency) Assume that $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth}\rangle)$ is the agglomeration of LDFNs. If all $\daleth^{\varsigma}{}_{\beth}$ are equal, i.e., $\daleth^{\varsigma}{}_{\beth} = \daleth^{\varsigma}$ for all j, then*

$$LDFWG(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) = \daleth^{\varsigma}.$$

Proof From Definition 17, we have

$$\begin{aligned}\text{LDFWG}(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) &= \daleth^{\varsigma}{}_1^{\mathfrak{P}^{\gamma}{}_1} \otimes \daleth^{\varsigma}{}_2^{\mathfrak{P}^{\gamma}{}_2} \otimes \ldots, \otimes \daleth^{\varsigma}{}_n^{\mathfrak{P}^{\gamma}{}_n}\\ &= \daleth^{\varsigma\,\mathfrak{P}^{\gamma}{}_1} \otimes \daleth^{\varsigma\,\mathfrak{P}^{\gamma}{}_2} \otimes \ldots, \otimes \daleth^{\varsigma\,\mathfrak{P}^{\gamma}{}_n}\\ &= \daleth^{\varsigma}.\end{aligned}$$

Corollary 2 *If $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth}\rangle)$ $j = (1, 2, \ldots n)$ is the agglomeration of largest LDFNs, i.e., $\daleth^{\varsigma}{}_{\beth} = (\langle 1, 0\rangle, \langle 1, 0\rangle)$ for all j, then*

$$LDFWG(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) = (\langle 1, 0\rangle, \langle 1, 0\rangle).$$

Proof We can easily obtain Corollary similar to Theorem 10.

Theorem 18 *Assume that $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth}\rangle)$ and $F^{\gamma}{}_{\beth} = (\langle \phi_{\beth}, \varphi_{\beth}\rangle, \langle \mathscr{K}_{\beth}, \mathscr{M}_{\beth}\rangle)$ are two families of LDFNs. If $r > 0$ and $F^{\gamma} = (\langle \zeta^{\tau}{}_{F^{\gamma}}, \eta^{\upsilon}{}_{F^{\gamma}}\rangle, \langle \mathscr{J}^{\aleph}{}_{F^{\gamma}}, \mathscr{C}^{\gamma}{}_{F^{\gamma}}\rangle)$ is an LDFN, then:*

1. *$LDFWG(\daleth^{\varsigma}{}_1 \oplus F^{\gamma}, \daleth^{\varsigma}{}_2 \oplus F^{\gamma}, \ldots \daleth^{\varsigma}{}_n \oplus F^{\gamma}) = LDFWG(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) \oplus F^{\gamma}$*

2. $LDFWG(r\daleth^{\varsigma}{}_1, r\daleth^{\varsigma}{}_2, \ldots r\daleth^{\varsigma}{}_n) = r\ LDFWG(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n)$
3. $LDFWG(\daleth^{\varsigma}{}_1 \oplus \digamma^{\gamma}{}_1, \daleth^{\varsigma}{}_2 \oplus \digamma^{\gamma}{}_2, \ldots \daleth^{\varsigma}{}_n \oplus \digamma^{\gamma}{}_n) = LDFWG(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) \oplus LDFWG(\digamma^{\gamma}{}_1, \digamma^{\gamma}{}_2, \ldots \digamma^{\gamma}{}_n)$
4. $LDFWG(r\daleth^{\varsigma}{}_1 \oplus \digamma^{\gamma}, r\daleth^{\varsigma}{}_2 \oplus \digamma^{\gamma}, \ldots \oplus r\daleth^{\varsigma}{}_n \oplus \digamma^{\gamma}) = r\ LDFWG(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) \oplus \digamma^{\gamma}$

Proof The proof of this theorem is the same as Theorem 15.

Theorem 19 *(Monotonicity) Assume that* $\daleth^{\varsigma}{}_{\gimel} = (\langle \zeta^{\tau}{}_{\gimel}, \eta^{\upsilon}{}_{\gimel}\rangle, \langle \mathscr{J}^{\aleph}{}_{\gimel}, \mathscr{C}^{\gamma}{}_{\gimel}\rangle)$ *and* $\daleth^{\varsigma}{}^{*}_{\gimel} = (\langle \zeta^{\tau}{}^{*}_{\gimel}, \eta^{\upsilon}{}^{*}_{\gimel}\rangle, \langle \mathscr{J}^{\aleph}{}^{*}_{\gimel}, \mathscr{C}^{\gamma}{}^{*}_{\gimel}\rangle)$ *are the agglomerations of LDFNs. If* $\zeta^{\tau}{}^{*}_{\gimel} \geq \zeta^{\tau}{}_{\gimel}$, $\eta^{\upsilon}{}^{*}_{\gimel} \leq \eta^{\upsilon}{}_{\gimel}$, $\mathscr{J}^{\aleph}{}^{*}_{\gimel} \geq \mathscr{J}^{\aleph}{}_{\gimel}$, *and* $\mathscr{C}^{\gamma}{}^{*}_{\gimel} \leq \mathscr{C}^{\gamma}{}_{\gimel}$ *for all* j, *then*

$$LDFWG(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) \leq LDFWG(\daleth^{\varsigma}{}^{*}_1, \daleth^{\varsigma}{}^{*}_2, \ldots \daleth^{\varsigma}{}^{*}_n)$$

Proof Here, $\eta^{\upsilon}{}^{*}_{\gimel} \geq \eta^{\upsilon}{}_{\gimel}$ and $\zeta^{\tau}{}^{*}_{\gimel} \leq \zeta^{\tau}{}_{\gimel}$ for all j. If $\eta^{\upsilon}{}^{*}_{\gimel} \geq \eta^{\upsilon}{}_{\gimel}$:

$\Leftrightarrow \eta^{\upsilon}{}^{*}_{\gimel} \geq \eta^{\upsilon}{}_{\gimel} \Leftrightarrow 1 - \eta^{\upsilon}{}^{*}_{\gimel} \leq 1 - \eta^{\upsilon}{}_{\gimel}$
$\Leftrightarrow (1 - \eta^{\upsilon}{}^{*}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \leq (1 - \eta^{\upsilon}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}$
$\Leftrightarrow \overline{\prod}_{\gimel=1}^{n}(1 - \eta^{\upsilon}{}^{*}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \leq \overline{\prod}_{\gimel=1}^{n}(1 - \eta^{\upsilon}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}$
$\Leftrightarrow 1 - \overline{\prod}_{\gimel=1}^{n}(1 - \eta^{\upsilon}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \leq 1 - \overline{\prod}_{\gimel=1}^{n}(1 - \eta^{\upsilon}{}^{*}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}$.

And:

$\mathscr{C}^{\gamma}{}^{*}_{\gimel} \geq \mathscr{C}^{\gamma}{}_{\gimel}$ and $\mathscr{J}^{\aleph}{}^{*}_{\gimel} \leq \mathscr{J}^{\aleph}{}_{\gimel}$ for all j. If $\mathscr{C}^{\gamma}{}^{*}_{\gimel} \geq \mathscr{C}^{\gamma}{}_{\gimel}$
$\Leftrightarrow \mathscr{C}^{\gamma}{}^{*}_{\gimel} \geq \mathscr{C}^{\gamma}{}_{\gimel} \Leftrightarrow 1 - \mathscr{C}^{\gamma}{}^{*}_{\gimel} \leq 1 - \mathscr{C}^{\gamma}{}_{\gimel}$
$\Leftrightarrow (1 - \mathscr{C}^{\gamma}{}^{*}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \leq (1 - \mathscr{C}^{\gamma}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}$
$\Leftrightarrow \overline{\prod}_{\gimel=1}^{n}(1 - \mathscr{C}^{\gamma}{}^{*}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \leq \overline{\prod}_{\gimel=1}^{n}(1 - \mathscr{C}^{\gamma}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}$
$\Leftrightarrow 1 - \overline{\prod}_{\gimel=1}^{n}(1 - \mathscr{C}^{\gamma}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \leq 1 - \overline{\prod}_{\gimel=1}^{n}(1 - \mathscr{C}^{\gamma}{}^{*}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}$.

Now:

$\zeta^{\tau}{}^{*}_{\gimel} \leq \zeta^{\tau}{}_{\gimel} \Leftrightarrow (\zeta^{\tau}{}^{*}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \leq (\zeta^{\tau}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}$
$\Leftrightarrow \overline{\prod}_{\gimel=1}^{n}(\zeta^{\tau}{}^{*}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \leq \overline{\prod}_{\gimel=1}^{n}(\zeta^{\tau}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}$.

And:

$\mathscr{J}^{\aleph}{}^{*}_{\gimel} \leq \mathscr{J}^{\aleph}{}_{\gimel}$
$\Leftrightarrow (\mathscr{J}^{\aleph}{}^{*}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \leq (\mathscr{J}^{\aleph}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}$.
$\Leftrightarrow \overline{\prod}_{\gimel=1}^{n}(\mathscr{J}^{\aleph}{}^{*}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \leq \overline{\prod}_{\gimel=1}^{n}(\mathscr{J}^{\aleph}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}$.
Let

$$\overline{\daleth^{\varsigma}} = \text{LDFWG}(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n)$$

and

$$\overline{\daleth^{\varsigma *}} = \text{LDFWG}(\daleth^{\varsigma}{}^{*}_1, \daleth^{\varsigma}{}^{*}_2, \ldots \daleth^{\varsigma}{}^{*}_n).$$

We get that $\overline{\daleth^{\varsigma *}} \geq \overline{\daleth^{\varsigma}}$. So,

$$\text{LDFWG}(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) \leq \text{LDFWG}(\daleth^{\varsigma *}_1, \daleth^{\varsigma *}_2, \ldots \daleth^{\varsigma *}_n).$$

3.4 LDFOWG Operator

Definition 19 Consider $\daleth^{\varsigma}{}_{\gimel} = (\langle \zeta^{\tau}{}_{\gimel}, \eta^{\upsilon}{}_{\gimel} \rangle, \langle \mathscr{J}^{\aleph}{}_{\gimel}, \mathscr{C}^{\gamma}{}_{\gimel} \rangle)$ is the agglomeration of LDFNs and $LDFOWG : \$^n \to \$$ be a mapping

$$\text{LDFOWG}(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) = \daleth^{\varsigma}{}^{\mathfrak{P}^{\gamma}{}_1}_{\sigma(1)} \otimes \daleth^{\varsigma}{}^{\mathfrak{P}^{\gamma}{}_2}_{\sigma(2)} \otimes \ldots, \otimes \daleth^{\varsigma}{}^{\mathfrak{P}^{\gamma}{}_n}_{\sigma(n)}, \tag{14.13}$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$, such that $\daleth^{\varsigma}{}_{\sigma(r-1)} \geq \daleth^{\varsigma}{}_{\sigma(r)}$, for any r. Then the mapping LDFOWG is called LDFOWG operator, where $(\mathfrak{P}^{\gamma}{}_1, \mathfrak{P}^{\gamma}{}_2, \ldots, \mathfrak{P}^{\gamma}{}_n)$ be the WV with the constraint $\mathfrak{P}^{\gamma}{}_i > 0$ and $\sum_{i=1}^{n} \mathfrak{P}^{\gamma}{}_i = 1$.

We may also consider LDFOWG using the theorem below based on LDFNs operational law.

Theorem 20 *Assume that $\daleth^{\varsigma}{}_{\gimel} = (\langle \zeta^{\tau}{}_{\gimel}, \eta^{\upsilon}{}_{\gimel} \rangle, \langle \mathscr{J}^{\aleph}{}_{\gimel}, \mathscr{C}^{\gamma}{}_{\gimel} \rangle)$ is the agglomeration of LDFNs, and we can find LDFOWG by LDFOWG$(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n)$*

$$= \left(\left\langle \overline{\prod}_{\gimel=1}^{n} \zeta^{\tau}{}^{\mathfrak{P}^{\gamma}{}_{\gimel}}_{\sigma(j)}, 1 - \overline{\prod}_{\gimel=1}^{n} (1 - \eta^{\upsilon}{}_{\sigma(j)})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle, \right.$$
$$\left. \left\langle \overline{\prod}_{\gimel=1}^{n} \mathscr{J}^{\aleph}{}^{\mathfrak{P}^{\gamma}{}_{\gimel}}_{\sigma(j)}, 1 - \overline{\prod}_{\gimel=1}^{n} (1 - \mathscr{C}^{\gamma}{}_{\sigma(j)})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle \right). \tag{14.14}$$

Theorem 21 (Monotonicity) *Assume that $\daleth^{\varsigma}{}_{\gimel} = (\langle \zeta^{\tau}{}_{\gimel}, \eta^{\upsilon}{}_{\gimel} \rangle, \langle \mathscr{J}^{\aleph}{}_{\gimel}, \mathscr{C}^{\gamma}{}_{\gimel} \rangle)$ and $\daleth^{\varsigma *}_{\gimel} = (\langle \zeta^{\tau *}_{\gimel}, \eta^{\upsilon *}_{\gimel} \rangle, \langle \mathscr{J}^{\aleph *}_{\gimel}, \mathscr{C}^{\gamma *}_{\gimel} \rangle)$ are the agglomerations of LDFNs. If $\zeta^{\tau *}_{\gimel} \geq \zeta^{\tau}{}_{\gimel}$, $\eta^{\upsilon *}_{\gimel} \leq \eta^{\upsilon}{}_{\gimel}$, $\mathscr{J}^{\aleph *}_{\gimel} \geq \mathscr{J}^{\aleph}{}_{\gimel}$, and $\mathscr{C}^{\gamma *}_{\gimel} \leq \mathscr{C}^{\gamma}{}_{\gimel}$ for all j, then*

$$LDFOWG(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) \leq LDFOWG(\daleth^{\varsigma *}_1, \daleth^{\varsigma *}_2, \ldots \daleth^{\varsigma *}_n).$$

Theorem 22 *Assume that $\daleth^{\varsigma}{}_{\gimel} = (\langle \zeta^{\tau}{}_{\gimel}, \eta^{\upsilon}{}_{\gimel} \rangle, \langle \mathscr{J}^{\aleph}{}_{\gimel}, \mathscr{C}^{\gamma}{}_{\gimel} \rangle)$ and $\digamma^{\gamma}{}_{\gimel} = (\langle \phi_{\gimel}, \varphi_{\gimel} \rangle, \langle \mathscr{K}_{\gimel}, \mathscr{M}_{\gimel} \rangle)$ are two families of LDFNs. If $r > 0$ and $\digamma^{\gamma} = (\langle \zeta^{\tau}{}_{\digamma^{\gamma}}, \eta^{\upsilon}{}_{\digamma^{\gamma}} \rangle, \langle \mathscr{J}^{\aleph}{}_{\digamma^{\gamma}}, \mathscr{C}^{\gamma}{}_{\digamma^{\gamma}} \rangle)$ is an LDFN, then:*

1. *$LDFOWG(\daleth^{\varsigma}{}_1 \oplus \digamma^{\gamma}, \daleth^{\varsigma}{}_2 \oplus \digamma^{\gamma}, \ldots \daleth^{\varsigma}{}_n \oplus \digamma^{\gamma}) = LDFOWG(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) \oplus \digamma^{\gamma}$*
2. *$LDFOWG(r\daleth^{\varsigma}{}_1, r\daleth^{\varsigma}{}_2, \ldots r\daleth^{\varsigma}{}_n) = r \; LDFOWG(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n)$*
3. *$LDFOWG(\daleth^{\varsigma}{}_1 \oplus \digamma^{\gamma}{}_1, \daleth^{\varsigma}{}_2 \oplus \digamma^{\gamma}{}_2, \ldots \daleth^{\varsigma}{}_n \oplus \digamma^{\gamma}{}_n) = LDFOWG(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) \oplus LDFOWG(\digamma^{\gamma}{}_1, \digamma^{\gamma}{}_2, \ldots \digamma^{\gamma}{}_n)$*

4. *LDFOWG*$(r\daleth^{\varsigma}{}_1 \oplus F^{\gamma}, r\daleth^{\varsigma}{}_2 \oplus F^{\gamma}, \ldots \oplus r\daleth^{\varsigma}{}_n \oplus F^{\gamma}) = r$*LDFOWG*$(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) \oplus F^{\gamma}$

4 Proposed Methodology Based on Developed AOs

Let $\mathscr{T}^{\beth} = \{\mathscr{T}^{\beth}{}_1, \mathscr{T}^{\beth}{}_2, \ldots, \mathscr{T}^{\beth}{}_m\}$ and $\breve{\mathscr{G}}^{\zeta} = \{\breve{\mathscr{G}}_1^{\zeta}, \breve{\mathscr{G}}_2^{\zeta}, \ldots, \breve{\mathscr{G}}_n^{\zeta}\}$ be the alternatives and criterion, respectively. DM offered his judgement matrix $D = (\aleph_{ij}^{\Bbbk})_{m\times n}$, in which $\aleph_{ij}^{\Bbbk}$ stands for the alternate $\mathscr{T}^{\beth}{}_i \in \mathscr{T}^{\beth}$ as per the parameter $\breve{\mathscr{G}}_{\beth}^{\zeta} \in \breve{\mathscr{G}}^{\zeta}$ by DM. The matrix D has converted into "normalized matrix" by the given formula $Y = (\varsigma^{\vartheta\wp}{}_{ij})_{m\times n}$,"

$$(\varsigma^{\vartheta\wp}{}_{ij})_{m\times n} = \begin{cases} (\aleph_{ij}^{\Bbbk})^c; & j \in \tau_c \\ \aleph_{ij}^{\Bbbk}; & j \in \tau_b, \end{cases} \tag{14.15}$$

where $(\aleph_{ij}^{\Bbbk})^c$ denotes the compliment of $\aleph_{ij}^{\Bbbk}$.

The MCDM will be updated to include the suggested operators, which will make the previously described processes necessary.

Algorithm

Step 1:
Acquire the judgment matrix $D = (\aleph_{ij}^{\Bbbk})_{m\times n}$ based on LDFNs from DMs.

$$\begin{array}{c} \\ \mathscr{T}^{\beth}{}_1 \\ \mathscr{T}^{\beth}{}_2 \\ \vdots \\ \mathscr{T}^{\beth}{}_m \end{array} \begin{array}{ccc} \breve{\mathscr{G}}_1 & \breve{\mathscr{G}}_2 & \breve{\mathscr{G}}_n \\ \left[\begin{array}{c} (\langle\zeta^{\tau}{}_{11}, \eta^{\upsilon}{}_{11}\rangle, \langle\mathscr{J}^{\aleph}{}_{11}, \mathscr{C}^{\gamma}{}_{11}\rangle) \\ (\langle\zeta^{\tau}{}_{21}, \eta^{\upsilon}{}_{21}\rangle, \langle\mathscr{J}^{\aleph}{}_{21}, \mathscr{C}^{\gamma}{}_{21}\rangle) \\ \vdots \\ (\langle\zeta^{\tau}{}_{m1}, \eta^{\upsilon}{}_{m1}\rangle, \langle\mathscr{J}^{\aleph}{}_{m1}, \mathscr{C}^{\gamma}{}_{m1}\rangle) \end{array} \right. & \begin{array}{c} (\langle\zeta^{\tau}{}_{12}, \eta^{\upsilon}{}_{12}\rangle, \langle\mathscr{J}^{\aleph}{}_{12}, \mathscr{C}^{\gamma}{}_{12}\rangle) \\ (\langle\zeta^{\tau}{}_{22}, \eta^{\upsilon}{}_{22}\rangle, \langle\mathscr{J}^{\aleph}{}_{22}, \mathscr{C}^{\gamma}{}_{22}\rangle) \\ \vdots \\ (\langle\zeta^{\tau}{}_{m2}, \eta^{\upsilon}{}_{m2}\rangle, \langle\mathscr{J}^{\aleph}{}_{m2}, \mathscr{C}^{\gamma}{}_{m2}\rangle) \end{array} & \begin{array}{cc} \cdots\cdots & (\langle\zeta^{\tau}{}_{1n}, \eta^{\upsilon}{}_{1n}\rangle, \langle\mathscr{J}^{\aleph}{}_{1n}, \mathscr{C}^{\gamma}{}_{1n}\rangle) \\ \cdots\cdots & (\langle\zeta^{\tau}{}_{2n}, \eta^{\upsilon}{}_{2n}\rangle, \langle\mathscr{J}^{\aleph}{}_{2n}, \mathscr{C}^{\gamma}{}_{2n}\rangle) \\ \ddots\ddots & \vdots \\ \cdots\cdots & (\langle\zeta^{\tau}{}_{mn}, \eta^{\upsilon}{}_{mn}\rangle, \langle\mathscr{J}^{\aleph}{}_{mn}, \mathscr{C}^{\gamma}{}_{mn}\rangle) \end{array} \left. \right] \end{array}.$$

Step 2:
There is no need for normalization if all indicators are of the same kind. The matrix D has amended to "transforming response matrix, $Y = (\varsigma^{\vartheta\wp}{}_{ij})_{m\times n}$" by Eq. 14.15.

Step 3:
Aggregate $\mathscr{R}_{ij}^{S}$ for all alternates $\mathscr{T}\daleth_i$ by utilizing the LDFWA (LDFWG) operator.
$\mathscr{R}^{S}{}_{ij} = LDFWA(\varsigma^{\vartheta\wp}{}_{i1}, \varsigma^{\vartheta\wp}{}_{i2}, \ldots \varsigma^{\vartheta\wp}{}_{in})$ or
$\mathscr{R}^{S}{}_{ij} = LDFWG(\varsigma^{\vartheta\wp}{}_{i1}, \varsigma^{\vartheta\wp}{}_{i2}, \ldots \varsigma^{\vartheta\wp}{}_{in})$.
Step 4:
Compute the score against all the alternatives.
Step 5:
The SF was used to classify the alternatives, and the most appropriate option was chosen.

5 MCDM Example

Multi-criteria decision-making (MCDM) is a useful tool for agricultural decision-making as it allows for the consideration of multiple conflicting objectives and constraints. These may include economic, environmental, and social factors. The use of MCDM can lead to more sustainable and efficient farming practices, as well as improved decision-making for farmers and policymakers.

Some specific applications of MCDM in agriculture include:

- Land use planning: MCDM can be used to evaluate and compare different land use options, such as crop rotation, irrigation systems, and conservation practices. This can help farmers and policymakers make more informed decisions about how to use land resources in a sustainable and efficient way.
- Crop selection: MCDM can be used to evaluate and compare different crop options, taking into account factors such as yield, profitability, water usage, and environmental impact. This can help farmers make more informed decisions about which crops to grow, leading to increased productivity and sustainability.
- Livestock management: MCDM can be used to evaluate and compare different livestock management options, such as feed management, breeding strategies, and disease control. This can help farmers make more informed decisions about how to raise and manage livestock in a sustainable and efficient way.
- Water management: MCDM can be used to evaluate and compare different water management options, such as irrigation systems, water storage, and conservation practices. This can help farmers and policymakers make more informed decisions about how to use water resources in a sustainable and efficient way.
- Climate change mitigation: MCDM can be used to evaluate and compare different mitigation options, such as crop rotation, irrigation systems, and conservation practices. This can help farmers and policymakers make more informed decisions about how to adapt to and mitigate the impacts of climate change.

It is important to note that MCDM is not a one-size-fits-all solution and that the specific method used will depend on the specific problem being addressed and the available data. Additionally, it is important to involve stakeholders in the decision-making process to ensure that the results are socially acceptable.

MCDM is a useful tool for agricultural decision-making as it allows for the consideration of multiple conflicting objectives and constraints. Its applications in agriculture include land use planning, crop selection, livestock management, water management, and climate change mitigation. It can lead to more sustainable and efficient farming practices, as well as improved decision-making for farmers and policymakers. However, it is important to use appropriate method and involving stakeholders in the decision-making process.

Agriculture is a significant contributor to Pakistan's economy, accounting for 18.9 percent of the country's gross domestic product and employing 42.3 percent of the labor force. In addition to this, it is a significant source of revenues from international commerce, and it encourages growth in a variety of other areas. To boost development in this field, the public authority is focusing on aiding small and marginalized ranchers and pushing limited scope creative solutions. The sixth population and housing census that was conducted in Pakistan in 2017 revealed that the country's overall population is expanding at a pace of 2.4 percent on an annual basis. Demand for goods produced by agriculture is expected to rise as a result of the fast population expansion. The current administration is centered on advancing this area and has begun various measures, for example, crop expansion, decreasing increase rates, proficient utilization of water, and advancement of high worth yields including biotechnology, agribusiness credit advancement, subsidized manure costs, and modest power for negritude wells. As a result, this current area's exhibition expanded complicated after undergoing moderate and slowed expansion over the previous 13 years.

Consider the decision-making challenge of determining the best agricultural land. Assume the agglomeration of choices, $\mathscr{T}^{\daleth}{}_1$, $\mathscr{T}^{\daleth}{}_2$, $\mathscr{T}^{\daleth}{}_3$, and $\mathscr{T}^{\daleth}{}_4$, also considering four criterions, $\wp^{\Re}{}_1$= irrigation, $\wp^{\Re}{}_2$= cost, $\wp^{\Re}{}_3$=soil, and $\wp^{\Re}{}_4$= processing industry and market. Assuming that the criteria were weighted as (0.25, 0.40, 0.20, 0.15).

Algorithm

5.1 With LDFWA Operator

Step 1:
Obtain matrix $D = (\aleph_{ij}^{\Bbbk})_{m\times n}$ by DM, which is shown in Table 14.2.
Step 2:
In this case, $\mathscr{G}_{\mathbf{2}}^{\breve{\zeta}}$ criteria are cost type criteria that all are the benefits types, so there is need of normalization. Normalized LDF-decision matrix is given in Table 14.1.
Step 3:
Aggregate the LDF values $\mathscr{R}^{S}{}_{ij}$ for all $\mathscr{T}^{\daleth}{}_i$ using LDFWA operator, given in Table 14.3.

Table 14.1 Normalized LDF-decision matrix

		$\breve{\mathscr{G}}^{\zeta}_{\mathbf{1}}$	$\breve{\mathscr{G}}^{\zeta}_{\mathbf{2}}$	$\breve{\mathscr{G}}^{\zeta}_{\mathbf{3}}$	$\breve{\mathscr{G}}^{\zeta}_{\mathbf{4}}$
	$\mathscr{T}^{\daleth}{}_1$	$(\langle 0.50, 0.85\rangle, \langle 0.30, 0.10\rangle)$	$(\langle 0.70, 0.45\rangle, \langle 0.20, 0.25\rangle)$	$(\langle 0.65, 0.75\rangle, \langle 0.45, 0.25\rangle)$	$(\langle 0.85, 0.80\rangle, \langle 0.40, 0.20\rangle)$
	$\mathscr{T}^{\daleth}{}_2$	$(\langle 0.80, 0.90\rangle, \langle 0.45, 0.15\rangle)$	$(\langle 0.65, 0.45\rangle, \langle 0.35, 0.55\rangle)$	$(\langle 0.75, 0.45\rangle, \langle 0.40, 0.30\rangle)$	$(\langle 0.65, 0.85\rangle, \langle 0.45, 0.35\rangle)$
	$\mathscr{T}^{\daleth}{}_3$	$(\langle 0.35, 0.65\rangle, \langle 0.50, 0.20\rangle)$	$(\langle 0.95, 0.65\rangle, \langle 0.65, 0.25\rangle)$	$(\langle 0.45, 0.90\rangle, \langle 0.30, 0.45\rangle)$	$(\langle 0.55, 0.95\rangle, \langle 0.50, 0.30\rangle)$
	$\mathscr{T}^{\daleth}{}_4$	$(\langle 0.50, 0.50\rangle, \langle 0.50, 0.25\rangle)$	$(\langle 0.55, 0.90\rangle, \langle 0.40, 0.50\rangle)$	$(\langle 0.45, 0.65\rangle, \langle 0.35, 0.50\rangle)$	$(\langle 0.35, 0.65\rangle, \langle 0.30, 0.20\rangle)$

Table 14.2 Rating given by DM

		$\breve{\mathscr{G}}^{\zeta}_{\mathbf{1}}$	$\breve{\mathscr{G}}^{\zeta}_{\mathbf{2}}$	$\breve{\mathscr{G}}^{\zeta}_{\mathbf{3}}$	$\breve{\mathscr{G}}^{\zeta}_{\mathbf{4}}$
	$\mathscr{T}^{\daleth}{}_1$	$(\langle 0.50, 0.85\rangle, \langle 0.30, 0.10\rangle)$	$(\langle 0.45, 0.70\rangle, \langle 0.25, 0.20\rangle)$	$(\langle 0.65, 0.75\rangle, \langle 0.45, 0.25\rangle)$	$(\langle 0.85, 0.80\rangle, \langle 0.40, 0.20\rangle)$
	$\mathscr{T}^{\daleth}{}_2$	$(\langle 0.80, 0.90\rangle, \langle 0.45, 0.15\rangle)$	$(\langle 0.45, 0.65\rangle, \langle 0.55, 0.35\rangle)$	$(\langle 0.75, 0.45\rangle, \langle 0.40, 0.30\rangle)$	$(\langle 0.65, 0.85\rangle, \langle 0.45, 0.35\rangle)$
	$\mathscr{T}^{\daleth}{}_3$	$(\langle 0.35, 0.65\rangle, \langle 0.50, 0.20\rangle)$	$(\langle 0.65, 0.95\rangle, \langle 0.25, 0.65\rangle)$	$(\langle 0.45, 0.90\rangle, \langle 0.30, 0.45\rangle)$	$(\langle 0.55, 0.95\rangle, \langle 0.50, 0.30\rangle)$
	$\mathscr{T}^{\daleth}{}_4$	$(\langle 0.50, 0.50\rangle, \langle 0.50, 0.25\rangle)$	$(\langle 0.90, 0.55\rangle, \langle 0.50, 0.40\rangle)$	$(\langle 0.45, 0.65\rangle, \langle 0.35, 0.50\rangle)$	$(\langle 0.35, 0.65\rangle, \langle 0.30, 0.20\rangle)$

Table 14.3 LDF-aggregated values $\mathscr{R}^{\mathcal{S}}{}_i$

$\mathscr{R}^{\mathcal{S}}{}_1$	$(\langle 0.596248, 0.760098\rangle, \langle 0.32997, 0.175855\rangle)$
$\mathscr{R}^{\mathcal{S}}{}_2$	$(\langle 0.769462, 0.522578\rangle, \langle 0.523542, 0.612701\rangle)$
$\mathscr{R}^{\mathcal{S}}{}_3$	$(\langle 0.503278, 0.624946\rangle, \langle 0.708147, 0.613116\rangle)$
$\mathscr{R}^{\mathcal{S}}{}_4$	$(\langle 0.482460, 0.581847\rangle, \langle 0.532108, 0.399725\rangle)$

Step 4:
Compute the score for all LDF-aggregated values $\mathscr{R}^{\mathcal{S}}{}_i$.

$$\breve{\mathscr{C}}^{\beth}(\mathscr{R}^{\mathcal{S}}{}_1) = 0.497566$$

$$\breve{\mathscr{C}}^{\beth}(\mathscr{R}^{\mathcal{S}}{}_2) = 0.539431$$

$$\breve{\mathscr{C}}^{\beth}(\mathscr{R}^{\mathcal{S}}{}_3) = 0.493341$$

$$\breve{\mathscr{C}}^{\beth}(\mathscr{R}^{\mathcal{S}}{}_4) = 0.508249$$

Step 5:
Ranks according to SFs.

$$\mathscr{R}^{\mathcal{S}}{}_2 \succ \mathscr{R}^{\mathcal{S}}{}_4 \succ \mathscr{R}^{\mathcal{S}}{}_1 \succ \mathscr{R}^{\mathcal{S}}{}_3.$$

So,

$$\mathscr{T}^{\beth}{}_2 \succ \mathscr{T}^{\beth}{}_4 \succ \mathscr{T}^{\beth}{}_1 \succ \mathscr{T}^{\beth}{}_3$$

$\mathscr{T}^{\beth}{}_2$ is the best alternative among all other alternatives.

5.2 With LDFWG Operator

Step 1:
Obtain matrix $D = (\aleph_{ij}^{\Bbbk})_{m\times n}$ by DM, which is shown in Table 14.4.
Step 2:
In this case, $\breve{\mathscr{G}}_{\mathbf{2}}^{\zeta}$ criteria are cost type criteria that all are the benefits types, so there is need of normalization. Normalized LDF-decision matrix is given in Table 14.5.
Step 3:
Aggregate the LDF values $\mathscr{R}^{\mathcal{S}}{}_{ij}$ for all $\mathscr{T}^{\beth}{}_i$ using LDFWG operator, given in Table 14.6.
Step 4:
Compute the score for all LDF-aggregated values $\mathscr{R}^{\mathcal{S}}{}_i$.

$$\breve{\mathscr{C}}^{\beth}(\mathscr{R}^{\mathcal{S}}{}_1) = 0.476266$$

Table 14.4 Rating given by DM

		$\breve{\mathscr{G}}^{\zeta}_{\mathbf{1}}$	$\breve{\mathscr{G}}^{\zeta}_{\mathbf{2}}$	$\breve{\mathscr{G}}^{\zeta}_{\mathbf{3}}$	$\breve{\mathscr{G}}^{\zeta}_{\mathbf{4}}$
	$\mathscr{T}^{\daleth}{}_1$	$(\langle 0.50, 0.85\rangle, \langle 0.30, 0.10\rangle)$	$(\langle 0.45, 0.70\rangle, \langle 0.25, 0.20\rangle)$	$(\langle 0.65, 0.75\rangle, \langle 0.45, 0.25\rangle)$	$(\langle 0.85, 0.80\rangle, \langle 0.40, 0.20\rangle)$
	$\mathscr{T}^{\daleth}{}_2$	$(\langle 0.80, 0.90\rangle, \langle 0.45, 0.15\rangle)$	$(\langle 0.45, 0.65\rangle, \langle 0.55, 0.35\rangle)$	$(\langle 0.75, 0.45\rangle, \langle 0.40, 0.30\rangle)$	$(\langle 0.65, 0.85\rangle, \langle 0.45, 0.35\rangle)$
	$\mathscr{T}^{\daleth}{}_3$	$(\langle 0.35, 0.65\rangle, \langle 0.50, 0.20\rangle)$	$(\langle 0.65, 0.95\rangle, \langle 0.25, 0.65\rangle)$	$(\langle 0.45, 0.90\rangle, \langle 0.30, 0.45\rangle)$	$(\langle 0.55, 0.95\rangle, \langle 0.50, 0.30\rangle)$
	$\mathscr{T}^{\daleth}{}_4$	$(\langle 0.50, 0.50\rangle, \langle 0.50, 0.25\rangle)$	$(\langle 0.90, 0.55\rangle, \langle 0.50, 0.40\rangle)$	$(\langle 0.45, 0.65\rangle, \langle 0.35, 0.50\rangle)$	$(\langle 0.35, 0.65\rangle, \langle 0.30, 0.20\rangle)$

Table 14.5 Normalized LDF-decision matrix

		$\breve{\mathscr{G}}^{\zeta}_{\mathbf{1}}$	$\breve{\mathscr{G}}^{\zeta}_{\mathbf{2}}$	$\breve{\mathscr{G}}^{\zeta}_{\mathbf{3}}$	$\breve{\mathscr{G}}^{\zeta}_{\mathbf{4}}$
	$\mathscr{T}^{\daleth}{}_1$	$(\langle 0.50, 0.85\rangle, \langle 0.30, 0.10\rangle)$	$(\langle 0.70, 0.45\rangle, \langle 0.20, 0.25\rangle)$	$(\langle 0.65, 0.75\rangle, \langle 0.45, 0.25\rangle)$	$(\langle 0.85, 0.80\rangle, \langle 0.40, 0.20\rangle)$
	$\mathscr{T}^{\daleth}{}_2$	$(\langle 0.80, 0.90\rangle, \langle 0.45, 0.15\rangle)$	$(\langle 0.65, 0.45\rangle, \langle 0.35, 0.55\rangle)$	$(\langle 0.75, 0.45\rangle, \langle 0.40, 0.30\rangle)$	$(\langle 0.65, 0.85\rangle, \langle 0.45, 0.35\rangle)$
	$\mathscr{T}^{\daleth}{}_3$	$(\langle 0.35, 0.65\rangle, \langle 0.50, 0.20\rangle)$	$(\langle 0.95, 0.65\rangle, \langle 0.65, 0.25\rangle)$	$(\langle 0.45, 0.90\rangle, \langle 0.30, 0.45\rangle)$	$(\langle 0.55, 0.95\rangle, \langle 0.50, 0.30\rangle)$
	$\mathscr{T}^{\daleth}{}_4$	$(\langle 0.50, 0.50\rangle, \langle 0.50, 0.25\rangle)$	$(\langle 0.55, 0.90\rangle, \langle 0.40, 0.50\rangle)$	$(\langle 0.45, 0.65\rangle, \langle 0.35, 0.50\rangle)$	$(\langle 0.35, 0.65\rangle, \langle 0.30, 0.20\rangle)$

Table 14.6 LDF-aggregated values $\mathscr{R}^{S}{}_{i}$

$\mathscr{R}^{S}{}_{1}$	$(\langle 0.547045, 0.771117\rangle, \langle 0.315797, 0.18666\rangle)$
$\mathscr{R}^{S}{}_{2}$	$(\langle 0.581468, 0.547835\rangle, \langle 0.469927, 0.700454\rangle)$
$\mathscr{R}^{S}{}_{3}$	$(\langle 0.442722, 0.812834\rangle, \langle 0.547528, 0.796085\rangle)$
$\mathscr{R}^{S}{}_{4}$	$(\langle 0.461491, 0.670541\rangle, \langle 0.503649, 0.468701\rangle)$

$$\breve{\mathscr{C}}^{\daleth}(\mathscr{R}^{S}{}_{2}) = 0.480777$$

$$\breve{\mathscr{C}}^{\daleth}(\mathscr{R}^{S}{}_{3}) = 0.345333$$

$$\breve{\mathscr{C}}^{\daleth}(\mathscr{R}^{S}{}_{4}) = 0.456474.$$

Step 5:
Ranks according to SFs.

$$\mathscr{R}^{S}{}_{2} \succ \mathscr{R}^{S}{}_{1} \succ \mathscr{R}^{S}{}_{4} \succ \mathscr{R}^{S}{}_{3}.$$

So,

$$\mathscr{T}^{\daleth}{}_{2} \succ \mathscr{T}^{\daleth}{}_{1} \succ \mathscr{T}^{\daleth}{}_{4} \succ \mathscr{T}^{\daleth}{}_{3}$$

$\mathscr{T}^{\daleth}{}_{2}$ is the best alternative among all other alternatives.

6 Conclusion

MCDM is a significant real-world decision issue, and its most fundamental and essential research is the expression of imprecise information. IFSs, PFSs, and q-ROFSs are all effective methods for handling fuzzy information. LDFSs are more generic than IFS, PFS, and q-ROFS due to their ability to loosen the severe limitations of IFS, PFS, and q-ROFS by considering RPs. MCDM is a crucial subfield in operation research. This assignment's techniques mostly rely on the nature of the issue being evaluated. Our everyday occurrences include unpredictability, imprecision, and obscurity. Existing structures were constructed on the basis of the concept that decision-makers (DMs) consider specific limitations while assessing various choices and qualities. However, this kind of situation makes it difficult for DMs to allocate MSDs and NMSDs; therefore, they do so with different constraints. LDFS is a novel method to uncertainty and decision-making issues that incorporates pairs of RPs versus MSDs and NMSDs in order to loosen these limits. We have used LDFSs to assess the validity of DMs' knowledge in the basic framework and to remove any distortion in the decision analysis. The significant advantage of including RPs into the examination is to reduce the likelihood of theoretical knowledge-based MSD and NMSD-related mistakes. In addition, we have developed a number of AOs, including the LDFWA operator and the LDFWG operator. Numerous intriguing aspects of the suggested operators are investigated, and their illustration is convincingly shown.

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