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Fuzzy Optimization, Decision-making and Operations Research

Theory and Applications

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ISBN 978-3-031-35667-4 ISBN 978-3-031-35668-1 (eBook)
https://doi.org/10.1007/978-3-031-35668-1

Mathematics Subject Classification: 90B50, 05C72, 90B05, 90B06, 03B52

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

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Preface

Fuzzy set theory is a powerful mathematical tool for dealing with uncertainty connected with the imprecision of states, perceptions, preferences, etc. With the rise of an intelligent era in human society, fuzzy sets and related applications have become increasingly active research topics in various fields. These are widely scattered over many disciplines, such as algebraic structures, artificial intelligence, computer science, control engineering, data mining, decision analysis, expert systems, management science, non-classical logic, operations research, pattern recognition, and robotics, among others.

Fuzzy optimization is a well-known topic in manufacturing and management organizations in artificial intelligence, so establishing general and operable fuzzy optimization methods is essential in both theory and application. In 1965, Zadeh originated the concept of fuzzy sets, which formed the foundation for describing and processing uncertain information. Later, much progress was made in both theory and application. The fuzzy numbers, an essential part of the fuzzy set, are prevalent in describing uncertain phenomena in actual problems. It has been suggested for many fuzzy optimizations, control, data analysis, etc.

Due to the advancement of fuzzy logic/mathematics, decision-making problems have become more realistic. Several sophisticated and intelligent systems have been developed to solve them. Lots of fuzzy operators are developed to solve fuzzy decision-making problems. Since the real world is full of uncertainty, many real-life problems are modeled as problems of operations research, and lots of problems in inventory management are solved using fuzzy optimization techniques along with game theory, project management problems, linear and nonlinear problems, and many others. On the other hand, many optimizations and decision-making problems are modeled as fuzzy graphs. Recently, many daily life problems have been solved using the concept of fuzzy graph theory.

The book starts with some fundamental concepts of fuzzy logic, fuzzy optimization problems, and decision-making problems, which are used in the subsequent chapters. The chapters include the recent development of fuzzy optimization techniques, fuzzy decision-making methods, different fuzzy operators and their applications, the use of graph theory to solve optimization problems, some problems in operations research in a fuzzy environment, etc. Each chapter also discusses some keys and representative applications. Fuzzy optimization and decision-making paved the way for professionals to build many rule-based expert systems. In the literature, many papers are available on the proposed title, but only a few books are available on the same topic. The authors have made a real contribution to putting things together into a comprehensive book.

The book offers an excellent reference guide for advanced undergraduate and graduate students, researchers, and professionals in mathematics, engineering, and computer science, and an inspiring read for all researchers interested in new developments in fuzzy logic, fuzzy optimization, decision-making, and operations research.

A brief contribution of each chapter is given in the following.

In Chap. 1, the fundamental issues of classical optimization and fuzzy optimization are discussed. There is clarity about the feasible and optimal solutions for classical optimization problems, but the conventional concepts need to be validated for fuzzy optimization, and hence new definitions are proposed. The single objective and multi-objective optimization problems for classical and fuzzy environments are discussed. Some modern optimization techniques, viz. genetic algorithms, particle swarm optimization, neural network, etc., are applied to solve such problems. Many parameters are associated with optimization problems and may contain multiple objective functions; some conflict. For such problems, Pareto's optimal solution is determined. Sometimes it is necessary to find the combined effect of the parameters in the solution. To combine the parameters, we need an aggregation process. Recently, many excellent aggregation methods have been available in the literature. Apart from the aggregation process, the t-norms and t-conorms-based operators are used to solve decision-making problems. Some of these operators are studied.

In Chap. 2, the natural notion of adequacy is formulated. In practice, there is often a need to describe the relation $y = f(x)$ between two quantities in the algorithmic form: e.g., the control value y needs to be described corresponding to the given input x, or to predict the future value y based on the current value x. In many such cases, expert knowledge is available about the desired dependence, but experts can only describe their knowledge by using imprecise ("fuzzy") words from a natural language. Methodologies for transforming such knowledge into an algorithm $y = f(x)$ are known as fuzzy methodologies. Several fuzzy methodologies exist; a natural question is: which of them is the most adequate? In this chapter, the natural notion of adequacy is formulated. If the expert rules are formulated based on some functions $y = f(x)$, then the methodology should reconstruct this function as accurately as possible. We show that none of the existing fuzzy methodologies is the most adequate in this sense, and we describe a new fuzzy methodology that is the most adequate.

Chapter 3 deals with measurement-related ideas which are used for decision-making problems. Ultimately, all our knowledge about the world comes from observations and measurements. An important part of this knowledge comes directly from observations and measurements. For example, when a person becomes sick, we can measure this person's body temperature, blood pressure, etc. and thus, usually get a good understanding of the problem. In addition, a significant part of our knowledge comes from experts who – inspired by previous observations and measurements – supplement the measurement results with their estimates. For example, a skilled medical doctor can supplement the measurement results with his/her experience-based intuition. Measurements have existed for several millennia, and many effective techniques have been developed for processing measurement results. In contrast, processing and expert opinions are reasonably new fields with many open problems. A natural idea is thus to see if measurement-related ideas can help to use expert knowledge as well. In this chapter, we discuss three case studies where such help turned out to be possible.

Chapter 4 is devoted to fuzzy control theory. One of the main objectives of fuzzy control is to translate expert rules – formulated in imprecise ("fuzzy") words from natural language – into a precise control strategy. This translation is usually done in two steps. First, we apply a fuzzy control methodology to get a rough approximation of the expert's control strategy, and then we tune the resulting fuzzy control system. The first step (getting a rough approximation) is well analyzed. Having an expert's intuitive understanding enables us to use soft computing techniques to perform in this step. At this first step, we only use expert rules. Then, we test the resulting control on a real or simulated system and tune the resulting control based on the results of this testing. This second (tuning) step is much more difficult: we no longer have any expert understanding of which tuning is better, and therefore, soft computing techniques are not that helpful. In this chapter, we proposed a particular case of the tuning problem as a traditional optimization problem and solve it by using traditional ("hard computing") techniques. In a practical industrial control example, we show that the resulting fusion of soft computing (for a rough approximation) and hard computing (for tuning) leads to high-quality control.

Chapter 5 deals with the framework of the transportation problem. Such problems are structured on the basis of parameters like supply, demand, cost, and quantity. Noys, the tech-savvy consumers globally enjoy the ease and comfortso-calledo called "delivery apps" which make these parameters uncertain and imprecise and hence crisp parameters are unable to handle or represent such situations. The more flexible and generalized fuzzy number, namely the interval-valued intuitionistic fuzzy number, can come in handy to the decision-maker for efficient representation of all these parameters. However, the purpose of the decision-maker is not only to minimize the transportation cost while delivering the article of trade but also to minimize the other associated costs. Many authors have worked with fully intuitionistic fuzzy multi-objective transportation problems with standard linear objective functions with and without using interval-valued intuitionistic fuzzy numbers.

This work proposes a comprehensive novel fully interval-valued intuitionistic fuzzy multi-objective indefinite quadratic transportation problem. The indefinite quadratic objective function being a product of two linear factors is capable of minimizing each of the factors simultaneously. The authenticity of the model is exhibited through a real-life problem scripted by the food industry. The first objective minimizes the transportation cost and the depreciation cost simultaneously. In the second objective, simultaneous minimization of the packaging cost and the associated wastage cost is targeted. The problem is solved through a proposed and existing methodology. The results obtained are discussed, and future work concludes the chapter.

Chapter 6 discusses project management in a fuzzy environment. Fuzzy logic is a very powerful tool to handle non-random uncertain problems. On the other hand, in project management, many issues are uncertain, particularly the time duration to complete an activity. It is very difficult to estimate the exact completion time of an activity, i.e., total completion time of a project. It happens due to the lack of material available in time, availability of labor, etc. So, in this chapter, an activity's time duration is assumed as a triangular fuzzy number (TFN). By considering the TFN as the weight of an activity, the possible completion time of a project is determined using the network analysis technique. An algorithm is designed to find such time. An example also illustrates the algorithm.

In Chap. 7, the concept of generalized Hukuhara (gH)-global subdifferential for interval-valued function (IVF) is proposed. To define this concept, we propose the notions of gH-lower and gH-upper global directional derivatives for IVFs. A few results on the characteristics of gH-lower and gH-upper global subdifferential are studied. Next, a result on the gH-directional derivative of the maximum of comparable IVFs is derived. In the sequel, the gH-lower subdifferential is compared with gH-Fréchet subdifferential, gH-proximal subdifferential, and gH-subdifferential for IVFs. Thereafter, a necessary and sufficient condition for obtaining an efficient solution to an interval optimization problem (IOP) with the help of gH-lower global subdifferential is given.

Chapter 8 discusses a new approach for determining an initial basic feasible solution for both balanced and unbalanced transportation problems. The transportation problem exposes its complexity and inconsistency. When the total of all sources' supplies equals the sum of all destinations' requests, it is called a balanced transportation problem. When the total of all sources' supplies does not meet the sum of all destinations' demands, it is called an unbalanced transportation problem. Here we present a novel approach for determining an initial basic feasible solution for both balanced and unbalanced transportation problems. Appears to involve the max-min method, which has resulted from the transportation problem. It has various approaches to solving the transportation problem. This chapter finds a suitable defuzzification method to convert hexagonal fuzzy numbers to crisp numbers to get the minimum cost.

In Chap. 9, matrix games are investigated in view of healthcare problems. Data privacy and cyber threats are becoming increasingly common in healthcare organizations. Defending against these uncertain digital attacks is today's highest challenge. Cyber security aims to prevent the theft and damage of all categories of classified data. These uncertain problems can be designed as matrix games, with attackers and defenders considered as players. Concerning fuzzy sets/intuitionistic fuzzy sets or ordinary intervals, interval-valued picture fuzzy numbers can be a great way to deal with such uncertain circumstances. Using a matrix game, this chapter examines cyber threat-related issues in the healthcare sector, and interval-valued picture fuzzy numbers are utilized as the players' payoffs. Two nonlinear mathematical programming models with multi-objective functions are constructed and converted into linear programming models with crisp objectives utilizing the weighted average approach. The LINGO platform solves the problems, and the optimal solutions are obtained. In the cyber threat issue, we present a practical example of how to choose the optimal strategies for the medical data controller of a healthcare institution. For the value 0.5 of the parameter, the numerical example shows that the defender can successfully defend the digital attack by 44–59% and failed by 6–14%. There exists an indeterminacy of countering the attack by 7–16%.

Chapter 10 discusses the minimization of span in $L(3,1)$-labeling for a particular type of intersection graph. The interval graph is a very useful subclass of intersection graphs. This class of graphs have many applications for solving real-life problems. One such real-life problem is $L(3,1)$-labeling of the graph. In this chapter, $L(3,1)$-labeling of the interval graph is considered. The $L(3,1)$-labeling of a graph $G=(V,E)$ is a function $\tau : V \rightarrow \{0,1,2,\ldots\}$ so that $|\tau(x)-\tau(y)| \geq 3$ if $d(x,y)=1$ and $|\tau(x)-\tau(y)| \geq 1$ if $d(x,y)=2$. The $L(3,1)$-labeling number $\lambda_{3,1}(G)$ of G is the smallest non-negative integer m, so G has a $L(3,1)$-labeling of span m. In this chapter, $L(3,1)$-labeling of interval graphs is studied and has found good results. It is shown that $\lambda_{3,1}(G) \leq 4\Delta - 1$ for the interval graph, where Δ is the maximum degree of the graph G. Also, an algorithm is designed to label an interval graph by $L(3,1)$-labeling. The running time of the proposed algorithm is also calculated.

Chapter 11 proposes a novel class of neutrosophic sets, combining intuitionistic, neutrosophic, and Pythagorean neutrosophic sets. In addition to identifying their most significant characteristics, their connections and separations are discussed. Comparing them with other neutrosophic sets in the literature demonstrates their significance. This example shows that every NSS is a GNSS, but the converse is not true. Additionally, the pre- and post-images of GNSS are discussed by demonstrating the theory behind these sets with sickness diagnosis as an example. The newly developed category of sets has a wider range of applications than earlier neutrosophic sets by comparing its outcomes with those reported in the literature. Generalized neutrosophic sets will have more applications than other sets.

In Chap. 12, the balanced neutrosophic graph is studied and applied to solve a real-life problem. The neutrosophic graph is an extension of the intuitionistic fuzzy graph. A neutrosophic graph is a necessary tool for handling real-life problems, so studying neutrosophic graphs is welcomed. In this study, a balanced neutrosophic graph is used, and in this graph, we propose an application of alliances of some information technology companies.

In Chap. 13, Hamy mean (HM) information is used to solve decision-making problems. To derive the best preference from the collection of preferences, decision-making information is one of the most important and dominant techniques to evaluate most problems in real-life dilemmas. HM information is also used for aggregating the bundled information into a singleton set computed based on algebraic laws. The main influence of this theory is to propose Dombi operational laws for complex intuitionistic fuzzy (CIF) information and try to construct the theory of HM information based on Dombi t-norm and t-conorm under the consideration of CIF information, called CIF Dombi HM (CIFDHM), CIF weighted Dombi HM (CIFWDHM), CIF Dombi dual HM (CIFDDHM), and CIF weighted Dombi dual HM (CIFWDDHM) operators. Further, some valuable properties and results for the presented information in the investigated analysis are studied. Moreover, MADM “multi-attribute decision-making” information is utilized in this manuscript based on pioneered operators and some examples are given to justify the worth and dominancy of the evaluated information. Finally, comparing the evaluated information with some other old or prevailing information enhances the derived operators’ quality.

Linear Diophantine fuzzy set (LDFS) is studied in Chap. 14. The LDFS is an integral part of the decision-making process under uncertain environments; because of its amazing quality of having a vast portrayal zone for authorized doublets, the LDFS theory expands the region of fuzzy information that may be obtained by using reference parameters. Because the real world is inaccurate, and there needs to be more knowledge, assessing and picking the best option can be challenging and unexpectedly difficult. The primary goal is to guide decision-makers through the process of selecting the best option inside a linear Diophantine fuzzy context. The four new aggregation operators (AOs): the “linear Diophantine fuzzy weighted average (LDFWA) operator, linear Diophantine fuzzy ordered weighted average (LDFOWA) operator, linear Diophantine fuzzy weighted geometric (LDFWG) operator, and linear Diophantine fuzzy ordered weighted geometric (LDFOWG) operator.” The proposed model is then validated using a clear example of linear Diophantine fuzzy content. This demonstrates the utility and applicability of the suggested strategy.

Chapter 15 deals with some decision-making that comes from the real world. Uncertainty is inherent in decision-making (DM) problems that occur in real-world situations. Numerous methods have been devised, but the idea of a fuzzy set (FS) has shown to be the most effective. When it comes to solving DM problems, like multi-criteria decision-making (MCDM), FS has shown to be highly ground-breaking. There has been made more advancement in this area. Herzberg’s two-factor theory inspired the newly proposed hyperbolic fuzzy set (HFS). In this study, a novel HFS-based scoring function is presented. Based on HFS, the Minkowski distance is introduced. Last but not least, a TOPSIS method is constructed based on HFS using the distance measure with some applications.

Chapter 16 deals with the TOPSIS method to solve decision-making problems. The technique for order preference by similarity to ideal solution (TOPSIS) is celebrated for solving decision-making problems. The Pythagorean fuzzy set generalizes the intuitionistic fuzzy set, which can deal with uncertain and incomplete information more appropriately. The motivation of the study is to develop an improved TOPSIS technique in a trapezoidal Pythagorean fuzzy environment, which can handle real-life decision-making problems more effectively and robustly. This chapter defines trapezoidal Pythagorean fuzzy numbers and discusses their various algebraic properties. Then, new distance functions have been defined for the trapezoidal Pythagorean fuzzy environment, which calculates the relative closeness coefficient. Furthermore, an aggregation operator is introduced. This trapezoidal Pythagorean fuzzy weighted arithmetic operator is used to develop an improved TOPSIS technique based on new distance functions under the trapezoidal Pythagorean fuzzy environment. The newly developed TOPSIS technique has been demonstrated and applied to select the best institute/college. Finally, comparative and sensitivity analyses are given to discuss the advantages and reliability of the proposed technique.

Chapter 17 describes the theory of prioritized Aczel–Alsina aggregation operators for complex intuitionistic fuzzy (CIF) information. Prioritized aggregation operators are very famous and reliable because they can help us aggregate information collection into a singleton set. Furthermore, the derived theory of Aczel and Alsina has received valuable and dominant attention from many scholars. In this chapter, the theory of prioritized Aczel–Alsina aggregation operators for CIF information, such as CIF-prioritized Aczel–Alsina averaging (CIFPAAA), CIF-prioritized Aczel–Alsina ordered averaging (CIFPAAOA), CIF-prioritized Aczel–Alsina geometric (CIFPAAG), and CIF-prioritized Aczel–Alsina ordered geometric (CIFPAAOG) operators, are discussed. Moreover, various properties and special cases of the derived work are also examined. Additionally, we expose a MADM "multi-attribute decision-making" technique under the consideration of derived operators. Finally, we illustrated various examples for comparing proposed and existing operators to show the supremacy and validity of the invented theory.

In Chap. 18, a new approach of intuitionistic fuzzy distance is developed and used to predict maternal complications. The problem of maternal complications is prevalent in developing nations. Every pregnant woman is entitled to good reproductive health to enhance safe delivery. In this work, we applied the intuitionistic fuzzy distance measure approach to predict maternal complications. A new approach of intuitionistic fuzzy distance is developed and characterized by certain theoretical results. Owing to the prevalence of maternal complications in developing nations, we deployed the developed intuitionistic fuzzy distance approach for the prediction of maternal complications to avoid maternal mortality during childbirth. Also, there is an improved value in the probability results compared to the previous methods.

Chapter 19 investigates fuzzy pre-predicator model. The current mathematical model is an understanding of the mathematical explanation of different global environmental challenges. The modeling of prey-predator dynamics has garnered the most attention from scientists and ecologists in recent years. The majority of ecologists who study the subject make the assumption that ecological parameters are well understood. The unpredictability in the model can arise for several reasons, including human error, faulty data supply, climatic changes, and other environmental elements, etc. altering the real situation. The fuzzy fractional diabetes model is studied in Caputo's sense, where the initial populations are taken to be a fuzzy number, to address this issue. The proposed method, known as the gH (generalized Hukuhara), derived the idea to clarify the fuzzy suggested system. The leading model is converted into a set of differential equations with a parametric form of when this approach is used in the fuzzy prey-predator system. Here, only two scenarios are analyzed in which prey $G(t)$ and predator $I(t)$ populations are both gH type-I and gH type-II differentiable. The stability conditions of non-negative feasible steady states have been examined in a fuzzy environment. Finally, thorough numerical simulations are performed to validate all the analytical results.

T-spherical fuzzy sets (T-SFSs) have fascinated the desire of researchers in a wide range of domains and it is discussed in Chap. 20. The striking framework of the T-SFS is keen to offer the larger inclination domain for modeling ambiguous information deploying the degrees of membership, neutral and non-membership. Further, T-SFSs prevail over the theories of spherical and Pythagorean fuzzy sets owing to their broader space, adjustable parameter, flexible structure, and influential design. The information measures a significant part of the literature and are crucial and beneficial tools widely applied in making decisions, mining data, diagnosing medical things, and recognizing patterns. This chapter aims to expand the literature on T-SFSs by introducing many innovative T-spherical fuzzy sets' information measures: distance, similarity, entropy, and inclusion. We investigate the relationship between distance, similarity, entropy, and inclusion measures for T-spherical fuzzy sets. Another achievement of this research is to establish a systematic transformation of information measures, measure distance, measure similarity, measure entropy, and measure inclusion for the T-SFSs. To accomplish this aim, a new formula for information measures of T-SFSs has been provided. To demonstrate the criteria of the measures, we employ them to recognize patterns, building materials, and diagnosis of the medical things. Additionally, a comparison between traditional and novel similarity measures is described in terms of counterintuitive cases. The outcomes demonstrate that the innovative information measures do not include any absurd cases.

Chapter 21 proposes a decision-making process which is used in the disaster management system and medical diagnosis using Pythagorean fuzzy information. Pythagorean fuzzy correlation coefficient (PFCC) is a trustworthy information measure to determine sundry real-world decision-making problems. Some authors have worked on methods for calculating PFCC, notwithstanding some limitations which bother accuracy and reliability. In this chapter, two methods for calculating PFCC are developed in a quest to obtain more reliable methods. The methods are adorned with the traditional attributes of the Pythagorean fuzzy set (PFS) to forestall any possibility of exclusive error. Some theoretic results based on the new methods are buttressed in consonant with the attributes of the classical correlation coefficient. To demonstrate the new methods' resourcefulness, real-world problems like disaster control and medical diagnosis are resolved using Pythagorean fuzzy data. The attractiveness of the new methods is portrayed in comparative analysis involving other methods of PFCC to justify the relevance of the new methods as reliable PFCC methods.

In Chap. 22, the q-Rung neutrosophic interval-valued soft set (q-Rung NSIVSS) is a generalization of the interval-valued fuzzy soft set (IVFSS), and the fuzzy soft set (FSS) is discussed extensively. The q-Rung NSIVSS aggregation was discussed through TOPSIS aggregated operation (AO). The TOPSIS method is an effective method for multi-criteria group decision-making (MCGDM), which is an extension of FSS. The objective is to find an ideal positive and negative solution based on q-Rung NSIVSS, aggregating TOPSIS, using a score function. Optimal alternatives are presented to determine closeness values. To strengthen our conclusions, we provide practical examples. This results in the outcome of the models for which q is provided. Comparing the existing models to those that have been proposed allows us to measure the validity and usefulness of the models under consideration. The most recent discoveries have a great deal of fascination and interest.

Chapter 23 deals with the MADM problems using a cosine trigonometric neutrosophic normal interval-valued set. Some novel methods to solve multiple attribute decision-making (MADM) problems using a cosine trigonometric neutrosophic normal interval-valued set (CTri-NNIVS) are introduced. A new concept of cosine trigonometric neutrosophic interval-valued set (NIVS) is introduced, generalizing trigonometric neutrosophic set and NIVS. Our discussion focuses on aggregate operations. We introduce a novel topic of cosine trigonometric neutrosophic normal interval-valued (CTri-NNIV) weighted averaging (CTriNNIVWA), CTri-NNIV weighted geometric (CTri-NNIVWG), CTri-NNIV generalized weighted averaging (CTri-GNNIVWA), and CTri-NNIV generalized weighted geometric (CTri-GNNIVWG). In addition, an algorithmic interaction with the MADM through these operators is designed. The appropriateness of the Hamming distance was discussed, which are discussed in more detail in the examples. As a final part of this chapter, some of the properties of these sets using different operations are discussed. Furthermore, the effectiveness and reliability of the models are demonstrated through comparison with existing models. Finally, the relationship is extended by showing some motivating details and attractive results for operators using CTri-NNIVWA, CTri-NNIVWG, CTri-GNNIVWA, and CTri-GNNIVWG.

Chapter 24 defines a new score function for interval-valued spherical fuzzy (IVFS) sets. The IVSF sets are effective for dealing with uncertainty due to their broader space. The score functions are commonly used to distinguish the IVSF sets. But, the existing score functions of IVSF sets need to distinguish the IVSF sets properly. So, this chapter aims to introduce a noble score function within the IVSF context and apply it to multi-attribute group decision-making (MAGDM). Also, we establish that the proposed score function easily overcomes the limitations of the current score functions. Then, we develop Euclidean distance measures in the context of an IVSF environment. Next, we extend the weighted distance-based approximation (WDBA) under IVSF information to solve MAGDM problems. The analytical hierarchy process to estimate the weights of decision experts is utilized. We develop the entropy method to determine the attribute weights, utilizing the proposed score function. Then, we solve a supplier selection problem for a renowned textile manufacturing company to illustrate the practicality and efficiency of the proposed model. In the numerical example, we consider five suppliers and the six important attributes by taking the opinions of three decision experts. According to the findings of this study, the first supplier is the best for the textile manufacturing industry. Finally, we compared the results to several existing methods to demonstrate the feasibility of our model.

In Chap. 25, a cubic fuzzy graph is investigated and solved a fuzzy optimization problem. These graphs stand as a fuzzy graph type with two fuzzy membership and interval-valued membership values, which is a combination of two different fuzzy values that allow the ambiguous and uncertain variables modeling, where the same format expression is not feasible. In this research, we introduce two important parameters related to the vertices of a graph, i.e., the dominating set and vertex covering in a cubic fuzzy graph, and some of their features were considered. These concepts are developed on some special cubic fuzzy graphs. Accordingly, the domination and vertex covering numbers are shown as real numbers. This has caused a better comparison in the results, and the effect of the resulting number has been seen to be the same in different memberships. Also, some features of dominating set and vertex covering have been studied in complete cubic fuzzy graphs. Finally, an application of these concepts in a decision-making problem is presented.

Chapter 26 studies a production inventory model where the manufacturing system manufactures only perfect quality products within a fractional part of the production duration in the initial stage of production because of in-control state in this stage but produces a mixture of imperfect and perfect quality products within the remaining part of the production time as it becomes out-of-control state due to several factors such as labors, machinery breakdown, etc. During the out-of-control state, to sustain the system's reliability, a development cost depending on the reliability parameter of the system as well as time has been incorporated. Here, the material cost of the product depends on the reliability of the product. And the production cost depends on material cost, development cost, and tool or die cost. Again, customer requirement depends on product reliability, selling price, and advertisement to stimulate customer. Under these considerations, a profit function of the model has been constructed to investigate the feasibility of our model optimizing the parameters connected with the reliability of a production process. Ultimately, a numerical illustration has been made to study the practicability of the model. A sensitivity analysis has been done to show the impact of various parameters involved in the profit function. From the numerical discussion, it is noticed that product reliability impacts raising the demand and, henceforth, increasing the profit. It is also seen that the manufacturing reliability parameter λ reduces as the selling price rises, which explores that the demand is reduced for more selling prices. Again, the decrease in demand rate implies that the production rate is decreased, for which the production reliability of the system is increased.

Chapter 27 considers the stock management system, which is the most significant consideration in the industry of deteriorating goods such as drugs, pharmaceuticals, eatables, and blood. It is important to use preservation technology to keep them useful for a long time. Many inventory models consider deterioration and use preservation technologies to monitor deterioration. This chapter is a contribution involving continuous and instantaneous degradation in which demand is assumed to be exponential, and shortages are not permitted. Preservation technology is used to keep deterioration under control. In most cases, it is optional to make an exact determination of the various costs that are linked with the model. As a result, to get a rough approximation of the results, we first used a method that needed to be more precise than others, called the fuzzy approach. Then the problem is made crisp using the defuzzification technique. In both the crisp and fuzzy scenarios, a mathematical formulation is created and solved with a differentiation tool to determine the procedure that achieves the result with the lowest total cost. For justification of the model, numerical illustrations are provided. Managerial perspectives and parameter observations are obtained by keeping one parameter constant while modifying the others. The effect of changes in total cost, economic order quantity, and optimal time on input parameters is investigated.

Chapter 28 mainly focuses on the economic ordered quantity (EOQ) model under both crisp and uncertain scenarios. Here the uncertainty or the vagueness is clearly described using triangular intuitionistic fuzzy numbers. The chapter has developed the removal area technique to de-intuitificate the triangular intuitionistic fuzzy number. In this respect, an EOQ model is considered where price and stock depend on demand with backlogging, shortages, and inflation. The model has considered two situations for the trade credit period. First, if the supplier arrives to collect the money before the stock end, and second, if the supplier arrives after the completion of the stock. The model is optimized under both situations, and the result is developed for different periods of the arrival of the supplier. A numerical simulation has been performed to check the optimality of the model in different situations. In this chapter, a comparative study is made for both crisp and intuitionistic values, and it is observed that the model works well by applying the de-intuitification technique. Also, a sensitivity analysis is carried out to understand the effect of the key parameters under an optimal situation.

Chapter 29 is about a new approach to developing a general EOQ model where the demand rate is considered a Triangular Cloudy Fuzzy Neutrosophic (TCFN) set. In this chapter, the various membership functions/grades of the components of a TCFN set are defined, and fuzzify the model. Then we find the ultimate score value of the neutrosophic fuzzy elements when the components are dependent on each other and associated with the standard and non-standard fuzzy set. Then the defuzzify technique is applied to the model to get the equivalent classical problem of the neutrosophic model, and we find the solution using a nonlinear optimization procedure. Then we clearly compare the results of different models like Crisp, general fuzzy, and TCFN through a table for proper understanding. At the end of the whole study, we came to the conclusion that considering the TCFN environment for solving any inventory problem is very appropriate and realistic for a decision-maker in the progression of a business. Also, sensitivity analysis and graphical illustrations are made to properly justify the new fuzzy approach.

In Chap. 30, the EOQ model with a fixed goal to reduce the costs as much as possible has been modeled as imprecise decision-making with an acceptance-rejection dilemma in the manager's mind. The intuitionistic fuzzy set theory extends the notion of fuzzy set theory in a more generalized way, including the sense of belongingness (acceptance) and non-belongingness (rejection). A decision-making phenomenon may go through a vague situation with the dilemma of acceptance and rejection. An economic order quantity (EOQ) model having a fixed goal to reduce the costs as much as possible can be a model of imprecise decision-making with an acceptance-rejection dilemma in the manager's mind. Thus, an EOQ model may be considered in an intuitionistic fuzzy uncertain environment.

In a situation where the parameters and decision variables are imprecise in the intuitionistic fuzzy type, it is better to describe the uncertain model with the help of intuitionistic fuzzy calculus. Thus, this chapter aims to discuss the classical EOQ model in uncertain intuitionistic fuzzy phenomena. The intuitionistic fuzzy differential equation approach is considered to describe the fuzzy model. Here, the demand and costs are taken as triangular intuitionistic fuzzy numbers (TIFN). Also, a new defuzzification technique is established in this chapter which is used to compare the results obtained for crisp and intuitionistic fuzzy models. The intuitionistic fuzzy environment with (i) differentiability of $q(t)$ appears to offer the best result for the cost minimization goal among the three discussed approaches, whereas the intuitionistic fuzzy environment with (ii) differentiability of q(t) reveals the worst result, according to the numerical analysis.

Chapter 31 discusses the solution of the second-order linear difference equation in an Intuitionistic fuzzy environment. Fuzzy intuitionistic sets illustrate the idea of ambiguity using degrees of belongingness and non-belongingness. Discrete changes in parameters are represented using difference equations. The extension principle scheme solves the intuitionistic fuzzy linear difference equation. It is detailed and covers all scenarios for solving a second-order linear difference equation with intuitionistic-valued beginning information. An appropriate application and numerical examples are given to demonstrate the suggested theory. To the author's knowledge, the second-order difference equation is solved in an intuitionistic fuzzy environment. The coefficients and initial conditions are taken as triangular intuitionistic fuzzy numbers.

Cooperative games with transferable utilities are studied in Chap. 32. This a probabilistic framework: call them probabilistic games. In this setup, each coalition has some probability of formation and the worth of the grand coalition is the expectation over its sub-coalitions due to this probability distribution. We propose the Shapley function for the class of probabilistic games. A special sub-class of probabilistic games is studied, and the Shapley function for this subclass is characterized. In this special subclass, players make coalitions sequentially. Prior knowledge about their compatibility with one another in a preceding coalition is used to predict the worth of the succeeding coalitions. This is a natural assumption and needs to be studied in the literature.

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Chapter 1
Fundamentals of Fuzzy Optimization and Decision-Making Problems

Madhumangal Pal, Chiranjibe Jana, and Anushree Bhattacharya

1 Optimization Problems

Optimization is derived from the Latin word *Optimus*, meaning "the best." The optimization problems are very important types of problems of mathematics, engineering, and many other files, which find the "best" solution among all possible solutions. Generally, optimization problems occur in mathematics, computer science, economics, business, etc. These problems are generally divided into two categories based on the nature of the variables. The variables are classified as continuous and discrete.

The general form of a continuous optimization problem is

$$\text{Find } X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ which Optimize } f(X).$$

subject to the constraint

$$g_j(X) \leq 0,\ j = 1, 2, \ldots, m$$
$$\text{and } h_i(X) = 0,\ i = 1, 2, \ldots, p$$
$$X \geq 0.$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is the objective function to be optimized based on n number of variables represented by the vector X,
$g_j(x) \leq 0$ are the inequality constraints,
$h_i(x) = 0$ are the equality constraints, and obviously
$m \geq 0$ and $p \geq 0$.

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C. Jana et al. (eds.), *Fuzzy Optimization, Decision-making and Operations Research*, https://doi.org/10.1007/978-3-031-35668-1_1

If $m = p = 0$, i.e., the problem has no constraints. This kind of problem is known as an unconstrained optimization problem, and finding a solution to this type of problem is generally straightforward.

The optimization is either maximization or minimization. The expression "Optimize $f(X)$" is known as the objective function. This function may be single or multiple. Any maximization problem can be transferred into a minimization problem by multiplying the objective function by (-1). Therefore, the standard optimization problem is considered as minimization problem. Also, the objective function(s) or constraint(s) may be linear or nonlinear. If all the objective functions and constraints are linear, the problem is called a linear problem. If at least one of the objective functions or constraints is nonlinear, then the problem is called nonlinear. Depending on the type (linear or nonlinear) of the objective functions and/or constraints, the optimization problems are classified into different categories. Some optimization problems are stated below.

- *Convex programming.* This problem investigates the situation when the objective function is concave (maximization) or convex (minimization) and the set of constraint is convex. This problem is considered as a special case of nonlinear programming or a generalization/extension of linear or convex quadratic programming.
- *Linear programming (LP).* This is a special form of convex programming problem. In this type of optimization problem, the objective function is linear, and also all the constraints are linear with inequalities and/or equalities. The set of constraints forms a bounded or unbounded convex polyhedron.
- *Nonlinear programming.* This type of problem investigates the problem where the constraints or objective function or both contain nonlinear terms. This problem is not necessarily a convex program.
- *Integer programming (IP).* The integer programming problem (IPP) is a special type of linear program, where some or all variables are integers. If the values of all variables are integers, then the problem is said to be pure IPP. If at least one variable is not an integer, the problem is called mixed IPP. This is not a convex program. Generally, finding a solution of an IPP is much more difficult than the LP problem.
- *Quadratic programming.* In this problem, the objective function is a quadratic expression of the variables, and the feasible region is defined by linear equalities and inequalities. This is a type of convex programming for a specific quadratic forms.
- *Fractional programming.* This optimization problem determines the optimum value of the ratios of two nonlinear functions.
- *Second-order cone programming (SOCP).* This is a special type of convex program that includes a specific type of quadratic programs.
- *Semidefinite programming problem (SDP).* This is a very important subfield of the convex optimization problem in which the variables are semidefinite matrices. This is a generalization of convex quadratic and linear programming.

- *Conic programming.* This is a general form of convex programming problem. With the proper type of cone, the LP, SDP, and SOCP are nothing but conic programs.
- *Combinatorial optimization.* This optimization problem deals with problems in which the set of feasible solutions is discrete, and if it is not discrete, then it can be converted to a discrete feasible set.
- *Stochastic programming.* When some of the parameters or constraints are random uncertain, then the problem is formulated as a stochastic programming problem. This type of formulation is required when the random (noisy) function has to measure or when the inputs are random in the search process.
- *Robust optimization.* This is similar to a stochastic programming problem. In this formulation, uncertainty is captured in the data underlying the optimization problem. This problem aims to find the solutions when all possible realizations of the uncertainties can be defined with the help of an uncertainty set.
- *Heuristics and metaheuristics.* This is a search-based method. Generally, this method does not give the guarantee for the optimal solution which is to be determined. Basically, heuristic methods are used to find approximate solutions for a large class of complex and large optimization problems.
- *Constraint satisfaction.* In this type of problem, the objective function is constant. Generally, the constraint satisfaction problem occurs in artificial intelligence, especially in automated reasoning.
- *Geometric programming.* This is an optimization method. In this method, the equality constraints are in the form of monomials, and the objective function and the inequality constraints are in the form of posynomials and are converted into a convex programming problem.
- *Calculus of variations.* This method is very good and determines the best way to achieve some goal. Generally, it finds a surface or a curve, which optimizes a specific problem.
- *Optimal control theory.* This is the extended theory of calculus of variations. In addition, some control variables or policies are optimized.
- *Dynamic programming.* Dynamic programming is used to solve a large problem by breaking it up into smaller subproblems. The method is used to solve conventional problems as well as stochastic optimization problems. The original problem is solved stage-wise using the Bellman principle.

These are some optimization problems that frequently occur in mathematics as well as economics, computer science, management, etc. Many other optimization problems are also available in different fields.

Like different optimization problems, many optimization methods are developed based on the nature and type of the objective function(s) and constraints and the type of objective function(s), constraints, and variables.

2 Development of Optimization Problems

Several optimization methods have been developed since the days of Newton. The simplest optimization problem contains only one objective function with a single variable and no constraints. To find the optimal solution of this problem, the method based on differential calculus was due to the contribution of Newton and Leibnitz. The calculus of variations were laid by Bernoulli, Lagrange, Euler, and Weierstrass. The method of optimization for constrained problem was due to Lagrange. Cauchy introduced the method of steepest descent for unconstrained optimization problems.

The development of optimization techniques was related with "Blackelt Circus" during the second world war. The development of optimization theory was accelerated due to the availability of high-speed computers during the middle of the twentieth century.

In 1960s, massive developments in numerical optimization processes for unconstrained optimization problems were done in the UK. In 1947, Dantzig developed the simplex method for solving linear programming problems. In 1957, Bellman studied the principle of optimality for the dynamic programming method. These methods are used for the constrained optimization problem. In 1951, the necessary and sufficient conditions for the optimal solution were defined by Kuhn and Tucker. In the early 1960s, Zoutendijk and Rosen investigated nonlinear programming problems. Carroll, Fiacco, and Mecormick presented some techniques for constrained optimization to be reduced to unconstrained optimization. During the 1960s, Duffin, Zener, and Peterson proposed the geometric programming method. Gomory developed the method for solving IPP and mixed IPP. Dantzing, Charnes, and Cooper developed chance-constrained programming for independent and customarily distributed parameters and variables. In 1928, Charnes and Cooper studied the goal programming method for solving multi-objective optimization problems. Von Neumann laid down the foundation of game theory, and the critical path method and PERT for project management problems using network analysis were developed between 1957 and 1958.

3 Classical Optimization Techniques

In these techniques, it is assumed that the given function to be optimized is twice differentiable with respect to variables and, also, the derivatives are continuous. There are three fundamental problems that are solved by the classical optimization methods, i.e.:

(i) the function contains only one variable
(ii) the function contains multiple variables without constraint
(iii) the function contains multiple variables with equality or/and inequality constraints.

Lagrange's multipliers are used for problems (single-valued or multi-valued) with equality constraints. The well-known Kuhn-Tucker conditions are applied to determine the optimum solutions for problems with inequality constraints. These methods generate a system of nonlinear simultaneous equations that are hard to solve.

One of the famous single linear objective functions and multiple linear inequality constraint problems is the LP problem. To solve this problem, many methods are developed, viz., simplex method, revised simplex method, dual-simplex method, two-phase method, decomposition principle, Karmarkar method, etc.

It is obvious that the solution LP is easy, compared to nonlinear programming problems. Two types of methods are developed for the one-dimensional minimization method: elimination or search method and interpolation method. The famous search methods are the golden section method, Fibonacci search method, unrestricted search method, exhaustive search method, etc. On the other hand, in the interpolation method, the derivative-free technique is the quadratic interpolation method, and the derivative-based method is a cubic interpolation, direct root method, etc.

The unconstrained minimization problems can be solved by direct search methods (derivative-free) and descent methods (with derivative). The direct search methods include the random search method, pattern search methods (Powell's method, Hooke and Jeeves method), Rosenbrock's method, etc., whereas descent methods include steepest descent method, conjugate gradient method, Newton's method, variable metric method, etc.

If the problem contains at least one constraint, then two types of methods are available, viz., direct and indirect. The direct methods are Heuristic search methods, constraint approximation methods, method of feasible directions, etc. The indirect methods are penalty function methods, exterior penalty function method, interior penalty function method, etc.

3.1 Multi-Objective Programming Problems

In many real-life situations, it is seen that the objective functions are more than one. The above methods mainly apply to optimizing a single objective function.

Multi-objective optimization, also called multicriteria optimization, multi-objective programming problem, vector optimization problem, multi-attribute optimization problem or Pareto optimization, etc., is a widely used programming problem of multiple criteria decision-making that is concerned with mathematical optimization problems involving more than one objective function to be optimized together. This kind of problem occurs in many fields of science, business, economics, engineering, logistics, etc., where optimal decisions are to be taken based on the different criteria, and most of the time, the criterion are of conflicting nature. Minimizing travelling cost while minimizing travel time and maximizing comfort while minimizing cost for a car are examples of multi-objective optimization problems with conflicting objective functions. In general, for a nontrivial multi-objective problem, multiple solutions exist that optimize all the objective functions simultaneously. For this situation, the objective functions are called conflicting. In this case, a particular type of solution is known as Pareto optimal solution. Suppose the value of some objective functions cannot be improved without degrading the value of other objective functions. In that case, such a solution is known as Pareto optimal, nondominated, Pareto efficient, or non-inferior. For these types of problems, multiple Pareto optimal solutions may exist, all of which are considered equally significant. The multi-objective problems occur in many different situations with specific goals and philosophies.

For these problems, new methods are devised. For multi-objective optimization problems, generally, attempt is made to transform the multi-objective problem into a single-objective problem. The widespread techniques are:

(i) Weighted sum method
(ii) Weighted product method
(iii) Global criteria method
(iv) Fuzzy programming method
(v) Weighted min-max method
(vi) Goal programming method

Recently, some methods have been developed to find optimal solutions of large and complex optimization problems. These methods are completely computer dependent and follow the behaviors of biological systems and animals. In the following section, four such methods, viz., (i) genetic algorithm, (ii) particle swarm optimization, (iii) ant-colony optimization, and (iv) neural network, are discussed.

4 Modern Optimization Techniques

4.1 Neural Network

For the very first, the main theoretical base with an efficient preliminary idea for contemporary neural networks was independently developed by Alexander Bain [2] (1873) and William James [16] (1890). For the first time, in 1943, neural networks were introduced with the help of Walter Pitts and Warren McCullough [40], who are researchers in Chicago. Deep learning algorithms have their heart for simulated neural networks, and machine learning also has artificial neural networks (ANNs) as a subset. Inspired by the structure of the human brain and the way of mimicking of biological units called neurons between one another, this name is followed for the neural network. A circuit or a network constructed with biological neurons is called a neural network, and so artificial neurons or nodes will be composed to form an artificial neural network. The weights between the nodes in a network of artificial neurons are taken as the same as the connections of the biological neuron in the modelling process. A positive weight presents an excitatory connection, where a negative value marks an inhibitory connection. All existing inputs are altered with the help of weight, and then all are summed up. The activation executes all the process as a linear combination. An activation function controls the final amplitude of the output of the system. As an example, it is generally seen that output is obtained with its acceptance range between 0 and 1; sometimes, the said range could be between −1 and 1.

Different types of node layers are present in artificial neural networks (ANNs): firstly, a layer for input data, then one or more hidden layers to be performed, and lastly a layer for output data. A node activates when the corresponding output is more than the predefined threshold value, and then the data is sent to the next layer of that particular network. Otherwise, there is no data flow to that network's next layer.

Figure 1.1 shows the whole system of multi-layer in ANNs.

Alexander Bain [2] and William James [16] together independently proposed the idea of theoretical preliminaries and basic concepts for contemporary neural networks. They showed in their study the entire interactions between neurons within the human brain resulting in thoughts and body activity. In the network of an ANN, an independent linear regression model is present for each individual node, and the models are composed of data for input, output, and a bias (a threshold value). The formula is as follows:

$$\sum m_i z_i + bias = m_1 z_1 + m_2 z_2 + m_3 z_3 + bias.$$

$Output = g(x)$, which is 1 if $\sum m_i z_i + b \geq 0$ and it is 0 if $\sum m_i z_i + b < 0$.

After determining a layer for input, all the weights for that network are assigned. The importance of any variable for a network will be evaluated with the help of these assigned weights; that is, greater values will reflect more contributions significantly

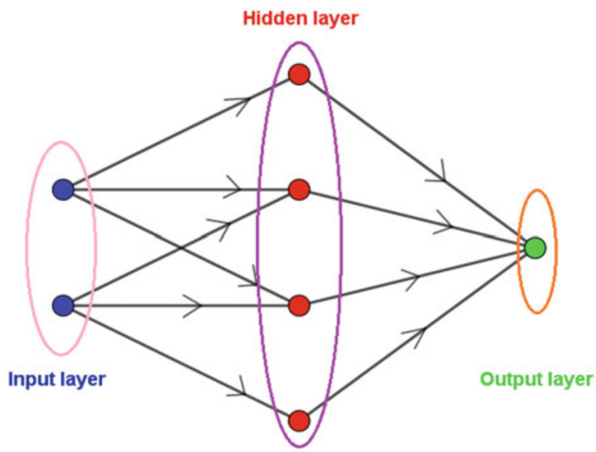

Fig. 1.1 Representation of ANNs

to the output compared to other data for inputs. After multiplying all data for inputs with their corresponding weights, all are to be summed up. Afterword, an activation function will pass the data and determine the output. If the output value dominates the threshold value, then the node is activated, and data is passed to the next layer in the network. The output of this resulting node is taken as an input for the next node. This process of passing data from one layer to the next makes this neural network a feed-forward network.

Neural networks can be classified into different types and applied in various situations. They are:

1. **Feed-forward neural networks:** These types of networks are often known as MLPs and are a multi-layer perceptron. It is very important to mark that many sigmoid neurons are to be comprised to construct such type of network considering the fact that many real-world problems are found as nonlinear types.
2. **Convolution neural networks:** The structure is quite close to feed-forward networks. Computer vision, image recognition, and pattern recognition can be done with the help of this network.
3. **Recurrent neural networks:** In this network, feedback loops act as identifiers. These types of learning algorithms are leveraged to make predictions about future outcomes like sales forecasting or stock market predictions based on time-series data.

There are various types of recent improvements in ANN. It is well-known that theory of BCM, Synaptic plasticity mechanisms are very important in biophysical models for clear understanding and these are widely used in neuroscience as well as in computer science.

New research is ongoing to better understand the algorithms of the computational process used in the brain. Among many mechanisms of data processing techniques, some recent biological evidence for neural back-propagation and radial basis networks have been obtained.

4.2 Genetic Algorithm

In the year of 1992, John Holland, with his collaborators [13], for the first time have introduced the concept of the genetic algorithm (GA) for adaptation in natural and artificial systems. In their work, GA is developed as a model of biological evolution on the basis of Charles Darwin's theory of natural selection. GA is one of the larger evolutionary algorithms, which is a meta-heuristic technique, and the motivation of the natural selection technique invents it. In the application area of operations research and computer science, GA is a very fruitful tool. On the basis of the mechanisms for biologically inspired operators like crossover, mutation, and selection, high-quality solutions to search problems and optimization problems are generated by the use of GA. The theory of evolution guarantees that replacing the weak individuals, the fittest (sometimes known as chromosomes) will survive, and GA is fully based on this theory. By using mutations of genes and crossover process, replacement of individuals will occur, and the best chromosomes among the population will be called as parent chromosomes, whereas child chromosomes are those new chromosomes, which are obtained by manipulation and cloning. In this manner, by evolving all the individuals present in the population, more and more fit, and good individuals are selected.

Starting by choosing a randomly generated individual of a population, an iterative process is going on for this evolution, and a generation is termed for the population obtained in each iteration. For solving an optimization problem by using GA, the worth of the objective function is taken as a fitness value. If a level of satisfaction has been reached or the number of generations which to be generated is highest, then the process of GA is usually terminated for that population. The requirements of a GA are:

(A) The solution domain is to be represented by a genetic form,
(B) To obtain the solution domain, a function for indicating fitness will be set up.

Sometimes, a problem contains a large number of possible solutions and depending on the nature of the problem, the population size is varied. To define a fitness expression for some problems, it is tough and sometimes looks impossible; in these situations, to find the fitness function value of a phenotype, a simulation process or interactive genetic algorithms may be used.

There are mainly four basic steps for the execution process of GA. These steps are illustrated as follows.

1. Initialization
At the first stage, allowing the whole search space of possible solutions, an initial population is randomly generated, although, sometimes, it is seen that the area for finding optimal solutions is fixed up and the considered primary solution is intentionally initiated in that particular region.

2. Selection
To reproduce a new generation, selecting a part of the existing population is mandatory for every successive iteration. Based on the fitness process using fitness function, individual solutions are selected for finding fitter solutions, which are more likely to be opted for survival. As the existing processes are very time-taking, a method is to be liked, which involves a random sample of the population. Over the genetic representation, the fitness function is defined, and the quality of the represented solution is measured with the help of this function. Depending on the types of problems, a function to measure the fitness of an objective is selected.

3. Genetic Operators
Reproducing a second-generation population for the individual solutions from those selected is the next step. This step will be performed with the help of crossover and genetic mutation operator. For each new solution to be obtained, a pair of “parent” solutions is found for breeding from the pool selected earlier. Combining some basic shares of different characteristics of the “parent” solutions and using the methods

of mutations, and cross-over, we have obtained a "child" solution. For every new child, new parents opt, and until a new population of solutions with the appropriate size is produced, the process is continued. A different generation with respect to the initial one may result, made by new chromosomes by the said procedure. Various types of opinions exist in the literature for genetic operators depending on the importance of mutation versus crossover concept.

4. Heuristics

To make the process of calculation more easier or faster or robust, other heuristics may be used. The diversity in a population is encouraged by the special heuristic crossover between solutions corresponding to the candidates, and these are similar in their nature. Also, this process has an important role in preventing a convergence to a less optimal solution.

Usually, the generalization process is repeated until a termination condition has been satisfied. Some of the obvious terminating conditions are:

(i) Satisfying minimum criterion, a solution is obtained.
(ii) With the predefined number of generations, a fixed quantity is obtained.
(iii) Fixed amount of money or time for computation, i.e., allocated budget is held out.
(iv) Manual inspection.
(v) The highest-ranking value of fitness function for the solution has been reached, or a decision is made that no better results can be produced by any successive iteration process.

There are variants of chromosome representation techniques for GA in literature.

(a) Simple Chromosome Representation

In this representation, every chromosome is presented by a bit string, and it is the most straightforward algorithm in all variations.

(b) Elitism

Unalterably, a practical variant of the generation process is to take the best characteristics from the present generation to carry over to the next generation for constructing a new population. In the deduction of solutions using the GA technique, the quality should not be compromised from generation to generation.

(c) Parallel Implementations

It comes in two different ways. A population is assumed on every node of the computer, and among the nodes, there is a migration of individuals in coarse-grained parallel GAs.

(d) Adaptive GAs

One of the promising and significant variants of GA with adaptive parameters is adaptive GAs. The obtained convergence speed of GA and the degree of solution accuracy are to be determined by the probabilities of crossover and mutation.

Also, the parent fields of GA are stochastic optimization, evolutionary algorithms, and evolutionary computing.

4.3 Ant Colony Optimization

To overcome the limitations of the already existing mathematical model in terms of size and computational time complexity, to get better quality solutions to optimization problems, an ant colony optimization method has been introduced. Dorigo [9] has proposed the ant colony optimization (ACO) as a meta-heuristic on the basis of the realistic behavior of ants in our daily life in the year 1992. Solving computational problems in computer science and operations research, ACO is a probabilistic technique that can be modified to obtain good graph paths. Inspired by the actions of real ants, artificial ants are designed, and they stand for the multi-agent method's parameters. Often in such models of ACO, in real life, biological ants communicate to each other by the help of a pheromone, and this particular technique is used on the whole. With the real-life optimization techniques for vehicle routing and Internet routing, a mechanism is made for choosing numerous tasks of optimization, which involves some sort of graph by combining the local search algorithms and artificial ants.

In the broader sense of view, a model-based search is performed by ACO, and it reflects some similarities to the estimation of any distribution algorithm. Initially, based on searching practices for the shortest path between a stock of food and its colony, the proposed ACO was developed for finding an optimal path in a graph. When there are two choices for different routes, one of which is shorter than the other for searching their food of a colony of ants, then they randomly choose one of the routes. On the other hand, choosing the shorter route to reach their food, it is very easy to those particular ants to go back and forth more frequently between their foods and anthill.

In ACO algorithms, for the purpose finding out good solutions corresponding to a supplied optimization problem, artificial ants work as simple computational agents. To apply the ACO algorithm, firstly the considered problem has to be converted into a problem to find the shortest path on a weighted graph. Every ant of the colony has stochastically constructed a solution in the first step of every iteration, i.e., the ordering of edges in the graph must be maintained. The second step of the method involves comparing different paths picked up by various ants. Finally, in the last step, an upgradation of the pheromone levels for every edge is done.

In ACO algorithm for weighted graphs, a very important task is the edge selection process. For moving through the graph, there is a need to construct a solution by every ant in the colony. Depending on the pheromone level as well as the corresponding length of every edge, an ant will choose the next edge for its journey. In the completion time for finding solutions for every ant, decreasing or increasing the trail levels associated with being chosen as a part of "bad" or "good" solutions, the trails are usually up.

At every step of the ACO algorithm associated with a perfect transitional solution, there must be a move from state x to state y of each ant. By computing a set $A_\delta(x)$ consisting of all feasible solutions from its current state, each ant δ will move to one of these possibilities. The attractiveness η_{xy} of the moves has been computed with the help of advantages of some heuristic indication for that move, and the past skilled experience for making that specific move is indicated by the trail level ζ_{xy} of the move. Depending on the combination of these two values, the probability p_{xy}^{δ} of moving from state x to state y has been determined for ant δ.

Generally, the δ-th movement of an ant from state x to state y with the following expression of probability,

$$p_{xy}^{\delta} = \frac{(\zeta_{xy}^{\alpha})(\eta_{xy}^{\beta})}{\sum_{z \in allowed_x} (\zeta_{xz}^{\alpha})(\eta_{xz}^{\beta})},$$

where ζ_{xy} is the deposited amount of pheromone to transit from state x to y. Also, α is a parameter to control the influence of ζ_{xy}; η_{xy} having the value of $\alpha > 0$, which is the desirability of state transition xy (a prior knowledge, typically $\frac{1}{d_{xy}}$, where d is the distance); $\beta \geq 1$ is a parameter to control the influence of η_{xy}; ζ_{xz} and η_{xz} represent the trail level and attractiveness for the other possible state transitions.

There are various types of ACO algorithms. They are:

(i) Ant colony system,
(ii) Elitist ant system,
(iii) ACO involving recursion,
(iv) Ant colony that is orthogonal and continuous,
(v) Parallel ant colony optimization,
(vi) Ant system that uses optimization on the basis of ranking,
(vii) Ant system that involves max-min optimization technique.

ACO algorithms have its applicability in many fields of ranging problems from quadratic assignment to protein folding, combinatorial optimization problem, and routing vehicles, and a lot of modified techniques have been chosen for stochastic problems, dynamic problems in real variables, and parallel implementations. To find near-optimal solutions for traveling salesman problem, the ACO algorithm is also a very useful method. Another fields and topics for the application of ACO are:

(a) For picking up and making a delivery in the problem of vehicle routing,
(b) Scheduling problems for different shops and corresponding jobs,
(c) Resource-constrained project scheduling problem,
(d) Capacitated vehicle routing problem,
(e) Ordering problem which is sequential in nature,
(f) Generalized assignment problem,
(g) Frequency assignment problem,
(h) Set cover problem,
(i) Maximum independent set problem.

4.4 Particle Swarm Optimization

Particle swarm optimization (PSO) is one of the most useful bio-inspired algorithms. This technique is based on population behavior and simulation method of some flock of birds or fish [19]. In the solution space, searching for an optimal solution, it is an excellent and simple technique. The need for only the objective function reflects the different nature of this technique from other optimization algorithms. Also, it is independent of the gradient or any differential representation of the objective. The process of PSO algorithm to the considered optimization problem guarantees that a candidate solution must maintain the characteristics of a swarm.

In 1995, Kennedy and Eberhart proposed PSO method firstly. As per the belief of a sociologist, a school of fish or a flock of birds can be benefited by the experience of all other members of the group. By way of explanation, in the time of flying of a group of birds, they are searching for food in at random manner, for instance, to obtain the best hunt for the entire flock, their overall discoveries for this purpose can be shared by all members. Any decision-maker can simulate the gesture of a congregation of birds, and one can also imagine that in a high-dimensional space for finding an optimal solution, every bird should be very helpful. In the space, the best solution is the obtained best solution by the group. On a multi-dimensional vector space, for a defined function, one can find out the minimum or maximum by using PSO in the best way.

On the basis of the position and velocity of an individual particle in a search space, PSO algorithm searches optimally possible solutions, which are updated in each iteration according to the particle's best position, i.e., personal best position, and the other is the entire swarm's best-known position named as global best.

In 1998, Shi and Eberhart [35] demonstrated good performance for single-objective problems by PSO algorithm. After that, for the conflicting multi-objective problems in which possible candidate solutions can be obtained in the form of Pareto form, the MOPSO was proposed by Coello and Lechuga in 2002. MOPSO is constructed on the primary technique used in PSO. The position and velocity vector equations of PSO remain the same in MOPSO. Now, let us illustrate the PSO in detail.

In PSO, each candidate solution is referred to as a particle. Each particle represents an n-dimensional point in the search space if the considered optimization problem has n variables. A fitness function is used to evaluate the fitness or quality of a particle in the swarm. The way of choosing the optimal solution for a particle is highly quantified by the fitness function. Depending on two things (one is the distance from the best particle of the swarm [20], and the other is the distance of a particular particle from its own personal best position), each particle has modified its position and flown through search space. Actually, to measure how close the candidate of the swarm is to the global optimum, a fitness function has been applied. The following information is maintained by each particle i:

(i) p_i indicating the present position of the individual node;
(ii) v_i representing the present velocity of the individual node;
(iii) x_i standing for the personal best position of the individual node.

In the swarm, a particle i has visited so far and attained various positions. One particular position where the fitness function has the highest value for that particle is referred to as the personal best position of the same. The personal best position, therefore, serves as a kind of experience and memory, which helps in the future to find the global best for the entire swarm. If χ denotes the objective function, then at a time step t, the personal best of a particle is updated as the following function:

$$x_i(t+1) = \begin{cases} x_i(t) & \text{if} \quad \chi(p_i(t+1) \geq \chi(x_i(t))) \\ p_i(t+1) & \text{if} \quad \chi(p_i(t+1) < \chi(x_i(t))). \end{cases}$$

The exchange of information between members of a flock or swarm is one of the unique principles of PSO. This technique is beneficial to build a memory for adjusting the personal best positions of the particles toward the global best by determining the best particle (s), or position (s) in the swarm. The ring and star topologies are the very first social topologies. It is remarkably seen that communication between all existing particles is allowed in the star topology. Generally, the best PSO is referred to the resulting algorithm. On the other hand, overlapping neighborhoods of particles are defined by the ring topology. Particles in a neighborhood communicate to identify the best in that neighborhood. All neighborhood particles then adjust toward neighborhood best or local best particle. The obtained algorithm is generally known as the best PSO. Recently, more complex social topologies have been investigated by Kennedy and Mendes [21], of which the von Neumann topology can be considered as an efficient alternative.

From the entire swarm, the best particle is found with the help of the following expression in the best PSO model,

$$\hat{x}(t) \in \{x_0, x_1, x_2, \ldots, x_s\} = min\{\chi(x_0(t)), \chi(x_1(t)), \ldots, \chi(x_s(t))\},$$

where s is the total number of particles in the swarm.

In reality, many parameters, functions, etc. are not certain due to the incomplete availability of the information. The uncertainty may be random or non-random. The random uncertainties deal with probability theory and have many theories and relatively old. The non-random uncertainly deals with fuzzy mathematics/logic, and few theories were developed after the invention of fuzzy sets in 1965. The region of non-random uncertainty is large than the random uncertainty. A common region of random and non-random uncertainty is also seen in some real-life problems. For non-random optimization or fuzzy optimization, several methods have been devised during the last four decades.

5 Fuzzy Optimization

A fuzzy optimization problem (FOP) is a mathematical problem/model involving transitional uncertainty and/or information deficiency uncertainty. Some authors refer to these terms as uncertainties, ambiguity, and vagueness. The transitional uncertainty belongs to fuzzy set theory. On the other hand, uncertainty occurs due to information deficiency and is the subject of possibility theory. The term "tall" is the transitional uncertainty, while "tall man" is the information deficiency.

Before discussing the fuzzy optimization, we will define the fuzzy optimization problem and its difficulties.

The general FOP is considered as follows:

The fuzzy objective function is

$$\widetilde{optimize} \quad \tilde{z} = \tilde{f}(\tilde{c}, x) \tag{1.1}$$

subject to fuzzy constraints

$$\tilde{g}_i(x, \tilde{a}_i) \quad \tilde{R}_i \quad \tilde{b}_i \quad i \in \Lambda = \{1, 2, \ldots, m\} \tag{1.2}$$

where $\tilde{R}_i$ and $i \in \Lambda$ are the fuzzy relations and the tilde$\tilde{}$represents fuzzy and/or possibilistic quantities or relationships that is to be clearly defined in the problem. In case of fuzzy uncertainty the tilde, $\tilde{}$ notation is used, while for possibilistic uncertainty, this notation is changed to circumflex$\hat{}$. In Eq. (1.1), $\tilde{c}$ is a known vector of uncertainty parameters, and x is the unknown variables whose values are to be determined by the problem. These variables are called the "decision variables" because the objective function's optimum value depends on these variables, and their values are to be determined. The variable x is generally a certain quantity, but in some cases, it may be uncertain. For example, the quantity x may be the amount of quantity to be transported from a manufacturing point to a destination (demand point or stockist point, etc.) point, or x may be the travel time from one point to another point, etc.

A particular case of Eq. (1.1) is a fuzzy linear programming problem (FLP problem), where the functions f and g_i are linear. The FLP problem is denoted as

$$\widetilde{maximize} \quad \tilde{z} = \tilde{c}_1 x_1 \tilde{+} \tilde{c}_2 x_2 \tilde{+} \cdots \tilde{+} \tilde{c}_n x_n \tag{1.3}$$

$$\text{subject to} \tag{1.4}$$

$$\tilde{a}_{i1} x_1 \tilde{+} \tilde{a}_{i2} x_2 \tilde{+} \cdots \tilde{+} \tilde{a}_{in} x_n \quad \tilde{R}_i \quad \tilde{b}_i, \quad i \in \Lambda \tag{1.5}$$

$$x_j \geq 0, \ j \in \{1, 2, \ldots, n\}. \tag{1.6}$$

Let $\Lambda = \{1, 2, \ldots, m\}$, $m > 1$, Λ be called the index set. Also, let $F = \{\mu_i : i \in \Lambda\}$ be the set of membership functions of a fuzzy subset R^n. Suppose X be a subset of R^n in which $supp(\mu_i) \subset X$ for all $i \in \Lambda$. The members of X are said to be decision variables.

A decision variable x_{WP} is called a weak Pareto maximum (WPM), if there exists no $y \in X$, such that

$$\mu_i(x_{WP}) < \mu_i(y) \quad \text{for all } i \in \Lambda. \tag{1.7}$$

A decision variable x_P is called a Pareto maximum (PM), if there exists no $y \in X$, such that

$$\mu_i(x_P) \leq \mu_i(y) \quad \text{for all } i \in \Lambda, \tag{1.8}$$
$$\mu_i(x_{WP}) < \mu_i(y) \quad \text{for all } i \in \Lambda.$$

A decision x_{SP} is called a strong Pareto maximum (SPM), if there is no $y \in X$ and $y \neq x_{SP}$, such that

$$\mu_i(x_{SP}) \leq \mu_i(y) \quad \text{for all } i \in \Lambda. \tag{1.9}$$

The set of all PM, WPM, and SPM are denoted, respectively, by X_P, X_{WP}, and X_{SP}. The elements of $X_{WP} \cup X_P \cup X_{SP}$ are known as Pareto optimal solutions or decisions.

There is an excellent inclusive relationship among these three solutions, i.e.,

$$X_{SP} \subset X_P \subset X_{WP}.$$

Fuzzy optimization, which deals with both fuzzy and possibilistic uncertainty, is one of the newest optimization areas of research. This is a parallel field of stochastic optimization, but here non-random uncertainty is considered. The work on fuzzy optimization started in 1970 after publication of the fundamental work of Bellman and Zadeh [4]. After three years, another paper was published on fuzzy optimization by Tanaka et al. [1, 37]. Tanaka, Okuda, and Asai operationalize the theoretical approach developed by Bellman and Zadeh. Independently, in 1974, Zimmermann was presented a paper at the ORSA/TIMS conference in Puerto Rico [51] that operationalized the Bellman and Zadeh approach and also largely simplified and clarified fuzzy optimization, and hence the Zimmermann approach become a standard work in this time. In this same period, a good description of fuzzy optimization is written in the book by Negoita and Ralescu [31]. In 1976, Negoita and Sularia [32] described a set containment approach for a fuzzy optimization problem. Two books with edited papers [6, 18] and several authored books on fuzzy optimization are published [3, 22, 27, 28, 33, 36]. Four decades of fuzzy optimization research have developed a wide range of applications. The following researches cover applications [6, 18] and [3, 10, 11, 14, 15, 17, 23–25, 34, 38, 41, 42]. In [26], Lodwick and Untiedt presented a good survey of fuzzy optimization problems.

6 Formulation of Fuzzy Mathematical Programming Problem

Let $F(\mathbf{R})$ be the set of all fuzzy quantities. Recall the terms of Eq. (1.1). Let the functions f, g_i be such that $f : \mathbf{R}^n \times \mathbf{C} \rightarrow \mathbf{R}$, $g_i : \mathbf{R}^n \times \mathbf{P}_i \rightarrow \mathbf{R}$, where $\mathbf{C}, \mathbf{P}_i$ are sets of parameters. Also, the membership functions for the fuzzy parameters $\widetilde{c}, \widetilde{a}_i, \widetilde{b}_i$ are defined as $\mu_{\widetilde{c}} : \mathbf{C} \rightarrow [0, 1]$, $\mu_{\widetilde{a}_i} : \mathbf{P}_i \rightarrow [0, 1]$ and $\mu_{\widetilde{b}_i} : \mathbf{R} \rightarrow [0, 1]$, for all $i \in \Lambda$.

$\widetilde{R}_i$, $i \in \Lambda$ are the fuzzy relations used to compare both sides of the constraints.

The finding of maximization/minimization in fuzzy optimization is not a simple task and needs some new mathematics. The more difficult issue is that the fuzzy values of the objective function are not linearly ordered, and to find the optimum value of the objective function, a suitable ordering on $F(\mathbf{R})$ is required. Also, a new concept of a "feasible solution" and "optimal solution" is required.

For a given $x \in \mathbf{R}^n$ and $\widetilde{a}_i \in F(\mathbf{P}_i)$ by Zadeh's extension principle, $\widetilde{g}_i(x, \widetilde{a}_i)$ is a fuzzy extension of $g_i(x, .)$, and its membership value is given by

$$\mu_{\widetilde{g}_i}(x, \widetilde{a}_i)(s) = \begin{cases} \sup\{\mu_{\widetilde{a}_i}(a) : a \in \mathbf{P}_i, g_i(x, a) = s\} & \text{if } g_i^{-1}(x, s) \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \tag{1.10}$$

for each $s \in \mathbf{R}$, where $g_i^{-1}(x, s) = \{a \in \mathbf{P}_i : g_i(x, a) = s\}$.

The fuzzy relation $\widetilde{R}_i$ for comparing two elements of $F(\mathbf{R})$ are the extensions of valued relations on $\mathbf{R}$, i.e., the conventional inequality $\leq$ and $\geq$.

Ultimately, $\widetilde{g}_i(x, \widetilde{a}_i) \in F(\mathbf{R})$, and this quantity is to be "compared" with fuzzy quantity $\widetilde{b}_i$ over the relation $\widetilde{R}_i$ for all $i \in \Lambda$. This is known as a fuzzy comparison.

As a particular case, if T is a t-norm and $\widetilde{R}_i$ is a T-fuzzy extension of the relation R_i, then the membership function of the ith constraints is given by

$$\begin{aligned} \mu_{\widetilde{R}_i}\left(\widetilde{g}_i(x, \widetilde{a}_i), \widetilde{b}_i\right) &= \sup\{T\left(\mu_{R_i}(p, q), T(\mu_{\widetilde{g}_i(x,\widetilde{a}_i)}(p), \mu_{\widetilde{b}_i}(q))\right) : p, q, \in \mathbf{R}\} \\ &= \sup\{T\left(\mu_{\widetilde{g}_i(x,\widetilde{a}_i)}(p), \mu_{\widetilde{b}_i}(q)\right) : p \ R_i \ q\} \end{aligned} \tag{1.11}$$

6.1 *Feasible Solutions of Optimization Problem*

Using the extension principle, the fuzzy relation $\widetilde{R}_i$ is considered the usual equality relation "=" or inequality relations "$\leq$" and "$\geq$". It is observed that T-fuzzy extensions of relational operators (=, $\leq$, $\geq$) have some disadvantages, and based on the problem, many fuzzy relations are defined for comparison of fuzzy numbers.

A subset $\widetilde{X} \in \mathbf{R}^n$ whose membership function is denoted by $\mu_{\widetilde{X}}$ for all $x \in \mathbf{R}^n$ and is defined as

$$\mu_{\widetilde{X}}(x) = A\left(\mu_{\widetilde{R}_1}(\widetilde{g}_1(x, \widetilde{a}_1, \widetilde{b}_1), \ldots, \mu_{\widetilde{R}_m}(\widetilde{g}_m(x, \widetilde{a}_m, \widetilde{b}_m))\right) \tag{1.12}$$

which is said to be the feasible solution of the FMP problem stated in Eq. (1.1).

A vector $x \in [\widetilde{X}]_\alpha$, $\alpha \in [0, 1]$, is said to be α-feasible solution of the problem Eq. (1.1).

A vector $\overline{x} \in \mathbf{R}^n$ such that $\mu_{\widetilde{X}}(\overline{x})$ = height of $(\widetilde{X})$ is known as max-feasible solution. (The height of a fuzzy set is the maximum membership value of an element of the set).

It is observed that the feasible solution of an FMP is a fuzzy set. The interpretation of $\mu_{\widetilde{X}}(x)$ depends on the nature of the uncertain parameters involved in the FMP. The membership value of $\widetilde{X}_i$ is given by

$$\mu_{\widetilde{X}_i}(x) = \mu_{\widetilde{R}_i}(\widetilde{g}_i(x, \widetilde{a}_i), \widetilde{b}_i).$$

The fuzzy set $\widetilde{X}$ can be considered as the ith fuzzy constraint. Note that the aggregation operator A aggregates all the fuzzy constraints into the feasible solution (1.12).

7 Decision-Making Problem

In a decision-making problem, generally, multiple alternatives and multiple criteria (they may be conflicting in nature) are provided; among them, we have to choose the best alternative(s). The criteria play the role of constraints. The problem may have one or more objective functions. Suppose the set of alternatives is $A = \{a_1, a_2, \ldots\}$ (it may be finite or infinite), and the set of criteria is $C = \{c_1, \ldots, c_n\}$ and a group of (people) experts $G = \{e_1, \ldots, e_m\}$.

Many different methods are devised to solve such types of problems based on alternatives, criteria, and experts. These methods have different names. Such names are *multi-criteria decision-making (MCDM)* and *multi-criteria decision aid (MCDA)*. The MCDA approach concentrates on the tools that help a decision-maker understand, capture, and analyze the distinction among the alternatives. On the other hand, in MCDM, it is assumed that the decision-making method can be formalized and concentrated on tools to describe this method. Thus, MCDM is a kind of descriptive method.

Within MCDM and MCDA, there are two main classified problems, viz., multi-objective decision-making (MODM) and multi-attribute decision-making (MADM), two main areas can be distinguished. In MODM, the number of alternatives is infinite, i.e., the region of alternatives is a continuum. On the other hand, in MADM, the number of alternatives is finite. MADM is also referred to as MCDM. In MODM problems, there are one or more (usually more than one) functions which are to be optimized to satisfy the criteria/constraints, and these problems are generally solved using the available optimization methods.

In both types of problems (MCDM and MADM), during decision-making, a set of rules are formed based on the predefined set of alternatives (generally finite) and a set of criteria. It is assumed that a preference is defined over the alternatives. Many techniques have been developed for assigning preferences on each criterion in the last few years. The most used approaches are:

1. *Utility functions:* Utility functions are defined over alternatives, and it has a fixed range. The larger value associated with an alternative is more preferable.
2. *Preference relations:* Preference relations are binary, which satisfies a pair of alternatives and identifies which one is preferred over the other.

Let us consider an example to explain the terms alternatives, criteria, preferences, etc.

Suppose we would like to buy a mobile phone. The alternatives are $A = \{$iPhone, Samsung$\}$, and criteria as per buyer need $C = \{$price, security, camera's quality$\}$.

Now, we represent the buyer's preference for the alternatives with the help of utility functions, i.e., with the help of the functions $U_{\text{price}}, U_{\text{security}}, U_{\text{camera's quality}}$ or with the help of preference relations, i.e., $R_{\text{price}}, R_{\text{security}}, R_{\text{camera's quality}}$.

For explanation, let us assume that

Utility functions:

iPhone: $U_{\text{price}} = 0.7, U_{\text{security}} = 0.9, U_{\text{camera's quality}} = 0.4$
Samsung: $U_{\text{price}} = 0.5, U_{\text{security}} = 0.4, U_{\text{camera's quality}} = 0.8$

Preference relations

- Price: $R_{\text{price}}(iPhone, Samsung), \neg R_{\text{price}}(Samsung, iPhone)$
- Security: $R_{\text{security}}(iPhone, Samsung), \neg R_{\text{security}}(Samsung, iPhone)$
- Security: $R_{\text{camera's quality}}(Samsung, iPhone), \neg R_{\text{camera's quality}}(iPhone, Samsung)$

To find the best choice, a new function is defined. Let $D_c(x, y)$ be such function. This function prefers x to y with retrospect to the criteria c. The decision can be taken by constructing an aggregate preference function/operator D_c that combines different criteria. Once this function/operator D_c is built, the selection of the best alternative can easily be done.

Many such functions/operators based on t-norms and t-conorms are now available in literature. Few of them are discussed in the next section. The new functions which combine the utility functions and the preference relations are known as aggregation functions or aggregation operators.

8 *t*-Norms and *t*-Conorms-Based Operators

Recently, a lot of triangular norms (t-norms) and triangular conorms (t-conorms or s-norm) have been defined by several researchers, and based on these, several types of decision-making problems have been solved. These norms are used as operators.

The t-norms and t-conorms are two fundamental operations, and these operations generalize the logical conjunction and logical disjunction to fuzzy logic and are used to solve multi-criteria decision-making problems. Any expression can't be a t-norms or t-conorms; they must satisfy some criteria. The formal definitions are given below.

The t-norm is a binary operation $T : [0, 1] \times [0, 1] \to [0, 1]$, which satisfies the following conditions:

(i) $T(\alpha, \beta) = T(\beta, \alpha)$ (commutativity)
(ii) $T(\alpha, T(\beta, \gamma)) = T(T(\alpha, \beta), \gamma)$ (associativity)
(iii) $\beta \le \gamma \Rightarrow T(\alpha, \beta) \le T(\alpha, \gamma)$ (monotonicity)
(iv) $T(\alpha, 1) = \alpha$ (neutral element 1)

The t-norm is a commutative, associative, monotone binary operation.

Examples of three fundamental t-norms:

$T_M(\alpha, \beta) = \min(\alpha, \beta)$ (minimum or Gödel t-norm)
$T_P(\alpha, \beta) = \alpha\dot{\beta}$ (product t-norm)
$T_L(\alpha, \beta) = \max(\alpha + \beta - 1, 0)$ (Lukasiewicz t-norm)

Another interesting t-norm defined by M.J. Frank in the late 1970s is a parametric family for the parameter $0 \le \mu \le \infty$ defined as

$$T^F_\mu(\alpha, \beta) = \begin{cases} T_M(\alpha, \beta), & \text{if } \mu = 0 \\ T_P(\alpha, \beta), & \text{if } \mu = 1 \\ T_L(\alpha, \beta), & \text{if } \mu = \infty \\ \log_\mu\left(1 + \frac{(\mu^\alpha - 1)(\mu^\beta - 1)}{\mu - 1}\right), & \text{otherwise.} \end{cases}$$

The triangular conorm S (also known as t-conorm or s-norm) is defined below:

(i) $S(\alpha, \beta) = S(\beta, \alpha)$ (commutativity)
(ii) $S(\alpha, S(\beta, \gamma)) = S(S(\alpha, \beta), \gamma)$ (associativity)
(iii) $\beta \le \gamma \Rightarrow S(\alpha, \beta) \le S(\alpha, \gamma)$ (monotonicity)
(iv) $S(\alpha, 0) = \alpha$ (neutral element 0).

Note that the neutral element for s-norm is 0, unlike 1 of t-norm, but all other conditions remain the same.

The very standard t-conorms are:

$S_M(\alpha, \beta) = \max(\alpha, \beta)$ (maximum or Gödel t-conorm)
$S_P(\alpha, \beta) = \alpha + \beta - \alpha\dot{\beta}$ (product t-conorm, probabilistic sum)
$S_L(\alpha, \beta) = \min(\alpha + \beta, 1)$ (Lukasiewicz t-conorm, bounded sum)).

There is a relationship between t-norm and t-conorms; if T is a t-norm, then $S(\alpha, \beta) = 1 - T(1 - \alpha, 1 - \beta)$ is a t-conorm, and vice versa. That is, (T, S) is a dual pair of a t-norm and a t-conorm.

An element α is called idempotents of a t-norm T if $T(\alpha, \alpha) = \alpha$. The trivial idempotents elements are 0 and 1. A t-norm is said to be Archimedean if each element of the sequence $\alpha_n, n \in \mathbf{N}, \alpha_1 < 1$, and $\alpha_{n+1} = T(\alpha_n, \alpha_n)$ converges to 0. If, for a continuous t-norm, there is no idempotents between 0 and 1, then this t-norm is called Archimedean. A continuous Archimedean t-norm is said to be strict if $T(\alpha, \alpha) > 0$ for all $\alpha > 0$. If a continuous Archimedean t-norm is not strict, it is nilpotent.

If T^* is a t-norm and $h : [0, 1] \to [0, 1]$ is an increasing bijective function, then

$$T(\alpha, \beta) = h^{-1}(T^*(h(\alpha), h(\beta)))$$

is a t-norm.

The frequently used operators to solve decision-making problems are due to Dombi, Hamacher, Yager, Einstein, and many others.

8.1 Dombi Operations

Let α and β be any two real numbers. The Dombi norms and conorms are defined as:

$$Dom(\alpha, \beta) = \frac{1}{1 + \{(\frac{1-\alpha}{\alpha})^{\varrho} + (\frac{1-\beta}{\beta})^{\varrho}\}^{1/\varrho}} \tag{1.13}$$

$$Dom^c(\alpha, \beta) = 1 - \frac{1}{1 + \{(\frac{\alpha}{1-\alpha})^{\varrho} + (\frac{\beta}{1-\beta})^{\varrho}\}^{1/\varrho}} \tag{1.14}$$

where $\varrho \geq 1$ is a parameter and $(\alpha, \beta) \in [0, 1] \times [0, 1]$.

8.2 Hamacher Operations

Let α and β be any two real numbers with a parameter ϱ. The Hamacher norms and conorms are defined as follows:

$$H_T(\alpha, \beta) = \frac{\alpha\beta}{\varrho + (1 - \varrho)(\alpha + \beta - \alpha\beta)} \tag{1.15}$$

$$H_S(\alpha, \beta) = \frac{\alpha + \beta - (2 - \varrho)\alpha\beta}{1 - (1 - \varrho)\alpha\beta} \tag{1.16}$$

8.3 Einstein Operators

The Einstein operations serve as examples of t-norms and t-conorms, including the Einstein product and Einstein sum. These are their definitions:

Definition 1 ([49]) Einstein product $\bigotimes$ and Einstein sum $\bigoplus$ between two real numbers α and β are defined below.

$$\alpha \oplus_E \beta = \frac{\alpha + \beta}{1 + \alpha.\beta} \tag{1.17}$$

$$\alpha \otimes_E \beta = \frac{\alpha.\beta}{1 + (1-\alpha).(1-\beta)} \tag{1.18}$$

where for all $(\alpha, \beta) \in [0, 1] \times [0, 1]$.

Let $f(\alpha)$ be the function assigned to the conjunctive operator $k(\alpha, \beta)$ and $g(\alpha)$ be another function assigned to the disjunctive operator $d(\alpha, \beta)$. Zadeh mentioned that for the disjunctive (conjunctive) operators, a series of several operators can be constructed, whose limit is the max (min) operator. Suppose $f(\alpha)$ is a given operator; using the concept of negation operator, a function $g(\alpha)$ can be given from which a disjunctive operator can be generated. There is a good connection between disjunctive, conjunctive, and negation operators, which construct the necessary and sufficient condition for DeMorgan identity. Then, by using any two operators, another operator can be designed. On the construction principle, Hamacher's conditions belong to the DeMorgan class and Yager's operator system. For satisfying the DeMorgan identity, the necessary and sufficient conditions are provided as $f_{\mathfrak{R}}(\alpha)$, which can be designed for every $f(\alpha)$, so that for the derived $K_{\mathfrak{R}}(\alpha, \beta)$ and $d_{\mathfrak{R}}(\alpha, \beta)$, $\lim\limits_{\mathfrak{R}\to\infty} K_{\mathfrak{R}}(\alpha, \beta) = \min(\alpha, \beta)$, and $\lim\limits_{\mathfrak{R}\to\infty} d_{\mathfrak{R}}(\alpha, \beta) = \max(\alpha, \beta)$. As Yager's operator is not reducible, for every $\mathfrak{R}$, there exists a λ, such that $K_{\mathfrak{R}}(\alpha, \beta) = 0$ in case $\alpha < \lambda$ and $\beta < \lambda$.

From the general construction, the measurement of fuzziness can be done. Fuzzy operators can be appropriately used in many applications of different fuzzy measurements. When fuzzy logic/mathematics is used in a decision-making problem, the optimum value is only defined along with its degree, i.e., the optimum value can't be measured indeed. It is very helpful to design a system in which one can conclude the sharpness of decisions from the sharpness of the applied operators and sets.

Table 1.1 provides the characteristics of the three useful operators defined by Hamacher [12], Yager [46], and Dombi [8]. Here negation operator is $n(\alpha) = 1 - \alpha$.

1. Dombi operator satisfies all the basic properties such as continuous, conjunctive, and disjunctive, while Yager's operator does not satisfies all these.

2. The form of Hamacher operator can be obtained by substituting the parameter $\Re$ (in Table 1.1) for $1/\Re$ in the case of conjunctive operator and for $1/(\Re^{'}+1)$ in case of disjunctive operator. In the resulting transformed form, the condition for satisfying the DeMorgan identity is $1/\Re = 1/(\Re^{'}+1)$. This is equivalent to the results of Hamacher when $n(\alpha) = 1-\alpha$.
3. It can be shown that Yager's formula is equivalent to $(i)\ 1-\min\left(1, \left((1-\alpha)^{\Re}+(1-\beta)^{\Re}\right)^{1/\Re}\right)$ and $(ii)\ \min\left(1, (\alpha^{\Re}+\beta^{\Re}\right)^{1\Re}\Big)$.
4. In the case of all three operators if $\Re \le \Re^{'}$, then $k_{\Re}(\alpha,\beta) \le k_{\Re'}(\alpha,\beta)$, and $d_{\Re}(\alpha,\beta) \ge d_{\Re'}(\alpha,\beta)$.
5. In the case of all three operators, $\lim\limits_{\Re\to 0} k_{\Re}(\alpha,\beta) = 0$, and $\lim\limits_{\Re\to 0} d_{\Re}(\alpha,\beta) = 1$.

The above discussion shows that Dombi operators are more general than Hamacher and Yager operators.

8.4 Power Averaging (PA) Operator

Suppose $\alpha_j,\ j = 1, 2, \ldots, n$ is a set of crisp numbers. The PA operator [47] is a mapping $PA\colon (\mathbb{R}^+)^n \to \mathbb{R}^+$ defined as:

$$PA(\alpha_1, \alpha_2, \ldots, \alpha_n) = \frac{\sum_{j=1}^{n}(1+T(\alpha_j))\alpha_j}{\sum_{j=1}^{n}(1+T(\alpha_j))},$$

where $T(\alpha_j) = \sum_{i=1, j\neq i}^{n} supp(\alpha_j, \beta_i),\ j = 1, 2, \ldots, n$ and $supp(\alpha_j, \alpha_i)$ is the support degree of α_j from α_i. It satisfies the criteria as follows: (i) $supp(\alpha_j, \alpha_i) \in [0, 1]$, (ii) $supp(\alpha_j, \alpha_i) = supp(\alpha_i, \alpha_j)$, and (iii) $supp(\alpha_i, \alpha_j) \ge supp(\alpha_k, \alpha_l)$ if $|\alpha_i - \alpha_j| < |\alpha_k - \alpha_l|$.

8.5 Prioritized Average (PA) Operator

The prioritized average operator (PA) is the first time defined by Yager [48]. Let $D = \{D_1, D_2 \ldots, D_r\}$ be a set of attributes, which have a prioritized relation between the attributes by the linear ordering $D_1 \succ D_2 \succ D_3 \succ, \ldots, \succ D_r$, which imply that D_ϕ has a higher prioritized than D_θ, if $\phi < \theta$. The value of $D_s(p)$ is the

Table 1.1 Three important operators

Conjunction					
Authors	$f(\alpha) =$	$f^{-1}(\alpha) =$	$k_{\mathfrak{R}}(\alpha, \beta) =$	$k_1(\alpha, \beta) =$	$\lim\limits_{\mathfrak{R}\to\infty} k_{\mathfrak{R}}(\alpha, \beta) =$
H. Hamacher	$\frac{e^{-\alpha}}{\mathfrak{R}+(1-\mathfrak{R})e^{-\alpha}}$	$-\ln \frac{\mathfrak{R}\alpha}{1+(\mathfrak{R}-1)\alpha}$	$\frac{\mathfrak{R}\alpha\beta}{1-(1-\mathfrak{R})(\alpha+\beta-\alpha\beta)}$	$\alpha\beta$	$\frac{\alpha\beta}{\alpha+\beta-\alpha\beta}$
R.R. Yager	$\begin{cases} 1-\alpha^{1/\mathfrak{R}}, \\ \text{if } \alpha < 1 \\ 0, \\ \text{if } \alpha \geq 1 \end{cases}$	$\begin{cases} (1-\alpha)^{\mathfrak{R}}, \\ \text{if } \alpha < 1 \\ 0, \\ \text{if } \alpha \geq 1 \end{cases}$	$\begin{cases} 1-\left((1-\alpha)^{\mathfrak{R}}+(1-\beta)^{\mathfrak{R}}\right)^{1/\mathfrak{R}}, \\ \text{if } (1-\alpha)^{\mathfrak{R}}+(1-\beta)^{\mathfrak{R}} < 1 \\ 0, \\ \text{if } (1-\alpha)^{\mathfrak{R}}+(1-\beta)^{\mathfrak{R}} \geq 1 \end{cases}$	$\max(0, \alpha+\beta-1)$	$\min(\alpha, \beta)$
J.Dombi	$\frac{1}{1+\alpha^{1/\mathfrak{R}}}$	$(\frac{1}{\alpha}-1)^{\mathfrak{R}}$	$\frac{1}{1+\left((\frac{1}{\alpha}-1)^{\mathfrak{R}}+(\frac{1}{\beta}-1)^{\mathfrak{R}}\right)^{1/\mathfrak{R}}}$	$\frac{\alpha\beta}{\alpha+\beta-\alpha\beta}$ if $\alpha = 0, \beta = 0$	$\min(\alpha, \beta)$
Disjunction					
	$g(\alpha) =$	$g^{-1}(\alpha) =$	$d_{\mathfrak{R}}(\alpha, \beta) =$	$d_1(\alpha, \beta) =$	$\lim\limits_{\mathfrak{R}\to\infty} d_{\mathfrak{R}}(\alpha, \beta) =$
H. Hamacher	$\frac{\mathfrak{R}(1-e^{-\alpha})}{\mathfrak{R}+(1-\mathfrak{R})e^{-\alpha}}$	$-\ln \frac{\mathfrak{R}(1-\alpha)}{\mathfrak{R}-(\mathfrak{R}-1)\alpha}$	$\frac{\mathfrak{R}(\alpha+\beta)+\alpha\beta(1-2\mathfrak{R})}{\mathfrak{R}+\alpha\beta(1-\mathfrak{R})}$	$\alpha+\beta-\alpha\beta$	$\frac{\alpha+\beta-2\alpha\beta}{1-\alpha\beta}$
R.R. Yager	$\begin{cases} \alpha^{1/\mathfrak{R}}, \\ \text{if } \alpha < 1 \\ 1, \\ \text{if } \alpha \geq 1 \end{cases}$	$\begin{cases} \alpha^{\mathfrak{R}}, \\ \text{if } \alpha < 1 \\ 1, \\ \text{if } \alpha \geq 1 \end{cases}$	$\begin{cases} \left(\alpha^{\mathfrak{R}}+\beta^{\mathfrak{R}}\right)^{1/\mathfrak{R}}, \\ \text{if } \alpha^{\mathfrak{R}}+\beta^{\mathfrak{R}} < 1 \\ 1, \\ \text{if } \alpha^{\mathfrak{R}}+\beta^{\mathfrak{R}} \geq 1 \end{cases}$	$\min(1, \alpha+\beta)$	$\max(\alpha, \beta)$
J.Dombi	$\frac{1}{1+\alpha^{-1/\mathfrak{R}}}$	$(\frac{1}{\alpha}-1)^{-\mathfrak{R}}$	$\frac{1}{1+\left((\frac{1}{\alpha}-1)^{-\mathfrak{R}}+(\frac{1}{\beta}-1)^{-\mathfrak{R}}\right)^{-1/\mathfrak{R}}}$	$\frac{\alpha+\beta-2\alpha\beta}{1-\alpha\beta}$ if $\alpha = 1, \beta = 1$	$\max(\alpha, \beta)$

performance of any alternative p under attribute D_ϕ, which satisfies $D_\phi(p) \in [0, 1]$. If

$$PA(D_r(p)) = \sum_{s=1}^{r} \psi_s D_s(p) \tag{1.19}$$

where $\psi_g = \frac{\hbar_g}{\sum_{g=1}^{r} \hbar_g}$, $\hbar_g = \prod_{b=1}^{g-1} D_b(p)$ $(b = 1, 2, \ldots, \zeta)$, $\hbar = 1$. Then PA is called the average operator. The PA [48] generally used input arguments have exact values.

8.6 Bonferroni Mean (BM) Operator

Bonferroni mean, first proposed by Bonferroni, can enable aggregation between the max, min operators, and the logical "or" and "and" operators [5]. The following is a definition of BM:

Definition 2 ([5]) Let $p, q \geq 0$ and α_i $i = 1, 2, \ldots, n$ be a collection of non-negative real numbers, and its aggregation functions defined as follows:

$$BM^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(\frac{1}{n(n-1)} \sum_{i,j=1, i \neq j}^{n} \alpha_i^p \alpha_j^q \right)^{\frac{1}{p+q}}. \tag{1.20}$$

are called Bonferroni mean (BM) operator.

Definition 3 ([50]) Suppose $p,\ q > 0$, and α_i, $i = 1, 2, \ldots, n$ are a collection of non-negative real numbers, then the geometric Bonferroni mean (GBM) operator is defined as follows:

$$GBM^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(\frac{1}{p+q} \prod_{i,j=1, i \neq j}^{n} (p\alpha_i + q\alpha_j) \right)^{\frac{1}{n(n-1)}}. \tag{1.21}$$

8.7 Maclaurin Symmetric Mean (MSM) Operator

The Maclaurin symmetric mean (MSM) operator is a good tool for collecting information on the interrelationship between the multi-input arguments.

Definition 4 Assume $\alpha_1, \alpha_2, \ldots, \alpha_n$ and there is a set of non-zero real numbers. The formula of MSM operator is given below.

$$MSM^{(m)}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(\frac{\sum\limits_{1<i_1<i_2<\ldots<i_k<g_r} \left(\prod\limits_{j=1}^{m} \alpha_j \right)}{C_{g_r}^{m}} \right)^{\frac{1}{m}} \tag{1.22}$$

where m is a parameter, $m = 1, 2, \ldots, g_r$, g_r showing the number of attributes in each partition p_r, i_1, i_2 and i_k are the set m derived integers $\{1, 2, \ldots, g_r\}$ of integers, the binomial coefficient $C_{g_r}^{m}$, whose expression is $C_{g_r}^{m} = \frac{g_r!}{m!(g_r-m)!}$. The characteristics of the MSM operator are given below.

(1) If $\alpha_i \leq \beta_i$, $i = 1, 2, \ldots, n$, then $MSM^{(m)}(\alpha_1, \alpha_2, \ldots, \alpha_k) \leq MSM^{(m)}$ $(\beta_1, \beta_2, \ldots, \beta_k)$
(2) $\min\limits_j\{\beta_j\} \leq MSM^m(\beta_1, \beta_2, \ldots, \beta_m) \leq \max\limits_j\{\beta_j\}$.

8.8 Frank Aggregation Operator

The triangle norms have been thoroughly investigated, starting from Zadeh presented max and min operations as a pair of triangular norm and triangular conorm. As tools for working with fuzzy sets, we can make use of a variety of triangular norms and associated triangular norms, including the product t-norm and probabilistic sum t-conorm [45], Lukasiewicz t-norm and t-conorm [7], Einstein t-norm and t-conorm [44], Hamacher t-norm and t-conorm [30], and triangle norms and triangular conorms, which are instances of Frank operations, which include the Frank product and Frank sum.

Definition 5 Frank t-norm and t-conorm is defined below:

$$\alpha \oplus \beta = 1 - log_\delta \left(1 + (\frac{\delta^{1-\alpha} - 1)(\delta^{1-\beta} - 1)}{\delta - 1} \right) \qquad \forall\, (\alpha, \beta) \in [0, 1] \times [0, 1] \tag{1.23}$$

$$\alpha \otimes \beta = log_\delta \left(1 + (\frac{\delta^{\alpha} - 1)(\delta^{\beta} - 1)}{\delta - 1} \right) \qquad \forall\, (\alpha, \beta) \in [0, 1] \times [0, 1] \tag{1.24}$$

The following characteristics of the Frank product and Frank sum are highlighted [43].

$$(\alpha \oplus \beta) + (\alpha \otimes \beta) = \alpha + \beta \tag{1.25}$$

$$\frac{\partial(\alpha \oplus \beta)}{\partial \alpha} + \frac{\partial(\alpha \otimes \beta)}{\partial \beta} = 1. \tag{1.26}$$

(1) If $\delta \to 1$, then $\alpha \oplus \beta \to \alpha + \beta - \alpha\beta$; the probabilistic product and probabilistic sum are reduced from the Frank product and Frank sum
(2) If $\delta \to \infty$, then $(\alpha \oplus \beta) \to \min(\alpha + \beta, 1)$, and $(\alpha \otimes \beta) \to \max(0, \alpha + \beta - 1)$; the Frank product and Frank sum are reduced to the Lukasiewicz product and Lukasiewicz sum, respectively.

8.9 Heronian Mean (HM) Operator

Definition 6 ([29]) If $H : [0, 1]^n \to [0, 1]$ and satisfies

$$H(\alpha_1, \alpha_2, \ldots, \alpha_n) = \frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{\alpha_i \alpha_j} \tag{1.27}$$

then the operator H is called Heronian mean (HM) operator.

Definition 7 ([29]) If $H^{p,q} : [0, 1]^n \to [0, 1]$ and $p, q \geq 0$ satisfies

$$H^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(\frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i^p \alpha_j^q \right)^{\frac{1}{p+q}} \tag{1.28}$$

then $H^{p,q}$ is called Heronian mean (HM) operator with parameter (p, q).

The HM operator possesses the following basic properties:

(1) Idempotency. $H^{p,q}(\alpha, \alpha, \ldots, \alpha) = \alpha$
(2) Monotonicity. If $\alpha_i \leq \alpha_j$ for all j, then $H^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq H^{p,q}(\beta_1, \beta_2, \ldots, \beta_n)$, where α_i and β_i are arbitrary real numbers.
(3) Bounded $\min\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \leq H^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \max\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$.

For different values of p, q, we find different forms of $H^{p,q}$ are as follows:

(1) If $p = q$, then $H^{p,p}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(\frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=1}^{n} (\alpha_i \alpha_j)^p \right)^{\frac{1}{2p}}$
(2) If $p = q = \frac{1}{2}$, then $H^{\frac{1}{2},\frac{1}{2}}(\alpha_1, \alpha_2, \ldots, \alpha_n) = H(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(\frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{\alpha_i \alpha_j} \right)^{\frac{1}{2p}}$
(3) If $p = q = 1$, then $H^{1,1}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(\frac{1}{n} \sum_{i=1}^{n} \nu_i \alpha_i \right)^{\frac{1}{2}}$ where, $\nu_i = \frac{1}{n+1}(\alpha_i + \sum_{j=1}^{n} \alpha_j)$

9 Aggregation Operators

Recently, many aggregation functions/operators have been defined. These functions combine all different data in a single value. Suppose $\alpha_1, \alpha_2, \ldots, \alpha_n$ is a set of parameters, and the functions aggregate all these parameters $\mathfrak{A}$ in a given domain $\mathfrak{D}$. These functions satisfy several interesting properties.

(i) Idempotency: $\mathfrak{A}(\alpha, \alpha, \ldots, \alpha) = \alpha$ for all $\alpha \in \mathfrak{D}$
(ii) Monotonicity: $\mathfrak{A}(\alpha_1, \alpha_2, \ldots, \alpha_n) \geq \mathfrak{A}(\alpha_1', \alpha_2', \ldots, \alpha_n')$ for all $\alpha_i \geq \alpha_i'$.
(iii) Symmetry: Let π be any permutation on $\{1, 2, \ldots, n\}$. $\mathfrak{A}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \mathfrak{A}(\pi(\alpha_1), \pi(\alpha_2), \ldots, \pi(\alpha_n))$.

There are some variations on the definition of aggregation operator. The monotonicity condition is applicable only when the $\geq$ operator is defined on the domain. Also, some authors define the aggregation operators in such a way that idempotency and monotonicity conditions are satisfied in the boundary of the domain only, i.e., if the domain is $[0, 1]$, idempotency condition satisfies only at 0 and 1, i.e., $\mathfrak{A}(0, 0, \ldots, 0) = 0$, and $\mathfrak{A}(1, 1, \ldots, 1) = 1$.

9.1 Aggregation Operators for Numerical Data

There are two very simple and well-known such operators arithmetic mean (AM) and the weighted mean (WM). In AM, no extra information is required, and in this aggregation, all the given data have equal importance. On the other hand, in WM, weights are assigned to the given data to indicate the importance or weight of the corresponding data. In WM, the aggregation is made with the given weight vector. These two aggregation operators are defined below.

Definition 8 Let $\mathfrak{A}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a set of n attributes or any type of data in $\mathbb{R}$. Then the AM is a function $AM : \mathbb{R}^n \to \mathbb{R}$, which is defined as

$$AM(\alpha_1, \alpha_2, \ldots, \alpha_n) = \frac{1}{n} \sum_{i=1}^{n} \alpha_i. \tag{1.29}$$

Definition 9 Let $\mathfrak{A}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a set of n data in $\mathbb{R}$ and the weight vector be $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$, where $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n} \omega_i = 1$. Then the WM is a function $WM : \mathbb{R}^n \to \mathbb{R}$ defined as

$$WM(\alpha_1, \alpha_2, \ldots, \alpha_n) = \sum_{i=1}^{n} \alpha_i \omega_i. \tag{1.30}$$

Similar to WM, another operator called ordered weighting averaging (OWA) operator is defined below.

Definition 10 Let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$, where $\omega_i \in [0, 1]$, and $\sum_{i=1}^{n} \omega_i = 1$ be the weight vector. Also, let $\{\pi(1), \pi(2), \ldots, \pi(n)\}$ be a permutation over $\{1, 2, \ldots, n\}$ such that $\alpha_{\pi(i-1)} \geq \alpha_{\pi(i)}$ for all $i = 1, 2, \ldots, n$ (this ordered implies that the data are ordered concerning the permutation π). The OWA is a mapping $OWA : \mathbb{R}^n \to \mathbb{R}$ defined as

$$OWA(\alpha_1, \alpha_2, \ldots, \alpha_n) = \sum_{i=1}^{n} \alpha_{\pi(i)}\, \omega_i. \tag{1.31}$$

Notice that both the operators WM and OWA are similar. In OWA, the data is considered as ordered, and then the weights are multiplied with it. In a MCDM problem, the weights in the WM are assigned to the criteria, while in the OWA, they are associated with the data itself or to the relative position of one value with respect to the other values. In this means, the decision-maker can introduce different types of prior knowledge or different types of information in the aggregation method.

The OWA and WM are similar and are a linear combination of the data and the weights. But in OWA, there is a concept of the ordering of the data; the weights' meanings are different for different aspects. That is, the weight vector ω in WM represents the importance of the criteria, while the weight vector in OWA represents the compensation degree. For further explanation, two symbols are used to represent two types of weighting vectors. For the sake of simplicity, the weight vectors ω and p represent the weights corresponding to the OWA and WM, respectively. The properties of both types of weight vectors are the same. In some MCDM problems, it is required to simultaneously incorporate the weights for the different criteria and a certain degree of compensation. For this purpose, a new operator called WOWA (weighted OWA) is introduced. In these operators, two types of operators are used together.

Definition 11 Let ω and p be two n-dimensional weight vectors. The weighted ordered weighted averaging (WOWA) [39] is a mapping $WOWA : \mathbb{R}^n \to \mathbb{R}$ defined as

$$WOWA_{\omega,p}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \sum_{i=1}^{n} W_i\, \alpha_{\pi(i)}, \tag{1.32}$$

where π is a permutation over $\{1, 2, \ldots, n\}$, and the weight W_i is defined as

$$W_i = W^*(\sum_{k \leq i} p_{\pi(k)}) - W^*(\sum_{k<i} p_{\pi(k)})$$

where W^* is a non-decreasing function that is obtained from interpolating the points

$$\{(1/n, U_1), (2/n, U_2), (3/n, U_3), \ldots, (n/n, U_n), (0, 0)\}, \ U_i = \sum_{k \leq i} \omega_k.$$

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Chapter 2
What Is the Most Adequate Fuzzy Methodology?

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1 Outline

Question In many practical control situations, we do not have the exact model of a system that we need to control, but we have the experience of successful expert human controllers. Human controllers often formulate their experience by using imprecise ("fuzzy") words from natural language like "small." How can we translate this expert knowledge into a precise control strategy for an automatic controller?

A similar problem emerges when we want to use expert rules to predict the future state of the worlds.

To translate imprecise expert statements into precise form, Lotfi Zadeh invented a special methodology that he called *fuzzy* (see, e.g., [1, 2, 4–6, 8]). In this methodology, we start by describing each natural-language term A (e.g., "small") by a function that assigns:

- to each possible value x of the corresponding quantity,
- a degree $\mu_A(x)$ from the interval [0, 1] to which, in the controller's opinion, this value satisfies the corresponding property (e.g., the degree to which the value x is small).

This function is known as a *membership function* or, alternatively, as a *fuzzy set*.

Once we have fuzzy sets corresponding to all relevant natural-language terms and we have all natural-language if-then rules provided by the human controllers, we

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C. Jana et al. (eds.), *Fuzzy Optimization, Decision-making and Operations Research*, https://doi.org/10.1007/978-3-031-35668-1_2

need to transform this information into a precise control strategy. There are several different methods for generating such a strategy. A natural question is: Which method should we select? In other words, which method is the most adequate?

What We Do in This Chapter To answer the above question, a natural requirement is that if the expert's if-then rules describe—in fuzzy terms—an actual control strategy $y = f(x)$, then the fuzzy methodology should return exactly this strategy. Somewhat surprisingly, it turns out that the existing fuzzy methodologies—including the very popular Mamdani approach—do not satisfy this requirement. In this chapter, we show that this requirement actually leads to a new methodology, a methodology that we describe and analyze.

Structure of This Chapter In Sect. 2, we briefly recall what a fuzzy methodology is and which fuzzy methodologies are typically used in practical applications. In Sect. 3, we describe a natural criterion for deciding which fuzzy methodology is the most adequate, and we show that from the viewpoint of this criterion, none of the current methodologies are perfect. In Sect. 4, we describe a methodology which is the most adequate according to our natural criterion—and analyze some properties of this methodology.

2 What Is Fuzzy Methodology: A Brief Reminder

Need for Expert Knowledge In many practical situations, we want to make a decision, for example:

- we want to decide what control to apply to a system,
- we want to decide what the patient's disease is and what dose of what medicine should be the best for this patient,
- we want to predict tomorrow's weather.

In many such situations, we do not have an accurate model of the system, and thus, we cannot formulate this problem in precise terms. What we usually do have is the experience of experts:

- we have the experience of human expert controllers who control a plant,
- we have the experience of expert medical doctors who are good in diagnosing and treating the patient,
- we have the experience of expert meteorologists who can predict tomorrow's weather in their region with high accuracy.

It is therefore desirable to use this expert knowledge to design an automatic controller and/or an automatic expert system.

Using Expert Knowledge Is Not Easy Most experts are willing to share their expertise, but the problem is that experts often cannot describe their knowledge in precise terms. Instead, they formulate this knowledge in terms of if-then rules that use imprecise ("fuzzy") words from natural language.

For example, many people know how to drive. So, at first glance, it may seem to be an easy task to design a self-driving car: just use the experience of good human drivers. However, this is not so easy. An automatic controller would need to know what control to apply in each situation. For example, if a car is going on a freeway with the speed of 100 km per hour and a car in front of it—which is 10 m ahead—slows down to 95 km per hour, what should we do? A natural human answer is "break a little bit," but what the automatic controller needs is with how many Newtons of force to push the brake pedal and for how many milliseconds—and most human drivers cannot provide these numbers.

Fuzzy Methodology: First Step To perform this challenging task, i.e., to extract precise knowledge from the imprecise expert knowledge, Lotfi Zadeh invented a new methodology that he called fuzzy. This methodology starts with providing a precise description of all natural-language words used by experts.

For this purpose:

- for each such word A and for each possible value x of the corresponding quantity,
- we ask the expert to mark, on a scale from 0 to 1, to what extent the value x has the corresponding property (e.g., to what extent x is small).

The intent is that:

- mark 1 corresponds to the case when the expert is absolutely sure that x satisfies this property,
- mark 0 means that the expert is absolutely sure that x does not satisfy this property
- marks between 0 and 1 correspond to intermediate cases.

The resulting function $A(x)$ that assigns the degree to each value x is called a *membership function* or a *fuzzy set*.

Comment Of course, there are infinitely many real numbers x, and we can only ask finitely many questions to the expert. So, in practice:

- we ask the expert a finite number of questions, about finitely many values $x_1, \ldots, x_n$, and then
- we use interpolation/extrapolation to estimate the values $A(x)$ for all other values x.

In particular, if we ask the expert to provide:

- the value M for which this user is absolutely sure that this property is satisfied (i.e., that $A(M) = 1$), and
- the values $\underline{m}$ and $\overline{m}$ such that outside the interval $[\underline{m}, \overline{m}]$, the property is *not* satisfied (i.e., $A(x) = 0$),

and use linear interpolation, then we get a frequently used *triangular* membership function.

If instead of a single value M we get the whole interval $\left[\underline{M}, \overline{M}\right]$ on which the property A is satisfied, i.e., for which $A(M) = 1$ for all values M from this interval, and we use linear interpolation, then we get trapezoid membership functions.

Fuzzy Methodology Beyond the First Step: What We Have After the first step, to determine the desired dependence $y = f(x)$, we have several expert if-then rules

$$\text{If } x \text{ is } A_1 \text{ then } y \text{ is } B_1.$$

$$\text{If } x \text{ is } A_2 \text{ than } y \text{ is } B_2.$$

$$\ldots$$

$$\text{If } x \text{ is } A_k \text{ then } y \text{ is } B_k.$$

where A_i and B_i are natural-language terms that are described by membership functions $A_i(x)$ and $B_i(y)$. Based on this information, we want to generate a function $y = f(x)$ that adequately describes these rules.

Example To illustrate our ideas, let us consider a simple example of controlling a thermostat by turning a knob.

- If we turn the knob to the right, the temperature increases.
- If we turn it to the left, the temperature decreases.

In this example:

- the desired control variable y is the angle on which we turn the knob
- the input x is the difference $x \stackrel{\text{def}}{=} T - T_0$ between the actual temperature T and the desired temperature T_0.

If the temperature is close to the desired one, i.e., if the difference x is close to 0, then we should not change anything, i.e., the control y should be negligible. So, we arrive at the first rule:

If x is negligible, then y should be negligible.

If the temperature is slightly higher than desired, then we should turn the knob to the left a little bit. So, we arrive at the second rule:

If x is small positive, then y should be small negative.

Similarly, if the temperature is slightly lower than desired, then we should turn the knob to the right a little bit. So, we arrive at the second rule:

If x is small negative, then y should be small positive.

We can add more rules, but for simplicity, let us only consider these three rules. The restriction to these three rules makes sense in situations when the control is almost perfect, and we experience only small deviations from the desired temperature.

Also, for simplicity, let us consider simple triangular membership functions corresponding to "negligible," "small positive," and "small negative." We will denote them, correspondingly, by $N(x)$, $SP(x)$, and $SN(x)$. Based on our experience, we assume that:

- for "negligible": the value $M = 0$ is definitely negligible, and values outside the interval $[-5, 5]$ are definitely not negligible;
- for "small positive": the value $M = 5$ is definitely small positive, and values outside the interval [0, 10] are definitely not small positive: value smaller than 0 are not positive, and values larger than 10 are not small;
- for "small negative": the value $M = -5$ is definitely small negative, and values outside the interval $[-10, 0]$ are definitely not small negative—values smaller than -10 are not small, and values larger than 0 are not negative.

In this case, linear interpolation leads to the following triangular membership functions, see Fig. 2.1, 2.2 and 2.3.

Fuzzy Methodologies Beyond the First Step: Examples Let us list the most frequently used fuzzy methodologies, i.e., methodologies for transforming fuzzy rules into a precise function $y = f(x)$.

Fuzzy Methodology Beyond the First Step: Mamdani Approach One of the most widely used approaches was originally proposed by Mamdani and is, thus, known as Mamdani approach. In this approach, we first take into account that for a given value x, the value y is reasonable (R) if:

- either the first rule is applicable, i.e., x is A_1 *and* y is B_1,
- *or* the second rule is applicable, i.e., x is A_2 *and* y is B_2.

We can symbolically describe it as follows:

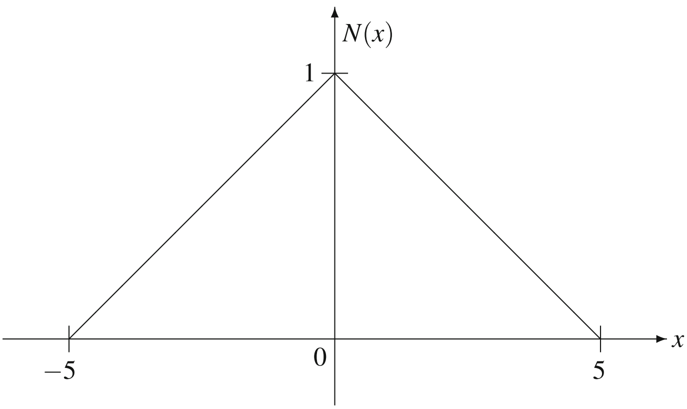

Fig. 2.1 Membership function corresponding to "negligible"

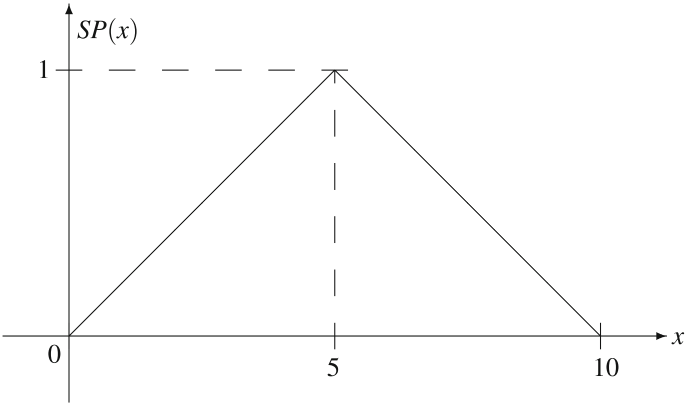

Fig. 2.2 Membership function corresponding to "small positive"

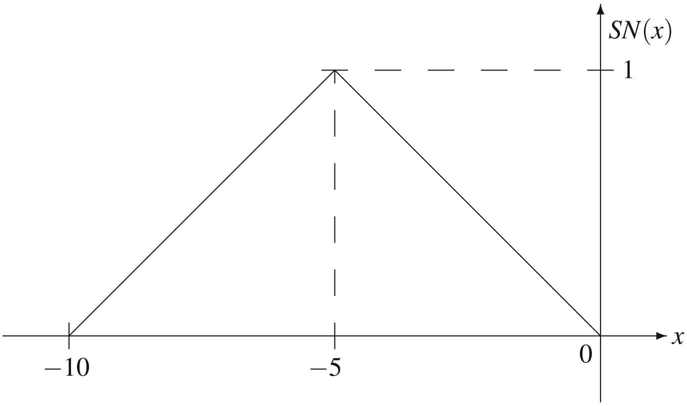

Fig. 2.3 Membership function corresponding to "small negative"

$$R(y) \Leftrightarrow (A_1(x) \,\&\, B_1(y)) \vee (A_2(x) \,\&\, B_2(y)) \vee \ldots$$

To give this formula a numerical meaning, we need to provide the numerical meaning to the "and"- and "or"-operations, i.e., in effect, to extend the "and"- and "or"-operations of the usual 2-valued logic (with the values "false" (0) are "true" (1)) to the whole interval [0, 1]. From the computational viewpoint, the simplest such extensions are min and max. Thus, we arrive at the following membership function for "reasonable":

$$R(y) = \max(\min(A_1(x), B_1(y)), \min(A_2(x), B_2(y)), \ldots)$$

Our ultimate objective is to come up with a single value $\overline{y}$. A reasonable way to come up with this value is to minimize the weighted squared difference between this value and possible values y, weighted by the degree to which y is possible, i.e., to minimize the following expression:

$$\int R(y) \cdot (\overline{y} - y)^2 \, dy.$$

To find the minimizing value $\overline{y}$, we can differentiate this expression with respect to $\overline{y}$ and equate the derivative to 0. As a result, we get the following expression:

$$\overline{y} = \frac{\int y \cdot R(y) \, dy}{\int R(y) \, dy}.$$

This expression is known as *centroid defuzzification*.

Fuzzy Methodology Beyond the First Step: Takagi-Sugeno Approach An alternative approach is that we replace each y-membership function $B_i(y)$ by the result of its defuzzification, for example, by the centroid value

$$y_i = \frac{\int y \cdot B_i(y) \, dy}{\int B_i(y) \, dy}.$$

In effect, we ignore the fuzziness of y in the rules and consider the following simplified rules:

If x is A_1 then $y = y_1$.

If x is A_2 than $y = y_2$.

...

If x is A_k then $y = y_k$.

These rules can be treated the same way as in the previous approach, the only difference is that now the conclusions of each rule are not fuzzy. In this case, the value $R(y)$ is only different from 0 when y coincides with each of the points y_i, and for each of these values, we have $R(y_i) = A_i(x)$. Thus, the centroid formula leads to

$$\overline{y} = \frac{\sum_{i=1}^{k} A_i(x) \cdot y_i}{\sum_{i=1}^{k} A_i(x)}.$$

3 How to Decide Which Fuzzy Methodology Is the Most Adequate

Idea Fuzzy methodology transforms rules and membership functions into an exact control strategy $f(x)$; see Fig. 2.4.

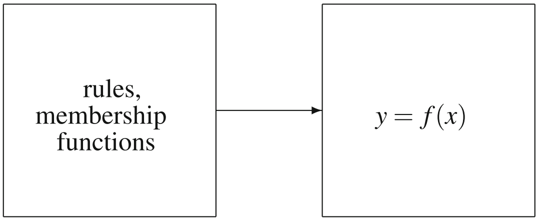

Fig. 2.4 Fuzzy methodology

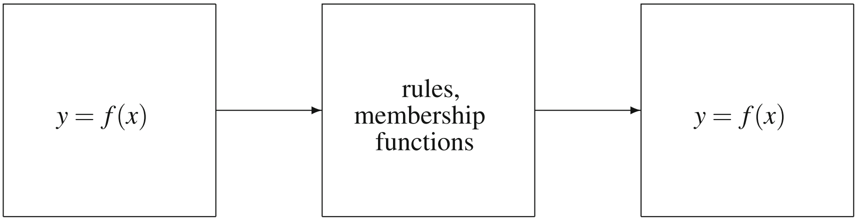

Fig. 2.5 Fuzzy methodology: ideal case

Suppose now that we start with the actual function $y = f(x)$. As we have mentioned, fuzzy techniques deal with situations when the experts cannot explicitly describe this function. Instead, they formulate rules based on this function. In this case, a natural requirement is that once we process these rules, we should get back the original function $y = f(x)$. This is what we should have in the ideal case; see Fig. 2.5.

The closer the reconstructed function to the original function, the more adequate the fuzzy methodology—this is a natural idea of gauging adequacy of different methodologies.

What Do We Mean by Rules Generated by a Function? Suppose that we know the function $y = f(x)$ and that we have fuzzy information about x: namely, that x is A_i for some property A_i, which is described by a membership function $A_i(x)$. What can we then say about y? How can we describe the corresponding membership function $B_i(y)$?

The answer to this question is well-known in fuzzy research: it is provided by the so-called Zadeh's extension principle. This answer can be easily explained. Indeed, in this case, a real number Y is a possible value of the quantity y if there exists a value X which is a possible value of the quantity x and for which $f(X) = Y$. The degree to which X is a possible value of the quantity X is determined by the corresponding membership function $A_i(x)$ and is, thus, equal to $A_i(X)$. If there is only one X for which $f(X) = Y$—this value X is then denoted by $X = f^{-1}(Y)$—then $A_i(X) = A_i(f^{-1}(Y))$ is exactly the degree $B_i(Y)$ to which Y is a possible value of y. So, in this case, we have

$$B_i(y) = A_i(f^{-1}(x)). \tag{2.1}$$

What if there are several different values of X for which $f(X) = Y$? This happens, e.g., when $f(x) = x^2$, then for each Y, there are two such values X: $X = \sqrt{Y}$ and $X = -\sqrt{Y}$. In this case, Y is possible if either we have the first of these values X or the second of these values X. The simplest way to estimate the degree to which an "or"-statement $A \vee B$ is true based in the degrees a and b to which individual statements A and B are true is to use maximum $\max(a, b)$. Thus, we get

$$B_i(y) = \max\{A_i(x) : f(x) = y\}. \tag{2.2}$$

This is exactly the formula that was first produced by Zadeh himself and is, thus, called Zadeh's extension principle. This membership function will be denoted as $B_i = f(A_i)$.

In these terms, the fuzzy methodology is most adequate if, based on the rules

if x is A_i then y is B_i, where $B_i = f(A_i)$,

we should be able to reconstruct the original function $f(x)$.

Important Comment In the following text, we will use the known fact that for reasonable membership functions $A_i(x)$—namely, for all the functions that first continuously increase from 0 to 1 and then continuously decrease from 1 to 0—Zadeh's extension principle can be reformulated in terms of α-cuts, i.e., sets $\mathbf{A}_i(\alpha) \stackrel{\text{def}}{=} \{x : A_i(x) \geq \alpha\}$ and $\mathbf{B}_i(\alpha) \stackrel{\text{def}}{=} \{y : B_i(x) \geq \alpha\}$ for all $\alpha \in (0, 1]$. Namely, we have

$$\mathbf{B}_i(\alpha) = f(\mathbf{A}_i(\alpha)),$$

where for each set S, by $f(S)$, we mean

$$f(S) \stackrel{\text{def}}{=} \{f(x) : x \in S\}.$$

Are Existing Fuzzy Methodologies Most Adequate? A natural question is: Are the existing fuzzy methodologies—e.g., the ones described above—most adequate in this natural sense? Our answer is No. Let us explain this answer.

Mamdani Methodology Is Not the Most Adequate (in the Above Sense) Let us explain, on a simple example, that Mamdani methodology is not the most adequate, i.e., that it does not reconstruct the original function $y = f(x)$.

Let us consider the above membership functions $N(x)$, $SP(x)$, and $SN(x)$ and a simple function $f(x) = -x$. In this case, as one can easily check, we have $f(N) = N$, $f(SP) = SN$, and $f(SN) = SP$. Thus, the rules generated by this function take exactly the form described in the previous section:

If x is N then y is N.

If x is SP then y is SN.

If x is SN, then y is SP.

Let us consider a small negative value $x = -\varepsilon$, where $\varepsilon > 0$. In this case,

$$N(x) = 1 - \frac{\varepsilon}{5},\ P(x) = \frac{\varepsilon}{5},\ \text{and } SN(x) = 0.$$

Thus, the reasonable value $R(y)$ is described by the formula

$$R(y) = \max\left(\min\left(N(y), 1 - \frac{\varepsilon}{5}\right), \min\left(SP(y), \frac{\varepsilon}{5}\right)\right).$$

The functions $\min(N(y), 1 - \varepsilon/5)$ and $\min(SP(y), \varepsilon/5)$ can be represented as follows, see Fig. 2.6 and 2.7.

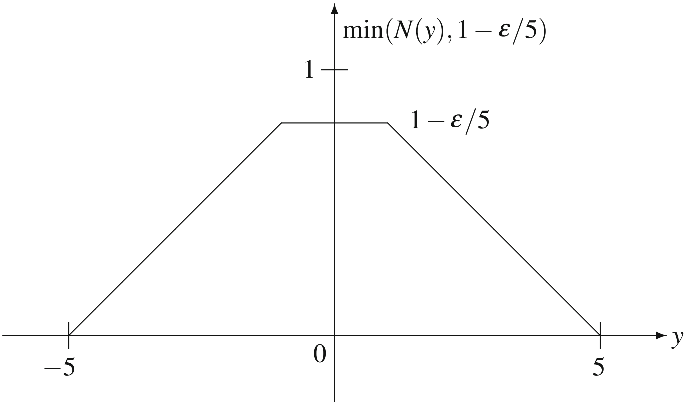

Fig. 2.6 Mamdani approach: $\min(N(y), 1 - \varepsilon/5)$

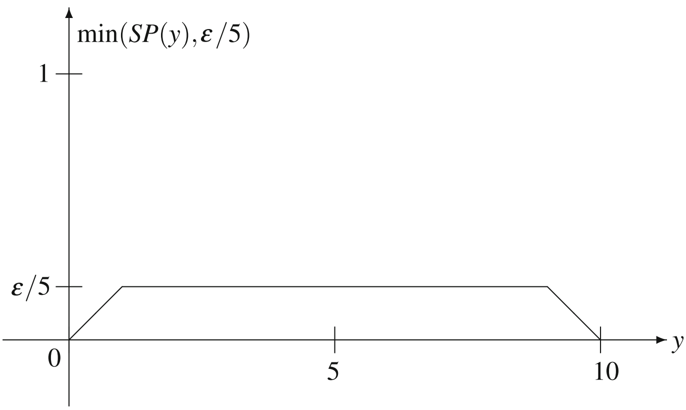

Fig. 2.7 Mamdani approach: $\min(SP(y), \varepsilon/5$

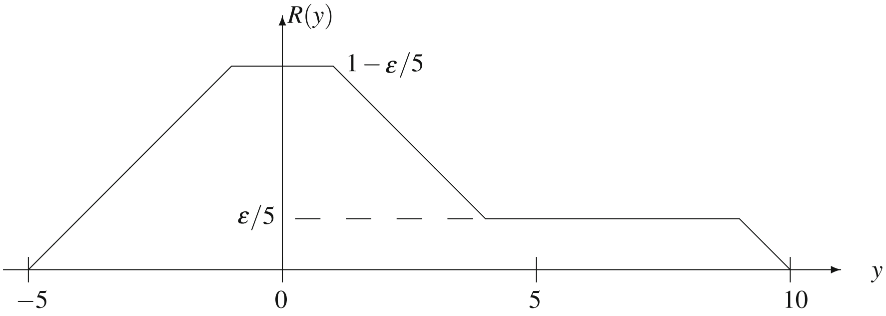

Fig. 2.8 Mamdani approach: resulting membership function for y

Thus, the desired function $R(y)$—which is the maximum of these functions—takes the following form: see Fig. 2.8.

The result of the centroid defuzzification is the ratio of two integrals, so let us estimate these integrals. Let us first estimate the denominator $\int R(y)\,dy$. When ε tends to 0, the function $R(y)$ tends to $N(y)$, for which $\int N(y)\,dy$ is the area of the corresponding triangle with height 1 and base $5-(-5)=10$, i.e.,

$$\frac{1}{2}\cdot 10\cdot 1=5.$$

Thus, the denominator is equal to $5+O(\varepsilon)$.

The integral in the numerator can be represented as the sum of the parts: the symmetric part $R_{\text{sym}}(y)=R_{\text{sym}}(-y)$ corresponding to values from $y=-5$ to $y=5$ and the remaining part $r(y)\stackrel{\text{def}}{=}R(y)-R_{\text{sym}}(y)$. For the symmetric part $R_{\text{sym}}(y)$, the integral $\int y\cdot R_{\text{sym}}(y)\,dy$ is 0—since for each $y>0$, contributions of the terms corresponding to y and to $-y$ cancel each other. Thus, the numerator is equal to $\int y\cdot r(y)\,dy$. For almost all the values y from $y=5$ to $y=10$, we have $r(y)=\varepsilon/5$, thus in the first approximation

$$\int y\cdot r(y)\,dy=\int_5^{10} y\cdot\frac{\varepsilon}{5}\,dy+o(\varepsilon)=\frac{\varepsilon}{5}\cdot\frac{1}{2}\cdot y^2\Big|_5^{10}+o(\varepsilon)=$$

$$\frac{\varepsilon}{5}\cdot\frac{1}{2}\cdot(10^2-5^2)+o(\varepsilon)=7.5\cdot\varepsilon+o(\varepsilon).$$

Thus, the desired ratio is equal to

$$\overline{y}=\frac{\int y\cdot R(y)\,dy}{\int R(y)\,dy}=\frac{7.5\cdot\varepsilon+o(\varepsilon)}{5+O(\varepsilon)}=1.5\cdot\varepsilon+o(\varepsilon).$$

This is clearly different from the original value

$$f(x) = f(-\varepsilon) = \varepsilon.$$

Takago-Sugeno Approach Is Not the Most Adequate It so happens that for the above example when $f(x) = -x$ and we have $N(x)$, $SP(x)$, and $SN(x)$, Takagi-Sugeno approach reconstructs the original function. However, for any nonlinear function $f(x)$, e.g., for $f(x) = -x+x^3$, this approach won't reconstruct the original function.

Indeed, the function reconstructed by this methodology is a linear combination of the membership functions corresponding to x. On the interval $[0, 5]$, all the membership functions are linear, so their linear combination is also linear—and thus, cannot be equal to any nonlinear function.

Remaining Problem Since none of the existing methodologies is the most adequate, we need to come up with a new most adequate fuzzy methodology.

4 Toward the Most Adequate Fuzzy Methodology

What Is Given: Reminder We are given fuzzy rules of the type

If x is A_i then y is B_i,

for $i = 1, \ldots, k$, and we know the membership functions $A_i(x)$ and $B_i(y)$ describing these rules.

What We Want: Reminder We want to make sure that when, for some function $f(x)$, we have $B_i = f(A_i)$ for all i, i.e., we have $\mathbf{B}_i(\alpha) = f(\mathbf{A}_i(\alpha))$ for all i and for all α, then this methodology should reconstruct the function $f(x)$. This prompts the following seemingly natural definition.

A Seemingly Natural Idea Let us return a function $f(x)$ for which, for all i and for all α, we have

$$\mathbf{B}_i(\alpha) = f(\mathbf{A}_i(\alpha)).$$

A Problem with This Idea Expert knowledge is usually approximate. As a result, the membership function B_i may be slightly different from $f(A_i)$. In this case, we may not have a function $f(x)$ for which, in the above equation, we have exact equality.

A Natural Solution to This Problem and the Resulting Description of the New Fuzzy Methodology In view of the approximate character of expert knowledge, let us look for a function $f(x)$ for which

$$\mathbf{B}_i(\alpha) \approx f(\mathbf{A}_i(\alpha)).$$

We can interpret these approximate equalities, e.g., by using the usual least squares approach (see, e.g., [7]):

$$\sum_{i,\alpha} d^2(\mathbf{B}_i(\alpha), f(\mathbf{A}_i(\alpha))) \to \min,$$

where the distance between the two intervals $[\underline{a}, \overline{a}]$ and $[\underline{b}, \overline{b}]$ can be defined, e.g., as the Euclidean distance between the corresponding 2-D points $(\underline{a}, \overline{a})$ and $(\underline{b}, \overline{b})$:

$$d^2([\underline{a}, \overline{a}], [\underline{b}, \overline{b}]) = (\underline{a} - \underline{b})^2 + (\overline{a} - \overline{b})^2.$$

Case of Monotonicity In the control situation that we used as an example, the desired function $f(x)$ is decreasing. In general, situations in which the function $f(x)$ is increasing or decreasing are ubiquitous. In such situation, the above minimization problem can be simplified.

To describe this simplification, let us denote the endpoint of the interval $\mathbf{A}_i(\alpha)$ by $\underline{A}_i(\alpha)$ and $\overline{A}_i(\alpha)$, so that

$$\mathbf{A}_i(\alpha) = [\underline{A}_i(\alpha), \overline{A}_i(\alpha)].$$

Similarly, let us denote the endpoint of the interval $\mathbf{B}_i(\alpha)$ by $\underline{B}_i(\alpha)$ and $\overline{B}_i(\alpha)$, so that

$$\mathbf{B}_i(\alpha) = [\underline{B}_i(\alpha), \overline{B}_i(\alpha)].$$

In these terms, we can explicitly describe the expression for the range $f(\mathbf{A}_i(\alpha))$:

- If the function $f(x)$ is increasing, then

$$f(\mathbf{A}_i(\alpha)) = f([\underline{A}_i(\alpha), \overline{A}_i(\alpha)]) = [f(\underline{A}_i(\alpha)), f(\overline{A}_i(\alpha))].$$

- If the function $f(x)$ is decreasing, then

$$f(\mathbf{A}_i(\alpha)) = f([\underline{A}_i(\alpha), \overline{A}_i(\alpha)]) = [f(\overline{A}_i(\alpha)), f(\underline{A}_i(\alpha))].$$

In this case, the minimized expression becomes simpler:

- If we know that the function $f(x)$ is increasing, then, according to the proposed methodology, we should select the function $f(x)$ that minimizes the expression

$$\sum_{i,\alpha} (\underline{B}_i(\alpha) - f(\underline{A}_i(\alpha)))^2 + \sum_{i,\alpha} (\overline{B}_i(\alpha) - f(\overline{A}_i(\alpha)))^2.$$

- If we know that the function $f(x)$ is decreasing, then, according to the proposed methodology, we should select the function $f(x)$ that minimizes the expression

$$\sum_{i,\alpha}(\underline{B}_i(\alpha) - f(\overline{A}_i(\alpha)))^2 + \sum_{i,\alpha}(\overline{B}_i(\alpha) - f(\underline{A}_i(\alpha)))^2.$$

Often, we look for a function $f(x)$ as a linear combination of functions from the given basis, i.e., as an expression

$$f(x) = C_1 \cdot e_1(x) + \ldots + C_m \cdot e_m(x),$$

where the functions $e_j(x)$ are given and the coefficients C_j need to be determined. For example, we can take $e_1(x) = 1$, $e_2(x) = x$, and $e_j(x) = x^{j-1}$; in this case, we are looking for a polynomial function $f(x)$. In this case, the above minimized expression becomes quadratic in terms of the unknown coefficients C_j. Thus, differentiating with respect to each of these coefficient and equating the derivatives to 0, we get an easy-to-solve system of linear equations for finding C_j.

Case of Several Inputs Sometimes, we have rules whose conditions involve several inputs $x_1, \ldots, x_n$, i.e., rules of the type

If x_1 is A_{i1} and ... and x_n is A_{in} then y is B_i.

Based on these rules, we need to find an appropriate function $y = f(x_1, \ldots, x_n)$.

In this case, Zadeh's extension principle takes the following form:

$$\mathbf{B}_i(\alpha) = f(\mathbf{A}_{i1}(\alpha), \ldots, \mathbf{A}_{in}(\alpha)),$$

where for every tuple of sets $S_1, \ldots, S_n$, the range $f(S_1, \ldots, S_n)$ means

$$f(S_1, \ldots, S_n) \stackrel{\text{def}}{=} \{f(x_1, \ldots, x_n) : x_1 \in S_1, \ldots, \text{ and } x_n \in S_n\}.$$

In this case, according to the proposed methodology, we should select the function $f(x_1, \ldots, x_n)$ for which

$$\mathbf{B}_i(\alpha) \approx f(\mathbf{A}_{i1}(\alpha), \ldots, \mathbf{A}_{in}(\alpha))$$

for all i and for all α, i.e., for example, for which the following expression attains the smallest possible value:

$$\sum_{i,\alpha} d^2(\mathbf{B}_i(\alpha), f(\mathbf{A}_{i1}(\alpha), \ldots, \mathbf{A}_{in}(\alpha))) \to \min.$$

Is This New Methodology Indeed the Most Adequate? Of course, by definition, if $f(A_i) = B_i$ for all i, then $f(x)$ is *one* of the functions satisfying the above condition.

Is this the only function with this property? Not necessarily: if all membership functions are constant on some interval $[\underline{x}, \overline{x}]$—e.g., if we consider trapezoid functions—then all we can extract from the given information is the range of the function $f(x)$ on this interval, but we cannot uniquely determine how exactly the function $f(x)$ behaves on this interval:

- this function can be linear on this interval,
- it can be nonlinear on this interval,

the membership functions $B_i(x)$ will be the same.

However, if we take into account that in control situations similar to the one described above, the function $f(x)$ is either strictly increasing or strictly decreasing, then we can prove that the above exception is the only case when we cannot uniquely reconstruct the original function $f(x)$: in all other cases, the function $f(x)$ can be uniquely reconstructed.

Proposition 1 *Let $A_1(x), \ldots, A_n(x)$ be continuous membership functions on an interval $[\underline{X}, \overline{X}]$ such that for every value x—except maybe a finite set of values—one of these membership functions is either strictly increasing or strictly decreasing in some neighborhood of this point. If two continuous functions $f(x)$ and $g(x)$ are both increasing or both decreasing, and we have $f(A_i) = g(A_i)$ for all i, then for all $x \in [\underline{X}, \overline{X}]$, we have $f(x) = g(x)$.*

Proof Let us show how to prove this proposition for the case when both functions $f(x)$ and $g(x)$ are increasing; the proof for the case when both functions are decreasing is similar. Let us take a point x from the given interval, and let us prove that $f(x) = g(x)$. Let $A_i(x)$ be the membership function, which is either strictly increasing or strictly decreasing in the vicinity of the point x. As before, let us denote the endpoints of the interval $\mathbf{A}_i(\alpha)$ by $\underline{A}_i(\alpha)$ and $\overline{A}_i(\alpha)$, so that

$$\mathbf{A}_i(\alpha) = [\underline{A}_i(\alpha), \overline{A}_i(\alpha)].$$

Since the function $f(x)$ is increasing, we have

$$f(\mathbf{A}_i(\alpha)) = f([\underline{A}_i(\alpha), \overline{A}_i(\alpha)]) = [f(\underline{A}_i(\alpha)), f(\overline{A}_i(\alpha))].$$

Again, without losing generality, we can assume that x belongs to the increasing part of $A_i(x)$. In this case, the values $f(\underline{A}_i(\alpha))$ strictly increase with α, so there exists a value α for which $\underline{A}_i(\alpha) = x$. For this value α, we have

$$f(\mathbf{A}_i(\alpha)) = [f(x), f(\overline{A}_i(\alpha))].$$

Similarly, we have

$$g(\mathbf{A}_i(\alpha)) = [g(x), g(\overline{A}_i(\alpha))].$$

Since we have $f(A_i) = g(A_i)$, we thus have

$$f(\mathbf{A}_i(\alpha)) = g(\mathbf{A}_i(\alpha))$$

for all α, therefore

$$[f(x), f(\overline{A}_i(\alpha))] = [g(x), g(\overline{A}_i(\alpha))]$$

and hence, $f(x) = g(x)$.

The equality $f(x) = g(x)$ is thus proven for all points x with the exception of finite many points. For each remaining point, this equality can be proved by continuity—since each of these points is a limit of nearby points, which are not in this finite list. The proposition is proven.

Discussion For analytical functions—i.e., functions that can be expanded in Taylor series in the neighborhood of each point—we can have even stronger results.

Proposition 2 *Let $A_1(x), \ldots, A_n(x)$ be continuous membership functions on an interval $[\underline{X}, \overline{X}]$, and on an interval $[\underline{x}, \overline{x}] \subseteq [\underline{X}, \overline{X}]$, one of these membership functions is either strictly increasing or strictly decreasing. If two analytical functions $f(x)$ and $g(x)$ are both increasing or both decreasing, and we have $f(A_i) = g(A_i)$ for all i, then for all $x \in [\underline{X}, \overline{X}]$, we have $f(x) = g(x)$.*

Proof Similarly to the proof of Proposition 1, we can conclude that the functions $f(x)$ and $g(x)$ coincide on the interval $[\underline{x}, \overline{x}]$. It is known that if two analytical functions coincide on some interval, then they are equal everywhere. The proposition is proven.

Discussion: We Should be Cautious When Trying to Extend This Result to Functions of Several Variables For functions of two or more variables, the new methodology leads to reasonable results if we restrict ourselves to a finite-parametric family of functions—e.g., to linear combinations of known functions

$$f(x_1, \ldots, x_n) = C_1 \cdot e_1(x_1, \ldots, x_n) + \ldots + C_m \cdot e_m(x_1, \ldots, x_n),$$

where

$$e_1(x_1, \ldots, x_n), \ldots, e_1(x_1, \ldots, x_n)$$

are given functions and $C_1, \ldots, C_m$ are the coefficients that need to be determined.

However, it should be mentioned that, in contrast to the 1-D case, if we do not impose any such restriction, then, in general, the proposed minimization does not determine a unique function $f(x_1, \ldots, x_n)$. Indeed, the desired criterion only described the ranges $[\underline{y}_i(\alpha), \overline{y}_i(\alpha)]$ of the function $f(x_1, x_2, \ldots)$ on all α-cuts for all rules $i = 1, \ldots, k$. So, all we have is $2k$ functions of one variable $\underline{y}_i(\alpha)$ and $\overline{y}_i(\alpha)$, and this information is not sufficient to uniquely determine a function of two or more variables.

What About Type 2? Up to now, we only considered what is usually called *type 1* fuzzy sets, when for each property A and for each value x, the degree to which the value x satisfies this property is described by a real number. In practice, just like when experts cannot describe the exact values of the corresponding physical quantities, they cannot meaningfully describe their degree of confidence by a single number. It is more realistic to ask the experts to express each of their degrees of confidence by an interval of possible values or even by a fuzzy subset of the interval [0, 1]. The function assigning an interval or a fuzzy set to each value x is known as, correspondingly, interval-valued fuzzy sets and type 2 fuzzy sets (see, e.g., [4]).

For rules in which properties A_i and B_i are described by such sets, it is also possible to formulate a similar criterion—since both Zadeh's extension principle and its α-cut reformulation can also be naturally extended to the interval-valued and type 2 fuzzy cases (see, e.g., [3]).

Acknowledgments This work was supported in part by the National Science Foundation grants 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science) and HRD-1834620 and HRD-2034030 (CAHSI Includes) and by the AT&T Fellowship in Information Technology.

It was also supported by the program of the development of the Scientific-Educational Mathematical Center of Volga Federal District No. 075-02-2020-1478 and by a grant from the Hungarian National Research, Development and Innovation Office (NRDI).

The authors are greatly thankful to the anonymous referees to valuable suggestions.

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Chapter 3
How Measurement-Related Ideas Can Help Us Use Expert Knowledge When Making Decisions: Three Case Studies

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1 Introduction

1.1 Using Expert Knowledge Is Important, But How?

To make an adequate decision, we need to have as much information about the corresponding situation as possible. For example, in pavement engineering, we need to decide–based on the available annual budget–which road segments need to be repaired this year and which can be used one more year without repair. To make this decision, we need to have an accurate information about the state of different road segments.

In general, large amount of information comes from measurements. However, in many areas, it is crucial to also use expert knowledge, for example:

- With all modern medical tests and measurements, doctor's intuition is still crucial.
- In spite of all the successes of self-driving cars, it is still not possible to fully replace a human driver.

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C. Jana et al. (eds.), *Fuzzy Optimization, Decision-making and Operations Research*, https://doi.org/10.1007/978-3-031-35668-1_3

It is therefore important to supplement measurement results with expert estimates.

The problem with this is that while measurement techniques provide us with statistically justified estimates of the values of the corresponding quantities, expert estimates usually do not come with such justifications. Because of this, practitioners are often reluctant to use expert estimates in their decision-making.

It is therefore desirable to make expert estimates statistically justified. In this chapter, we show, on three case studies, how this can be done.

1.2 How Can Experts Help?

In order to explain how expert knowledge can be made statistically justified, let us first recall how expert information can supplement measurement results. To do that, let us recall that in measurement practice:

- we come up with a parametric model of the corresponding class of phenomena,
- we test this model—to make sure that it provides an adequate description of the phenomena
- we use measurements to estimate the parameters corresponding to a given situation.

How can experts help?

- experts can (and often do) provide such a model
- experts can (and do) provide estimates of the corresponding parameters.

1.3 Why Is This Useful?

In terms of a model: the currently used model often comes from a semi-empirical study. Such curve-fitting models are not very convincing, this can be overfitting. Experts' knowledge and intuition can help separate explainable models from curve-fitting results.

In terms of expert estimations: experts may not be accurate as measurements, but they are often faster and cheaper to use. They also supplement measurement results, this making the resulting estimates more accurate.

1.4 But How Exactly Can We Use Expert Knowledge to Supplement Measurement Results?

From the common-sense viewpoint, expert knowledge is useful. But how can we include expert estimates into a measurement-based framework, with its precise justifications?

A natural idea is to treat an expert as a measuring instrument and to calibrate the expert similarly to how we calibrate measuring instruments. Thus, we can get a statistically justified estimate for the accuracy of expert-generated numbers.

Moreover, we can use this calibration to improve the expert's estimates. This is similar to how, once we know the instrument's bias, we can subtract it and get more accurate results.

1.5 Three Case Studies

To illustrate the above general ideas, we provide three case studies.

- In the first case study, we show that application of usual linear calibration to experts can be helpful.
- In the second case study, we provide an example of useful nonlinear calibration.
- The third case study explains how expert knowledge can make semi-empirical models more convincing.

Comment Preliminary results of the three test studies first appeared in [3, 44, 46].

2 First Case Study: Measurement-Type "Calibration" of Expert Estimates Improves Their Accuracy and Their Usability: Pavement Engineering

2.1 Experts Are Often Used for Estimation

Sometimes, experts are used because no measuring instruments can replace these experts. For example, in dermatology, estimates of a skilled expert are often a more accurate result than the results of applying algorithms to measurement results. This is one of the main reasons why, in spite of numerous experts systems, human doctors are still needed and still valued.

In other cases, in principle, we can use automatic systems, but experts are still much cheaper to use. An example of such situation is pavement engineering. In principle, we can use an expensive automatic vision-based system to gauge the condition of the pavement. However, it is much cheaper–and faster–to use human raters.

2.2 Expert Estimates Are Often Very Imprecise

Humans rarely have a skill of accurately evaluating the values of different quantities. For example, it is well known that humans drastically overestimate small probabilities. Correspondingly, humans underestimate the probabilities which are close to 1 (see, e.g., [16] and references therein).

Since most people's estimates are very inaccurate, it is difficult to find good expert estimators. It is well known that there is a high competition to get into medical schools. Even in pavement engineering, finding a good rater is difficult.

2.3 It Is Difficult to Find Good Experts: Example from Pavement Engineering

Roads are extremely important for our civilization, and most of them are very heavily used. Because of this use, pavements deteriorate, and they need maintenance and repairs. Keeping roads in good shape is expensive. It is therefore important to be able to get a good understanding of the state of different road segments.

- On the one hand, if we underestimate the seriousness of a pavement fault and do not repair it, the road will deteriorate further, and future repairs will be much more expensive.
- On the other hand, if we overestimate the seriousness of a fault and spend resources repairing it, we thus waste resources that could have been used more productively.

Many different types of fault occur: cracks, potholes, swellings and depressions, etc. All these faults decrease the pavement quality, so the quality of a pavement depends on the degree to which all these faults occur. Engineers have accumulated a lot of records of road segments with different faults, records that show how these faults evolved with time and how long it took for these road segments to become unusable (and need repairs). Based on these records, they have come up with a complex formula that uses the overall lengths and orientation of the cracks, the area and depth of potholes, the area and height of swellings, and other numerical quantities into a single characteristic. This characteristic helps gauge how long the road segment will survive under the given traffic volume. This characteristic is known as the *Pavement Condition Index (PCI)*. The complex algorithm for computing PCI is described in the standard [7] (see also [10, 40–42]).

As we have mentioned, in principle, it is possible to measure all these characteristics and compute the resulting value of the PCI index, but this would be very expensive. Because of this, in practice, pavement engineering relies on expert raters who can estimate PCI without performing all these measurements. Candidate raters are tested to make sure that they indeed provide accurate PCI estimates. To gauge the accuracy of a rater candidate, many locations across the USA use criteria developed by the Metropolitan Transportation Commission (MTC) of California [29].

A crucial part of the rater certification is a field survey exam. In this exam, a rater evaluates 24 test sites that have been previously evaluated by expert raters. Candidate's PCI values are then compared with the PCI values of the expert rater.

The expert's values are taken as the ground truth (GT). To certify, the rater must satisfy the following two criteria:

- at least for 50% of the evaluated sites, the difference should not exceed 8 points
- at least for 88% of the evaluated sites, the difference should not exceed 18 points.

MTC provided a sample of 18 typical candidates. Out of these candidates, only five (28%) satisfied both criteria and, thus, passed the exam and can be used as raters.

Problems

- What can we do to increase the number of available experts?
- And for those who have been selected as experts, can we improve the accuracy of their estimates?

2.4 Calibration

We are interested in situations when expert serve, in effect, as measuring instruments.

Measuring instruments are usually much more accurate then human experts. Still, they are sometimes not very accurate. Even when they are originally reasonably accurate, in time, their accuracy decreases.

When the measuring instrument becomes not very accurate, we do not necessarily throw it away. For example, supposed that before we step on the scales, the scales already show 10 pounds. We do not necessarily throw away these scales: instead, we adjust the starting point.

When a household device for measuring blood pressure starts producing weird results, the manufacturers do not advise the customers to throw it away and to buy a new one; they advise the customers to come to a doctor's office and to calibrate the customer's instrument.

In general, calibration is a routine procedure for measuring instruments (see, e.g., [45]). In this procedure, we measure the same quantities:

- by using our measuring instruments—resulting in the values $x_1, \ldots, x_n$
- by using a much more accurate ("standard") measuring instrument—resulting in the values $s_1, \ldots, s_n$.

In many cases–like in the above scales example–the main problem is the bias. We compensate for the bias by subtracting the estimated value. The resulting corrected values $x_i + b$ are closer to the ground-truth s_i. A reasonable way to estimate the bias is to use the least squares method [45, 48]: $\sum\limits_{i=1}^{n} ((x_i + b) - s_i)^2 \to \min$.

In some cases, there is also a relative systematic error, when each value is under- or overestimated by a certain percentage. To compensate for this under- and overestimation, we need to multiply by an appropriate constant. For example, if all the values are overestimated by 10%, then each ground-truth value s_i is replaced by the biased value $s_i + 0.1 \cdot s_i = 1.1 \cdot s_i$. To compensate for this relative bias, we thus need to multiply all the measurement results by $1/1.1$.

In general, to compensate for the relative bias, we need to replace the original measurement results x_i by corrected values $a \cdot x_i$ for some a. To compensate for both absolute and relative biases, we replace x_i with $a \cdot x_i + b$.

The values a and b can be found by the least squares method:

$$\sum_{i=1}^{n} ((a \cdot x_i + b) - s_i)^2 \to \min .$$

After that, instead of using the original measurement result x produced by the measuring instrument, we calibrate it into a more accurate value

$$x' = a \cdot x + b.$$

In addition to such a linear calibration, it is sometimes beneficial to use nonlinear calibration. Sometimes, a quadratic or cubic calibration is used—which leads to more accurate measurement results. In many practical situations, it is also beneficial to use fractional-linear re-scaling $x' = \dfrac{a \cdot x + b}{1 + c \cdot x}$; see, e.g., [18–20, 26, 27, 31].

2.5 Idea: Let Us Calibrate Experts

A natural idea is that since experts serve as measuring instruments, we can similarly calibrate the experts. Namely, instead of using the original expert estimates:

- we first re-scale the original expert estimates in accordance with the appropriate calibration function
- we use these re-scaled values instead of the original expert estimates.

As a result–just like for measuring instruments–we will hopefully get more accurate estimates.

In some situations, when for some experts their original estimates were not very accurate, we may end up with re-scaled estimates of acceptable quality, so we can use these experts.

2.6 Such Calibration Is Indeed Helpful

A good example of the efficiency of such calibration is expert's estimations of small probabilities. According to Kahneman and Tversky [16], these estimates e_i are way off.

However, the values $e'_i = a \cdot \sin^2(b \cdot e_i)$ are much more accurate (see, e.g., [21–24]). Namely, for $p_i < 20\%$:

- the worst-case difference between the original estimates e_i and the actual probabilities was 8.6%—more than 40% of the original probability value
- the worst-case difference between the re-scaled estimates e'_i and the probabilities p_i is 0.7%—which is 3.5% of the original probability value, and is, thus, an order of magnitude more accurate.

2.7 We Applied Our Idea to Pavement Engineering

We started with the 18 rater candidates from the original MTC sample. In the original test, only five of these candidates passed the exam: rater candidates R6, R8, R9, R14, and R15.

Originally, the rater's ratings r_i were compared with the 24 corresponding ground-truth values s_i. Instead, we first found the values a and b that minimize the sum of the squares $\sum_{i=1}^{24}((a \cdot r_i + b) - s_i)^2$. Then, we used the re-scaled values $r'_i = a \cdot r_i + b$ to compare with the ground truth.

2.8 As a Result, More Experts Are Selected

Based on the re-scaled ratings, four more candidates passed the test: candidates R1, R3, R5, and R11. This means that these four folks can now be used for rating pavement conditions.

Of course, instead of using their original ratings r_i, we first need to re-scale these ratings to $r'_i = a \cdot r_i + b$ for this rater's a and b. As a result, we can accept nine raters. Thus, the acceptance rate is now no longer $5/18 \approx 28\%$; it is $9/18 = 50\%$.

2.9 For Most Originally Selected Experts, Re-scaling Leads to More Accurate Estimates

After re-scaling, one of the originally accepted candidates–R9–no longer fits. For this rater, we can use his original ratings.

For the remaining four originally selected raters, re-scaling improves the accuracy of their estimates:

- for R6, the mean square rating error decreases from 11.21 points to 10.01 points—a decrease of 9.9%;
- for R8, the mean square rating error decreases from 10.00 points to 8.66 points—a decrease of 6.4%;

- for R14, the mean square rating error decreases from 8.62 to 6.95 points—a decrease of 19.4%
- for R15, the mean square rating error decreases from 6.47 points to 6.21 points—a decrease of 4.0%.

Comment Similarly good results were consistently achieved for several other groups of rater candidates.

3 Second Case Study: Relationship Between Measurement Results and Expert Estimates of Cumulative Quantities, on the Example of Pavement Roughness

3.1 Cumulative Quantities

Many physical quantities can be measured directly, e.g., we can directly measure mass, acceleration, and force. However, we are often interested in *cumulative* quantities that combine values corresponding to different moments of time and/or different locations. For example, when we are studying public health or pollution or economic characteristics, we are often interested in characteristics describing the whole city, the whole region, and the whole country.

3.2 Formulation of the Problem

Cumulative characteristics are not easy to measure. To measure each such characteristic, we need to perform a large number of measurements and then use an appropriate algorithm to combine these results into a single value.

Such measurements are complicated. So, we often have to supplement the measurement results with expert estimates. To process such data, it is desirable to describe both estimates in the same scale:

- to estimate the actual value of the corresponding quantity based on the expert estimate
- vice versa, to estimate the expert estimate based on the actual value of the quantity.

3.3 Case Study: Estimating Pavement Roughness

Estimating road roughness is an important problem. Indeed, road pavements need to be maintained and repaired. Both maintenance and repair are expensive. So, to make a good decision on which road segments to repair this year, it is desirable to estimate the pavement roughness as accurately as possible.

- If we overestimate the road roughness, we will waste money on "repairing" an already good road.
- If we underestimate the road roughness, the road segment will be left unrepaired and deteriorate further. As a result, the cost of future repair will skyrocket.

The standard way to measure the pavement roughness is to use the International Roughness Index (IRI) (see, e.g., [6, 11, 12, 47]). This measure of roughness is recommended by the US standards [6, 11, 12].

Crudely speaking, IRI describes the effect of the pavement roughness on a standardized model of a vehicle. Measuring IRI is not easy, because the real vehicles differ from this standardized model. As a result, we measure roughness by some instruments and use these measurements to estimate IRI. For example, we can:

- perform measurements by driving an available vehicle along this road segment,
- extract the local roughness characteristics from the effect of the pavement on this vehicle
- estimate the effect of the same pavement on the standardized vehicle.

In view of this difficulty, in many cases, practitioners rely on expert estimates of the pavement roughness. The corresponding measure–estimated on a scale from 0 to 5–is known as the present serviceability rating (PSR) (see, e.g., [5, 13]).

3.4 Empirical Relation Between Measurement Results and Expert Estimates

The empirical relation between PSR and IRI is described by the formula:

$$\mathrm{PSR} = 5 \cdot \exp(-0.0041 \cdot \mathrm{IRI}).$$

This formula was first proposed by B. Al-Omari and M. Darter in [4], and it still remains actively used in pavement engineering (see, e.g., [8, 13, 38, 39]). It works much better than many previously proposed alternative formulas, such as

$$\mathrm{PSR} = a + b \cdot \sqrt{\mathrm{IRI}}$$

proposed in [30]. However, it is not clear why namely this formula works so well.

3.5 What We Do in This Section

We propose a possible explanation for the above empirical formula. This explanation will be general: it will apply to all possible cases of cumulative quantities.

We will come up with a general formula $y = f(x)$ that describes how a subjective estimate y of a cumulative quantity depends on the result x of its measurement.

As a case study, we will use gauging road roughness.

3.6 Main Idea

In general, the numerical value of a *subjective estimate* depends on the scale. In road roughness estimates, we usually use a 0–5 scale. In other applications, it may be more customary to use 0–10 or 0–1 scale.

A usual way to transform between the two scales is to multiply all the values by a corresponding factor. For example, to transform from 0–10 to 0–1 scale, we multiply all the values by $\lambda = 0.1$. In other transitions, we can use transformations $y \to \lambda \cdot y$ with different re-scaling factors λ.

There is no major advantage in selecting a specific scale. So, subjective estimates are defined modulo such a re-scaling transformation $y \to \lambda \cdot y$.

At first glance, the result of *measuring* a cumulative quantity may look uniquely determined. However, a detailed analysis shows that there is some non-uniqueness here as well. Indeed, the result of a cumulative measurement comes from combining values measured at different moments of time and/or values corresponding to different spatial locations. For each individual measurement, the probability of a sensor's malfunction may be low. However, often, we perform a large number of measurements. So, some of them bound to be caused by such malfunctions and are, thus, outliers.

It is well known that even a single outlier can drastically change the average. So, to avoid such influence, the usual algorithms first filter out possible outliers. This filtering is not an exact science; we can set up slightly different thresholds for detecting an outlier, slightly different threshold for allowed number of remaining outliers, etc.

We may get a computation result that only takes actual signals into account. With a different setting, we may get a different result, affected by a few outliers.

Let's denote the average value of an outlier is L and the average number of such outliers is n. Then, the second scheme, in effect, adds a constant $n \cdot L$ to the cumulative value computed by the first scheme.

Yes, there is also some random deviation. However, when the number n is reasonably large, then, due to the large numbers theorem, these deviations average out, and we get approximately the mean value (see, e.g., [48])—just like when we flip a coin many (N) times, the overall number of times when it falls head will be close to $0.5 \cdot N$.

So, the measured value of a cumulative quantity is defined modulo an addition of some value:

$$x \to x + a \text{ for some constant } a.$$

3.7 Motivation for Invariance

We do not know exactly what is the ideal threshold, so we have no reason to select a specific shift as ideal. It is therefore reasonable to require that the desired formula $y = f(x)$ not depend on the choice of such a shift, i.e., that the corresponding dependence not change if we simply replace x with $x' = x + a$.

Of course, we cannot just require that $f(x) = f(x+a)$ for all x and all a. Indeed, in this case, the function $f(x)$ will simply be a constant, but y increases with x. But this is clearly not how invariance is usually defined. For example, for many physical interactions, there is no fixed unit of time. So, formulas should not change if we simply change a unit for measuring time: $t' = \lambda \cdot t$. The formula $d = v \cdot t$ relating the distance d, the velocity v, and the time t should not change. We want to make this formula true when time is measured in the new units. So, we may need to also appropriately change the units of other related quantities.

In the above example, we need to appropriately change the unit for measuring velocity, so that not only time units are changed, e.g., from hours to second, but velocities are also changed from km/hour to km/sec.

So, if we re-scale x, the formula $y = f(x)$ should remain valid if we appropriately re-scale y. As we have mentioned earlier, possible re-scalings of the subjective estimate y have the form $y \to y' = \lambda \cdot y$. Thus, for each a, there exists $\lambda(a)$ (depending on a) for which $y = f(x)$ implies that $y' = f(x')$, where

$$x' \stackrel{\text{def}}{=} x + a \text{ and } y' \stackrel{\text{def}}{=} \lambda \cdot y.$$

Definition A monotonic function $f(x)$ is called *unit-invariant* if for every real number a, there exists a positive real number $\lambda(a)$ for which, for each x and y:

- if $y = f(x)$,
- then $y' = f(x')$, where $x' \stackrel{\text{def}}{=} x + a$ and $y' \stackrel{def}{=} \lambda(a) \cdot y$.

Proposition *A function $f(x)$ is unit-invariant if and only if it has the form*

$$f(x) = C \cdot \exp(-b \cdot x) \textit{ for some } C \textit{ and } b.$$

Comment For road roughness, this result explains the empirical formula.

Proof It is easy to check that every function $y = f(x) = C \cdot \exp(-b \cdot x)$ is indeed unit-invariant.

Indeed, for each a, we have

$$f(x') = f(x + a) = C \cdot \exp(-b \cdot (x + a)) =$$

$$C \cdot \exp(-b \cdot x - b \cdot a) = \lambda(a) \cdot C \cdot \exp(-b \cdot x).$$

Here we denoted $\lambda(a) \stackrel{\text{def}}{=} \exp(-b \cdot a)$. Thus here, indeed, $y = f(x)$ implies that $y' = f(x')$.

Vice versa, let us assume that the function $f(x)$ is unit-invariant. Then, for each a, the condition $y = f(x)$ implies that $y' = f(x')$, i.e., that $\lambda(a) \cdot y = f(x + a)$. Substituting $y = f(x)$ into this equality, we conclude that $f(x + a) = \lambda(a) \cdot f(x)$. It is known (see, e.g., [2]) that every monotonic solution of this functional equation has the form

$$f(x) = C \cdot \exp(-b \cdot x) \text{ for some } C \text{ and } b.$$

The proposition is proven.

3.8 Conclusions of This Section

In pavement engineering, to make a good decision, it is important to accurately gauge the quality of road segments. Such estimates help us decide how to best distribute the available resources between different road segments. So, proper and timely maintenance is performed on road segments whose quality has deteriorated, thus, to avoid future costly repairs of untreated road segments.

The standard way to gauge the quality of a road segment is International Roughness Index (IRI). It requires a large amount of costly measurements. As a result, it is not practically possible to regularly measure IRI of all road segments. So, IRI measurements are usually restricted to major roads.

For local roads, we need to have an indirect way to estimate their quality. To estimate the quality of a road segment, we combine user estimates of different segment properties into a single index known as Present Serviceability Rating (PSR).

There is an empirical formula relating IRI and PSR. However, one of the limitations of this formula is that it is purely heuristic. This formula lacks a theoretical explanation, and thus, the practitioners may be not fully trusting its results. In this section, we provide such a theoretical explanation. We hope that the resulting increased trust in this formula will help enhance its use. Thus, it will help make road management decisions.

4 Third Case Study: Normalization-Invariant Fuzzy Logic Operations Explain Empirical Success of Student Distributions in Describing Measurement Uncertainty

4.1 Traditional Engineering Approach to Measurement Uncertainty

Traditionally, in engineering applications, it is assumed that each measurement error is normally distributed (see, e.g., [45]).

This assumption makes perfect sense from the practical viewpoint: it has been shown that for the majority of measuring instruments, the measurement error is indeed normally distributed (see, e.g., [36, 37]). It also makes sense from the theoretical viewpoint, since in many cases, the measurement error comes from a joint effect of many independent small components, and according to the Central Limit Theorem (see, e.g., [48]), for the large number of components, the resulting distribution is indeed close to Gaussian.

Another explanation: we only have partial information about the distribution. Often, we only know the first and the second moments. The first moment–mean–represents a bias. If we know the bias, we can always subtract it from the measurement result. Thus, re-calibrated measuring instrument will have 0 mean. So, we can always safely assume that the mean is 0. Then, the second moment is simply the variance $V = \sigma^2$.

There are many distributions with 0 mean and given σ. For example, we can have a distribution in which we have σ and $-\sigma$ with probability 1/2 each. However, such a distribution creates a false certainty—that no other values of x are possible. Out of all such distributions, it makes sense to select the one which maximally preserves the uncertainty.

Uncertainty can be gauged by average number of binary questions needed to determine x with accuracy ε. It is described by *entropy* $S = -\int \rho(x)\cdot\log_2(\rho(x))\,dx$ (see, e.g., [15, 33]). Out of all distributions $\rho(x)$ with mean 0 and given σ, the entropy is the largest for normal $\rho(x)$.

4.2 Need for Heavy-Tailed Distributions

For the normal distribution,

$$\rho(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma}\cdot\exp\left(-\frac{x^2}{2\sigma^2}\right).$$

The "tails"–values corresponding to large $|x|$–are very light, practically negligible.

Often, $\rho(x)$ decreases much slower, as $\rho(x) \sim c \cdot x^{-\alpha}$; see, e.g., [25, 43]. We cannot have $\rho(x) = c \cdot x^{-\alpha}$, since $\int_0^\infty x^{-\alpha}\,dx = +\infty$, and we want $\int \rho(x)\,dx = 1$.

Often, the measurement error is well represented by a Student distribution $\rho_S(x) = (a + b \cdot x^2)^{-\nu}$. This is true in geodesy and in other applications as well. This distribution is even recommended by the International Organization for Standardization (ISO) [14].

4.3 What We Do

How to explain the empirical success of Student's distribution $\rho_S(x)$? In this section, we show that a natural fuzzy-logic-based ([9, 17, 28, 34, 35, 49]) formalization of commonsense requirements leads to $\rho_S(x)$.

Our idea is to use the fact that uncertainty means that the first value is possible and the second value is possible, etc. Let's select $\rho(x)$ with the largest degree to which all the values are possible.

It is reasonable to use fuzzy logic to describe degrees of possibility. An expert marks his/her degree by selecting a number from the interval [0, 1].

4.4 Need for Normalization

For "small," we are absolutely sure that 0 is small: $\mu_{\text{small}}(0) = 1$ and $\max\limits_x \mu_{\text{small}}(x) = 1$. For "medium," there is no x with $\mu_{\text{med}}(x) = 1$, so $\max\limits_x \mu_{\text{med}}(x) < 1$.

A usual way to deal with such situations is to *normalize* $\mu(x)$ into $\mu'(x) = \dfrac{\mu(x)}{\max\limits_y \mu(y)}$. Normalization is also performed when we get additional information.

Example: suppose that we knew that x is small and then we learn an additional information—that $x \geq 5$. Then, $\mu_{\text{new}}(x) = \mu_{\text{small}}(x)$ for $x \geq 5$ and $\mu_{\text{new}}(x) = 0$ for $x < 5$, and $\max\limits_x \mu_{\text{new}}(x) < 1$. So, to get a normalized function, we need to normalize these values $\mu_{\text{new}}(x)$.

Normalization is also needed when experts use probabilities to come up with the degrees. Indeed, the larger the $\rho(x)$, the more probable it is to observe a value close to x. Thus, it is reasonable to take the degrees $\mu(x)$ proportional to $\rho(x)$: $\mu(x) = c \cdot \rho(x)$. Normalization leads to $\mu(x) = \dfrac{\rho(x)}{\max\limits_y \rho(y)}$. Vice versa, if we have the result $\mu(x)$ of normalizing a pdf, we can reconstruct $\rho(x)$ as $\rho(x) = \dfrac{\mu(x)}{\int \mu(y)\,dy}$.

4.5 How to Combine Degrees

For each x, we get a degree to which x is possible. We want to compute the degree to which x_1 is possible *and* x_2 is possible, etc. So, we need to apply an "and"-operation (t-norm) to the corresponding degrees.

A natural idea is to use normalization-invariant t-norms. We can compute the normalized degree of confidence in a statement $A \,\&\, B$ in two different ways:

- we can normalize $f_\&(a, b)$ to $\lambda \cdot f_\&(a, b)$;
- we can first normalize a and b and then apply an "and"-operation: $f_\&(\lambda \cdot a, \lambda \cdot b)$.

It's reasonable to require that we get the same estimate: $f_\&(\lambda \cdot a, \lambda \cdot b) = \lambda \cdot f_\&(a, b)$.

It is known that strict Archimedean t-norms $f_\&(a, b) = f^{-1}(f(a) + f(b))$ are universal approximators (see, e.g., [32]). So, we can safely assume that $f_\&$ is strict Archimedean:

$$c = f_\&(a, b) \Leftrightarrow f(c) = f(a) + f(b).$$

Thus, invariance means that $f(c) = f(a) + f(b)$ implies $f(\lambda \cdot c) = f(\lambda \cdot a) + f(\lambda \cdot b)$. So, for every λ, the transformation $T : f(a) \to f(\lambda \cdot a)$ is additive: $T(A + B) = T(A) + T(B)$.

It is known (see, e.g., [1, 2]) that every monotonic additive function is linear. Thus, $f(\lambda \cdot a) = c(\lambda) \cdot f(a)$ for all a and λ. For monotonic $f(a)$, this implies $f(a) = C \cdot a^{-\alpha}$ (see, e.g., [32]). So, $f(c) = f(a) + f(b)$ implies $C \cdot c^{-\alpha} = C \cdot a^{-\alpha} + C \cdot b^{-\alpha}$, and $c = f_\&(a, b) = (a^{-\alpha} + b^{-\alpha})^{-1/\alpha}$.

4.6 Deriving Student Distribution

We want to maximize the degree

$$f_\&(\mu(x_1), \mu(x_2), \ldots) = ((\mu(x_1))^{-\alpha} + (\mu(x_2))^{-\alpha} + \ldots)^{-1/\alpha}.$$

The function $a \mapsto a^{-\alpha}$ is decreasing. So, maximizing $f_\&(\mu(x_1), \ldots)$ is equivalent to minimizing the sum $(\mu(x_1))^{-\alpha} + (\mu(x_2))^{-\alpha} + \ldots$ In the limit, this sum tends to $I \stackrel{\text{def}}{=} \int (\mu(x))^{-\alpha}\, dx$. So, we minimize I under constrains $\int x \cdot \rho(x)\, dx = 0$ and $\int x^2 \cdot \rho(x)\, dx = \sigma^2$, where $\rho(x) = \dfrac{\mu(x)}{\int \mu(y)\, dy}$. Thus, we minimize $\int (\mu(x))^{-\alpha}\, dx$ under constraints

$$\int x \cdot \mu(x)\, dx = 0 \text{ and } \int x^2 \cdot \mu(x)\, dx - \sigma^2 \cdot \int \mu(x)\, dx = 0.$$

Lagrange multiplier method leads to minimizing

$$\int (\mu(x))^{-\alpha}\,dx + \lambda_1 \cdot \int x \cdot \mu(x)\,dx +$$

$$\lambda_2 \cdot \left(\int x^2 \cdot \mu(x)\,dx - \sigma^2 \cdot \int \mu(x)\,dx \right) \to \min .$$

Equating the derivative w.r.t. $\mu(x)$ to 0, we get:

$$-\alpha \cdot (\mu(x))^{-\alpha-1} + \lambda_1 \cdot x + \lambda_2 \cdot x^2 - \lambda_2 \cdot \sigma^2 = 0.$$

Thus, $\mu(x) = (a_0 + a_1 \cdot x + a_2 \cdot x^2)^{-\nu}$, for $\nu = 1/(\alpha + 1)$.

For $\rho(x) = c \cdot \mu(x)$, we get $\rho(x) = c \cdot (a_0 + a_1 \cdot x + a_2 \cdot x^2)^{-\nu}$. So, $\rho(x) = c \cdot (a_2 \cdot (x - x_0)^2 + c_1)^{-\nu}$. This $\rho(x)$ is symmetric w.r.t. x_0, so, the mean is x_0. We know that the mean is 0, so $x_0 = 0$, and $\rho(x) = \text{const} \cdot (1 + a_2 \cdot x^2)^{-\nu}$: exactly Student's $\rho_S(x)$!

Example Hamacher t-norm (see, e.g., [9, 17, 28, 34, 35]) has the following form:

$$f_\&(a, b) = \frac{a \cdot b}{a + b - a \cdot b}.$$

For small a and b, it is asymptotically equivalent to

$$\frac{a \cdot b}{a + b} = \frac{1}{\dfrac{1}{a} + \dfrac{1}{b}} = (a^{-1} + b^{-1})^{-1}.$$

In this case, we have $\alpha = 1$, so $\nu = 1/2$, and the corresponding probability distribution has–asymptotically–the form $\rho(x) = \text{const} \cdot (1 + a_2 \cdot x^2)^{-1/2}$.

5 Conclusions and Future Work

5.1 Why Do We Need to Use Expert Knowledge

One of the main objectives of science and engineering is to help us make important decisions. To make a reasonable decision, we need to have a good knowledge of the corresponding situation, i.e., a good knowledge of the values of the quantities that describe the situation. The more information we have, the better decision we can make.

In many practical situations, this information comes from measurements. For this information, measurement techniques provide justified statistical estimates of the quantities of interest. In addition to measurement results, we often have expert estimates. These estimates provide an additional information about the corresponding quantities.

5.2 Challenges Related to the Use of Expert Knowledge

One of the main challenges related to the use of expert knowledge is that, in contrast to measurement results, expert estimates are usually not statistically justified. Because of this, practitioners are often reluctant to use them.

5.3 Measurement-Related Ideas Can Help

In view of the above challenge, it is desirable to utilize measurement-related ideas –ideas that lead to statistically justified conclusions– to process expert knowledge. In this chapter, we provided three case studies explaining how this can be done.

One way to solve this problem is to calibrate an expert—the same way we calibrate measuring instruments. In the first two case studies, we showed that such a calibration indeed leads to useful results.

The third case study provides an example of another use of expert knowledge in knowledge processing: namely, expert knowledge can be used to make semi-empirical measurement models more explainable—and, thus, more reliable.

5.4 Future Work

In this chapter, we barely scratched the surface. There are many effective measurement-related techniques and ideas, and we hope that after proper modifications, these ideas can be used to process expert knowledge as well.

6 Auxiliary Results for Sect. 2

6.1 First Auxiliary Result: Why 50%?

In the MTC procedure, as the first threshold, we consider the accuracy with which we should have at least 50% of the measurements. In other words, we compare the median of the empirical distribution with some threshold. But why 50%? Why not select a value corresponding to, say, 40% or 60%?

The only explanation that MTC provides is that selecting 50% leads to empirically the best results. But why? Here is our explanation.

We want to find a parameter describing the distribution of expert's approximation errors. This may be the standard deviation, and this may be some other appropriate parameter. We want the relative accuracy with which we determine these parameters to be as good as possible.

We estimate this parameter based on a frequency f that corresponds to some probability p. It is known (see, e.g., [48]) that after n observations, $f - p$ is approximately normally distributed, with 0 mean and

$$\sigma[p] = \sqrt{\frac{p \cdot (1-p)}{n}}.$$

We can measure the relative accuracy both:

- with respect to the probability p of the original event
- with respect to the probability $1 - p$ of the opposite event.

We want both relative accuracies to be as small as possible. The relative accuracy with which we can find the desired probability p is equal to

$$\frac{\sigma[p]}{p} = \sqrt{\frac{1-p}{n \cdot p}} = \sqrt{\frac{1}{n} \cdot \left(\frac{1}{p} - 1\right)}.$$

Similarly, the relative accuracy with which we can find the probability $1 - p$ is equal to

$$\frac{\sigma[p]}{1-p} = \sqrt{\frac{p}{n \cdot (1-p)}} = \sqrt{\frac{1}{n} \cdot \left(\frac{1}{1-p} - 1\right)}.$$

We need to make sure that the largest of these two values is as small as possible. One can check that the largest of these two values is

$$\sqrt{\frac{1}{n} \cdot \left(\max\left(\frac{1}{p}, \frac{1}{1-p}\right) - 1\right)} =$$

$$\sqrt{\frac{1}{n} \cdot \left(\frac{1}{\min(p, 1-p)} - 1\right)}.$$

This expression is a decreasing function of $\min(p, 1 - p)$. Thus, for the relative standard deviation to be as small as possible, $\min(p, 1 - p)$ must be as large as possible.

This expression grows from 0 to 0.5 when p increases from 0 to 0.5 and then decreases to 0. Thus, its maximum is attained when $p = 0.5$—and this is exactly what MTC recommends. So, we have a theoretical explanation for this empirically successful recommendation.

6.2 Why 88%

There are many different independent reasons why an expert estimate may differ from the actual value, so the expert uncertainty can be represented as a sum of a large number of small independent random variables. It is known–see, e.g., [48]–that, under reasonable condition, the distribution of such a sum is close to normal. This result is known as the central limit theorem. Thus, we can safely assume that the distribution of expert uncertainty is normal.

For a normal distribution with 0 mean, if the probability for the value to be within ± 8 is 50%, then the probability for the value to be within ± 18 is indeed close to 88%. This explains the second part of the MTC test.

Comment In both cases, our explanations seem to be simple and natural. We would not be surprised if it turns out that, when selecting the corresponding numbers, the authors of the MTC test were inspired not only by the empirical evidence but also by similar simple theoretical ideas.

Acknowledgments This work was supported in part by the National Science Foundation grants 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science) and HRD-1242122 (Cyber-ShARE Center of Excellence).

The authors are greatly thankful to Svetlana Prokopchina for the encouragement and to the anonymous referees for valuable suggestions.

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Chapter 4
On Fusion of Soft and Hard Computing: Traditional ("Hard Computing") Optimal Rescaling Techniques Simplify Fuzzy Control

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1 Introduction

1.1 Fuzzy Control: One of the Most Successful Soft Computing Techniques

In most industrial applications, we want to control the corresponding industrial processes in such a way as to maximize the output within certain (physical and economical) restrictions. When the corresponding mathematical description is linear, we can use well-known optimal control techniques to find the optimal control strategy. In reality, however, most industrial processes are nonlinear. For nonlinear control problems, the situation is much more complicated: there are good recipes which often work, but, alas, there is still no general methods of generating an optimal (or even a reasonably good) control (see, e.g., [10]). (For a formal proof that the corresponding optimization problems are computationally difficult (NP-hard), see, e.g., [6] and references therein.)

If for a certain industrial process no known technique leads to a good quality control, what can we do? Usually, the very fact that this process is actually used in the industry means that this process is reasonably well controlled by human controllers. Therefore, if we want to automate this control, we must somehow

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C. Jana et al. (eds.), *Fuzzy Optimization, Decision-making and Operations Research*, https://doi.org/10.1007/978-3-031-35668-1_4

transform the knowledge of these expert controllers (operators) into an automatic control strategy.

The necessity for such a transformation was one of the main motivations behind one of the most successful soft computing techniques—fuzzy control. Specifically, our goal is to describe a function which takes the sensor inputs $x_1, \ldots, x_n$ (numbers) and generates the (numerical) value of the control effort u. Unfortunately, expert operators cannot formulate their expertise in these terms. Instead, they describe their control strategy by using uncertain ("fuzzy") statements of the type "if the obstacle is straight ahead, the distance to it is small, and the velocity of the car is medium, press the brakes hard." Fuzzy control is a methodology which translates such statements into precise formulas for control. Fuzzy control was started by L. Zadeh and E. H. Mamdani [3, 8, 22, 23] in the framework of Zadeh's *fuzzy set theory* [21]. For the state of fuzzy systems and fuzzy control, the reader is referred, e.g., to [2, 5, 7, 9, 13–17].

1.2 Tuning Is Necessary

Fuzzy control methodology usually consists of two steps:

- First, we apply a routine fuzzy control methodology to get a rough *approximation* to the expert's control strategy.
- Then, we test the resulting control on real or simulated system and *tune* the resulting fuzzy control system based on the results of this testing.

The first step usually starts with assigning membership functions to all the terms that the expert uses in his rules (in our sample phrase, these words are "small," "medium," and "hard"). Most software packages for fuzzy control are based on (usually triangular) membership functions whose domains have equally spaced endpoints. For example, we can fix a neutral value N (usually, $N = 0$) and a number Δ and take:

- "negligible" with the domain $[N - \Delta, N + \Delta]$;
- "small positive" with the domain $[N, N + 2\Delta]$;
- "medium positive" with the domain $[N + \Delta, N + 3\Delta]$, etc.

Correspondingly:

- "small negative" has the domain $[N - 2\Delta, N]$;
- "medium negative" has the domain $[N - 3\Delta, N - \Delta]$, etc.

Once an interval $[a - \Delta, a + \Delta]$ is given, then we can take a triangular membership function $\mu(x)$ which:

- is equal to 0 outside this interval;
- is equal to 1 for $x = a$
- is linear on each of the intervals $[a - \Delta, a]$ and $[a, a + \Delta]$.

Usually, when we test the resulting control on a real or simulated system, this control is not perfect, so a further tuning is necessary based on the results of this testing.

1.3 Usually, Soft Computing Techniques Are Used For Tuning, But This May Not Be the Best Idea

Often, soft computing techniques such as neural networks or genetic algorithms are used for tuning fuzzy control. This is done mainly by tuning the corresponding membership functions. The results are usually reasonable, but this tuning often takes lots of time; for example, several thousand iterations are typical for neural networks.

How come soft computing techniques are so good for getting a rough approximation, but these same techniques are not so good for improving (tuning) this approximation? The explanation is very simple:

- On the first step (getting a rough approximation), the fact that we have an expert's intuitive understanding enables us to use soft computing techniques to perform this step.
- In contrast, on the second (tuning) step, we no longer have any expert understanding of which tuning is better; as a result, soft computing techniques are not that helpful.

1.4 Natural Idea: Let's Use Hard Computing for Tuning

Since soft computing techniques do not work that well for tuning, we propose to supplement them with more traditional ("hard computing") optimization techniques. In this chapter, we show that we can formulate an important particular case of the tuning problem as a traditional optimization problem and solve it by using traditional ("hard computing") techniques. We also show, on a practical industrial control example, that the resulting fusion of

- soft computing (for a rough approximation)
- hard computing (for tuning)

does lead to a high-quality control.

Comment Preliminary results of our research first appeared in [20].

2 Rescaling: An Important Particular Case of Tuning

2.1 Rescaling: Physical Motivations

In some cases, there are physical reasons why the use of membership functions with equally spaced domains does not work well. For example, if the control variable u is always positive (e.g., if we control the flow of some substance into a reactor), then negative values (that will be eventually generated by an equal spacing method) simply make no sense.

A natural idea is to choose another scale $\widetilde{u} = f(u)$ to represent the control variable u, so that equal spacing will work fine for $\widetilde{u}$. This idea is in good accordance with our common-sense description of physical processes; let us give a few examples.

From the physical viewpoint, it is quite possible to describe the strength of an *earthquake* by its energy, but when we talk about its consequences, it is much more convenient to use a logarithmic scale (called *Richter scale*).

Nonlinear scales are used to describe amplifiers and noise (decibels, in electrical engineering).

A nonlinear scale is used to describe hardness of minerals in geosciences, etc.

(For a general survey of different scales and rescalings, see [18].)

In our case, we want to design such a scale that for $f(u)$ the equally spaced endpoints $N - k \cdot \Delta$ and $N + k \cdot \Delta$ would make sense for all integers k. Therefore, we are looking for a function $f(u)$, whose domain is the set of all positive values and whose range is the set of all possible real numbers. In mathematical notations, f must map $(0, \infty)$ onto $(-\infty, \infty)$. There are lots of such functions, and evidently not all of them will improve the control. So we arrive at the following problem:

2.2 The Main Problem: Informally

Which rescaling $f : (0, \infty) \to (-\infty, \infty)$ *should we choose?*

2.3 What We Are Planning to Do

In this chapter, we do the following:

- first, we formulate the problem of choosing the best rescaling function $f(u)$ as a mathematical optimization problem;
- then, we solve this optimization problem under some reasonable optimality criteria; as a result, we get an optimal function $f(u)$;
- finally, we show that the use of this optimal rescaling function really improves fuzzy control.

3 Toward the Use of Hard Computing: Motivations of the Proposed Formal Description of the Problem

3.1 Why Is This Problem Difficult?

We want to find a scaling function $f(u)$ that is the best in some reasonable sense. In other words, we want to find a scaling function for which some characteristic $I(f)$ attains the value that corresponds to the best performance of the resulting fuzzy control.

As examples of such characteristics, we can take:

- an average running time of the algorithm,
- smoothness of the resulting control;
- stability of the resulting control, etc.

A seemingly natural approach is to describe this characteristic in precise terms and solve the corresponding optimization problem. Alas, life is not so simple. The problem is that even for the simplest linear plants (controlled systems), we do not know how to compute any of these possible characteristics for a give rescaling $f(u)$. How can we find $f(u)$ for which $I(f)$ is optimal if we cannot compute $I(f)$ even for a single function $f(u)$? There does not seem to be a likely answer.

However, we will show that this problem is solvable (and give the solution).

Comment To solve this problem, we use a general idea described in the book [12]; this book also contains applications of similar optimization methods to other soft computing techniques such as fuzzy logic, neural networks, genetic algorithms, etc.

3.2 Some Rescalings Preserve Equal Spacing

Let us first show that not all physically meaningful rescalings help.

Indeed, in order to get numerical values of the variable u (e.g., of the spatial coordinate x), we must fix a starting point (origin) and a measuring unit (e.g., meter). In principle, we could as well choose feet to describe length.

If we change the unit, then some things change, e.g., the *numerical values* of all the coordinates change: x meters are equal to $\lambda \cdot x$ feet, where λ is the number of feet in 1 meter.

On the other hand, some things do not change, e.g., when we change the measuring unit, equally spaced intervals remain equally spaced.

Similarly, we could choose a different initial point for measuring the x coordinate. If we take, as a new initial point, a point which previously had a coordinate x_0 (so that now its coordinate is 0), then, similarly, on one hand, the numerical values of the points' coordinates change from x to $x - x_0$; on the other hand, intervals that had equal length in the old scale (x) will still have equal length if we measure then in the new scale ($x - x_0$).

We can also change *both* the measuring unit and the starting point. This way we arrive at a transformation $x \to \lambda \cdot x + x_0$.

Summarizing: if x is a reasonable scale–in the sense that equally spaced membership functions lead to a reasonably good control–then the same is true for an arbitrary scale of the type $\lambda \cdot x + x_0$, where $\lambda > 0$, and x_0 is a real number. The reason is that if we have a sequence of equally spaced intervals $[N + k \cdot \Delta, N + (k+1) \cdot \Delta]$, then these intervals will remain equally spaces after these linear rescalings $x \to \lambda \cdot x + x_0$: namely, these intervals will turn into intervals $[\widetilde{N} + k \cdot \widetilde{\Delta}, \widetilde{N} + (k+1) \cdot \widetilde{\Delta}]$, where $\widetilde{N} = \lambda \cdot N + x_0$ and $\widetilde{\Delta} = \lambda \Delta$.

3.3 We Must Choose a Family of Scaling Functions, Not a Single Function

Let us now consider a scale u for which equal spacing does not work. Assume that $u \to f(u)$ is a transformation after which equal spacing becomes applicable. This means that if we use $f(u)$ as a new scale, then equal spacings work fine. But as we have just shown, for any $\lambda > 0$ and x_0 equal spacing will also work fine for the scale $\lambda \cdot f(u) + x_0$.

Therefore, if $f(u)$ is a function that transforms the initial scale into a scale, for which equal spacing works fine, then for every $\lambda > 0$ and x_0, the function $\widetilde{f}(u) = \lambda \cdot f(u) + x_0$ has the same desired property.

This means that there is no way to pick one function $f(u)$, because with any function $f(u)$, the whole family of functions $\lambda \cdot f(u) + x_0$ has the same property. Therefore, desired functions form a *2-parametric family* $\{\lambda \cdot f(u) + x_0\}_{\lambda > 0, x_0}$. Hence, instead of choosing a single function, we must formulate a problem of choosing a family.

3.4 Which Family Is the Best?

Among all such families, we want to choose the best one. In formalizing what "the best" means, we follow the general idea described in [12]. The criteria to choose may be computational simplicity, stability, or smoothness of the resulting control, etc.

In mathematical optimization problems, numerical criteria are most frequently used, where to every family, we assign some value expressing its performance and choose a family for which this value is maximal.

However, it is not necessary to restrict ourselves to such numeric criteria only. For example, if we have several different families that lead to the same average stability characteristics T, we can choose between them the one that leads to the maximal smoothness characteristics P. In this case, the actual criterion that we use to compare two families is not numerical, but more complicated. For example, we may say that a family Φ_1 is better than the family Φ_2 if and only if either $T(\Phi_1) < T(\Phi_2)$ or $T(\Phi_1) = T(\Phi_2)$ and $P(\Phi_1) < P(\Phi_2)$.

A criterion can be even more complicated. What a criterion *must* do is to allow us for every pair of families to tell whether the first family is better with respect to this criterion (we'll denote it by $\Phi_2 < \Phi_1$) or the second is better ($\Phi_1 < \Phi_2$) or these families have the same quality in the sense of this criterion (we'll denote it by $\Phi_1 \sim \Phi_2$).

3.5 The Criterion for Choosing the Best Family Must Be Consistent

Of course, it is necessary to demand that these choices be consistent: e.g., if $\Phi_1 < \Phi_2$ and $\Phi_2 < \Phi_3$, then $\Phi_1 < \Phi_3$.

3.6 The Criterion Must Be Final

Another natural demand is that this criterion must be *final* in the sense that it must choose a *unique* optimal family (i.e., a family that is better with respect to this criterion than any other family).

The reason for this demand is very simple:

If a criterion does not choose any family at all, then it is of no use.

If several different families are "the best" according to this criterion, then we still have a problem choosing the absolute "best" family. Therefore, we need some additional criterion for that choice.

For example, if several families turn out to have the same stability characteristics, we can choose, among them, a family with maximal smoothness. So what we actually do in this case is abandon that criterion for which there were several "best" families and consider a new "composite" criterion instead: Φ_1 is better than Φ_2 according to this new criterion if either it was better according to the old criterion, or, according to the old criterion, they had the same quality, and Φ_1 is better than Φ_2 according to the additional criterion.

In other words, if a criterion does not allow us to choose a unique best family, it means that this criterion is not ultimate; we have to modify it until we arrive at a final criterion that will have that property.

3.7 The Criterion Must Be Reasonably Invariant

We have already discussed the effect of changing units in a new scale $f(u)$. But it is also possible to change units in the original scale, in which the control u is described. If we use a unit that is c times smaller, then a control whose numeric value in the original scale was u will now have the numeric value cu. For example, if we initially measured the flux of a substance (e.g., rocket fuel) into the reactor by kg/sec, we can now switch to lb/sec.

Comment There is no physical sense in changing the starting point for u, because we consider the control variable that takes only positive values, and so 0 is a fixed value, corresponding to the minimal possible control.

We are looking for the universal rescaling method, which will be applicable to any reasonable situation (we do not want it to be adjustable to the situation, because the whole purpose of this rescaling is to avoid time-consuming adjustments). Suppose now that we first used kg/sec, compared two different scaling functions $f(u)$ and $\widetilde{f}(u)$, and it turned out that $f(u)$ is better (or, to be more precise, that the family $\Phi = \{\lambda \cdot f(u) + x_0\}$ is better than the family $\widetilde{\Phi} = \{\lambda \cdot \widetilde{f}(u) + x_0\}$). It sounds reasonable to expect that the relative quality of the two scaling functions should not depend on what units we used for u. So we expect that when we apply the same methods, but with the values of control expressed in lb/sec, then the results of applying $f(u)$ will still be better than the results of applying $\widetilde{f}(u)$.

The result of applying the function $f(u)$ to the control in lb/sec can be expressed in old units (kg/sec) as $f(c \cdot u)$, where c is a ratio of these two units. So the result of applying the rescaling function $f(u)$ to the data in new units (lb/sec) coincides with the result of applying a new scaling function $f_c(u) = f(c \cdot u)$ to the control in old units (kg/sec). So, we conclude that if $f(u)$ is better than $\widetilde{f}(u)$, then $f_c(u)$ must be better than $\widetilde{f_c}(u)$, where $f_c(u) = f(c \cdot u)$, and $\widetilde{f_c}(u) = \widetilde{f}(c \cdot u)$. This must be true for every c because we could use not only kg/sec or lb/sec but arbitrary units as well.

Now we are ready for the formal definitions.

4 Definitions and the Main Result

Definition 1 By a *rescaling function* (or a *rescaling*, for short), we mean a strictly monotonic function that maps the set of all positive real numbers $(0, \infty)$ onto the set of all real numbers $(-\infty, +\infty)$.

Definition 2 We say that two rescalings $f(u)$ and $\widetilde{f}(u)$ are *equivalent* if $\widetilde{f}(u) = \lambda \cdot f(u) + x_0$ for some positive constant λ and for some real number x_0.

Comment As we have already mentioned, if we apply two equivalent rescalings, we will get two scales that are either both leading to a good control or are both inadequate.

Definition 3 By a *family*, we mean the set of functions $\{\lambda \cdot f(u) + x_0\}$, where $f(u)$ is a fixed rescaling, λ runs over all positive real numbers, and x_0 runs over all real numbers. The set of all families will be denoted by S.

Definition 4 A pair of relations $(<, \sim)$ is called *consistent* if it satisfies the following conditions:

- if $F < G$ and $G < H$, then $F < H$;
- $F \sim F$;
- if $F \sim G$, then $G \sim F$;
- if $F \sim G$ and $G \sim H$, then $F \sim H$;
- if $F < G$ and $G \sim H$, then $F < H$;
- if $F \sim G$ and $G < H$, then $F < H$;
- if $F < G$, then it is not true that $G < F$ or $F \sim G$.

Definition 5 Assume a set A is given. Its elements will be called *alternatives*. By an *optimality criterion*, we mean a consistent pair $(<, \sim)$ of relations on the set A of all alternatives. If $G < F$, we say that F is *better* than G; if $F \sim G$, we say that the alternatives F and G are *equivalent* with respect to this criterion.

Definition 6 We say that an alternative F is *optimal* (or the *best*) with respect to a criterion $(<, \sim)$ if for every other alternative G, either $G < F$ or $F \sim G$.

Definition 7 We say that a criterion is *final* if there exists an optimal alternative, and this optimal alternative is unique.

Comment In the present chapter, we consider optimality criteria on the set S of all families.

Definition 8 By a *result of a unit change* in a function $f(u)$ to a unit that is $c > 0$ times smaller, we mean a function $f_c(u) = f(c \cdot u)$.

Definition 9 By the *result of a unit change* in a family Φ by $c > 0$, we mean the set of all the functions that are obtained by this unit change from $f \in \Phi$. This result will be denoted by $c \cdot \Phi$.

Definition 10 We say that an optimality criterion on F is *unit-invariant* if for every two families Φ and $\widetilde{\Phi}$ and for every number $c > 0$ the following two conditions are true:

- if $\widetilde{\Phi} < \Phi$, then $c \cdot \widetilde{\Phi} < c \cdot \Phi$;
- if $\Phi \sim \widetilde{\Phi}$, then $c \cdot \Phi \sim c \cdot \widetilde{\Phi}$.

Theorem 1 *If a family Φ is optimal in the sense of some optimality criterion that is final and unit-invariant, then every rescaling $f(u)$ from Φ is equivalent to $f(u) = \ln(u)$.*

Comments

- This result means that the optimal rescalings are of the type $\gamma \cdot \ln(u) + \alpha$ for some real numbers $\gamma > 0$ and α.
- For reader's convenience, the proof is given in the last section.
- This is not just a theoretical result: it came from the experience of one of the authors (HFV) to design a control for chemical reaction within a constant volume, non-adiabatic, continuously stirred tank reactor (CSTR). The model that describes the CSTR is described by the following system of differential equations (see, e.g., [11]):

$$\dot{x}_1 = -x_1 + D \cdot a \cdot (1 - x_1) \cdot \exp\left(\frac{x_2}{1 + \frac{x_2}{\gamma}}\right);$$

$$\dot{x}_2 = -x_2 + B \cdot D \cdot a \cdot (1 - x_1) \cdot \exp\left(\frac{x_2}{1 + \frac{x_2}{\gamma}}\right) - u \cdot (x_2 - x_c),$$

  where:

  - x_1 is the conversion rate;
  - x_2 is the (dimensionless) temperature
  - u is the (dimensionless) heat transfer coefficient.

  The objective of the control is to stabilize the system (i.e., bring it closer to the equilibrium point).

  Without the rescaling, we got a fuzzy control whose quality was even worse than that of a PID controller. However, when we applied a logarithmic rescaling $x_2 \to X = \ln(x_2)$ and used membership functions with equal spacing for X, the resulting control became comparable to the results of applying the intelligent "gain scheduled" (nonlinear) PID controller [4, 11]. In other words, we got the control that was as good as the one generated by the state-of-art traditional control theory with respect to stability and controllability of the plant. Moreover, it turned out that the resulting control is computationally simpler than the traditional fuzzy control. The details of this case study were published in [19].

5 Proof of the Main Result

The idea of this proof is as follows: first we prove that the optimal family is unit-invariant (in Part 1), and from that, in Part 2, we conclude that an arbitrary function f from Φ satisfies a certain functional equation; the solutions to this equation are known, and this completes the proof.

1. Let us first prove that the optimal family Φ_{opt} exists and is *unit-invariant* in the sense that $\Phi_{opt} = c \cdot \Phi_{opt}$ for all $c > 0$.

Indeed, we assumed that the optimality criterion is final; therefore, there exists a unique optimal family Φ_{opt}. Let's now prove that this optimal family is unit-invariant (this proof is practically the same as in [12]). The fact that Φ_{opt} is optimal means that for every other Φ, either $\Phi < \Phi_{opt}$ or $\Phi_{opt} \sim \Phi$. If $\Phi_{opt} \sim \Phi$ for some $\Phi \neq \Phi_{opt}$, then from the definition of the optimality criterion, we can easily deduce that Φ is also optimal, which contradicts the fact that there is only one optimal family. So for every Φ, either $\Phi < \Phi_{opt}$ or $\Phi_{opt} = \Phi$.

Take an arbitrary c, and apply this conclusion to $\Phi = c \cdot \Phi_{opt}$. If $c \cdot \Phi_{opt} = \Phi < \Phi_{opt}$, then from the invariance of the optimality criterion (condition ii)), we conclude that $\Phi_{opt} < c^{-1} \cdot \Phi_{opt}$, and that conclusion contradicts the choice of Φ_{opt} as the optimal family. So $\Phi = c \cdot \Phi_{opt} < \Phi_{opt}$ is impossible, and therefore $\Phi_{opt} = \Phi$, i.e., $\Phi_{opt} = c \cdot \Phi_{opt}$, and the optimal family is really unit-invariant.

2. Let us now deduce the actual form of the functions $f(u)$ from the optimal family Φ_{opt}.

If $f(u)$ is such a function, then the result $f(c \cdot u)$ of changing the unit of u to a c times smaller unit belongs to $c \cdot \Phi_{opt}$; so, due to Part 1 of this proof, the function $f(c \cdot u)$ also belongs to the family Φ_{opt}.

By the definition of a family, all its functions can be obtained from each other by a linear transformation $\lambda \cdot f(u) + x_0$; therefore, $f(c \cdot u) = \lambda \cdot f(u) + x_0$ for some λ and x_0. The corresponding values λ and x_0 depend on c; so, we arrive at the following functional equation for $f(u)$:

$$f(c \cdot u) = \lambda(c) \cdot f(u) + x_0(c).$$

In the survey on functional equations [1], the solutions of this equation are not explicitly given, but a for a similar functional equation

$$f(x + y) = f(x) \cdot h(y) + k(y),$$

all solutions are enumerated in Corollary 1 to Theorem 1 from Section 3.1.2 of [1]: they are $f(x) = \gamma \cdot x + \alpha$ and $f(x) = \gamma \cdot \exp(c \cdot x) + \alpha$, where $\gamma \neq 0$, $c \neq 0$, and α are arbitrary constants. To use this result, let us reduce our equation to the one with known solutions.

The only difference between these two equations is that we have a product and we need a sum. There is a well-known way to reduce product to a sum: turn to logarithms, because $\ln(a \cdot b) = \ln(a) + \ln(b)$. Let us introduce new variables $X = \ln(u)$ and $Y = \ln(c)$. In terms of these new variables, $u = \exp(X)$ and $c = \exp(Y)$. Substituting these values into our functional equation and taking into consideration that

$$\exp(X) \cdot \exp(Y) = \exp(X + Y),$$

we conclude that

$$F(X + Y) = H(Y) \cdot F(X) + K(Y),$$

where we denoted

$$F(X) \stackrel{\text{def}}{=} f(\exp(X)), \quad H(Y) \stackrel{\text{def}}{=} \lambda(\exp(Y)),$$

$$K(Y) \stackrel{\text{def}}{=} x_0(\exp(Y)).$$

So, according to the above-cited result, either

$$F(X) = \gamma \cdot X + \alpha,$$

or $F(X) = \gamma \cdot \exp(c \cdot X) + \alpha$.

From $F(X) = f(\exp(X))$, we conclude that $f(u) = F(\ln(u))$; therefore, either $f(u) = \gamma \cdot \ln(u) + \alpha$ or $f(u) = \gamma \cdot \exp(c \cdot \ln(u)) + \alpha = \gamma \cdot u^c + \alpha$. In the second case, the function $f(u)$ maps $(0, \infty)$ onto the interval (α, ∞), and we defined a rescaling as a function whose values run over all possible real numbers. So the second case is impossible, and $f(u) = \gamma \cdot \ln(u) + \alpha$, which means that $f(u)$ is equivalent to a logarithm. Q.E.D.

6 Conclusions

One of the most successful examples of soft computing is fuzzy control. One of the important steps in designing a fuzzy control is the choice of the membership functions for all the terms that the experts use. This choice strongly influences the quality of the resulting control.

For simple controlled systems, it is sufficient to have equally spaced membership functions, i.e., functions that have similar shape (usually triangular or trapezoid) and are located in intervals of equal length

$$\ldots, [N - \Delta, N + \Delta], [N, N + 2\Delta], [N + \Delta, N + 3\Delta], \ldots$$

For complicated systems, this choice does not lead to a good fuzzy control, so it is necessary to tune the membership functions. This tuning is usually done by using soft computing techniques such as neural networks or genetic algorithms. Such tuning is, however, a very time-consuming procedure. We show that traditional ("hard computing") optimization techniques lead to a faster tuning.

Specifically, we consider the case when equally spaced membership functions are inadequate because the control variable u can take only positive values. Such situations occur, for example, when we control the flux of the substances into a chemical reactor (e.g., the flux of fuel into an engine). Our idea is to “rescale” this variable, i.e., to use a new variable $\widetilde{u} = f(u)$, and to choose a function $f(u)$ in such a way that we can apply membership functions, which are equally spaced in $\widetilde{u}$.

We give a mathematical proof that the optimal rescaling is logarithmic ($f(u) = a \cdot \ln(u) + b$). We also show on a real-life example of a nonlinear chemical reactor that the resulting fuzzy control, without any further tuning of membership functions, can be comparable in quality with the best state-of-art nonlinear controls of traditional control theory.

Acknowledgments The authors are greatly thankful to the anonymous referees for valuable suggestions.

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Chapter 5
A Novel Fully Interval-Valued Intuitionistic Fuzzy Multi-objective Indefinite Quadratic Transportation Problem with an Application to Cost and Wastage Management in the Food Industry

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Abbreviations

DM	Decision-maker
TP	Transportation problem
IF	Intuitionistic fuzzy
IFN	Intuitionistic fuzzy number
IFTP	Intuitionistic fuzzy transportation problem
IV	Interval-valued
IVIFN	Interval-valued intuitionistic fuzzy number
TIFN	Triangular intuitionistic fuzzy number
IVTIF	Interval-valued triangular intuitionistic fuzzy
IVTIFN	Interval-valued triangular intuitionistic fuzzy number
MOTP	Multi-objective transportation problem
FIVIFMOIQTP	Fully interval-valued intuitionistic fuzzy multi-objective indefinite quadratic transportation problem
MF	Membership function
NMF	Nonmembership function
OF	Objective function

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C. Jana et al. (eds.), *Fuzzy Optimization, Decision-making and Operations Research*, https://doi.org/10.1007/978-3-031-35668-1_5

1 Introduction

With the advent of mobile-based apps, many have started ordering food, groceries, household items, tech items, and many other commodities from their phones. Today's customer wants the deliveries with the blink of an eye—whenever and wherever. The ease and the comfort which these apps provide is one side of the picture, i.e., the consumer side. On the other side, the provider or the in charge of these deliveries is always uncertain in judgment of the supply, the demand, and the amount of the commodity to be delivered.

Talking about the cost of transportation, there are many factors which effect these costs like route and climate conditions, infrastructure requirements, duration, fuel costs, etc. Merely transporting the commodity is not the only agenda, but doing it safely, without any damage, is also important. However, due to some unavoidable reasons like expiry, leakage, breakage, negligence, etc., there is an associated depreciation cost (damage cost) during transportation. A deeper dig into the market situation also tells us that the DM not only has to minimize the transportation and depreciation cost but also has to focus on other objectives. For effective decision-making during transportation, understanding the packaging budget and packaging cost is also important. This includes design and prototyping cost, material, printing, die-cutting, embossing, lamination, etc. Different types of packaging material are used, for example, cardboard boxes, bubble wraps, Styrofoam, air-filled pillows, etc. But with packaging comes a lot of wastage due to poor packaging design, ill-trained staff, poor infrastructure, etc. which adds up to what we call as the wastage cost. Also, once the package reaches the customer, the packaging material is mostly discarded straightaway. This material which may or may not be biodegradable dramatically acts as pollutants such as litter, landfill space, water pollutant, etc.

Transport managers after identifying these vulnerable areas must work in sync to optimize the transportation and packaging process such that the needs of the customers are also met and damage to the climate is also mitigated. Minimal and efficient packaging will reduce material use, hence lesser packaging cost and minimal waste and better space utilization in the transport vehicle leading to overall lesser fuel usage and lesser transportation cost. Thus, it is imperative to have a transportation model which aims at minimal and efficient packaging along with minimized transportation and damage cost. The parameters of this model, *viz.*, supply, demand, cost, and quantity, must be of such a nature which is able to handle uncertainties and impreciseness efficiently.

Gauging the above situation in an entirety, the authors are motivated to sculpt the above situation into a novel fully intuitionistic fuzzy interval-valued multi-objective indefinite quadratic TP. Parameters here such as supply, demand, cost coefficients, and the variables by nature being imprecise and unconfirmed all are taken to be IVIFNs. We take the objectives to be indefinite quadratic where each function is a product of two linear factors and hence simultaneous minimization of each factor is achieved. In the first objective, simultaneous minimization of the transportation and depreciation cost is targeted, while the second focuses on simultaneous minimization of packaging cost and the associated wastage cost. Applicability of the model is shown through a TP pertaining to the food industry.

2 Literature Review

As discussed in the above section, the parameters of a real-life TP with crisp parameters are unable to handle situations of ambiguity and uncertainty. Therefore, appropriate non-crisp parameters must be used. Since conception, fuzzy sets introduced by Zadeh [15] have been extensively used in the field of TPs as they associate with each element of the set the membership degree (acceptance level) [2, 24, 29].

In 1986, [13] proposed the IF set which is more reliable than the fuzzy set as the former also associates the degree of nonmembership (nonacceptance level) along with the degree of membership. Solution of IFTP using linear programming was given in [7], while [6] gave the concept of pareto-optimal solution for fixed-charge solid TP under IF environment. Zero-point maximum allocation method was proposed by Sharma [20] to solve IFTP. Not only restricted to TPs, the application of TIFNs in bi-matrix games can also be seen in [22, 23].

To answer how to handle the situations when even these degrees are not available as exact values because of some hesitation, the idea of IVIF sets was established by Atanassov [14] where the more flexible approach of depicting membership and nonmembership values by intervals was adopted. Bharati and Singh [33] solved TP under IVIF environment. Recently, a multi-objective multi-item four-dimensional green TP in IVIF environment was solved in [8], and IVIFTP of type 2 was solved by Choudhary and Yadav [1].

For more than one clashing objective, an MOTP is considered. Diaz [12] described efficient solutions of MOTP in 1979. Nomani et al. [18] gave a weighted approach based on goal programming strategy to solve MOTP, while [10] solved rough MOTP. Fuzzy approach was exploited by Li and Lai [16] to solve MOTP. Ahmad and Adhami [9] solved multi-objective nonlinear TP with fuzzy parameters.

The formulation of the objective functions whether single objective or multi-objective wholly depends on the requirements of the DM whether he considers fixed-charge or fractional or any other form of objective function. Bhatia [11] formulated indefinite quadratic solid TP. A bilevel TP with indefinite quadratic objective functions for vaccine transportation was formulated by Singh et al. [4]. Recently, [31] solved TP for IV trapezoidal IFNs, while [5] solved neutrosophic multi-objective green four-dimensional fixed-charge TP. Joshi et al. [35] solved multi-objective linear fractional TP under neutrosophic environment.

Literature related to work of researchers on fully IFTP with different types of OFs is shown in Table 5.1.

Table 5.1 Literature review

Author and citation	Fully IF	Type of IFN used	Interval-valued IFN used	Type of objective function	MOTP
Mahmoodirad et al. [3]	✓	Triangular		Standard	
El Sayed and Abo-Sinna [17]	✓	Trapezoidal		Fractional	✓
Ghosh et al. [32]	✓	Triangular		Fixed-charge solid	
Kumar and Hussain [27]	✓	Triangular		Standard	
Mahajan and Gupta [34]	✓	Triangular		Standard	✓
Bagheri et al. [19]	✓	Triangular		Standard	✓
Anukokila and Radhakrishnan [25]	✓	Trapezoidal		Fractional	
Jalil et al. [28]	✓	Triangular		Solid	✓
Giri et al. [26]	✓	Triangular		Fixed-charge multi-item solid	
Malik and Gupta [21]	✓	Triangular	✓	Standard	✓
Proposed work	✓	Triangular	✓	Indefinite quadratic	✓

Here onward, the paper proceeds as follows: In Sect. 3, we briefly state the preliminaries which includes basic definitions and some arithmetic operations. Section 4 establishes the FIVIFMOIQTP. Section 5 gives the proposed solution methodology. Section 6 exhibits the practicality of the model through a solved numerical example. In Sect. 7, we solve the numerical using another methodology. Section 8 gives the conclusion and future work.

3 Preliminary

Definition 1 (IF Set) [13] An IF set $\hat{S}'$ in Y (a universe of discourse) is a triplet defined by $\{(y, \mu_{\hat{S}'}(y), \nu_{\hat{S}'}(y)) : y \in Y\}$, where $\mu_{\hat{S}'} : Y \to [0, 1]$and $\nu_{\hat{S}'} : Y \to [0, 1]$, respectively, depict the MF and NMF of the element $y \in Y$ being in $\hat{S}'$. They satisfy the relation $0 \leq \mu_{\hat{S}'}(y) + \nu_{\hat{S}'}(y) \leq 1 \forall y \in Y$. The value of the expression $1 - \mu_{\hat{S}'}(y) - \nu_{\hat{S}'}(y)$ is called the hesitancy degree of the element $y \in Y$ to $\hat{S}'$.

Definition 2 (IVIF Set) [14] An IVIF set $\hat{S} = \{(y, \mu_{\hat{S}}(y), \nu_{\hat{S}}(y)) : y \in Y\}$, where $\mu_{\hat{S}} : Y \to I[0, 1]$and $\nu_{\hat{S}} : Y \to I[0, 1]$, respectively, depict the interval-valued MF and NMF s.t. $Sup(\mu_{\hat{S}}(y)) + Sup(\nu_{\hat{S}}(y)) \leq 1 \ \forall \ y \in Y$. Here, $I[0, 1] = \{[u, v] : 0 \leq u < v \leq 1\}$.

Definition 3 (IVTIFN) [1] Denoted by $\hat{C} = (c_1^U, c_1^L, c_2, c_3^L, c_3^U), (c_1'^L, c_1'^U, c_2, c_3'^U, c_3'^L)$, an IVTIFN with its degrees of membership and nonmembership is defined as follows:

$$\mu^L_{\hat{C}}(y) = \begin{cases} 1 & y = c_2, \\ \frac{y-c^L_1}{c_2-c^L_1} & \text{if } c^L_1 < y < c_2, \\ \frac{c^L_3-y}{c^L_3-c_2} & \text{if } c_2 < y < c^L_3, \\ 0 & \text{otherwise} \end{cases}$$

&

$$\mu^U_{\hat{C}}(y) = \begin{cases} 1 & y = c_2, \\ \frac{y-c^U_1}{c_2-c^U_1} & \text{if } c^U_1 < y < c_2, \\ \frac{c^U_3-y}{c^U_3-c_2} & \text{if } c_2 < y < c^U_3, \\ 0 & \text{otherwise} \end{cases}$$

$$\nu^L_{\hat{C}}(y) = \begin{cases} 0 & y = c_2, \\ \frac{c_2-y}{c_2-c'^L_1} & \text{if } c'^L_1 < y < c_2, \\ \frac{y-c_2}{c'^L_3-c_2} & \text{if } c_2 < y < c'^L_3, \\ 1 & \text{otherwise} \end{cases}$$

&

$$\nu^U_{\hat{C}}(y) = \begin{cases} 0 & y = c_2, \\ \frac{c_2-y}{c_2-c'^U_1} & \text{if } c'^U_1 < y < c_2, \\ \frac{y-c_2}{c'^U_3-c_2} & \text{if } c_2 < y < c'^U_3, \\ 1 & \text{otherwise} \end{cases}$$

where $c'^L_1 \le c'^U_1 \le c^U_1 \le c^L_1 \le c_2 \le c^L_3 \le c^U_3 \le c'^U_3 \le c'^L_3$. Graphical representation is given in Fig. 5.1.

Remark 1 A IVTIFN $\hat{C}$ can be reduced to a triangular IFN $(c^L_1, c_2, c^L_3), (c'^L_1, c_2, c'^L_3)$ if $c'^L_1 = c'^U_1, c^U_1 = c^L_1, c^L_3 = c^U_3, c'^U_3 = c'^L_3$.

Definition 4 (Arithmetic Operations on IVTIFNs) Malik and Gupta [21] Let $\hat{C} = (c^U_1, c^L_1, c_2, c^L_3, c^U_3), (c'^L_1, c'^U_1, c_2, c'^U_3, c'^L_3)$ and $\hat{D} = (d^U_1, d^L_1, d_2, d^L_3, d^U_3)$, $(d'^L_1, d'^U_1, d_2, d'^U_3, d'^L_3)$ be two IVTIFNs, then

(a) $\hat{C} \oplus \hat{D} = (c^U_1 + d^U_1, c^L_1 + d^L_1, c_2 + d_2, c^L_3 + d^L_3, c^U_3 + d^U_3), (c'^L_1 + d'^L_1, c'^U_1 + d'^U_1, c_2 + d_2, c'^U_3 + d'^U_3, c'^L_3 + d'^L_3)$.
(b) $\hat{C} \ominus \hat{D} = (c^U_1 - d^U_1, c^L_1 - d^L_1, c_2 - d_2, c^L_3 - d^L_3, c^U_3 - d^U_3), (c'^L_1 - d'^L_1, c'^U_1 - d'^U_1, c_2 - d_2, c'^U_3 - d'^U_3, c'^L_3 - d'^L_3)$.

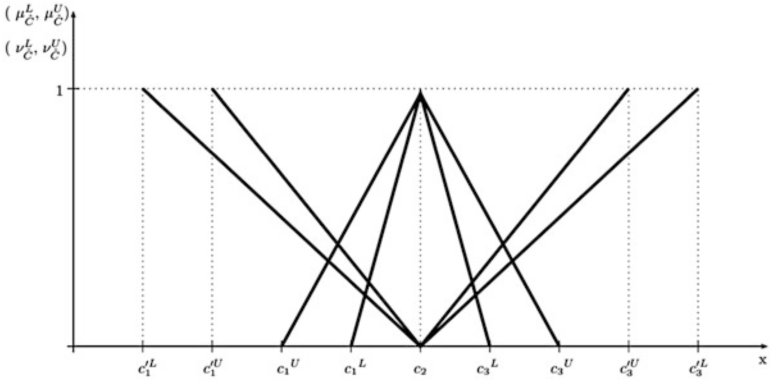

Fig. 5.1 Interval-valued triangular IFN

(c) $k\hat{C} = \begin{cases} \{(kc_1^U, kc_1^L, kc_2, kc_3^L, kc_3^U), \\ (kc_1'^L, kc_1'^U, kc_2, kc_3'^U, kc_3'^L)\}, & if k \geq 0, \\ \{(kc_3^U, kc_3^L, kc_2, kc_1^L, kc_1^U), \\ (kc_3'^L, kc_3'^U, kc_2, kc_1'^U, kc_1'^L)\}, & if k < 0. \end{cases}$

(d) If $c_1'^L \geq 0, d_1'^L \geq 0$, then
$\hat{C} \otimes \hat{D} = (c_1^U d_1^U, c_1^L d_1^L, c_2 d_2, c_3^L d_3^L, c_3^U d_3^U), (c_1'^L d_1'^L, c_1'^U d_1'^U, c_2 d_2, c_3'^U d_3'^U, c_3'^L d_3'^L)$.

Definition 5 (Equality of Two IVTIFNs) Malik and Gupta [21] Two IVTIFNs, $\hat{C} = (c_1^U, c_1^L, c_2, c_3^L, c_3^U), (c_1'^L, c_1'^U, c_2, c_3'^U, c_3'^L)$ and $\hat{D} = (d_1^U, d_1^L, d_2, d_3^L, d_3^U)$, $(d_1'^L, d_1'^U, d_2, d_3'^U, d_3'^L)$ are considered equal, i.e., $\hat{C} \simeq \hat{D}$ iff
$c_1^U = d_1^U, c_1^L = d_1^L, c_2 = d_2, c_3^L = d_3^L, c_3^U = d_3^U, c_1'^L = d_1'^L, c_1'^U = d_1'^U, c_3'^U = d_3'^U, c_3'^L = d_3'^L$.

Definition 6 (Expected Value) Bharati and Singh [33] Let $\hat{C} = (c_1^U, c_1^L, c_2, c_3^L, c_3^U), (c_1'^L, c_1'^U, c_2, c_3'^U, c_3'^L)$ be a IVTIFN, then its expected value is given by the following formula:
$Eval(\hat{C}) = \frac{(c_1^U + c_1^L + c_1'^L + c_1'^U + 8c_2 + c_3^L + c_3^U + c_3'^U + c_3'^L)}{16}$
where $Eval$ is a real-valued linear function defined over the set of all IVIFNs over $\mathbb{R}$.

Definition 7 (Ordering of Two IVTIFNs) Bharati and Singh [33] Let $\hat{C}$ and $\hat{D}$ be two IVTIFNs. Then,

(a) $\hat{C} \preceq \hat{D} \Leftrightarrow Eval(\hat{C}) \leq Eval(\hat{D})$
(b) $\hat{C} \succeq \hat{D} \Leftrightarrow Eval(\hat{C}) \geq Eval(\hat{D})$

Definition 8 (Efficient Point/Pareto-Optimal Solution) Chhibber et al. [6] A point $\hat{\hat{y}}$ is termed as an efficient solution for which no value of $\hat{y}$ exists in the feasible region such that $\hat{Z}_k(\hat{y}) \preceq \hat{Z}_k(\hat{\hat{y}})\ \forall\ k$ and $\hat{Z}_k(\hat{y}) \prec \hat{Z}_k(\hat{\hat{y}})$ for at least one k.

4 Fully Interval-Valued Intuitionistic Fuzzy Multi-objective Indefinite Quadratic Transportation Problem (FIVIFMOIQTP)

As discussed previously, real-life situations are not well represented by crisp parameters. In this case, we prefer representing them by interval-valued intuitionistic fuzzy numbers. Also, single objectives are not sufficient to describe the aims and purposes of the DM, and hence an MOTP is preferred. An indefinite quadratic objective function being a product of two linear factors can simultaneously minimize each factor. An MOTP in which all the parameters such as supply, demand, cost parameters, and variables are all IVIFNs and the objectives are indefinite quadratic in nature is an FIVIFMOIQTP. It can be mathematically formulated as:

(FIVIFMOIQTP)

$$\text{Min } \hat{Z}_1(y) = \hat{Z}_{11}(y)\hat{Z}_{12}(y) = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} \hat{p}_{ij} \otimes \hat{y}_{ij}\right)\left(\sum_{i=1}^{m}\sum_{j=1}^{n} \hat{q}_{ij} \otimes \hat{y}_{ij}\right),$$

$$\text{Min}\hat{Z}_2(y) = \hat{Z}_{21}(y)\hat{Z}_{22}(y) = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} \hat{e}_{ij} \otimes \hat{y}_{ij}\right)\left(\sum_{i=1}^{m}\sum_{j=1}^{n} \hat{f}_{ij} \otimes \hat{y}_{ij}\right),$$

$$\vdots$$

$$\text{Min}\hat{Z}_K(y) = \hat{Z}_{K1}(y)\hat{Z}_{K2}(y) = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} \hat{u}_{ij} \otimes \hat{y}_{ij}\right)\left(\sum_{i=1}^{m}\sum_{j=1}^{n} \hat{v}_{ij} \otimes \hat{y}_{ij}\right).$$

subject to

$$\sum_{j=1}^{n} \hat{y}_{ij} = \hat{s}_i \quad ; \quad \sum_{i=1}^{m} \hat{y}_{ij} = \hat{d}_j \quad ; \hat{y}_{ij} \geq \hat{0} \ \ i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n.$$

where, $\hat{s}_i = (s_{i1}^U, s_{i1}^L, s_{i2}, s_{i3}^L, s_{i3}^U), (s_{i1}'^L, s_{i1}'^U, s_{i2}, s_{i3}'^U, s_{i3}'^L)$ is the IVTIF supply at the i^{th} source and $\hat{d}_j = (d_{j1}^U, d_{j1}^L, d_{j2}, d_{j3}^L, d_{j3}^U), (d_{j1}'^L, d_{j1}'^U, d_{j2}, d_{j3}'^U, d_{j3}'^L)$ is the IVTIF demand at the j^{th} destination. $\hat{c}_{ij} = (c_{ij1}^U, c_{ij1}^L, c_{ij2}, c_{ij3}^L, c_{ij3}^U), (c_{ij1}'^L, c_{ij1}'^U, c_{ij2}, c_{ij3}'^U, c_{ij3}'^L)$ is the IVTIF cost parameters when unit quantity is transported from source i to destination j.

$\hat{y}_{ij} = (y^U_{ij1}, y^L_{ij1}, y_{ij2}, y^L_{ij3}, y^U_{ij3}), (y'^L_{ij1}, y'^U_{ij1}, y_{ij2}, y'^U_{ij3}, y'^L_{ij3})$ is the IVTIF quantity of the goods that is transported from source *i* to destination *j*. The feasible region of the above problem is a non-empty and bounded set in which each of the linear factor is assumed to be positive. Each objective function is both quasi-convex and quasi-concave on the feasible region, and thus we obtain the optimal solution at its extreme point.

Feasibility conditions: $\hat{y}_{ij} \geq 0$ and $\sum_i \hat{s_i} = \sum_j \hat{d}_j$.

Applying the $Eval$ function over all the IVTIF components to convert them into crisp numbers, we use Definitions 4 and 5. We get a crisp TP, *viz.*, FIVIFMOIQTP-1.
(FIVIFMOIQTP-1)

$$\text{Min } EVal\left(\hat{Z}_1(y)\right) = EVal\left(\hat{Z}_{11}(y)\hat{Z}_{12}(y)\right)$$
$$= EVal\left(\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\hat{p}_{ij} \otimes \hat{y}_{ij}\right)\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\hat{q}_{ij} \otimes \hat{y}_{ij}\right)\right),$$

$$\text{Min } EVal\left(\hat{Z}_2(y)\right) = EVal\left(\hat{Z}_{21}(y)\hat{Z}_{22}(y)\right)$$
$$= EVal\left(\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\hat{e}_{ij} \otimes \hat{y}_{ij}\right)\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\hat{f}_{ij} \otimes \hat{y}_{ij}\right)\right),$$
$$\vdots$$

$$\text{Min } EVal\left(\hat{Z}_K(y)\right) = EVal\left(\hat{Z}_{K1}(y)\hat{Z}_{K2}(y)\right)$$
$$= EVal\left(\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\hat{u}_{ij} \otimes \hat{y}_{ij}\right)\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\hat{v}_{ij} \otimes \hat{y}_{ij}\right)\right).$$

subject to

$$\sum_j y^U_{ij1} = s^U_{i1},\ \ \sum_j y^L_{ij1} = s^L_{i1},\ \ \sum_j y_{ij2} = s_{i2},\ \ \sum_j y^L_{ij3} = s^L_{i3},\ \ \sum_j y^U_{ij3} = s^U_{i3},$$
$$\sum_j y'^L_{ij1} = s'^L_{i1},\ \ \sum_j y'^U_{ij1} = s'^U_{i1},$$

$$\sum_j y'^U_{ij3} = s'^U_{i3}, \ \sum_j y'^L_{ij3} = s'^L_{i3}, \ \sum_i y^U_{ij1} = d^U_{i1}, \ \sum_i y^L_{ij1} = d^L_{i1}, \ \sum_i y_{ij2} = d_{i2},$$
$$\sum_i y^L_{ij3} = d^L_{i3}, \ \sum_i y^U_{ij3} = d^U_{i3},$$

$$\sum_i y'^L_{ij1} = d'^L_{i1}, \ \sum_i y'^U_{ij1} = d'^U_{i1}, \ \sum_i y'^U_{ij3} = d'^U_{i3}, \ \sum_i y'^L_{ij3} = d'^L_{i3},$$

$$y'^L_{ij1} \geq 0, \ y'^U_{ij1} - y'^L_{ij1} \geq 0, \ y^U_{ij1} - y'^U_{ij1} \geq 0, \ y^L_{ij1} - y^U_{ij1} \geq 0,$$
$$y_{ij2} - y^L_{ij1} \geq 0, \ y^L_{ij3} - y_{ij2} \geq 0, \ y^U_{ij3} - y^L_{ij3} \geq 0,$$

$$y'^U_{ij3} - y^U_{ij3} \geq 0, \ y'^L_{ij3} - y'^U_{ij3} \geq 0, \ i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n.$$

Hereafter, we denote $EVal\left(\hat{Z}_k(y)\right) = \hat{\hat{Z}}_k(y) \forall k = 1, 2, \ldots, K$.

Theorem 1 *The efficient solution of (FIVIFMOIQTP-1) is the efficient solution of (FIVIFMOIQTP).*

Proof Let $\hat{y}^* = (y^{*U}_{ij1}, y^{*L}_{ij1}, y^*{}_{ij2}, y^{*L}_{ij3}, y^{*U}_{ij3}), (y^{*\prime L}_{ij1}, y^{*\prime U}_{ij1}, y^*{}_{ij2}, y^{*\prime U}_{ij3}, y^{*\prime L}_{ij3}) \forall i, j$ be an efficient solution of FIVIFMOIQTP-1. From feasibility conditions, it follows that

$$\sum_j y^{*U}_{ij1} = s^U_{i1}, \ \sum_j y^{*L}_{ij1} = s^L_{i1}, \ \sum_j y^*{}_{ij2} = s_{i2}, \ \sum_j y^{*L}_{ij3} = s^L_{i3},$$
$$\sum_j y^{*U}_{ij3} = s^U_{i3}, \ \sum_j y^{*\prime L}_{ij1} = s'^L_{i1}, \ \sum_j y^{*\prime U}_{ij1} = s'^U_{i1},$$

$$\sum_j y^{*\prime U}_{ij3} = s'^U_{i3}, \ \sum_j y^{*\prime L}_{ij3} = s'^L_{i3}, \ \sum_i y^{*U}_{ij1} = d^U_{i1}, \ \sum_i y^{*L}_{ij1} = d^L_{i1},$$
$$\sum_i y^*{}_{ij2} = d_{i2}, \ \sum_i y^{*L}_{ij3} = d^L_{i3}, \ \sum_i y^{*U}_{ij3} = d^U_{i3},$$

$$\sum_i y^{*\prime L}_{ij1} = d'^L_{i1}, \ \sum_i y^{*\prime U}_{ij1} = d'^U_{i1}, \ \sum_i y^{*\prime U}_{ij3} = d'^U_{i3}, \ \sum_i y^{*\prime L}_{ij3} = d'^L_{i3},$$

$$y^{*\prime L}_{ij1} \geq 0, \ y^{*\prime U}_{ij1} - y^{*\prime L}_{ij1} \geq 0, \ y^{*U}_{ij1} - y^{*\prime U}_{ij1} \geq 0, \ y^{*L}_{ij1} - y^{*U}_{ij1} \geq 0,$$
$$y^*{}_{ij2} - y^{*L}_{ij1} \geq 0, \ y^{*L}_{ij3} - y^*{}_{ij2} \geq 0,$$

$$y^{*U}_{ij3} - y^{*L}_{ij3} \geq 0, \quad y^{*\prime U}_{ij3} - y^{*U}_{ij3} \geq 0, \quad y^{*\prime L}_{ij3} - y^{*\prime U}_{ij3} \geq 0,$$
$$i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, n.$$

The above constraints imply

$$\sum_{j=1}^{n} \hat{y}^{*}_{ij} = \hat{s}_i \;\; ; \;\; \sum_{i=1}^{m} \hat{y}^{*}_{ij} = \hat{d}_j \;\; ; \;\; \hat{y}^{*}_{ij} \succeq \hat{0} \;\; i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, n.$$

which in turn imply that $\hat{y}^*$ is a feasible solution of FIVIFMOIQTP. Now, as $\hat{y}^*$ is an efficient solution of FIVIFMOIQTP-1, $\nexists$ any IVTIFN $\hat{\hat{y}}$ s.t. $Eval(\hat{Z}_k(\hat{\hat{y}})) \leq Eval(\hat{Z}_k(\hat{y}^*)) \; \forall \; k$ and $Eval(\hat{Z}_k(\hat{\hat{y}})) < Eval(\hat{Z}_k(\hat{y}^*))$ for at least one k. Thus, Definition 7 yields that $\hat{y}^*$ is an efficient solution of FIVIFMOIQTP as well.

5 Proposed Solution Methodology

In this section, we present the following approach to solve (FIVIFMOIQTP). The following steps are to be followed:

Step 1 Divide the problem FIVIFMOIQTP-1 into k subproblems where each subproblem is a single objective TP with k^{th} OF, *viz.*, $EVal\left(\hat{Z}_K(y)\right) = \hat{\hat{Z}}_K(y)$ and the constraint sets of (FIVIFMOIQTP-1) above. In the end, we will obtain k optimal solutions, Y_k corresponding to the above k subproblems.

Step 2 Find $\hat{\hat{Z}}_K(Y_k) \; \forall \; k$ and construct the following payoff matrix:

$$\begin{bmatrix} \hat{\hat{Z}}_1(Y_1) & \hat{\hat{Z}}_2(Y_1) & \hat{\hat{Z}}_3(Y_1) & \cdots & \hat{\hat{Z}}_k(Y_1) \\ \hat{\hat{Z}}_1(Y_2) & \hat{\hat{Z}}_2(Y_2) & \hat{\hat{Z}}_3(Y_2) & \cdots & \hat{\hat{Z}}_k(Y_2) \\ \hat{\hat{Z}}_1(Y_3) & \hat{\hat{Z}}_2(Y_3) & \hat{\hat{Z}}_3(Y_3) & \cdots & \hat{\hat{Z}}_k(Y_3) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \hat{\hat{Z}}_1(Y_k) & \hat{\hat{Z}}_2(Y_k) & \hat{\hat{Z}}_3(Y_k) & \cdots & \hat{\hat{Z}}_k(Y_k) \end{bmatrix}$$

Remark 2 The diagonal values of this payoff matrix give the OF values of the k^{th} OF, *viz.*, $\hat{\hat{Z}}_k(y)$ at its corresponding optimal point Y_k, i.e., $\hat{\hat{Z}}_k(Y_k)$. The non-diagonal values give the OF value $\hat{\hat{Z}}_k(Y_r)$ for $r, k = 1, 2, \ldots, K, r \neq k$. In this payoff matrix, we collect all the $k \times k$ OF values calculated not only at their respective optimal points Y_k but on the optimal points $Y_r, r \neq k$. Thus, the k^{th} column of the matrix represents the range of values attained by any k^{th} objective function.

Step 3 Using the payoff matrix, find the upper bound(least acceptable achievement level $\hat{\hat{U}}_k$) and the lower bound(most acceptable achievement level $\hat{\hat{L}}_k$) for formulating the linear MFs for each of the k^{th} objective function. $\hat{\hat{L}}_k = Min\{\hat{\hat{Z}}_k(Y_r)\}$ and $\hat{\hat{U}}_k = Max\{\hat{\hat{Z}}_k(Y_r)\}$ where $1 \le r \le K$.

Step 4 Formulate the upper and lower linear MFs, *viz.*, $\hat{\hat{\mu}}_k^U(\hat{\hat{Z}}_k(y))$ and $\hat{\hat{\mu}}_k^L(\hat{\hat{Z}}_k(y))$, for each k^{th} OF using

$$\hat{\hat{\mu}}_k^U(\hat{\hat{Z}}_k(y)) = \begin{cases} 1 & \hat{\hat{Z}}_k(y) \le \hat{\hat{L}}_k^{\mu}, \\ \frac{\hat{\hat{U}}_k^{\mu} - \hat{\hat{Z}}_k(y)}{\hat{\hat{U}}_k^{\mu} - \hat{\hat{L}}_k^{\mu}} & \hat{\hat{L}}_k^{\mu} < \hat{\hat{Z}}_k(y) < \hat{\hat{U}}_k^{\mu}, \\ 0 & \hat{\hat{Z}}_k(y) \ge \hat{\hat{U}}_k^{\mu}. \end{cases}$$

&

$$\hat{\hat{\mu}}_k^L(\hat{\hat{Z}}_k(y)) = \begin{cases} 1 & \hat{\hat{Z}}_k(Y) \le \hat{\hat{L}}_k^{\mu}, \\ \eta\left(\frac{\hat{\hat{U}}_k^{\mu} - \hat{\hat{Z}}_k(y)}{\hat{\hat{U}}_k^{\mu} - \hat{\hat{L}}_k^{\mu}}\right) & \hat{\hat{L}}_k^{\mu} < \hat{\hat{Z}}_k(y) < \hat{\hat{U}}_k^{\mu}, \\ 0 & \hat{\hat{Z}}_k(y) \ge \hat{\hat{U}}_k^{\mu}. \end{cases}$$

Formulate the upper and lower linear NMFs, *viz.*, $\hat{\hat{\nu}}_k^U(\hat{\hat{Z}}_k(y))$ and $\hat{\hat{\nu}}_k^L(\hat{\hat{Z}}_k(y))$ for each k^{th} OF, using

$$\hat{\hat{\nu}}_k^U(\hat{\hat{Z}}_k(y)) = \begin{cases} 0 & \hat{\hat{Z}}_k(y) \le \hat{\hat{L}}_k^{\nu}, \\ \frac{\hat{\hat{Z}}_k(y) - \hat{\hat{L}}_k^{\nu}}{\hat{\hat{U}}_k^{\nu} - \hat{\hat{L}}_k^{\nu}} & \hat{\hat{L}}_k^{\nu} < \hat{\hat{Z}}_k(y) < \hat{\hat{U}}_k^{\nu}, \\ 1 & \hat{\hat{Z}}_k(y) \ge \hat{\hat{U}}_k^{\nu}. \end{cases}$$

&

$$\hat{\hat{\nu}}_k^L(\hat{\hat{Z}}_k(y)) = \begin{cases} 0 & \hat{\hat{Z}}_k(Y) \le \hat{\hat{L}}_k^{\nu}, \\ \eta\left(\frac{\hat{\hat{Z}}_k(y) - \hat{\hat{L}}_k^{\nu}}{\hat{\hat{U}}_k^{\nu} - \hat{\hat{L}}_k^{\nu}}\right) & \hat{\hat{L}}_k^{\nu} < \hat{\hat{Z}}_k(y) < \hat{\hat{U}}_k^{\nu}, \\ 1 & \hat{\hat{Z}}_k(y) \ge \hat{\hat{U}}_k^{\nu}. \end{cases}$$

where $\eta \in [0, 1]$, $k = 1, 2, \ldots, K$. $\hat{\hat{U}}_k^{\mu} = \hat{\hat{U}}_k^{\nu} = \hat{\hat{U}}_k$, $\hat{\hat{L}}_k^{\mu} = \hat{\hat{L}}_k$ and $\hat{\hat{L}}_k^{\nu} = \hat{\hat{L}}_k^{\mu} + t(\hat{\hat{U}}_k^{\mu} - \hat{\hat{L}}_k^{\mu})$, $t \in (0, 1)$. Here, t denotes the tolerance level which signifies the extent to which the DM is ready to compromise and accept. We can see that the acceptance and the rejection degrees are represented as intervals, *viz.*, $[\hat{\hat{\mu}}^L(y), \hat{\hat{\mu}}^U(y)]$ and $[\hat{\hat{\nu}}^L(y), \hat{\hat{\nu}}^U(y)]$, respectively.

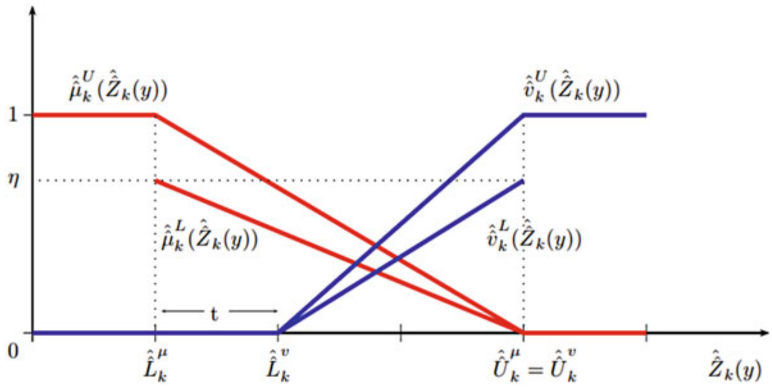

Fig. 5.2 Lower and upper linear MF and NMFs

Graphically, the above functions can be represented as in Fig. 5.2.

Step 5 We propose the following model to solve (FIVIFMOIQTP):
(FIVIFMOIQTP-2)

$$Max\ \ [\psi\lambda + (1-\psi)\xi - \psi\kappa - (1-\psi)\rho]$$

s.t.

$$\begin{aligned}
\hat{\mu}_k^U(\hat{\hat{Z}}_k(y)) &\geq \psi\lambda + (1-\psi)\xi, \\
\hat{\mu}_k^L(\hat{\hat{Z}}_k(y)) &\geq \lambda, \\
\hat{\nu}_k^U(\hat{\hat{Z}}_k(y)) &\leq \psi\kappa + (1-\psi)\rho, \\
\hat{\nu}_k^L(\hat{\hat{Z}}_k(y)) &\leq \kappa, \\
\psi\lambda + (1-\psi)\xi + \psi\kappa + (1-\psi)\rho &\leq 1, \\
\xi + \rho \leq 1; \quad \xi \geq \lambda; \quad \rho \geq \kappa; \quad \lambda \geq \kappa; \quad \xi &\geq \rho, \\
0 \leq \psi \leq 1; \quad 0 \leq \eta \leq 1; \quad 0 < t < 1; \quad \kappa &\geq 0, \\
\text{and all the constraints of (FIVIFMOIQTP-1).}
\end{aligned}$$

where λ is the minimum acceptance degree and κ is the maximum rejection.

Remark 3 (FIVIFMOIQTP-2) can also be simplified further to obtain
(FIVIFMOIQTP-3)

$$Max\ \ [\psi(\lambda - \kappa) + (1-\psi)(\xi - \rho)]$$

s.t.

$$\hat{\hat{U}}_k^{\mu} - \hat{\hat{Z}}_k(y) \geq (\hat{\hat{U}}_k^{\mu} - \hat{\hat{L}}_k^{\mu})\,(\psi\lambda + (1-\psi)\xi)\,,$$

$$\eta\left(\hat{\hat{U}}_k^{\mu} - \hat{\hat{Z}}_k(y)\right) \geq \left(\hat{\hat{U}}_k^{\mu} - \hat{\hat{L}}_k^{\mu}\right)\lambda,$$

$$\hat{\hat{Z}}_k(y) - \hat{\hat{L}}_k^{\nu} \leq (\hat{\hat{U}}_k^{\nu} - \hat{\hat{L}}_k^{\nu})\,(\psi\kappa + (1-\psi)\rho)\,,$$

$$\eta\left(\hat{\hat{Z}}_k(y) - \hat{\hat{L}}_k^{\nu}\right) \leq \left(\hat{\hat{U}}_k^{\nu} - \hat{\hat{L}}_k^{\nu}\right)\kappa,$$

$$\psi\lambda + (1-\psi)\xi + \psi\kappa + (1-\psi)\rho \leq 1,$$

$$\xi + \rho \leq 1; \quad \xi \geq \lambda; \quad \rho \geq \kappa; \quad \lambda \geq \kappa; \quad \xi \geq \rho,$$

$$0 \leq \psi \leq 1; \quad 0 \leq \eta \leq 1; \quad 0 < t < 1; \quad \kappa \geq 0,$$

and all the constraints of (FIVIFMOIQTP-1).

Step 6 The obtained optimal solution $y^{opt} = (y_{ij1}^U, y_{ij1}^L, y_{ij2}, y_{ij3}^L, y_{ij3}^U)$, $(y_{ij1}'^L, y_{ij1}'^U, y_{ij2}, y_{ij3}'^U, y_{ij3}'^L)$ can be substituted in the each OF, *viz.*, $\hat{Z}_K(y)\ \forall\ k$ to get the optimal OF value of the proposed (FIVIFMOIQTP).

6 Numerical Example

The application of the model is validated through the following numerical example taken from the food industry:

Nowadays, we order food items like juices (bottled/tetra-packed), dairy products like milk/tofu/cheese, frozen meat (temperature-controlled), etc. through any delivery app. The DM in order to optimize delivery to all the customers in a satisfactory and a legible time identifies a suitable (nearest) supply store which can serve requisite destinations efficiently. Consider two supply centers $SC1$ and $SC2$ and three destinations $D1$, $D2$, and $D3$. As any delivery app is used by multiple customers at any point of time, the DM is always unsure of the supply, the demand, and the quantity to be transported. Thus, we represent them by IVTIFNs.

Order placed has to go through packaging and delivery. The objectives under focus are:

- To simultaneously minimize the transportation cost $\hat{Z}_{11}(y)$ (due to fuel cost, travel duration, road conditions, etc.) and the damage cost $\hat{Z}_{12}(y)$ (due to spillage, negligence, breakage, etc.)
- To simultaneously minimize the packaging cost $\hat{Z}_{21}(y)$(due to cartons, bill receipt, bubble wraps, etc.) and the associated wastage cost $\hat{Z}_{22}(y)$(due to faulty packaging, poor designing, ill-trained staff, excessive filling, etc.)

Table 5.2 Related data—FIVIFMOIQTP

Sources \ Destinations	$D1$	$D2$	$D3$	Availabilities
$SC1$	$\hat{p}_{11}, \hat{q}_{11}$ $\hat{e}_{11}, \hat{f}_{11}$	$\hat{p}_{12}, \hat{q}_{12}$ $\hat{e}_{12}, \hat{f}_{12}$	$\hat{p}_{13}, \hat{q}_{13}$ $\hat{e}_{13}, \hat{f}_{13}$	$\hat{s}_1$
$SC1$	$\hat{p}_{21}, \hat{q}_{21}$ $\hat{e}_{21}, \hat{f}_{21}$	$\hat{p}_{22}, \hat{q}_{22}$ $\hat{e}_{22}, \hat{f}_{22}$	$\hat{p}_{23}, \hat{q}_{23}$ $\hat{e}_{23}, \hat{f}_{23}$	$\hat{s}_2$
Demands	$\hat{d}_1$	$\hat{d}_2$	$\hat{d}_3$	$\hat{s}_1 + \hat{s}_2 = \hat{d}_1 + \hat{d}_2 + \hat{d}_3$

Thus, we model an MOTP with minimization of two objective functions $\hat{Z}_1(y)$ and $\hat{Z}_2(y)$ which are indefinite quadratic in nature, *i.e.*, product of two linear factors and hence mathematically formulated as $\hat{Z}_1(y) = \hat{Z}_{11}(y)\hat{Z}_{12}(y)$ and $\hat{Z}_2(y) = \hat{Z}_{21}(y)\hat{Z}_{22}(y)$ as stated above. The cost coefficients by nature being imprecise and undetermined in nature are taken to be IVTIFNs.

Collectively, we can say that a FIVIFMOIQTP has been formulated. Mathematically stating,

$$\text{Min } \hat{Z}_1(y) = \hat{Z}_{11}(y)\hat{Z}_{12}(y) = \left(\sum_{i=1}^{2}\sum_{j=1}^{3} \hat{p}_{ij} \otimes \hat{y}_{ij}\right)\left(\sum_{i=1}^{2}\sum_{j=1}^{3} \hat{q}_{ij} \otimes \hat{y}_{ij}\right),$$

$$\text{Min } \hat{Z}_2(y) = \hat{Z}_{21}(y)\hat{Z}_{22}(y) = \left(\sum_{i=1}^{2}\sum_{j=1}^{3} \hat{e}_{ij} \otimes \hat{y}_{ij}\right)\left(\sum_{i=1}^{2}\sum_{j=1}^{3} \hat{f}_{ij} \otimes \hat{y}_{ij}\right)$$

subject to

$$\sum_{j=1}^{3} \hat{y}_{ij} = \hat{s}_i \quad ; \quad \sum_{i=1}^{2} \hat{y}_{ij} = \hat{d}_j \quad ; \hat{y}_{ij} \geq \hat{0}.$$

The related data is shown in Table 5.2

The values for $\hat{p}_{ij}$, $\hat{q}_{ij}$, $\hat{e}_{ij}$, and $\hat{f}_{ij}$ for $i = 1, 2$ and $j = 1, 2, 3$ are as follows:

$\hat{p}_{11} = (15, 10, 24, 29, 34), (4, 11, 24, 37, 40)$, $\quad \hat{s}_1 = (120, 130, 140, 150, 160)$ $(100, 110, 140, 170, 180)$,
$\hat{p}_{12} = (35, 44, 49, 57, 60), (29, 34, 49, 64, 69)$, $\quad \hat{s}_2 = (60, 70, 80, 90, 100)$, $(40, 50, 80, 110, 120)$,
$\hat{p}_{13} = (74, 79, 84, 89, 94), (64, 69, 84, 97, 100)$,
$\hat{p}_{21} = (34, 39, 44, 49, 54), (24, 29, 44, 59, 64)$, $\quad \hat{d}_1 = (50, 60, 70, 75, 80)$, $(40, 45, 70, 85, 90)$,

$\hat{p}_{22} = (49, 54, 59, 64, 71), (39, 44, 59, 74, 79)$, $\hat{d}_2 = (52, 55, 60, 65, 70), (30, 40, 60, 75, 85)$,
$\hat{p}_{23} = (24, 29, 34, 39, 44), (14, 19, 34, 49, 54)$, $\hat{d}_3 = (78, 85, 90, 100, 110), (70, 75, 90, 120, 125)$,

$\hat{q}_{11} = (5, 7, 9, 11, 13), (1, 3, 9, 15, 17)$,
$\hat{q}_{12} = (19, 24, 29, 34, 39), (9, 14, 29, 44, 49)$,
$\hat{q}_{13} = (21, 22, 23, 24, 25), (19, 20, 23, 26, 27)$,
$\hat{q}_{21} = (29, 34, 39, 44, 49), (19, 24, 39, 54, 59)$,
$\hat{q}_{22} = (11, 15, 19, 23, 27), (3, 7, 19, 31, 39)$,
$\hat{q}_{23} = (24, 34, 40, 54, 64), (10, 18, 40, 76, 80)$,
$\hat{e}_{11} = (4, 5, 6, 7, 8), (2, 3, 6, 9, 12)$,
$\hat{e}_{12} = (13, 15, 17, 19, 21), (9, 11, 17, 24, 26)$,
$\hat{e}_{13} = (16, 21, 26, 31, 36), (6, 11, 26, 41, 46)$,
$\hat{e}_{21} = (19, 20, 21, 22, 23), (17, 18, 21, 24, 25)$,
$\hat{e}_{22} = (10, 13, 16, 19, 22), (4, 7, 16, 25, 28)$,
$\hat{e}_{23} = (27, 29, 31, 33, 35), (23, 25, 31, 37, 39)$,
$\hat{f}_{11} = (7, 9, 11, 13, 15), (3, 5, 11, 17, 19)$,
$\hat{f}_{12} = (21, 26, 31, 36, 41), (11, 16, 31, 46, 51)$,
$\hat{f}_{13} = (23, 24, 25, 26, 27), (21, 22, 25, 28, 29)$,
$\hat{f}_{21} = (31, 36, 41, 46, 51), (21, 26, 41, 56, 61)$,
$\hat{f}_{22} = (13, 17, 21, 25, 29), (5, 9, 21, 33, 37)$,
$\hat{f}_{23} = (26, 36, 46, 56, 66), (6, 18, 46, 70, 80)$.

Solution process as per the algorithm is as follows:

Step 1 Divide the above problem into two subproblems where each subproblem is a single objectve TP with OF as $EVal\left(\hat{Z}_1(y)\right) = \hat{\hat{Z}}_1(y)$ and $EVal\left(\hat{Z}_2(y)\right) = \hat{\hat{Z}}_2(y)$ and the constraint sets of (FIVIFMOIQTP-1) above. In the end, we will obtain two optimal solutions, Y_1 and Y_2 corresponding to the above two subproblems.

Step 2 Construct the payoff matrix by finding the values of $\hat{\hat{Z}}_1(Y_1), \hat{\hat{Z}}_2(Y_1)$, $\hat{\hat{Z}}_1(Y_2)$, and $\hat{\hat{Z}}_2(Y_2)$.

Step 3 From the payoff matrix, we obtain $\hat{\hat{U}}_1^{\mu} = \hat{\hat{U}}_1^{\nu} = 56942010$, $\hat{\hat{U}}_2^{\mu} = \hat{\hat{U}}_2^{\nu} = 26783470$, $\hat{\hat{L}}_1^{\mu} = 56287590$, $\hat{\hat{L}}_2^{\mu} = 20700150$, $\hat{\hat{L}}_1^{\nu} = 56287590 + t(654420)$, $\hat{\hat{L}}_2^{\nu} = 20700150 + t(6083320)$, $0 < t < 1$.

Step 4 Formulate upper and lower linear MFs and NMFs:

$$\hat{\mu}_1^U(\hat{\hat{Z}}_1(Y)) = \begin{cases} 1 & \hat{\hat{Z}}_1(Y) \le \hat{\hat{L}}_1^{\mu}, \\ \frac{56942010 - \hat{\hat{Z}}_1(Y)}{654420} & \hat{\hat{L}}_1^{\mu} < \hat{\hat{Z}}_1(Y) < \hat{\hat{U}}_1^{\mu}, \\ 0 & \hat{\hat{Z}}_1(Y) \ge \hat{\hat{U}}_1^{\mu}. \end{cases}$$

$$\hat{\mu}_1^L(\hat{\hat{Z}}_1(Y)) = \begin{cases} 1 & \hat{\hat{Z}}_1(Y) \le \hat{\hat{L}}_1^{\mu}, \\ \eta\left(\frac{56942010-\hat{\hat{Z}}_1(Y)}{654420}\right) & \hat{\hat{L}}_1^{\mu} < \hat{\hat{Z}}_1(Y) < \hat{\hat{U}}_1^{\mu}, \\ 0 & \hat{\hat{Z}}_1(Y) \ge \hat{\hat{U}}_1^{\mu}. \end{cases}$$

$$\hat{\nu}_1^U(\hat{\hat{Z}}_1(Y)) = \begin{cases} 0 & \hat{\hat{Z}}_1(Y) \le \hat{\hat{L}}_1^{\nu}, \\ \frac{\hat{\hat{Z}}_1(Y)-[56287590+t(654420)]}{(1-t)654420} & \hat{\hat{L}}_1^{\nu} < \hat{\hat{Z}}_1(Y) < \hat{\hat{U}}_1^{\nu}, \\ 1 & \hat{\hat{Z}}_1(Y) \ge \hat{\hat{U}}_1^{\nu}. \end{cases}$$

$$\hat{\nu}_1^L(\hat{\hat{Z}}_1(Y)) = \begin{cases} 0 & \hat{\hat{Z}}_1(Y) \le \hat{\hat{L}}_1^{\nu}, \\ \eta\frac{\hat{\hat{Z}}_1(Y)-[56287590+t(654420)]}{(1-t)654420} & \hat{\hat{L}}_1^{\nu} < \hat{\hat{Z}}_1(Y) < \hat{\hat{U}}_1^{\nu}, \\ 1 & \hat{\hat{Z}}_1(Y) \ge \hat{\hat{U}}_1^{\nu}. \end{cases}$$

$$\hat{\mu}_2^U(\hat{\hat{Z}}_2(Y)) = \begin{cases} 1 & \hat{\hat{Z}}_2(Y) \le \hat{\hat{L}}_2^{\mu}, \\ \frac{26783470-\hat{\hat{Z}}_2(Y)}{6083320} & \hat{\hat{L}}_2^{\mu} < \hat{\hat{Z}}_2(Y) < \hat{\hat{U}}_2^{\mu}, \\ 0 & \hat{\hat{Z}}_2(Y) \ge \hat{\hat{U}}_2^{\mu}. \end{cases}$$

$$\hat{\mu}_2^L(\hat{Z}_2(Y)) = \begin{cases} 1 & \hat{\hat{Z}}_2(Y) \le \hat{\hat{L}}_2^{\mu}, \\ \eta\left(\frac{26783470-\hat{\hat{Z}}_2(Y)}{6083320}\right) & \hat{\hat{L}}_2^{\mu} < \hat{\hat{Z}}_2(Y) < \hat{\hat{U}}_2^{\mu}, \\ 0 & \hat{\hat{Z}}_2(Y) \ge \hat{\hat{U}}_2^{\mu}. \end{cases}$$

$$\hat{\nu}_2^U(\hat{Z}_2(Y)) = \begin{cases} 0 & \hat{\hat{Z}}_2(Y) \le \hat{\hat{L}}_2^{\nu}, \\ \frac{\hat{\hat{Z}}_2(Y)-[20700150+t(6083320)]}{(1-t)6083320} & \hat{\hat{L}}_2^{\nu} < \hat{\hat{Z}}_2(Y) < \hat{\hat{U}}_2^{\nu}, \\ 1 & \hat{\hat{Z}}_2(Y) \ge \hat{\hat{U}}_2^{\nu}. \end{cases}$$

$$\hat{\nu}_2^L(\hat{Z}_2(Y)) = \begin{cases} 0 & \hat{\hat{Z}}_2(Y) \le \hat{\hat{L}}_2^{\nu}, \\ \eta\frac{\hat{\hat{Z}}_2(Y)-[20700150+t(6083320)]}{(1-t)6083320} & \hat{\hat{L}}_2^{\nu} < \hat{\hat{Z}}_2(Y) < \hat{\hat{U}}_2^{\nu}, \\ 1 & \hat{\hat{Z}}_2(Y) \ge \hat{\hat{U}}_2^{\nu}. \end{cases}$$

Step 5 The final crisp model to be solved is

$$Max \ \ [\psi(\lambda-\kappa)+(1-\psi)(\xi-\rho)] \tag{5.1}$$

s.t.

$$\hat{\hat{Z}}_1(Y) + 654420(\psi\lambda + (1-\psi)\xi) \le 56942010,$$

$$\hat{\hat{Z}}_2(Y) + 6083320(\psi\lambda + (1-\psi)\xi) \le 26783470, \eta\hat{\hat{Z}}_1(Y) + 654420\lambda \le \eta.56942010,$$

$$\eta\hat{\hat{Z}}_2(Y) + 6083320\lambda \leq \eta.26783470,$$

$$\hat{\hat{Z}}_1(Y) - t(654420) - (1-t)(\psi\kappa + (1-\psi)\rho)654420 \leq 56287590,$$

$$\hat{\hat{Z}}_2(Y) - t(6083320) - (1-t)(\psi\kappa + (1-\psi)\rho)6083320 \leq 20700150,$$

$$\eta\hat{\hat{Z}}_1(Y) - \eta t(654420) - \kappa(1-t)654420 \leq \eta.56287590,$$

$$\eta\hat{\hat{Z}}_2(Y) - \eta t(6083320) - \kappa(1-t)6083320 \leq \eta.20700150,$$

$$\psi\lambda + (1-\psi)\xi + \psi\kappa + (1-\psi)\rho \leq 1,$$

$$\xi + \rho \leq 1; \quad \xi \geq \lambda; \quad \xi \geq \rho; \quad \rho \geq \kappa \geq 0, \quad \lambda \geq \kappa \geq 0,$$

$$0 \leq \psi \leq 1; \quad 0 \leq \eta \leq 1; \quad t = 0.11,$$

$$\sum_j y^U_{1j1} = 120, \ \sum_j y^L_{1j1} = 130, \ \sum_j y_{1j2} = 140, \ \sum_j y^L_{1j3} = 150,$$

$$\sum_j y^U_{1j3} = 160,$$

$$\sum_j y'^L_{1j1} = 100, \sum_j y'^U_{1j1} = 110, \sum_j y'^U_{1j3} = 170, \ \sum_j y'^L_{1j3} = 180,$$

$$\sum_i y^U_{i11} = 50, \ \sum_i y^L_{i11} = 60, \ \sum_i y_{i12} = 70, \ \sum_i y^L_{i13} = 75,$$

$$\sum_i y^U_{i13} = 80,$$

$$\sum_i y'^L_{i11} = 40, \ \sum_i y'^U_{i11} = 45, \ \sum_i y'^U_{i13} = 85, \ \sum_i y'^L_{i13} = 90,$$

$$\sum_j y^U_{2j1} = 60, \ \sum_j y^L_{2j1} = 70, \ \sum_j y_{2j2} = 80, \ \sum_j y^L_{2j3} = 90,$$

$$\sum_j y^U_{2j3} = 100, \ \sum_j y'^L_{2j1} = 40, \ \sum_j y'^U_{2j1} = 50,$$

$$\sum_j y'^U_{2j3} = 110, \ \sum_j y'^L_{2j3} = 120, \ \sum_i y^U_{i21} = 52, \ \sum_i y^L_{i21} = 55,$$

$$\sum_i y_{i22} = 60, \ \sum_i y^L_{i23} = 65, \ \sum_i y^U_{i23} = 70,$$

$$\sum_i y'^L_{i21} = 30, \ \sum_i y'^U_{i21} = 40, \ \sum_i y'^U_{i23} = 75, \ \sum_i y'^L_{i23} = 85,$$

$$\sum_i y_{i31}^U = 78, \quad \sum_i y_{i31}^L = 85, \quad \sum_i y_{i32} = 90, \quad \sum_i y_{i33}^L = 100,$$

$$\sum_i y_{i33}^U = 110,$$

$$\sum_i y_{i31}'^L = 70, \quad \sum_i y_{i31}'^U = 75, \quad \sum_i y_{i33}'^U = 120, \quad \sum_i y_{i33}'^L = 125,$$

$$y_{1j1}'^L \geq 0, \quad y_{1j1}'^U - y_{1j1}'^L \geq 0, \quad y_{1j1}^U - y_{1j1}'^U \geq 0, \quad y_{1j1}^L - y_{1j1}^U \geq 0,$$

$$y_{1j2} - y_{1j1}^L \geq 0, \quad y_{1j3}^L - y_{1j2} \geq 0, \quad y_{1j3}^U - y_{1j3}^L \geq 0, \quad y_{1j3}'^U - y_{1j3}^U \geq 0,$$

$$y_{1j3}'^L - y_{1j3}'^U \geq 0, \quad j = 1, 2, 3,$$

$$y_{2j1}'^L \geq 0, \quad y_{2j1}'^U - y_{2j1}'^L \geq 0, \quad y_{2j1}^U - y_{2j1}'^U \geq 0, \quad y_{2j1}^L - y_{2j1}^U \geq 0,$$

$$y_{2j2} - y_{2j1}^L \geq 0, \quad y_{2j3}^L - y_{2j2} \geq 0, \quad y_{2j3}^U - y_{2j3}^L \geq 0, \quad y_{2j3}'^U - y_{2j3}^U \geq 0,$$

$$y_{2j3}'^L - y_{2j3}'^U \geq 0, \quad j = 1, 2, 3.$$

The values of the OF (5.1) for different values of η, ψ, and $t = 0.11$ are shown in Table 5.3.

Step 6 It is clear from Table 5.3 that the maximum value of the OF (5.1) is obtained at $(\eta, \psi) = (0.9, 0.1)$.

We obtain the optimal solution as $y^{opt} =$
$\hat{y}_{11} = (y_{111}^U, y_{111}^L, y_{112}, y_{113}^L, y_{113}^U), (y_{111}'^L, y_{111}'^U, y_{112}, y_{113}'^U, y_{113}'^L) =$ (50, 60, 70, 75, 80), (40, 45, 70, 85, 90),
$\hat{y}_{12} = (y_{121}^U, y_{121}^L, y_{122}, y_{123}^L, y_{123}^U), (y_{121}'^L, y_{121}'^U, y_{122}, y_{123}'^U, y_{123}'^L) =$ (2, 2, 2, 2, 2), (0, 0, 2, 2, 2),
$\hat{y}_{13} = (y_{131}^U, y_{131}^L, y_{132}, y_{133}^L, y_{133}^U), (y_{131}'^L, y_{131}'^U, y_{132}, y_{133}'^U, y_{133}'^L) =$ (68, 68, 68, 73, 78), (60, 65, 68, 83, 88),
$\hat{y}_{21} = (y_{211}^U, y_{211}^L, y_{212}, y_{213}^L, y_{213}^U), (y_{211}'^L, y_{211}'^U, y_{212}, y_{213}'^U, y_{213}'^L) =$ (0, 0, 0, 0, 0), (0, 0, 0, 0, 0),

Table 5.3 OF values for different values of η, ψ, and $t = 0.11$

$\psi \downarrow$ \ $\eta \rightarrow$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	0.91	0.92	0.93	0.94	0.95	0.9599992	0.9700001	0.9800001	**0.9900001**
0.2	0.82	0.8400001	0.8600001	0.8800001	0.9000001	0.9200003	0.9400001	0.9600001	0.9800002
0.3	0.73	0.7600001	0.7900002	0.8200001	0.8500001	0.8800004	0.9100002	0.9400002	0.9700003
0.4	0.6400001	0.6800002	0.7200003	0.7600001	0.8000002	0.8400006	0.8800002	0.9200002	0.9600003
0.5	0.5500001	0.6000002	0.6500004	0.7000002	0.7500002	0.8000007	0.8500003	0.9000003	0.9500004
0.6	0.4600001	0.5200003	0.5800004	0.6400002	0.7000002	0.7600004	0.8200003	0.8800004	0.9400004
0.7	0.3700001	0.4400001	0.5100005	0.5800002	0.6500003	0.7200010	0.7900004	0.8600004	0.9300005
0.8	0.2800001	0.3600004	0.4400006	0.5200003	0.6000003	0.6800012	0.7600004	0.8400005	0.9200006
0.9	0.1900001	0.2800004	0.3700007	0.4600003	0.5500004	0.6400013	0.7300005	0.8200006	0.9100006

Table 5.4 OF values for the solved numerical

Objective function	Meaning	Optimal value represented as IFN
$\hat{Z}_{11}$	Minimized transportation cost	(8542, 10015, 11660, 13871, 16408), (5310, 6930, 11660, 18539, 21093)
$\hat{Z}_{12}$	Minimized depreciation cost	(2506, 3337, 4234, 5552, 6952), (1370, 1895, 4234, 8596, 10201)
$\hat{Z}_{21}$	Minimized packaging cost	(2084, 2940, 3832, 4914, 6106), (790, 1380, 3832, 7410, 8947)
$\hat{Z}_{22}$	Minimized wastage cost	(2866, 3737, 4762, 6032, 7472), (1590, 2257, 4762, 8860, 10395)
$\hat{Z}_1$	$\hat{Z}_{11}\hat{Z}_{12}$	(8542, 10015, 11660, 13871, 16408), (5310, 6930, 11660, 18539, 21093). (2506, 3337, 4234, 5552, 6952), (1370, 1895, 4234, 8596, 10201)
$\hat{Z}_2$	$\hat{Z}_{21}\hat{Z}_{22}$	(2084, 2940, 3832, 4914, 6106), (790, 1380, 3832, 7410, 8947). (2866, 3737, 4762, 6032, 7472), (1590, 2257, 4762, 8860, 10395)

$\hat{y}_{22} = (y^U_{221}, y^L_{221}, y_{222}, y^L_{223}, y^U_{223}), (y'^L_{221}, y'^U_{221}, y_{222}, y'^U_{223}, y'^L_{223}) = (50, 53, 58, 63, 68), (30, 40, 58, 73, 83),$
$\hat{y}_{23} = (y^U_{231}, y^L_{231}, y_{232}, y^L_{233}, y^U_{233}), (y'^L_{231}, y'^U_{231}, y_{232}, y'^U_{233}, y'^L_{233}) = (10, 17, 22, 27, 32), (10, 10, 22, 37, 37).$
Substituting these values in the objective functions, we obtain the value $\hat{Z}_1 = 56287590$ and $\hat{Z}_2 = 20700100$ of each objective function. Individual OF values represented as IFNs are tabulated in Table 5.4.

7 Discussion

We have also solved our numerical problem using the weight-based approach as well [30]. The model used is
(FIVIFMOIQTP-4)

$$Max\ \ [\psi(\lambda-\kappa)+(1-\psi)(\xi-\rho)-(w_1(d_1^+ + d_1^-)+w_2(d_2^+ + d_2^-)+\cdots + w_k(d_k^+ + d_k^-))] \tag{5.2}$$

s.t.

$$\hat{\hat{Z}}_k(y) - d_k^+ + d_k^- = \hat{\hat{G}}_k,$$

$$d_k^+, d_k^-, k = 1, 2, \ldots, K,$$

$$w_1 + w_2 + \ldots + w_k = 1,$$

along with the constraints of (FIVIFMOIQTP-3).

where $w_1, w_2, \ldots, w_K$ are the weights based on the priority levels of objectives such that their sum equals one. $\hat{\hat{G}}_1, \hat{\hat{G}}_2, \ldots, \hat{\hat{G}}_K$ are the aspiration levels of the k^{th} objective function. d_k^+, d_k^- are the deviation variables associated with the OF $\hat{\hat{Z}}_k(y)$. The methodology of solving the numerical is the same as in Sect. 6 except for the usage of above model FIVIFMOIQTP-4 instead of FIVIFMOIQTP-3.

Thus, proceeding as before, the value of the OF (5.2) of the above weight-based model for different values of (η, ψ) and using $w_1 = 0.2$, $w_2 = 0.8$, and $t = 0.3993$ is shown in Table 5.5.

From Table 5.5, it is clear that the maximum value of the OF (5.2) is obtained for $\eta = 0.9$, $\psi = 0.1$.

Optimal compromise solution so obtained is $y^{opt} =$
$\hat{y}_{11} = (y_{111}^U, y_{111}^L, y_{112}, y_{113}^L, y_{113}^U), (y_{111}'^L, y_{111}'^U, y_{112}, y_{113}'^U, y_{113}'^L) =$ (50, 60, 70, 75, 80), (40, 45, 70, 85, 90),
$\hat{y}_{12} = (y_{121}^U, y_{121}^L, y_{122}, y_{123}^L, y_{123}^U), (y_{121}'^L, y_{121}'^U, y_{122}, y_{123}'^U, y_{123}'^L) =$ (8, 8, 8, 8, 8), (6, 6, 8, 8, 8),
$\hat{y}_{13} = (y_{131}^U, y_{131}^L, y_{132}, y_{133}^L, y_{133}^U), (y_{131}'^L, y_{131}'^U, y_{132}, y_{133}'^U, y_{133}'^L) =$ (62, 62, 62, 67, 72), (54, 59, 62, 77, 82),
$\hat{y}_{21} = (y_{211}^U, y_{211}^L, y_{212}, y_{213}^L, y_{213}^U), (y_{211}'^L, y_{211}'^U, y_{212}, y_{213}'^U, y_{213}'^L) =$ (0, 0, 0, 0, 0), (0, 0, 0, 0, 0),
$\hat{y}_{22} = (y_{221}^U, y_{221}^L, y_{222}, y_{223}^L, y_{223}^U), (y_{221}'^L, y_{221}'^U, y_{222}, y_{223}'^U, y_{223}'^L) =$ (44, 47, 52, 57, 62), (24, 34, 52, 67, 77),
$\hat{y}_{23} = (y_{231}^U, y_{231}^L, y_{232}, y_{233}^L, y_{233}^U), (y_{231}'^L, y_{231}'^U, y_{232}, y_{233}'^U, y_{233}'^L) =$ (16, 23, 28, 33, 38), (16, 16, 28, 43, 43).

Table 5.5 OF values for different values of η, ψ, and $t = 0.3993$, $w_1 = 0.2$, $w_2 = 0.8$

$\eta \rightarrow$ / $\psi \downarrow$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	Infeasible	Infeasible	-1681988	Infeasible	-1681982	-1681988	Infeasible	-1681982	**-1681982**
0.2	Infeasible	-1681988	-1681988	Infeasible	-1689182	-1681988	Infeasible	-1681982	-1681982
0.3	Infeasible	-1684131	-1681988	Infeasible	-1681982	-1681988	Infeasible	-1681982	-1681982
0.4	Infeasible	-1716505	-1704617	Infeasible	-1682140	-1681988	Infeasible	-1681982	-1681982
0.5	Infeasible	-1747537	-1732330	Infeasible	-1703934	-1690656	Infeasible	-1681982	-1681982
0.6	Infeasible	-1777315	-1758700	Infeasible	-1724351	-1708457	Infeasible	-1681982	Infeasible
0.7	Infeasible	-1805927	-1783832	-1763072	-1743520	-1725064	Infeasible	Infeasible	Infeasible
0.8	-1865150	-1833448	-1807816	Infeasible	-1761553	-1740588	Infeasible	Infeasible	Infeasible
0.9	-1902436	-1863887	-1830767	Infeasible	-1778557	-1755133	Infeasible	-1712759	-1693522

Table 5.6 OF values for the solved numerical

Objective function	Meaning	Optimal value represented as IFN
$\hat{Z}_{11}$	Minimized transportation cost	(8158, 9655, 11300, 13529, 16042), (4950, 6570, 11300, 18191, 20757)
$\hat{Z}_{12}$	Minimized depreciation cost	(2572, 3463, 4396, 5798, 7258), (1352, 1925, 4396, 8974, 14108)
$\hat{Z}_{21}$	Minimized packaging cost	(2168, 3000, 3868, 4926, 6094), (922, 1488, 3868, 7380, 8893)
$\hat{Z}_{22}$	Minimized wastage cost	(2932, 3863, 4948, 6278, 7778), (1536, 2213, 4948, 9190, 10785)
$\hat{Z}_1$	$\hat{Z}_{11}\hat{Z}_{12}$	(8158, 9655, 11300, 13529, 16042), (4950, 6570, 11300, 18191, 20757). (2572, 3463, 4396, 5798, 7258), (1352, 1925, 4396, 8974, 14108)
$\hat{Z}_2$	$\hat{Z}_{21}\hat{Z}_{22}$	(2168, 3000, 3868, 4926, 6094), (922, 1488, 3868, 7380, 8893). (2932, 3863, 4948, 6278, 7778), (1536, 2213, 4948, 9190, 10785)

Substituting these values in the objective functions, we obtain the value $\hat{Z}_1 = 56688120$, $\hat{Z}_2 = 21636420$ of each objective function. Individual OF values represented as IFNs are tabulated in Table 5.6.

8 Conclusion and Future Work

Uncertainty and lack of judgment have motivated the researchers to use fuzzy sets and its extensions in the domain of transportation problems. Many have worked on fuzzy sets, IF sets, and single objective/multi-objective TP. In this work, we have tried to formulate and solve a FIVIFMOIQTP. The benefit of using an IVIFN to represent cost, supply, demand, and quantity is that the degree of acceptance and rejection is represented as an interval rather than crisp which gives more flexibility and wider range of choice to the DM. An indefinite quadratic objective function can simultaneously minimize each of its linear factors. Last but not the least incorporating multiple objectives makes the model multifaceted and ready to use in any domain of the transportation sector with conflicting objectives. The model is also validated using a numerical example and is solved using two approaches with the help of the software $LINGO$19.0 using Intel Processor i5 with 8 GB RAM on 64-bit Windows OS. The results are discussed as well. Thus, we have presented a comprehensive model which can purposely be reused as per the need of the DM. For example, instead of all the parameters as IVIF, the DM can be selective in choosing some parameters as crisp; instead of multiple objectives, single objective can be taken; and the same solution methodology serves. Some other fields where this model fits are the clothing industry and the medical industry where garments, medicines, etc. are packed and transported.

The work can be extended to a bilevel FIVIFMOIQTP where a different objective function can be taken at the lower level like fractional or fixed-charge as per the choice of the DM. A real-life transportation model with more objective functions, sources, and destinations can also be formulated and solved. However, the proposed work comes with a limitation that the model will be difficult to handle in case pentagonal IFNs or octagonal IFNs are used in place of TIFNs.

Acknowledgments The authors are readily grateful to the reviewers for their valuable suggestions, incorporating which has helped us in improving the quality of the paper to a great extent.

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Chapter 6
Project Management Using Network Analysis in Fuzzy Environment

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1 Introduction

A project is a well-defined sequence of tasks that must be completed to attain a certain outcome. It consists of jobs, tasks, or activities, all that be completed to finish the project. On the other hand, project management involves skills, methods, knowledge, experience, etc. to meet the certain goals as per requirement of the project acceptance criteria along with specified parameters. It has some final deliverable restriction to a finite time period, budget, and many other related issues. Construction of a building, shopping mall highway, and bridge and setting up a new mobile network, power plant, research and development work, and production and sales of new products are a few examples of the project. A project involves many interrelated task (or activities), and all the task should be finished within the specific time period, with a specific order (or sequence), and require resources such as manpower, materials, money, space, facilities, etc. The main aim of a project is to perform all the activities involved in the project in a proper sequence such as the following:

- Complete the project in a predefined time
- Minimize the following:
  - Total project completion time
  - Total completion time for a specified cost
  - Total cost for a predefined time
  - Total project completion cost
  - Idle resources

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C. Jana et al. (eds.), *Fuzzy Optimization, Decision-making and Operations Research*, https://doi.org/10.1007/978-3-031-35668-1_6

Therefore, at the beginning, it is required to prepare a sequence of the tasks for scheduling and managing all the tasks. The techniques for scheduling, planning, and managing complex and large projects are known as network planning, network analysis, and network scheduling methods. Since a project is a collection of activities and an appropriate sequence of activities is beneficial, a project is generally represented as a network (a weighted directed graph). This process has the advantage for visualization the entire project containing several tasks. These are easily be defined with the help of parameters, such as cost of a task, its completion time, and its starting time. The order of the task must be defined.

Two fundamental planning and control process are available in a network analysis to complete a predetermined project. These are PERT (project evaluation and review technique) and CPM (critical path method).

The PERT was developed in 1957 by the US Navy's Polaris Nuclear Submarine Missile development project. Later, this method is used in different industries/organizations. The PERT is successfully applied in the organization of the Winter Olympics in 1968. This technique has been used for planning different types of projects, mainly, when the duration of the activity is probabilistic. This project management method determines the real time to complete the entire project. This technique is appropriate for complex projects with many non-routine tasks and large projects with complex requirements. Until now, more than 200 papers have been published related to the PERT, and even then more research is required on this area.

In 1957 [12], the critical path method (CPM) was developed by the largest chemical company known as DuPont Company. Two mathematicians developed a method whose target was to avoid the added costs related to the scheduling of shutting down and restarting plants. The CPM is originally a scheduling algorithm which identifies the critical tasks/activities in the longest sequence of tasks in a network. The critical tasks are vital for attending the project deadline. The CPM is suitable when a project consists of several interconnected and interdependent tasks, repetitive tasks, or projects with strict timelines and deadlines (e.g., building of a bridge, software development, etc.).

Other project management techniques are

1. Work breakdown structure (WBS): In this method, a big project divided the whole work into small tasks which are easily manageable, and this process is known as a work breakdown structure.
2. Scrum: Scrum is a very common technique in the Agile methodology. In this method, the project is divided into a series of cycles as sprints.
3. Scaled Agile framework (SAFe): This process is basically the Agile project management implemented at scale.
4. Kanban: In this method, the work flow of a project is divided into small tasks. All these tasks are then organized and displayed such that everybody on the team knows the progress of the project.
5. Gantt: A Gantt chart is basically a bar chart which displayed the project tasks over time. It is helpful as it shows which specific work needs to be finished on a specified time.

6. Waterfall: This is an organized and linear approach for managing and controlling the projects.
7. PEP (performance evaluation program): In this method, evaluation can help to identify areas of application that need improvement and determine whether the project is achieving its goals or objectives.

2 Stages of Project Management

The tasks associated in a project are performed in three stages/phases which are planning, scheduling, and control.

2.1 Planning

In this stage, the objectives of the project are settled, and the assumptions related to the project are identified. Also, in this stage, all the tasks or activities or jobs are listed, and these must be completed to finish the entire project.

2.2 Scheduling

In this process, the activities are ordered according to their appearance. The following steps are performed in this phase:

1. The starting and ending time for each task
2. The critical path on which the tasks need special care
3. The float and slack times for other paths which are not critical

2.3 Controlling

The controlling stage is performed after completion of the previous two stages. This stage involves the following tasks:

1. Preparing progress reports periodically
2. Identifying the progress of the project
3. Analyzing the status
4. Taking decisions for resource allocation, modification, crashing, etc.

3 Advantages of Network Analysis

Several methods are mentioned at the beginning of this chapter for project management. But the network analysis technique has the following advantages for project management:

This method shows the interrelationship among activities in the project. It also provides a complete idea of controlling the sequence of performance of the tasks. It is evident that the pictorial approach is a better process for clarifying verbal instructions. This method identifies activities which are critical for a project. The manager can forecast the required exact resources from the network corresponding to the project and also maintain the resource allocation to attend the critical condition and to maintain or minimize the total project completion cost. This method also integrates all project components to whatever detail the management desires. It connects time to cost, which allows a monetary value which is placed when changes are needed.

3.1 Basic Components of a Network and Some Terminologies

In a network, the events/nodes and activities (also known as jobs and tasks) are the fundamental components.

3.2 Event or Node

A node or an event or a point is a particular moment that depicts the start or end of a single or multiple tasks. This is a position of decision or accomplishment. The start and end points of a task are displayed by two nodes, generally known as the tail and head events, respectively. Generally, a small circle represents an event; other symbols may also be used, viz., rectangles, hexagons, etc. These geometric symbols are numbered for unique representations of an activity. Reaching an event means the work has been completed up to that event.

3.3 Activity or Task

Two nodes are joined by a directed arc or edge representing an activity or a task that consumes time duration, money, material, or other types of resources.

An activity is identified with two nodes called tail (starting event) and head (ending event). Usually, two integers, i and j, represent an activity (i, j) where (i, j) is a directed edge and hence (i, j) is an ordered pair. The integers i and j

represent the tail and head of the activity (i, j) (see in Fig. 6.1). Upper case alphabets generally denote the activities.

Depending on the preference and nature, the activities are classified as follows:

1. *Predecessor Activity:* An activity (i, j) is called predecessor activity if it finished before starting of all activities those are end at j.
2. *Successor Activity:* An activity (i, j) is called successor activity if it starts immediately after finishing all the activities those are end at i.
3. *Dummy Activity:* In a network, there is only one source node and only one destination node. Sometimes, it happened that one or more activities finished in a nondestination node or there is no successor activity. In this case, a new activity is incorporated in the network which joins such type of activity with a destination node or any other suitable node; this activity is called dummy activity. The time or cost or other resources of this activity is considered as zero. The dotted line depicts it in the network diagram.

3.4 Merge and Burst Events

If more than one task/activity ends in an event, then the event is called a merge event. If one or more activities start from an event, this event is called a burst event (see Fig. 6.2).

3.5 Network

A network is the complete diagrammatical representation of a project where directed edges (arrows) and nodes are connected logically and sequentially that represent activities and events. This is nothing but a directed graph with edge weights.

Fig. 6.1 Activity and events/nodes

Fig. 6.2 Merge and burst events

3.6 Path

A connected chain of activities and nodes of a network whose starting and ending nodes are different is known as a path.

4 Common Errors in a Network

The construction of a network is not an easy task. There are some systematic rules to the construction of a network. The network must be free from looping, dangling, and redundancy.

4.1 Looping (Cycling)

The network diagram must be drawn in such a way that it is loopless. If the diagram contains one or more loops, then the project cannot be completed and repeated for infinite time, cost, etc. Also, if there is a loop in the network, it is impossible to find a path from the starting node to the end node. A looping network is given in Fig. 6.3.

4.2 Dangling

It is mentioned that no activities can be ended to a node other than destination node. This activity disconnects the network and this situation is called as dangling. It must be avoided. In this situation, a dummy activity is incorporated to maintain the connectivity of the network (see Fig. 6.4).

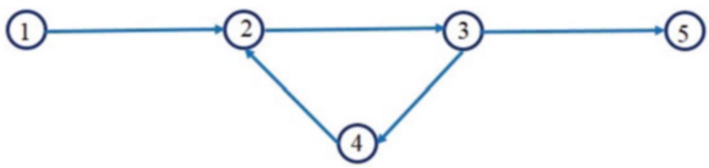


Fig. 6.3 Looping

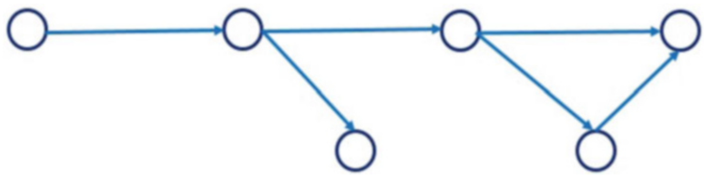


Fig. 6.4 Dangling

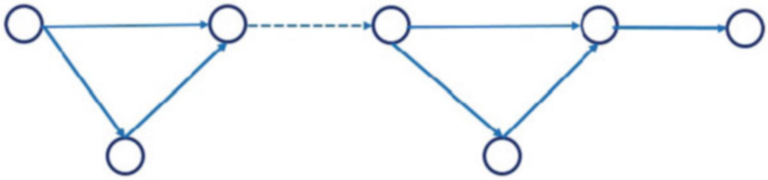


Fig. 6.5 Redundancy

4.3 Redundancy

If in between two nodes there is only a dummy activity, then this dummy activity can be removed by merging the end nodes. This is called redundancy (see Fig. 6.5).

5 Rules to Construct Network

Generally, the following rules are followed to construct a network:

1. A unique arrow or directed edge represents each activity.
2. Crossing between the edges must be avoided. Try to draw straight edges.
3. Each activity has unique tail and head nodes.
4. After completion of all activities preceding a node, a new node initiated.
5. A node occurs only once, i.e., that is, the network is free from any loop.
6. An activity succeeding in a node cannot be started until all the activities merge into this event.
7. The nodes are numbered with different integers, i.e., two nodes have two distinct integers. Therefore, an activity is identified uniquely and unambiguity. The number of the starting node of an activity must be less than that of the ending node.
8. For any two nodes, there either is no activity or has only one activity.
9. Dummy activity must be avoided. It increases the size of the network. So, dummy activity will be included only if it is extremely required. That is, the unnecessary use of dummy activities will increase the complexity of the network.
10. The project network has unique entry (source) node and unique terminal (destination) node.

5.1 Numbering the Events

After the construction of the network as per the occurrence of the activities, all the nodes are labeled by numbers. There are specific rules for such numbering due to D. R. Fulkerson. This numbering reflects the flow of the network:

(i) Every node has a unique number.
(ii) Numbering of nodes must be done sequentially from left to right.
(iii) In a network, there is only one starting event, and the number assigned to that event is 1.
(iv) Delete all edges which are originating from the starting node 1. This will generate at least one initial node (no activities merge with it).
(v) Assign numbers 2, 3, ..., etc. to these initial nodes.
(vi) Remove all edges which are originated from these numbered events. This again creates new initial nodes.
(vii) Repeat steps (v) and (vi) until all the nodes get numbered.

The first step in network analysis is the construction of the network using the above rules. The next step is to prepare a planning schedule such that the total project completion time is minimum.

The following notations and symbols are used in the analysis of the network:

(i) The earliest occurrence time E_i of each event i, earliest finishing time F_{ij}, and latest beginning time B_{ij} of each activity (i, j) given by the edge e_{ij} and latest allowable occurrence time of each event i.
(ii) Slack $s_i = L_i - E_i$ for every event i. An event i is said to be a *critical event* if the slack s_i corresponding to it is zero. An activity between two critical events given by two adjacent nodes of the network is called a *critical activity*.
(iii) The *critical path* of a project, the network, is the path between the starting event and the final event, consisting of the critical events and the critical activities. The computation of critical path is a vital step in the sense that the total finishing time of the project will vary with the time duration of the activities on the critical path and with no others.

6 Project Management in Uncertain Environment

A network is a composition of two sets, set of vertices and edges. It is already mentioned that a network is nothing but a graph with edges or nodes or both that have weights. In fuzzy project management, the nodes (events) are certain and it is denoted as V; the set of edges (activities) $\mathscr{E}$ is a subset of $V \times V$ and the weight (time duration) of the edge (i, j) is uncertain, here TFN; and it is denoted by $\widetilde{t}_{ij}$. Finally, a fuzzy network is denoted as $\mathscr{S} = (V, \mathscr{E}, t)$. The nodes are numbered as $1, 2, \ldots, n$; n represents the total number of nodes in the network $\mathscr{S}$. For any edge (i, j), as per construction process, $i < j$.

Dubois and Prade [8] first analyzed the decision-making problems, viz., shortest-path problem and PERT/CPM, taking the edge weights as fuzzy numbers. Nayeem and Pal [20] developed a unified approach to solving fuzzy shortest-path problems with interval numbers and triangular fuzzy numbers (TFNs) as edge weights. Chanas and Kamburowski [2] developed a method for solving fuzzy PERT using the concept of extended addition and strong level sets. Later, Buckley [1] worked on fuzzy PERT. Mares et al. [13, 14] introduced the convolution law to aggregate the fuzzy numbers. In their method, all possible paths are generated to find the critical path, and obviously, it is very time-consuming. Chanas *et al.* [3, 4] investigated network flow problems where each edge has a fuzzy arc length. Chanas and Zielinski [5] nicely introduced the CPM in fuzzy networks. In the late 1970s ([21]), the fuzzy PERT or the fuzzy CPM was studied, where fuzzy numbers are used to mention the weight of the activities. Many papers have been published on fuzzy PERT; some of them are as follows: [1–3, 6, 9, 11, 16–19, 23–25]. The approach to the subject in all these papers is similar. Mazlum and Güneri [15] studied CPM, PERT, and project management using fuzzy logic.

In project management system, the weights of the activities are time duration, costs, etc. All these parameters are appropriately represented by fuzzy numbers. In this chapter, the TFNs are considered to represent the time duration, cost, etc. of the activities.

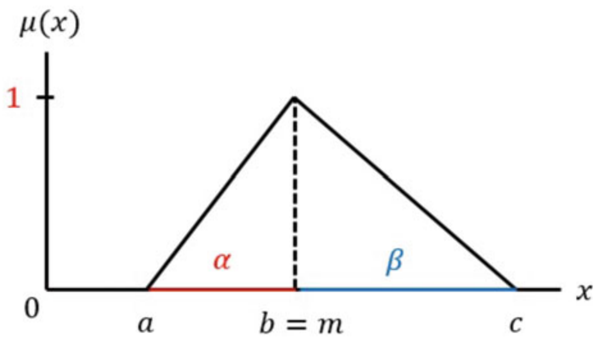
Suppose the time taken to complete a task is about 30 min. That means the chance to complete the said task in 30 min is very high. It may be completed in 28 min or it may take 33 min. The chance to complete the task from 33 min to 28 min gradually decreases. The same thing happens from 33 min to 33 min. This type of type duration can be represented by (28, 30, 33). Generally, a TFN is written in the form (a, b, c). $(b-a)$ and $(c-b)$ are referred as left and right spreads. If the spreads are less, uncertainty becomes less. A TFN is also written in a mean-spread form as $\langle m, \alpha, \beta \rangle$ where $m = b, \alpha = b - a,$ and $\beta = c - b$. Mathematically, a TFN is written as

$$\mu_A(x) = \begin{cases} \frac{x-a}{b-a}, \text{ if } a \le x \le b \\ 1, \text{ if } x = b \\ \frac{c-x}{c-b}, \text{ if } b \le x \le c \end{cases}$$

The value of the function $\mu_A(x)$ lies between 0 and 1. The value of the function $\mu_A(x)$ is called a membership value.

The diagrammatic representation is shown in Fig. 6.6.

Fig. 6.6 Diagram of triangular fuzzy number



6.1 Arithmetic of TFNs

Let $P = \langle m, \alpha, \beta\rangle$ and $Q = \langle m, \gamma, \delta\rangle$ be two TFNs.

If $m > 0$, then the $P > 0$; if $m < 0$, $P < 0$; and if $m = 0$, then $P = 0$.

Then the basic addition and subtraction are defined as

$P + Q = \langle m + n, \alpha + \gamma, \beta + \delta\rangle$

$P - Q = \langle m - n, \alpha + \delta, \beta + \gamma\rangle$

$P - P = \langle 0, \alpha + \beta, \beta + \alpha\rangle$.

Multiplication by a scalar k is given by

$$k.P = \begin{cases} \langle km, k\alpha, k\beta\rangle, & \text{if } k > 0 \\ \langle km, -k\beta, -k\alpha\rangle, & \text{if } k < 0 \end{cases}$$

Product of two TFNs is not an exact TFN, but it is very close to a TFN shown below:

$$P.Q \simeq \begin{cases} \langle mn, m\gamma + n\alpha, m\delta + n\beta\rangle, & \text{if } P, Q > 0 \\ \langle mn, n\alpha - m\delta, n\beta - m\gamma\rangle, & \text{if } P < 0, Q > 0 \\ \langle mn, -n\beta - m\delta, -n\alpha - m\gamma\rangle, & \text{if } P < 0, Q < 0 \end{cases}$$

The inverse and division are defined as follows:

If $m \neq 0$, then

$$P^{-1} = \langle m, \alpha, \beta\rangle^{-1} = \langle m^{-1}, \beta m^{-2}, \alpha m^{-2}\rangle$$

$$\frac{P}{Q} = P.Q^{-1} = \langle m, \alpha, \beta\rangle.\langle n^{-1}, \delta n^{-2}, \gamma n^{-2}\rangle \simeq \left\langle \frac{m}{n}, \frac{m\delta + n\alpha}{n^2}, \frac{m\gamma + n\beta}{n^2} \right\rangle.$$

The nth power of the TFN P is given by

$$P^n = \langle m, \alpha, \beta\rangle^n \simeq \begin{cases} \langle m^n, -nm^{n-1}\beta, -nm^{n-1}\alpha\rangle, & \text{when } n \text{ is negative} \\ \langle m^n, nm^{n-1}\alpha, nm^{n-1}\beta\rangle, & \text{when } n \text{ is positive} \end{cases}$$

The fuzzy arithmetic is not the same as the arithmetic of real or complex numbers. For arithmetic on TFNs, the following interesting issues are observed.

(i) Subtraction between the same TFNs produces a symmetric TFN whose mean value is zero, and the spreads are the sum of both the spreads of computed TFN, i.e., $P - Q = \langle 0, a, a\rangle$.

(ii) The division of a TFN by itself is also a symmetric TFN having a mean value one. Inverse and division of a TFN do not exist if its mean value is zero, i.e., $P/Q = \langle 1, b, b\rangle$.

(iii) The arithmetics operations addition and multiplication on TFNs follow the commutative and associative laws.

(iv) The distributive laws do not always hold.
(v) It may be noted that

  (a) $\langle m, \alpha, \beta\rangle.\langle 0, 0, 0\rangle = \langle 0, 0, 0\rangle$.
  (b) $\langle 0, \alpha, \beta\rangle.\langle 0, \gamma, \delta\rangle = \langle 0, 0, 0\rangle$.

Several arithmetic operations are mentioned above. Similarly, the logical operations, i.e., comparisons of two TFNs, are also available. For this purpose, many methods are proposed by researchers.

6.2 OERI and Acceptance Index

Comparison between two TFNs is not an easy task, and it is required in many places, particularly in decision-making. Several methods are available for this purpose. The overall existence ranking index (OERI) is one of the best methods. In this method, the deconvolution technique is used to avoid the double inclusion of fuzzy uncertainties in the fuzzy PERT system.

In this chapter, the acceptance index is used for ranking TFNs. Sengupta and Pal [22] proposed an acceptance index for ranking of interval numbers. Nayeem and Pal [20] developed a similar technique ranking of TFNs. This ranking index is much more efficient and mathematically equivalent to the OERI for the ranking of TFNs. Also, the deconvolution process is used here.

Definition 1 Suppose $\mu_P^{-1}(t)$ and $\mu_Q^{-1}(t)$ be two inverse membership functions of the fuzzy numbers P and Q, along with a given existence level t. The OERI for the numbers P and Q is given below:

$$OERI(P, Q) = OERI(P) - OERI(Q), \text{ where}$$

$$OERI(P) = \int_0^1 [\chi(t)L(t)\mu_{P_L}^{-1}(t) + (1-\chi(t))R(t)\mu_{P_R}^{-1}(t)]dt, \quad (6.1)$$

where $L(t)$ and $R(t)$ are the subjective weights for the left and right parts of P. $\chi(t)$ and $1-\chi(t)$, $\chi(t) \in [0, 1]$ represent the subjective weights at the existence level t.

The subjective weighting functions $L(t)$, $R(t)$, and $\chi(t)$ are considered by the decision-maker based on the problem and own opinion. In [7], various aspects of weighting functions are discussed. Depending on the decision-maker's attitude, different weighting functions are available.

In the indifferent point of view, $\chi(t) = \frac{1}{2}$ and $L(t) = R(t) = 1$.

In the optimistic point of view, $\chi(t) = 1 - \frac{1}{2}t$ and $L(t) = R(t) = 1$.

In case of optimistic point of view, $\chi(t) = 1 - \frac{1}{2}t$ and $L(t) = R(t) = 1$.

In case of a pessimistic point of view, $\chi(t) = \frac{1}{2}t$ and $L(t) = R(t) = 1$.

For comparison of two TFNs, many methods are proposed, among them the following method introduced by Nayeem and Pal is more realistic. Nayeem and Pal defined the acceptability index denoted by $\mathscr{A}$-index.

Definition 2 The acceptability index, i.e., $\mathscr{A}$-index of the proposition " $\widetilde{p} = \langle p, \alpha, \beta \rangle$ is preferred to $\widetilde{q} = \langle q, \gamma, \delta \rangle$" is given by

$$\mathscr{A}(\widetilde{p} \prec \widetilde{q}) = \frac{q - p}{\beta + \gamma}. \tag{6.2}$$

Using this $\mathscr{A}$-index, the following ranking orders are defined.

Definition 3 If $\mathscr{A}(\widetilde{p} \prec \widetilde{q}) \geq 1$, then $\widetilde{p}$ is called totally dominating over $\widetilde{q}$ in minimization sense, and for maximization sense, the fact is converse, and this is denoted by $\widetilde{p} \prec \widetilde{q}$.

Definition 4 If $0 < \mathscr{A}(\widetilde{p} \prec \widetilde{q}) < 1$, then in the sense of minimization, $\widetilde{p}$ is called "partially dominating" over $\widetilde{q}$ and in the sense of maximization $\widetilde{q}$ is called 'partially dominating' over $\widetilde{p}$. This phenomena is denoted by $\widetilde{p} \prec_P \widetilde{q}$.

On the other hand, a pessimistic decision-maker would prefer the number left end whose *support* is smaller than that of the other and an optimistic decision-maker would prefer the number right end of whose support is greater than that of the other.

Now, the membership functions of a TFN $\widetilde{P} = \langle p, \alpha, \beta \rangle$ are given by

$$\mu_{P_L}(t) = \frac{t - (p - \alpha)}{\alpha} \quad \text{for} \quad p - \alpha \leq t \leq p$$

$$\text{and } \mu_{P_R}(t) = \frac{(p + \beta) - t}{\beta} \quad \text{for} \quad p \leq t \leq p + \beta$$

so that $\mu_{P_L}^{-1}(t) = (p - \alpha) + \alpha t$ and $\mu_{P_R}^{-1}(t) = (p + \beta) - \beta t$.

The $OERI(P)$ proposed by Chang and Lee for the appropriate choice of weighting functions is given by

$$\frac{1}{2}\int_0^1 (p + \alpha + \alpha t + p + \beta - \beta t)dt = p - \frac{\alpha - \beta}{4}$$

in case of indifferent view.

Similarly, for optimistic and pessimistic views, it takes the values $OERI(P) = p - \frac{5\alpha - \beta}{12}$ and $p - \frac{\alpha - 5\beta}{12}$, respectively.

Thus, $OERI(Q)$ for the TFN $\widetilde{Q} = \langle q, \gamma, \delta \rangle$ is given by $q - \frac{\gamma - \delta}{4}, q - \frac{5\gamma - \delta}{12}, q - \frac{\gamma - 5\delta}{12}$, respectively, for the three views.

Thus, for indifferent view, $\widetilde{P} < \widetilde{Q}$ if

$OERI(\widetilde{P}) < OERI(\widetilde{Q})$

$$, i.e..\ p-\frac{\alpha-\beta}{4}<q-\frac{\gamma-\delta}{4}$$
$$, i.e.,\ \ p-q<\frac{(\alpha-\beta)-(\gamma-\delta)}{4}$$
$$i.e.\ p-q<\frac{\alpha+\delta}{4}-\frac{\beta+\gamma}{4}$$

$$, i.e., \frac{q-p}{\beta+\gamma}>\frac{1}{4}\left(1-\frac{\alpha+\delta}{\beta+\gamma}\right). \tag{6.3}$$

Similarly, for the optimistic view, $\widetilde{P}<\widetilde{Q}$ if

$$\frac{q-p}{\beta+\gamma}>\frac{1}{12}\left(1-5\frac{\alpha+\delta}{\beta+\gamma}\right) \tag{6.4}$$

and for pessimistic view, $\widetilde{P}<\widetilde{Q}$ if

$$\frac{q-p}{\beta+\gamma}>\frac{1}{12}\left(5-\frac{\alpha+\delta}{\beta+\gamma}\right). \tag{6.5}$$

In this approach, $\widetilde{P}\prec\widetilde{Q}$ if

$$\frac{q-p}{\beta+\gamma}\geq 1 \tag{6.6}$$

and $\widetilde{P}\prec_P\widetilde{Q}$ if

$$0<\frac{q-p}{\beta+\gamma}<1 \tag{6.7}$$

The choice depends on the decision-makers' view of optimism or pessimism in case of

$$\frac{q-p}{\beta+\gamma}=0,\ i.e.,\ \ p=q. \tag{6.8}$$

From Eqs. (6.6), (6.7), and (6.8), it is obvious that the importance is given on the points p and q where the membership value is 1 attained by $\widetilde{P}$ and $\widetilde{Q}$, respectively. From Eq. (6.6), it is clear that when $\widetilde{P}\prec\widetilde{Q}$, $p+\beta$ lies left to $q-\gamma$, and (6.7) guarantees that p always lies on the left of q when $\widetilde{P}\prec_P\widetilde{Q}$.

But since $\dfrac{\alpha+\delta}{\beta+\gamma}$ is a positive quantity, it must be noted from (6.3), (6.4), and (6.5) that the right-hand side of each of the inequalities may not be a nonnegative quantity. Thus, $\widetilde{P}<\widetilde{Q}$ even if $p>q$. Again, if (6.6) holds, all of (6.3), (6.4), and (6.5) hold. Also, the time to compare two TFNs is less than other proposed methods.

6.3 Deconvolution

The extended subtraction for two L-R fuzzy numbers $\widetilde{P} = (p_m, p_L, p_R)_{LR}$ and $\widetilde{Q} = (q_m, q_L, q_R)_{LR}$ is defined as $\widetilde{P} \ominus \widetilde{Q} = (p_m - q_m, p_L + q_R, p_R + q_L)_{LR}$.

Notice that during subtraction, the uncertainty will increase, even both the TFNs are the same. If we add two TFNs $\widetilde{P}$ and $\widetilde{Q}$ and then subtract $\widetilde{Q}$ from the sum, then the mean value is the same as the mean value of $\widetilde{P}$, but spreads are added two times. Extended subtraction method is used to avoid double inclusion. For this purpose, the deconvolution approach discussed in [10] is used in the backward pass calculation in fuzzy CPM.

The backward extended subtraction or deconvolution (denoted by $[-]$) for two L-R fuzzy numbers $\widetilde{P} = (p_m, p_L, p_R)_{LR}$ and $\widetilde{Q} = (q_m, q_L, q_R)_{LR}$ is defined as $\widetilde{P}[-]\widetilde{Q} = (p_m - q_m, p_L - q_L, p_R - q_R)_{LR}$.

There is a big controversy in the deconvolution of fuzzy numbers. Note that the left and/or right spreads may be negative, and as per the definition of TFNs, these spreads must be nonnegative. It is obvious that if any one spreads is negative, then there is a question about the existence of the TFN. Particularly, this number does not exist. But, here, we are interested in comparing or ranking the fuzzy numbers. So for this application point of view, the existence or nonexistence of the fuzzy number can be ignored if an appropriate ranking method is used.

There is a drawback to use deconvolution in fuzzy numbers. The result obtained by applying the deconvolution is not necessary a valid fuzzy number. But we are only concentrated in the ranking or comparison of the results; the problem of existence or nonexistence does not matter if an appropriate ranking process is used.

During deconvolution, a spread may become negative, and this spread can be transferred to an equivalent nonnegative spread using the following method:

(a) The mean value, i.e., the point where the fuzzy number attains the membership value 1, is transformed to $l_m = \int_w [\mu_{A_L}^{-1}(w) - \mu_{B_L}^{-1}(w)]dw$.
(b) The left and right spreads are transformed to $l_L = l_m - (a_m - b_m)$ and $l_R = (a_R - b_R) - l_L$, respectively.

(ii) If $a_L - b_L \geq 0$ and $a_R - b_R < 0$, then

(a) The mode is transformed to $r_m = \int_w [\mu_{A_R}^{-1}(w) - \mu_{B_R}^{-1}(w)]dw$.
(b) The left and right spreads are transformed to $r_R = (a_m - b_m) - r_m$ and $r_L = (a_L - b_L) - r_R$, respectively.

For triangular fuzzy numbers, l_m and r_m are given by $l_m = (a_m - b_m) - \frac{1}{2}(a_L - b_L)$ and $r_m = (a_m - b_m) + \frac{1}{2}(a_R - b_R)$ so that $l_L = -\frac{1}{2}(a_L - b_L)$, $l_R = (a_R - b_R) + \frac{1}{2}(a_L - b_L)$, and $r_L = (a_L - b_L) + \frac{1}{2}(a_R - b_R)$, $r_R = -\frac{1}{2}(a_R - b_R)$.

If $(a_L - b_L)$ and $(a_R - b_R)$ are both negative, then the mean value remains unchanged, and the left spread and right spread are changed to $-(a_R - b_R)$ and $-(a_L - b_L)$, respectively.

7 Critical Path Analysis in Fuzzy Environment

After construction of the network, the analysis of the time, cots, etc. are needed for planning of different activities of the project. In fuzzy environment, the activities are also fuzzy, which means their time durations or costs are fuzzy numbers. Here, TFNs are assumed as the weighs activities' weights (time duration, cost, etc.) that the activities are certain, but their weights are uncertain. The events are certain. No weight is associated with any event, so events are completely certain. So in the project management system, the constructed network is certain; only the weights on the edges are uncertain, in this case, TFNs. This problem can be solved using the existing method for the crisp network by defuzzifing the weights on the edges to the crisp numbers. But the original information on the edges will be lost during defuzzification. So, we need a separate method which will handle the fuzzy numbers. In this chapter, such a new method is discussed.

The main objective of the network analysis is the preparation of the planning schedule of the project. In such a planning schedule, the following factors are included:

(i) Project finishing time.
(ii) Earliest starting time of each activity.
(iii) Latest starting time of an activity without delaying.
(iv) Float for all activities. This is the time duration by which the completion of an activity can be delayed, but the total completion time of the project remains the same.
(v) Computation of critical activities and the corresponding critical path.

7.1 Notation

Some useful notations are listed below:

- e_{ij} is the (i, j)th edge or activity.
- $\widetilde{E}_i$ is the earliest occurrence time of the event i, i.e., it is the earliest time at which the event i can occur without changing the total project completion time.
- $\widetilde{L}_i$ is the latest occurrence time in which the event i can start without affecting the total project completion time.
- $\widetilde{t_{ij}}$ time duration of the activity e_{ij}.
- $\widetilde{LS}_{ij}$ is the latest starting time of the activity e_{ij}
- $\widetilde{EF}_{ij}$ is the earliest finishing time of activity e_{ij}.

The proposed method to find a critical path has two steps, viz., (a) forward pass calculations and (b) backward pass calculations.

7.2 Forward Pass Calculations

In this step, the computation starts from the starting event (s, numbered as 1), continues to the events in an increasing number of event, and ends at the destination event (t). At each event, the earliest starting and ending times are determined for each activity. The steps are written below:

The main steps for forward pass calculation are mentioned below:
Initially, $\widetilde{E}_1 = 0, i = l$.
The earlier occurrence time of the event j for all nodes j is given by

$$\widetilde{E}_j = \widetilde{\max}_i\{\widetilde{EF}_{ij}\} = \widetilde{\max}_i\{\widetilde{E}_i + \widetilde{t}_{ij}\}$$

for all immediate predecessor activities of the node j.

Notice that to find $\widetilde{E}_j$, addition of TFNs and the finding of maximum of TFNs are involved. These steps involved new (fuzzy) arithmetic and comparison operations. The final value, i.e., $\widetilde{E}_n$, gives the length of the critical path, which is nothing but the completion time of the project. But the critical activities and critical path are determined by backward pass calculations.

7.3 Backward Pass Calculations

In this step, the calculation starts from the terminal event. The process is repeated through the event in a decreasing order and ends at initial event 1. At each node (event), the latest allowable occurrence time $\widetilde{L}_i$ for all i is computed.

The main computations in this step are the following:
Initially, $\widetilde{L}_n = \widetilde{E}_n$, $j = n$:

$$\widetilde{L}_i = \widetilde{\min}_i\{\widetilde{LS}_{ij}\} = \widetilde{\min}_j\{\widetilde{L}_j[-]\widetilde{t}_{ij}\}$$

for all immediate successor activities of the node i.

The critical nodes can be determined by calculating the values of $\widetilde{L}_i$ for all i. If $\widetilde{E}_i = \widetilde{L}_i$ for some $i = 1, 2, \ldots, n$, then node i is critical. However, the computation of $\widetilde{L}_i$ for all i is not enough to find critical activities and critical paths. For this purpose, we need to find out more information, viz., floats and slack.

7.4 Computation of Floats and Slack Times

The floats are of three types, viz., total float, free float and independent float. These parameters depend on the earliest and latest event times, and a formal definition is also given. After drawing the network and labeling, the earliest and latest times for each event are calculated. Then the next step is to compute the slack time of each event and floats of each activity. The float suggests us how much time one can delay the activity by retaining the same project completion time.

The total float for an activity e_{ij} is denoted by TF_{ij} and is determined by $\widetilde{TF}_{ij} = \widetilde{LS}_{ij} - \widetilde{ES}_{ij}$ or $\widetilde{TF}_{ij} = \widetilde{LF}_{ij} - \widetilde{EF}_{ij}$ or $\widetilde{TF}_{ij} = \widetilde{L}_j - (\widetilde{E}_i + \widetilde{t}_{ij})$ as $\widetilde{L}_j = \widetilde{LF}_{ij}$ and $\widetilde{EF}_{ij} = \widetilde{E}_i + \widetilde{t}_{ij}$.

For any activity, the free float is either zero or any positive number less than total float. Free float has one interesting use. It is used for rescheduling activities with minimum changing of earlier schedule.

If $\widetilde{IF}_{ij}$ is negative, then it is taken as zero.

In terms of different types of times and floats, the critical event and critical activity are defined as follows:
An event i is called **critical** if its slack is zero, i.e., $\widetilde{E}_i = \widetilde{L}_i$.
An activity e_{ij} is called critical if its total float is zero, i.e., $\widetilde{LS}_{ij} = \widetilde{ES}_{ij}$ or $\widetilde{LF}_{ij} = \widetilde{EF}_{ij}$.
If an activity is not critical, it is called noncritical.

8 The Algorithm

This section presents an algorithm for determining the critical path for a network whose edge weights are TFNs. The forward process involves extended addition TFNs, and the backward process involves using backward extended subtraction, i.e., deconvolution of TFNs.

Let $G = (V, \mathscr{E})$ be a network corresponding to the given project. The nodes are numbered as $v_1, v_2, \ldots, v_n$ and $\mathscr{E}$ is the set of edges e_{ij}. The weight of the edge e_{ij} is denoted as $\widetilde{t}_{ij}$, and it is a TFN. It is already mentioned that $\widetilde{E}_i$ and $\widetilde{L}_i$ are the earliest occurrence time and latest allowable occurrence time of the event i. Again, $\widetilde{EF}_{ij}$ and $\widetilde{LS}_{ij}$ are the earliest finishing time and latest starting time of the edge e_{ij}:

ALGORITHM FUZZYPROJ$(G, n, s, t, \mathscr{P})$
Input: A project network $G = (V, \mathscr{E})$ with n nodes. s and t represent the source and sink nodes, and $[\widetilde{t}_{ij}]_{n\times n}$ is the adjacency matrix of G.
Output: A critical path $\mathscr{P}$ from s to t.
Step 1: //forward pass calculation//
Step 1.1: Initially, $j \leftarrow s$, $\widetilde{E}_j = \widetilde{0}$, where, $\widetilde{0} = \langle 0, 0, 0\rangle$, the zero TFN and $V^* = V - \{s\}$.
Step 1.2: while $V^* \neq \phi$ do

Calculate $\widetilde{EF}_{ij} = \widetilde{E}_j \oplus \widetilde{t}_{ji}$ for all j such that $e_{ji} \in \mathscr{E}$.
Find $\widetilde{EF}_{ij^*} = \widetilde{\max}_{j \in P(i)} \widetilde{EF}_{ij}$, where $P(i)$ is the set of events immediately preceding the node i.
Set $\widetilde{E}_i \leftarrow \widetilde{EF}_{ij^*}$ and $V^* \leftarrow V^* - \{i\}$.
endwhile;

Step 2: //backward pass computation//
Step 2.1: Initially, $j \leftarrow t$ and $\widetilde{L}_j \leftarrow \widetilde{E}_j$, $V^* = \{t\}$.
Step 2.2: While $V^* \neq V$ do
Find $\widetilde{LS}_{ij} = \widetilde{L}_j[-]\widetilde{t}_{ij}$ for all j such that $e_{ij} \in \mathscr{E}$.
Compute d$\widetilde{LS}_{ij^*} = \widetilde{\min}_{j \in S(i)} \widetilde{LS}_{ij}$, where $S(i)$ is the set of events immediately following the node i. Update $\widetilde{L}_i \leftarrow \widetilde{LS}_{ij^*}$ and $V^* \leftarrow V^* \cup \{i\}$.
endwhile;

Step 3: //calculation of slack//
for all $i \in V$ do
$\widetilde{SK}_i = \widetilde{L}_i[-]\widetilde{E}_i$.
endfor;

Step 4: //computation of critical path//
Step 4.1: $i \leftarrow s$, $\mathscr{P} \leftarrow \phi$.
Step 4.2: while $i \neq t$ do
Compute d$\widetilde{SK}_{j^*} = \widetilde{\min}_{j \in N(i)} \widetilde{SK}_j$, where $N(i)$ is the neighbor of i.
Set $\mathscr{P} \leftarrow \mathscr{P} \cup e_{ij^*}$ and $i \leftarrow j^*$.
endwhile;

END FUZZYPROJ

9 An Illustrative

To illustrate the proposed process, we consider a project network shown in Fig. 6.7. In this network, the duration of the activities is considered as TFNs. The same procedure applies to other types of fuzzy numbers also.

If the left and right spreads are set to zero, then using CPM for a crisp network, the critical path is obtained as **1 → 3 → 4 → 6 → 7**.

Applying our proposed algorithm, one can determine the critical path as follows:

Initially, $\widetilde{E}_1 = \langle 0, 0, 0 \rangle$.
$\widetilde{E}_2 = \widetilde{E}_1 \oplus \widetilde{t}_{ij} = \langle 0, 0, 0 \rangle \oplus \langle 20, 2, 3 \rangle = \langle 20, 2, 3 \rangle$,
$\widetilde{E}_3 = \widetilde{E}_1 \oplus \widetilde{t}_{ij} = \langle 0, 0, 0 \rangle \oplus \langle 23, 4, 5 \rangle = \langle 23, 4, 5 \rangle$.
So, $\widetilde{EF}_{41} = \widetilde{E}_1 \oplus \widetilde{t}_{14} = \langle 0, 0, 0 \rangle \oplus \langle 8, 2, 4 \rangle = \langle 8, 2, 4 \rangle$
and $\widetilde{EF}_{43} = \widetilde{E}_3 \oplus \widetilde{t}_{34} = \langle 23, 4, 5 \rangle \oplus \langle 16, 3, 4 \rangle = \langle 39, 7, 9 \rangle$.

There are two paths to reach the node 4.
Now, $\mathscr{A}(\widetilde{EF}_{41} \prec \widetilde{EF}_{43}) = \frac{39-8}{4+7} > 1$. So, $\widetilde{EF}_{43}$ totally dominates $\widetilde{EF}_{41}$.

Now, the earliest fuzzy starting time $\widetilde{E}_4$ for the node 4 is obtained as follows:
$\widetilde{E}_4 = \widetilde{max}\{\widetilde{EF}_{41}, \widetilde{EF}_{43}\} = \langle 39, 7, 9 \rangle$.

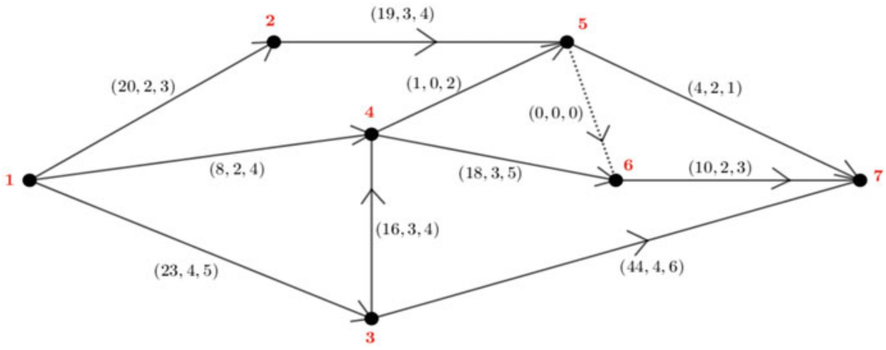


Fig. 6.7 A network with TFNs as project duration

$\widetilde{EF}_{52} = \widetilde{E}_2 \oplus \widetilde{t}_{25} = \langle 20, 2, 3\rangle \oplus \langle 19, 3, 4\rangle = \langle 39, 5, 7\rangle$

$\widetilde{EF}_{54} = \widetilde{E}_4 \oplus \widetilde{t}_{45} = \langle 39, 7, 9\rangle \oplus \langle 1, 0, 2\rangle = \langle 40, 7, 11\rangle$.

Now, $\mathscr{A}(\widetilde{EF}_{52} \prec \widetilde{EF}_{54}) = \frac{40-39}{7+7} < 1$. So, $\widetilde{EF}_{52} \prec_P \widetilde{EF}_{54}$.

$\widetilde{E}_5 = \widetilde{max}\{\widetilde{EF}_{54}, \widetilde{EF}_{52}\} = \widetilde{EF}_{52} = \langle 40, 7, 11\rangle$.

$\widetilde{EF}_{65} = \widetilde{E}_5 \oplus \widetilde{t}_{56} = \langle 40, 7, 11\rangle \oplus \langle 0, 0, 0\rangle = \langle 40, 7, 11\rangle$

$\widetilde{EF}_{64} = \widetilde{E}_4 \oplus \widetilde{t}_{46} = \langle 39, 7, 9\rangle \oplus \langle 18, 3, 5\rangle = \langle 57, 10, 14\rangle$.

Now, $\mathscr{A}(\widetilde{EF}_{65} \prec \widetilde{EF}_{64}) = \frac{57-40}{11+10} < 1$. So, $\widetilde{EF}_{65} \prec_P \widetilde{EF}_{64}$.

$\widetilde{E}_6 = \widetilde{max}\{\widetilde{EF}_{65}, \widetilde{EF}_{64}\} = \widetilde{EF}_{64} = \langle 57, 10, 14\rangle$.

$\widetilde{EF}_{73} = \widetilde{E}_3 \oplus \widetilde{t}_{37} = \langle 23, 4, 5\rangle \oplus \langle 44, 4, 6\rangle = \langle 67, 8, 11\rangle$

$\widetilde{EF}_{76} = \widetilde{E}_6 \oplus \widetilde{t}_{67} = \langle 57, 10, 14\rangle \oplus \langle 10, 2, 3\rangle = \langle 67, 12, 17\rangle$. $\widetilde{EF}_{75} = \widetilde{E}_5 \oplus \widetilde{t}_{57} = \langle 40, 7, 11\rangle \oplus \langle 4, 2, 1\rangle = \langle 44, 9, 12\rangle$.

Obviously, $\widetilde{EF}_{75} \prec \widetilde{EF}_{73}$ and $\widetilde{EF}_{75} \prec \widetilde{EF}_{76}$.

But, $\mathscr{A}(\widetilde{EF}_{73} \prec \widetilde{EF}_{76}) = 0$.

In this case, the mean values of $\widetilde{EF}_{73}$ and $\widetilde{EF}_{76}$ are both the same, but spreads are different. Here, two cases arise.

For optimistic point of view, $\widetilde{E}_7 = \langle 67, 12, 17\rangle$, and for pessimistic point of view, $\widetilde{E}_7 = \langle 67, 8, 11\rangle$.

Using these two views, the $\widetilde{L}_j$s are calculated.

Optimistic Case

In this case, $\widetilde{E}_7 = \langle 67, 12, 17\rangle$.

Now, $\widetilde{LS}_{67} = \widetilde{L}_7[-]\widetilde{t}_{67} = \langle 67, 12, 17\rangle[-]\langle 10, 2, 3\rangle = \langle 57, 10, 14\rangle$

$\widetilde{L}_6 = \widetilde{LS}_{67} = \langle 57, 10, 14\rangle$.

$\widetilde{LS}_{37} = \widetilde{L}_7[-]\widetilde{t}_{37} = \langle 67, 12, 17\rangle[-]\langle 44, 4, 6\rangle = \langle 23, 8, 11\rangle$

$\widetilde{L}_3 = \widetilde{LS}_{37} = \langle 23, 8, 11\rangle$

$\widetilde{LS}_{57} = \widetilde{L}_7[-]\widetilde{t}_{57} = \langle 67, 12, 17\rangle[-]\langle 4, 2, 1\rangle = \langle 63, 10, 16\rangle$

$\widetilde{LS}_{56} = \widetilde{L}_6[-]\widetilde{t}_{56} = \langle 57, 10, 14\rangle[-]\langle 0, 0, 0\rangle = \langle 57, 10, 14\rangle$

$\widetilde{L}_5 = \widetilde{min}\{\widetilde{LS}_{57}, \widetilde{LS}_{56}\} = \widetilde{LS}_{37} = \langle 57, 10, 14\rangle$

$\widetilde{LS}_{46} = \widetilde{L}_6[-]\widetilde{t}_{46} = \langle 57, 10, 14\rangle[-]\langle 18, 3, 5\rangle = \langle 39, 7, 9\rangle$

$\widetilde{LS}_{45} = \widetilde{L}_5[-]\widetilde{t}_{45} = \langle 57, 10, 14\rangle[-]\langle 1, 0, 2\rangle = \langle 56, 10, 12\rangle$

$\widetilde{L}_4 = \widetilde{min}\{\widetilde{LS}_{46}, \widetilde{LS}_{45}\} = \widetilde{LS}_{46} = \langle 39, 7, 9\rangle$
$\widetilde{LS}_{37} = \widetilde{L}_7[-]\widetilde{t}_{37} = \langle 67, 12, 17\rangle[-]\langle 44, 4, 6\rangle = \langle 23, 8, 11\rangle$
$\widetilde{LS}_{34} = \widetilde{L}_4[-]\widetilde{t}_{34} = \langle 39, 7, 9\rangle[-]\langle 16, 3, 4\rangle = \langle 23, 4, 5\rangle$
Here, the mean values of these two TFNs are the same, but their spreads are different. In optimistic point of view,
$\widetilde{L}_3 = \langle 23, 4, 5\rangle$
$\widetilde{LS}_{25} = \widetilde{L}_5[-]\widetilde{t}_{25} = \langle 57, 10, 14\rangle[-]\langle 19, 3, 4\rangle = \langle 38, 7, 10\rangle$
$\widetilde{L}_2 = \langle 38, 7, 10\rangle$
$\widetilde{LS}_{12} = \widetilde{L}_2[-]\widetilde{t}_{12} = \langle 38, 7, 10\rangle[-]\langle 20, 2, 3\rangle = \langle 18, 5, 7\rangle$
$\widetilde{LS}_{13} = \widetilde{L}_3[-]\widetilde{t}_{13} = \langle 23, 4, 5\rangle[-]\langle 23, 4, 5\rangle = \langle 0, 0, 0\rangle$
$\widetilde{LS}_{14} = \widetilde{L}_4[-]\widetilde{t}_{14} = \langle 39, 7, 9\rangle[-]\langle 8, 2, 4\rangle = \langle 31, 5, 5\rangle$
$\widetilde{L}_1 = \widetilde{min}\{\widetilde{LS}_{12}, \widetilde{LS}_{13}, \widetilde{LS}_{14}\} = \widetilde{LS}_{13} = \langle 0, 0, 0\rangle$

Slakes for Each Node $\widetilde{SK}_1 = \widetilde{L}_1[-]\widetilde{E}_1 = \langle 0, 0, 0\rangle[-]\langle 0, 0, 0\rangle = \langle 0, 0, 0\rangle$
$\widetilde{SK}_2 = \widetilde{L}_2[-]\widetilde{E}_2 = \langle 38, 7, 10\rangle[-]\langle 20, 2, 3\rangle = \langle 18, 5, 7\rangle$
$\widetilde{SK}_3 = \widetilde{L}_3[-]\widetilde{E}_3 = \langle 23, 4, 5\rangle[-]\langle 23, 4, 5\rangle = \langle 0, 0, 0\rangle$
$\widetilde{SK}_4 = \widetilde{L}_4[-]\widetilde{E}_4 = \langle 39, 7, 9\rangle[-]\langle 39, 7, 9\rangle = \langle 0, 0, 0\rangle$
$\widetilde{SK}_5 = \widetilde{L}_5[-]\widetilde{E}_5 = \langle 57, 10, 14\rangle[-]\langle 40, 7, 11\rangle = \langle 17, 3, 3\rangle$
$\widetilde{SK}_6 = \widetilde{L}_2[-]\widetilde{E}_6 = \langle 57, 10, 14\rangle[-]\langle 57, 10, 14\rangle = \langle 0, 0, 0\rangle$
$\widetilde{SK}_7 = \widetilde{L}_7[-]\widetilde{E}_7 = \langle 67, 12, 17\rangle[-]\langle 67, 12, 17\rangle = \langle 0, 0, 0\rangle$.

Thus, the critical path is $1 \to 3 \to 4 \to 6 \to 7$. This critical path drawn in bold line is shown in Fig. 6.8.

Pessimistic Case In this case, $\widetilde{E}_7 = \langle 67, 8, 11\rangle$.
Now, $\widetilde{LS}_{67} = \widetilde{L}_7[-]\widetilde{t}_{67} = \langle 67, 8, 11\rangle[-]\langle 10, 2, 3\rangle = \langle 57, 6, 8\rangle$
$\widetilde{L}_6 = \widetilde{LS}_{67} = \langle 57, 6, 8\rangle$.
$\widetilde{LS}_{57} = \widetilde{L}_7[-]\widetilde{t}_{57} = \langle 67, 8, 11\rangle[-]\langle 4, 2, 1\rangle = \langle 63, 6, 10\rangle$
$\widetilde{LS}_{56} = \widetilde{L}_6[-]\widetilde{t}_{56} = \langle 57, 6, 8\rangle[-]\langle 0, 0, 0\rangle = \langle 57, 6, 8\rangle$
$\widetilde{L}_5 = \widetilde{min}\{\widetilde{LS}_{57}, \widetilde{LS}_{56}\} = \widetilde{LS}_{37} = \langle 57, 6, 8\rangle$
$\widetilde{LS}_{46} = \widetilde{L}_6[-]\widetilde{t}_{46} = \langle 57, 6, 8\rangle[-]\langle 18, 3, 5\rangle = \langle 39, 3, 3\rangle$

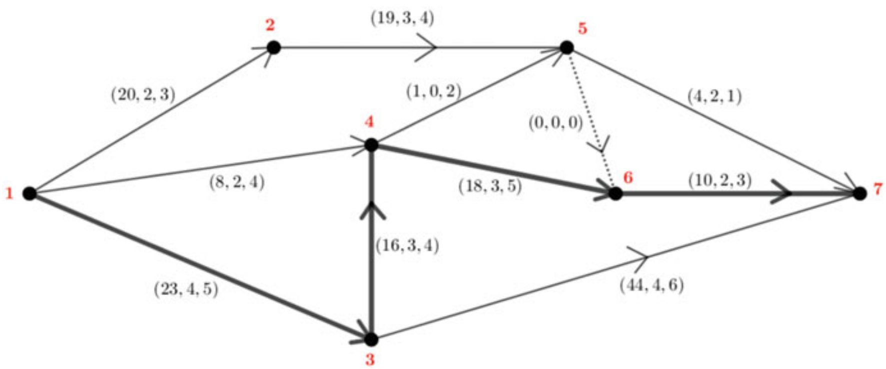


Fig. 6.8 The optimistic critical path for the network of Fig. 6.7

$\widetilde{LS}_{45} = \widetilde{L}_5[-]\widetilde{t}_{45} = \langle 57, 6, 8\rangle[-]\langle 1, 0, 2\rangle = \langle 56, 6, 6\rangle$
$\widetilde{L}_4 = \widetilde{min}\{\widetilde{LS}_{46}, \widetilde{LS}_{45}\} = \widetilde{LS}_{46} = \langle 39, 3, 3\rangle$
$\widetilde{LS}_{37} = \widetilde{L}_7[-]\widetilde{t}_{37} = \langle 67, 8, 11\rangle[-]\langle 44, 4, 6\rangle = \langle 23, 4, 5\rangle$
$\widetilde{LS}_{34} = \widetilde{L}_4[-]\widetilde{t}_{34} = \langle 39, 3, 3\rangle[-]\langle 16, 3, 4\rangle = \langle 23, 0, -1\rangle$
Here, the mean values of these two TFNs are the same. Note that the right spread is negative and the left one is 0. So, using the deconvolution technique, this TFN is replaced by $\langle 23, 1, 0\rangle$. In a pessimistic point of view,
$\widetilde{L}_3 = \langle 23, 4, 5\rangle$
$\widetilde{LS}_{25} = \widetilde{L}_5[-]\widetilde{t}_{25} = \langle 38, 3, 4\rangle$
$\widetilde{L}_2 = \langle 38, 3, 4\rangle$
$\widetilde{LS}_{12} = \widetilde{L}_2[-]\widetilde{t}_{12} = \langle 38, 3, 4\rangle[-]\langle 20, 2, 3\rangle = \langle 18, 1, 1\rangle$
$\widetilde{LS}_{13} = \widetilde{L}_3[-]\widetilde{t}_{13} = \langle 23, 4, 5\rangle[-]\langle 23, 4, 5\rangle = \langle 0, 0, 0\rangle$
$\widetilde{LS}_{14} = \widetilde{L}_4[-]\widetilde{t}_{14} = \langle 39, 3, 3\rangle[-]\langle 8, 2, 4\rangle = \langle 31, 1, -1\rangle = \langle 30.5, 0.5, 0.5\rangle$ by deconvolution
$\widetilde{L}_1 = \widetilde{min}\{\widetilde{LS}_{12}, \widetilde{LS}_{13}, \widetilde{LS}_{14}\} = \widetilde{LS}_{13} = \langle 0, 0, 0\rangle$

Slakes for Each Node $\widetilde{SK}_1 = \widetilde{L}_1[-]\widetilde{E}_1 = \langle 0, 0, 0\rangle[-]\langle 0, 0, 0\rangle = \langle 0, 0, 0\rangle$
$\widetilde{SK}_2 = \widetilde{L}_2[-]\widetilde{E}_2 = \langle 38, 3, 4\rangle[-]\langle 20, 2, 3\rangle = \langle 18, 1, 1\rangle$
$\widetilde{SK}_3 = \widetilde{L}_3[-]\widetilde{E}_3 = \langle 23, 4, 5\rangle[-]\langle 23, 4, 5\rangle = \langle 0, 0, 0\rangle$
$\widetilde{SK}_4 = \widetilde{L}_4[-]\widetilde{E}_4 = \langle 39, 3, 3\rangle[-]\langle 39, 7, 9\rangle = \langle 0, -4, -6\rangle = \langle 0, 6, 4\rangle$ by deconvolution
$\widetilde{SK}_5 = \widetilde{L}_5[-]\widetilde{E}_5 = \langle 57, 6, 8\rangle[-]\langle 40, 7, 11\rangle = \langle 17, -1, -3\rangle = \langle 17, 3, 1\rangle$
$\widetilde{SK}_6 = \widetilde{L}_2[-]\widetilde{E}_6 = \langle 57, 6, 8\rangle[-]\langle 57, 10, 14\rangle == \langle 0, -4, -6\rangle = \langle 0, 6, 4\rangle$
$\widetilde{SK}_7 = \widetilde{L}_7[-]\widetilde{E}_7 = \langle 67, 8, 11\rangle[-]\langle 67, 8, 11\rangle = \langle 0, 0, 0\rangle$.

Thus, another critical path is $1 \to 3 \to 7$. This critical path is drawn in a blue bold line shown in Fig. 6.9.

See that for this simple network, there are two alternative paths whose lengths are $\langle 67, 12, 17\rangle$ and $\langle 67, 8, 11\rangle$. In these two TFNs, the mean values are the same, while spreads are different. The proposed method is also valid for crisp weights of

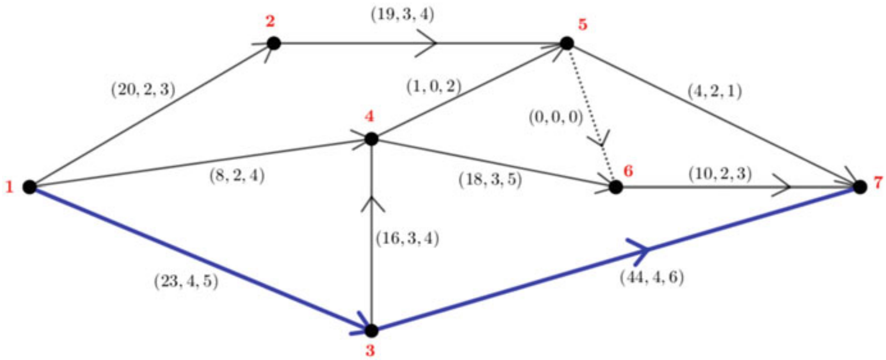


Fig. 6.9 The pessimistic critical path for the network of Fig. 6.7

the edges. If both the weights are set to zero, the entire problem becomes a problem on a crisp graph. This is the beauty of working with fuzzy numbers.

In a crisp network, all the weights of the edges are real numbers. But in this problem, the weights of the edges are considered as TFNs. On the other hand, in a probabilistic network, the duration on the activities is represented in terms of three expected times, viz., optimistic, pessimistic, and most likely. In this representation, three different times (real numbers) are considered. So, for a probabilistic network, the length of the critical path is not certain; it is expected. But in the fuzzy network, the length of the critical path is a fuzzy number, which gives the possibility to complete the project.

10 Conclusion

In reality, the time duration to complete an activity is, in general, uncertain. So, here, the duration of an activity is taken as a triangular fuzzy number. For this project, the possible completion time is calculated. The most probable time and the maximum time needed to finish the project are determined. Also, the minimum time to complete the project is computed. The same method is applied to other types of fuzzy numbers, such as interval numbers, trapezoidal fuzzy numbers, pentagonal fuzzy numbers, etc.

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Chapter 7
Generalized Hukuhara Global Subdifferentiability in Interval Optimization Problems

Anshika, Krishan Kumar, and Debdas Ghosh

1 Introduction

Various fields of optimization such as nonlinear applied analysis, variational analysis, etc., have faced significant evolution over the last few decades. Convex analysis [23] plays a vital role in the development of nonlinear optimization. However, the local properties of nonsmooth nonconvex functions are not necessarily global. Due to these drawbacks of convex analysis, various subdifferentials, along with different directional derivatives, asymptotic functions, and conjugation procedures were defined for different classes of nonconvex functions (see [1, 4, 22, 24]).

The first fundamentals on interval arithmetic were discussed by Moore [21] in 1966. However, there were a few demerits of the proposed arithmetic difference (see [16]). Due to this, for the difference of compact intervals, Hukuhara [16] introduced a new rule known as H-difference or the Hukuhara difference of intervals. However, this definition suffers certain drawbacks (see [5]). Despite of that, Wu [26] presented the notions of limit, continuity, and differentiability of IVFs. After that, Markov [20] introduced a new nonstandard subtraction which helped in removing the deformity of H-differentiability and proposed the generalized calculus on intervals. Subsequently, Stefaninni and Bede [25] defined a concept of gH-difference or generalized Hukuhara difference of intervals which provides an additive inverse for all pairs of compact intervals (which fails in previously defined differences).

The calculus of IVFs performs an important part in observing the optimality and smoothness of an IVF. At first, Hukuhara [16] used H-difference of intervals to define the notion of differentiability of IVFs. However, the H-differentiability

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C. Jana et al. (eds.), *Fuzzy Optimization, Decision-making and Operations Research*, https://doi.org/10.1007/978-3-031-35668-1_7

possesses certain demerits (see [5]). After the commencement of the gH-difference of IVFs, Stefaninni and Bede [25] proposed the idea of gH-differentiability for IVFs. Thereafter, Chalco-Cano et al. [5] proposed various concepts on the calculus of IVFs. One can note that the linear orderedness property fails for intervals. Therefore, the ordering of intervals is necessary to study the arithmetic of intervals. Due to this, Ishibuchi and Tanaka [17] observed partial ordering structures and Ghosh et al. [12] studied the variable ordering relations for intervals with their application to IOPs. To extend the calculus on IOPs further, the concepts of gH-directional derivative, gH-Fréchet derivative, and gH-Gâteaux derivative of IVFs were introduced in [10]. Further, the notions of gH-partial derivative and gH-gradient for IVF were discussed in [8]. Ghosh et al. [8] derived the KKT conditions for unconstrained IOPs by considering the geometrical significance of the solutions. Many researchers have extended this calculus of IVFs for instance [7, 9, 13, 18, 19] and references therein.

In the analysis of nonsmooth IOPs, it has been observed that very few calculi have been developed using gH-subdifferentiability for IVFs (see [2, 3, 18, 19]). However, these theories involve the help of convex IVFs, which is insufficient in providing adequate information on the local or global efficient solution of nonconvex nonsmooth IVFs. Motivated from this, we present the theory on gH-global subdifferentiability of IVFs. Towards this, we define the notions of gH-lower and gH-upper global directional derivatives, which help in defining the notions of gH-lower and gH-upper global subdifferentials for IVFs. Also, we present a comparison of gH-global subdifferentials with gH-subdifferential, gH-proximal subdifferential, and gH-Fréchet subdifferential for IVFs.

From the literature on IVFs and IOPs (see [2, 6, 8, 10, 11, 19]), it has been observed that there is no separate calculus to deal with the class of nonconvex IVFs. Also, it can be seen that if the objective function contains the set of minimizers, then the Dini directional derivatives fail to provide optimality conditions for IVFs [14, 15]. In this chapter, we propose the notions of gH-lower and gH-upper global directional derivatives for IVFs, which provides a global view with respect to gH-Dini directional derivatives. Next, we have shown that the gH-global directional derivatives possess some important properties from gH-directional derivatives (see Remark 2). In the sequel, we have defined the concepts of gH-lower and gH-upper global subdifferentials for IVFs. It has been proved that the gH-upper global subdifferential equals the gH-upper Dini subdifferential. Finally, with the help of gH-lower global subdifferential for IVFs, we have presented the optimality conditions to estimate efficient solutions to nonsmooth IOPs.

The whole work is arranged in the following order. Section 2 covers some calculus rules and basic tools for IVFs. In the next Section 3, the gH-lower and gH-upper global directional derivatives for IVFs are proposed. Several important characteristics of gH-lower and gH-upper global directional derivatives are given in the sequel. The same section presents a relation on the gH-global directional derivative of the maximum IVF. In the next Section 4, the notions of gH-lower and gH-upper subdifferentials are discussed with the help of gH-lower and gH-upper global directional derivatives for IVFs. Next, we have proved that the gH-upper global subdifferential equals the upper gH-Dini subdifferential for IVFs. Several properties on gH-global subdifferentiability are developed along with calculus rules and various comparisons have been performed with other subdifferentials for nonconvex IVFs. After that, Section 5 provides the necessary and sufficient conditions for ensuring the global efficient solutions to an IOP. Finally, Section 6 discusses the conclusion and future approaches for the study.

2 Preludes

The following notations are used throughout the chapter:

- $\mathbb{R}$ and $\mathbb{R}_+$ denote the set of real numbers and the set of nonnegative real numbers, respectively
- Bold capital letters refer to the elements of $I(\mathbb{R})$
- Bold capital letters with a cap refers to the elements of $I(\mathbb{R})^n$
- $I(\mathbb{R})$ refers to the set of all closed and bounded intervals
- $\overline{I(\mathbb{R})} = I(\mathbb{R}) \cup \{-\infty, +\infty\}$
- $\mathscr{B}(0, 1) = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ denotes the closed unit ball in $\mathbb{R}^n$ centered at origin
- $\mathbf{0}$ represents the interval $[0, 0]$.

2.1 Fundamental Operations and Dominance Relations on Intervals

Consider two intervals $\mathbf{L} = [\underline{l}, \bar{l}]$ and $\mathbf{M} = \left[\underline{m}, \overline{m}\right]$. Then, the addition and the difference between the two intervals are defined by

$$\mathbf{L} \oplus \mathbf{M} = \left[\, \underline{l} + \underline{m},\ \bar{l} + \overline{m}\, \right],\ \mathbf{L} \ominus \mathbf{M} = \left[\, \underline{l} - \overline{m},\ \bar{l} - \underline{m}\, \right],\ \text{respectively.}$$

Similarly, the product of an interval $\mathbf{L}$ with a real number δ is defined by

$$\delta \odot \mathbf{L} = \mathbf{L} \odot \delta = \begin{cases} [\delta \underline{l},\ \delta \bar{l}], & \text{if } \delta \geq 0 \\ [\delta \bar{l},\ \delta \underline{l}], & \text{if } \delta < 0. \end{cases}$$

The norm [21] of an interval $\mathbf{L} = [\underline{l}, \bar{l}] \in I(\mathbb{R})$ and an interval vector $\widehat{\mathbf{L}} = (\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_n)^\top \in I(\mathbb{R})^n$ is defined by

$$\|\mathbf{L}\|_{I(\mathbb{R})} = \max\{|\underline{l}|, |\bar{l}|\} \text{ and } \|\widehat{\mathbf{L}}\|_{I(\mathbb{R})^n} = \sum_{i=1}^{n} \|\mathbf{L}_i\|_{I(\mathbb{R})},\ \text{respectively.}$$

A real number l, or more appropriately, the singleton set $\{l\}$, can be represented by the interval $[l, l]$. In this case, interval $\mathbf{L} = [l, l]$ is called a degenerate interval.

Definition 1 **(gH-difference of intervals [25])** Let $\mathbf{L}$, $\mathbf{M} \in I(\mathbb{R})$ such that $\mathbf{L} = [\underline{l}, \overline{l}]$ and $\mathbf{M} = [\underline{m}, \overline{m}]$. Then, the gH-difference between $\mathbf{L}$ and $\mathbf{M}$, denoted by $\mathbf{L} \ominus_{gH} \mathbf{M}$, is defined by

$$\mathbf{L} \ominus_{gH} \mathbf{M} = \left[\min\{\underline{l} - \underline{m}, \overline{l} - \overline{m}\},\ \max\{\underline{l} - \underline{m}, \overline{l} - \overline{m}\}\right].$$

For the product space $I(\mathbb{R})^n = I(\mathbb{R}) \times I(\mathbb{R}) \times \cdots \times I(\mathbb{R})$ (n times), the algebraic operations are defined as follows. For two elements $\widehat{\mathbf{L}} = (\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_n)^\top$ and $\widehat{\mathbf{M}} = (\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_n)^\top$ of $I(\mathbb{R})^n$, the operation $\widehat{\mathbf{L}} \star \widehat{\mathbf{M}}$ is defined by

$$\widehat{\mathbf{L}} \star \widehat{\mathbf{M}} = (\mathbf{L}_1 \star \mathbf{M}_1, \mathbf{L}_2 \star \mathbf{M}_2, \ldots, \mathbf{L}_n \star \mathbf{M}_n)^\top,$$

where $\star \in \{\oplus, \ominus, \ominus_{gH}\}$.

Definition 2 **(Dominance of intervals [27])** Let $\mathbf{L} = [\underline{l}, \overline{l}]$ and $\mathbf{M} = [\underline{m}, \overline{m}] \in I(\mathbb{R})$.

(i) If $\underline{l} \le \underline{m}$ and $\overline{l} \le \overline{m}$, then $\mathbf{M}$ is said to be dominated by $\mathbf{L}$ and it is denoted by $\mathbf{L} \preceq \mathbf{M}$;

(ii) If $\mathbf{L} \preceq \mathbf{M}$ and $\mathbf{L} \neq \mathbf{M}$, then $\mathbf{M}$ is said to be strictly dominated by $\mathbf{L}$ it is denoted by $\mathbf{L} \prec \mathbf{M}$. Equivalently, $\mathbf{L} \prec \mathbf{M}$ if and only if
‘$\underline{l} < \underline{m}$ and $\overline{l} \le \overline{m}$’ or ‘$\underline{l} \le \underline{m}$ and $\overline{l} < \overline{m}$’ or ‘$\underline{l} < \underline{m}$ and $\overline{l} < \overline{m}$’;

(iii) If neither $\mathbf{L} \preceq \mathbf{M}$ nor $\mathbf{M} \preceq \mathbf{L}$, we say that none of $\mathbf{L}$ and $\mathbf{M}$ dominates the other, or $\mathbf{L}$ and $\mathbf{M}$ are not comparable. Equivalently, $\mathbf{L}$ and $\mathbf{M}$ are not comparable if either ‘$\underline{l} < \underline{m}$ and $\overline{l} > \overline{m}$’ or ‘$\underline{l} > \underline{m}$ and $\overline{l} < \overline{m}$’.

2.2 *Calculus of IVFs*

Throughout the article, assume that $\mathscr{S}$ is a nonempty subset of $\mathbb{R}^n$, unless stated otherwise.

Definition 3 **(gH-continuous IVF [8])** Let $\boldsymbol{\Theta} : \mathscr{S} \to I(\mathbb{R})$ be an IVF on $\mathscr{S}$. Let $\bar{r} \in \mathscr{S}$, and $w \in \mathbb{R}^n$ with $\bar{r} + w \in \mathscr{S}$. The IVF $\boldsymbol{\Theta}$ is said to be gH-continuous at $\bar{r}$ if

$$\lim_{\|w\| \to 0} \left(\boldsymbol{\Theta}(\bar{r} + w) \ominus_{gH} \boldsymbol{\Theta}(\bar{r})\right) = \mathbf{0}.$$

Definition 4 **(gH-derivative [25])** The gH-derivative of an IVF $\boldsymbol{\Theta} : \mathbb{R} \to I(\mathbb{R})$ at $\bar{r} \in \mathbb{R}$ is defined by

$$\boldsymbol{\Theta}'(\bar{r}) = \lim_{w \to 0} \tfrac{1}{w} \odot (\boldsymbol{\Theta}(\bar{r} + w) \ominus_{gH} \boldsymbol{\Theta}(\bar{r})),\ \text{provided the limit exists.}$$

Definition 5 (Proper IVF) Let $\mathscr{S} \subseteq \mathbb{R}^n$ and $\boldsymbol{\Theta} : \mathscr{S} \to \overline{I(\mathbb{R})}$ be an extended IVF. Then, $\boldsymbol{\Theta}$ is called a proper IVF if there exists $\bar{r} \in \mathscr{S}$ such that

$$\boldsymbol{\Theta}(\bar{r}) \prec +\infty \text{ and } -\infty \prec \boldsymbol{\Theta}(r) \text{ for all } r \in \mathscr{S},$$

Definition 6 (Domain of an IVF) Let $\mathscr{S} \subseteq \mathbb{R}^n$. For an extended IVF $\boldsymbol{\Theta} : \mathscr{S} \to \overline{I(\mathbb{R})}$, the domain of $\boldsymbol{\Theta}$, denoted as dom $\boldsymbol{\Theta}$, is defined by

$$\text{dom } \boldsymbol{\Theta} = \{r \in \mathscr{S} : \|\boldsymbol{\Theta}(r)\|_{I(\mathbb{R})} \prec +\infty\}.$$

Lemma 1 (See [6]) *Let* $\mathbf{L}, \mathbf{M} \in I(\mathbb{R})$.

(i) If $\mathbf{L} \nprec \mathbf{M}$, *then* $\mathbf{L} \ominus_{gH} \mathbf{M} \nprec \mathbf{0}$
(ii) If $\mathbf{L} \prec \mathbf{M}$, *then* $\mathbf{L} \ominus_{gH} \mathbf{M} \prec \mathbf{0}$.

Lemma 2 (See [3]) *Let* $\mathbf{L}$, $\mathbf{M}$, *and* $\mathbf{N}$ *be three elements of* $I(\mathbb{R})$. *If* $\mathbf{L} \preceq \mathbf{M} \implies \mathbf{L} \ominus_{gH} \mathbf{N} \preceq \mathbf{M} \ominus_{gH} \mathbf{N}$.

Lemma 3 *Let* $\mathbf{L}, \mathbf{M} \in I(\mathbb{R})$. *If* $\mathbf{L} \prec \mathbf{M} \oplus \epsilon$, *then* $\mathbf{L} \ominus_{gH} \mathbf{M} \prec \epsilon$.

Proof Let $\mathbf{L} = [\underline{l}, \bar{l}]$, $\mathbf{M} = [\underline{m}, \overline{m}]$. Since $\mathbf{L} \preceq \mathbf{M} \oplus \epsilon$, then

$$\underline{l} \leq \underline{m} + \epsilon \text{ and } \bar{l} \leq \overline{m} + \epsilon \implies \underline{l} - \underline{m} \leq \epsilon \text{ and } \bar{l} - \overline{m} \leq \epsilon.$$

Therefore, $\mathbf{L} \ominus_{gH} \mathbf{M} \prec \epsilon$.

Definition 7 (Minimum and Maximum of intervals [3]) Let $\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_p$ be elements of $I(\mathbb{R})$ with $\mathbf{Z}_1 \preceq \mathbf{Z}_2 \preceq \cdots \preceq \mathbf{Z}_p$. Then,

$$\max\{\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_p\} = \mathbf{Z}_p \text{ and } \min\{\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_p\} = \mathbf{Z}_1.$$

Definition 8 (Supremum and infimum of a subset of $\overline{I(\mathbb{R})}$ [18]) Let $\mathbf{S} \subseteq \overline{I(\mathbb{R})}$. An interval $\bar{\mathbf{A}} \in I(\mathbb{R})$ is called an upper bound of $\mathbf{S}$ if $\mathbf{B} \preceq \bar{\mathbf{A}}$ for all $\mathbf{B}$ in $\mathbf{S}$. Also, an upper bound $\bar{\mathbf{A}}$ of $\mathbf{S}$ is called a supremum of $\mathbf{S}$ if

$$\bar{\mathbf{A}} \preceq \mathbf{C} \text{ for all upper bounds } \mathbf{C} \text{ of } \mathbf{S} \in I(\mathbb{R}).$$

Similarly, an interval $\bar{\mathbf{A}} \in I(\mathbb{R})$ is called a lower bound of $\mathbf{S}$ if $\bar{\mathbf{A}} \preceq \mathbf{B}$ for all $\mathbf{B}$ in $\mathbf{S}$ and a lower bound $\bar{\mathbf{A}}$ of $\mathbf{S}$ is called an infimum of $\mathbf{S}$ if

$$\mathbf{C} \preceq \bar{\mathbf{A}} \text{ for all lower bounds } \mathbf{C} \text{ of } \mathbf{S} \in I(\mathbb{R}).$$

Definition 9 (Supremum and infimum of an IVF [18]) Let $\mathscr{P} \subseteq \mathscr{S}$ and $\boldsymbol{\Theta} : \mathscr{P} \to \overline{I(\mathbb{R})}$ be an extended IVF. Then, the supremum of an IVF $\boldsymbol{\Theta}$ is defined by

$$\sup_{r \in \mathscr{P}} \boldsymbol{\Theta}(r) = \sup\{\boldsymbol{\Theta}(r) : r \in \mathscr{P}\}.$$

Similarly, the infimum of $\boldsymbol{\Theta}$, denoted by $\inf_{r\in\mathscr{P}} \boldsymbol{\Theta}(r)$ and defined by

$$\inf_{r\in\mathscr{P}} \boldsymbol{\Theta}(r) = \inf\{\boldsymbol{\Theta}(r) : r \in \mathscr{P}\}.$$

Lemma 4 (See [2]) *Let $\mathscr{P} \subseteq \mathscr{S}$ and $\boldsymbol{\Theta} : \mathscr{P} \to \overline{I(\mathbb{R})}$ be an extended IVF. Then, for $\mathscr{P}_1, \mathscr{P}_2 \subseteq \mathscr{P}$ with $\mathscr{P}_1 \subseteq \mathscr{P}_2$ and $\delta \geq 0$,*

(i) $\inf_{r\in\mathscr{P}_2} \boldsymbol{\Theta}(r) \preceq \inf_{r\in\mathscr{P}_1} \boldsymbol{\Theta}(r)$,
(ii) $\sup_{r\in\mathscr{P}_1} \boldsymbol{\Theta}(r) \preceq \sup_{r\in\mathscr{P}_2} \boldsymbol{\Theta}(r)$,
(iii) $\inf_{r\in\mathscr{P}} (\delta \odot \boldsymbol{\Theta})(r) = \delta \odot \inf_{r\in\mathscr{P}} \boldsymbol{\Theta}(r)$, *and*
(iv) $\sup_{r\in\mathscr{P}} (\delta \odot \boldsymbol{\Theta})(r) = \delta \odot \sup_{r\in\mathscr{P}} \boldsymbol{\Theta}(r)$.

Definition 10 (Sequence in $I(\mathbb{R})^n$ [11]) An IVF $\widehat{\boldsymbol{\Theta}} : \mathbb{N} \to I(\mathbb{R})^n$ is called a sequence in $I(\mathbb{R})^n$.

Definition 11 (Convergence of a sequence in $I(\mathbb{R})^n$ [11]) A sequence of interval vectors $\{\widehat{\mathbf{G}}_k\}$ in $I(\mathbb{R})^n$ is said to be convergent to an interval vector $\widehat{\mathbf{G}} \in I(\mathbb{R})^n$ if for every $\epsilon > 0$, there exists an $p \in \mathbb{N}$ such that

$$\|\widehat{\mathbf{G}}_k \ominus_{gH} \widehat{\mathbf{G}}\|_{I(\mathbb{R})^n} < \epsilon \text{ for each } k \geq p.$$

The interval vector $\widehat{\mathbf{G}}$ denotes limit of the sequence of interval vectors $\{\widehat{\mathbf{G}}_k\}$ and we have $\lim_{k\to\infty} \widehat{\mathbf{G}}_k = \widehat{\mathbf{G}}$.

Remark 1 (*See [11]*) If a sequence of interval vectors $\{\widehat{\mathbf{G}}_k\}$ in $I(\mathbb{R})^n$ converges to some interval vector $\widehat{\mathbf{G}} \in I(\mathbb{R})^n$, where $\widehat{\mathbf{G}}_k = (\mathbf{G}_{1_k}, \mathbf{G}_{2_k}, \ldots, \mathbf{G}_{n_k})^\top$ and $\widehat{\mathbf{G}} = (\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_n)^\top$, then the sequence $\{\mathbf{G}_{j_k}\}$ in $I(\mathbb{R})$ converges to $\mathbf{G}_j \in I(\mathbb{R})$ for each $j = 1, 2, \ldots, n$.

Lemma 5 (See [3]) *Let $\{\mathbf{L}_k\}$ and $\{\mathbf{M}_k\}$ be two sequences in $I(\mathbb{R})^n$ and $\lim_{k\to\infty} \mathbf{L}_k = \mathbf{L}$ and $\lim_{k\to\infty} \mathbf{M}_k = \mathbf{M}$. If $\mathbf{L}_k \preceq \mathbf{M}_k$ for each k, then $\mathbf{L} \preceq \mathbf{M}$.*

Lemma 6 *Let $\{\mathbf{L}_k\}$ and $\{\mathbf{M}_k\}$ be two sequences in $I(\mathbb{R})^n$ and $\liminf_{k\to\infty} \mathbf{L}_k = \mathbf{L}$ and $\liminf_{k\to\infty} \mathbf{M}_k = \mathbf{M}$. If $\mathbf{L}_k \preceq \mathbf{M}_k$ for each k, then $\mathbf{L} \preceq \mathbf{M}$.*

Proof Let $\mathbf{L}_k \preceq \mathbf{M}_k$ for all k, then we have

$$\begin{aligned}
& \underline{l}_k \leq \underline{m}_k \text{ and } \overline{l}_k \leq \overline{m}_k \\
\Longrightarrow\ & \liminf_{k\to\infty} \underline{l}_k \leq \liminf_{k\to\infty} \underline{m}_k \text{ and } \liminf_{k\to\infty} \overline{l}_k \leq \liminf_{k\to\infty} \overline{m}_k \\
\Longrightarrow\ & \underline{l} \leq \underline{m} \text{ and } \overline{l} \leq \overline{m} \\
\Longrightarrow\ & \mathbf{L} \preceq \mathbf{M}.
\end{aligned}$$

Definition 12 **(gH-subgradient [10])** Let $\boldsymbol{\Theta} : \mathscr{S} \to \overline{I(\mathbb{R})}$ be a convex IVF on a convex set $\mathscr{S} \subseteq \mathbb{R}^n$. An element $\widehat{\mathbf{G}} \in I(\mathbb{R})^n$ is called a gH-subgradient of $\boldsymbol{\Theta}$ at $\bar{r} \in \mathscr{S}$ if

$$(r - \bar{r})^\top \odot \widehat{\mathbf{G}} \preceq \boldsymbol{\Theta}(r) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}) \text{ for each } r \in \mathscr{S}.$$

Assembling all gH-subgradients of $\boldsymbol{\Theta}$ at $\bar{r} \in \mathscr{S}$ together forms a set called gH-subdifferential of $\boldsymbol{\Theta}$ at $\bar{r}$ and is denoted by $\partial\boldsymbol{\Theta}(\bar{r})$.

Definition 13 **(gH-Dini lower and gH-Dini upper directional derivatives)** Let $\boldsymbol{\Theta} : \mathscr{S} \to \overline{I(\mathbb{R})}$ be an IVF on $\mathscr{S}$. Then, the gH-Dini lower and gH-Dini upper directional derivatives of $\boldsymbol{\Theta}$ at $\bar{r}$ in the direction $w \in \mathbb{R}^n$ are given by

$$\boldsymbol{\Theta}_{\mathscr{D}}(\bar{r})(w) = \liminf_{t \downarrow 0} \tfrac{1}{t} \odot (\boldsymbol{\Theta}(\bar{r} + tw) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}))$$
$$\text{and } \boldsymbol{\Theta}^{\mathscr{D}}(\bar{r})(w) = \limsup_{t \downarrow 0} \tfrac{1}{t} \odot (\boldsymbol{\Theta}(\bar{r} + tw) \ominus_{gH} \boldsymbol{\Theta}(\bar{r})), \text{ respectively,}$$

provided the limits exist.

Definition 14 **(gH-lower and gH-upper Dini subdifferentials)** Let $\boldsymbol{\Theta} : \mathscr{S} \to \overline{I(\mathbb{R})}$ be a proper IVF and $\bar{r} \in \text{dom } \boldsymbol{\Theta}$. Then, the gH-lower and gH-upper Dini subdifferentials of $\boldsymbol{\Theta}$ at $\bar{r} \in \text{dom } \boldsymbol{\Theta}$ are defined by

$$\partial_{\mathscr{D}}\boldsymbol{\Theta}(\bar{r}) = \{\widehat{\mathbf{G}} \in I(\mathbb{R})^n : w^\top \odot \widehat{\mathbf{G}} \preceq \boldsymbol{\Theta}_{\mathscr{D}}(\bar{r})(w) \text{ for all } w \in \mathbb{R}^n\} \text{ and}$$
$$\partial^{\mathscr{D}}\boldsymbol{\Theta}(\bar{r}) = \{\widehat{\mathbf{G}} \in I(\mathbb{R})^n : w^\top \odot \widehat{\mathbf{G}} \preceq \boldsymbol{\Theta}^{\mathscr{D}}(\bar{r})(w) \text{ for all } w \in \mathbb{R}^n\}, \text{ respectively,}$$

provided the limits exist.

Definition 15 **(gH-proximal subdifferentiability for IVF)** Let $\boldsymbol{\Theta} : \mathscr{S} \to \overline{I(\mathbb{R})}$ be an IVF on $\mathscr{S}$. Then, the gH-proximal subdifferential of $\boldsymbol{\Theta}$ at $\bar{r} \in \mathscr{S}$ is defined by

$$\partial_{\mathscr{P}}\boldsymbol{\Theta}(\bar{r}) = \Big\{\widehat{\mathbf{G}} \in I(\mathbb{R})^n : \exists\, M>0, \delta>0 \text{ such that } (r - \bar{r})^\top \odot \widehat{\mathbf{G}} \ominus_{gH} M\|r - \bar{r}\|^2 \preceq \boldsymbol{\Theta}(r) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}) \text{ for every } r \in \mathscr{B}(\bar{r}, \delta)\Big\}.$$

Definition 16 **(gH-Fréchet subdifferentiability for IVF)** Let $\boldsymbol{\Theta} : \mathscr{S} \to \overline{I(\mathbb{R})}$ be an IVF on $\mathscr{S}$. Then, the gH-Fréchet subdifferential of $\boldsymbol{\Theta}$ at $\bar{r} \in \mathscr{S}$ is defined by

$$\partial_{\mathscr{F}}\boldsymbol{\Theta}(\bar{r}) = \left\{\widehat{\mathbf{G}} \in I(\mathbb{R})^n : \mathbf{0} \preceq \liminf_{\substack{r \to 0 \\ r \neq 0}} \tfrac{1}{\|r - \bar{r}\|_{\mathscr{S}}} \odot (((\boldsymbol{\Theta}(r) \ominus_{gH} \boldsymbol{\Theta}(\bar{r})) \ominus_{gH} (r - \bar{r})^\top \odot \widehat{\mathbf{G}})\right\}.$$

3 gH-global directional derivative for IVFs

Definition 17 **(gH-upper and gH-lower global directional derivatives for IVF)** Let $\boldsymbol{\Theta} : \mathscr{S} \rightarrow \overline{I(\mathbb{R})}$ be a proper IVF on $\mathscr{S}$. Then, for every $\lambda > 0$, the gH-upper and gH-lower global directional derivatives of $\boldsymbol{\Theta}$ at $\bar{r} \in \text{dom } \boldsymbol{\Theta}$ in the direction $w \in \mathbb{R}^n$ are given by

$$\boldsymbol{\Theta}^{\mathscr{G}_\lambda}(\bar{r})(w) = \sup_{0<t\leq\lambda} \tfrac{1}{t} \odot (\boldsymbol{\Theta}(\bar{r} + tw) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}))$$

$$\text{and } \boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})(w) = \inf_{0<t\leq\lambda} \tfrac{1}{t} \odot (\boldsymbol{\Theta}(\bar{r} + tw) \ominus_{gH} \boldsymbol{\Theta}(\bar{r})), \text{ respectively, provided the limits exist.}$$

Example 1 We calculate the gH-upper and gH-lower global directional derivatives at $\bar{r} = 0$ for the IVF $\boldsymbol{\Theta} : [-1, 1] \rightarrow I(\mathbb{R})$ given by

$$\boldsymbol{\Theta}(r) = [4r^2 - 2r, 2r^2 + 1].$$

The gH-upper global directional derivative of $\boldsymbol{\Theta}$ at $\bar{r} = 0$, $\lambda = 1$, and $w \in \mathbb{R}^n$ is

$$\begin{aligned}
\boldsymbol{\Theta}^{\mathscr{G}_\lambda}(0)(w) &= \sup_{0<t\leq 1} \tfrac{1}{t} \odot (\boldsymbol{\Theta}(tw) \ominus_{gH} \boldsymbol{\Theta}(0)) \\
&= \sup_{0<t\leq 1} \tfrac{1}{t} \odot ([4t^2w^2 - 2tw, 2t^2w^2 + 1] \ominus_{gH} [0, 1]) \\
&= \sup_{0<t\leq 1} \tfrac{1}{t} \odot [\min\{4t^2w^2 - 2tw, 2t^2w^2\}, \max\{4t^2w^2 - 2tw, 2t^2w^2\}] \\
&= \sup_{0<t\leq 1} [4tw^2 - 2w, 2tw^2] \\
&= [4w^2 - 2w, 2w^2],
\end{aligned}$$

and the gH-upper global directional derivative of $\boldsymbol{\Theta}$ at $\bar{r} = 0$, $\lambda = 1$, and $w \in \mathbb{R}^n$ is

$$\begin{aligned}
\boldsymbol{\Theta}_{\mathscr{G}_\lambda}(0)(w) &= \inf_{0<t\leq 1} \tfrac{1}{t} \odot (\boldsymbol{\Theta}(tw) \ominus_{gH} \boldsymbol{\Theta}(0)) \\
&= \inf_{0<t\leq 1} \tfrac{1}{t} \odot ([4t^2w^2 - 2tw, 2t^2w^2 + 1] \ominus_{gH} [0, 1]) \\
&= \inf_{0<t\leq 1} \tfrac{1}{t} \odot [\min\{4t^2w^2 - 2tw, 2t^2w^2\}, \max\{t^2w^2 - 2tw, 2t^2w^2\}] \\
&= \inf_{0<t\leq 1} [4tw^2 - 2w, 2tw^2] \\
&= [-2, 0] \odot w.
\end{aligned}$$

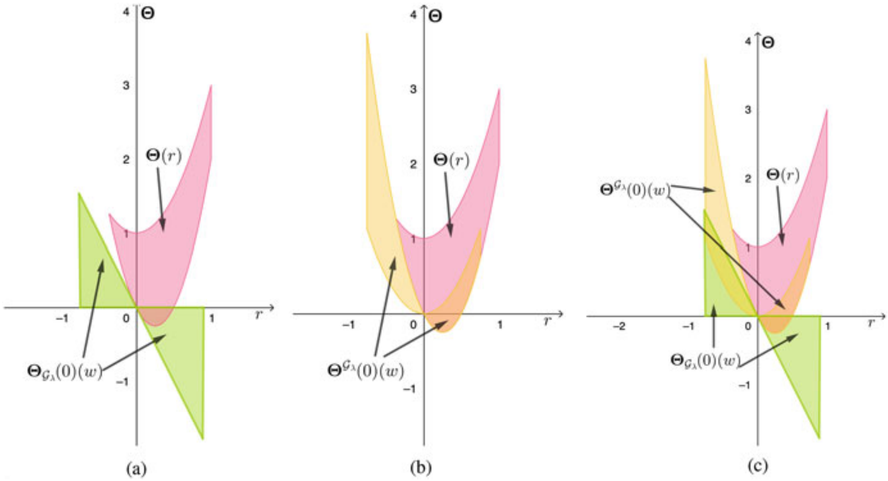

Fig. 7.1 Geometrical view of gH-upper and gH-lower global directional derivatives of IVF $\boldsymbol{\Theta}$ at $\bar{r} = 0$

A geometrical view of gH-global directional derivatives of $\boldsymbol{\Theta}$ is given in Fig. 7.1. The IVFs $\boldsymbol{\Theta}_{\mathscr{G}_\lambda}(0)(w)$ and $\boldsymbol{\Theta}^{\mathscr{G}_\lambda}(0)(w)$ are given in Fig. 7.1a and Fig. 7.1b, respectively. A combined view of gH-lower and gH-upper global directional derivatives of $\boldsymbol{\Theta}$ at $\bar{r} = 0$ is shown in Fig. 7.1c.

Lemma 7 *Let $\boldsymbol{\Theta} : \mathscr{S} \to \overline{I(\mathbb{R})}$ be an IVF and $\bar{r} \in dom\ \boldsymbol{\Theta}$. Then, for every $\lambda > 0$ and $w \in \mathbb{R}^n$, we have*

$$(\alpha \odot \boldsymbol{\Theta})^{\mathscr{G}_\lambda}(\bar{r})(w) = \alpha \odot \boldsymbol{\Theta}^{\mathscr{G}_\lambda}(\bar{r})(w), \text{ where } \alpha \geq 0.$$

Proof Note that for any $\lambda > 0$ and $w \in \mathbb{R}^n$, we have

$$\begin{aligned}(\alpha \odot \boldsymbol{\Theta})^{\mathscr{G}_\lambda}(\bar{r})(w) &= \sup_{0<t\leq\lambda} \tfrac{1}{t} \odot ((\alpha \odot \boldsymbol{\Theta})(\bar{r} + tw) \ominus_{gH} (\alpha \odot \boldsymbol{\Theta})(\bar{r}))\\ &= \sup_{0<t\leq\lambda} \tfrac{\alpha}{t} \odot \left(\boldsymbol{\Theta}(\bar{r} + tw) \ominus_{gH} \boldsymbol{\Theta}(\bar{r})\right)\\ &= \alpha \odot \left(\sup_{0<t\leq\lambda} \tfrac{1}{t} \odot \left(\boldsymbol{\Theta}(\bar{r} + tw) \ominus_{gH} \boldsymbol{\Theta}(\bar{r})\right)\right) \text{ from (iv) of Lemma 4}\\ &= \alpha \odot \boldsymbol{\Theta}^{\mathscr{G}_\lambda}(\bar{r})(w).\end{aligned}$$

Remark 2 It is to be observe that if $\boldsymbol{\Theta} : \mathscr{S} \to \overline{I(\mathbb{R})}$ is an IVF on $\mathscr{S}$ and $\bar{r} \in \operatorname{dom} \boldsymbol{\Theta}$, then for every $\lambda > 0$, $\gamma > 0$, the required assertions hold:

(i) If $\gamma \geq \lambda$, then from Definition 8, Definition 9, Definition 17, and Lemma 4, and for any $w \in \mathbb{R}^n$

$$\sup_{0<t\leq\lambda} \tfrac{1}{t} \odot (\boldsymbol{\Theta}(\bar{r}+tw) \ominus_{gH} \boldsymbol{\Theta}(\bar{r})) \preceq \sup_{0<t\leq\gamma} \tfrac{1}{t} \odot (\boldsymbol{\Theta}(\bar{r}+tw) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}))$$
$$\implies \boldsymbol{\Theta}^{\mathscr{G}_\lambda}(\bar{r})(w) \preceq \boldsymbol{\Theta}^{\mathscr{G}_\gamma}(\bar{r})(w)$$
$$\text{and } \inf_{0<t\leq\gamma} \tfrac{1}{t} \odot (\boldsymbol{\Theta}(\bar{r}+tw) \ominus_{gH} \boldsymbol{\Theta}(\bar{r})) \preceq \inf_{0<t\leq\lambda} \tfrac{1}{t} \odot (\boldsymbol{\Theta}(\bar{r}+tw) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}))$$
$$\implies \boldsymbol{\Theta}_{\mathscr{G}_\gamma}(\bar{r})(w) \preceq \boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})(w).$$

(ii) In view of Definition 13, and Definition 17, for any $w \in \mathbb{R}^n$, we have

$$\sup_{\lambda>0} \boldsymbol{\Theta}^{\mathscr{G}_\lambda}(\bar{r})(w) = \lim_{\lambda\to+\infty} \boldsymbol{\Theta}^{\mathscr{G}_\lambda}(\bar{r})(w) = \sup_{t>0} \tfrac{1}{t} \odot (\boldsymbol{\Theta}(\bar{r}+tw) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}))$$
$$\text{and } \inf_{\lambda>0} \boldsymbol{\Theta}^{\mathscr{G}_\lambda}(\bar{r})(w) = \limsup_{t\downarrow 0} \tfrac{1}{t} \odot (\boldsymbol{\Theta}(\bar{r}+tw) \ominus_{gH} \boldsymbol{\Theta}(\bar{r})) = \boldsymbol{\Theta}^{\mathscr{D}}(\bar{r})(w).$$

In the similar manner, for any $w \in \mathbb{R}^n$, we have

$$\sup_{\lambda>0} \boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})(w) = \liminf_{t\downarrow 0} \boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})(w) = \boldsymbol{\Theta}_{\mathscr{D}}(\bar{r})(w)$$
$$\text{and } \inf_{\lambda>0} \boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})(w) = \lim_{\lambda\to+\infty} \boldsymbol{\Theta}_{\mathscr{G}_\lambda} = \inf_{t>0} \tfrac{1}{t} \odot \boldsymbol{\Theta}(\bar{r}+tw) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}).$$

(iii) In view of Definition 8, Definition 9, Definition 13, Definition 17, and Lemma 4, for any $w \in \mathbb{R}^n$, we have

$$\inf_{\lambda>0} \boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})(w) \preceq \boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})(w) \preceq \sup_{\lambda>0} \boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})(w)$$
$$\implies \boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})(w) \preceq \sup_{\lambda>0} \boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})(w)$$
$$\implies \boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})(w) \preceq \boldsymbol{\Theta}_{\mathscr{D}}(\bar{r})(w) \text{ from (ii) of Remark 2}$$

and

$$\inf_{\lambda>0} \boldsymbol{\Theta}^{\mathscr{G}_\lambda}(\bar{r})(w) \preceq \boldsymbol{\Theta}^{\mathscr{G}_\lambda}(\bar{r})(w) \preceq \sup_{\lambda>0} \boldsymbol{\Theta}^{\mathscr{G}_\lambda}(\bar{r})(w)$$
$$\implies \inf_{\lambda>0} \boldsymbol{\Theta}^{\mathscr{G}_\lambda}(\bar{r})(w) \preceq \boldsymbol{\Theta}^{\mathscr{G}_\lambda}(\bar{r})(w)$$
$$\implies \boldsymbol{\Theta}^{\mathscr{D}}(\bar{r})(w) \preceq \boldsymbol{\Theta}^{\mathscr{G}_\lambda}(\bar{r})(w) \text{ from (ii) of Remark 2.}$$

Therefore, in view of the above relations, we conclude that $\boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})(w) \preceq \boldsymbol{\Theta}_{\mathscr{D}}(\bar{r})(w) \preceq \boldsymbol{\Theta}^{\mathscr{D}}(\bar{r})(w) \preceq \boldsymbol{\Theta}^{\mathscr{G}_\lambda}(\bar{r})(w)$.

Lemma 8 *Let $\boldsymbol{\Theta} : \mathscr{S} \to \overline{I(\mathbb{R})}$ be an IVF and $\bar{r} \in dom\ \boldsymbol{\Theta}$. Then, for every $\lambda > 0$ and $w \in \mathbb{R}^n$*

$$\boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})(\alpha w) = \alpha \odot \boldsymbol{\Theta}_{\mathscr{G}_{\lambda\alpha}}(\bar{r})(w), \text{ where } \alpha > 0.$$

Proof Note that for any $\lambda > 0$ and $w \in \mathbb{R}^n$, we have

$$\begin{aligned}
\boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})(\alpha w) &= \inf_{0<t\le\lambda} \tfrac{1}{t} \odot (\boldsymbol{\Theta}(\bar{r} + t(\alpha w)) \ominus_{gH} \boldsymbol{\Theta}(\bar{r})) \\
&= \inf_{0<t\le\lambda} \tfrac{\alpha}{\alpha t} \odot \big(\boldsymbol{\Theta}(\bar{r} + t(\alpha w)) \ominus_{gH} \boldsymbol{\Theta}(\bar{r})\big) \text{ for } \alpha > 0 \\
&= \alpha \odot \left(\inf_{0<t\le\lambda} \tfrac{1}{\alpha t} \odot (\boldsymbol{\Theta}(\bar{r} + t(\alpha w)) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}))\right) \text{ from (iii) of} \\
&\qquad \text{Lemma 4 and } \alpha > 0 \\
&= \alpha \odot \left(\inf_{0<t\alpha\le\lambda\alpha} \tfrac{1}{\alpha t} \odot (\boldsymbol{\Theta}(\bar{r} + (t\alpha) w) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}))\right) \text{ for } \alpha > 0 \\
&= \alpha \odot \left(\inf_{0<\gamma\le\lambda\alpha} \tfrac{1}{\alpha t} \odot (\boldsymbol{\Theta}(\bar{r} + \gamma w) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}))\right) \text{ for } \alpha > 0 \\
&= \alpha \odot \boldsymbol{\Theta}_{\mathscr{G}_{\lambda\alpha}}(\bar{r})(w).
\end{aligned}$$

Therefore, $\boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})(\alpha w) = \alpha \odot \boldsymbol{\Theta}_{\mathscr{G}_{\lambda\alpha}}(\bar{r})(w)$ for all $\lambda > 0$, $\alpha > 0$ and $w \in \mathbb{R}^n$.

Theorem 1 (gH-global directional derivative of the maximum IVF) *Let $\mathscr{S} \subseteq \mathbb{R}^n$ and A be any finite set of indices. Let for every $i \in A$, $\boldsymbol{\Theta}^i : \mathscr{S} \to \overline{I(\mathbb{R})}$ be a gH-continuous IVF such that $\boldsymbol{\Theta}^i_{\mathscr{G}_\lambda}(\bar{r})(w)$ exists for each $\bar{r} \in \mathscr{S}$. Let for every $r \in \mathscr{S}$, the set $\{\boldsymbol{\Theta}^i(r) : i \in A\}$ is a set of comparable intervals and define*

$$\boldsymbol{\Theta}(r) = \max_{i\in A} \boldsymbol{\Theta}^i(r).$$

Then, for any $\bar{r} \in \mathscr{S}$ and $w \in \mathscr{S}$,

$$\boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})(w) = \max_{i\in I(\bar{r})} \boldsymbol{\Theta}^i_{\mathscr{G}_\lambda}(\bar{r})(w), \text{ where } I(\bar{r}) = \{i \in A : \boldsymbol{\Theta}_i(\bar{r}) = \boldsymbol{\Theta}(\bar{r})\}. \tag{7.1}$$

Proof Let $\bar{r} \in \mathscr{S}$ and $w \in \mathscr{S}$ such that $\bar{r} + tw \in \mathscr{S}$ for $t > 0$. Then,

$\boldsymbol{\Theta}^i(\bar{r} + tw) \preceq \boldsymbol{\Theta}(\bar{r} + tw)$ for each $i \in A$

or, $\boldsymbol{\Theta}^i(\bar{r} + tw) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}) \preceq \boldsymbol{\Theta}(\bar{r} + tw) \ominus_{gH} \boldsymbol{\Theta}(\bar{r})$ from Lemma 2, for each $i \in A$

or, $\boldsymbol{\Theta}^i(\bar{r} + tw) \ominus_{gH} \boldsymbol{\Theta}^i(\bar{r}) \preceq \boldsymbol{\Theta}(\bar{r} + tw) \ominus_{gH} \boldsymbol{\Theta}(\bar{r})$ for each $i \in I(\bar{r})$

or, $\inf\limits_{0<t\leq\lambda} \frac{1}{t} \odot (\boldsymbol{\Theta}^i(\bar{r}+tw) \ominus_{gH} \boldsymbol{\Theta}^i(\bar{r})) \preceq \inf\limits_{0<t\leq\lambda} \frac{1}{t} \odot (\boldsymbol{\Theta}(\bar{r}+tw) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}))$

for each $i \in I(\bar{r})$

or, $\max \boldsymbol{\Theta}^i_{\mathscr{G}_\lambda}(\bar{r})(w) \preceq \boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})(w)$ from (iii) of Lemma 4, for each $i \in I(\bar{r})$. (7.2)

Conversely. let us consider a neighbourhood $\mathscr{N}(\bar{r})$ such that $I(r) \subset I(\bar{r})$ for each $r \in \mathscr{N}(\bar{r})$. Assume contrarily that there exists a sequence $\{r_k\}$ in $\mathscr{S}$ with $r_k \to \bar{r}$ such that $I(r_k) \not\subset I(\bar{r})$. Choose $i_k \in I(r_k)$ but $i_k \notin I(\bar{r})$. Since $I(r_k)$ is closed, $i_k \to \bar{i} \in I(r_k)$. By gH-continuity of $\boldsymbol{\Theta}^i$, we have

$$\boldsymbol{\Theta}^{\bar{i}}(r_k) = \boldsymbol{\Theta}(r_k) \implies \boldsymbol{\Theta}^{\bar{i}}(\bar{r}) = \boldsymbol{\Theta}(\bar{r}) \text{ as } k \to \infty,$$

which is a contradiction to $i_k \notin I(\bar{r})$. Thus, $I(r) \subset I(\bar{r})$ for all $r \in \mathscr{N}(\bar{r})$.
Consider $\{t_k\} \subset \mathbb{R}_+$, $t_k \to t$ and $\bar{r}+t_k w \in \mathscr{N}(\bar{r})$ for all $w \in \mathscr{S}$ and $t \in (0, \lambda]$, $\lambda > 0$. Then,

$\boldsymbol{\Theta}^i(\bar{r}) \preceq \boldsymbol{\Theta}(\bar{r})$ for all $i \in A$

or, $\boldsymbol{\Theta}(\bar{r}+t_k w) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}) \preceq \boldsymbol{\Theta}(\bar{r}+t_k w) \ominus_{gH} \boldsymbol{\Theta}^i(\bar{r})$ from Lemma 2, for all $i \in A$

or, $\boldsymbol{\Theta}(\bar{r}+t_k w) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}) \preceq \boldsymbol{\Theta}^i(\bar{r}+t_k w) \ominus_{gH} \boldsymbol{\Theta}^i(\bar{r})$ for all $i \in I(\bar{r}+t_k w)$

$$\text{or, } \inf_{0<t\leq\lambda} \left(\lim_{k\to\infty} \frac{1}{t_k} \odot (\boldsymbol{\Theta}(\bar{r}+t_k w) \ominus_{gH} \boldsymbol{\Theta}(\bar{r})) \right)$$

$$\preceq \inf_{0<t\leq\lambda} \left(\lim_{k\to\infty} \frac{1}{t_k} \odot (\boldsymbol{\Theta}^i(\bar{r}+t_k w) \ominus_{gH} \boldsymbol{\Theta}^i(\bar{r})) \right)$$

from Lemma 5 and for all $i \in I(\bar{r})$

or, $\boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})(w) \preceq \max \boldsymbol{\Theta}^i_{\mathscr{G}_\lambda}(\bar{r})(w)$ for all $i \in I(\bar{r})$. (7.3)

From (7.2) and (7.3), we get $\boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})(w) = \max\limits_{i\in I(\bar{r})} \boldsymbol{\Theta}^i_{\mathscr{G}_\lambda}(\bar{r})(w)$.

4 *gH*-global subdifferentiability for IVFs

Definition 18 (**gH-lower and gH-upper global subdifferentials**) Let $\boldsymbol{\Theta} : \mathscr{S} \to \overline{I(\mathbb{R})}$ be a proper IVF and $\bar{r} \in \text{dom}\, \boldsymbol{\Theta}$. Then, for every $\lambda > 0$, the gH-lower and gH-upper global subdifferentials of $\boldsymbol{\Theta}$ at $\bar{r} \in \text{dom}\, \boldsymbol{\Theta}$ are defined by

$$\partial_{\mathscr{G}_\lambda} \boldsymbol{\Theta}(\bar{r}) = \{\widehat{\mathbf{G}} \in I(\mathbb{R})^n : w^\top \odot \widehat{\mathbf{G}} \preceq \boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})(w) \text{ for all } w \in \mathscr{B}(0,1)\}$$

$$\text{and } \partial^{\mathscr{G}_\lambda} \boldsymbol{\Theta}(\bar{r}) = \{\widehat{\mathbf{G}} \in I(\mathbb{R})^n : w^\top \odot \widehat{\mathbf{G}} \preceq \boldsymbol{\Theta}^{\mathscr{G}_\lambda}(\bar{r})(w) \text{ for all } w \in \mathscr{B}(0,1)\},$$

respectively.

Example 2 In this example, we calculate the gH-lower and gH-upper global subgradients of an IVF $\boldsymbol{\Theta} : [-3, 3] \to I(\mathbb{R})$ given by

$$\boldsymbol{\Theta}(r) = \left[\sin|r|, \tfrac{|r|}{6} + 1\right] \text{ at } \bar{r} = 0 \text{ and } \lambda = 1.$$

Let us assume that there exists a $\mathbf{G} \in \partial_{\mathscr{G}_\lambda} \boldsymbol{\Theta}(0)$. Then, for all $w \in \mathscr{B}(0, 1)$ and for $\lambda = 1$, we have

$$\begin{aligned}
& w \odot \mathbf{G} \preceq \boldsymbol{\Theta}_{\mathscr{G}_\lambda}(0)(w) \\
\Longrightarrow\ & w \odot \mathbf{G} \preceq \inf_{0<t\leq 1} \tfrac{1}{t} \odot (\boldsymbol{\Theta}(tw) \ominus_{gH} \boldsymbol{\Theta}(0)) \\
\Longrightarrow\ & w \odot \mathbf{G} \preceq \inf_{0<t\leq 1} \tfrac{1}{t} \odot \left[\sin|tw|, \tfrac{|tw|}{6}\right] \\
\Longrightarrow\ & w \odot \mathbf{G} \preceq \left[\inf_{0<t\leq 1} \tfrac{1}{t} \odot \sin|tw|, \tfrac{|w|}{6}\right] \\
\Longrightarrow\ & w \odot \mathbf{G} \preceq \left[0, \tfrac{1}{6}\right] \odot |w|.
\end{aligned}$$

Therefore, there arise the following cases:

Case (i) For $w \geq 0$, we have

$$w \odot \mathbf{G} \preceq \left[0, \tfrac{1}{6}\right] \odot w \implies \mathbf{G} \preceq \left[0, \tfrac{1}{6}\right].$$

Case (ii) For $w < 0$, we have

$$w \odot \mathbf{G} \preceq \left[0, \tfrac{1}{6}\right] \odot (-w) \implies \left[-\tfrac{1}{6}, 0\right] \preceq \mathbf{G}.$$

Hence, in view of Case (i) and Case (ii), we get

$$\partial_{\mathscr{G}_\lambda} \boldsymbol{\Theta}(0) = \left\{\mathbf{G} \in I(\mathbb{R}) : \left[-\tfrac{1}{6}, 0\right] \preceq \mathbf{G} \preceq \left[0, \tfrac{1}{6}\right]\right\}.$$

Now, assume that there exists an $\mathbf{S} \in \partial^{\mathscr{G}_\lambda} \boldsymbol{\Theta}(0)$. Then, for all $w \in \mathscr{B}(0, 1)$ and for $\lambda = 1$, we have

$$\begin{aligned}
& w \odot \mathbf{S} \preceq \boldsymbol{\Theta}_{\mathscr{G}_\lambda}(0)(w) \\
\Longrightarrow\ & w \odot \mathbf{S} \preceq \sup_{0<t\leq 1} \tfrac{1}{t} \odot (\boldsymbol{\Theta}(tw) \ominus_{gH} \boldsymbol{\Theta}(0)) \\
\Longrightarrow\ & w \odot \mathbf{S} \preceq \sup_{0<t\leq 1} \tfrac{1}{t} \odot \left[\tfrac{|tw|}{6}, \sin|tw|\right]
\end{aligned}$$

$$\implies \quad w \odot \mathbf{S} \preceq \left[\frac{|w|}{6}, \sup_{0<t\leq 1} \frac{1}{t} \odot \sin|tw| \right]$$

$$\implies \quad w \odot \mathbf{S} \preceq \left[\tfrac{1}{6}, 1\right] \odot |w|.$$

Therefore, there arise the following cases:

Case (i) For $w \geq 0$, we have

$$w \odot \mathbf{S} \preceq \left[\tfrac{1}{6}, 1\right] \odot w \implies \mathbf{S} \preceq \left[\tfrac{1}{6}, 1\right].$$

Case (ii) For $w < 0$, we have

$$w \odot \mathbf{S} \preceq \left[1, \tfrac{1}{6}\right] \odot (-w) \implies \left[-1, -\tfrac{1}{6}, \right] \preceq \mathbf{S}.$$

Hence, in view of Case (i) and Case (ii), we get

$$\partial_{\mathscr{G}_\lambda} \boldsymbol{\Theta}(0) = \left\{ \mathbf{S} \in I(\mathbb{R}) : \left[-1, -\tfrac{1}{6}\right] \preceq \mathbf{S} \preceq \left[\tfrac{1}{6}, 1\right] \right\}.$$

The geometrical view of gH-lower and gH-upper global subdifferentiability of $\boldsymbol{\Theta}$ of Example 2 is given in Fig. 7.2. The IVF $\boldsymbol{\Theta}$ is shown by the green region. For $\lambda = 1$ and at $r = 0$, the two possible gH-lower global subgradients of $\boldsymbol{\Theta}$ are denoted by $\mathbf{G}_1$ and $\mathbf{G}_2$ in Fig. 7.2a and the two possible gH-upper global subgradients of $\boldsymbol{\Theta}$ are denoted by $\mathbf{S}_1$ and $\mathbf{S}_2$ in Fig. 7.2b. A combined view of both gH-lower and gH-upper subgradients of $\boldsymbol{\Theta}$ is shown in Fig. 7.2c.

Theorem 2 *Let $\boldsymbol{\Theta} : \mathscr{S} \to \overline{I(\mathbb{R})}$ be an IVF and $\bar{r} \in dom\, \boldsymbol{\Theta}$. Then, for every $\lambda > 0$, the gH-upper global subdifferential equal to the gH-upper Dini subdifferential of $\boldsymbol{\Theta}$ at $\bar{r} \in dom\, \boldsymbol{\Theta}$, i.e.,*

$$\partial^{\mathscr{G}_\lambda} \boldsymbol{\Theta}(\bar{r}) = \partial^{\mathscr{D}} \boldsymbol{\Theta}(\bar{r}).$$

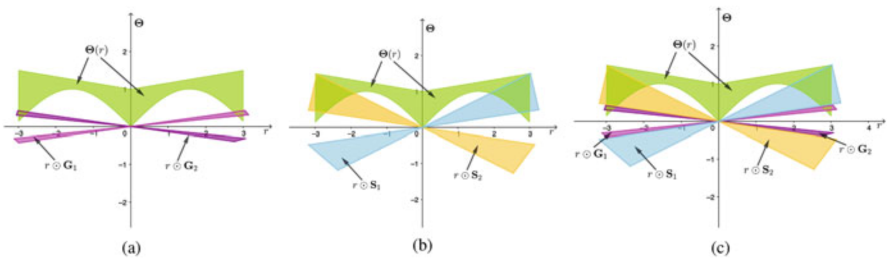

Fig. 7.2 Geometrical view of gH-lower and gH-upper subgradients of $\boldsymbol{\Theta}$ at $r = 0$

Proof Let there exists a $\widehat{\mathbf{G}} \in \partial^{\mathscr{G}_\lambda} \boldsymbol{\Theta}(\bar{r})$. Then, for every $\lambda > 0$ and for all $w \in \mathscr{B}(0, 1)$, we have

$$\begin{aligned}
&\iff w^\top \odot \widehat{\mathbf{G}} \preceq \boldsymbol{\Theta}^{\mathscr{G}_\lambda}(\bar{r})(w) \\
&\iff w^\top \odot \widehat{\mathbf{G}} \preceq \sup_{0<t\leq\lambda} \left(\tfrac{1}{t} \odot (\boldsymbol{\Theta}(\bar{r} + tw) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}))\right) \\
&\iff w^\top \odot \widehat{\mathbf{G}} \preceq \inf_{\lambda>0} \left(\sup_{0<t\leq\lambda} \left(\tfrac{1}{t} \odot (\boldsymbol{\Theta}(\bar{r} + tw) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}))\right)\right) \\
&\iff w^\top \odot \widehat{\mathbf{G}} \preceq \boldsymbol{\Theta}^{\mathscr{D}}(\bar{r})(w) \text{ from (ii) of Remark 2.}
\end{aligned}$$

Therefore, we conclude that $\partial^{\mathscr{G}_\lambda} \boldsymbol{\Theta}(\bar{r}) = \partial^{\mathscr{D}} \boldsymbol{\Theta}(\bar{r})$.

Theorem 3 *Let $\boldsymbol{\Theta} : \mathscr{S} \to \overline{I(\mathbb{R})}$ be an IVF and $\bar{r} \in dom\ \boldsymbol{\Theta}$. Then, for any $\delta > 0$ and for every $w \in \mathscr{B}(0, \delta)$, we have*

$$w^\top \odot \widehat{\mathbf{G}} \preceq \boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})(w) \text{ for all } \widehat{\mathbf{G}} \in \partial_{\mathscr{G}_{\lambda\delta}} \boldsymbol{\Theta}(\bar{r}) \text{ and for all } \lambda > 0.$$

Proof Note that for every $w \in \mathscr{B}(0, \delta)$, we can find a $w \in \mathscr{B}(0, 1)$ such that $w = p\delta$ with $\delta > 0$. Then, for every $\lambda > 0$ and from Lemma 8, we have

$$\boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})(w) = \boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})(p\delta) = \delta \odot \boldsymbol{\Theta}_{\mathscr{G}_{\lambda\delta}}(\bar{r})(p). \tag{7.4}$$

Let $\widehat{\mathbf{G}} \in \partial_{\mathscr{G}_{\lambda\delta}} \boldsymbol{\Theta}(\bar{r})$. Then, for every $\lambda > 0$ and for all $w \in \mathscr{B}(0, 1)$, we have

$$\begin{aligned}
&p^\top \odot \widehat{\mathbf{G}} \preceq \boldsymbol{\Theta}_{\mathscr{G}_{\lambda\delta}}(\bar{r})(p) \\
\implies &(\delta p)^\top \odot \widehat{\mathbf{G}} \preceq \delta \odot \boldsymbol{\Theta}_{\mathscr{G}_{\lambda\delta}}(\bar{r})(p) \\
\implies &w^\top \odot \widehat{\mathbf{G}} \preceq \delta \odot \boldsymbol{\Theta}_{\mathscr{G}_{\lambda\delta}}(\bar{r})(w) \text{ since } w = p\delta \\
\implies &w^\top \odot \widehat{\mathbf{G}} \preceq \boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})(w) \text{ from (7.4).}
\end{aligned}$$

Thus, $w^\top \odot \widehat{\mathbf{G}} \preceq \boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})(p)$ for all $\widehat{\mathbf{G}} \in \partial_{\mathscr{G}_{\lambda\delta}} \boldsymbol{\Theta}(\bar{r})$ and for all $\lambda > 0$.

Lemma 9 *Let $\boldsymbol{\Theta} : \mathscr{S} \to I(\mathbb{R})$ be an IVF and $\bar{r} \in dom\ \boldsymbol{\Theta}$. Then, for $\lambda > 0$ the following relations hold:*

(i) $\partial_{\mathscr{G}_\lambda}(\delta \odot \boldsymbol{\Theta})(\bar{r}) = \delta \odot \partial_{\mathscr{G}_\lambda} \boldsymbol{\Theta}(\bar{r})$, *where* $\delta > 0$.
(ii) If $\lambda_1 \geq \lambda_2 \geq 0$, *then* $\partial_{\mathscr{G}_{\lambda_1}} \boldsymbol{\Theta}(\bar{r}) \subseteq \partial_{\mathscr{G}_{\lambda_2}} \boldsymbol{\Theta}(\bar{r})$.

Proof (i) Let $\widehat{\mathbf{G}} \in \delta \odot \partial_{\mathscr{G}_\lambda} \boldsymbol{\Theta}(\bar{r})$ with $\lambda > 0$. Then, we can write $\widehat{\mathbf{G}} = \delta \odot \widehat{\mathbf{G}}'$, where $\widehat{\mathbf{G}}' \in \partial_{\mathscr{G}_\lambda} \boldsymbol{\Theta}(\bar{r})$. Thus, for all $w \in \mathscr{B}(0, 1)$, we have

$$w^\top \odot \widehat{\mathbf{G}}' \preceq \boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})(w)$$

$$\implies w^\top \odot \left(\frac{1}{\delta} \odot \widehat{\mathbf{G}}\right) \preceq \boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})(w) \text{ for } \delta > 0$$

$$\implies w^\top \odot \widehat{\mathbf{G}} \preceq \delta \odot \inf_{0<t\leq\lambda} \left(\frac{1}{t} \odot (\boldsymbol{\Theta}(\bar{r} + tw) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}))\right) \text{ for } \delta > 0$$

$$\implies w^\top \odot \widehat{\mathbf{G}} \preceq \inf_{0<t\leq\lambda} \left(\delta \odot \frac{1}{t} \odot (\boldsymbol{\Theta}(\bar{r} + tw) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}))\right) \text{ from (iii) of Lemma 4 and } \delta > 0$$

$$\implies w^\top \odot \widehat{\mathbf{G}} \preceq \inf_{0<t\leq\lambda} \left(\frac{1}{t} \odot ((\delta \odot \boldsymbol{\Theta})(\bar{r} + tw) \ominus_{gH} (\delta \odot \boldsymbol{\Theta})(\bar{r}))\right) \text{ for } \delta>0$$

$$\implies w^\top \odot \widehat{\mathbf{G}} \preceq (\delta \odot \boldsymbol{\Theta})_{\mathscr{G}_\lambda}(\bar{r})(w)$$

$$\implies \widehat{\mathbf{G}} \in \partial_{\mathscr{G}_\lambda}(\delta \odot \boldsymbol{\Theta})(\bar{r}).$$

Conversely, assume that $\widehat{\mathbf{G}} \in \partial_{\mathscr{G}_\lambda}(\delta \odot \boldsymbol{\Theta})(\bar{r})$. Then, for $\lambda > 0$ and $w \in \mathscr{B}(0, 1)$, we have

$$w^\top \odot \widehat{\mathbf{G}} \preceq (\delta \odot \boldsymbol{\Theta})_{\mathscr{G}_\lambda}(\bar{r})(w)$$

$$\implies w^\top \odot \widehat{\mathbf{G}} \preceq \inf_{0<t\leq\lambda} \left(\frac{1}{t} \odot (\delta \odot \boldsymbol{\Theta}(\bar{r} + tw) \ominus_{gH} \delta \odot \boldsymbol{\Theta}(\bar{r}))\right)$$

$$\implies \frac{1}{\delta} \odot (w^\top \odot \widehat{\mathbf{G}}) \preceq \inf_{0<t\leq\lambda} \left(\frac{1}{t} \odot (\boldsymbol{\Theta}(\bar{r} + tw) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}))\right)$$

$$\implies \frac{1}{\delta} \odot (w^\top \odot \widehat{\mathbf{G}}) \preceq \boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})(w)$$

$$\implies \frac{1}{\delta} \odot \widehat{\mathbf{G}} \in \partial_{\mathscr{G}_\lambda} \boldsymbol{\Theta}(\bar{r})$$

$$\implies \widehat{\mathbf{G}} \in \delta \odot \partial_{\mathscr{G}_\lambda} \boldsymbol{\Theta}(\bar{r}).$$

Thus, we conclude that $\partial_{\mathscr{G}_\lambda}(\delta \odot \boldsymbol{\Theta})(\bar{r}) = \delta \odot \partial_{\mathscr{G}_\lambda} \boldsymbol{\Theta}(\bar{r})$ where $\delta > 0$.

(ii) Let $\widehat{\mathbf{G}} \in \partial_{\mathscr{G}_{\lambda_1}} \boldsymbol{\Theta}(\bar{r})$ such that for $\lambda_1 > 0$ and $w \in \mathscr{B}(0, 1)$, we have

$$w^\top \odot \widehat{\mathbf{G}} \preceq \boldsymbol{\Theta}_{\mathscr{G}_{\lambda_1}}(\bar{r})(w)$$

$$\text{or, } w^\top \odot \widehat{\mathbf{G}} \preceq \boldsymbol{\Theta}_{\mathscr{G}_{\lambda_1}}(\bar{r})(w) \preceq \boldsymbol{\Theta}_{\mathscr{G}_{\lambda_2}}(\bar{r})(w) \text{ from (ii) of Remark 2}$$

$$\text{or, } \widehat{\mathbf{G}} \in \partial_{\mathscr{G}_{\lambda_2}} \boldsymbol{\Theta}(\bar{r}).$$

Therefore, $\partial_{\mathscr{G}_{\lambda_1}} \boldsymbol{\Theta}(\bar{r}) \subseteq \partial_{\mathscr{G}_{\lambda_2}} \boldsymbol{\Theta}(\bar{r})$.

Theorem 4 *Let $\boldsymbol{\Theta} : \mathscr{S} \to I(\mathbb{R})$ be an IVF and $\bar{r} \in dom\, \boldsymbol{\Theta}$. Let an IVF $\mathbf{M}(\bar{r}) = \boldsymbol{\Theta}(a\bar{r} + b)$ with $a, b \in \mathbb{R}$ and $a > 0$. Then, for $\lambda > 0$, we have*

$$\partial_{\mathscr{G}_\lambda} \mathbf{M}(\bar{r}) = a \odot \partial_{\mathscr{G}_{a\lambda}} \boldsymbol{\Theta}(\bar{r}).$$

Proof Let $\widehat{\mathbf{G}} \in \partial_{\mathscr{G}_\lambda} \mathbf{M}(\bar{r})$ such that for $\lambda > 0$ and $w \in \mathscr{B}(0, 1)$, we have

$$
\begin{aligned}
& w^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{M}_{\mathscr{G}_\lambda}(\bar{r})(w) \\
\iff & w^\top \odot \widehat{\mathbf{G}} \preceq \inf_{0<t\le\lambda} \left(\tfrac{1}{t} \odot (\mathbf{M}(\bar{r}+tw) \ominus_{gH} \mathbf{M}(\bar{r}))\right) \\
\iff & w^\top \odot \widehat{\mathbf{G}} \preceq \inf_{0<t\le\lambda} \left(\tfrac{1}{t} \odot (\boldsymbol{\Theta}(a(\bar{r}+tw)+b) \ominus_{gH} \boldsymbol{\Theta}(a\bar{r}+b))\right) \\
\iff & w^\top \odot \widehat{\mathbf{G}} \preceq \inf_{0<t\le\lambda} \left(\tfrac{1}{t} \odot (\boldsymbol{\Theta}(a\bar{r}+b+atw) \ominus_{gH} \boldsymbol{\Theta}(a\bar{r}+b))\right) \\
\iff & w^\top \odot \widehat{\mathbf{G}} \preceq \boldsymbol{\Theta}_{\mathscr{G}_\lambda}(a\bar{r}+b)(aw) \\
\iff & w^\top \odot \widehat{\mathbf{G}} \preceq a \odot \boldsymbol{\Theta}_{\mathscr{G}_{a\lambda}}(a\bar{r}+b)(w) \text{ from Lemma 8} \\
\iff & \tfrac{1}{a} \odot (w^\top \odot \widehat{\mathbf{G}}) \preceq \boldsymbol{\Theta}_{\mathscr{G}_{a\lambda}}(\bar{r})(w) \\
\iff & \tfrac{1}{a} \odot \widehat{\mathbf{G}} \in \partial_{\mathscr{G}_{a\lambda}}\boldsymbol{\Theta}(\bar{r}) \\
\iff & \widehat{\mathbf{G}} \in a \odot \partial_{\mathscr{G}_{a\lambda}}\boldsymbol{\Theta}(\bar{r}).
\end{aligned}
$$

Therefore, we conclude that $\partial_{\mathscr{G}_\lambda}\mathbf{M}(\bar{r}) = a \odot \partial_{\mathscr{G}_{a\lambda}}\boldsymbol{\Theta}(\bar{r})$ for $\lambda > 0$.

Theorem 5 *Let $\boldsymbol{\Theta}, \mathbf{K} : \mathscr{S} \to \overline{I(\mathbb{R})}$ be two proper IVFs and $\bar{r} \in$ dom $\boldsymbol{\Theta} \cap$ dom $\mathbf{K}$. If $\boldsymbol{\Theta} = \mathbf{K}$ in an open neighbourhood of $\bar{r} \in \mathscr{S}$, then there exists an λ_0 such that $\partial_{\mathscr{G}_{\lambda_0}}\boldsymbol{\Theta}(\bar{r}) = \partial_{\mathscr{G}_{\lambda_0}}\mathbf{K}(\bar{r})$ for every $r \in \mathscr{B}(\bar{r}, \lambda_0)$.*

Proof Let $\bar{r} \in$ dom $\boldsymbol{\Theta} \cap$ dom $\mathbf{K}$ and J be an open neighbourhood of $\bar{r}$ and $\boldsymbol{\Theta}(r) = \mathbf{K}(r)$ for every $r \in J$. Then, there exists $\lambda_0 > 0$ such that

$$\boldsymbol{\Theta}(r) = \mathbf{K}(r) \text{ for all } r \in \mathscr{B}(\bar{r}, \lambda_0).$$

Note that $\mathscr{B}(\bar{r}, \lambda_0) \subseteq \mathscr{B}(\bar{r}, 2\lambda_0) \subseteq J$. Then, for all $r \in \mathscr{B}(\bar{r}, \lambda_0)$, we have $r + tw \in \mathscr{B}(\bar{r}, 2\lambda_0)$ for all $t \in (0, \lambda_0]$ and $w \in \mathscr{B}(0, 1)$. Thus, for every $r \in \mathscr{B}(\bar{r}, \lambda_0)$ and for each $w \in \mathscr{B}(0, 1)$

$$
\begin{aligned}
\boldsymbol{\Theta}_{\mathscr{G}_{\lambda_0}}(r)(w) &= \inf_{0<t\le\lambda_0} \left(\tfrac{1}{t} \odot (\boldsymbol{\Theta}(r+tw) \ominus_{gH} \boldsymbol{\Theta}(r))\right) \\
&= \inf_{0<t\le\lambda_0} \left(\tfrac{1}{t} \odot (\mathbf{K}(r+tw) \ominus_{gH} \mathbf{K}(r))\right) = \mathbf{K}_{\mathscr{G}_{\lambda_0}}(r)(w).
\end{aligned}
$$

In view of the above relation, we can conclude that $\partial_{\mathscr{G}_{\lambda_0}}\boldsymbol{\Theta}(\bar{r}) = \partial_{\mathscr{G}_{\lambda_0}}\mathbf{K}(\bar{r})$ for all $r \in \mathscr{B}(\bar{r}, \lambda_0)$.

Theorem 6 (Comparison with other subdifferentials) *Let $\boldsymbol{\Theta} : \mathscr{S} \to I(\mathbb{R})$ be an IVF on $\mathscr{S}$ and $\bar{r} \in$ dom $\boldsymbol{\Theta}$. Then, for $\lambda > 0$, we have*

$$\partial\boldsymbol{\Theta}(\bar{r}) \subseteq \partial_{\mathscr{G}_\lambda}\boldsymbol{\Theta}(\bar{r}) \subseteq \partial_{\mathscr{P}}\boldsymbol{\Theta}(\bar{r}) \subseteq \partial_{\mathscr{F}}\boldsymbol{\Theta}(\bar{r}).$$

Proof First inclusion: Let there exists a $\widehat{\mathbf{G}} \in \partial\boldsymbol{\Theta}(\bar{r})$. Then, for all $r \in \mathscr{S}$, we have

$$
\begin{aligned}
& (r-\bar{r})^{\top} \odot \widehat{\mathbf{G}} \preceq \boldsymbol{\Theta}(r) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}) \\
\implies & (tw)^{\top} \odot \widehat{\mathbf{G}} \preceq \boldsymbol{\Theta}(\bar{r}+tw) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}) \text{ for all } w \in (0,\lambda],\ tw \\
& = (r-\bar{r}) \in \mathbb{R}^n, \lambda > 0 \\
\implies & w^{\top} \odot \widehat{\mathbf{G}} \preceq \inf_{0<t\leq\lambda} \left(\tfrac{1}{t} \odot (\boldsymbol{\Theta}(\bar{r}+tw) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}))\right) \text{ for all } w \in \mathbb{R}^n, \lambda > 0 \\
\implies & w^{\top} \odot \widehat{\mathbf{G}} \preceq \boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r}) \text{ for all } w \in \mathscr{B}(0,1), \lambda > 0.
\end{aligned}
$$

Therefore, $\widehat{\mathbf{G}} \in \partial_{\mathscr{G}_\lambda}\boldsymbol{\Theta}(\bar{r})$ for all $\lambda > 0$. Thus, $\partial\boldsymbol{\Theta}(\bar{r}) \subseteq \partial_{\mathscr{G}_\lambda}\boldsymbol{\Theta}(\bar{r})$ for $\lambda > 0$.

Second inclusion: Let there exists a $\widehat{\mathbf{G}} \in \partial_{\mathscr{G}_\lambda}\boldsymbol{\Theta}(\bar{r})$. Then, for every $\lambda > 0$ and for all $w \in \mathscr{B}(0,1)$, we obtain

$$
\begin{aligned}
& w^{\top} \odot \widehat{\mathbf{G}} \preceq \boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r}) \\
\implies & w^{\top} \odot \widehat{\mathbf{G}} \preceq \inf_{0<t\leq\lambda} \left(\tfrac{1}{t} \odot (\boldsymbol{\Theta}(\bar{r}+tw) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}))\right) \\
\implies & (tw)^{\top} \odot \widehat{\mathbf{G}} \preceq \boldsymbol{\Theta}(\bar{r}+tw) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}) \text{ for all } t \in (0,\lambda], \|u\| \leq 1 \\
\implies & (r-\bar{r})^{\top} \odot \widehat{\mathbf{G}} \preceq \boldsymbol{\Theta}(r) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}) \text{ for all } r \in \mathscr{B}(\bar{r},\lambda).
\end{aligned}
$$

It is to be observed that for any $M > 0$, the following relation holds

$$
(r-\bar{r}) \odot \widehat{\mathbf{G}} \ominus_{gH} M\|r-\bar{r}\|^2 \preceq (r-\bar{r}) \odot \widehat{\mathbf{G}}.
$$

Thus, in view of the above relation, we get

$$
(r-\bar{r})^{\top} \odot \widehat{\mathbf{G}} \ominus_{gH} M\|r-\bar{r}\|^2 \preceq \boldsymbol{\Theta}(r) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}) \text{ for all } r \in \mathscr{B}(\bar{r},\lambda),\ M > 0.
$$

Therefore, $\widehat{\mathbf{G}}(\bar{r}) \in \partial_{\mathscr{P}}\boldsymbol{\Theta}(\bar{r})$. Thus, $\partial_{\mathscr{G}_\lambda}\boldsymbol{\Theta}(\bar{r}) \subseteq \partial_{\mathscr{P}}\boldsymbol{\Theta}(\bar{r})$.

Third inclusion: Let there exists $\widehat{\mathbf{G}} \in \partial_{\mathscr{P}}\boldsymbol{\Theta}(\bar{r})$. Then, there exists an $M > 0$ and $\lambda > 0$ such that for all $r \in \mathscr{B}(\bar{r},\lambda)$, we have

$$
\begin{aligned}
& (r-\bar{r})^{\top} \odot \widehat{\mathbf{G}} \ominus_{gH} M\|r-\bar{r}\|^2 \preceq \boldsymbol{\Theta}(r) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}) \\
\implies & -M\|r-\bar{r}\|^2 \preceq (\boldsymbol{\Theta}(r) \ominus_{gH} \boldsymbol{\Theta}(\bar{r})) \ominus_{gH} (r-\bar{r})^{\top} \odot \widehat{\mathbf{G}} \text{ from Lemma 3} \\
\implies & -M\|r-\bar{r}\| \preceq \tfrac{1}{\|r-\bar{r}\|} \odot ((\boldsymbol{\Theta}(r) \ominus_{gH} \boldsymbol{\Theta}(\bar{r})) \ominus_{gH} (r-\bar{r})^{\top} \odot \widehat{\mathbf{G}}) \\
\implies & \liminf_{\substack{r\to 0\\ r\neq 0}}(-M\|r-\bar{r}\|) \preceq \liminf_{\substack{x\to 0\\ r\neq 0}} \tfrac{1}{\|r-\bar{r}\|} \odot ((\boldsymbol{\Theta}(r) \ominus_{gH} \boldsymbol{\Theta}(\bar{r})) \ominus_{gH} (r-\bar{r})^{\top} \odot \widehat{\mathbf{G}}) \\
& \text{from Lemma 6} \\
\implies & \mathbf{0} \preceq \liminf_{\substack{r\to 0\\ r\neq 0}} \tfrac{1}{\|r-\bar{r}\|} \odot ((\boldsymbol{\Theta}(r) \ominus_{gH} \boldsymbol{\Theta}(\bar{r})) \ominus_{gH} (r-\bar{r})^{\top} \odot \widehat{\mathbf{G}}).
\end{aligned}
$$

Therefore, $\widehat{\mathbf{G}} \in \partial_{\mathscr{F}} \boldsymbol{\Theta}(\bar{r})$. Thus, $\partial_{\mathscr{P}} \boldsymbol{\Theta}(\bar{r}) \subseteq \partial_{\mathscr{F}} \boldsymbol{\Theta}(\bar{r})$.

Remark 3 First inclusion: It is to be noted that the first inclusion in Theorem 6 may be strict. For instance, consider the IVF $\boldsymbol{\Theta} : \mathbb{R} \to I(\mathbb{R})$ define by

$$\boldsymbol{\Theta}(r) = \begin{cases} \left[-1, -\frac{1}{2}\right] \odot r \oplus [1, 2]], & r > 0 \\ [0, 0], & r \leq 0. \end{cases}$$

Let us check $\partial \boldsymbol{\Theta}(\bar{r})$ and $\partial_{\mathscr{G}_\lambda} \boldsymbol{\Theta}(\bar{r})$, where $\bar{r} = 0$ and $\lambda = 1$. Let us assume that there exists a $\mathbf{G} \in \partial \boldsymbol{\Theta}(0)$. Then, for each $r \in \mathbb{R}$, we have

$$\begin{aligned} (r - 0) \odot \mathbf{G} &\preceq \boldsymbol{\Theta}(r) \ominus_{gH} \boldsymbol{\Theta}(0) \\ r \odot \mathbf{G} &\preceq \begin{cases} \left[-1, -\frac{1}{2}\right] \odot r \oplus [1, 2], & r > 0 \\ [0, 0], & r \leq 0. \end{cases} \end{aligned} \tag{7.5}$$

Since $\mathbf{G} \in I(\mathbb{R})$, therefore we have

Case (a) For $\mathbf{0} \prec \mathbf{G}$, the relation

$$r \odot \mathbf{G} \preceq \left[-1, -\tfrac{1}{2}\right] \odot r \oplus [1, 2] \text{ does not hold for any } r > 1.$$

Case (b) For $\mathbf{G} \prec \mathbf{0}$, the relation

$$r \odot \mathbf{G} \preceq [0, 0] \text{ does not hold for any } r < 0.$$

Case (c) If $\mathbf{G}$ and $\mathbf{0}$ are not comparable, then the relation

$$r \odot \mathbf{G} \preceq [0, 0] \text{ does not hold for any } r < 0.$$

Case (d) For $\mathbf{G} = \mathbf{0}$, the relation

$$\mathbf{0} \preceq \left[-1, -\tfrac{1}{2}\right] \odot r \oplus [1, 2] \text{ does not hold for any } r > 1.$$

In view of Case (a), Case (b), Case (c), and Case (d), it can be observed that there is no $\mathbf{G}$, which satisfies relation 7.5. Thus, we conclude that $\partial \boldsymbol{\Theta}(0) = \emptyset$. Now, the gH-lower global derivative of $\boldsymbol{\Theta}$ at $\bar{r} = 0$ and $\lambda = 1$ is given by

$$
\begin{aligned}
\boldsymbol{\Theta}_{\mathscr{G}_\lambda}(0)(w) &= \inf_{0<t\le 1} \frac{1}{t} \odot (\boldsymbol{\Theta}(tw) \ominus_{gH} \boldsymbol{\Theta}(0)) \\
&= \begin{cases} \inf\limits_{0<t\le 1} \frac{1}{t} \odot \left(\left[-1, -\frac{1}{2}\right] \odot (tw) \oplus [1,2]\right), & w > 0 \\ \mathbf{0}, & w \le 0 \end{cases} \\
&= \begin{cases} \left[-1, -\frac{1}{2}\right] \odot w \oplus [1,2], & w > 0 \\ \mathbf{0}, & w \le 0. \end{cases}
\end{aligned}
$$

Next, let $\mathbf{G} \in \partial_{\mathscr{G}_\lambda} \boldsymbol{\Theta}(0)$ such that for every $w \in \mathscr{B}(0,1)$, we have

$$w \odot \mathbf{G} \preceq \boldsymbol{\Theta}_{\mathscr{G}_\lambda}(0)(w).$$

Since $\mathbf{G} \in I(\mathbb{R})$, therefore we have the following cases:

Case (i) If $\mathbf{0} \prec \mathbf{G}$, then the relation

$$w \odot \mathbf{G} \preceq \left[-1, -\frac{1}{2}\right] \odot w \oplus [1,2] \text{ does not hold for } w = 1.$$

Case (ii) If $\mathbf{G} \prec \mathbf{0}$, then the relation

$$w \odot \mathbf{G} \preceq [0,0] \text{ does not hold for any } -1 \le w < 0.$$

Case (iii) If $\mathbf{G}$ and $\mathbf{0}$ are not comparable, then the relation

$$w \odot \mathbf{G} \preceq [0,0] \text{ does not hold for any } -1 \le w < 0.$$

Case (iv) If $\mathbf{G} = \mathbf{0}$, then the relation

$$\mathbf{0} \preceq \left[-1, -\frac{1}{2}\right] \odot w \oplus [1,2] \text{ holds for every } w \in \mathscr{B}(0,1).$$

In view of Case (i), Case (ii), Case (iii), Case (iv), we get $\partial_{\mathscr{G}_\lambda} \boldsymbol{\Theta}(0) = \{[0,0]\}$. Thus, we conclude that $\partial \boldsymbol{\Theta}(0) \subset \partial_{\mathscr{G}_\lambda} \boldsymbol{\Theta}(0)$.

Second inclusion: It can be observed that the second inclusion in Theorem 6 may be strict. For instance, consider the IVF $\boldsymbol{\Theta} : \mathbb{R} \to I(\mathbb{R})$ define by

$$\boldsymbol{\Theta}(r) = \begin{cases} [1,2], & r > 0 \\ [0,0], & r \le 0. \end{cases}$$

Let us check $\partial_{\mathscr{G}_\lambda} \boldsymbol{\Theta}(\bar{r})$ and $\partial_{\mathscr{P}} \boldsymbol{\Theta}(\bar{r})$ at $\bar{r} = 0$ and $\lambda > 0$. The gH-lower global derivative of $\boldsymbol{\Theta}$ at $\bar{r} = 0$ is given by

$$\boldsymbol{\Theta}_{\mathscr{G}_\lambda}(0)(w) = \inf_{0<t\leq\lambda} \tfrac{1}{t} \odot (\boldsymbol{\Theta}(tw) \ominus_{gH} \boldsymbol{\Theta}(0)) = \begin{cases} \inf\limits_{0<t\leq\lambda} \frac{1}{\lambda} \odot [1, 2], & w > 0 \\ \mathbf{0}, & w \leq 0 \end{cases}$$

$$= \begin{cases} \frac{1}{\lambda} \odot [1, 2], & w > 0 \\ \mathbf{0}, & w \leq 0. \end{cases}$$

Next, let $\mathbf{G} \in \partial_{\mathscr{G}_\lambda} \boldsymbol{\Theta}(0)$ such that for $w \in \mathscr{B}(0, 1)$, we have

$$\begin{aligned} & w \odot \mathbf{G} \preceq \boldsymbol{\Theta}_{\mathscr{G}_\lambda}(0)(w) \\ \implies\ & w \odot \mathbf{G} \preceq \begin{cases} \frac{1}{\lambda} \odot [1, 2], & 0 < w \leq 1 \\ \mathbf{0}, & -1 \leq w \leq 0. \end{cases} \end{aligned}$$

There arise the following two cases:

Case (a) For $-1 \leq w \leq 0$, we have

$$w \odot \mathbf{G} \preceq [0, 0] \implies \mathbf{0} \preceq \mathbf{G}.$$

Case (b) For $0 < w \leq 1$, we have

$$w \odot \mathbf{G} \preceq \tfrac{1}{\lambda} \odot [1, 2] \implies w\underline{g} \leq \frac{1}{\lambda} \text{ and } w\overline{g} \leq \frac{1}{2\lambda} \implies \underline{g} \leq \frac{1}{w\lambda} \text{ and } \overline{g} \leq \frac{1}{2w\lambda}.$$

Since $\lambda > 0$ is arbitrary and $w \in (0, 1]$, therefore we get $\mathbf{G} \preceq \frac{1}{\lambda} \odot [1, 2]$.

Therefore, in view of Case (a) and Case (b), we obtain $\partial_{\mathscr{G}_\lambda} \boldsymbol{\Theta}(0) = \{\mathbf{G} \in I(\mathbb{R}) : \mathbf{0} \preceq \mathbf{G} \preceq \frac{1}{\lambda} \odot [1, 2]\}$.

Now, let $\mathbf{G} \in \partial_{\mathscr{P}} \boldsymbol{\Theta}(0)$. Then, take $M = 1$ and $\lambda > 0$ such that for all $r \in \mathscr{B}(0, \lambda)$, we have

$$(r - 0) \odot \mathbf{G} \ominus_{gH} |r - 0|^2 \preceq \boldsymbol{\Theta}(r) \ominus_{gH} \boldsymbol{\Theta}(0).$$

There arise the following two cases:

Case (i) For $-\lambda \leq r \leq 0$, we have

$$r \odot \mathbf{G} \ominus_{gH} r^2 \preceq [0, 0] \implies [-\lambda, -\lambda] \preceq \mathbf{G}.$$

Case (ii) For $0 < r \leq -\lambda$, we have

$$r \odot \mathbf{G} \ominus_{gH} [r^2, r^2] \preceq \frac{1}{\lambda} \odot [1, 2] \implies r\underline{g} - r^2 \leq \frac{1}{\lambda} \text{ and } r\underline{g} - r^2 \leq \frac{2}{\lambda}$$
$$\implies \underline{g} \leq \frac{2}{\sqrt{\lambda}} \text{ and } \overline{g} \leq \frac{2\sqrt{2}}{\sqrt{\lambda}}$$
$$\implies \mathbf{G} \preceq \left[\frac{2}{\sqrt{\lambda}}, \frac{2\sqrt{2}}{\sqrt{\lambda}}\right].$$

Therefore, in view of Case (i) and Case (ii), we obtain

$$\partial_{\mathscr{P}} \boldsymbol{\Theta}(0) = \left\{ \mathbf{G} \in I(\mathbb{R}) : [-\lambda, -\lambda] \preceq \mathbf{G} \preceq \left[\frac{2}{\sqrt{\lambda}}, \frac{2\sqrt{2}}{\sqrt{\lambda}}\right] \right\}.$$

Thus, we conclude that $\partial_{\mathscr{G}_\lambda} \boldsymbol{\Theta}(0) \subset \partial_{\mathscr{P}} \boldsymbol{\Theta}(0)$.

5 Application in Nonsmooth Nonconvex Optimization

In this section, we investigate the efficient solutions to the following IOP:

$$\inf_{x \in \mathscr{S}} \boldsymbol{\Theta}(r), \tag{7.6}$$

where $\boldsymbol{\Theta} : \mathscr{S} \to I(\mathbb{R})$ is an IVF on $\mathscr{S}$.

Definition 19 (Efficient solution [8]) Let $\boldsymbol{\Theta} : \mathscr{S} \to I(\mathbb{R})$ be an IVF on $\mathscr{S}$. Then, $\bar{r} \in \mathscr{S}$ is an efficient solution to the IOP (7.6) if and only if

$$\boldsymbol{\Theta}(r) \not\prec \boldsymbol{\Theta}(\bar{r}) \text{ for all } r \in \mathscr{S}.$$

Theorem 7 *Let $\boldsymbol{\Theta} : \mathscr{S} \to I(\mathbb{R})$ be an IVF on $\mathscr{S}$ and $\bar{r} \in \mathscr{S}$ be an efficient point of IOP (7.6). Then, for every $\lambda > 0$ and $w \in \mathbb{R}^n$,*

$$\boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})(w) \not\prec \mathbf{0}.$$

Proof Let $\bar{r} \in \mathscr{S}$ is an efficient point of IOP (7.6). Therefore, for every $r \in \mathscr{S}$, we have

$$\begin{aligned} & \boldsymbol{\Theta}(r) \not\prec \boldsymbol{\Theta}(\bar{r}) \\ \implies & \boldsymbol{\Theta}(\bar{r} + tw) \not\prec \boldsymbol{\Theta}(\bar{r}) \text{ for each } t > 0 \text{ and } w \in \mathbb{R}^n \\ \implies & \boldsymbol{\Theta}(\bar{r} + tw) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}) \not\prec \mathbf{0} \text{ for all } t > 0 \text{ and } w \in \mathbb{R}^n \text{ from (i) of Lemma 1} \end{aligned}$$

$$\implies \inf_{0<t\leq\lambda} \tfrac{1}{t} \odot (\boldsymbol{\Theta}(\bar{r}+tw) \ominus_{gH} \boldsymbol{\Theta}(\bar{r})) \nprec \mathbf{0} \text{ for all } t>0 \text{ and } w \in \mathbb{R}^n$$

$$\implies \boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})(w) \nprec \mathbf{0} \text{ for all } t>0 \text{ and } w \in \mathbb{R}^n.$$

To prove conversely, let us assume contrarily that there exists an $r \in \mathscr{S}$ such that

$$\boldsymbol{\Theta}(r) \prec \boldsymbol{\Theta}(\bar{r})$$

$$\implies \boldsymbol{\Theta}(\bar{r}+tw) \prec \boldsymbol{\Theta}(\bar{r}) \text{ for all } t>0 \text{ and } w \in \mathbb{R}^n$$

$$\implies \boldsymbol{\Theta}(\bar{r}+tw) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}) \prec \mathbf{0} \text{ for all } t>0 \text{ and } w \in \mathbb{R}^n \text{ from (ii) of Lemma 1}$$

$$\implies \inf_{0<t\leq\lambda} \tfrac{1}{t} \odot (\boldsymbol{\Theta}(\bar{r}+tw) \ominus_{gH} \boldsymbol{\Theta}(\bar{r})) \prec \mathbf{0} \text{ for all } t>0 \text{ and } w \in \mathbb{R}^n$$

$$\implies \boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})(w) \prec \mathbf{0} \text{ for all } t>0 \text{ and } w \in \mathbb{R}^n.$$

This is clearly a contradiction to the assumption that $\boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})(w) \nprec \mathbf{0}$ for all $t>0$ and $w \in \mathbb{R}^n$. Hence, $\bar{r}$ is an efficient point of the IOP (7.6).

Theorem 8 *Let $\boldsymbol{\Theta} : \mathscr{S} \to I(\mathbb{R})$ be an IVF and $\bar{r} \in dom(\boldsymbol{\Theta})$. If $\widehat{\mathbf{0}} \in \partial_{\mathscr{G}_\lambda}\boldsymbol{\Theta}(\bar{r})$ for some $\bar{r} \in \mathscr{S}$ and $\boldsymbol{\Theta} > 0$, where $\widehat{\mathbf{0}} \in I(\mathbb{R})^n$, then $\bar{r}$ is an efficient solution to the IOP (7.6).*

Proof Let $\widehat{\mathbf{0}} \in \partial_{\mathscr{G}_\lambda}\boldsymbol{\Theta}(\bar{r})$. Thus, for $\lambda > 0$ and for all $w \in \mathscr{B}(0,1)$, we have

$$w^\top \odot \widehat{\mathbf{0}} \preceq \boldsymbol{\Theta}_{\mathscr{G}_\lambda}(\bar{r})$$

$$\implies \mathbf{0} \preceq \inf_{0<t\leq\lambda} \tfrac{1}{t} \odot (\boldsymbol{\Theta}(\bar{r}+tw) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}))$$

$$\implies \mathbf{0} \preceq \boldsymbol{\Theta}(\bar{r}+tw) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}) \text{ for all } t \in (0,\lambda],\ \|u\| \leq 1$$

$$\implies \mathbf{0} \preceq \boldsymbol{\Theta}(r) \ominus_{gH} \boldsymbol{\Theta}(\bar{r}) \text{ for all } r \in \mathscr{B}(\bar{r},\lambda)$$

$$\implies \boldsymbol{\Theta}(\bar{r}) \preceq \boldsymbol{\Theta}(r) \text{ for all } r \in \mathscr{B}(\bar{r},\lambda).$$

Thus, we get $\boldsymbol{\Theta}(r) \nprec \boldsymbol{\Theta}(\bar{r})$. Hence, $\bar{r}$ is an efficient solution to the IOP (7.6).

Example 3 In this, we exemplify a verification of the result in Theorem 8. Consider the IOP:

$$\min_{(r_1,r_2)\in\mathscr{S}\subseteq\mathbb{R}^2} \boldsymbol{\Theta}(r_1,r_2) = [1,2] \odot |r_1-1| \oplus [3,4] \odot |r_2| \oplus [2,3]. \tag{7.7}$$

The IVF $\boldsymbol{\Theta}$ with $\underline{\theta}(r_1,r_2) = |r_1-1|+3|r_2|+2$ and $\overline{\theta}(r_1,r_2) = 2|r_1-1|+4|r_2|+3$ are depicted with red colour and multi colour in Fig. 7.3. In view of Fig. 7.3, observed that $(\bar{r}_1,\bar{r}_2) = (1,0)$ is a weak efficient solution to the IOP (7.7). Observing that

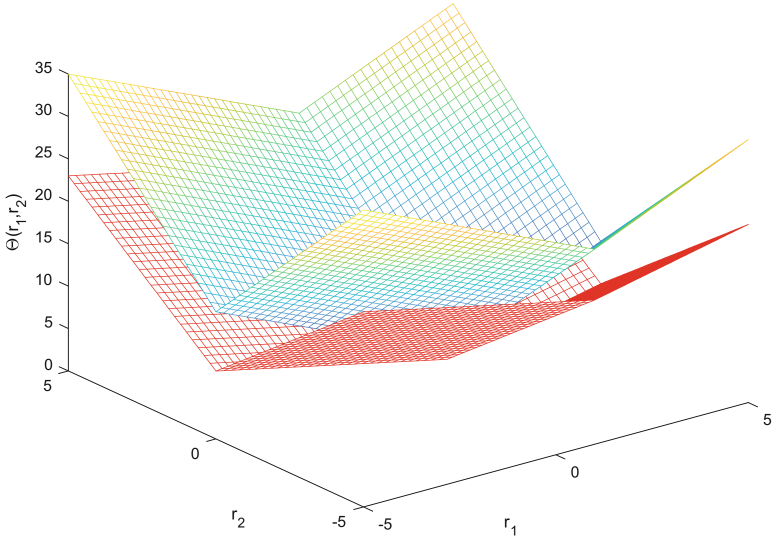

Fig. 7.3 The IVF $\boldsymbol{\Theta}$ of Example 3

$$\mathbf{0} = w_1 \odot \mathbf{0} \oplus w_2 \odot \mathbf{0} \preceq \boldsymbol{\Theta}_{\mathscr{G}_\lambda}(1,0)(w) \text{ for all } \lambda > 0 \text{ and } w \in \mathscr{B}(0,1)$$

$$\implies \mathbf{0} \preceq \inf_{0<t\leq 1} \frac{1}{t} \odot ([1,2] \odot |tw_1| \oplus [3,4] \odot |tw_2|) \text{ for all } \lambda{>}0 \text{ and } w{\in}\mathscr{B}(0,1)$$

$$\implies \mathbf{0} \preceq [1,2] \odot |w_1| \oplus [3,4] \odot |w_2| \text{ for all } \lambda > 0 \text{ and } w \in \mathscr{B}(0,1).$$

Therefore, we have $\widehat{\mathbf{0}} = (\mathbf{0}, \mathbf{0}) \in \partial \boldsymbol{\Theta}_{\mathscr{G}_\lambda}(1,0)$.

6 Conclusion and future directions

In this chapter, the notions of gH-lower and gH-upper global directional derivative (Definition 17) and the concepts of gH-lower and gH-upper global subdifferentials (Definition 18) has been introduced. A relation on the gH-global directional derivative of the maximum IVF is reported (Theorem 1). Next, we have observed that the upper gH-global subdifferential equals the upper gH-Dini subdifferential for IVFs (Theorem 2). Comparison with other subdifferentials has been performed (Theorem 6). Finally, two applications of proposed concepts in nonsmooth IOPs are given (Theorem 7 and Theorem 8).

In the future, we shall attempt to find the compactness of gH-lower global subdifferential and optimality conditions for constrained and unconstrained interval optimization problems. The proposed results are expected to be useful for algorithm purposes in nonsmooth programming. Also, a rule of the mean value theorem on the gH-global subdifferentiability of IVFs can be presented.

As an another approach, one may try to extend the proposed results to the difference of two approximately star-shaped and gap gH-continuous IVFs. Towards this, let $\boldsymbol{\Theta}, \mathbf{P} : \mathscr{S} \to \overline{I(\mathbb{R})}$ be two IVFs and let $\mathbf{T} = \boldsymbol{\Theta} \ominus_{gH} \mathbf{P}$ be an IVF, which is finite at $\bar{r} \in \mathscr{S}$ and $\mathscr{S}$ be a nonempty subset of $\mathbb{R}^n$. Consider a constrained IOP:

$$\min\ \mathbf{T}(r) \text{ subject to } r \in \mathscr{C}, \tag{7.8}$$

where $\mathscr{C}$ is a nonempty closed convex subset of $\mathscr{S}$. Assume that $\boldsymbol{\Theta}$ is approximately convex and gH-continuous IVF at $\bar{r} \in \mathscr{C}$. Then, a necessary condition for $\bar{r}$ to be an efficient point to the IOP (7.8) on $\mathscr{C}$ is given by

$$\partial_{\mathscr{G}_\lambda} \mathbf{P}(\bar{r}) \subseteq \partial_{\mathscr{G}_\lambda} \boldsymbol{\Theta}(\bar{r}) \oplus \mathscr{N}_{\mathscr{C}}(\bar{r}),$$

where $\mathscr{N}_{\mathscr{C}}(\bar{r})$ is normal cone of $\mathscr{C}$ at $\bar{r}$. The above defined relation will help in finding the necessary condition for a max-efficient solution to the following maximization IOP:

$$\max \mathbf{P}(r) \text{ subject to } r \in \mathscr{C}.$$

Further work can be performed to propose a gH-global subgradient method and its convergence to solve the unconstrained IOPs.

Acknowledgments We express our gratitude to the anonymous reviewers and the editors for their valuable comments and suggestions to improve the quality of the paper. D. Ghosh acknowledges the research grant MATRICS (MTR/2021/000696) from SERB, India to carry out this research work.

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Chapter 8
Role of Hexagonal Fuzzy Numbers While Applying the Max-Min Concept to a Transportation Problem

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1 Introduction

Transportation problem, which is a large network that is constructed in LPP, occurs in a variety of situations. The problem's fundamental notion is to determine the lowest total transportation cost to meet destination demand using supply at the origin. The transportation problem will be used in several situations, including production, capital, scheduling, location, stock management, and employee management, among others. Many approaches were developed recently to discover the best answer to the transportation challenge. Diverse sources contribute to varying terminals in such a method that the transportation cost is decreased in the transportation problem. Three strategies can be used to obtain a basic workable answer: (1) North West Corner method, (2) Least Cost method, and (3) Vogel's Approximation method.

According to the literature, the Vogel Approximation Method (VAM) approach is the best of the three. The Modified Distribution Method (MODI) approach is used to determine whether the transportation problem is optimal. The transportation issue can be divided into two categories: balanced and unbalanced. A balanced transportation problem occurs when the number of sources equals the number of demands. If not, it is referred to as an unbalanced transportation problem. If the

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C. Jana et al. (eds.), *Fuzzy Optimization, Decision-making and Operations Research*, https://doi.org/10.1007/978-3-031-35668-1_8

supply of an item is more than the demand, a dummy column should be added to make the situation balanced. If the demand exceeds the supply, the dummy row should be added to transform the imbalanced problem into a balanced transportation problem.

Transportation theory is a study of optimal transportation and resource allocation. Gaspard Monge, a French mathematician, formulated the Transportation theory in 1781. In our everyday life, we face a variety of challenges in decision-making, such as estimating the cost of money. Real Life problems in the uncertainty theory, established by L.A. Zadeh, is highly beneficial for dealing with a large number of data [1]. For real-time situations, we must identify the maximum or minimum optimum solution. Companies transport their products from the site of manufacture to the place of consumption. While each manufacturing site has a finite supply, each client has a distinct need that must be addressed. At this stage, transportation models are utilized to identify the lowest-cost shipping plan that will suit the customer's needs while staying within certain constraints.

Amarpeet Kaul and Amit Kumar [2] developed a new strategy for handling a transportation fuzzy problem based on the guess that the result is uncertain by transportation costs. Chen S [3] investigated whether the membership function for regular fuzzy numbers is constrained, and he suggested the idea of a generalized fuzzy number. Jain [4] was the first to introduce the ranking of normal fuzzy numbers. Klir [5] demonstrated a thorough understanding of fuzzy concept and suggested the optimum candidate approach to solve the transportation problem and used the centroid ranking methodology to hexagonal fuzzy numbers [6–9]. Nagoor Gani& Abdul Razak [10] found a two-stage fuzzy transportation cost minimizing in trapezoidal fuzzy numbers for supplies and demands. Bellman & Zadeh [11] developed a fuzzy set theory for decision-making for the first time. Ranking fuzzy numbers using interval values was introduced by Liou and Wang [12]. Zimmermann [13] shows that fuzzy linear programming solutions are always efficient.

The Transportation problem is concerned with the transportation of items from several locations of origin, such as factories, to multiple points of demand, such as destinations. Capacity or availability refers to a source's capacity to produce items, whereas needs refer to a fixed element. This chapter tries to find the suitable defuzzification method to convert hexagonal fuzzy numbers to find the optimum cost.

2 Preliminaries

2.1 *Definition: (Fuzzy Set) [13]*

Let X be a nonempty set. A fuzzy set of A of X is defined as

$\overline{A} = \{(x, \mu_A(x)) / x \in X\}$, $\mu_A(x)$ is called membership function.

2.2 Definition: (Fuzzy Number) [13]

A fuzzy number is a generalization of a regular real number. Set of each possible value has between 0 and 1.

There exist at least one x ϵ R with $\mu_A(x) = 1$

$\mu_A(x)$ is piecewise continuous

2.3 Definition: Hexagonal Fuzzy Number [6]

The hexagonal fuzzy number is denoted as A, where a_1, a_2, a_3, a_4, a_5, and a_6 are real numbers.

$$\mu_A(x) = \begin{cases} 0 \le x < a1 \\ \frac{1}{2}\frac{(x-a1)}{(a2-a1)}, a1 \le x \le a2 \\ \frac{1}{2} + \frac{1}{2}\frac{(x-a2)}{(a3-a2)}, a2 \le x \le a3 \\ 1, a3 \le x \le a4 \\ 1 - \frac{1}{2}\frac{(x-a4)}{(a5-a4)}, a4 \le x \le a5 \\ \frac{1}{2}\frac{(a6-x)}{(a6-a5)}, a5 \le x \le a6 \\ x > a6 \end{cases}$$

The graphical representation of different forms of hexagonal fuzzy number are shown in [14] (Figs. 8.1, 8.2, 8.3, and 8.4).

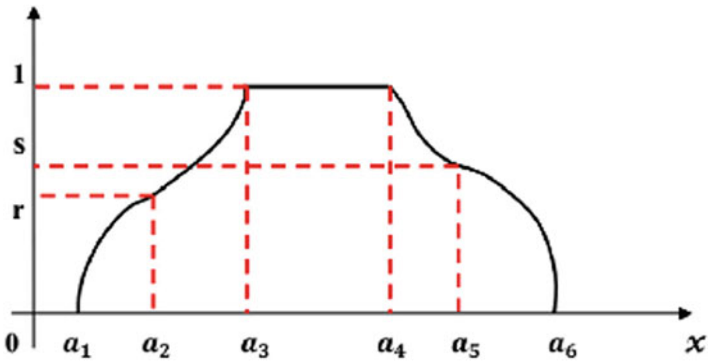

Fig. 8.1 Nonlinear hexagonal fuzzy number having asymmetry [14]

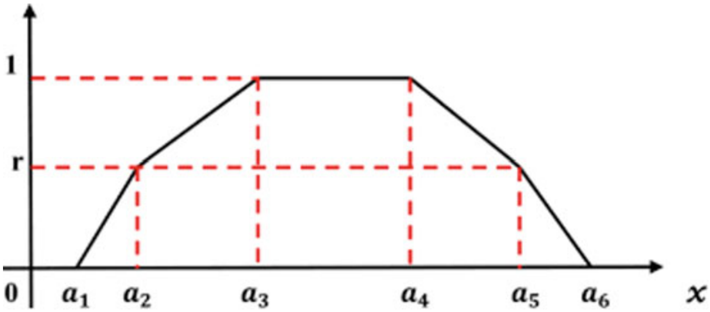

Fig. 8.2 Linear hexagonal fuzzy number having symmetry [14]

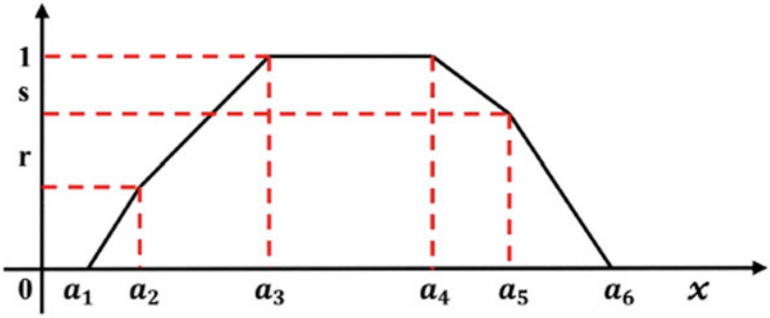

Fig. 8.3 Linear hexagonal fuzzy number having asymmetry [14]

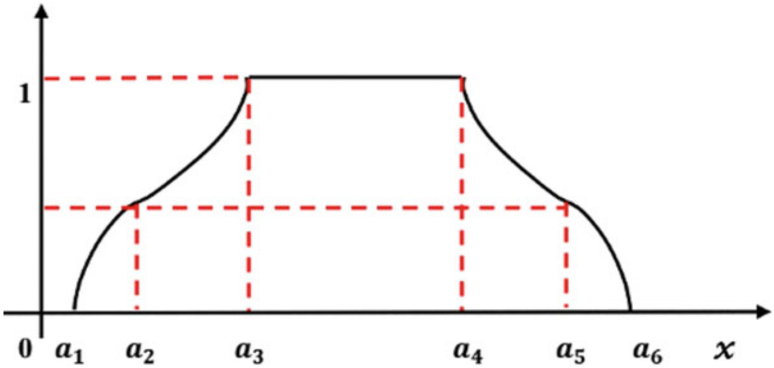

Fig. 8.4 Nonlinear hexagonal fuzzy number having symmetry [12]

2.4 Definition: (Positive and Negative) [6]

A hexagonal fuzzy number is $A_H = (a_1, a_2, a_3, a_4, a_5, a_6)$ is positive if $a_i > 0$ for i = 1,2,3..6 and it is negative if $a_i < 0$ for i = 1,2,3..6

2.5 Arithmetic Operation [6]

If (A) = (m, n, o, p, q, r) and (*B*) = (s, t, u, v, w, x) are two fuzzy number.

Addition: (A) + (B) = (m_+ s, n_+ t, o_+ u, p_+ v, q_+ w, r_+x)
Subtraction: (*A*) − (B) = (m_-x, n_- w, o_- v , p_- u, q_-t, r_- s)
Multiplication: (A) * (*B*) = (m*s, n ∗ t, o* u, p ∗ v, q*w, r*x)

2.6 Ranking of Hexagonal Fuzzy Number [15, 16]

A ranking function R: F (R) ->R, which maps each fuzzy number into a real number where a natural order exists, is an effective method for comparing fuzzy numbers. F (R) is a set of fuzzy numbers defined on a set of real numbers. A_H = (a1,a2,a3,a4,a5,a6) and for any two hexagonal fuzzy numbers

We can compare B_H = (b1,b2,b3,b4,b5,b6) to the following:

(i) $A_H = B_H \Leftrightarrow R(A_H) = R(B_H)$
(ii) $A_H \geq B_H \Leftrightarrow R(A_H) \geq R(B_H)$
(iii) $A_H \leq B_H \Leftrightarrow R(A_H) \leq R(B_H)$

2.7 Mathematical Analysis of Fuzzy Transportation Problem [7]

Consider a transportation problem in m origins and n destinations.

Let ai, (ai ≥ 0) represent the source availability i, and bj, (bj ≥ 0), represent the destination j. Let cij denote the cost from source i to destination j. The transportation cost from source i to destination j is xij. The challenge then becomes deciding the most cost-effective method of the possible amounts of all sources to fulfill demand at the destination while keeping overall transportation costs to a minimum. The mathematical description of transportation with parameters in the situation when total supply equals total demand is defined by

$$\text{Min } z = \sum\nolimits_{i=1}^{m} \sum_{j=1}^{n} ..c_{ij}x_{ij}$$

$$\text{Subject to } \sum_{j=1}^{n} .x_{ij} = a_i;\ i = 1, 2, \ldots m$$

$$\sum_{j=1}^{m} .x_{ij} = b_j; j = 1, 2, \ldots n$$

$$\sum_{j=1}^{m} .a_i = \sum_{j=1}^{n} .b_j; i = 1, 2, \ldots m, j = 1, 2, \ldots n \text{ and } X_{ij} \geq 0$$

3 Max Min Method –Algorithm

All kinds of transportation problems may be solved using the provided strategy. The procedure of this method for finding an IBFS is shown below.

Step 1
The table can form and check if the sum of the demand equals the sum of the supply before proceeding to step 2.
Step 2
We convert the fuzzy cost into crisp value using a ranking technique centroid of centroid method for the specified transportation problem.
Step 3
Divide the cost matrix's column count by the row-by-row deviation between the maximum and minimum to place on the corresponding right side of the column.
Step 4
Divide the cost matrix's row count by the column-by-column deviation between the maximum and minimum to place on the corresponding bottom of the row.
Step 5
We calculate the corresponding minimal cost value for the greatest of the produced values and then assign that cell of the given matrix. If more than one maximum value occurs then break the choice arbitrarily.
Step 6
Procedures are repeated again and again. Repeat 1 to 5 until all of the allocations have been completed.

4 Numerical Example

The fuzzy transportation problem is solved using a variety of ranking method techniques.

Example
Consider the below hexagonal fuzzy transportation problem (Table 8.1).

Table 8.1 Hexagonal fuzzy number [8]

	D1	D2	D3	Supply
S1	(3,5,7,9,10,12)	(3,7,11,15,19,24)	(11,14,17,21,25,30)	(7,9,11,13,16,20)
S2	(7,9,11,14,18,22)	(3,5,7,9,10,12)	(5,7,10,13,17,21)	(6,8,11,14,19,25)
S3	(2,3,4,6,7,9)	(5,7,8,11,14,17)	(7,9,11,14,18,22)	(9,11,13,15,18,20)
Demand	(6,9,12,15,20,25)	(6,7,9,11,13,16)	(10,12,14,16,20,24)	

Table 8.2 Converted fuzzy number to crisp number using the above ranking method 1 [1, 2]

	D1	D2	D3	Supply
S1	30	52	78	50
S2	54	30	48	54
S3	20	42	54	58
Demand	58	40	64	Balanced

Table 8.3 Allocation as per the proposed max-min method [1, 2]

	D1	D2	D3	**Supply**	(max-min)/3
S1	30 **50**	50	78	~~50~~	**16**
S2	54	30	48	**54**	8
S3	20	42	54	**58**	11.33
Demand	~~58~~ **8**	~~40~~	**64**		
(max-min)/3	11.33	7.33	10		

4.1 Ranking Method 1

Fuzzy transportation table is converted into crisp transportation table by using a Ranking method 1 (Table 8.2).

$$\mathbf{R}\,(\mathbf{A_H}) = \mathbf{2}\,(\mathbf{a_2} + \mathbf{a_5})$$

$R(a_{11}) = 2(5 + 10) = 30$
$R(a_{11}) = 30,\ R(a_{12}) = 52,\ R(a_{13}) = 78$
$R(a_{21}) = 54,\ R(a_{22}) = 30,\ R(a_{23}) = 48$
$R(a_{31}) = 20,\ R(a_{32}) = 42,\ R(a_{33}) = 54$

By applying the max-min Method, one can find the maximum of generated values and its equivalent minimum cost value for each cost in the prepared matrix. In the case of more than one, one can choose any of the maximum resultant benefits as shown in (Table 8.3)

One can observe that the maximum of the treatment is appropriate and so is the corresponding minimum value. So, one can assign the cost cell to the supplied value. If there are many maximum result values, we can select any of them (Table 8.4).

The same procedure will be repeated till the final allotment is reached (Table 8.5).

Table 8.4 Allocation as per the proposed max-min method [1, 2]

	D1	D2	D3	**Supply**	(max-min)/3
S1	$30^{\mathbf{50}}$	52	78	~~50~~	–
S2	54	30	48	**54**	8
S3	$20^{\mathbf{8}}$	42	54	~~58~~ **50**	11.33
Demand	~~58~~ 8	**40**	**64**		
(max-min)/2	17	7.33	10		

Table 8.5 Solution [1, 2]

	D1	D2	D3	**Supply**
S1	$30^{\mathbf{50}}$	52	78	**50**
S2	54	$30^{\mathbf{40}}$	$48^{\mathbf{14}}$	**54**
S3	$20^{\mathbf{8}}$	42	$54^{\mathbf{50}}$	**58**
Demand	**58**	**40**	**64**	

The Transportation Cost
= (30*50) + (30*40) + (48*14) + (20*8) + (54*50)
= 1500 + 1200 + 672 + 160 + 2700
= 6232

Table 8.6 Converted fuzzy number to crisp number using the above ranking Method 2 [1, 2]

	D1	D2	D3	Supply
S1	15.5	26.5	39	24.83
S2	26.67	15.5	23.83	26.33
S3	10	20.83	26.67	29.17
Demand	28.83	19.83	31.67	

Table 8.7 Allocation as per the proposed max-min method [1, 2]

	D1	D2	D3	**Supply**	(max-min)/3
S1	$15.5^{24.83}$	26.5	39	~~24.83~~	**7.83**
S2	26.67	15.5	23.83	26.33	3.72
S3	10	20.83	26.67	29.17	5.56
Demand	~~28.83~~ 4	19.83	31.67		
(max-min)/3	5.56	3.67	5.06		

4.2 Ranking Method 2

Fuzzy transportation table is converted into crisp transportation table by using a Ranking method 2 (Tables 8.6 and 8.7).

$$\mathbf{R}\,(\mathbf{A_H}) = \frac{\mathbf{7}\,\boldsymbol{a_2} - \mathbf{2}\,\boldsymbol{a_1} + \boldsymbol{a_3} + \boldsymbol{a_4} + \mathbf{6}\,\boldsymbol{a_5} - \boldsymbol{a_6}}{\mathbf{6}}$$

$R(a_{11}) = \frac{7(5)-2(3)+7+9+6(10)-12}{6} = 15.5$
$R(a_{11}) = 15.5$, $R(a_{12}) = 26.5$, $R(a_{13}) = 39$
$R(a_{21}) = 26.67$, $R(a_{22}) = 15.5$, $R(a_{23}) = 23.83$
$R(a_{31}) = 10$, $R(a_{32}) = 20.83$, $R(a_{33}) = 26.67$

Table 8.8 Allocation as per the proposed max-min method [1, 2]

	D1	D2	D3	**Supply**	(max-min)/3
S1	15.5 $^{24.83}$	26.5	39	~~24.83~~	**7.83**
S2	26.67	15.5	23.83	26.33	3.72
S3	10 ^{4}	20.83	26.67	~~29.17~~ 25.17	5.56
Demand	~~4 0~~	19.83	31.67		
(max-min)/2	8.34	2.67	1.42		

Table 8.9 Application of proposed max-min method [1, 2]

	D1	D2	D3	Supply
S1	15.5 $^{\mathbf{24.83}}$	26.5	39	24.83
S2	26.67	15.5 $^{\mathbf{19.83}}$	23.83 $^{\mathbf{6.5}}$	26.33
S3	10 $^{\mathbf{4}}$	20.83	26.67 $^{\mathbf{25.17}}$	29.17
Demand	28.83	19.83	31.67	

The Transportation Cost
= (15.5*24.83) + (15.5*19.83) + (23.83*6.5) + (10*4) + (26.67*25.17)
= 384.865 + 307.365 + 154.895 + 40 + 671.2839
= 1558.4089

Table 8.10 Converted fuzzy number to crisp number using the above ranking method 3 [1, 2]

	D1	D2	D3	**Supply**
S1	71.5	119.78	177.22	**113.06**
S2	119.47	71.5	108.47	**120.69**
S3	46.44	91.36	119.47	**130.17**
Demand	**128.64**	**92.89**	**142.39**	

We find the maximum of the treatment is appropriate, as well as the corresponding minimum value, and then assign the cost cell to the supplied value. If there are many maximum result values, we can select any of them (Table 8.8).

The same procedure will be repeated till the final allotment is reached (Table 8.9).

4.3 Ranking Method 3

Fuzzy transportation table is converted into crisp transportation table by using ranking method 3 (Tables 8.10 and 8.11).

$$\mathbf{R}\,(\mathbf{A_H}) = \left(\frac{\mathbf{2}a_1 + \mathbf{4}\,a_2 + \mathbf{9}\,a_3 + \mathbf{9}\,a_4 + \mathbf{4}\,a_5 + \mathbf{2}\,a_6}{\mathbf{6}} * \frac{\mathbf{11}}{\mathbf{6}}\right)$$

$$\mathbf{R}\,(\mathbf{a_{11}}) = \left(\frac{\mathbf{2\,(3) + 4\,(5) + 9\,(7) + 9\,(9) + 4\,(10) + 2\,(12)}}{\mathbf{6}} * \frac{\mathbf{11}}{\mathbf{6}}\right) = \mathbf{71.5}$$

Table 8.11 Allocation as per the proposed max-min method [1, 2]

	D1	D2	D3	**Supply**	(max-min)/3
S1	$71.5^{113.06}$	119.78	177.22	~~113.06~~	35.24
S2	119.47	71.5	108.47	**120.69**	15.99
S3	46.44	91.36	119.47	**130.17**	24.34
Demand	~~128.64~~ **15.58**	**92.89**	**142.39**		
(max-min)/3	24.34	16.09	22.92		

Table 8.12 Allocation as per the proposed max-min method [1, 2]

	D1	D2	D3	**Supply**	(max-min)/3
S1	$71.5^{113.06}$	119.78	177.22	~~113.06~~	–
S2	119.47	71.5	108.47	**120.69**	15.99
S3	$46.44^{15.58}$	91.36	119.47	**130.17**	24.34
Demand	~~128.64 15.58~~	**92.89**	**142.39**		
(max-min)/2	36.52	9.93	5.5		

Table 8.13 Application of proposed max-min method [1, 2]

	D1	D2	D3	**Supply**
S1	$71.5^{\mathbf{113.06}}$	119.78	177.22	**113.06**
S2	119.47	$71.5^{\mathbf{92.89}}$	$108.47^{\mathbf{27.8}}$	**120.69**
S3	$46.44^{\mathbf{15.58}}$	91.36	$119.47^{\mathbf{114.59}}$	**130.17**
Demand	**128.64**	**92.89**	**142.39**	

The Transportation Cost
= (71.5*113.06) + (71.5*92.89) + (108.47*27.8) + (46.44*15.58) + (119.47*114.59)
= 8083.79 + 6641.635 + 3015.466 + 723.5352 + 13690.07
= 32154.4935

$R(a_{11}) = 71.5, R(a_{12}) = 119.78, R(a_{13}) = 177.22$
$R(a_{21}) = 119.47, R(a_{22}) = 71.5, R(a_{23}) = 108.47$
$R(a_{31}) = 46.44, R(a_{32}) = 91.36, R(a_{33}) = 119.47$

We find the maximum of the treatment is appropriate, as well as the corresponding minimum value, and then assign the cost cell to the supplied value. If there are many maximum result values, we can select any of them (Tables 8.12 and 8.13).

4.4 *Ranking Method 4*

Fuzzy transportation table is converted to crisp transportation table by using ranking method 4 (Table 8.14).

$$\mathbf{R}\,(\mathbf{A_H}) = (\mathbf{a_6} - \mathbf{a_1})$$

Table 8.14 Fuzzy number conversion to crisp number through ranking method 4 [1, 2]

	D1	D2	D3	Supply
S1	9	21	19	13
S2	15	9	16	19
S3	7	12	15	11
Demand	19	10	14	Balanced

Table 8.15 Allocation as per the proposed max-min method [1, 2]

	D1	D2	D3	**Supply**	(max-min)/3
S1	9	21	19	13	**3.33**
S2	15	9 [10]	16	~~19~~ 9	2.33
S3	7	12	15	11	2.67
Demand	19	~~10~~	14		
(max-min)/3	2.67	4	1.33		

Table 8.16 Allocation as per the proposed max-min method [1, 2]

	D1	D2	D3	**Supply**	(max-min)/3
S1	9 [13]	21	19	~~13~~	**3.33**
S2	15	9 [10]	16	~~19~~ 9	2.33
S3	7	12	15	11	2.67
Demand	~~19~~ 6	~~10~~	14		
(max-min)/2	2.67	4	1.33		

$R(a_{11}) = 12 - 3 = 9$
$R(a_{11}) = 9, R(a_{12}) = 21, R(a_{13}) = 19$
$R(a_{21}) = 15, R(a_{22}) = 9, R(a_{23}) = 16$
$R(a_{31}) = 7, R(a_{32}) = 12, R(a_{33}) = 15$

Find for every maximum generated value through the max-min method, the equivalent minimum cost value in the prepared matrix. In the case of more than one equivalent values, one can break the choice arbitrarily (Table 8.15).

We find the maximum of the treatment is appropriate, as well as the corresponding minimum value, and then assign the cost cell to the supplied value. If there are many maximum result values, we can select any of them (Table 8.16).

The same procedure will be repeated till the final allotment is reached (Table 8.17).

4.5 *Ranking Method 5*

Fuzzy transportation table is converted to crisp transportation table by using ranking method 5 (Table 8.18).

Table 8.17 Solution [1, 2]

	D1	D2	D3	**Supply**
S1	9 13	21	19	13
S2	15	9 10	16 9	19
S3	7 6	12	15 5	11
Demand	19	10	14	

The Transportation Cost
= (9*13) + (9*10) + (16*9) + (7*6) + (15*5)
= 117 + 90 + 144 + 42 + 75
= 468

Table 8.18 Fuzzy number conversion to crisp number through ranking method 4 [1, 2]

	D1	D2	D3	Supply
S1	7.67	13.17	19.67	12.67
S2	13.5	7.67	12.17	13.83
S3	5.17	10.33	13.5	14.33
Demand	14.5	10.33	16	Balanced

Table 8.19 Allocation as per the proposed max-min method [1, 2]

	D1	D2	D3	**Supply**	(max-min)/3
S1	7.67 12.67	13.17	19.67	~~12.67~~	**4**
S2	13.5	7.67	12.17	13.83	1.94
S3	5.17	10.33	13.5	14.33	2.78
Demand	~~14.5~~ 1.83	10.33	16		
(max-min)/3	2.78	1.83	2.5		

$$\mathbf{R}\left(\mathbf{A_H}\right) = \left(\mathbf{a_1} + \mathbf{a_2} + \mathbf{a_3} + \mathbf{a_5} + \mathbf{a_6}\right)/\mathbf{6}$$

$R(a_{11}) = (3 + 5 + 7 + 9 + 10 + 12)/6 = 7.67$
$R(a_{11}) = 7.67,\ R(a_{12}) = 13.17,\ R(a_{13}) = 19.67$
$R(a_{21}) = 13.5,\ R(a_{22}) = 7.67,\ R(a_{23}) = 12.17$
$R(a_{31}) = 5.17,\ R(a_{32}) = 10.33,\ R(a_{33}) = 13.5$

Find for every maximum generated value through the max-min method, the equivalent minimum cost value in the prepared matrix. In the case of more than one equivalent values, one can break the choice arbitrarily (Table 8.19).

We find the maximum of the treatment is appropriate, as well as the corresponding minimum value, and then assign the cost cell to the supplied value. If there are many maximum result values, we can select any of them (Table 8.20).

The same procedure will be repeated till the final allotment is reached (Table 8.21).

Table 8.20 Allocation as per the proposed max-min method [1, 2]

	D1	D2	D3	**Supply**	(max-min)/3
S1	$7.67^{12.67}$	13.17	19.67	~~12.67~~	–
S2	13.5	7.67	12.17	13.83	1.94
S3	$5.17^{1.83}$	10.33	13.5	~~14.33 12.5~~	2.78
Demand	~~14.5 1.83~~	10.33	16		
(max-min)/2	4.17	1.33	0.67		

Table 8.21 Solution [1, 2]

	D1	D2	D3	**Supply**
S1	$7.67^{12.67}$	13.17	19.67	12.67
S2	13.5	$7.67^{10.33}$	$12.17^{3.5}$	13.83
S3	$5.17^{1.83}$	10.33	$13.5^{12.5}$	14.33
Demand	14.5	10.33	16	

The Transportation Cost
= (7.67*12.67) + (7.67*10.33) + (12.17*3.5) + (5.17*1.83) + (13.5*12.5)
= 97.1789 + 79.2311 + 42.595 + 9.461 + 168.75)
= 397.2161

Table 8.22 Fuzzy number conversion to crisp number through ranking method 4 [1, 2]

	D1	D2	D3	**Supply**
S1	2.15	3.64	5.42	**3.47**
S2	2.95	2.15	3.36	**3.75**
S3	1.42	2.82	2.95	**3.96**
Demand	**3.96**	**2.84**	**4.38**	

Table 8.23 Allocation as per the proposed max-min method [1, 2]

	D1	D2	D3	**Supply**	(max-min)/3
S1	$2.15^{3.47}$	3.64	5.42	~~3.47~~	1.09
S2	2.95	2.15	3.36	**3.75**	0.40
S3	1.42	2.82	2.95	**3.96**	0.51
Demand	~~3.96~~ **0.49**	**2.84**	**4.38**		
(max-min)/3	0.51	0.50	0.82		

4.6 Ranking Method 6

Fuzzy transportation table is converted to crisp transportation table by using ranking method **6** (Tables 8.22 and 8.23).

$$\mathbf{R}\,(\mathbf{CC}) = \left(\frac{\mathbf{2}a_1 + \mathbf{3}\,a_2 + \mathbf{4}\,a_3 + \mathbf{4}\,a_4 + \mathbf{3}\,a_5 + \mathbf{2}\,a_6}{\mathbf{18}} * \frac{\mathbf{5}}{\mathbf{18}} \right)$$

Table 8.24 Allocation as per the proposed max-min method [1, 2]

	D1	D2	D3	**Supply**	(max-min)/3
S1	2.15 $^{3.47}$	3.64	5.42	~~3.47~~	–
S2	2.95	2.15	3.36	**3.75**	0.40
S3	1.42 $^{0.49}$	2.82	2.95	~~3.96~~ **3.47**	0.51
Demand	~~3.96 0.49~~	**2.84**	**4.38**		
(max-min)/2	0.77	0.33	0.2		

Table 8.25 Allocation as per the proposed max-min method [1, 2]

	D1	D2	D3	Supply
S1	2.15 $^{\mathbf{3.47}}$	3.64	5.42	**3.47**
S2	2.95	2.15 $^{\mathbf{2.84}}$	3.36$^{\mathbf{0.91}}$	**3.75**
S3	1.42 $^{\mathbf{0.49}}$	2.82	2.95$^{\mathbf{3.47}}$	**3.96**
Demand	**3.96**	**2.84**	**4.38**	

The Transportation Cost
= (2.15*3.47) + (2.15*2.84) + (3.36*0.91) + (1.42*0.49) + (2.95*3.47)
= 27.5564

Table 8.26 A Comparison of existing and proposed methods

Existing method	Transportation cost
Ranking method 1	6232
Ranking method 2	1558.4089
Ranking method 3	3254.4935
Ranking method 4	468
Ranking method 5	397.2161
Ranking method 6	27.5564

$$\mathbf{R}\,(\mathbf{a}_{11}) = \left(\frac{\mathbf{2\,(3) + 3\,(5) + 4\,(7) + 4\,(9) + 3\,(10) + 2\,(12)}}{\mathbf{18}} * \frac{\mathbf{5}}{\mathbf{18}}\right) = \mathbf{2.15}$$

$R(a_{11}) = 2.15$, $R(a_{12}) = 3.64$, $R(a_{13}) = 5.42$
$R(a_{21}) = 2.95$, $R(a_{22}) = 2.15$, $R(a_{23}) = 3.36$
$R(a_{31}) = 1.42$, $R(a_{32}) = 2.82$, $R(a_{33}) = 2.95$

We find the maximum of the treatment is appropriate, as well as the corresponding minimum value, and then assign the cost cell to the supplied value. If there are many maximum result values, we can select any of them (Tables 8.24, 8.25, and 8.26).

5 Conclusion

We conclude the optimum solution of a fuzzy transportation using hexagonal fuzzy numbers to use a ranking technique based on the ranking method 6 to provide crisp values, and we recommend that the fuzzy hexagonal transportation problem be solved using the proposed max-min method based on the numerical results. This approach is quite easy when compared to all other existing ways, and it also achieves the minimum transportation cost.

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Chapter 9
Development of an Interval Picture Fuzzy Matrix Game-Based Approach to Combat Cyberthreats in the Healthcare Sector

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1 Introduction

By using a matrix game (*MG*) [14], rational decision-making can be analyzed in real-life situations. In real-world situations, game problems often involve imperfect information, incorrect data, and disagreements between opponents. As a result, researchers face the greatest challenge when it comes to combating uncertainty in real-world situations.

Every cyberattack drives huge waste of information. To counter an attack, the system must be secure sensibly and constructively. Nevertheless, the complexity of attacks makes securing the system a challenging task. It is possible to use the *MG* theory to control such scenarios, and the digital attackers and system protectors can be interpreted as players in the game. Protectors and attackers pay off in terms of how much destruction is caused by the attackers and how much success is achieved by defending against them. Due to the uncertainty and vagueness of attack and defense mechanisms, decision-makers face a great difficulty to estimate their payoffs precisely.

1.1 Motivation and Objectives

The general model of *MG* uses crisp numbers; however, there always occur some degree of imprecision in the opposition's predictions in the game problem. Fuzzy

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C. Jana et al. (eds.), *Fuzzy Optimization, Decision-making and Operations Research*, https://doi.org/10.1007/978-3-031-35668-1_9

sets (FSs) represent uncertainty better than crisp data. For each element in FS, a degree of membership (DOM) is allotted. To describe the complicated scenarios, *MG* problems in uncertain environments utilize FSs as payoffs. Atanassov [17] modified the version of FSs and first proposed the concepts of intuitionistic fuzzy sets (IFSs). For each element, the DOM and the degree of non-membership (DON) are both considered by an IFS. The IFS explores uncertainty more accurately and explicitly than the FS.

In some cases, the available facts are always accompanied by contradictory, uncertain, and undefined data. It is to be noted that the key concepts of the degree of neutrality are absent in FS and IFS theory. Picture fuzzy set (PiFS) [2] is the extension of FSs and IFSs. PiFS can model uncertainty using DOM, DON, and the degree of indeterminacy (DOI) of elements.

The interval-valued picture fuzzy sets (IVPiFSs) were submitted for the purpose of communicating issues with a set of real numbers in the interval [0,1]. In some cases, the DOM, DON, and DOI of a statement cannot be completely described in real life but are represented by possible intervals. IVPiFS was conceptualized, and the set-theoretic operators of IVPiFS were discussed in Cuong and Hai [3]. The next instance of an uncertain situation suggests the superior application of IVPiFS in respect of FS or IFS.

Suppose a medical diagnostic center **Delta** diagnose patients with fully automatic machines using artificial intelligence and delivers patients' information online. So, there is always a huge data containing confidential information processed digitally during a span of time. Hackers choose this period to disorganize the digital system and snatch important data from the particular network site of **Delta**. In order to counter attacks from hackers, the center deployed a digital security agency (DSA). As a result, DSA adopts some crucial strategies, like x_1, use of properly trained staff on cybersecurity; x_2, use of layered protection system; and x_3, use of updated software. There are many ways in which hackers can cause damage. DSA considers mainly three different ways such as y_1, phishing; y_2, malware; and y_3, malicious mobile applications, from which the attack comes frequently. So, y_1, y_2, and y_3 may be treated as strategies for hackers. *MG* can be applied to deal this problem, where DSA and hackers are, respectively, Player-I ($\mathbf{Pl_1}$) and Player-II ($\mathbf{Pl_2}$).

Distinct digital strikes destruct health-related information. Accordingly, the DSA wishes to know whether any information has been lost by hackers as a result of different attack strategies, mainly considering strategies y_1, y_2, and y_3. DSA consulted with different experts (say A_1, A_2, A_3) for their opinions. As a result, they presented their perspectives based on the strategies used by the DSA during different attacks. Uncertain information always contains some perception of "indeterminate" concepts other than "truth" and "falsity." In such a situation, interval-valued picture fuzzy numbers (IVPiFNs) can be used in place of fuzzy numbers (FNs) or intuitionistic fuzzy numbers (IFNs). Let the expert "A_3" quantify the loss in the form of an IVPiFN as $\langle[0.6, 0.7], [0.08, 0.12], [0.06, 0.1]\rangle$, for the choice of the strategy x_1 and y_1, respectively, by DSA and the hacker. This explains that DSA has 60% to 70% positive chance of facing damage, whereas a chance of 6% to 10% occurs to not get any damage. Also, DSA has an indeterminacy of 8% to

12% chance to encounter damage according to "A_3." These examples demonstrate the relevance of using IVPiFN in the context of *MG* problems when cyberthreats are considered.

The above example illustrates the difficulty of estimating players' payoffs when there is ambiguous data, differing vague conditions, etc. To grip this condition, the payoffs can be considered as IVPiFNs, and it makes the problem more reliable. Consequently, uncertainty in *MG*s can be countered using IVPiFN. The objectives of the present study are augmented as follows:

(i) To define the concept of *MG* problems using IVPiFNs as payoff elements.
(ii) To formulate a novel solution methodology of the *MG* with IVPiFN payoffs.
(iii) To apply the concept of interval-valued picture fuzzy matrix game for real-world complex problems.

1.2 Research Gaps

IVPiFS empowers a considerable range to express inexplicit circumstances. Even though *MG* theories have advanced in various directions, no work has been done to solve *MG* problems using IVPiFN payoffs. For practical necessity, we explore a *MG* where IVPiFNs are taken as payoffs.

1.3 Contributions

As part of this chapter, we construct a *MG* that has IVPiFNs as payoffs and defines the solution concept to the problem. The applicability and effectiveness of the discussed approach are demonstrated by a numerical example of cyberthreats in the healthcare sector. In this chapter, we explore how the results obtained are physically significant. The present work is mainly composed of the following contributions:

(i) IVPiFNs are used as payoffs to counter *MG* problems. As a result of this study, the idea of a reasonable solution, as well as the solution to the *MG* with IVPiFN payoffs, is developed. Also, this chapter defines the concept of the score function and accuracy function of IVPiFN.
(ii) A novel approach is established to get the optimal strategies. Also, we proved that the gain floor of the maximizing player cannot exceed the loss ceiling of the minimizing player. In addition, we show that there always exists a solution to the game problem where payoffs are taken as IVPiFNs.
(iii) It is preferable to get optimal solutions in the shape of IVPiFNs for each player.
(iv) To manifest the validity and applicability of the proposed approach, it is applied to counter cyberthreat issues in the healthcare sector and find the optimal strategies for the medical data controller.

The rest of this present work can be summed up as follows. Some preliminaries on IVPiFS and the order relation of intervals are recalled in Sect. 2. The concept of *MG* with IVPiFN payoffs is discussed in Sect. 3. The formulation of the mathematical models and the solution methodology of the game problem are developed in Sect. 4. A numerical example relating to the cyberthreat in the healthcare sector is illustrated in Sect. 5. The chapter is concluded in Sect. 6.

1.4 Literature Review

In this section, we present a survey of the related literature mainly in three directions: (i) *MG* with fuzzy payoffs, (ii) *MG* with intuitionistic fuzzy payoffs, and (iii) applications of PiFNs.

MG is concerned with the study of conflicting situations that aims to capture behavior in strategic situations. There are various kinds of mathematical games which have been extensively studied and successfully applied in many fields. Nonetheless, it is hard to evaluate payoffs precisely in game circumstances due to imprecise information and uncertain comprehension of circumstances.

(i) In recent years, numerous researchers examined and analyzed fuzzy matrix games (*FMGs*). To solve *FMGs*, Bigdeli et al. [10] utilized nearby interval estimation of FNs and solved security games. Based on the Mehar method, Verma and Kumar [35] derived the optimal strategy for players. Seikh et al. [20] developed an approach to counter the hesitant *FMGs*. Seikh et al. [21] explored a methodology of solving *MG* with dense fuzzy payoffs. Zheng et al. [40] studied a fuzzy multi-objective programming approach for *MG*s with payoffs of fuzzy rough numbers. Brikaa et al. [19] utilized the Mehar approach to solve *MG*s with triangular dual hesitant fuzzy payoffs.

(ii) The IFS is a generalized way to express impreciseness in the payoffs. *MG* with IFN entries is solved by Ambika methods developed by Verma and Kumar [36]. Naqvi et al. [7] solved game problems where symmetric triangular I-fuzzy numbers are treated as payoff matrix entries. Verma and Aggarwal [30] discussed basic results and solution methodologies for game problems where payoffs are taken as linguistic IFNs. Kha et al. [13] developed a novel equilibrium solution concept for intuitionistic fuzzy *MG* considering the proportion mix of possibility and necessity expectations. Zheng et al. [41] studied a resolving indeterminacy approach to solving multi-criteria zero-sum *MG* with intuitionistic fuzzy goals. Seikh and Dutta [25] executed a technique using intuitionistic fuzzy optimization for solving problems arising in *MGs*. Li [6] used interval-valued intuitionistic fuzzy sets (IVIFSs) as payoffs in a game problem and studied the solution procedure of the game. Xia [27] considered the interval-valued intuitionistic fuzzy numbers (IVIFNs) as players' payoff in a *MG* and developed a solution methodology depending on Archimedean t-conorm and t-norm.

(iii) In the literature, a lot of work have been done using PiFSs. For example, Cuong [2] established some properties of PiFSs and defined the distance measures between PiFSs. Cuong and Hai [3] defined different operations on PiFSs, like negations, conjunctions, and disjunctions. Wei [9] developed the procedure to find the similarity between PiFSs. Jana et al. [5] addressed Dombi aggregation operators for PiFSs and used them to deal with the MADM process. Sing [29] studied the correlation coefficient of PiFS. Wang and Li [31] conceptualized the picture hesitant fuzzy set and applied it to decision-making problems. Seikh and Mandal [24] proposed some picture fuzzy aggregation operators based on Frank t-norm and t-conorm. Wang et al. [18] developed a multi-criteria decision-making framework for risk ranking of energy performance contracting projects under picture fuzzy environment. Meksavang et al. [28] utilized the ordered weighted distance operators and aggregate the picture fuzzy information in a modified VIKOR technique.

However, our study in this chapter is significantly different from other aforesaid works. Table 9.1 shows how our present work is different from the existing models.

2 Preliminaries

This section recaps some fundamental definitions and preliminary concepts related to IVPiFNs. The score function and accuracy function of IVPiFN are also defined here.

Definition 1 (Interval-Valued Picture Fuzzy Number (IVPiFN) [3]) Let $\Omega(\neq \phi)$ be the universe. An interval-valued picture fuzzy set (IVPiFS) $\tilde{\tilde{P}}$ in Ω is defined as $\tilde{\tilde{P}} = \left\{\langle \xi, \tau_{\tilde{\tilde{P}}}(\xi), \omega_{\tilde{\tilde{P}}}(\xi), \nu_{\tilde{\tilde{P}}}(\xi)\rangle | \xi \in \Omega\right\}$, where $\tau_{\tilde{\tilde{P}}}(\xi)$, $\omega_{\tilde{\tilde{P}}}(\xi)$, and $\nu_{\tilde{\tilde{P}}}(\xi)$ are subsets of [0, 1], and $\sup(\tau_{\tilde{\tilde{P}}}(\xi)) + \sup(\omega_{\tilde{\tilde{P}}}(\xi)) + \sup(\nu_{\tilde{\tilde{P}}}(\xi)) \leq 1$.

The intervals $\tau_{\tilde{\tilde{P}}}(\xi)$, $\omega_{\tilde{\tilde{P}}}(\xi)$, and $\nu_{\tilde{\tilde{P}}}(\xi)$ are, respectively, the DOM, DOI, and DON of $\xi \in \Omega$. Thus, the IVPiFS may be concisely written as $\tilde{\tilde{P}} = \left\{\langle \xi, [\tau^-(\xi), \tau^+(\xi)], [\omega^-(\xi), \omega^+(\xi)], [\nu^-(\xi), \nu^+(\xi)]\rangle | \xi \in \Omega\right\}$, where $0 \leq \tau^-(\xi) \leq \tau^+(\xi) \leq 1$, $0 \leq \omega^-(\xi) \leq \omega^+(\xi) \leq 1$, $0 \leq \nu^-(\xi) \leq \nu^+(\xi) \leq 1$ and $\tau^+(\xi) + \omega^+(\xi) + \nu^+(\xi) \leq 1$.

For any element $\xi \in \Omega$, the triplet $\langle \tau_{\tilde{\tilde{P}}}(\xi), \omega_{\tilde{\tilde{P}}}(\xi), \nu_{\tilde{\tilde{P}}}(\xi)\rangle$ is known as IVPiFN. For convenience, $\langle \tau_{\tilde{\tilde{P}}}(\xi), \omega_{\tilde{\tilde{P}}}(\xi), \nu_{\tilde{\tilde{P}}}(\xi)\rangle$ is often represented by $\langle [\tau^-, \tau^+], [\omega^-, \omega^+], [\nu^-, \nu^+]\rangle$, where $[\tau^-, \tau^+]$, $[\omega^-, \omega^+]$, and $[\nu^-, \nu^+]$ are subsets of [0,1] and $\tau^+ + \omega^+ + \nu^+ \leq 1$.

Definition 2 (Operations for IVPiFNs [3]) Assume that $\tilde{\tilde{P}}_1 = \langle [\tau_1^-, \tau_1^+], [\omega_1^-, \omega_1^+], [\nu_1^-, \nu_1^+]\rangle$ and $\tilde{\tilde{P}}_2 = \langle [\tau_2^-, \tau_2^+], [\omega_2^-, \omega_2^+], [\nu_2^-, \nu_2^+]\rangle$ are two IVPiFNs and $\alpha > 0$ is a real number. Then

Table 9.1 Comparison of the presented model with the existing models in the literature

Articles	Types of payoff values	Zero-sum/non-zero-sum game	Method used/Approach	Application area
Seikh and Dutta [26]	Interval-valued neutrosophic number	Zero sum	Linear programming approach	Counter cybersecurity
An and Li [15]	Intuitionistic fuzzy number	Non-Zero sum r	Linear programming approach	Company development strategy choice problem
Seikh et al. [21]	Dense fuzzy set	Zero sum	Linear programming approach	Media share problem
Xia [27]	Interval-valued intuitionistic fuzzy number	Zero sum	Based on weighted average operator	Problem of production right of a product
Naqvi et al. [7]	Triangular I-fuzzy number	Zero sum	Aspiration level approach	Voting share problem
Seikh et al. [20]	Hesitant fuzzy set	Zero sum	Lexicographic method	Market share problem
Mi et al. [39]	Probabilistic linguistic information	Zero sum	Based on the linear interpolation method	Forest management
Xue et al. [38]	Hesitant fuzzy set	Zero sum	Ambika method	Counter-terrorism issue
Seikh et al. [22]	Type-2 fuzzy variable	Zero sum	Based on type reduction	Plastic ban problem
Verma and Aggarwal [30]	Linguistic intuitionistic fuzzy numbers	Zero sum	Linear programming approach	3D printer marketing problem
Karmakar et al. [34]	Type-2 intuitionistic fuzzy set	Zero sum	Composite relative degree of payoffs	Bio-gas implementation
Fei and Li [37]	Interval number	Non-zero sum	Bilinear programming approach	Tourism planning management
The proposed approach	Interval-valued picture fuzzy number	Zero sum	Weighted average approach	Counter-cyberthreat issue in healthcare sector

(i) $\tilde{\tilde{P}}_1 \subseteq \tilde{\tilde{P}}_2 \iff \tau_1^- \le \tau_2^-,\ \tau_1^+ \le \tau_2^+,\ \omega_1^- \ge \omega_2^-,\ \omega_1^+ \ge \omega_2^+,\ \nu_1^- \ge \nu_2^-,\ \nu_1^+ \ge \nu_2^+$.

(ii) $\tilde{\tilde{P}}_1 = \tilde{\tilde{P}}_2 \iff \tau_1^- = \tau_2^-,\ \tau_1^+ = \tau_2^+,\ \omega_1^- = \omega_2^-,\ \omega_1^+ = \omega_2^+,\ \nu_1^- = \nu_2^-,\ \nu_1^+ = \nu_2^+$.

(iii) $\tilde{\tilde{P}}_1 + \tilde{\tilde{P}}_2 = \langle [\tau_1^- + \tau_2^- - \tau_1^- \tau_2^-, \tau_1^+ + \tau_2^+ - \tau_1^+ \tau_2^+], [\omega_1^- \omega_2^-, \omega_1^+ \omega_2^+], [\nu_1^- \nu_2^-, \nu_1^+ \nu_2^+] \rangle$.

(iv) $\alpha \tilde{\tilde{P}}_1 = \langle [1 - (1 - \tau_1^-)^\alpha, 1 - (1 - \tau_1^+)^\alpha], [(\omega_1^-)^\alpha, (\omega_1^+)^\alpha], [(\nu_1^-)^\alpha, (\nu_1^+)^\alpha] \rangle$.

Definition 3 (Score Function and Accuracy Function of IVPiFN) For the IVPiFN $\tilde{\tilde{P}} = \langle[\tau^-, \tau^+], [\omega^-, \omega^+], [\nu^-, \nu^+]\rangle$, the score function $\Upsilon(\tilde{\tilde{P}})$ and the accuracy function $\Psi(\tilde{\tilde{P}})$ can be obtained as follows:

$$\Upsilon(\tilde{\tilde{P}}) = \frac{1}{4}(2 + \tau^- + \tau^+ - 2\,\omega^- - 2\,\omega^+ - \nu^- - \nu^+),$$

$$\Psi(\tilde{\tilde{P}}) = \frac{1}{2}\{\tau^- + \tau^+ - \omega^+(1-\tau^+) - \omega^-(1-\tau^-) - \nu^+(1-\omega^+) - \nu^-(1-\omega^-)\}.$$

For two IVPiFNs $\tilde{\tilde{P}}_1$ and $\tilde{\tilde{P}}_2$, the ranking order relation is constructed as follows:

(i) $\Upsilon(\tilde{\tilde{P}}_1) > \Upsilon(\tilde{\tilde{P}}_2) \Rightarrow \tilde{\tilde{P}}_1 \succ_p \tilde{\tilde{P}}_2$;
(ii) $\Upsilon(\tilde{\tilde{P}}_1) = \Upsilon(\tilde{\tilde{P}}_2)$, and $\Psi(\tilde{\tilde{P}}_1) > \Psi(\tilde{\tilde{P}}_2) \Rightarrow \tilde{\tilde{P}}_1 \succ_p \tilde{\tilde{P}}_2$;
(iii) $\Upsilon(\tilde{\tilde{P}}_1) = \Upsilon(\tilde{\tilde{P}}_2)$, and $\Psi(\tilde{\tilde{P}}_1) = \Psi(\tilde{\tilde{P}}_2) \Rightarrow \tilde{\tilde{P}}_1 \simeq_p \tilde{\tilde{P}}_2$.

Here "$\succ_p$" and "$\simeq_p$" express the usual meaning of "larger than" and "equal to," respectively, in picture fuzzy environment.

The following order relation of interval numbers is used to develop the solution methodology.

Definition 4 (Order Relation for Ordinary Interval Numbers [11]) Let $\hat{I}_1 = [I_1{}^-, I_1{}^+]$ and $\hat{I}_2 = [I_2{}^-, I_2{}^+]$ be two intervals. Then $\hat{I}_1 \le \hat{I}_2 \Leftrightarrow I_1{}^- \le I_2{}^-$, and $I_1{}^+ \le I_2{}^+$.

2.1 Notations

The following notations are used to develop the model.

Units	Illustrations
$\tilde{\tilde{P}} = \langle[\tau^-, \tau^+], [\omega^-, \omega^+], [\nu^-, \nu^+]\rangle$	IVPiFN
$\Upsilon(\tilde{\tilde{P}})$	Score function of $\tilde{\tilde{P}}$
$\Psi(\tilde{\tilde{P}})$	Accuracy function of $\tilde{\tilde{P}}$
$\tilde{\tilde{E}}(\bar{\mathbf{a}}, \bar{\mathbf{b}})$	Expected payoff for Player-I
$\bar{\mathbf{a}}_*$	maximin strategy for player-I
$\bar{\mathbf{b}}_*$	minimax strategy for player-II
$\tilde{\tilde{\eta}}_*$	Gain-floor of Player-I
$\tilde{\tilde{\zeta}}_*$	Loss-ceiling of Player II
$\langle[\tau_{mn}{}^-, \tau_{mn}{}^+], [\omega_{mn}{}^-, \omega_{mn}{}^+], [\nu_{mn}{}^-, \nu_{mn}{}^+]\rangle$	Pay-offs with IVPiFNs

3 Matrix Games with IVPiFN Payoffs

Let $\Theta_1 = \{1, 2, \ldots, y\}$ and $\Theta_2 = \{1, 2, \ldots, z\}$ be two index sets and $S_1 = \{\epsilon_m, m \in \Theta_1\}$ and $S_2 = \{\beta_n, n \in \Theta_2\}$ contain all the possible pure strategies for **Pl₁** and **Pl₂**, respectively.

Suppose **Pl₁** gains a payoff representing an IVPiFN

$$\langle(\epsilon_m, \beta_n), [\tau^-(\epsilon_m, \beta_n), \tau^+(\epsilon_m, \beta_n)], [\omega^-(\epsilon_m, \beta_n), \omega^+(\epsilon_m, \beta_n)],$$
$$[\nu^-(\epsilon_m, \beta_n), \nu^+(\epsilon_m, \beta_n)]\rangle,$$

for selecting the pure strategies ϵ_m and β_n by **Pl₁** and **Pl₂** respectively, which is shortly written as $\langle[\tau^-, \tau^+], [\omega^-, \omega^+], [\nu^-, \nu^+]\rangle$. Here, the negative of the IVPiFN $\langle[\tau^-, \tau^+], [\omega^-, \omega^+], [\nu^-, \nu^+]\rangle$ is the payoff for **Pl₂**.

Thus, a *MG* with IVPiFN entries is compactly written as

$$\tilde{N} = (\langle[\tau_{mn}{}^-, \tau_{mn}{}^+], [\omega_{mn}{}^-, \omega_{mn}{}^+], [\nu_{mn}{}^-, \nu_{mn}{}^+]\rangle)_{y\times z}.$$

The matrix $\tilde{N}$ is usually conceived as the payoff matrix for **Pl₁**. From now on, the two-person *MG* with payoff matrix $\tilde{N}$ is supposed to call as an interval-valued picture fuzzy matrix game (IVPiFMG) $\tilde{N}$.

Suppose that **Pl₁** and **Pl₂** select $\epsilon_m \in S_1$ and $\beta_n \in S_2$ with probability a_m and b_n, respectively. If $\sum_{m=1}^{y} a_m = 1$ and $\sum_{n=1}^{z} b_n = 1$, then $\bar{\mathbf{a}} = (a_1, a_2, \ldots, a_y)$ and $\bar{\mathbf{b}} = (b_1, b_2, \ldots, b_z)$ are said to be the mixed strategies for **Pl₁** and **Pl₂**, respectively. Let $\Delta_1 = \left\{\bar{\mathbf{a}} = (a_1, a_2, \ldots, a_y) \in \Re_+^y : \sum_{m=1}^{y} a_m = 1\right\}$ and $\Delta_2 = \left\{\bar{\mathbf{b}} = (b_1, b_2, \ldots, b_z) \in \Re_+^z : \sum_{n=1}^{z} b_n = 1\right\}$ be the collections of all possible mixed strategies for **Pl₁** and **Pl₂**, respectively.

For $(\bar{\mathbf{a}}, \bar{\mathbf{b}}) \in \Delta_1 \times \Delta_2$, the expected payoff $(\tilde{\tilde{E}}(\bar{\mathbf{a}}, \bar{\mathbf{b}}))$ for **Pl₁** can be enumerated as

$$\begin{aligned}
\tilde{\tilde{E}}(\bar{\mathbf{a}}, \bar{\mathbf{b}}) = \bar{\mathbf{a}}^T \tilde{N} \bar{\mathbf{b}} &= \sum_{m=1}^{y}\sum_{n=1}^{z} \langle[\tau_{mn}{}^-, \tau_{mn}{}^+], [\omega_{mn}{}^-, \omega_{mn}{}^+], [\nu_{mn}{}^-, \nu_{mn}{}^+]\rangle a_m b_n \\
&= \Bigg\langle\Bigg[1 - \prod_{n=1}^{z}\prod_{m=1}^{y}(1-\tau_{mn}{}^-)^{a_m b_n}, 1 - \prod_{n=1}^{z}\prod_{m=1}^{y}(1-\tau_{mn}{}^+)^{a_m b_n}\Bigg], \\
&\Bigg[\prod_{n=1}^{z}\prod_{m=1}^{y}(\omega_{mn}{}^-)^{a_m b_n}, \prod_{n=1}^{z}\prod_{m=1}^{y}(\omega_{mn}{}^+)^{a_m b_n}\Bigg], \Bigg[\prod_{n=1}^{z}\prod_{m=1}^{y}(\nu_{mn}{}^-)^{a_m b_n}, \\
&\prod_{n=1}^{z}\prod_{m=1}^{y}(\nu_{mn}{}^+)^{a_m b_n}\Bigg]\Bigg\rangle.
\end{aligned} \tag{9.1}$$

It is clear that $\tilde{\tilde{E}}(\bar{\mathbf{a}}, \bar{\mathbf{b}})$ is an IVPiFN.

Following Owen [8] and Definition 2, if for some $(\bar{\mathbf{a}}_0, \bar{\mathbf{b}}_0) \in \Delta_1 \times \Delta_2$, such that

$$\bar{\mathbf{a}}_0^T \tilde{N} \bar{\mathbf{b}}_0 = \max_{\bar{\mathbf{a}} \in \Delta_1} \min_{\bar{\mathbf{b}} \in \Delta_2} \{\bar{\mathbf{a}}^{\mathbf{T}} \tilde{N} \bar{\mathbf{b}}\} = \min_{\bar{\mathbf{b}} \in \Delta_2} \max_{\bar{\mathbf{a}} \in \Delta_1} \{\bar{\mathbf{a}}^{\mathbf{T}} \tilde{N} \bar{\mathbf{b}}\},$$

then $\bar{\mathbf{a}}_0$ and $\bar{\mathbf{b}}_0$ are the optimal strategies for $\mathbf{Pl_1}$ and $\mathbf{Pl_2}$, respectively. In this case, $\bar{\mathbf{a}}_0^T \tilde{N} \bar{\mathbf{b}}_0$ is considered to be the value of IVPiFMG $\tilde{N}$.

By using Definition 2 and Eq. (9.1), we can convert the optimization problem of $\bar{\mathbf{a}}_0^T \tilde{N} \bar{\mathbf{b}}_0$ to a mathematical programming problem with multi-objective functions χ, δ, and κ, where

$$\chi = \left[1 - \prod_{n=1}^{z} \prod_{m=1}^{y} (1 - \tau_{mn}{}^{-})^{a_m b_n}, 1 - \prod_{n=1}^{z} \prod_{m=1}^{y} (1 - \tau_{mn}{}^{+})^{a_m b_n}\right],$$

$$\delta = \left[\prod_{n=1}^{z} \prod_{m=1}^{y} (\omega_{mn}{}^{-})^{a_m b_n}, \prod_{n=1}^{z} \prod_{m=1}^{y} (\omega_{mn}{}^{+})^{a_m b_n}\right],$$

$$\text{and } \kappa = \left[\prod_{n=1}^{z} \prod_{m=1}^{y} (\nu_{mn}{}^{-})^{a_m b_n}, \prod_{n=1}^{z} \prod_{m=1}^{y} (\nu_{mn}{}^{+})^{a_m b_n}\right].$$

A reasonable solution for *FMG* was conceptualized by Bector et al. . Following is an extension of the idea of a reasonable solution for IVPiFMG.

Definition 5 If for $\bar{\mathbf{a}}_* \in \Delta_1$ and $\bar{\mathbf{b}}_* \in \Delta_2$, $\bar{\mathbf{a}}_* \tilde{N} \bar{\mathbf{b}} \subset \tilde{\tilde{P}}_1$ and $\bar{\mathbf{a}} \tilde{N} \bar{\mathbf{b}}_* \supset \tilde{\tilde{P}}_2$ hold for any IVPiFNs $\tilde{\tilde{P}}_1$, $\tilde{\tilde{P}}_2$ and for any $\bar{\mathbf{a}} \in \Delta_1$ and $\bar{\mathbf{b}} \in \Delta_2$, then $\bar{\mathbf{a}}_*$, and $\bar{\mathbf{b}}_*$ are, respectively, the reasonable values for $\mathbf{Pl_1}$ and $\mathbf{Pl_2}$. The triplet $(\bar{\mathbf{a}}_*, \bar{\mathbf{b}}_*, \tilde{\tilde{P}}_1, \tilde{\tilde{P}}_2)$ represents the reasonable solution of the IVPiFMG $\tilde{N}$.

The concept of the reasonable solution, as defined above, and the concept of the solution of IVPiFMG $\tilde{N}$ are different. The notion of the solution of IVPiFMG $\tilde{N}$ is defined below.

Definition 6 Let $\tilde{\tilde{P}}_{1*} \in L_1$ and $\tilde{\tilde{P}}_{2*} \in L_2$, where L_1 and L_2 are the collections of reasonable values for $\mathbf{Pl_1}$ and $\mathbf{Pl_2}$, respectively. If there does not exist such $\tilde{\tilde{P}}_1{}' \in L_1$ ($\tilde{\tilde{P}}_1{}' \neq \tilde{\tilde{P}}_{1*}$) and $\tilde{\tilde{P}}_2{}' \in L_2$ ($\tilde{\tilde{P}}_2{}' \neq \tilde{\tilde{P}}_{2*}$) for which $\tilde{\tilde{P}}_{1*} \subset \tilde{\tilde{P}}_1{}'$ and $\tilde{\tilde{P}}_{2*} \supset \tilde{\tilde{P}}_2{}'$, then $(\bar{\mathbf{a}}_*, \bar{\mathbf{b}}_*, \tilde{\tilde{P}}_{1*}, \tilde{\tilde{P}}_{2*})$ is considered to be the solution of the IVPiFMG $\tilde{N}$.

Here $\bar{\mathbf{a}}_*$ is the maximin strategy for $\mathbf{Pl_1}$ and $\bar{\mathbf{b}}_*$ is the minimax strategy for $\mathbf{Pl_2}$. $\tilde{\tilde{P}}_{1*}$ and $\tilde{\tilde{P}}_{2*}$ are recognized as the values of the IVPiFMG for $\mathbf{Pl_1}$ and $\mathbf{Pl_2}$, respectively.

For $\mathbf{Pl_1}$, the minimum of the expected values $\tilde{\tilde{\eta}}$ can be determined as

$$\tilde{\tilde{\eta}} = \left\langle \left[\sigma^-, \sigma^+\right], \left[\gamma^-, \gamma^+\right], \left[\varrho^-, \varrho^+\right] \right\rangle = \min_{\bar{\mathbf{b}} \in \Delta_2} \tilde{\tilde{E}}(\bar{\mathbf{a}}, \bar{\mathbf{b}})$$

$$= \Bigg\langle \min_{\bar{\mathbf{b}} \in \Delta_2} \left\{ \left[1 - \prod_{n=1}^{z} \prod_{m=1}^{y} (1 - \tau_{mn}{}^-)^{a_m b_n}, 1 - \prod_{n=1}^{z} \prod_{m=1}^{y} (1 - \tau_{mn}{}^+)^{a_m b_n} \right] \right\},$$

$$\max_{\bar{\mathbf{b}} \in \Delta_2} \left\{ \left[\prod_{n=1}^{z} \prod_{m=1}^{y} (\omega_{mn}{}^-)^{a_m b_n}, \prod_{n=1}^{z} \prod_{m=1}^{y} (\omega_{mn}{}^+)^{a_m b_n} \right] \right\},$$

$$\max_{\bar{\mathbf{b}} \in \Delta_2} \left\{ \left[\prod_{n=1}^{z} \prod_{m=1}^{y} (\nu_{mn}{}^-)^{a_m b_n}, \prod_{n=1}^{z} \prod_{m=1}^{y} (\nu_{mn}{}^+)^{a_m b_n} \right] \right\} \Bigg\rangle.$$

It implies that $\tilde{\tilde{\eta}}$ is a function of $\bar{\mathbf{a}}$. In order to maximize $\tilde{\tilde{\eta}}$, $\mathbf{Pl_1}$ should choose a mixed strategy $\bar{\mathbf{a}}_* \in \Delta_1$ and obtain

$$\tilde{\tilde{\eta}}_* = \left\langle \left[\sigma_*{}^-, \sigma_*{}^+\right], \left[\gamma_*{}^-, \gamma_*{}^+\right], \left[\varrho_*{}^-, \varrho_*{}^+\right] \right\rangle = \max_{\bar{\mathbf{a}} \in \Delta_1} \min_{\bar{\mathbf{b}} \in \Delta_2} \tilde{\tilde{E}}(\bar{\mathbf{a}}, \bar{\mathbf{b}})$$

$$= \Bigg\langle \max_{\bar{\mathbf{a}} \in \Delta_1} \min_{\bar{\mathbf{b}} \in \Delta_2} \left\{ \left[1 - \prod_{n=1}^{z} \prod_{m=1}^{y} (1 - \tau_{mn}{}^-)^{a_m b_n}, 1 - \prod_{n=1}^{z} \prod_{m=1}^{y} (1 - \tau_{mn}{}^+)^{a_m b_n} \right] \right\},$$

$$\min_{\bar{\mathbf{a}} \in \Delta_1} \max_{\bar{\mathbf{b}} \in \Delta_2} \left\{ \left[\prod_{n=1}^{z} \prod_{m=1}^{y} (\omega_{mn}{}^-)^{a_m b_n}, \prod_{n=1}^{z} \prod_{m=1}^{y} (\omega_{mn}{}^+)^{a_m b_n} \right] \right\},$$

$$\min_{\bar{\mathbf{a}} \in \Delta_1} \max_{\bar{\mathbf{b}} \in \Delta_2} \left\{ \left[\prod_{n=1}^{z} \prod_{m=1}^{y} (\nu_{mn}{}^-)^{a_m b_n}, \prod_{n=1}^{z} \prod_{m=1}^{y} (\nu_{mn}{}^+)^{a_m b_n} \right] \right\} \Bigg\rangle. \tag{9.2}$$

Here, $\bar{\mathbf{a}}_*$ and $\tilde{\tilde{\eta}}_*$ are, respectively, the maximin strategy and the gain floor for $\mathbf{Pl_1}$.

For $\mathbf{Pl_2}$, the maximum of the expected loss ($\tilde{\tilde{\zeta}}$) can be determined as follows:

$$\tilde{\tilde{\zeta}} = \left\langle \left[\xi^-, \xi^+\right], \left[\eta^-, \eta^+\right], \left[\psi^-, \psi^+\right] \right\rangle$$

$$= \max_{\bar{\mathbf{a}} \in \Delta_1} \tilde{\tilde{E}}(\bar{\mathbf{a}}, \bar{\mathbf{b}})$$

$$= \Bigg\langle \max_{\bar{\mathbf{a}} \in \Delta_1} \left\{ \left[1 - \prod_{n=1}^{z} \prod_{m=1}^{y} (1 - \tau_{mn}{}^-)^{a_m b_n}, 1 - \prod_{n=1}^{z} \prod_{m=1}^{y} (1 - \tau_{mn}{}^+)^{a_m b_n} \right] \right\},$$

$$\min_{\bar{\mathbf{a}} \in \Delta_1} \left\{ \left[\prod_{n=1}^{z} \prod_{m=1}^{y} (\omega_{mn}{}^-)^{a_m b_n}, \prod_{n=1}^{z} \prod_{m=1}^{y} (\omega_{mn}{}^+)^{a_m b_n} \right] \right\},$$

$$\min_{\bar{\mathbf{a}}\in\Delta_1}\left\{\left[\prod_{n=1}^{z}\prod_{m=1}^{y}({\nu_{mn}}^{-})^{a_m b_n},\prod_{n=1}^{z}\prod_{m=1}^{y}({\nu_{mn}}^{+})^{a_m b_n}\right]\right\}\Big\rangle.$$

Clearly $\tilde{\tilde{\zeta}}$ depends on $\bar{\mathbf{b}}$. For this reason, $\mathbf{Pl_2}$ should consider a mixed strategy $\bar{\mathbf{b}}_* \in \Delta_2$, so that $\tilde{\tilde{\zeta}}$ will be minimum. So, $\mathbf{Pl_2}$ try to obtain $\tilde{\tilde{\zeta}}_*$, where

$$\begin{aligned}
\tilde{\tilde{\zeta}}_* &= \left\langle\left[{\xi_*}^{-},{\xi_*}^{+}\right],\left[{\eta_*}^{-},{\eta_*}^{+}\right],\left[{\psi_*}^{-},{\psi_*}^{+}\right]\right\rangle\\
&= \min_{\bar{\mathbf{b}}\in\Delta_2}\max_{\bar{\mathbf{a}}\in\Delta_1}\tilde{\tilde{E}}(\bar{\mathbf{a}},\bar{\mathbf{b}})\\
&= \Big\langle\min_{\bar{\mathbf{b}}\in\Delta_2}\max_{\bar{\mathbf{a}}\in\Delta_1}\left\{\left[1-\prod_{n=1}^{z}\prod_{m=1}^{y}(1-{\tau_{mn}}^{-})^{a_m b_n},1-\prod_{n=1}^{z}\prod_{m=1}^{y}(1-{\tau_{mn}}^{+})^{a_m b_n}\right]\right\},\\
&\qquad\max_{\bar{\mathbf{b}}\in\Delta_2}\min_{\bar{\mathbf{a}}\in\Delta_1}\left\{\left[\prod_{n=1}^{z}\prod_{m=1}^{y}({\omega_{mn}}^{-})^{a_m b_n},\prod_{n=1}^{z}\prod_{m=1}^{y}({\omega_{mn}}^{+})^{a_m b_n}\right]\right\},\\
&\qquad\max_{\bar{\mathbf{b}}\in\Delta_2}\min_{\bar{\mathbf{a}}\in\Delta_1}\left\{\left[\prod_{n=1}^{z}\prod_{m=1}^{y}({\nu_{mn}}^{-})^{a_m b_n},\prod_{n=1}^{z}\prod_{m=1}^{y}({\nu_{mn}}^{+})^{a_m b_n}\right]\right\}\Big\rangle. \qquad (9.3)
\end{aligned}$$

Here, $\bar{\mathbf{b}}_*$ and $\tilde{\tilde{\zeta}}_*$ are, respectively, the minimax strategy and the loss ceiling for $\mathbf{Pl_2}$.

As in a crisp game situation, the next theorem proves that the gain floor ($\tilde{\tilde{\eta}}_*$) of $\mathbf{Pl_1}$ cannot exceed the loss ceiling ($\tilde{\tilde{\zeta}}_*$) of $\mathbf{Pl_2}$.

Theorem 1 *For IVPiFMG, the relation* $\tilde{\tilde{\eta}}_* \subseteq \tilde{\tilde{\zeta}}_*$ *is valid.*

Proof It is clear that for any $\bar{\mathbf{a}} \in \Delta_1$ and $\bar{\mathbf{b}} \in \Delta_2$,

$$\begin{aligned}
\min_{\bar{\mathbf{b}}\in\Delta_2}\left\{1-\prod_{n=1}^{z}\prod_{m=1}^{y}(1-{\tau_{mn}}^{-})^{a_m b_n}\right\} &\leq 1-\prod_{n=1}^{z}\prod_{m=1}^{y}(1-{\tau_{mn}}^{-})^{a_m b_n}\\
&\leq \max_{\bar{\mathbf{a}}\in\Delta_1}\left\{1-\prod_{n=1}^{z}\prod_{m=1}^{y}(1-{\tau_{mn}}^{-})^{a_m b_n}\right\}.
\end{aligned}$$

Hence,

$$\min_{\bar{\mathbf{b}}\in\Delta_2}\left\{1-\prod_{n=1}^{z}\prod_{m=1}^{y}(1-{\tau_{mn}}^{-})^{a_m b_n}\right\} \leq \min_{\bar{\mathbf{b}}\in\Delta_2}\max_{\bar{\mathbf{a}}\in\Delta_1}\left\{1-\prod_{n=1}^{z}\prod_{m=1}^{y}(1-{\tau_{mn}}^{-})^{a_m b_n}\right\}.$$

Therefore we have,

$$\max_{\bar{\mathbf{a}}\in\Delta_1}\min_{\bar{\mathbf{b}}\in\Delta_2}\left\{1-\prod_{n=1}^{z}\prod_{m=1}^{y}(1-\tau_{mn}{}^{-})^{a_m b_n}\right\}\leq\min_{\bar{\mathbf{b}}\in\Delta_2}\max_{\bar{\mathbf{a}}\in\Delta_1}\left\{1-\prod_{n=1}^{z}\prod_{m=1}^{y}(1-\tau_{mn}{}^{-})^{a_m b_n}\right\}. \tag{9.4}$$

Similarly, it also implies that

$$\max_{\bar{\mathbf{a}}\in\Delta_1}\min_{\bar{\mathbf{b}}\in\Delta_2}\left\{1-\prod_{n=1}^{z}\prod_{m=1}^{y}(1-\tau_{mn}{}^{+})^{a_m b_n}\right\}\leq\min_{\bar{\mathbf{b}}\in\Delta_2}\max_{\bar{\mathbf{a}}\in\Delta_1}\left\{1-\prod_{n=1}^{z}\prod_{m=1}^{y}(1-\tau_{mn}{}^{+})^{a_m b_n}\right\}. \tag{9.5}$$

Using Definition (2) and Eqs. (9.4) and (9.5), we have

$$\begin{aligned}&\max_{\bar{\mathbf{a}}\in\Delta_1}\min_{\bar{\mathbf{b}}\in\Delta_2}\left\{\left[1-\prod_{n=1}^{z}\prod_{m=1}^{y}(1-\tau_{mn}{}^{-})^{a_m b_n},\,1-\prod_{n=1}^{z}\prod_{m=1}^{y}(1-\tau_{mn}{}^{+})^{a_m b_n}\right]\right\}\\&\quad\leq\min_{\bar{\mathbf{b}}\in\Delta_2}\max_{\bar{\mathbf{a}}\in\Delta_1}\left\{\left[1-\prod_{n=1}^{z}\prod_{m=1}^{y}(1-\tau_{mn}{}^{-})^{a_m b_n},\,1-\prod_{n=1}^{z}\prod_{m=1}^{y}(1-\tau_{mn}{}^{+})^{a_m b_n}\right]\right\}.\end{aligned} \tag{9.6}$$

Again for any $\bar{\mathbf{a}}\in\Delta_1$ and $\bar{\mathbf{b}}\in\Delta_2$, we have

$$\max_{\bar{\mathbf{b}}\in\Delta_2}\left\{\prod_{n=1}^{z}\prod_{m=1}^{y}(\omega_{mn}{}^{-})^{a_m b_n}\right\}\geq\prod_{n=1}^{z}\prod_{m=1}^{y}(\omega_{mn}{}^{-})^{a_m b_n}\geq\min_{\bar{\mathbf{a}}\in\Delta_1}\left\{\prod_{n=1}^{z}\prod_{m=1}^{y}(\omega_{mn}{}^{-})^{a_m b_n}\right\}$$

and

$$\max_{\bar{\mathbf{b}}\in\Delta_2}\left\{\prod_{n=1}^{z}\prod_{m=1}^{y}(\nu_{mn}{}^{-})^{a_m b_n}\right\}\geq\prod_{n=1}^{z}\prod_{m=1}^{y}(\nu_{mn}{}^{-})^{a_m b_n}\geq\min_{\bar{\mathbf{a}}\in\Delta_1}\left\{\prod_{n=1}^{z}\prod_{m=1}^{y}(\nu_{mn}{}^{-})^{a_m b_n}\right\}.$$

Therefore, we have

$$\min_{\bar{\mathbf{a}}\in\Delta_1}\max_{\bar{\mathbf{b}}\in\Delta_2}\left\{\prod_{n=1}^{z}\prod_{m=1}^{y}(\omega_{mn}{}^{-})^{a_m b_n}\right\}\geq\max_{\bar{\mathbf{b}}\in\Delta_2}\min_{\bar{\mathbf{a}}\in\Delta_1}\left\{\prod_{n=1}^{z}\prod_{m=1}^{y}(\omega_{mn}{}^{-})^{a_m b_n}\right\} \tag{9.7}$$

and

$$\min_{\bar{\mathbf{a}}\in\Delta_1}\max_{\bar{\mathbf{b}}\in\Delta_2}\left\{\prod_{n=1}^{z}\prod_{m=1}^{y}(\nu_{mn}{}^{-})^{a_m b_n}\right\}\geq\max_{\bar{\mathbf{b}}\in\Delta_2}\min_{\bar{\mathbf{a}}\in\Delta_1}\left\{\prod_{n=1}^{z}\prod_{m=1}^{y}(\nu_{mn}{}^{-})^{a_m b_n}\right\}. \tag{9.8}$$

In a similar fashion, it follows that

$$\min_{\bar{\mathbf{a}}\in\Delta_1}\max_{\bar{\mathbf{b}}\in\Delta_2}\left\{\prod_{n=1}^{z}\prod_{m=1}^{y}({\omega_{mn}}^{+})^{a_m b_n}\right\}\geq\max_{\bar{\mathbf{b}}\in\Delta_2}\min_{\bar{\mathbf{a}}\in\Delta_1}\left\{\prod_{n=1}^{z}\prod_{m=1}^{y}({\omega_{mn}}^{+})^{a_m b_n}\right\} \tag{9.9}$$

and

$$\min_{\bar{\mathbf{a}}\in\Delta_1}\max_{\bar{\mathbf{b}}\in\Delta_2}\left\{\prod_{n=1}^{z}\prod_{m=1}^{y}({\nu_{mn}}^{+})^{a_m b_n}\right\}\geq\max_{\bar{\mathbf{b}}\in\Delta_2}\min_{\bar{\mathbf{a}}\in\Delta_1}\left\{\prod_{n=1}^{z}\prod_{m=1}^{y}({\nu_{mn}}^{+})^{a_m b_n}\right\}. \tag{9.10}$$

From Eqs. (9.7) and (9.9), we can write

$$\begin{aligned}&\min_{\bar{\mathbf{a}}\in\Delta_1}\max_{\bar{\mathbf{b}}\in\Delta_2}\left\{\left[\prod_{n=1}^{z}\prod_{m=1}^{y}({\omega_{mn}}^{-})^{a_m b_n},\prod_{n=1}^{z}\prod_{m=1}^{y}({\omega_{mn}}^{+})^{a_m b_n}\right]\right\}\\&\quad\geq\max_{\bar{\mathbf{b}}\in\Delta_2}\min_{\bar{\mathbf{a}}\in\Delta_1}\left\{\left[\prod_{n=1}^{z}\prod_{m=1}^{y}({\omega_{mn}}^{-})^{a_m b_n},\prod_{n=1}^{z}\prod_{m=1}^{y}({\omega_{mn}}^{+})^{a_m b_n}\right]\right\}.\end{aligned} \tag{9.11}$$

Again, using Eqs. (9.8) and (9.10), we have,

$$\begin{aligned}&\min_{\bar{\mathbf{a}}\in\Delta_1}\max_{\bar{\mathbf{b}}\in\Delta_2}\left\{\left[\prod_{n=1}^{z}\prod_{m=1}^{y}({\nu_{mn}}^{-})^{a_m b_n},\prod_{n=1}^{z}\prod_{m=1}^{y}({\nu_{mn}}^{+})^{a_m b_n}\right]\right\}\\&\quad\geq\max_{\bar{\mathbf{b}}\in\Delta_2}\min_{\bar{\mathbf{a}}\in\Delta_1}\left\{\left[\prod_{n=1}^{z}\prod_{m=1}^{y}({\nu_{mn}}^{-})^{a_m b_n},\prod_{n=1}^{z}\prod_{m=1}^{y}({\nu_{mn}}^{+})^{a_m b_n}\right]\right\}.\end{aligned} \tag{9.12}$$

Using Eqs. (9.6), (9.11) and (9.12), it follows that

$$\begin{aligned}&\max_{\bar{\mathbf{a}}\in\Delta_1}\min_{\bar{\mathbf{b}}\in\Delta_2}\left\{\left\langle\left[1-\prod_{n=1}^{z}\prod_{m=1}^{y}(1-{\tau_{mn}}^{-})^{a_m b_n},1-\prod_{n=1}^{z}\prod_{m=1}^{y}(1-{\tau_{mn}}^{+})^{a_m b_n}\right],\right.\right.\\&\left[\prod_{n=1}^{z}\prod_{m=1}^{y}({\omega_{mn}}^{-})^{a_m b_n},\prod_{n=1}^{z}\prod_{m=1}^{y}({\omega_{mn}}^{+})^{a_m b_n}\right],\left[\prod_{n=1}^{z}\prod_{m=1}^{y}({\nu_{mn}}^{-})^{a_m b_n},\right.\\&\left.\left.\left.\prod_{n=1}^{z}\prod_{m=1}^{y}({\nu_{mn}}^{+})^{a_m b_n}\right]\right\rangle\right\}\\&\leq\min_{\bar{\mathbf{b}}\in\Delta_2}\max_{\bar{\mathbf{a}}\in\Delta_1}\left\{\left\langle\left[1-\prod_{n=1}^{z}\prod_{m=1}^{y}(1-{\tau_{mn}}^{-})^{a_m b_n},\right.\right.\right.\\&\left.1-\prod_{n=1}^{z}\prod_{m=1}^{y}(1-{\tau_{mn}}^{+})^{a_m b_n}\right],\end{aligned}$$

$$\left[\prod_{n=1}^{z}\prod_{m=1}^{y}(\omega_{mn}^{-})^{a_m b_n}, \prod_{n=1}^{z}\prod_{m=1}^{y}(\omega_{mn}^{+})^{a_m b_n}\right], \left[\prod_{n=1}^{z}\prod_{m=1}^{y}(\nu_{mn}^{-})^{a_m b_n}, \prod_{n=1}^{z}\prod_{m=1}^{y}(\nu_{mn}^{+})^{a_m b_n}\right]\Big\rangle\Big\}.$$

Using Definition 2, we have, $\tilde{\tilde{\eta}}_* \subseteq \tilde{\tilde{\zeta}}_*$. □

4 Mathematical Model and Solution Approach for IVPiFMG

The following sections describe the formulation of the model and the solution procedure for IVPiFMG. The solution procedure has been developed based on the work of [6, 26].

For $\mathbf{Pl_1}$, the mathematical programming model with multi-objective functions as given in Eq. (9.13) is constructed considering Eq. (9.2) and Definitions 5 and 6.

$$\begin{aligned}
&\max\left\{\left[\sigma^-, \sigma^+\right]\right\}, \min\left\{\left[\gamma^-, \gamma^+\right]\right\}, \min\left\{\left[\varrho^-, \varrho^+\right]\right\}\\
\text{subject to }&\left[1-\prod_{n=1}^{z}\prod_{m=1}^{y}(1-\tau_{mn}^{-})^{a_m b_n}, 1-\prod_{n=1}^{z}\prod_{m=1}^{y}(1-\tau_{mn}^{+})^{a_m b_n}\right]\\
&\geq\left[\sigma^-, \sigma^+\right], \text{ for any } \bar{\mathbf{b}} \in \Delta_2\\
&\left[\prod_{n=1}^{z}\prod_{m=1}^{y}(\omega_{mn}^{-})^{a_m b_n}, \prod_{n=1}^{z}\prod_{m=1}^{y}(\omega_{mn}^{+})^{a_m b_n}\right]\leq\left[\gamma^-, \gamma^+\right], \text{ for any } \bar{\mathbf{b}}\in\Delta_2\\
&\left[\prod_{n=1}^{z}\prod_{m=1}^{y}(\nu_{mn}^{-})^{a_m b_n}, \prod_{n=1}^{z}\prod_{m=1}^{y}(\nu_{mn}^{+})^{a_m b_n}\right]\leq\left[\varrho^-, \varrho^+\right], \text{ for any } \bar{\mathbf{b}}\in\Delta_2\\
&\sum_{m=1}^{y} a_m = 1, \quad a_m \geq 0, \qquad m \in \Theta_1\\
&\sigma^- \geq 0, \sigma^+ \geq 0, \; \gamma^- \geq 0, \gamma^+ \geq 0, \; \varsigma^- \geq 0, \varsigma^+ \geq 0,\\
&0 \leq \sigma^+ + \gamma^+ + \varsigma^+ \leq 1,
\end{aligned} \tag{9.13}$$

where $\sigma^- = \min_{\bar{\mathbf{b}}\in\Delta_2}\left\{1-\prod_{n=1}^{z}\prod_{m=1}^{y}(1-\tau_{mn}^{-})^{a_m b_n}\right\}$, $\sigma^+ = \min_{\bar{\mathbf{b}}\in\Delta_2}\left\{1-\prod_{n=1}^{z}\prod_{m=1}^{y}(1-\tau_{mn}^{+})^{a_m b_n}\right\}$,

$\gamma^- = \max_{\bar{\mathbf{b}}\in\Delta_2}\left\{\prod_{n=1}^{z}\prod_{m=1}^{y}(\omega_{mn}{}^-)^{a_m b_n}\right\}$, $\gamma^+ = \max_{\bar{\mathbf{b}}\in\Delta_2}\left\{\prod_{n=1}^{z}\prod_{m=1}^{y}(\omega_{mn}{}^+)^{a_m b_n}\right\}$, $\varrho^- = \max_{\bar{\mathbf{b}}\in\Delta_2}\left\{\prod_{n=1}^{z}\prod_{m=1}^{y}(\nu_{mn}{}^-)^{a_m b_n}\right\}$
and $\varrho^+ = \max_{\bar{\mathbf{b}}\in\Delta_2}\left\{\prod_{n=1}^{z}\prod_{m=1}^{y}(\nu_{mn}{}^+)^{a_m b_n}\right\}$.

By solving Eq. (9.13), we can obtain the maximin strategy $\bar{\mathbf{a}}_*$ and the gain floor $\tilde{\tilde{\eta}}_* = \left\langle\left[\sigma_*{}^-, \sigma_*{}^+\right], \left[\gamma_*{}^-, \gamma_*{}^+\right], \left[\varrho_*{}^-, \varrho_*{}^+\right]\right\rangle$ for $\mathbf{Pl_1}$.

It is evident that the maximization of $[\sigma^-, \sigma^+]$ in Eq. (9.13) is similar to the minimization of $[1-\sigma^+, 1-\sigma^-]$, and it is also similar to the minimization of $[\ln(1-\sigma^+), \ln(1-\sigma^-)]$ for $0 \le \sigma^-, \sigma^+ < 1$. So, the maximization of $[\sigma^-, \sigma^+]$ is equivalent to the minimization of $[\ln(1-\sigma^+), \ln(1-\sigma^-)]$ for $0 \le \sigma^-, \sigma^+ < 1$.

In a similar manner, we can write the minimization of $[\gamma^-, \gamma^+]$, and $[\varrho^-, \varrho^+]$ are, respectively, equivalent to the minimization of $[\ln\gamma^-, \ln\gamma^+]$ and $[\ln\varrho^-, \ln\varrho^+]$, for $0 < \gamma^-, \gamma^+, \varrho^-, \varrho^+ < 1$.

Now as the objective functions $[\ln\gamma^-, \ln\gamma^+]$ and $[\ln\varrho^-, \ln\varrho^+]$ have the similar importance, the average of these two functions $\left[\frac{\ln\gamma^- + \ln\varrho^-}{2}, \frac{\ln\gamma^+ + \ln\varrho^+}{2}\right]$ is taken into consideration.

Therefore, the three objective functions in Eq. (9.13) can be aggregated using the weighted average method as follows:

$$\min\Big[\mu\ \ln(1-\sigma^+) + (1-\mu)\Big(\frac{\ln\gamma^- + \ln\varrho^-}{2}\Big),\ \mu\ \ln(1-\sigma^-) + (1-\mu)\Big(\frac{\ln\gamma^+ + \ln\varrho^+}{2}\Big)\Big], \tag{9.14}$$

where $\mu \in [0,1]$ is a weight value, which is to be chosen by the players for optimization.

Using the interval order relation as defined in Definition 4, the constraints in Eq. (9.13) can be written as follows:

$$\prod_{n=1}^{z}\prod_{m=1}^{y}(1-\tau_{mn}{}^-)^{a_m b_n} \le 1-\sigma^-,\quad \prod_{n=1}^{z}\prod_{m=1}^{y}(1-\tau_{mn}{}^+)^{a_m b_n} \le 1-\sigma^+,$$

$$\prod_{n=1}^{z}\prod_{m=1}^{y}(\omega_{mn}{}^-)^{a_m b_n} \le \gamma^-,\quad \prod_{n=1}^{z}\prod_{m=1}^{y}(\omega_{mn}{}^+)^{a_m b_n} \le \gamma^+,$$

$$\prod_{n=1}^{z}\prod_{m=1}^{y}(\nu_{mn}{}^-)^{a_m b_n} \le \varrho^-,\quad \prod_{n=1}^{z}\prod_{m=1}^{y}(\nu_{mn}{}^+)^{a_m b_n} \le \varrho^+,$$

which are equivalent to the following inequalities:

$$\sum_{n=1}^{z}\sum_{m=1}^{y} a_m b_n \ln(1-\tau_{mn}{}^{-}) \leq \ln(1-\sigma^{-}),\ \ \sum_{n=1}^{z}\sum_{m=1}^{y} a_m b_n \ln(1-\tau_{mn}{}^{+}) \leq \ln(1-\sigma^{+}),$$

$$\sum_{n=1}^{z}\sum_{m=1}^{y} a_m b_n \ln(\omega_{mn}{}^{-}) \leq \ln\gamma^{-},\ \ \sum_{n=1}^{z}\sum_{m=1}^{y} a_m b_n \ln(\omega_{mn}{}^{+}) \leq \ln\gamma^{+},$$

$$\sum_{n=1}^{z}\sum_{m=1}^{y} a_m b_n \ln(\nu_{mn}{}^{-}) \leq \ln\varrho^{-},\ \ \sum_{n=1}^{z}\sum_{m=1}^{y} a_m b_n \ln(\nu_{mn}{}^{+}) \leq \ln\varrho^{+},$$

for $0 < \tau_{mn}{}^{-}, \tau_{mn}{}^{+}, \omega_{mn}{}^{-}, \omega_{mn}{}^{+}, \nu_{mn}{}^{-}, \nu_{mn}{}^{+}, \sigma^{-}, \sigma^{+}, \gamma^{-}, \gamma^{+}, \varrho^{-}, \varrho^{+} < 1$.

Therefore, using the weighted average method, the constraints in Eq. (9.13) can be aggregated as follows:

$$\begin{aligned}
&\sum_{n=1}^{z}\sum_{m=1}^{y}\Big[\mu\ \ln(1-\tau_{mn}{}^{+}) + (1-\mu)\Big(\frac{\ln(\omega_{mn}{}^{-}) + \ln(\nu_{mn}{}^{-})}{2}\Big)\Big]a_m b_n \\
&\qquad\leq \mu\ \ln(1-\sigma^{+}) + (1-\mu)\Big(\frac{\ln\gamma^{-} + \ln\varrho^{-}}{2}\Big), \text{ for any } \bar{\mathbf{b}} \in \Delta_2 \\
&\sum_{n=1}^{z}\sum_{m=1}^{y}\Big[\mu\ \ln(1-\tau_{mn}{}^{-}) + (1-\mu)\Big(\frac{\ln(\omega_{mn}{}^{+}) + \ln(\nu_{mn}{}^{+})}{2}\Big)\Big]a_m b_n \\
&\qquad\leq \mu\ \ln(1-\sigma^{-}) + (1-\mu)\Big(\frac{\ln\gamma^{+} + \ln\varrho^{+}}{2}\Big), \text{ for any } \bar{\mathbf{b}} \in \Delta_2 \\
&\sum_{m=1}^{y} a_m = 1,\ \ a_m \geq 0,\ m \in \Theta_1 \\
&\sigma^{-} \geq 0, \sigma^{+} \geq 0,\ \gamma^{-} > 0, \gamma^{+} > 0,\ \varrho^{-} > 0, \varrho^{+} > 0,\ \sigma^{+} + \gamma^{+} + \varrho^{+} \leq 1.
\end{aligned} \tag{9.15}$$

Using Eqs. (9.14) and (9.15), Eq. (9.13) is changed into the following Eq. (9.16).

$$\min\Big\{\Big[\mu\ \ln(1-\sigma^{+}) + (1-\mu)\Big(\frac{\ln\gamma^{-} + \ln\varrho^{-}}{2}\Big),\ \mu\ \ln(1-\sigma^{-}) + (1-\mu)\Big(\frac{\ln\gamma^{+} + \ln\varrho^{+}}{2}\Big)\Big]\Big\}$$

$$\text{subject to } \sum_{n=1}^{z}\sum_{m=1}^{y}\Big[\mu\ \ln(1-\tau_{mn}{}^{+})+(1-\mu)\Big(\frac{\ln(\omega_{mn}{}^{-})+\ln(\nu_{mn}{}^{-})}{2}\Big)\Big]a_m b_n$$
$$\leq \mu\ \ln(1-\sigma^{+})+(1-\mu)\Big(\frac{\ln\gamma^{-}+\ln\varrho^{-}}{2}\Big),\ \text{for any } \bar{\mathbf{b}}\in\Delta_2$$
$$\sum_{n=1}^{z}\sum_{m=1}^{y}\Big[\mu\ \ln(1-\tau_{mn}{}^{-})+(1-\mu)\Big(\frac{\ln(\omega_{mn}{}^{+})+\ln(\nu_{mn}{}^{+})}{2}\Big)\Big]a_m b_n$$
$$\leq \mu\ \ln(1-\sigma^{-})+(1-\mu)\Big(\frac{\ln\gamma^{+}+\ln\varrho^{+}}{2}\Big),\ \text{for any } \bar{\mathbf{b}}\in\Delta_2$$
$$\sum_{m=1}^{y}a_m=1,\ \ a_m\geq 0,\ m\in\Theta_1$$
$$\sigma^{-}\geq 0,\sigma^{+}\geq 0,\ \gamma^{-}>0,\gamma^{+}>0,\ \varrho^{-}>0,$$
$$\varrho^{+}>0,\ \sigma^{+}+\gamma^{+}+\varrho^{+}\leq 1. \tag{9.16}$$

Let $\theta^{-}=\mu\ \ln(1-\sigma^{+})+(1-\mu)\Big(\frac{\ln\gamma^{-}+\ln\varrho^{-}}{2}\Big)$ and $\theta^{+}=\mu\ \ln(1-\sigma^{-})+(1-\mu)\Big(\frac{\ln\gamma^{+}+\ln\varrho^{+}}{2}\Big)$. Then, $\theta^{-}\leq 0$ and $\theta^{+}\leq 0$ as $\mu\in[0,1]$, $0<1-\sigma^{+}\leq 1$, $0<\gamma^{-}\leq 1$, $0<\varrho^{-}\leq 1$, $0<1-\sigma^{-}\leq 1$, $0<\gamma^{+}\leq 1$, and $0<\varrho^{+}\leq 1$.

Thus, Eq. (9.16) changes to Eq. (9.17) as follows:

$$\min\Big\{\Big[\theta^{-},\theta^{+}\Big]\Big\}$$
$$\text{subject to } \sum_{n=1}^{z}\sum_{m=1}^{y}\Big[\mu\ \ln(1-\tau_{mn}{}^{+})+(1-\mu)\Big(\frac{\ln(\omega_{mn}{}^{-})+\ln(\nu_{mn}{}^{-})}{2}\Big)\Big]a_m b_n$$
$$\leq\ \theta^{-},\ \text{for any } \bar{\mathbf{b}}\in\Delta_2$$
$$\sum_{n=1}^{z}\sum_{m=1}^{y}\Big[\mu\ \ln(1-\tau_{mn}{}^{-})+(1-\mu)\Big(\frac{\ln(\omega_{mn}{}^{+})+\ln(\nu_{mn}{}^{+})}{2}\Big)\Big]a_m b_n$$
$$\leq\ \theta^{+},\ \text{for any } \bar{\mathbf{b}}\in\Delta_2$$
$$\sum_{m=1}^{y}a_m=1,\ \ a_m\geq 0,\ m\in\Theta_1$$
$$\theta^{-}\leq 0,\theta^{+}\leq 0, \tag{9.17}$$

except for $\tau_{mn}{}^{-}=1$, $\tau_{mn}{}^{+}=1$, $\omega_{mn}{}^{-}=0$, $\omega_{mn}{}^{+}=0$, $\nu_{mn}{}^{-}=0$, and $\nu_{mn}{}^{+}=0$.

Due to the finite nature of Δ_2, it creates perception to consider only the extreme points of Δ_2 within the constraints of Eq. (9.17). Therefore, Eq. (9.17) changes to the following Eq. (9.18):

$$\begin{aligned}
&\min\left\{\left[\theta^{-},\theta^{+}\right]\right\}\\
\text{subject to}\quad &\sum_{m=1}^{y}\left[\mu\ \ln(1-\tau_{mn}{}^{+})+(1-\mu)\left(\frac{\ln(\omega_{mn}{}^{-})+\ln(\nu_{mn}{}^{-})}{2}\right)\right]a_m\\
&\leq\ \theta^{-},\ \ n\in\Theta_2\\
&\sum_{m=1}^{y}\left[\mu\ \ln(1-\tau_{mn}{}^{-})+(1-\mu)\left(\frac{\ln(\omega_{mn}{}^{+})+\ln(\nu_{mn}{}^{+})}{2}\right)\right]a_m\\
&\leq\ \theta^{+},\ \ n\in\Theta_2\\
&\sum_{m=1}^{y}a_m=1,\ \ a_m\geq 0,\ m\in\Theta_1\\
&\theta^{-}\leq 0,\theta^{+}\leq 0,
\end{aligned}\tag{9.18}$$

Here Eq. (9.18) is an interval-valued programming problem and can be solved by the method proposed in [11]. We follow the methodology proposed in [6] to deal with the interval-valued objective function of Eq. (9.18). Let $\theta=\theta^{-}+\theta^{+}$. As $\theta^{-}\leq 0$ and $\theta^{+}\leq 0$, $\theta\leq 0$. Hence, Eq. (9.18) changed into the following Eq. (9.19):

$$\begin{aligned}
&\min\left\{\theta\right\}\\
\text{subject to}\quad &\sum_{m=1}^{y}\left[\mu\ \ln(1-\tau_{mn}{}^{+})+(1-\mu)\left(\frac{\ln(\omega_{mn}{}^{-})+\ln(\nu_{mn}{}^{-})}{2}\right)+\mu\ \ln(1-\tau_{mn}{}^{-})\right.\\
&\qquad\left.+\,(1-\mu)\left(\frac{\ln(\omega_{mn}{}^{+})+\ln(\nu_{mn}{}^{+})}{2}\right)\right]a_m\leq\theta,\ \ n\in\Theta_2\\
&\sum_{m=1}^{y}a_m=1,\ \ a_m\geq 0,\ m\in\Theta_1\\
&\theta\leq 0
\end{aligned}\tag{9.19}$$

except for $\tau_{mn}{}^{-}=1$, $\tau_{mn}{}^{+}=1$, $\omega_{mn}{}^{-}=0$, $\omega_{mn}{}^{+}=0$, $\nu_{mn}{}^{-}=0$, and $\nu_{mn}{}^{+}=0$.

Clearly, the optimal strategies for $\mathbf{Pl_1}$ will be obtained by solving Eq. (9.19) for distinct $\mu\in[0,1]$ using existing simplex methods.

Earlier, for $\mathbf{Pl_2}$, the following mathematical programming problem with multi-objective functions as given in Eq. (9.20) is formulated to obtain the minimax strategy $\bar{\mathbf{q}}_*$ and the loss-ceiling $\tilde{\tilde{\zeta}}_*=\left\langle\left[\xi_*{}^{-},\xi_*{}^{+}\right],\left[\eta_*{}^{-},\eta_*{}^{+}\right],\left[\psi_*{}^{-},\psi_*{}^{+}\right]\right\rangle$ considering Eq. (9.3) and Definitions 5 and 6.

$$\min\left\{\left[\xi^-,\xi^+\right]\right\},\max\left\{\left[\eta^-,\eta^+\right]\right\},\max\left\{\left[\psi^-,\psi^+\right]\right\}$$

$$\text{subject to}\quad \left[1-\prod_{n=1}^{z}\prod_{m=1}^{y}(1-\tau_{mn}{}^{-})^{a_m b_n},1-\prod_{n=1}^{z}\prod_{m=1}^{y}(1-\tau_{mn}{}^{+})^{a_m b_n}\right]$$

$$\leq\left[\xi^-,\xi^+\right],\text{ for any }\bar{\mathbf{a}}\in\Delta_1$$

$$\left[\prod_{n=1}^{z}\prod_{m=1}^{y}(\omega_{mn}{}^{-})^{a_m b_n},\prod_{n=1}^{z}\prod_{m=1}^{y}(\omega_{mn}{}^{+})^{a_m b_n}\right]\geq\left[\eta^-,\eta^+\right],\text{ for any }\bar{\mathbf{a}}\in\Delta_1$$

$$\left[\prod_{n=1}^{z}\prod_{m=1}^{y}(\nu_{mn}{}^{-})^{a_m b_n},\prod_{n=1}^{z}\prod_{m=1}^{y}(\nu_{mn}{}^{+})^{a_m b_n}\right]\geq\left[\psi^-,\psi^+\right],\text{ for any }\bar{\mathbf{a}}\in\Delta_1$$

$$\sum_{n=1}^{z}b_n=1,\quad b_n\geq 0,\ n\in\Theta_2$$

$$\xi^-\geq 0,\xi^+\geq 0,\ \eta^-\geq 0,\eta^+\geq 0,\ \psi^-\geq 0,\psi^+\geq 0,$$

$$0\leq\xi^++\eta^++\psi^+\leq 1,\tag{9.20}$$

where $\xi^-=\max\limits_{\bar{\mathbf{a}}\in\Delta_1}\left\{1-\prod\limits_{n=1}^{z}\prod\limits_{m=1}^{y}(1-\tau_{mn}{}^{-})^{a_m b_n}\right\}$, $\xi^+=\max\limits_{\bar{\mathbf{a}}\in\Delta_1}\left\{1-\prod\limits_{n=1}^{z}\prod\limits_{m=1}^{y}(1-\tau_{mn}{}^{+})^{a_m b_n}\right\}$,
$\eta^-=\min\limits_{\bar{\mathbf{a}}\in\Delta_1}\left\{\prod\limits_{n=1}^{z}\prod\limits_{m=1}^{y}(\omega_{mn}{}^{-})^{a_m b_n}\right\}$, $\eta^+=\min\limits_{\bar{\mathbf{a}}\in\Delta_1}\left\{\prod\limits_{n=1}^{z}\prod\limits_{m=1}^{y}(\omega_{mn}{}^{+})^{a_m b_n}\right\}$, $\psi^-=\min\limits_{\bar{\mathbf{a}}\in\Delta_1}\left\{\prod\limits_{n=1}^{z}\prod\limits_{m=1}^{y}(\nu_{mn}{}^{-})^{a_m b_n}\right\}$
and $\psi^+=\min\limits_{\bar{\mathbf{a}}\in\Delta_1}\left\{\prod\limits_{n=1}^{z}\prod\limits_{m=1}^{y}(\nu_{mn}{}^{+})^{a_m b_n}\right\}$.

Previously, using the weighted average approach, the objective functions and the constraints of Eq. (9.20) can be aggregated and transformed into the following Eq. (9.21):

$$\max\left\{\left[\mu\ \ln(1-\xi^+)+(1-\mu)\left(\frac{\ln\eta^-+\ln\psi^-}{2}\right),\mu\ \ln(1-\xi^-)\right.\right.$$

$$\left.\left.+(1-\mu)\left(\frac{\ln\eta^++\ln\psi^+}{2}\right)\right]\right\}$$

$$\text{subject to}\quad\sum_{n=1}^{z}\sum_{m=1}^{y}\left[\mu\ \ln(1-\tau_{mn}{}^{+})+(1-\mu)\left(\frac{\ln(\omega_{mn}{}^{-})+\ln(\nu_{mn}{}^{-})}{2}\right)\right]a_m b_n$$

$$\geq\mu\ \ln(1-\xi^+)+(1-\mu)\left(\frac{\ln\eta^-+\ln\psi^-}{2}\right),\quad\text{for any }\bar{\mathbf{a}}\in\Delta_1$$

$$\sum_{n=1}^{z}\sum_{m=1}^{y}\left[\mu\ \ln((1-\tau_{mn}{}^{-})+(1-\mu)\left(\frac{\ln(\omega_{mn}{}^{+})+\ln(\nu_{mn}{}^{+})}{2}\right)\right]a_m b_n$$
$$\geq \mu\ \ln(1-\xi^{-})+(1-\mu)\left(\frac{\ln\eta^{+}+\ln\psi^{+}}{2}\right),\quad \text{for any } \bar{\mathbf{a}}\in\Delta_1$$
$$\sum_{n=1}^{z} b_n=1,\quad b_n\geq 0,\ n\in\Theta_2$$
$$\xi^{-}\geq 0,\xi^{+}\geq 0,\ \eta^{-}\geq 0,\eta^{+}\geq 0,\ \psi^{-}\geq 0,\psi^{+}\geq 0,$$
$$0\leq \xi^{+}+\eta^{+}+\psi^{+}\leq 1. \tag{9.21}$$

Let $\phi^{-}=\mu\ \ln(1-\xi^{+})+(1-\mu)\left(\frac{\ln\eta^{-}+\ln\psi^{-}}{2}\right)$, and $\phi^{+}=\mu\ \ln(1-\xi^{-})+(1-\mu)\left(\frac{\ln\eta^{+}+\ln\psi^{+}}{2}\right)$. Then, $\phi^{-}\leq 0$ and $\phi^{+}\leq 0$ as $\mu\in[0,1]$, $0<1-\xi^{+}\leq 1$, $0<\eta^{-}\leq 1$, $0<\psi^{-}\leq 1$, $0<1-\xi^{-}\leq 1$, $0<\eta^{+}\leq 1$, and $0<\psi^{+}\leq 1$. Then, Eq. (9.21) changes to Eq. (9.22).

$$\max\left\{\left[\phi^{-},\phi^{+}\right]\right\}$$
$$\text{subject to}\quad \sum_{n=1}^{z}\sum_{m=1}^{y}\left[\mu\ \ln(1-\tau_{mn}{}^{+})+(1-\mu)\left(\frac{\ln(\omega_{mn}{}^{-})+\ln(\nu_{mn}{}^{-})}{2}\right)\right]a_m b_n$$
$$\geq \phi^{-},\quad \text{for any } \bar{\mathbf{a}}\in\Delta_1$$
$$\sum_{n=1}^{z}\sum_{m=1}^{y}\left[\mu\ \ln((1-\tau_{mn}{}^{-})+(1-\mu)\left(\frac{\ln(\omega_{mn}{}^{+})+\ln(\nu_{mn}{}^{+})}{2}\right)\right]a_m b_n$$
$$\geq \phi^{+},\quad \text{for any } \bar{\mathbf{a}}\in\Delta_1$$
$$\sum_{n=1}^{z} b_n=1,\quad b_n\geq 0,\ n\in\Theta_2$$
$$\phi^{-}\leq 0,\phi^{+}\leq 0. \tag{9.22}$$

Due to the finite nature of Δ_1, a perception is created to examine only the extreme points of Δ_1 within the constraints of Eq. (9.22). As a result, Eq. (9.22) can be transformed into Eq. (9.23) as follows:

$$\max\left\{\left[\phi^{-},\phi^{+}\right]\right\}$$
$$\text{subject to}\quad \sum_{n=1}^{z}\left[\mu\ \ln(1-\tau_{mn}{}^{+})+(1-\mu)\left(\frac{\ln(\omega_{mn}{}^{-})+\ln(\nu_{mn}{}^{-})}{2}\right)\right]b_n\geq\phi^{-},\ m\in\Theta_1$$

$$\sum_{n=1}^{z}\Big[\mu\ \ln((1-\tau_{mn}{}^{-})+(1-\mu)\Big(\frac{\ln(\omega_{mn}{}^{+})+\ln(\nu_{mn}{}^{+})}{2}\Big)\Big]b_n\geq\phi^{+},\quad m\in\Theta_1$$

$$\sum_{n=1}^{z} b_n = 1,\ \ b_n \geq 0,\ n\in\Theta_2$$

$$\phi^{-}\leq 0, \phi^{+}\leq 0. \tag{9.23}$$

Based on the work of [6], we investigate the remaining solution method.

Let $\phi = \phi^{-} + \phi^{+}$. As $\phi^{-} \leq 0$ and $\phi^{+} \leq 0$, $\phi \leq 0$. Therefore, Eq. (9.23) reduced to Eq. (9.24).

$$\max\Big\{\phi\Big\}$$

$$\text{subject to}\ \sum_{n=1}^{z}\Big[\mu\ \ln(1-\tau_{mn}{}^{+})+(1-\mu)\Big(\frac{\ln(\omega_{mn}{}^{-})+\ln(\nu_{mn}{}^{-})}{2}\Big)+\mu\ln(1-\tau_{mn}{}^{-})$$

$$+(1-\mu)\Big(\frac{\ln(\omega_{mn}{}^{+})+\ln(\nu_{mn}{}^{+})}{2}\Big)\Big]b_n\geq\phi,\quad m\in\Theta_1$$

$$\sum_{n=1}^{z} b_n = 1,\ \ b_n \geq 0,\ n\in\Theta_2$$

$$\phi\leq 0, \tag{9.24}$$

except for $\tau_{mn}{}^{-} = 1$, $\tau_{mn}{}^{-} = 1$, $\omega_{mn}{}^{-} = 0$, $\omega_{mn}{}^{+} = 0$, $\nu_{mn}{}^{-} = 0$, and $\nu_{mn}{}^{+} = 0$.

It is to be noted that the optimal strategies for $\mathbf{Pl_2}$ will be obtained by solving Eq. (9.24) for distinct $\mu \in [0, 1]$ by way of existing simplex methods.

In Theorem 2, we establish that there always exists a solution to an IVPiFMG.

Theorem 2 *There always exists a solution* $(\bar{\mathbf{a}}_{*}, \bar{\mathbf{b}}_{*}, \bar{\mathbf{a}}_{*}^{T}\tilde{N}\bar{\mathbf{b}}_{*})$ *for the IVPiFMG* $\tilde{N}$*, for different* $\mu \in (0, 1)$.

Proof Clearly, the problems as in Eqs. (9.19) and (9.24) are considered to be the primal-dual problems for different values of $\mu \in (0, 1)$, in the game problem where the payoff matrix is

$$\Big(\mu\ \ln(1-\tau_{mn}{}^{+})+(1-\mu)\Big(\frac{\ln(\omega_{mn}{}^{-})+\ln(\nu_{mn}{}^{-})}{2}\Big)+\mu\ \ln(1-\tau_{mn}{}^{-})$$

$$+(1-\mu)\Big(\frac{\ln(\omega_{mn}{}^{+})+\ln(\nu_{mn}{}^{+})}{2}\Big)\Big)_{y\times z},$$

for any value of $\mu \in (0, 1)$.

According to Owen [8], it follows that Eqs. (9.19) and (9.24) always provide optimal solutions $(\bar{\mathbf{a}}_*, \theta_*)$ and $(\bar{\mathbf{b}}_*, \phi_*)$, respectively. So, for any given weight $\mu \in (0, 1)$, IVPiFMG $\tilde{N}$ has a solution $(\bar{\mathbf{a}}_*, \bar{\mathbf{b}}_*, \bar{\mathbf{a}}_*^T \tilde{N} \bar{\mathbf{b}}_*)$. □

The solution methodology can be described in the following steps:

Step 1: Consider an IVPiFMG.
Step 2: Construct Eqs. (9.13) and (9.20) for $\mathbf{Pl_1}$ and $\mathbf{Pl_2}$, respectively, with interval-valued multi-objective functions.
Step 3: Applying the weighted average method, transform Eqs. (9.13) and (9.20) into Eqs. (9.19) and (9.24), respectively, with crisp objective function.
Step 4: Solve Eqs. (9.19) and (9.24) in LINGO platform for different $\mu \in (0, 1)$, and obtain the optimal strategies $\bar{\mathbf{a}}_*$ and $\bar{\mathbf{b}}_*$ for $\mathbf{Pl_1}$ and $\mathbf{Pl_2}$, respectively.

5 Numerical Illustration

Medical institutions are concerned about the increased risk of cyberthreats. As the healthcare sector continues to serve life-critical facilities to improve treatment with new technologies, cyberattackers are looking to exploit vulnerabilities that are coupled with these changes. Cyberattackers in the healthcare sector can have ramifications beyond financial loss and breach of privacy, as the loss of patient data can put lives at risk. Here, we consider the following instance of game problem between digital attacker and the protector of the system in the medical sector and execute the corresponding optimal solutions applying the presented methodology. Also, the obtained results are analyzed.

Assume that the medical data controller (MDC) wants to prevent hackers from gaining access to patients' information. Any malicious strike can be used by hackers to steal data. In real time, it can be challenging to predict the paths of digital strikes. The job of the MDC is to defend against such digital attacks. MDC considers three main paths from which the attack comes frequently such as ρ_1, ransomware; ρ_2, healthcare-related mobile applications; and ρ_3, medical devices. This implies that ρ_1, ρ_2, and ρ_3 may be the main strategies of the cyberattackers to disturb the personal health-related data of the patients. To control such attacks, the MDC takes main strategies such as σ_1, use of updated software and operating systems; σ_2, training of staff periodically; and σ_3, proper records disposal. This type of problem can be mitigated using *MG*, where MDC and hackers are referred to as defenders and attackers, respectively. It is possible to consider the MDC and cyberattackers as Player-I ($\mathbf{Pl_1}$) and Player-II ($\mathbf{Pl_2}$), respectively, in the context of the game problem. It may be taken that the defender's defense success rate is the payoff for $\mathbf{Pl_1}$. Considering all the strategies defined above, the MDC is unable to forecast defense success rates precisely because there is always some level of uncertainty in cybersecurity. As a result of the MDC's estimates, the outcomes are referred to as follows in terms of linguistic terms:

Table 9.2 Associated IVPiFN with linguistic term

Linguistic term	IVPiFN
Very High	$\langle[0.75, 0.85], [0.05, 0.1], [0.01, 0.04]\rangle$
High	$\langle[0.65, 0.8], [0.01, 0.05], [0.01, 0.03]\rangle$
Slightly High	$\langle[0.45, 0.6], [0.1, 0.2], [0.05, 0.15]\rangle$
Medium	$\langle[0.15, 0.25], [0.2, 0.3], [0.3, 0.4]\rangle$
Slightly Low	$\langle[0.25, 0.4], [0.1, 0.2], [0.2, 0.35]\rangle$
Low	$\langle[0.08, 0.1], [0.25, 0.35], [0.4, 0.55]\rangle$
Very Low	$\langle[0.01, 0.03], [0.3, 0.4], [0.49, 0.56]\rangle$

$$\begin{array}{cc} & \begin{array}{ccc} \rho_1 & \rho_2 & \rho_3 \end{array} \\ \tilde{N} = \begin{array}{c} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{array} & \begin{pmatrix} \text{Very High} & \text{Slightly High} & \text{Medium} \\ \text{Low} & \text{High} & \text{Slightly High} \\ \text{Slightly Low} & \text{Very Low} & \text{High} \end{pmatrix}. \end{array}$$

The corresponding link between linguistic terms and IVPiFNs is enlisted in Table 9.2. In this case, the payoff matrix $\tilde{N}$ is formed following Table 9.2.

$$\begin{array}{cc} & \begin{array}{ccc} \rho_1 & \rho_2 & \rho_3 \end{array} \\ \tilde{N} = \begin{array}{c} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{array} & \begin{pmatrix} \langle[0.75, 0.85], [0.05, 0.1], [0.01, 0.04]\rangle & \langle[0.45, 0.6], [0.1, 0.2], [0.05, 0.15]\rangle & \langle[0.15, 0.25], [0.2, 0.3], [0.3, 0.4]\rangle \\ \langle[0.08, 0.1], [0.25, 0.35], [0.4, 0.55]\rangle & \langle[0.65, 0.8], [0.01, 0.05], [0.01, 0.03]\rangle & \langle[0.45, 0.6], [0.1, 0.2], [0.05, 0.15]\rangle \\ \langle[0.25, 0.4], [0.1, 0.2], [0.2, 0.35]\rangle & \langle[0.01, 0.03], [0.3, 0.4], [0.49, 0.56]\rangle & \langle[0.65, 0.8], [0.01, 0.05], [0.01, 0.03]\rangle \end{pmatrix}. \end{array}$$

Here, the payoff $\langle[0.75, 0.85], [0.05, 0.1], [0.01, 0.04]\rangle$ indicates the defense success rate according to the strategies σ_1 and ρ_1 chosen by $\mathbf{Pl_1}$ and $\mathbf{Pl_2}$, respectively. In this scenario, $\mathbf{Pl_1}$ can defend against the attack successfully by 75% to 85% and failed to defend successfully by 1% to 4%. There is an indeterminacy of 5% to 10% for $\mathbf{Pl_1}$ when it comes to the defense success rate. There are also equivalent clarifications for other payoffs of $\tilde{N}$.

5.1 The Solution Procedure and Result Discussion

In this section, we follow Eqs. (9.19) and (9.24) as discussed earlier to obtain the optimal solutions for $\mathbf{Pl_1}$ and $\mathbf{Pl_2}$, respectively. Thus, Eqs. (9.25) and (9.26) are formulated for $\mathbf{Pl_1}$ and $\mathbf{Pl_2}$, respectively.

$$\min\{\theta\}$$

$$\text{subject to} \quad \left[\mu \ln 0.0375 + \left(\frac{1-\mu}{2}\right) \ln 0.000002\right] a_1$$

$$
\begin{aligned}
&+\Big[\mu \ln 0.828 + \Big(\frac{1-\mu}{2}\Big) \ln 0.01925\Big] a_2 \\
&+\Big[\mu \ln 0.45 + \Big(\frac{1-\mu}{2}\Big) \ln 0.0014\Big] a_3 \le \theta, \\
&\Big[\mu \ln 0.22 + \Big(\frac{1-\mu}{2}\Big) \ln 0.00015\Big] a_1 \\
&+\Big[\mu \ln 0.07 + \Big(\frac{1-\mu}{2}\Big) \ln 0.00000015\Big] a_2 \\
&+\Big[\mu \ln 0.9603 + \Big(\frac{1-\mu}{2}\Big) \ln 0.032928\Big] a_3 \le \theta, \\
&\Big[\mu \ln 0.6375 + \Big(\frac{1-\mu}{2}\Big) \ln 0.0072\Big] a_1 \\
&+\Big[\mu \ln 0.22 + \Big(\frac{1-\mu}{2}\Big) \ln 0.00015\Big] a_2 \\
&+\Big[\mu \ln 0.07 + \Big(\frac{1-\mu}{2}\Big) \ln 0.00000015\Big] a_3 \le \theta, \\
&\qquad a_1 + a_2 + a_3 = 1, \quad a_1, a_2, a_3 \ge 0, \\
&\theta \le 0, \qquad (9.25)
\end{aligned}
$$

and,

$$
\begin{aligned}
&\max\{\phi\} \\
\text{subject to } &\Big[\mu \ln 0.0375 + \Big(\frac{1-\mu}{2}\Big) \ln 0.000002\Big] b_1 \\
&+\Big[\mu \ln 0.22 + \Big(\frac{1-\mu}{2}\Big) \ln 0.00015\Big] b_2 \\
&+\Big[\mu \ln 0.6375 + \Big(\frac{1-\mu}{2}\Big) \ln 0.0072\Big] b_3 \ge \phi, \\
&\Big[\mu \ln 0.828 + \Big(\frac{1-\mu}{2}\Big) \ln 0.01925\Big] b_1 \\
&+\Big[\mu \ln 0.07 + \Big(\frac{1-\mu}{2}\Big) \ln 0.00000015\Big] b_2 \\
&+\Big[\mu \ln 0.22 + (\frac{1-\mu}{2}\Big) \ln 0.00015\Big] b_3 \ge \phi, \\
&\Big[\mu \ln 0.45 + \Big(\frac{1-\mu}{2}\Big) \ln 0.0014\Big] b_1 \\
&+\Big[\mu \ln 0.9603 + \Big(\frac{1-\mu}{2}\Big) \ln 0.032928\Big] b_2
\end{aligned}
$$

$$+\left[\mu \ln 0.07 + \left(\frac{1-\mu}{2}\right)\ln 0.00000015\right]b_3 \geq \phi,$$
$$b_1 + b_2 + b_3 = 1, \quad b_1, b_2, b_3 \geq 0.$$
$$\phi \leq 0 \tag{9.26}$$

To obtain the optimal solutions, Eqs. (9.25) and (9.26) are solved for different values of $\mu \in (0, 1)$ using LINGO software, version 17.0 in a machine with processor- Intel-Core (TM) i3, RAM-4GB. The obtained optimal solutions are shown in Table 9.3.

In Table 9.3, maximin strategies ($\bar{\mathbf{a}}_*$) for $\mathbf{Pl_1}$ and minimax strategies ($\bar{\mathbf{b}}_*$) for $\mathbf{Pl_2}$ are enlisted, and the corresponding expected payoffs ($\tilde{\tilde{E}}(\bar{\mathbf{a}}_*, \bar{\mathbf{b}}_*) = \bar{\mathbf{a}}_*^T \tilde{N} \bar{\mathbf{b}}_*$) are obtained. For instance, when $\mu = 0.5$, $\bar{\mathbf{a}}_* = (0.4249, 0.2819, 0.2932)^T$ is the maximin strategy for $\mathbf{Pl_1}$, and $\bar{\mathbf{b}}_* = (0.3227, 0.2794, 0.3979)^T$ is the minimax strategy for $\mathbf{Pl_2}$. $\tilde{\tilde{E}}(\bar{\mathbf{a}}_*, \bar{\mathbf{b}}_*) = \langle[0.4492, 0.5915], [0.0775, 0.1654], [0.0658, 0.1465]\rangle$ is the corresponding expected payoff for $\mathbf{Pl_1}$. Also, this result implies that the defender consider the strategy σ_1,σ_2, and σ_3 with probabilities 0.4249, 0.2819, and 0.2932, respectively, and the attacker choose strategy ρ_1, ρ_2, and ρ_3, respectively, for optimal solution. There is a 44.92% to 59.15% chance that the defender can successfully defend the attack in this case. Furthermore, the defender cannot successfully defend the attack by 6.58% to 14.65% and is indeterminate about defending it successfully by 7.75% to 16.54%.

Table 9.3 Optimal solutions of Eqs. (9.25) and (9.26)

Value of μ	$\bar{\mathbf{a}}_*^T, \bar{\mathbf{b}}_*^T$	$\tilde{\tilde{E}}(\bar{\mathbf{a}}_*, \bar{\mathbf{b}}_*)$	$\Upsilon(\tilde{\tilde{E}}(\bar{\mathbf{a}}_*, \bar{\mathbf{b}}_*))$
0.1	(0.4593,0.2534,0.2873), (0.3552,0.2793,0.3656)	⟨[0.4554, 0.5964], [0.0794, 0.1666], ⟩ ⟨[0.0653, 0.1463]⟩	0.5871
0.2	(0.4518,0.2594,0.2888), (0.3480,0.2795,0.3725)	⟨[0.4539, 0.5951], [0.0790, 0.1664], ⟩ ⟨[0.0655, 0.1465]⟩	0.5866
0.3	(0.4437,0.2661,0.2902), (0.3403,0.2797,0.38)	⟨[0.4523, 0.5939], [0.0786, 0.1662], ⟩ ⟨[0.0656, 0.1466]⟩	0.5861
0.4	(0.4347,0.2736,0.2917), (0.3319,0.2797,0.3884)	⟨[0.4507, 0.5926], [0.0781, 0.1658], ⟩ ⟨[0.0658, 0.1466]⟩	0.5858
0.5	(0.4249,0.2819,0.2932), (0.3227,0.2794,0.3979)	⟨[0.4492, 0.5915], [0.0775, 0.1654], ⟩ ⟨[0.0658, 0.1465]⟩	0.5857
0.6	(0.4141,0.2913,0.2946), (0.3126,0.2787,0.4087)	⟨[0.4477, 0.5904], [0.0769, 0.1649],]⟩ ⟨[0.0658, 0.1463]⟩	0.5856
0.7	(0.4021,0.3021,0.2958), (0.3015,0.2776,0.4209)	⟨[0.4463, 0.5895], [0.0761, 0.1643], ⟩ ⟨[0.0657, 0.1459]⟩	0.5859
0.8	(0.3887,0.3145,0.2968), (0.2893,0.2755,0.4352)	⟨[0.4451, 0.5889], [0.0752, 0.1636], ⟩ ⟨[0.0655, 0.1454]⟩	0.5864
0.9	(0.3737,0.3291,0.2972), (0.2759,0.2723,0.4518)	⟨[0.4441, 0.5885], [0.0742, 0.1627], ⟩ ⟨[0.0651, 0.1446]⟩	0.5873

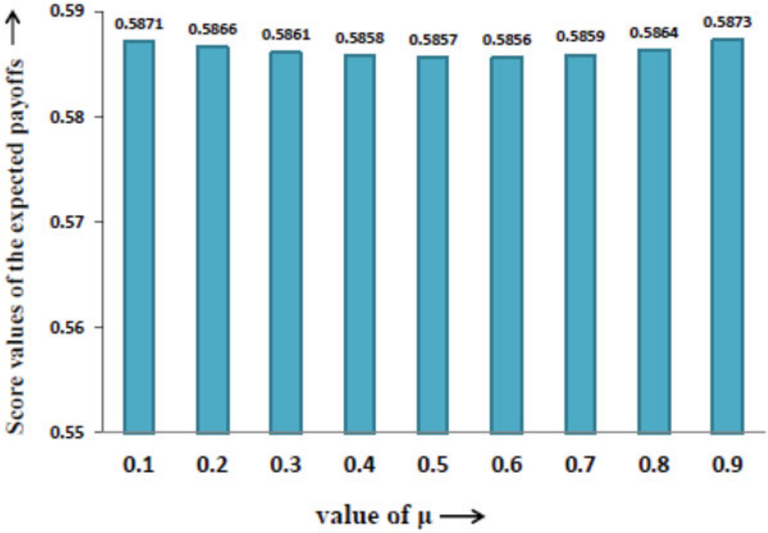

Fig. 9.1 The values of score function of the expected payoffs for distinct μ

There is an effect of changing μ value on the score function values of the expected payoffs as reflected in Table 9.3 and Fig. 9.1. There is no significant impact of a small change in μ on the score function values of the expected payoffs. This suggests the sensitivity of the discussed method.

5.2 Comparison of the Proposed Approach

Xia [27] developed *MG* problems considering payoffs as IVIFNs based on the weighted average operator. Naqvi et al. [7] studied *MG* with triangular I-fuzzy numbers as payoff entries. Verma and Aggarwal [30] utilized linguistic IFNs as payoffs and applied the linear programming approach to counter *MG* problems. Karmakar et al. [34] considered type-2 IFSs as payoffs of the players and used the idea of composite relative degree to solve *MG* problems. In real scenarios, the views of decision-makers have multiple answers, like acceptance, unbiased, non-acceptance, and refusal, and those cannot be answered using an interval-valued fuzzy set or IVIFS or interval numbers. In practical considerations, the DOM, DON, and DOI of a statement cannot be completely described, but they can be modeled as intervals. A set of numbers in the real unit interval is used in the IVPiFS to communicate issues. The proposed approach gives a direction to counter the cyberthreat issue in the healthcare sector as a game problem, where the digital attackers and the defenders of such attacks are taken as players. The respective payoffs of the players are considered as IVPiFN. A novel approach is developed to solve the game problem with IVPiFN payoffs. So, the proposed approach to solving IVPiFMG is an extension of the existing works [7, 27, 30, 34].

The obtained optimal solutions suggest adopting mixed strategies rather than pure strategies. The medical data controller can use the present study to make decisions so that they can minimize the damage to important health information. The recommended strategies may vary to some extent depending on what the players decide to change in the evaluation data of the payoff matrix.

6 Conclusion

In many areas of game theory, *MGs* with uncertain facts are a major research field. IVPiFN considers more information in the form of interval-valued DOM, DOI, and DON than FS/IFS. Our aim in this chapter is to develop the solution process of IVPiFMG for dealing with cyberthreat-related issues. As a first attempt, this work solves *MG* problems in which attackers and defenders are modeled as players, with IVPiFNs representing the payoffs. First, the formal representation of the IVPiFMG problem is developed. Two different problems with multi-objective functions are countered using the weighted average approach, and the optimal solutions are obtained. The idea of reasonable solution and the solution of IVPiFMG are conceptualized in this chapter. Also, the score function and the accuracy function for IVPiFN are defined. It is proved that for IVPiFMG, $\mathbf{Pl}_1'$s gain-floor does not exceed $\mathbf{Pl}_2'$s loss-ceiling. Also, we have shown that IVPiFMG always has a solution. We illustrate the applicability of this methodology by presenting a numerical example. Optimal solutions are discussed in terms of their physical significance.

The discussed methodology has a few limitations as well. As the proposed method considers the formation of problems with multi-objective functions, it cannot determine the solution to the game problem directly. Furthermore, this proposed methodology has the limitation that the optimal solutions are extremely dependent on the choice of parameter μ, which is difficult for the decision-maker to choose the best solution. It should also be noted that the physical significance of the obtained results is discussed depending on the defined score function. But different score functions may produce different results.

The proposed methodology may be used to solve non-zero-sum games in the future. The present study is between two persons, and future research can focus on solving multi-player games in picture fuzzy environment. Also, the present study is dependent on the value of the parameter μ. So, to choose the player's best preferences scientifically, further study is needed. The proposed method can be extended by utilizing the other extended version of fuzzy language such as neutrosophic sets [12] and pythagorean fuzzy set [32, 33]. A lot of real-life problems can be addressed using the proposed approach, such as ecological management [16], plastic ban problem [22], telecom market share problem [23], biogas-plant implementation problem [34], smart transportation [4], cybersecurity-related problem [26], and management problem [1].

Acknowledgments The authors would like to thank the anonymous reviewer for valuable comments and constructive suggestions to improve the quality of the presentation of the chapter.

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Chapter 10
Minimization of Span in $L(3, 1)$-Labeling for a Particular Type of Intersection Graphs

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1 Introduction

Graph labelling problems are widely investigated due to their practical significance. The FAP stands for the "frequency assignment problem," which entails allocating frequencies to all radio transmitters in a way that prevents interference by assigning frequencies to each transmitter that are far enough apart from one another. This problem is formulated as a graph vertex coloring problem by Hale [10]. The major goal of this issue is to reduce the span (highest label used). $\lambda_{3,1}$-number of G is the smallest span over all feasible labelling functions of the $L(3, 1)$-labeling and is indicated by the symbol $\lambda_{3,1}(G)$. $L(3, 1)$-labeling of a graph $G = (V, E)$ is a function $\tau : V \rightarrow \{0, 1, 2, \ldots\}$ so that $|\tau(x) - \tau(y)| \geq 3$ if $d(x, y) = 1$ and $|\tau(x) - \tau(y)| \geq 1$ if $d(x, y) = 2$. After completion of L(3,1)-labeling of the graphs, the highest label is the span of the graph, denoted by λ.

FAP has been studied in the following papers: [15, 16, 22, 26–35]. We have focused our attention on $L(3, 1)$-labeling of paths and interval graphs. For any graph G, $\lambda_{0,1}(G) \leq \Delta^2 - \Delta$ [11], and $\lambda_{2,1}(G) \leq \Delta^2 + \Delta - 2$ [5]. In [12], Khan et al. have shown that $\Delta - 1 \leq \lambda_{0,1}(G) \leq \Delta$ for the cactus graph. In [17] intersection graphs are discussed. The problem is simple for a path P_n with n vertices. It is easily verified that $\lambda_{0,1}(P_1)$=$\lambda_{0,1}(P_2) = 0$, $\lambda_{0,1}(P_n) = 1$ for $n \geq 3$ [13]. When the starting

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C. Jana et al. (eds.), *Fuzzy Optimization, Decision-making and Operations Research*, https://doi.org/10.1007/978-3-031-35668-1_10

and ending vertices of P_n are merged, then P_n becomes C_{n-1}. For path $\lambda_{1,1}(P_2) = 1$ and $\lambda_{1,1}(P_n) = 2$ for each $n \geq 3$, and $\lambda_{1,1}(C_n)$ is 2 if n is a multiple of 3, and it is 3 otherwise [1]. In [4] Calamoneri et al. proved that $\lambda_{h,k}(G) \leq \max(h, 2k)\Delta + hw$ for a circular-arc graph (CA-graph). In [19] Paul et al. show that $\lambda_{2,1}(G) \leq \Delta + w$ for interval graph (I-graph) and that $\lambda_{2,1}(G) \leq \Delta + 3w$ for CA-graph. Also, Paul et al. have studied $L(0, 1)$-labeling on I-graph [18] and $L(2, 1)$-labeling of I-graph [19]. Recently, Amanathulla et al. proved that $\lambda_{0,1}(G) \leq \Delta$ and $\lambda_{1,1}(G) \leq 2\Delta$ [23] and $\lambda_{3,2,1}(G) \leq 9\Delta - 6$ and $\lambda_{4,3,2,1}(G) \leq 16\Delta - 12$ [24] for CA-graphs. They also have invested $L(h_1, h_2, \ldots, h_m)$-labeling of I-graphs [25].

The upper bound of $L(h, k)$-labeling of various types of graphs is displayed in the table below.

Graphs	$L(h, k)$-labeling numbers
General graphs	$0 \leq \lambda_{0,1} \leq \Delta^2 - \Delta$ [2]
	$\Delta \leq \lambda_{1,1} \leq \Delta^2$ [37]
	$\Delta + 1 \leq \lambda_{2,1} \leq \Delta^2 + \Delta - 2$ [8, 9]
Paths	$\lambda_{0,1}(P_n)$ =0 or 1 [13]
	$\lambda_{1,1}(P_n)$ =1 or 2 [1]
	$\lambda_{2,1}(P_n)$ =2, 3 or 4 [9]
Cycles	For $n \geq 3$, $\lambda_{0,1}(C_n)$ =1 or 2 [2]
	For $n \geq 3$, $\lambda_{1,1}(C_n)$ =2 or 3 [1]
	For $n \geq 3$, $\lambda_{2,1}(C_n) = 4$ [9]
Complete	$\lambda_{1,1}(k_n) = n - 1$ [6]
Complete bipartite	$\lambda_{1,1}(K_{m,n}) = m + n - 1$ [6]
Planar	$\lambda_{1,1}(G) \leq \lceil \frac{5}{3}\Delta + 1 \rceil + 77$ [14]
	$\lambda_{2,1}(G) \leq 2\Delta + 35$ [36]
	$\lambda_{2,1}(G) \leq \frac{5}{3}\Delta + 95$ [14]
	$\lambda_{h,k}(G) \leq k\lceil \frac{5}{3}\Delta \rceil + 18h + 77k - 18$ [14]
Interval	$\lambda_{2,1}(G) \leq \Delta + w$ [19]
Circular-arc	$\lambda_{h,k}(G) \leq \max\{h, 2k\}\Delta + h\omega$ [4]
	$\lambda_{2,1}(G) \leq \Delta + 3w$ [19]
Permutation	$\lambda_{0,1}(G) \leq 2\Delta - 2$ [3]
	$\lambda_{0,1}(G) \leq \Delta - 1$ [21]
	$\lambda_{1,1}(G) \leq 3\Delta - 2$ [3]
	$\lambda_{2,1}(G) \leq max\{4\Delta - 2, 5\Delta - 8\}$ [20]
	$\lambda_{2,1}(G) \leq 5\Delta - 2$ [3]

Very recently, Ghosh et al. have studied $L(3, 1)$-labeling of some simple graphs [7]. The applications of $L(3, 1)$-labeling in real life motivate us to consider $L(3, 1)$-labeling problems on I-graphs, and we obtain good results for it, which is $\lambda_{3,1}(G) \leq 4\Delta - 1$ for I-graph G. Beside this, we have designed an efficient algorithm for labeling an I-graph by $L(3, 1)$-labeling. Additionally, the proposed algorithm's execution time is calculated.

The remaining part of the chapter is divided into the following sections. Section 2 presents a few notations and definitions. The $L(3, 1)$-labeling problem of I-graphs is covered in Sect. 3. Section 4 is for the conclusion.

2 Preliminaries and Notations

In this chapter, we consider the graph as simple and finite. A graph G, where $G = (V, E)$, $V = \{v_1, v_2, \ldots, v_n\}$, is called a path, denoted by P_n, iff $(v_i, v_{i+1}) \in E$, where $1 \leq i \leq n-1$ (see Fig. 10.2). I-graph is an important subclass of intersection graph.

Definition 1 (I-Graph) The undirected graph $G = (V, E)$ is a I-graph if the vertex set V can be put into one-to-one correspondence with a set of intervals I on the real line R such that two vertices are adjacent in G iff their corresponding intervals have non-empty intersection.

It is to be noticed that a vertex v_j and an interval I_j are one and the same thing. Figure 10.1 displayed an interval representation and its corresponding I-graph.

Notations Let G be an I-graph with n vertices and $I = \{I_1, I_2, \ldots, I_n\}$, be the corresponding set of intervals. Here we give some notations that are used in this article.

1. $L(I_p)$: the collection of labels that are used to label the interval I_p before labeling I_p, for any $I_p \in I$.
2. $L^1_{31}(I_p)$: the set of $L(3, 1)$-labels that are used to label the vertices at distance 1 from I_p, before labeling I_p, for any $I_p \in I$.
3. $L^2_{31}(I_p)$: the set of $L(3, 1)$-labels that are used to label the vertices at distance 2 from I_p, before labeling I_p, for any $I_p \in I$.
4. $L^{vl}_{31}(1, I_p)$: the collection of valid labels that can be used to label I_p before labeling I_p and meet the adjacency requirement of $L(3, 1)$-labeling, for any interval $I_p \in I$.
5. $L^{vl}_{31}(2, I_p)$: the collection of valid labels that can be used to label I_p before labeling I_p and meet the requirement of $L(3, 1)$-labeling, for any interval $I_p \in I$.
6. τ_j: the label of the interval I_j, for any interval $I_j \in I$.
7. L: the label set.

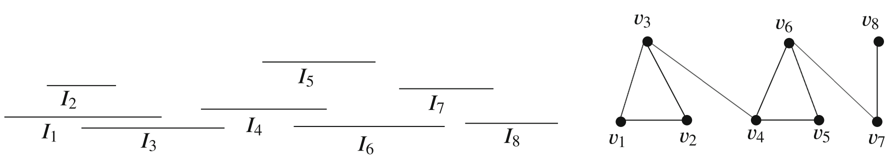

Fig. 10.1 A set of intervals and its associated I-graph

3 $L(3, 1)$-Labeling of I-Graphs

Several lemmas connected to the suggested problem are presented in this section. Additionally, an algorithm is created for $L(3, 1)$-labeling of I-graphs.

Lemma 1 *For any I-graph G, $|L^2_{31}(I_p)| \leq \Delta - 1$, for any interval $I_p \in I$ of G.*

Proof Let G be a n-vertex I-graph. Starting with the interval on the left, we label the graph. Let v_p be the vertex corresponding to the interval I_p of the I-graph G. Suppose in a stage the intervals $I_1, I_2, \ldots, I_{p-1}$ (for $p = 2, 3, \ldots, n$) are already labeled by $L(3, 1)$-labeling and the remaining intervals are not label.

Let $|L^2_{31}(I_p)| = q$. This means that the intervals at a distance of two from the interval I_p were labelled using separate $L(2, 1)$-labels a total of q times. There is an interval I_α (in Fig. 10.2) that is adjacent to at most Δ intervals of G since Δ is the degree of the I-graph G. In Fig. 10.3, I_α is adjacent to $I_k, I_\beta, I_{k_{21}}, I_{k_{22}}$. There are several intervals ($I_\beta, I_{k_{21}}, I_{k_{22}}$ in Fig. 10.3) that are two distances from I_k, but there is at least one interval (I_p in Fig. 10.3) that is not two distances from I_p. Because of this, $q \leq \Delta - 1$, i.e., $|L^2_{31}(I_p)| \leq \Delta - 1$.

Observation 1 *For an I-graph G, $L^i(I_p) \subseteq L(I_p)$, for every interval I_p of G and $i = 1, 2$.*

Observation 2 *For an I-graph G, $|L^1_{31}(I_p)| \leq \Delta$, for every interval $I_p \in I$ of G.*

Theorem 1 *For any I-graph G, the $L(3, 1)$-labeling number $\lambda_{3,1}(G)$ is at most $4\Delta - 1$.*

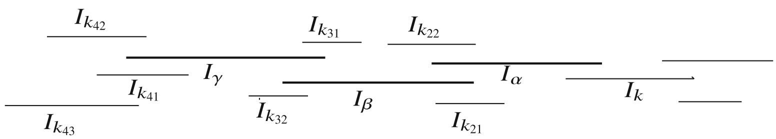

Fig. 10.2 A set of intervals

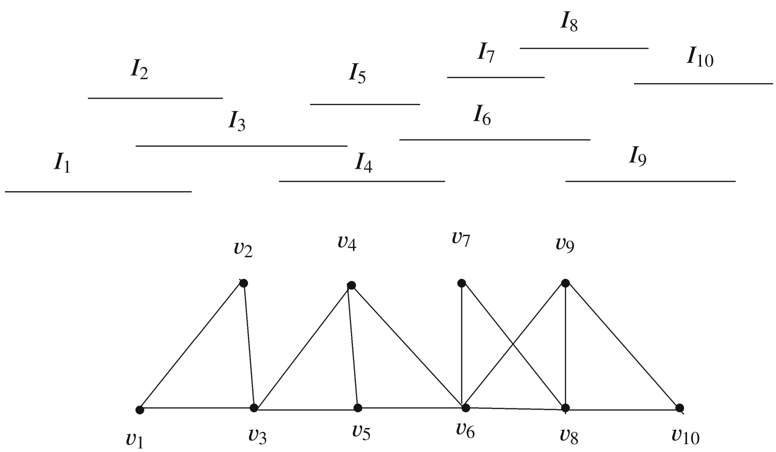

Fig. 10.3 An interval representation and the associated I-graph

Proof Let the graph G have n vertices and let $I = \{I_1, I_2, \ldots, I_n\}$, where I_p is the interval corresponding to the v_p G. In accordance with their ascending subscripts, we will label the intervals.

Assume that $L(I_p) = \{0, 1 \ldots, 4\Delta - 1\}$ and that $I_p \in I$. We can say that $|L(I_p)| = 4\Delta$ if the set $L(I_p)$ is sufficient to label all the intervals corresponding to all the vertices of the graph G meeting $L(3, 1)$-labeling criterion. We take into account a scenario in which the intervals $I_1, I_2, \ldots, I_{p-1}$, for $p = 2, 3 \ldots, n$ are already labeled and $I_p, I_{p+1}, \ldots, I_n$ are not labeled. We wish to label the interval l I_p in this instance. The fact that $|L^1_{31}(I_p)| \leq \Delta$ is known (by Observation 2). As a result, the requirement of distance one of $L(3, 1)$-labeling is satisfied in the worst scenario, where $4\Delta - 3\Delta = \Delta$ labels of the set $L(I_p)$ are accessible.

Since $|L^2_{31}(I_p)| \leq \Delta - 1$ (By Lemma 1), it follows that in the worst case, at least one label of the set $L(I_p)$ that satisfies the $L(3, 1)$-labeling condition is accessible. We can label any interval of the I-graph G satisfying the $L(3, 1)$-labeling requirement by using simply the label of the set $L(I_p)$ because I_p is arbitrarily chosen. If we label the interval I_p by $L(3, 1)$-labeling and take $L(I_p)$ such that $L(I_p) \subset \{0, 1 \ldots, 4\Delta-1\}$, then it follows that the set $L(I_p)$ may or may not contain a label meeting the $L(3, 1)$-labeling requirement. Therefore, $\lambda_{3,1}(G) \leq 4\Delta-1$. The $L(3, 1)$-labeling number for I-graph is therefore $4\Delta - 1$ at most.

3.1 Algorithm for L(3,1)-Labeling of I-Graphs

This section describes the algorithm we created to compute the set. We suppose that certain intervals (those with indexes of $p < j$) are labelled by $L(3, 1)$-labeling and some intervals (those with indexes of $p \geq j$) are not labelled.

Algorithm $VLL31$
Input: $I_p, L^1_{31}(I_p), L^2_{31}(I_p)$ for $p = 2, 3, \ldots, n$.
Output: $L^{vl}_{31}(k, I_p)$ for $k = 1, 2$; $p = 2, 3, \ldots, n$.

Step 1: for $i = 1$ to s, where $s = max\{L(I_p)\} + 3$
    for $j = 1$ to $|L^1_{31}(I_p)|$
    let l_j be jth element of $L^1_{31}(I_p)$
    if $|i - l_j| \geq 3$, then add i to the set $L^{vl}_{31}(1, I_p)$;
    end for;
  end for;
Step 2: for $m = 1$ to $|L^{vl}_{31}(1, I_p)|$
    for $n = 1$ to $|L^2_{31}(I_p)|$
    Let r_m and s_n be the elements of $L^{vl}_{31}(1, I_p)$ and $L^2_{31}(I_p)$, respectively;
    if $|r_m - s_n| \geq 1$, then add r_m to the set $L^{vl}_{31}(2, I_k)$;
    end for;
  end for;
end $VLL31$.

Lemma 2 *Algorithm $VLL31$ computes the set $L_{31}^{vl}(k, I_p)$ accurately for $k = 1, 2$ and its running time is $O(\Delta^2)$.*

Proof Each element $i \in L_{31}^{vl}(1, I_p)$ differs from r_l by at least 3 for each $r_l \in L_{31}^1(I_p)$ according to algorithm $VLL31$. So $|i - r_l| \geq 3$ for each $i \in L_{31}^{vl}(1, I_p)$ and for each $l_r \in L_{31}^1(I_p)$. Therefore, the set $L_{31}^{vl}(1, I_p)$ is correctly computed by algorithm $VLL31$ for each $I_k \in I$, $p = 2, 3, \ldots, n$. Again, according to the above algorithm, every element l_α of $L_{31}^{vl}(2, I_p)$ differs from l_β by at least 1 for each $l_\beta \in L_{31}^2(I_p)$. So $|l_m - q_n| \geq 2$ for all $l_m \in L_{31}^{vl}(2, I_p)$ and for all $q_n \in L_{31}^1(I_p)$, and $|l_m - q_n| \geq 1$ for all $l_m \in L_{31}^{vl}(2, I_p)$ and for all $q_n \in L_{31}^2(I_p)$. Hence, algorithm $VLL31$ correctly computes $L_{31}^{vl}(k, I_p)$ for every $k = 1, 2$.

Since, $|L|$ be cardinality of the label set L, so $|L^i(I_p)| \leq |L|$ for $i = 1, 2$ and $I_p \in I$, and also $r \leq 4\Delta + 2$, where $r = max\{L(I_p)\} + 3$. So $L_{31}^{vl}(1, I_p)$ is computed by using at most $(4\Delta + 2)|L|$ times, i.e., using $O(\Delta|L|)$ times. Again, $|L_{31}^{vl}(2, I_n)| \leq (4\Delta + 2)$, so $L_{31}^{vl}(2, I_p)$ is computed using at most $(4\Delta + 2)|L|$ times, i.e., using $O(\Delta|L|)$ times. Since $|L| \leq 4\Delta + 2$, the overall time complexity for algorithm $VLL31$ is $O(\Delta^2)$.

Lemma 3 *For every I-graph G, $L_{31}^{vl}(1, I_p)$ is the largest non-empty set of labels meeting the requirement at distance one of $L(3, 1)$-labeling, where $l \leq s$ for all $l \in L_{31}^{vl}(1, I_p)$, $s = max\{L(I_p)\} + 3$, for any $I_p \in I$.*

Proof Since $L_{31}^1(I_p) \subseteq L(I_p)$ (by Observation 1) and $s = max\{L(I_p)\} + 3$, so $|s - l_i| \geq 3$ for any $l_i \in L_{31}^1(I_p)$. So $s \in L_{31}^{vl}(1, I_p)$, so, the set $L_{31}^{vl}(1, I_p)$ is non empty. Let A be arbitrary labeled set satisfying adjacency condition of $L(3, 1)$-labeling, where $l \leq s$ for any $l \in A$, $s = max\{L(I_p)\} + 3$. Let a be an element in the set A. Then $|a - l_i| \geq 3$ for any $l_i \in L_{31}^1(I_p)$. So, $a \in L_{31}^{vl}(1, I_p)$. Thus, $a \in A$ implies $a \in L_{31}^{vl}(1, I_p)$. Therefore, $A \subseteq L_{31}^{vl}(1, I_p)$. Since A is arbitrary, $L_{31}^{vl}(1, I_p)$ is the largest non-empty set of labels meeting the adjacency requirement of $L(3, 1)$-labeling, such that $l \leq s$ for all $l \in L_{31}^{vl}(1, I_p)$ and $s = max\{L(I_p)\} + 3$, for any $I_p \in I$.

Lemma 4 *For every I-graph G, $L_{31}^{vl}(2, I_p)$ is the largest non-empty set of labels meeting $L(3, 1)$-labeling requirement, where $l \leq s$ for all $l \in L_{31}^{vl}(1, I_p)$ and $s = max\{L(I_p)\} + 3$, for any $I_p \in I$.*

Proof Since $L^i(I_p) \subseteq L(I_p)$, for $i = 1, 2$ (by Observation 1), $s = max\{L(I_p)\} + 3$ and so $|s - l_q| \geq 3$ for any $l_q \in L^i(I_p)$, $i = 1, 2$, i.e., $|s - l_q| \geq 3$ for all $l_q \in L_{31}^1(I_p)$ and $|s - l_q| \geq 1$ for all $l_q \in L_{31}^2(I_p)$. So s is the valid $L(3, 1)$-label of I_p, and $s \in L_{31}^{vl}(2, I_p)$. This implies that $L_{31}^{vl}(2, I_p)$ is a non-empty set. Let A be an arbitrary set of labels satisfying $L(3, 1)$-labeling conditions, where $l \leq s$ for all $l \in A$ and $s = max\{L(I_p)\} + 3$. Also, let $a \in A$. Then $|a - l_q| \geq 3$ for any $l_q \in L_{31}^1(I_p)$, and $|a - l_t| \geq 1$ for any $l_t \in L_{31}^2(I_p)$. Thus, $a \in L_{31}^{vl}(2, I_p)$. Therefore, $a \in A$ implies $a \in L_{31}^{vl}(2, I_p)$. So $A \subseteq L_{31}^{vl}(2, I_p)$. Since A is arbitrary, $L_{31}^{vl}(2, I_p)$ is the maximal non-empty set of labels meeting $L(3,1)$-labeling requirement, $l \leq s$ for every $l \in L_{31}^{vl}(2, I_p)$ and $s = max\{L(I_p)\} + 3$, for every $I_p \in I$.

Algorithm L31

Input: The set $I = \{I_1, I_2, \ldots, I_n\}$ and $L_{31}^{vl}(t, I_p)$ for $p = 2, 3, \ldots, n$ and $t = 1, 2$.
Output: τ_p, the $L(3,1)$-label of I_p, $p = 1, 2, \ldots, n$.

Step 1: (Initialization)
$\tau_1 = 0$;
$L(I_2) = \{0\}$;
Step 2: for $p = 2$ to $n - 1$
$\tau_p = min\{L_{31}^{vl}(2, I_p)\}$;
$L(I_{p+1}) = L(I_p) \cup \{\tau_p\}$;
end for;
Step 3: $\tau_n = min\{L_{31}^{vl}(2, I_n)\}$;
Step 4: $L = L(I_n) \cup \{\tau_n\}$;
end $L31$.

Theorem 2 *Any I-graph is correctly labelled by the algorithm $L31$ using $L(3,1)$-labeling.*

Proof Let G be any n vertices I-graph. Let $I = \{I_1, I_2, \ldots, I_n\}$, also let $\tau_1 = 0$, $L(I_2) = \{0\}$. If $n = 1$, then $L(I_2)$ is sufficient for labeling the graph, and obviously, $\lambda_{3,1}(G) = 0$.

If $n > 1$, then $L(I_2)$ is insufficient to label the graph G by $L(3,1)$-labeling, because $L(I_2)$ only has one label and additional labels are needed in this situation. We take into account a case in which the intervals $I_1, I_2, \ldots, I_{p-1}$ are already labeled for $p = 2, 3, \ldots, n$. In such case, we want to label I_p by labelling with $L(3,1)$. We are aware that $L_{31}^{vl}(1, I_p)$ is the maximal non-empty set that meets the criterion of distance one of $L(3,1)$-labeling and $L_{31}^{vl}(2, I_p)$ is the maximal non-empty set satisfying $L(3,1)$-labeling, where $l \leq s$ for all $l \in L_{31}^{vl}(k, I_p)$ and $s = max\{L(I_p)\} + 3$ for any $I_p \in I$ and $k = 1, 2$ (by Lemma 3 and Lemma 4). There is once more no label that is $l \leq s$ and $l \notin L_{31}^{vl}(2, I_p)$ that satisfies the $L(3,1)$-labeling condition. Since I_p is less than or equal to s and satisfies the $L(3,1)$-labeling criterion, the set of labels $L_{31}^{vl}(2, I_p)$ is the only valid label for I_p. Since we want to label the interval I_p by $L(3,1)$-labeling, so $\tau_p = q$, where $q = min\{L_{31}^{vl}(2, I_p)\}$. Since no label less than q meets the $L(3,1)$-labeling criterion, q is the least label for I_p. Since I_p is arbitrary, the graph G can be labelled using the fewest labels that meet the $L(3,1)$-labeling requirement, with the result that $\lambda_{3,1}(G) = max\{L(I_n) \cup \{\tau_n\}\}$.

Theorem 3 *The time complexity of Algorithm $L31$ is $O(n\Delta^2)$.*

Proof In Algorithm $L31$, our target is to find the least possible label for every interval I_p. By our algorithm τ_p, the $L(3,1)$-label of I_p can be computed if

$L_{31}^{vl}(2, I_p)$ is computed. Now by Lemma 2, $L_{31}^{vl}(2, I_p)$ can be computed in $O(\Delta^2)$ time. Since we need to find $L_{31}^{vl}(3, I_p)$ for $p = 2, 3, \ldots, n$, so the running time of Algorithm $L31$ is $O((n-1)\Delta^2)$, i.e., $O(n\Delta^2)$.

3.2 Illustration of the Algorithm $L31$

Let us consider an I-graph with 10 vertices (see Fig. 10.3), and label this graph by Algorithm $L31$ (see Fig. 10.4).

For this graph, the set of intervals, $I = \{I_1, I_2, \ldots, I_{10}\}$ and $\Delta = 5$. τ_j, the label of the vertex I_j, for $j = 1, 2, \ldots, 10$.

Initialized $\tau_1 = 0$, $L(I_2) = \{0\}$.

Iteration 1: For $k = 2$.
$L_{31}^1(I_2) = \{0\}$, $L_{31}^2(I_2) = \emptyset$
$L_{31}^{vl}(1, I_2) = \{3\}$, $L_{31}^{vl}(2, I_2) = \{3\}$
Therefore, $\tau_2 = \min\{L_{31}^{vl}(2, I_2)\} = 3$ and
$L(I_3) = L(I_2) \cup \{\tau_2\} = \{0\} \cup \{3\} = \{0, 3\}$.
Iteration 2: For $k = 3$.
$L_{31}^1(I_3) = \{0, 3\}$, $L_{31}^2(I_3) = \emptyset$
$L_{31}^{vl}(1, I_3) = \{6\}$, $L_{31}^{vl}(2, I_3) = \{6\}$
Therefore, $\tau_3 = \min\{L_{31}^{vl}(2, I_3)\} = 6$ and
$L(I_4) = L(I_3) \cup \{\tau_3\} = \{0, 3\} \cup \{6\} = \{0, 3, 6\}$.

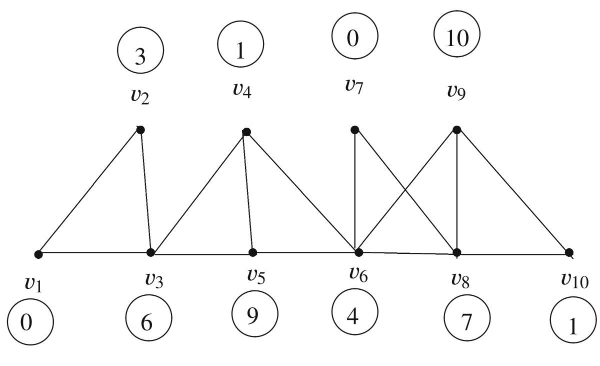

Fig. 10.4 The I-graph in Fig. 10.3 labeled by $L(3, 1)$-labeling. The label of the associated vertices is represented by the number inside the circle

Iteration 3: For $k = 4$.
$L_{31}^{1}(I_4) = \{6\}$, $L_{31}^{2}(I_2) = \{0, 3\}$
$L_{31}^{vl}(1, I_4) = \{0, 1, 2, 3, 9\}$, $L_{31}^{vl}(2, I_4) = \{1, 2, 9\}$
Therefore, $\tau_4 = \min\{L_{31}^{vl}(2, I_4)\} = 1$ and
$L(I_5) = L(I_4) \cup \{\tau_4\} = \{0, 1, 3, 6\}$.
Iteration 4: For $k = 5$.
$L_{31}^{1}(I_5) = \{1, 6\}$, $L_{31}^{2}(I_5) = \{0, 3\}$
$L_{31}^{vl}(1, I_5) = \{9\}$, $L_{31}^{vl}(2, I_5) = \{9\}$
Therefore, $\tau_5 = \min\{L_{31}^{vl}(2, I_5)\} = 9$ and
$L(I_6) = L(I_5) \cup \{\tau_5\} = \{0, 1, 3, 6, 9\}$.
Iteration 5: For $k = 6$.
$L_{31}^{1}(I_6) = \{1, 9\}$, $L_{31}^{2}(I_6) = \{6\}$
$L_{31}^{vl}(1, I_6) = \{4, 5, 6, 12\}$, $L_{31}^{vl}(2, I_6) = \{4, 5, 12\}$
Therefore, $\tau_6 = \min\{L_{31}^{vl}(2, I_6)\} = 4$ and
$L(I_7) = L(I_6) \cup \{\tau_6\} = \{0, 1, 3, 4, 6, 9\}$.
Iteration 6: For $k = 7$.
$L_{31}^{1}(I_7) = \{4\}$, $L_{31}^{2}(I_7) = \{1, 9\}$
$L_{31}^{vl}(1, I_7) = \{0, 1, 7, 8, 9, 10, 11, 12\}$, $L_{31}^{vl}(2, I_7) = \{0, 7, 8, 10, 11, 12\}$
Therefore, $\tau_7 = \min\{L_{31}^{vl}(2, I_7)\} = 0$ and
$L(I_8) = L(I_7) \cup \{\tau_7\} = \{0, 1, 3, 4, 6, 9\}$.
Iteration 7: For $k = 8$.
$L_{31}^{1}(I_8) = \{0, 4\}$, $L_{31}^{2}(I_8) = \{1, 9\}$
$L_{31}^{vl}(1, I_8) = \{7, 8, 9, 10, 11, 12\}$, $L_{31}^{vl}(2, I_8) = \{7, 8, 10, 11, 12\}$
Therefore, $\tau_8 = \min\{L_{31}^{vl}(2, I_8)\} = 7$ and
$L(I_9) = L(I_8) \cup \{\tau_8\} = \{0, 1, 3, 4, 6, 7, 9\}$.
Iteration 8: For $k = 9$.
$L_{31}^{1}(I_9) = \{4, 7\}$, $L_{31}^{2}(I_9) = \{0, 1, 9\}$
$L_{31}^{vl}(1, I_9) = \{0, 1, 10, 11, 12\}$, $L_{31}^{vl}(2, I_9) = \{10, 11, 12\}$
Therefore, $\tau_9 = \min\{L_{31}^{vl}(2, I_9)\} = 10$ and
$L(I_{10}) = L(I_9) \cup \{\tau_9\} = \{0, 1, 3, 4, 6, 7, 9, 10\}$.
Iteration 9: For $k = 10$.
$L_{31}^{1}(I_{10}) = \{7, 10\}$, $L_{31}^{2}(I_{10}) = \{0, 4\}$
$L_{31}^{vl}(1, I_{10}) = \{0, 1, 2, 3, 4, 13\}$, $L_{31}^{vl}(2, I_{10}) = \{1, 2, 3, 13\}$
Therefore, $\tau_{10} = \min\{L_{31}^{vl}(2, I_{10})\} = 1$.

Below are the vertices and the labels of the associated vertices:

Vertices	I_1	I_2	I_3	I_4	I_5	I_6	I_7	I_8	I_9	I_{10}
$L(3, 1)$-labels	0	3	6	1	9	4	0	7	10	1

4 Conclusion

There are just a few classes of graphs for which the $L(3, 1)$-labeling result is accessible, despite the fact that $L(3, 1)$-labeling issues have been researched in the past. A decent upper bound for $L(3, 1)$-labeling is obviously desirable for other classes of graphs. We explored $L(3, 1)$-labeling of I-graphs in this chapter and established that $\lambda_{3,1}(G) \leq 4\Delta - 1$ for I-graphs G. Additionally, we have developed a successful technique for labelling an I-graph using the $L(3, 1)$-labeling. The proposed algorithm has a $O(n\Delta^2)$ running time. There is a scope for the researcher to develop a new upper bound on the problem because our result is not exact.

Acknowledgments The authors are greatly thankful to the anonymous referees and editors for their valuable suggestions.

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Chapter 11
Generalized Neutrosophic Sets and Their Applications for Aggregated Operators Based on Diagnostic Disease Problem

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1 Introduction

According to [12] the separation between Pythagorean fuzzy sets is investigated, while [23] analyzes the Hamming and Euclidean distances and their application to fuzzy sets. As a result of Cantor's work and [4], set theory was established between 1874 and 1888. In the past, the set was used differently than it is now. Classical set theory deals with these sets. In fuzzy set (FS) theory, crisp sets cannot address problems involving uncertainty. According to Zadeh's 1965 article [31], the fuzzy set theory was first introduced in 1965. In many daily life domains, uncertainty exists, which Zadeh dealt with for the first time by using fuzzy sets (FSs). Fuzzy sets can partially contain the elements of the entire universe. Each element of a set has a measure of membership in the set. Fuzzy sets were first described as $alpha$ level sets by H. T. Nguyen in 1978 after introducing the concept of α level sets [13]. It is now possible to compute linguistic variables using FSs, calculate linguistic probabilities, perform fuzzy number math, and expand relations' domains according to the extension principle. In addition, it was discovered that this set method was simpler to use than functional approach strategies. To explore additional uses for the FSs, Zadeh created the notion of linguistic variables in [32]. There are FS applications in almost all branches of mathematics nowadays [1, 2]. A FS, however, does not consider the problem of determining whether an element belongs to a set or not. To accommodate the intuitionistic FS (IFS), Atanassov devised the concept of IFS [3] to deal with the uncertainty of information and transform it into a more

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C. Jana et al. (eds.), *Fuzzy Optimization, Decision-making and Operations Research*, https://doi.org/10.1007/978-3-031-35668-1_11

accurate form, IFS with two independent capabilities, namely, truth presentation and false enrollment values. To handle partial, ambiguous, and contradictory data, Smarandache expanded FSs to neutrosophic sets (NSs) with three enrollment qualities: truth, indeterminacy, and falsehood. Atanassov's IFS hypothesis, L-FS hypothesis, and fuzzy set hypothesis are all extended by the multi-FS hypothesis. Pythagorean fuzzy set (PFS) introduced by Yager [29] and their application to multi-criteria decision-making problems [30]. The PFSs were generalized to Pythagorean soft rough sets by Hussain et al. [9]; other works can be found in [33] and [34].

A recent study by Fei et al. [8] employed Pythagorean fuzzy numbers (PFNs) and interval-valued Pythagorean fuzzy numbers (IVPFNs) to describe information with greater uncertainties in multi-criteria decision-making (MCDM). According to Cruz et al., the PFS can also be generalized through a single positive integer [5]. The literature categorizes FSs into two categories: PFSs and IFSs. They are connected even though they are used independently. It is not possible to use these sets in a wide variety of ways. It is possible for IFSs to participate in functions exhibiting linear geometrical characteristics, whereas PFSs require their discourse functions to meet Pythagorean criteria. Functions can behave linearly or quadratically in some cases. There is a clear demonstration of the limitations of Example 1 on both the IFS and PFS levels. In this work, we demonstrate their increasingly broad class, not only combining their consolidated and distinctive properties but also going beyond them. Our definition of a novel class of sets is based on two provided non-negative integers, m and p. In addition to combining the characteristics of the IFS and the PFS, they also possess several unique traits of their own. GNSSs are currently called generalized neutronosophic sets (GNSSs). These are generalizations of the many IFS and PFS classes. The example clearly illustrates the rationale and justification for including GNSSs 1. It is used to study the mathematical treatment of many physical processes and systems using GNSSs. Currently, fuzzy sets are being generalized in a way that will be helpful and useful to research models of these systems. A novel theory of logic and sets called neurophilosophic logic has been proposed recently. Neutosophy is the study of the neutral mind and serves as the main line of demarcation between IFS and FS. NSS was introduced by Smarandache [28]. It assesses each claim's level of veracity, ambiguity, and truthfulness. In the NSS set, every aspect of the cosmos has a level of certainty, ambiguity, and untruth between [0, 1]. It has been proven philosophically that NSS exists. Smarandache et al. [11] first described the Pythagorean neutrosophic interval-valued set (PNSIVS).

A single-valued NSS is applied to context analysis [27] and medical diagnostics, according to [25]. According to Ejegwa, extended distance measures for IFSs, such as Hamming, Euclidean, normalized Hamming, and normalized Euclidean distances, can be applied to both multi-attribute and multi-criteria decision-making situations [6]. Based on what has already been published, we find that the majority of the distance functions for PNSNIVSs are generalizations of Pythagorean neutrosophic interval-valued sets (PNSIVS). MCDM approaches are used to evaluate the items against a number of contradictory quantitative and/or qualitative criteria, as described in [22]. As opposed to offering advice about which option is best, these strategies help decision-makers choose which option best suits their needs [24]. The uniqueness of MCDM approaches lies in their elimination of trial-and-error methodologies. A setting defined by cutting-edge competition and unique, complicated high-tech solutions relies on them for evaluating, choosing, categorizing, and prioritizing goals. A MCDM experiment designed and produced, as well as evaluated and selected, the most suitable approaches [10]. The goal of the MCDM is to select the most desirable option from a constrained set of options based on predefined characteristics or criteria [26]. In a recent study, Palanikumar et al. [15–21] studied many algebraic structures and its applications for MCDM. To present the concept of GNSSs, their characteristics, and their applications methodically, we divided our work into six sections. Section 1 describes how our research will be presented. In Sect. 2, we discuss the concept of GNSSs, their characteristics, and their applications in a methodical way. The definition of and properties of GNSSs are discussed in Sect. 3. In Sect. 4, we examine the generalized neutrosophic relations (GNSRs) defined by the GNSSs and demonstrate how these relations can be applied.

2 Preliminary

In this section, we will quickly go over a few of the foundational words we will need for our future studies.

Definition 1 Let X be a non-empty set; a FS G in X is characterized by a membership function $\Xi_G : X \to [0, 1]$ such that

$$\Xi_G(\varepsilon) = \begin{cases} 1 & if \quad \varepsilon \in X, \\ 0 & if \quad \varepsilon \notin X, \\ (0,1) & if\ \varepsilon\ lies\ between\ in\ X. \end{cases} \tag{11.1}$$

We write $G = \{\langle \varepsilon, \Xi_G(\varepsilon)\rangle | \varepsilon \in X\}$. The set X is the universe of the discourse.

Definition 2 An IFS G in X is defined as a set of the form

$$G = \{\langle \varepsilon, \Xi_G(\varepsilon), \Upsilon_G(\varepsilon)\rangle | \varepsilon \in X\} \tag{11.2}$$

characterized by a membership function $\Xi_G(\varepsilon) : X \to [0, 1]$ and a non-membership function $\Upsilon_G(\varepsilon) : X \to [0, 1]$. These functions define, respectively, the degree of *membership* and the degree of *non-membership* of an element $\varepsilon \in X$ to G, which is a subset of X. Moreover, for every $\varepsilon \in X$, $0 \leq \Xi_G(\varepsilon) + \Upsilon_G(\varepsilon) \leq 1$, and we define for each G in X,

$$\Pi_G(\varepsilon) = 1 - \Xi_G(\varepsilon) - \Upsilon_G(\varepsilon) \tag{11.3}$$

as the hesitation margin of ε in X for the fuzzy set G. $\Pi_G(\varepsilon)$ measures the degree of indeterminacy of $\varepsilon \in X$, to the set G, and its value lies in [0, 1]. $\Pi_G(\varepsilon)$ expresses the lack of knowledge whether $\varepsilon \in X$ or $\varepsilon \notin X$. In this way,

$$\Xi_G(\varepsilon) + \Upsilon_G(\varepsilon) + \Pi_G(\varepsilon) = 1. \tag{11.4}$$

Definition 3 ([7]) Let X be the universe; a PFS G in X is an object of the form

$$G = \{\langle \varepsilon, \Xi_G(\varepsilon), \Upsilon_G(\varepsilon)\rangle | \varepsilon \in X\} \tag{11.5}$$

where the functions $\Xi_G(\varepsilon) : X \to [0, 1]$ and $\Upsilon_G(\varepsilon) : X \to [0, 1]$ are, respectively, the membership and non-membership functions, satisfying the property,

$$0 \leq (\Xi_G(\varepsilon))^2 + (\Upsilon_G(\varepsilon))^2 \leq 1 \tag{11.6}$$

$\forall\ \varepsilon \in X$. In this case, the *hesitation margin* of ε in X is defined by

$$\Pi_G(\varepsilon) = \sqrt{1 - (\Xi_G(\varepsilon)^2 + \Upsilon_G(\varepsilon) + \Omega_G(\varepsilon))^2}. \tag{11.7}$$

The value $\Pi_G(\varepsilon)$ explains whether $\varepsilon \in X$ or $\varepsilon \notin X$. So,

$$(\Xi_G(\varepsilon))^2 + (\Upsilon_G(\varepsilon))^2 + (\Pi_G(\varepsilon))^2 = 1. \tag{11.8}$$

Generally, for all $\varepsilon \in X$, $\Xi_X(\varepsilon) = 1$, and $\Xi_\emptyset(\varepsilon) = 0$, but in the presence of uncertainty, the value $\Xi_G(\varepsilon)$ models to what extent ε is in G. That is, Ξ_G is also called the possibility function of G. All the usual sets without a membership function are called crisp sets.

3 Generalized Neutrosophic Sets

Definition 4 Let X be the universal, and m, p and n are the non-negative integers; then a generalized neutrosophic set G in X is defined as

$$G = \{\langle \varepsilon, \Xi_G(\varepsilon), \Upsilon_G(\varepsilon), \Omega_G(\varepsilon)\rangle : \varepsilon \in X\}; \tag{11.9}$$

the functions $\Xi_G(\varepsilon) : X \to [0, 1]$, $\Upsilon_G(\varepsilon) : X \to [0, 1]$, and $\Omega_G(\varepsilon) : X \to [0, 1]$ specify the degree of truth, the degree of indeterminacy, and the degree of false membership of the element $\varepsilon \in X$ to G, respectively, which is a subset of X, and $\forall \varepsilon \in X$:

$$0 \leq \big(\Xi_G(\varepsilon)\big)^m + \big(\Upsilon_G(\varepsilon)\big)^p + \big(\Omega_G(\varepsilon)\big)^n \leq 2. \tag{11.10}$$

In this case, the generalized neutrosophic set index of ε in X is defined as

$$\varPi_G(\varepsilon) = \sqrt[lcm(m,p,n)]{2 - \left(\left(\varXi_G(\varepsilon)\right)^m + \left(\varUpsilon_G(\varepsilon)\right)^p + \left(\varOmega_G(\varepsilon)\right)^n\right)}, \tag{11.11}$$

$\varPi_G(\varepsilon) \in [0, 1]$ and

$$\left(\varXi_G(\varepsilon)\right)^m + \left(\varUpsilon_G(\varepsilon)\right)^p + \left(\varOmega_G(\varepsilon)\right)^n + \left(\varPi_G(\varepsilon)\right)^{lcm(m,p,n)} = 2. \tag{11.12}$$

When $\varPi_G(\varepsilon) = 0$, then $\varXi_G(\varepsilon)^m + \varUpsilon_G(\varepsilon)^p + \varOmega_G(\varepsilon)^n = 1$. $lcm(m, p, n)$ means the *least common multiple* of m, p, and n. Here m, p, and n are non-negative integers. Let $\varPi$ denote the uncertainty or a lack of commitment associated with the membership, indeterminacy, and non-membership degrees of $\varepsilon \in X$. The generalized neutrosophic sets over the sets X will be represented by $GNSS_{(m,p,n)}(\varepsilon)$.

Example 1 The value of a membership function $\varXi_G$ at a point ε of the set G is $\varXi_G(\varepsilon) = 0.9$, the degree of indeterminacy function $\varXi_G$ at a point ε of the set G is $\varXi_G(\varepsilon) = 0.8$, and the non-membership function $\varUpsilon_G$ at ε is $\varUpsilon_G(\varepsilon) = 0.95$. This problem is not studied by both the NSSs and PNSs, respectively, on the grounds that $\varXi_G(\varepsilon) + \varUpsilon_G(\varepsilon) + \varOmega_G(\varepsilon) > 2$, and $(\varXi_G(\varepsilon))^2 + (\varUpsilon_G(\varepsilon))^2 + (\varOmega_G(\varepsilon))^2 < 2$; however, this problem is studied by the generalized neutrosophic sets as $(\varXi_G(\varepsilon))^m + (\varUpsilon_G(\varepsilon))^p + (\varOmega_G(\varepsilon))^n < 2$, for $m \geq 2$ and $p > 2$ and $n > 2$ *vice versa*, e.g., $(\varXi_G(\varepsilon))^3 + (\varUpsilon_G(\varepsilon))^3 + (\varOmega_G(\varepsilon))^3 < 2$.

Remark 1

1. Because multiple values of m, p, and n can be used to serve the needs of different users, this class of sets is known as the generalized neutrosophic sets.
2. The classical set or the crisp set is what we obtain when m, p, and n are all 0 .
3. The intuitionistic fuzzy set(IFSs) is what remains when one of the integers m, p, and n is 0 and the other is 1. Without loss of generality, if $m = 1$ and $p = 0$ $n = 1$, then Eqs. 11.11 and 11.12 leave a non-empty set G with a membership function $\varXi_G$ and non-membership function called the IFSs. Also, if $m = 1$ and $p = 1$ $n = 1$, then Eqs. 11.11 and 11.12 leave with a membership function $\varXi_G$, indeterminacy function, and non-membership function called the NSSs. Hence, NSSs are a particular instance of generalized neutrosophic sets.
4. When $p = 0$, $n = 0$, and m is any positive integer, Eq. 11.11 arrived, $\varXi_G(\varepsilon)^m = 1$, which implies $\varXi_G(\varepsilon) = 1$ that each element of X is in G.
5. The well-known neutrosophic sets (NSSs) result when m, p and n are all 1.
6. The case of the Pythagorean neutrosophic sets (PyNSSs) arises when m, p and n are all equal to 2.
7. We get the generalized neutrosophic sets (GNSS (m,p,n)) when m, p and n have different positive values.

Scope of generalized neutrosophic sets for selected values of $\varXi$, $\varUpsilon$,$\varOmega$ and $\varPi$. We can make decision from Table 11.1.

We can make decision from Table 11.2.

Table 11.1 Comparison of different sets of the class of GNSS

Sets	GNSSs Form	m	p	n	Π
NSSs	$GFS_{(1,1,1)}$	1	1	1	0.8062
PYNSSs	$GFS_{(2,2,2)}$	2	2	2	0.5937
GNSSs	$GFS_{(2,3,2)}$	2	3	2	0.5544
GNSSs	$GFS_{(3,2,2)}$	3	2	2	0.5210
GNSSs	$GFS_{(3,3,3)}$	3	3	3	0.9919

Table 11.2 Difference among different generalized neutrosophic sets

$GNSS_{(1,1,1)}(NSS)$	$GNSS_{(2,2,2)}(PyNSS)$	$GNSS_{(2,3,2)}$
$\Xi + \Upsilon + \Omega \leq 2$	$\Xi + \Upsilon + \Omega \leq 2$ or $\Xi + \Upsilon + \Omega \geq 2$	same as in $GNSS_{(2,2,2)}(PyNSS)$
$0 \leq (\Xi + \Upsilon + \Omega) \leq 2$	$0 \leq \Xi^2 + \Upsilon^2 + \Omega^2 \leq 2$	$0 \leq \Xi^m + \Upsilon^p + \Omega^n \leq 2$
$\Pi = 2 - (\Xi + \Upsilon + \Omega)$	$\Pi = \sqrt{2 - (\Xi^2 + \Upsilon^2 + \Omega^2)}$	$\Pi = \sqrt[lcm(m,p,n)]{2 - (\Xi^m + \Upsilon^p + \Omega^n)}$
$\Pi + \Xi + \Upsilon + \Omega = 2$	$\Pi^2 + \Xi^2 + \Upsilon^2 + \Omega^2 = 2$	$\Pi^{lcm(m,p,n)} + \Xi^m + \Upsilon^p + \Omega^n = 2$

Theorem 1 *Every NSSs is expressed in terms of the $GNSS$ for suitable values of m, p, and n.*

Proof Very obvious, as is evident from Remarks (1) to (6).

Every $GNSS_{(m,p,n)}(\varepsilon)$ is $GNSS_{(l,q,r)}(\varepsilon)$, where $l \geq m$ and $q \geq p$,$r \geq n$, but the converse does not follow by the following example.

Example 2 Let $G \in GNSS_{m,p,n}(\{\varepsilon\})$. If $\Xi_G(\varepsilon) = 0.69$, $\Upsilon_G(\varepsilon) = 0.8$ and $\Omega_G(\varepsilon) = 0.94$, then for $m = 2$, $p = 2$,$n = 2$, $(0.69)^2 + (0.8)^2 + (0.94)^2 < 2$. Moreover, for $m = 3$, $p = 3$ and $n = 3$ or $n = 3$, $(0.69)^m + (0.8)^p + (0.94)^n < 2$. This implies that $G \in GNSS_{(m,p,n)}(\varepsilon)$ for $m \geq 2$, $p \geq 2$ and $n > 2$, but G is not NSSs.

Theorem 2 *Let $X = \{\varepsilon_i\}$ be the universal $i = 1, \cdots, n$, and $H \in GNSS_{(m,p,n)}(\varepsilon)$. If $\Pi_H(\varepsilon_i) = 0$, then*

1. $|\Xi_H(\varepsilon_i)| = \sqrt[m]{|2 - (\Upsilon_H(\varepsilon_i)^p + \Omega_H(\varepsilon_i)^n)|}$.
2. $|\Upsilon_H(\varepsilon_i)| = \sqrt[p]{|2 - (\Xi_H(\varepsilon_i)^m + \Omega_H(\varepsilon_i)^n)|}$.
3. $|\Upsilon_H(\varepsilon_i)| = \sqrt[n]{|2 - (\Xi_H(\varepsilon_i)^m + \Upsilon_H(\varepsilon_i)^p)|}$.

Proof

1. Since $\Pi_G(\varepsilon_i) = 0$, so $\Xi_G(\varepsilon)^m + \Upsilon_G(\varepsilon)^p + \Omega_G(\varepsilon)^n = 2$. This implies that $\Xi_G(\varepsilon)^m = 2 - \Upsilon_G(\varepsilon)^p - \Omega_G(\varepsilon)^n$, $|\Xi_G(\varepsilon)|^m = |2 - \Upsilon_G(\varepsilon)^p - \Omega_G(\varepsilon)^n|$. Hence, $|\Xi_G(\varepsilon)| = \sqrt[m]{|2 - \Upsilon_G(\varepsilon)^p - \Omega_G(\varepsilon)^n|}$. Similarly to prove other parts.

Example 3 Suppose $G \in GNSS_{(m,p,n)}(\varepsilon)$ and $\Upsilon_G(\varepsilon_i) = 0.85$,$\Omega_G(\varepsilon_i) = 0.78$. If $m = 2$, $p = 3$,$n = 2$, then $|\Xi_G(\varepsilon_i)| = \sqrt[2]{|\left((0.85)^3\right) + ((0.76)^2\big) - 2|} = 0.89$. Thus, $(\Xi_G(x_i))^2 + (\Upsilon_G(\varepsilon_i))^3 + (\Omega_G(\varepsilon_i))^2 = 2$ implies that $\Pi_G(\varepsilon_i) = 0$.

3.1 Basic Operations of $GNSSs$

Now, we define some important operations for union, intersection, complementation, sum, and product based on $GNSSs$.

Definition 5 Let $G \in GNSS_{(m,p,n)}(\varepsilon)$; then the complement of G is defined as $G^c = \langle \varepsilon, \Omega_G(\varepsilon), \Upsilon_G(\varepsilon), \Xi_G(\varepsilon)\rangle$, for $\varepsilon \in X$.

Definition 6 If G and H are two sets in $GNSS_{(m,p,n)}(\varepsilon)$, then their union and intersection are defined as follows:

1. $G \cup H = \left\{\left\langle \varepsilon, \max\left(\Xi_G(\varepsilon), \Xi_H(\varepsilon)\right), \min\left(\Upsilon_G(\varepsilon), \Upsilon_H(\varepsilon)\right), \min\left(\Omega_G(\varepsilon), \Omega_H(\varepsilon)\right)\right\rangle : \varepsilon \in X\right\}$.
2. $G \cap H = \left\{\left\langle \varepsilon, \min\left(\Xi_G(\varepsilon), \Xi_H(\varepsilon)\right), \max\left(\Upsilon_G(\varepsilon), \Upsilon_H(\varepsilon)\right), \max\left(\Omega_G(\varepsilon), \Omega_H(\varepsilon)\right)\right\rangle : \varepsilon \in X\right\}$.

Definition 7 Let $G \in GNSS_{(m,p,n)}(\varepsilon)$; then the score function of G is defined as

$$s(G) = (\Xi_G(\varepsilon))^m + (\Upsilon_G(\varepsilon))^p - (\Omega_G(\varepsilon))^n,$$

$s(G) \in [-1, 1]$, for any positive integers m, p, and n.

Definition 8 Let $G \in GNSS_{(m,p,n)}(\varepsilon)$; then the accuracy function of G is defined as

$$a(G) = \left(\Xi_G(\varepsilon)\right)^m + \left(\Upsilon_G(\varepsilon)\right)^p + \left(\Omega_G(\varepsilon)\right)^n$$

for $a(G) \in [0, 1]$, for any positive integers m, p, and n.

Theorem 3 *Let $G \in GNSS_{(m,p,n)}(\varepsilon)$; then the following relations validate $\forall \varepsilon \in X$:*

1. $s(G) = 2 \iff$

  (a) $\Xi_G(\varepsilon) = \sqrt[m]{|(-\Upsilon_G(\varepsilon))^p + \Omega_G(\varepsilon))^n) + 2)|}$
  (b) $\Upsilon_G(\varepsilon) = \sqrt[p]{|(-\Xi_G(\varepsilon))^m + \Omega_G(\varepsilon))^n) + 2)|}$
  (c) $\Omega_G(\varepsilon) = \sqrt[n]{|(\Xi_G(\varepsilon))^m + \Upsilon_G(\varepsilon))^p) - 2)|}$

2. $s(G) = -2 \iff$

  (a) $\Xi_G(\varepsilon) = \sqrt[m]{|(-\Upsilon_G(\varepsilon))^p + \Omega_G(\varepsilon))^n - 2)|}$
  (b) $\Upsilon_G(\varepsilon) = \sqrt[p]{|(-\Xi_G(\varepsilon))^m + \Omega_G(\varepsilon))^n - 2)|}$
  (c) $\Omega_G(\varepsilon) = \sqrt[n]{|(\Xi_G(\varepsilon))^m + \Upsilon_G(\varepsilon))^n + 2|)}$

Proof

1. We suppose that $s(G) = 2$. Now, $2 = \left(\Xi_G(\varepsilon)\right)^m + \left(\Upsilon_G(\varepsilon)\right)^p - \left(\Omega_G(\varepsilon)\right)^n$. Therefore,
   $\left(\Upsilon_G(\varepsilon)\right)^p = (\left(- \Xi_G(\varepsilon)\right)^m + \left(\Omega_G(\varepsilon)\right)^n) + 2$ and $\left|(\Upsilon_G(\varepsilon)\right)^p| = |((- \Xi_G(\varepsilon))^m| + |\left(\Omega_G(\varepsilon)\right)^n)| + 2$.

Thus $\Upsilon_G(\varepsilon) = \sqrt[p]{|(-\Xi_G(\varepsilon))^m + (\Omega_G(\varepsilon))^n) + 2|}, \quad \forall \varepsilon \in X$.
The converse follows immediately.

2. Suppose $s(G) = -2$, then, $-2 = (\Xi_G(\varepsilon))^m + ((\Upsilon_G(\varepsilon))^p - (\Omega_G(\varepsilon))^n), \Rightarrow (\Upsilon_G(\varepsilon))^p = ((-\Xi_G(\varepsilon))^m + (\Omega_G(\varepsilon))^n) - 2. \Rightarrow |(\Upsilon_G(\varepsilon))^p| = (|(-\Xi_G(\varepsilon))^m| + |(\Omega_G(\varepsilon))^n)| - 2$.
Thus, $\Upsilon_G(\varepsilon) = \sqrt[p]{|(-\Xi_G(\varepsilon))^m + (\Omega_G(\varepsilon))^n + 2|}, \ \forall \varepsilon \in X$.
Converse of the theorem follows immediately.

Theorem 4 *Let $G \in GNSS_{(m,p,n)}(\varepsilon)$; then, $\forall \varepsilon \in X$; hence, the following statements are valid:*

1. $a(G) = 2 \iff \Pi_G(\varepsilon) = 0$,
2. (a) $a(G) = 0 \iff |\Xi_G(\varepsilon)| = |\Upsilon_G(\varepsilon)^p + \Omega_G(\varepsilon)^n|^{1/m}$
(b) $a(G) = 0 \iff |\Upsilon_G(\varepsilon)| = |\Xi_G(\varepsilon)^m + \Omega_G(\varepsilon)^n|^{1/p}$
(c) $a(G) = 0 \iff |\Omega_G(\varepsilon)| = |\Xi_G(\varepsilon)^m + \Upsilon_G(\varepsilon)^p|^{1/n}$.

Proof

1. Suppose $a(G) = 2$, Now,

$$(\Xi_G(\varepsilon))^m + (\Upsilon_G(\varepsilon))^p + (\Omega_G(\varepsilon))^n = 2,$$

Since

$$\Pi_G(\varepsilon) = \sqrt[lcm(m,p,n)]{2 - [\Xi_G(\varepsilon)^m + \Upsilon_G(\varepsilon)^p + \Omega_G(\varepsilon)^n]},$$

so $\Pi_G(\varepsilon) = 0$. Conversely, assume that $\Pi_G(\varepsilon) = 0$, then,

$$(\Xi_G(\varepsilon))^m + (\Upsilon_G(\varepsilon))^p + (\Omega_G(\varepsilon))^n = 2 \iff a(G) = 2.$$

2. Suppose $a(G) = 0$.Now, $\Xi_G(\varepsilon)^m = -\Upsilon_G(\varepsilon)^p - \Omega_G(\varepsilon)^n \iff |\Xi_G(\varepsilon)| = |\Upsilon_G(\varepsilon)^p + \Omega_G(\varepsilon)^n|^{1/m}$.

Definition 9 Let $G, H \in GNSS_{(m,p,n)}(\varepsilon)$. Then $G = H \iff \Xi_G(\varepsilon) = \Xi_H(\varepsilon)$, $\Upsilon_G(\varepsilon) = \Upsilon_H(\varepsilon)$, and$\Omega_G(\varepsilon) = \Omega_H(\varepsilon)$, $\forall \varepsilon \in X$. $G \subseteq H \iff \Xi_G(\varepsilon) \leq \Xi_H(\varepsilon)$ and $\Upsilon_G(\varepsilon) \geq \Upsilon_H(\varepsilon)$ and $\Omega_G(\varepsilon) \geq \Omega_H(\varepsilon)$, $\forall \varepsilon \in X$. $G \subset H \iff G \subseteq H$ and $G \neq H$.

Definition 10 Let $G, H \in GNSS_{(m,p,n)}(\varepsilon)$; then G and H are said to be comparable if $G \subseteq H$ or $H \subseteq G$.

Theorem 5 *Let $G, H \in GNSS_{(m,p,n)}(\varepsilon)$; then the following properties hold:*

1. $s(G) = s(H) \ (a(G) = a(H)) \iff G = H \ (G = H)$,
2. $s(G) \leq s(H) \ (a(G) \leq a(H)) \iff G \subseteq H \ (G \subseteq H)$,
3. $s(G) < s(H) \ (a(G) < a(H)) \iff G \subseteq H$ *and* $G \neq H$ $(G \subseteq H$ *and* $G \neq H)$.

Proof Straightforward.

4 Relations on Generalized Neutrosophic Sets(GNSSs)

Generalized neutrosophic relations (GNSRs) are generalized neutrosophic sets R. In generalized neutrosophic sets, $G \times H$ is defined as $G \subseteq X$ to $H \subseteq Y$. In Cartesian terms, a relation is a non-empty subset of $X \times Y$. As a result, every component of $G \times H$ is mapped to [0, 1] by way of the relation R. A GNSS can be viewed as a generalization of a GFS or an NSS. In addition to neutrosophic relations, the concept has been extended to GNSSs.

Definition 11 Let $h : X \to Y$ be the neutrosophic function such that $L \in GNSS_{(m,p,n)}(\varepsilon)$ and $M \in GNSS_{(m,p,n)}(\eta)$, then

1. the image of L denoted by $h(L)$ is a $GNSS_{(m,p,n)}$ of Y characterized, respectively, by the membership, indeterminacy, and non-membership functions as follows:

$$\Xi_{h(L)}(\eta) = \begin{cases} \Cup_{\varepsilon \in h^{-1}(\eta)} \Xi_L(\varepsilon), & h^{-1}(\eta) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

$$\Upsilon_{h(L)}(\eta) = \begin{cases} \Cap_{\varepsilon \in h^{-1}(\eta)} \Upsilon_L(\varepsilon), & h^{-1}(\eta) \neq \emptyset, \\ 1, & \text{otherwise,} \end{cases}$$

$$\Omega_{h(L)}(\eta) = \begin{cases} \Cap_{\varepsilon \in h^{-1}(\eta)} \Omega_L(\varepsilon), & h^{-1}(\eta) \neq \emptyset, \\ 1, & \text{otherwise,} \end{cases}$$

for each $\eta \in Y$. Also,

$$0 \leq \Xi_{h(L)}(\eta)^m + \Upsilon_{h(L)}(\eta)^p + \Omega_{h(L)}(\eta)^n \leq 2.$$

In this case, the degree of refusal of $\eta \in Y$ to $h(L)$ is defined as

$$\Pi_h(L)(\eta) = \sqrt[lcm(m,p,n)]{2 - \Big(\Big(\Xi_{h(L)}(\eta))^m + (\Upsilon_{h(L)}(\eta))^p + (\Omega_{h(L)}(\eta)^n)\Big)\Big)}. \tag{11.13}$$

2. the inverse image of M denoted by $h^{-1}(M)$ is a $GNSS_{(m,p,n)}$ of X characterized, respectively, by the membership, indeterminacy, and non-membership functions: $\Xi_{h^{-1}(M)}(\varepsilon) = \Xi_M\big(h(\varepsilon)\big)$, and $\Upsilon_{h^{-1}(M)}(\varepsilon) = \Upsilon_M\big(h(\varepsilon)\big)$, $\Omega_{h^{-1}(M)}(\varepsilon) = \Omega_M\big(h(\varepsilon)\big)$, $\forall \varepsilon \in X$.
   Also, the sum of $\big(\Xi_{h^{-1}(M)}(\varepsilon)\big)^m$, $\big(\Upsilon_{h^{-1}(M)}(\varepsilon)\big)^p$ and $\big(\Omega_{h^{-1}(M)}(\varepsilon)\big)^n$ in [0, 1]. The degree of refusal of $\varepsilon \in X$ to $h^{-1}(M)$ is given by

$$\Pi_{h^{-1}(M)}(\varepsilon) = \sqrt[lcm(m,p,n)]{2 - \left((\Xi_{h^{-1}(M)}(\varepsilon))^m + (\Upsilon_{h^{-1}(M)}(\varepsilon))^p + (\Omega_{h^{-1}(M)}(\varepsilon))^n\right)}. \tag{11.14}$$

Definition 12 Let X and Y be any two non-empty sets. Then a (GNSR) from X to Y is denoted by $R(\varepsilon \to \eta)$, and it is defined as follows:

$$0 \leq (\Xi_R(\varepsilon, \eta))^m + (\Upsilon_R(\varepsilon, \eta))^p + (\Omega_R(\varepsilon, \eta))^n \leq 2. \tag{11.15}$$

The degree of refusal (ε, η) of the product $X \times Y$ to R is defined as

$$\Pi_R(\varepsilon, \eta) = \sqrt[lcm(m,p,n)]{2 - \left(\left(\Xi_R(\varepsilon, \eta)\right)^m + (\Upsilon_R(\varepsilon, \eta))^p + (\Omega_R(\varepsilon, \eta))^n\right)}. \tag{11.16}$$

For our convenience $R(\varepsilon \to \eta)$ by R.

Definition 13 Suppose that $L \in GNSS_{(m,p,n)}(\varepsilon)$. Then we define the relation R and set L is a $GNSS_{(m,p,n)}$ M of Y is denoted by $M = R * L$, and it is defined as follows:

$$\Xi_M(\eta) = \max_{\varepsilon}(\min[\Xi_L(\varepsilon), \Xi_R(\varepsilon, \eta)]), \tag{11.17}$$

and

$$\Upsilon_M(\eta) = \min_{\varepsilon}(\max[\Upsilon_L(\varepsilon), \Upsilon_R(\varepsilon, \eta)]), \tag{11.18}$$

$$\Omega_M(\eta) = \min_{\varepsilon}(\max[\Omega_L(\varepsilon), \Omega_R(\varepsilon, \eta)]), \tag{11.19}$$

$\forall\, \varepsilon \in X$ and $\eta \in Y$.Thus M satisfies $GNSS$ and the degree of refusal as usual.

Definition 14 We define the max-min-max composition of two $GNSRs$ $Q(\varepsilon \to \eta)$ and $R(\eta \to \sigma)$, denoted by $R * Q$ as a $GNSR$ from the set X to Z. The relation $R * Q$ is defined as follows:

$$\Xi_{R*Q}(\varepsilon, \sigma) = \max_{\eta}\Big(\min[\Xi_Q(\varepsilon, \eta), \Xi_R(\eta, \sigma)]\Big), \tag{11.20}$$

and

$$\Upsilon_{R*Q}(\varepsilon, \sigma) = \min_{\eta}\Big(\max[\Upsilon_Q(\varepsilon, \eta), \Upsilon_R(\eta, \sigma)]\Big), \tag{11.21}$$

$$\Omega_{R*Q}(\varepsilon, \sigma) = \min_{\eta}\Big(\max[\Omega_Q(\varepsilon, \eta), \Omega_R(\eta, \sigma)]\Big), \tag{11.22}$$

$\forall (\varepsilon, \sigma) \in X \times Z$ and $\forall \eta \in Y$, satisfying the condition

$$0 \leq (\Xi_{R*Q}(\varepsilon, \sigma))^m + \big((\Upsilon_{R*Q}(\varepsilon, \sigma)\big)^p + \big(\Omega_{R*Q}(\varepsilon, \sigma)\big)^n \leq 2. \tag{11.23}$$

The refusal degree of the point (ε, σ) of $X \times Z$ to $R * Q$ is defined as

$$\Pi_{R*Q}(\varepsilon, \sigma) = \sqrt[lcm(m,p,n)]{2 - \Big((\Xi_{R*Q}(\varepsilon, \sigma))^m + \big(\Upsilon_{R*Q}(\varepsilon, \sigma)\big)^p + \big(\Omega_{R*Q}(\varepsilon, \sigma)\big)^n\Big)}. \tag{11.24}$$

Theorem 6 *Let $R_1 \in GNSR(L, M)$ and $R_2 \in GNSR(M, N)$; then composition $R_1 * R_2$ is an $GNSR$ from L to N.*

Proof Straightforward.

There are several algorithms in the literature to calculate the $R * Q$ We define as

$$\begin{aligned} R * Q = \Xi_{R*Q}(\varepsilon, \sigma) + (\Upsilon_{R*Q}(\varepsilon, \sigma)(\Pi_{R*Q}(\varepsilon, \sigma)) \\ - \Omega_{R*Q}(\varepsilon, \sigma)\Pi_{R*Q}(\varepsilon, \sigma), \forall(\varepsilon, \sigma) \in X \times Z. \end{aligned} \tag{11.25}$$

4.1 Diagnostic Disease Problem

1. Clinical diagnosis(D_1):
   A clinical examination is performed instead of diagnostic testing to make a diagnosis based only on symptoms and physical manifestations. Diagnosing a disease, condition, or injury is done based on a patient's signs and symptoms, medical history, and physical exam findings. Additional tests, such as blood tests, imaging tests, and biopsies, may be conducted once a clinical diagnosis has been established.
2. Laboratory diagnosis(D_2):
   The diagnosis depends more on the results of tests or lab work than on the patient's physical examination. When diagnosing an infectious disease, for instance, signs and symptoms, laboratory findings, and the organism's traits must all be considered.
3. Principal diagnosis(D_3):
   It is the medical condition that most accurately describes the patient's major complaint or treatment needs. There are many patients who have multiple diagnoses. A doctor can only make a primary diagnosis after performing all the required tests and examinations. It clarifies why a patient was initially admitted to the hospital. A primary diagnosis differs from an admitted diagnostic, which describes the patient's condition at the time of admission without relying on formal testing.

4. Admitting diagnosis(D_4):
   The diagnosis that most accurately describes the patient's complaint or what treatment they require. Multiple diagnoses are common among patients. A doctor can only provide a primary diagnosis after all the necessary tests and examinations have been completed. In addition, it clarifies the main reason a patient was admitted to the hospital in the first place. Primary diagnoses differ from admitted diagnoses, which characterize a patient's condition at the time of hospital admission without formal testing.
5. Diagnostic imaging(D_5):
   The single medical diagnosis that most accurately describes a patient's main complaint or what treatments they need. Many patients have multiple diagnoses. After all tests and examinations have been completed, a doctor can only make a primary diagnosis. A patient's primary reason for being admitted to the hospital is clarified. In contrast to an admitting diagnosis, a primary diagnosis describes a patient's condition at the time of admission without using formal testing.

Diagonize the following diseases by the above five diagnoses.

1. Urinary tract infection
   Since clinicians frequently base their diagnoses on a single symptom or sign, it is crucial that they are aware of the pretest probability. In addition to identifying the pathogenic microorganism(s) and determining their reaction to different medications, bacterial cultures continue to be essential for the diagnosis of UTIs. Urinary tract infections can be detected quickly and easily using MRI and CT scans because of their excellent image quality.
2. Pneumonia
   To diagnose pneumonia, a clinical history, physical examination, and/or laboratory tests are typically used. In most clinical recommendations, chest X-rays (CXRs), which can differentiate pneumonia from other respiratory tract illnesses, are the most reliable method for diagnosing pneumonia. On radiographs, pus and infectious material fill the alveoli of the airways. The air bronchogram becomes more confluent as the infection worsens. As a result, air-filled bronchi pass through pus-filled alveoli. There are some individuals who are more likely to need hospitalization for pneumonia than others. Seek immediate medical attention if you have a condition such as heart disease, asthma, kidney, or endocrine problems.
3. Renal failure
   Doppler ultrasound and other high-frequency sound wave imaging techniques can be used to assess kidney and arterial function. Using this technology, blood artery blockages can be detected and their severity determined. CT scans produce detailed images of renal arteries using an X-ray machine linked to a computer. Blood flow can be visualized using a dye injection. In MRA, radio waves and strong magnetic fields are used to provide exact 3D images of the kidneys and renal arteries. A dye injection can be used to observe how your blood flows. MRA creates detailed 3D images of the kidneys and renal arteries using radio waves and powerful magnetic fields. With a unique type of X-ray examination

that helps identify the obstruction, it is sometimes possible to open the narrowed portion of the renal arteries with a balloon or stent. Maintain a healthy lifestyle, reduce salt consumption, eat wholesome meals, and exercise frequently if you have moderately or severely high blood pressure. A serum creatinine level can be used to determine kidney impairment. In spite of the rarity of nephrotic levels, vascular renal disease is typically associated with low-to-moderate proteinuria.

4. Crohn's disease
   A colonoscopy involves a doctor examining the interior of your colon and rectum with an endoscope, a long, flexible, narrow tube with a light on one end. Additionally, your doctor might examine your ileum for signs of Crohn's disease. In a hospital setting or in an outpatient setting, a qualified professional performs a colonoscopy. You will receive written instructions from your doctor on how to prepare for bowel surgery at home. The blood test is one of the lab tests used to diagnose Crohn's disease. In order to check for changes in red blood cells, a medical professional may take a sample of your blood for analysis. The X-ray machine has the shape of a tunnel, and you will sit on a table that slides into it. CT scans can detect Crohn's disease and its symptoms. Intestinal endoscopy is the most accurate way to diagnose Crohn's disease and exclude other conditions such as ulcerative colitis, diverticular disease, or cancer. It is common for people with Crohn's disease to live active, fulfilling lives. To get enough calories, Crohn's disease patients typically need to modify their diets.

Here $C_i = \{C_1, C_2, C_3, C_4, C_5\}$, where C_1 = Chemist1,C_2 = Chemist2, C_3 = Chemist3, C_4 = Chemist4, and C_5 = Chemist5. They want to diagnose four disease given in the set as $M = \{D_1, D_2, D_3, D_4, D_5\}$, where D_1= clinical diagnosis, D_2 = laboratory diagnosis, D_3 = principal diagnosis, D_4 = admitting diagnosis, and D_5 = diagnostic imaging, each from a diagnostic lab. Chemists choose the diagnosis by prioritizing their choices since their decisions are based on their choices. It is the responsibility of each chemist to prioritize the four diseases in a diagnosis according to the satisfaction he/she feels, based on the set.

Let $F = \{$Urinary tract, infection, Pneumonia, Renal failure, Crohn's disease$\}$. Diagnosis is based on accuracy, satisfaction, good, or fair. Data are provided in decision Table 3, which is the Cartesian product of the neutrosophic relation between C and F denoted by $C * F$. Each entry of the decision table is of the form:

$$r_{ij} = \langle \varXi_{ij}, \varUpsilon_{ij}, \varOmega_{ij}, \varPi_{ij} \rangle, \ i = 1, 2, 3, \ldots, 5, \ j = 1, 2, 3, 4.$$

Let r_{ij} consist of further four sub-entries $\varXi_{ij}, \varUpsilon_{ij}, \varOmega_{ij}$, and $\varPi_{ij}$. Let $\varXi_{ij}$ be the membership value denoting the points allotted by the chemist c_i to the disease f_j of a diagnosis. The second sub-entry $\varUpsilon_{ij}$ is the non-membership value showing the points left by the chemist to the diseases f_j. The third sub-entry $\varPi_{ij}$ denotes the degree of the refusal of the chemist, c_i, in the diseases f_j. More explicitly, e.g., $\varXi_{ij} = 0.85$ means that the chemist c_i gives 0.85 measure of choice to disease f_j of a diagnosis. Similarly, $\varUpsilon_{ij} = 0.75$ means that the chemist c_i expresses 0.75 measure of inaccuracy for f_j,$\varOmega_{ij} = 0.3$ means that the chemist c_i expresses 0.3 measure of indeterminacy for f_j, and $\Pi_{ij} = 0.1$ means that the chemist remains 0.1 measure undecided about the diagnosis f_j.

All decision tables do not have a fourth sub-entry for our convenience. Only the first three sub-entries are written. Assume that each entry of a table contains the fourth sub-entry Π_{ij} evaluated by Eq. 11.11. Table 11.2 contains the diagnosis data for the diseases, which is a neutrosophic relation of the sets F and M as $F * M$. The entry $t_{jk} = \langle \Xi'_{jk}, \Upsilon'_{jk}, \Omega'_{jk}, \Pi'_{jk} \rangle$, $j = 1, 2, 3, 4$, $k = 1, 2, 3, \ldots, 5$. The entry Ξ'_{jk} denotes the level of the membership of the disease f_j in the diagnosism_k, e.g., $\Xi'_{jk} = 0.65$ means that the disease f_j is 0.65 degree present in the diagnosism_k. In the same way, Υ'_{jk} shows the degree of the non-membership of disease f_j in the diagnosism_k, for example, if $\Upsilon'_{jk} = 0.7$, it means that the disease f_j is 0.7 degree not available in the diagnosism_k. In the same way, Ω'_{jk} shows the degree of the non-membership of disease f_j in the diagnosis m_k, for example, if $\Omega'_{jk} = 0.45$, it means that the disease f_j is 0.7 degree indeterminant in the diagnosis m_k. Similarly, Π_{jk} denotes the degree of the refusal of the disease f_j in the diagnosisM_k; $\Pi_{jk} = 0.2$ means that there is no decision of the attribute G_j about the diagnosis m_k. For our own convenience, we confined each entry to the matrix T to contain $\Xi'_j k$, $\Upsilon'_j k$ and $\Omega'_j k$.

We find the values of the matrices Π_{C*F} and Π_{F*M} given by Eq. 11.11 for $\{m, p, n\} = \{(1, 1, 1), (2, 2, 2), (3, 3, 3)\}$ and also the values of the matrices Π_{C*F} and Π_{F*M} given by Eq. 11.11 for $\{m, p, n\} = \{(2, 2, 3), (2, 3, 2), (3, 2, 2), (2, 3, 3), (3, 2, 3), (3, 3, 2)\}$. Using Eqs. 11.20–11.22, we find the matrix given by Eq. 11.25 whose entries are given in Table 11.5 and 11.6. To ensure that our results are consistent with the GNSS values, we have also calculated the values using the two neutrosophic set methods, NSSs and PyNSSs. There are two ways in which we make decisions based on Table 11.3 and 11.4.

Table 11.3 Chemist's value to the diseases

$Q(C \to F)$	Urinary tract infection	Pneumonia	Renal failure	Crohns Disease
Chemist-1	⟨0.7, 0.4, 0.6⟩	⟨0.7, 0.5, 0.4⟩	⟨0.7, 0.66, 0.7⟩	⟨0.6, 0.2, 0.4⟩
Chemist-2	⟨0.8, 0.3, 0.4⟩	⟨0.8, 0.6, 0.43⟩	⟨0.6, 0.56, 0.8⟩	⟨0.7, 0.4, 0.3⟩
Chemist-3	⟨0.8, 0.5, 0.5⟩	⟨0.9, 0.7, 0.7⟩	⟨0.5, 0.45, 0.92⟩	⟨0.8, 0.4, 0.2⟩
Chemist-4	⟨0.6, 0.2, 0.3⟩	⟨0.76, 0.6, 0.43⟩	⟨0.7, 0.63, 0.7⟩	⟨0.5, 0.5, 0.2⟩
Chemist-5	⟨0.8, 0.4, 0.5⟩	⟨0.77, 0.66, 0.4⟩	⟨0.5, 0.45, 0.8⟩	⟨0.7, 0.4, 0.1⟩

Table 11.4 Generalized neutrosophic relation of the sets F and M denoted by $R(F \to M)$

$R(F \to M)$	D_1	D_2	D_3	D_4	D_5
Urinary tract infection	⟨0.7, 0.5, 0.6⟩	⟨0.7, 0.5, 0.3⟩	⟨0.8, 0.6, 0.5⟩	⟨0.7, 0.7, 0.4⟩	⟨0.7, 0.7, 0.4⟩
Pneumonia	⟨0.8, 0.3, 0.4⟩	⟨0.8, 0.6, 0.43⟩	⟨0.7, 0.5, 0.3⟩	⟨0.6, 0.4, 0.5⟩	⟨0.6, 0.8, 0.45⟩
Renal failure	⟨0.7, 0.5, 0.5⟩	⟨0.9, 0.6, 0.4⟩	⟨0.8, 0.6, 0.5⟩	⟨0.5, 0.2, 0.56, ⟩	⟨0.5, , 0.2, 0.56⟩
Crohn's disease	⟨0.6, 0.2, 0.6⟩	⟨0.7, 0.5, 0.3⟩	⟨0.8, 0.5, 0.2⟩	⟨0.6, 0.4.0.35⟩	⟨0.6, 0.4, 0.35⟩

Table 11.5 Diagnosis value shown for each chemist

$R * Q$	D_1	D_2	D_3	D_4	D_5	Set Type
Chemist-1	0.68	0.72	0.72	0.6856	0.6856	$GNSS_{(1,1,1)}$
	0.62	0.8960	0.78	0.6466	0.6462	$GNSS_{(2,2,2)}$
	0.5320	0.9279	0.9279	0.5959	0.5502	$GNSS_{(3,3,3)}$
Chemist-2	0.80	0.84	0.8084	0.7056	0.7056	$GNSS_{(1,1,1)}$
	0.80	0.9411	0.8380	0.7258	0.7279	$GNSS_{(2,2,2)}$
	0.80	0.94	0.94	0.7641	0.7810	$GNSS_{(3,3,3)}$
Chemist-3	0.80	0.8960	0.80	0.7056	0.7056	$GNSS_{(1,1,1)}$
	0.80	0.9411	0.80	0.7457	0.7698	$GNSS_{(2,2,2)}$
	0.80	0.80	0.80	0.8904	0.9065	$GNSS_{(3,3,3)}$
Chemist-4	0.76	0.8620	0.72	0.6096	0.6096	$GNSS_{(1,1,1)}$
	0.76	0.9228	0.78	0.6372	0.6373	$GNSS_{(2,2,2)}$
	0.76	0.9279	0.9279	0.6710	0.7079	$GNSS_{(3,3,3)}$
Chemist-5	0.7560	0.9420	0.80	0.7080	0.7080	$GNSS_{(1,1,1)}$
	0.7246	0.9273	0.80	0.7464	0.7578	$GNSS_{(2,2,2)}$
	0.6852	0.80	0.80	0.8476	0.8701	$GNSS_{(3,3,3)}$

1. **From left to top:** The decision illustrates the chemist's satisfaction with the diagnosis. Chemist-1 is content with his choice of first diagnosis and then lab diagnosis. Chemist-2's expertise lies in clinical diagnosis, laboratory diagnosis, and principal diagnosis. Chemist-3 is satisfied with clinical diagnosis, laboratory diagnosis, and principal diagnosis. Chemist-4 is satisfied with the laboratory diagnosis. According to Chemist-5, he is content with both the primary diagnosis and the tertiary diagnosis.
2. **From top to left:** The decision tells how much a particular diagnosis is appropriate. Chemist-2 and Chemist-3 are satisfied with the clinical diagnosis. Laboratory diagnosis suits the needs of Chemist-5 and Chemist-4. Chemist-2 is satisfied with the principal diagnosis. According to Chemist-5, admitting diagnosis is satisfactory. Chemist-5 and Chemist-3 are satisfied with the diagnostic image.

Let the max-min-max composition of generalized neutrosophic relations Q and R be denoted by $R * Q$ and defined in Eq. 11.25 for diagnostic disease problem.

4.2 Problem: Diabetes in Different Age Sectors

1. Severe autoimmune diabetes (X_1):
   In this type of diabetes, the immune system produces antibodies that attack beta cells (the cells that produce insulin). It is called an autoimmune response when such a response occurs in an infant. People with type 1 diabetes should closely monitor their blood glucose levels as part of their diabetes management plan. In addition to insulin injections or insulin pumps, this plan calls for daily insulin injections.

Table 11.6 Diagnosis value for each chemist

$R * Q$	D_1	D_2	D_3	D_4	D_5	Set Type-
Chemist-1	0.5763	0.8159	0.8159	0.6153	0.6024	$GNSS_{(2,2,3)}$
	0.5859	0.8294	0.8294	0.6184	0.6092	$GNSS_{(2,3,2)}$
	0.5329	0.9552	0.9552	0.5946	0.5122	$GNSS_{(2,3,3)}$
	0.5749	0.8317	0.8317	0.6133	0.5974	$GNSS_{(3,2,2)}$
	0.5280	0.9591	0.9591	0.5923	0.4908	$GNSS_{(3,2,3)}$
	0.5322	0.9854	0.9854	0.5936	0.5037	$GNSS_{(3,3,2)}$
Chemist-2	0.80	0.8367	0.8367	0.7334	0.7342	$GNSS_{(2,2,3)}$
	0.80	0.8399	0.8399	0.7350	0.7355	$GNSS_{(2,3,2)}$
	0.80	0.8848	0.8848	0.7612	0.7787	$GNSS_{(2,3,3)}$
	0.80	0.72	0.72	0.6856	0.7361	$GNSS_{(3,2,2)}$
	0.80	0.8801	0.8801	0.7651	0.7795	$GNSS_{(3,2,3)}$
	0.80	0.8862	0.8862	0.7623	0.7821	$GNSS_{(3,3,2)}$
Chemist-3	0.80	0.80	0.80	0.6456	0.7507	$GNSS_{(2,2,3)}$
	0.80	0.80	0.80	0.6618	0.7540	$GNSS_{(2,3,2)}$
	0.80	0.80	0.80	0.8326	0.8329	$GNSS_{(2,3,3)}$
	0.80	0.80	0.80	0.6681	0.7540	$GNSS_{(3,2,2)}$
	0.80	0.80	0.80	0.8326	0.8329	$GNSS_{(3,2,3)}$
	0.80	0.80	0.80	0.8403	0.8403	$GNSS_{(3,3,2)}$
Chemist-4	0.76	0.8159	0.8159	0.6593	0.6725	$GNSS_{(2,2,3)}$
	0.76	0.8294	0.8294	0.6601	0.6749	$GNSS_{(2,3,2)}$
	0.76	0.9552	0.9552	0.6727	0.7514	$GNSS_{(2,3,3)}$
	0.76	0.8317	0.8317	0.6601	0.6749	$GNSS_{(3,2,2)}$
	0.76	0.9591	0.9591	0.6727	0.7514	$GNSS_{(3,2,3)}$
	0.76	0.9845	0.9845	0.6731	0.7558	$GNSS_{(3,3,2)}$
Chemist-5	0.73	0.80	0.80	0.7216	0.7527	$GNSS_{(2,2,3)}$
	0.7031	0.80	0.80	0.7312	0.7563	$GNSS_{(2,3,2)}$
	0.6840	0.80	0.80	0.8250	0.8361	$GNSS_{(2,3,3)}$
	0.6982	0.80	0.80	0.7302	0.7559	$GNSS_{(3,2,2)}$
	0.6816	0.80	0.80	0.8245	0.8308	$GNSS_{(3,2,3)}$
	0.6828	0.80	0.80	0.8291	0.8385	$GNSS_{(3,3,2)}$

2. Severe insulin-deficient diabetes (X_2):
   Many of the characteristics of type 1 diabetes patients were present in patients with this type of diabetes, such as young age, thinness, and inadequate insulin levels. A significant difference was that there were no antibodies in their blood, proving that it was not the immune system that caused their illness. Damaged insulin-producing cells were to blame for insufficient insulin synthesis in people with SIDD. There was a higher risk of visual loss in this group. People with this type of diabetes may also take oral drugs despite controlling their diabetes in the same way as those with type 1.

3. Severe insulin-resistant diabetes (X_3):
   Unlike other varieties of diabetes, this type of diabetes is characterized by insulin resistance, which occurs when the body doesn't respond to its own insulin properly. Insulin resistance is worsened when people with SIRD are overweight. Type 3 diabetes is associated with a heightened risk of kidney impairment. Similarly, diabetes management options for patients with SIRD were found to be less effective than those for the other five subgroups. In the way, research result in new diagnostic techniques and more stringent treatments for these patients.
4. Mild obesity-related diabetes (X_4):
   People with this less severe form of diabetes are characterized by extreme obesity and insulin resistance. It is believed that fat causes this less severe form of diabetes, which is associated with less severe insulin resistance.
5. Mild age-related diabetes (X_5):
   Diabetes in MARD patients was milder and older than in middle-aged patients with diabetes. There is a high prevalence of this type of diabetes, according to the report.

The following list the types of sectors affected by the disease and the types of diabetes they are suffered from.

1. Infant:
   In addition to being irritable or agitated or having a seizure, the infant may also have breathing difficulties. Due to the hazards associated with diabetes, most diabetic infants will be closely monitored throughout their first few hours of life. They will receive regular heel sticks to assess their blood sugar levels. The likelihood of developing insulin resistance later in life is higher for term babies who are small for gestational age. Normal plasma levels of insulin cannot sufficiently stimulate the absorption of glucose by peripheral tissues in cases of insulin resistance (IR). It is possible to develop moderate diabetes caused by obesity (MOD) when you are overweight or obese but do not have insulin resistance. A medical disorder such as diabetes is often discovered later in life and presents with milder symptoms than other diseases.
2. Toddler:
   Your child's body may not be able to use the sugar in his or her bloodstream to produce energy. There may be a lack of energy in the muscles and organs of your child, resulting in extreme hunger and unexplained weight loss. The blood glucose of type 1 diabetics must be injected into their cells to be converted into energy. Depending on your child's needs, the care team will customize your insulin schedule. Sugar (glucose) is used differently by your child's body in this chronic condition. Obesity and a sedentary lifestyle both increase the risk of developing type 2 diabetes.
3. Young:
   An increasing proportion of young individuals with latent autoimmune diabetes in the young (LADA) experience an initial insulin-free period. A medical disorder such as diabetes is diagnosed later in life and manifests with fewer

symptoms. Diabetes with severe insulin deficiency (SIDD) is a cluster 1 subtype. When diagnosed, these patients have a low BMI and are young. However, GADA is not present. There is no clear explanation for why these people have beta cell dysfunction. Chronic hyperglycemia is caused by abnormal carbohydrate, protein, and fat metabolism. It is caused either by a deficiency in insulin action or secretion or by both, which characterizes the heterogeneous group of diseases known as diabetes. The most common type of diabetes, type 2 diabetes (T2D), has a range of pathophysiology, ranging from severe insulin resistance to normal insulin sensitivity.

4. Adult:
Latent autoimmune diabetes in adults (LADA) develop gradually. As with type 1 diabetes, LADA occurs when your pancreas stops producing insulin due to some sort of "insult" that gradually damages the insulin-producing cells. Severe insulin resistance is characterized by significant hyperinsulinemia and reduced glucose sensitivity to both endogenous and exogenous insulin. As a result, insulin has a markedly diminished effect on physiology. It is common for people with SIRD to have high levels of insulin resistance. This means that their cells do not respond to insulin even though their bodies produce it. Additionally, they tend to be overweight. Those with mild age-related diabetes (MARD) tend to be older than those with other types of diabetes. Keeping their blood sugar under control is not a problem for them. It is the most prevalent type of diabetes, accounting for about 40 percent of cases.
5. Pregnant women:
High blood sugar during pregnancy can affect both the mother and unborn child. Pregnancy category B includes regular insulin (U-100 and U-500), insulin apart, insulin lispro (U-100 and U-200), NPH, and insulin determined. The FDA has deemed these insulins low risk during pregnancy based on human data. It is unknown at what blood sugar level insulin injections should be started. In contrast, if the fasting blood sugar level is higher than 105 mg/dl or if the level 2 hours after a meal is higher than 120 mg/dl twice, many doctors will prescribe insulin. Insulin is the drug of choice for treating hyperglycemia in gestational diabetes mellitus. Metformin and glyburide are two drugs.

Patients are given in the set $G = \{P_1, P_2, P_3, P_4, P_5\}$, where $P_1 = Patient1$, $P_2 = Patient2$, $P_3 = Patient3$, $P_4 = Patient4$, and$P_5 = Patient5$. The different types of diabetes were given in the set: $P = \{X_1, X_2, X_3, X_4, X_5\}$, where X_1= severe autoimmune diabetes (SAID), X_2= severe insulin-deficient diabetes (SIDD), X_3= severe insulin-resistant diabetes (SIRD), X_4= mild obesity-related diabetes (MOD), and X_5= mild age-related diabetes (MARD). The diabetes are suffered in accordance with the relevancy of the respective five age sector as $U = \{\text{Infant,Toddler,Young,Adult,Pregnancy women}\}$. The generalized neutro-

Table 11.7 Different age to the respective patients

	Infant	Toddler	Young	Adult	Pregnancy women
P_1	⟨0.9, 0.7, 0.7⟩	⟨0.9, 0.7, 0.7⟩	⟨0.5, 0.45, 0.92⟩	⟨0.8, 0.4, 0.2⟩	⟨0.9, 0.7, 0.7⟩
P_2	⟨0.8, 0.6, 0.43⟩	⟨0.8, 0.6, 0.43⟩	⟨0.6, 0.56, 0.8⟩	⟨0.7, 0.4, 0.3⟩	⟨0.8, 0.6, 0.43⟩
P_3	⟨0.9, 0.7, 0.7⟩	⟨0.9, 0.7, 0.7⟩	⟨0.5, 0.45, 0.92⟩	⟨0.8, 0.4, 0.2⟩	⟨0.9, 0.7, 0.7⟩
P_4	⟨0.76, 0.6, 0.43⟩	⟨0.76, 0.6, 0.43⟩	⟨0.7, 0.63, 0.7⟩	⟨0.5, 0.5, 0.2⟩	⟨0.76, 0.6, 0.43⟩
P_5	⟨0.8, 0, 6, 0.43⟩	⟨0.8, 0.6, 0.43⟩	⟨0.6, 0.56, 0.8⟩	⟨0.7, 0.4, 0.3⟩	⟨0.8, 0.6, 0.43⟩

Table 11.8 The value of the diabetes to the respective age sector

$R(P \to U)$	X_1)	X_2	X_3	X_4	X_5
Infant	⟨0.7, 0.5, 0.3⟩	⟨0.3, 0.6, 0.4⟩	⟨0.7, 0.5, 0.3⟩	⟨0.8, 0.6, 0.4⟩	⟨0.8, 0.6, 0.4⟩
Toddler	⟨0.7, 0.5, 0.3⟩	⟨0.8, 0.6, 0.4⟩	⟨0.7, 0.5, 0.3⟩	⟨0.8, 0.6, 0.4⟩	⟨0.8, 0.6, 0.4⟩
Young	⟨0.8, 0.6, 0.5⟩	⟨0.9, 0.6, 0.4⟩	⟨0.8, 0.6, 0.5⟩	⟨0.9, 0.6, 0.4⟩	⟨0.9, 0.6, 0.4⟩
Adult	⟨0.8, 0.5, 0.2⟩	⟨0.7, 0.5, 0.3⟩	⟨0.8, 0.5, 0.2⟩	⟨0.7, 0.5, 0.3⟩	⟨0.7, 0.5, 0.3⟩
Pregnancy Women	⟨0.7, 0.5, 0.1⟩	⟨0.8, 0.6, 0.0⟩	⟨0.7, 0.5, 0.1⟩	⟨0.8, 0.6, 0.0⟩	⟨0.8, 0.6, 0⟩

sophic relation of the sets G and P are denoted by $Q(G \to P)$. We can make decision from Table 11.7.

Generalized neutrosophic relation of the sets P and U denoted by $R(P \to U)$. We can make decision from Table 11.8.

We find the values of the matrices Π_{C*F} and Π_{F*M} given by Eq. 11.11 for $\{m, p, n, \} = \{(1, 1, 1), (2, 2, 2), (3, 3, 3)\}$. After this, using Eqs. 11.20 and 11.22, we find the matrix given by Eq. 11.25 whose entries are given in Table 11.9. Along with the values of the GNSSs, we have also calculated the values by the two neutrosophic set methods $NSSs$ and $PyNSSs$ for the certainty that our results coincide with them. This fact is evident from Table 11.9. We can make decision from Table 11.9.

1. **From left to top:** This is based on particular patients suffering from different types of diabetes. Patient-1 is suffering from X_4 and X_5. Patient-2 is suffering from X_2. Patient-3 is suffering from X_2. Patient-4 is suffering from X_2,X_4, and X_5. Patient-5 is suffering from X_2,X_4, and X_5.
2. **From top to left:** This is based on the type of diabetes affecting the patient. X_1 affects Patient-1 and Patient-3; X_2 affects Patient-1 and Patient-3; X_3 affects Patient-1 and Patient-3; and X_4 and X_5 affect all patients.

We hope that this problem gives better results for more practical and real-world problems. Max-min-max composition of generalized neutrosophic relations Q and R is denoted by $R * Q$ and defined in Eq. 11.25 for diabetes problem.

Table 11.9 Value of diabetes to the respective patients

$R * Q$	X_1	X_2	X_3	X_4	X_5	Set Type-
Patient-1	0.85	0.88	0.85	0.84	0.84	$GNSS_{(1,1,1)}$
	0.85	0.9647	0.85	0.9411	0.9411	$GNSS_{(2,2,2)}$
	0.85	0.9573	0.85	0.9815	0.9815	$GNSS_{(2,3,2)}$
Patient-2	0.8080	0.84	0.8084	0.84	0.84	$GNSS_{(1,1,1)}$
	0.8330	0.9411	0.8330	0.9411	0.9411	$GNSS_{(2,2,2)}$
	0.8321	0.9815	0.8321	0.9815	0.9815	$GNSS_{(2,3,2)}$
Patient-3	0.85	0.88	0.85	0.84	0.84	$GNSS_{(1,1,1)}$
	0.85	0.9647	0.85	0.9411	0.9411	$GNSS_{(2,2,2)}$
	0.85	0.9573	0.85	0.9815	0.9815	$GNSS_{(2,3,2)}$
Patient-4	0.81	0.84	0.81	0.84	0.84	$GNSS_{(1,1,1)}$
	0.8437	0.9411	0.8437	0.9411	0.9411	$GNSS_{(2,2,2)}$
	0.8169	0.9815	0.8169	0.9815	0.9815	$GNSS_{(2,3,2)}$
Patient-5	0.80	0.84	0.80	0.84	0.84	$GNSS_{(1,1,1)}$
	0.8330	0.9411	0.8330	0.9411	0.9411	$GNSS_{(2,2,2)}$
	0.8321	0.9815	0.8321	0.9815	0.9815	$GNSS_{(2,3,2)}$

5 Conclusions

The following are the main conclusions of our work:

1. Through the introduction of the GNSFSs, problems with uncertainty that do not fall under the FSs, IFSs, and PyFSs can be studied.
2. To date, GNSSs are the most generalized form of FSs. As shown, the type of the set is determined by the neutrosophic numerical values of m, p, and n.
3. By changing the values of their indices, GNSSs perform all the functions that NSSs and PyNSSs can perform; additionally, as shown in this demonstration, their canvas of apps is quite broad.
4. Because GNSSs can be used for any purpose by merely changing the values of m, p, and n, they are appropriately called versatile NSSs.
5. The max-min-max composition was used to examine two of the most prevalent real-world applications of MCDM: employee postings using GNSFSs and consumer satisfaction. It was found that the results of the GSNSFs, INSFSs, and PyNSFSs were all in agreement.
6. There are formulas for calculating distances between various GNSFSs using Euclidean, normalized Euclidean, Hamming, and normalized Euclidean distances in the medical field. Literature findings were in agreement with the findings of this study.
7. As with any other field, GNSFSs have limitations. It is critical to select the numerical values for the indices m, p, and n based on the kind of problem being researched. GNSFSs can still be inspected for m, p, and n to any non-negative real number, even if the values of these parameters are restricted to merely non-negative integers.

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Chapter 12
An Application of Neutrosophic Graph in Decision-Making Problem for Alliances of Companies

Jasminara Khatun and Sk Amanathulla

1 Introduction

Rosenfeld [14] first defined fuzzy graph (FG) when fuzzy relations are taken into account on fuzzy sets in 1975. The extension of FG is an intuitionistic fuzzy graph (IFG). Neutrosophic models give the system more flexibility, compatibility, and precision than the IF model. In 2006, the idea of neutrosophic set (NS) was invented by Smarandache [23] that would be a generalization of IFS. Cuong added extra components that determine the degree of neutral membership in addition to IFS. IFS give an element's membership and non-membership degree, while NS give the truth, indeterminacy, and falsity membership degree of an element. NS used in several fields, including computer science, chemistry, economics, engineering, mathematics, etc.

1.1 Review of Literature

After Rosenfeld [14], the FG theory is going on with its different branches, such as fuzzy planar graphs [19, 21], fuzzy threshold graph [20], balanced interval-valued FG [12], highly irregular interval-valued FG [11], and m-step fuzzy competition graphs [18]. FG coloring has been introduced by Samanta et al. [22]. Some

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C. Jana et al. (eds.), *Fuzzy Optimization, Decision-making and Operations Research*, https://doi.org/10.1007/978-3-031-35668-1_12

problems regarding interval-valued FG have been studied by Pramanik et al. [6–10]. Voskoglou et al. [27] have introduced fuzzy hypergraphs.

In 2015, Sahoo et al. [16] studied IF competition graph. Akram et al. have studied several problems, like strong IFG [1], IF hypergraphs [2], IF cycles and IF trees [3], and IF planar graphs [4]. Balanced IFG is introduced by Karunambigai et al. [5]. Sahoo et al. introduced IF tolerance graph with its application [17] and various types of products on IFG [15, 16]. Recently, Amanathulla et al. studied a few problems which are the extension of fuzzy graphs [13, 24–26].

1.2 Motivation

NGs add a new dimension to traditional graph theory, which is an extended version of IFG. FG and IFG have various applications in reality. There are various problems those cannot be solved neither by FG nor by IFG. These problems can be handled by NG. These motivates us to consider NG in this article and obtained significant results. An application of balanced NG for alliances of companies is also given in this chapter.

The rest portion of the chapter is arranged as follows. Some preliminaries are given in Sect. 2. In Sect. 3, NG and some related terms are defined. In Sect. 4, balanced NG have been studied. An application of balanced NG is given in Sect. 5. In Sect. 6, concluding remarks are made.

2 Preliminaries

An extension of FG is IFG. An IFG is defined below.

Definition 1 An IFG $G = (A, \sigma, \mu)$ where $\sigma = (\sigma_1, \sigma_2)$, $\mu = (\mu_1, \mu_2)$ and

(i) $A = \{h_1, h_2, \ldots, h_n\}$ where $\sigma_1 : T \to [0, 1]$ and $\sigma_2 : T \to [0, 1]$ are the membership and non-membership degree of the node $h_k \in T$, respectively, and $0 \le \sigma_1(h_k) + \sigma_2(h_k) \le 1$ for every $h_k \in T$ $(k = 1, 2, \ldots, n)$.

(ii) $\mu_1 : A \times A \to [0, 1]$ and $\mu_2 : A \times A \to [0, 1]$, where $\mu_1(h_i, h_j)$ and $\mu_2(h_i, h_j)$ are, respectively, the membership and non-membership degree of the edge (h_i, h_j) so that $\mu_1(h_i, h_j) \le \sigma_1(h_i) \wedge \sigma_1(h_j)$ and $\mu_2(h_i, h_j) \le \sigma_2(h_i) \vee \sigma_2(h_j)$, $0 \le \mu_1(h_i, h_j) + \mu_2(h_i, h_j) \le 1$ for each (h_i, h_j).

In Fig. 12.1, an IFG has been given.

A neutrosophic set (NS) is an extended version of IFS. The formal definition of NS is given below.

Definition 2 A NS N on an universal set U is $N = \{(n, t_N(n), i_N(n), f_N(n)) : n \in N\}$, where $t_N(n) \in [0, 1]$, $i_N(n) \in [0, 1]$, and $f_N(n) \in [0, 1]$ are, respectively, the degree of truth, indeterminacy, and falsity membership of n in N which satisfies $0 \le t_N(n) + i_N(n) + f_N(n) \le 3$ for each $n \in N$.

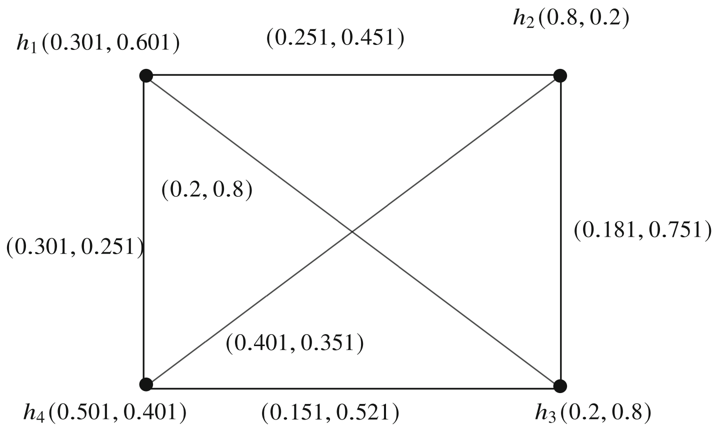

Fig. 12.1 An IFG

3 Neutrosophic Graph

This part contains some definition and important properties of neutrosophic graph.

Definition 3 A NG is defined by $G = (A, B, W_A, W_B)$, where $A = \{h_1, h_2, \ldots, h_n\}$ be the node set, $W_A = (t_A, i_A, f_A)$ and $W_B = (t_B, i_B, f_B)$ are two NS such that

(i) $t_A : A \to [0, 1]$, $i_A : A \to [0, 1]$ and $f_A : A \to [0, 1]$, respectively, denote the degree of truth, indeterminacy, and falsity membership functions of a node $h_i \in A$ and $0 \le t_A(h_i) + i_A(h_i) + f_A(h_i) \le 3$ for any $h_i \in A, i = 1, 2, \ldots, n$.

(ii) $t_B : A \times A \to [0, 1]$, $i_B : A \times A \to [0, 1]$ and $f_B : A \times A \to [0, 1]$, where $t_B(h_i, h_j)$, $i_B(h_i, h_j)$, and $f_B(h_i, h_j)$, respectively, denote the truth, indeterminacy, and falsity membership value of the edge (h_i, h_j), so that $t_B(h_i, h_j) \le t_A(h_i) \wedge t_A(h_j)$, $i_B(h_i, h_j) \le i_A(h_i) \wedge i_A(h_j)$, $f_B(h_i, h_j) \le f_A(h_i) \vee f_A(h_j)$, and $0 \le t_B(h_i, h_j) + i_B(h_i, h_j) + f_B(h_i, h_j) \le 3$ for all $(h_i, h_j) \in A \times A, i = 1, 2, \ldots, n$.

A PFG is shown in Fig. 12.2.

Definition 4 A NG $G = (A, B, W_A, W_B)$, where $A = \{h_1, h_2, \ldots, h_n\}$ be the node set, $W_A = (t_A, i_A, f_A)$, $W_B = (t_B, i_B, f_B)$ is called a complete NG if $t_B(h_i, h_j) = t_A(h_i) \wedge t_A(h_j)$, $i_B(h_i, h_j) = i_A(h_i) \wedge i_A(h_j)$, $f_B(h_i, h_j) = f_A(h_i) \vee f_A(h_j)$ for every $h_i, h_j \in A$.

A complete NG is shown in Fig. 12.3.

Definition 5 A NG $G = (A, B, W_A, W_B)$, where $A = \{h_1, h_2, \ldots, h_n\}$ be the node set, $W_A = (t_A, i_A, f_A)$, $W_B = (t_B, i_B, f_B)$ is said to be a strong NG if $t_B(h_i, h_j) = t_A(h_i) \wedge t_A(h_j), i_B(h_i, h_j) = i_A(h_i) \wedge i_A(h_j)$, $f_B(h_i, h_j) = f_A(h_i) \vee f_A(h_j)$ for all $(h_i, h_j) \in B$.

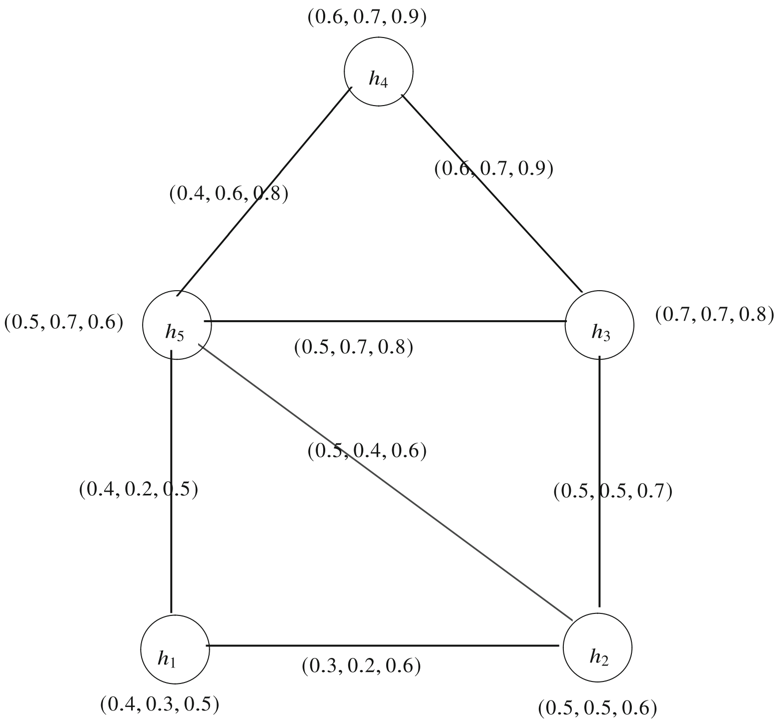

Fig. 12.2 A NG

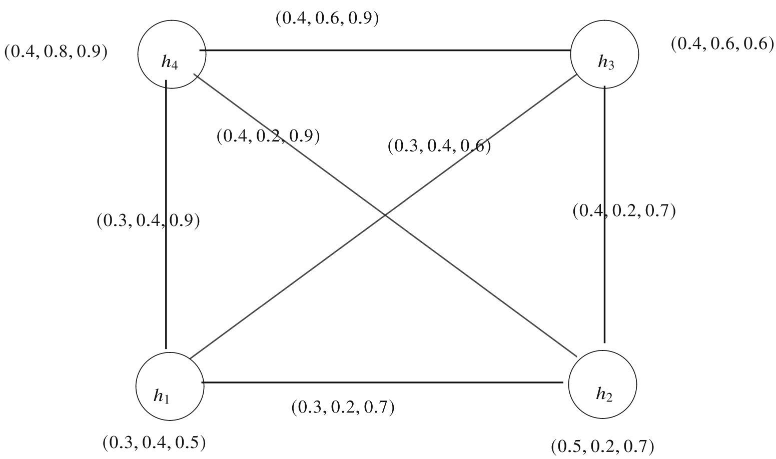

Fig. 12.3 A complete NG

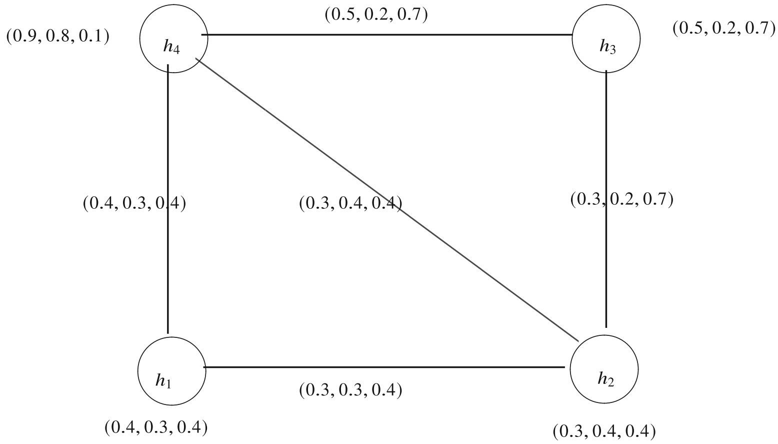

Fig. 12.4 A strong NG

A strong NG is shown in Fig. 12.4.

Definition 6 Let $G = (A, B, W_A, W_B)$ be a NG, where $W_A = (t_A, i_A, f_A)$, $W_B = (t_B, i_B, f_B)$. Then the degree of a node a of the NG is defined by $d(a) = (d_t, d_i, d_f)$, where $d_t(a) = \sum_{a \neq b} t_B(a, b)$, $d_i(a) = \sum_{a \neq b} i_B(a, b)$, $d_f(a) = \sum_{a \neq b} f_B(a, b)$.

For the NG in Fig. 12.4, $d(h_1) = (0.7, 0.6, 0.8)$, $d(h_2) = (0.9, 0.9, 1.5)$, $d(h_3) = (0.8, 0.4, 1.4)$, $d(h_4) = (1.2, 0.9, 1.5)$.

Definition 7 Let $G = (A, B, W_A, W_B)$ be a NG, where $W_A = (t_A, i_A, f_A)$, $W_B = (t_B, i_B, f_B)$. Then order of G is defined by $O(G) = (O_t(G), O_i(G), O_f(G))$, where $O_t(G) = \sum_{a \in A} t_A(a)$, $O_i(G) = \sum_{a \in A} i_A(a)$, $O_f(G) = \sum_{a \in A} f_A(a)$.

For the NG G in Fig. 12.4, the order of G is $O(G) = (2.1, 1.7, 1.6)$.

Definition 8 Let $G = (A, B, W_A, W_B)$ be a NG, where $W_A = (t_A, i_A, f_A)$, $W_B = (t_B, i_B, f_B)$. The size of G is defined by $S(G) = (S_t(G), S_i(G), S_f(G))$, where $S_t(G) = \sum_{a \neq b} t_B(a, b)$, $S_i(G) = \sum_{a \neq b} i_B(a, b)$, $S_f(G) = \sum_{a \neq b} f_B(a, b)$.

The size of the NG in Fig. 12.4 is $S(G) = (1.8, 1.4, 2.6)$.

Definition 9 For the NG $G = (A, B, W_A, W_B)$, where $W_A = (t_A, i_A, f_A)$, $W_B = (t_B, i_B, f_B)$, an edge (a, b), $a, b \in A$ is called independent strong edge if $\frac{1}{2}[t_A(a) \wedge t_A(b)] < i_B(a, b)$, $\frac{1}{2}[i_A(a) \vee i_A(b)] > i_B(a, b)$ and $\frac{1}{2}[f_A(a) \vee f_A(b)] > f_B(a, b)$, otherwise it is refer to as a weak edge.

In Fig. 12.5, the edges (h_1, h_2) and (h_2, h_3) are independent strong, but the edge (h_1, h_3) is not independent strong. The edge (h_1, h_3) is a weak edge.

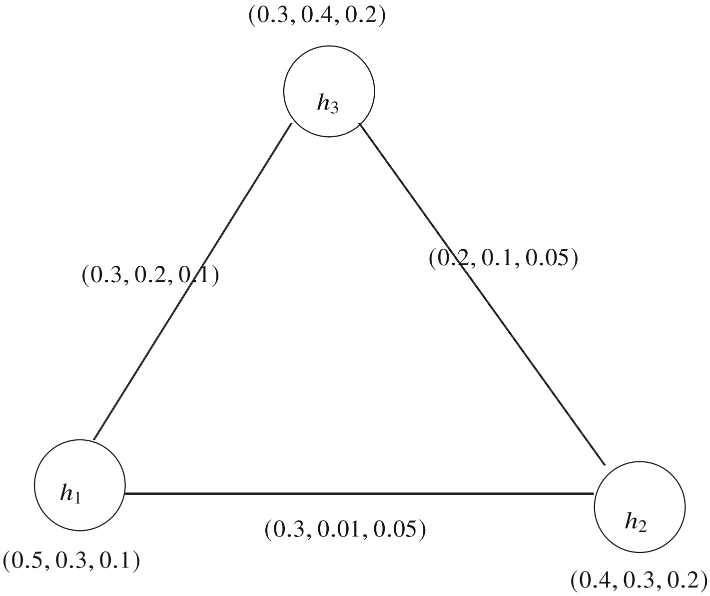

Fig. 12.5 A NG

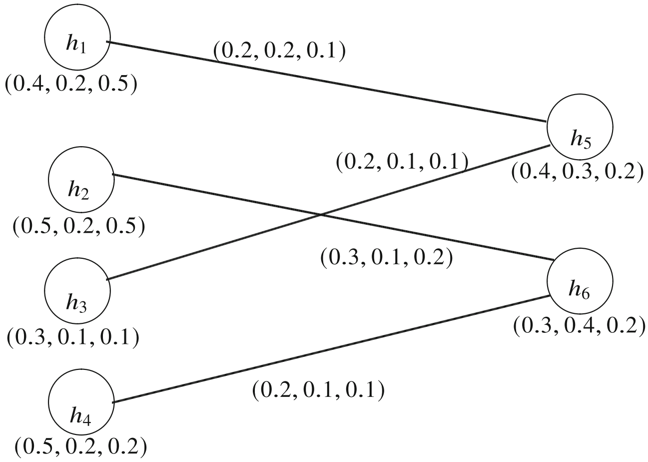

Fig. 12.6 A bipartite NG

Definition 10 A NG $G = (A, B, W_A, W_B)$, where $W_A = (t_A, i_A, f_A)$, $W_B = (t_B, i_B, f_B)$ is called a bipartite NG if the node set A can be decomposed into two non-empty sets h_1 and h_2 so that $t_B(h_i, h_j) = 0$, $i_B(h_i, h_j) = 0$, $f_B(h_i, h_j) = 0$ if $h_i, h_j \in A_1$ or $h_i, h_j \in A_2$.

A bipartite graph is shown in Fig. 12.6.

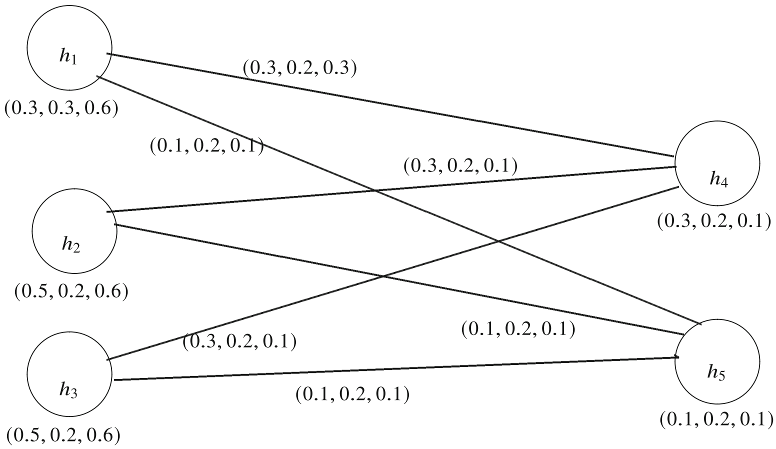

Fig. 12.7 A complete bipartite NG

Definition 11 A bipartite NG $G = (A, B, W_A, W_B)$, where $W_A = (t_A, i_A, f_A)$, $W_B = (t_B, i_B, f_B)$ called a complete bipartite NG if $t_B(h_i, h_j) = t_A(h_i) \wedge t_A(h_j)$, $i_B(h_i, h_j) = i_A(h_i) \wedge i_A(h_j)$, $f_B(h_i, h_j) = f_A(h_i) \vee f_A(h_j)$ for every $h_i \in A_1$ and $h_j \in A_2$.

A complete bipartite graph is shown in Fig. 12.7.

Definition 12 A path in a NG G is a sequence of nodes $h_1, h_2, \ldots, h_n$ which satisfies one of the conditions that are listed below:
(i) $t_B(h_i, h_{i+1}) > 0, i_B(h_i, h_{i+1}) = 0, f_B(h_i, h_{i+1}) = 0$ for all i.
(ii) $t_B(h_i, h_{i+1}) > 0, i_B(h_i, h_{i+1}) = 0, f_B(h_i, h_{i+1}) > 0$ for all i.
(iii) $t_B(h_i, h_{i+1}) > 0, i_B(h_i, h_{i+1}) > 0, f_B(h_i, h_{i+1}) = 0$ for all i.
(i) $t_B(h_i, h_{i+1}) > 0, i_B(h_i, h_{i+1}) > 0, f_B(h_i, h_{i+1}) > 0$ for all i.

Now we will define the concepts of homomorphism and isomorphism of NG, as well as describe some properties of homomorphism and isomorphism of NG.

Definition 13 Let $G_1 = (A_1, B_1, W_{A_1}, W_{B_1})$, $G_2 = (A_2, B_2, W_{A_2}, W_{B_2})$ where $W_{A_i} = (t_{A_i}, i_{A_i}, f_{A_i})$, $W_{B_i} = (t_{B_i}, i_{B_i}, f_{B_i})$ for $i = 1, 2$. A homomorphism between G_1 and G_2 is a mapping $\eta : A_1 \to A_2$ satisfying the following conditions:

(i) $t_{A_1}(h_i) \le t_{A_2}(\eta(h_i))$, $i_{A_1}(h_i) \le i_{A_2}(\eta(h_i))$, $f_{A_1}(h_i) \ge f_{A_2}(\eta(h_i))$ for all $h_i \in A_1$.
(ii) $t_{B_1}(h_i, h_j) \le t_{B_2}(\eta(h_i), \eta(h_j))$, $i_{A_1}(h_i, h_j) \le i_{B_2}(\eta(h_i), \eta(h_j))$, $f_{A_1}(h_i, h_j) \ge f_{B_2}(\eta(h_i), \eta(b_j))$ for all $(h_i, h_j) \in B_1$.

Definition 14 Let $G_1 = (A_1, B_1, W_{A_1}, W_{B_1})$, $G_2 = (A_2, B_2, W_{A_2}, W_{B_2})$ where $W_{A_i} = (t_{A_i}, i_{A_i}, f_{A_i})$, $W_{B_i} = (t_{B_i}, i_{B_i}, f_{B_i})$ for $i = 1, 2$. An isomorphism between G_1 and G_2 is a mapping $\eta : A_1 \to A_2$ having the following requirements:

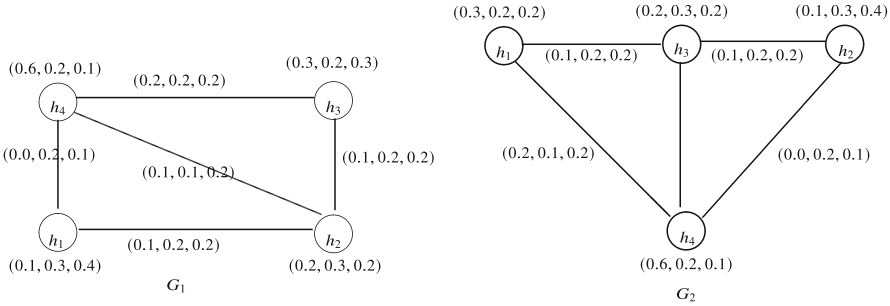

Fig. 12.8 Two isomorphic NG

(i) $t_{A_1}(h_i) = t_{A_2}(\eta(h_i))$, $i_{A_1}(h_i) = i_{A_2}(\eta(h_i))$, $f_{A_1}(h_i) = f_{A_2}(\eta(h_i))$ for all $h_i \in A_1$.

(ii) $t_{B_1}(h_i, h_j) = t_{B_2}(\eta(h_i), \eta(h_j))$, $i_{A_1}(h_i, h_j) = i_{B_2}(\eta(h_i), \eta(h_j))$, $f_{A_1}(h_i, h_j) = f_{B_2}(\eta(h_i), \eta(b_j))$ for all $(h_i, h_j) \in B_1$.

In Fig. 12.8, the NGs G_1 and G_2 are isomorphic.

Definition 15 Let G_1 and G_2 be two NGs. A weak isomorphism $\eta : G_1 \to G_2$ is a bijective mapping $\eta : A_1 \to A_2$ which satisfies the conditions that are outlined below:

(i) η is a homomorphism.

(ii) $t_{A_1}(h_i) = t_{A_2}(\eta(h_i))$, $i_{A_1}(h_i) = i_{A_2}(\eta(h_i))$, $f_{A_1}(h_i) = f_{A_2}(\eta(h_i))$ for all $h_i \in A_1$.

Definition 16 An automorphism of a NG $G = (A, B, W_A, W_B)$, where $W_A = (t_A, i_A, f_A)$, $W_B = (t_B, i_B, f_B)$ is an isomorphism of G onto itself.

4 Balanced Neutrosophic Graph

In this part, the definition and some properties of balanced neutrosophic graph (BNG) have presented. An algorithm to check BNG is also proposed in this section.

Definition 17 $G = (A, B, W_A, W_B)$, where $W_A = (t_A, i_A, f_A)$, $W_B = (t_B, i_B, f_B)$ be a NG. Then the weight of G is defined by $w(G) = (w_t(G), w_i(G), w_f(G))$, where $w_t(G) = \sum_{(h_i,h_j)\in B} t_A(h_i) \wedge t_A(h_j)$ $w_i(G) = \sum_{(h_i,h_j)\in B} i_A(h_i) \wedge i_A(h_j)$ $w_f(G) = \sum_{(h_i,h_j)\in B} f_A(h_i) \vee f_A(h_j)$

Definition 18 Let $G = (A, B, W_A, W_B)$, where $W_A = (t_A, i_A, f_A)$, $W_B = (t_B, i_B, f_B)$ be a NG. Then the density of G is defined by $\rho(G) = (\rho_t(G), \rho_i(G), \rho_f(G))$, where $\rho_t(G) = \frac{S_t(G)}{w_t(G)}$, $\rho_i(G) = \frac{S_i(G)}{w_i(G)}$, $\rho_f(G) = \frac{S_f(G)}{w_f(G)}$ for all $h_i, h_j \in A$. All the components $\rho_t(G)$, $\rho_i(G)$, and $\rho_f(G)$ lie between 0 and 3.

Definition 19 $G = (A, B, W_A, W_B)$, where $W_A = (t_A, i_A, f_A)$, $W_B = (t_B, i_B, f_B)$ be a NG. An intense neutrosophic subgraph S is a subgraph of G where $A(S) \subseteq A(G)$ and $B(S) \subseteq B(G)$ and $\rho(S) \leq \rho(G)$.

Now $\rho(S) \leq \rho(G)$ holds if $\rho_\mu(S) \leq \rho_\mu(G)$, $\rho_\eta(S) \leq \rho_\eta(G)$, $\rho_\nu(S) \leq \rho_\nu(G)$.

Definition 20 $G = (A, B, W_A, W_B)$, where $W_A = (t_A, i_A, f_A)$, $W_B = (t_B, i_B, f_B)$ be a NG. A feeble neutrosophic subgraph S is a subgraph of G where $A(S) \subseteq A(G)$, $B(S) \subseteq B(G)$, and $\rho(S) > \rho(G)$.

Definition 21 A NG $G = (A, B, W_A, W_B)$, where $W_A = (t_A, i_A, f_A)$, $W_B = (t_B, i_B, f_B)$ is called a balanced NG if all subgraphs are intense in G, i.e., $\rho(S) \leq \rho(G)$ for any subgraph S of G. $\rho(S) \leq \rho(G)$ holds if $\rho_t(S) \leq \rho_t(G)$, $\rho_i(S) \leq \rho_i(G)$, and $\rho_f(S) \leq \rho_f(G)$.

Now we consider a NG $G = (A, B, W_A, W_B)$, where $W_A = (t_A, i_A, f_A)$, $M_B = (t_B, i_B, f_B)$ and $A = \{h_1, h_2, h_3, h_4, h_5\}$, $B = \{(h_1, h_2), (h_1, h_3), (h_1, h_4), (h_2, h_3), (h_2, h_5)\}$ in Fig. 12.9 and check to see if it is balanced. We know that the size of G is $S_G = (S_p(G), S_i(G), S_f(G))$. For this NG, $S_p(G) = \sum_{h_i \neq h_j} p_B(h_i, h_j) = 1.2$, $S_i(G) = \sum_{h_i \neq h_j} i_B(h_i, h_j) = 0.95$, $S_f(G) = \sum_{h_i \neq h_j} f_B(h_i, h_j) = 1.95$. Again, the weight of the NG G is $w(G) = (w_p(G), w_i(G), w_f(G))$.

Then $w_p(G) = \sum_{(h_i,h_j)\in B} p_A(h_i) \wedge p_A(h_j) = 1.5$

$w_i(G) = \sum_{(h_i,h_j)\in B} i_A(h_i) \wedge i_A(h_j) = 1.9$

$w_f(G) = \sum_{(h_i,h_j)\in B} f_A(h_i) \wedge f_A(h_j) = 3.0$

So, the density of the NG G is $\rho(G) = (\rho_p(G), \rho_i(G), \rho_f(G))$, where $\rho_p(G) = \frac{S_p(G)}{w_p(G)} = 0.8$, $\rho_i(G) = \frac{S_i(G)}{w_i(G)} = 0.5$, $\rho_i(G) = \frac{S_f(G)}{w_f(G)} = 0.65$. So, $\rho(G) = (0.8, 0.5, 0.65)$. From Table 12.1, we see that the subgroup's density S_i is $(0.80, 0.50, 0.65)$ for $i = 1, 2, \ldots, 19, 21, 23$ and that of S_j is $(0, 0, 0)$ for $j = 20, 22, 24, 25, 26$. Here, all $\rho(S_r) \leq \rho(G)$ for every subgraph S_r of G, shown in Table 12.1.

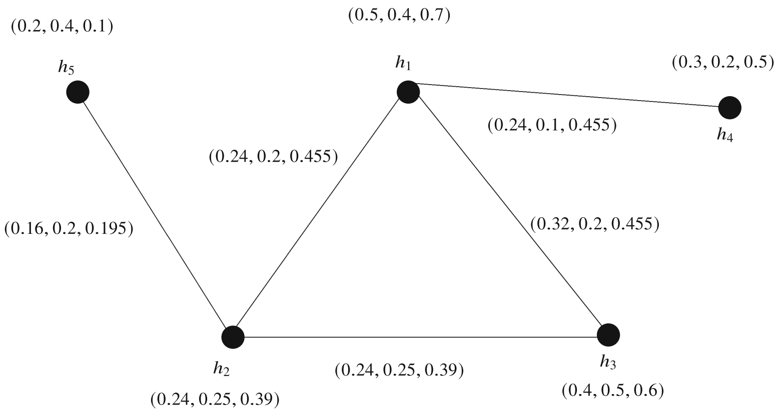

Fig. 12.9 A NG

Table 12.1 Density of all subgraph of the PFG in Fig. 12.9

Subgraph Vertex set	Density	
S_1	$\{h_1, h_2, h_3, h_4, h_5\}$	(0.8, 0.5, 0.65)
S_2	$\{h_1, h_2, h_3, h_4\}$	(0.8, 0.5, 0.65)
S_3	$\{h_1, h_2, h_3, h_5\}$	(0.8, 0.5, 0.65)
S_4	$\{h_1, h_2, h_4, h_5\}$	(0.8, 0.5, 0.65)
S_5	$\{h_1, h_3, h_4, h_5\}$	(0.8, 0.5, 0.65)
S_6	$\{h_2, h_3, h_4, h_5\}$	(0.8, 0.5, 0.65)
S_7	$\{h_1, h_2, h_3\}$	(0.8, 0.5, 0.65)
S_8	$\{h_1, h_2, h_4\}$	(0.8, 0.5, 0.65)
S_9	$\{h_1, h_2, h_5\}$	(0.8, 0.5, 0.65)
S_{10}	$\{h_1, h_3, h_4\}$	(0.8, 0.5, 0.65)
S_{11}	$\{h_1, h_3, h_5\}$	(0.8, 0.5, 0.65)
S_{12}	$\{h_1, h_4, h_5\}$	(0.8, 0.5, 0.65)
S_{13}	$\{h_2, h_3, h_4\}$	(0.8, 0.5, 0.65)
S_{14}	$\{h_2, h_3, h_5\}$	(0.8, 0.5, 0.65)
S_{15}	$\{h_2, h_4, h_5\}$	(0.8, 0.5, 0.65)
S_{16}	$\{h_3, h_4, h_5\}$	(0.8, 0.5, 0.65)
S_{17}	$\{h_1, h_2\}$	(0.8, 0.5, 0.65)
S_{18}	$\{h_1, h_3\}$	(0.8, 0.5, 0.65)
S_{19}	$\{h_1, h_4\}$	(0.8, 0.5, 0.65)
S_{20}	$\{h_1, h_5\}$	(0, 0, 0)
S_{21}	$\{h_2, h_3\}$	(0.8, 0.5, 0.65)
S_{22}	$\{h_2, h_4\}$	(0, 0, 0)
S_{23}	$\{h_2, h_5\}$	(0.8, 0.5, 0.65)
S_{24}	$\{h_3, h_4\}$	(0, 0, 0)
S_{25}	$\{h_3, h_5\}$	(0, 0, 0)
S_{26}	$\{h_4, h_5\}$	(0, 0, 0)

Hence, the NG G is balanced.

Observation 1 *A NG $G = (A, B, W_A, W_B)$, where $W_A = (t_A, i_A, f_A)$, $W_B = (t_B, i_B, f_B)$ is balanced iff $p_B(h_i, h_j) = \min\{p_A(h_i), p_A(h_j)\} \times \mu_1$, $i_B(h_i, h_j) = \min\{i_A(h_i), i_A(h_j)\} \times \mu_2$, $f_B(h_i, h_j) = \max\{f_A(h_i), f_A(h_j)\} \times \mu_3$ for all $(h_i, h_j) \in B$, where $\rho(G) = (\mu_1, \mu_2, \mu_3)$.*

Observation 2 *If NG $G = (A, B, W_A, W_B)$, where $W_A = (t_A, i_A, f_A)$, $W_B = (t_B, i_B, f_B)$ be a balanced NG and S be any subgraph of G then $\rho(S) = \rho(G)$ or $\rho(S) = (0, 0, 0)$.*

4.1 An Algorithm

In this part an algorithm for checking a balanced NG is proposed. Using this algorithm, one can check whether a NG is balanced or not.

Algorithm $JBNS$
Input: A NG G.
Output: G is balanced NG or not balanced NG.

Step 1: Compute,

$$\mu_1 = \frac{\sum_{h_i \neq h_j} t_B(h_i, h_j)}{\sum_{(h_i,h_j)\in B} t_A(h_i) \wedge t_A(h_j)}$$

$$\mu_2 = \frac{\sum_{h_i \neq h_j} i_B(h_i, h_j)}{\sum_{(h_i,h_j)\in B} i_A(h_i) \wedge i_A(h_j)} \text{ and}$$

$$\mu_3 = \frac{\sum_{h_i \neq h_j} f_B(h_i, h_j)}{\sum_{(h_i,h_j)\in B} f_A(h_i) \wedge f_A(h_j)}$$

where the truth, indeterminacy, and falsity membership degree of the vertex h_i are $t_A(h_i), i_A(h_i), and f_A(h_i)$, respectively, and $t_B(h_i, h_j), i_B(h_i, h_j), and f_B(h_i, h_j)$ are, respectively, the truth, indeterminacy, and falsity membership degree of the edge joining the vertices h_i and h_j and μ_1, μ_2, μ_3 are, respectively, the t-density, i-density, and f-density of G, i.e., $\rho(G) = (\mu_1, \mu_2, \mu_3)$.
Step 2: for $i = 1$ to n
for $j = 1$ to n $(i \neq j)$
if $(t_B(h_i, h_j), i_B(h_i, h_j), f_B(h_i, h_j)) = (0, 0, 0)$
or $(t_B(h_i, h_j), i_B(h_i, h_j), f_B(h_i, h_j))$
$= (\mu_1[t_A(h_i) \wedge t_A(h_j)], \mu_2[i_A(h_i) \wedge i_A(h_j)], \mu_3[f_A(h_i) \wedge f_A(h_j))$
then $opt = 10$;
else
$opt = 20$;
Step 3: if $opt = 10$, then G is a balanced NG.
else G is not a balanced NG.
end $JBNG$.

5 Application of Balanced Neutrosophic Graph in Business Alliance

In this part, an application of balanced NG to alliance their business for six IT companies are presented. The main goal of the proposed application is to identify potential business partners who could work together under the conditions outlined below.

Here we take six information technology companies, Wipro (W), HCL Technology ($HCLT$), Infosys (I), Indiamart International (II), Route Mobile (RM), and Tata Consultancy Service (TCS). Any business can collaborate with one or more other businesses. Now we consider six companies as six vertices and draw a NG where an edge connects the alliance business between two organizations. For instance, if Wipro (W) allianced with HCL Technology ($HCLT$), then there is an edge between W and $HCLT$. Vertices and edges' membership functions are taken into consideration as follows.

For vertices:

1. Each company's operational efficiency and financial stability were referred to as a vertex truth membership degree.
2. Each company's market positioning was referred to as an indeterminacy membership degree of the vertex.
3. The poor management strategy of each companies referred to as the falsity membership degree of the vertex.

For Edges:

1. Alliances between two companies that are doing well are called the truth membership degree of every edge.
2. Alliance between two companies with no growth referred as a indeterminacy membership degree of every edge.
3. A partnership between two businesses is to be deemed unsuccessful and to have a false membership degree on each edge.

The degree of memberships of every vertex is shown in Fig. 12.10 and that of for edges are shown in Table 12.2. From Fig. 12.10, we observe that the membership value of the vertices W and TCS are (1.8, 0.3, 0.6) and (1.4, 0.5, 0.6), respectively. Also, from Table 12.2, we see that the membership value of the edge between the vertices W and TCS is $(1.0, 0.12, 0.08)$. We may do the same for other vertices and edges to determine their membership values from Fig. 12.10 and Table 12.2, respectively.

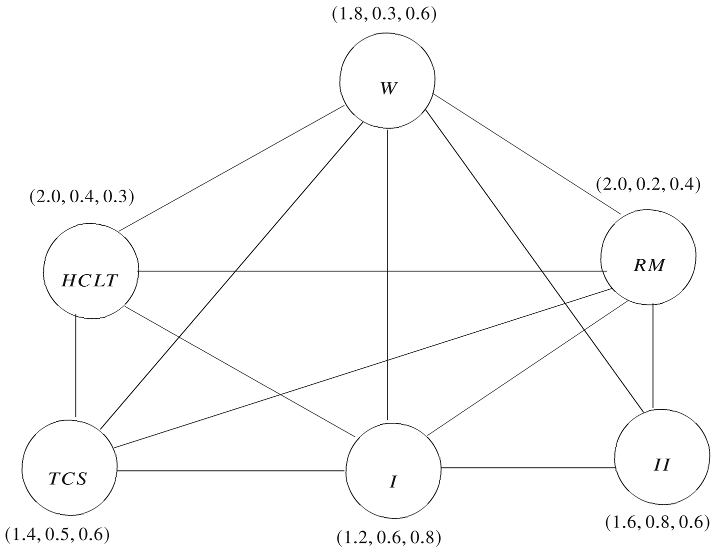

Fig. 12.10 A NG corresponding to six companies

Table 12.2 Membership values of edges

	W	$HCLT$	I
W	–	(1.35, 0.135, 0.12)	(0.9, 0.135, 0.16)
$HCLT$	(1.35, 0.135, 0.12)	–	(0.9, 0.18, 0.16)
I	(0.9, 0.135, 0.16)	(0.9, 0.18, 0.16)	–
II	(1.2, 0.135, 0.12)	–	(0.9, 0.27, 0.16)
RM	(1.35, 0.09, 0.12)	(1.5, 0.0.09, 0.08)	(0.9, 0.09, 0.16)
TCS	(1.0, 0.12, 0.08)	(1.1, 0.18, 0.14)	(0.85, 0.23, 0.14)
	II	RM	TCS
W	(1.2, 0.135, 0.12)	(1.35, 0.09, 0.12)	(1.0, 0.12, 0.14)
$HCLT$	–	(1.5, 0.09, 0.08)	(1.1, 0.18, 0.14)
I	(0.9, 0.27, 0.16)	(0.9, 0.09, 0.16)	(0.85, 0.23, 0.14)
II	–	(1.2, , 0.09, 0.12)	–
RM	(1.2, 0.09, 0.12)	–	(0.95, 0.1, 0.16)
TCS	–	(0.95, 0.1, 0.16)	–

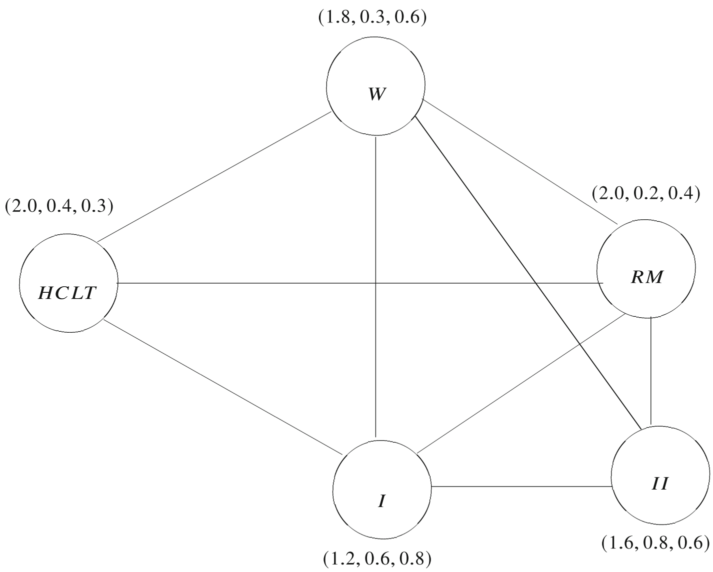

Fig. 12.11 Largest balanced neutrosophic subgraph of the NG in Fig. 12.10

Here, the density, i.e., the business relationship rate of the NG in Fig. 12.10, is (0.75, 0.45, 0.2). From the graph in Fig. 12.11, $S = \{W, HCLT, I, II, IM\}$ is largest neutrosophic subgraph, where the relationship rate of each company is equal for each pair of vertices. Therefore, the subgraph $S = \{W, HCLT, I, II, IM\}$ is balanced (see Fig. 12.11). Hence, these five companies, namely, Wipro, HCL Technology, Infosys, Indiamart International, and Route Mobile, can be allianced properly. As a result, many businesses can use our example to align with the above-described tactics.

6 Conclusion

NG is a needful tool to solve real-life problems, so the study related to NG is clearly welcome. In this study, we introduced a new subgraph of NG, called balanced NG, and using this graph, we present an application of alliances of some Information Technology Companies.

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Chapter 13
Dombi Hamy Mean Operators Based on Complex Intuitionistic Fuzzy Uncertainty and Their Application in Multi-Attribute Decision-Making

Tahir Mahmood and Zeeshan Ali

1 Introduction

MADM technique is one of the fundamental parts of the decision-making tool, and because of its structure, certain people have utilized it in the field of computer networks, road signals, image segmentations, and magical diagnosis. Ambiguity and complexity are part of life, and certain people have affected them because of their involvement. Decision-making information is also one of the superior and valuable techniques used for evaluating this information which contained a lot of problems and ambiguity. But in certain valuable situations, experts have lost a lot of information during decision-making procedures under the consideration of classical information, because of ambiguity and complexity. For evaluating the above queries, the major framework of the fuzzy set (FS) was formulated by Zadeh [1]. FS is one of the big achievements in the environment of mathematical society by modifying the theory of classical information. Because in classical information, we have only two possibilities like "0" or "1." But instead of classical information, we have a lot of possibilities in the presence of FS, and due to this reason, various scholars have proposed and utilized it in the field of decision-making techniques [2–4]. Similarly, FS has contained a lot of problems because FS has skipped the falsity grade which is an important part of our daily life problems. For this, Atanassov [5, 6] pioneered the intuitionistic FS (IFS) which is the modified version of FS. Because FS contained only one grade and the theory of IFS contained the two grades, called truth and falsity information with a strategy: $0 \leq \zeta_{K_R}(c) + \vartheta_{K_R}(c) \leq 1$. But instead of FS information, we have a lot of possibilities in the presence of IFS, and due to this

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C. Jana et al. (eds.), *Fuzzy Optimization, Decision-making and Operations Research*, https://doi.org/10.1007/978-3-031-35668-1_13

reason, various scholars have proposed and utilized it in the field of decision-making techniques [7–12].

No doubt the theory of FS and IFS has a lot of applications in the field of computer science, software engineering, management science, and economics. But some experts have mentioned some limitation of the FS and IFS that the traditional FS and IFS has managed one-dimension information, but in various real-life problems, many experts have dealt with those types of information which will be computed or given in the shape of two-dimension information, for instance, if anyone decided to buy a new car, for this he visited car enterprise for meeting the owner of the car companies. The owner provided information regarding each car in the shape, (i) brand name and (ii) brand price, which represented the amplitude and phase term of the truth grade and theory of FS and IFS has failed. For evaluating the above queries, the major framework of the complex FS (CFS) was formulated by Ramot et al. [13]. CFS is one of the big achievements in the environment of mathematical society by modifying the theory of FS information. Because in FS information, we have only one-dimension information in a one-term set. But instead of FS information, we have two-dimension information in the presence of CFS, and due to this reason, various scholars have proposed and utilized it in the field of decision-making techniques [14, 15]. Similarly, CFS has contained a lot of problems because CFS has skipped the falsity grade which is an important part of our daily life problems. For this, Alkouri and Salleh [16] pioneered the complex IFS (CIFS) which is the modified version of CFS. Because CFS contained only one grade and the theory of CIFS contained the two grades, called truth and falsity information with a strategy: $0 \leq \zeta_{K_R}(c) + \vartheta_{K_R}(c) \leq 1, 0 \leq \zeta_{K_I}(c) + \vartheta_{K_I}(c) \leq 1$. But instead of CFS information, we have a lot of possibilities in the presence of CIFS, and due to this reason, various scholars have proposed and utilized it in the field of decision-making techniques [17–20].

To derive the best preference from the collection of preferences, decision-making information is one of the most important and dominant techniques to evaluate most problems in real-life problems. Additionally, HM information is also used for aggregating the bundled information into a singleton set, computed based on algebraic laws. Under the consideration of the above valuable discussion, we noticed that Li et al. [21] derived the Dombi HM operator for IFS and Wu et al. [22] pioneered the major information of Dombi HM operators based on interval-valued IFSs, where the HM operators [23] and Dombi operational laws [24] are also very effective and dominant compared to prevailing information [25–27]. Keeping the advantages of the information, we aim to evaluate the following scenarios:

1. To derive Dombi operational laws for CIF information.
2. To construct the theory of CIFDHM, CIFWDHM, CIFDDHM, and CIFWD-DHM operators.
3. To evaluate some valuable properties and results for the presented information in the investigated analysis.
4. To illustrate MADM information based on pioneered operators and given some examples to justify the worth and dominancy of the evaluated information.

5. To compare the evaluated information with some other old or prevailing information to enhance the quality of the derived operators.

The summary of this analysis is computed in the shape: in Sect 2. In Sect. 3, we derived the Dombi operational laws for CIF information and also constructed the theory of CIFDHM, CIFWDHM, CIFDDHM, and CIFWDDHM operators. Further, we evaluated some valuable properties and results for the presented information in the investigated analysis. In Sect. 4, MADM "multi-attribute decision-making" information is utilized in this manuscript based on pioneered operators and given some examples to justify the worth and dominancy of the evaluated information. Finally, we compared the evaluated information with some other old or prevailing information to enhance the quality of the derived operators. Concluding and final remarks are part of Sect. 5.

2 Preliminaries

The main influence of this section is to revise the theory of HM operator, Dombi t-norm and t-conorm, CIF information, and their algebraic information.

Definition 1 ([23]) The computed information is described below:

$$HM^{x}(K_1, K_2, \ldots, K_n) = \frac{\sum_{1 \le i_1 \le i_2, \ldots, \le i_x \le n} (\prod_{j=1}^{x} (K_{i_j}))^{1/x}}{C_n^x} \tag{13.1}$$

Represented the Hamy mean information with binomial coefficient $C_{n^x} = n!/x!(n-x)!$.

Definition 2 ([24]) The computed information is described below:

$$T_{D,\Gamma}(Y, Z) = \frac{1}{(1 + ((\frac{(1-Y)}{Y})^{\Gamma} + (\frac{(1-Z)}{Z})^{\Gamma})^{1/\Gamma})} \tag{13.2}$$

$$T^{*}_{D,\Gamma}(Y, Z) = 1 - \frac{1}{(1 + ((\frac{(Y)}{1-Y})^{\Gamma} + (\frac{(Z)}{1-Z})^{\Gamma})^{1/\Gamma})} \tag{13.3}$$

Represented the Dombi T-norm and Dombi T-conorm.

Definition 3 ([16]) The computed information is described below:

$$K = ((\zeta_K(c), \vartheta_K(c)) : c \in C) \tag{13.4}$$

Represented the complex intuitionistic fuzzy (CIF) set, where $\zeta_K(c) = (\zeta_{K_R}(c), \zeta_{K_I}(c)), \vartheta_K(c) = (\vartheta_{K_R}(c), \vartheta_{K_I}(c))$, shows the complex shape of truth and falsity information with $0 \le \zeta_{K_R}(c) + \vartheta_{K_R}(c) \le 1, 0 \le \zeta_{K_I}(c) + \vartheta_{K_I}(c) \le 1$. Further, we revise the refusal information, represented by $\eta_K(c) = (\eta_{K_R}(c), \eta_{K_I}(c)) = (1 - (\zeta_{K_R}(c) + \vartheta_{K_R}(c)), 1 - (\zeta_{K_I}(c) + \vartheta_{K_I}(c)))$, and the final and last shape of the CIF number, represented by $K_j = ((\zeta_{K_{R_j}}, \zeta_{K_{I_j}}), (\vartheta_{K_{R_j}}, \vartheta_{K_{I_j}}))$, $j = 1, 2, .., n$.

Definition 4 ([16]) To evaluate some algebraic information, we use some CIF information $K_j = ((\zeta_{K_{R_j}}, \zeta_{K_{I_j}}), (\vartheta_{K_{R_j}}, \vartheta_{K_{I_j}}))$, $j = 1, 2.$; then we have

$$K_1 \oplus K_2 = ((\zeta_{K_{R_1}} + \zeta_{K_{R_2}} - \zeta_{K_{R_1}} * \zeta_{K_{R_2}}, \zeta_{K_{I_1}} + \zeta_{K_{I_2}} - \zeta_{K_{I_1}} * \zeta_{K_{I_2}}), (\vartheta_{K_{R_1}} * \vartheta_{K_{R_2}}, \vartheta_{K_{I_1}} * \vartheta_{K_{I_2}}))) \tag{13.5}$$

$$K_1 \otimes K_2 = ((\zeta_{K_{R_1}} * \zeta_{K_{R_2}}, \zeta_{K_{I_1}} * \zeta_{K_{I_2}}), (\vartheta_{K_{R_1}} + \vartheta_{K_{R_2}} - \vartheta_{K_{R_1}} * \vartheta_{K_{R_2}}, \vartheta_{K_{I_1}} + \vartheta_{K_{I_2}} - \vartheta_{K_{I_1}} * \vartheta_{K_{I_2}}))) \tag{13.6}$$

$$\Gamma K_j = ((1 - (1 - \zeta_{K_{R_j}})^{\Gamma}, 1 - (1 - \zeta_{K_{I_j}})^{\Gamma}), (\vartheta_{K_{R_j}}^{\Gamma}, \vartheta_{K_{I_j}}^{\Gamma})) \tag{13.7}$$

$$K_j^{\Gamma} = ((\zeta_{K_{R_j}}^{\Gamma}, \zeta_{K_{I_j}}^{\Gamma}), (1 - (1 - \vartheta_{K_{R_j}})^{\Gamma}, 1 - (1 - \vartheta_{K_{I_j}})^{\Gamma})) \tag{13.8}$$

Definition 5 ([16]) To evaluate the theory of score and accuracy information, we use some CIF information $K_j = ((\zeta_{K_{R_j}}, \zeta_{K_{I_j}}), (\vartheta_{K_{R_j}}, \vartheta_{K_{I_j}}))$, $j = 1, 2.$; then we have

$$\nabla_{SV}(K_j) = \frac{1}{2} * (\zeta_{K_{R_j}} + \zeta_{K_{I_j}} - \vartheta_{K_{R_j}} - \vartheta_{K_{R_j}}) \in [-1, 1] \tag{13.9}$$

$$\nabla_{AV}(K_j) = \frac{1}{2} * (\zeta_{K_{R_j}} + \zeta_{K_{I_j}} + \vartheta_{K_{R_j}} + \vartheta_{K_{R_j}}) \in [0, 1] \tag{13.10}$$

To accommodate the relation between any two CIF information, we compute some properties, such that

1. Obtained $\nabla_{SV}(K_1)$ less then $\nabla_{SV}(K_2)$, when K_1 less then K_2;
2. Obtained $\nabla_{SV}(K_1)$ greater then $\nabla_{SV}(K_2)$, when K_1 greater then K_2;
3. Obtained $\nabla_{SV}(K_1) = \nabla_{SV}(K_2)$, then

i. Obtained $\nabla_{AV}(K_1)$ less then $\nabla_{AV}(K_2)$, when K_1 less then K_2;
ii. Obtained $\nabla_{AV}(K_1)$ greater then $\nabla_{AV}(K_2)$, when K_1 greater then K_2;

3 Dombi Hamy Mean Operators for CIF Information

The main influence of this section is to derive the theory of CIFDHM, CIFWDHM, CIFDDHM, and CIFWDDHM operators and evaluate some valuable properties and results.

Definition 6 To evaluate some Dombi information, we use some CIF information $K_j = ((\zeta_{K_{R_j}}, \zeta_{K_{I_j}}), (\vartheta_{K_{R_j}}, \vartheta_{K_{I_j}}))$, $j = 1, 2.$; then we have

$$K_1 \oplus K_2 = ((1 - \frac{1}{(1 + ((\frac{(\zeta_{K_{R_1}})}{1-\zeta_{K_{R_1}}})^{\Gamma} + (\frac{(\zeta_{K_{R_2}})}{1-\zeta_{K_{R_2}}})^{\Gamma})^{1/\Gamma})},$$
$$1 - \frac{1}{(1 + ((\frac{(\zeta_{K_{I_1}})}{1-\zeta_{K_{I_1}}})^{\Gamma} + (\frac{(\zeta_{K_{I_2}})}{1-\zeta_{K_{I_2}}})^{\Gamma})^{1/\Gamma})}),$$
$$(\frac{1}{(1 + ((\frac{(1-\vartheta_{K_{R_1}})}{\vartheta_{K_{R_1}}})^{\Gamma} + (\frac{(1-\vartheta_{K_{R_2}})}{\vartheta_{K_{R_2}}})^{\Gamma})^{1/\Gamma})}, \frac{1}{(1 + ((\frac{(1-\vartheta_{K_{I_1}})}{\vartheta_{K_{I_1}}})^{\Gamma} + (\frac{(1-\vartheta_{K_{I_2}})}{\vartheta_{K_{I_2}}})^{\Gamma})^{1/\Gamma})})) \tag{13.11}$$

$$K_1 \otimes K_2 = ((\frac{1}{(1 + ((\frac{(1-\zeta_{K_{R_1}})}{\zeta_{K_{R_1}}})^{\Gamma} + (\frac{(1-\zeta_{K_{R_2}})}{\zeta_{K_{R_2}}})^{\Gamma})^{1/\Gamma})},$$
$$\frac{1}{(1 + ((\frac{(1-\zeta_{K_{I_1}})}{\zeta_{K_{I_1}}})^{\Gamma} + (\frac{(1-\zeta_{K_{I_2}})}{\zeta_{K_{I_2}}})^{\Gamma})^{1/\Gamma})}),$$
$$(1 - \frac{1}{(1 + ((\frac{(\vartheta_{K_{R_1}})}{1-\vartheta_{K_{R_1}}})^{\Gamma} + (\frac{(\vartheta_{K_{R_2}})}{1-\vartheta_{K_{R_2}}})^{\Gamma})^{1/\Gamma})},$$
$$1 - \frac{1}{(1 + ((\frac{(\vartheta_{K_{I_1}})}{1-\vartheta_{K_{I_1}}})^{\Gamma} + (\frac{(\vartheta_{K_{I_2}})}{1-\vartheta_{K_{I_2}}})^{\Gamma})^{1/\Gamma})})) \tag{13.12}$$

$$\Gamma K_1 = ((1 - \frac{1}{(1 + (\Gamma(\frac{(\zeta_{K_{R_1}})}{1-\zeta_{K_{R_1}}})^{\Gamma})^{1/\Gamma})}, 1 - \frac{1}{(1 + (\Gamma(\frac{(\zeta_{K_{I_1}})}{1-\zeta_{K_{I_1}}})^{\Gamma})^{1/\Gamma})}),$$
$$(\frac{1}{(1 + (\Gamma(\frac{(1-\vartheta_{K_{R_1}})}{\vartheta_{K_{R_1}}})^{\Gamma})^{1/\Gamma})}, \frac{1}{(1 + (\Gamma(\frac{(1-\vartheta_{K_{I_1}})}{\vartheta_{K_{I_1}}})^{\Gamma})^{1/\Gamma})})) \tag{13.13}$$

$$K_1^{\Gamma} = ((\frac{1}{(1+(\Gamma(\frac{(1-\zeta_{K_{R_1}})}{\zeta_{K_{R_1}}})^{\Gamma})^{1/\Gamma})}, \frac{1}{(1+(\Gamma(\frac{(1-\zeta_{K_{I_1}})}{\zeta_{K_{I_1}}})^{\Gamma})^{1/\Gamma})}),$$
$$(1-\frac{1}{(1+(\Gamma(\frac{(\vartheta_{K_{R_1}})}{1-\vartheta_{K_{R_1}}})^{\Gamma})^{1/\Gamma})}, 1-\frac{1}{(1+(\Gamma(\frac{(\vartheta_{K_{I_1}})}{1-\vartheta_{K_{I_1}}})^{\Gamma})}))\tag{13.14}$$

Definition 7 The computed information is described below:

$$CIFDHM^x(K_1, K_2, \ldots, K_n) = \frac{\bigoplus_{1\le i_1\le i_2,\ldots,\le i_x\le n}(\otimes_{j=1}^{x}(K_{i_j}))^{1/x}}{C_n^x}\tag{13.15}$$

Represented the CIFDHM information with binomial coefficient $C_{n^x} = n!/x!(n-x)!$.

Theorem 1 *To evaluate the information in Eq. (13.15) with the help of Dombi operational laws, we prove that the resultant information of Eq. (13.15) is again CIF information, such that*

$$CIFDHM^x(K_1, K_2, \ldots, K_n)$$
$$= ((1-\frac{1}{1+((\frac{x}{C_n^x})*(\sum_{1\le i_1\le i_2\le,\ldots,\le i_x\le n}(\frac{1}{\sum_{j=1}^{x}(\frac{1-\zeta_{K_{R_{i_j}}}}{\zeta_{K_{R_{i_j}}}})^{\Gamma}})))^{(1/\Gamma)}},$$
$$1-\frac{1}{1+((\frac{x}{C_n^x})*(\sum_{1\le i_1\le i_2\le,\ldots,\le i_x\le n}(\frac{1}{\sum_{j=1}^{x}(\frac{1-\zeta_{K_{I_{i_j}}}}{\zeta_{K_{I_{i_j}}}})^{\Gamma}})))^{(1/\Gamma)}}),$$
$$(\frac{1}{1+((\frac{x}{C_n^x})*(\sum_{1\le i_1\le i_2\le,\ldots,\le i_x\le n}(\frac{1}{\sum_{j=1}^{x}(\frac{\vartheta_{K_{R_{i_j}}}}{1-\vartheta_{K_{R_{i_j}}}})^{\Gamma}})))^{(1/\Gamma)}},$$
$$\frac{1}{1+((\frac{x}{C_n^x})*(\sum_{1\le i_1\le i_2\le,\ldots,\le i_x\le n}(\frac{1}{\sum_{j=1}^{x}(\frac{\vartheta_{K_{I_{i_j}}}}{1-\vartheta_{K_{I_{i_j}}}})^{\Gamma}})))^{(1/\Gamma)}}))\tag{13.16}$$

Proof Information is given in Def. (13.6) and Def. (13.7); we have

$$\otimes_{j=1}^{x} K_{i_j} = ((\frac{1}{1+(\sum_{j=1}^{x}(\frac{1-\zeta_{K_{R_{i_j}}}}{\zeta_{K_{R_{i_j}}}})^{\Gamma})^{(1/\Gamma)}}, \frac{1}{1+(\sum_{j=1}^{x}(\frac{1-\zeta_{K_{I_{i_j}}}}{\zeta_{K_{I_{i_j}}}})^{\Gamma})^{(1/\Gamma)}}),$$

$$(1-\frac{1}{1+(\sum_{j=1}^{x}(\frac{\vartheta_{K_{R_{i_j}}}}{1-\vartheta_{K_{R_{i_j}}}})^{\Gamma})^{(1/\Gamma)}},1-\frac{1}{1+(\sum_{j=1}^{x}(\frac{\vartheta_{K_{I_{i_j}}}}{1-\vartheta_{K_{I_{i_j}}}})^{\Gamma})^{(1/\Gamma)}}))$$

$$(\otimes_{j=1}^{x}K_{i_j})^{\frac{1}{x}}=((\frac{1}{1+((\frac{1}{x})*(\sum_{j=1}^{x}(\frac{1-\zeta_{K_{R_{i_j}}}}{\zeta_{K_{R_{i_j}}}})^{\Gamma}))^{(1/\Gamma)}},$$

$$\frac{1}{1+(\frac{1}{x})*\sum_{j=1}^{x}(\frac{1-\zeta_{K_{I_{i_j}}}}{\zeta_{K_{I_{i_j}}}})^{\Gamma})^{(1/\Gamma)}}),$$

$$(1-\frac{1}{1+(\frac{1}{x})*\sum_{j=1}^{x}(\frac{\vartheta_{K_{R_{i_j}}}}{1-\vartheta_{K_{R_{i_j}}}})^{\Gamma})^{(1/\Gamma)}},1-\frac{1}{1+(\frac{1}{x})*\sum_{j=1}^{x}(\frac{\vartheta_{K_{I_{i_j}}}}{1-\vartheta_{K_{I_{i_j}}}})^{\Gamma})^{(1/\Gamma)}}))$$

$$\oplus_{1\le i_1\le i_2,\ldots,\le i_x\le n}(\otimes_{j=1}^{x}(K_{i_j}))^{1/x}$$

$$=((1-\frac{1}{1+((\sum_{1\le i_1\le i_2\le,\ldots,\le i_x\le n}(\frac{x}{\sum_{j=1}^{x}(\frac{1-\zeta_{K_{R_{i_j}}}}{\zeta_{K_{R_{i_j}}}})^{\Gamma}})))^{(1/\Gamma)}},$$

$$1-\frac{1}{1+((\sum_{1\le i_1\le i_2\le,\ldots,\le i_x\le n}(\frac{x}{\sum_{j=1}^{x}(\frac{1-\zeta_{K_{I_{i_j}}}}{\zeta_{K_{I_{i_j}}}})^{\Gamma}})))^{(1/\Gamma)}}),$$

$$(\frac{1}{1+((\sum_{1\le i_1\le i_2\le,\ldots,\le i_x\le n}(\frac{x}{\sum_{j=1}^{x}(\frac{\vartheta_{K_{R_{i_j}}}}{1-\vartheta_{K_{R_{i_j}}}})^{\Gamma}})))^{(1/\Gamma)}},$$

$$\frac{1}{1+((\sum_{1\le i_1\le i_2\le,\ldots,\le i_x\le n}(\frac{x}{\sum_{j=1}^{x}(\frac{\vartheta_{K_{I_{i_j}}}}{1-\vartheta_{K_{I_{i_j}}}})^{\Gamma}})))^{(1/\Gamma)}}))$$

$$\frac{\oplus_{1\le i_1\le i_2,\ldots,\le i_x\le n}(\otimes_{j=1}^{x}(K_{i_j}))^{1/x}}{C_n^x}$$

$$= ((1 - \frac{1}{1 + ((\frac{x}{C_n^x}) * (\sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n} (\frac{1}{\sum_{j=1}^{x} (\frac{1-\zeta_{K_{R_{i_j}}}}{\zeta_{K_{R_{i_j}}}})^{\Gamma}})))^{(1/\Gamma)}},$$

$$1 - \frac{1}{1 + ((\frac{x}{C_n^x}) * (\sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n} (\frac{1}{\sum_{j=1}^{x} (\frac{1-\zeta_{K_{I_{i_j}}}}{\zeta_{K_{I_{i_j}}}})^{\Gamma}})))^{(1/\Gamma)}}),$$

$$(\frac{1}{1 + ((\frac{x}{C_n^x}) * (\sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n} (\frac{1}{\sum_{j=1}^{x} (\frac{\vartheta_{K_{R_{i_j}}}}{1-\vartheta_{K_{R_{i_j}}}})^{\Gamma}})))^{(1/\Gamma)}},$$

$$\frac{1}{1 + ((\frac{x}{C_n^x}) * (\sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n} (\frac{1}{\sum_{j=1}^{x} (\frac{\vartheta_{K_{I_{i_j}}}}{1-\vartheta_{K_{I_{i_j}}}})^{\Gamma}})))^{(1/\Gamma)}}))$$

$$CIFDHM^x(K_1, K_2, \ldots, K_n) = \frac{\bigoplus_{1 \le i_1 \le i_2, \ldots, \le i_x \le n} (\bigotimes_{j=1}^{x} (K_{i_j}))^{1/x}}{C_n^x} =$$

$$((1 - \frac{1}{1 + ((\frac{x}{C_n^x}) * (\sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n} (\frac{1}{\sum_{j=1}^{x} (\frac{1-\zeta_{K_{R_{i_j}}}}{\zeta_{K_{R_{i_j}}}})^{\Gamma}})))^{(1/\Gamma)}},$$

$$1 - \frac{1}{1 + ((\frac{x}{C_n^x}) * (\sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n} (\frac{1}{\sum_{j=1}^{x} (\frac{1-\zeta_{K_{I_{i_j}}}}{\zeta_{K_{I_{i_j}}}})^{\Gamma}})))^{(1/\Gamma)}}),$$

$$(\frac{1}{1 + ((\frac{x}{C_n^x}) * (\sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n} (\frac{1}{\sum_{j=1}^{x} (\frac{\vartheta_{K_{R_{i_j}}}}{1-\vartheta_{K_{R_{i_j}}}})^{\Gamma}})))^{(1/\Gamma)}},$$

$$\frac{1}{1 + ((\frac{x}{C_n^x}) * (\sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n} (\frac{1}{\sum_{j=1}^{x} (\frac{\vartheta_{K_{I_{i_j}}}}{1-\vartheta_{K_{I_{i_j}}}})^{\Gamma}})))^{(1/\Gamma)}}))$$

Proposition 1 *When* $K_j = K = ((\zeta_{K_R}, \zeta_{K_I}), (\vartheta_{K_R}, \vartheta_{K_I}))$, $j = 1, 2, \ldots, n$, *then*

$$CIFDHM^x(K_1, K_2, \ldots, K_n) = K \tag{13.17}$$

Proof Assumed that $K_j = K = ((\zeta_{K_R}, \zeta_{K_I}), (\vartheta_{K_R}, \vartheta_{K_I})), j = 1, 2, \ldots, n$, then by using Eq. (16), we have

$$CIFDHM^x(K_1, K_2, \ldots, K_n)$$

$$= ((1 - \frac{1}{1 + ((\frac{x}{C_n^x}) * (\sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n}(\frac{1}{\sum_{j=1}^{x}(\frac{1-\zeta_{K_{R_{i_j}}}}{\zeta_{K_{R_{i_j}}}})^{\Gamma}})))^{(1/\Gamma)}},$$

$$1 - \frac{1}{1 + ((\frac{x}{C_n^x}) * (\sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n}(\frac{1}{\sum_{j=1}^{x}(\frac{1-\zeta_{K_{I_{i_j}}}}{\zeta_{K_{I_{i_j}}}})^{\Gamma}})))^{(1/\Gamma)}}),$$

$$(\frac{1}{1 + ((\frac{x}{C_n^x}) * (\sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n}(\frac{1}{\sum_{j=1}^{x}(\frac{\vartheta_{K_{R_{i_j}}}}{1-\vartheta_{K_{R_{i_j}}}})^{\Gamma}})))^{(1/\Gamma)}},$$

$$\frac{1}{1 + ((\frac{x}{C_n^x}) * (\sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n}(\frac{1}{\sum_{j=1}^{x}(\frac{\vartheta_{K_{I_{i_j}}}}{1-\vartheta_{K_{I_{i_j}}}})^{\Gamma}})))^{(1/\Gamma)}}))$$

$$= ((1 - \frac{1}{1 + ((\frac{x}{C_n^x}) * (\sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n}(\frac{1}{\sum_{j=1}^{x}(\frac{1-\zeta_{K_R}}{\zeta_{K_R}})^{\Gamma}})))^{(1/\Gamma)}},$$

$$1 - \frac{1}{1 + ((\frac{x}{C_n^x}) * (\sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n}(\frac{1}{\sum_{j=1}^{x}(\frac{1-\zeta_{K_I}}{\zeta_{K_I}})^{\Gamma}})))^{(1/\Gamma)}}),$$

$$(\frac{1}{1 + ((\frac{x}{C_n^x}) * (\sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n}(\frac{1}{\sum_{j=1}^{x}(\frac{\vartheta_{K_R}}{1-\vartheta_{K_R}})^{\Gamma}})))^{(1/\Gamma)}},$$

$$\frac{1}{1 + ((\frac{x}{C_n^x}) * (\sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n}(\frac{1}{\sum_{j=1}^{x}(\frac{\vartheta_{K_I}}{1-\vartheta_{K_I}})^{\Gamma}})))^{(1/\Gamma)}}))$$

$$((1 - \frac{1}{1 + (((\frac{1}{(\frac{1-\zeta_{K_R}}{\zeta_{K_R}})^{\Gamma}})))^{(1/\Gamma)}},$$

$$1-\frac{1}{1+(((\frac{1}{(\frac{1-\zeta_{K_I}}{\zeta_{K_I}})^{\Gamma}})))^{(1/\Gamma)}}),$$

$$(\frac{1}{1+(((\frac{1}{(\frac{\vartheta_{K_R}}{1-\vartheta_{K_R}})^{\Gamma}})))^{(1/\Gamma)}},$$

$$\frac{1}{1+(((\frac{1}{(\frac{\vartheta_{K_I}}{1-\vartheta_{K_I}})^{\Gamma}})))^{(1/\Gamma)}}))$$

$$=((\zeta_{K_R},\zeta_{K_I}),(\vartheta_{K_R},\vartheta_{K_I}))=K$$

Proposition 2 *When* $K_j=((\zeta_{K_{R_j}},\zeta_{K_{I_j}}),(\vartheta_{K_{R_j}},\vartheta_{K_{I_j}}))\leq K_j^*=((\zeta^*_{K_{R_j}},\zeta^*_{K_{I_j}}),(\vartheta^*_{K_{R_j}},\vartheta^*_{K_{I_j}}))$, $j=1,2,\ldots,n$, *then*

$$CIFDHM^x(K_1,K_2,\ldots,K_n)\leq CIFDHM^x(K_1^*,K_2^*,\ldots,K_n^*) \tag{13.18}$$

Proof Considered the theory $K_j\leq K_j^*$, then it means that $\zeta_{K_{R_j}}\leq\zeta^*_{K_{R_j}},\zeta_{K_{I_j}}\leq\zeta^*_{K_{I_j}}$ and $\vartheta_{K_{R_j}}\geq\vartheta^*_{K_{R_j}},\vartheta_{K_{I_j}}\geq\vartheta^*_{K_{I_j}}$, then we have

$$\zeta_{K_{R_j}}\leq\zeta^*_{K_{R_j}}\implies 1-\zeta_{K_{R_j}}\geq 1-\zeta^*_{K_{R_j}}\implies\frac{1-\zeta_{K_{R_j}}}{\zeta_{K_{R_j}}}\geq\frac{1-\zeta^*_{K_{R_j}}}{\zeta^*_{K_{R_j}}}$$

$$\implies(\frac{1-\zeta_{K_{R_j}}}{\zeta_{K_{R_j}}})^{\Gamma}\geq(\frac{1-\zeta^*_{K_{R_j}}}{\zeta^*_{K_{R_j}}})^{\Gamma}$$

$$\implies\sum_{j=1}^{x}(\frac{1-\zeta_{K_{R_j}}}{\zeta_{K_{R_j}}})^{\Gamma}\geq\sum_{j=1}^{x}(\frac{1-\zeta^*_{K_{R_j}}}{\zeta^*_{K_{R_j}}})^{\Gamma}$$

$$\implies\frac{1}{\sum_{j=1}^{x}(\frac{1-\zeta_{K_{R_j}}}{\zeta_{K_{R_j}}})^{\Gamma}}\geq\frac{1}{\sum_{j=1}^{x}(\frac{1-\zeta^*_{K_{R_j}}}{\zeta^*_{K_{R_j}}})^{\Gamma}}$$

$$\implies\sum_{1\leq i_1\leq i_2\leq,\ldots,\leq i_x\leq n}\frac{1}{\sum_{j=1}^{x}(\frac{1-\zeta_{K_{R_j}}}{\zeta_{K_{R_j}}})^{\Gamma}}\geq$$

$$\sum_{1\leq i_1\leq i_2\leq,\ldots,\leq i_x\leq n} \frac{1}{\sum_{j=1}^{x}(\frac{1-\zeta^*_{K_{R_j}}}{\zeta^*_{K_{R_j}}})^{\Gamma}}$$

$$\implies \frac{x}{C_n^x} * \sum_{1\leq i_1\leq i_2\leq,\ldots,\leq i_x\leq n} \frac{1}{\sum_{j=1}^{x}(\frac{1-\zeta_{K_{R_j}}}{\zeta_{K_{R_j}}})^{\Gamma}}$$

$$\geq \frac{x}{C_n^x} * \sum_{1\leq i_1\leq i_2\leq,\ldots,\leq i_x\leq n} \frac{1}{\sum_{j=1}^{x}(\frac{1-\zeta^*_{K_{R_j}}}{\zeta^*_{K_{R_j}}})^{\Gamma}}$$

$$\implies (\frac{x}{C_n^x} * \sum_{1\leq i_1\leq i_2\leq,\ldots,\leq i_x\leq n} \frac{1}{\sum_{j=1}^{x}(\frac{1-\zeta_{K_{R_j}}}{\zeta_{K_{R_j}}})^{\Gamma}})^{\frac{1}{\Gamma}}$$

$$\geq (\frac{x}{C_n^x} * \sum_{1\leq i_1\leq i_2\leq,\ldots,\leq i_x\leq n} \frac{1}{\sum_{j=1}^{x}(\frac{1-\zeta^*_{K_{R_j}}}{\zeta^*_{K_{R_j}}})^{\Gamma}})^{\frac{1}{\Gamma}}$$

$$\implies 1 + (\frac{x}{C_n^x} * \sum_{1\leq i_1\leq i_2\leq,\ldots,\leq i_x\leq n} \frac{1}{\sum_{j=1}^{x}(\frac{1-\zeta_{K_{R_j}}}{\zeta_{K_{R_j}}})^{\Gamma}})^{\frac{1}{\Gamma}}$$

$$\geq 1 + (\frac{x}{C_n^x} * \sum_{1\leq i_1\leq i_2\leq,\ldots,\leq i_x\leq n} \frac{1}{\sum_{j=1}^{x}(\frac{1-\zeta^*_{K_{R_j}}}{\zeta^*_{K_{R_j}}})^{\Gamma}})^{\frac{1}{\Gamma}}$$

$$\implies \frac{1}{1 + (\frac{x}{C_n^x} * \sum_{1\leq i_1\leq i_2\leq,\ldots,\leq i_x\leq n} \frac{1}{\sum_{j=1}^{x}(\frac{1-\zeta_{K_{R_j}}}{\zeta_{K_{R_j}}})^{\Gamma}})^{\frac{1}{\Gamma}}}$$

$$\geq \frac{1}{1 + (\frac{x}{C_n^x} * \sum_{1\leq i_1\leq i_2\leq,\ldots,\leq i_x\leq n} \frac{1}{\sum_{j=1}^{x}(\frac{1-\zeta^*_{K_{R_j}}}{\zeta^*_{K_{R_j}}})^{\Gamma}})^{\frac{1}{\Gamma}}}$$

$$\implies 1 - \frac{1}{1 + (\frac{x}{C_n^x} * \sum_{1\leq i_1\leq i_2\leq,\ldots,\leq i_x\leq n} \frac{1}{\sum_{j=1}^{x}(\frac{1-\zeta_{K_{R_j}}}{\zeta_{K_{R_j}}})^{\Gamma}})^{\frac{1}{\Gamma}}}$$

$$\leq 1 - \frac{1}{1 + (\frac{x}{C_n^x} * \sum_{1\leq i_1\leq i_2\leq,\ldots,\leq i_x\leq n} \frac{1}{\sum_{j=1}^{x}(\frac{1-\zeta^*_{K_{R_j}}}{\zeta^*_{K_{R_j}}})^{\Gamma}})^{\frac{1}{\Gamma}}}$$

$$\implies 1 - \frac{1}{1 + (\frac{x}{C_n^x} * \sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n} \frac{1}{\sum_{j=1}^{x} (\frac{1-\zeta_{K_{I_j}}}{\zeta_{K_{I_j}}})^{\Gamma}})^{\frac{1}{\Gamma}}}$$

$$\le 1 - \frac{1}{1 + (\frac{x}{C_n^x} * \sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n} \frac{1}{\sum_{j=1}^{x} (\frac{1-\zeta^*_{K_{I_j}}}{\zeta^*_{K_{I_j}}})^{\Gamma}})^{\frac{1}{\Gamma}}}$$

Further,

$$\vartheta_{K_{R_j}} \ge \vartheta^*_{K_{R_j}} \implies \frac{\vartheta_{K_{R_j}}}{1 - \vartheta_{K_{R_j}}} \ge \frac{\vartheta^*_{K_{R_j}}}{1 - \vartheta^*_{K_{R_j}}}$$

$$\implies \sum_{j=1}^{x} (\frac{\vartheta_{K_{R_j}}}{1 - \vartheta_{K_{R_j}}})^{\Gamma} \ge \sum_{j=1}^{x} (\frac{\vartheta^*_{K_{R_j}}}{1 - \vartheta^*_{K_{R_j}}})^{\Gamma}$$

$$\implies \frac{1}{\sum_{j=1}^{x} (\frac{\vartheta_{K_{R_j}}}{1-\vartheta_{K_{R_j}}})^{\Gamma}} \ge \frac{1}{\sum_{j=1}^{x} (\frac{\vartheta^*_{K_{R_j}}}{1-\vartheta^*_{K_{R_j}}})^{\Gamma}}$$

$$\implies \sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n} \frac{1}{\sum_{j=1}^{x} (\frac{\vartheta_{K_{R_j}}}{1-\vartheta_{K_{R_j}}})^{\Gamma}}$$

$$\ge \sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n} \frac{1}{\sum_{j=1}^{x} (\frac{\vartheta^*_{K_{R_j}}}{1-\vartheta^*_{K_{R_j}}})^{\Gamma}}$$

$$\implies (\frac{x}{C_n^x}) * \sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n} \frac{1}{\sum_{j=1}^{x} (\frac{\vartheta_{K_{R_j}}}{1-\vartheta_{K_{R_j}}})^{\Gamma}}$$

$$\ge (\frac{x}{C_n^x}) * \sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n} \frac{1}{\sum_{j=1}^{x} (\frac{\vartheta^*_{K_{R_j}}}{1-\vartheta^*_{K_{R_j}}})^{\Gamma}}$$

$$\implies ((\frac{x}{C_n^x}) * \sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n} \frac{1}{\sum_{j=1}^{x} (\frac{\vartheta_{K_{R_j}}}{1-\vartheta_{K_{R_j}}})^{\Gamma}})^{1/\Gamma}$$

$$\ge ((\frac{x}{C_n^x}) * \sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n} \frac{1}{\sum_{j=1}^{x} (\frac{\vartheta^*_{K_{R_j}}}{1-\vartheta^*_{K_{R_j}}})^{\Gamma}})^{1/\Gamma}$$

$$\implies 1 + ((\frac{x}{C_n^x}) * \sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n} \frac{1}{\sum_{j=1}^{x} (\frac{\vartheta_{K_{R_j}}}{1-\vartheta_{K_{R_j}}})^{\Gamma}})^{1/\Gamma}$$

$$\ge 1 + ((\frac{x}{C_n^x}) * \sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n} \frac{1}{\sum_{j=1}^{x} (\frac{\vartheta^*_{K_{R_j}}}{1-\vartheta^*_{K_{R_j}}})^{\Gamma}})^{1/\Gamma}$$

$$\implies \frac{1}{1 + ((\frac{x}{C_n^x}) * \sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n} \frac{1}{\sum_{j=1}^{x} (\frac{\vartheta_{K_{R_j}}}{1-\vartheta_{K_{R_j}}})^{\Gamma}})^{1/\Gamma}}$$

$$\ge \frac{1}{1 + ((\frac{x}{C_n^x}) * \sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n} \frac{1}{\sum_{j=1}^{x} (\frac{\vartheta^*_{K_{R_j}}}{1-\vartheta^*_{K_{R_j}}})^{\Gamma}})^{1/\Gamma}}$$

$$\implies \frac{1}{1 + ((\frac{x}{C_n^x}) * \sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n} \frac{1}{\sum_{j=1}^{x} (\frac{\vartheta_{K_{I_j}}}{1-\vartheta_{K_{I_j}}})^{\Gamma}})^{1/\Gamma}}$$

$$\ge \frac{1}{1 + ((\frac{x}{C_n^x}) * \sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n} \frac{1}{\sum_{j=1}^{x} (\frac{\vartheta^*_{K_{I_j}}}{1-\vartheta^*_{K_{I_j}}})^{\Gamma}})^{1/\Gamma}}$$

Then by using the information in Eqs. (13.9) and (13.10), we can easily obtain the required information, such

$$CIFDHM^x(K_1, K_2, \ldots, K_n) \le CIFDHM^x(K_1^*, K_2^*, \ldots, K_n^*).$$

Proposition 3 *When* $K_j^- = ((min_j \zeta_{K_{R_j}}, min_j \zeta_{K_{I_j}}), (max_j \vartheta_{K_{R_j}}, max_j \vartheta_{K_{I_j}}))$, $K_j^+ = ((max_j \zeta_{K_{R_j}}, max_j \zeta_{K_{I_j}}), (min_j \vartheta_{K_{R_j}}, min_j \vartheta_{K_{I_j}}))$, $j = 1, 2, \ldots, n$, *then*

$$K_j^- \le CIFDHM^x(K_1, K_2, \ldots, K_n) \le K_j^+ \tag{13.19}$$

Proof Assumed that $K_j^- = ((min_j \zeta_{K_{R_j}}, min_j \zeta_{K_{I_j}}), (max_j \vartheta_{K_{R_j}}, max_j \vartheta_{K_{I_j}}))$, $K_j^+ = ((max_j \zeta_{K_{R_j}}, max_j \zeta_{K_{I_j}}), (min_j \vartheta_{K_{R_j}}, min_j \vartheta_{K_{I_j}}))$, $j = 1, 2, \ldots, n$, then by using Propositions 1 and 2, we have

$$CIFDHM^x(K_1, K_2, \ldots, K_n) \le CIFDHM^x(K_1^+, K_2^+, \ldots, K_n^+) = K_j^+$$

$$CIFDHM^{x}(K_1, K_2, \ldots, K_n) \geq CIFDHM^{x}(K_1^-, K_2^-, \ldots, K_n^-) = K_j^-$$

Then, by using the above information, we have

$$K_j^- \leq CIFDHM^{x}(K_1, K_2, \ldots, K_n) \leq K_j^+.$$

Definition 8 The computed information is described below:

$$CIFWDHM^{x}(K_1, K_2, \ldots, K_n) = \begin{cases} \frac{\oplus_{1\leq i_1 \leq i_2, \ldots, \leq i_x \leq n}(1-\sum_{j=1}^{x}(\beth_j))(\otimes_{j=1}^{x}(K_{i_j}))^{1/x}}{C_n^x} & 1 \leq x < n \\ \otimes_{j=1}^{x}(K_j)^{\frac{1-\beth_j}{1-n}} & x = n \end{cases} \quad (13.20)$$

Represented the CIFWDHM information with binomial coefficient $C_n^x = \frac{n!}{x!(n-x)!}$ With weight vector $\sum_{j=1}^{x}(\beth_j) = 1$, where $\beth_j \in [0, 1]$.

Proposition 4 *To evaluate the information in Eq. (13.20) with the help of Dombi operational laws, we prove that the resultant information of Eq. (13.20) is again CIF information, such that//*

$$CIFWDHM^{x}(K_1, K_2, \ldots, K_n)$$

$$= \frac{\oplus_{1\leq i_1 \leq i_2, \ldots, \leq i_x \leq n}(1-\sum_{j=1}^{x}(\beth_j))(\otimes_{j=1}^{x}(K_{i_j}))^{1/x}}{C_n^x}$$

$$=((1-\frac{1}{1+((\frac{x}{C_n^x})*(\sum_{1\leq i_1 \leq i_2 \leq, \ldots, \leq i_x \leq n}(1-\sum_{j=1}^{x}(\beth_j))(\frac{1}{\sum_{j=1}^{x}(\frac{1-\zeta_{K_{R_{i_j}}}}{\zeta_{K_{R_{i_j}}}})^{\Gamma}})))^{(1/\Gamma)}},$$

$$1-\frac{1}{1+((\frac{x}{C_n^x})*(\sum_{1\leq i_1 \leq i_2 \leq, \ldots, \leq i_x \leq n}(1-\sum_{j=1}^{x}(\beth_j))(\frac{1}{\sum_{j=1}^{x}(\frac{1-\zeta_{K_{I_{i_j}}}}{\zeta_{K_{I_{i_j}}}})^{\Gamma}})))^{(1/\Gamma)}}),$$

$$(\frac{1}{1+((\frac{x}{C_n^x})*(\sum_{1\leq i_1 \leq i_2 \leq, \ldots, \leq i_x \leq n}(1-\sum_{j=1}^{x}(\beth_j))(\frac{1}{\sum_{j=1}^{x}(\frac{\vartheta_{K_{R_{i_j}}}}{1-\vartheta_{K_{R_{i_j}}}})^{\Gamma}})))^{(1/\Gamma)}},$$

$$\frac{1}{1+((\frac{x}{C_n^x})*(\sum_{1\leq i_1 \leq i_2 \leq, \ldots, \leq i_x \leq n}(1-\sum_{j=1}^{x}(\beth_j))(\frac{1}{\sum_{j=1}^{x}(\frac{\vartheta_{K_{I_{i_j}}}}{1-\vartheta_{K_{I_{i_j}}}})^{\Gamma}})))^{(1/\Gamma)}})) \quad (13.21)$$

$$CIFWDHM^{x}(K_1, K_2, \ldots, K_n) = \otimes_{j=1}^{x}(K_j)^{\frac{1-\beth_j}{1-n}}$$
$$= ((1 - \frac{1}{1 + (\sum_{j=1}^{x}(\frac{1-\beth_j}{1-n})(\frac{1-\zeta_{K_{R_j}}}{\zeta_{K_{R_j}}})^{(\Gamma)})^{\frac{1}{\Gamma}}},$$
$$1 - \frac{1}{1 + (\sum_{j=1}^{x}(\frac{1-\beth_j}{1-n})(\frac{1-\zeta_{K_{I_j}}}{\zeta_{K_{I_j}}})^{(\Gamma)})^{\frac{1}{\Gamma}}}),$$
$$(\frac{1}{1 + (\sum_{j=1}^{x}(\frac{1-\beth_j}{1-n})(\frac{\vartheta_{K_{R_j}}}{1-\vartheta_{K_{R_j}}})^{(\Gamma)})^{\frac{1}{\Gamma}}}, \frac{1}{1 + (\sum_{j=1}^{x}(\frac{1-\beth_j}{1-n})(\frac{\vartheta_{K_{I_j}}}{1-\vartheta_{K_{I_j}}})^{(\Gamma)})^{\frac{1}{\Gamma}}})) \tag{13.22}$$

Proof Omitted.

Definition 9 The computed information is described below:

$$CIFDDHM^{x}(K_1, K_2, \ldots, K_n) = (\otimes_{1 \le i_1 \le i_2, \ldots, \le i_x \le n}(\frac{\oplus_{j=1}^{x}(K_{i_j})}{x}))^{\frac{1}{C_n^x}} \tag{13.23}$$

Represented the CIFDDHM information with binomial coefficient $C_n^x = \frac{n!}{x!(n-x)!}$.

Proposition 5 *To evaluate the information in Eq. (13.23) with the help of Dombi operational laws, we prove that the resultant information of Eq. (13.23) is again CIF information, such that*

$$CIFDDHM^{x}(K_1, K_2, \ldots, K_n)$$
$$= ((\frac{1}{1 + ((\frac{x}{C_n^x}) * (\sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n}(\frac{1}{\sum_{j=1}^{x}(\frac{\zeta_{K_{R_{i_j}}}}{1-\zeta_{K_{R_{i_j}}}})^{\Gamma}})))^{(1/\Gamma)}},$$
$$\frac{1}{1 + ((\frac{x}{C_n^x}) * (\sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n}(\frac{1}{\sum_{j=1}^{x}(\frac{\zeta_{K_{I_{i_j}}}}{1-\zeta_{K_{I_{i_j}}}})^{\Gamma}})))^{(1/\Gamma)}}),$$
$$(1 - \frac{1}{1 + ((\frac{x}{C_n^x}) * (\sum_{1 \le i_1 \le i_2 \le, \ldots, \le i_x \le n}(\frac{1}{\sum_{j=1}^{x}(\frac{1-\vartheta_{K_{R_{i_j}}}}{\vartheta_{K_{R_{i_j}}}})^{\Gamma}})))^{(1/\Gamma)}},$$

$$1-\frac{1}{1+((\frac{x}{C_n^x})*(\sum_{1\leq i_1\leq i_2\leq,\ldots,\leq i_x\leq n}(\frac{1}{\sum_{j=1}^{x}(\frac{1-\vartheta_{K_{I_{i_j}}}}{\vartheta_{K_{I_{i_j}}}})^{\Gamma}})))^{(1/\Gamma)}}))\tag{13.24}$$

Proof Omitted.

Definition 10 The computed information is described below:

$$CIFWDDHM^{x}(K_1,K_2,\ldots,K_n)=\begin{cases}\frac{\otimes_{1\leq i_1\leq i_2,\ldots,\leq i_x\leq n}(1-\sum_{j=1}^{x}(\beth_j))(\oplus_{j=1}^{x}(K_{i_j}))^{1/x}}{C_n^x} & 1\leq x<n\\ \oplus_{j=1}^{x}\frac{1-\beth_j}{1-n}(K_j) & x=n\end{cases}\tag{13.25}$$

Represented the CIFWDHM information with binomial coefficient $C_n^x=\frac{n!}{x!(n-x)!}$ With weight vector $\sum_{j=1}^{x}(\beth_j)=1$, where $\beth_j\in[0,1]$.

Proposition 6 *To evaluate the information in Eq. (13.25) with the help of Dombi operational laws, we prove that the resultant information of Eq. (13.25) is again CIF information, such that*

$$CIFWDDHM^{x}(K_1,K_2,\ldots,K_n)$$
$$=\frac{\otimes_{1\leq i_1\leq i_2,\ldots,\leq i_x\leq n}(1-\sum_{j=1}^{x}(\beth_j))(\oplus_{j=1}^{x}(K_{i_j}))^{1/x}}{C_n^x}$$
$$=((\frac{1}{1+((\frac{x}{C_n^x})*(\sum_{1\leq i_1\leq i_2\leq,\ldots,\leq i_x\leq n}(1-\sum_{j=1}^{x}(\beth_j))(\frac{1}{\sum_{j=1}^{x}(\frac{\zeta_{K_{R_{i_j}}}}{1-\zeta_{K_{R_{i_j}}}})^{\Gamma}})))^{(1/\Gamma)}},$$
$$\frac{1}{1+((\frac{x}{C_n^x})*(\sum_{1\leq i_1\leq i_2\leq,\ldots,\leq i_x\leq n}(1-\sum_{j=1}^{x}(\beth_j))(\frac{1}{\sum_{j=1}^{x}(\frac{\zeta_{K_{I_{i_j}}}}{1-\zeta_{K_{I_{i_j}}}})^{\Gamma}})))^{(1/\Gamma)}}),$$
$$(1-\frac{1}{1+((\frac{x}{C_n^x})*(\sum_{1\leq i_1\leq i_2\leq,\ldots,\leq i_x\leq n}(1-\sum_{j=1}^{x}(\beth_j))(\frac{1}{\sum_{j=1}^{x}(\frac{1-\vartheta_{K_{R_{i_j}}}}{\vartheta_{K_{R_{i_j}}}})^{\Gamma}})))^{(1/\Gamma)}},$$
$$1-\frac{1}{1+((\frac{x}{C_n^x})*(\sum_{1\leq i_1\leq i_2\leq,\ldots,\leq i_x\leq n}(1-\sum_{j=1}^{x}(\beth_j))(\frac{1}{\sum_{j=1}^{x}(\frac{1-\vartheta_{K_{I_{i_j}}}}{\vartheta_{K_{I_{i_j}}}})^{\Gamma}})))^{(1/\Gamma)}}))\tag{13.26}$$

$$
\begin{aligned}
&CIFWDDHM^{x}(K_1, K_2, \ldots, K_n)\\
&= \oplus_{j=1}^{x} \frac{1-\beth_j}{1-n}(K_j) = ((\frac{1}{1+(\sum_{j=1}^{x}(\frac{1-\beth_j}{1-n})(\frac{\zeta_{K_{R_j}}}{1-\zeta_{K_{R_j}}})^{(\Gamma)})^{\frac{1}{\Gamma}}},\\
&\frac{1}{1+(\sum_{j=1}^{x}(\frac{1-\beth_j}{1-n})(\frac{\zeta_{K_{I_j}}}{1-\zeta_{K_{I_j}}})^{(\Gamma)})^{\frac{1}{\Gamma}}}),\\
&(1-\frac{1}{1+(\sum_{j=1}^{x}(\frac{1-\beth_j}{1-n})(\frac{1-\vartheta_{K_{R_j}}}{\vartheta_{K_{R_j}}})^{(\Gamma)})^{\frac{1}{\Gamma}}},\\
&1-\frac{1}{1+(\sum_{j=1}^{x}(\frac{1-\beth_j}{1-n})(\frac{1-\vartheta_{K_{I_j}}}{\vartheta_{K_{I_j}}})^{(\Gamma)})^{\frac{1}{\Gamma}}})) \qquad (13.27)
\end{aligned}
$$

Proof Omitted.

Proposition 7 *When* $K_j = K = ((\zeta_{K_R}, \zeta_{K_I}), (\vartheta_{K_R}, \vartheta_{K_I}))$, $j = 1, 2, \ldots, n$, *then*

$$CIFWDDHM^{x}(K_1, K_2, \ldots, K_n) = K \qquad (13.28)$$

Proof Omitted.

Proposition 8 *When* $K_j = ((\zeta_{K_{R_j}}, \zeta_{K_{I_j}}), (\vartheta_{K_{R_j}}, \vartheta_{K_{I_j}})) \leq K_j^* = ((\zeta^*_{K_{R_j}}, \zeta^*_{K_{I_j}}), (\vartheta^*_{K_{R_j}}, \vartheta^*_{K_{I_j}}))$, $j = 1, 2, \ldots, n$, *then*

$$CIFWDDHM^{x}(K_1, K_2, \ldots, K_n) \leq CIFWDDHM^{x}(K_1^*, K_2^*, \ldots, K_n^*) \qquad (13.29)$$

Proof Omitted.

Proposition 9 *When* $K_j^- = ((min_j\zeta_{K_{R_j}}, min_j\zeta_{K_{I_j}}), (max_j\vartheta_{K_{R_j}}, max_j\vartheta_{K_{I_j}}))$, $K_j^+ = ((max_j\zeta_{K_{R_j}}, max_j\zeta_{K_{I_j}}), (min_j\vartheta_{K_{R_j}}, min_j\vartheta_{K_{I_j}}))$, $j = 1, 2, \ldots, n$, *then*

$$K_j^- \leq CIFWDDHM^{x}(K_1, K_2, \ldots, K_n) \leq K_j^+ \qquad (13.30)$$

Proof Omitted.

4 Multi-Attribute Decision-Making Problem

Here, we derive the theory of MADM technique based on proposed operators for CIF information which is to enhance the worth of the derived information.

To evaluate some practical life problems, we assumed that the information of alternatives $K_{AL-1}, K_{AL-2}, \ldots, K_{AL-m}$, and for this, we need to discuss the family of attributes $K_1, K_2, \ldots, K_n$ with the mathematics information $\beth_j$, $j = 1, 2, \ldots, n$ with a strategy $\sum_{j=1}^{n} \beth_j = 1$. Under the consideration of the above information, we computed a matrix by including the CIF numbers represented by $D = [r_{ij}]_{m \times n}$, such that $K_j = ((\zeta_{K_{R_j}}, \zeta_{K_{I_j}}), (\vartheta_{K_{R_j}}, \vartheta_{K_{I_j}}))$, $j = 1, 2, .., n$. represented the CIF set, where $\zeta_K(c) = (\zeta_{K_R}(c), \zeta_{K_I}(c))$, $\vartheta_K(c) = (\vartheta_{K_R}(c), \vartheta_{K_I}(c))$ shows the complex shape of truth and falsity information with $0 \leq \zeta_{K_R}(c) + \vartheta_{K_R}(c) \leq 1$, $0 \leq \zeta_{K_I}(c) + \vartheta_{K_I}(c) \leq 1$. Further, we revise the refusal information, represented by $\eta_K(c) = (\eta_{K_R}(c), \eta_{K_I}(c)) = (1 - (\zeta_{K_R}(c) + \vartheta_{K_R}(c)), 1 - (\zeta_{K_I}(c) + \vartheta_{K_I}(c)))$. Using the information explained above, we use the below procedure to evaluate some real-life problems:

Step 1: Collect the theory of CIF information and put it in a closed matrix.

Step 2: Aggregate the information by using the theory of CIFDHM, CIFWDHM, CIFDDHM, and CIFWDDHM operators to evaluate the information of the suggested matrix.

Step 3: Evaluate the score value of the aggregated information.

Step 4: Rank all information based on their score information to illustrate the best preference.

Using the pioneered procedure, we illustrated some practical information to enhance the quality and quantity of the presented information.

4.1 Illustrated Example

An enterprise decided to invest someone money in different companies; for this, the owner of the company makes a group of experts for finding the best investment company. The experts of the company visit different types of companies represented as alternatives, called K_{AL-1}, car company; K_{AL-2}, mobile company; K_{AL-3}, laptop company; K_{AL-4}, property company; and K_{AL-5}, construction company. Based on the four attributes, we try to take a decision, on which company is the best for investment; the information about each attribute is available in the shape: K_1, comfort zone; K_2, safety zone; K_3, benefits; and K_4, social impact by using weight vectors 0.4,0.3,0.1, and 0.2. Using the information explained above, we use the below procedure to evaluate some real-life problems.

Table 13.1 Representation of the CIF information

	K_1	K_2	K_3
K_{AL-1}	$((0.5, 0.6), (0.3, 0.2))$	$((0.51, 0.61), (0.31, 0.21))$	$((0.52, 0.62), (0.32, 0.22))$
K_{AL-2}	$((0.7, 0.3), (0.1, 0.3))$	$((0.71, 0.31), (0.11, 0.31))$	$((0.72, 0.32), (0.12, 0.32))$
K_{AL-3}	$((0.6, 0.7), (0.2, 0.1))$	$((0.61, 0.71), (0.21, 0.11))$	$((0.62, 0.72), (0.22, 0.12))$
K_{AL-4}	$((0.4, 0.4), (0.4, 0.4))$	$((0.41, 0.41), (0.41, 0.41))$	$((0.42, 0.42), (0.42, 0.42))$
K_{AL-5}	$((0.3, 0.2), (0.1, 0.1))$	$((0.31, 0.21), (0.11, 0.11))$	$((0.32, 0.22), (0.12, 0.12))$
	K_4		
K_{AL-1}	$((0.5, 0.6), (0.3, 0.2))$		
K_{AL-2}	$((0.7, 0.3), (0.1, 0.3))$		
K_{AL-3}	$((0.6, 0.7), (0.2, 0.1))$		
K_{AL-4}	$((0.4, 0.4), (0.4, 0.4))$		
K_{AL-5}	$((0.3, 0.2), (0.1, 0.1))$		

Table 13.2 Representation of the aggregated information

	$CIFDHM$	$CIFWDHM$
K_{AL-1}	$(((0.0930, 0.5434), (0.1166, 0.0058)))$	$(((0.0160, 0.1589), (0.4541, 0.0358)))$
K_{AL-2}	$(((0.9469, 0.0067), (0.0006, 0.1166)))$	$(((0.7391, 0.00011), (0.00043, 0.4541)))$
K_{AL-3}	$(((0.5434, 0.9469), (0.0005, 0.0005)))$	$(((0.1589, 0.7391), (0.0358, 0.0004)))$
K_{AL-4}	$(((0.0090, 0.0090), (0.6401, 0.6401)))$	$(((0.00144, 0.00144), (0.9180, 0.9180)))$
K_{AL-5}	$(((0.00067, 0.0003), (0.00006, 0.0006)))$	$(((0.00011, 0.0004), (0.00043, 0.00043)))$
	$CIFDDHM$	$CIFWDDHM$
K_{AL-1}	$(((0.9994, 0.9999), (0.0006, 0.0001)))$	$(((0.9921, 0.9993), (0.0001, 0.0002)))$
K_{AL-2}	$(((0.9999, 0.1362), (0.0001, 0.0006)))$	$(((0.9999, 0.4541), (0.00001, 0.000011)))$
K_{AL-3}	$(((0.9999, 0.9999), (0.0001, 0.0001)))$	$(((0.9993, 0.9999), (0.0001, 0.0003)))$
K_{AL-4}	$(((0.9064, 0.9064), (0.0090, 0.0090)))$	$(((0.9180, 0.9180), (0.0014, 0.0014)))$
K_{AL-5}	$(((0.1362, 0.0054), (0.003, 0.003)))$	$(((0.4541, 0.0358), (0.0004, 0.0004)))$

Key 1: Collect the theory of CIF information and put it in the shape of Table 13.1.
Key 2: Aggregate the information by using the theory of CIFDHM, CIFWDHM, CIFDDHM, and CIFWDDHM operators to evaluate the information of the suggested matrix given in Table 13.2.
Key 3: Evaluate the score value of the aggregated information given in Table 13.3.
Key 4: Rank all information based on their score information to illustrate the best preference described in Table 13.4.

By utilizing the pioneered procedure on the information in Table 13.1, we noticed that the proposed information is given the same ranking information and the best preference is K_{AL-3} according to the CIFDHM, CIFWDHM, CIFDDHM, and CIFWDDHM operators.

Table 13.3 Representation of the score information

	$CIFDHM$	$CIFWDHM$
K_{AL-1}	0.257	−0.1575
K_{AL-1}	0.4154	0.1423
K_{AL-1}	0.7422	0.4308
K_{AL-1}	−0.6311	−0.9166
K_{AL-1}	0.00028	−0.00037
	$CIFDDHM$	$CIFWDDHM$
K_{AL-1}	0.99935	0.9956
K_{AL-1}	0.5677	0.7269
K_{AL-1}	0.99998	0.99963
K_{AL-1}	0.8974	0.9166
K_{AL-1}	0.07086	0.2449

Table 13.4 Represented the ranking information

Methods	Score values
CIFDHM operator	$K_{AL-3} \geq K_{AL-2} \geq K_{AL-1} \geq K_{AL-5} \geq K_{AL-4}$
CIFWDHM operator	$K_{AL-3} \geq K_{AL-2} \geq K_{AL-5} \geq K_{AL-1} \geq K_{AL-4}$
CIFDDHM operator	$K_{AL-3} \geq K_{AL-1} \geq K_{AL-4} \geq K_{AL-2} \geq K_{AL-5}$
CIFWDDHM operator	$K_{AL-3} \geq K_{AL-1} \geq K_{AL-4} \geq K_{AL-2} \geq K_{AL-5}$

4.2 Comparative Analysis

The main theme of this theory is to show the supremacy of the evaluated information with the help of comparative analysis which is a valuable part of every manuscript. For comparative analysis, we use the information computed based on FSs, IFSs, CFSs, and CIFSs. Under the consideration of the above valuable discussion, we noticed that Li et al. [21] derived the Dombi HM operator for IFS and Wu et al. [22] pioneered the major information of Dombi HM operators based on interval-valued IFSs, Garg and Rani [25] derived the aggregation operators, Garg and Rani [26] pioneered the averaging-geometric aggregation information, and Garg and Rani [27] generalized geometric information based on CIFSs. The main comparative analysis of the proposed and prevailing theories based on the information given in Table 13.1 is available in Table 13.5.

Li et al. [21] derived that the Dombi HM operator for IFS has failed to evaluate the CIF types of information given in Table 13.1, because the information derived by Li et al. [21] was computed based on IFS which is a special case of the pioneered information. Similarly, Wu et al. [22] derived the major information of Dombi HM operators based on interval-valued IFSs that has failed to evaluate the CIF types of information given in Table 13.1, because the information derived by Wu et al. [22] was computed based on interval-valued IFS which is different from the pioneered information. Further, the theory of Garg and Rani [25] derived the aggregation operators, Garg and Rani [26] pioneered the averaging-geometric aggregation information, and Garg and Rani [27] generalized geometric information

Table 13.5 Comparative information is obtained from the information in Table 13.1.

Methods	Score values
Li et al. [21]	∗ ∗ ∗ ∗ ∗ ∗ ∗ ∗ ∗ ∗∗
Wu et al. [22]	∗ ∗ ∗ ∗ ∗ ∗ ∗ ∗ ∗ ∗∗
Garg and Rani [25]	0.3004, 0.30054, 0.50063, 0.00027, 0.15065
Garg and Rani [26]	0.41, 0.414, 0.6163, 0.1027, 0.2565
Garg and Rani [27]	0.2114, 0.2114, 0.41163, 0.01127, 0.0617
CIFDHM operator	0.257, 0.4154, 0.7422, −0.6311, 0.00028
CIFWDHM operator	−0.1575, 0.1423, 0.4308, −0.9166, −0.00037
CIFDDHM operator	0.99935, 0.5677, 0.99998, 0.8974, 0.07086
CIFWDDHM operator	0.9956, 0.7269, 0.99963, 0.9166, 0.2449
Methods	Ranking Values
Li et al. [21]	∗ ∗ ∗ ∗ ∗ ∗ ∗ ∗ ∗ ∗∗
Wu et al. [22]	∗ ∗ ∗ ∗ ∗ ∗ ∗ ∗ ∗ ∗∗
Garg and Rani [25]	$K_{AL-3} \geq K_{AL-2} \geq K_{AL-1} \geq K_{AL-5} \geq K_{AL-4}$
Garg and Rani [26]	$K_{AL-3} \geq K_{AL-2} \geq K_{AL-1} \geq K_{AL-5} \geq K_{AL-4}$
Garg and Rani [27]	$K_{AL-3} \geq K_{AL-2} \geq K_{AL-1} \geq K_{AL-5} \geq K_{AL-4}$
CIFDHM operator	$K_{AL-3} \geq K_{AL-2} \geq K_{AL-1} \geq K_{AL-5} \geq K_{AL-4}$
CIFWDHM operator	$K_{AL-3} \geq K_{AL-2} \geq K_{AL-5} \geq K_{AL-1} \geq K_{AL-4}$
CIFDDHM operator	$K_{AL-3} \geq K_{AL-1} \geq K_{AL-4} \geq K_{AL-2} \geq K_{AL-5}$
CIFWDDHM operator	$K_{AL-3} \geq K_{AL-1} \geq K_{AL-4} \geq K_{AL-2} \geq K_{AL-5}$

based on CIFSs that are provided the same ranking information in the shape, where K_{AL-3} is the best preference.

Therefore, the pioneered information based on the CIF set is massive dominant and flexible compared to old or prevailing information.

5 Conclusion

The main impact of this analysis is to present the theory of Dombi operational under the consideration of complex intuitionistic fuzzy information. Furthermore, we described the theory of CIFDHM, CIFWDHM, CIFDDHM, and CIFWDDHM operators. Moreover, we evaluated some valuable properties and results for the presented information in the investigated analysis. Under consideration of the above information, we derived MADM information and gave some examples to justify the worth and dominancy of the evaluated information. In last, we compared the evaluated information with some other old or prevailing information to enhance the quality of the evaluated operators.

When an expert provides three-dimension information such as yes, no, and abstinence, then the theory of the CIF set has failed because the current theory can deal only with yes and no types of information but avoid the abstinence information, so, in such type of situation, the theory of CIF set has been failed.

In the coming times, we revise the theory of aggregation operators [28], Aczel-Alsina operators [29, 30], similarity measures [31, 32], Dombi operational laws [33], spherical fuzzy and complex spherical fuzzy information [34–40], and decision-making [41–44] which is to utilize it in the environment of decision-making, computer network, image segmentation, and medical diagnosis to improve the value of the derived information.

Acknowledgments Write Acknowledgment for financial assistance, reviewers, etc.

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Chapter 14
Linear Diophantine Fuzzy Information Aggregation with Multi-criteria Decision-Making

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1 Introduction

The act of selecting steps to take after compiling relevant information and analyzing the relative merits of several potential solutions is known as decision-making. Establishing pertinent information and identifying available alternatives are the first two stages of a decision-making process that should be carried out according to a step-by-step methodology. Depending on the objectives and available options, the decision-making process may be either strategically, tactically, or operational. Since the beginning of the twentieth century, one of the most significant challenges faced by society has been confusing and inaccurate information. In several areas, such as economics, administration, psychology, mathematics, engineering, cognitive systems, and autonomous systems, data aggregation is a crucial phase in the decision-making process. Knowledge of the alternative has traditionally been conceptualized by individuals as a restricted amount or linguistic number. On the other hand, it is difficult to synthesize the information due to the substantial ambiguity involved. The multi-criteria decision-making (MCDM) approach is a frequently used intellectual activity instrument whose primary objective is to pick from a restricted number of possibilities based on the details provided by decision-makers (DMs). The MCDM approach, on the other hand, is prone to becoming ambiguous and inaccurate. This is because it integrates the complexity of human reasoning skills, making it difficult for DMs to engage in the review process in an accurate manner. In addition to addressing the issue of uncertainty, Zadeh [1] was a pioneer in developing fuzzy set theory. It is imperative that a solution be found for this issue. Atanassov [2] developed the "intuitionistic fuzzy set (IFS)."

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C. Jana et al. (eds.), *Fuzzy Optimization, Decision-making and Operations Research*, https://doi.org/10.1007/978-3-031-35668-1_14

Yager [3–5] introduced "Pythagorean fuzzy set (PFS)" as an extended form of IFS. Yager added some generalizations to the IFS and PFS, and he developed the concept of the "q-rung orthopair fuzzy set (q-ROFS)" [6]. A constraint of the q-ROFS is that the sum of qth membership degree (MSD) power and non-membership degree (NMSD) power might be equal to or less than one. Riaz and Hashmi established the notion of the linear Diophantine fuzzy set (LDFS) [17]. After the advent of this notion, a number of academics were drawn to it and began working in this field.

Xu et al. [7–9] gave some Aos related to IFS. Wei et al. [11], Feng et al. [14], Mahmood et al. [10], Zhang et al. [12], Zhao et al. [13], Garg [15], and Rahman et al. [16] introduced many AOs for different extensions of fuzzy sets. Some work related to AOs and graph structures can be seen in [18, 19]. Extensive work related to bipolar fuzzy set is given in [20, 21]. Feng et al. [22] proposed some novel score functions related to orthopair fuzzy set. Senapati and Yager proposed Fermatean fuzzy set as the extension of IFS [23]. Smarandache proposed a novel idea of neutrosophic set [24, 25]. Farid and Riaz introduced some Einstein interacting geometric AOs for q-ROFSs [26]. Many AOs for "linear Diophantine fuzzy numbers" are given in [27, 28]. Ashraf et al. proposed some distance metric for cubic picture fuzzy set [29, 30]. Saha et al. [31, 32] introduced some hybrid AOs for different extensions of fuzzy set. Wei and Zhang [33] gave some single-valued neutrosophic Bonferroni power AOs. Riaz et al. proposed a number of AOs, including Einstein prioritized [35], interactive [36], hybrid [34], and prioritized with PDs [37]. Some extra-ordinary work related to proposed work is given in [38–41]. Ejegwa and Davvaz proposed the improved composite relation for q-ROFSs [42]. Ejegwa and Ahemen introduced some enhanced IF similarity measures [43]. Ejegwa et al. described the Pythagorean fuzzy correlation approach from a statistical standpoint [44]. Jana et al. [45] gave the notion of picture fuzzy Dombi AOs. Naeem et al. [46] presented some features related to topology in m-polarity PFSs. Peng et al. [47] proposed upgraded "single valued neutrosophic number" (SVNN) operations and established their associated AOs. Nancy and Garg [48] established AOs by employing Frank operations. Liu et al. [49] developed some AOs for SVNNs based on "Hamacher operations." Farid and Riaz [50] proposed Einstein interactive AOs for SVNNs. Zhang et al. [51] provided the AOs in the context of an "interval-valued neutrosophic set." Wu et al. [52] developed the prioritized AOs with SVNNs. Wei [53] proposed some similarity measures, Singh [54] idea of correlation coefficients, and Son [55] gave some clustering method for picture fuzzy set.

Multi-criteria decision-making (MCDM) is a method used to evaluate and select the best option among a set of alternatives based on multiple criteria. It is a powerful tool for decision-makers, as it allows for the consideration of multiple factors that may have an impact on the success of a decision. MCDM has been applied in a wide range of fields, including agriculture, where it can be used to make important decisions related to crop selection, land use, irrigation systems, and more.

One of the main advantages of MCDM in agriculture is that it takes into account the multiple and often conflicting objectives that farmers and other stakeholders may have. For example, when selecting a crop to plant, a farmer may consider factors such as expected yield, market demand, and pest resistance. Each of these factors may have different levels of importance to the farmer, and MCDM allows for the weighting of these factors to reflect this. Additionally, MCDM can be used to evaluate the trade-offs between different factors, such as the relationship between yield and water use efficiency.

Another important use of MCDM in agriculture is in land use planning. MCDM can be used to evaluate different land use options and determine the best option based on multiple criteria such as economic profitability, environmental sustainability, and social acceptability. This can be particularly useful in situations where there is a need to balance competing interests such as urbanization and agricultural production.

MCDM can also be used in irrigation systems. In this case, the farmer can evaluate different irrigation options based on criteria such as water use efficiency, cost, and impact on the environment. Additionally, MCDM can be used to evaluate the trade-offs between different irrigation options, such as the relationship between cost and water use efficiency. This can be particularly useful in areas where water is scarce, and farmers need to make decisions about how to use water resources in the most efficient and sustainable way.

Furthermore, MCDM can be used in the context of climate change, where farmers need to make decisions about crop selection, irrigation systems, and land use in the face of changing weather patterns, rising temperatures, and increased water scarcity. MCDM allows for the consideration of multiple factors such as crop resilience, water use efficiency, and environmental impact, which can help farmers make more informed decisions about how to adapt to changing conditions.

MCDM is an important tool for decision-making in agriculture. It allows for the consideration of multiple and often conflicting objectives, and it can be used to evaluate the trade-offs between different factors. This makes MCDM a valuable tool for farmers and other stakeholders in the agricultural sector, as it can help them make more informed decisions that balance economic profitability, environmental sustainability, and social acceptability. The main objectives of the manuscript are as follows:

- Some basic AOs are proposed for the aggregation of linear Diophantine fuzzy information.
- The essential properties of proposed AOs are also examined.
- Decision-making algorithm based on proposed Aos is also explained.
- Numerical example related to agriculture land selection is also given to show the practical implication of proposed algorithm.

This format is maintained for the remainder of the paper. In the second portion, we will talk about some essential LDFS concepts. The third section offers several potential AOs for LDFNs. In Sect. 4, an MCDM framework is shown for the recommended AOs. Section 5 has a test scenario with numerical information. The most important findings from the research are discussed in the sixth section.

2 Preliminary

In this part, we will go over some of the most fundamental aspects of LDFS.

Definition 1 ([17]) An LDFS R^r in X can be characterized by

$$R^r = \left\{\left(\Xi, \langle \zeta^{\tau}{}_{R^r}(\Xi), \eta^{\upsilon}{}_{R^r}(\Xi)\rangle, \langle \mathscr{J}^{\aleph}{}_{R^r}(\Xi), \mathscr{C}^{\gamma}{}_{R^r}(\Xi)\rangle\right) : \Xi \in X\right\},$$

where $\zeta^{\tau}{}_{R^r}(\Xi), \eta^{\upsilon}{}_{R^r}(\Xi), \mathscr{J}^{\aleph}{}_{R^r}(\Xi), \mathscr{C}^{\gamma}{}_{R^r}(\Xi) \in [0, 1]$ are the MSD, the NMSD, and the corresponding reference parameters (RPs), respectively. Moreover,

$$0 \leq \mathscr{J}^{\aleph}{}_{R^r}(\Xi) + \mathscr{C}^{\gamma}{}_{R^r}(\Xi) \leq 1,$$

and

$$0 \leq \mathscr{J}^{\aleph}{}_{R^r}(\Xi)\zeta^{\tau}{}_{R^r}(\Xi) + \mathscr{C}^{\gamma}{}_{R^r}(\Xi)\eta^{\upsilon}{}_{R^r}(\Xi) \leq 1$$

for all $\Xi \in X$. The LDFS

$$R^r_X = \{(\Xi, \langle 1, 0\rangle, \langle 1, 0\rangle) : \Xi \in X\}$$

is recognized the "absolute LDFS" in X. The LDFS

$$R^r_\phi = \{(\Xi, \langle 0, 1\rangle, \langle 0, 1\rangle) : \Xi \in X\}$$

is recognized the "null LDFS" in X.

Modeling or categorization certain structures can be accomplished with the help of the RPs. We are able to describe a wide variety of systems by altering the fundamental significance of the RPs. Moreover, $\eta_{R^r}(\Xi)\pi_{R^r}(\Xi) = 1-(\mathscr{J}^{\aleph}{}_{R^r}(\Xi)\zeta^{\tau}{}_{R^r}(\Xi)+ \mathscr{C}^{\gamma}{}_{R^r}(\Xi)\eta^{\upsilon}{}_{R^r}(\Xi))$ is called the "indeterminacy degree" and its corresponding RP of Ξ to R^r.

It is very evident that our suggested conception is more appropriate and advanced, and it includes a range of RPs. This procedure is applicable to a wide range of projects, including those in the fields of industry, medicine, cognitive computing, and MCDM.

Definition 2 ([17]) A "linear Diophantine fuzzy number" (LDFN) is the form of $\daleth^{\varsigma} = (\langle \zeta^{\tau}{}_{\hbar^{\kappa}}, \eta^{\upsilon}{}_{\daleth^{\varsigma}}\rangle, \langle \mathscr{J}^{\aleph}{}_{\daleth^{\varsigma}}, \mathscr{C}^{\gamma}{}_{\daleth^{\varsigma}}\rangle)$ having the given characteristics:

(1) $0 \leq \zeta^{\tau}{}_{\daleth^{\varsigma}}, \eta^{\upsilon}{}_{\daleth^{\varsigma}}, \mathscr{J}^{\aleph}{}_{\daleth^{\varsigma}}, \mathscr{C}^{\gamma}{}_{\daleth^{\varsigma}} \leq 1.$
(2) $0 \leq \mathscr{J}^{\aleph}{}_{\daleth^{\varsigma}} + \mathscr{C}^{\gamma}{}_{\daleth^{\varsigma}} \leq 1.$
(3) $0 \leq \mathscr{J}^{\aleph}{}_{\daleth^{\varsigma}}\zeta^{\tau}{}_{\daleth^{\varsigma}} + \mathscr{C}^{\gamma}{}_{\daleth^{\varsigma}}\eta^{\upsilon}{}_{\daleth^{\varsigma}} \leq 1.$

Definition 3 ([17]) Consider $\daleth^{\varsigma} = (\langle \zeta^{\tau}{}_{\daleth^{\varsigma}}, \eta^{\upsilon}{}_{\daleth^{\varsigma}} \rangle, \langle \mathscr{J}^{\aleph}{}_{\daleth^{\varsigma}}, \mathscr{C}^{\gamma}{}_{\daleth^{\varsigma}} \rangle)$ is the LDFN, and then the "score function" (SF) $\breve{¶}(\daleth^{\varsigma})$ is defined by $✠(\daleth^{\varsigma}) : LDFN(X) \to [-1, 1]$ and given by

$$✠(\daleth^{\varsigma}) = \frac{1}{2}[(\zeta^{\tau}{}_{\daleth^{\varsigma}} - \eta^{\upsilon}{}_{\daleth^{\varsigma}}) + (\mathscr{J}^{\aleph}{}_{\daleth^{\varsigma}} - \mathscr{C}^{\gamma}{}_{\daleth^{\varsigma}}),]$$

where $LDFN(X)$ is the collection of LDFNs on X.

Definition 4 ([17]) Consider $\daleth^{\varsigma} = (\langle \zeta^{\tau}{}_{\daleth^{\varsigma}}, \eta^{\upsilon}{}_{\daleth^{\varsigma}} \rangle, \langle \mathscr{J}^{\aleph}{}_{\daleth^{\varsigma}}, \mathscr{C}^{\gamma}{}_{\daleth^{\varsigma}} \rangle)$ is the LDFN, and then the "accuracy function" is defined by $\psi : LDFN(X) \to [0, 1]$ and given as

$$\psi(\daleth^{\varsigma}) = \frac{1}{2}\Big[\Big(\frac{\zeta^{\tau}{}_{\daleth^{\varsigma}} + \eta^{\upsilon}{}_{\daleth^{\varsigma}}}{2}\Big) + (\mathscr{J}^{\aleph}{}_{\daleth^{\varsigma}} + \mathscr{C}^{\gamma}{}_{\daleth^{\varsigma}})]$$

Definition 5 ([17]) Let $\daleth^{\varsigma}{}_1 = (\langle \zeta^{\tau}{}_1, \eta^{\upsilon}{}_1 \rangle, \langle \mathscr{J}^{\aleph}{}_1, \mathscr{C}^{\gamma}{}_1 \rangle)$ be an LDFN and $\mathfrak{X} > 0$. Then:

- $\daleth^{\varsigma c}_1 = (\langle \eta^{\upsilon}{}_1, \zeta^{\tau}{}_1 \rangle, \langle \mathscr{C}^{\gamma}{}_1, \mathscr{J}^{\aleph}{}_1 \rangle)$.
- $\mathfrak{X}\daleth^{\varsigma}{}_1 = \Big(\langle 1 - (1 - \zeta^{\tau}{}_1)^{\mathfrak{X}}, \eta^{\upsilon \mathfrak{X}}_1 \rangle, \langle 1 - (1 - \mathscr{J}^{\aleph}{}_1)^{\mathfrak{X}}, \mathscr{C}^{\gamma \mathfrak{X}}_1 \rangle\Big)$.
- $\daleth^{\varsigma \mathfrak{X}}_1 = \Big(\langle \zeta^{\tau \mathfrak{X}}_1, 1 - (1 - \eta^{\upsilon}{}_1)^{\mathfrak{X}} \rangle, \langle \mathscr{J}^{\aleph \mathfrak{X}}_1, 1 - (1 - \mathscr{C}^{\gamma}{}_1)^{\mathfrak{X}} \rangle\Big)$.

Definition 6 ([17]) Let $\daleth^{\varsigma}{}_i = (\langle \zeta^{\tau}{}_i, \eta^{\upsilon}{}_i \rangle, \langle \mathscr{J}^{\aleph}{}_i, \mathscr{C}^{\gamma}{}_i \rangle)$ be two LDFNs with $i = 1, 2$. Then:

- $\daleth^{\varsigma}{}_1 \subseteq \daleth^{\varsigma}{}_2 \Leftrightarrow \zeta^{\tau}{}_1 \le \zeta^{\tau}{}_2, \eta^{\upsilon}{}_2 \le \eta^{\upsilon}{}_1, \mathscr{J}^{\aleph}{}_1 \le \mathscr{J}^{\aleph}{}_2, \mathscr{C}^{\gamma}{}_2 \le \mathscr{C}^{\gamma}{}_1$.
- $\daleth^{\varsigma}{}_1 = \daleth^{\varsigma}{}_2 \Leftrightarrow \zeta^{\tau}{}_1 = \zeta^{\tau}{}_2, \eta^{\upsilon}{}_1 = \eta^{\upsilon}{}_2, \mathscr{J}^{\aleph}{}_1 = \mathscr{J}^{\aleph}{}_2, \mathscr{C}^{\gamma}{}_1 = \mathscr{C}^{\gamma}{}_2$.
- $\daleth^{\varsigma}{}_1 \oplus \daleth^{\varsigma}{}_2 = (\langle \zeta^{\tau}{}_1 + \zeta^{\tau}{}_2 - \zeta^{\tau}{}_1\zeta^{\tau}{}_2, \eta^{\upsilon}{}_1\eta^{\upsilon}{}_2 \rangle, \langle \mathscr{J}^{\aleph}{}_1 + \mathscr{J}^{\aleph}{}_2 - \mathscr{J}^{\aleph}{}_1\mathscr{J}^{\aleph}{}_2, \mathscr{C}^{\gamma}{}_1\mathscr{C}^{\gamma}{}_2 \rangle)$.
- $\daleth^{\varsigma}{}_1 \otimes \daleth^{\varsigma}{}_2 = (\langle \zeta^{\tau}{}_1\zeta^{\tau}{}_2, \eta^{\upsilon}{}_1 + \eta^{\upsilon}{}_2 - \eta^{\upsilon}{}_1\eta^{\upsilon}{}_2 \rangle, \langle \mathscr{J}^{\aleph}{}_1\mathscr{J}^{\aleph}{}_2, \mathscr{C}^{\gamma}{}_1 + \mathscr{C}^{\gamma}{}_2 - \mathscr{C}^{\gamma}{}_1\mathscr{C}^{\gamma}{}_2 \rangle)$.

Definition 7 ([17]) Let $\daleth^{\varsigma}{}_i = (\langle \zeta^{\tau}{}_i, \eta^{\upsilon}{}_i \rangle, \langle \mathscr{J}^{\aleph}{}_i, \mathscr{C}^{\gamma}{}_i \rangle)$ be the assemblage of LDFNs with $i \in \Delta$. Then:

- $\bigcup_{i \in \Delta} \daleth^{\varsigma}{}_i = \Big(\langle \sup_{i \in \Delta} \zeta^{\tau}{}_i, \inf_{i \in \Delta} \eta^{\upsilon}{}_i \rangle, \langle \sup_{i \in \Delta} \mathscr{J}^{\aleph}{}_i, \inf_{i \in \Delta} \mathscr{C}^{\gamma}{}_i \rangle\Big)$.
- $\bigcap_{i \in \Delta} \daleth^{\varsigma}{}_i = \Big(\langle \inf_{i \in \Delta} \zeta^{\tau}{}_i, \sup_{i \in \Delta} \eta^{\upsilon}{}_i \rangle, \langle \inf_{i \in \Delta} \mathscr{J}^{\aleph}{}_i, \sup_{i \in \Delta} \mathscr{C}^{\gamma}{}_i \rangle\Big)$.

There are many AOs for the aggregation of LDFNs, namely, Einstein AOs [28], prioritized AOs [27], and fairly AOs [56].

Definition 8 ([28]) Consider $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth} \rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth} \rangle)$ the agglomeration of LDFNs and $\Re^{\Im} = (\Re^{\Im}{}_1, \Re^{\Im}{}_2, \ldots, \Re^{\Im}{}_n)^T$ be the weight vector (WV) with $\sum_{\beth=1}^{n} \Re^{\Im}{}_{\beth} = 1$. Then "linear Diophantine fuzzy Einstein weighted average

(LDFEWA) operator” is defined as

$$LDFEWA(\hbar_1^{\kappa}, \hbar_2^{\kappa}, \hbar_3^{\kappa}, \ldots, \hbar_n^{\kappa}) =$$

$$\sum_{\beth=1}^{n} \Re^{\Im}{}_{\Lambda} \daleth^{\varsigma}{}_{\beth} = \Re^{\Im}{}_1 \cdot_{\mathcal{E}} \hbar_1^{\kappa} \oplus_{\mathcal{E}} \Re^{\Im}{}_2 \cdot_{\mathcal{E}} \hbar_2^{\kappa} \oplus_{\mathcal{E}} \Re^{\Im}{}_3 \cdot_{\mathcal{E}} \gamma_3 \oplus_{\mathcal{E}} \ldots \oplus_{\mathcal{E}} \Re^{\Im}{}_n \cdot_{\mathcal{E}} \hbar_n^{\kappa}.$$

In LDFEWA operator, we use $\Re^{\Im}$ as a WV and $\daleth^{\varsigma}{}_{\beth}$ are the LDFNs, where $\beth = 1, 2, \ldots, n$.

Theorem 1 ([28]) *Let* $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth} \rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth} \rangle)$ *be an agglomeration of LDFNs and* $\Re^{\Im} = (\Re^{\Im}{}_1, \Re^{\Im}{}_2, \ldots, \Re^{\Im}{}_n)^T$ *be the WV with* $\sum_{\beth=1}^{n} \Re^{\Im}{}_{\beth} = 1$. *Then the LDFEWA operator can also be written as*

$$\begin{aligned} &LDFEWA(\hbar_1^{\kappa}, \hbar_2^{\kappa}, \ldots, \hbar_n^{\kappa}) \\ &= \Bigg(\Bigg\langle \frac{\prod_{\beth=1}^{n} (1 + \zeta^{\tau}{}_{\beth})^{\Re^{\Im}{}_{\beth}} - \prod_{\beth=1}^{n} (1 - \zeta^{\tau}{}_{\beth})^{\Re^{\Im}{}_{\beth}}}{\prod_{\beth=1}^{n} (1 + \zeta^{\tau}{}_{\beth})^{\Re^{\Im}{}_{\beth}} + \prod_{\beth=1}^{n} (1 - \zeta^{\tau}{}_{\beth})^{\Re^{\Im}{}_{\beth}}}, \\ &\frac{2 \prod_{\beth=1}^{n} \eta^{\upsilon}{}_{\beth}^{\Re^{\Im}{}_{\beth}}}{\prod_{\beth=1}^{n} (2 - \eta^{\upsilon}{}_{\beth})^{\Re^{\Im}{}_{\beth}} + \prod_{\beth=1}^{n} (\eta^{\upsilon}{}_{\beth})^{\Re^{\Im}{}_{\beth}}} \Bigg\rangle, \\ &\Bigg\langle \frac{\prod_{\beth=1}^{n} (1 + \mathscr{J}^{\aleph}{}_{\beth})^{\Re^{\Im}{}_{\beth}} - \prod_{\beth=1}^{n} (1 - \mathscr{J}^{\aleph}{}_{\beth})^{\Re^{\Im}{}_{\beth}}}{\prod_{\beth=1}^{n} (1 + \mathscr{J}^{\aleph}{}_{\beth})^{\Re^{\Im}{}_{\beth}} + \prod_{\beth=1}^{n} (1 - \mathscr{J}^{\aleph}{}_{\beth})^{\Re^{\Im}{}_{\beth}}}, \\ &\frac{2 \prod_{\beth=1}^{n} \mathscr{C}^{\gamma}{}_{\beth}^{\Re^{\Im}{}_{\beth}}}{\prod_{\beth=1}^{n} (2 - \mathscr{C}^{\gamma}{}_{\beth})^{\Re^{\Im}{}_{\beth}} + \prod_{\beth=1}^{n} (\mathscr{C}^{\gamma}{}_{\beth})^{\Re^{\Im}{}_{\beth}}} \Bigg\rangle \Bigg). \end{aligned}$$

Definition 9 ([28]) Consider $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth} \rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth} \rangle)$ is the agglomeration of LDFNs and $\Re^{\Im} = (\Re^{\Im}{}_1, \Re^{\Im}{}_2, \ldots, \Re^{\Im}{}_n)^T$ be the WV with $\sum_{\beth=1}^{n} \Re^{\Im}{}_{\beth} = 1$. Then “linear Diophantine fuzzy Einstein weighted geometric (LDFEWG) operator” is defined as

$$LDFEWG(\hbar_1^{\kappa}, \hbar_2^{\kappa}, \hbar_3^{\kappa}, \ldots, \hbar_n^{\kappa}) =$$

$$\prod_{\beth=1}^{n} \Re^{\Im}{}_{\beth} \daleth^{\varsigma}{}_{\beth} = \Re^{\Im}{}_1 \cdot_{\mathcal{E}} \hbar_1^{\kappa} \otimes_{\mathcal{E}} \Re^{\Im}{}_2 \cdot_{\mathcal{E}} \hbar_2^{\kappa} \otimes_{\mathcal{E}} \Re^{\Im}{}_3 \cdot_{\mathcal{E}} \hbar_3^{\kappa} \otimes_{\mathcal{E}} \ldots \otimes_{\mathcal{E}} \Re^{\Im}{}_n \cdot_{\mathcal{E}} \hbar_n^{\kappa}.$$

Theorem 2 *[[28]] Let* $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth} \rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth} \rangle)$ *be the agglomeration of LDFNs and* $\Re^{\Im} = (\Re^{\Im}{}_1, \Re^{\Im}{}_2, \ldots, \Re^{\Im}{}_n)^T$ *be the WV with* $\sum_{\beth=1}^{n} \Re^{\Im}{}_{\beth} = 1$. *Then*

LDFEWG operator can also be written as

$$LDFEWG(\hbar_1^{\kappa}, \hbar_2^{\kappa}, \ldots, \hbar_n^{\kappa})$$
$$= \Bigg(\Bigg\langle \frac{2\prod_{\beth=1}^{n} {\zeta^{\tau}}_{\beth}^{\Re\Im_{\beth}}}{\prod_{\beth=1}^{n}(2-{\zeta^{\tau}}_{\beth})^{\Re\Im_{\beth}} + \prod_{\beth=1}^{n}({\zeta^{\tau}}_{\beth})^{\Re\Im_{\beth}}},$$
$$\frac{\prod_{\beth=1}^{n}(1+{\eta^{\upsilon}}_{\beth})^{\Re\Im_{\beth}} - \prod_{\beth=1}^{n}(1-{\eta^{\upsilon}}_{\beth})^{\Re\Im_{\beth}}}{\prod_{\beth=1}^{n}(1+{\eta^{\upsilon}}_{\beth})^{\Re\Im_{\beth}} + \prod_{\beth=1}^{n}(1-{\eta^{\upsilon}}_{\beth})^{\Re\Im_{\beth}}}\Bigg\rangle,$$
$$\Bigg\langle \frac{2\prod_{\beth=1}^{n} {\mathscr{J}^{\aleph}}_{\beth}^{\Re\Im_{\beth}}}{\prod_{\beth=1}^{n}(2-{\mathscr{J}^{\aleph}}_{\beth})^{\Re\Im_{\beth}} + \prod_{\beth=1}^{n}({\mathscr{J}^{\aleph}}_{\beth})^{\Re\Im_{\beth}}},$$
$$\frac{\prod_{\beth=1}^{n}(1+{\mathscr{C}^{\gamma}}_{\beth})^{\Re\Im_{\beth}} - \prod_{\beth=1}^{n}(1-{\mathscr{C}^{\gamma}}_{\beth})^{\Re\Im_{\beth}}}{\prod_{\beth=1}^{n}(1+{\mathscr{C}^{\gamma}}_{\beth})^{\Re\Im_{\beth}} + \prod_{\beth=1}^{n}(1-{\mathscr{C}^{\gamma}}_{\beth})^{\Re\Im_{\beth}}}\Bigg\rangle\Bigg)$$

Definition 10 ([27]) Assume that ${\daleth^{\varsigma}}_{\beth} = (\langle {\zeta^{\tau}}_{\beth}, {\eta^{\upsilon}}_{\beth}\rangle, \langle {\mathscr{J}^{\aleph}}_{\beth}, {\mathscr{C}^{\gamma}}_{\beth}\rangle)$ is the agglomeration of LDFNs, and LDFPWA : $\$^n \to \$$ is the mapping. If

$$\text{LDFPWA}({\daleth^{\varsigma}}_1, {\daleth^{\varsigma}}_2, \ldots {\daleth^{\varsigma}}_n) = \frac{\breve{\hbar}_1}{\sum_{\beth=1}^{n}\breve{\hbar}_{\beth}}{\daleth^{\varsigma}}_1 \oplus \frac{\breve{\hbar}_2}{\sum_{\beth=1}^{n}\breve{\hbar}_{\beth}}{\daleth^{\varsigma}}_2 \oplus \ldots, \oplus \frac{\breve{\hbar}_n}{\sum_{\beth=1}^{n}\breve{\hbar}_{\beth}}{\daleth^{\varsigma}}_n, \quad (14.1)$$

then the mapping LDFPWA is called "linear Diophantine fuzzy prioritized weighted averaging (LDFPWA) operator," where $\breve{\hbar}_{\beth} = \overline{\prod}_{k=1}^{j-1}\mathscr{H}({\daleth^{\varsigma}}_k)$ $(j = 2\ldots, n)$, $\breve{\hbar}_1 = 1$, and $\mathscr{H}({\daleth^{\varsigma}}_k)$ is the expectation score function of kth LDFN.

Theorem 3 ([27]) *Assuming that* ${\daleth^{\varsigma}}_{\beth} = (\langle {\zeta^{\tau}}_{\beth}, {\eta^{\upsilon}}_{\beth}\rangle, \langle {\mathscr{J}^{\aleph}}_{\beth}, {\mathscr{C}^{\gamma}}_{\beth}\rangle)$ *is the agglomeration of LDFNs, we can find LDFPWA by*

$$LDFPWA({\daleth^{\varsigma}}_1, {\daleth^{\varsigma}}_2, \ldots {\daleth^{\varsigma}}_n)$$
$$= \Bigg(\Bigg\langle 1 - \overline{\prod}_{\beth=1}^{n}(1-{\zeta^{\tau}}_{\beth})^{\frac{\breve{\hbar}_{\beth}}{\sum_{\beth=1}^{n}\breve{\hbar}_{\beth}}}, \overline{\prod}_{\beth=1}^{n}{\eta^{\upsilon}}_{\beth}^{\frac{\breve{\hbar}_{\beth}}{\sum_{\beth=1}^{n}\breve{\hbar}_{\beth}}}\Bigg\rangle,$$
$$\Bigg\langle 1 - \overline{\prod}_{\beth=1}^{n}(1-{\mathscr{J}^{\aleph}}_{\beth})^{\frac{\breve{\hbar}_{\beth}}{\sum_{\beth=1}^{n}\breve{\hbar}_{\beth}}}, \overline{\prod}_{\beth=1}^{n}{\mathscr{C}^{\gamma}}_{\beth}^{\frac{\breve{\hbar}_{\beth}}{\sum_{\beth=1}^{n}\breve{\hbar}_{\beth}}}\Bigg\rangle\Bigg). \quad (14.2)$$

Definition 11 ([27]) Assume that ${\daleth^{\varsigma}}_{\beth} = (\langle {\zeta^{\tau}}_{\beth}, {\eta^{\upsilon}}_{\beth}\rangle, \langle {\mathscr{J}^{\aleph}}_{\beth}, {\mathscr{C}^{\gamma}}_{\beth}\rangle)$ is the agglomeration of LDFNs and $LDFPWG : \$^n \to \$$ is the mapping. If

$$\text{LDFPWG}({\daleth^{\varsigma}}_1, {\daleth^{\varsigma}}_2, \ldots {\daleth^{\varsigma}}_n) = {\daleth^{\varsigma}}_1^{\frac{\breve{\hbar}_1}{\sum_{\beth=1}^{n}\breve{\hbar}_{\beth}}} \otimes {\daleth^{\varsigma}}_2^{\frac{\breve{\hbar}_2}{\sum_{\beth=1}^{n}\breve{\hbar}_{\beth}}} \otimes \ldots, \otimes {\daleth^{\varsigma}}_n^{\frac{\breve{\hbar}_n}{\sum_{\beth=1}^{n}\breve{\hbar}_{\beth}}}, \quad (14.3)$$

then the mapping LDFPWG is called “linear Diophantine fuzzy prioritized weighted geometric (LDFPWG) operator.”

Theorem 4 ([27]) *Assuming that* $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth}\rangle)$ *is the agglomeration of LDFNs, we can find LDFPWG by*

$$
\begin{aligned}
& LDFPWG(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{n}) \\
& = \left(\left\langle \prod_{\beth=1}^{n} \zeta^{\tau}{}_{\beth}^{\frac{\check{\hbar}_{\beth}}{\sum_{\beth=1}^{n} \check{\hbar}_{\beth}}}, 1 - \prod_{\beth=1}^{n} (1 - \eta^{\upsilon}{}_{\beth})^{\frac{\check{\hbar}_{\beth}}{\sum_{\beth=1}^{n} \check{\hbar}_{\beth}}} \right\rangle, \right. \\
& \left. \left\langle \prod_{\beth=1}^{n} \mathscr{J}^{\aleph}{}_{\beth}^{\frac{\check{\hbar}_{\beth}}{\sum_{\beth=1}^{n} \check{\hbar}_{\beth}}}, 1 - \prod_{\beth=1}^{n} (1 - \mathscr{C}^{\gamma}{}_{\beth})^{\frac{\check{\hbar}_{\beth}}{\sum_{\beth=1}^{n} \check{\hbar}_{\beth}}} \right\rangle \right).
\end{aligned}
\tag{14.4}
$$

Definition 12 ([56]) Let $\daleth^{\varsigma}{}_{\beth} = \left(\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth}\rangle\right)$ be the agglomeration of LDFNs and LDFFWA: $\mathscr{F}^{n} \rightarrow \mathscr{F}$ be a n dimension mapping. If

$$
\text{LDFFWA}(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{e}) = \left(\Re^{\Im}{}_{1} * \daleth^{\varsigma}{}_{1} \tilde{\oplus} \Re^{\Im}{}_{2} * \daleth^{\varsigma}{}_{2} \tilde{\oplus} \ldots, \tilde{\oplus} \Re^{\Im}{}_{e} * \daleth^{\varsigma}{}_{e} \right),
\tag{14.5}
$$

then the mapping LDFFWA is called “linear Diophantine fuzzy fairly weighted averaging (LDFFWA) operator,” and here $\Re^{\Im}{}_{i}$ is the weight vector (WV) of $\daleth^{\varsigma}{}_{i}$ with $\Re^{\Im}{}_{i} > 0$ and $\sum_{i=1}^{e} \Re^{\Im}{}_{i} = 1$.

Theorem 5 ([56]) *Let* $\daleth^{\varsigma}{}_{\beth} = \left(\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth}\rangle\right)$ *be the agglomeration of LDFNs, and we can also find LDFFWA by*

$$
\begin{aligned}
& LDFFWA(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots, \daleth^{\varsigma}{}_{e}) \\
& = \left(\begin{array}{l}
\left\langle \frac{1}{2} \frac{\prod_{i=1}^{e} (\zeta^{\tau}{}_{i})^{\Re^{\Im}{}_{i}}}{\prod_{i=1}^{e} (\zeta^{\tau}{}_{i})^{\Re^{\Im}{}_{i}} + \prod_{i=1}^{e} (\eta^{\upsilon}{}_{i})^{\Re^{\Im}{}_{i}}} \times \left(1 + \prod_{i=1}^{e} \left(2 - \zeta^{\tau}{}_{i} - \eta^{\upsilon}{}_{i}\right)^{\Re^{\Im}{}_{i}}\right), \right. \\
\left. \frac{1}{2} \frac{\prod_{i=1}^{e} (\eta^{\upsilon}{}_{i})^{\Re^{\Im}{}_{i}}}{\prod_{i=1}^{e} (\zeta^{\tau}{}_{i})^{\Re^{\Im}{}_{i}} + \prod_{i=1}^{e} (\eta^{\upsilon}{}_{i})^{\Re^{\Im}{}_{i}}} \times \left(1 + \prod_{i=1}^{e} \left(2 - \zeta^{\tau}{}_{i} - \eta^{\upsilon}{}_{i}\right)^{\Re^{\Im}{}_{i}}\right) \right\rangle, \\
\left\langle \frac{\prod_{i=1}^{e} (\mathscr{J}^{\aleph}{}_{i})^{\Re^{\Im}{}_{i}}}{\prod_{i=1}^{e} (\mathscr{J}^{\aleph}{}_{i})^{\Re^{\Im}{}_{i}} + \prod_{i=1}^{e} (\mathscr{C}^{\gamma}{}_{i})^{\Re^{\Im}{}_{i}}} \times \left(1 - \prod_{i=1}^{e} \left(1 - \mathscr{J}^{\aleph}{}_{i} - \mathscr{C}^{\gamma}{}_{i}\right)^{\Re^{\Im}{}_{i}}\right), \right. \\
\left. \frac{\prod_{i=1}^{e} (\mathscr{C}^{\gamma}{}_{i})^{\Re^{\Im}{}_{i}}}{\prod_{i=1}^{e} (\mathscr{J}^{\aleph}{}_{i})^{\Re^{\Im}{}_{i}} + \prod_{i=1}^{e} (\mathscr{C}^{\gamma}{}_{i})^{\Re^{\Im}{}_{i}}} \times \left(1 - \prod_{i=1}^{e} \left(1 - \mathscr{J}^{\aleph}{}_{i} - \mathscr{C}^{\gamma}{}_{i}\right)^{\Re^{\Im}{}_{i}}\right) \right\rangle,
\end{array} \right)
\end{aligned}
$$

where $\Re^{\Im}{}_{i}$ *is the WV of* $\daleth^{\varsigma}{}_{i}$ *with* $\Re^{\Im}{}_{i} > 0$ *and* $\sum_{i=1}^{e} \Re^{\Im}{}_{i} = 1$.

Definition 13 ([56]) Let $\daleth^{\varsigma}{}_{\beth} = \left(\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth}\rangle\right)$ be the agglomeration of LDFNs and LDFFOWA: $\mathscr{F}^{n} \rightarrow \mathscr{F}$ be a n dimension mapping. If

$$\text{LDFFOWA}(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_e)$$
$$= \left(\Re^{\Im}{}_1 * \daleth^{\varsigma}{}_{\tau(1)} \tilde{\oplus} \Re^{\Im}{}_2 * \daleth^{\varsigma}{}_{\tau(2)} \tilde{\oplus} \ldots, \tilde{\oplus} \Re^{\Im}{}_e * \daleth^{\varsigma}{}_{\tau(e)}\right), \tag{14.6}$$

then the mapping LDFFOWA is called "linear Diophantine fuzzy fairly ordered weighted averaging (LDFFOWA) operator," and here $\Re^{\Im}{}_i$ is the WV of $\daleth^{\varsigma}{}_i$ with $\Re^{\Im}{}_i > 0$ and $\sum_{i=1}^{e} \Re^{\Im}{}_i = 1$.
$\zeta^{\tau} : 1, 2, 3, \ldots\ldots, n \rightarrow 1, 2, 3, \ldots\ldots, n$ is a permutation map s.t. $\daleth^{\varsigma}{}_{\tau(i-1)} \geq \daleth^{\varsigma}{}_{\tau(i)}$.

Theorem 6 ([56]) *Let $\daleth^{\varsigma}{}_{\beth} = \left(\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth}\rangle\right)$ be the agglomeration of LDFNs, and we can also find LDFFOWA by*

$$LDFFOWA(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots, \daleth^{\varsigma}{}_e)$$

$$= \begin{pmatrix} \left\langle \frac{1}{2} \frac{\prod_{i=1}^{e}\left(\zeta^{\tau}{}_{\tau(i)}\right)^{\Re^{\Im}{}_i}}{\prod_{i=1}^{e}\left(\zeta^{\tau}{}_{\tau(i)}\right)^{\Re^{\Im}{}_{\tau(i)}} + \prod_{i=1}^{e}\left(\eta^{\upsilon}{}_{\tau(i)}\right)^{\Re^{\Im}{}_i}} \times \left(1 + \prod_{i=1}^{e}\left(2 - \zeta^{\tau}{}_{\tau(i)} - \eta^{\upsilon}{}_{\tau(i)}\right)^{\Re^{\Im}{}_i}\right), \right. \\ \left. \frac{1}{2} \frac{\prod_{i=1}^{e}\left(\eta^{\upsilon}{}_{\tau(i)}\right)^{\Re^{\Im}{}_i}}{\prod_{i=1}^{e}\left(\zeta^{\tau}{}_{\tau(i)}\right)^{\Re^{\Im}{}_i} + \prod_{i=1}^{e}\left(\eta^{\upsilon}{}_{\tau(i)}\right)^{\Re^{\Im}{}_i}} \times \left(1 + \prod_{i=1}^{e}\left(2 - \zeta^{\tau}{}_{\tau(i)} - \eta^{\upsilon}{}_{\tau(i)}\right)^{\Re^{\Im}{}_i}\right) \right\rangle, \\ \left\langle \frac{\prod_{i=1}^{e}\left(\mathscr{J}^{\aleph}{}_{\tau(i)}\right)^{\Re^{\Im}{}_i}}{\prod_{i=1}^{e}\left(\mathscr{J}^{\aleph}{}_{\tau(i)}\right)^{\Re^{\Im}{}_{\tau(i)}} + \prod_{i=1}^{e}\left(\mathscr{C}^{\gamma}{}_{\tau(i)}\right)^{\Re^{\Im}{}_i}} \times \left(1 - \prod_{i=1}^{e}\left(1 - \mathscr{J}^{\aleph}{}_{\tau(i)} - \mathscr{C}^{\gamma}{}_{\tau(i)}\right)^{\Re^{\Im}{}_i}\right), \right. \\ \left. \frac{\prod_{i=1}^{e}\left(\mathscr{C}^{\gamma}{}_{\tau(i)}\right)^{\Re^{\Im}{}_i}}{\prod_{i=1}^{e}\left(\mathscr{J}^{\aleph}{}_{\tau(i)}\right)^{\Re^{\Im}{}_i} + \prod_{i=1}^{e}\left(\mathscr{C}^{\gamma}{}_{\tau(i)}\right)^{\Re^{\Im}{}_i}} \times \left(1 - \prod_{i=1}^{e}\left(1 - \mathscr{J}^{\aleph}{}_{\tau(i)} - \mathscr{C}^{\gamma}{}_{\tau(i)}\right)^{\Re^{\Im}{}_i}\right) \right\rangle \end{pmatrix},$$

where $\Re^{\Im}{}_i$ is the WV of $\daleth^{\varsigma}{}_i$ with $\Re^{\Im}{}_i > 0$ and $\sum_{i=1}^{e} \Re^{\Im}{}_i = 1$.

Definition 14 ([28]) Consider $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth}\rangle)$ the agglomeration of LDFNs and $\Re^{\Im} = (\Re^{\Im}{}_1, \Re^{\Im}{}_2, \ldots, \Re^{\Im}{}_n)^T$ be the weight vector (WV) with $\sum_{\beth=1}^{n} \Re^{\Im}{}_{\beth} = 1$. Then "linear Diophantine fuzzy Einstein weighted average (LDFEWA) operator" is defined as

$$LDFEWA(\hbar_1^{\kappa}, \hbar_2^{\kappa}, \hbar_3^{\kappa}, \ldots, \hbar_n^{\kappa}) =$$
$$\sum_{\beth=1}^{n} \Re^{\Im}{}_{\Lambda} \daleth^{\varsigma}{}_{\beth} = \Re^{\Im}{}_1 \cdot_{\mathcal{E}} \hbar_1^{\kappa} \oplus_{\mathcal{E}} \Re^{\Im}{}_2 \cdot_{\mathcal{E}} \hbar_2^{\kappa} \oplus_{\mathcal{E}} \Re^{\Im}{}_3 \cdot_{\mathcal{E}} \gamma_3 \oplus_{\mathcal{E}} \ldots \oplus_{\mathcal{E}} \Re^{\Im}{}_n \cdot_{\mathcal{E}} \hbar_n^{\kappa}.$$

In LDFEWA operator, we use $\Re^{\Im}$ as a WV and $\daleth^{\varsigma}{}_{\beth}$ are the LDFNs, where $\beth = 1, 2, \ldots, n$.

Theorem 7 ([28]) *Let* $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth}\rangle)$ *be an agglomeration of LDFNs and* $\Re^{\Im} = (\Re^{\Im}{}_1, \Re^{\Im}{}_2, \ldots, \Re^{\Im}{}_n)^T$ *be the WV with* $\sum_{\beth=1}^{n} \Re^{\Im}{}_{\beth} = 1$. *Then the LDFEWA operator can also be written as*

$$LDFEWA(\hbar_1^{\kappa}, \hbar_2^{\kappa}, \ldots, \hbar_n^{\kappa}) = \Bigg(\Bigg\langle \frac{\prod_{\beth=1}^{n}(1+\zeta^{\tau}{}_{\beth})^{\Re^{\Im}{}_{\beth}} - \prod_{\beth=1}^{n}(1-\zeta^{\tau}{}_{\beth})^{\Re^{\Im}{}_{\beth}}}{\prod_{\beth=1}^{n}(1+\zeta^{\tau}{}_{\beth})^{\Re^{\Im}{}_{\beth}} + \prod_{\beth=1}^{n}(1-\zeta^{\tau}{}_{\beth})^{\Re^{\Im}{}_{\beth}}}, \frac{2\prod_{\beth=1}^{n} {\eta^{\upsilon}{}_{\beth}}^{\Re^{\Im}{}_{\beth}}}{\prod_{\beth=1}^{n}(2-\eta^{\upsilon}{}_{\beth})^{\Re^{\Im}{}_{\beth}} + \prod_{\beth=1}^{n}(\eta^{\upsilon}{}_{\beth})^{\Re^{\Im}{}_{\beth}}} \Bigg\rangle, \Bigg\langle \frac{\prod_{\beth=1}^{n}(1+\mathscr{J}^{\aleph}{}_{\beth})^{\Re^{\Im}{}_{\beth}} - \prod_{\beth=1}^{n}(1-\mathscr{J}^{\aleph}{}_{\beth})^{\Re^{\Im}{}_{\beth}}}{\prod_{\beth=1}^{n}(1+\mathscr{J}^{\aleph}{}_{\beth})^{\Re^{\Im}{}_{\beth}} + \prod_{\beth=1}^{n}(1-\mathscr{J}^{\aleph}{}_{\beth})^{\Re^{\Im}{}_{\beth}}}, \frac{2\prod_{\beth=1}^{n} {\mathscr{C}^{\gamma}{}_{\beth}}^{\Re^{\Im}{}_{\beth}}}{\prod_{\beth=1}^{n}(2-\mathscr{C}^{\gamma}{}_{\beth})^{\Re^{\Im}{}_{\beth}} + \prod_{\beth=1}^{n}(\mathscr{C}^{\gamma}{}_{\beth})^{\Re^{\Im}{}_{\beth}}} \Bigg\rangle\Bigg).$$

Definition 15 ([28]) Consider $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth}\rangle)$ is the agglomeration of LDFNs and $\Re^{\Im} = (\Re^{\Im}{}_1, \Re^{\Im}{}_2, \ldots, \Re^{\Im}{}_n)^T$ be the WV with $\sum_{\beth=1}^{n} \Re^{\Im}{}_{\beth} = 1$. Then "linear Diophantine fuzzy Einstein weighted geometric (LDFEWG) operator" is defined as

$$LDFEWG(\hbar_1^{\kappa}, \hbar_2^{\kappa}, \hbar_3^{\kappa}, \ldots, \hbar_n^{\kappa}) = \prod_{\beth=1}^{n} \Re^{\Im}{}_{\beth} \daleth^{\varsigma}{}_{\beth} = \Re^{\Im}{}_1 \cdot_{\mathcal{E}} \hbar_1^{\kappa} \otimes_{\mathcal{E}} \Re^{\Im}{}_2 \cdot_{\mathcal{E}} \hbar_2^{\kappa} \otimes_{\mathcal{E}} \Re^{\Im}{}_3 \cdot_{\mathcal{E}} \hbar_3^{\kappa} \otimes_{\mathcal{E}} \ldots \otimes_{\mathcal{E}} \Re^{\Im}{}_n \cdot_{\mathcal{E}} \hbar_n^{\kappa}.$$

Theorem 8 ([28]) *Let* $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth}\rangle)$ *be the agglomeration of LDFNs and* $\Re^{\Im} = (\Re^{\Im}{}_1, \Re^{\Im}{}_2, \ldots, \Re^{\Im}{}_n)^T$ *be the WV with* $\sum_{\beth=1}^{n} \Re^{\Im}{}_{\beth} = 1$. *Then LDFEWG operator can also be written as*

$$LDFEWG(\hbar_1^{\kappa}, \hbar_2^{\kappa}, \ldots, \hbar_n^{\kappa}) = \Bigg(\Bigg\langle \frac{2\prod_{\beth=1}^{n} {\zeta^{\tau}{}_{\beth}}^{\Re^{\Im}{}_{\beth}}}{\prod_{\beth=1}^{n}(2-\zeta^{\tau}{}_{\beth})^{\Re^{\Im}{}_{\beth}} + \prod_{\beth=1}^{n}(\zeta^{\tau}{}_{\beth})^{\Re^{\Im}{}_{\beth}}}, \frac{\prod_{\beth=1}^{n}(1+\eta^{\upsilon}{}_{\beth})^{\Re^{\Im}{}_{\beth}} - \prod_{\beth=1}^{n}(1-\eta^{\upsilon}{}_{\beth})^{\Re^{\Im}{}_{\beth}}}{\prod_{\beth=1}^{n}(1+\eta^{\upsilon}{}_{\beth})^{\Re^{\Im}{}_{\beth}} + \prod_{\beth=1}^{n}(1-\eta^{\upsilon}{}_{\beth})^{\Re^{\Im}{}_{\beth}}} \Bigg\rangle,$$

$$\left\langle \frac{2\prod_{\beth=1}^{n} \mathscr{I}^{\aleph}{}_{\beth}^{\mathfrak{R}^{\mathfrak{I}}{}_{\beth}}}{\prod_{\beth=1}^{n}(2-\mathscr{I}^{\aleph}{}_{\beth})^{\mathfrak{R}^{\mathfrak{I}}{}_{\beth}}+\prod_{\beth=1}^{n}(\mathscr{I}^{\aleph}{}_{\beth})^{\mathfrak{R}^{\mathfrak{I}}{}_{\beth}}}, \right.$$
$$\left. \frac{\prod_{\beth=1}^{n}(1+\mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{R}^{\mathfrak{I}}{}_{\beth}}-\prod_{\beth=1}^{n}(1-\mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{R}^{\mathfrak{I}}{}_{\beth}}}{\prod_{\beth=1}^{n}(1+\mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{R}^{\mathfrak{I}}{}_{\beth}}+\prod_{\beth=1}^{n}(1-\mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{R}^{\mathfrak{I}}{}_{\beth}}} \right\rangle \Bigg).$$

AOs are used in a variety of fields to summarize and analyze large sets of data. They are commonly used in business and finance to summarize financial data, in computer science and programming to analyze log files and performance metrics, and in data science and machine learning to extract insights from large datasets.

In business and finance, AOs are used to summarize financial data such as sales and revenue. For example, a company may use an operator to calculate the total revenue for a particular product or product line. This information can then be used to make decisions about pricing, production, and marketing.

In computer science and programming, AOs are used to analyze log files and performance metrics. For example, a web developer may use an operator to calculate the average response time of a web server or the number of requests per second. This information can be used to identify performance bottlenecks and optimize the performance of the system.

In data science and machine learning, AOs are used to extract insights from large datasets. For example, a data scientist may use an operator to calculate the average of a particular variable in a dataset. This information can be used to identify patterns and trends in the data, which can inform decisions about which variables to include in a model or which groups to target in a marketing campaign.

In the field of natural language processing, AOs are used to extract insights from text data. For example, a researcher may use an operator to calculate the most common words or phrases in a dataset of text. This information can be used to identify topics or themes in the data, which can inform decisions about which algorithms to use for text classification or sentiment analysis.

In bioinformatics, AOs are used to summarize and analyze large sets of genetic data. For example, a researcher may use an operator to calculate the frequency of a particular genetic variant in a population. This information can be used to identify genetic risk factors for diseases and inform drug development.

In general, AOs are a powerful tool for extracting insights from large sets of data. They can be used to summarize data, identify patterns and trends, and inform decisions across a wide range of fields.

3 Linear Diophantine Fuzzy Aggregation Operators

In this section, we discussed “linear Diophantine fuzzy weighted average (LDFWA) operator, linear Diophantine fuzzy ordered weighted average (LDFOWA) operator, linear Diophantine fuzzy weighted geometric (LDFWG) operator and linear Diophantine fuzzy weighted ordered geometric (LDFOWG) operator."

3.1 LDFWA Operator

Definition 16 Consider $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth}\rangle)$ is the agglomeration of LDFNs, and LDFWA : $\$^{n} \rightarrow \$$ be the mapping.

$$\text{LDFWA}(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{n}) = \mathfrak{P}^{\gamma}{}_{1}\daleth^{\varsigma}{}_{1} \oplus \mathfrak{P}^{\gamma}{}_{2}\daleth^{\varsigma}{}_{2} \oplus \ldots, \oplus \mathfrak{P}^{\gamma}{}_{n}\daleth^{\varsigma}{}_{n}. \tag{14.7}$$

Then LDFWA is known as LDFWA operator, where $(\mathfrak{P}^{\gamma}{}_{1}, \mathfrak{P}^{\gamma}{}_{2}, \ldots, \mathfrak{P}^{\gamma}{}_{n})$ be the weight vector (WV) with the constraint $\mathfrak{P}^{\gamma}{}_{\beth} > 0$ and $\sum_{h=1}^{n} \mathfrak{P}^{\gamma}{}_{\beth} = 1$.

We also evaluate LDFWA operator by the following theorem.

Theorem 9 *Consider* $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth}\rangle)$ *is the agglomeration of LDFNs, and we can find LDFWA by*

$$\begin{aligned} &LDFWA(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{n}) \\ &= \left(\left\langle 1 - \overline{\prod}_{\beth=1}^{n} (1 - \zeta^{\tau}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}, \overline{\prod}_{\beth=1}^{n} \eta^{\upsilon}{}_{\beth}^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle, \right. \\ &\left. \left\langle 1 - \overline{\prod}_{\beth=1}^{n} (1 - \mathscr{J}^{\aleph}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}, \overline{\prod}_{\beth=1}^{n} \mathscr{C}^{\gamma}{}_{\beth}^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle \right). \end{aligned} \tag{14.8}$$

Proof It is quite simple for the first assertion to come before Definition 17 and Theorem 13. The following instances demonstrate this point further:

$$\begin{aligned} &\text{LDFWA}(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{n}) \\ &= \left(\mathfrak{P}^{\gamma} 1 \daleth^{\varsigma}{}_{1} \oplus \mathfrak{P}^{\gamma}{}_{2}\daleth^{\varsigma}{}_{2} \oplus \ldots, \mathfrak{P}^{\gamma}{}_{n}\daleth^{\varsigma}{}_{n} \right) \\ &= \left(\left\langle 1 - \overline{\prod}_{\beth=1}^{n} (1 - \zeta^{\tau}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}, \overline{\prod}_{\beth=1}^{n} \eta^{\upsilon}{}_{\beth}^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle, \right. \\ &\left. \left\langle \overline{\prod}_{\beth=1}^{n} (1 - \mathscr{J}^{\aleph}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}, \overline{\prod}_{\beth=1}^{n} \mathscr{C}^{\gamma}{}_{\beth}^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle \right). \end{aligned}$$

In order to demonstrate the validity of this theorem, we turned to mathematics induction.

For $n = 2$

$$\mathfrak{P}^{\gamma}{}_{1}\daleth^{\varsigma}{}_{1}=\left(\left\langle 1-(1-\zeta^{\tau}{}_{1})^{\mathfrak{P}^{\gamma}{}_{1}},\eta^{\upsilon\mathfrak{P}^{\gamma}{}_{1}}_{1}\right\rangle,\left\langle 1-(1-\mathscr{J}^{\aleph}{}_{1})^{\mathfrak{P}^{\gamma}{}_{1}},\mathscr{C}^{\gamma\mathfrak{P}^{\gamma}{}_{1}}_{1}\right\rangle\right)$$

$$\mathfrak{P}^{\gamma}{}_{2}\daleth^{\varsigma}{}_{2}=\left(\left\langle 1-(1-\zeta^{\tau}{}_{2})^{\mathfrak{P}^{\gamma}{}_{1}},\eta^{\upsilon\mathfrak{P}^{\gamma}{}_{1}}_{2}\right\rangle,\left\langle 1-(1-\mathscr{J}^{\aleph}{}_{2})^{\mathfrak{P}^{\gamma}{}_{1}},\mathscr{C}^{\gamma\mathfrak{P}^{\gamma}{}_{1}}_{2}\right\rangle\right).$$

Then

$$\begin{aligned}
&\mathfrak{P}^{\gamma}{}_{1}\daleth^{\varsigma}{}_{1}\oplus\mathfrak{P}^{\gamma}{}_{2}\daleth^{\varsigma}{}_{2}\\
&=\left(\left\langle 1-(1-\zeta^{\tau}{}_{1})^{\mathfrak{P}^{\gamma}{}_{1}},\eta^{\upsilon\mathfrak{P}^{\gamma}{}_{1}}_{1}\right\rangle,\left\langle 1-(1-\mathscr{J}^{\aleph}{}_{1})^{\mathfrak{P}^{\gamma}{}_{1}},\mathscr{C}^{\gamma\mathfrak{P}^{\gamma}{}_{1}}_{1}\right\rangle\right)\oplus\\
&\quad\left(\left\langle 1-(1-\zeta^{\tau}{}_{2})^{\mathfrak{P}^{\gamma}{}_{1}},\eta^{\upsilon\mathfrak{P}^{\gamma}{}_{1}}_{2}\right\rangle,\left\langle 1-(1-\mathscr{J}^{\aleph}{}_{2})^{\mathfrak{P}^{\gamma}{}_{1}},\mathscr{C}^{\gamma\mathfrak{P}^{\gamma}{}_{1}}_{2}\right\rangle\right)\\
&=\left(\left\langle 1-(1-\zeta^{\tau}{}_{1})^{\mathfrak{P}^{\gamma}{}_{1}}+1-(1-\zeta^{\tau}{}_{2})^{\mathfrak{P}^{\gamma}{}_{1}}-\left((1-(1-\zeta^{\tau}{}_{1})^{\mathfrak{P}^{\gamma}{}_{1}}\right)\right.\right.\\
&\quad\left.\left((1-(1-\zeta^{\tau}{}_{2})^{\mathfrak{P}^{\gamma}{}_{1}}\right),\eta^{\upsilon\mathfrak{P}^{\gamma}{}_{1}}_{1}.\eta^{\upsilon\mathfrak{P}^{\gamma}{}_{1}}_{2}\right\rangle,\left\langle 1-(1-\mathscr{J}^{\aleph}{}_{1})^{\mathfrak{P}^{\gamma}{}_{1}}+1-(1-\mathscr{J}^{\aleph}{}_{2})^{\mathfrak{P}^{\gamma}{}_{1}}\right.\\
&\quad\left.\left.-\left((1-(1-\mathscr{J}^{\aleph}{}_{1})^{\mathfrak{P}^{\gamma}{}_{1}}\right)\left(1-(1-\mathscr{J}^{\aleph}{}_{2})^{\mathfrak{P}^{\gamma}{}_{1}}\right),\mathscr{C}^{\gamma\mathfrak{P}^{\gamma}{}_{1}}_{1}.\mathscr{C}^{\gamma\mathfrak{P}^{\gamma}{}_{1}}_{2}\right\rangle\right)
\end{aligned}$$

$$\begin{aligned}
&=\left(\left\langle 1-(1-\zeta^{\tau}{}_{1})^{\mathfrak{P}^{\gamma}{}_{1}}(1-\zeta^{\tau}{}_{2})^{\mathfrak{P}^{\gamma}{}_{1}},\eta^{\upsilon\mathfrak{P}^{\gamma}{}_{1}}_{1}.\eta^{\upsilon\mathfrak{P}^{\gamma}{}_{1}}_{2}\right\rangle,\right.\\
&\quad\left.\left\langle 1-(1-\mathscr{J}^{\aleph}{}_{1})^{\mathfrak{P}^{\gamma}{}_{1}}(1-\mathscr{J}^{\aleph}{}_{2})^{\mathfrak{P}^{\gamma}{}_{1}},\mathscr{C}^{\gamma\mathfrak{P}^{\gamma}{}_{1}}_{1}.\mathscr{C}^{\gamma\mathfrak{P}^{\gamma}{}_{1}}_{2}\right\rangle\right)
\end{aligned}$$

$$\begin{aligned}
&=\left(\left\langle 1-\overline{\prod}_{\gimel=1}^{2}(1-\zeta^{\tau}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}},\overline{\prod}_{\gimel=1}^{2}\eta^{\upsilon\mathfrak{P}^{\gamma}{}_{\gimel}}_{\gimel}\right\rangle,\right.\\
&\quad\left.\left\langle 1-\overline{\prod}_{\gimel=1}^{2}(1-\mathscr{J}^{\aleph}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}},\overline{\prod}_{\gimel=1}^{2}\mathscr{C}^{\gamma\mathfrak{P}^{\gamma}{}_{\gimel}}_{\gimel}\right\rangle\right).
\end{aligned}$$

This demonstrates that Eq. (14.10) is correct for the value of n equal to two; now assume that Eq. (14.10) is accurate for the value of n equal to k, i.e.,

$$\text{LDFWA}(\daleth^{\varsigma}{}_{1},\daleth^{\varsigma}{}_{2},\ldots\daleth^{\varsigma}{}_{k})$$

$$= \left(\left\langle 1-\overline{\prod}_{\beth=1}^{k}(1-\zeta^{\tau}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}, \overline{\prod}_{\beth=1}^{k}\eta^{\upsilon}{}_{\beth}^{\mathfrak{P}^{\gamma}{}_{\beth}}\right\rangle,\right.$$
$$\left.\left\langle 1-\overline{\prod}_{\beth=1}^{k}(1-\mathscr{J}^{\aleph}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}, \overline{\prod}_{\beth=1}^{k}\mathscr{C}^{\gamma}{}_{\beth}^{\mathfrak{P}^{\gamma}{}_{\beth}}\right\rangle\right).$$

Now that $n = k + 1$, according to the operational laws that govern LDFNs, we obtain

$$\text{LDFWA}(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{k+1}) = \text{LDFWA}(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{k}) \oplus \mathfrak{P}^{\gamma}{}_{\beth}\daleth^{\varsigma}{}_{k+1}$$
$$= \left(\left\langle 1-\overline{\prod}_{\beth=1}^{k}(1-\zeta^{\tau}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}, \overline{\prod}_{\beth=1}^{k}\eta^{\upsilon}{}_{\beth}^{\mathfrak{P}^{\gamma}{}_{\beth}}\right\rangle,\right.$$
$$\left.\left\langle 1-\overline{\prod}_{\beth=1}^{k}(1-\mathscr{J}^{\aleph}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}, \overline{\prod}_{\beth=1}^{k}\mathscr{C}^{\gamma}{}_{\beth}^{\mathfrak{P}^{\gamma}{}_{\beth}}\right\rangle\right)\oplus$$
$$\left(\left\langle 1-(1-\zeta^{\tau}{}_{k+1})^{\mathfrak{P}^{\gamma}{}_{k+1}}, \eta^{\upsilon}{}_{k+1}^{\mathfrak{P}^{\gamma}{}_{k+1}}\right\rangle, \left\langle 1-(1-\mathscr{J}^{\aleph}{}_{k+1})^{\mathfrak{P}^{\gamma}{}_{k+1}}, \mathscr{C}^{\gamma}{}_{k+1}^{\mathfrak{P}^{\gamma}{}_{k+1}}\right\rangle\right)$$
$$= \left(\left\langle 1-\overline{\prod}_{\beth=1}^{k}(1-\zeta^{\tau}{}_{k})^{\mathfrak{P}^{\gamma}{}_{\beth}} + 1-(1-\zeta^{\tau}{}_{k+1})^{\mathfrak{P}^{\gamma}{}_{k+1}}\right.\right.$$
$$-\left(1-\overline{\prod}_{\beth=1}^{k}(1-\zeta^{\tau}{}_{k})^{\mathfrak{P}^{\gamma}{}_{\beth}}\right)\left(1-(1-\zeta^{\tau}{}_{k+1})^{\mathfrak{P}^{\gamma}{}_{k+1}}\right),$$
$$\left.\overline{\prod}_{\beth=1}^{k}\eta^{\upsilon}{}_{k}^{\mathfrak{P}^{\gamma}{}_{\beth}}.\eta^{\upsilon}{}_{k+1}^{\mathfrak{P}^{\gamma}{}_{k+1}}\right\rangle, \left\langle 1-\overline{\prod}_{\beth=1}^{k}(1-\mathscr{J}^{\aleph}{}_{k})^{\mathfrak{P}^{\gamma}{}_{\beth}} + 1-(1-\mathscr{J}^{\aleph}{}_{k+1})^{\mathfrak{P}^{\gamma}{}_{k+1}}\right.$$
$$-\left(1-\overline{\prod}_{\beth=1}^{k}(1-\mathscr{J}^{\aleph}{}_{k})^{\mathfrak{P}^{\gamma}{}_{\beth}}\right)\left(1-(1-\mathscr{J}^{\aleph}{}_{k+1})^{\mathfrak{P}^{\gamma}{}_{k+1}}\right),$$
$$\left.\left.\overline{\prod}_{\beth=1}^{k}\mathscr{C}^{\gamma}{}_{k}^{\mathfrak{P}^{\gamma}{}_{\beth}}.\mathscr{C}^{\gamma}{}_{k+1}^{\mathfrak{P}^{\gamma}{}_{k+1}}\right\rangle\right)$$
$$= \left(\left\langle 1-\overline{\prod}_{\beth=1}^{k}(1-\zeta^{\tau}{}_{k})^{\mathfrak{P}^{\gamma}{}_{\beth}}(1-\zeta^{\tau}{}_{k+1})^{k+1}, \overline{\prod}_{\beth=1}^{k}\eta^{\upsilon}{}_{k}^{\mathfrak{P}^{\gamma}{}_{\beth}}.\eta^{\upsilon}{}_{k+1}^{\mathfrak{P}^{\gamma}{}_{k+1}}\right\rangle,\right.$$
$$\left.\left\langle 1-\overline{\prod}_{\beth=1}^{k}(1-\mathscr{J}^{\aleph}{}_{k})^{\mathfrak{P}^{\gamma}{}_{\beth}}(1-\mathscr{J}^{\aleph}{}_{k+1})^{k+1}, \overline{\prod}_{\beth=1}^{k}\mathscr{C}^{\gamma}{}_{k}^{\mathfrak{P}^{\gamma}{}_{\beth}}.\mathscr{C}^{\gamma}{}_{k+1}^{\mathfrak{P}^{\gamma}{}_{k+1}}\right\rangle\right)$$

$$= \left(\left\langle 1-\overline{\prod}_{\beth=1}^{k+1}(1-\zeta^{\tau}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}, \overline{\prod}_{\beth=1}^{k+1}\eta^{\upsilon}{}_{\beth}^{\mathfrak{P}^{\gamma}{}_{\beth}}\right\rangle,\right.$$
$$\left.\left\langle 1-\overline{\prod}_{\beth=1}^{k+1}(1-\mathscr{J}^{\aleph}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}, \overline{\prod}_{\beth=1}^{k+1}\mathscr{C}^{\gamma}{}_{\beth}^{\mathfrak{P}^{\gamma}{}_{\beth}}\right\rangle\right).$$

This shows that for $n = k + 1$,, Eq. (14.10) holds. Then,

$$\begin{aligned}&\text{LDFWA}(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n)\\&= \left(\left\langle 1 - \overline{\prod}_{\beth=1}^{n}(1-\zeta^{\tau}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}, \overline{\prod}_{\beth=1}^{n}\eta^{\upsilon}{}_{\beth}^{\mathfrak{P}^{\gamma}{}_{\beth}}\right\rangle,\right.\\&\left.\left\langle 1 - \overline{\prod}_{\beth=1}^{n}(1-\mathscr{J}^{\aleph}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}, \overline{\prod}_{\beth=1}^{n}\mathscr{C}^{\gamma}{}_{\beth}^{\mathfrak{P}^{\gamma}{}_{\beth}}\right\rangle\right).\end{aligned}$$

The next couple of paragraphs will discuss a few of the beneficial qualities that LDFWA operator has.

Theorem 10 (Idempotency) *Assume that* $\daleth^{\varsigma}{}_{\beth} = (\langle\zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth}\rangle, \langle\mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth}\rangle)$ *is the agglomeration of LDFNs, where* $\breve{\hbar}_{\beth} = \overline{\prod}_{k=1}^{j-1}\mathscr{H}(\daleth^{\varsigma}{}_k)$ $(j = 2\ldots, n)$, $\breve{\hbar}_1 = 1$, *and* $\mathscr{H}(\daleth^{\varsigma}{}_k)$ *is the expectation SF of kth LDFN. If all* $\daleth^{\varsigma}{}_{\beth}$ *are equal, i.e.,* $\daleth^{\varsigma}{}_{\beth} = \daleth^{\varsigma}$ *for all* j*, then*

$$LDFWA(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) = \daleth^{\varsigma}.$$

Proof From Definition 17, we have

$$\begin{aligned}\text{LDFWA}(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) &= \mathfrak{P}^{\gamma}{}_1\daleth^{\varsigma}{}_1 \oplus \mathfrak{P}^{\gamma}{}_2\daleth^{\varsigma}{}_2 \oplus \ldots, \oplus\mathfrak{P}^{\gamma}{}_n\daleth^{\varsigma}{}_n\\&= \mathfrak{P}^{\gamma}{}_1\daleth^{\varsigma} \oplus \mathfrak{P}^{\gamma}{}_2\daleth^{\varsigma} \oplus \ldots, \oplus\mathfrak{P}^{\gamma}{}_n\daleth^{\varsigma}\\&= (\mathfrak{P}^{\gamma}{}_1 + \mathfrak{P}^{\gamma}{}_2 + \ldots + \mathfrak{P}^{\gamma}{}_n)\daleth^{\varsigma}\\&= \daleth^{\varsigma}.\end{aligned}$$

Corollary 1 *If* $\daleth^{\varsigma}{}_{\beth} = (\langle\zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth}\rangle, \langle\mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth}\rangle)$, $j = (1, 2, \ldots n)$ *is the agglomeration of largest LDFNs, i.e.,* $\daleth^{\varsigma}{}_{\beth} = \langle(1, 0), (1, 0)\rangle$ *for all* j*, then*

$$LDFWA(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) = \langle(1, 0), (1, 0)\rangle.$$

Proof We can easily obtain Corollary similar to Theorem 10.

Theorem 11 (Monotonicity) *Assume that* $\daleth^{\varsigma}{}_{\beth} = (\langle\zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth}\rangle, \langle\mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth}\rangle)$ *and* $\daleth^{\varsigma}{}_{\beth}^{*} = (\langle\zeta^{\tau}{}_{\beth}^{*}, \eta^{\upsilon}{}_{\beth}^{*}\rangle, \langle\mathscr{J}^{\aleph}{}_{\beth}^{*}, \mathscr{C}^{\gamma}{}_{\beth}^{*}\rangle)$ *are the agglomerations of LDFNs. If* $\zeta^{\tau}{}_{\beth}^{*} \geq \zeta^{\tau}{}_{\beth}$, $\eta^{\upsilon}{}_{\beth}^{*} \leq \eta^{\upsilon}{}_{\beth}$, $\mathscr{J}^{\aleph}{}_{\beth}^{*} \geq \mathscr{J}^{\aleph}{}_{\beth}$, *and* $\mathscr{C}^{\gamma}{}_{\beth}^{*} \leq \mathscr{C}^{\gamma}{}_{\beth}$ *for all* j*, then*

$$LDFWA(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) \leq LDFWA(\daleth^{\varsigma}{}_1^{*}, \daleth^{\varsigma}{}_2^{*}, \ldots \daleth^{\varsigma}{}_n^{*}).$$

Proof Here, $\zeta^{\tau}{}_{\beth}^{*} \geq \zeta^{\tau}{}_{\beth}$ and $\eta^{\upsilon}{}_{\beth}^{*} \leq \eta^{\upsilon}{}_{\beth}$ for all j. If $\zeta^{\tau}{}_{\beth}^{*} \geq \zeta^{\tau}{}_{\beth}$:

$\Leftrightarrow \zeta^{\tau}{}^{*}_{\beth} \geq \zeta^{\tau}{}_{\beth} \Leftrightarrow 1 - \zeta^{\tau}{}^{*}_{\beth} \leq 1 - \zeta^{\tau}{}_{\beth}$
$\Leftrightarrow (1 - \zeta^{\tau}{}^{*}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \leq (1 - \zeta^{\tau}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}$
$\Leftrightarrow \overline{\prod}_{\beth=1}^{n}(1 - \zeta^{\tau}{}^{*}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \leq \overline{\prod}_{\beth=1}^{n}(1 - \zeta^{\tau}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}$
$\Leftrightarrow 1 - \overline{\prod}_{\beth=1}^{n}(1 - \zeta^{\tau}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \leq 1 - \overline{\prod}_{\beth=1}^{n}(1 - \zeta^{\tau}{}^{*}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}$

Again:

$\mathscr{J}^{\aleph}{}^{*}_{\beth} \geq \mathscr{J}^{\aleph}{}_{\beth}$ and $\mathscr{C}^{\gamma}{}^{*}_{\beth} \leq \mathscr{C}^{\gamma}{}_{\beth}$ for all j. If $\mathscr{J}^{\aleph}{}^{*}_{\beth} \geq \mathscr{J}^{\aleph}{}_{\beth}$,
$\Leftrightarrow \mathscr{J}^{\aleph}{}^{*}_{\beth} \geq \mathscr{J}^{\aleph}{}_{\beth} \Leftrightarrow 1 - \mathscr{J}^{\aleph}{}^{*}_{\beth} \leq 1 - \mathscr{J}^{\aleph}{}_{\beth}$
$\Leftrightarrow (1 - \mathscr{J}^{\aleph}{}^{*}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \leq (1 - \mathscr{J}^{\aleph}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}$
$\Leftrightarrow \overline{\prod}_{\beth=1}^{n}(1 - \mathscr{J}^{\aleph}{}^{*}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \leq \overline{\prod}_{\beth=1}^{n}(1 - \mathscr{J}^{\aleph}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}$
$\Leftrightarrow 1 - \overline{\prod}_{\beth=1}^{n}(1 - \mathscr{J}^{\aleph}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \leq 1 - \overline{\prod}_{\beth=1}^{n}(1 - \mathscr{J}^{\aleph}{}^{*}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}$

Now:

$\eta^{\upsilon}{}^{*}_{\beth} \leq \eta^{\upsilon}{}_{\beth}$
$\Leftrightarrow (\eta^{\upsilon}{}^{*}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \leq (\eta^{\upsilon}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}$
$\Leftrightarrow \overline{\prod}_{\beth=1}^{n}(\eta^{\upsilon}{}^{*}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \leq \overline{\prod}_{\beth=1}^{n}(\eta^{\upsilon}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}$

And:

$\mathscr{C}^{\gamma}{}^{*}_{\beth} \leq \mathscr{C}^{\gamma}{}_{\beth}$
$\Leftrightarrow (\mathscr{C}^{\gamma}{}^{*}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \leq (\mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}$
$\Leftrightarrow \overline{\prod}_{\beth=1}^{n}(\mathscr{C}^{\gamma}{}^{*}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \leq \overline{\prod}_{\beth=1}^{n}(\mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}$

Let

$$\overline{\daleth^{\varsigma}} = \text{LDFWA}(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{n})$$

and

$$\overline{\daleth^{\varsigma *}} = \text{LDFWA}(\daleth^{\varsigma}{}^{*}_{1}, \daleth^{\varsigma}{}^{*}_{2}, \ldots \daleth^{\varsigma}{}^{*}_{n})$$

We get that $\overline{\daleth^{\varsigma *}} \geq \overline{\daleth^{\varsigma}}$. So,

$$\text{LDFWA}(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{n}) \leq \text{LDFWA}(\daleth^{\varsigma}{}^{*}_{1}, \daleth^{\varsigma}{}^{*}_{2}, \ldots \daleth^{\varsigma}{}^{*}_{n}).$$

Theorem 12 *Assume that* $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth}\rangle)$ *and* $\digamma^{\gamma}{}_{\beth} = (\langle \phi_{\beth}, \varphi_{\beth}\rangle, \langle \mathscr{K}_{\beth}, \mathscr{M}_{\beth}\rangle)$ *are two families of LDFNs. If* $r > 0$ *and* $\digamma^{\gamma} = (\langle \zeta^{\tau}{}_{\digamma^{\gamma}}, \eta^{\upsilon}{}_{\digamma^{\gamma}}\rangle, \langle \mathscr{J}^{\aleph}{}_{\digamma^{\gamma}}, \mathscr{C}^{\gamma}{}_{\digamma^{\gamma}}\rangle)$ *is an LDFN, then:*

1. *LDFWA*$(\daleth^{\varsigma}{}_{1} \oplus \digamma^{\gamma}, \daleth^{\varsigma}{}_{2} \oplus \digamma^{\gamma}, \ldots \daleth^{\varsigma}{}_{n} \oplus \digamma^{\gamma}) =$ *LDFWA*$(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{n}) \oplus \digamma^{\gamma}$
2. *LDFWA*$(r\daleth^{\varsigma}{}_{1}, r\daleth^{\varsigma}{}_{2}, \ldots r\daleth^{\varsigma}{}_{n}) = r$ *LDFWA*$(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{n})$

3. $LDFWA(\daleth^{\varsigma}{}_1 \oplus F^{\gamma}{}_1, \daleth^{\varsigma}{}_2 \oplus F^{\gamma}{}_2, \ldots \daleth^{\varsigma}{}_n \oplus F^{\gamma}{}_n) = LDFWA(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) \oplus LDFWA(F^{\gamma}{}_1, F^{\gamma}{}_2, \ldots F^{\gamma}{}_n)$
4. $LDFWA(r\daleth^{\varsigma}{}_1 \oplus F^{\gamma}, r\daleth^{\varsigma}{}_2 \oplus F^{\gamma}, \ldots \oplus r\daleth^{\varsigma}{}_n \oplus F^{\gamma}) = r\ LDFWA(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) \oplus F^{\gamma}$

Proof Here, we just proof 1 and 3.
1. Since,

$$\daleth^{\varsigma}{}_{\gimel} \oplus F^{\gamma} = \left(\left(1-(1-\zeta^{\tau}{}_{\gimel})(1-\zeta^{\tau}{}_{F^{\gamma}}), \eta^{\upsilon}{}_{\gimel}\eta^{\upsilon}{}_{F^{\gamma}}\right), \left(1-(1-\mathscr{J}^{\aleph}{}_{\gimel})(1-\mathscr{J}^{\aleph}{}_{F^{\gamma}}), \mathscr{C}^{\gamma}{}_{\gimel}\mathscr{C}^{\gamma}{}_{F^{\gamma}}\right)\right).$$

By Theorem 13,

$$\mathrm{LDFWA}(\daleth^{\varsigma}{}_1 \oplus F^{\gamma}, \daleth^{\varsigma}{}_2 \oplus F^{\gamma}, \ldots \daleth^{\varsigma}{}_n \oplus F^{\gamma})$$

$$= \left(\left\langle (1-\overline{\prod}_{\gimel=1}^{n}\left((1-\zeta^{\tau}{}_{\gimel})(1-\zeta^{\tau}{}_{F^{\gamma}})\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \overline{\prod}_{\gimel=1}^{n}\left(\eta^{\upsilon}{}_{F^{\gamma}}\eta^{\upsilon}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}\right\rangle, \left\langle (1-\overline{\prod}_{\gimel=1}^{n}\left((1-\mathscr{J}^{\aleph}{}_{\gimel})(1-\mathscr{J}^{\aleph}{}_{F^{\gamma}})\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \overline{\prod}_{\gimel=1}^{n}\left(\mathscr{C}^{\gamma}{}_{F^{\gamma}}\mathscr{C}^{\gamma}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}\right\rangle\right)$$

$$= \left(\left\langle (1-\left(1-\zeta^{\tau}{}_{F^{\gamma}}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}\overline{\prod}_{\gimel=1}^{n}\left(1-\zeta^{\tau}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \left(\eta^{\upsilon}{}_{F^{\gamma}}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}\overline{\prod}_{\gimel=1}^{n}\left(\eta^{\upsilon}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}\right\rangle, \left\langle (1-\left(1-\mathscr{J}^{\aleph}{}_{F^{\gamma}}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}\overline{\prod}_{\gimel=1}^{n}\left(1-\mathscr{J}^{\aleph}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \left(\mathscr{C}^{\gamma}{}_{F^{\gamma}}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}\overline{\prod}_{\gimel=1}^{n}\left(\mathscr{C}^{\gamma}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}\right\rangle\right)$$

$$= \left(\left\langle (1-\left(1-\zeta^{\tau}{}_{F^{\gamma}}\right)\overline{\prod}_{\gimel=1}^{n}\left(1-\zeta^{\tau}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \left(\eta^{\upsilon}{}_{F^{\gamma}}\right)\overline{\prod}_{\gimel=1}^{n}\left(\eta^{\upsilon}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}\right\rangle, \left\langle (1-\left(1-\mathscr{J}^{\aleph}{}_{F^{\gamma}}\right)\overline{\prod}_{\gimel=1}^{n}\left(1-\mathscr{J}^{\aleph}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \left(\mathscr{C}^{\gamma}{}_{F^{\gamma}}\right)\overline{\prod}_{\gimel=1}^{n}\left(\mathscr{C}^{\gamma}{}_{\gimel}\right)^{\mathfrak{P}^{\gamma}{}_{\gimel}}\right\rangle\right).$$

Now, by operational laws of LDFNs,

$$\mathrm{LDFWA}(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) \oplus F^{\gamma}$$

$$= \left(\left\langle (1-\overline{\prod}_{\gimel=1}^{n}(1-\zeta^{\tau}{}_{\gimel})^{\mathfrak{P}^{\gamma}{}_{\gimel}}, \overline{\prod}_{\gimel=1}^{n}\eta^{\upsilon}{}_{\gimel}^{\mathfrak{P}^{\gamma}{}_{\gimel}}\right\rangle,\right.$$

$$\left\langle (1-\overline{\prod}_{\beth=1}^{n}(1-\mathscr{J}^{\aleph}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}, \overline{\prod}_{\beth=1}^{n}\mathscr{C}^{\gamma}{}_{\beth}^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle \oplus$$

$$(\langle \zeta^{\tau}{}_{F^{\gamma}}, \eta^{\upsilon}{}_{F^{\gamma}}\rangle, \langle \mathscr{J}^{\aleph}{}_{F^{\gamma}}, \mathscr{C}^{\gamma}{}_{F^{\gamma}}\rangle)\Big)$$

$$= \left(\left\langle (1-\left(1-\zeta^{\tau}{}_{F^{\gamma}}\right)\overline{\prod}_{\beth=1}^{n}\left(1-\zeta^{\tau}{}_{\beth}\right)^{\mathfrak{P}^{\gamma}{}_{\beth}}, \left(\eta^{\upsilon}{}_{F^{\gamma}}\right)\overline{\prod}_{\beth=1}^{n}\left(\eta^{\upsilon}{}_{\beth}\right)^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle,\right.$$

$$\left.\left\langle (1-\left(1-\mathscr{J}^{\aleph}{}_{F^{\gamma}}\right)\overline{\prod}_{\beth=1}^{n}\left(1-\mathscr{J}^{\aleph}{}_{\beth}\right)^{\mathfrak{P}^{\gamma}{}_{\beth}}, \left(\mathscr{C}^{\gamma}{}_{F^{\gamma}}\right)\overline{\prod}_{\beth=1}^{n}\left(\mathscr{C}^{\gamma}{}_{\beth}\right)^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle\right).$$

Thus,

$$\text{LDFWA}(\daleth^{\varsigma}{}_{1}\oplus F^{\gamma}, \daleth^{\varsigma}{}_{2}\oplus F^{\gamma}, \ldots \daleth^{\varsigma}{}_{n}\oplus F^{\gamma}) = \text{LDFWA}(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{n})\oplus F^{\gamma}.$$

3. According to Theorem 13,

$$\text{q-ROFWA}(\daleth^{\varsigma}{}_{1}\oplus F^{\gamma}{}_{2}, \daleth^{\varsigma}{}_{2}\oplus F^{\gamma}{}_{2}, \ldots \daleth^{\varsigma}{}_{n}\oplus F^{\gamma}{}_{n})$$

$$= \left(\left\langle 1-\overline{\prod}_{\beth=1}^{n}\left((1-\zeta^{\tau}{}_{\beth})(1-\phi_{\beth})\right)^{\mathfrak{P}^{\gamma}{}_{\beth}}, \overline{\prod}_{\beth=1}^{n}\left(\varphi_{\beth}\eta^{\upsilon}{}_{\beth}\right)^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle,\right.$$

$$\left.\left\langle 1-\overline{\prod}_{\beth=1}^{n}\left((1-\mathscr{J}^{\aleph}{}_{\beth})(1-\mathscr{K}_{\beth})\right)^{\mathfrak{P}^{\gamma}{}_{\beth}}, \overline{\prod}_{\beth=1}^{n}\left(\mathscr{M}_{\beth}\mathscr{C}^{\gamma}{}_{\beth}\right)^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle\right)$$

$$= \left(\left\langle 1-\overline{\prod}_{\beth=1}^{n}\left(1-\phi_{\beth}\right)^{\mathfrak{P}^{\gamma}{}_{\beth}}\overline{\prod}_{\beth=1}^{n}\left(1-\zeta^{\tau}{}_{\beth}\right)^{\mathfrak{P}^{\gamma}{}_{\beth}},\right.\right.$$

$$\left.\overline{\prod}_{\beth=1}^{n}\left(\varphi_{\beth}\right)^{\mathfrak{P}^{\gamma}{}_{\beth}}\overline{\prod}_{\beth=1}^{n}\left(\eta^{\upsilon}{}_{\beth}\right)^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle,$$

$$\left\langle 1-\overline{\prod}_{\beth=1}^{n}\left(1-\mathscr{K}_{\beth}\right)^{\mathfrak{P}^{\gamma}{}_{\beth}}\overline{\prod}_{\beth=1}^{n}\left(1-\mathscr{J}^{\aleph}{}_{\beth}\right)^{\mathfrak{P}^{\gamma}{}_{\beth}},\right.$$

$$\left.\left.\overline{\prod}_{\beth=1}^{n}\left(\mathscr{M}_{\beth}\right)^{\mathfrak{P}^{\gamma}{}_{\beth}}\overline{\prod}_{\beth=1}^{n}\left(\mathscr{C}^{\gamma}{}_{\beth}\right)^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle\right).$$

Now,

$$\text{LDFWA}(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{n}) \oplus \text{LDFWA}(F^{\gamma}{}_{1}, F^{\gamma}{}_{2}, \ldots F^{\gamma}{}_{n})$$

$$= \left(\left\langle 1-\overline{\prod}_{\beth=1}^{n}(1-\zeta^{\tau}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}, \overline{\prod}_{\beth=1}^{n}\eta^{\upsilon}{}_{\beth}^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle,\right.$$

$$\left.\left\langle 1-\overline{\prod}_{\beth=1}^{n}(1-\mathscr{J}^{\aleph}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}, \overline{\prod}_{\beth=1}^{n}\mathscr{C}^{\gamma}{}_{\beth}^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle\right)\oplus$$

$$\left(\left\langle 1-\overline{\prod}_{\daleth=1}^{n}(1-\phi_\daleth)^{\mathfrak{P}^\gamma{}_\daleth}, \overline{\prod}_{\daleth=1}^{n}\varphi_\daleth^{\mathfrak{P}^\gamma{}_\daleth}\right\rangle,\right.$$
$$\left.\left\langle 1-\overline{\prod}_{\daleth=1}^{n}(1-\mathscr{K}_\daleth)^{\mathfrak{P}^\gamma{}_\daleth}, \overline{\prod}_{\daleth=1}^{n}\mathscr{M}_\daleth^{\mathfrak{P}^\gamma{}_\daleth}\right\rangle\right)$$
$$=\left(\left\langle 1-\overline{\prod}_{\daleth=1}^{n}\left(1-\phi_\daleth\right)^{\mathfrak{P}^\gamma{}_\daleth}\overline{\prod}_{\daleth=1}^{n}\left(1-\zeta^\tau{}_\daleth\right)^{\mathfrak{P}^\gamma{}_\daleth},\right.\right.$$
$$\left.\overline{\prod}_{\daleth=1}^{n}\left(\varphi_\daleth\right)^{\mathfrak{P}^\gamma{}_\daleth}\overline{\prod}_{\daleth=1}^{n}\left(\eta^\upsilon{}_\daleth\right)^{\mathfrak{P}^\gamma{}_\daleth}\right\rangle,$$
$$\left\langle 1-\overline{\prod}_{\daleth=1}^{n}\left(1-\mathscr{K}_\daleth\right)^{\mathfrak{P}^\gamma{}_\daleth}\overline{\prod}_{\daleth=1}^{n}\left(1-\mathscr{J}^\aleph{}_\daleth\right)^{\mathfrak{P}^\gamma{}_\daleth},\right.$$
$$\left.\left.\overline{\prod}_{\daleth=1}^{n}\left(\mathscr{M}_\daleth\right)^{\mathfrak{P}^\gamma{}_\daleth}\overline{\prod}_{\daleth=1}^{n}\left(\mathscr{C}^\gamma{}_\daleth\right)^{\mathfrak{P}^\gamma{}_\daleth}\right\rangle\right).$$

Thus,

$$\begin{aligned}&\text{LDFWA}(\daleth^\varsigma{}_1 \oplus F^\gamma{}_2, \daleth^\varsigma{}_2 \oplus F^\gamma{}_2, \ldots \daleth^\varsigma{}_n \oplus F^\gamma{}_n)\\ &\quad= \text{LDFWA}(\daleth^\varsigma{}_1, \daleth^\varsigma{}_2, \ldots \daleth^\varsigma{}_n) \oplus \text{LDFWA}(F^\gamma{}_1, F^\gamma{}_2, \ldots F^\gamma{}_n).\end{aligned}$$

3.2 LDFOWA Operator

Definition 17 Consider $\daleth^\varsigma{}_\daleth = (\langle\zeta^\tau{}_\daleth, \eta^\upsilon{}_\daleth\rangle, \langle\mathscr{J}^\aleph{}_\daleth, \mathscr{C}^\gamma{}_\daleth\rangle)$ is the agglomeration of LDFNs, and LDFOWA : $\$^n \to \$$ be the mapping.

$$\text{LDFOWA}(\daleth^\varsigma{}_1, \daleth^\varsigma{}_2, \ldots \daleth^\varsigma{}_n) = \mathfrak{P}^\gamma{}_1\daleth^\varsigma{}_{\sigma(1)} \oplus \mathfrak{P}^\gamma{}_2\daleth^\varsigma{}_{\sigma(2)} \oplus \ldots, \oplus\mathfrak{P}^\gamma{}_n\daleth^\varsigma{}_{\sigma(n)}, \tag{14.9}$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$, such that $\daleth^\varsigma{}_{\sigma(r-1)} \geq \daleth^\varsigma{}_{\sigma(r)}$, for any r. Then LDFOWA is known as LDFOWA operator, where $(\mathfrak{P}^\gamma{}_1, \mathfrak{P}^\gamma{}_2, \ldots, \mathfrak{P}^\gamma{}_n)$ be the WV with the constraint $\mathfrak{P}^\gamma{}_\daleth > 0$ and $\sum_{h=1}^{n}\mathfrak{P}^\gamma{}_\daleth = 1$.

We might also think about LDFOWA by employing the theorem following.

Theorem 13 *Consider* $\daleth^\varsigma{}_\daleth = (\langle\zeta^\tau{}_\daleth, \eta^\upsilon{}_\daleth\rangle, \langle\mathscr{J}^\aleph{}_\daleth, \mathscr{C}^\gamma{}_\daleth\rangle)$ *is the agglomeration of LDFNs, and we can find LDFWA by*

$$LDFOWA(\daleth^\varsigma{}_1, \daleth^\varsigma{}_2, \ldots \daleth^\varsigma{}_n)$$

$$= \left(\left\langle 1 - \overline{\prod}_{\beth=1}^{n} (1 - \zeta^{\tau}{}_{\sigma(j)})^{\mathfrak{P}^{\gamma}\beth}, \overline{\prod}_{\beth=1}^{n} \eta^{\upsilon}{}_{\sigma(j)}^{\mathfrak{P}^{\gamma}\beth} \right\rangle, \right.$$

$$\left. \left\langle 1 - \overline{\prod}_{\beth=1}^{n} (1 - \mathscr{J}^{\aleph}{}_{\sigma(j)})^{\mathfrak{P}^{\gamma}\beth}, \overline{\prod}_{\beth=1}^{n} \mathscr{C}^{\gamma}{}_{\sigma(j)}^{\mathfrak{P}^{\gamma}\beth} \right\rangle \right). \tag{14.10}$$

Theorem 14 (Monotonicity) *Assume that* $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth} \rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth} \rangle)$ *and* $\daleth^{\varsigma *}_{\beth} = (\langle \zeta^{\tau *}_{\beth}, \eta^{\upsilon *}_{\beth} \rangle, \langle \mathscr{J}^{\aleph *}_{\beth}, \mathscr{C}^{\gamma *}_{\beth} \rangle)$ *are the agglomerations of LDFNs. If* $\zeta^{\tau *}_{\beth} \geq \zeta^{\tau}{}_{\beth}$, $\eta^{\upsilon *}_{\beth} \leq \eta^{\upsilon}{}_{\beth}$, $\mathscr{J}^{\aleph *}_{\beth} \geq \mathscr{J}^{\aleph}{}_{\beth}$, *and* $\mathscr{C}^{\gamma *}_{\beth} \leq \mathscr{C}^{\gamma}{}_{\beth}$ *for all* j, *then*

$$LDFOWA(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) \leq LDFOWA(\daleth^{\varsigma *}_1, \daleth^{\varsigma *}_2, \ldots \daleth^{\varsigma *}_n).$$

Proof This is the same as Theorem 14.

Theorem 15 *Assume that* $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth} \rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth} \rangle)$ *and* $F^{\gamma}{}_{\beth} = (\langle \phi_{\beth}, \varphi_{\beth} \rangle, \langle \mathscr{K}_{\beth}, \mathscr{M}_{\beth} \rangle)$ *are two families of LDFNs. If* $r > 0$ *and* $F^{\gamma} = (\langle \zeta^{\tau}{}_{F^{\gamma}}, \eta^{\upsilon}{}_{F^{\gamma}} \rangle, \langle \mathscr{J}^{\aleph}{}_{F^{\gamma}}, \mathscr{C}^{\gamma}{}_{F^{\gamma}} \rangle)$ *is an LDFN, then:*

1. $LDFOWA(\daleth^{\varsigma}{}_1 \oplus F^{\gamma}, \daleth^{\varsigma}{}_2 \oplus F^{\gamma}, \ldots \daleth^{\varsigma}{}_n \oplus F^{\gamma}) = LDFOWA(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) \oplus F^{\gamma}$
2. $LDFOWA(r\daleth^{\varsigma}{}_1, r\daleth^{\varsigma}{}_2, \ldots r\daleth^{\varsigma}{}_n) = r\ LDFOWA(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n)$
3. $LDFOWA(\daleth^{\varsigma}{}_1 \oplus F^{\gamma}{}_1, \daleth^{\varsigma}{}_2 \oplus F^{\gamma}{}_2, \ldots \daleth^{\varsigma}{}_n \oplus F^{\gamma}{}_n) = LDFOWA(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) \oplus LDFOWA(F^{\gamma}{}_1, F^{\gamma}{}_2, \ldots F^{\gamma}{}_n)$
4. $LDFOWA(r\daleth^{\varsigma}{}_1 \oplus F^{\gamma}, r\daleth^{\varsigma}{}_2 \oplus F^{\gamma}, \ldots \oplus r\daleth^{\varsigma}{}_n \oplus F^{\gamma}) = rLDFOWA(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) \oplus F^{\gamma}$

3.3 LDFWG Operator

Definition 18 Consider $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth} \rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth} \rangle)$ is the agglomeration of LDFNs and $LDFWG : \$^n \to \$$ be a mapping.

$$\text{LDFWG}(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) = \daleth^{\varsigma}{}_1^{\mathfrak{P}^{\gamma}{}_1} \otimes \daleth^{\varsigma}{}_2^{\mathfrak{P}^{\gamma}{}_2} \otimes \ldots, \otimes \daleth^{\varsigma}{}_n^{\mathfrak{P}^{\gamma}{}_n}. \tag{14.11}$$

Then the mapping LDFWG is called LDFWG operator, where $(\mathfrak{P}^{\gamma}{}_1, \mathfrak{P}^{\gamma}{}_2, \ldots, \mathfrak{P}^{\gamma}{}_n)$ be the WV with the constraint $\mathfrak{P}^{\gamma}{}_i > 0$ and $\sum_{i=1}^{n} \mathfrak{P}^{\gamma}{}_i = 1$.

We may also consider LDFWG using the theorem below based on LDFNs operational law.

Theorem 16 *Assume that* $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth} \rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth} \rangle)$ *is the agglomeration of LDFNs, and we can find LDFWG by* $LDFWG(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n)$

$$= \left(\left\langle \overline{\prod}_{\beth=1}^{n} \zeta^{\tau}{}_{\beth}^{\mathfrak{P}^{\gamma}\beth}, 1 - \overline{\prod}_{\beth=1}^{n} (1 - \eta^{\upsilon}{}_{\beth})^{\mathfrak{P}^{\gamma}\beth} \right\rangle, \right.$$

$$\left\langle \overline{\prod}_{\daleth=1}^{n} \mathscr{I}^{\aleph_{\daleth}^{\mathfrak{P}^{\gamma}{}_{\daleth}}}, 1-\overline{\prod}_{\daleth=1}^{n} (1-\mathscr{C}^{\gamma}{}_{\daleth})^{\mathfrak{P}^{\gamma}{}_{\daleth}} \right\rangle\Bigg). \tag{14.12}$$

Proof It is quite simple for the first assertion to come before Definition 19 and Theorem 20. The following instances demonstrate this point further:

$$\begin{aligned}
&\text{LDFWG}(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{n}) \\
&= \daleth^{\varsigma}{}_{1}^{\mathfrak{P}^{\gamma}{}_{1}} \otimes \daleth^{\varsigma}{}_{2}^{\mathfrak{P}^{\gamma}{}_{2}} \otimes \ldots, \otimes \daleth^{\varsigma}{}_{n}^{\mathfrak{P}^{\gamma}{}_{n}} \\
&= \Bigg(\left\langle \overline{\prod}_{\daleth=1}^{n} \zeta^{\tau}{}_{\daleth}^{\mathfrak{P}^{\gamma}{}_{\daleth}}, 1-\overline{\prod}_{\daleth=1}^{n} (1-\eta^{\upsilon}{}_{\daleth})^{\mathfrak{P}^{\gamma}{}_{\daleth}} \right\rangle, \\
&\quad \left\langle \overline{\prod}_{\daleth=1}^{n} \mathscr{I}^{\aleph}{}_{\daleth}^{\mathfrak{P}^{\gamma}{}_{\daleth}}, 1-\overline{\prod}_{\daleth=1}^{n} (1-\mathscr{C}^{\gamma}{}_{\daleth})^{\mathfrak{P}^{\gamma}{}_{\daleth}} \right\rangle \Bigg).
\end{aligned}$$

In order to demonstrate the validity of this theorem, we turned to mathematics induction.

For $n = 2$

$$\daleth^{\varsigma}{}_{1}^{\mathfrak{P}^{\gamma}{}_{1}} = \left(\left\langle \zeta^{\tau}{}_{1}^{\mathfrak{P}^{\gamma}{}_{1}}, 1-(1-\eta^{\upsilon}{}_{1})^{\mathfrak{P}^{\gamma}{}_{1}} \right\rangle, \left\langle \mathscr{I}^{\aleph}{}_{1}^{\mathfrak{P}^{\gamma}{}_{1}}, 1-(1-\mathscr{C}^{\gamma}{}_{1})^{\mathfrak{P}^{\gamma}{}_{1}} \right\rangle \right)$$

$$\daleth^{\varsigma}{}_{2}^{\mathfrak{P}^{\gamma}{}_{2}} = \left(\left\langle \zeta^{\tau}{}_{2}^{\mathfrak{P}^{\gamma}{}_{1}}, 1-(1-\eta^{\upsilon}{}_{\daleth})^{\mathfrak{P}^{\gamma}{}_{1}} \right\rangle, \left\langle \mathscr{I}^{\aleph}{}_{2}^{\mathfrak{P}^{\gamma}{}_{1}}, 1-(1-\mathscr{C}^{\gamma}{}_{\daleth})^{\mathfrak{P}^{\gamma}{}_{1}} \right\rangle \right).$$

Then

$$\daleth^{\varsigma}{}_{1}^{\mathfrak{P}^{\gamma}{}_{1}} \otimes \daleth^{\varsigma}{}_{2}^{\mathfrak{P}^{\gamma}{}_{2}}$$

$$\begin{aligned}
&= \left(\left\langle \zeta^{\tau}{}_{1}^{\mathfrak{P}^{\gamma}{}_{1}}, 1-(1-\eta^{\upsilon}{}_{1})^{\mathfrak{P}^{\gamma}{}_{1}} \right\rangle, \left\langle \mathscr{I}^{\aleph}{}_{1}^{\mathfrak{P}^{\gamma}{}_{1}}, 1-(1-\mathscr{C}^{\gamma}{}_{1})^{\mathfrak{P}^{\gamma}{}_{1}} \right\rangle \right) \otimes \\
&\quad \left(\left\langle \zeta^{\tau}{}_{2}^{\mathfrak{P}^{\gamma}{}_{1}}, 1-(1-\eta^{\upsilon}{}_{\daleth})^{\mathfrak{P}^{\gamma}{}_{1}} \right\rangle, \left\langle \mathscr{I}^{\aleph}{}_{2}^{\mathfrak{P}^{\gamma}{}_{1}}, 1-(1-\mathscr{C}^{\gamma}{}_{\daleth})^{\mathfrak{P}^{\gamma}{}_{1}} \right\rangle \right) \\
&= \Bigg(\Bigg\langle \zeta^{\tau}{}_{1}^{\mathfrak{P}^{\gamma}{}_{1}}.\zeta^{\tau}{}_{2}^{\mathfrak{P}^{\gamma}{}_{1}}, 1-(1-\eta^{\upsilon}{}_{1})^{\mathfrak{P}^{\gamma}{}_{1}}+1-(1-\eta^{\upsilon}{}_{\daleth})^{\mathfrak{P}^{\gamma}{}_{1}} - \left(1-(1-\eta^{\upsilon}{}_{1})^{\mathfrak{P}^{\gamma}{}_{1}}\right) \\
&\quad \left(1-(1-\eta^{\upsilon}{}_{\daleth})^{\mathfrak{P}^{\gamma}{}_{1}}\right) \Bigg\rangle, \Bigg\langle \mathscr{I}^{\aleph}{}_{1}^{\mathfrak{P}^{\gamma}{}_{1}}.\mathscr{I}^{\aleph}{}_{2}^{\mathfrak{P}^{\gamma}{}_{1}}, 1-(1-\mathscr{C}^{\gamma}{}_{1})^{\mathfrak{P}^{\gamma}{}_{1}}+1 \\
&\quad -(1-\mathscr{C}^{\gamma}{}_{\daleth})^{\mathfrak{P}^{\gamma}{}_{1}} - \left(1-(1-\mathscr{C}^{\gamma}{}_{1})^{\mathfrak{P}^{\gamma}{}_{1}}\right)\left(1-(1-\mathscr{C}^{\gamma}{}_{\daleth})^{\mathfrak{P}^{\gamma}{}_{1}}\right) \Bigg\rangle \Bigg)
\end{aligned}$$

$$= \left(\left\langle \zeta^{\tau \mathfrak{P}^{\gamma}{}_{1}}_{1} . \zeta^{\tau \mathfrak{P}^{\gamma}{}_{1}}_{2}, 1 - (1 - \eta^{\upsilon}{}_{1})^{\mathfrak{P}^{\gamma}{}_{1}} (1 - \eta^{\upsilon}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{1}} \right\rangle, \right.$$

$$\left. \left\langle \mathscr{I}^{\aleph \mathfrak{P}^{\gamma}{}_{1}}_{1} . \mathscr{I}^{\aleph \mathfrak{P}^{\gamma}{}_{1}}_{2}, 1 - (1 - \mathscr{C}^{\gamma}{}_{1})^{\mathfrak{P}^{\gamma}{}_{1}} (1 - \mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{1}} \right\rangle \right)$$

$$= \left(\left\langle \overline{\prod}_{\beth=1}^{2} \zeta^{\tau \mathfrak{P}^{\gamma}{}_{\beth}}_{\beth}, 1 - \overline{\prod}_{\beth=1}^{2} (1 - \eta^{\upsilon}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle, \right.$$

$$\left. \left\langle \overline{\prod}_{\beth=1}^{2} \mathscr{I}^{\aleph \mathfrak{P}^{\gamma}{}_{\beth}}_{\beth}, 1 - \overline{\prod}_{\beth=1}^{2} (1 - \mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle \right).$$

This shows that Eq. (14.14) is true for $n = 2$, and now assume that Eq. (14.14) holds for $n = k$, i.e.,

$$\text{LDFWG}(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{k})$$

$$= \left(\left\langle \overline{\prod}_{\beth=1}^{k} \zeta^{\tau \mathfrak{P}^{\gamma}{}_{\beth}}_{\beth}, 1 - \overline{\prod}_{\beth=1}^{k} (1 - \eta^{\upsilon}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle, \right.$$

$$\left. \left\langle \overline{\prod}_{\beth=1}^{k} \mathscr{I}^{\aleph \mathfrak{P}^{\gamma}{}_{\beth}}_{\beth}, 1 - \overline{\prod}_{\beth=1}^{k} (1 - \mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle \right).$$

Now $n = k + 1$, and by operational laws of LDFNs, we have

$$\text{LDFWG}(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{k+1}) = \text{LDFWG}(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{k}) \otimes \daleth^{\varsigma \mathfrak{P}^{\gamma}{}_{\beth}}_{k+1}$$

$$= \left(\left\langle \overline{\prod}_{\beth=1}^{k} \zeta^{\tau \mathfrak{P}^{\gamma}{}_{\beth}}_{\beth}, 1 - \overline{\prod}_{\beth=1}^{k} (1 - \eta^{\upsilon}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle, \right.$$

$$\left. \left\langle \overline{\prod}_{\beth=1}^{k} \mathscr{I}^{\aleph \mathfrak{P}^{\gamma}{}_{\beth}}_{\beth}, 1 - \overline{\prod}_{\beth=1}^{k} (1 - \mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle \right) \otimes$$

$$\left(\left\langle \zeta^{\tau \mathfrak{P}^{\gamma}{}_{k+1}}_{k+1}, 1 - (1 - \eta^{\upsilon}{}_{k+1})^{\mathfrak{P}^{\gamma}{}_{k+1}} \right\rangle, \left\langle \mathscr{I}^{\aleph \mathfrak{P}^{\gamma}{}_{k+1}}_{k+1}, 1 - (1 - \mathscr{C}^{\gamma}{}_{k+1})^{\mathfrak{P}^{\gamma}{}_{k+1}} \right\rangle \right)$$

$$= \left(\left\langle \overline{\prod}_{\beth=1}^{k} \zeta^{\tau \mathfrak{P}^{\gamma}{}_{\beth}}_{k} . \zeta^{\tau \mathfrak{P}^{\gamma}{}_{k+1}}_{k+1}, 1 - \overline{\prod}_{\beth=1}^{k} (1 - \eta^{\upsilon}{}_{k})^{\mathfrak{P}^{\gamma}{}_{\beth}} (1 - \eta^{\upsilon}{}_{k+1})^{k+1} \right\rangle, \right.$$

$$\left. \left\langle \overline{\prod}_{\beth=1}^{k} \mathscr{I}^{\aleph \mathfrak{P}^{\gamma}{}_{\beth}}_{k} . \mathscr{I}^{\aleph \mathfrak{P}^{\gamma}{}_{k+1}}_{k+1}, 1 - \overline{\prod}_{\beth=1}^{k} (1 - \mathscr{C}^{\gamma}{}_{k})^{\mathfrak{P}^{\gamma}{}_{\beth}} (1 - \mathscr{C}^{\gamma}{}_{k+1})^{k+1} \right\rangle \right)$$

$$= \left(\left\langle \overline{\prod}_{\beth=1}^{k+1} \zeta^{\tau}{}_{\beth}^{\mathfrak{P}^{\gamma}{}_{\beth}}, 1 - \overline{\prod}_{\beth=1}^{k+1} (1 - \eta^{\upsilon}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle, \right.$$
$$\left. \left\langle \overline{\prod}_{\beth=1}^{k+1} \mathscr{J}^{\aleph}{}_{\beth}^{\mathfrak{P}^{\gamma}{}_{\beth}}, 1 - \overline{\prod}_{\beth=1}^{k+1} (1 - \mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle \right).$$

This shows that for $n = k + 1$, Eq. (14.10) holds. Then,

$$\text{LDFWG}(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n)$$
$$= \left(\left\langle \overline{\prod}_{\beth=1}^{n} \zeta^{\tau}{}_{\beth}^{\mathfrak{P}^{\gamma}{}_{\beth}}, 1 - \overline{\prod}_{\beth=1}^{n} (1 - \eta^{\upsilon}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle, \right.$$
$$\left. \left\langle \overline{\prod}_{\beth=1}^{n} \mathscr{J}^{\aleph}{}_{\beth}^{\mathfrak{P}^{\gamma}{}_{\beth}}, 1 - \overline{\prod}_{\beth=1}^{n} (1 - \mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \right\rangle \right).$$

A few of LDFWG's promising properties are described below.

Theorem 17 *(Idempotency) Assume that* $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth} \rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth} \rangle)$ *is the agglomeration of LDFNs. If all* $\daleth^{\varsigma}{}_{\beth}$ *are equal, i.e.,* $\daleth^{\varsigma}{}_{\beth} = \daleth^{\varsigma}$ *for all* j*, then*

$$LDFWG(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) = \daleth^{\varsigma}.$$

Proof From Definition 17, we have

$$\begin{aligned} \text{LDFWG}(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) &= \daleth^{\varsigma}{}_{1}^{\mathfrak{P}^{\gamma}{}_1} \otimes \daleth^{\varsigma}{}_{2}^{\mathfrak{P}^{\gamma}{}_2} \otimes \ldots, \otimes \daleth^{\varsigma}{}_{n}^{\mathfrak{P}^{\gamma}{}_n} \\ &= \daleth^{\varsigma\mathfrak{P}^{\gamma}{}_1} \otimes \daleth^{\varsigma\mathfrak{P}^{\gamma}{}_2} \otimes \ldots, \otimes \daleth^{\varsigma\mathfrak{P}^{\gamma}{}_n} \\ &= \daleth^{\varsigma}. \end{aligned}$$

Corollary 2 *If* $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth} \rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth} \rangle)$ $j = (1, 2, \ldots n)$ *is the agglomeration of largest LDFNs, i.e.,* $\daleth^{\varsigma}{}_{\beth} = (\langle 1, 0 \rangle, \langle 1, 0 \rangle)$ *for all* j*, then*

$$LDFWG(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) = (\langle 1, 0 \rangle, \langle 1, 0 \rangle).$$

Proof We can easily obtain Corollary similar to Theorem 10.

Theorem 18 *Assume that* $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth} \rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth} \rangle)$ *and* $F^{\gamma}{}_{\beth} = (\langle \phi_{\beth}, \varphi_{\beth} \rangle, \langle \mathscr{K}_{\beth}, \mathscr{M}_{\beth} \rangle)$ *are two families of LDFNs. If* $r > 0$ *and* $F^{\gamma} = (\langle \zeta^{\tau}{}_{F^{\gamma}}, \eta^{\upsilon}{}_{F^{\gamma}} \rangle, \langle \mathscr{J}^{\aleph}{}_{F^{\gamma}}, \mathscr{C}^{\gamma}{}_{F^{\gamma}} \rangle)$ *is an LDFN, then:*

1. $LDFWG(\daleth^{\varsigma}{}_1 \oplus F^{\gamma}, \daleth^{\varsigma}{}_2 \oplus F^{\gamma}, \ldots \daleth^{\varsigma}{}_n \oplus F^{\gamma}) = LDFWG(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) \oplus F^{\gamma}$

2. $LDFWG(r\daleth^{\varsigma}{}_{1}, r\daleth^{\varsigma}{}_{2}, \ldots r\daleth^{\varsigma}{}_{n}) = r\ \ LDFWG(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{n})$
3. $LDFWG(\daleth^{\varsigma}{}_{1}\oplus\digamma^{\gamma}{}_{1}, \daleth^{\varsigma}{}_{2}\oplus\digamma^{\gamma}{}_{2}, \ldots \daleth^{\varsigma}{}_{n}\oplus\digamma^{\gamma}{}_{n}) = LDFWG(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{n})$ $\oplus\, LDFWG(\digamma^{\gamma}{}_{1}, \digamma^{\gamma}{}_{2}, \ldots \digamma^{\gamma}{}_{n})$
4. $LDFWG(r\daleth^{\varsigma}{}_{1}\oplus\digamma^{\gamma}, r\daleth^{\varsigma}{}_{2}\oplus\digamma^{\gamma}, \ldots \oplus r\daleth^{\varsigma}{}_{n}\oplus\digamma^{\gamma}) = r\ LDFWG(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{n}) \oplus \digamma^{\gamma}$

Proof The proof of this theorem is the same as Theorem 15.

Theorem 19 *(Monotonicity) Assume that* $\daleth^{\varsigma}{}_{\beth} = (\langle \zeta^{\tau}{}_{\beth}, \eta^{\upsilon}{}_{\beth}\rangle, \langle \mathscr{J}^{\aleph}{}_{\beth}, \mathscr{C}^{\gamma}{}_{\beth}\rangle)$ *and* $\daleth^{\varsigma}{}^{*}_{\beth} = (\langle \zeta^{\tau}{}^{*}_{\beth}, \eta^{\upsilon}{}^{*}_{\beth}\rangle, \langle \mathscr{J}^{\aleph}{}^{*}_{\beth}, \mathscr{C}^{\gamma}{}^{*}_{\beth}\rangle)$ *are the agglomerations of LDFNs. If* $\zeta^{\tau}{}^{*}_{\beth} \geq \zeta^{\tau}{}_{\beth}$, $\eta^{\upsilon}{}^{*}_{\beth} \leq \eta^{\upsilon}{}_{\beth}$, $\mathscr{J}^{\aleph}{}^{*}_{\beth} \geq \mathscr{J}^{\aleph}{}_{\beth}$, *and* $\mathscr{C}^{\gamma}{}^{*}_{\beth} \leq \mathscr{C}^{\gamma}{}_{\beth}$ *for all* j*, then*

$$LDFWG(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{n}) \leq LDFWG(\daleth^{\varsigma}{}^{*}_{1}, \daleth^{\varsigma}{}^{*}_{2}, \ldots \daleth^{\varsigma}{}^{*}_{n})$$

Proof Here, $\eta^{\upsilon}{}^{*}_{\beth} \geq \eta^{\upsilon}{}_{\beth}$ and $\zeta^{\tau}{}^{*}_{\beth} \leq \zeta^{\tau}{}_{\beth}$ for all j. If $\eta^{\upsilon}{}^{*}_{\beth} \geq \eta^{\upsilon}{}_{\beth}$:

$\Leftrightarrow \eta^{\upsilon}{}^{*}_{\beth} \geq \eta^{\upsilon}{}_{\beth} \Leftrightarrow 1-\eta^{\upsilon}{}^{*}_{\beth} \leq 1-\eta^{\upsilon}{}_{\beth}$
$\Leftrightarrow (1-\eta^{\upsilon}{}^{*}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \leq (1-\eta^{\upsilon}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}$
$\Leftrightarrow \prod_{\beth=1}^{n}(1-\eta^{\upsilon}{}^{*}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \leq \prod_{\beth=1}^{n}(1-\eta^{\upsilon}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}$
$\Leftrightarrow 1-\prod_{\beth=1}^{n}(1-\eta^{\upsilon}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \leq 1-\prod_{\beth=1}^{n}(1-\eta^{\upsilon}{}^{*}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}$.

And:

$\mathscr{C}^{\gamma}{}^{*}_{\beth} \geq \mathscr{C}^{\gamma}{}_{\beth}$ and $\mathscr{J}^{\aleph}{}^{*}_{\beth} \leq \mathscr{J}^{\aleph}{}_{\beth}$ for all j. If $\mathscr{C}^{\gamma}{}^{*}_{\beth} \geq \mathscr{C}^{\gamma}{}_{\beth}$
$\Leftrightarrow \mathscr{C}^{\gamma}{}^{*}_{\beth} \geq \mathscr{C}^{\gamma}{}_{\beth} \Leftrightarrow 1-\mathscr{C}^{\gamma}{}^{*}_{\beth} \leq 1-\mathscr{C}^{\gamma}{}_{\beth}$
$\Leftrightarrow (1-\mathscr{C}^{\gamma}{}^{*}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \leq (1-\mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}$
$\Leftrightarrow \prod_{\beth=1}^{n}(1-\mathscr{C}^{\gamma}{}^{*}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \leq \prod_{\beth=1}^{n}(1-\mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}$
$\Leftrightarrow 1-\prod_{\beth=1}^{n}(1-\mathscr{C}^{\gamma}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \leq 1-\prod_{\beth=1}^{n}(1-\mathscr{C}^{\gamma}{}^{*}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}$.

Now:

$\zeta^{\tau}{}^{*}_{\beth} \leq \zeta^{\tau}{}_{\beth} \Leftrightarrow (\zeta^{\tau}{}^{*}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \leq (\zeta^{\tau}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}$
$\Leftrightarrow \prod_{\beth=1}^{n}(\zeta^{\tau}{}^{*}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \leq \prod_{\beth=1}^{n}(\zeta^{\tau}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}$.

And:

$\mathscr{J}^{\aleph}{}^{*}_{\beth} \leq \mathscr{J}^{\aleph}{}_{\beth}$
$\Leftrightarrow (\mathscr{J}^{\aleph}{}^{*}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \leq (\mathscr{J}^{\aleph}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}$.
$\Leftrightarrow \prod_{\beth=1}^{n}(\mathscr{J}^{\aleph}{}^{*}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}} \leq \prod_{\beth=1}^{n}(\mathscr{J}^{\aleph}{}_{\beth})^{\mathfrak{P}^{\gamma}{}_{\beth}}$.
Let

$$\overline{\daleth^{\varsigma}} = \mathrm{LDFWG}(\daleth^{\varsigma}{}_{1}, \daleth^{\varsigma}{}_{2}, \ldots \daleth^{\varsigma}{}_{n})$$

and

$$\overline{\daleth^{\varsigma *}} = \mathrm{LDFWG}(\daleth^{\varsigma}{}^{*}_{1}, \daleth^{\varsigma}{}^{*}_{2}, \ldots \daleth^{\varsigma}{}^{*}_{n}).$$

We get that $\overline{\daleth^{\varsigma *}} \geq \overline{\daleth^{\varsigma}}$. So,

$$\text{LDFWG}(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) \leq \text{LDFWG}(\daleth^{\varsigma *}_1, \daleth^{\varsigma *}_2, \ldots \daleth^{\varsigma *}_n).$$

3.4 LDFOWG Operator

Definition 19 Consider $\daleth^{\varsigma}{}_{\gimel} = (\langle \zeta^{\tau}{}_{\gimel}, \eta^{\upsilon}{}_{\gimel} \rangle, \langle \mathscr{J}^{\aleph}{}_{\gimel}, \mathscr{C}^{\gamma}{}_{\gimel} \rangle)$ is the agglomeration of LDFNs and $LDFOWG : \$^n \to \$$ be a mapping

$$\text{LDFOWG}(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) = \daleth^{\varsigma}{}^{\mathfrak{P}^{\gamma}{}_1}_{\sigma(1)} \otimes \daleth^{\varsigma}{}^{\mathfrak{P}^{\gamma}{}_2}_{\sigma(2)} \otimes \ldots, \otimes \daleth^{\varsigma}{}^{\mathfrak{P}^{\gamma}{}_n}_{\sigma(n)}, \tag{14.13}$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$, such that $\daleth^{\varsigma}{}_{\sigma(r-1)} \geq \daleth^{\varsigma}{}_{\sigma(r)}$, for any r. Then the mapping LDFOWG is called LDFOWG operator, where $(\mathfrak{P}^{\gamma}{}_1, \mathfrak{P}^{\gamma}{}_2, \ldots, \mathfrak{P}^{\gamma}{}_n)$ be the WV with the constraint $\mathfrak{P}^{\gamma}{}_i > 0$ and $\sum_{i=1}^{n} \mathfrak{P}^{\gamma}{}_i = 1$.

We may also consider LDFOWG using the theorem below based on LDFNs operational law.

Theorem 20 *Assume that* $\daleth^{\varsigma}{}_{\gimel} = (\langle \zeta^{\tau}{}_{\gimel}, \eta^{\upsilon}{}_{\gimel} \rangle, \langle \mathscr{J}^{\aleph}{}_{\gimel}, \mathscr{C}^{\gamma}{}_{\gimel} \rangle)$ *is the agglomeration of LDFNs, and we can find LDFOWG by LDFOWG*$(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n)$

$$= \left(\left\langle \overline{\prod}_{\gimel=1}^{n} \zeta^{\tau}{}^{\mathfrak{P}^{\gamma}{}_{\gimel}}_{\sigma(j)}, 1 - \overline{\prod}_{\gimel=1}^{n} (1 - \eta^{\upsilon}{}_{\sigma(j)})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle, \right.$$
$$\left. \left\langle \overline{\prod}_{\gimel=1}^{n} \mathscr{J}^{\aleph}{}^{\mathfrak{P}^{\gamma}{}_{\gimel}}_{\sigma(j)}, 1 - \overline{\prod}_{\gimel=1}^{n} (1 - \mathscr{C}^{\gamma}{}_{\sigma(j)})^{\mathfrak{P}^{\gamma}{}_{\gimel}} \right\rangle \right). \tag{14.14}$$

Theorem 21 (Monotonicity) *Assume that* $\daleth^{\varsigma}{}_{\gimel} = (\langle \zeta^{\tau}{}_{\gimel}, \eta^{\upsilon}{}_{\gimel} \rangle, \langle \mathscr{J}^{\aleph}{}_{\gimel}, \mathscr{C}^{\gamma}{}_{\gimel} \rangle)$ *and* $\daleth^{\varsigma *}_{\gimel} = (\langle \zeta^{\tau *}_{\gimel}, \eta^{\upsilon *}_{\gimel} \rangle, \langle \mathscr{J}^{\aleph *}_{\gimel}, \mathscr{C}^{\gamma *}_{\gimel} \rangle)$ *are the agglomerations of LDFNs. If* $\zeta^{\tau *}_{\gimel} \geq \zeta^{\tau}{}_{\gimel}$, $\eta^{\upsilon *}_{\gimel} \leq \eta^{\upsilon}{}_{\gimel}$, $\mathscr{J}^{\aleph *}_{\gimel} \geq \mathscr{J}^{\aleph}{}_{\gimel}$, *and* $\mathscr{C}^{\gamma *}_{\gimel} \leq \mathscr{C}^{\gamma}{}_{\gimel}$ *for all j, then*

$$LDFOWG(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) \leq LDFOWG(\daleth^{\varsigma *}_1, \daleth^{\varsigma *}_2, \ldots \daleth^{\varsigma *}_n).$$

Theorem 22 *Assume that* $\daleth^{\varsigma}{}_{\gimel} = (\langle \zeta^{\tau}{}_{\gimel}, \eta^{\upsilon}{}_{\gimel} \rangle, \langle \mathscr{J}^{\aleph}{}_{\gimel}, \mathscr{C}^{\gamma}{}_{\gimel} \rangle)$ *and* $\digamma^{\gamma}{}_{\gimel} = (\langle \phi_{\gimel}, \varphi_{\gimel} \rangle, \langle \mathscr{K}_{\gimel}, \mathscr{M}_{\gimel} \rangle)$ *are two families of LDFNs. If* $r > 0$ *and* $\digamma^{\gamma} = (\langle \zeta^{\tau}{}_{\digamma^{\gamma}}, \eta^{\upsilon}{}_{\digamma^{\gamma}} \rangle, \langle \mathscr{J}^{\aleph}{}_{\digamma^{\gamma}}, \mathscr{C}^{\gamma}{}_{\digamma^{\gamma}} \rangle)$ *is an LDFN, then:*

1. *LDFOWG*$(\daleth^{\varsigma}{}_1 \oplus \digamma^{\gamma}, \daleth^{\varsigma}{}_2 \oplus \digamma^{\gamma}, \ldots \daleth^{\varsigma}{}_n \oplus \digamma^{\gamma}) =$ *LDFOWG*$(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) \oplus \digamma^{\gamma}$
2. *LDFOWG*$(r\daleth^{\varsigma}{}_1, r\daleth^{\varsigma}{}_2, \ldots r\daleth^{\varsigma}{}_n) = r$ *LDFOWG*$(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n)$
3. *LDFOWG*$(\daleth^{\varsigma}{}_1 \oplus \digamma^{\gamma}{}_1, \daleth^{\varsigma}{}_2 \oplus \digamma^{\gamma}{}_2, \ldots \daleth^{\varsigma}{}_n \oplus \digamma^{\gamma}{}_n) =$ *LDFOWG*$(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) \oplus$ *LDFOWG*$(\digamma^{\gamma}{}_1, \digamma^{\gamma}{}_2, \ldots \digamma^{\gamma}{}_n)$

4. *LDFOWG*$(r\daleth^{\varsigma}{}_1 \oplus F^{\gamma}, r\daleth^{\varsigma}{}_2 \oplus F^{\gamma}, \ldots \oplus r\daleth^{\varsigma}{}_n \oplus F^{\gamma}) = r$*LDFOWG*$(\daleth^{\varsigma}{}_1, \daleth^{\varsigma}{}_2, \ldots \daleth^{\varsigma}{}_n) \oplus F^{\gamma}$

4 Proposed Methodology Based on Developed AOs

Let $\mathscr{T}^{\beth} = \{\mathscr{T}^{\beth}{}_1, \mathscr{T}^{\beth}{}_2, \ldots, \mathscr{T}^{\beth}{}_m\}$ and $\breve{\mathscr{G}}^{\zeta} = \{\breve{\mathscr{G}}^{\zeta}_1, \breve{\mathscr{G}}^{\zeta}_2, \ldots, \breve{\mathscr{G}}^{\zeta}_n\}$ be the alternatives and criterion, respectively. DM offered his judgement matrix $D = (\aleph^{\Bbbk}_{ij})_{m\times n}$, in which $\aleph^{\Bbbk}_{ij}$ stands for the alternate $\mathscr{T}^{\beth}{}_i \in \mathscr{T}^{\beth}$ as per the parameter $\breve{\mathscr{G}}^{\zeta}_{\beth} \in \breve{\mathscr{G}}^{\zeta}$ by DM. The matrix D has converted into "normalized matrix" by the given formula $Y = (\varsigma^{\vartheta\wp}{}_{ij})_{m\times n}$,"

$$(\varsigma^{\vartheta\wp}{}_{ij})_{m\times n} = \begin{cases} (\aleph^{\Bbbk}_{ij})^c; & j \in \tau_c \\ \aleph^{\Bbbk}_{ij}; & j \in \tau_b, \end{cases} \tag{14.15}$$

where $(\aleph^{\Bbbk}_{ij})^c$ denotes the compliment of $\aleph^{\Bbbk}_{ij}$.

The MCDM will be updated to include the suggested operators, which will make the previously described processes necessary.

Algorithm

Step 1:
Acquire the judgment matrix $D = (\aleph^{\Bbbk}_{ij})_{m\times n}$ based on LDFNs from DMs.

$$\begin{array}{c} \\ \mathscr{T}^{\beth}{}_1 \\ \mathscr{T}^{\beth}{}_2 \\ \vdots \\ \mathscr{T}^{\beth}{}_m \end{array} \begin{array}{c} \begin{array}{cc} \breve{\mathscr{G}}_1 & \breve{\mathscr{G}}_2 \end{array} \\ \left[\begin{array}{cc} (\langle\zeta^{\tau}{}_{11}, \eta^{\upsilon}{}_{11}\rangle, \langle\mathscr{J}^{\aleph}{}_{11}, \mathscr{C}^{\gamma}{}_{11}\rangle) & (\langle\zeta^{\tau}{}_{12}, \eta^{\upsilon}{}_{12}\rangle, \langle\mathscr{J}^{\aleph}{}_{12}, \mathscr{C}^{\gamma}{}_{12}\rangle) \\ (\langle\zeta^{\tau}{}_{21}, \eta^{\upsilon}{}_{21}\rangle, \langle\mathscr{J}^{\aleph}{}_{21}, \mathscr{C}^{\gamma}{}_{21}\rangle) & (\langle\zeta^{\tau}{}_{22}, \eta^{\upsilon}{}_{22}\rangle, \langle\mathscr{J}^{\aleph}{}_{22}, \mathscr{C}^{\gamma}{}_{22}\rangle) \\ \vdots & \vdots \\ (\langle\zeta^{\tau}{}_{m1}, \eta^{\upsilon}{}_{m1}\rangle, \langle\mathscr{J}^{\aleph}{}_{m1}, \mathscr{C}^{\gamma}{}_{m1}\rangle) & (\langle\zeta^{\tau}{}_{m2}, \eta^{\upsilon}{}_{m2}\rangle, \langle\mathscr{J}^{\aleph}{}_{m2}, \mathscr{C}^{\gamma}{}_{m2}\rangle) \end{array}\right. \end{array}$$

$$\begin{array}{c} \breve{\mathscr{G}}_n \\ \left.\begin{array}{cc} \cdots\cdots & (\langle\zeta^{\tau}{}_{1n}, \eta^{\upsilon}{}_{1n}\rangle, \langle\mathscr{J}^{\aleph}{}_{1n}, \mathscr{C}^{\gamma}{}_{1n}\rangle) \\ \cdots\cdots & (\langle\zeta^{\tau}{}_{2n}, \eta^{\upsilon}{}_{2n}\rangle, \langle\mathscr{J}^{\aleph}{}_{2n}, \mathscr{C}^{\gamma}{}_{2n}\rangle) \\ \ddots\ddots & \vdots \\ \cdots\cdots & (\langle\zeta^{\tau}{}_{mn}, \eta^{\upsilon}{}_{mn}\rangle, \langle\mathscr{J}^{\aleph}{}_{mn}, \mathscr{C}^{\gamma}{}_{mn}\rangle) \end{array}\right]. \end{array}$$

Step 2:
There is no need for normalization if all indicators are of the same kind. The matrix D has amended to "transforming response matrix, $Y = (\varsigma^{\vartheta\wp}{}_{ij})_{m\times n}$" by Eq. 14.15.

Step 3:
Aggregate $\mathscr{R}^{S}_{ij}$ for all alternates $\mathscr{T}^{\daleth}{}_i$ by utilizing the LDFWA (LDFWG) operator.
$\mathscr{R}^{S}{}_{ij} = LDFWA(\varsigma^{\vartheta\wp}_{i1}, \varsigma^{\vartheta\wp}_{i2}, \dots \varsigma^{\vartheta\wp}_{in})$ or
$\mathscr{R}^{S}{}_{ij} = LDFWG(\varsigma^{\vartheta\wp}_{i1}, \varsigma^{\vartheta\wp}_{i2}, \dots \varsigma^{\vartheta\wp}_{in})$.
Step 4:
Compute the score against all the alternatives.
Step 5:
The SF was used to classify the alternatives, and the most appropriate option was chosen.

5 MCDM Example

Multi-criteria decision-making (MCDM) is a useful tool for agricultural decision-making as it allows for the consideration of multiple conflicting objectives and constraints. These may include economic, environmental, and social factors. The use of MCDM can lead to more sustainable and efficient farming practices, as well as improved decision-making for farmers and policymakers.

Some specific applications of MCDM in agriculture include:

- Land use planning: MCDM can be used to evaluate and compare different land use options, such as crop rotation, irrigation systems, and conservation practices. This can help farmers and policymakers make more informed decisions about how to use land resources in a sustainable and efficient way.
- Crop selection: MCDM can be used to evaluate and compare different crop options, taking into account factors such as yield, profitability, water usage, and environmental impact. This can help farmers make more informed decisions about which crops to grow, leading to increased productivity and sustainability.
- Livestock management: MCDM can be used to evaluate and compare different livestock management options, such as feed management, breeding strategies, and disease control. This can help farmers make more informed decisions about how to raise and manage livestock in a sustainable and efficient way.
- Water management: MCDM can be used to evaluate and compare different water management options, such as irrigation systems, water storage, and conservation practices. This can help farmers and policymakers make more informed decisions about how to use water resources in a sustainable and efficient way.
- Climate change mitigation: MCDM can be used to evaluate and compare different mitigation options, such as crop rotation, irrigation systems, and conservation practices. This can help farmers and policymakers make more informed decisions about how to adapt to and mitigate the impacts of climate change.

It is important to note that MCDM is not a one-size-fits-all solution and that the specific method used will depend on the specific problem being addressed and the available data. Additionally, it is important to involve stakeholders in the decision-making process to ensure that the results are socially acceptable.

MCDM is a useful tool for agricultural decision-making as it allows for the consideration of multiple conflicting objectives and constraints. Its applications in agriculture include land use planning, crop selection, livestock management, water management, and climate change mitigation. It can lead to more sustainable and efficient farming practices, as well as improved decision-making for farmers and policymakers. However, it is important to use appropriate method and involving stakeholders in the decision-making process.

Agriculture is a significant contributor to Pakistan's economy, accounting for 18.9 percent of the country's gross domestic product and employing 42.3 percent of the labor force. In addition to this, it is a significant source of revenues from international commerce, and it encourages growth in a variety of other areas. To boost development in this field, the public authority is focusing on aiding small and marginalized ranchers and pushing limited scope creative solutions. The sixth population and housing census that was conducted in Pakistan in 2017 revealed that the country's overall population is expanding at a pace of 2.4 percent on an annual basis. Demand for goods produced by agriculture is expected to rise as a result of the fast population expansion. The current administration is centered on advancing this area and has begun various measures, for example, crop expansion, decreasing increase rates, proficient utilization of water, and advancement of high worth yields including biotechnology, agribusiness credit advancement, subsidized manure costs, and modest power for negritude wells. As a result, this current area's exhibition expanded complicated after undergoing moderate and slowed expansion over the previous 13 years.

Consider the decision-making challenge of determining the best agricultural land. Assume the agglomeration of choices, $\mathscr{T}^{\daleth}{}_1$, $\mathscr{T}^{\daleth}{}_2$, $\mathscr{T}^{\daleth}{}_3$, and $\mathscr{T}^{\daleth}{}_4$, also considering four criterions, $\wp^{\Re}{}_1$= irrigation, $\wp^{\Re}{}_2$= cost, $\wp^{\Re}{}_3$=soil, and $\wp^{\Re}{}_4$= processing industry and market. Assuming that the criteria were weighted as (0.25, 0.40, 0.20, 0.15).

Algorithm

5.1 With LDFWA Operator

Step 1:
Obtain matrix $D = (\aleph_{ij}^{\Bbbk})_{m\times n}$ by DM, which is shown in Table 14.2.
Step 2:
In this case, $\mathscr{G}_{\mathbf{2}}^{\breve{\zeta}}$ criteria are cost type criteria that all are the benefits types, so there is need of normalization. Normalized LDF-decision matrix is given in Table 14.1.
Step 3:
Aggregate the LDF values $\mathscr{R}^{\mathcal{S}}{}_{ij}$ for all $\mathscr{T}^{\daleth}{}_i$ using LDFWA operator, given in Table 14.3.

Table 14.1 Normalized LDF-decision matrix

		$\breve{\mathscr{G}}^{\zeta}_{\mathbf{1}}$	$\breve{\mathscr{G}}^{\zeta}_{\mathbf{2}}$	$\breve{\mathscr{G}}^{\zeta}_{\mathbf{3}}$	$\breve{\mathscr{G}}^{\zeta}_{\mathbf{4}}$
	$\mathscr{T}^{\daleth}{}_{1}$	$(\langle 0.50, 0.85\rangle, \langle 0.30, 0.10\rangle)$	$(\langle 0.70, 0.45\rangle, \langle 0.20, 0.25\rangle)$	$(\langle 0.65, 0.75\rangle, \langle 0.45, 0.25\rangle)$	$(\langle 0.85, 0.80\rangle, \langle 0.40, 0.20\rangle)$
	$\mathscr{T}^{\daleth}{}_{2}$	$(\langle 0.80, 0.90\rangle, \langle 0.45, 0.15\rangle)$	$(\langle 0.65, 0.45\rangle, \langle 0.35, 0.55\rangle)$	$(\langle 0.75, 0.45\rangle, \langle 0.40, 0.30\rangle)$	$(\langle 0.65, 0.85\rangle, \langle 0.45, 0.35\rangle)$
	$\mathscr{T}^{\daleth}{}_{3}$	$(\langle 0.35, 0.65\rangle, \langle 0.50, 0.20\rangle)$	$(\langle 0.95, 0.65\rangle, \langle 0.65, 0.25\rangle)$	$(\langle 0.45, 0.90\rangle, \langle 0.30, 0.45\rangle)$	$(\langle 0.55, 0.95\rangle, \langle 0.50, 0.30\rangle)$
	$\mathscr{T}^{\daleth}{}_{4}$	$(\langle 0.50, 0.50\rangle, \langle 0.50, 0.25\rangle)$	$(\langle 0.55, 0.90\rangle, \langle 0.40, 0.50\rangle)$	$(\langle 0.45, 0.65\rangle, \langle 0.35, 0.50\rangle)$	$(\langle 0.35, 0.65\rangle, \langle 0.30, 0.20\rangle)$

Table 14.2 Rating given by DM

		$\breve{\mathscr{G}}^{\zeta}_{\mathbf{1}}$	$\breve{\mathscr{G}}^{\zeta}_{\mathbf{2}}$	$\breve{\mathscr{G}}^{\zeta}_{\mathbf{3}}$	$\breve{\mathscr{G}}^{\zeta}_{\mathbf{4}}$
	$\mathscr{T}^{\daleth}{}_1$	$(\langle 0.50, 0.85\rangle, \langle 0.30, 0.10\rangle)$	$(\langle 0.45, 0.70\rangle, \langle 0.25, 0.20\rangle)$	$(\langle 0.65, 0.75\rangle, \langle 0.45, 0.25\rangle)$	$(\langle 0.85, 0.80\rangle, \langle 0.40, 0.20\rangle)$
	$\mathscr{T}^{\daleth}{}_2$	$(\langle 0.80, 0.90\rangle, \langle 0.45, 0.15\rangle)$	$(\langle 0.45, 0.65\rangle, \langle 0.55, 0.35\rangle)$	$(\langle 0.75, 0.45\rangle, \langle 0.40, 0.30\rangle)$	$(\langle 0.65, 0.85\rangle, \langle 0.45, 0.35\rangle)$
	$\mathscr{T}^{\daleth}{}_3$	$(\langle 0.35, 0.65\rangle, \langle 0.50, 0.20\rangle)$	$(\langle 0.65, 0.95\rangle, \langle 0.25, 0.65\rangle)$	$(\langle 0.45, 0.90\rangle, \langle 0.30, 0.45\rangle)$	$(\langle 0.55, 0.95\rangle, \langle 0.50, 0.30\rangle)$
	$\mathscr{T}^{\daleth}{}_4$	$(\langle 0.50, 0.50\rangle, \langle 0.50, 0.25\rangle)$	$(\langle 0.90, 0.55\rangle, \langle 0.50, 0.40\rangle)$	$(\langle 0.45, 0.65\rangle, \langle 0.35, 0.50\rangle)$	$(\langle 0.35, 0.65\rangle, \langle 0.30, 0.20\rangle)$

Table 14.3 LDF-aggregated values $\mathscr{R}^{\mathcal{S}}{}_i$

$\mathscr{R}^{\mathcal{S}}{}_1$	$(\langle 0.596248, 0.760098\rangle, \langle 0.32997, 0.175855\rangle)$
$\mathscr{R}^{\mathcal{S}}{}_2$	$(\langle 0.769462, 0.522578\rangle, \langle 0.523542, 0.612701\rangle)$
$\mathscr{R}^{\mathcal{S}}{}_3$	$(\langle 0.503278, 0.624946\rangle, \langle 0.708147, 0.613116\rangle)$
$\mathscr{R}^{\mathcal{S}}{}_4$	$(\langle 0.482460, 0.581847\rangle, \langle 0.532108, 0.399725\rangle)$

Step 4:
Compute the score for all LDF-aggregated values $\mathscr{R}^{\mathcal{S}}{}_i$.

$$\breve{\mathscr{C}}^{\daleth}(\mathscr{R}^{\mathcal{S}}{}_1) = 0.497566$$

$$\breve{\mathscr{C}}^{\daleth}(\mathscr{R}^{\mathcal{S}}{}_2) = 0.539431$$

$$\breve{\mathscr{C}}^{\daleth}(\mathscr{R}^{\mathcal{S}}{}_3) = 0.493341$$

$$\breve{\mathscr{C}}^{\daleth}(\mathscr{R}^{\mathcal{S}}{}_4) = 0.508249$$

Step 5:
Ranks according to SFs.

$$\mathscr{R}^{\mathcal{S}}{}_2 \succ \mathscr{R}^{\mathcal{S}}{}_4 \succ \mathscr{R}^{\mathcal{S}}{}_1 \succ \mathscr{R}^{\mathcal{S}}{}_3.$$

So,

$$\mathscr{T}^{\daleth}{}_2 \succ \mathscr{T}^{\daleth}{}_4 \succ \mathscr{T}^{\daleth}{}_1 \succ \mathscr{T}^{\daleth}{}_3$$

$\mathscr{T}^{\daleth}{}_2$ is the best alternative among all other alternatives.

5.2 With LDFWG Operator

Step 1:
Obtain matrix $D = (\aleph_{ij}^{\Bbbk})_{m\times n}$ by DM, which is shown in Table 14.4.
Step 2:
In this case, $\breve{\mathscr{G}}_{\mathbf{2}}^{\zeta}$ criteria are cost type criteria that all are the benefits types, so there is need of normalization. Normalized LDF-decision matrix is given in Table 14.5.
Step 3:
Aggregate the LDF values $\mathscr{R}^{\mathcal{S}}{}_{ij}$ for all $\mathscr{T}^{\daleth}{}_i$ using LDFWG operator, given in Table 14.6.
Step 4:
Compute the score for all LDF-aggregated values $\mathscr{R}^{\mathcal{S}}{}_i$.

$$\breve{\mathscr{C}}^{\daleth}(\mathscr{R}^{\mathcal{S}}{}_1) = 0.476266$$

Table 14.4 Rating given by DM

	$\breve{\mathscr{G}}_{\mathbf{1}}$	$\breve{\mathscr{G}}_{\mathbf{2}}$	$\breve{\mathscr{G}}_{\mathbf{3}}$	$\breve{\mathscr{G}}_{\mathbf{4}}$
$\mathscr{T}^{\daleth}_1$	$(\langle 0.50, 0.85\rangle, \langle 0.30, 0.10\rangle)$	$(\langle 0.45, 0.70\rangle, \langle 0.25, 0.20\rangle)$	$(\langle 0.65, 0.75\rangle, \langle 0.45, 0.25\rangle)$	$(\langle 0.85, 0.80\rangle, \langle 0.40, 0.20\rangle)$
$\mathscr{T}^{\daleth}_2$	$(\langle 0.80, 0.90\rangle, \langle 0.45, 0.15\rangle)$	$(\langle 0.45, 0.65\rangle, \langle 0.55, 0.35\rangle)$	$(\langle 0.75, 0.45\rangle, \langle 0.40, 0.30\rangle)$	$(\langle 0.65, 0.85\rangle, \langle 0.45, 0.35\rangle)$
$\mathscr{T}^{\daleth}_3$	$(\langle 0.35, 0.65\rangle, \langle 0.50, 0.20\rangle)$	$(\langle 0.65, 0.95\rangle, \langle 0.25, 0.65\rangle)$	$(\langle 0.45, 0.90\rangle, \langle 0.30, 0.45\rangle)$	$(\langle 0.55, 0.95\rangle, \langle 0.50, 0.30\rangle)$
$\mathscr{T}^{\daleth}_4$	$(\langle 0.50, 0.50\rangle, \langle 0.50, 0.25\rangle)$	$(\langle 0.90, 0.55\rangle, \langle 0.50, 0.40\rangle)$	$(\langle 0.45, 0.65\rangle, \langle 0.35, 0.50\rangle)$	$(\langle 0.35, 0.65\rangle, \langle 0.30, 0.20\rangle)$

Table 14.5 Normalized LDF-decision matrix

		$\breve{\mathscr{G}}^{\zeta}_{\mathbf{1}}$	$\breve{\mathscr{G}}^{\zeta}_{\mathbf{2}}$	$\breve{\mathscr{G}}^{\zeta}_{\mathbf{3}}$	$\breve{\mathscr{G}}^{\zeta}_{\mathbf{4}}$
	$\mathscr{T}^{\daleth}_1$	$(\langle 0.50, 0.85\rangle, \langle 0.30, 0.10\rangle)$	$(\langle 0.70, 0.45\rangle, \langle 0.20, 0.25\rangle)$	$(\langle 0.65, 0.75\rangle, \langle 0.45, 0.25\rangle)$	$(\langle 0.85, 0.80\rangle, \langle 0.40, 0.20\rangle)$
	$\mathscr{T}^{\daleth}_2$	$(\langle 0.80, 0.90\rangle, \langle 0.45, 0.15\rangle)$	$(\langle 0.65, 0.45\rangle, \langle 0.35, 0.55\rangle)$	$(\langle 0.75, 0.45\rangle, \langle 0.40, 0.30\rangle)$	$(\langle 0.65, 0.85\rangle, \langle 0.45, 0.35\rangle)$
	$\mathscr{T}^{\daleth}_3$	$(\langle 0.35, 0.65\rangle, \langle 0.50, 0.20\rangle)$	$(\langle 0.95, 0.65\rangle, \langle 0.65, 0.25\rangle)$	$(\langle 0.45, 0.90\rangle, \langle 0.30, 0.45\rangle)$	$(\langle 0.55, 0.95\rangle, \langle 0.50, 0.30\rangle)$
	$\mathscr{T}^{\daleth}_4$	$(\langle 0.50, 0.50\rangle, \langle 0.50, 0.25\rangle)$	$(\langle 0.55, 0.90\rangle, \langle 0.40, 0.50\rangle)$	$(\langle 0.45, 0.65\rangle, \langle 0.35, 0.50\rangle)$	$(\langle 0.35, 0.65\rangle, \langle 0.30, 0.20\rangle)$

Table 14.6 LDF-aggregated values $\mathscr{R}^{S}{}_{i}$

$\mathscr{R}^{S}{}_{1}$	$(\langle 0.547045, 0.771117\rangle, \langle 0.315797, 0.18666\rangle)$
$\mathscr{R}^{S}{}_{2}$	$(\langle 0.581468, 0.547835\rangle, \langle 0.469927, 0.700454\rangle)$
$\mathscr{R}^{S}{}_{3}$	$(\langle 0.442722, 0.812834\rangle, \langle 0.547528, 0.796085\rangle)$
$\mathscr{R}^{S}{}_{4}$	$(\langle 0.461491, 0.670541\rangle, \langle 0.503649, 0.468701\rangle)$

$$\breve{\mathscr{C}}^{\daleth}(\mathscr{R}^{S}{}_{2}) = 0.480777$$

$$\breve{\mathscr{C}}^{\daleth}(\mathscr{R}^{S}{}_{3}) = 0.345333$$

$$\breve{\mathscr{C}}^{\daleth}(\mathscr{R}^{S}{}_{4}) = 0.456474.$$

Step 5:
Ranks according to SFs.

$$\mathscr{R}^{S}{}_{2} \succ \mathscr{R}^{S}{}_{1} \succ \mathscr{R}^{S}{}_{4} \succ \mathscr{R}^{S}{}_{3}.$$

So,

$$\mathscr{T}^{\daleth}{}_{2} \succ \mathscr{T}^{\daleth}{}_{1} \succ \mathscr{T}^{\daleth}{}_{4} \succ \mathscr{T}^{\daleth}{}_{3}$$

$\mathscr{T}^{\daleth}{}_{2}$ is the best alternative among all other alternatives.

6 Conclusion

MCDM is a significant real-world decision issue, and its most fundamental and essential research is the expression of imprecise information. IFSs, PFSs, and q-ROFSs are all effective methods for handling fuzzy information. LDFSs are more generic than IFS, PFS, and q-ROFS due to their ability to loosen the severe limitations of IFS, PFS, and q-ROFS by considering RPs. MCDM is a crucial subfield in operation research. This assignment's techniques mostly rely on the nature of the issue being evaluated. Our everyday occurrences include unpredictability, imprecision, and obscurity. Existing structures were constructed on the basis of the concept that decision-makers (DMs) consider specific limitations while assessing various choices and qualities. However, this kind of situation makes it difficult for DMs to allocate MSDs and NMSDs; therefore, they do so with different constraints. LDFS is a novel method to uncertainty and decision-making issues that incorporates pairs of RPs versus MSDs and NMSDs in order to loosen these limits. We have used LDFSs to assess the validity of DMs' knowledge in the basic framework and to remove any distortion in the decision analysis. The significant advantage of including RPs into the examination is to reduce the likelihood of theoretical knowledge-based MSD and NMSD-related mistakes. In addition, we have developed a number of AOs, including the LDFWA operator and the LDFWG operator. Numerous intriguing aspects of the suggested operators are investigated, and their illustration is convincingly shown.

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Chapter 15
Hyperbolic Fuzzy TOPSIS Method for Multi-Criteria Decision-Making Problems

Palash Dutta and Abhilash Kangsha Banik

1 Significance of the Work

Multi-criteria decision-making (MCDM) problem has been a fascinating topic for researchers to solve decision-making (DM) problems. It requires the decision-makers to assign values to solve it. Although q-ROFS was introduced to get more flexibility of taking membership and nonmembership values, it cannot perform in situations such as taking membership value as 1 and a nonzero nonmembership value (e.g. (1,0.2)) and vice versa. The solution to this lies in the HFS approach where such situations do not cause trouble as even the extreme values (1,1) could be considered and thus can be viewed as an improvisation over q-ROFS approach. So, based on the HFS approach, new methodologies could be created to solve MCDM problem where decision-makers get more independence of choosing values. In this work, we propose a new TOPSIS methodology based on HFS background to solve MCDM problems. A new score function and Minkowski distance based on HFS background are proposed to be used in the methodology.

2 Introduction

MCDM problems have been developed to assist decision-makers in dealing with decision-making (DM) problems involving multiple criteria. The presence of uncertainty in DM problem makes it more difficult for decision-makers to identify the ideal solution. The use of distance measure has been crucial in solving DM

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C. Jana et al. (eds.), *Fuzzy Optimization, Decision-making and Operations Research*, https://doi.org/10.1007/978-3-031-35668-1_15

problems with uncertainty. A new understanding of the DM process has emerged as a result of Zadeh's [41] introduction of fuzzy sets (FS). Fuzzy approaches have proven to be a superior choice to conventional forms in solving MCDM problems with the presence of uncertainty. To tackle the MCDM problem, Hwang and Yoon [19] suggested the TOPSIS (technology for order preference by similarity to ideal solution) approach. Chen [9] extended TOPSIS to solve DM problem in a fuzzy environment. Fuzzy inferior ratio approach was put up by Hadi-Vencheh and Mirjaberi [17] for DM problems.

The scope of FS is limited as just membership degree (MD) is present. So, intuitionistic fuzzy (IF) set (IFS) [1–4] was put forward by Atanassov by adding nonmembership degree (NMD) in addition to the already existing membership degree (MD), such that their sum is limited by 1. The conception of vague sets was developed by Gau and Buehrer [14] as an addition to FST. A score function was created by Chen and Tan based on sets [8]. Later, vague set was noted as being comparable to that of IFSs by Bustince and Burillo [5]. On the basis of these, several score functions have been developed based on these ideas [13, 20, 29]. In order to solve DM problems with IF values (IFV), Xu [33] created aggregation operators (AOs) like (IFWA). In order to solve MADM problems, Xu and Yager [34] have introduced AO like (IFWG). DM problems can be successfully resolved by distance measures utilizing IFSs. By incorporating the two elements of MD and NMD, Bustince and Burillo [6] proposed distance measurements using IFS. Later, Szmidt and Kacprzyk further developed new distance measures using a third element called hesitation degree (HD) [28]. Using the Hausdorff metric, Grzegorzewski provided some distance measures [16] using MD and NMD. The Hausdorff distances were further expanded by Yang and Chiclana [40] using MD, NMD, and HD. In order to handle DM problems in an IF environment, Boran et al. [7] suggested a TOPSIS technique to solve DM problems.

Even though IFS is applicable, it fails when the total of MD and NMD becomes greater than 1. If $A = (0.6, 0.5) \in$ IFS, for instance, then $0.6 + 0.5 > 1$ and IFS fails. As a more potent tool than IFS, Yager later created Pythagorean fuzzy (PF) set (PFS) theory [35, 36, 38], which ensures that the $\text{MD}^2+\text{NMD}^2 \neq 1$. Due to greater independence in selecting MD and NMD, the PFS concept has now been used to handle a variety of DM situations. A score function was introduced to solve MCDM problems based on TOPSIS method that Zhang and Xu [42] proposed. A new score function was also put forth by Wu and Wei [31], who used it in their suggested DM problem. With regard to the proposed MCDM method, Ma and Xu [22] provided a new score function and suggested new AOs. In this regard, some new scoring functions were also created [23, 24, 26]. In order to solve MCDM issues, Chen [10] developed a Minkowski distance measure for PFS utilizing a distance parameter called m. For $m = 1$, $m = 2$, and $m \to \infty$, the distance measure yields normalized Hamming distance, normalized Euclidean distance, and normalized Hausdorff distance, respectively.

However, PFS again fails when MD^2+NMD^2 becomes more than 1. If $A = (0.8, 0.7) \in$ PFS, for instance, then $0.8^2 + 0.7^2 > 1$ and PFS fails. As a generalization of the IFS and PFS, Yager formulated the q-rung orthopair fuzzy set (q-ROFS) [39], where MD^q+NMD^q is constrained to 1. More independence in choosing membership and nonmembership values is provided by raising q's powers. Therefore, it becomes IFS for q=1 and PFS for q=2. A new score function based on q-ROFS was proposed by Liu and Wang, along with AOs $q - ROFWA$ and $q - ROFWG$, and their applications to DM [21]. Peng et al. [25] developed a new score function that also took the degree of hesitation into account and used it to solve MCDM issues. Similarly, several score functions were created in this regard as improvements over pre-existing ones by various scholars [15, 30, 32]. Du [11], employing a distance parameter m for q-ROFS, developed a Minkowski distance measure that yields normalized Hamming distance for $m = 1$, normalized Euclidean distance for $m = 2$, and normalized Hausdorff distance for $m \to \infty$. The q-ROF TOPSIS method for the green supplier selection problem was proposed by Pinar et al. [27].

Centred on the outcome that variables causing job satisfaction among employees differ from those producing dissatisfaction, Herzberg suggested a two-factor model of motivation [18]. It was supported by information that Herzberg gathered through his interviews with 203 engineers and accountants in the Pittsburgh region. Motivators, such as challenging work, recognition, and responsibility, which provide satisfaction, and hygiene factors, such as salary and job security, which, while they do not directly contribute to higher satisfaction, nonetheless result in dissatisfaction when absent, were the two variables that were observed.

Now, taking inspiration from the two-factor theory, when the MD is 1 and the NMD is not zero and vice versa, the q-ROFS approach fails. Consider the case when $A = (1, 0.1) \in$ q-ROFS, it fails since $1^q + 0.1^q > 1$, for any q. Here, hyperbolic fuzzy set (HFS) developed by Dutta and Borah [12] steps in to play a more independent role in the selection of optimistic degree (OD) and pessimistic degree (PD). Using HFS, we can take the OD as 1 and also a nonzero PD (and vice versa) which is unlikely in all the existing forms such as IFS, PFS, and the more generalized q-ROFS.

2.1 Motivation of the Study

MCDM problems require finding the best alternative from a set of alternatives that meet certain criteria. Traditionally, the information of the alternatives was provided via crisp numbers. But, due to the presence of uncertainty, crisp numbers cannot give the precise information. The introduction of fuzzy MCDM techniques based on fuzzy set has remarkably addressed such issues. The developments such as IFS, PFS, and q-ROFS have aided in solving MCDM problems. But the information of the alternatives given via IFS in the form of MD and NMD is limited as MD+NMD cannot be greater than 1. So, there is lesser flexibility in choosing MD and NMD. Its upgraded approach PFS gives more flexibility in choosing MD and NMD. And, finally, the q-ROFS gives the most flexibility among them in choosing MD and NMD. But it cannot take MD=1 and a nonzero NMD as MD^q+NMD^q > 1.

The HFS approach has been remarkable in this regard where the values taken are more flexible than the q-ROFS. So, there is a requirement of new techniques to be formulated based on HFS background to solve MCDM problems. This forms the basic motivation of our study.

2.2 Structure of the Study

The overview of the study is as follows: The Sect. 2 comprises the general idea of DM processes using IFS, PFS, and q-ROFS; also a literature review on previous studies is provided which is in the introduction part. The following is a summary of the paper. Preliminary definitions of FS, IFS, PFS, and q-ROFS are discussed in Sect. 3. Fundamental concept of HFS is presented in Sect. 4 after which certain operators are described. A novel score function based on HFS has been proposed later in Sect. 5 along with a few properties. The disadvantage of the proposed score function has been demonstrated by discussing the drawbacks of the existing score functions. The Minkowski distance utilizing HFS has been suggested in Sect. 6 together with the Hamming, Euclidean, and Hausdorff distances. The TOPSIS technique has been suggested in Sect. 7 as a means of resolving MCDM issues. A few MCDM problems are implemented in Sect. 8 to demonstrate the shortcomings of the exiting approaches and the applicability of the suggested approach. Finally, an appropriate conclusion and future scope are provided in Sect. 9.

3 Preliminaries

Here in the segment, we discuss some basic definitions of the FS, IFS, PFS, and q-ROFS.

Definition 3.1 (Fuzzy Set [41]) Let $Y = \{y_j : j = 1, 2, \ldots, n\}$ be a finite universe of discourse. Then a fuzzy set $\mathscr{F}$ is defined by $\mathscr{F} = \{\langle y_j, \widetilde{\mathscr{P}_{\mathscr{F}}}(y_j)\rangle; y_j \in Y\}$ where the function $\widetilde{\mathscr{P}_{\mathscr{F}}}(y_j) : Y \to [0, 1]$ defines the degree of membership.

Definition 3.2 (Intuitionistic Fuzzy Set [2]) An IFS $\mathscr{I}$ on a finite universe of discourse $Y = \{y_j : j = 1, 2, \ldots, n\}$ is defined by $\mathscr{I} = \{\langle y_j, \widetilde{\mathscr{P}_{\mathscr{I}}}(y_j), \widetilde{\mathscr{Q}_{\mathscr{I}}}(y_j)\rangle; y_j \in Y\}$ where the functions $\widetilde{\mathscr{P}_{\mathscr{I}}}(y_j) : Y \to [0, 1]$ define the MD and $\widetilde{\mathscr{Q}_{\mathscr{I}}}(y_j) : Y \to [0, 1]$ define the NMD such that $0 \leq \widetilde{\mathscr{P}_{\mathscr{Q}}}(y_j) + \widetilde{\mathscr{Q}_{\mathscr{I}}}(y_j) \leq 1$.

The HD is defined by $\widetilde{\mathscr{R}_{\mathscr{I}}}(y_j) = 1 - [\widetilde{\mathscr{P}_{\mathscr{I}}}(y_j) + \widetilde{\mathscr{Q}_{\mathscr{I}}}(y_j)]$ and $\widetilde{\mathscr{P}_{\mathscr{I}}}(y_j) + \widetilde{\mathscr{Q}_{\mathscr{I}}}(y_j) + \widetilde{\mathscr{R}_{\mathscr{I}}}(y_j) = 1$.

Definition 3.3 ([2]) Let us consider $\mathscr{I}_1 = \{\langle y_j, \widetilde{\mathscr{P}_{\mathscr{I}_1}}(y_j), \widetilde{\mathscr{Q}_{\mathscr{I}_1}}(y_j)\rangle; y_j \in Y\}$ and $\mathscr{I}_2 = \{\langle y_j, \widetilde{P}_{\mathscr{I}_2}(y_j), \widetilde{\mathscr{Q}_{\mathscr{I}_2}}(y_j)\rangle; y_j \in Y\}$ be two IFSs defined in Y. Then, $\mathscr{I}_1 \subseteq \mathscr{I}_2$ iff $\widetilde{\mathscr{P}_{\mathscr{I}_1}}(y_j) \leq \widetilde{\mathscr{P}_{\mathscr{I}_2}}(y_j)$ and $\widetilde{\mathscr{Q}_{\mathscr{I}_1}}(y_j) \geq \widetilde{\mathscr{Q}_{\mathscr{I}_2}}(y_j)$.

Definition 3.4 (Pythagorean Fuzzy Set [37]) An PFS P on a finite universe of discourse $Y = \{y_j : j = 1, 2, \ldots, n\}$ is defined by $P = \{\langle y_j, \widetilde{\mathscr{P}}_P(y_j), \widetilde{\mathscr{Q}}_P(y_j)\rangle; y_j \in Y\}$ where the functions $\widetilde{\mathscr{P}}_P(y_j) : Y \to [0, 1]$ define the MD and $\widetilde{\mathscr{Q}}_P(y_j) : Y \to [0, 1]$ define the NMD such that $0 \leq (\widetilde{\mathscr{P}}_P(y_j))^2 + (\widetilde{\mathscr{Q}}_P(y_j))^2 \leq 1$.

The HD is defined by $\widetilde{\mathscr{R}}_P(y_j) = [1 - \{(\widetilde{\mathscr{P}}_P(y_j))^2 + (\widetilde{\mathscr{Q}}_P(y_j))^2\}]^{1/2}$ and $\widetilde{\mathscr{R}}_P(Y) \in [0, 1]$ such that $(\widetilde{\mathscr{P}}_P(y_j))^2 + (\widetilde{\mathscr{Q}}_P(y_j))^2 + (\widetilde{\mathscr{R}}_P(y_j))^2 = 1$.

Definition 3.5 ([37]) Let us consider $P_1 = \{\langle y_j, \widetilde{\mathscr{P}}_{P_1}(y_j), \widetilde{\mathscr{Q}}_{P_1}(y_j)\rangle; y_j \in Y\}$ and $P_2 = \{\langle y_j, \widetilde{\mathscr{P}}_{P_2}(y_j), \widetilde{\mathscr{Q}}_{P_2}(y_j)\rangle; y_j \in Y\}$ be two PFSs defined in Y. Then, $P_1 \subseteq P_2$ iff $\widetilde{\mathscr{P}}_{P_1}(y_j) \leq \widetilde{\mathscr{P}}_{P_2}(y_j)$ and $\widetilde{\mathscr{Q}}_{P_1}(y_j) \geq \widetilde{\mathscr{Q}}_{P_2}(y_j)$.

Definition 3.6 (q-Rung Orthopair Fuzzy Set [39]) An q-ROFS Q on a finite universe of discourse $Y = \{y_j : j = 1, 2, \ldots, n\}$ is defined by $Q = \{\langle y_j, \widetilde{\mathscr{P}}_Q(y_j), \widetilde{\mathscr{Q}}_Q(y_j)\rangle; y_j \in Y\}$ where the functions $\widetilde{\mathscr{P}}_Q(y_j) : Y \to [0, 1]$ define the MD and $\widetilde{\mathscr{Q}}_Q(y_j) : Y \to [0, 1]$ define the NMD such that $0 \leq (\widetilde{\mathscr{P}}_Q(y_j))^q + (\widetilde{\mathscr{Q}}_Q(y_j))^q \leq 1$.

The HD is defined by $\widetilde{\mathscr{R}}_Q(y_j) = [1 - \{(\widetilde{\mathscr{P}}_Q(y_j))^q + (\widetilde{\mathscr{Q}}_Q(y_j))^q\}^{1/q}]$ and $\widetilde{\mathscr{R}}_Q(y_j) \in [0, 1]$ such that $(\widetilde{\mathscr{P}}_Q(y_j))^q + (\widetilde{\mathscr{Q}}_Q(y_j))^q + (\widetilde{\mathscr{R}}_Q(y_j))^q = 1$.

Definition 3.7 ([39]) Let us consider $Q_1 = \{\langle y_j, \widetilde{\mathscr{P}}_{Q_1}(y_j), \widetilde{\mathscr{Q}}_{Q_1}(y_j)\rangle; y_j \in Y\}$ and $Q_2 = \{\langle y_j, \widetilde{\mathscr{P}}_{Q_2}(y_j), \widetilde{\mathscr{Q}}_{Q_2}(y_j)\rangle; Y \in Y\}$ be two q-ROFSs defined in Y. Then, $Q_1 \subseteq Q_2$ iff $\widetilde{\mathscr{P}}_{Q_1}(Y) \leq \widetilde{\mathscr{P}}_{Q_2}(Y)$ and $\widetilde{\mathscr{Q}}_{Q_1}(Y) \geq \widetilde{\mathscr{Q}}_{Q_2}(Y)$.

4 Hyperbolic Fuzzy Set

Here in this segment, the rudimentary idea of HFS with definitions and properties are provided.

4.1 *The Idea of HFS*

A comprehensive study of the formulation of HFS is done by Dutta and Borah [12] where a hyperbola is considered from the general equation of the conic section to define the concept of HFS.

Figure 15.1 shows the graphical analysis of HFS. In the above figure, we consider the points under the graph $xy \leq 1$. If we consider the OD and PD as its axes, we get a squared portion that lies fully under the graph $xy \leq 1$. So the term hyperbolic fuzzy set is justified as $xy = 1$ represents a hyperbola which is obtained from the hyperbola $\dfrac{(x^2 - y^2)}{2} = 1$ with an angle of rotation 45°.

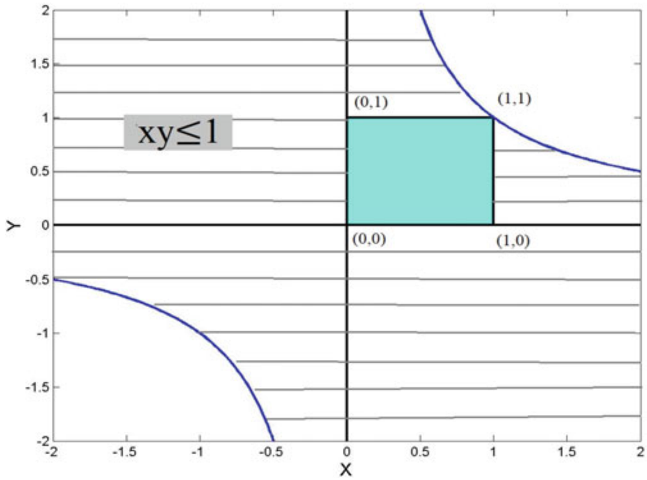

Fig. 15.1 Geometric interpretation of HFS

Table 15.1 Difference between IFS, PFS, q-ROFS, and HFS

IFS	PFS	q-ROFS	HFS
$0 \leq \mathscr{P} + \mathscr{Q} \leq 1$	$0 \leq \mathscr{P}^2 + \mathscr{Q}^2 \leq 1$	$0 \leq \mathscr{P}^q + \mathscr{Q}^q \leq 1$	$0 \leq \mathscr{P}\mathscr{Q} \leq 1$
$\mathscr{P} + \mathscr{Q} \not> 1$	$\mathscr{P}^2 + \mathscr{Q}^2 \not> 1$	$\mathscr{P}^q + \mathscr{Q}^q \not> 1$	$\mathscr{P}^q + \mathscr{Q}^q \leq 1$ or $\mathscr{P}^q + \mathscr{Q}^q > 1$

4.2 Basic Definition

Dutta and Borah [12] Let $Y = \{y_j : i = 1, 2, \ldots, n\}$ be a finite universe of discourse. We can define a hyperbolic fuzzy set $\mathscr{H}$ by $\mathscr{H} = \{\langle y_j, \widetilde{\mathscr{P}_{\mathscr{H}}}(y_j), \widetilde{\mathscr{Q}_{\mathscr{H}}}(y_j)\rangle; y_j \in Y\}$ where the functions $\widetilde{\mathscr{P}_{\mathscr{H}}}(y_j) : Y \to [0, 1]$ and $\widetilde{\mathscr{Q}_{\mathscr{H}}}(y_j) : Y \to [0, 1]$ define the optimistic degree (OD) and pessimistic degree (PD), respectively, of the element $y_j \in Y$ to $\mathscr{H}$, which is a subset of Y, and for every $y_j \in Y, 0 \leq \widetilde{\mathscr{P}_{\mathscr{H}}}(y_j) \times \widetilde{\mathscr{Q}_{\mathscr{H}}}(y_j) \leq 1$.

An HFS $\mathscr{H}$ is a pair of values (α, β) such that $\alpha, \beta \in$ [0,1] and $\alpha \times \beta \leq 1$. Here, $\alpha = \mathscr{P}_{\mathscr{H}}$ is the degree of optimism in $\mathscr{H}$ and $\beta = \mathscr{Q}_{\mathscr{H}}$ is the degree of pessimism in $\mathscr{H}$.

We observe that for $\alpha, \beta \in [0,1]$, then $\alpha \times \beta \leq (\alpha)^q + (\beta)^q \leq 1$ for any $q \in$N. So, we can conclude that HFS provides a wider range of OD and PDs over q-ROFS.

The following table (Table 15.1) shows the difference between IFS, PFS, q-ROFS, and HFS:

Now, we can define the operations as follows:

Definition 4.1 ([12]) Let $\mathscr{H}_1 = \{\langle y_j, \widetilde{\mathscr{P}_{\mathscr{H}_1}}(y_j), \widetilde{\mathscr{Q}_{\mathscr{H}_1}}(y_j)\rangle; Y \in Y\}$ and $\mathscr{H}_2 = \{\langle y_j, \widetilde{\mathscr{P}_{\mathscr{H}_2}}(y_j), \widetilde{\mathscr{Q}_{\mathscr{H}_2}}(y_j)\rangle; y_j \in Y\}$ be two HFSs defined in Y. Then, the following operations can be defined as follows:

(i) $\mathscr{H}_1 \subseteq \mathscr{H}_2$ iff $\widetilde{\mathscr{P}_{\mathscr{H}_1}}(y_j) \leq \widetilde{\mathscr{P}_{\mathscr{H}_2}}(y_j)$ and $\widetilde{\mathscr{Q}_{\mathscr{H}_1}}(y_j) \geq \widetilde{\mathscr{Q}_{\mathscr{H}_2}}(y_j)$
(ii) $\mathscr{H}_1^c(\text{or } \overline{\mathscr{H}}) = \{\langle y_j, 1 - \widetilde{\mathscr{P}_{\mathscr{H}_1}}(y_j), 1 - \widetilde{\mathscr{Q}_{\mathscr{H}_1}}(y_j)\rangle; y_j \in Y\}$
(iii) $\mathscr{H}_1 \cup \mathscr{H}_2 = \{\langle y_j, max(\widetilde{\mathscr{P}_{\mathscr{H}_1}}(y_j), \widetilde{\mathscr{P}_{\mathscr{H}_2}}(y_j)), min(\widetilde{\mathscr{Q}_{\mathscr{H}_1}}(y_j), \widetilde{\mathscr{Q}_{\mathscr{H}_2}}(y_j))\rangle; y_j \in Y\}$
(iv) $\mathscr{H}_1 \cap \mathscr{H}_2 = \{\langle y_j, min(\widetilde{\mathscr{P}_{\mathscr{H}_1}}(y_j), \widetilde{\mathscr{P}_{\mathscr{H}_2}}(y_j)), max(\widetilde{\mathscr{Q}_{\mathscr{H}_1}}(y_j), \widetilde{\mathscr{Q}_{\mathscr{H}_2}}(y_j))\rangle; y_j \in Y\}$
(v) $\mathscr{H}_1 + \mathscr{H}_2 = \{\langle y_j, (\widetilde{\mathscr{P}_{\mathscr{H}_1}}(y_j) + \widetilde{\mathscr{P}_{\mathscr{H}_2}}(y_j) - \widetilde{\mathscr{P}_{\mathscr{H}_1}}(y_j)\widetilde{\mathscr{P}_{\mathscr{H}_2}}(y_j)), (\widetilde{\mathscr{Q}_{\mathscr{H}_1}}(y_j)\widetilde{\mathscr{Q}_{\mathscr{H}_2}}(y_j))\rangle; y_j \in Y\}$
(vi) $\mathscr{H}_1 \times \mathscr{H}_2 = \{\langle y_j, (\widetilde{\mathscr{P}_{\mathscr{H}_1}}(y_j)\widetilde{\mathscr{P}_{\mathscr{H}_2}}(y_j)), (\widetilde{\mathscr{Q}_{\mathscr{H}_1}}(y_j) + \widetilde{\mathscr{Q}_{\mathscr{H}_2}}(y_j) - \widetilde{\mathscr{Q}_{\mathscr{H}_1}}(y_j)\widetilde{\mathscr{Q}_{\mathscr{H}_2}}(y_j))\rangle; y_j \in Y\}$

4.3 Necessity and Possibility Operators

Here, we will define two operators over the HFSs that transform an HFS into a fuzzy set. We have the following definition:

Definition Let $\mathscr{H}$ be an HFS; then we can define the following as:

(i) Necessity Operator: $\Box\mathscr{H} = \{\langle y_j, \widetilde{\mathscr{P}_{\mathscr{H}}}(y_j), 1 - \widetilde{\mathscr{P}_{\mathscr{H}}}(y_j)\rangle; y_j \in Y\}$
(ii) Possibility Operator: $\diamond\mathscr{H} = \{\langle y_j, 1 - \widetilde{\mathscr{Q}_{\mathscr{H}}}(y_j), \widetilde{\mathscr{Q}_{\mathscr{H}}}(y_j)\rangle; y_j \in Y\}$

Proposition *For every HFS $\mathscr{H}$, we have:*

(i) $\Box\Box\mathscr{H} = \Box\mathscr{H}$
(ii) $\diamond\diamond\mathscr{H} = \diamond\mathscr{H}$
(iii) $\diamond\Box\mathscr{H} = \Box\mathscr{H}$
(iv) $\Box\diamond\mathscr{H} = \diamond\mathscr{H}$
(v) $\overline{\Box\overline{\mathscr{H}}} = \Box\mathscr{H}$
(vi) $\overline{\diamond\overline{\mathscr{H}}} = \diamond\mathscr{H}$

Clearly, for $\Box\mathscr{H} = \{\langle y_j, \widetilde{\mathscr{P}_{\mathscr{H}}}(y_j), 1 - \widetilde{\mathscr{P}_{\mathscr{H}}}(y_j)\rangle; y_j \in Y\}$.

We have (i) $\Box\Box\mathscr{H} = \{\langle y_j, \widetilde{\mathscr{P}_{\mathscr{H}}}(y_j), 1 - (1 - \widetilde{\mathscr{P}_{\mathscr{H}}}(y_j))\rangle; y_j \in Y\} = \mathscr{H}$.
Similarly, (ii) $\diamond\diamond\mathscr{H} = \diamond\mathscr{H}$.
Also, (iii) $\diamond\Box\mathscr{H} = \{\langle y_j, \widetilde{\mathscr{P}_{\mathscr{H}}}(y_j), 1 - \widetilde{\mathscr{P}_{\mathscr{H}}}(y_j)\rangle; y_j \in Y\} = \mathscr{H}$.
Similarly, (iv) $\Box\diamond\mathscr{H} = \diamond\mathscr{H}$.
Now, (v) $\overline{\Box\overline{\mathscr{H}}} = \overline{\Box\{\langle y_j, 1 - \widetilde{\mathscr{P}_{\mathscr{H}}}(y_j), 1 - \widetilde{\mathscr{Q}_{\mathscr{H}}}(y_j)\rangle; y_j \in Y\}}$

$$= \overline{\{\langle y_j, 1 - \widetilde{\mathscr{P}_{\mathscr{H}}}(y_j), \widetilde{\mathscr{P}_{\mathscr{H}}}(y_j)\rangle; y_j \in Y\}}$$
$$= \{\langle y_j, \widetilde{\mathscr{P}_{\mathscr{H}}}(y_j), 1 - \widetilde{\mathscr{P}_{\mathscr{H}}}(y_j)\rangle; y_j \in Y\} = \Box\mathscr{H}$$

Similarly, (vi) $\overline{\diamond\overline{\mathscr{H}}} = \diamond\mathscr{H}$.

Theorem *For two HFSs $\mathscr{H}_1$ and $\mathscr{H}_2$:*

(i) $\Box(\mathscr{H}_1 \cap \mathscr{H}_2) = \Box\mathscr{H}_1 \cap \Box\mathscr{H}_2$
(ii) $\Box(\mathscr{H}_1 \cup \mathscr{H}_2) = \Box\mathscr{H}_1 \cup \Box\mathscr{H}_2$
(iii) $\diamond(\mathscr{H}_1 \cap \mathscr{H}_2) = \diamond\mathscr{H}_1 \cap \diamond\mathscr{H}_2$
(iv) $\diamond(\mathscr{H}_1 \cup \mathscr{H}_2) = \diamond\mathscr{H}_1 \cup \diamond\mathscr{H}_2$
(v) $\overline{\Box(\overline{\mathscr{H}_1} + \overline{\mathscr{H}_2})} = \Box\mathscr{H}_1 \times \Box\mathscr{H}_2$
(vi) $\overline{\Box(\overline{\mathscr{H}_1} \times \overline{\mathscr{H}_1})} = \Box\mathscr{H}_1 + \Box\mathscr{H}_2$
(vii) $\overline{\diamond(\overline{\mathscr{H}_1} + \overline{\mathscr{H}_1})} = \diamond\mathscr{H}_1 \times \diamond\mathscr{H}_1$
(viii) $\overline{\diamond(\overline{\mathscr{H}_1} \times \overline{\mathscr{H}_2})} = \diamond\mathscr{H}_1 + \diamond\mathscr{H}_2$

Proof (i) $\Box(\mathscr{H}_1 \cap \mathscr{H}_2) = \{\langle y_j, min(\widetilde{\mathscr{P}_{\mathscr{H}_1}}(Y), \widetilde{\mathscr{P}_{\mathscr{H}_2}}(Y)), max(\widetilde{\mathscr{Q}_{\mathscr{H}_1}}(Y), \widetilde{\mathscr{Q}_{\mathscr{H}_2}}(Y))\rangle; Y \in Y\}$

$=\{\langle y_j, min(\widetilde{\mathscr{P}_{\mathscr{H}_1}}(y_j), \widetilde{\mathscr{P}_{\mathscr{H}_2}}(y_j)), 1 - min(\widetilde{\mathscr{P}_{\mathscr{H}_1}}(y_j), \widetilde{\mathscr{P}_{\mathscr{H}_2}}(y_j))\rangle; y_j \in Y\}$

$=\{\langle y_j, min(\widetilde{\mathscr{P}_{\mathscr{H}_1}}(y_j), \widetilde{\mathscr{P}_{\mathscr{H}_2}}(y_j)), max(1 - \widetilde{\mathscr{P}_{\mathscr{H}_1}}(y_j), 1 - \widetilde{\mathscr{P}_{\mathscr{H}_2}}(y_j))\rangle; y_j \in Y\}$

Also, $\Box\mathscr{H}_1 \cap \Box\mathscr{H}_2 = \{\langle y_j, \widetilde{\mathscr{P}_{\mathscr{H}_1}}(y_j), 1 - \widetilde{\mathscr{P}_{\mathscr{H}_1}}(y_j)\rangle; y_j \in Y\} \cap \{\langle y_j, \widetilde{\mathscr{P}_{\mathscr{H}_2}}(y_j), 1 - \widetilde{\mathscr{P}_{\mathscr{H}_2}}(y_j)\rangle; y_j \in Y\}$

$=\{\langle y_j, min(\widetilde{\mathscr{P}_{\mathscr{H}_1}}(y_j), \widetilde{\mathscr{P}_{\mathscr{H}_2}}(y_j)), max(1 - \widetilde{\mathscr{P}_{\mathscr{H}_1}}(y_j), 1 - \widetilde{\mathscr{P}_{\mathscr{H}_2}}(y_j))\rangle; y_j \in Y\}$

Hence, $\Box(\mathscr{H}_1 \cap \mathscr{H}_2) = \Box\mathscr{H}_1 \cap \Box\mathscr{H}_2$.

Similarly, (ii) $\Box(\mathscr{H}_1 \cup \mathscr{H}_2) = \Box\mathscr{H}_1 \cup \Box\mathscr{H}_2$.
(iii) $\diamond(\mathscr{H}_1 \cap \mathscr{H}_2) = \diamond\mathscr{H}_1 \cap \diamond\mathscr{H}_2$.
(iv) $\diamond(\mathscr{H}_1 \cup \mathscr{H}_2) = \diamond\mathscr{H}_1 \cup \diamond\mathscr{H}_2$.

Further, it can be easily proven:
(v) $\overline{\Box(\overline{\mathscr{H}_1} + \overline{\mathscr{H}_2})} = \Box\mathscr{H}_1 \times \Box\mathscr{H}_2$
(vi) $\overline{\Box(\overline{\mathscr{H}_1} \times \overline{\mathscr{H}_1})} = \Box\mathscr{H}_1 + \Box\mathscr{H}_2$
(vii) $\overline{\diamond(\overline{\mathscr{H}_1} + \overline{\mathscr{H}_1})} = \diamond\mathscr{H}_1 \times \diamond\mathscr{H}_1$
(viii) $\overline{\diamond(\overline{\mathscr{H}_1} \times \overline{\mathscr{H}_2})} = \diamond\mathscr{H}_1 + \diamond\mathscr{H}_2$

5 Score Function Based on HFS

Here in this segment, a novel score function based on HFS is proposed along with some properties. The basic idea is to show its advantages over the existing score functions based on IFS, PFS, and q-ROFS.

5.1 Existing Score Functions

Ranking of fuzzy values by comparing them has been a notable feature of decision-making. With the development of IFS, score functions have been developed to compare the magnitude of two IF values. A score function for IF values was proposed by Chen and Tan [8] as follows:

Definition 5.1 Let $\mathscr{I} = \{\langle y_j, \widetilde{\mathscr{P}_\mathscr{I}}(y_j), \widetilde{\mathscr{Q}_\mathscr{I}}(y_j)\rangle; y_j \in Y\}$ be IF values; then the score function of $\mathscr{I}$ can be defined as follows:
(a) $S_{CT}(\mathscr{I}) = \widetilde{\mathscr{P}_\mathscr{I}}(Y) - \widetilde{\mathscr{Q}_\mathscr{I}}(Y)$ such that $S_{CT}(\mathscr{I}) \in [-1, 1]$.

Using the score function, a comparison law for IF values was introduced as follows:

Definition 5.2 Let $\mathscr{I}_1 = \{\langle y_j, \widetilde{\mathscr{P}_{\mathscr{I}_1}}(y_j), \widetilde{\mathscr{Q}_{\mathscr{I}_1}}(y_j)\rangle; y_j \in Y\}$ and $\mathscr{I}_2 = \{\langle y_j, \widetilde{\mathscr{P}_{\mathscr{I}_2}}(y_j), \widetilde{\mathscr{Q}_{\mathscr{I}_2}}(y_j)\rangle; y_j \in Y\}$ be two IFNs defined in Y. Let $S(\mathscr{I}_1)$ and $S(\mathscr{I}_2)$ be the scores of $\mathscr{I}_1$ and $\mathscr{I}_2$, respectively, then:

(1) If $S(\mathscr{I}_1) < S(\mathscr{I}_2)$, then $\mathscr{I}_1 < \mathscr{I}_2$
(2) If $S(\mathscr{I}_1) > S(\mathscr{I}_2)$, then $\mathscr{I}_1 > \mathscr{I}_2$
(3) If $S(\mathscr{I}_1) = S(\mathscr{I}_2)$, then $\mathscr{I}_1 = \mathscr{I}_2$

Some other score functions have been developed using IF values.
(b) $S_{WZL} = \dfrac{3\widetilde{\mathscr{P}_\mathscr{I}} - \widetilde{\mathscr{Q}_\mathscr{I}} - 1}{2}$ such that $S_{WZL} \in [-1, 1]$ (Wang et al. [29]).
(c) S_{LW}=$\widetilde{\mathscr{P}_\mathscr{I}}$+$\widetilde{\mathscr{P}_\mathscr{I}}$(1-$\widetilde{\mathscr{P}_\mathscr{I}}$-$\widetilde{\mathscr{Q}_\mathscr{I}}$) such that $S_{LW} \in [0,1]$ (Liu and Wang [20]).
(d) $S_G = \dfrac{e^{\widetilde{\mathscr{P}_\mathscr{I}} - \widetilde{\mathscr{Q}_\mathscr{I}}}}{\widetilde{\mathscr{R}_\mathscr{I}} + 1}$ such that $S_G \in [e^{-1}, e]$ (Gao and Liu [13]).

Again, with the development of PFS and then q-ROFS, new score functions have been developed with the comparison law similar to the IF comparison laws as shown above. All the existing score functions based on PF and q-ROF values are shown as follows:
(e) $S_{ZX} = (\widetilde{\mathscr{P}_P})^2 - (\widetilde{\mathscr{Q}_P})^2$ such that $S_{ZX} \in [-1, 1]$ (Zhang and Xu [42]).
(f) $S_{mx} = \begin{cases} \sqrt{(\widetilde{\mathscr{P}_P})^2 - (\widetilde{\mathscr{Q}_P})^2}, \ for, \ \widetilde{\mathscr{P}_P} \geq \widetilde{\mathscr{Q}_P} \\ -\sqrt{(\widetilde{\mathscr{Q}_P})^2 - (\widetilde{\mathscr{P}_P})^2}, \ for, \ \widetilde{\mathscr{Q}_P} \geq \widetilde{\mathscr{P}_P} \end{cases}$ such that $S_{mx} \in [0, 1]$ (Ma and Xu [22]).
(g) $S_{WW} = \dfrac{1 + (\widetilde{\mathscr{P}_P})^2 - (\widetilde{\mathscr{Q}_P})^2}{2}$ such that $S_{WW} \in [0, 1]$ (Wu and Wei [31]).
(h) $S_{PD} = \dfrac{e^{(\widetilde{\mathscr{P}_P})^2 - (\widetilde{\mathscr{Q}_P})^2}}{(\widetilde{\mathscr{R}_P})^2 + 1}$ such that $S_{PD} \in [e^{-1}, e]$ (Peng and Dai [24]).
(i) $S_{Pe} = (\widetilde{\mathscr{P}_P})^2 - (\widetilde{\mathscr{Q}_P})^2 + \left[\dfrac{e^{(\widetilde{\mathscr{P}_P})^2 - (\widetilde{\mathscr{Q}_P})^2}}{e^{(\widetilde{\mathscr{P}_P})^2 - (\widetilde{\mathscr{Q}_P})^2} + 1} - \dfrac{1}{2}\right](\widetilde{\mathscr{R}_P})^2$ such that $S_{Pe} \in [0, 1]$ (Peng [26]).
(j) $S_{PZL} = \widetilde{\mathscr{P}_P}^2 - \widetilde{\mathscr{Q}_P}^2 - ln(1 + \widetilde{\mathscr{R}_P}^2)$ such that $S_{PZD} \in [-1, 1]$ (Peng, Zhang, and Luo [23]).
(k) $S_{LU} = (\widetilde{\mathscr{P}_Q})^q - (\widetilde{\mathscr{Q}_Q})^q$ such that $S_{LU} \in [0, 1]$ (Liu and Wang [21]).
(l) $S_{We} = \dfrac{\widetilde{\mathscr{P}_Q}^q - \widetilde{\mathscr{Q}_Q}^q + 1}{2}$ such that $S_{We} \in [0, 1]$ (Wei et al. [30]).
(m) $S_{Pq} = \widetilde{\mathscr{P}_Q}^q - \widetilde{\mathscr{Q}_Q}^q + \left[\dfrac{e^{\widetilde{\mathscr{P}_Q}^q - \widetilde{\mathscr{Q}_Q}^q}}{e^{\widetilde{\mathscr{P}_Q}^q - \widetilde{\mathscr{Q}_Q}^q} + 1} - \dfrac{1}{2}\right]\widetilde{\mathscr{R}_Q}^q$ such that $S_{Pq} \in [0, 1]$ (Peng [25]).

(n) $S_X = \begin{cases} (\widetilde{\mathscr{P}_Q}^q - \widetilde{\mathscr{Q}_Q}^q)^{1/q}, for, \widetilde{\mathscr{P}_Q} \geq \widetilde{\mathscr{Q}_Q} \\ -(\widetilde{\mathscr{Q}_Q}^q - \widetilde{\mathscr{P}_Q}^q)^{1/q}, for, \widetilde{\mathscr{Q}_Q} \geq \widetilde{\mathscr{P}_Q} \end{cases}$ such that $S_X \in [0, 1]$ (Xing et al. [32]).

(o) $S_{GC} = \dfrac{e^{\widetilde{\mathscr{P}_Q}^q - \widetilde{\mathscr{Q}_Q}^q}}{\widetilde{\mathscr{R}_Q}^q + 1}$ such that $S_{GC} \in [e^{-1}, e]$ (Garg and Chen [15]).

5.2 *Novel Score Function Based on HFS*

Now, we propose a new score function that can be explained as follows:

Definition 5.3 Let $\mathscr{H} = \{\langle y_j, \widetilde{\mathscr{P}_{\mathscr{H}}}(y_j), \widetilde{\mathscr{Q}_{\mathscr{H}}}(y_j)\rangle; y_j \in Y\}$ be a HFN; then the score function is denoted as

$$S_A(\mathscr{H}) = \frac{2\widetilde{\mathscr{P}_{\mathscr{H}}}(y_j) - 2\widetilde{\mathscr{Q}_{\mathscr{H}}}(y_j) + 2 + \widetilde{\mathscr{P}_{\mathscr{H}}}(y_j) \times \widetilde{\mathscr{Q}_{\mathscr{H}}}(y_j)}{4}, \qquad S_A \in [0, 1]. \tag{15.1}$$

The 3-D geometric interpretation of the score function is shown in Fig. 15.2. In Fig. 15.2, we can understand the change of S_A with change in $\widetilde{\mathscr{P}}$ and $\widetilde{\mathscr{Q}}$. Then, we considered the variation of S_A with increase in $\widetilde{\mathscr{P}}$ taking $\widetilde{\mathscr{Q}}$ constant and increase in $\widetilde{\mathscr{Q}}$ taking $\widetilde{\mathscr{P}}$ constant. In Fig. 15.3, where $\mathscr{H} = \{\widetilde{\mathscr{P}}, \widetilde{\mathscr{Q}}\}$, we can see that taking $\widetilde{\mathscr{Q}} = 0, 0.2, 0.4, 0.6, 0.8, 1.0$, the value of S_A increases w.r.t the increase of $\widetilde{\mathscr{P}}$ for the values of $\widetilde{\mathscr{Q}}$. Similarly, in Fig. 15.3, where $\mathscr{H} = \{\widetilde{\mathscr{P}}, \widetilde{\mathscr{Q}}\}$, we can see that taking $\widetilde{\mathscr{P}} = 0, 0.2, 0.4, 0.6, 0.8, 1.0$, the value of S_A decreases w.r.t the increase of $\widetilde{\mathscr{Q}}$ for the values of $\widetilde{\mathscr{P}}$. In Fig. 15.4 we considered the variation of $\frac{\partial S_A}{\partial \widetilde{\mathscr{P}}}(\widetilde{\mathscr{P}}, \widetilde{\mathscr{Q}})$ w.r.t $\widetilde{\mathscr{Q}}$ and then we considered the variation of $\frac{\partial S_A}{\partial \widetilde{\mathscr{Q}}}(\widetilde{\mathscr{P}}, \widetilde{\mathscr{Q}})$ w.r.t $\widetilde{\mathscr{P}}$.

Proposition 1 *For a HFN $\mathscr{H} = \{\widetilde{\mathscr{P}}, \widetilde{\mathscr{Q}}\}$, S_A increases monotonically with the increase of $\widetilde{\mathscr{P}}$ and decreases monotonically with the increase of $\widetilde{\mathscr{Q}}$.*

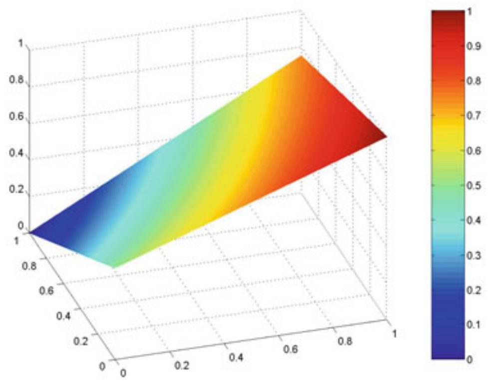

Fig. 15.2 3-D geometric interpretation of score function

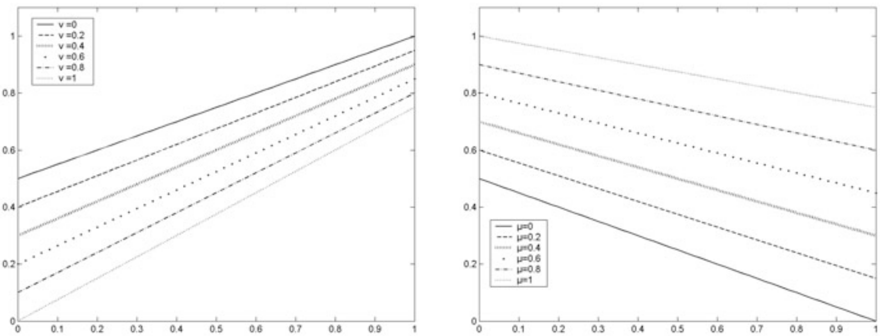

Fig. 15.3 Variation of $S_A(\widetilde{\mathscr{P}}, \widetilde{\mathscr{Q}})$ w.r.t (i) (left) $\widetilde{\mathscr{P}}$ and (ii) (right) $\widetilde{\mathscr{Q}}$

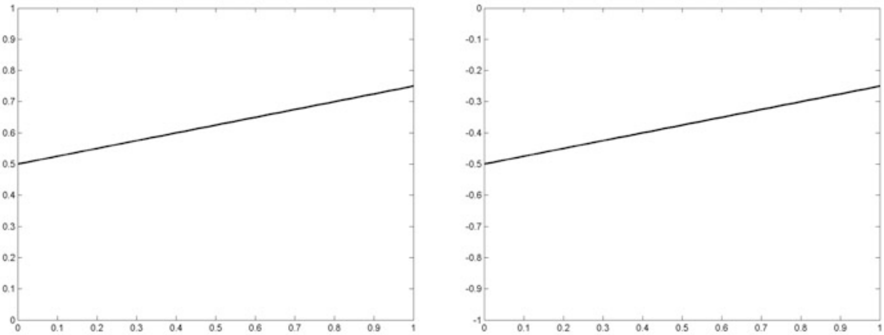

Fig. 15.4 (i) (Left) Variation of $\frac{\partial S_A}{\partial \widetilde{\mathscr{P}}}(\widetilde{\mathscr{P}}, \widetilde{\mathscr{Q}})$ w.r.t $\widetilde{\mathscr{Q}}$ and (ii) (right) variation of $\frac{\partial S_A}{\partial \widetilde{\mathscr{Q}}}(\widetilde{\mathscr{P}}, \widetilde{\mathscr{Q}})$ w.r.t $\widetilde{\mathscr{P}}$

Proof $\frac{\partial S_A}{\partial \widetilde{\mathscr{P}}} = \frac{2+\widetilde{\mathscr{Q}}}{4} \geq 0$ which is obtained as first partial derivative of S_A w.r.t $\widetilde{\mathscr{P}}$ based on (1).

Similarly, by first partial derivative of S_A w.r.t $\widetilde{\mathscr{Q}}$, we get $\frac{\partial S_A}{\partial \widetilde{\mathscr{Q}}} = \frac{-2+\widetilde{\mathscr{P}}}{4} \leq 0$.

As a result, we can see that S_A increases monotonically with the increase of $\widetilde{\mathscr{P}}$ and decreases monotonically with the increase of $\widetilde{\mathscr{P}}$.

Proposition 2 *For a HFN* $\mathscr{H} = \{\widetilde{\mathscr{P}}, \widetilde{\mathscr{Q}}\}$, S_A *holds the following:*

(a)$S_A(\mathscr{H}) = 0$ *iff* $\mathscr{H} = (0, 1)$; $S_A(\mathscr{H}) = 1$ *iff* $\mathscr{H} = (1, 0)$ *and (b)* $0 \leq S_A(\mathscr{H}) \leq 1$.

Proof

(a) Based on Proposition 1, it can be easily seen that if we just take $\widetilde{\mathscr{P}}$ and $\widetilde{\mathscr{Q}}$ into consideration, $S_A(\mathscr{H})$ can have the min value ($\mathscr{H}$ = (0, 1)) or max value ($\mathscr{H}$ = (1, 0)). In other words, $S_A(\mathscr{H})_{min}$ =0 and $S_A(\mathscr{H})_{max}$ = 1.
(b) Based on (a) $0 \leq S_A(\mathscr{H}) \leq 1$.

Proposition 3 *Let $\mathscr{H}_i = \{\widetilde{\mathscr{P}}_i, \widetilde{\mathscr{Q}}_i\}(i = 1, 2)$ be two HFNs. If $\widetilde{\mathscr{P}}_1 \geq \widetilde{\mathscr{P}}_2$ and $\widetilde{\mathscr{Q}}_1 \leq \widetilde{\mathscr{Q}}_2$, then $S_A(\mathscr{H}_1) \geq S_A(\mathscr{H}_2)$.*

Proof (a) Based on Proposition 1, it can be easily seen that $S_A(\mathscr{H})$ increases monotonically when the value of $\widetilde{\mathscr{P}}$ increases and decreases monotonically when the value of $\widetilde{\mathscr{Q}}$ increases.

Therefore, if $\widetilde{\mathscr{P}}_1 \geq \widetilde{\mathscr{P}}_2$ and $\widetilde{\mathscr{Q}}_1 \leq \widetilde{\mathscr{Q}}_2$, then $S_A(\mathscr{H}_1) \geq S_A(\mathscr{H}_2)$.

Proposition 4 *Let $\mathscr{H}_i = \{\widetilde{\mathscr{P}}_i, \widetilde{\mathscr{Q}}_i\}(i = 1, 2)$ be two HFNs; then:*

(a) If $\mathscr{H}_1 \subseteq \mathscr{H}_2$, then, $S_A(\mathscr{H}_1) \leq S_A(\mathscr{H}_2)$ and $S_A(\mathscr{H}_2^c) \leq S_A(\mathscr{H}_1^c)$
(b) $S_A(\mathscr{H}_1 \times \mathscr{H}_2) \leq S_A(\mathscr{H}_1 \cap \mathscr{H}_2) \leq S_A(\mathscr{H}_1), S_A(\mathscr{H}_2) \leq S_A(\mathscr{H}_1 \cup \mathscr{H}_2) \leq S_A(\mathscr{H}_1 + \mathscr{H}_2)$

Proof

(a) Based on Proposition 3, $\mathscr{H}_1 \subseteq \mathscr{H}_2 \widetilde{\mathscr{P}}_1 \leq \widetilde{\mathscr{P}}_2$ and $\widetilde{\mathscr{Q}}_1 \geq \widetilde{\mathscr{Q}}_2$; then, $S_A(\mathscr{H}_1) \leq S_A(\mathscr{H}_2)$.
Also, $\mathscr{H}_1 \subseteq \mathscr{H}_2 \implies \mathscr{H}_2^c \subseteq \mathscr{H}_1^c$; then $S_A(\mathscr{H}_2^c) \leq S_A(\mathscr{H}_1^c)$.
(b) We know $(\mathscr{H}_1 \times \mathscr{H}_2) \subseteq (\mathscr{H}_1 \cap \mathscr{H}_2) \subseteq \mathscr{H}_1, \mathscr{H}_2 \subseteq (\mathscr{H}_1 \cup \mathscr{H}_2) \subseteq (\mathscr{H}_1 + \mathscr{H}_2)$.
Hence, $S_A(\mathscr{H}_1 \times \mathscr{H}_2) \leq S_A(\mathscr{H}_1 \cap \mathscr{H}_2) \leq S_A(\mathscr{H}_1), S_A(\mathscr{H}_2) \leq S_A(\mathscr{H}_1 \cup \mathscr{H}_2) \leq S_A(\mathscr{H}_1 + \mathscr{H}_2)$.

Proposition 5 *Let $\mathscr{H}_i = \{\widetilde{\mathscr{P}}_i, \widetilde{\mathscr{Q}}_i\}(i = 1, 2, 3)$ be three HFNs, such that $\mathscr{H}_1 \subseteq \mathscr{H}_2 \subseteq \mathscr{H}_3$; then:*
(a)$S_A(\mathscr{H}_1 \times \mathscr{H}_2) \leq S_A(\mathscr{H}_1 \times \mathscr{H}_3)$, (b)$S_A(\mathscr{H}_1 \cap \mathscr{H}_2) \leq S_A(\mathscr{H}_1 \cap \mathscr{H}_3)$
(c)$S_A(\mathscr{H}_1 \cup \mathscr{H}_2) \leq S_A(\mathscr{H}_1 \cup \mathscr{H}_3)$, d)$S_A(\mathscr{H}_1 + \mathscr{H}_2) \leq S_A(\mathscr{H}_1 + \mathscr{H}_3)$

Proof We know $\mathscr{H}_1 \subseteq \mathscr{H}_2 \subseteq \mathscr{H}_3$; then, $\mathscr{H}_1 \times \mathscr{H}_2 \subseteq \mathscr{H}_1 \times \mathscr{H}_3, \mathscr{H}_1 \cap \mathscr{H}_2 \subseteq \mathscr{H}_1 \cap \mathscr{H}_3, \mathscr{H}_1 \cup \mathscr{H}_2 \subseteq \mathscr{H}_1 \cup \mathscr{H}_3, \mathscr{H}_1 + \mathscr{H}_2 \subseteq \mathscr{H}_1 + \mathscr{H}_3$.
Hence, (a) $S_A(\mathscr{H}_1 \times \mathscr{H}_2) \leq S_A(\mathscr{H}_1 \times \mathscr{H}_3)$, (b) $S_A(\mathscr{H}_1 \cap \mathscr{H}_2) \leq S_A(\mathscr{H}_1 \cap \mathscr{H}_3)$, (c) $S_A(\mathscr{H}_1 \cup \mathscr{H}_2) \leq S_A(\mathscr{H}_1 \cup \mathscr{H}_3)$, and (d)$S_A(\mathscr{H}_1 + \mathscr{H}_2) \leq S_A(\mathscr{H}_1 + \mathscr{H}_3)$.

5.3 Drawbacks of the Existing Score Functions

Now, to discuss the drawbacks of the existing score function, we consider the following cases:

Case 1 The drawbacks of the score functions based on IFS (except S_G) can be seen in Table 15.2.

In case of S_{CT}, drawback occurs whenever $\widetilde{\mathscr{P}_{\mathscr{I}}}(y_j) = \widetilde{\mathscr{Q}_{\mathscr{I}}}(y_j)$. Also for S_{WZL}, whenever $\mathscr{P}_{\mathscr{I}}(y_j) = 3\widetilde{\mathscr{Q}_{\mathscr{I}}}(y_j)$ it fails. In case of S_{LW}, it fails whenever $\mathscr{P}_{\mathscr{I}}(y_j) = 0$.

Case 2 Considering all the existing score functions, we consider three fuzzy profiles which are given by $\mathscr{A}_1 = (0, 0), \mathscr{A}_2 = (0.4, 0.4), \mathscr{A}_3 = (0.5, 0.5)$. Finding the score values of $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3$ for the different score functions, we

Table 15.2 Comparison of score functions based on IFS

Score function	$\mathscr{I}_1$=(0,0)	$\mathscr{I}_2$=(0,0.1)	$\mathscr{I}_3$=(0.1,0.3)	$\mathscr{I}_4$=(0.5,0.5)	Ranking
S_{CT}	0	−0.1	−0.2	0	$\mathscr{I}_4 = \mathscr{I}_1 > \mathscr{I}_2 > \mathscr{I}_3$
S_{WZL}	−0.5	−0.55	−0.5	0	$\mathscr{I}_4 > \mathscr{I}_1 = \mathscr{I}_3 > \mathscr{I}_2$
S_{LW}	0	0	0.16	0.5	$\mathscr{I}_4 > \mathscr{I}_3 > \mathscr{I}_1 = \mathscr{I}_2$
S_G	0.5	0.4762	0.5117	1	$\mathscr{I}_4 > \mathscr{I}_3 > \mathscr{I}_1 > \mathscr{I}_2$

Table 15.3 Comparison of score functions

Score function	$\mathscr{A}_1 = (0, 0)$	$\mathscr{A}_2 = (0.4, 0.4)$	$\mathscr{A}_3 = (0.5, 0.5)$	Ranking
S_{CT}	0	0	0	$\mathscr{A}_1 = \mathscr{A}_2 = \mathscr{A}_3$
S_{WZL}	−0.5	−0.1	0	$\mathscr{A}_1 < \mathscr{A}_2 < \mathscr{A}_3$
S_{LW}	0	0.48	0.5	$\mathscr{A}_1 < \mathscr{A}_2 < \mathscr{A}_3$
S_G	0.5	0.83	1	$\mathscr{A}_1 < \mathscr{A}_2 < \mathscr{A}_3$
S_{ZX}	0	0	0	$\mathscr{A}_1 = \mathscr{A}_2 = \mathscr{A}_3$
S_{mx}	0	0	0	$\mathscr{A}_1 = \mathscr{A}_2 = \mathscr{A}_3$
S_{WW}	0.5	0.5	0.5	$\mathscr{A}_1 = \mathscr{A}_2 = \mathscr{A}_3$
S_{PD}	0.5	0.5953	1	$\mathscr{A}_1 < \mathscr{A}_2 < \mathscr{A}_3$
S_{Pe}	0	0	0	$\mathscr{A}_1 = \mathscr{A}_2 = \mathscr{A}_3$
S_{PZL}	−0.6932	−0.5188	−0.4055	$\mathscr{A}_1 < \mathscr{A}_2 < \mathscr{A}_3$
S_{LU}	0	0	0	$\mathscr{A}_1 = \mathscr{A}_2 = \mathscr{A}_3$
S_{We}	0.5	0.5	0.5	$\mathscr{A}_1 = \mathscr{A}_2 = \mathscr{A}_3$
S_{Pq}	0	0	0	$\mathscr{A}_1 = \mathscr{A}_2 = \mathscr{A}_3$
S_X	0	0	0	$\mathscr{A}_1 = \mathscr{A}_2 = \mathscr{A}_3$
S_{GC}(q=3)	0.5	0.5342	0.5714	$\mathscr{A}_1 < \mathscr{A}_2 < \mathscr{A}_3$
S_A	0.5	0.54	0.5625	$\mathscr{A}_1 < \mathscr{A}_2 < \mathscr{A}_3$

can see the drawbacks in Table 15.3. Here, we can see the drawbacks of $S_{CT}, S_{ZX}, S_{mx}, S_{WW}, S_{Pe}, S_{LU}, S_{We}, S_{Pq}, S_X$. For example, in case of S_{CT}, we can see that $S_{CT}(\mathscr{A}_1) = S_{CT}(\mathscr{A}_2) = S_{CT}(\mathscr{A}_3) = 0$. Similar, drawbacks can be seen in $S_{ZX}, S_{mx}, S_{WW}, S_{Pe}, S_{LU}, S_{We}, S_{Pq}, S_{Pq}, S_X$.

Case 3 Again, S_{GC} can be seen as a generalization of S_G and S_{PD} which is of the form $S_{GC} = \dfrac{e^{(\widetilde{\mathscr{P}}_P)^q - (\widetilde{\mathscr{Q}}_P)^q}}{(\widetilde{\mathscr{R}}_P)^q + 1}$ such that $S_{GC} \in [e^{-1}, e]$.

Let $S_{GC}(\mathscr{H}_1, \mathscr{H}_2) = \mathscr{H}$; then for the pair $\left((\frac{1+ln\mathscr{H}}{2})^{\frac{1}{q}}, (\frac{1-ln\mathscr{H}}{2})^{\frac{1}{q}}\right)$, we have $S_{GC} = \dfrac{e^{(\frac{1+ln\mathscr{H}}{2} - \frac{1-ln\mathscr{H}}{2})}}{2 - \frac{1+ln\mathscr{H}}{2} - \frac{1-ln\mathscr{H}}{2}} = \dfrac{e^{ln\mathscr{H}}}{1} = \mathscr{H}$ i.e. for the different pairs $(\mathscr{H}_1, \mathscr{H}_2)$ and $\left((\frac{1+ln\mathscr{H}}{2})^{\frac{1}{q}}, (\frac{1-ln\mathscr{H}}{2})^{\frac{1}{q}}\right)$, the score function gives the same values.

For example, taking q=1, S_{GC} becomes S_G.

Let $\mathscr{A}_1 = (0.5, 0.3)$; then $S(\mathscr{A}_1) = 1.0178356318$.

Table 15.4 Comparison of score functions

Table	Profile 1	Profile 2	Profile 3	Profile 4
$\mathscr{A}_1$	(0.6,0.4)	(0.8,0.6)	(0.8,0.7)	(0.9,0.7)
$\mathscr{A}_2$	(0.3,0.35)	(0.3,0.35)	(0.3,0.35)	(0.3,0.35)
$\mathscr{A}_3$	(0.15,0.21)	(0.15,0.21)	(0.15,0.21)	(0.15,0.21)
1-ROFS	S($\mathscr{A}_1$)=0.2			
IFS	S($\mathscr{A}_2$)=-0.5			
	S($\mathscr{A}_3$)=-0.6			
2-ROFS		S($\mathscr{A}_1$)=0.28		
PFS		S($\mathscr{A}_2$)=-0.0325		
		S($\mathscr{A}_3$)=-0.0216		
3-ROFS			S($\mathscr{A}_1$)=0.169	
			S($\mathscr{A}_2$)=-0.01589	
			S($\mathscr{A}_3$)=-0.0059	
4-ROFS				S($\mathscr{A}_1$)=0.28
				S($\mathscr{A}_2$)=-0.0325
				S($\mathscr{A}_3$)=-0.0216
Ranking	$\mathscr{A}_1 > \mathscr{A}_2 > \mathscr{A}_3$	$\mathscr{A}_1 > \mathscr{A}_3 > \mathscr{A}_2$	$\mathscr{A}_1 > \mathscr{A}_3 > \mathscr{A}_2$	$\mathscr{A}_1 > \mathscr{A}_3 > \mathscr{A}_2$
Under	S($\mathscr{A}_1$)=0.61	S($\mathscr{A}_1$)=0.67	S($\mathscr{A}_1$)=0.665	S($\mathscr{A}_1$)=0.7075
HFS	S($\mathscr{A}_2$)=0.51375	S($\mathscr{A}_2$)=0.51375	S($\mathscr{A}_2$)=0.51375	S($\mathscr{A}_2$)=0.51375
	S($\mathscr{A}_3$)=0.492875	S($\mathscr{A}_3$)=0.492875	S($\mathscr{A}_3$)=0.492875	S($\mathscr{A}_3$)=0.492875
Ranking	$\mathscr{A}_1 > \mathscr{A}_2 > \mathscr{A}_3$	$\mathscr{A}_1 > \mathscr{A}_2 > A_3$	$\mathscr{A}_1 > A_2 > \mathscr{A}_3$	$\mathscr{A}_1 > \mathscr{A}_2 > \mathscr{A}_3$

Now, for $\mathscr{A}_2 = (0.5088392216, 0.4911607784)$, we have $S(\mathscr{A}_2) = 1.0178356318$, i.e. for different pairs $\mathscr{A}_1$ and $\mathscr{A}_2$, we have the same score value.

Again, for q=2, S_{GC} becomes S_{PD}.

Let $\mathscr{A}_1 = (0.5, 0.3)$; then $S(\mathscr{A}_1) = 0.70693425963$.

Now, for $\mathscr{A}_2 = (0.57148158222, 0.82061489213)$, we have $S(\mathscr{A}_2) = 0.70693425963$, i.e. for different pairs $\mathscr{A}_1$ and $\mathscr{A}_2$, we have the same score value.

Such type of drawback occurs in S_{GC}, S_G, S_{PD}.

Case 4 Again, we consider, the score function S_{LU}. We consider different fuzzy profiles in Table 15.4. For 1-ROFS, we consider $\mathscr{A}_1 = (0.6, 0.4), \mathscr{A}_2 = (0.3, 0.35), \mathscr{A}_3 = (0.15, 0.21)$. Here, we get the ranking order as $\mathscr{A}_1 > \mathscr{A}_2 > \mathscr{A}_3$. For 2-ROFS, we consider $\mathscr{A}_1 = (0.8, 0.6), \mathscr{A}_2 = (0.3, 0.35), \mathscr{A}_3 = (0.15, 0.21)$. Here, we get the ranking order as $\mathscr{A}_1 > \mathscr{A}_3 > \mathscr{A}_2$. Similarly, for 3-ROFS, we consider $\mathscr{A}_1 = (0.8, 0.7), \mathscr{A}_2 = (0.3, 0.35), \mathscr{A}_3 = (0.15, 0.21)$. Here, we get the ranking order as $\mathscr{A}_1 > \mathscr{A}_3 > \mathscr{A}_2$. And, for 4-ROFS, we consider $\mathscr{A}_1 = (0.9, 0.7), \mathscr{A}_2 = (0.3, 0.35), \mathscr{A}_3 = (0.15, 0.21)$. Here, we get the ranking order as $\mathscr{A}_1 > \mathscr{A}_3 > \mathscr{A}_2$. So, we can see that the ranking order of $\mathscr{A}_2$ and $\mathscr{A}_3$ changes on increasing the value of q. So, we get different ranking order for the same values for different values of q. But, in case of our proposed score function, we get the same order.

Considering all the above findings, we can see that the proposed score function can serve as a better alternative to all the existing score functions.

6 Distance Measure Based on HFS

In this section, the Minkowski distance is defined under hyperbolic fuzzy environment.

Definition 6.1 Let $\mathscr{H}_1 = \{\langle y_j, \widetilde{\mathscr{P}_{\mathscr{H}_1}}(y_j), \widetilde{\mathscr{Q}_{\mathscr{H}_1}}(y_j)\rangle; y_j \in Y\}$ and $\mathscr{H}_2 = \{\langle y_j, \widetilde{\mathscr{P}_{\mathscr{H}_2}}(y_j), \widetilde{\mathscr{Q}_{\mathscr{H}_2}}(y_j)\rangle; y_j \in Y\}$ be two HFSs defined in Y. The distance measure d between $\mathscr{H}_1$ and $\mathscr{H}_2$ is a function d:HFS × HFS→ [0, 1] which satisfies the following:

1. $0 \leq d(\mathscr{H}_1, \mathscr{H}_2) \leq 1$.
2. $d(\mathscr{H}_1, \mathscr{H}_2) = 0$ iff $\mathscr{H}_1 = \mathscr{H}_1$.
3. $d(\mathscr{H}_1, \mathscr{H}_2) = d(\mathscr{H}_2, \mathscr{H}_1)$.
4. If $\mathscr{H}_1, \mathscr{H}_2, \mathscr{H}_3 \in$ HFS(Y) such that $\mathscr{H}_1 \subseteq \mathscr{H}_2 \subseteq \mathscr{H}_3$, then, $d(\mathscr{H}_1, \mathscr{H}_3) \geq d(\mathscr{H}_1, \mathscr{H}_2)$ and $d(\mathscr{H}_1, \mathscr{H}_3) \geq d(\mathscr{H}_2, \mathscr{H}_3)$.

Now, the Minkowski distance based on HFS can be defined as follows:

Definition 6.2 Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two HFSs defined in Y. Then the Minkowski distance between $\mathscr{H}_1$ and $\mathscr{H}_2$ is described as:

$$d(\mathscr{H}_1, \mathscr{H}_2)_M = \Big[\frac{1}{2}\sum_{i=1}^{n}\{|\widetilde{\mathscr{P}_{\mathscr{H}_1}}(y_j) - \widetilde{\mathscr{P}_{\mathscr{H}_2}}(y_j)|^m + |\widetilde{\mathscr{Q}_{\mathscr{H}_1}}(y_j) - \widetilde{\mathscr{Q}_{\mathscr{H}_2}}(y_j)|^m + |\widetilde{\mathscr{R}_{\mathscr{H}_1}}(y_j) - \widetilde{\mathscr{R}_{\mathscr{H}_2}}(y_j)|^m\}\Big]^{\frac{1}{m}}, \quad for, m \geq 1 \tag{15.2}$$

where $\widetilde{\mathscr{R}_{\mathscr{H}_1}}(y_j) = 1 - \widetilde{\mathscr{P}_{\mathscr{H}_1}}(y_j) \times \widetilde{\mathscr{Q}_{\mathscr{H}_1}}(y_j)$ and $\widetilde{\mathscr{R}_{\mathscr{H}_2}}(y_j) = 1 - \widetilde{\mathscr{P}_{\mathscr{H}_2}}(y_j) \times \widetilde{\mathscr{Q}_{\mathscr{H}_2}}(y_j)$.

We define normalized Minkowski distance = (Minkowski Distance)/n, i.e.

$$d(\mathscr{H}_1, \mathscr{H}_2)_{nM} = \Big[\frac{1}{2n}\Big\{\sum_{i=1}^{n}\{|\widetilde{\mathscr{P}_{\mathscr{H}_1}}(y_j) - \widetilde{\mathscr{P}_{\mathscr{H}_2}}(y_j)|^m + |\widetilde{\mathscr{Q}_{\mathscr{H}_1}}(y_j) - \widetilde{\mathscr{Q}_{\mathscr{H}_2}}(y_j)|^m + |\widetilde{\mathscr{R}_{\mathscr{H}_1}}(y_j) - \widetilde{\mathscr{R}_{\mathscr{H}_2}}(y_j)|^m\}\Big\}\Big]^{\frac{1}{m}} \tag{15.3}$$

Clearly, normalized Minkowski distance satisfies properties 1–3 of distance measure.

For property 4, $|\widetilde{\mathscr{P}_{\mathscr{H}_1}}(y_j) - \widetilde{\mathscr{P}_{\mathscr{H}_2}}(y_j)|^m \leq |\widetilde{\mathscr{P}_{\mathscr{H}_1}}(y_j) - \widetilde{\mathscr{P}_{\mathscr{H}_3}}(y_j)|^m$.

Also, $|\widetilde{\mathscr{Q}_{\mathscr{H}_1}}(y_j) - \widetilde{\mathscr{Q}_{\mathscr{H}_2}}(y_j)|^m \leq |\widetilde{\mathscr{Q}_{\mathscr{H}_1}}(y_j) - \widetilde{\mathscr{Q}_{\mathscr{H}_3}}(y_j)|^m$.

And $|\widetilde{\mathscr{R}_{\mathscr{H}_1}}(y_j) - \widetilde{\mathscr{R}_{\mathscr{H}_2}}(y_j)|^m \leq |\widetilde{\mathscr{R}_{\mathscr{H}_1}}(y_j) - \widetilde{\mathscr{R}_{\mathscr{H}_3}}(y_j)|^m$.

$d(\mathscr{H}_1, \mathscr{H}_3) \geq d(\mathscr{H}_1, \mathscr{H}_2)$.

Similarly, $d(\mathscr{H}_1, \mathscr{H}_3) \geq d(\mathscr{H}_2, \mathscr{H}_3)$.

Hence, normalized Minkowski distance is a distance measure for HFS $\mathscr{H}_1$ and $\mathscr{H}_2$.

Now, for $m = 1$, it is normalized Hamming distance which can be described as

$$d(\mathscr{H}_1, \mathscr{H}_2)_{nH} = \Big[\frac{1}{2n}\Big\{\sum_{i=1}^{n}\{|\widetilde{\mathscr{P}_{\mathscr{H}_1}}(y_j) - \widetilde{\mathscr{P}_{\mathscr{H}_2}}(y_j)| + |\widetilde{\mathscr{Q}_{\mathscr{H}_1}}(y_j) - \widetilde{\mathscr{Q}_{\mathscr{H}_2}}(y_j)| + |\widetilde{\mathscr{R}_{\mathscr{H}_1}}(y_j) - \widetilde{\mathscr{R}_{\mathscr{H}_2}}(y_j)|\}\Big\}\Big]. \tag{15.4}$$

For $m = 2$, it is normalized Euclidean distance which can be described as

$$d(\mathscr{H}_1, \mathscr{H}_2)_{nE} = \Big[\frac{1}{2n}\Big\{\sum_{i=1}^{n}\{(\widetilde{\mathscr{P}_{\mathscr{H}_1}}(y_j) - \widetilde{\mathscr{P}_{\mathscr{H}_2}}(y_j))^2 + (\widetilde{\mathscr{Q}_{\mathscr{H}_1}}(y_j) - \widetilde{\mathscr{Q}_{\mathscr{H}_2}}(y_j))^2 + (\widetilde{\mathscr{R}_{\mathscr{H}_1}}(y_j) - \widetilde{\mathscr{R}_{\mathscr{H}_2}}(y_j))^2\}\Big\}\Big]^{\frac{1}{2}}. \tag{15.5}$$

For $m \to \infty$, it is normalized Hausdorff (Chebyshev) distance which can be described as

$$d(\mathscr{H}_1, \mathscr{H}_2)_{nC} = \Big[\frac{1}{n}\Big\{\sum_{i=1}^{n} max\{|\widetilde{\mathscr{P}_{\mathscr{H}_1}}(y_j) - \widetilde{\mathscr{P}_{\mathscr{H}_2}}(y_j)|, |\widetilde{\mathscr{Q}_{\mathscr{H}_1}}(y_j) - \widetilde{\mathscr{Q}_{\mathscr{H}_2}}(y_j)|, |\widetilde{\mathscr{R}_{\mathscr{H}_1}}(y_j) - \widetilde{\mathscr{R}_{\mathscr{H}_2}}(y_j)|\}\Big\}\Big]. \tag{15.6}$$

Now, we can define the weighted Minkowski distance based on HFS as follows:

Definition 6.3 Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two HFSs defined in Y. Then the Minkowski distance between $\mathscr{H}_1$ and $\mathscr{H}_2$ is described as:

$$d(\mathscr{H}_1, \mathscr{H}_2)_{wM} = \Big[\frac{1}{2}\sum_{i=1}^{n} w_i\{|\widetilde{\mathscr{P}_{\mathscr{H}_1}}(y_j) - \widetilde{\mathscr{P}_{\mathscr{H}_2}}(y_j)|^m + |\widetilde{\mathscr{Q}_{\mathscr{H}_1}}(y_j) - \widetilde{\mathscr{Q}_{\mathscr{H}_2}}(y_j)|^m + |\widetilde{\mathscr{R}_{\mathscr{H}_1}}(y_j) - \widetilde{\mathscr{R}_{\mathscr{H}_2}}(y_j)|^m\}\Big]^{\frac{1}{m}},$$

$$where, m \geq 1. \tag{15.7}$$

7 Solving MCDM Problem Based on HFS Using TOPSIS

In this section the TOPSIS approach is introduced for solving MCDM problem in HFS environment. This approach is ideal for handling MCDM issues in HFS environment. Also included is a brief description of the proposed method's algorithm.

7.1 Description of the MCDM Problem with HFNs

We take a MCDM problem comprising of m alternatives $Y = \{Y_1, Y_2, \ldots, Y_m\}$, $(m \geq 2)$ and n criteria $C = \{C_1, C_2, \ldots, C_n\}$ whose weight vectors are $w = (w_1, w_2, \ldots, w_n)^T$ which meet the condition $0 \leq w_j \leq 1$ and $\sum_{j=1}^{n} w_j = 1$. We consider an expert which expresses the evaluation values of the alternative $Y_k (k = 1, 2, \ldots m)$ over the criteria $C_l (l = 1, 2, \ldots n)$ in the form of HFN, $\mathscr{H}_{kl} = (\widetilde{\mathscr{P}}_{kl}, \widetilde{\mathscr{Q}}_{kl})$ where $\widetilde{\mathscr{P}}_{kl}$ is the optimistic degree and $\widetilde{\mathscr{Q}}_{kl}$ is the pessimistic degree. Hence the formulated hyperbolic fuzzy decision matrix is given by $R = (\mathscr{H}_{kl})_{m\times n}$. The following matrix form can succinctly describe the MCDM problem with HFNs:

$$R = (C_j(y_j))_{m\times n} = \begin{bmatrix} \mathscr{H}(\widetilde{\mathscr{P}_{11}}, \widetilde{\mathscr{Q}_{11}}) & \mathscr{H}(\widetilde{\mathscr{P}_{12}}, \widetilde{\mathscr{Q}_{12}}) & \ldots\ldots & \mathscr{H}(\widetilde{\mathscr{P}_{1n}}, \widetilde{\mathscr{Q}_{1n}}) \\ \mathscr{H}(\widetilde{\mathscr{P}_{21}}, \widetilde{\mathscr{Q}_{21}}) & \mathscr{H}(\widetilde{\mathscr{P}_{22}}, \widetilde{\mathscr{Q}_{22}}) & \ldots\ldots & \mathscr{H}(\widetilde{\mathscr{P}_{2n}}, \widetilde{\mathscr{Q}_{2n}}) \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ \mathscr{H}(\widetilde{\mathscr{P}_{m1}}, \widetilde{\mathscr{Q}_{m1}}) & \mathscr{H}(\widetilde{\mathscr{P}_{m2}}, \widetilde{\mathscr{Q}_{m2}}) & \ldots\ldots & \mathscr{H}(\widetilde{\mathscr{P}_{mn}}, \widetilde{\mathscr{Q}_{mn}}) \end{bmatrix} \tag{15.8}$$

7.2 Algorithm of the Proposed Method

Centred on above discussion, the subsequent steps are presented to define the TOPSIS method under HFS environment:

Step 1. Create the decision matrix $R = (\mathscr{H}_{kl})_{m\times n}$ where $\mathscr{H}_{kl} (k = 1, 2, .., m, l = 1, 2, .., n)$ are the assessments of the alternative $Y_k \in Y$ in relation to the criterion $C_l \in C$.

Step 2. For matching the practicality of the values, decision matrix $R = (\mathscr{H}_{kl})_{m\times n}$ is transformed to normalized form $N = (\mathscr{H}_{kl})_{m\times n}$ which is given by

$$\mathscr{H}_{kl} = \begin{cases} (\widetilde{\mathscr{P}}_{kl}, \widetilde{\mathscr{Q}}_{kl}), & for,\ benefit\ type\ criteria \\ (1 - \widetilde{\mathscr{P}}_{kl}, 1 - \widetilde{\mathscr{Q}}_{kl}), & for,\ cost\ type\ criteria \end{cases}$$

Step 3. Find the positive ideal alternative (PIA) and negative ideal alternative (NIA) which are given by

$$\begin{aligned} Y^{+} &= \{C_l, max\ S(\mathscr{H}_{kl})|l = 1, 2, .., n\} \\ &= \{\langle C_1, \mathscr{H}(\widetilde{\mathscr{P}_1}^{+}, \widetilde{\mathscr{Q}_1}^{+})\rangle, \langle C_2, \mathscr{H}(\widetilde{\mathscr{P}_2}^{+}, \widetilde{\mathscr{Q}_2}^{+})\rangle, ., \\ &\langle C_n, \mathscr{H}(\widetilde{\mathscr{P}_n}^{+}, \widetilde{\mathscr{Q}_n}^{+})\rangle\} \end{aligned} \tag{15.9}$$

$$\begin{aligned} Y^{-} &= \{C_l, min\ S(\mathscr{H}_{kl})|l = 1, 2, .., n\} \\ &= \{\langle C_1, \mathscr{H}(\widetilde{\mathscr{P}_1}^{-}, \widetilde{\mathscr{Q}_1}^{-})\rangle, \langle C_2, \mathscr{H}(\widetilde{\mathscr{P}_2}^{-}, \widetilde{\mathscr{Q}_2}^{-})\rangle, ., \\ &\langle C_n, \mathscr{H}(\widetilde{\mathscr{P}_n}^{-}, \widetilde{\mathscr{Q}_n}^{-})\rangle\} \end{aligned} \tag{15.10}$$

Step 4. Determine the distances between the alternative Y_k and the hyperbolic fuzzy PIS Y^{+} as well as the hyperbolic fuzzy NIS Y^{-}, respectively, using Equations

$$\begin{aligned} D(Y_k, Y^{+}) &= \Big[\frac{1}{2}\sum_{i=1}^{n} w_l\{|\widetilde{\mathscr{P}}_{kl} - \widetilde{\mathscr{P}}_l^{+}| + |\widetilde{\mathscr{Q}}_{kl} - \widetilde{\mathscr{Q}}_l^{+}| + |\widetilde{\mathscr{R}}_{kl} - \widetilde{\mathscr{R}}_l^{+}|\}\Big], \\ &where, k = 1, 2, \ldots, m \end{aligned} \tag{15.11}$$

$$\begin{aligned} D(Y_k, Y^{-}) &= \Big[\frac{1}{2}\sum_{i=1}^{n} w_l\{|\widetilde{\mathscr{P}}_{kl} - \widetilde{\mathscr{P}}_l^{-}| + |\widetilde{\mathscr{Q}}_{kl} - \widetilde{\mathscr{Q}}_l^{-}| + |\widetilde{\mathscr{R}}_{kl} - \widetilde{\mathscr{R}}_l^{-}|\}\Big], \\ &where, k = 1, 2, \ldots, m \end{aligned} \tag{15.12}$$

Step 5. Calculate the revised closeness $\zeta(Y_k)$ of the alternative $Y_k(k = 1, 2, \ldots, m)$ which is given by

$$\zeta(Y_k) = \frac{D_{max}(Y_k, Y^{-})}{D(Y_k, Y^{-})} - \frac{D(Y_k, Y^{+})}{D_{min}(Y_k, Y^{+})} \tag{15.13}$$

Step 6. Define the ranking order of the alternatives based on the revised closeness $\zeta(Y_k)$ obtained from Step 5. The alternatives are arranged in ascending order w.r.t the smaller values of $\zeta(Y_k)(k = 1, 2, \ldots, m)$. The best alternative is thus defined by

$$Y^{*} = \{Y_k : (k : \zeta^{*}(Y_k) = \max_{1 \le k \le m} \zeta(Y_k))\} \tag{15.14}$$

8 Illustrate Examples

In this section, we use the proposed approach in the following examples which are given below.

Example 8.1 We consider a MCDM problem with three alternatives X_1, X_2, and X_3 which are valued by an expert under four criteria A_1, A_2, A_3, and A_4 and list them in terms of IFNs (i.e. when q = 1 of q-ROFNs). All the criteria are benefit-type criteria. The values are noted in Table 15.5. The attribute weight is fitted to be $w = (0.25, 0.25, 0.25, 0.25)$. Ranking order of different methods is noted in Table 15.6.

We use the proposed algorithm to solve the MCDM problem.

According to Step 1, the decision matrix is given in Table 15.5.

In Step 2, as all the criteria are benefit-type criteria, there is no need of normalization and thus the decision matrix remains the same.

For Step 3, the PIA X^+ and NIA X^- are given as below:

Using the score function, we get $X^+ = \{(1, 0), (0.7, 0.2), (1, 0), (0.7, 0.3)\}$ and $X^- = \{(0.6, 0.4), (0.5, 0.5), (0.6, 0.2), (0.5, 0.4)\}$.

For Step 4, we have $D(X_1, X^+) = 0.0159$, $D(X_2, X^+) = 0.0262$, $D(X_3, X^+) = 0.0838$, $D(X_1, X^-) = 0.0691$, $D(X_2, X^-) = 0.0688$, $D(X_3, X^-) = 0$.

So, $D_{min}(X_i, X^+) = D(X_1, X^+) = 0.0159$ and $D_{max}(X_i, X^-) = D(X_1, X^-) = 0.0691$.

Now, for Step 5, $\zeta(X_1) = 0$, $\zeta(X_2) = -0.6516$, $\zeta(X_3) = -5.2549$.

Finally using Step 6, we have $X_1 > X_2 > X_3$.

Upon comparing the results with the IFWA and IFWG methods [33, 34] as displayed in Table 15.6, it becomes evident that using the IFWA/IFWG approach leads to the selection of both X_1 and X_2 as the best alternatives. However, through our proposed method, X_1 emerges as the clear best alternative, effectively mitigating any confusion that may arise from having multiple options with the same highest score.

Example 8.2 From Example 8.1, it can be seen that the methods are inadequate under the constraint $\widetilde{\mathscr{P}} + \widetilde{\mathscr{Q}} \leq 1$. Now, to exemplify the advantages of the proposed

Table 15.5 Decision matrix of Example 8.1

R_1	A_1	A_2	A_3	A_4
X_1	(1.0,0)	(0.7,0.2)	(0.8,0.1)	(0.6,0.3)
X_2	(0.8,0.2)	(0.6,0.1)	(1.0,0)	(0.7,0.3)
X_3	(0.6,0.4)	(0.5,0.5)	(0.6,0.2)	(0.5,0.4)

Table 15.6 Ranking results with different methods for Example 8.1

	Score	Values of operators		
Methods	X_1	X_2	X_3	Ranking
$IFWA$	1	1	0.6470	$X_1 = X_2 > X_3$
$IFWG$	0.8320	0.8320	0.6344	$X_1 = X_2 > X_3$

Table 15.7 Decision matrix of Example 8.2

R_2	A_1	A_2	A_3	A_4
X_1	(1.0,0)	(0.9,0.2)	(0.8,0.1)	(0.6,0.3)
X_2	(0.8,0.2)	(0.6,0.1)	(1.0,0)	(0.9,0.3)
X_3	(0.6,0.4)	(0.5,0.5)	(0.6,0.2)	(0.9,0.4)

Table 15.8 Ranking results with different methods for Example 8.1

	Score	Values of operators		
Methods	X_1	X_2	X_3	Ranking
$IFWA$	Cannot be evaluated			No
$IFWG$	Cannot be evaluated			No
$PFWA$	1	1	0.7421	$X_1 = X_2 > X_3$
$PFWG$	0.8492	0.8492	0.7660	$X_1 = X_2 > X_3$

method, slight adjustments to the attribute values are done by replacing the values of a_{12}, a_{24}, and a_{34} with (0.9,0.2), (0.9,0.3), and (0.9,0.4), respectively, to the data of Example 8.1 such that it falls under PFS (q = 2 of q-ROFS). The revised decision matrix is given in Table 15.7.

Now, similar to the above problem, we solve the MCDM problem. Using the score function, we get $X^+ = \{(1.0, 0), (0.9, 0.2), (1.0, 0), (0.9, 0.3)\}$ and $X^- = \{(0.6, 0.4), (0.5, 0.5), (0.6, 0.2), (0.6, 0.3)\}$.

$D(X_1, X^+) = 0.0241$, $D(X_2, X^+) = 0.0338$, $D(X_3, X^+) = 0.0850$
$D(X_1, X^-) = 0.0672$, $D(X_2, X^-) = 0.0713$, $D(X_3, X^-) = 0.0181$

So, $D_{min}(X_i, X^+) = D(X_1, X^+) = 0.0241$ and $D_{max}(X_i, X^-) = D(X_2, X^-) = 0.0713$.

$$\zeta(X_1) = -0.0570, \zeta(X_2) = -0.4026, \zeta(X_3) = -3.2781$$

So, $X_1 > X_2 > X_3$.

Upon comparing the results with different methods [22, 33, 34] presented in Table 15.8, it becomes evident that the IFWA/IFWG method could not provide any results. However, when utilizing the PFWA/PFWG method, both X_1 and X_2 are identified as the best alternatives. In contrast, our proposed method designates X_1 as the superior alternative, effectively resolving any confusion that may arise from having multiple options with the same highest rank.

Example 8.3 From Examples 8.1 and 8.2, it can be seen that the methods are inadequate under the constraint $\widetilde{\mathscr{P}} + \widetilde{\mathscr{Q}} \leq 1$ and $\widetilde{\mathscr{P}}^2 + \widetilde{\mathscr{Q}}^2 \leq 1$, respectively. Now, to exemplify the advantages of the proposed method, further adjustments to the attribute values are done by replacing the values of a_{12}, a_{21}, and a_{34} with (0.9,0.7), (0.8,0.7), and (0.9,0.5), respectively, to the data of Example 8.2 such that it falls under q-ROFS (q = 3). The revised decision matrix is given in Table 15.9.

Table 15.9 Decision matrix of Example 8.2

R_3	A_1	A_2	A_3	A_4
X_1	(1.0,0)	(0.9,0.7)	(0.8,0.1)	(0.6,0.3)
X_2	(0.8,0.7)	(0.6,0.1)	(1.0,0)	(0.9,0.3)
X_3	(0.6,0.4)	(0.5,0.5)	(0.6,0.2)	(0.9,0.5)

Table 15.10 Ranking results with different methods for Example 8.1

	Score	Values of operators		
Methods	X_1	X_2	X_3	Ranking
$IFWA$	Cannot be evaluated			No
$IFWG$	Cannot be evaluated			No
$PFWA$	Cannot be evaluated			No
$PFWG$	Cannot be evaluated			No
$q-ROFWA$	1	1	0.7430	$X_1 = X_2 > X_3$
$q-ROFWG$	0.8925	0.8925	0.7968	$X_1 = X_2 > X_3$

Now, similar to the above problem, we solve the MCDM problem. Using the score function, we get $X^+ = \{(1.0, 0), (0.6, 0.1), (1.0, 0), (0.9, 0.3)\}$ and $X^- = \{(0.6, 0.4), (0.5, 0.5), (0.6, 0.2), (0.6, 0.3)\}$.

$D(X_1, X^+) = 0.0700$, $D(X_2, X^+) = 0.0456$, $D(X_3, X^+) = 0.0884$
$D(X_1, X^-) = 0.0737$, $D(X_2, X^-) = 0.0819$, $D(X_3, X^-) = 0.0241$

So, $D_{min}(X_i, X^+) = D(X_2, X^+) = 0.0456$ and $D_{max}(X_i, X^-) = D(X_2, X^-) = 0.0819$.

$$\zeta(X_1) = -0.6335, \zeta(X_2) = 0, \zeta(X_3) = -1.6445$$

So, $X_2 > X_1 > X_3$.

Upon comparing the results with different methods [21, 22, 33, 34] presented in Table 15.10, it is evident that both the IFWA/IFWG and PFWA/PFWG methods could not produce any results. In contrast, when employing the $q-ROFWA/q-ROFWG$ method, both X_1 and X_2 are identified as the best alternatives. However, our proposed method designates X_2 as the superior choice, effectively resolving any confusion that may arise from having multiple options with the same top rank.

9 Conclusion and Future Scope

MCDM is a vital component of DM process. IFS, PFS, and q-ROFS have all been used to construct various methodologies. However, these approaches have some restrictions on the choice of MD and NMD. However, because the HFS idea allows for more independence in choosing the values, it is a superior option to q-ROFS. Here, a new score function is defined using the HFS notion. Our suggested score function has been compared to a number of other score functions that are already in use. It is clear that the score functions that are currently in use have certain drawbacks. Therefore, compared to existing forms, our suggested scoring function is a preferable option. As a generalization of Hamming, Euclidean, and Hausdorff distance, the Minkowski distance measure based on HFS was also introduced. The TOPSIS approach to the MCDM problem has also been explored, and its applicability is demonstrated by using the distance measure to solve the problem. Therefore, the MCDM that has been illustrated here can be solved using the way we have suggested.

The concept of HFS is remarkable in solving DM problems due to its advantages over existing forms in taking MD and NMD. A lot of scope is there in decision-making using HFS. New distance and similarity measures can be developed using HFS aiding in solving DM problems.

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Chapter 16
Advanced TOPSIS-Based College Selection MCGDM Problem in Trapezoidal Pythagorean Fuzzy Environment

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1 Introduction

Uncertainty-based decision-making (DM) methods are one of the most essential theories in our regular life. Traditionally, it is usually imagined that the decision details that access the objects are stated in the form of crisp numbers. But in most cases, a decision in realistic circumstances is taken in an environment where the objectives and restrictions are usually imprecise in nature and therefore takes numerous steps to attain the final target. To resolve these types of problems, Professor L. A. Zadeh [1] manifested the thought of fuzzy sets (FS) in 1965. Normally, fuzziness occurs when the boundary of a piece of data is not specified in a human's mind. The innovation of fuzzy sets creates a new branch in mathematics that plays an essential role in engineering, modern science, medical sectors, and the technical field of research. Further, researchers incorporated the structure of various fuzzy numbers such as triangular [2], trapezoidal [3], pentagonal [4], hexagonal [5], etc. But, the conception of fuzzy set cannot grab the idea of non-membership portions. Thus, the concept of FS was extended by Prof. Atanassov [6] after introducing the idea of intuitionistic fuzzy set (IFS) in 1986. IFS can capture two membership functions: i) truth (μ) and false (λ) of an uncertain number. Further, Liu and Yuan [7] and Ban [8] initiated the conception of triangular and trapezoidal intuitionistic fuzzy sets respectively in the research domain that contributes a significant role in the uncertainty area. Wang [9, 10] introduced some aggregation operators such as triangular intuitionistic weighted average (TIWA), triangular intuitionistic

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C. Jana et al. (eds.), *Fuzzy Optimization, Decision-making and Operations Research*, https://doi.org/10.1007/978-3-031-35668-1_16

order weighted average (TIOWA), triangular intuitionistic hybrid weighted average (TIHWA), and triangular intuitionistic weighted geometric (TIWG) that are very useful to tackle MCGDM problems. Further, Xu [11] proposed the generalized weighted average operator grounded on the intuitionistic trapezoidal fuzzy numbers; Atanassov and Gargov [12] demonstrate the idea of interval-valued IFS; Gou et al. [13] aim to develop the idea of exponential operation law for IFNs. Normally, there are several cases, we can observe that $\mu + \lambda > 1$ in IFS. To resolve this limitation, Yager [14] developed the idea of Pythagorean fuzzy set (PFS) in which he considered the idea as $\mu^2 + \lambda^2 \leq 1$. So, it is obvious that PFS is more efficient than IFS as PFS can be used more accurately and sufficiently to solve uncertainty problems than IFS. Further, Yager [15] also developed some important aggregation operators based on PFN; Zhang and Xu [16] derived the classical TOPSIS method for MCDM problems with PyFN environment; Garg [17] defined some average aggregation operator and geometric aggregation operator under PyF environment; Zhou and Chen [18] manifested Euclidean distance, Hamming distance [19], the Hausdorff metric [20], and so on that are used to measure the distance between two IFNs. Later on, Prof. G. W. Wei [21, 22] introduced a new distance function that is known as Wei's distance function that is mainly used in picture IFSs. Abbas et al. [23] started the idea of cubic Pythagorean fuzzy number and applied it to tackle the MADM problem. Garg [24] gave the idea of generalized Pythagorean fuzzy geometric interactive aggregation operators using Einstein operations and applied it to decision-making problems. Verma and Merigo [25] introduced the two new generalized similarity measures between Pythagorean fuzzy sets based on cosine and cotangent functions. Apart from these, several other researchers [26–33] have studied decision-making problems by introducing several novel operational laws and aggregation operators in different fuzzy environments.

Recently, various methods such as TOPSIS [34], VIKOR [35], AHP [36], and ELECTRE [37] have been developed to solve the MCDM problems. Among these techniques, the TOPSIS method manifested by Hwang and Yoon [34] becomes very effective and popular strategy for evaluating the MCDM problem. The main concept of the TOPSIS technique is to select the alternative that has minimum distance from the positive ideal solution (PIS) and maximum distance from the negative ideal solution (NIS). After the invention of the TOPSIS strategy, it has been widely applied to different uncertainty fields. Chen [38] manifested the TOPSIS technique for MCDM problems in fuzzy environment. Boran et al. [39] extended the TOPSIS method to solve the MCDM problem in intuitionistic fuzzy environment. Ye [40] put forwarded TOPSIS technique in the interval-valued intuitionistic fuzzy numbers for virtual enterprise partner selection problem. Ou et al. [41] introduced linguistic intuitionistic fuzzy set TOPSIS strategy to solve MCDM problems. Zhang and Xu [16] derived the classical TOPSIS method for MCDM problems with the PyFN environment. Han et al. [42] developed a TOPSIS strategy in the linguistic Pythagorean fuzzy based on novel entropy and distance measure and applied it to solve the MADM problem. Umer et al.[43] extended the TOPSIS technique in the interval type-2 Pythagorean fuzzy numbers for a selection of solar tracking system problems.

From the above literature, it is clear that the TOPSIS technique is a popular effective and vital technique for solving MCDM/MCGDM problems. On the other hand, TrPFNs is a generalized Pythagorean fuzzy number that can grab the uncertainty in a suitable way. So far, TOPSIS technique is developed for Pythagorean fuzzy numbers and interval type-2 Pythagorean fuzzy numbers but not for TrPFNs. To enrich this research gap, we have introduced the improved TOPSIS technique for TrPFNs that has been applied to an interesting real-life problem namely selection of best college based on several criteria.

In this research article, we draw our attention to the field of trapezoidal Pythagorean fuzzy numbers (TrPFNs) that are generated from TrPFSs. We have discussed some important definitions related to TrPFN and then its algebraic properties and some important theorem on TrPFN. After that, we define three distance functions that are Euclidean distance, Hamming distance, and Wei's distance functions based on two TrPFNs in which we relate two TrPFNs as a real number so that we rank them. In general, we used these three types of distance functions so that we can conclude our results more precisely. Later on, we discussed aggregation operators that are trapezoidal Pythagorean fuzzy weighted arithmetic (TrPyFWA) aggregation operators, and then we discussed some properties related to this operator. These operators are used in the TOPSIS method to solve the MCGDM problem in the trapezoidal Pythagorean fuzzy number environment. Finally, we have done the sensitivity analysis and comparative analysis to show the efficiency and reliability of our proposed technique.

1.1 Advantages and Limitation of the Proposed Method

Advantage: Here, we developed an improved TOPSIS technique based on a newly introduced novel distance function defined on trapezoidal Pythagorean fuzzy number environment. First of all, it is simple but very much rational technique to choose the best alternative. Second, we have used TrP_yFNs, where the uncertainty can be put in robust way. So in our proposed technique, we get comprehensible and rational result.

Limitation: In this proposed technique, each alternative needs the same number of criteria because we cannot apply underlying aggregation operators if alternatives have a different number of criteria. This is a limitation of the proposed technique.

2 Some Important Definition and Mathematical Preliminaries

Here, some rudimentary definitions and operation laws associated to IFSs, TIFSs, and PyFSs have been discussed.

Definition 2.1 ([6]) Let U be the universal set. Then IFSs are characterized as

$$\mathcal{I} = \{\langle k; \mu(k), \lambda(k)\rangle : \ where\ k \in U\ and\ \mu, \lambda : U \to [0, 1]\}.$$

Here, μ is known as membership function and λ is known as non-membership function of an element of the IFS $\mathcal{K}$. For convenience, we present the pair as $\widetilde{I} = \{(\mu, \lambda);\ where\ \mu, \lambda \in [0, 1] : \mu + \lambda \leq 1\}$ and called it intuitionistic fuzzy number (IFN).

Example 2.1 Let $\left\langle 0.6, 0.3 \right\rangle$ and $\left\langle 0.6, 0.7 \right\rangle$ be two ordered pair numbers. Since $0.6 + 0.3 = 0.9 < 1$, then $\left\langle 0.6, 0.7 \right\rangle$ is an IFN but since $0.6 + 0.7 = 1.3 \not\geq 1$, then $\left\langle 0.6, 0.7 \right\rangle$ is not an IFN.

Definition 2.2 ([14]) Let U be the universal set. Then

$$\mathcal{P} = \{\langle k; \mu(k), \lambda(k)\rangle : \ where\ k \in U\ and\ \mu, \lambda : U \to [0, 1]\}$$

is called Pythagorean fuzzy set on U if it satisfies the condition $\mu^2(k) + \lambda^2(k) \leq 1;\ \forall k \in U$. For convenience, Pythagorean fuzzy number $(PyFN)$ is represented as follows: $\widetilde{P} = \{(\mu, \lambda);\ where\ \mu, \lambda \in [0, 1] : \mu^2 + \lambda^2 \leq 1\}$.

Example 2.2 In the above Example 2.1, we have seen that $\left\langle 0.6, 0.7 \right\rangle$ is not an IFN. But, here $0.6^2 + 0.7^2 = 0.85 < 1$. This implies that $\left\langle 0.6, 0.7 \right\rangle$ is $PyFN$.

Definition 2.3 ([8]) Let U be the universal set. Then

$$\mathcal{T} = \{\langle k; \mu(k), \lambda(k)\rangle : \ where\ k \in U\ and\ \mu, \lambda : U \to [0, 1]\}$$

is said to be trapezoidal intuitionistic fuzzy set on U. Here, $\mu\ and\ \lambda$ are trapezoidal fuzzy numbers defined as

$$\mu(k) = (F(k), G(k), T(k), V(k))$$

$$\lambda(k) = (f(k), g(k), t(k), v(k))$$

with the condition $V(k) + v(k) \leq 1\ \ \forall k \in U$. For convenience, we represent the pair as here trapezoidal intuitionistic fuzzy number is defined as $\widetilde{T} = \left\{\left\langle \left(\left(F, G, T, V\right), \left((f, g, t, v)\right)\right\rangle,\ where\ F, G, T, V, f, g, t, v : U \to [0, 1]\right\}$ with condition satisfied that $V + v \leq 1$.

Example 2.3 Let $\left\langle (0.2.0.6, 0.3, 0.6), (0.1.0.5, 0.3, 0.3) \right\rangle$ and $\left\langle (0.1.0.4, 0.3, 0.6), (0.2.0.5, 0.4, 0.7) \right\rangle$ be any two trapezoidal fuzzy numbers. Since $0.6 + 0.3 = 0.9 < 1$, then $\left\langle (0.2.0.6, 0.3, 0.6), (0.1.0.5, 0.3, 0.3) \right\rangle$ is a trapezoidal intuitionistic fuzzy number. But, $0.6 + 0.7 = 1.3 \not\geq 1$. Then, $\Big\langle(0.1.0.4, 0.3, 0.6), (0.2.0.5, 0.4, 0.7)\Big\rangle$ is not a trapezoidal intuitionistic fuzzy number.

3 Trapezoidal Pythagorean Fuzzy Sets

Definition 3.1 Let U be a universal set. Then

$$\mathcal{S} = \{\langle k, \mu(k), \lambda(k)\rangle : k \in U\}$$

is called trapezoidal Pythagorean fuzzy set, where $\mu(k)$ *and* $\lambda(k)$ are trapezoidal fuzzy number defined as

$$\mu(k) = \big(F(k), G(k), T(k), V(k)\big)$$

$$\lambda(k) = \big(f(k), g(k), t(k), v(k)\big)$$

such that $V^2(k) + v^2(k) \leq 1$. For convenience, we represent the pair as $\widetilde{S} = \Big\{\big\langle\big(\big(F, G, T, V\big), \big((f, g, t, v)\big)\big\rangle;\ where\ F, G, T, V, f, g, t, v : U \to [0, 1],\ V^2 + v^2 \leq 1\Big\}$ and called it trapezoidal Pythagorean fuzzy number (TrP_yFN).

Example 3.1 In Example 2.3, we have already noticed that $\Big\langle(0.1.0.4, 0.3, 0.6), (0.2.0.5, 0.4, 0.7)\Big\rangle$ is not a trapezoidal intuitionistic fuzzy number. But, since $0.6^2 + 0.7^2 = .85 < 1$, then $\Big\langle(0.1.0.4, 0.3, 0.6), (0.2.0.5, 0.4, 0.7)\Big\rangle$ is a trapezoidal Pythagorean fuzzy number.

Remark 3.1 From the above examples, we can conclude that the range of trapezoidal Pythagorean fuzzy environment will be more wider than trapezoidal intuitionistic fuzzy environment.

Now, we will discuss some important propositions and theorems related to TrP_yFN.

Proposition 3.1 *Let* $\widetilde{T_1} = \langle(F_1, G_1, T_1, V_1), (f_1, g_1, t_1, v_1)\rangle$ *And* $\widetilde{T_2} = \langle(F_2, G_2, T_2, V_2), (f_2, g_2, t_2, v_2)\rangle$ *be any two trapezoidal Pythagorean fuzzy numbers. Here, we have defined their operational rules as follows:*

(i) $$\widetilde{T}_1 \oplus \widetilde{T}_2 = \Bigg\langle\Bigg(\sqrt{F_1^2 + F_2^2 - F_1^2F_2^2}, \sqrt{G_1^2 + G_2^2 - G_1^2G_2^2}, \sqrt{T_1^2 + T_2^2 - T_1^2T_2^2}, \sqrt{V_1^2 + V_2^2 - V_1^2V_2^2}\Bigg), \Big(f_1f_2, g_1g_2, t_1t_2, v_1v_2\Big)\Bigg\rangle$$

(ii) $\widetilde{T}_1 \otimes \widetilde{T}_2 = \left\langle \left(F_1F_2, G_1G_2, T_1T_2, V_1V_2 \right), \left(\sqrt{f_1^2 + f_2^2 - f_1^2 f_2^2}, \sqrt{g_1^2 + g_2^2 - g_1^2 g_2^2}, \sqrt{t_1^2 + t_2^2 - t_1^2 t_2^2}, \sqrt{v_1^2 + v_2^2 - v_1^2 v_2^2} \right) \right\rangle$

(iii) $\psi \widetilde{T}_1 = \left\langle \left(\sqrt{1-(1-F_1^2)^{\psi}}, \sqrt{1-(1-G_1^2)^{\psi}}, \sqrt{1-(1-T_1^2)^{\psi}}, \sqrt{1-(1-V_1^2)^{\psi}} \right), \left(f_1^{\psi}, g_1^{\psi}, t_1^{\psi}, v_1^{\psi} \right) \right\rangle$

(iv) $\widetilde{T}_1^{\psi} = \left\langle \left(F_1^{\psi}, G_1^{\psi}, T_1^{\psi}, V_1^{\psi} \right), \left(\sqrt{1-(1-f_1^2)^{\psi}}, \sqrt{1-(1-g_1^2)^{\psi}}, \sqrt{1-(1-t_1^2)^{\psi}}, \sqrt{1-(1-v_1^2)^{\psi}} \right) \right\rangle$

Proposition 3.2 *Let* $\widetilde{T} = \langle (F, G, T, V), (f, g, t, v) \rangle$, $\widetilde{T}_1 = \langle (F_1, G_1, T_1, V_1), (f_1, g_1, t_1, v_1) \rangle$, *and* $\widetilde{T}_2 = \langle (F_2, G_2, T_2, V_2), (f_2, g_2, t_2, v_2) \rangle$ *be three* TrP_yFNs*; then:*

(i) $\widetilde{T}_1 \cup \widetilde{T}_2 = \left\langle \left(max\{F_1, F_2\}, max\{G_1, G_2\}, max\{T_1, T_2\}, max\{V_1, V_2\} \right), \left(min\{f_1, f_2\}, min\{g_1, g_2\}, min\{t_1, t_2\}, min\{v_1, v_2\} \right) \right\rangle$

(ii) $\widetilde{T}_1 \cap \widetilde{T}_2 = \left\langle \left(min\{F_1, F_2\}, min\{G_1, G_2\}, min\{T_1, T_2\}, min\{V_1, V_2\} \right), \left(max\{f_1, f_2\}, max\{g_1, g_2\}, max\{t_1, t_{}\}, max\{v_1, v_2\} \right) \right\rangle$

(iii) $\widetilde{T}^c = \langle (f, g, t, v), (F, G, T, V) \rangle$

Theorem 3.1 *Let* $\widetilde{T} = \langle (F, G, T, V), (f, g, t, v) \rangle$, $\widetilde{T}_1 = \langle (F_1, G_1, T_1, V_1), (f_1, g_1, t_1, v_1) \rangle$, *and* $\widetilde{T}_2 = \langle (F_2, G_2, T_2, V_2), (f_2, g_2, t_2, v_2) \rangle$ *be three* TrP_yFNs*; then:*

(i) $\widetilde{T}_1 \cup \widetilde{T}_2 = \widetilde{T}_2 \cup \widetilde{T}_1$
(ii) $\widetilde{T}_1 \cap \widetilde{T}_2 = \widetilde{T}_2 \cap \widetilde{T}_1$
(iii) $\phi(\widetilde{T}_1 \cup \widetilde{T}_2) = \phi\widetilde{T}_1 \cup \phi\widetilde{T}_2$
(iv) $(\widetilde{T}_1 \cup \widetilde{T}_2)^{\phi} = \widetilde{T}_1^{\phi} \cup \widetilde{T}_2^{\phi}$
(v) $(\widetilde{T}^c)^{\phi} = (\phi\widetilde{T})^c$
(vi) $\phi(\widetilde{T}^c) = (\widetilde{T}^{\phi})^c$

(i) Proof: From proposition 3.2, we have

$$\widetilde{T}_1 \cup \widetilde{T}_2 = \left\langle \left(max\{F_1, F_2\}, max\{G_1, G_2\}, max\{T_1, T_2\}, max\{V_1, V_2\} \right), \left(min\{f_1, f_2\}, min\{g_1, g_2\}, min\{t_1, t_2\}, min\{v_1, v_2\} \right) \right\rangle$$

$$= \left\langle \left(max\{F_2, F_1\}, max\{G_2, G_1\}, max\{T_2, T_1\}, max\{V_2, V_1\} \right), \left(min\{f_2, f_1\}, min\{g_2, g_1\}, min\{t_2, t_1\}, min\{v_2, v_1\} \right) \right\rangle = \widetilde{T}_2 \cup \widetilde{T}_1$$

(ii) Proof: From proposition 3.2, similarly, we can deduce the result.
(iii) Proof: Now, using the above two Proposition 3.1 (iii) and Proposition 3.2 (i), we get

$$\phi(\widetilde{T}_1 \cup \widetilde{T}_2)$$

$$= \phi\Big\langle\Big(max\{F_1, F_2\}, max\{G_1, G_2\}, max\{T_1, T_2\}, max\{V_1, V_2\}\Big),$$

$$\Big(min\{f_1, f_2\}, min\{g_1, g_2\}, min\{t_1, t_2\}, min\{v_1, v_2\}\Big)\Big\rangle$$

$$= \Big\langle\Big(\sqrt{1-(1-max\{F_1^2, F_2^2\})^{\phi}}, \sqrt{1-(1-max\{G_1^2, G_2^2\})^{\phi}},$$

$$\sqrt{1-(1-max\{T_1^2, T_2^2\})^{\phi}}, \sqrt{1-(1-max\{V_1^2, V_2^2\})^{\phi}}\Big),$$

$$\Big(min\{f_1^{\phi}, f_2^{\phi}\}, min\{g_1^{\phi}, g_2^{\phi}\}, min\{t_1^{\phi}, t_2^{\phi}\}, , min\{v_1^{\phi}, v_2^{\phi}\}\Big)\Big\rangle$$

$$= \Big\langle\Big(max\Big\{\sqrt{1-(1-F_1^2)^{\phi}}, \sqrt{1-(1-F_2^2)^{\phi}}\Big\}, max\Big\{\sqrt{1-(1-G_1^2)^{\phi}},$$

$$\sqrt{1-(1-G_2^2)^{\phi}}\Big\}, max\Big\{\sqrt{1-(1-T_1^2)^{\phi}}, \sqrt{1-(1-T_2^2)^{\phi}}\Big\},$$

$$max\Big\{\sqrt{1-(1-V_1^2)^{\phi}}, \sqrt{1-(1-V_2^2)^{\phi}}\Big\}\Big),$$

$$\Big(min\{f_1^{\phi}, f_2^{\phi}\}, min\{g_1^{\phi}, g_2^{\phi}\}, min\{t_1^{\phi}, t_2^{\phi}\}, , min\{v_1^{\phi}, v_2^{\phi}\}\Big)\Big\rangle = \phi\widetilde{T}_1 \cup \phi\widetilde{T}_2.$$

(iv) Proof: This proof follows from the previous one.
(v) Proof: From Proposition 3.1 (iv) and Proposition 3.2 (iii), we have

$$(\widetilde{T}^c)^{\phi}$$

$$= \big\langle(F, G, T, V), (f, g, t, v)\big\rangle^{\phi}$$

$$= \Big\langle\Big(f^{\phi}, g^{\phi}, t^{\phi}, v^{\phi}\Big), \Big(\sqrt{1-(1-F^2)^{\phi}}, \sqrt{1-(1-G^2)^{\phi}}, \sqrt{1-(1-T^2)^{\phi}},$$

$$\sqrt{1-(1-V^2)^{\phi}}\Big)\Big\rangle = (\phi\widetilde{T})^c$$

(vi) Proof: The proof is similar to the previous one.

Theorem 3.2 *Let* $\widetilde{T} = \big\langle(F, G, T, V), (f, g, t, v)\big\rangle$, $\widetilde{T}_1 = \big\langle(F_1, G_1, T_1, V_1), (f_1, g_1, t_1, v_1)\big\rangle$, $\widetilde{T}_2 = \big\langle(F_2, G_2, T_2, V_2), (f_2, g_2, t_2, v_2)\big\rangle$ *be three* TrP_yFNs*; then:*

(i) $\widetilde{T}_1 \oplus \widetilde{T}_2 = \widetilde{T}_2 \oplus \widetilde{T}_1$
(ii) $\widetilde{T}_1 \otimes \widetilde{T}_2 = \widetilde{T}_2 \otimes \widetilde{T}_1$

(iii) $\phi(\widetilde{T}_1 \oplus \widetilde{T}_2) = \phi\widetilde{T}_1 \oplus \phi\widetilde{T}_2$
(iv) $(\phi_1\widetilde{T} \oplus \phi_2\widetilde{T}) = (\phi_1 + \phi_2)T$
(v) $(\widetilde{T}_1 \otimes \widetilde{T}_2)^{\phi} = (\widetilde{T}_1^{\phi} \otimes \widetilde{T}_2^{\phi})$
(vi) $(\widetilde{T}^{\phi_1} \otimes \widetilde{T}^{\phi_2}) = \widetilde{T}^{\phi_1+\phi_2}$

(i) Proof: From Proposition 3.1 (i), we get

$$\begin{aligned}
&\widetilde{T}_1 \oplus \widetilde{T}_2 \\
&= \Big\langle \Big(\sqrt{F_1^2 + F_2^2 - F_1^2F_2^2}, \sqrt{G_1^2 + G_2^2 - G_1^2G_2^2}, \sqrt{T_1^2 + T_2^2 - T_1^2T_2^2}, \\
&\qquad \sqrt{V_1^2 + V_2^2 - V_1^2V_2^2}\Big), \Big(f_1f_2, g_1g_2, t_1t_2, v_1v_2\Big)\Big\rangle \\
&= \Big\langle \Big(\sqrt{F_2^2 + F_1^2 - F_2^2F_1^2}, \sqrt{G_2^2 + G_1^2 - G_2^2G_1^2}, \sqrt{T_2^2 + T_1^2 - T_2^2T_1^2}, \\
&\qquad \sqrt{V_2^2 + V_1^2 - V_2^2V_1^2}\Big), \Big(f_2f_1, g_2g_1, t_2t_1, v_2v_1\Big)\Big\rangle \\
&= \widetilde{T}_2 \oplus \widetilde{T}_1
\end{aligned}$$

(ii) Proof: The proof follows from the previous theorem by using Proposition 3.1 (ii).
(iii) Proof: From Proposition 3.1 (iii), we have

$$\begin{aligned}
&\phi\widetilde{T}_1 \oplus \phi\widetilde{T}_2 \\
&= \Big\langle \Big(\sqrt{1 - (1 - F_1^2)^{\phi}}, \sqrt{1 - (1 - G_1^2)^{\phi}}, \sqrt{1 - (1 - T_1^2)^{\phi}}, \\
&\qquad \sqrt{1 - (1 - V_1^2)^{\phi}}\Big), \Big(f_1^{\phi}, g_1^{\phi}, t_1^{\phi}, v_1^{\phi}\Big)\Big\rangle \oplus \\
&\quad \Big\langle \Big(\sqrt{1 - (1 - F_2^2)^{\phi}}, \sqrt{1 - (1 - G_2^2)^{\phi}}, \sqrt{1 - (1 - T_2^2)^{\phi}}, \\
&\qquad \sqrt{1 - (1 - V_2^2)^{\phi}}\Big), \Big(f_2^{\phi}, g_2^{\phi}, t_2^{\phi}, v_2^{\phi}\Big)\Big\rangle \\
&= \Big\langle \Big(\Big(\sqrt{1 - (1 - F_1^2)^{\phi} + 1 - (1 - F_2^2)^{\phi} - (1 - (1 - F_1^2)^{\phi})(1 - (1 - F_2^2)^{\phi})}\Big), \\
&\qquad \Big(\sqrt{1 - (1 - G_1^2)^{\phi} + 1 - (1 - G_2^2)^{\phi} - (1 - (1 - G_1^2)^{\phi})(1 - (1 - G_2^2)^{\phi})}\Big), \\
&\qquad \Big(\sqrt{1 - (1 - T_1^2)^{\phi} + 1 - (1 - T_2^2)^{\phi} - (1 - (1 - T_1^2)^{\phi})(1 - (1 - T_2^2)^{\phi})}\Big),
\end{aligned}$$

$$\left(\sqrt{1-(1-V_1^2)^{\phi}+1-(1-V_2^2)^{\phi}-(1-(1-V_1^2)^{\phi})(1-(1-V_2^2)^{\phi})}\right)\Big),$$

$$\Big(\Big(f_1 f_2)^{\phi}, \Big(g_1 g_2)^{\phi}, \Big(t_1 t_2)^{\phi}, \Big(v_1 v_2)^{\phi}\Big)\Big\rangle$$

$$= \phi(\widetilde{T}_1 \oplus \widetilde{T}_2).$$

(iv) Proof: The proof is similar to the previous one.
(v) Proof: From Propositions 3.1 (ii) and 3.1 (iv), we get

$$\widetilde{T}_1^{\phi} \otimes \widetilde{T}_2^{\phi}$$

$$= \Big\langle \Big(F_1^{\phi}, G_1^{\phi}, T_1^{\phi}, V_1^{\phi}\Big), \Big(\sqrt{1-(1-f_1^2)^{\phi}}, \sqrt{1-(1-g_1^2)^{\phi}},$$

$$\sqrt{1-(1-t_1^2)^{\phi}}, \sqrt{1-(1-v_1^2)^{\phi}}\Big)\Big\rangle \otimes$$

$$\Big\langle \Big(F_2^{\phi}, G_2^{\phi}, T_2^{\phi}, V_2^{\phi}\Big), \Big(\sqrt{1-(1-f_2^2)^{\phi}}, \sqrt{1-(1-g_2^2)^{\phi}},$$

$$\sqrt{1-(1-t_2^2)^{\phi}}, \sqrt{1-(1-v_2^2)^{\phi}}\Big)\Big\rangle$$

$$= \Big\langle \Big(F_1^{\phi} F_2^{\phi}, G_1^{\phi} G_2^{\phi}, T_1^{\phi} T_2^{\phi}, V_1^{\phi} V_2^{\phi}\Big),$$

$$\Big(\Big(\sqrt{1-(1-f_1^2)^{\phi}+1-(1-f_2^2)^{\phi}-(1-(1-f_1^2)^{\phi})(1-(1-f_2^2)^{\phi})}\Big),$$

$$\Big(\sqrt{1-(1-g_1^2)^{\phi}+1-(1-g_2^2)^{\phi}-(1-(1-g_1^2)^{\phi})(1-(1-g_2^2)^{\phi})}\Big),$$

$$\Big(\sqrt{1-(1-t_1^2)^{\phi}+1-(1-t_2^2)^{\phi}-(1-(1-t_1^2)^{\phi})(1-(1-t_2^2)^{\phi})}\Big),$$

$$\Big(\sqrt{1-(1-v_1^2)^{\phi}+1-(1-v_2^2)^{\phi}-(1-(1-v_1^2)^{\phi})(1-(1-v_2^2)^{\phi})}\Big)\Big)\Big\rangle$$

$$= \Big\langle \Big((F_1 F_2)^{\phi}, (G_1 G_2)^{\phi}, (T_1 T_2)^{\phi}, (V_1 V_2)^{\phi}\Big), \Big(\Big(\sqrt{1-(1-f_1^2-f_2^2-f_1^2 f_2^2)^{\phi}}\Big),$$

$$\Big(\sqrt{1-(1-g_1^2-g_2^2-g_1^2 g_2^2)^{\phi}}\Big), \Big(\sqrt{1-(1-t_1^2-t_2^2-t_1^2 t_2^2)^{\phi}}\Big),$$

$$\Big(\sqrt{1-(1-v_1^2-v_2^2-v_1^2 v_2^2)^{\phi}}\Big)\Big)\Big\rangle$$

$$= (\widetilde{T}_1 \otimes \widetilde{T}_2)^{\phi}$$

(vi) Proof: The proof is similar to the previous one.

Theorem 3.3 *Let* $\widetilde{T} = \langle (F, G, T, V), (f, g, t, v) \rangle$, $\widetilde{T}_1 = \langle (F_1, G_1, T_1, V_1), (f_1, g_1, t_1, v_1) \rangle$, *and* $\widetilde{T}_2 = \langle (F_2, G_2, T_2, V_2), (f_2, g_2, t_2, v_2) \rangle$ *be three* TrP_yFNs*; then:*

(i) $\phi\widetilde{T}_1 \ominus \phi\widetilde{T}_2 = \phi(\widetilde{T}_1 \ominus \widetilde{T}_2)$, *if* $F_1 \geq F_2,\ G_1 \geq G_2,\ T_1 \geq T_2,\ V_1 \geq V_2,\ f_1 \leq min(f_2, \frac{f_2\pi_1}{\pi_2}),\ g_1 \leq min(g_2, \frac{g_2\pi_1}{\pi_2}),\ t_1 \leq min(t_2, \frac{t_2\pi_1}{\pi_2}),\ v_1 \leq min(v_2, \frac{v_2\pi_1}{\pi_2})$.

(ii) $(\widetilde{T}_1 \oslash \widetilde{T}_2)^{\phi} = \widetilde{T}_1^{\phi} \oslash \widetilde{T}_2^{\phi}$, *if* $F_1 \leq min(F_2, \frac{F_2\pi_1}{\pi_2}),\ G_1 \leq min(G_2, \frac{G_2\pi_1}{\pi_2}),\ T_1 \leq min(T_2, \frac{T_2\pi_1}{\pi_2}),\ V_1 \leq min(V_2, \frac{V_2\pi_1}{\pi_2}),\ f_1 \geq f_2,\ g_1 \geq g_2,\ t_1 \geq t_2,\ v_1 \geq v_2$.

(iii) $\phi_1\widetilde{T} \ominus \phi_2\widetilde{T} = (\phi_1 - \phi_2)\widetilde{T}$, *if* $\phi_1 \geq \phi_2$.

(iv) $\widetilde{T}^{\phi_1} \oslash \widetilde{T}^{\phi_2} = \widetilde{T}^{\phi_1 - \phi_2}$.

(i) Proof: Given that $F_1 \geq F_2,\ G_1 \geq G_2,\ T_1 \geq T_2,\ V_1 \geq V_2,\ f_1 \leq min(f_2, \frac{f_2\pi_1}{\pi_2}),\ g_1 \leq min(g_2, \frac{g_2\pi_1}{\pi_2}),\ t_1 \leq min(t_2, \frac{t_2\pi_1}{\pi_2}),\ v_1 \leq min(v_2, \frac{v_2\pi_1}{\pi_2})$.

So, $f_1\pi_2 \leq f_2\pi_1,\ g_1\pi_2 \leq g_2\pi_1,\ t_1\pi_2 \leq t_2\pi_1,\ v_1\pi_2 \leq v_2\pi_1$

$$\Longrightarrow f_1^2 f_2^2 + f_1^2\pi_2^2 \leq f_1^2 f_2^2 + f_2^2\pi_1^2,\ g_1^2 g_2^2 + g_1^2\pi_2^2 \leq g_1^2 g_2^2 + g_2^2\pi_1^2,\ t_1^2 t_2^2 + t_1^2\pi_2^2 \leq t_1^2 t_2^2 + t_2^2\pi_1^2,\quad v_1^2 v_2^2 + v_1^2\pi_2^2 \leq v_1^2 v_2^2 + v_2^2\pi_1^2$$

$$\Longrightarrow \frac{f_1^2}{f_2^2} \leq \frac{f_2^2+\pi_1^2}{f_1^2+\pi_2^2},\ \frac{g_1^2}{g_2^2} \leq \frac{g_2^2+\pi_1^2}{g_1^2+\pi_2^2},\ \frac{t_1^2}{t_2^2} \leq \frac{t_2^2+\pi_1^2}{t_1^2+\pi_2^2},\ \frac{v_1^2}{v_2^2} \leq \frac{v_2^2+\pi_1^2}{v_1^2+\pi_2^2}$$

$$\Longrightarrow \left(\frac{f_1^2}{f_2^2}\right)^{\phi} \leq \left(\frac{f_2^2+\pi_1^2}{f_1^2+\pi_2^2}\right)^{\phi},\ \left(\frac{g_1^2}{g_2^2}\right)^{\phi} \leq \left(\frac{g_2^2+\pi_1^2}{g_1^2+\pi_2^2}\right)^{\phi},\ \left(\frac{t_1^2}{t_2^2}\right)^{\phi} \leq \left(\frac{t_2^2+\pi_1^2}{t_1^2+\pi_2^2}\right)^{\phi},\ \left(\frac{v_1^2}{v_2^2}\right)^{\phi} \leq \left(\frac{v_2^2+\pi_1^2}{v_1^2+\pi_2^2}\right)^{\phi}$$

$$\Longrightarrow 1-\left(\frac{f_2^2+\pi_1^2}{f_1^2+\pi_2^2}\right)^{\phi}+\left(\frac{f_1^2}{f_2^2}\right)^{\phi} \leq 1;\ 1-\left(\frac{g_2^2+\pi_1^2}{g_1^2+\pi_2^2}\right)^{\phi}+\left(\frac{g_1^2}{g_2^2}\right)^{\phi} \leq 1;\ 1-\left(\frac{t_2^2+\pi_1^2}{t_1^2+\pi_2^2}\right)^{\phi}+\left(\frac{t_1^2}{t_2^2}\right)^{\phi} \leq 1;$$

$$1-\left(\frac{v_2^2+\pi_1^2}{v_1^2+\pi_2^2}\right)^{\phi}+\left(\frac{v_1^2}{v_2^2}\right)^{\phi} \leq 1$$

$$\Longrightarrow \sqrt{\left(1-\left(\frac{f_2^2+\pi_1^2}{f_1^2+\pi_2^2}\right)^{\phi}\right)^2}+\left(\frac{f_1^2}{f_2^2}\right)^{\phi} \leq 1;\ \sqrt{\left(1-\left(\frac{g_2^2+\pi_1^2}{g_1^2+\pi_2^2}\right)^{\phi}\right)^2}+\left(\frac{g_1^2}{g_2^2}\right)^{\phi} \leq 1;$$
$$\sqrt{\left(1-\left(\frac{t_2^2+\pi_1^2}{t_1^2+\pi_2^2}\right)^{\phi}\right)^2}+\left(\frac{t_1^2}{t_2^2}\right)^{\phi} \leq 1;\ \sqrt{\left(1-\left(\frac{v_2^2+\pi_1^2}{v_1^2+\pi_2^2}\right)^{\phi}\right)^2}+\left(\frac{v_1^2}{v_2^2}\right)^{\phi} \leq 1$$

$$\Longrightarrow \left(\sqrt{1-\left(\frac{1-F_1^2}{1-F_2^2}\right)^{\phi}}\right)^2+\left(\frac{f_1^{\phi}}{f_2^{\phi}}\right)^2 \leq 1;\ \left(\sqrt{1-\left(\frac{1-G_1^2}{1-G_2^2}\right)^{\phi}}\right)^2+\left(\frac{g_1^{\phi}}{g_2^{\phi}}\right)^2 \leq 1;$$
$$\left(\sqrt{1-\left(\frac{1-T_1^2}{1-T_2^2}\right)^{\phi}}\right)^2+\left(\frac{t_1^{\phi}}{t_2^{\phi}}\right)^2 \leq 1;\ \left(\sqrt{1-\left(\frac{1-V_1^2}{1-V_2^2}\right)^{\phi}}\right)^2+\left(\frac{v_1^{\phi}}{v_2^{\phi}}\right)^2 \leq 1.$$

Then,

$$\phi(\widetilde{T}_1 \ominus \widetilde{T}_2)$$
$$= \phi\left\langle\left(\sqrt{\frac{F_1^2 - F_2^2}{1 - F_2^2}}, \sqrt{\frac{G_1^2 - G_2^2}{1 - G_2^2}}, \sqrt{\frac{T_1^2 - T_2^2}{1 - T_2^2}}, \sqrt{\frac{V_1^2 - V_2^2}{1 - V_2^2}}\right), \left(\frac{f_1}{f_2}, \frac{g_1}{g_2}, \frac{t_1}{t_2}, \frac{v_1}{v_2}\right)\right\rangle$$
$$= \left\langle\left(\sqrt{1 - \left(1 - \frac{F_1^2 - F_2^2}{1 - F_2^2}\right)^{\phi}}, \sqrt{1 - \left(1 - \frac{G_1^2 - G_2^2}{1 - G_2^2}\right)^{\phi}}, \sqrt{1 - \left(1 - \frac{T_1^2 - T_2^2}{1 - T_2^2}\right)^{\phi}},\right.\right.$$
$$\left.\left.\sqrt{1 - \left(1 - \frac{V_1^2 - V_2^2}{1 - V_2^2}\right)^{\phi}}\right), \left(\frac{f_1^{\phi}}{f_2^{\phi}}, \frac{g_1^{\phi}}{g_2^{\phi}}, \frac{t_1^{\phi}}{t_2^{\phi}}, \frac{v_1^{\phi}}{v_2^{\phi}}\right)\right\rangle.$$

Now,

$$\phi\widetilde{T}_1 \ominus \phi\widetilde{T}_2$$
$$= \left\langle\left(\sqrt{1 - (1 - F_1^2)^{\phi}}, \sqrt{1 - (1 - G_1^2)^{\phi}}, \sqrt{1 - (1 - T_1^2)^{\phi}}, \sqrt{1 - (1 - V_1^2)^{\phi}}\right),\right.$$
$$\left.\left(f_1^{\phi}, g_1^{\phi}, t_1^{\phi}, v_1^{\phi}\right)\right\rangle \ominus$$
$$\left\langle\left(\sqrt{1 - (1 - F_2^2)^{\phi}}, \sqrt{1 - (1 - G_2^2)^{\phi}}, \sqrt{1 - (1 - T_2^2)^{\phi}}, \sqrt{1 - (1 - V_2^2)^{\phi}}\right),\right.$$
$$\left.\left(f_2^{\phi}, g_2^{\phi}, t_2^{\phi}, v_2^{\phi}\right)\right\rangle$$
$$= \left\langle\left(\sqrt{\frac{1 - (1 - F_1^2)^{\phi} - (1 - (1 - F_2^2)^{\phi}}{1 - (1 - (1 - F_2^2)^{\phi})}}, \sqrt{\frac{1 - (1 - G_1^2)^{\phi} - (1 - (1 - G_2^2)^{\phi})}{1 - (1 - (1 - G_2^2)^{\phi})}},\right.\right.$$
$$\left.\sqrt{\frac{1 - (1 - T_1^2)^{\phi} - (1 - (1 - T_2^2)^{\phi})}{1 - (1 - (1 - T_2^2)^{\phi})}}, \sqrt{\frac{1 - (1 - V_1^2)^{\phi} - (1 - (1 - V_2^2)^{\phi})}{1 - (1 - (1 - V_2^2)^{\phi})}}\right),$$
$$\left.\left(\frac{f_1^{\phi}}{f_2^{\phi}}, \frac{g_1^{\phi}}{g_2^{\phi}}, \frac{t_1^{\phi}}{t_2^{\phi}}, \frac{v_1^{\phi}}{v_2^{\phi}}\right)\right\rangle$$
$$= \phi(\widetilde{T}_1 \ominus \widetilde{T}_2).$$

(ii) Proof: The proof is similar to the previous one.
(iii) Proof: As $\phi_1 \geq \phi_2$, then we get

$$\phi_1\widetilde{T} \ominus \phi_2\widetilde{T}$$

$$=\Bigg\langle\Bigg(\sqrt{1-(1-F^2)^{\phi_1}}, \sqrt{1-(1-G^2)^{\phi_1}}, \sqrt{1-(1-T^2)^{\phi_1}}, \sqrt{1-(1-V^2)^{\phi_1}}\Bigg),$$

$$\Big(f^{\phi_1}, g^{\phi_1}, t^{\phi_1}, v^{\phi_1}\Big)\Bigg\rangle\ominus$$

$$\Bigg\langle\Bigg(\sqrt{1-(1-F^2)^{\phi_2}}, \sqrt{1-(1-G^2)^{\phi_2}}, \sqrt{1-(1-T^2)^{\phi_2}}, \sqrt{1-(1-V^2)^{\phi_2}}\Bigg),$$

$$\Big(f^{\phi_2}, g^{\phi_2}, t^{\phi_2}, v^{\phi_2}\Big)\Bigg\rangle$$

$$=\Bigg\langle\Bigg(\sqrt{\frac{1-(1-F^2)^{\phi_1}-(1-(1-F^2)^{\phi_2})}{1-(1-(1-F^2)^{\phi_2}}}, \sqrt{\frac{1-(1-G^2)^{\phi_1}-(1-(1-G^2)^{\phi_2})}{1-(1-(1-G^2)^{\phi_2})}},$$

$$\sqrt{\frac{1-(1-T^2)^{\phi_1}-(1-(1-T^2)^{\phi_2})}{1-(1-(1-T^2)^{\phi_2})}},$$

$$\sqrt{\frac{1-(1-V^2)^{\phi_1}-(1-(1-V^2)^{\phi_2})}{1-(1-(1-V^2)^{\phi}_2)}}\Bigg), \Bigg(\frac{f^{\phi_1}}{f^{\phi_2}}, \frac{g^{\phi_1}}{g^{\phi_2}}, \frac{t^{\phi_1}}{t^{\phi_1}}, \frac{v^{\phi_1}}{v^{\phi_2}}\Bigg)\Bigg\rangle$$

$$=\Bigg\langle\Bigg(\sqrt{1-(1-F^2)^{\phi_1-\phi_2}}, \sqrt{1-(1-G^2)^{\phi_1-\phi_2}}, \sqrt{1-(1-T^2)^{\phi_1-\phi_2}},$$

$$\sqrt{1-(1-V^2)^{\phi_1-\phi_2}}\Bigg),$$

$$\Big(f^{\phi_1-\phi_2}, g^{\phi_1-\phi_2}, t^{\phi_1-\phi_2}, v^{\phi_1-\phi_2}\Big)\Bigg\rangle$$

$$=(\phi_1-\phi_2)\widetilde{T}.$$

(iv) Proof: The proof follows from the previous one.

Theorem 3.4 *Let* $\widetilde{T}_1 = \langle(F_1, G_1, T_1, V_1), (f_1, g_1, t_1, v_1)\rangle$, $\widetilde{T}_2 = \langle(F_2, G_2, T_2, V_2), (f_2, g_2, t_2, v_2)\rangle$ *be two* TrP_yFNs*; then:*

(i) $\widetilde{T}_1^c \bigcup \widetilde{T}_2^c = (\widetilde{T}_1 \bigcap \widetilde{T}_2)^c$

(ii) $\widetilde{T}_1^c \bigcap \widetilde{T}_2^c = (\widetilde{T}_1 \bigcup \widetilde{T}_2)^c$

(iii) $\widetilde{T}_1^c \bigoplus \widetilde{T}_2^c = (\widetilde{T}_1 \bigotimes \widetilde{T}_2)^c$

(iv) $\widetilde{T}_1^c \bigotimes \widetilde{T}_2^c = (\widetilde{T}_1 \bigotimes \widetilde{T}_2)^c$

(v) $\widetilde{T}_1^c \ominus \widetilde{T}_2^c = (\widetilde{T}_1 \oslash \widetilde{T}_2)^c$, *if* $f_1 \geq f_2$, $g_1 \geq g_2$, $t_1 \geq t_2$, $v_1 \geq v_2$ *and* $F_1 \leq min\left(F_2, \frac{F_2\pi_1}{\pi_2}\right)$,$G_1 \leq min\left(G_2, \frac{G_2\pi_1}{\pi_2}\right)$, $T_1 \leq min\left(T_2, \frac{T_2\pi_1}{\pi_2}\right)$, $V_1 \leq min\left(V_2, \frac{V_2\pi_1}{\pi_2}\right)$

(vi) $\widetilde{T}_1^c \oslash \widetilde{T}_2^c = (\widetilde{T}_1 \ominus \widetilde{T}_2)^c$, *if* $F_1 \geq F_2$, $G_1 \geq G_2$, $T_1 \geq T_2$, $V_1 \geq V_2$ *and* $f_1 \leq min\left(f_2, \frac{f_2\pi_1}{\pi_2}\right)$, $g_1 \leq min\left(g_2, \frac{g_2\pi_1}{\pi_2}\right)$, $t_1 \leq min\left(t_2, \frac{t_2\pi_1}{\pi_2}\right)$

(i) Proof: We have from definition,

$$\begin{aligned}
&\widetilde{T}_1^c \bigcup \widetilde{T}_2^c \\
&= \Big\langle (f_1, g_1, t_1, v_1), (F_1, G_1, T_1, V_1) \Big\rangle \bigcup \Big\langle (f_2, g_2, t_2, v_2), (F_2, G_2, T_2, V_2) \Big\rangle \\
&= \Big\langle \Big(max\{f_1, f_2\}, max\{g_1, g_2\}, max\{t_1, t_2\}, max\{v_1, v_2\} \Big), \\
&\quad \Big(min\{F_1, F_2\}, min\{G_1, G_2\}, min\{T_1, T_2)\}, min\{V_1, V_2\} \Big) \Big\rangle \\
&= \Big\langle \Big(min\{F_1, F_2\}, min\{G_1, G_2\}, min\{T_1, T_2\}, min\{V_1, V_2\} \Big), \\
&\quad \Big(max\{f_1, f_2\}, max\{g_1, g_2\}, max\{t_1, t_2\}, max\{v_1, v_2\} \Big) \Big\rangle^c \\
&= (\widetilde{T}_1 \bigcap \widetilde{T}_2)^c.
\end{aligned}$$

(ii) Proof: The proof follows from the previous one.
(iii) Proof: From the definition, we get

$$\begin{aligned}
&\widetilde{T}_1^c \bigoplus \widetilde{T}_2^c \\
&= \langle (f_1, g_1, t_1, v_1), (F_1, G_1, T_1, V_1) \rangle \bigoplus \langle (f_2, g_2, t_2, v_2), (F_2, G_2, T_2, V_2) \rangle \\
&= \Big\langle \Big(\sqrt{f_1^2 + f_2^2 - f_1^2 f_2^2}, \sqrt{g_1^2 + g_2^2 - g_1^2 g_2^2}, \sqrt{t_1^2 + t_2^2 - t_1^2 t_2^2}, \sqrt{v_1^2 + v_2^2 - v_1^2 v_2^2} \Big), \\
&\quad \Big(F_1 F_2, G_1 G_2, T_1 T_2, V_1 V_2 \Big) \Big\rangle \\
&= \Big\langle \Big(F_1 F_2, G_1 G_2, T_1 T_2, V_1 V_2 \Big), \Big(\sqrt{f_1^2 + f_2^2 - f_1^2 f_2^2}, \sqrt{g_1^2 + g_2^2 - g_1^2 g_2^2}, \\
&\quad \sqrt{t_1^2 + t_2^2 - t_1^2 t_2^2}, \sqrt{v_1^2 + v_2^2 - v_1^2 v_2^2} \Big) \Big\rangle^c \\
&= (\widetilde{T}_1 \bigotimes \widetilde{T}_2)^c.
\end{aligned}$$

(iv) Proof: The proof is similar to the previous one.

(v) Proof: As $f_1 \geq f_2$, $g_1 \geq g_2$, $t_1 \geq t_2$, $v_1 \geq v_2$ and $F_1 \leq min\left(F_2, \frac{F_2\pi_1}{\pi_2}\right)$, $G_1 \leq min\left(G_2, \frac{G_2\pi_1}{\pi_2}\right)$,

$T_1 \leq min\left(T_2, \frac{T_2\pi_1}{\pi_2}\right)$, $V_1 \leq min\left(V_2, \frac{V_2\pi_1}{\pi_2}\right)$, then from definition we get

$$\begin{aligned}
&\widetilde{T}_1^c \ominus \widetilde{T}_2^c \\
=&\langle(f_1, g_1, t_1, v_1), (F_1, G_1, T_1, V_1)\rangle \ominus \langle(f_2, g_2, t_2, v_2), (F_2, G_2, T_2, V_2)\rangle \\
=&\left\langle\left(\sqrt{\frac{f_1^2 - f_2^2}{1 - f_2^2}}, \sqrt{\frac{g_1^2 - g_2^2}{1 - g_2^2}}, \sqrt{\frac{t_1^2 - t_2^2}{1 - t_2^2}}, \sqrt{\frac{v_1^2 - v_2^2}{1 - v_2^2}}\right), \left(\frac{F_1}{F_2}, \frac{G_1}{G_2}, \frac{T_1}{T_2}, \frac{V_1}{V_2}\right)\right\rangle \\
=&\left\langle\left(\frac{F_1}{F_2}, \frac{G_1}{G_2}, \frac{T_1}{T_2}, \frac{V_1}{V_2}\right), \left(\sqrt{\frac{f_1^2 - f_2^2}{1 - f_2^2}}, \sqrt{\frac{g_1^2 - g_2^2}{1 - g_2^2}}, \sqrt{\frac{t_1^2 - t_2^2}{1 - t_2^2}}, \sqrt{\frac{v_1^2 - v_2^2}{1 - v_2^2}}\right)\right\rangle^c \\
=& (\widetilde{T}_1 \oslash \widetilde{T}_2)^c.
\end{aligned}$$

(vi) Proof: The proof is similar to the previous one.

Theorem 3.5 *Let* $\widetilde{T}_1 = \langle(F_1, G_1, T_1, V_1), (f_1, g_1, t_1, v_1)\rangle$, $\widetilde{T}_2 = \langle(F_2, G_2, T_2, V_2), (f_2, g_2, t_2, v_2)\rangle$ *be two* TrP_yFNs*; then:*

(i) $\left(\widetilde{T}_1 \bigcup \widetilde{T}_2\right) \bigcap \widetilde{T}_2 = \widetilde{T}_2$

(ii) $\left(\widetilde{T}_1 \bigcap \widetilde{T}_2\right) \bigcup \widetilde{T}_2 = \widetilde{T}_2$

(iii) $\left(\widetilde{T}_1 \ominus \widetilde{T}_2\right) \bigoplus \widetilde{T}_2 = \widetilde{T}_1$, *if* $F_1 \geq F_2$, $G_1 \geq G_2$, $T_1 \geq T_2$, $V_1 \geq V_2$ *and* $f_1 \leq min\left(f_2, \frac{f_2\pi_1}{\pi_2}\right)$, $g_1 \leq min\left(g_2, \frac{g_2\pi_1}{\pi_2}\right)$, $t_1 \leq min\left(t_2, \frac{t_2\pi_1}{\pi_2}\right)$, $v_1 \leq min\left(v_2, \frac{v_2\pi_1}{\pi_2}\right)$

(iv) $\left(\widetilde{T}_1 \oslash \widetilde{T}_2\right) \bigotimes \widetilde{T}_2 = \widetilde{T}_1$, *if* $f_1 \geq f_2$, $g_1 \geq g_2$, $t_1 \geq t_2$, $v_1 \geq v_2$ *and* $F_1 \leq min\left(F_2, \frac{F_2\pi_1}{\pi_2}\right)$, $G_1 \leq min\left(G_2, \frac{G_2\pi_1}{\pi_2}\right)$, $T_1 \leq min\left(T_2, \frac{T_2\pi_1}{\pi_2}\right)$, $V_1 \leq min\left(V_2, \frac{V_2\pi_1}{\pi_2}\right)$

(i) Proof: Taking LHS, we get

$$(\widetilde{T}_1 \bigcup \widetilde{T}_2) \bigcap \widetilde{T}_2$$

$$= \Big\langle \Big(max\{F_1, F_2\}, max\{G_1, G_2\}, max\{T_1, T_2\}, max \max\{V_1, V_2\} \Big),$$

$$\Big(min\{f_1, f_2\}, \min\{g_1, g_2\}, min\{t_1, t_2\}, \min\{v_1, v_2\} \Big) \Big\rangle$$

$$\bigcap \Big\langle (F_2, G_2, T_2, V_2), (f_2, g_2, t_2, v_2) \Big\rangle$$

$$= \Big\langle \Big(min\{max\{F_1, F_2\}, F_2\}, min\{max\{G_1, G_2\}, G_2\},$$

$$min\{max\{T_1, T_2\}, T_2\}, \{max\{V_1, V_2\}, V_2\} \Big),$$

$$\Big(max\{min\{f_1, f_2\}, f_2\}, max\{min\{g_1, g_2\}, g_2\}, \max\{min\{t_1, t_2\}, t_2\},$$

$$max\{min\{v_1, v_2\}, v_2\} \Big) \Big\rangle$$

$$= \Big\langle (F_2, G_2, T_2, V_2), (f_2, g_2, t_2, v_2) \Big\rangle$$

$$= \widetilde{T}_2.$$

(ii) Proof: The proof is similar to the previous one.
(iii) Proof: Now from the given LHS, we get

$$(\widetilde{T}_1 \ominus \widetilde{T}_2) \bigoplus \widetilde{T}_2$$

$$= \Big\langle \Big(\sqrt{\frac{F_1^2 - F_2^2}{1 - F_2^2}}, \sqrt{\frac{G_1^2 - G_2^2}{1 - G_2^2}}, \sqrt{\frac{T_1^2 - T_2^2}{1 - T_2^2}}, \sqrt{\frac{V_1^2 - V_2^2}{1 - V_2^2}} \Big), \Big(\frac{f_1}{f_2}, \frac{g_1}{g_2}, \frac{t_1}{t_2}, \frac{v_1}{v_2} \Big) \Big\rangle \bigoplus$$

$$\Big\langle \Big(F_2, G_2, T_2, V_2 \Big), \Big(f_2, g_2, t_2, v_2 \Big) \Big\rangle$$

$$= \Big\langle \sqrt{\Big(\sqrt{\frac{F_1^2 - F_2^2}{1 - F_2^2}} \Big)^2 + F_2^2 - \Big(\sqrt{\frac{F_1^2 - F_2^2}{1 - F_2^2}} \Big)^2 F_2^2},$$

$$\sqrt{\Big(\sqrt{\frac{G_1^2 - G_2^2}{1 - G_2^2}} \Big)^2 + G_2^2 - \Big(\sqrt{\frac{G_1^2 - G_2^2}{1 - G_2^2}} \Big)^2 G_2^2},$$

$$\sqrt{\left(\sqrt{\frac{T_1^2 - T_2^2}{1 - T_2^2}}\right)^2 + T_2^2 - \left(\sqrt{\frac{T_1^2 - T_2^2}{1 - T_2^2}}\right)^2 T_2^2},$$

$$\sqrt{\left(\sqrt{\frac{V_1^2 - V_2^2}{1 - V_2^2}}\right)^2 + V_2^2 - \left(\sqrt{\frac{V_1^2 - V_2^2}{1 - V_2^2}}\right)^2 V_2^2},$$

$$\left(\frac{f_1}{f_2} f_2, \frac{g_1}{g_2} g_2, \frac{t_1}{t_2} t_2, \frac{v_1}{v_2} v_2\right)\bigg\rangle$$

$$= \bigg\langle\Big(F_1, G_1, T_1, V_1\Big), \Big(f_1, g_1, t_1, v_1\Big)\bigg\rangle$$

$$= \widetilde{T}_1.$$

(iv) Proof: The proof is similar to the previous one.

Definition 3.2 Let $\widetilde{T}_1 = \langle (F_1, G_1, T_1, V_1), (f_1, g_1, t_1, v_1) \rangle$ and $\widetilde{T}_2 = \langle (F_2, G_2, T_2, V_2), (f_2, g_2, t_2, v_2) \rangle$ be any two trapezoidal Pythagorean fuzzy numbers. Then, the distance between $\widetilde{T}_1$ and $\widetilde{T}_2$ is given by

$$d_{Eu}(\widetilde{T}_1, \widetilde{T}_2) = \sqrt{\begin{array}{c}(F_2 - F_1)^2 + (G_2 - G_1)^2 + (T_2 - T_1)^2 + (V_2 - V_1)^2 \\ +(f_2 - f_1)^2 + (g_2 - g_1)^2 + (t_2 - t_1)^2 + (v_2 - v_1)^2\end{array}}$$

$$d_H(\widetilde{T}_1, \widetilde{T}_2) = \mid F_2 - F_1 \mid + \mid G_2 - G_1 \mid + \mid T_2 - T_1 \mid + \mid V_2 - V_1 \mid + \mid f_2 - f_1 \mid + \mid g_2 - g_1 \mid + \mid t_2 - t_1 \mid + \mid v_2 - v_1 \mid$$

$$d_W(\widetilde{T}_1, \widetilde{T}_2) = 1 - cos\left[\frac{\pi}{4}\Big\{ \mid F_2 - F_1 \mid + \mid G_2 - G_1 \mid + \mid T_2 - T_1 \mid + \mid V_2 - V_1 \mid + \mid f_2 - f_1 \mid + \mid g_2 - g_1 \mid + \mid t_2 - t_1 \mid + \mid v_2 - v_1 \mid \Big\}\right].$$

4 Aggregation Operator

Definition 4.1 Let $\widetilde{T}_k = \langle (F_k, G_k, T_k, V_k), (f_k, g_k, t_k, v_k) \rangle$ $(k = 1, 2, \cdots, n)$ be any collection of TrP_yFNs. Then trapezoidal Pythagorean fuzzy weighted arithmetic (TrP_yFWA) aggregation operator is defined as follows:

$$TrP_yFWA(\widetilde{T}_1, \widetilde{T}_2, \cdots, \widetilde{T}_n)$$
$$= \phi_1 \widetilde{T}_1 \bigoplus \phi_2 \widetilde{T}_2 \bigoplus \cdots \bigoplus \phi_n \widetilde{T}_n,$$

where ϕ_k's are weights of $\widetilde{T}_k$ $(k = 1, 2, \cdots, n)$, respectively, and $\sum_{k=1}^{n} \phi_k = 1$.

Theorem 4.1 *Let $\widetilde{T}_k = \langle (F_k, G_k, T_k, V_k), (f_k, q_k, t_k, v_k)\rangle$ $(k = 1, 2, \cdots, n)$ be any collection of TrP_yFNs. Then, their aggregated value using the TrP_yFWA aggregation operator is also a trapezoidal Pythagorean fuzzy number and is given by*

$$TrP_yFWA(\widetilde{T}_1, \widetilde{T}_2, \cdots, \widetilde{T}_n)$$
$$= \Bigg\langle \Bigg(\sqrt{1 - \prod_{r=1}^{n} (1 - F_r^2)^{\phi_r}}, \sqrt{1 - \prod_{r=1}^{n} (1 - G_r^2)^{\phi_r}}, \sqrt{1 - \prod_{r=1}^{n} (1 - T_r^2)^{\phi_r}},$$
$$\sqrt{1 - \prod_{r=1}^{n} (1 - V_r^2)^{\phi_r}} \Bigg),$$
$$\Bigg(\prod_{i=1}^{n} f_r{}^{\phi_r}, \prod_{i=1}^{n} q_r{}^{\phi_r}, \prod_{i=1}^{n} t_r{}^{\phi_r}, \prod_{i=1}^{n} v_r{}^{\phi_r} \Bigg) \Bigg\rangle, \cdots\cdots\cdots\cdots\cdots\cdots\cdots (1)$$

where $\phi_r s$ are weights of $\widetilde{T}_r$ $(r = 1, 2, \cdots, n)$ and $\sum_{r=1}^{n} \phi_r = 1$.

Proof If we take $r = 2$, then

$$\phi_1 \widetilde{T}_1 \bigoplus \phi_2 \widetilde{T}_2$$
$$= \Bigg\langle \Bigg(\sqrt{1 - (1 - F_1^2)^{\phi_1}}, \sqrt{1 - (1 - G_1^2)^{\phi_1}}, \sqrt{1 - (1 - T_1^2)^{\phi_1}}, \sqrt{1 - (1 - V_1^2)^{\phi_1}} \Bigg),$$
$$\Bigg(f_1^{\phi_1}, g_1^{\phi_1}, t_1^{\phi_1}, v_1^{\phi_1} \Bigg) \Bigg\rangle \bigoplus$$
$$\Bigg\langle \Bigg(\sqrt{1 - (1 - F_2^2)^{\phi_2}}, \sqrt{1 - (1 - G_2^2)^{\phi_2}}, \sqrt{1 - (1 - T_2^2)^{\phi_2}}, \sqrt{1 - (1 - V_2^2)^{\phi_2}} \Bigg),$$
$$\Bigg(f_2^{\phi_2}, g_2^{\phi_2}, t_2^{\phi_2}, v_2^{\phi_2} \Bigg) \Bigg\rangle$$
$$= \Bigg\langle \Bigg(\sqrt{1 - (1 - F_1^2)^{\phi_1} + 1 - (1 - F_2^2)^{\phi_2} - (1 - (1 - F_1^2)^{\phi_1})(1 - (1 - F_2^2)^{\phi_2})},$$
$$\sqrt{1 - (1 - G_1^2)^{\phi_1} + 1 - (1 - G_2^2)^{\phi_2} - (1 - (1 - G_1^2)^{\phi_1})(1 - (1 - G_2^2)^{\phi_2})},$$
$$\sqrt{1 - (1 - T_1^2)^{\phi_1} + 1 - (1 - T_2^2)^{\phi_2} - (1 - (1 - T_1^2)^{\phi_1})(1 - (1 - T_2^2)^{\phi_2})},$$
$$\sqrt{1 - (1 - V_1^2)^{\phi_1} + 1 - (1 - V_2^2)^{\phi_2} - (1 - (1 - V_1^2)^{\phi_1})(1 - (1 - V_2^2)^{\phi_2})}, \Bigg),$$
$$\Bigg(f_1^{\phi_1} f_2^{\phi_2}, g_1^{\phi_1} g_2^{\phi_2}, t_1^{\phi_1} t_2^{\phi_2}, v_1^{\phi_1} v_2^{\phi_2} \Bigg) \Bigg\rangle$$

$$= \left\langle \left(\sqrt{1-(1-{F_1}^2)^{\phi_1}(1-{F_2}^2)^{\phi_2}}, \sqrt{1-(1-{G_1}^2)^{\phi_1}(1-{G_2}^2)^{\phi_2}}, \sqrt{1-(1-{T_1}^2)^{\phi_1}(1-{T_2}^2)^{\phi_2}}, \sqrt{1-(1-{V_1}^2)^{\phi_1}(1-{V_2}^2)^{\phi_2}} \right), \left(f_1^{\phi_1} f_2^{\phi_2}, g_1^{\phi_1} g_2^{\phi_2}, t_1^{\phi_1} t_2^{\phi_2}, v_1^{\phi_1} v_2^{\phi_2} \right) \right\rangle.$$

Let it hold for $r = n$ (induction hypothesis) i.e.,

$$\phi_1 \widetilde{T}_1 \bigoplus \phi_2 \widetilde{T}_2 \bigoplus \cdots \bigoplus \phi_n \widetilde{T}_n$$

$$= \left\langle \left(\sqrt{1-\prod_{r=1}^{n}(1-F_r^2)^{\phi_r}}, \sqrt{1-\prod_{r=1}^{n}(1-G_r^2)^{\phi_r}}, \sqrt{1-\prod_{r=1}^{n}(1-T_r^2)^{\phi_r}}, \sqrt{1-\prod_{r=1}^{n}(1-V_r^2)^{\phi_r}} \right), \left(\prod_{i=1}^{n} {f_r}^{\phi_r}, \prod_{i=1}^{n} {g_r}^{\phi_r}, \prod_{i=1}^{n} {t_r}^{\phi_r}, \prod_{i=1}^{n} {v_r}^{\phi_r} \right) \right\rangle.$$

Now, we have to show that it also holds for $r = n + 1$

$$\phi_1 \widetilde{T}_1 \bigoplus \phi_2 \widetilde{T}_2 \bigoplus \cdots \bigoplus \phi_n \widetilde{T}_n \bigoplus \psi_{n+1} \widetilde{T}_{n+1}$$

$$= \left\langle \left(\sqrt{1-\prod_{r=1}^{n}(1-F_r^2)^{\phi_r} + 1-(1-{F_{n+1}}^2)^{\phi_{n+1}} - \left(1-\prod_{r=1}^{n}(1-F_r^2)^{\phi_r}\right)\left(1-(1-{F_{n+1}}^2)^{\phi_{n+1}}\right)}, \right. \right.$$

$$\sqrt{1-\prod_{r=1}^{n}(1-G_r^2)^{\phi_r} + 1-(1-{G_{n+1}}^2)^{\phi_{n+1}} - \left(1-\prod_{r=1}^{n}(1-G_r^2)^{\phi_r}\right)\left(1-(1-{G_{n+1}}^2)^{\phi_{n+1}}\right)},$$

$$\sqrt{1-\prod_{r=1}^{n}(1-T_r^2)^{\phi_r} + 1-(1-{T_{n+1}}^2)^{\phi_{n+1}} - \left(1-\prod_{r=1}^{n}(1-T_r^2)^{\phi_r}\right)\left(1-(1-{T_{n+1}}^2)^{\phi_{n+1}}\right)},$$

$$\left. \sqrt{1-\prod_{r=1}^{n}(1-V_r^2)^{\phi_r} + 1-(1-{V_{n+1}}^2)^{\phi_{n+1}} - \left(1-\prod_{r=1}^{n}(1-V_r^2)^{\phi_r}\right)\left(1-(1-{V_{n+1}}^2)^{\phi_{n+1}}\right)} \right),$$

$$\left. \left(\prod_{i=1}^{n} {f_r}^{\phi_r} {f_{n+1}}^{\phi_{n+1}}, \prod_{i=1}^{n} {g_r}^{\phi_r} {g_{n+1}}^{\phi_{n+1}}, \prod_{i=1}^{n} {t_r}^{\phi_r} {t_{n+1}}^{\phi_{n+1}}, \prod_{i=1}^{n} {v_r}^{\phi_r} {v_{n+1}}^{\phi_{n+1}} \right) \right\rangle$$

$$= \left\langle \left(\sqrt{1-\prod_{r=1}^{n+1}(1-F_r^2)^{\phi_r}}, \sqrt{1-\prod_{r=1}^{n+1}(1-G_r^2)^{\phi_r}}, \sqrt{1-\prod_{r=1}^{n+1}(1-T_r^2)^{\phi_r}}, \sqrt{1-\prod_{r=1}^{n+1}(1-V_r^2)^{\phi_r}} \right), \left(\prod_{r=1}^{n+1} {f_r}^{\phi_r}, \prod_{r=1}^{n+1} {G_r}^{\phi_r}, \prod_{r=1}^{n+1} {t_r}^{\phi_r}, \prod_{r=1}^{n+1} {v_r}^{\phi_r} \right) \right\rangle.$$

As the theorem is also true for $r = n + 1$ and, hence, by mathematical induction, we get the result

$$TrP_yWAA(\widetilde{T}_1, \widetilde{T}_2, \cdots, \widetilde{T}_n)$$
$$= \Bigg\langle \Bigg(\sqrt{1 - \prod_{r=1}^{n}(1 - F_r^2)^{\phi_r}}, \sqrt{1 - \prod_{r=1}^{n}(1 - G_r^2)^{\phi_r}}, \sqrt{1 - \prod_{r=1}^{n}(1 - T_r^2)^{\phi_r}},$$
$$\sqrt{1 - \prod_{r=1}^{n}(1 - V_r^2)^{\phi_r}} \Bigg), \Bigg(\prod_{r=1}^{n} f_r{}^{\phi_r}, \prod_{r=1}^{n} g_r{}^{\phi_r}, \prod_{r=1}^{n} t_r{}^{\phi_r}, \prod_{r=1}^{n} v_r{}^{\phi_r} \Bigg) \Bigg\rangle.$$

Hence, the theorem is proved.

Example 4.1 Let $\widetilde{T}_1 = \Big\langle(0.4, 0.5, 0.55, 0.7), (0.3, 0.4, 0.5, 0.6)\Big\rangle$, $\widetilde{T}_2 = \Big\langle(0.4, 0.5, 0.55, 0.6), (0.45, 0.6, 0.7, 0.8)\Big\rangle$, and $\widetilde{T}_3 = \Big\langle(0.6, 0.7, 0.75, 0.8), (0.3, 0.4, 0.45, 0.5)\Big\rangle$ be three TrP_yNs and $\phi_1 = 0.35$, $\phi_2 = 0.3$, $\phi_3 = 0.35$ be the weight of the corresponding TrP_yFNs. Then

$$TrP_yWAA(\widetilde{T}_1, \widetilde{T}_2, \widetilde{T}_3)$$
$$= \Bigg\langle \Bigg(\sqrt{1 - \prod_{r=1}^{3}(1 - F_r^2)^{\phi_r}}, \sqrt{1 - \prod_{r=1}^{3}(1 - G_r^2)^{\phi_r}}, \sqrt{1 - \prod_{r=1}^{3}(1 - T_r^2)^{\phi_r}},$$
$$\sqrt{1 - \prod_{r=1}^{3}(1 - V_r^2)^{\phi_r}} \Bigg), \Bigg(\prod_{r=1}^{3} f_r{}^{\phi_r}, \prod_{r=1}^{3} g_r{}^{\phi_r}, \prod_{r=1}^{3} t_r{}^{\phi_r}, \prod_{r=1}^{3} v_r{}^{\phi_r} \Bigg) \Bigg\rangle$$
$$= \Big\langle(0.3314, 0.4136, 0.5083, 0.6487), (0.2772, 0.3793, 0.4655, 0.566)\Big\rangle.$$

Since $0.6487^2 + 0.566^2 = 0.7412 < 1$, then $TrP_yWAA(\widetilde{T}_1, \widetilde{T}_2, \widetilde{T}_3)$ is also a TrP_yFN.

4.1 Property of TrP_yFWA

Lemma 4.1.1 (Idempotency Properties of TrP_yFWA Operator)
Let $\widetilde{T}_r = \langle(F_r, G_r, T_r, V_r), (f_r, g_r, t_r, v_r)\rangle$ $(r = 1, 2, \cdots, n)$ *be any collection of* TrP_yFNs. *If each* $\widetilde{T}_r = \langle(F_r, G_r, T_r, V_r), (f_r, g_r, t_r, v_r)\rangle$ *is equal to* $T = \langle(F, G, T, V), (f, g, t, v)\rangle$ *for all* $(r = 1, 2, \cdots, n)$, *then*

$$TrP_yFWA(\widetilde{T}_1, \widetilde{T}_2, \cdots, \widetilde{T}_n) = \widetilde{T}.$$

Proof From definition, we get

$$TrP_yFWA(\widetilde{T}_1, \widetilde{T}_2, \cdots, \widetilde{T}_n)$$

$$= \Big\langle \Big(\sqrt{1 - \prod_{r=1}^{n} (1 - F_r^2)^{\phi_r}}, \sqrt{1 - \prod_{r=1}^{n} (1 - G_r^2)^{\phi_r}}, \sqrt{1 - \prod_{r=1}^{n} (1 - T_r^2)^{\phi_r}}, \sqrt{1 - \prod_{r=1}^{n} (1 - V_r^2)^{\phi_r}} \Big), \Big(\prod_{r=1}^{n} f_r^{\phi_r}, \prod_{r=1}^{n} g_r^{\phi_r}, \prod_{r=1}^{n} t_r^{\phi_r}, \prod_{r=1}^{n} v_r^{\phi_r} \Big) \Big\rangle$$

$$= \Big\langle \Big(\sqrt{1 - \prod_{r=1}^{n} (1 - F^2)^{\phi_r}}, \sqrt{1 - \prod_{r=1}^{n} (1 - G^2)^{\phi_r}}, \sqrt{1 - \prod_{r=1}^{n} (1 - T^2)^{\phi_r}}, \sqrt{1 - \prod_{r=1}^{n} (1 - V^2)^{\phi_r}} \Big), \Big(\prod_{r=1}^{n} f^{\phi_r}, \prod_{r=1}^{n} g^{\phi_r}, \prod_{r=1}^{n} t^{\phi_r}, \prod_{r=1}^{n} v^{\phi_r} \Big) \Big\rangle \text{ (since for each } \widetilde{T}_r = \widetilde{T})$$

$$= \Big\langle \Big(\sqrt{1 - (1 - F^2)^{\sum_{r=1}^{n} \phi_r}}, \sqrt{1 - (1 - G^2)^{\sum_{r=1}^{n} \phi_r}}, \sqrt{1 - (1 - T^2)^{\sum_{r=1}^{n} \phi_r}}, \sqrt{1 - (1 - V^2)^{\sum_{r=1}^{n} \phi_r}} \Big), \Big(f^{\sum_{r=1}^{n} \phi_r}, g^{\sum_{r=1}^{n} \phi_r}, t^{\sum_{r=1}^{n} \phi_r}, v^{\sum_{r=1}^{n} \phi_r} \Big) \Big\rangle$$

$$= \Big\langle \Big(\sqrt{1 - (1 - F^2)}, \sqrt{1 - (1 - G^2)}, \sqrt{1 - (1 - T^2)}, \sqrt{1 - (1 - V^2)} \Big), \Big(f, g, t, v \Big) \Big\rangle$$

$$= \langle (F, G, T, V), (f, g, t, v) \rangle = \widetilde{T}.$$

Lemma 4.1.2 (Boundedness Properties of TrP_yFWA Operator)

Let $\widetilde{T}_r = \langle (F_r, G_r, T_r, V_r), (f_r, g_r, t_r, v_r) \rangle$ $(r = 1, 2, \cdots, n)$ be any collection of TrP_yFNs. If $\widetilde{T}^- = \langle (F^-, G^-, T^-, V^-), (f^+, g^+, t^+, v^+) \rangle$ and $\widetilde{T}^+ = \langle (F^+, G^+, T^+, V^+), (f^-, g^-, t^-, v^-) \rangle$, where $F^- = min\{F_r : r = 1, 2, \cdots, n\}, G^- = min\{G_r : r = 1, 2, \cdots, n\}, T^- = min\{T_r : r = 1, 2, \cdots, n\}, V^- = min\{V_r : r = 1, 2, \cdots, n\}, f^- = min\{f_r : r = 1, 2, \cdots, n\}, g^- = min\{q_r : r = 1, 2, \cdots, n\}, t^- = min\{t_r : r = 1, 2, \cdots, n\}, v^- = min\{v_r : r = 1, 2, \cdots, n\}$, $F^+ = max\{F_r : r = 1, 2, \cdots, n\}, G^+ = max\{Z_r : r = 1, 2, \cdots, n\}, T^+ = max\{T_r : r = 1, 2, \cdots, n\}, V^+ = max\{V_r : r = 1, 2, \cdots, n\}, f^+ = max\{f_r : r = 1, 2, \cdots, n\}, g^+ = max\{g_r : r = 1, 2, \cdots, n\}, t^+ = max\{t_r : r = 1, 2, \cdots, n\}, v^+ = max\{v_r : r = 1, 2, \cdots, n\}$, then

$$\widetilde{T}^{-} \leq TrP_yFWA(\widetilde{T}_1, \widetilde{T}_2, \cdots, \widetilde{T}_n) \leq \widetilde{T}^{+}.$$

Proof From the given condition, we get

$$\begin{aligned}
&\widetilde{T}^{-} \leq \widetilde{T}_r \leq \widetilde{T}^{+} \\
\Longrightarrow\ &\phi_r \widetilde{T}^{-} \leq \phi_r \widetilde{T}_r \leq \phi_r \widetilde{T}^{+} \\
\Longrightarrow\ &\sum_{r=1}^{n} \phi_r \widetilde{T}^{-} \leq \sum_{r=1}^{n} \phi_r \widetilde{T}_r \leq \sum_{r=1}^{n} \phi_r \widetilde{T}^{+} \\
\Longrightarrow\ &\left(\sum_{r=1}^{n} \phi_r\right) \widetilde{T}^{-} \leq \sum_{r=1}^{n} \phi_r \widetilde{T}_r \leq \left(\sum_{r=1}^{n} \phi_r\right) \widetilde{T}^{+} \\
\Longrightarrow\ &\widetilde{T}^{-} \leq \sum_{r=1}^{n} \phi_r \widetilde{T}_r \leq \widetilde{T}^{+} \text{ since } \sum_{r=1}^{n} \phi_r = 1 \\
\Longrightarrow\ &\widetilde{T}^{-} \leq T_r PyWAA(\widetilde{T}_1, \widetilde{T}_2, \cdots, \widetilde{T}_n) \leq \widetilde{T}^{+}.
\end{aligned}$$

Lemma 4.1.3 (Monotonicity Properties of TrP_yFWA Operator) *Let $\widetilde{T}_r$ and $\widetilde{T}_r{}^*$ for $(r = 1, 2, \cdots, n)$ be any two collections of TrP_yFNs such that $\widetilde{T}_r \leq \widetilde{T}_r{}^*$ for any $(r = 1, 2, \cdots, n)$. Then*

$$T_r P_y FWA(\widetilde{T}_1, \widetilde{T}_2, \cdots, \widetilde{T}_n) \leq T_r P_y FWA(\widetilde{T}_1{}^*, \widetilde{T}_2{}^*, \cdots, \widetilde{T}_n{}^*).$$

Proof Given that,

$$\begin{aligned}
&\widetilde{T}_r \leq \widetilde{T}_r{}^* \\
\Longrightarrow\ &\phi_r \widetilde{T}_r \leq \phi_r \widetilde{T}^* \\
\Longrightarrow\ &\sum_{r=1}^{n} \phi_r \widetilde{T}_r \leq \sum_{r=1}^{n} \phi_r \widetilde{T}_r{}^* \\
\Longrightarrow\ &T_r P_y FWA(\widetilde{T}_1, \widetilde{T}_2, \cdots, \widetilde{T}_n) \leq T_r P_y FWA(\widetilde{T}_1{}^*, \widetilde{T}_2{}^*, \cdots, \widetilde{T}_n{}^*).
\end{aligned}$$

5 TOPSIS Strategy for MCGDM Based on TrP_yFN

In this section, we developed a framework for determining the ranking order of the alternatives under the trapezoidal Pythagorean fuzzy environment. TOPSIS (technique for order preference by similarity to ideal solution) [16, 38] is a celebrated technique in the decision-making field. The basic idea is that the chosen alternative should have the lowest distance from positive ideal solution (PIS) and the largest distance from negative ideal solution (NIS).

For MCGDM problem, let $X = X_1, X_2, \cdots, X_r$ be a set of alternatives, $Y = Y_1, Y_2, \cdots, Y_s$ be a set of criteria, and $Z = Z_1, Z_2, \cdots, Z_m$ be a set of distinct decision-makers. Here, we assume weight vector of the criteria is $\theta = (\theta_1, \theta_2, \cdots, \theta_s)^T$, where $\theta_i \geq 0$ and $\sum_{i=1}^{s} \theta_i = 1$. Also, the weight of the decision-makers is $\Omega = (\Omega_1, \Omega_2, \cdots, \Omega_t)^T$, where $\Omega_i \geq 0$ and $\sum_{i=1}^{m} \Omega_i = 1$. Here, we developed an improved TOPSIS method based on aggregation operator and distance functions to determine the ranking order of the alternative as follows:

Step 1: For a decision-maker $Z_k (k = 1, 2, \cdots, m)$, the evaluation values of the alternative $X_i (i = 1, 2, \cdots, r)$ under the criteria $Y_i (i = 1, 2, \cdots, s)$ are given in the form decision matrix as follows:

$$\mathbf{DM^k} = \begin{array}{c} \\ X_1 \\ X_2 \\ \vdots \\ X_r \end{array} \begin{array}{c} \begin{array}{ccc} Y_1 & Y_2 \cdots & Y_s \end{array} \\ \begin{pmatrix} \widetilde{T}_{11}^k & \widetilde{T}_{12}^k \cdots & \widetilde{T}_{1s}^k \\ \widetilde{T}_{21}^k & \widetilde{T}_{22}^k \cdots & \widetilde{T}_{2s}^k \\ \vdots & \vdots & \vdots \\ \widetilde{T}_{r1}^k & \widetilde{T}_{r2}^k \cdots & \widetilde{T}_{rs}^k \end{pmatrix} \end{array},$$

where each entity $\widetilde{T}_{ij}^k = \left\langle \left(F_{ij}^k, G_{ij}^k, T_{ij}^k, V_{ij}^k \right), \left(f_{ij}^k, g_{ij}^k, t_{ij}^k, v_{ij}^k \right) \right\rangle, (i = 1, 2, \cdots, r, j = 1, 2, \cdots, s, k = 1, 2, \cdots, m)$ of the decision matrix is TrP_yFN with the condition $(V_{ij}^k)^2 + (v_{ij}^k)^2 \leq 1$.

Step 2: In this step, we use our proposed aggregation operator TrP_yFWA on decision matrices to get the aggregated evaluation value of the alternative in the form matrix $DM = (\widetilde{T}_{ij})_{r \times s}$, where

$$\begin{aligned} \widetilde{T}_{ij} &= TrP_yFWA(\widetilde{T}_{ij}^1, \widetilde{T}_{ij}^2, \cdots, \widetilde{T}_{ij}^m) \\ &= \Omega_1 \widetilde{T}_{ij}^1 + \Omega_2 \widetilde{T}_{ij}^2 + \cdots + \Omega_m \widetilde{T}_{ij}^m \\ &= \Bigg\langle \Bigg(\sqrt{1 - \prod_{r=1}^{m} \left(1 - (F_{ij}^r)^2\right)^{\Omega_r}}, \sqrt{1 - \prod_{r=1}^{m} \left(1 - (G_{ij}^r)^2\right)^{\Omega_r}}, \\ &\quad \sqrt{1 - \prod_{r=1}^{m} \left(1 - (T_{ij}^r)^2\right)^{\Omega_r}}, \sqrt{1 - \prod_{r=1}^{m} \left(1 - (V_{ij}^r)^2\right)^{\Omega_r}} \Bigg), \end{aligned}$$

$$\left(\prod_{r=1}^{m}(f_{ij}^{r})^{\Omega_r}, \prod_{r=1}^{m}(g_{ij}^{r})^{\Omega_r}, \prod_{r=1}^{m}(t_{ij}^{r})^{\Omega_r}, \prod_{r=1}^{m}(v_{ij}^{r})^{\Omega_r}\right)\Big\rangle.$$

Therefore, the aggregated decision matrix (DM) is defined as follows:

$$\mathbf{DM} = \begin{array}{c} \\ X_1 \\ X_2 \\ \vdots \\ X_r \end{array} \begin{array}{c} \begin{array}{ccc} Y_1 & Y_2 \cdots & Y_s \end{array} \\ \begin{pmatrix} \widetilde{T}_{11} & \widetilde{T}_{12} \cdots & \widetilde{T}_{1s} \\ \widetilde{T}_{21} & \widetilde{T}_{22} \cdots & \widetilde{T}_{2s} \\ \vdots & \vdots & \vdots \\ \widetilde{T}_{r1} & \widetilde{T}_{r2} \cdots & \widetilde{T}_{rs} \end{pmatrix} \end{array},$$

where $\widetilde{T}_{ij} = \langle (F_{ij}, G_{ij}, T_{ij}, V_{ij}), (f_{ij}, g_{ij}, t_{ij}, v_{ij}) \rangle$ is the aggregated evaluation value of the alternative, $i = 1, 2, \cdots, r$, $j = 1, 2, \cdots s$.

Step 3: Now, we utilize criterion weights (θ_i) according to Proposition 3.1(iii) and get the weighted evaluation value as follows:

$$\mathbf{D} = \begin{array}{c} \\ X_1 \\ X_2 \\ \vdots \\ X_r \end{array} \begin{array}{c} \begin{array}{ccc} Y_1 & Y_2 \cdots & Y_s \end{array} \\ \begin{pmatrix} \widetilde{D}_{11} & \widetilde{D}_{12} \cdots & \widetilde{D}_{1s} \\ \widetilde{D}_{21} & \widetilde{D}_{22} \cdots & \widetilde{D}_{2s} \\ \vdots & \vdots & \vdots \\ \widetilde{D}_{r1} & \widetilde{D}_{r2} \cdots & \widetilde{D}_{rs} \end{pmatrix} \end{array},$$

where $\widetilde{D}_{ij} = \theta_j \widetilde{T}_{ij}$.

Step 4: Now, we calculate the PIS and NIS as follows:

$P^+ = \{p_1^+, p_2^+, \cdots, p_s^+\}$,

where $p_j^+ = \Big\langle (\max_i (F_{ij}), \max_i (G_{ij}), \max_i (T_{ij}), \max_i (V_{ij})), (\min_i (f_{ij}), \min_i (g_{ij}), \min_i (t_{ij}), \min_i (v_{ij}),) \Big\rangle$

$P^- = \{p_1^-, p_2^-, \cdots, p_s^-\}$,

where $p_j^- = \Big\langle (\min_i (F_{ij}), \min_i (G_{ij}), \min_i (T_{ij}), \min_i (V_{ij})), (\max_i (f_{ij}), \max_i (g_{ij}), \max_i (t_{ij}), \max_i (v_{ij})) \Big\rangle$.

The global PIS and NIS for TrP_yFN are given as

$p_j^+ = \Big\langle (1, 1, 1, 1), (0, 0, 0, 0) \Big\langle$ and $p_j^- = \Big\langle (0, 0, 0, 0), (1, 1, 1, 1) \Big\langle$.

Step 5: In this step, we find out the distance between the alternative A_i and PIS and NIS according to the equation (2) as follows:

$$\left.\begin{aligned} Q_i^+ &= d(A_i, P^+) = \sum_{j=1}^{s} d(p_j^+, \widetilde{D}_{ij}), \\ Q_i^- &= d(A_i, P^-) = \sum_{j=1}^{s} d(p_j^-, \widetilde{D}_{ij}) \end{aligned}\right\} \cdots\cdots\cdots\cdot (2)$$

Step 6: Here, we measure the relative closeness coefficient as follows:

$$S_i = \frac{Q_i^+}{Q_i^+ + Q_i^-}.$$

Step 7: In this step, we utilize the relative closeness coefficient to find the ranking order of the alternative. The smallest value of S_i gives the best alternative.

Remark 5.1 The various steps of the proposed TOPSIS-based MCGDM technique have been shown pictorially in Fig. 16.1.

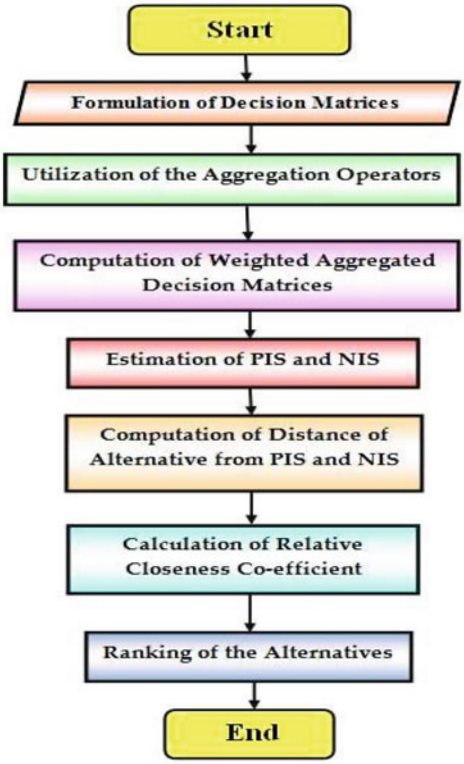

Fig. 16.1 Flowchart of the proposed TOPSIS technique

6 Illustrative Example

In this current era, we are very much concerned and doubtful in case of selection of the college for higher study. Several government and private institutes are present in our society for the higher studies in technical or general fields. Normally, question will arise which college/institute provides the best facilities such as quality of faculty member, eco-friendly campus, hostel facilities, discipline, good placement record, etc. But, the guardians are confused, and their mind is in dilemma to know all the answers properly such that they will select the best institute in their respective area. Thus, this becomes a fundamental and burning decision-making problem in imprecise arena. Here, we take some different experts like senior student of the corresponding institute, staff's view of the corresponding institute, and social media ranking information from web search to focus on the statistical data of the decision matrices. To resolve this burning issue, we consider an MCGDM problem in trapezoidal Pythagorean environment linked with three alternatives, namely: a) College-1, b) College-2, and c) College-3 in case of selection process. Also, we choose three different attributes: i) quality of faculty; ii) environment of the campus and facilities; and iii) placement in MNC as judgment factors of this given problem. Let us consider there are three experts as decision-makers Z_1= Senior student, Z_2= Staff, Z_3 = Web Results having weight function $\Omega = (0.35, 0.33, 0.32)$, and additionally, we consider dissimilar weight vector $\Theta = (0.35, 0.3, 0.35)$ linked with distinct attribute function. Here, we utilize the proposed TOPSIS-based MCGDM technique in the trapezoidal Pythagorean environment as follows:

Step 1: Here, we construct the decision matrices according to the decision-makers as follows:

$$\mathbf{DM^1} = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \end{array} \begin{array}{c} \begin{array}{ccc} Y_1 & Y_2 & Y_3 \end{array} \\ \begin{pmatrix} \langle(0.4, 0.5, 0.55, 0.7), (0.3, 0.4, 0.5, 0.6)\rangle & \langle(0.4, 0.5, 0.55, 0.6), (0.45, 0.6, 0.7, 0.8)\rangle & \langle(0.6, 0.7, 0.75, 0.8), (0.3, 0.4, 0.45, 0.5)\rangle \\ \langle(0.45, 0.5, 0.55, 0.6), (0.3, 0.4, 0.5, 0.6)\rangle & \langle(0.5, 0.55, 0.6, 0.65), (0.55, 0.6, 0.65, 0.7)\rangle & \langle(0.35, 0.4, 0.5, 0.6), (0.4, 0.5, 0.6, 0.7)\rangle \\ \langle(0.2, 0.3, 0.4, 0.5), (0.6, 0.7, 0.75, 0.8)\rangle & \langle(0.3, 0.4, 0.5, 0.6), (0.35, 0.4, 0.5, 0.6)\rangle & \langle(0.25, 0.3, 0.35, 0.4), (0.45, 0.5, 0.6, 0.7)\rangle \end{pmatrix} \end{array}$$

$$\mathbf{DM^2} = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \end{array} \begin{array}{c} \begin{array}{ccc} Y_1 & Y_2 & Y_3 \end{array} \\ \begin{pmatrix} \langle(0.2, 0.3, 0.4, 0.5), (0.35, 0.45, 0.5, 0.6)\rangle & \langle(0.3, 0.45, 0.5, 0.6), (0.45, 0.5, 0.55, 0.7)\rangle & \langle(0.5, 0.55, 0.6, 0.7), (0.4, 0.5, 0.6, 0.7)\rangle \\ \langle(0.3, 0.4, 0.5, 0.6), (0.2, 0.5, 0.6, 0.7)\rangle & \langle(0.5, 0.6, 0.7, 0.8), (0.25, 0.4, 0.55, 0.6)\rangle & \langle(0.1, 0.3, 0.5, 0.7), (0.2, 0.35, 0.4, 0.5)\rangle \\ \langle(0.4, 0.5, 0.6, 0.7), (0.25, 0.3, 0.35, 0.5)\rangle & \langle(0.3, 0.4, 0.5, 0.6), (0.4, 0.5, 0.6, 0.7)\rangle & \langle(0.4, 0.5, 0.6, 0.7), (0.4, 0.5, 0.6, 0.7)\rangle \end{pmatrix} \end{array}$$

$$\mathbf{DM^3} = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \end{array} \begin{array}{c} \begin{array}{ccc} Y_1 & Y_2 & Y_3 \end{array} \\ \begin{pmatrix} \langle(0.35, 0.4, 0.55, 0.7), (0.2, 0.3, 0.4, 0.5)\rangle & \langle(0.1, 0.2, 0.3, 0.4), (0.2, 0.3, 0.4, 0.5)\rangle & \langle(0.15, 0.2, 0.35, 0.4), (0.25, 0.3, 0.4, 0.5)\rangle \\ \langle(0.3, 0.4, 0.45, 0.5), (0.35, 0.4, 0.5, 0.55)\rangle & \langle(0.45, 0.5, 0.6, 0.75), (0.55, 0.6, 0.65, 0.7)\rangle & \langle(0.45, 0.5, 0.6, 0.65), (0.5, 0.6, 0.7, 0.75)\rangle \\ \langle(0.25, 0.3, 0.45, 0.5), (0.4, 0.5, 0.6, 0.7)\rangle & \langle(0.4, 0.5, 0.6, 0.7), (0.35, 0.4, 0.45, 0.6)\rangle & \langle(0.3, 0.4, 0.5, 0.6), (0.4, 0.5, 0.55, 0.8)\rangle \end{pmatrix} \end{array}.$$

Step 2: In this step, our first aim is to develop the single decision matrix (**DM**) by using the proposed aggregation operators. For this, we use the operator TrP_yFWA according to equation (1) and get

$$\mathbf{DM} = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \end{array} \begin{array}{c} \begin{array}{ccc} Y_1 & Y_2 & Y_3 \end{array} \\ \begin{pmatrix} \widetilde{C}_{11} & \widetilde{C}_{12} & \widetilde{C}_{13} \\ \widetilde{C}_{21} & \widetilde{C}_{22} & \widetilde{C}_{23} \\ \widetilde{C}_{31} & \widetilde{C}_{32} & \widetilde{C}_{33} \end{pmatrix} \end{array},$$

where $\widetilde{C}_{ij}$ is given below,

$$\widetilde{C}_{11} = \left[\langle(0.3314, 0.4136, 0.5083, 0.6487), (0.2772, 0.3793, 0.4655, 0.566)\rangle\right],$$

$$\widetilde{C}_{12} = \left[\langle(0.3016, 0.4144, 0.4713, 0.5494), (0.3471, 0.4526, 0.5405, 0.6586)\rangle\right],$$

$$\widetilde{C}_{13} = \left[\langle(0.4772, 0.5546, 0.6166, 0.6858), (0.3112, 0.3927, 0.4765, 0.5587)\rangle\right],$$

$$\widetilde{C}_{21} = \left[\langle(0.3620, 0.4389, 0.5043, 0.5716), (0.2757, 0.4306, 0.5310, 0.6140)\rangle\right],$$

$$\widetilde{C}_{22} = \left[\langle(0.4849, 0.5533, 0.6373, 0.7402), (0.4240, 0.5249, 0.6151, 0.6653)\rangle\right],$$

$$\widetilde{C}_{23} = \left[\langle(0.3382, 0.4102, 0.5358, 0.6522), (0.3418, 0.4712, 0.5514, 0.6404)\rangle\right],$$

$$\widetilde{C}_{31} = \left[\langle(0.2972, 0.3824, 0.4947, 0.5828), (0.3948, 0.4752, 0.5430, 0.6564)\rangle\right],$$

$$\widetilde{C}_{32} = \left[\langle(0.3362, 0.4358, 0.5358, 0.6363), (0.36580.4306, 0.5134, 0.6313)\rangle\right],$$

$$\widetilde{C}_{33} = \left[\langle(0.3232, 0.4097, 0.4980, 0.5890), (0.4168, 0.5000, 0.5835, 0.7306)\rangle\right].$$

Step 3: Now, we obtain weighted aggregated decision matrix (**D**) using the criterion weights (θ_i) according to Proposition 3.1(iii) as follows:

$$\mathbf{D} = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \end{array} \begin{array}{c} \begin{array}{ccc} Y_1 & Y_2 & Y_3 \end{array} \\ \begin{pmatrix} \widetilde{G}_{11} & \widetilde{G}_{12} & \widetilde{G}_{13} \\ \widetilde{G}_{21} & \widetilde{G}_{22} & \widetilde{G}_{23} \\ \widetilde{G}_{31} & \widetilde{G}_{32} & \widetilde{G}_{33} \end{pmatrix} \end{array},$$

where $\widetilde{G}_{ij}$ is given below,

$$\widetilde{G}_{11} = \left[\langle(0.1998, 0.2521, 0.3152, 0.4171), (0.6382, 0.7123, 0.7652, 0.8194)\rangle\right],$$

$$\widetilde{G}_{12} = \left[\langle(0.1679, 0.2344, 0.2694, 0.3197), (0.7280, 0.7883, 0.8315, 0.8822)\rangle\right],$$

$$\widetilde{G}_{13} = \left[\langle(0.2941, 0.3474, 0.3926, 0.4466), (0.6646, 0.7210, 0.7715, 0.8157)\rangle\right],$$

$$\widetilde{G}_{21} = \left[\langle(0.2190, 0.2686, 0.3124, 0.3596), (0.6370, 0.7446, 0.8013, 0.8431)\rangle\right],$$

$$\widetilde{G}_{22} = \left[\langle(0.2780, 0.3223, 0.3804, 0.4603), (0.7731, 0.8242, 0.8643, 0.8849)\rangle\right],$$

$$\widetilde{G}_{23} = \left[\langle(0.2040, 0.2499, 0.3342, 0.4198), (0.6868, 0.7685, 0.8119, 0.8556)\rangle\right],$$

$$\widetilde{G}_{31} = \left[\langle(0.1785, 0.2320, 0.3059, 0.3677), (0.7223, 0.7707, 0.8076, 0.8630)\rangle\right],$$

$$\widetilde{G}_{32} = \left[\langle(0.1880, 0.2475, 0.3107, 0.3797), (0.7396, 0.7766, 0.8187, 0.8711)\rangle\right],$$

$$\widetilde{G}_{33} = \left[\langle(0.1946, 0.2496, 0.3081, 0.3722), (0.7362, 0.7846, 0.8282, 0.8960)\rangle\right].$$

Step 4: In this step, we have calculated PIS and NIS from the above decision matrix **D** as follows:
$P^+ = \{p_1^+, p_2^+, p_3^+\}$, where

$$p_1^+ = \Big\langle(0.2190, 0.2686, 0.3152, 0.4171), (0.6370, 0.7123, 0.7652, 0.8194)\Big\rangle$$
$$p_2^+ = \Big\langle(0.2780, 0.3223, 0.3804, 0.4603), (0.728, 0.7766, 0.8187, 0.8711)\Big\rangle$$
$$p_3^+ = \Big\langle(0.2941, 0.3474, 0.3926, 0.4466), (0.6646, 0.7210, 0.7715, 0.8157)\Big\rangle,$$

and $P^- = \{p_1^-, p_2^-, p_3^-\}$, where

$$p_1^- = \Big\langle(0.1785, 0.2320, 0.3059, 0.3596), (0.7223, 0.7707, 0.8076, 0.8630)\Big\rangle$$
$$p_2^- = \Big\langle(0.1679, 0.2344, 0.2694, 0.3197), (0.7731, 0.7766, 0.8643, 0.8849)\Big\rangle$$
$$p_3^- = \Big\langle(0.1946, 0.2496, 0.3081, 0.3722), (0.7362, 0.7846, 0.8282, 0.8960)\Big\rangle.$$

Step 5: Here, we find out the distance between the alternatives A_i and PIS & NIS as follows:

$$Q_1^+ = 0.2302,\ Q_2^+ = 0.2012,\ Q_3^+ = 0.3095 \ \text{and}\ Q_1^- = 0.2693,$$
$$Q_2^- = 0.2706,\ Q_3^- = 0.097.$$

Step 6: Now, the relative closeness coefficient is calculated below as

$$S_1 = 0.4609,\ S_2 = 0.4265,\ S_3 = 0.7614.$$

Therefore, the raking order of the relative closeness coefficient is $S_2 < S_1 < S_3$. Therefore, the ranking order of the alternative is $X_3 < X_1 < X_2$. Hence, X_2 is the best option.

6.1 Sensitivity Analysis

In this section, we will observe how the ranking orders of the alternatives differ from each other in change of only the weight of the decision-makers (Ω_i) and other data remain unchanged. First, we take the same decision matrices (DM^k) according to the decision-makers, and then we converted it into a single decision matrix (**DM**) using the operator TrP_yFWA. After that, we will find the corresponding weighted aggregated decision matrix (**D**) using the criterion weights (θ_i) and proposition 3.1(iii). Then we will sort out the PIS and NIS from the weighted aggregated decision matrix (**D**). After that, we use three different distance functions, namely: Hamming distance, Wei's distance, and Euclidean distance to find out the distances between the alternatives A_i and PIS and NIS. Finally, we compute the value of relative closeness coefficients (S_i) to find out the ranking order of the alternatives. Here, the sensitivity analysis is performed by varying the decision-makers weights under the three different distance functions and noticed how the ranking orders of the alternatives are being affected. The results of sensitivity analysis are given in **Table 16.1**. In Fig. 16.2, we have shown the weights of the decision-makers. Figures 16.3, 16.4, and 16.5 show the change of ranking order under Euclidean, Hamming, and Wei's distance functions, respectively. From these figures, we see that the ranking orders under Euclidean distance and Wei's distance functions are more or less same. Here, X_2 is the best alternative under the both distance functions. But the ranking order under the Hamming distance function is different, and in this case, X_1 is the best alternative. Thus, we can conclude that our method is more or less stable.

6.2 Comparison Analysis

In order to justify the advantages and effectiveness of our proposed technique, here we are presenting a comparative analysis with the existing method given by Ye [40]. The decision information in the study of Ye [40] is taken in the form of interval-valued intuitionistic fuzzy numbers, whereas the information of our paper has been put in the form of TrP_yFN. As TrP_yFN is a generalization of trapezoidal intuitionistic fuzzy number, thus our current study can capture the underlying uncertainty in a more robust way. Moreover, the decision-making process proposed in this chapter not only solves the problem in the trapezoidal Pythagorean fuzzy environment but also in the trapezoidal intuitionistic fuzzy environment, whereas

Table 16.1 Sensitivity analysis under different distance functions

S.N.	Weight function	Distance Function	Relative Closeness Coefficient	Ranking of the Alternative
		Euclidean distance	$S_1 = 0.4217$, $S_2 = 0.4528$, $S_3 = 0.7666$	$X_3 < X_2 < X_1$
1	(0.40,0.30,0.30)	Hamming distance	$S_1 = 0.3925$, $S_2 = 0.4829$, $S_3 = 0.6960$	$X_3 < X_2 < X_1$
		Wei's distance	$S_1 = 0.4219$, $S_2 = 0.4524$, $S_3 = 0.7572$	$X_3 < X_2 < X_1$
		Euclidean distance	$S_1 = 0.4906$, $S_2 = 0.4048$, $S_3 = 0.6901$	$X_3 < X_1 < X_2$
2	(0.30,0.40,0.30)	Hamming distance	$S_1 = 0.4333$, $S_2 = 0.4580$, $S_3 = 0.6462$	$X_3 < X_2 < X_1$
		Wei's distance	$S_1 = 0.4902$, $S_2 = 0.4048$, $S_3 = 0.6897$	$X_3 < X_1 < X_2$
		Euclidean distance	$S_1 = 0.4745$, $S_2 = 0.415$, $S_3 = 0.6866$.	$X_3 < X_1 < X_2$
3	(0.30,0.30,0.40)	Hamming distance	$S_1 = 0.4057$, $S_2 = 0.4775$, $S_3 = 0.6529$	$X_3 < X_2 < X_1$
		Wei's distance	$S_1 = 0.4753$, $S_2 = 0.4147$, $S_3 = 0.6867$	$X_3 < X_1 < X_2$
		Euclidean distance	$S_1 = 0.4497$, $S_2 = 0.4312$, $S_3 = 0.7241$.	$X_3 < X_1 < X_2$
4	(0.35,0.30,0.35)	Hamming distance	$S_1 = 0.4005$, $S_2 = 0.4794$, $S_3 = 0.6746$	$X_3 < X_2 < X_1$
		Wei's distance	$S_1 = 0.4496$, $S_2 = 0.4313$, $S_3 = 0.7241$	$X_3 < X_1 < X_2$
		Euclidean distance	$S_1 = 0.47$, $S_2 = 0.421$, $S_3 = 0.7268$.	$X_3 < X_1 < X_2$
5	(0.33,0.33,0.34)	Hamming distance	$S_1 = 0.4117$, $S_2 = 0.4758$, $S_3 = 0.6742$	$X_3 < X_2 < X_1$
		Wei's distance	$S_1 = 0.4699$, $S_2 = 0.4211$, $S_3 = 0.7267$	$X_3 < X_1 < X_2$

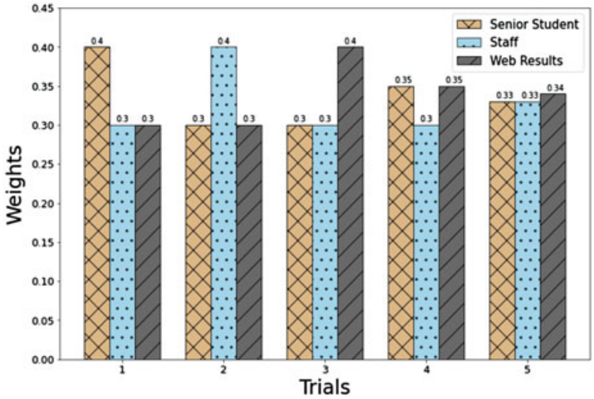

Fig. 16.2 Variation of decision-maker's weights

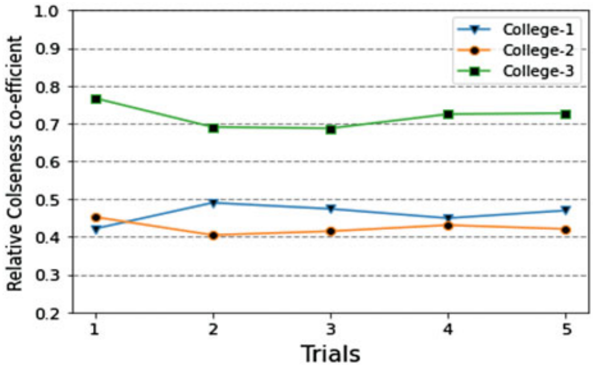

Fig. 16.3 Ranking order of the alternatives under Euclidean distance

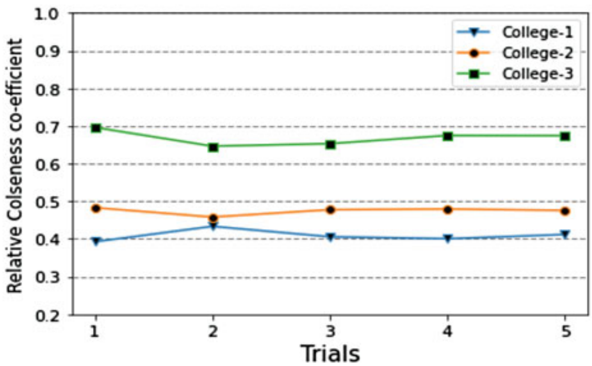

Fig. 16.4 Ranking order of the alternatives under Hamming distance

the technique in [40] is only applicable for the interval-valued intuitionistic fuzzy environment. These facts clearly show the advantages and effectiveness of our method.

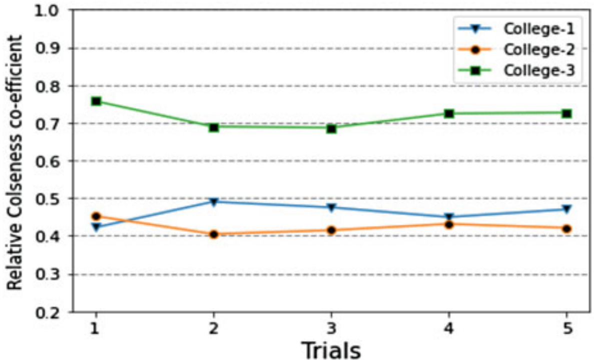

Fig. 16.5 Ranking order of the alternatives under Wei's distance

7 Conclusion

In this chapter, we have introduced the concept of TrP_yFNs and discussed some elementary propositions and relevant theorem on TrP_yFNs. Then, we have defined new distance functions for TrP_yFNs that use to measure relative closeness coefficient. Based on this TrP_yFN, we proposed the trapezoidal Pythagorean fuzzy weighted arithmetic (TrP_yFWA) operator. Then, we have discussed the idempotency, boundedness, and monotonicity properties of the proposed aggregation operator. Subsequently, by utilizing defined aggregation operator and distance functions, we developed an improved TOPSIS strategy for solving a numerical problem under trapezoidal Pythagorean fuzzy environment. Lastly, a numerical example was given to demonstrate the defined TOPSIS method. Finally, sensitivity analysis and comparison analysis were presented to the reliability and efficiency of the proposed technique.

In the future, we can introduce several operational laws and various MCDM/MCGDM techniques to grab the uncertainties in the trapezoidal Pythagorean fuzzy environment in a more rigorous way. Furthermore, researchers can also apply the concept of TrP_yFN in various research areas such as cloud computing, mobile computing problems, pattern recognition problems, big-data analysis, diagnosis problems, realistic mathematical modeling, etc. From this chapter, we can say that the newly defined TrP_yFN is another fuzzy number to tackle the uncertainty and vagueness theory.

Conflict of interest: The authors have no conflict of interest in this research article.
Ethical statement: The authors do not contain any studies with human participants or animals performed by any of the authors.
Authors contributions: All the authors contributed equally to this article.
Data availability statement: This is not applicable for this article since we did not provide any kind of real data in this article.

Acknowledgments In this article, the study of Tipu Sultan Haque is funded by Council of Scientific & Industrial Research (CSIR), India (File no.-08/003(0136)/2019-EMR-I).

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Chapter 17
Identification and Classification of Prioritized Aczel-Alsina Aggregation Operators Based on Complex Intuitionistic Fuzzy Information and Their Applications in Decision-Making Problem

Kifayat Ullah, Mehwish Sarfraz, Maria Akram, and Zeeshan Ali

1 Introduction

Pattern recognition, clustering analysis, artificial intelligence, and decision-making techniques play a very important and critical role in the environment of awkward and unreliable information. The theory of the MADM technique is one of the important subparts of the decision-making strategy, which is used for evaluating the best or finest decision from the collection of preferences. A lot of complications have occurred when expertly using classical information during decision-making procedures because, in the case of classical information, we have only two possibilities such as zero or one. Therefore, Zadeh [1] examined a well-known and valid idea of the fuzzy set (FS), where the theory of FS has covered more than two possibilities such as zero, one, and from $[0, 1]$. Various individuals have utilized the theory of FS in different fields, but in various situations, the theory of FS has failed. Therefore, Atanassov [2] derived the intuitionistic FS (IFS), which covered the truth and falsity grades with a condition that the sum of the truth grade and falsity grade must be contained in the unit interval. Furthermore, Garg and Rani [3] discovered the theory of distances measures, Jia and Wang [4] derived the Choquet integral aggregation operators, Gohain et al. [5] evaluated the similarity measures, Ecer [6] derived the MAIRCA technique, Panda and Nagwani [7] exposed the topic modeling under the presence of IFSs, Gohain et al. [8] also examined the distance measures, Jebadass and Balasubramaniam [9] evaluated the enhancement of color

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image under the presence of IFSs, and finally, Gu et al. [10] derived the theory of risk assessment for IFSs.

Because handling two-dimensional information at once is so crucial and difficult, using the phase term in the context of truth grade is a very difficult challenge for students. The aforementioned query has been brought up by several academic's numerous times. Additionally, there are numerous instances where two-dimensional information is involved. Therefore, the major idea of complex FS (CFS) was invented by Ramot et al. [11], where the CFS contained only one grade in the form of a complex number whose real and unreal parts belong to the unit interval. Various individuals have utilized the theory of CFS in different fields, but in various situations, the theory of CFS has failed. Therefore, Alkouri and Salleh [12] extended or modified the theory of complex IFS (CIFS). Furthermore, Garg and Rani [13] derived the information measures, Garg and Rani [14] discovered the theory of correlation coefficient measures, and Rajareega et al. [15] exposed the theory of weighted distance measures.

Prioritized aggregation operators are very famous and reliable because they can help us to aggregate the collection of information into a singleton set. Furthermore, the derived theory of Aczel and Alsina [16] has received valuable and dominant attention from many scholars. Moreover, Senapati et al. [17] derived the theory of AA aggregation operators (AAAOs) for IFSs, Senapati et al. [18] exposed the theory of geometric AAAOs for IFSs, and finally, Mahmood et al. [19] examined the theory of AAAOs for CIFSs. Moreover, prioritized aggregation operators (PAOs) were examined by Yager [20] in 2008. Additionally, Yu and Xu [21] derived the idea of PAOs for IFSs. Many scholars have derived different types of operators based on fuzzy set and their extensions which is very awkward. It can be highly difficult and unclear for scholars to combine any two different types of operators or measurements using CIF data. The questions above are extremely technical and pertinent, because it is a very difficult work for scholars to provide the answer to the aforementioned facts. Therefore, the major theme of this analysis is listed below:

1. To derive the theory of CIFPAAA, CIFPAAOA, CIFPAAG, and CIFPAAOG operators
2. To explore various properties and special cases of the derived work
3. To expose an MADM technique under the consideration of derived operators
4. To illustrate various examples for determining the comparison between proposed and existing operators to show the supremacy and validity of the invented theory

This manuscript is summarized or constructed in the following ways: In Sect. 2, we revised the theory of CIFSs and their operational laws. In Sect. 3, we proposed the CIFPAAA, CIFPAAOA, CIFPAAG, and CIFPAAOG operators and their properties. In Sect. 4, we exposed an MADM technique under the consideration of derived operators. In Sect. 5, we illustrated various examples for determining the comparison between proposed and existing operators to show the supremacy and validity of the invented theory. Some concluding remarks are given in Sect. 6.

2 Preliminaries

Here, our major theme is to revise the idea of CIFS and their operational laws.

Definition 1 [12] The CIFS in X represented by β is given by

$$\beta = \left\{ \left(\gamma_\beta (\xi)\, e^{i2\pi(\theta_\gamma(\xi))}, \delta_\beta (\xi)\, e^{i2\pi(\theta_\delta(\xi))} \right) : \xi \in X \right\} \tag{17.1}$$

where $\gamma_\beta (\xi)\, e^{i2\pi(\theta_\gamma(\xi))}$ and $\delta_\beta (\xi)\, e^{i2\pi(\theta_\delta(\xi))}$ represented the grades of the truth and falsity information with a characteristic: $0 \le \gamma_\beta(\xi) + \delta_\beta(\xi) \le 1$ and $0 \le \theta_\gamma(\xi) + \theta_\delta(\xi) \le 1$. Furthermore, we derive the theory of neutral grade $\pi_{CF}(\xi) = \pi_{RP}(\xi)\, e^{i2\pi(\pi_{IP}(\xi))} = \left(1 - \left(\gamma_\beta(\xi) + \delta_\beta(\xi)\right)\right) e^{i2\pi(1-(\theta_\gamma(\xi)+\theta_\delta(\xi)))}$ with simple $\beta_j = \left(\gamma_{\beta_j} e^{i2\pi\left(\theta_{\gamma_j}\right)}, \delta_{\beta_j} e^{i2\pi\left(\theta_{\delta_j}\right)} \right), j = 1, 2, \ldots, \vartheta$.

Definition 2 [19] For any CIFN $\beta_j = \left(\gamma_{\beta_j} e^{i2\pi\left(\theta_{\gamma_j}\right)}, \delta_{\beta_j} e^{i2\pi\left(\theta_{\delta_j}\right)} \right), j = 1$, we have stated the score and accuracy functions, such that

$$Sco(\beta_1) = \frac{1}{2}\left(\gamma_{\beta_1} + \theta_{\gamma_1} - \delta_{\beta_1} - \theta_{\delta_1}\right) \in [-1, 1] \tag{17.2}$$

$$Acc(\beta_1) = \frac{1}{2}\left(\gamma_{\beta_1} + \theta_{\gamma_1} + \delta_{\beta_1} + \theta_{\delta_1}\right) \in [0, 1] \tag{17.3}$$

For the above theory, we have the following characteristics, such as: when $Sco(\beta_1) < Sco(\beta_2) \Rightarrow \beta_1 < \beta_2$; when $Sco(\beta_1) = Sco(\beta_2) \Rightarrow \beta_1 = \beta_2$; when $Acc(\beta_1) < Acc(\beta_2) \Rightarrow \beta_1 < \beta_2$; when $Acc(\beta_1) = Acc(\beta_2) \Rightarrow \beta_1 = \beta_2$.

Definition 3 [19] For any two CIFNs $\beta_j = \left(\gamma_{\beta_j} e^{i2\pi\left(\theta_{\gamma_j}\right)}, \delta_{\beta_j} e^{i2\pi\left(\theta_{\delta_j}\right)} \right), j = 1, 2$, we have stated the Aczel-Alsina operational laws, such that

$$\beta_1 \oplus \beta_2 = \begin{pmatrix} \left(1 - e^{-\left(\left(-\ln\left(1-\gamma_{\beta_1}\right)\right)^\eta + \left(-\ln\left(1-\gamma_{\beta_2}\right)\right)^\eta\right)^{\frac{1}{\eta}}}\right) \\ e^{2\pi i\left(1-e^{-\left(\left(-\ln\left(1-\theta_{\gamma_1}\right)\right)^\eta + \left(-\ln\left(1-\theta_{\gamma_2}\right)\right)^\eta\right)^{\frac{1}{\eta}}}\right)}, \\ \left(e^{-\left(\left(-\ln\left(\delta_{\beta_1}\right)\right)^\eta + \left(-\ln\left(\delta_{\beta_2}\right)\right)^\eta\right)^{1/\eta}}\right) \\ e^{2\pi i\left(e^{-\left(\left(-\ln\left(\theta_{\delta_1}\right)\right)^\eta + \left(-\ln\left(\theta_{\delta_2}\right)\right)^\eta\right)^{1/\eta}}\right)} \end{pmatrix} \tag{17.4}$$

$$\beta_1 \otimes \beta_2 = \begin{pmatrix} \left(e^{-\left(\left(-\ln\left(\gamma_{\beta_1}\right)\right)^{\eta}+\left(-\ln\left(\gamma_{\beta_2}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right) \\ e^{2\pi i\left(e^{-\left(\left(-\ln\left(\theta_{\delta_1}\right)\right)^{\eta}+\left(-\ln\left(\theta_{\delta_2}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right)}, \\ \left(1-e^{-\left(\left(-\ln\left(1-\delta_{\beta_1}\right)\right)^{\eta}+\left(-\ln\left(1-\delta_{\beta_2}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right) \\ e^{2\pi i\left(1-e^{-\left(\left(-\ln\left(1-\theta_{\gamma_1}\right)\right)^{\eta}+\left(-\ln\left(1-\theta_{\gamma_2}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right)} \end{pmatrix} \tag{17.5}$$

$$\Phi\beta_1 = \begin{pmatrix} \left(1-e^{-\left(\Phi\left(-\ln\left(1-\gamma_{\beta_1}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right) e^{2\pi i\left(1-e^{-\left(\Phi\left(-\ln\left(1-\theta_{\gamma_1}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right)} \\ \left(e^{-\left(\Phi\left(-\ln\left(\delta_{\beta_1}\right)\right)^{\eta}\right)^{1/\eta}}\right) e^{2\pi i\left(e^{-\left(\Phi\left(-\ln\left(\theta_{\delta_1}\right)\right)^{\eta}\right)^{1/\eta}}\right)}, \end{pmatrix} \tag{17.6}$$

$$\beta_1^{\Phi} = \begin{pmatrix} \left(e^{-\left(\Phi\left(-\ln\left(\gamma_{\beta_1}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right) e^{2\pi i\left(e^{-\left(\Phi\left(-\ln\left(\theta_{\gamma_1}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right)}, \\ \left(1-e^{-\left(\Phi\left(-\ln\left(1-\delta_{\beta_1}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right) e^{2\pi i\left(1-e^{-\left(\Phi\left(-\ln\left(1-\theta_{\delta_1}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right)} \end{pmatrix} \tag{17.7}$$

3 Prioritized Aczel-Alsina Aggregation Operators for CIFSs

In this section, we examine the theory of CIFPAAA, CIFPAAOA, CIFPAAG, CIFPAAOG operators and their valuable properties.

Definition 4 Here, we expose the theory of the CIFPAAA operator for any collection of CIFNs $\beta_j = \left(\gamma_{\beta_j} e^{i2\pi\left(\theta_{\gamma_j}\right)}, \delta_{\beta_j} e^{i2\pi\left(\theta_{\delta_j}\right)}\right)$, $j = 1, 2, \ldots, \vartheta$, we have

$$CIFPAAA\left(\beta_1, \beta_2, \ldots \beta_{\vartheta}\right) = \oplus_{j=1}^{\vartheta} \frac{T_j}{\sum_{j=1}^{\vartheta} T_j} \beta_j \tag{17.8}$$

Theorem 1 To use the theory in Eq. (17.8), we prove that they again give us a CIFN, such as

$$
\begin{aligned}
&CIFPAAA\left(\beta_1, \beta_2, \ldots \beta_\vartheta\right) \\
&= \left(\begin{array}{c}
\left(1 - e^{-\left(\sum_{j=1}^{\vartheta} \frac{T_j}{\sum_{j=1}^{k+1} T_j} \left(-\ln\left(1-\gamma_{\beta_j}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right) e^{2\pi i \left(1-e^{-\left(\sum_{j=1}^{\vartheta} \frac{T_j}{\sum_{j=1}^{k+1} T_j} \left(-\ln\left(1-\theta_{\gamma_j}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right)}, \\
\left(e^{-\left(\sum_{j=1}^{\vartheta} \frac{T_j}{\sum_{j=1}^{k+1} T_j} \left(-\ln\left(\delta_{\beta_j}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right) e^{2\pi i \left(e^{-\left(\sum_{j=1}^{\vartheta} \frac{T_j}{\sum_{j=1}^{k+1} T_j} \left(-\ln\left(\theta_{\delta_j}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right)}
\end{array} \right)
\end{aligned}
\tag{17.9}
$$

Proof We aim to derive the theory in Eq. (17.9), such as, if $\vartheta = 2$, then

$$
\begin{aligned}
&\frac{T_1}{\sum_{j=1}^{\vartheta} T_j} \beta_1 \\
&= \left(\begin{array}{c}
\left(1 - e^{-\left(\frac{T_1}{\sum_{j=1}^{\vartheta} T_j} \left(-\ln\left(1-\gamma_{\beta_1}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right) e^{2\pi i \left(1-e^{-\left(\frac{T_1}{\sum_{j=1}^{\vartheta} T_j} \left(-\ln\left(1-\theta_{\gamma_1}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right)}, \\
\left(e^{-\left(\frac{T_1}{\sum_{j=1}^{\vartheta} T_j} \left(-\ln\left(\delta_{\beta_1}\right)\right)^{\eta}\right)^{1/\eta}}\right) e^{2\pi i \left(e^{-\left(\frac{T_1}{\sum_{j=1}^{\vartheta} T_j} \left(-\ln\left(\theta_{\delta_1}\right)\right)^{\eta}\right)^{1/\eta}}\right)}
\end{array} \right)
\end{aligned}
$$

$$\frac{T_2}{\sum_{j=1}^{\vartheta} T_j}\beta_2$$

$$= \left(\begin{array}{c} \left(1-e^{-\left(\frac{T_2}{\sum_{j=1}^{\vartheta} T_j}\left(-\ln\left(1-\gamma_{\beta_2}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right) e^{2\pi i\left(1-e^{-\left(\frac{T_2}{\sum_{j=1}^{\vartheta} T_j}\left(-\ln\left(1-\theta_{\gamma_2}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right)}, \\ \left(e^{-\left(\frac{T_2}{\sum_{j=1}^{\vartheta} T_j}\left(-\ln\left(\delta_{\beta_2}\right)\right)^{\eta}\right)^{1/\eta}}\right) e^{2\pi i\left(e^{-\left(\frac{T_2}{\sum_{j=1}^{\vartheta} T_j}\left(-\ln\left(\theta_{\delta_2}\right)\right)^{\eta}\right)^{1/\eta}}\right)} \end{array} \right)$$

$$CIFPAAA\left(\beta_1, \beta_2\right) = \frac{T_1}{\sum_{j=1}^{2} T_j}\beta_1 \oplus \frac{T_2}{\sum_{j=1}^{2} T_j}\beta_2$$

$$= \left(\begin{array}{c} \left(1-e^{-\left(\frac{T_1}{\sum_{j=1}^{\vartheta} T_j}\left(-\ln\left(1-\gamma_{\beta_1}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right) e^{2\pi i\left(1-e^{-\left(\frac{T_1}{\sum_{j=1}^{\vartheta} T_j}\left(-\ln\left(1-\theta_{\gamma_1}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right)}, \\ \left(e^{-\left(\frac{T_1}{\sum_{j=1}^{\vartheta} T_j}\left(-\ln\left(\delta_{\beta_1}\right)\right)^{\eta}\right)^{1/\eta}}\right) e^{2\pi i\left(e^{-\left(\frac{T_1}{\sum_{j=1}^{\vartheta} T_j}\left(-\ln\left(\theta_{\delta_1}\right)\right)^{\eta}\right)^{1/\eta}}\right)} \end{array} \right)$$

$$\oplus \left(\begin{array}{c} \left(1-e^{-\left(\frac{T_2}{\sum_{j=1}^{\vartheta} T_j}\left(-\ln\left(1-\gamma_{\beta_2}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right) e^{2\pi i\left(1-e^{-\left(\frac{T_2}{\sum_{j=1}^{\vartheta} T_j}\left(-\ln\left(1-\theta_{\gamma_2}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right)}, \\ \left(e^{-\left(\frac{T_2}{\sum_{j=1}^{\vartheta} T_j}\left(-\ln\left(\delta_{\beta_2}\right)\right)^{\eta}\right)^{1/\eta}}\right) e^{2\pi i\left(e^{-\left(\frac{T_2}{\sum_{j=1}^{\vartheta} T_j}\left(-\ln\left(\theta_{\delta_2}\right)\right)^{\eta}\right)^{1/\eta}}\right)} \end{array} \right)$$

$$= \left(\begin{array}{c} \left(1 - e^{-\left(\frac{T_1}{\sum_{j=1}^{2} T_j} \left(-\ln\left(1-\gamma_{\beta_1}\right)\right)^{\eta} + \frac{T_2}{\sum_{j=1}^{2} T_j} \left(-\ln\left(1-\gamma_{\beta_2}\right)\right)^{\eta} \right)^{\frac{1}{\eta}}} \right) \\ e^{2\pi i \left(1 - e^{-\left(\frac{T_1}{\sum_{j=1}^{2} T_j} \left(-\ln\left(1-\theta_{\gamma_1}\right)\right)^{\eta} + \frac{T_2}{\sum_{j=1}^{2} T_j} \left(-\ln\left(1-\theta_{\gamma_2}\right)\right)^{\eta} \right)^{\frac{1}{\eta}}} \right)}, \\ \left(e^{-\left(\frac{T_1}{\sum_{j=1}^{2} T_j} \left(-\ln\left(\delta_{\beta_1}\right)\right)^{\eta} + \frac{T_2}{\sum_{j=1}^{2} T_j} \left(-\ln\left(\delta_{\beta_2}\right)\right)^{\eta} \right)^{1/\eta}} \right) \\ e^{2\pi i \left(e^{-\left(\frac{T_1}{\sum_{j=1}^{2} T_j} \left(-\ln\left(\theta_{\delta_1}\right)\right)^{\eta} + \frac{T_2}{\sum_{j=1}^{2} T_j} \left(-\ln\left(\theta_{\delta_2}\right)\right)^{\eta} \right)^{1/\eta}} \right)} \end{array} \right)$$

$$= \left(\begin{array}{c} \left(1 - e^{-\left(\sum_{j=1}^{\vartheta} \frac{T_j}{\sum_{j=1}^{2} T_j} \left(-\ln\left(1-\gamma_{\beta_j}\right)\right)^{\eta} \right)^{\frac{1}{\eta}}} \right) \\ e^{2\pi i \left(1 - e^{-\left(\sum_{j=1}^{\vartheta} \frac{T_j}{\sum_{j=1}^{2} T_j} \left(-\ln\left(1-\theta_{\gamma_j}\right)\right)^{\eta} \right)^{\frac{1}{\eta}}} \right)}, \\ \left(e^{-\left(\sum_{j=1}^{\vartheta} \frac{T_j}{\sum_{j=1}^{2} T_j} \left(-\ln\left(\delta_{\beta_j}\right)\right)^{\eta} \right)^{1/\eta}} \right) \\ e^{2\pi i \left(e^{-\left(\sum_{j=1}^{\vartheta} \frac{T_j}{\sum_{j=1}^{2} T_j} \left(-\ln\left(\theta_{\delta_j}\right)\right)^{\eta} \right)^{1/\eta}} \right)} \end{array} \right)$$

For the value of $\vartheta = 2$, we get the correct answer, further, we assume that the data in Eq. (17.9) is also valid for $\vartheta = k$, such as

$$CIFPAAA\left(\beta_1, \beta_2, \ldots \beta_k\right) = \oplus_{j=1}^{k} \frac{T_j}{\sum_{j=1}^{k} T_j}\left(\beta_j\right)$$

$$= \left(\begin{array}{c} \left(1 - e^{-\left(\sum_{j=1}^{k} \frac{T_j}{\sum_{j=1}^{k} T_j}\left(-\ln\left(1-\gamma_{\beta_j}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right) e^{2\pi i\left(1-e^{-\left(\sum_{j=1}^{k} \frac{T_j}{\sum_{j=1}^{k} T_j}\left(-\ln\left(1-\theta_{\gamma_j}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right)}, \\ \left(e^{-\left(\sum_{j=1}^{k} \frac{T_j}{\sum_{j=1}^{k} T_j}\left(-\ln\left(\delta_{\beta_j}\right)\right)^{\eta}\right)^{1/\eta}}\right) e^{2\pi i\left(e^{-\left(\sum_{j=1}^{k} \frac{T_j}{\sum_{j=1}^{k} T_j}\left(-\ln\left(\theta_{\delta_j}\right)\right)^{\eta}\right)^{1/\eta}}\right)} \end{array} \right)$$

Then, we expose that the data in Eq. (17.9) is also valid for $\vartheta = k + 1$, such as

$$CIFPAAA\left(\beta_1, \beta_2, \ldots \beta_{k+1}\right) = \oplus_{j=1}^{k} \frac{T_j}{\sum_{j=1}^{k+1} T_j}\left(\beta_j\right) \oplus \frac{T_{k+1}}{\sum_{j=1}^{k+1} T_j}\left(\beta_{k+1}\right)$$

$$= \left(\begin{array}{c} \left(1 - e^{-\left(\sum_{j=1}^{k} \frac{T_j}{\sum_{j=1}^{k+1} T_j}\left(-\ln\left(1-\gamma_{\beta_j}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right) e^{2\pi i\left(1-e^{-\left(\sum_{j=1}^{k} \frac{T_j}{\sum_{j=1}^{k+1} T_j}\left(-\ln\left(1-\theta_{\gamma_j}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right)}, \\ \left(e^{-\left(\sum_{j=1}^{k} \frac{T_j}{\sum_{j=1}^{k+1} T_j}\left(-\ln\left(\delta_{\beta_j}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right) e^{2\pi i\left(e^{-\left(\sum_{j=1}^{k} \frac{T_j}{\sum_{j=1}^{k+1} T_j}\left(-\ln\left(\theta_{\delta_j}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right)} \end{array} \right)$$

$$\oplus\left(\begin{array}{c}\left(1-e^{-\left(\frac{T_{k+1}}{\sum_{j=1}^{k+1}T_j}\left(-\ln\left(1-\gamma_{\beta_{k+1}}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right)\\ e^{2\pi i\left(1-e^{-\left(\frac{T_{k+1}}{\sum_{j=1}^{k+1}T_j}\left(-\ln\left(1-\theta_{\gamma_{k+1}}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right)},\\ \left(e^{-\left(\frac{T_{k+1}}{\sum_{j=1}^{k+1}T_j}\left(-\ln\left(1-\delta_{\beta_{k+1}}\right)\right)^{\eta}\right)^{1/\eta}}\right)\\ e^{2\pi i\left(e^{-\left(\frac{T_{k+1}}{\sum_{j=1}^{k+1}T_j}\left(-\ln\left(1-\theta_{\delta_{k+1}}\right)\right)^{\eta}\right)^{1/\eta}}\right)}\end{array}\right)$$

$$=\left(\begin{array}{c}\left(1-e^{-\left(\sum_{j=1}^{k+1}\frac{T_j}{\sum_{j=1}^{k+1}T_j}\left(-\ln\left(1-\gamma_{\beta_j}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right)\\ e^{2\pi i\left(1-e^{-\left(\sum_{j=1}^{k+1}\frac{T_j}{\sum_{j=1}^{k+1}T_j}\left(-\ln\left(1-\theta_{\gamma_j}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right)},\\ \left(e^{-\left(\sum_{j=1}^{k+1}\frac{T_j}{\sum_{j=1}^{k+1}T_j}\left(-\ln\left(\delta_{\beta_j}\right)\right)^{\eta}\right)^{1/\eta}}\right)\\ e^{2\pi i\left(e^{-\left(\sum_{j=1}^{k+1}\frac{T_j}{\sum_{j=1}^{k+1}T_j}\left(-\ln\left(\theta_{\delta_j}\right)\right)^{\eta}\right)^{1/\eta}}\right)}\end{array}\right)$$

Hence, we obtain our required result which is valid for all positive integers.

Preposition 1 (Idempotency) If $\beta_j = \left(\gamma_\beta e^{i2\pi(\theta_\gamma)}, \delta_\beta e^{i2\pi(\theta_\delta)}\right) = \beta$, then

$$CIFPAAA\,(\beta_1, \beta_2, \ldots, \beta_\vartheta) = \beta \tag{17.10}$$

Preposition 2 (Monotonicity) If $\beta_j \leq \beta_j^*$, then

$$CIFPAAA(\beta_1, \beta_2, \ldots, \beta_\vartheta) \leq CIFPAAA\left(\beta_1^*, \beta_2^*, \ldots, \beta_\vartheta^*\right) \tag{17.11}$$

Preposition 3 (Boundedness) If $\beta_j^- = \left(\min\limits_j \gamma_{\beta_j} e^{i2\pi\left(\min\limits_j \theta_{\gamma_j}\right)}, \max\limits_j \delta_{\beta_j} e^{i2\pi\left(\max\limits_j \theta_{\delta_j}\right)}\right)$, and $\beta_j^+ = \left(\max\limits_j \gamma_{\beta_j} e^{i2\pi\left(\max\limits_j \theta_{\gamma_j}\right)}, \min\limits_j \delta_{\beta_j} e^{i2\pi\left(\min\limits_j \theta_{\delta_j}\right)}\right)$, $j = 1, 2, \ldots, \vartheta$, then

$$\beta_j^- \leq CIFPAAA(\beta_1, \beta_2, \ldots, \beta_\vartheta) \leq \beta_j^+ \tag{17.12}$$

Definition 5 Here, we expose the theory of the CIFPAAOA operator for any collection of CIFNs $\beta_j = \left(\gamma_{\beta_j} e^{i2\pi\left(\theta_{\gamma_j}\right)}, \delta_{\beta_j} e^{i2\pi\left(\theta_{\delta_j}\right)}\right)$, $j = 1, 2, \ldots, \vartheta$, we have

$$CIFPAAOA(\beta_1, \beta_2, \ldots \beta_\vartheta) = \oplus_{j=1}^{\vartheta} \frac{T_j}{\sum_{j=1}^{\vartheta} T_j} \beta_{o(j)} \tag{17.13}$$

Where, $o(j) \leq o(j-1)$.

Theorem 2 To use the theory in Eq. (17.13), we prove that they again give us a CIFN, such as

$$CIFPAAOA(\beta_1, \beta_2, \ldots \beta_\vartheta)$$

$$= \left(\begin{array}{c} \left(1 - e^{-\left(\sum_{j=1}^{\vartheta} \frac{T_j}{\sum_{j=1}^{k+1} T_j}\left(-\ln\left(1-\gamma_{\beta_{o(j)}}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right) \\ e^{2\pi i\left(1-e^{-\left(\sum_{j=1}^{\vartheta} \frac{T_j}{\sum_{j=1}^{k+1} T_j}\left(-\ln\left(1-\theta_{\gamma_{o(j)}}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right)}, \\ \left(e^{-\left(\sum_{j=1}^{\vartheta} \frac{T_j}{\sum_{j=1}^{k+1} T_j}\left(-\ln\left(\delta_{\beta_{o(j)}}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right) \\ e^{2\pi i\left(e^{-\left(\sum_{j=1}^{\vartheta} \frac{T_j}{\sum_{j=1}^{k+1} T_j}\left(-\ln\left(\theta_{\delta_{o(j)}}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right)} \end{array} \right) \tag{17.14}$$

Preposition 4 (Idempotency) If $\beta_j = \left(\gamma_\beta e^{i2\pi(\theta_\gamma)}, \delta_\beta e^{i2\pi(\theta_\delta)}\right) = \beta$, then

$$CIFPAAOA\,(\beta_1, \beta_2, \ldots, \beta_\vartheta) = \beta \tag{17.15}$$

Preposition 5 (Monotonicity) If $\beta_j \leq \beta_j^*$, then

$$CIFPAAOA\,(\beta_1, \beta_2, \ldots, \beta_\vartheta) \leq CIFPAAOA\left(\beta_1^*, \beta_2^*, \ldots, \beta_\vartheta^*\right) \tag{17.16}$$

Preposition 6 (Boundedness) If $\beta_j^- = \left(\min_j \gamma_{\beta_j} e^{i2\pi\left(\min_j \theta_{\gamma_j}\right)}, \max_j \delta_{\beta_j} e^{i2\pi\left(\max_j \theta_{\delta_j}\right)}\right)$, and $\beta_j^+ = \left(\max_j \gamma_{\beta_j} e^{i2\pi\left(\max_j \theta_{\gamma_j}\right)}, \min_j \delta_{\beta_j} e^{i2\pi\left(\min_j \theta_{\delta_j}\right)}\right)$, $j = 1, 2, \ldots, \vartheta$, then

$$\beta_j^- \leq CIFPAAOA\,(\beta_1, \beta_2, \ldots, \beta_\vartheta) \leq \beta_j^+ \tag{17.17}$$

Definition 6 Here, we expose the theory of the CIFPAAG operator for any collection of CIFNs $\beta_j = \left(\gamma_{\beta_j} e^{i2\pi\left(\theta_{\gamma_j}\right)}, \delta_{\beta_j} e^{i2\pi\left(\theta_{\delta_j}\right)}\right)$, $j = 1, 2, \ldots, \vartheta$, we have

$$CIFPAAG\,(\beta_1, \beta_2, \ldots \beta_\vartheta) = \otimes_{j=1}^{\vartheta} \beta_j^{\frac{T_j}{\sum_{j=1}^{\vartheta} T_j}} \tag{17.18}$$

Theorem 3 To use the theory in Eq. (17.18), we prove that they again give us a CIFN, such as

$$CIFPAAG\,(\beta_1, \beta_2, \ldots \beta_\vartheta)$$
$$= \left(\begin{array}{c} \left(e^{-\left(\sum_{j=1}^{\vartheta} \frac{T_j}{\sum_{j=1}^{k+1} T_j}\left(-\ln\left(\gamma_{\beta_j}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}} \right) e^{2\pi i \left(e^{-\left(\sum_{j=1}^{\vartheta} \frac{T_j}{\sum_{j=1}^{k+1} T_j}\left(-\ln\left(\theta_{\gamma_j}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}} \right)}, \\ \left(1 - e^{-\left(\sum_{j=1}^{\vartheta} \frac{T_j}{\sum_{j=1}^{k+1} T_j}\left(-\ln\left(1-\delta_{\beta_j}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}} \right) e^{2\pi i \left(1 - e^{-\left(\sum_{j=1}^{\vartheta} \frac{T_j}{\sum_{j=1}^{k+1} T_j}\left(-\ln\left(1-\theta_{\delta_j}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}} \right)} \end{array} \right) \tag{17.19}$$

Preposition 7 (Idempotency) If $\beta_j = \left(\gamma_\beta e^{i2\pi(\theta_\gamma)}, \delta_\beta e^{i2\pi(\theta_\delta)}\right) = \beta$, then

$$CIFPAAG(\beta_1, \beta_2, \ldots, \beta_\vartheta) = \beta \tag{17.20}$$

Preposition 8 (Monotonicity) If $\beta_j \leq \beta_j^*$, then

$$CIFPAAG(\beta_1, \beta_2, \ldots, \beta_\vartheta) \leq CIFPAAG\left(\beta_1^*, \beta_2^*, \ldots, \beta_\vartheta^*\right) \tag{17.21}$$

Preposition 9 (Boundedness) If $\beta_j^- = \left(\min_j \gamma_{\beta_j} e^{i2\pi\left(\min_j \theta_{\gamma_j}\right)}, \max_j \delta_{\beta_j} e^{i2\pi\left(\max_j \theta_{\delta_j}\right)}\right)$, and $\beta_j^+ = \left(\max_j \gamma_{\beta_j} e^{i2\pi\left(\max_j \theta_{\gamma_j}\right)}, \min_j \delta_{\beta_j} e^{i2\pi\left(\min_j \theta_{\delta_j}\right)}\right)$, $j = 1, 2, \ldots, \vartheta$, then

$$\beta_j^- \leq CIFPAAG(\beta_1, \beta_2, \ldots, \beta_\vartheta) \leq \beta_j^+ \tag{17.22}$$

Definition 7 Here, we expose the theory of the CIFPAAOG operator for any collection of CIFNs $\beta_j = \left(\gamma_{\beta_j} e^{i2\pi\left(\theta_{\gamma_j}\right)}, \delta_{\beta_j} e^{i2\pi\left(\theta_{\delta_j}\right)}\right)$, $j = 1, 2, \ldots, \vartheta$, we have

$$CIFPAAOG(\beta_1, \beta_2, \ldots \beta_\vartheta) = \otimes_{j=1}^{\vartheta} \beta_{o(j)}^{\frac{T_j}{\sum_{j=1}^{\vartheta} T_j}} \tag{17.23}$$

Where, $o(j) \leq o(j-1)$.

Theorem 4 To use the theory in Eq. (17.23), we prove that they again give us a CIFN, such as

$$CIFPAAOG(\beta_1, \beta_2, \ldots \beta_\vartheta) = \left(\begin{array}{c} \left(e^{-\left(\sum_{j=1}^{\vartheta} \frac{T_j}{\sum_{j=1}^{k+1} T_j} \left(-\ln\left(\gamma_{\beta_{o(j)}}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}} \right) e^{2\pi i \left(e^{-\left(\sum_{j=1}^{\vartheta} \frac{T_j}{\sum_{j=1}^{k+1} T_j} \left(-\ln\left(\theta_{\gamma_{o(j)}}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}} \right)}, \\ \left(1 - e^{-\left(\sum_{j=1}^{\vartheta} \frac{T_j}{\sum_{j=1}^{k+1} T_j} \left(-\ln\left(1-\delta_{\beta_{o(j)}}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}} \right) e^{2\pi i \left(1-e^{-\left(\sum_{j=1}^{\vartheta} \frac{T_j}{\sum_{j=1}^{k+1} T_j} \left(-\ln\left(1-\theta_{\delta_{o(j)}}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}} \right)} \end{array} \right) \tag{17.24}$$

Preposition 10 (Idempotency) If $\beta_j = \left(\gamma_\beta e^{i2\pi(\theta_\gamma)}, \delta_\beta e^{i2\pi(\theta_\delta)}\right) = \beta$, then

$$CIFPAAOG(\beta_1, \beta_2, \ldots, \beta_\vartheta) = \beta \tag{17.25}$$

Preposition 11 (Monotonicity) If $\beta_j \leq \beta_j^*$, then

$$CIFPAAOG(\beta_1, \beta_2, \ldots, \beta_\vartheta) \leq CIFPAAOG\left(\beta_1^*, \beta_2^*, \ldots, \beta_\vartheta^*\right) \tag{17.26}$$

Preposition 12 (Boundedness) If $\beta_j^- = \left(\min_j \gamma_{\beta_j} e^{i2\pi\left(\min_j \theta_{\gamma_j}\right)}, \max_j \delta_{\beta_j} e^{i2\pi\left(\max_j \theta_{\delta_j}\right)}\right)$, and $\beta_j^+ = \left(\max_j \gamma_{\beta_j} e^{i2\pi\left(\max_j \theta_{\gamma_j}\right)}, \min_j \delta_{\beta_j} e^{i2\pi\left(\min_j \theta_{\delta_j}\right)}\right)$, $j = 1, 2, \ldots, \vartheta$, then

$$\beta_j^- \leq CIFPAAOG(\beta_1, \beta_2, \ldots, \beta_\vartheta) \leq \beta_j^+ \tag{17.27}$$

4 MADM Methods for CIFNs

The MADM method is an important component of the decision-making process that is used to assess the best or finest decision from the information gathered. Here, we have shown how the MADM approach works when derived operators for CIFSs are present. We also revealed an MADM method when derived operators were taken into account. Lastly, we provided several examples for comparing the suggested and existing operators in order to demonstrate the superiority and viability of the conceived theory.

Consider the collection of alternatives and their attributes such as $k = \{x_1, x_2, \ldots x_m\}$ and $C = \{c_1, c_2, \ldots, c_n\}$, with a priority degree $c_1 > c_2 > c_3, \ldots, c_n$. To compute a decision matrix $K^q = \left(K_{ij}^q\right)_{mxn}$ with $\beta_j = \left(\gamma_{\beta_j} e^{i2\pi\left(\theta_{\gamma_j}\right)}, \delta_{\beta_j} e^{i2\pi\left(\theta_{\delta_j}\right)}\right)$, $j = 1, 2, \ldots, \vartheta$, noticed that the theory of $\gamma_\beta(\xi) e^{i2\pi(\theta_\gamma(\xi))}$ and $\delta_\beta(\xi) e^{i2\pi(\theta_\delta(\xi))}$ represented the theory of truth and falsity information with a characteristic: $0 \le \gamma_\beta(\xi) + \delta_\beta(\xi) \le 1$ and $0 \le \theta_\gamma(\xi) + \theta_\delta(\xi) \le 1$. Furthermore, we derive the theory of neutral grade $\pi_{CF}(\xi) = \pi_{RP}(\xi) e^{i2\pi(\pi_{IP}(\xi))} = \left(1 - \left(\gamma_\beta(\xi) + \delta_\beta(\xi)\right)\right) e^{i2\pi\left(1 - \left(\theta_\gamma(\xi) + \theta_\delta(\xi)\right)\right)}$. Therefore, here, we compute a decision-making procedure for evaluating the best decision from the collection of preferences, such as:

Step 1 Compute or arrange a decision matrix by including the CIFNs. When we have cost type of data, then we need to evaluate it with the help of the below idea, such as:

$$r_{ij}^q = \begin{cases} k_{ij}^q, \, for \; benefit \; attribute \; c_j \\ k_{ij}^q, \, for \cos t \; attribute \; c_j \end{cases}$$

In the case of benefit, no need to normalize.

Step 2 Aggregate the information in the matrix with the help of the below one, such as

$$CIFPAAA(\beta_1, \beta_2, \ldots \beta_\vartheta) = \left(\begin{array}{c} \left(1 - e^{-\left(\sum_{j=1}^{\vartheta} \frac{T_j}{\sum_{j=1}^{k+1} T_j}\left(-\ln\left(1-\gamma_{\beta_j}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}} \right) e^{2\pi i \left(1 - e^{-\left(\sum_{j=1}^{\vartheta} \frac{T_j}{\sum_{j=1}^{k+1} T_j}\left(-\ln\left(1-\theta_{\gamma_j}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right)}, \\ \left(e^{-\left(\sum_{j=1}^{\vartheta} \frac{T_j}{\sum_{j=1}^{k+1} T_j}\left(-\ln\left(\delta_{\beta_j}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right) e^{2\pi i \left(e^{-\left(\sum_{j=1}^{\vartheta} \frac{T_j}{\sum_{j=1}^{k+1} T_j}\left(-\ln\left(\theta_{\delta_j}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right)} \end{array} \right)$$

Or

$$CIFPAAG(\beta_1, \beta_2, \ldots \beta_\vartheta) = \left(\begin{array}{c} \left(e^{-\left(\sum_{j=1}^{\vartheta} \frac{T_j}{\sum_{j=1}^{k+1} T_j}\left(-\ln\left(\gamma_{\beta_j}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right) e^{2\pi i \left(e^{-\left(\sum_{j=1}^{\vartheta} \frac{T_j}{\sum_{j=1}^{k+1} T_j}\left(-\ln\left(\theta_{\gamma_j}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right)}, \\ \left(1 - e^{-\left(\sum_{j=1}^{\vartheta} \frac{T_j}{\sum_{j=1}^{k+1} T_j}\left(-\ln\left(1-\delta_{\beta_j}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right) e^{2\pi i \left(1 - e^{-\left(\sum_{j=1}^{\vartheta} \frac{T_j}{\sum_{j=1}^{k+1} T_j}\left(-\ln\left(1-\theta_{\delta_j}\right)\right)^{\eta}\right)^{\frac{1}{\eta}}}\right)} \end{array} \right)$$

Step 3 Evaluate the best or finest preference with the help of the below score information, such as

$$Sco(\beta_i) = \gamma_{\gamma_i} - \delta_{\beta_i} \ \ i = 1, 2, \ldots m$$

Step 4 Evaluate the ranking information based on the derived score values.

Furthermore, we aim to expose various numerical examples for showing the reliability and effectiveness of the derived theory.

4.1 Numerical Example

A company wants to launch different types of software in a market. The representation of the different software is stated as follows: $\partial\beta_{AL-1}$, $\partial\beta_{AL-2}$, $\partial\beta_{AL-3}$, $\partial\beta_{AL-4}$, $\partial\beta_{AL-5}$, which are represented as alternatives. The theoretical representation of each alternative is of the form, such as Math Type, MS office, LaTeX, MATLAB, and Mathematica. To choose the best one from the above software, we use the following features or criteria such as β_1, β_2, β_3, and β_4, where the theoretical representation of each criterion follows: the price of the software, the version of the software, and finally, feedback from the users. Therefore, here, we compute a decision-making procedure for evaluating the best decision from the collection of preferences, such as:

Step 1 Compute or arrange a decision matrix (using the information in Table 17.1) by including the CIFNs. When we have cost type of data, then we need to evaluate it with the help of the below idea, such as:

$$r_{ij}^{q} = \begin{cases} k_{ij}^{q}, for\ benefit\ attribute\ c_j \\ k_{ij}^{q}, for\ \cos t\ attribute\ c_j \end{cases}$$

In the case of benefit, no need to normalize, but the data in Table 17.1 is not needed to be normalized.

Step 2 Aggregate the information in the matrix by using the theory of the CIFPAAA operator and CIFPAAG operator, see Table 17.2.

Step 3 Evaluate the best or finest preference with the help of score information, see Table 17.3.

Step 4 Evaluate the ranking information based on the derived score values, see Table 17.4.

The finest decision is $\partial\beta_{AL-3}$ according to the theory of the CIFPAAA operator and CIFPAAG operator. Furthermore, we are doing a comparison between the proposed work and the existing operator under the consideration of the above numerical examples to show the reliability and effectiveness of the derived theory.

Table 17.1 Original decision information matrix

	β_1	β_2	β_3	β_4
$\partial\boldsymbol{\beta_{AL}-1}$	$(0.4e^{i2\pi(0.5)}, 0.3e^{i2\pi(0.2)})$	$(0.41e^{i2\pi(0.51)}, 0.31e^{i2\pi(0.21)})$	$(0.42e^{i2\pi(0.52)}, 0.32e^{i2\pi(0.22)})$	$(0.43e^{i2\pi(0.53)}, 0.33e^{i2\pi(0.23)})$
$\partial\boldsymbol{\beta_{AL}-2}$	$(0.4e^{i2\pi(0.1)}, 0.1e^{i2\pi(0.3)})$	$(0.41e^{i2\pi(0.11)}, 0.11e^{i2\pi(0.31)})$	$(0.42e^{i2\pi(0.12)}, 0.12e^{i2\pi(0.32)})$	$(0.43e^{i2\pi(0.13)}, 0.13e^{i2\pi(0.33)})$
$\partial\boldsymbol{\beta_{AL}-3}$	$(0.8e^{i2\pi(0.7)}, 0.1e^{i2\pi(0.1)})$	$(0.81e^{i2\pi(0.71)}, 0.11e^{i2\pi(0.11)})$	$(0.82e^{i2\pi(0.72)}, 0.12e^{i2\pi(0.12)})$	$(0.83e^{i2\pi(0.73)}, 0.13e^{i2\pi(0.13)})$
$\partial\boldsymbol{\beta_{AL}-4}$	$(0.7e^{i2\pi(0.3)}, 0.1e^{i2\pi(0.2)})$	$(0.71e^{i2\pi(0.31)}, 0.11e^{i2\pi(0.21)})$	$(0.72e^{i2\pi(0.32)}, 0.12e^{i2\pi(0.22)})$	$(0.73e^{i2\pi(0.33)}, 0.13e^{i2\pi(0.23)})$
$\partial\boldsymbol{\beta_{AL}-5}$	$(0.4e^{i2\pi(0.4)}, 0.1e^{i2\pi(0.3)})$	$(0.41e^{i2\pi(0.41)}, 0.11e^{i2\pi(0.31)})$	$(0.42e^{i2\pi(0.42)}, 0.12e^{i2\pi(0.32)})$	$(0.43e^{i2\pi(0.43)}, 0.13e^{i2\pi(0.33)})$

Table 17.2 Aggregated information matrix

	CIFPAAA	*CIFPAAG*
$\partial\boldsymbol{\beta}_{\boldsymbol{AL}-\mathbf{1}}$	$(0.4134e^{i2\pi(0.5134)}, 0.312e^{i2\pi(0.2119)})$	$(0.4121e^{i2\pi(0.5121)}, 0.3135e^{i2\pi(0.2137)})$
$\partial\boldsymbol{\beta}_{\boldsymbol{AL}-\mathbf{2}}$	$(0.40058e^{i2\pi(0.10066)}, 0.10045e^{i2\pi(0.30049)})$	$(0.4005e^{i2\pi(0.10045)}, 0.10066e^{i2\pi(0.30059)})$
$\partial\boldsymbol{\beta}_{\boldsymbol{AL}-\mathbf{3}}$	$(0.8193e^{i2\pi(0.7192)}, 0.1174e^{i2\pi(0.1174)})$	$(0.8177e^{i2\pi(0.7179)}, 0.1198e^{i2\pi(0.1198)})$
$\partial\boldsymbol{\beta}_{\boldsymbol{AL}-\mathbf{4}}$	$(0.7165e^{i2\pi(0.3165)}, 0.1145e^{i2\pi(0.215)})$	$(0.715e^{i2\pi(0.3151)}, 0.1173e^{i2\pi(0.2167)})$
$\partial\boldsymbol{\beta}_{\boldsymbol{AL}-\mathbf{5}}$	$(0.4134e^{i2\pi(0.4134)}, 0.1114e^{i2\pi(0.312)})$	$(0.4121e^{i2\pi(0.4121)}, 0.1114e^{i2\pi(0.3135)})$

Table 17.3 Score matrix

	CIFPAAA	*CIFPAAG*
$\partial\boldsymbol{\beta}_{AL-1}$	0.20015	0.1985
$\partial\boldsymbol{\beta}_{AL-2}$	0.00515	0.00485
$\partial\boldsymbol{\beta}_{AL-3}$	0.6518	0.648
$\partial\boldsymbol{\beta}_{AL-4}$	0.3518	0.3481
$\partial\boldsymbol{\beta}_{AL-5}$	0.20017	0.1981

Table 17.4 Ranking matrix

Methods	Ranking results
CIFPAAA	$\partial\beta_{AL-3} \leq \partial\beta_{AL-4} \leq \partial\beta_{AL-5} \leq \partial\beta_{AL-1} \leq \partial\beta_{AL-2}$
CIFPAAG	$\partial\beta_{AL-3} \leq \partial\beta_{AL-4} \leq \partial\beta_{AL-1} \leq \partial\beta_{AL-5} \leq \partial\beta_{AL-2}$

5 Comparative Analysis

To enhance the capability and worth of the invented operators, we concentrate to derive the comparison between derived work and various existing works, for this, we try to collect various prevailing information such as Senapati et al. [17] derived the theory of AAAOs for IFSs, Senapati et al. [18] exposed the theory of geometric AAAO for IFSs, and finally, Mahmood et al. [19] examined the theory of AAAO for CIFSs. Additionally, Yu and Xu [21] derived the idea of PAOs for IFSs. Under the presence of the information in Table 17.1, the comparative information is available in Table 17.5.

The CIFPAAA operator, the CIFPAAG operator, and $\partial\beta_{AL3}$ in the theory of Mahmood et al. [19], where the best decision is once again the same. Furthermore, we also noticed that the theory of Senapati et al. [17], Senapati et al. [18], and the theory of Yu and Xu [21] have a lot of limitations because of their structure. Senapati et al. [17], Senapati et al. [18], and Yu and Xu [21] derived their theories based on IFS which is a special case of the invented theory.

Hence, our proposed model is massively reliable, and dominant compared to others because the proposed operators are calculated based on CIFSs which is the modified form of FSs, IFSs, and CFSs.

6 Conclusion

The main influence or theme of this manuscript is stated below:

1. With the help of two different structures such as prioritized averaging aggregation operators and Aczel-Alsina t-norm and t-conorm, we exposed the idea of CIFPAAA and CIFPAAOA operators.

Table 17.5 Comparison information matrix

Methods	Score values	Ranking values
Senapati et al. [17]	× × × × × × × × × × × × × ×	× × × × × × × × × × × × × ×
Senapati et al. [18]	× × × × × × × × × × × × × ×	× × × × × × × × × × × × × ×
Mahmood et al. [19]	0.200008,0.00513,0.6513,0.3511,0.2001	$\partial\beta_{AL3} \leq \partial\beta_{AL4} \leq \partial\beta_{AL1} \leq \partial\beta_{AL5} \leq \partial\beta_{AL2}$
	0.1991,0.004870.6485,0.3488,0.1989	$\partial\beta_{AL3} \leq \partial\beta_{AL4} \leq \partial\beta_{AL1} \leq \partial\beta_{AL5} \leq \partial\beta_{AL2}$
Yu and Xu [21]	× × × × × × × × × × × × × ×	× × × × × × × × × × × × × ×
CIFPAAA operator	0.20015,0.00515,0.6518,0.3518,0.20017	$\partial\beta_{AL3} \leq \partial\beta_{AL4} \leq \partial\beta_{AL5} \leq \partial\beta_{AL1} \leq \partial\beta_{AL2}$
CIFPAAG operator	0.1985,0.00485,0.648,0.3481,0.1981	$\partial\beta_{AL3} \leq \partial\beta_{AL4} \leq \partial\beta_{AL1} \leq \partial\beta_{AL5} \leq \partial\beta_{AL2}$

2. With the help of two different structures such as prioritized geometric aggregation operators and Aczel-Alsina t-norm and t-conorm, we exposed the idea of CIFPAG and CIFPAAOG operators.
3. Some important properties and special cases of the derived work are also examined.
4. We illustrated the procedure of the MADM technique under the consideration of derived operators for CIF information.
5. Finally, we illustrated various examples for determining the comparison between proposed and existing operators.

In the future, we aim to derive various new ideas and then try to utilize them in the field of game theory, machine learning, software engineering, computer science, artificial intelligence, neural networks, clustering analysis, pattern recognition, medical diagnosis, and decision-making [22–24] to enhance the worth of the derived theory.

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Chapter 18
Intuitionistic Fuzzy Approach for Predicting Maternal Outcomes

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1 Introduction

On daily basis, both health and nonhealth organizations make significant efforts in reducing newborn and maternal deaths. However, WHO in (2019) [1] reported that between 1990 and 2020, the newborn mortality rate was almost halved from 37 to 17 deaths per 1000 births. Again, from 2000 to 2017, the global maternal mortality rate fell by nearly 38%. Despite these efforts, mothers within the bearing age are still experiencing death in unacceptably large numbers, which could be associated with complications during pregnancy or childbirth.

The complication in pregnancy occurs from time to time and can affect the mother's health, the baby's health, or both. Often some women within the bearing age may have some health problems that arise during pregnancy (such as preeclampsia, eclampsia, bleeding, ectopic pregnancy, miscarriage or fetal loss, and many others), while other women may have health problems before they became pregnant (such as unsafe abortion, infections, high blood pressure, and convulsions, sometimes followed by a coma), which could lead to complications. It is of great importance for women within the bearing age to receive good healthcare before and during pregnancy to reduce the risk of pregnancy complications, which may lead to morbidity or mortality.

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C. Jana et al. (eds.), *Fuzzy Optimization, Decision-making and Operations Research*, https://doi.org/10.1007/978-3-031-35668-1_18

Conversely, due to the uncertainties in maternal reproductive health, there is a need to introduce a fuzzy approach, which can help to handle uncertainties. Zadeh [2] introduced a fuzzy set to handle vagueness and ambiguity. Fuzzy set is capable of modeling human knowledge and intelligence. Nonetheless, it has some limitations, because in it, only one membership function is for prediction. To properly handle uncertainties in real world problems, an intuitionistic fuzzy set (IFS) was introduced by Atanassov [3] because to a large extent, it can reliably model most real-life situations.

The need for a robust intuitionistic fuzzy (IF) approach was introduced in this chapter based on the following major concerns in terms of complications related to mothers within the bearing age: (i) complication among women within the bearing age poses lots of concerns to developing and less-developed countries with about 50–100% [4] higher than those witnessed in developed countries, (ii) predicting pregnant mothers remains a challenge because its physical/clinical presentation could be confusing to some health personnel since it can be related to other health complications such as infections, urinary tract infection, obesity and weight gain, and hypertension, (iii) it is of great importance for developing countries to embrace new methods for predicting maternal complication, (iv) some patients may find it difficult to express how they feel, thereby making it difficult for the physician to provide the right diagnosis to the patient, and (v) complication among mothers within the bearing age could be misdiagnosed most especially in low-to-middle income countries (LMICs) due to quick access to local birth attendance and lack of funding to access good health facilities.

This study recommends the use of an intuitionistic fuzzy (IF) approach to enhance decision-making for the prediction of maternal complications because IF approach is capable of handling the hesitation between patient and doctor in predicting maternal outcomes, which in most cases could be challenging in terms of the investigations like laboratory analysis and ultrasound and their proposed diagnoses outcomes. The IF method makes use of a membership function and nonmembership function with the possibility of the existence of hesitation margin to enrich decision-making in most real-life situations. There is no doubt that the health sector is not left alone, as such, some researchers have adopted intuitionistic methods to predict tropical diseases. Furthermore, maternal health remains a source of concern in the health system. Thus, this work seeks to contribute to the body of knowledge in reducing complications that may arise during and after pregnancy, as well as improve maternal healthcare. The concept of the intuitionistic fuzzy method gives a better predicting result than the conventional fuzzy sets.

2 Review of Related Works

Several works has been tailored toward adoption of IF methods to improve medical decisions, De et al. [5] and Ejegwa and Onasanya [6] applied intuitionistic fuzzy sets in medical diagnosis and suggested that an improved intuitionistic fuzzy composite relation will yield a better decision. The works in [7, 8] showcased the strength of intuitionistic fuzzy sets for the prediction of medical diagnosis, results were obtained from the lowest value from their computations. Ahn et al. [8] applied interval-valued IFS to medical diagnosis of headache. Ref. [9] compared the switch between type-2 fuzzy sets and intuitionistic fuzzy sets to proffer a better medical diagnosis. Luo and Zhao [10] conducted a study using distance measure to compute intuitionistic fuzzy sets to improve medical diagnosis. Dhiman et al. [11] adopted intuitionistic fuzzy set to extract expert knowledge for predicting medical diagnosis. Sulaiman et al. [12] applied weighted similarity on intuitionistic fuzzy set for medical diagnostics. Subsequently, intuitionistic fuzzy sets have been used in various domains: electoral system [13], pattern recognitions [14–21], image processing [5, 22–27], environmental management [28], and to predict decision-making [29–36].

Intuitionistic fuzzy sets offer various measurement approaches, including the utilization of similarity and distance measures. These measures serve to represent the proximity or dissimilarity between different Intuitionistic Fuzzy Sets (IFSs). Lui [37] investigated cosine similarity measure with hybrid intuitionistic fuzzy information to enhance medical diagnosis. Pramanik and Mondal [38] studied intuitionistic fuzzy distance approaches with applications. Furthermore, Çağman and Deli [39] recommended similarity measures of two IFSs, which was applied to medical diagnosis to improve clinical results and other competing diagnosis. Gohain [40] proposed two new similarity measures for intuitionistic fuzzy sets and its various applications to pattern recognition, medical diagnosis, and the decision-making problem of face mask selection for the novel COVID-19 virus [41]. Garg and Kumar [42] proposed some novel similarity measures to measure the relative strength of the different IFSs.

However, distance measure between IFSs to a large extent is a method reliable for better interpretations. Several scholars have suggested methods to strengthen output results obtained by scholars. Dutta and Goala [43] used an advanced distance measure on IFSs to suggest the disease that a patient may be suffering from, while the work of Mahanta and Panda [44] showcased the importance of distance measure for intuitionistic fuzzy sets with diverse applications. Xiao [45] proposed a new distance measure between IFSs based on the Jensen-Shannon divergence, the new IFS distance measure cannot only satisfy the axiomatic definition of distance measure but also has nonlinear characteristics. A novel distance measure for cubic intuitionistic fuzzy sets was used to predict medical diagnosis and pattern recognition [46]. Though several scholars have experimented using IFSs, it is imperative to discuss the application of IF method in the prediction of maternal health outcomes.

Our research centered on devising an Intuitionistic Fuzzy (IF) method by building upon the frameworks introduced by Szmidt and Kacprzk [30] and enhancing their intuitionistic fuzzy distances, as suggested by Ejegwa et al [47] Subsequently, we developed a novel intuitionistic fuzzy method through these modifications. The remaining part of this work is arranged as follows: Sect. 3 compares methodologies of IF methods (distance methods to be precise) and the basic notions of each of the methods, Sect. 4 presents numerical experiment and discussion based on real maternity data set, and Sect. 5 presents the summary of the work with a concluding remark and drawbacks of the work.

3 Methodology

3.1 Intuitionistic Fuzzy Sets and Their Distance Measures

We will discuss the concept of IFSs distance method generated by Szmidt and Kacprzk [30] and then present a modified approach of Szmidt and Kacprzk's [47] methods. We take Υ as a nonempty set throughout this chapter.

Definition 3.1 [50] A fuzzy set $\hat{A}$ drawn from γ could be written as $\hat{A} = \{\langle e, \Phi_{\hat{A}}(e)\rangle : e \in \gamma\}$, where $\Phi_{\tilde{A}}(e), \gamma \to [0, 1]$ is the membership function of $\hat{A}$. Fuzzy set is a collection of objects with graded membership.

Definition 3.2 [3] An IFS represented by $\tilde{B}$ in Υ is an object of the form

$$\tilde{B} = \{\langle e, \Phi_{\tilde{B}}(e), \Psi_{\tilde{B}}(e)\langle : e \in \gamma\}$$

where the function $\Phi_{\tilde{B}}(e), \Psi_{\tilde{B}}(e) : \gamma \to [0, 1]$ define the degree of membership and degree of nonmembership of the element, $e \in \Upsilon$ to $\tilde{B}$, and for every $e \in \Upsilon$, $e \in \gamma, \Phi_{\tilde{B}}(e), \Psi_{\tilde{B}}(e) \in [0, 1]$. Further, we have $\Omega_{\tilde{B}}(e) = 1 - \Phi_{\tilde{B}}(e) - \Psi_{\tilde{B}}(e)$, which is called the intuitionistic fuzzy set index or hesitation margin of $e \in \widetilde{B}$. The function, $\Omega_{\tilde{B}}(e)$, describes the degree at which e belongs to $\tilde{B}$ or not. According to [51–53], we observe that every fuzzy set is an IFS but the converse does not hold, and that

$$\Omega_{\tilde{B}}(e) + \Phi_{\tilde{B}}(e) + \Psi_{\tilde{B}}(e) = 1.$$

The advantage of IFS over fuzzy sets is the introduction of additional degrees of freedom (nonmemberships and hesitation margins) into the set description to give room to handle imprecise knowledge, which may lead to describe many real problems in a more adequate way.

Definition 3.3 [3] If $\tilde{B}_1$, and $\tilde{B}_2$ inΥ, then for all $e \in \Upsilon$, we have

(i) $\widetilde{B}_1 = \widetilde{B}_2$ iff $\Phi_{\widetilde{B}_1}(e) = \Phi_{\widetilde{B}_2}(e)$ and $\Psi_{\widetilde{B}_1}(e) = \Psi_{\widetilde{B}_2}$
(ii) $\widetilde{B}_1 \subseteq \widetilde{B}_2$ iff $\Phi_{\widetilde{B}_1}(e) \leq \Phi_{\widetilde{B}_2}(e)$ and $\Psi_{\widetilde{B}_1}(e) \geq \Psi_{\widetilde{B}_2}$
(iii) $\widetilde{B}_1 \preceq \widetilde{B}_2$ iff $\Phi_{\widetilde{B}_1}(e) \preceq \Phi_{\widetilde{B}_2}(e)$ and $\Psi_{\widetilde{B}_1}(e) \preceq \Psi_{\widetilde{B}_1}$
(iv) $\widetilde{B}_1 = \left\{\left\langle e, \Psi_{\widetilde{B}_1}(e), \Phi_{\widetilde{B}_1}(e)\right\rangle : e \in \gamma\right\}$, $\tilde{B}_2 = \widetilde{B}_2 = \left\{\left\langle e, \Psi_{\widetilde{B}_2}(e), \Phi_{\widetilde{B}_2}(e)\right\rangle : e \in \gamma\right\}$

(v) $\widetilde{\mathrm{B}}_1 \cup \widetilde{\mathrm{B}}_2 = \left\{ \left\langle e, \max\left\{ \Phi_{\widetilde{\mathrm{B}}_1}(e), \Phi_{\widetilde{\mathrm{B}}_2}(e) \right\}, \min\left\{ \Psi_{\widetilde{\mathrm{B}}_1}(e), \Psi_{\widetilde{\mathrm{B}}_2}(e) \right\rangle : e \in \gamma \right\}$

(vi) $\widetilde{\mathrm{B}}_1 \cap \widetilde{\mathrm{B}}_2 = \left\{ \left\langle e, \min\left\{ \Phi_{\widetilde{\mathrm{B}}_1}(e), \Phi_{\widetilde{\mathrm{B}}_2}(e) \right\}, \max\left\{ \Psi_{\widetilde{\mathrm{B}}_1}(e), \Psi_{\widetilde{\mathrm{B}}_2}(e) \right\rangle : e \in \gamma \right\}$.

Definition 3.4 [54] The distance measure between IFSs, $\tilde{\mathrm{B}}$, $\tilde{\mathrm{B}}_1$, and $\tilde{\mathrm{B}}_2$ in Υ is a function, $\mho$: *IFS* × *IFS* → [0, 1], which satisfies the following axioms:

C1 $\mho\left(\tilde{\mathrm{B}}_1, \tilde{\mathrm{B}}_2\right) \in [0, 1]$

C2 $\mho\left(\tilde{\mathrm{B}}_1, \tilde{\mathrm{B}}_2\right) = 0$ if and only if $\tilde{\mathrm{B}}_{\mathbf{1}} = \tilde{\mathrm{B}}_{\mathbf{2}}$

C3 $\mho\left(\tilde{\mathrm{B}}_1, \tilde{\mathrm{B}}_2\right) = \mho\left(\tilde{\mathrm{B}}_2, \tilde{\mathrm{B}}_1\right)$

C4 $\mho\left(\tilde{\mathrm{B}}_1, \tilde{\mathrm{B}}\right) + \mho\left(\widetilde{\tilde{\mathrm{B}}, \mathrm{B}_2}\right) \geq \mho\left(\tilde{\mathrm{B}}_1, \tilde{\mathrm{B}}_2\right)$

C5 if $\tilde{\mathrm{B}}_1 \subseteq \tilde{\mathrm{B}}_2 \subseteq \tilde{\mathrm{B}}$, then $\mho\left(\tilde{\mathrm{B}}_1, \tilde{\mathrm{B}}\right) \geq \mho\left(\tilde{\mathrm{B}}_1, \tilde{\mathrm{B}}_2\right)$ and $\mho\left(\tilde{\mathrm{B}}_1, \tilde{\mathrm{B}}\right) \geq \mho\left(\widetilde{\tilde{\mathrm{B}}, \mathrm{B}_2}\right)$.

Distance measure is a term that describes the difference between IFSs and can be considered as a dual concept of similarity measure. We recall the four distance measures proposed in [48–51] between IFSs, which were partly based on the geometric interpretation of IFSs with good geometric properties.

IFSs Methods Developed by Szmidt and Kacprzyk

Suppose there are two IFSs, $\tilde{\mathrm{B}}_1$ and $\tilde{\mathrm{B}}_2$ in $\Upsilon = \{\mathrm{e}_1, \mathrm{e}_2 \ldots, \mathrm{e}_n\}$, given by

$$\widetilde{\mathrm{B}}_1 = \left\{ \left\langle e_j, \Phi_{\widetilde{\mathrm{B}}_1}\left(e_j\right), \Psi_{\widetilde{\mathrm{B}}_1}\left(e_j\right), \Omega_{\widetilde{\mathrm{B}}_1}\left(e_j\right) \right\rangle : e_j \in \mathrm{Y} \right\}$$

and

$$\widetilde{\mathrm{B}}_2 = \left\{ \left\langle e_j, \Phi_{\widetilde{\mathrm{B}}_2}\left(e_j\right), \Psi_{\widetilde{\mathrm{B}}_2}\left(e_j\right), \Omega_{\widetilde{\mathrm{B}}_2}\left(e_j\right) \right\rangle : e_j \in \mathrm{Y} \right\}, \quad \forall \mathrm{j} = 1, 2, \ldots, \mathrm{n}.$$

Then the following are distance measures between A and B [49, 50]:
The Hamming distance

$$\mho_1\left(\widetilde{\mathrm{B}}_1, \widetilde{\mathrm{B}}_2\right) = \frac{\sum_{j=1}^{n}\left(\left|\Phi_{\widetilde{\mathrm{B}}_1}\left(e_j\right) - \Phi_{\widetilde{\mathrm{B}}_2}\left(e_j\right)\right| + \left|\Psi_{\widetilde{\mathrm{B}}_1}\left(e_j\right) - \Psi_{\widetilde{\mathrm{B}}_2}\left(e_j\right)\right| + \left|\Omega_{\widetilde{\mathrm{B}}_1}\left(e_j\right) - \Omega_{\widetilde{\mathrm{B}}_2}\left(e_j\right)\right| \right)}{2} \tag{18.1}$$

The Euclidean distance

$$\mho_2\left(\widetilde{\mathrm{B}}_1,\widetilde{\mathrm{B}}_2\right)=\sqrt{\frac{\sum_{j=1}^{n}\left[\left(\Phi_{\widetilde{\mathrm{B}}_1}\left(e_j\right)-\Phi_{\widetilde{\mathrm{B}}_2}\left(e_j\right)\right)^2+\left(\Psi_{\widetilde{\mathrm{B}}_1}\left(e_j\right)-\Psi_{\widetilde{\mathrm{B}}_2}\left(e_j\right)\right)^2+\left(\Omega_{\widetilde{\mathrm{B}}_1}\left(e_j\right)-\Omega_{\widetilde{\mathrm{B}}_2}\left(e_j\right)\right)^2\right]}{2}} \tag{18.2}$$

The normalized Hamming distance

$$\mho_3\left(\widetilde{\mathrm{B}}_1,\widetilde{\mathrm{B}}_2\right)=\frac{\sum_{j=1}^{n}\left(|\Phi_{\widetilde{\mathrm{B}}_1}\left(e_j\right)-\Phi_{\widetilde{\mathrm{B}}_2}\left(e_j\right)|+|\Psi_{\widetilde{\mathrm{B}}_1}\left(e_j\right)-\Psi_{\widetilde{\mathrm{B}}_2}\left(e_j\right)|+|\Omega_{\widetilde{\mathrm{B}}_1}\left(e_j\right)-\Omega_{\widetilde{\mathrm{B}}_2}\left(e_j\right)|\right)}{2n} \tag{18.3}$$

The normalized Euclidean distance

$$\mho_4\left(\widetilde{\mathrm{B}}_1,\widetilde{\mathrm{B}}_2\right)=\sqrt{\frac{\sum_{j=1}^{n}\left[\left(\Phi_{\widetilde{\mathrm{B}}_1}\left(e_j\right)-\Phi_{\widetilde{\mathrm{B}}_2}\left(e_j\right)\right)^2+\left(\Psi_{\widetilde{\mathrm{B}}_1}\left(e_j\right)-\Psi_{\widetilde{\mathrm{B}}_2}\left(e_j\right)\right)^2+\left(\Omega_{\widetilde{\mathrm{B}}_1}\left(e_j\right)-\Omega_{\widetilde{\mathrm{B}}_2}\left(e_j\right)\right)^2\right]}{2n}} \tag{18.4}$$

Recently, Ejegwa et al. [47] developed the following distance measures for IFSs by modifying the approaches in [30]:

$$\mho_5\left(\widetilde{\mathrm{B}}_1,\widetilde{\mathrm{B}}_2\right)=\frac{\sum_{j=1}^{n}\left(|\Phi_{\widetilde{\mathrm{B}}_1}\left(e_j\right)-\Phi_{\widetilde{\mathrm{B}}_2}\left(e_j\right)|+|\Psi_{\widetilde{\mathrm{B}}_1}\left(e_j\right)-\Psi_{\widetilde{\mathrm{B}}_2}\left(e_j\right)|+|\Omega_{\widetilde{\mathrm{B}}_1}\left(e_j\right)-\Omega_{\widetilde{\mathrm{B}}_2}\left(e_j\right)|\right)}{3} \tag{18.5}$$

$$\mho_6\left(\widetilde{\mathrm{B}}_1,\widetilde{\mathrm{B}}_2\right)=\sqrt{\frac{\sum_{j=1}^{n}\left[\left(\Phi_{\widetilde{\mathrm{B}}_1}\left(e_j\right)-\Phi_{\widetilde{\mathrm{B}}_2}\left(e_j\right)\right)^2+\left(\Psi_{\widetilde{\mathrm{B}}_1}\left(e_j\right)-\Psi_{\widetilde{\mathrm{B}}_2}\left(e_j\right)\right)^2+\left(\Omega_{\widetilde{\mathrm{B}}_1}\left(e_j\right)-\Omega_{\widetilde{\mathrm{B}}_2}\left(e_j\right)\right)^2\right]}{3}} \tag{18.6}$$

$$\mho_7\left(\widetilde{B}_1,\widetilde{B}_2\right)=\frac{\sum_{j=1}^{n}\left(|\Phi_{\widetilde{B}_1}\left(e_j\right)-\Phi_{\widetilde{B}_2}\left(e_j\right)|+|\Psi_{\widetilde{B}_1}\left(e_j\right)-\Psi_{\widetilde{B}_2}\left(e_j\right)|+|\Omega_{\widetilde{B}_1}\left(e_j\right)-\Omega_{\widetilde{B}_2}\left(e_j\right)|\right)}{3n} \tag{18.7}$$

$$\mho_8\left(\widetilde{B}_1,\widetilde{B}_2\right)=\sqrt{\frac{\sum_{j=1}^{n}\left[\left(\Phi_{\widetilde{B}_1}\left(e_j\right)-\Phi_{\widetilde{B}_2}\left(e_j\right)\right)^2+\left(\Psi_{\widetilde{B}_1}\left(e_j\right)-\Psi_{\widetilde{B}_2}\left(e_j\right)\right)^2+\left(\Omega_{\widetilde{B}_1}\left(e_j\right)-\Omega_{\widetilde{B}_2}\left(e_j\right)\right)^2\right]}{3n}} \tag{18.8}$$

Because the concept of similarity measure is the dual of distance measure, the similarity measures of the above distance measures can be obtained by $\overline{\overline{\mho}}\left(\widetilde{B}_1,\widetilde{B}_2\right)=1-\mho\left(\widetilde{B}_1,\widetilde{B}_2\right)$ [52]. Based on this, the similarity measures are as follows:

$$\bar{\mho}_1\left(\widetilde{B}_1,\widetilde{B}_2\right)=1-\frac{\sum_{j=1}^{n}\left(|\Phi_{\widetilde{B}_1}\left(e_j\right)-\Phi_{\widetilde{B}_2}\left(e_j\right)|+|\Psi_{\widetilde{B}_1}\left(e_j\right)-\Psi_{\widetilde{B}_2}\left(e_j\right)|+|\Omega_{\widetilde{B}_1}\left(e_j\right)-\Omega_{\widetilde{B}_2}\left(e_j\right)|\right)}{2} \tag{18.9}$$

$$\overline{\mho}_2\left(\widetilde{B}_1,\widetilde{B}_2\right)=1-\sqrt{\frac{\sum_{j=1}^{n}\left[\left(\Phi_{\widetilde{B}_1}\left(e_j\right)-\Phi_{\widetilde{B}_2}\left(e_j\right)\right)^2+\left(\Psi_{\widetilde{B}_1}\left(e_j\right)-\Psi_{\widetilde{B}_2}\left(e_j\right)\right)^2+\left(\Omega_{\widetilde{B}_1}\left(e_j\right)-\Omega_{\widetilde{B}_2}\left(e_j\right)\right)^2\right]}{2}} \tag{18.10}$$

$$\overline{\mho}_3\left(\widetilde{B}_1,\widetilde{B}_2\right)=1-\frac{\sum_{j=1}^{n}\left(|\Phi_{\widetilde{B}_1}\left(e_j\right)-\Phi_{\widetilde{B}_2}\left(e_j\right)|+|\Psi_{\widetilde{B}_1}\left(e_j\right)-\Psi_{\widetilde{B}_2}\left(e_j\right)|+|\Omega_{\widetilde{B}_1}\left(e_j\right)-\Omega_{\widetilde{B}_2}\left(e_j\right)|\right)}{2n} \tag{18.11}$$

$$\overline{\mho}_4\left(\widetilde{B}_1,\widetilde{B}_2\right)=1-\sqrt{\frac{\sum_{j=1}^{n}\left[\left(\Phi_{\widetilde{B}_1}\left(e_j\right)-\Phi_{\widetilde{B}_2}\left(e_j\right)\right)^2+\left(\Psi_{\widetilde{B}_1}\left(e_j\right)-\Psi_{\widetilde{B}_2}\left(e_j\right)\right)^2+\left(\Omega_{\widetilde{B}_1}\left(e_j\right)-\Omega_{\widetilde{B}_2}\left(e_j\right)\right)^2\right]}{2n}} \tag{18.12}$$

$$\overline{\mho}_5\left(\widetilde{B}_1,\widetilde{B}_2\right)=1-\frac{\sum_{j=1}^{n}\left(|\Phi_{\widetilde{B}_1}\left(e_j\right)-\Phi_{\widetilde{B}_2}\left(e_j\right)|+|\Psi_{\widetilde{B}_1}\left(e_j\right)-\Psi_{\widetilde{B}_2}\left(e_j\right)|+|\Omega_{\widetilde{B}_1}\left(e_j\right)-\Omega_{\widetilde{B}_2}\left(e_j\right)|\right)}{3} \tag{18.13}$$

$$\overline{\mho}_6\left(\widetilde{B}_1,\widetilde{B}_2\right)=1-\sqrt{\frac{\sum_{j=1}^{n}\left[\left(\Phi_{\widetilde{B}_1}\left(e_j\right)-\Phi_{\widetilde{B}_2}\left(e_j\right)\right)^2+\left(\Psi_{\widetilde{B}_1}\left(e_j\right)-\Psi_{\widetilde{B}_2}\left(e_j\right)\right)^2+\left(\Omega_{\widetilde{B}_1}\left(e_j\right)-\Omega_{\widetilde{B}_2}\left(e_j\right)\right)^2\right]}{3}} \tag{18.14}$$

$$\overline{\mho}_7\left(\widetilde{B}_1,\widetilde{B}_2\right)=1-\frac{\sum_{j=1}^{n}\left(|\Phi_{\widetilde{B}_1}\left(e_j\right)-\Phi_{\widetilde{B}_2}\left(e_j\right)|+|\Psi_{\widetilde{B}_1}\left(e_j\right)-\Psi_{\widetilde{B}_2}\left(e_j\right)|+|\Omega_{\widetilde{B}_1}\left(e_j\right)-\Omega_{\widetilde{B}_2}\left(e_j\right)|\right)}{3n} \tag{18.15}$$

$$\overline{\mho}_8\left(\widetilde{B}_1,\widetilde{B}_2\right)=1-\sqrt{\frac{\sum_{j=1}^{n}\left[\left(\Phi_{\widetilde{B}_1}\left(e_j\right)-\Phi_{\widetilde{B}_2}\left(e_j\right)\right)^2+\left(\Psi_{\widetilde{B}_1}\left(e_j\right)-\Psi_{\widetilde{B}_2}\left(e_j\right)\right)^2+\left(\Omega_{\widetilde{B}_1}\left(e_j\right)-\Omega_{\widetilde{B}_2}\left(e_j\right)\right)^2\right]}{3n}} \tag{18.16}$$

New Method of Intuitionistic Fuzzy Distance Measure

To improve reliable results and evade loss of information, we present the following method of intuitionistic fuzzy distance measure between IFSs:

$$\widetilde{B}_1=\left\{\left\langle e_j,\Phi_{\widetilde{B}_1}\left(e_j\right),\Psi_{\widetilde{B}_1}\left(e_j\right),\Omega_{\widetilde{B}_1}\left(e_j\right)\right\rangle : e_j\in \Upsilon\right\} \text{ and}$$

$$\widetilde{B}_2=\left\{\left\langle e_j,\Phi_{\widetilde{B}_2}\left(e_j\right),\Psi_{\widetilde{B}_2}\left(e_j\right),\Omega_{\widetilde{B}_2}\left(e_j\right)\right\rangle : e_j\in \Upsilon\right\},$$

in $\Upsilon=\{e_1,e_2\ldots,e_n\}$, $\forall j=1,2,\ldots,n$ by

$$\mho_{\omega}\left(\widetilde{\mathrm{B}}_1, \widetilde{\mathrm{B}}_2\right) = \frac{\sum_{j=1}^{n} \omega_j \begin{bmatrix} Av\Big\{\left|\Phi_{\widetilde{\mathrm{B}}_1}(e_j) - \Phi_{\widetilde{\mathrm{B}}_2}(e_j)\right|, \left|\Psi_{\widetilde{\mathrm{B}}_1}(e_j) - \Psi_{\widetilde{\mathrm{B}}_2}(e_j)\right|, \\ \left|\Omega_{\widetilde{\mathrm{B}}_1}(e_j) - \Omega_{\widetilde{\mathrm{B}}_2}(e_j)\right|\Big\} \end{bmatrix}}{n}, \tag{18.17}$$

where Av is average, and ω_j is the weights of the elements of Υ such that $\sum_{j=1}^{n} \omega_j = 1$ and $\omega_j \in [0, 1]$. This approach can be best described as weighted modified Hausdorff intuitionistic fuzzy distance approach. The similarity equivalent of the weighted modified Hausdorff intuitionistic fuzzy distance approach is given by

$$\overline{\mho}_{\omega}\left(\widetilde{\mathrm{B}}_1, \widetilde{\mathrm{B}}_2\right) = 1 - \frac{\sum_{j=1}^{n} \omega_j \begin{bmatrix} Av\Big\{\left|\Phi_{\widetilde{\mathrm{B}}_1}(e_j) - \Phi_{\widetilde{\mathrm{B}}_2}(e_j)\right|, \\ \left|\Psi_{\widetilde{\mathrm{B}}_1}(e_j) - \Psi_{\widetilde{\mathrm{B}}_2}(e_j)\right|, \left|\Omega_{\widetilde{\mathrm{B}}_1}(e_j) - \Omega_{\widetilde{\mathrm{B}}_2}(e_j)\right|\Big\} \end{bmatrix}}{n}. \tag{18.18}$$

By simplifying Eqs. (18.17) and (18.18), we obtain

$$\mho_{\omega}\left(\widetilde{\mathrm{B}}_1, \widetilde{\mathrm{B}}_2\right) = \frac{1}{3n}\sum_{j=1}^{n} \omega_j \begin{bmatrix} \left|\Phi_{\widetilde{\mathrm{B}}_1}(e_j) - \Phi_{\widetilde{\mathrm{B}}_2}(e_j)\right| + \left|\Psi_{\widetilde{\mathrm{B}}_1}(e_j) - \Psi_{\widetilde{\mathrm{B}}_2}(e_j)\right| \\ + \left|\Omega_{\widetilde{\mathrm{B}}_1}(e_j) - \Omega_{\widetilde{\mathrm{B}}_2}(e_j)\right| \end{bmatrix}, \tag{18.19}$$

$$\overline{\mho}_{\omega}\left(\widetilde{\mathrm{B}}_1, \widetilde{\mathrm{B}}_2\right) = 1 - \frac{1}{3n}\sum_{j=1}^{n} \omega_j \begin{bmatrix} \left|\Phi_{\widetilde{\mathrm{B}}_1}(e_j) - \Phi_{\widetilde{\mathrm{B}}_2}(e_j)\right| + \left|\Psi_{\widetilde{\mathrm{B}}_1}(e_j) - \Psi_{\widetilde{\mathrm{B}}_2}(e_j)\right| \\ + \left|\Omega_{\widetilde{\mathrm{B}}_1}(e_j) - \Omega_{\widetilde{\mathrm{B}}_2}(e_j)\right| \end{bmatrix}. \tag{18.20}$$

Now, we present some properties of Eqs. (18.17) and (18.18) as follows.

Proposition 3.1 *Suppose* $\tilde{\mathrm{B}}_1$ and $\tilde{\mathrm{B}}_2$ *are IFSs of* Υ*, then we have:*

(i) $\mho_{\omega}\left(\tilde{\mathrm{B}}_1, \tilde{\mathrm{B}}_2\right) = \mho_{\omega}\left(\tilde{\mathrm{B}}_2, \tilde{\mathrm{B}}_1\right)$

(ii) $\overline{\overline{\mho}}_{\omega}(\mathrm{B}_1, \mathrm{B}_2) = \overline{\overline{\mho}}_{\omega}(\mathrm{B}_2, \mathrm{B}_1)$.

Proof The proof of (i) follows because

$$\mho_{\omega}\left(\widetilde{\mathrm{B}}_1, \widetilde{\mathrm{B}}_2\right) = \frac{1}{3n}\sum_{j=1}^{n} \omega_j \begin{bmatrix} \left|\Phi_{\widetilde{\mathrm{B}}_1}(e_j) - \Phi_{\widetilde{\mathrm{B}}_2}(e_j)\right| + \left|\Psi_{\widetilde{\mathrm{B}}_1}(e_j) - \Psi_{\widetilde{\mathrm{B}}_2}(e_j)\right| \\ + \left|\Omega_{\widetilde{\mathrm{B}}_1}(e_j) - \Omega_{\widetilde{\mathrm{B}}_2}(e_j)\right| \end{bmatrix}$$

$$= \frac{1}{3n}\sum_{j=1}^{n}\omega_j\left[\begin{array}{c}\left|-\left[\Phi_{\widetilde{B}_2}\left(e_j\right)-\Phi_{\widetilde{B}_1}\left(e_j\right)\right]\right| + \quad \left|-\left[\Psi_{\widetilde{B}_2}\left(e_j\right)-\Psi_{\widetilde{B}_1}\left(e_j\right)\right]\right| \\ + \quad \left|-\left[\Omega_{\widetilde{B}_2}\left(e_j\right)-\Omega_{\widetilde{B}_1}\left(e_j\right)\right]\right|\end{array}\right]$$

$$= \frac{1}{3n}\sum_{j=1}^{n}\omega_j\left[\begin{array}{c}\left|\Phi_{\widetilde{B}_2}\left(e_j\right)-\Phi_{\widetilde{B}_1}\left(e_j\right)\right| + \quad \left|\Psi_{\widetilde{B}_2}\left(e_j\right)-\Psi_{\widetilde{B}_1}\left(e_j\right)\right| \\ + \quad \left|\Omega_{\widetilde{B}_2}\left(e_j\right)-\Omega_{\widetilde{B}_1}\left(e_j\right)\right|\end{array}\right]$$

$$= \mho_\omega\left(\tilde{B}_2, \tilde{B}_1\right).$$

The proof of (ii) is similar.

Theorem 3.2 Suppose $\tilde{B}_1$ and $\tilde{B}_2$ *are IFSs in* Υ, then the functions $\mho_\omega\left(\tilde{B}_1, \tilde{B}_2\right)$ and $\overline{\mho}_\omega\left(\widetilde{B}_1, \widetilde{B}_2\right)$ *satisfy*

(i) $\mho_\omega\left(\tilde{B}_1, \tilde{B}_2\right), \overline{\overline{\mho}}_\omega\left(\tilde{B}_1, \tilde{B}_2\right) \in [0, 1]$(ii) $\mho_\omega\left(\tilde{B}_1, \tilde{B}_2\right) = 0, \overline{\overline{\mho}}_\omega\left(\tilde{B}_1, \tilde{B}_2\right) = 0$ iff $\tilde{B}_1 = \tilde{B}_2$.

Proof Firstly, we show $\mho_\omega\left(\tilde{B}_1, \tilde{B}_2\right) \in [0, 1]$, which means $\mho_\omega\left(\tilde{B}_1, \tilde{B}_2\right) \geq 0$ and $\mho_\omega\left(\tilde{B}_1, \tilde{B}_2\right) \leq 1$. Because $\left|\phi_{\tilde{B}_1}\left(e_j\right)-\phi_{\tilde{B}_2}\left(e_j\right)\right| \geq 0$, and $\left|\omega_{\tilde{B_1}}\left(e_j\right)-\omega_{\tilde{B_2}}\left(e_j\right)\right| = 0$, it follows that $\mho_\omega\left(\tilde{B}_1, \tilde{B}_2\right) \geq 0$.

Next, we prove that $\mho_\omega\left(\tilde{B}_1, \tilde{B}_2\right) \leq 1$. Recall that

$$\mho_\omega\left(\widetilde{B}_1, \widetilde{B}_2\right) = \frac{1}{3n}\sum_{j=1}^{n}\omega_j\left[\begin{array}{c}\left|\Phi_{\widetilde{B}_1}\left(e_j\right)-\Phi_{\widetilde{B}_2}\left(e_j\right)\right| + \quad \left|\Psi_{\widetilde{B}_1}\left(e_j\right)-\Psi_{\widetilde{B}_2}\left(e_j\right)\right| \\ + \quad \left|\Omega_{\widetilde{B}_1}\left(e_j\right)-\Omega_{\widetilde{B}_2}\left(e_j\right)\right|\end{array}\right].$$

Then, $\mho_\omega\left(\widetilde{B}_1, \widetilde{B}_2\right) = \frac{1}{3n}\left[\begin{array}{l}\sum_{j=1}^{n}\omega_j\left|\Phi_{\widetilde{B}_1}\left(e_j\right)-\Phi_{\widetilde{B}_2}\left(e_j\right)\right| \\ \quad +\sum_{j=1}^{n}\omega_j\left|\Psi_{\widetilde{B}_1}\left(e_j\right)-\Psi_{\widetilde{B}_2}\left(e_j\right)\right| \\ + \quad \sum_{j=1}^{n}\omega_j\left|\Omega_{\widetilde{B}_1}\left(e_j\right)-\Omega_{\widetilde{B}_2}\left(e_j\right)\right|\end{array}\right]$

By substituting

$$\sum_{j=1}^{n}\omega_j\left|\Phi_{\widetilde{B}_1}\left(e_j\right)-\Phi_{\widetilde{B}_2}\left(e_j\right)\right| = \mathcal{A}, \quad \sum_{j=1}^{n}\omega_j\left|\Psi_{\widetilde{B}_1}\left(e_j\right)-\Psi_{\widetilde{B}_2}\left(e_j\right)\right| = \mathcal{B}, \text{ and}$$

$$\sum_{j=1}^{\mathrm{n}} \omega_{\mathrm{j}} \left|\phi_{\mathrm{n}_1}\left(e_{\mathrm{j}}\right) - \phi_{\mathrm{n}_1}\left(e_{\mathrm{j}}\right)\right| = \mathcal{C},$$

we have $\mho_\omega\left(\tilde{\mathrm{B}}_1, \tilde{\mathrm{B}}_2\right) = \frac{\mathcal{A}+\mathcal{B}+\mathcal{C}}{3n}$. Then $\mho_\omega\left(\tilde{\mathrm{B}}_1, \tilde{\mathrm{B}}_2\right) - 1 = \frac{\mathcal{A}+\mathcal{B}+\mathcal{C}}{3n} - 1$implies $\mho_\omega\left(\tilde{\mathrm{B}}_1, \tilde{\mathrm{B}}_2\right) - 1 = \frac{\mathcal{A}+\mathcal{B}+\mathcal{C}-3n}{3n}$. Thus, $\mho_\omega\left(\tilde{\mathrm{B}}_1, \tilde{\mathrm{B}}_2\right) - 1 = -\frac{(3n-\mathcal{A}-\mathcal{B}-\mathcal{C})}{3n}$, which implies that $\mho_\omega\left(\tilde{\mathrm{B}}_1, \tilde{\mathrm{B}}_2\right) - 1 \le 0$. Therefore, we have $\mho_\omega\left(\tilde{\mathrm{B}}_1, \tilde{\mathrm{B}}_2\right) \le 1$. Hence, $\mho_\omega\left(\tilde{\mathrm{B}}_1, \tilde{\mathrm{B}}_2\right) \in [0, 1]$. Similarly, $\overline{\mho}_\omega\left(\widetilde{\mathrm{B}}_1, \widetilde{\mathrm{B}}_2\right) \in [0, 1]$. Hence, (i) holds.

Next, we prove (ii). Suppose $\mho_\omega\left(\tilde{\mathrm{B}}_1, \tilde{\mathrm{B}}_2\right) = 0$, then $\sum_{j=1}^{n} \omega_j \left|\Phi_{\widetilde{\mathrm{B}}_1}\left(e_j\right) - \Phi_{\widetilde{\mathrm{B}}_2}\left(e_j\right)\right| = 0$, $\sum_{j=1}^{n} \omega_j \left|\Psi_{\widetilde{\mathrm{B}}_1}\left(e_j\right) - \Psi_{\widetilde{\mathrm{B}}_2}\left(e_j\right)\right| = 0$, and $\sum_{j=1}^{n} \omega_j \left|\Omega_{\widetilde{\mathrm{B}}_1}\left(e_j\right) - \Omega_{\widetilde{\mathrm{B}}_2}\left(e_j\right)\right| = 0$. Thus, $\Phi_{\widetilde{\mathrm{B}}_1}\left(e_j\right) = \Phi_{\widetilde{\mathrm{B}}_2}\left(e_j\right)$, $\Psi_{\widetilde{\mathrm{B}}_1}\left(e_j\right) = \Psi_{\widetilde{\mathrm{B}}_2}\left(e_j\right)$, and $\Omega_{\widetilde{\mathrm{B}}_1}\left(e_j\right) = \Omega_{\widetilde{\mathrm{B}}_2}\left(e_j\right)$. Hence, $\tilde{\mathrm{B}}_1 = \tilde{\mathrm{B}}_2$.

Conversely, assume $\tilde{\mathrm{B}}_1 = \tilde{\mathrm{B}}_2$. Then

$$\mho_\omega\left(\widetilde{\mathrm{B}}_1, \widetilde{\mathrm{B}}_2\right) = \frac{1}{3n}\sum_{j=1}^{n} \omega_j \left[\begin{array}{c} \left|\Phi_{\widetilde{\mathrm{B}}_1}\left(e_j\right) - \Phi_{\widetilde{\mathrm{B}}_2}\left(e_j\right)\right| + \left|\Psi_{\widetilde{\mathrm{B}}_1}\left(e_j\right) - \Psi_{\widetilde{\mathrm{B}}_2}\left(e_j\right)\right| \\ + \left|\Omega_{\widetilde{\mathrm{B}}_1}\left(e_j\right) - \Omega_{\widetilde{\mathrm{B}}_2}\left(e_j\right)\right| \end{array}\right]$$

$$= \frac{1}{3n}\sum\nolimits_{j=1}^{n} \omega_j\, (0 + 0 + 0) = 0.$$

Similarly, $\overline{\overline{\mho}}_\omega\left(\widetilde{\mathrm{B}}_1, \widetilde{\mathrm{B}}_2\right) = 0$ *iff* $\tilde{\mathrm{B}}_1 = \tilde{\mathrm{B}}_2$, which proves (ii).

Theorem 3.3 *If* $\tilde{\mathrm{B}}_1, \tilde{\mathrm{B}}_2$ and $\tilde{\mathrm{B}}_3$ *are IFSs in* Υ *with the containment,* $\tilde{\mathrm{B}}_1 \subseteq \tilde{\mathrm{B}}_2 \subseteq \tilde{\mathrm{B}}_3$. *Then following statements hold:*

(i) $\overline{\overline{\mho}}_\omega\left(\tilde{\mathrm{B}}_1, \tilde{\mathrm{B}}_3\right) \ge \overline{\overline{\mho}}_\omega\left(\tilde{\mathrm{B}}_1, \tilde{\mathrm{B}}_2\right)$.
(ii) $\overline{\overline{\mho}}_\omega\left(\tilde{\mathrm{B}}_1, \tilde{\mathrm{B}}_3\right) \ge \overline{\overline{\mho}}_\omega\left(\tilde{\mathrm{B}}_2, \tilde{\mathrm{B}}_3\right)$.
(iii) $\overline{\overline{\mho}}_\omega\left(\tilde{\mathrm{B}}_1, \tilde{\mathrm{B}}_3\right) \ge \max\left\{\overline{\overline{\mho}}_\omega\left(\tilde{\mathrm{B}}_1, \tilde{\mathrm{B}}_2\right), \overline{\overline{\mho}}_\omega\left(\tilde{\mathrm{B}}_2, \tilde{\mathrm{B}}_3\right)\right\}$.

Proof Suppose $\tilde{\mathrm{B}}_1 \subseteq \tilde{\mathrm{B}}_2 \subseteq \tilde{\mathrm{B}}_3$, then

$$\left|\Phi_{\widetilde{\mathrm{B}}_1}\left(e_j\right) - \Phi_{\widetilde{\mathrm{B}}_3}\left(e_j\right)\right| \ge \left|\Phi_{\widetilde{\mathrm{B}}_1}\left(e_j\right) - \Phi_{\widetilde{\mathrm{B}}_2}\left(e_j\right)\right|, \left|\Psi_{\widetilde{\mathrm{B}}_1}\left(e_j\right) - \Psi_{\widetilde{\mathrm{B}}_3}\left(e_j\right)\right|$$
$$\ge \left|\Psi_{\widetilde{\mathrm{B}}_1}\left(e_j\right) - \Psi_{\widetilde{\mathrm{B}}_2}\left(e_j\right)\right|, \text{and } \left|\Omega_{\widetilde{\mathrm{B}}_1}\left(e_j\right) - \Omega_{\widetilde{\mathrm{B}}_3}\left(e_j\right)\right| \ge \left|\Omega_{\widetilde{\mathrm{B}}_1}\left(e_j\right) - \Omega_{\widetilde{\mathrm{B}}_2}\left(e_j\right)\right|.$$

Consequently, we have

$$\frac{1}{3n}\sum_{j=1}^{n}\omega_j\left[\begin{array}{c}\left|\Phi_{\tilde{B}_1}(e_j)-\Phi_{\tilde{B}_3}(e_j)\right|+\left|\Psi_{\tilde{B}_1}(e_j)-\Psi_{\tilde{B}_3}(e_j)\right|\\+\left|\Omega_{\tilde{B}_1}(e_j)-\Omega_{\tilde{B}_3}(e_j)\right|\end{array}\right]\geq$$

$$\frac{1}{3n}\sum_{j=1}^{n}\omega_j\left[\begin{array}{c}\left|\Phi_{\tilde{B}_1}(e_j)-\Phi_{\tilde{B}_2}(e_j)\right|+\left|\Psi_{\tilde{B}_1}(e_j)-\Psi_{\tilde{B}_2}(e_j)\right|\\+\left|\Omega_{\tilde{B}_1}(e_j)-\Omega_{\tilde{B}_2}(e_j)\right|\end{array}\right].$$

Hence, $\mho_\omega\left(\tilde{B}_1,\tilde{B}_3\right)\geq\mho_\omega\left(\tilde{B}_1,\tilde{B}_2\right)$, which establishes (i). Using the same principle, (ii) is established. Since $\mho_\omega\left(\tilde{B}_1,\tilde{B}_3\right)\geq\mho_\omega\left(\tilde{B}_1,\tilde{B}_2\right)$ and $\mho_\omega\left(\tilde{B}_1,\tilde{B}_3\right)\geq\mho_\omega\left(\tilde{B}_2,\tilde{B}_3\right)$, it is easy to see that $\mho_\omega\left(\tilde{B}_1,\tilde{B}_3\right)\geq\max\left\{\mho_\omega\left(\tilde{B}_1,\tilde{B}_2\right),\mho_\omega\left(\tilde{B}_2,\tilde{B}_3\right)\right\}$. Hence (iii) is proved.

Corollary 3.4 *If* $\tilde{B}_1,\tilde{B}_2$ and $\tilde{B}_3$ *are IFSs in* Υ *with the containment,* $\tilde{B}_1\subseteq\tilde{B}_2\subseteq\tilde{B}_3$. *Then following statements hold:*

(i) $\overline{\overline{\mho}}_\omega(B_1, B_3)\leq B_u(B_1, B_2)$.
(ii) $\overline{\overline{\mho}}_\omega(D_1, B_2)\leq d_c(B_2, B_2)$.
(iii) $\overline{\overline{\mho}}_\omega(B_1, B_2)\leq\max\{\overline{\overline{\mho}}_\omega(B_3, B_2), \overline{\overline{\mho}}_\omega(B_2, B_2)\}$.

Proof The proofs of the statements are similar to Theorem 3.3.

4 Numerical Experiment and Discussion

4.1 Data Source/Descriptions

We obtained data from both tertiary and private health-care centers (St Luke Hospital Anua, UyoAkwa Ibom State, University of River State Teaching Hospital and Zion Medical Centre Ahoada, Rivers State) in the South East Zone of Nigeria. The data were taken from about 2000 patients, with 15 input features: Maternal BP (MBPM), Maternal Weight(MW), Hemoglobin Level (HL), Packed Cell Volume Level (PCVL), Pulse Rate (PR), Mode of Delivery (MOD), Malaria Frequency (MP), Hepatitis C (HC), Diabetes Status (DM), Herbal Ingestion (HI), Respiratory disorder (RD), Material Aga (MA), Ascorbic acid Level (ACC), Preeclampsia (PREE), and Antennal booking (AB). The seven output features are Miscarriage, Preterm (Contraction), Still birth, Placentae Previa, UTI, Full term, and Mortality. Ethical clearance was obtained in each of health centers, but patients' personnel details were not disclosed.

Table 18.1 Raw dataset for maternal sickness

	MBPM	MW	HL	PCVL	PR	MOD	MP	HC	DM	HI	RD	MA	ACC	PREE	AB
Miscarriage	2	85	13	36	0	1	1	0	0	0	0	34	0	0	1
Pre-Preterm (Contraction)	5	72	13	41	0	1	0	0	0	0	0	30	0	0	1
Still birth	4	58	12	36	0	1	0	0	0	0	0	28	0	0	1
Placentae Previa	2	85	13	36	0	1	1	0	0	0	0	34	0	0	1
UTI	1	0	0	100	0	2	0	0	0	0	20	26	0	0	1
FULL-TERM	1	55	13	27	0	1	1	0	0	0	0	23	0	0	1
Mortality	1	96	12	36	0	2	0	0	0	0	0	27	0	0	1

Suppose $M = \{M_1, M_2, M_3, M_4, M_5\}$is the set of maternal patients to be diagnosed for various types of maternal outcome, and$D$={MBPM, MW, HL, PCVL, PR, MOD, MP, HC, DM, HI, RD, MA, ACC, PREE, AB} is the set of symptoms of maternal outcome. The maternal patients are susceptible to the following outcomes: Miscarriage, Preterm (Contraction), Still birth, Placentae Previa, UTI, Full term, Mortality.

The symptoms of each of the maternal outcome are taken in intuitionistic fuzzy values as shown in the tables below.

Table 18.1 is the raw data obtained from field, before transformation into intuitionistic maternal dataset, maternal patients against the symptoms in Tables 18.2 and 18.3.

Table 18.2 presents the maternal patients against the symptoms in intuitionistic fuzzy format; this was achieved using a numerical method, fuzzier, and the dataset. However, we considered the symptoms of each of the maternal outcomes using Intuitionistic Fuzzy, because these maternal symptoms have lots of hesitation associated with them. During examination, some of the symptoms may not be seen, but may later surface as the pregnancy increases, thereby causing some level of delay. Most especially, it is the third level of delay, where the health experts may find it difficult to understand the best type of treatment or management to give to the patient. Table 18.2 is then transformed into maternal outcome and diagnosis to generate Table 18.3.

Table 18.3 represents the transformation of maternal outcome and diagnosis; this was achieved using the numerical method. Tables 18.2 and 18.3 are used to compute Table 18.4 using the best approach in [50] and Table 18.5 using the best approach in [47]. However, Table 18.6 presents the results from the new method of intuitionistic fuzzy distance measure, where the weight of the symptoms is given by a set

$$\omega = \{\mathbf{0.1}, \mathbf{0.1}, \mathbf{0.05}, \mathbf{0.05}, \mathbf{0.1}, \mathbf{0.05}, \mathbf{0.05}, \mathbf{0.05}, \mathbf{0.05}, \mathbf{0.05}, \mathbf{0.1}, \mathbf{0.05}, \mathbf{0.05}, \mathbf{0.1}, \mathbf{0.05}\}.$$

Table 18.2 Intuitionistic transformation of the maternal dataset

	MBPM	MW	HL	PCVL	PR	MOD	MP	HC	DM	HI	RD	MA	ACC	PREE	AB
M1	(0.4,0.5,0.1)	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.5, 0.4, 0.1)	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.5, 0.4, 0.1)
M2	(0.5, 0.4, 0.1)	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.5, 0.4, 0.1)
M3	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.5, 0.4, 0.1)	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.4,0.1,0.5)	(0.5, 0.4, 0.1)
M4	(0.4,0.2, 0.4)	(0.4,0.2,0.4)	(0.4,0.2,0.4)	(0.4,0.2,0.4)	(0.4,0.2,0.4)	(0.4, 0.4, 0.2)	(0.4,0.2,0.4)	(0.4,0.2,0.4)	(0.4,0.2, 0.4)	(0.4,0.2, 0.4)	(0.4,0.2, 0.4)	(0.4,0.2, 0.4)	(0.4,0.2, 0.4)	(0.4,0.2, 0.4)	(0.4, 0.4, 0.2)
M5	(0.2,0.7, 0.1)	(0.2,0.7, 0.1)	(0.2,0.7, 0.1)	(0.2,0.7, 0.1)	(0.2,0.7, 0.1)	(0.2,0.7, 0.1)	(0.2,0.7, 0.1)	(0.2,0.7, 0.1)	(0.2,0.7, 0.1)	(0.2,0.7, 0.1)	(0.2,0.7, 0.1)	(0.2,0.7, 0.1)	(0.2,0.7, 0.1)	(0.2,0.7, 0.1)	(0.7, 0.2, 0.1)

Table 18.3 Maternal outcome and diagnosis

	MBPM	MW	HL	PCVL	PR	MOD	MP	HC	DM	HI	RD	MA	ACC	PREE	AB
Miscarriage	(0.2, 0.3, 0.5)	(0.3,0.2,0.5)	(0.5, 0.2,0.3)	(0.5, 0.2,0.3)	(0.3,0.2,0.5)	(0.5, 0.2,0.3)	(0.5, 0.2,0.3)	(0.3,0.2,0.5)	(0.5, 0.2,0.3)	(0.5, 0.2,0.3)	(0.3,0.2,0.5)	(0.5, 0.2,0.3)	(0.3,0.2,0.5)	(0.5, 0.2,0.3)	(0.2, 0.3, 0.5)
Pre-Preterm (Contraction)	(0.3,0.2,0.5)	(0.5, 0.2,0.3)	(0.5, 0.2,0.3)	(0.2, 0.3, 0.5)	(0.3,0.2,0.5)	(0.3,0.2,0.5)	(0.5, 0.2,0.3)	(0.2, 0.3, 0.5)	(0.5, 0.2,0.3)	(0.5, 0.2,0.3)	(0.3,0.2,0.5)	(0.5, 0.2,0.3)	(0.3,0.2,0.5)	(0.3,0.2,0.5)	(0.5, 0.2,0.3)
Still birth	(0.3,0.2,0.5)	(0.5, 0.2,0.3)	(0.5, 0.2,0.3)	(0.5, 0.2,0.3)	(0.3,0.2,0.5)	(0.3,0.2,0.5)	(0.5, 0.2,0.3)	(0.3,0.2,0.5)	(0.5, 0.2,0.3)	(0.5, 0.2,0.3)	(0.3,0.2,0.5)	(0.2, 0.3, 0.5)	(0.3,0.2,0.5)	(0.5, 0.2,0.3)	(0.5, 0.2,0.3)
Placentae Previa	(0.2, 0.6, 0.2)	(0.2, 0.6, 0.2)	(0.2, 0.6, 0.2)	(0.2, 0.2,0.6)	(0.2, 0.6, 0.2)	(0.6, 0.2, 0.2)	(0.6, 0.2, 0.2)	(0.2, 0.6, 0.2)	(0.2, 0.6, 0.2)	(0.6, 0.2, 0.2)	(0.2, 0.6, 0.2)	(0.6, 0.2, 0.2)	(0.2, 0.6, 0.2)	(0.2, 0.6, 0.2)	(0.6, 0.2, 0.2)
UTI	(0.2, 0.7, 0.1)	(0.2, 0.7, 0.1)	(0.2, 0.7, 0.1)	(0.2, 0.7, 0.1)	(0.2, 0.7, 0.1)	(0.7,0.2,0.1)	(0.2, 0.7, 0.1)	(0.2, 0.7, 0.1)	(0.7,0.2,0.1)	(0.2, 0.1,0.7)	(0.7,0.2,0.1)	(0.2, 0.7, 0.1)	(0.2, 0.7, 0.1)	(0.2, 0.7, 0.1)	(0.7,0.2,0.1)
FULL-TERM	(0.2, 0.8, 0.0)	(0.2, 0.8, 0.0)	(0.2, 0.8, 0.0)	(0.2, 0.8, 0.0)	(0.2, 0.8, 0.0)	(0.8, 0.2, 0.0)	(0.2, 0.8, 0.0)	(0.2, 0.8, 0.0)	(0.2, 0.8, 0.0)	(0.2, 0.8, 0.0)	(0.2, 0.8, 0.0)	(0.8, 0.2, 0.0)	(0.2, 0.8, 0.0)	(0.2, 0.8, 0.0)	(0.8, 0.2, 0.0)
Mortality	(0.1, 0.0, 0.9)	(0.1, 0.0, 0.9)	(0.0, 0.1, 0.9)	(0.1, 0.0, 0.9)	(0.1, 0.0, 0.9)	(0.1, 0.0, 0.9)	(0.0, 0.1, 0.9)	(0.1, 0.0, 0.9)	(0.1, 0.0, 0.9)	(0.1, 0.0, 0.9)	(0.1, 0.0, 0.9)	(0.1, 0.0, 0.9)	(0.0, 0.1, 0.9)	(0.1, 0.0, 0.9)	(0.1, 0.0, 0.9)

Table 18.4 Distance values between patients and maternal outcomes using Szmidt and Kacprzk [50] methods

	Miscarriage	Preterm (Contraction)	Still birth	Placentae previa	UTI	Full term	Mortality
Szmidt and Kacprzk Distance measures							
M1	**0.11522**	**0.10656**	**0.11322**	0.27972	0.37962	0.46753	0.37562
M2	**0.12587**	**0.12055**	**0.12321**	0.27173	0.37796	0.45721	0.39161
M3	**0.11255**	**0.10390**	**0.11489**	0.27373	0.36663	0.44346	0.39161
M4	0.09457	0.09291	**0.08858**	0.41525	0.31968	0.41525	0.43556
M5	0.35798	0.34032	0.34166	0.18648	**0.09124**	0.15551	0.64336
Szmidt and Kacprzk Similarity Measures							
M1	0.88478	0.89344	**0.88678**	0.72028	0.62038	0.53247	0.62438
M2	0.87413	**0.87945**	0.87679	0.72827	0.62205	0.54279	0.60839
M3	0.88745	**0.89610**	0.88512	0.72627	0.63337	0.55654	0.60839
M4	0.90543	0.90709	**0.91142**	0.58475	0.68032	0.58475	0.56444
M5	0.64202	**0.65968**	0.65834	0.81352	0.90876	0.84449	0.35664

Table 18.5 Distance values between patients and maternal outcomes using Ejegwa et al. [47] method

	Miscarriage	Preterm (Contraction)	Still birth	Placentae previa	UTI	Full term	Mortality
Distance measures							
M1	0.11988	0.11544	0.11544	0.27084	0.35520	0.42846	0.28416
M2	0.12876	0.12876	0.12876	0.26196	0.34188	0.41514	0.30192
M3	0.11988	0.11544	0.11544	0.26640	0.36852	0.41514	0.30192
M4	0.08436	0.08436	0.07992	0.37074	0.28860	0.37074	0.35076
M5	0.32856	0.30192	0.30636	0.15540	0.07104	0.12210	0.51948
Distance Measures							
M1	0.88012	**0.88456**	0.88456	0.72916	0.64480	0.57154	0.71584
M2	**0.87124**	**0.87124**	0.87124	0.73804	0.65812	0.58486	0.69808
M3	0.88012	**0.88456**	0.88456	0.73360	0.63148	0.58486	0.69808
M4	0.91564	0.91564	**0.92008**	0.62926	0.71140	0.62926	0.64924
M5	0.67144	**0.69808**	0.69364	0.84460	0.92896	0.87790	0.48052

The weights of the symptoms are generated using assumptive approach.

Table 18.4 is the computation of the most appropriate method of Szmidt and Kacprzk [50] in terms of patients (M1, M2, M3, M4, and M5) with the likely maternal outcomes.

Table 18.5 is the computation using the most appropriate method of Ejegwa et al. [47] between patients (M1, M2, M3, M4, and M5) and the likely maternal outcomes.

Table 18.6 indicates the computation using the new method of intuitionistic fuzzy distance measure and its similarity measure in terms of patients (M1, M2, M3, M4, and M5) with the likely maternal outcomes.

Table 18.6 New method of intuitionistic fuzzy distance measure

	Miscarriage	Preterm (Contraction)	Still birth	Placentae previa	UTI	Full-term	Mortality
New method of intuitionistic fuzzy distance measure							
M1	0.00755	0.00733	0.00710	0.01909	0.02353	0.02919	0.01865
M2	0.00844	0.00866	0.00866	0.01820	0.02220	0.02786	0.02042
M3	0.00755	0.00733	0.00733	0.01887	0.02464	0.02786	0.02042
M4	0.00555	0.00555	0.00511	0.02520	0.01909	0.02520	0.02309
M5	0.02176	0.02042	0.02087	0.00888	0.00599	0.00722	0.03485
New method of intuitionistic fuzzy similarity measures							
M1	0.99245	0.99267	**0.99290**	0.98091	0.97647	0.97081	0.98135
M2	**0.99156**	0.99134	0.99134	0.98180	0.97780	0.97214	0.97958
M3	**0.99245**	0.99267	0.99267	0.98113	0.97536	0.97214	0.97958
M4	0.99445	0.99445	**0.99489**	0.97480	0.98091	0.97480	0.97691
M5	0.97824	**0.97958**	0.97913	0.99112	0.99401	0.99279	0.96515

However, Tables 18.4 shows the intuitionistic transformation of the maternal dataset to obtain the distance between patients and maternal outcomes, as well as maternal patients and diagnosis for each patient. We applied similarity measurement in Table 18.4 to validate the result, which gave Tables 18.5 and 18.6. These are the similarity measurement using Szmidt and Kacprzk [50], Ejegwa et al. [47], and new intuitionistic method. The decision making is considered in two forms: (a) horizontal decision concerning the maternal outcome, (b) vertical decision concerning patients for each of the methods. Decisions are made based on the greater value of relationships between patients and maternal outcomes for similarity and the least value of relationships between patients and maternal outcomes for distance.

From Table 18.4, we see that M1, M2, and M3 are predictable to suffer from contraction, M4 is predictable to suffer from still birth, and M5 is predictable to suffer from UTI. From Table 18.5, we see that M1 is predictable to suffer from contraction and still birth; M2 is predictable to suffer from miscarriage, contraction, and still birth; M3 is predictable to suffer from contraction and still birth; M4 is predictable to suffer from still birth; and M5 is predictable to suffer from UTI. From Table 18.6, we see that M1 is predictable to suffer from still birth, M2 is predictable to suffer from miscarriage, M3 is predictable to suffer from contraction and still birth, M4 is predictable to suffer from still birth, and M5 is predictable to suffer from UTI. Szmidt and Kacprzk [50] method, Ejegwa et al. [47] method, and the new method show that M4 and M5 are predictable to suffer from still birth and UTI, respectively.

By comparison, the new method is the most appropriate distance/similarity for IFSs because it yields the least/greatest measure. In the same vein, Fig. 18.1 gives a clear picture when compared with the result obtained using the two methods. Each of the methods could be used by physicians to predict the current condition of mothers within the bearing age, with probability values ranging from 0 to 1.

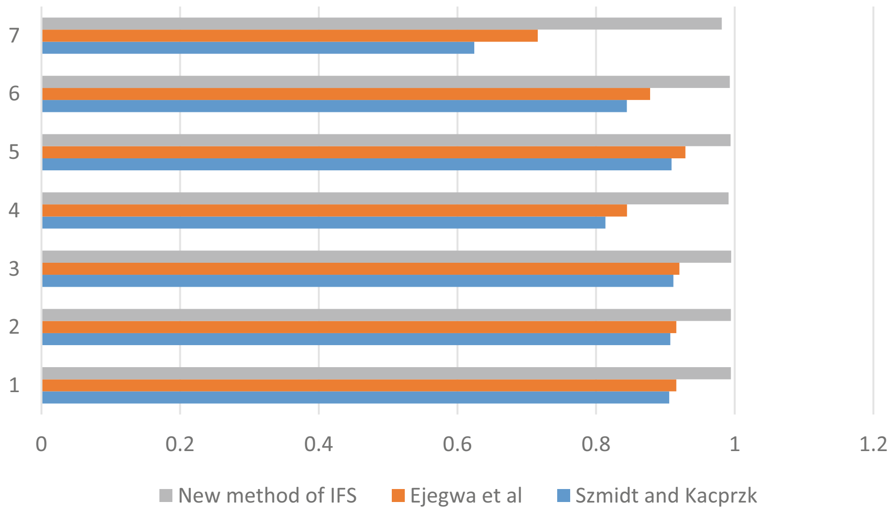

Fig. 18.1 Clustered bar for predicting maternal outcome using both methods

The third method have a slight difference in terms of predicting still-birth, preterm, miscarriage, and UTI, while for placentae previa and full-term outcome, it is clear that the third method produce significant differences in predicting the maternal outcomes. In predicting various maternal outcomes, third method showed a huge difference in predicting mortality as one of the outcomes among mother within the bearing age. The third method been considered in the works are still not perfect but have their limitations and it is based on this that health experts, exposure several measures and avenues available to them, in order to solve problem associated to maternal health and its outcomes. There is no doubt that some socio-cultural factors like early marriage/early childbearing, educational attainment, women's decision-making power, traditional obstetric care service, female genital mutilation, economic status, and access to health-care services should be addressed to reduce mortality among mothers within the bearing age. In line with this, the work of Marchieand Anyanwu [55] asserted that mortality among women within the bearing age can be addressed by encouraging and sensitizing mothers, also encouraging them to welcome early/prompt decisions to seek medical care in an emergency.

Figure 18.2 portrays the strength of each of the methods used in this work. However, the method of Szmidt and Kacprzk [50] can also be used to predict maternal outcomes among mothers within the bearing age. However, the method by Ejegwa et al. [47] was used to predict the maternal outcome within minimal probabilistic prediction in each of the outcomes used in this study. Also the new method connotes a strong linear representation, which is a clear evidence that the new method could be used to predict maternal outcomes with little or no error. The method helps to handle the uncertainty that may be associated with handling maternal outcomes. Nevertheless, the graph in Fig. 18.2 further illustrates that both

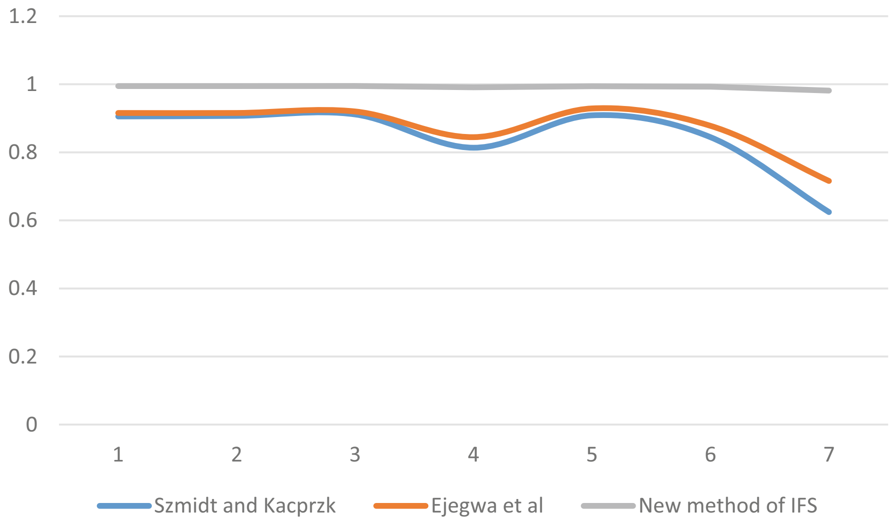

Fig. 18.2 Similarity line plot for predicting maternal outcome using three methods

methods have little similarity in terms of predicting outcomes like miscarriage, preterm, UTI, and stillbirth, while placentae previa and full term have slight differences, although they still have almost the same prediction. Thus, aside from mortality, both methods tend to have significant distance in terms of predication.

It's clear that the different methods used in the experiments show that the patients in the dataset might have to deal with pre-term contractions and stillbirths. It's a real-life situation that highlights some serious health issues that mothers in the reproductive age group could face. According to Malacova et al. [56], preterm and still birth are strongly interrelated and each of these conditions predisposes women to the other outcomes such as cerebral palsy, mental retardation, visual and hearing impairments, and poor health and growth. It is undeniable that mothers of babies facing these challenges undergo emotional stress and may encounter various health problems. This emotional strain can lead to negative feelings towards their babies during the early postnatal months, potentially hindering their ability to provide the necessary postnatal services and support. In addition, we have shown that there is a small but significant correlation between these adverse pregnancy outcomes: miscarriage, preterm, and still birth. Most especially, women who have experienced miscarriage are also more likely to have had a stillbirth. Subsequently, the works of Hure et al. [57] and Khalil et al. [58] pointed out that some factors that could lead to miscarriage, preterm delivery, and stillbirth, among others, are age at first birth, number of live births, smoking status, fertility problems, use of in vitro fertilization (IVF), and level of education. Without hesitation, the experience of miscarriage, preterm delivery, and stillbirth is generally regarded as very stressful for those involved.

5 Conclusion

Due to the complexity and uncertainty associated with maternal health, the intuitionistic fuzzy distance/similarity methods used in this study can be adopted to improve decision making toward maternal outcomes. Each of these maternal outcomes when not properly managed may lead to mortality among women within the bearing age. The methods have showcased the importance of applying intuitionistic fuzzy sets instead of fuzzy sets because of the introduction of additional degrees of freedom (nonmemberships and hesitation margins) into the set description. These approaches are adequate because they help deal with multiple outcomes such as yes, no, abstaining, and so on, which was applicable in our study where we have stillbirth, preterm, full-term, mortality, placentae previa, and UTI, respectively. This approach gives an additional possibility to represent imprecise knowledge to the physician to describe what is likely to be the patient outcome and also to proffer solution to the problem. In this work, we have demonstrated that intuitionistic fuzzy distance/similarity methods can be used to predict the maternal outcomes. The new method IFSs outperforms the methods by Ejegwa et al. [47] and that of Szmidt and Kacprzk [50]with a slight difference. In general, the work has contributed to the body of knowledge in terms of the following: (i) improving decision support, (ii) enhancing distance and similarity measures, (iii) comparative analysis to showcase the strength of the two methods, (iv) employing of distance and similarity measures to predict the maternal outcome, and (v) better performance rating between the two methods. Our dataset can be applied using technique for order performance by similarity to ideal solution (TOPSIS) method to enhance multi-attributes decision making for mother within the bearing age. This work is novel in terms of similarity-distance measures of IFS, and so can be applied in other variants of fuzzy sets with little modifications.

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Chapter 19
Study of Fuzzy Fractional Caputo Order Approach to Diabetes Model

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1 Introduction

Today, diabetes is a silent epidemic that is greatly increasing the burden of noncommunicable illnesses and is often stoked by decreased levels of exercise and an increase in the prevalence of obesity.

Currently, mathematics and biology are collaborating for a common good. Many diverse biological problems may be modeled mathematically. Among of them is modeling after diabetes and prediabetes. When levels of blood sugar are higher than usual but not high sufficient to be classified as diabetes, the condition is called prediabetes.

Diabetes is mostly a chronic illness that affects people. Diabetes is a metabolic syndrome that is often brought on by inherited and environmental factors, including obesity, increased blood pressure, a family history of cardiovascular disease, excessive triglycerides, etc. The following categories apply to diabetes:

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C. Jana et al. (eds.), *Fuzzy Optimization, Decision-making and Operations Research*, https://doi.org/10.1007/978-3-031-35668-1_19

- *Type-1 diabetes:* Type 1 diabetes is an autoimmune condition that is metabolism. Blood is filled with glucose from eating. Glucose has to pass a gate in the cell wall to enter the body's cells. These gates can be opened by insulin.
- *Type-2 diabetes:* Because all these gates are more challenging to open in type-2 diabetes, insulin must work harder to open them, which in turn requires a greater effort on the part of the pancreas.
- *Gestational diabetes:* Diabetes mellitus gestational diabetes is another name for the condition. It happens when a pregnant woman without diabetes has high levels of blood sugar.

Epidemiology refers to the study of how diseases spread inside a living thing in connection to its environment [1]. Numerical simulations may be used to study a disease' epidemiology. Contagious diseases including measles [2], rubella [3], HIV [4], dengue fever [5], TB [6], and more recently, Ebola [7] and the Zika virus [8], have all been the subject of research that have sought to forecast and mimic the spread of these diseases in the past. As research develops, mathematical modeling is being used to explore not just the spread of communicable illnesses as well as an expanding number of noncommunicable disorders. It is often possible to mimic medications as well as other environmental illnesses [9]. This is conceivable because of the qualities of how it disseminates, namely, via closer contact even as media disseminates. One of the "dispersion" aspects of hyperglycemia, a quasi illness, is the impact of social interactions on dietary adjustment. If a person's fasting blood sugar level is more than 126 mg/dL or even if their level of blood sugar is higher than 200 mg/dL 2 hours following eating, they are considered to have hyperglycemia. The glycemic connection has been described using a variety of formulations based on insulin concentrations and levels. However, while these approaches may indeed be beneficial in research, they all have restrictions with respect to estimating blood sugar levels in a real therapeutic scenario owing to the underlying demand for continuously modified inputs about parameter estimates such as glucose concentrations and insulin accessibility [10].

Diabetics who are healthy choose to lead uncomfortable lives in their daily activities. According to Hill et al. [11], the association between diabetics on an unhealthy diet and healthy volunteers might lead to behavioral "dissemination." The result of culture "spread" is prevalence. As high pervasiveness increases, diabetic predominance emerges. Calculating the percentage of likely interaction also requires determining the ratio of vulnerable people in a given individual group. In current history, several writers, including Boutayeb et al. [12], Mahata et al. [13], Pandit et al. [14], Makroglou et al. [15], have investigated various mathematical systems to represent diabetes and associated problems. The techniques of Caputo [16], Baleanu et al. [17], Singh et al. [18, 19], and Kumar et al. [20] are only a few examples as to how fractional extensions of mathematical systems of integer order describe the natural reality in a fairly methodical manner.

The arrangement of the chapter is as follows: Sect. 2 contains the basic concept. Model formation and stability analysis are smeared in Sect. 3 and Sect. 4. Numerical illustration is prepared in Sect. 5. Section 6 contains conclusion of the work.

2 Pre-requisite Concepts

Definition 1 [21] The Caputo fractional derivative of order $0 < \phi \leq 1$ for the function $u : C^n[0, \infty] \to \mathbb{R}$ is defined as ${}^{C}D_t^{\phi}\,(u(t)) = \frac{1}{\Gamma(n-\phi)} \int_0^t \frac{1}{(t-z)^{\phi+1-n}} \frac{d^n}{dz^n} u(z)dz$, where $C^n[0, \infty]$ is a n tines continuously differentiable function and the Gamma function is defined by $\Gamma(,)$ such that $n - 1 < \phi < n$.

Definition 2 [22] A triangular fuzzy number can be represented by three points $\widetilde{P_1} = (P_{11}, P_{12}, P_{13})$ and the membership function can be represented by

$$\mu_{\widetilde{R}}(x_1) = \begin{cases} 0, & x_1 \leq P_{11} \\ \frac{x_1 - P_{11}}{P_{12} - P_{11}}, & P_{11} \leq x_1 \leq P_{12} \\ 1, & x_1 = P_{12} \\ \frac{P_{13} - x_1}{P_{13} - P_{12}}, & P_{12} \leq x_1 \leq P_{13} \\ 0 & x_1 \geq P_{13} \end{cases}$$

Definition 3 [23 Generalized Hukuhara Derivative (gHD)] A fuzzy valued function $f_1 : (a, b) \to \mathbb{R}$ at p_0 is defined by,

$f_1{}'(p_0) = \lim\limits_{l \to 0} \frac{f_1(p_0+l) \ominus_{gh} f_1(p_0)}{l}$, $f_1{}'(x_0) \in \mathbb{R}$ satisfy the condition then G_h say that GH derivative at p_0.

Now, f_1(t) is (i)-gHD at p_0 when $[f'(p_0)]_\alpha = [f_{11}{}'(p_0, \alpha), f_{12}{}'(p_0, \alpha)]andf_1$(t) is (ii)-gHD at p_0 when $[f'(p_0)]_\alpha = [f_{12}{}'(p_0, \alpha), f_{11}{}'(p_0, \alpha)]$.

Definition 4 [24] Let F be an interval-valued function of type 2 defined on the domain $[0, \infty)$ given by $F(y) = [f_1(y), f_2(y]$. Then, $F$is said to be (i) Φ- gH differentiable of type-I, then $F^{\Phi}(y) = [f_1{}^{\Phi}(y), f_2{}^{\Phi}(y)\ ]$ and (ii) Φ- gH differentiable of type-II, then $F^{\Phi}(y) = [f_2{}^{\Phi}(y),\ f_1{}^{\Phi}(y)]$.

3 Model Formulation

As per Roy and Mahata et al. [23], the integer order system may be expressed as

$$D_t G(t) = -a_1 G(t) + a_2 I(t) + p,$$

$$D_t I(t) = -a_3 G(t) - a_4 I(t) + q, \tag{19.1}$$

with initial state $G(0) = G_0,\ I(0) = I_0$, where $D_t \equiv \frac{\mathrm{d}}{\mathrm{dt}}$ (Table 19.1).

Over the last several decades, fractional differential equations (FDEs) [25–28] have been used to investigate physical processes with greater accuracy and precision. Other medical and scientific study fields that utilize them include biology,

Table 19.1 Descriptions of the model parameters

Notation	Interpretations
G	The variation of glucose levels from their average physiological value
I	The insulin concentration's departure from its physiologically recommended mean value
p	Divided by the extracellular compartment value is the intravenous injection function I (Insuline)
q	Value obtained by dividing the intravenous injection function G (glucose) by the extracellular compartment
a_1	The ability of insulin to increase insulin concentration and its sensitivity to do so
a_2	The ability of pancreatic insulin to increase blood glucose levels
a_3	Hepatic glycogen store and tissue glucose uptake are both sensitive to increased insulin levels
a_4	The capacity of liver glycogen store to increase insulin concentration when tissue insulin is used

physics, and others. Modern calculus is an extension of traditional integer-order calculus. FDEs are becoming more and more popular for replicating real-world scenarios because of their distinct properties that DEs lack. FDEs are preferred to Des, since they are nonlocal and also have memory effects. Furthermore, the model's future scenario is often influenced by both its past and present states, which is why FDEs have drawn a lot of interest from academics in recent years. This is because FDEs, as opposed to integer-order models, can more accurately account for the retention and heritable properties of various materials and processes. Our mathematical model is reformulated using the Caputo fractional derivative for fractional index ϕ as:

$$
{}^{C}D_t^{\phi} G(t) = -a_1 G(t) + a_2 I(t) + p,
$$

$$
{}^{C}D_t^{\phi} I(t) = -a_3 G(t) - a_4 I(t) + q. \tag{19.2}
$$

Depending on how accurate the data or information is, FDEs or ODEs may be used to analyze the mathematical models of many physical situations. Fuzzy fractional operators are the right instruments to represent physical issues when there is uncertainty or fuzziness in the information of a model. The fuzzy set, which is a generalization of the crisp set, was introduced for the first time by L.A. Zadeh [29]. Later, fuzzy DEs and fuzzy FDEs were proposed [30–33]. Numerous practical issues made use of the concept of fuzzy FDEs. One study of the fractional relaxation oscillation DEs in a fuzzy notion was conducted by Armand et al. [26]. Rahamanet al. [34] studied the solution of an Economic Production Quantity model using the generalized Hukuhara derivative approach. By using the fuzzy Laplace transform (FLT), Ahmad et al. [35] studied the fuzzy fractional Fisher's equation. Fuzzy dispersive PDE has been researched using FLT in the literature. We used model

(19.2) under the fuzzy fractional Caputo derivative as follows, which was inspired by the above work:

$$ {}^{C}D_t^{\phi} G(t) = -a_1 G(t) + a_2 I(t) + p, $$

$$ {}^{C}D_t^{\phi} I(t) = -a_3 G(t) - a_4 I(t) + q. \tag{19.3} $$

with initial states $G(0) = \tilde{G}_0, \quad I(0) = \tilde{I}_0$.

We consider the following cases:

Case 1: When $\tilde{G}(t)$ and $\tilde{I}(t)$ are gHD type-I
Case 2: When $\tilde{G}(t)$and $\tilde{I}(t)$ are gHD type-II
Case 3: When $\tilde{G}(t)$ is gHD type-I and $\tilde{I}(t)$ is gHD type-II
Case 4: When $\tilde{G}(t)$ is gHD type-II and $\tilde{I}(t)$ is gHD type-II

Let us consider the initial conditions are fuzzy numbers.

Case 1: When $\tilde{G}(t)$, $\tilde{I}(t)$ are Φ-gH differentiable of type I The Eq. (19.3) changes to

$$ {}^{C}D_t^{\phi} G_1(t,\alpha) = -a_1 G_2(t,\alpha) + a_2 I_1(t,\alpha) + p $$

$$ {}^{C}D_t^{\phi} G_2(t,\alpha) = -a_1 G_1(t,\alpha) + a_2 I_2(t,\alpha) + p $$

$$ {}^{C}D_t^{\phi} I_1(t,\alpha) = -a_3 G_2(t,\alpha) - a_4 I_2(t,\alpha) + q \tag{19.4} $$

$$ {}^{C}D_t^{\phi} I_2(t,\alpha) = -a_3 G_1(t,\alpha) - a_4 I_1(t,\alpha) + q $$

Case 2: When $\tilde{G}(t)$, $\tilde{I}(t)$ are Φ-gH differentiable of type II The Eq. (19.3) changes to

$$ {}^{C}D_t^{\phi} G_1(t,\alpha) = -a_1 G_1(t,\alpha) + a_2 I_2(t,\alpha) + p $$

$$ {}^{C}D_t^{\phi} G_2(t,\alpha) = -a_1 G_2(t,\alpha) + a_2 I_1(t,\alpha) + p $$

$$ {}^{C}D_t^{\phi} I_1(t,\alpha) = -a_3 G_1(t,\alpha) - a_4 I_1(t,\alpha) + q \tag{19.5} $$

$$ {}^{C}D_t^{\phi} I_2(t,\alpha) = -a_3 G_2(t,\alpha) - a_4 I_2(t,\alpha) + q $$

4 Model Analysis

4.1 Existence of Equilibrium Point of the System (19.4)

The model (19.4) has only coexistence equilibrium point, $E_1{}^*(G_1{}^*,\ G_2{}^*,\ I_1{}^*,\ I_2{}^*)$, where $G_1{}^* = G_2{}^* = \frac{a_4 p + a_2 q}{a_2 \boldsymbol{a_3} + a_1 a_4}$, $I_1{}^* = \boldsymbol{I_2}{}^* = \frac{\boldsymbol{a_1 q - q a_3}}{\boldsymbol{a_2 a_3 + a_1 a_4}}$. The equilibrium point $\boldsymbol{E_1}{}^*$ is feasible if $\boldsymbol{a_1 q} > \boldsymbol{q a_3}$.

4.2 Stability Analysis of the System (19.4)

Theorem 1 The $E_1{}^*(G_1{}^*,\ G_2{}^*,\ I_1{}^*,\ I_2{}^*)$ of the system (19.2) is unstable.

Proof: Let the Jacobi matrix of the model (19.4) at $E_1{}^*(G_1{}^*, G_2{}^*, I_1{}^*, I_2{}^*)$ be

$$J_{E_1{}^*} = \begin{pmatrix} 0 & -a_1 & a_2 & 0 \\ -a_1 & 0 & 0 & a_2 \\ 0 & -a_3 & 0 & -a_4 \\ -a_3 & 0 & -a_4 & 0 \end{pmatrix}.$$

The characteristic equation becomes

$\lambda^4 + \mu_1 \lambda^3 + \mu_2 \lambda^2 + \mu_3 \lambda + \mu_4 = 0$, where the eigenvalue is λ.

where $\mu_1 = 0$, $\mu_2 = -(a_1{}^2 + a_4{}^2)$, $\mu_3 = 2(a_1 a_2 a_3 + a_2 a_3 a_4)$, $\mu_4 = a_1{}^2 a_4{}^2 - a_2{}^2 a_3{}^2$

Using stability condition of RH -criteria, the system (19.4) is unstable at $E_1{}^*(G_1{}^*,\ G_2{}^*,\ I_1{}^*,\ I_2{}^*)$.

4.3 Existence of Equilibrium Point of the System (19.5)

The model (19.5) has only coexistence equilibrium point, $E_1{}^{**}(G_1{}^{**}, G_2{}^{**}, I_1{}^{**}, I_2{}^{**})$ where $G_1{}^{**} = G_2{}^{**} = \frac{p a_4 + q a_2}{a_1 a_4 + a_2 a_3}$, $I_1{}^{**} = I_2{}^{**} = \frac{a_1 q - q a_3}{a_2 a_3 + a_1 a_4}$. The equilibrium point $E_1{}^*$ is feasible if $a_1 q > q a_3$.

4.4 Stability Analysis of the System (19.5)

Theorem 2 The $E_1{}^{**}(G_1{}^{**},\ G_2{}^{**},\ I_1{}^{**}, I_2{}^{**})$ of the system (19.2) is unstable.

Proof: Let the Jacobi matrix of the model (19.5) at $E_1{}^{**}(G_1{}^{**},\ G_2{}^{**},\ I_1{}^{**}, I_2{}^{**})$ be

$$J_{E_1^{**}} = \begin{pmatrix} -a_1 & 0 & 0\ a_2 \\ 0 & -a_1 & a_2\ 0 \\ -a_3 & 0 & -a_4\ 0 \\ 0 & -\ a_3 & 0\ -a_4 \end{pmatrix}.$$

The characteristic equation becomes
$\lambda^4 + r_1\lambda^3 + r_2\lambda^2 + r_3\lambda + r_4 = 0$, where the eigenvalue is λ.
where $r_1 = 2(a_1 + a_4)$, $r_2 = (a_1{}^2 + a_4{}^2 + 4a_1a_4)$, $r_3 = (2a_1{}^2a_4 + 2a_1a_4{}^2)$, $r_4 = a_2{}^2a_3{}^2 + a_1{}^2a_4{}^2$.
Using stability condition of RH -criteria, the system (19.5) is stable at $E_1{}^{**}(G_1{}^{**}, G_2{}^{**}, \ I_1{}^{**}, I_2{}^{**})$.

5 Numerical Illustrations

Numerical simulations are carried out with the help of MATLAB software to support the mathematical study of the model system (Tables 19.2, 19.3, and 19.4).

Figure 19.1 depicts the fuzzy solution of the model system (19.4) for $\phi = 0.96$ when $t \in [0, 20]$. In Fig. 19.1a, b, we observed that $G_1(t, \alpha) \le G_2(t, \alpha)$, $I_1(t, \alpha) \le I_2(t, \alpha)$ for $t \in [0, 20]$, which imply that strong fuzzy solution [23] exists. Also in Fig. 19.1c, we find that $G_1(t, \alpha) = G_2(t, \alpha)$, $I_1(t, \alpha) = I_2(t, \alpha)$ for

Table 19.2 Parameter values

Notation	Interpretations	Source
p	500	[23]
q	800	[23]
a_1	0.03	[23]
a_2	0.07	[23]
a_3	0.04	[23]
a_4	0.08	[23]

Table 19.3 The fuzzy solutions of the model system (19.4) for $t = 10$, $\Phi = 0.96$

α	$G_1(t,\alpha)$	$G_2(t,\alpha)$	$I_1(t,\alpha)$	$I_2(t,\alpha)$
0	5762.4	6114.6	4199.9	4585.3
0.1	5.7790	6.0960	4.2188	4.5656
0.2	5.7957	6.0774	4.2377	4.5460
0.3	5.8123	6.0589	4.2567	4.5264
0.4	5.8289	6.0403	4.2756	4.5068
0.5	5.8456	6.0217	4.2945	4.4871
0.6	5.8622	6.0031	4.3134	4.4675
0.7	5.8789	5.9845	4.3323	4.4479
0.8	5.8955	5.9659	4.3512	4.4282
0.9	5.9121	5.9474	4.3701	4.4086
1	5.9288	5.9288	4.3890	4.3890

Table 19.4 The fuzzy solutions of the model system (19.5) for $t = 10$, $\Phi = 0.96$

α	$G_1(t,\alpha)$	$G_2(t,\alpha)$	$I_1(t,\alpha)$	$I_2(t,\alpha)$
0	5.9070	5.9700	4.3737	4.4115
0.1	5.9092	5.9658	4.3752	4.4093
0.2	5.9114	5.9617	4.3768	4.4070
0.3	5.9136	5.9576	4.3783	4.4048
0.4	5.9157	5.9535	4.3798	4.4025
0.5	5.9179	5.9494	4.3813	4.4003
0.6	5.9201	5.9452	4.3829	4.3980
0.7	5.9222	5.9411	4.3844	4.3957
0.8	5.9244	5.9370	4.3859	4.3935
0.9	5.9266	5.9329	4.3875	4.3912
1	5.9288	5.9288	4.3890	4.3890

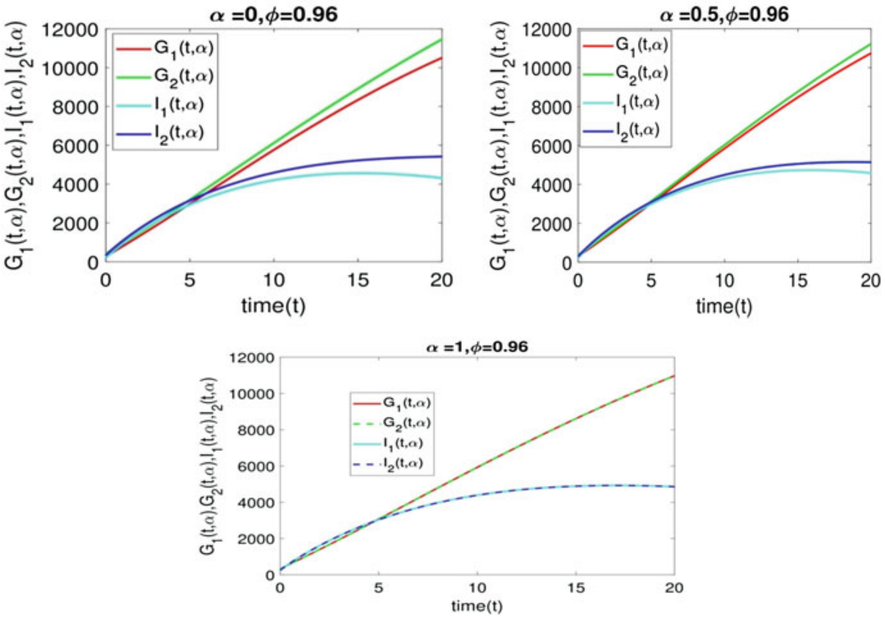

Fig. 19.1 Time series solution with $\phi = 0.96$ of the model (19.4) for $\alpha = 0, 0.5, 1$

$t \in [0, 20]$. Figure 19.1 depicts that the equilibrium point of the system (19.4) is unstable.

Figure 19.2 depicts the fuzzy solution of the model system (19.5) for $\phi = 0.96$ when $t \in [0, 100]$. In Fig. 19.2a, b, c, we find that $G_1(t,\alpha) = G_2(t,\alpha)$, $I_1(t,\alpha) = I_2(t,\alpha)$ for $t \in [0, 20]$, which imply that strong fuzzy solution [23] exists. Figure 19.2 depicts that the equilibrium point of the system (19.4) is stable.

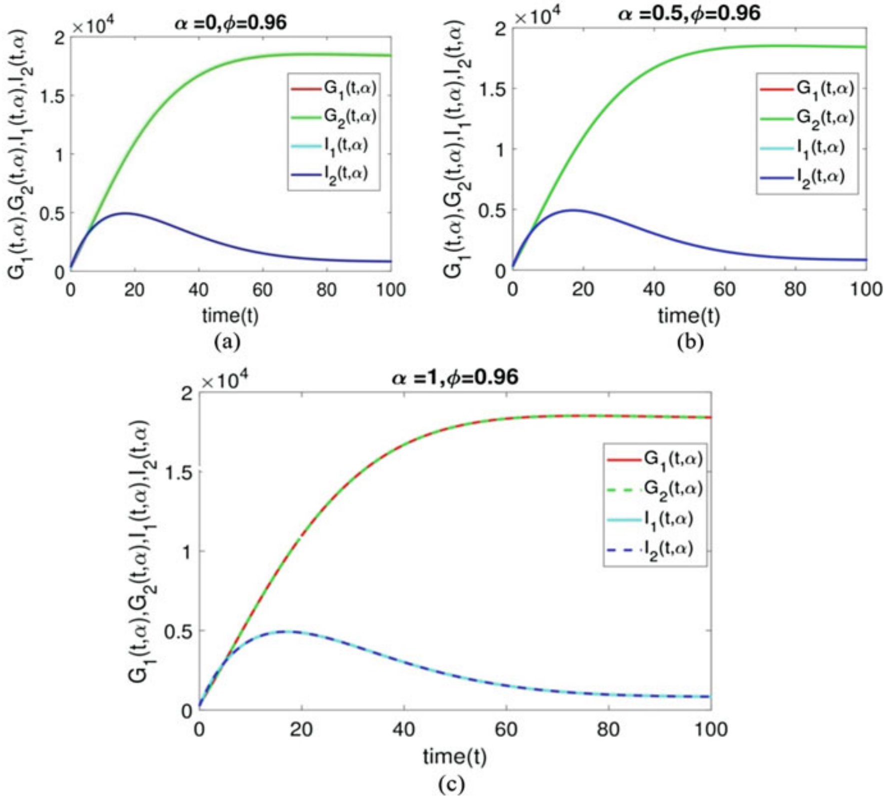

Fig. 19.2 Time series solution with $\phi = 0.96$ of the model (19.4) for $\alpha = 0, 0.5, 1$

To analyze the dynamical behavior of G_1, G_2, I_1, and I_2, the values of the parameters in Table 19.2 are employed. Figure 19.3a–c depict all classes' behavior over time for various fractional indices ϕ. Figure 19.3 shows that the number of all individuals increases when ϕ changes from 0.7 to 0.9. Figure 19.4 depicts the fuzzy solution of the model system for when $t \in [0, 6000]$ and $t \in [0, 5500]$. We observe that the outcomes of numerical results are fuzzy triangular functions.

6 Conclusion

Nowadays, mathematical modeling is a key issue in research and development since it has the potential to become a crucial instrument in the area of medicine. We have discussed the fuzzy fractional diabetic model in Caputo's meaning, where a fuzzy number is assumed to represent the initial populations. The generalized Hukuhara derivatives of type I and type II notion are used for the model's analysis.

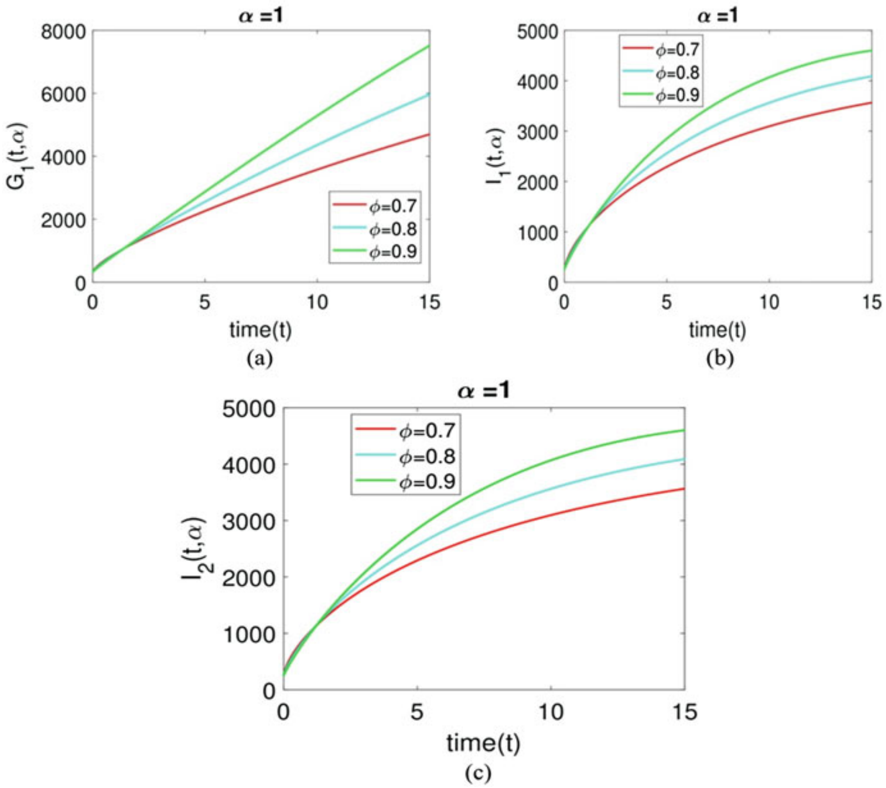

Fig. 19.3 Dynamical behavior of model system with $\alpha = 1$ for several fractional indices ϕ

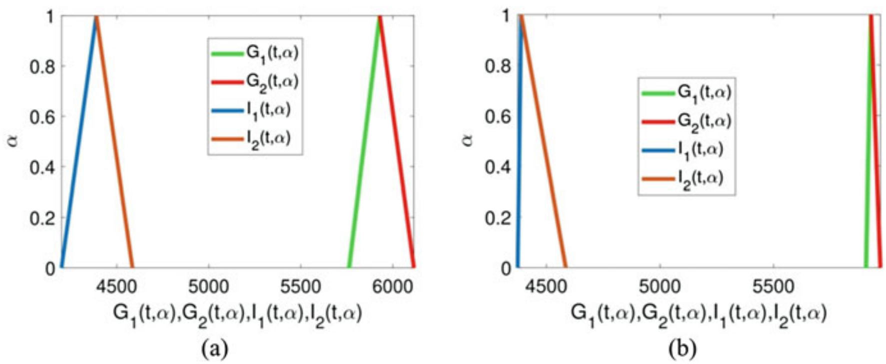

Fig. 19.4 Graphical representation of fuzzy solution

The stability analysis, which is crucial for a biological system, is carried out here for the model of the glucose-insulin regulation system in a fuzzy environment. The numerical solutions had been completed for potential scenarios, and relevant graphics had been used to explain their relevance. From numerical simulations, we observe that the outcomes of numerical results are fuzzy triangular functions. The outcomes of our suggested fuzzy fractional order model have been visually shown.

In our next research, we'll look at how chaotic systems are affected by the generalized Caputo operator. The new operator can also be used to bring additional dynamical properties and features to existing fuzzy fractional systems or equations that have practical applications.

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Chapter 20
Decision Analysis Framework Based on Information Measures of T-Spherical Fuzzy Sets

Shahzaib Ashraf and Attaullah

1 Introduction

A decision-making process is concerned for the selection of the best alternative in the available options. To tackle the making decision problems, decision-makers need to collect data from multiple sources, examine the data, and make final judgments. In complex problems, more than one criterion is effective to assess the performance of the alternatives for reliable decision-making. Therefore, in these conditions, decision-making is defined as multi-criteria decision-making (MCDM). MCDM approaches facilitate decision-makers to make reasonable judgments by taking into account a variety of decision criteria. As a result, the decision-maker is much more focused to choose practical and credible procedures for selecting the best options.

Classical procedures of crisp set theory were inept and inefficient in dealing with vagueness and uncertain data in decision-making challenges. In order to interact with such unpredictable circumstances, Zadeh [63] pioneered the fuzzy set (FS) approach, which evoked a dramatic change in several scientific and technological areas. The concept of FS has sparked significant attention owing to its capacity to contend with ambiguity and uncertainty. Over the recent decades, several perms of FSs have been reported by various researchers. Atanassov [7] introduced the intuitionistic fuzzy set (IFS), which is capable to handle with complexity and uncertainty with the help of truth and falsity degrees.

Yager [59] initiated the tremendous concept of Pythagorean fuzzy set (PyFS) as an extension of the IFS theory by broadening the space for membership grade

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C. Jana et al. (eds.), *Fuzzy Optimization, Decision-making and Operations Research*, https://doi.org/10.1007/978-3-031-35668-1_20

(MG) and non-membership grade (NMG) using a relatively lenient condition. These dominant features allow the PyFSs to perform more aptly and exceptionally than IFS. However, the graph of PyFS occupies a small portion of the available space owing to the restricted condition, and this significant model is inutile beyond that condition. Therefore, researchers desperately need a comprehensive framework that explains these challenges. Recently, Yager [58] introduced a new concept that accommodate both IFS and PyFS, named q-rung orthopair fuzzy set (q-ROFS).

Numerous researchers have demonstrated their preferences in this novel region of study. The majority of q-ROFS's applications have been examined in the background of making decision. The structure of the q-ROFS is adjustable owing to the adaptable parameter q that can be opted according to the nature of data and need of that particular problem. Thus, the wider space and flexible structure of q-ROFS allow it to dominate over the limited theories, including IFS and PyFS, that are bounded by a single, specified, and strict condition.

Despite their broad application, the aptitudes of IFSs, PyFSs, and q-ROFSs are limited to describe the satisfaction and dissatisfaction grades only. In short, their designs are inept to capture the neutral part of human judgments. For instance, when surveying a voter's conception on a party nominee, it may be favorable, unfavorable, neutral, or refusal. Moreover, public opinion on the social media's function in a democratic country varies significantly by ideology and political stance, including favorable, negative, and neutral observations. To manage this kind of ambiguous information, Cuong [13] derived the representation of picture fuzzy sets (PFSs), which consists of three indices, namely, positive MG $\wp_S(\mathrm{v}_i)$, neutral MG $I_S(\mathrm{v}_i)$ and negative MG $\mathscr{G}_S(\mathrm{v}_i)$ with the limitation that $0 \le \wp_S(\mathrm{v}_i) + I_S(\mathrm{v}_i) + \mathscr{G}_S(\mathrm{v}_i) \le 1$. Cuong and Kreinovich [14] further contributed to establish the theory of PFSs by providing several rudimentary operations and properties. The PFSs are observed to be incredibly applicable when engaging with circumstances that have neutral aspect in addition to occurrence and non-occurrence. Ashraf [3] presented the norm of algebraic which is based on novel aggregation operators for picture information. Tian et al. [49] derived the operators under picture fuzzy environment and explored their applications to decision-making. Despite the high accuracy and widespread application of PFSs, they become obsolete whenever the summation of all three grades is more than 1. To address this type of difficulty, Ashraf et al. [5] proffered spherical fuzzy sets (SFSs) as a powerful extension of PFSs with the constraint that $0 \le \wp_S^2(\mathrm{v}_i) + I_S^2(\mathrm{v}_i) + \mathscr{G}_S^2(\mathrm{v}_i) \le 1$. Shahzaib [6] showed the spherical fuzzy t-conorm and its t-norms, to established the negator of spherical, and provided some categorization of the spherical fuzzy t-norms and t-conorms that are helpful for the operator to aggregate the spherical fuzzy knowledge. Khan et al. [25] presented the generalized decision-making technique for the hospitalization and treatment of COVID-19 sufferers' issue under spherical hesitant fuzzy information. Many researchers contributed to develop the theory of SFSs. Gündoğdu and Kahraman [26] proposed TOPSIS methodology and discussed their application in DM under SF information. Akram et al. [1, 2] and Zahid et al. [64] presented the complex

SF VIKOR, TOPSIS, and ELECTRE models to handle the problem in making the decision. Ashraf [4] represents the novel operators using Dombi norm and SF information to address the uncertainty in DM. This idea empowers the researchers to deal with situations that are too sophisticated for the previous notions and finds extensive applications in real-life scenarios. Mahmood [31] generalized the idea of (SFS) to develop the T-spherical fuzzy set (T-SFS) by incorporating a parameter T that enables DMs to select membership grades from any point in the interval [0, 1] irrespective of the constraint. A T-SFS, being the most generalized accessible fuzzy framework, is capable to capture individual opinions on any imprecise event in an indefinitely proficient manner. The analysis of the outputs of PFS, SFS, and T-SFS unfolds the edge of T-SFS as compared to other fuzzy structures. Park [36] employing t-spherical fuzzy preference relations, the shopping online platform's item quality was assessed. Soft set theory is an expanded form of fuzzy set theory, which Molodtsov proposed in 1999 to address uncertainty parametrically. A soft set is a based on finite element family of sets; it is called "soft" because the set's boundary is determined by the parameters. Many researchers have contributed toward the fuzzification of the notion of soft set. Bipolar-valued fuzzy sets are an outgrowth of fuzzy sets in which the membership degree range is increased from [0, 1] to [−1, 1]. The degree of membership 0 in a bipolar-valued fuzzy set reveals that index is meaningless to the relating property, the occurrence degrees on (0, 1] reveal that index partially satisfy the estate, and the membership degrees on [−1, 0] indicate that elements partially fulfill the property. The idea of complex fuzzy set is given by Daniel Ramot in 2002. A mathematical foundation is the complicated fuzzy set that describes set membership using a complex number. The innate difficulty in grasping the concept of complex-valued membership is a significant impediment to realizing its maximum potential. Bipolar soft set is made up of two soft sets; one gives us true comments, and the other gives us incriminating comments. According to the bipolar philosophy, human judgment is based on both sides, positive and negative, and we select the one that is better. Bipolar complex fuzzy set (BCFS) is a merging of bipolar fuzzy set (BFS) that is used by decision analysts to elaborate the positive and unfavorable characteristics of a thing, complex fuzzy set which is used by decision analysts to manage two-dimensional information.

MADM is a prominent and trending topic of research in the field of fuzzy mathematics to figure out the most compatible solution for real-world complex problems. The MADM processes are normally assisted by measuring the similarity, also measuring the distance, inclusion measures, entropy measures, and its operators. The list of similarity measures has garnered considerable interest in recent decades due to its importance in DM, data mining, recognition of pattern, and diagnosis of the medical applications. Szmidt and Kacprzyk [44] performed the first investigation, extending well-known distance measures comparing methods used for normal fuzzy sets to those utilized for the IFS, like the distance measure and the hamming distance. However, Wang and Xin [51] highlighted the shortcoming of the Szmidt and Kacprzyk's [44] distance measures which were ineffective in certain situations. Therefore, several innovative pattern recognition distance measures were developed and implemented. Grzegorzewski [20] also extended Hamming, Euclidean, and their normalized versions to the IFS framework. Chen [9] later demonstrated that several flaws occurred in Grzegorzewski [20] by providing counterexamples. Hung and Yang [21] described three similar things that measured and extended the distance of Hausdorff for IFSs. Also other side, rather than extending well-established measures, various research established novels measure the similarity for IFSs. Mitchell [33] demonstrated the similarity measures, and Li and Cheng [62] improved the similarity measure for counterintuitive circumstances and validated it statistically. Similarly, Liang and Shi [30] provided examples to demonstrate the similarity measure for IFSs. Xu [55] formulated a series of IFS-based similarity measures and used them for the MADM problem employing IF information. Xu and Chen [56] combined and extended several weighted Hamming, Euclidean, and Hausdorff distances to present a set of more authentic and practical distance and similarity measures. Xu and Yager [57] constructed a resemblance calculate between IFSs and used it to MAGDM utilizing IF preference relations. Additionally to these research, several researchers examined the connections between IFS's distance, similarity, and entropy measurements. Zeng and Guo [65] analyzed the linked between distance(normalized), equality, and the entropy of the interval-valued fuzzy collections. It is further shown that utilizing the normalized distance of their axiomatic concepts, interval-valued fuzzy sets' similarity, inclusions, and entropy could be generated. Wei et al. [54] give a generalized entropy measure for IFSs. Additionally, a technique was developed for constructing similarity measures for IFS and PyFSs using entropy measures. Numerous studies (see [12–67] for more details) investigated measure information for IFSs and PyFSs and their transformations relationship. Various studies focused on the information measures for q-ROFSs. Du et al. [16] proposed Minkowski-type distance measures for q-ROFSs, including distances. To overcome the comparability problem, Peng et al. [40] innovated the use of a score formula for q-rung orthopair fuzzy numbers. Subsequently, they developed a measure of distance for q-ROFSs with multiple parameters and validated their findings with justification. Additionally, the numerous appealing characteristics of the resulting similarity and distance measures are inferred. Peng and Liu [38] examined the relationship between the distance, similarity, entropy, and inclusion measures for q-ROFSs. Khan et al. [24] deliberated distance and measure it for SFSs and its applicability to project selection. Rafiq et al. [43] investigated SFSs cosine similarity measurements and their applications in DM. Shishavan et al. [45] employed similarity measures of SFSs to diagnosis of the medical things and the selections of green supplier. Wei et al. [53] developed ten similarity measures between SFSs using the cosine function and implemented these similarity measures as well as recognition of pattern and diagnosis of medical using weighted similarity measurements.

The innovative features of T-SFS addressed the limitations of traditional fuzzy set theory including FS, IFS, PyFS, q-ROFS, PFS, and many more. It is worth noting that both PFS and SFS incorporate membership, neutral and non-membership grades, but their constraint was responsible for limiting the space of these membership grades. The T-SFS widens the space of these traditional models by introducing a flexible parameter within the constraint to provide more flexibility to the decision-makers for the assignment of membership grades. Similarly, the structure and space of membership grades in FS, IFS, PyFSs, and q-ROFSs are restricted to target only two aspects of the ambiguous information in terms of satisfaction and dissatisfaction. The shortcomings of existing information measures, including the occurrence of meaningless situations (i.e., dividing by zero) [37, 61], ineptness to avoid counterintuitive examples [8, 10, 11, 37, 61], generation of unreasonable outcomes [34, 41], and incompetency to classify the findings [10, 17, 27, 30, 52, 66], captivated us present new informational measure within the potent framework of T-SFSs. To overcome the drawbacks of the preceding literature, this study is focused to present a class of beneficial information measures under T-SFS, provide associated information measure formulations, and investigate their transformation relationships. Therefore, developing an information measure for T-SFSs is of major academic significance. The aforementioned constraints associated with the preceding models drove us to develop information measures based on T-SFSs, a modified version of SFSs.

Presented work are enlisted as below:

1. This study is devoted to introduce the axiomatic descriptions for T-SFS information measures along with their formulae and transformation relationships.
2. A major target goal of this project is to use the established distance measures ($D_1 - D_{13}$) to pattern recognition and medical diagnostics to demonstrate their feasibility and efficacy.
3. Further, we demonstrate the potency of the novel similarity measures by employing the our measures to recognition of pattern, construction materials, and diagnostics of medical things.
4. A comparison with existing measures of similarity for diagnosis of medical problems is accomplished to illustrate the efficiency of the developed similarity measures.
5. Additionally, we exhibit the applicability of the suggested T-SFS inclusion measures to pattern recognition with an example to verify the authenticity of new inclusion.

The remaining part of the article is structured as follows. In Sect. 2 some fundamental notions of HFS, IHFS, and q-ROFS define certain associated laws of operation. In Sect. 3, we introduced some new information measures along with their formulae and explored their transformation relationships for T-SFSs. Sections 4–6 illustrate the application of novel information measures to recognition of pattern and diagnosis of medical things and in building materials. Moreover, a correlating research has been presented between the proposed distance and similarity measures with existing ones. Section 7 summarizes the results of the investigation.

2 Preliminary

In the first section, we provide few relevant fundamental information of SFSs and T-SFSs along with some related operational laws. These core concepts will assist readers in comprehending the proposed framework.

Definition 1 ([5]) Let $\mathscr{H} = \{v_1, v_2, v_3, \ldots, v_n\}$ be a fixed set. Then the SFS S over $\mathscr{H}$ is defined as

$$S = \{\langle v_i, \wp_S(v_i), I_S(v_i), \mathscr{G}_S(v_i)\rangle | v_i \in \mathscr{H}\}$$

for each $\mathrm{v}_i \in \mathscr{H}$ the functions $\wp_S : \mathscr{H} \rightarrow [0, 1]$, $I_S : \mathscr{H} \rightarrow [0, 1]$ and $\mathscr{G}_{\mathscr{Q}} : \mathrm{v} \rightarrow [0, 1]$ shows the positive, neutral, and negative MG of v_i in $\mathscr{H}$, respectively. Also $\wp_S(\mathrm{v}_i)$, $I_S(\mathrm{v}_i)$, and $\mathscr{G}_S(\mathrm{v}_i)$ satisfy the following condition: $(\forall\, v_i \in \mathscr{H})$, $0 \leq (\mathscr{G}_{\mathscr{Q}}(\mathrm{v}_i))^2 + (I_S(v_i))^2 + (\wp_S(\mathrm{v}_i))^2 \leq 1$.

$\mu_S(\mathrm{v}_i) = \sqrt{1 - ((\wp_S(v_i))^2 + (I_S(v_i))^2 + (\mathscr{G}_S(v_i))^2)}$ referred to as the degree of refusal $\mathrm{v}_i \in \mathscr{H}$. The numbers in the form $(\wp_S(v_i), I_S(v_i), \mathscr{G}_S(v_i))$ are said to SFN and for every SFN can be shown by $r = (\wp_r, I_r, \mathscr{G}_r)$, where $\wp_r$, I_r, and $\mathscr{G}_r \in [0, 1]$, with condition that $0 \leq \wp_r^2 + I_r^2 + \mathscr{G}_r^2 \leq 1$.

Definition 2 ([31]) Let $\mathscr{H} = \{v_1, v_2, v_3, \ldots, v_n\}$ be a set of fixed. Then the T-spherical fuzzy set (T-SFS) T over $\mathscr{H}$ is defined as

$$T = \{\langle v_i, \wp_T(v_i), I_T(v_i), \mathscr{G}_T(v_i)\rangle | v_i \in \mathscr{H}\}$$

for each $\mathrm{v}_i \in \mathscr{H}$ the functions $\wp_T : \mathscr{H} \rightarrow [0, 1]$, $I_T : \mathrm{v} \rightarrow [0, 1]$ and $\mathscr{G}_T : \mathscr{H} \rightarrow [0, 1]$ represent the positive MG, neutral MG, and negative MG of v_i in $\mathscr{H}$, respectively. Also $\wp_T(\mathrm{v}_i)$, $I_T(\mathrm{v}_i)$, and $\mathscr{G}_T(\mathrm{v}_i)$ satisfy the following condition: $(\forall\, v_i \in \mathscr{H})$, $0 \leq (\mathscr{G}_T(\mathrm{v}_i))^t + (I_T(v_i))^t + (\wp_T(\mathrm{v}_i))^t \leq 1$, $(t \in Z)$.

$\mu_P(\mathrm{v}_i) = \sqrt[t]{1 - (\wp_T(v_i))^t + (I_T(v_i))^t + (\mathscr{G}_T(v_i))^t}$ is said to be refusal degree of $\mathrm{v}_i \in \mathscr{H}$. The numbers in the form $(\wp_S(v_i), I_S(v_i), \mathscr{G}_S(v_i))$ are said to T-SFN, and each T-SFN can be denoted by $r = (\wp_r, I_r, \mathscr{G}_r)$, where $\wp_r$, I_r, and $\mathscr{G}_r \in [0, 1]$, with condition that $0 \leq \wp_r^t + I_r^t + \mathscr{G}_r^t \leq 1$.

Definition 3 ([31]) Assume that $r_j = \left(\wp_{r_j}, I_{r_j}, \mathscr{G}_{r_j}\right)$ and $r_k = \left(\wp_{r_k}, I_{r_k}, \mathscr{G}_{r_k}\right)$ are every two t-SFNs, union, intersection, and its compliment defined as follows:

1. $r_j \subseteq r_k$ iff $\forall r \in \mathscr{H}, \wp_{r_j} \leq \wp_{r_k}, I_{r_j} \leq I_{r_k}$ and $\mathscr{G}_{r_j} \geq \mathscr{G}_{r_k}$;
2. $r_j = r_k$ iff $r_j \subseteq r_k and r_k \subseteq r_j$;
3. $r_j \cup r_k = \left(max(\wp_{r_j}, \wp_{r_k}), min(I_{r_j}, I_{r_k}), min(\mathscr{G}_{r_j}, \mathscr{G}_{r_k})\right)$;
4. $r_j \cap r_k = \left(min(\wp_{r_j}, \wp_{r_k}), min(I_{r_j}, I_{r_k}), max(\mathscr{G}_{r_j}, \mathscr{G}_{r_k})\right)$;
5. $r_j = \left(\mathscr{G}_{r_j}, I_{r_j}, \wp_{r_j}\right)$.

Definition 4 ([31]) Suppose that $r_j = \left(\wp_{r_j}, I_{r_j}, \mathscr{G}_{r_j}\right)$ and $r_k = \left(\wp_{r_k}, I_{r_k}, \mathscr{G}_{r_k}\right)$ are any two of SFNs and $\gamma \geq 0$. Operations of T-SFNs which is based on rigorous Archimedean triangular norm and conorm can be described as follows:

1. $r_j \oplus r_k = \left\{\sqrt{s^{-1}(s(\wp^t_{r_j}) + s(\wp^t_{r_k}))}, t^{-1}(t(I_{r_j}) + t(I_{r_k})), t^{-1}(t(\mathscr{G}_{r_j}) + t(\mathscr{G}_{r_k}))\right\}$;
2. $\gamma r_j = \left\{\sqrt{s^{-1}(\gamma s(\wp^t_{r_j})}, t^{-1}(\gamma t(I_{r_j})), t^{-1}(\gamma t(\mathscr{G}_{r_j}))\right\}$;
3. $r_j \otimes r_k = \left\{t^{-1}(s(\wp_{r_j}) + s(\wp_{r_k})), t^{-1}(t(I_{r_j}) + t(I_{r_k})), \sqrt{s^{-1}(t(\mathscr{G}^t_{r_j}) + t(\mathscr{G}^t_{r_k}))}\right\}$;
4. $r_j^{\gamma} = \left\{t^{-1}(\gamma t(\wp_{r_j}), t^{-1}(\gamma t(I_{r_j}), \sqrt{s^{-1} t(\mathscr{G}^t_{r_k})}\right\}$.

2.1 Compared Rules for T-SFNs

Now, we will discuss certain operations that factor heavily toward the ranking of SFNs.

Definition 5 Suppose $r_j = \left(\wp_{r_j}, I_{r_j}, \mathscr{G}_{r_j}\right)$ be any T-SFN. Then

1. The following defines the scoring function:

$$sc(r_j) = \frac{\left(\wp^t_{r_j} + 1 - I^t_{r_j} + 1 - \mathscr{G}^t_{r_j}\right)}{3} = \frac{\left(2 + \wp^t_{r_j} - I^t_{r_j} - \mathscr{G}^t_{r_j}\right)}{3}.$$

2. The following is a definition of the accuracy function:

$$ac = \wp^t_{r_j} - \mathscr{G}^t_{r_j}.$$

3. The following is a definition of the certainty function:

$$cr(r_j) = \wp_{r_j}.$$

Definition 6 Let $r_j = \left(\wp_{r_j}, I_{r_j}, \mathscr{G}_{r_j}\right)$ and $r_k = \left(\wp_{r_k}, I_{r_k}, \mathscr{G}_{r_k}\right)$ be any two T-SFSs. Then by using Definition 5, equating technique is characterized as follows:

1. if $sc(r_j) > sc(r_k)$, then $r_j > r_k$;
2. if $sc(r_j) = sc(r_k), ac(r_j) > ac(r_k)$ then $r_j > r_k$;
3. if $sc(r_j) = sc(r_k), ac(r_j) = ac(r_k)$ and $cr(r_j) > cr(r_k)$, then $r_j > r_k$;
4. if $sc(r_j) = sc(r_k), ac(r_j) = ac(r_k)$ and $cr(r_j) = cr(r_k)$, then $r_j = r_k$.

Definition 7 If $\mathscr{R}, \mathscr{U} \in \mathrm{T} - SFSs$. Then their fundamental operations are defined as follows:

1. $\mathscr{R}^c = \{\langle v_i, \mathscr{G}_{\mathscr{R}}(v_i), I_{\mathscr{R}}(v_i), \wp_{\mathscr{R}}(v_i)\rangle \,|v_i \in \mathscr{H}\}$;

2. $\mathscr{R} \subseteq \mathscr{U}$ iff for all $\mathrm{v}_i \in \mathscr{H}$, $\wp_{\mathscr{R}}(\mathrm{v}_i) \leq \wp_{\mathscr{U}}(\mathrm{v}_i)$,$I_{\mathscr{R}}(\mathrm{v}_i) \geq I_{\mathscr{U}}(\mathrm{v}_i)$ and $\mathscr{G}_{\mathscr{R}}(\mathrm{v}_i) \geq \mathscr{G}_{\mathscr{U}}(\mathrm{v}_i)$;
3. $\mathscr{R} = \mathscr{U}$ iff for all $\mathrm{v}_i \in \mathscr{H}$, $\wp_{\mathscr{R}}(\mathrm{v}_i) = \wp_{\mathscr{U}}(\mathrm{v}_i)$, $I_{\mathscr{R}}(\mathrm{v}_i) = I_{\mathscr{U}}(\mathrm{v}_i)$ and $\mathscr{G}_{\mathscr{R}}(\mathrm{v}_i) = \mathscr{G}_{\mathscr{U}}(\mathrm{v}_i)$;
4. $\Phi_{\mathscr{R}} = \{\langle v_i, 1, 0, 0\rangle \,|v_i \in \mathscr{H}\}$;
5. $\varnothing_{\mathscr{R}} = \{\langle v_i, 0, 1, 1\rangle \,|v_i \in \mathscr{H}\}$;
6. $\mathscr{R} \cap \mathscr{U} = \{v_i \in \mathscr{H}, \wp_{\mathscr{R}}(v_i) \wedge \wp_{\mathscr{U}}(v_i), I_{\mathscr{R}}(v_i) \vee I_{\mathscr{U}}(v_i)$, $\mathscr{G}_{\mathscr{R}}(v_i) \vee \mathscr{G}_{\mathscr{U}}(v_i)|v_i \in \mathscr{H}\}$;
7. $\mathscr{R} \cup \mathscr{U} = \{v_i \in \mathscr{H}, \wp_{\mathscr{R}}(v_i) \vee \wp_{\mathscr{U}}(v_i), I_{\mathscr{R}}(v_i) \wedge I_{\mathscr{U}}(v_i)$, $\mathscr{G}_{\mathscr{R}}(v_i) \wedge \mathscr{G}_{\mathscr{U}}(v_i)|v_i \in \mathscr{H}\}$;
8. $\mathscr{R} \oplus \mathscr{U} = \left\{ \begin{matrix} v_i \in \mathscr{H}, \sqrt[t]{\wp^t_{\mathscr{R}}(v_i) + \wp^t_{\mathscr{U}}(v_i) - \wp^t_{\mathscr{R}}(v_i)\wp^t_{\mathscr{U}}(v_i)}, I^t_{\mathscr{R}}(v_i) I^t_{\mathscr{U}}(v_i), \\ \mathscr{G}^t_{\mathscr{R}}(v_i)\mathscr{G}^t_{\mathscr{U}}(v_i)|v_i \in \mathscr{H} \end{matrix} \right\}$;
9. $\mathscr{R} \otimes \mathscr{U} = \left\{ \begin{matrix} v_i \in \mathscr{H}, \wp^t_{\mathscr{R}}(v_i)\wp^t_{\mathscr{U}}(v_i), \sqrt[t]{I^t_{\mathscr{R}}(v_i) + I^t_{\mathscr{U}}(v_i) \ - I^t_{\mathscr{R}}(v_i) I^t_{\mathscr{U}}(v_i)}, \\ \sqrt[t]{\mathscr{G}^t_{\mathscr{R}}(v_i) + \mathscr{G}^t_{\mathscr{U}}(v_i) - \mathscr{G}^t_{\mathscr{R}}(v_i)\mathscr{G}^t_{\mathscr{U}}(v_i)}|v_i \in \mathscr{H} \end{matrix} \right\}$;
10. $\mathscr{R} \ominus \mathscr{U} = \left\{ v_i \in \mathscr{H}, \frac{\wp^t_{\mathscr{R}}(v_i) - \wp^t_{\mathscr{U}}(v_i)}{1 - \wp^t_{\mathscr{U}}(v_i)}, \frac{I^t_{\mathscr{R}}(v_i)}{I^t_{\mathscr{U}}(v_i)}, \frac{\mathscr{G}^t_{\mathscr{R}}(v_i)}{\mathscr{G}^t_{\mathscr{U}}(v_i)}|v_i \in \mathscr{H} \right\}$ if $\wp_{\mathscr{R}}(\mathrm{v}_i) \geq \wp_{\mathscr{U}}(\mathrm{v}_i)$, $I_{\mathscr{R}}(\mathrm{v}_i) \leq \min\left\{ I_{\mathscr{U}}(v_i), \frac{I_{\mathscr{R}}(v_i)\pi_{\mathscr{R}}(v_i)}{\pi_{\mathscr{U}}(v_i)} \right\}$ and $\mathscr{G}_{\mathscr{R}}(\mathrm{v}_i) \leq \min\left\{ \mathscr{G}_{\mathscr{U}}(v_i), \frac{\mathscr{G}_{\mathscr{R}}(v_i)\pi_{\mathscr{R}}(v_i)}{\pi_{\mathscr{U}}(v_i)} \right\}$;
11. $\mathscr{R} \oslash \mathscr{U} = \left\{ v_i \in \mathscr{H}, \frac{\wp^t_{\mathscr{R}}(v_i)}{\wp^t_{\mathscr{U}}(v_i)}, \sqrt[t]{\frac{I^t_{\mathscr{R}}(v_i) + I^t_{\mathscr{U}}(v_i)}{1 - I^t_{\mathscr{U}}(v_i)}}, \sqrt[t]{\frac{\mathscr{G}^t_{\mathscr{R}}(v_i) + \mathscr{G}^t_{\mathscr{U}}(v_i)}{1 - \mathscr{G}^t_{\mathscr{U}}(v_i)}}|v_i \in \mathscr{H} \right\}$ if $\mathscr{G}_{\mathscr{R}}(\mathrm{v}_i) \geq \mathscr{G}_{\mathscr{U}}(\mathrm{v}_i)$, $I_{\mathscr{R}}(\mathrm{v}_i) \geq I_{\mathscr{U}}(\mathrm{v}_i)$ and $\wp_{\mathscr{R}}(\mathrm{v}_i) \leq \min\left\{ \wp_{\mathscr{U}}(v_i), \frac{\wp_{\mathscr{U}}(v_i)\pi_{\mathscr{R}}(v_i)}{\pi_{\mathscr{U}}(v_i)} \right\}$.

3 Certain Information Measures Between T-SFSs

This section presents the axiomatic framework of T-SFSs information measures as well as their related formulations. Simultaneously, their evolving connections are thoroughly examined.

3.1 Distance measures for T-SFSs

Let $\mathscr{R}$, $\mathscr{U}$, and $\mathscr{W}$ be three T-SFSs on $\mathscr{H}$. The measure of distance $D(\mathscr{R}, \mathscr{U})$ is an index D: T-SFS$(\mathscr{H}) \times T$-SFS$(\mathscr{H}) \to [0, 1]$, carrying the following features:

1. $0 \le D(\mathscr{R}, \mathscr{U}) \le 1$;
2. $D(\mathscr{R}, \mathscr{U}) = D(\mathscr{U}, \mathscr{R})$;
3. $D(\mathscr{R}, \mathscr{U}) = D(\mathscr{U}, \mathscr{R})$;
4. $D(\mathscr{R}, \mathscr{U}) = D(\mathscr{U}, \mathscr{R})$;
5. $D(\mathscr{R}, \mathscr{U}) = 0$ iff $\mathscr{R} = \mathscr{U}$;
6. $D(\mathscr{R}, \mathscr{R}^c) = 1$ iff $\mathscr{R}$ is a crisp set;
7. If $\mathscr{R} \subseteq \mathscr{U} \subseteq \mathscr{W}$, then $D(\mathscr{R}, \mathscr{U}) \le D(\mathscr{R}, \mathscr{W})$ and $D(\mathscr{U}, \mathscr{W}) \le D(\mathscr{R}, \mathscr{W})$.

Proposition 1 *Suppose $\mathscr{R}$ and $\mathscr{U}$ be two T-SFSs. Then $D_i(\mathscr{R}, \mathscr{U})$ $(i = 1, 2, \ldots, 13)$ are the measure of distance:*

1. $D_1(\mathscr{R}, \mathscr{U}) = \dfrac{1}{2|\mathscr{H}|} \sum_{v_i \in \mathscr{H}} \begin{pmatrix} |\wp^t_{\mathscr{R}}(v_i) - \wp^t_{\mathscr{U}}(v_i)| + |I^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{U}}(v_i)| \\ +|\mathscr{G}^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{U}}(v_i)| + |\pi^t_{\mathscr{R}}(v_i) - \pi^t_{\mathscr{U}}(v_i)| \end{pmatrix}$;
2. $D_2(\mathscr{R}, \mathscr{U}) = \dfrac{1}{2|\mathscr{H}|} \sum_{v_i \in \mathscr{H}} \begin{pmatrix} |\left(\wp^t_{\mathscr{R}}(v_i) - \wp^t_{\mathscr{U}}(v_i)\right) + \\ \left(I^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{U}}(v_i)\right) + \left(\mathscr{G}^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{U}}(v_i)\right) \end{pmatrix}$;
3. $D_3(\mathscr{R}, \mathscr{U}) = \dfrac{1}{4|X|} \begin{pmatrix} \sum_{v_i \in \mathscr{H}} \begin{pmatrix} |\wp^t_{\mathscr{R}}(v_i) - \wp^t_{\mathscr{U}}(v_i)| + |I^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{U}}(v_i)| + \\ |\mathscr{G}^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{U}}(v_i)| + |\mathscr{G}^t_{\mathscr{R}}(v_i) - \pi^t_{\mathscr{U}}(v_i)| \end{pmatrix} \\ + \sum_{v_i \in \mathscr{H}} \begin{pmatrix} |\left(\wp^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{R}}(v_i)\right) - I^t_{\mathscr{R}}(v_i) \\ + \left(\wp^t_{\mathscr{U}}(v_i) - I^t_{\mathscr{U}}(v_i) - \mathscr{G}^t_{\mathscr{U}}(v_i)\right)| \end{pmatrix} \end{pmatrix}$;
4. $D_4(\mathscr{R}, \mathscr{U}) = \dfrac{1}{|\mathscr{H}|} \sum_{v_i \in \mathscr{H}} \left(|\wp^t_{\mathscr{R}}(v_i) - \wp^t_{\mathscr{U}}(v_i)| \vee |I^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{U}}(v_i)| \vee |\left(\mathscr{G}^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{U}}(v_i)\right)|\right)$;
5. $D_5(\mathscr{R}, \mathscr{U}) = \dfrac{2}{|\mathscr{H}|} \sum_{v_i \in \mathscr{H}} \dfrac{|\wp^t_{\mathscr{R}}(v_i) - \wp^t_{\mathscr{U}}(v_i)| \vee |I^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{U}}(v_i)| \vee |\left(\mathscr{G}^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{U}}(v_i)\right)|}{1 + |\wp^t_{\mathscr{R}}(v_i) - \wp^t_{\mathscr{U}}(v_i)| \vee |I^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{U}}(v_i)| \vee |\left(\mathscr{G}^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{U}}(v_i)\right)|}$;
6. $D_6(\mathscr{R}, \mathscr{U}) = \dfrac{2 \sum_{v_i \in \mathscr{H}} |\wp^t_{\mathscr{R}}(v_i) - \wp^t_{\mathscr{U}}(v_i)| \vee |I^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{U}}(v_i)| \vee |\left(\mathscr{G}^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{U}}(v_i)\right)|}{\sum_{v_i \in \mathscr{H}} \left(1 + |\wp^t_{\mathscr{R}}(v_i) - \wp^t_{\mathscr{U}}(v_i)| \vee |I^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{U}}(v_i)| \vee |\left(\mathscr{G}^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{U}}(v_i)\right)|\right)}$;

7. $D_7(\mathscr{R},\ \mathscr{U}) = 1 - \alpha \dfrac{\sum_{v_i \in \mathscr{H}} \left(\wp^t_{\mathscr{R}}(v_i) \wedge \wp^t_{\mathscr{U}}(v_i)\right)}{\sum_{v_i \in \mathscr{H}} \left(\wp^t_{\mathscr{R}}(v_i) \vee \wp^t_{\mathscr{U}}(v_i)\right)} - \gamma \dfrac{\sum_{v_i \in \mathscr{H}} \left(I^t_{\mathscr{R}}(v_i) \wedge I^t_{\mathscr{U}}(v_i)\right)}{\sum_{v_i \in \mathscr{H}} \left(I^t_{\mathscr{R}}(v_i) \vee I^t_{\mathscr{U}}(v_i)\right)} - \beta \dfrac{\sum_{v_i \in \mathscr{H}} \left(\mathscr{G}^t_{\mathscr{R}}(v_i) \wedge \mathscr{G}^t_{\mathscr{U}}(v_i)\right)}{\sum_{v_i \in \mathscr{H}} \left(\mathscr{G}^t_{\mathscr{R}}(v_i) \vee \mathscr{G}^t_{\mathscr{U}}(v_i)\right)}$, $\alpha + \gamma + \beta = 1,\ \alpha, \gamma, \beta \in [0, 1]$;

8. $D_8(\mathscr{R},\ \mathscr{U}) = 1 - \dfrac{\alpha}{|\mathscr{H}|} \sum_{v_i \in \mathscr{H}} \dfrac{\left(\wp^t_{\mathscr{R}}(v_i) \wedge \wp^t_{\mathscr{U}}(v_i)\right)}{\left(\wp^t_{\mathscr{R}}(v_i) \vee \wp^t_{\mathscr{U}}(v_i)\right)} - \dfrac{\gamma}{|v_i|} \sum_{v_i \in \mathscr{H}} \dfrac{\left(I^t_{\mathscr{R}}(v_i) \wedge I^t_{\mathscr{U}}(v_i)\right)}{\left(I^t_{\mathscr{R}}(v_i) \vee I^t_{\mathscr{U}}(v_i)\right)} - \dfrac{\beta}{|X|} \sum_{v_i \in \mathscr{H}} \dfrac{\left(\mathscr{G}^t_{\mathscr{R}}(v_i) \wedge \mathscr{G}^t_{\mathscr{U}}(v_i)\right)}{\left(\mathscr{G}^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{U}}(v_i)\right)|}$, $\alpha + \gamma + \beta = 1,\ \alpha, \gamma, \beta \in [0, 1]$;

9. $D_9(\mathscr{R},\ \mathscr{U}) = 1 - \dfrac{1}{|\mathscr{H}|} \sum_{v_i \in \mathscr{H}} \dfrac{\left(\wp^t_{\mathscr{R}}(v_i) \wedge \wp^t_{\mathscr{U}}(v_i)\right) + \left(I^t_{\mathscr{R}}(v_i) \wedge I^t_{\mathscr{U}}(v_i)\right) + \left(\mathscr{G}^t_{\mathscr{R}}(v_i) \wedge \mathscr{G}^t_{\mathscr{U}}(v_i)\right)}{\left(\wp^t_{\mathscr{R}}(v_i) \vee \wp^t_{\mathscr{U}}(v_i)\right) + \left(I^t_{\mathscr{R}}(v_i) \vee I^t_{\mathscr{U}}(v_i)\right) + \left(\mathscr{G}^t_{\mathscr{R}}(v_i) \vee \mathscr{G}^t_{\mathscr{U}}(v_i)\right)}$;

10. $D_{10}(\mathscr{R},\ \mathscr{U}) = 1 - \dfrac{\sum_{v_i \in \mathscr{H}} \left(\wp^t_{\mathscr{R}}(v_i) \wedge \wp^t_{\mathscr{U}}(v_i)\right) + \left(I^t_{\mathscr{R}}(v_i) \wedge I^t_{\mathscr{U}}(v_i)\right) + \left(\mathscr{G}^t_{\mathscr{R}}(v_i) \wedge \mathscr{G}^t_{\mathscr{U}}(v_i)\right)}{\sum_{v_i \in \mathscr{H}} \left(\wp^t_{\mathscr{R}}(v_i) \vee \wp^t_{\mathscr{U}}(v_i)\right) + \left(I^t_{\mathscr{R}}(v_i) \vee I^t_{\mathscr{U}}(v_i)\right) + \left(\mathscr{G}^t_{\mathscr{R}}(v_i) \vee \mathscr{G}^t_{\mathscr{U}}(v_i)\right)}$;

11. $D_{11}(\mathscr{R},\ \mathscr{U}) = 1 - \dfrac{1}{|\mathscr{H}|} \sum_{v_i \in \mathscr{H}} \dfrac{\left(\wp^t_{\mathscr{R}}(v_i) \wedge \wp^t_{\mathscr{U}}(v_i)\right) + \left(I^t_{\mathscr{R}}(v_i) \wedge I^t_{\mathscr{U}}(v_i)\right) + \left(\mathscr{G}^t_{\mathscr{R}}(v_i) \wedge \mathscr{G}^t_{\mathscr{U}}(v_i)\right)}{\left(\wp^t_{\mathscr{R}}(v_i) \vee \wp^t_{\mathscr{U}}(v_i)\right) + \left(I^t_{\mathscr{R}}(v_i) \vee I^t_{\mathscr{U}}(v_i)\right) + \left(\mathscr{G}^t_{\mathscr{R}}(v_i) \vee \mathscr{G}^t_{\mathscr{U}}(v_i)\right)}$;

12. $D_{12}(\mathscr{R},\ \mathscr{U}) = 1 - \dfrac{\sum_{v_i \in \mathscr{H}} \left(\wp^t_{\mathscr{R}}(v_i) \wedge \wp^t_{\mathscr{U}}(v_i)\right) + \left((1 - I^t_{\mathscr{R}}(v_i)) \wedge (1 - I^t_{\mathscr{U}}(v_i))\right) + \left((1 - \mathscr{G}^t_{\mathscr{R}}(v_i)) \wedge (1 - \mathscr{G}^t_{\mathscr{U}}(v_i))\right)}{\sum_{v_i \in \mathscr{H}} \left(\wp^t_{\mathscr{R}}(v_i) \vee \wp^t_{\mathscr{U}}(v_i)\right) + \left((1 - I^t_{\mathscr{R}}(v_i)) \vee (1 - I^t_{\mathscr{U}}(v_i))\right) + \left((1 - \mathscr{G}^t_{\mathscr{R}}(v_i)) \vee (1 - \mathscr{G}^t_{\mathscr{U}}(v_i))\right)}$;

13. $D_{13}(\mathscr{R},\ \mathscr{U}) =$

$$\sqrt[t]{\begin{array}{l} \dfrac{1}{2|\mathscr{H}|(l_1 + 1)^t} \sum_{v_i \in \mathscr{H}} \left\{ \begin{array}{c} |\left(l_1 \left(\wp^t_{\mathscr{R}}(v_i)\right) - \wp^t_{\mathscr{U}}(v_i)\right) - \\ \left(\left(I^t_{\mathscr{R}}(v_i)\right) - I^t_{\mathscr{U}}(v_i)\right) - \left(\mathscr{G}^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{U}}(v_i)\right)|^t \end{array} \right\} \\ + \dfrac{1}{2|\mathscr{H}|(l_2 + 1)^t} \sum_{v_i \in \mathscr{H}} \left\{ \begin{array}{c} |\left(l_2 \left(\mathscr{G}^t_{\mathscr{R}}(v_i)\right) - \mathscr{G}^t_{\mathscr{U}}(v_i)\right) - \\ \left(\left(I^t_{\mathscr{R}}(v_i)\right) - I^t_{\mathscr{U}}(v_i)\right) - \left(\wp^t_{\mathscr{R}}(v_i) - \wp^t_{\mathscr{U}}(v_i)\right)|^t \end{array} \right\} \end{array}}$$

3.2 Similarity Measure for T-SFSs

Let $\mathscr{R}$, $\mathscr{U}$, and $\mathscr{W}$ be three T-SFSs on $\mathscr{H}$. A measure of similarity $S(\mathscr{R}, \mathscr{U})$ is a mapping S: T-SFS$(\mathscr{H}) \times T$-SFS$(\mathscr{H}) \rightarrow [0, 1]$ satisfying the following conditions:

1. $0 \leq S(\mathscr{R}, \mathscr{U}) \leq 1$;
2. $S(\mathscr{R}, \mathscr{U}) = S(\mathscr{U}, \mathscr{R})$;
3. $S(\mathscr{R}, \mathscr{U}) = S(\mathscr{U}, \mathscr{R})$;
4. $S(\mathscr{R}, \mathscr{U}) = S(\mathscr{U}, \mathscr{R})$;
5. $S(\mathscr{R}, \mathscr{U}) = 1$ iff $\mathscr{R} = \mathscr{U}$;
6. $S(\mathscr{R}, \mathscr{R}^c) = 0$ iff $\mathscr{R}$ is a crisp set;
7. If $\mathscr{R} \subseteq \mathscr{U} \subseteq \mathscr{W}$, then $S(\mathscr{R}, \mathscr{U}) \leq S(\mathscr{R}, \mathscr{W})$ and $S(\mathscr{U}, \mathscr{W}) \leq S(\mathscr{R}, \mathscr{W})$.

Theorem 1 *Let $\mathscr{R}$ and $\mathscr{U}$ be two T-SFSs. Then $S_i(\mathscr{R}, \mathscr{U})$ $(i = 1, 2, \ldots, 13)$ are similarity measures:*

1. $S_1(\mathscr{R},\ \mathscr{U}) = 1 - \frac{1}{2|\mathscr{H}|} \sum_{v_i \in \mathscr{H}} \begin{pmatrix} |\wp^t_{\mathscr{R}}(v_i) - \wp^t_{\mathscr{U}}(v_i)| + |I^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{U}}(v_i)| \\ +|\mathscr{G}^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{U}}(v_i)| + |\pi^t_{\mathscr{R}}(v_i) - \pi^t_{\mathscr{U}}(v_i)| \end{pmatrix}$;
2. $S_2(\mathscr{R},\ \mathscr{U}) = 1 - \frac{1}{2|\mathscr{H}|} \sum_{v_i \in \mathscr{H}} \begin{pmatrix} |\left(\wp^t_{\mathscr{R}}(v_i) - \wp^t_{\mathscr{U}}(v_i)\right) - \\ \left(I^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{U}}(v_i)\right) - \left(\mathscr{G}^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{U}}(v_i)\right) \end{pmatrix}$;
3. $S_3(\mathscr{R},\ \mathscr{U}) = 1 - \frac{1}{4|X|} \begin{pmatrix} \sum_{v_i \in \mathscr{H}} \begin{pmatrix} |\wp^t_{\mathscr{R}}(v_i) - \wp^t_{\mathscr{U}}(v_i)| + |I^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{U}}(v_i)| + \\ |\mathscr{G}^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{U}}(v_i)| + |\pi^t_{\mathscr{R}}(v_i) - \pi^t_{\mathscr{U}}(v_i)| \end{pmatrix} \\ + \sum_{v_i \in \mathscr{H}} \begin{pmatrix} |\left(\wp^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{R}}(v_i)\right) - I^t_{\mathscr{R}}(v_i) \\ + \left(\wp^t_{\mathscr{U}}(v_i) - I^t_{\mathscr{U}}(v_i) - \mathscr{G}^t_{\mathscr{U}}(v_i)\right)| \end{pmatrix} \end{pmatrix}$
4. $S_4(\mathscr{R},\ \mathscr{U}) = 1 - \frac{1}{|\mathscr{H}|} \sum_{v_i \in \mathscr{H}} \left(|\wp^t_{\mathscr{R}}(v_i) - \wp^t_{\mathscr{U}}(v_i)| \vee |I^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{U}}(v_i)| \vee |\left(\mathscr{G}^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{U}}(v_i)\right)|\right)$;
5. $S_5(\mathscr{R},\ \mathscr{U}) = \frac{1}{|\mathscr{H}|} \sum_{v_i \in \mathscr{H}} \frac{1 - |\wp^t_{\mathscr{R}}(v_i) - \wp^t_{\mathscr{U}}(v_i)| \vee |I^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{U}}(v_i)| \vee |\left(\mathscr{G}^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{U}}(v_i)\right)|}{1 + |\wp^t_{\mathscr{R}}(v_i) - \wp^t_{\mathscr{U}}(v_i)| \vee |I^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{U}}(v_i)| \vee |\left(\mathscr{G}^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{U}}(v_i)\right)|}$;
6. $S_6(\mathscr{R},\ \mathscr{U}) = \frac{\sum_{v_i \in \mathscr{H}} \left(1 - |\wp^t_{\mathscr{R}}(v_i) - \wp^t_{\mathscr{U}}(v_i)| \vee |I^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{U}}(v_i)| \vee |\left(\mathscr{G}^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{U}}(v_i)\right)|\right)}{\sum_{v_i \in \mathscr{H}} \left(1 + |\wp^t_{\mathscr{R}}(v_i) - \wp^t_{\mathscr{U}}(v_i)| \vee |I^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{U}}(v_i)| \vee |\left(\mathscr{G}^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{U}}(v_i)\right)|\right)}$;

7. $S_7(\mathscr{R},\ \mathscr{U}) = \alpha \dfrac{\sum_{v_i\in\mathscr{H}} \left(\wp^t_{\mathscr{R}}(v_i) \wedge \wp^t_{\mathscr{U}}(v_i)\right)}{\sum_{v_i\in\mathscr{H}} \left(\wp^t_{\mathscr{R}}(v_i) \vee \wp^t_{\mathscr{U}}(v_i)\right)} + \gamma \dfrac{\sum_{v_i\in\mathscr{H}} \left(I^t_{\mathscr{R}}(v_i) \wedge I^t_{\mathscr{U}}(v_i)\right)}{\sum_{v_i\in\mathscr{H}} \left(I^t_{\mathscr{R}}(v_i) \vee I^t_{\mathscr{U}}(v_i)\right)}$
$+\beta \dfrac{\sum_{v_i\in\mathscr{H}} \left(\mathscr{G}^t_{\mathscr{R}}(v_i) \wedge \mathscr{G}^t_{\mathscr{U}}(v_i)\right)}{\sum_{v_i\in\mathscr{H}} \left(\mathscr{G}^t_{\mathscr{R}}(v_i) \vee \mathscr{G}^t_{\mathscr{U}}(v_i)\right)}, \alpha + \gamma + \beta = 1, \alpha, \gamma, \beta \in [0, 1];$

8. $S_8(\mathscr{R},\ \mathscr{U}) = \dfrac{\alpha}{|\mathscr{H}|} \sum_{v_i\in\mathscr{H}} \dfrac{\left(\wp^t_{\mathscr{R}}(v_i) \wedge \wp^t_{\mathscr{U}}(v_i)\right)}{\left(\wp^t_{\mathscr{R}}(v_i) \vee \wp^t_{\mathscr{U}}(v_i)\right)} + \dfrac{\gamma}{|v_i|}$
$\sum_{v_i\in\mathscr{H}} \dfrac{\left(I^t_{\mathscr{R}}(v_i) \wedge I^t_{\mathscr{U}}(v_i)\right)}{\left(I^t_{\mathscr{R}}(v_i) \vee I^t_{\mathscr{U}}(v_i)\right)} + \dfrac{\beta}{|X|} \sum_{v_i\in\mathscr{H}} \dfrac{\left(\mathscr{G}^t_{\mathscr{R}}(v_i) \wedge \mathscr{G}^t_{\mathscr{U}}(v_i)\right)}{\left(\mathscr{G}^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{U}}(v_i)\right)},$
$\alpha + \gamma + \beta = 1, \alpha, \gamma, \beta \in [0, 1];$

9. $S_9(\mathscr{R},\ \mathscr{U}) =$
$\dfrac{1}{|\mathscr{H}|} \sum_{v_i\in\mathscr{H}} \dfrac{\left(\wp^t_{\mathscr{R}}(v_i) \wedge \wp^t_{\mathscr{U}}(v_i)\right) + \left(I^t_{\mathscr{R}}(v_i) \wedge I^t_{\mathscr{U}}(v_i)\right) + \left(\mathscr{G}^t_{\mathscr{R}}(v_i) \wedge \mathscr{G}^t_{\mathscr{U}}(v_i)\right)}{\left(\wp^t_{\mathscr{R}}(v_i) \vee \wp^t_{\mathscr{U}}(v_i)\right) + \left(I^t_{\mathscr{R}}(v_i) \vee I^t_{\mathscr{U}}(v_i)\right) + \left(\mathscr{G}^t_{\mathscr{R}}(v_i) \vee \mathscr{G}^t_{\mathscr{U}}(v_i)\right)};$

10. $S_{10}(\mathscr{R},\ \mathscr{U}) =$
$\dfrac{\sum_{v_i\in\mathscr{H}} \left(\wp^t_{\mathscr{R}}(v_i) \wedge \wp^t_{\mathscr{U}}(v_i)\right) + \left(I^t_{\mathscr{R}}(v_i) \wedge I^t_{\mathscr{U}}(v_i)\right) + \left(\mathscr{G}^t_{\mathscr{R}}(v_i) \wedge \mathscr{G}^t_{\mathscr{U}}(v_i)\right)}{\sum_{v_i\in\mathscr{H}} \left(\wp^t_{\mathscr{R}}(v_i) \vee \wp^t_{\mathscr{U}}(v_i)\right) + \left(I^t_{\mathscr{R}}(v_i) \vee I^t_{\mathscr{U}}(v_i)\right) + \left(\mathscr{G}^t_{\mathscr{R}}(v_i) \vee \mathscr{G}^t_{\mathscr{U}}(v_i)\right)};$

11. $$S_{11}(\mathscr{R},\ \mathscr{U}) = \frac{1}{|\mathscr{H}|} \sum_{v_i\in\mathscr{H}} \frac{\begin{array}{c}(\wp^t_{\mathscr{R}}(v_i)\wedge\wp^t_{\mathscr{U}}(v_i))+((1-I^t_{\mathscr{R}}(v_i))\wedge(1-I^t_{\mathscr{U}}(v_i)))\\+((1-\mathscr{G}^t_{\mathscr{R}}(v_i))\wedge(1-\mathscr{G}^t_{\mathscr{U}}(v_i)))\end{array}}{\begin{array}{c}(\wp^t_{\mathscr{R}}(v_i)\vee\wp^t_{\mathscr{U}}(v_i))+((1-I^t_{\mathscr{R}}(v_i))\wedge(1-I^t_{\mathscr{U}}(v_i)))\\+((1-\mathscr{G}^t_{\mathscr{R}}(v_i))\wedge(1-\mathscr{G}^t_{\mathscr{U}}(v_i)))\end{array}};$$

12. $$S_{12}(\mathscr{R},\ \mathscr{U}) = \frac{\begin{array}{c}\sum_{v_i\in\mathscr{H}}(\wp^t_{\mathscr{R}}(v_i)\wedge\wp^t_{\mathscr{U}}(v_i))+((1-I^t_{\mathscr{R}}(v_i))\wedge(1-I^t_{\mathscr{U}}(v_i)))\\+((1-\mathscr{G}^t_{\mathscr{R}}(v_i))\wedge(1-\mathscr{G}^t_{\mathscr{U}}(v_i)))\end{array}}{\begin{array}{c}\sum_{v_i\in\mathscr{H}}(\wp^t_{\mathscr{R}}(v_i)\vee\wp^t_{\mathscr{U}}(v_i))+((1-I^t_{\mathscr{R}}(v_i))\vee(1-I^t_{\mathscr{U}}(v_i)))\\+((1-\mathscr{G}^t_{\mathscr{R}}(v_i))\vee(1-\mathscr{G}^t_{\mathscr{U}}(v_i)))\end{array}};$$

13. $S_{13}(\mathscr{R},\ \mathscr{U}) =$
$$\sqrt[t]{\begin{array}{l}\dfrac{1}{2|\mathscr{H}|(l_1+1)^t} \sum_{v_i\in\mathscr{H}} \left\{\begin{array}{c}|\left(l_1\left(\wp^t_{\mathscr{R}}(v_i)\right) - \wp^t_{\mathscr{U}}(v_i)\right) - \\ \left(\left(I^t_{\mathscr{R}}(v_i)\right) - I^t_{\mathscr{U}}(v_i)\right) - \left(\mathscr{G}^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{U}}(v_i)\right)|^t\end{array}\right\} \\ +\dfrac{1}{2|\mathscr{H}|(l_2+1)^t} \sum_{v_i\in\mathscr{H}} \left\{\begin{array}{c}|\left(l_2\left(\mathscr{G}^t_{\mathscr{R}}(v_i)\right) - \mathscr{G}^t_{\mathscr{U}}(v_i)\right) - \\ \left(\left(I^t_{\mathscr{R}}(v_i)\right) - I^t_{\mathscr{U}}(v_i)\right) - \left(\wp^t_{\mathscr{R}}(v_i) - \wp^t_{\mathscr{U}}(v_i)\right)|^t\end{array}\right\}\end{array}}$$

Theorem 2 *For* $i = 1, 2, 3, \ldots, 13$, *if* $\alpha = \beta = 1/2$, *then the following hold:*

1. $S_i(\mathscr{R}, \mathscr{U}^c) = S_i(\mathscr{U}^c, \mathscr{R})$;
2. $S_i(\mathscr{R}, \mathscr{U}) = S_i(\mathscr{R} \cap \mathscr{U}, \mathscr{R} \cup \mathscr{U})$;
3. $S_i(\mathscr{R}, \mathscr{R} \cap \mathscr{U}) = S_i(\mathscr{R}, \mathscr{R} \cup \mathscr{U})$;
4. $S_i(\mathscr{R}, \mathscr{R} \cup \mathscr{U}) = S_i(\mathscr{R}, \mathscr{R} \cap \mathscr{U})$.

Theorem 3 *For* $i = 1, 2, \ldots, 6$, *the following hold:*

1. $S_i(\mathscr{R}, \mathscr{R} \otimes \mathscr{U}) = S_i(\mathscr{U}, \mathscr{R} \oplus \mathscr{U})$;
2. $S_i(\mathscr{R}, \mathscr{R} \oplus \mathscr{U}) = S_i(\mathscr{U}, \mathscr{R} \otimes \mathscr{U})$.

Theorem 4 *For i=1,4,5,6 and for all* $\mathrm{v}_i \in \mathscr{H}$, $\left(\wp^t_{\mathscr{R}}(v_i) + \wp^t_{\mathscr{U}}(v_i)\right) = 1$, $\left(I^t_{\mathscr{R}}(v_i)\right) + I^t_{\mathscr{U}}(\mathrm{v}_i) = 1$ *and* $\mathscr{G}^t_{\mathscr{R}}(\mathrm{v}_i) + \mathscr{G}^t_{\mathscr{U}}(\mathrm{v}_i) = 1$, *we have*

1. $S_i(\mathscr{R}, \mathscr{R} \oslash \mathscr{U}) = S_i(\mathscr{U}, \mathscr{R} \ominus \mathscr{U})$, $\wp^t_{\mathscr{R}}(\mathrm{v}_i) \leq \wp^t_{\mathscr{U}}(\mathrm{v}_i)$, $\left(I^t_{\mathscr{R}}(v_i)\right) \geq I^t_{\mathscr{U}}(\mathrm{v}_i)$ *and* $\left(\mathscr{G}^t_{\mathscr{R}}(v_i) \geq \mathscr{G}^t_{\mathscr{U}}(v_i)\right)$
2. $S_i(\mathscr{R}, \mathscr{R} \ominus \mathscr{U}) = S_i(\mathscr{U}, \mathscr{R} \oslash \mathscr{U})$, $\wp^t_{\mathscr{R}}(\mathrm{v}_i) \geq \wp^t_{\mathscr{U}}(\mathrm{v}_i)$, $\left(I^t_{\mathscr{R}}(v_i)\right) \leq I^t_{\mathscr{U}}(\mathrm{v}_i)$ *and* $\left(\mathscr{G}^t_{\mathscr{R}}(v_i) \leq \mathscr{G}^t_{\mathscr{U}}(v_i)\right)$

3.3 *Entropy for T-SFSs*

Let $\mathscr{R}$, and $\mathscr{U}$ be two T-SFSs on $\mathscr{H}$. A measure of entropy $E(\mathscr{R})$ is a function E: T-SFS$(\mathscr{H}) \to [0, 1]$ carries the given features:

1. $0 \leq E(\mathscr{R}) \leq 1$;
2. $E(\mathscr{R}) = 0$ iff $\mathscr{R}$ is a set of crisp;
3. $E(\mathscr{R}) = 1$ iff $\wp^t_{\mathscr{R}}(\mathrm{v}_i) = \mathscr{G}^t_{\mathscr{R}}(\mathrm{v}_i)$;
4. $E(\mathscr{R}) = E(\mathscr{R}^c)$;
5. If $E(\mathscr{R}) \leq E(\mathscr{U})$ if $\mathscr{R}$ is less fuzzy than $\mathscr{U}$, that is $\wp^t_{\mathscr{R}}(\mathrm{v}_i) \leq \wp^t_{\mathscr{U}}(\mathrm{v}_i) \leq I^t_{\mathscr{U}}(\mathrm{v}_i) \leq I^t_{\mathscr{R}}(\mathrm{v}_i) \leq \mathscr{G}^t_{\mathscr{U}}(\mathrm{v}_i) \leq \mathscr{G}^t_{\mathscr{R}}(\mathrm{v}_i)$ and $\mathscr{G}^t_{\mathscr{R}}(\mathrm{v}_i) \leq \mathscr{G}^t_{\mathscr{U}}(\mathrm{v}_i) \leq I^t_{\mathscr{R}}(\mathrm{v}_i) \leq I^t_{\mathscr{U}}(\mathrm{v}_i) \wp^t_{\mathscr{U}}(\mathrm{v}_i) \leq \wp^t_{\mathscr{R}}(\mathrm{v}_i)$

Theorem 5 *Let* $\mathscr{R}$ *be two T-SFSs. Then* $E_i(\mathscr{R}, \mathscr{U})$ $(i = 1, 2, \ldots, 12)$ *are entropy measures:*

1. $E_1(\mathscr{R}) = \sum_{v_i \in \mathscr{H}} \dfrac{\left(\pi^t_{\mathscr{R}}(v_i) + 1 - |\wp^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{R}}(v_i)\right)|}{\left(\pi^t_{\mathscr{R}}(v_i) + 1 + |\wp^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{R}}(v_i)\right)|}$;
2. $E_2(\mathscr{R}) = \dfrac{1}{2|\mathscr{H}|} \sum_{v_i \in \mathscr{H}} \dfrac{1 - |\wp^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{R}}(v_i)|}{1 + |\wp^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{R}}(v_i)|}$;
3. $E_3(\mathscr{R}) = \dfrac{\sum_{v_i \in \mathscr{H}} \left(1 - |\wp^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{R}}(v_i)|\right)}{\sum_{v_i \in \mathscr{H}} \left(1 + |\wp^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{R}}(v_i)|\right)}$;
4. $E_4(\mathscr{R}) = 1 - \dfrac{1}{|\mathscr{H}|} \sum_{v_i \in \mathscr{H}} \left(|\wp^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{R}}(v_i)|\right)$;
5. $E_5(\mathscr{R}) = \dfrac{\sum_{v_i \in \mathscr{H}} \left(\wp^t_{\mathscr{R}}(v_i) \wedge I^t_{\mathscr{R}}(v_i) \wedge \mathscr{G}^t_{\mathscr{R}}(v_i)\right)}{\sum_{v_i \in \mathscr{H}} \left(\wp^t_{\mathscr{R}}(v_i) \vee I^t_{\mathscr{R}}(v_i) \vee \mathscr{G}^t_{\mathscr{R}}(v_i)\right)}$;
6. $E_6(\mathscr{R}) = \dfrac{1}{|\mathscr{H}|} \sum_{v_i \in \mathscr{H}} \dfrac{\left(\wp^t_{\mathscr{R}}(v_i) \wedge I^t_{\mathscr{R}}(v_i) \wedge \mathscr{G}^t_{\mathscr{R}}(v_i)\right)}{\left(\wp^t_{\mathscr{R}}(v_i) \vee I^t_{\mathscr{R}}(v_i) \vee \mathscr{G}^t_{\mathscr{R}}(v_i)\right)}$;

7. $E_7(\mathscr{R}) = \frac{\sum_{v_i \in \mathscr{H}} \left(0.5\pi^t_{\mathscr{R}}(v_i) + \wp^t_{\mathscr{R}}(v_i) \wedge I^t_{\mathscr{R}}(v_i) \wedge \mathscr{G}^t_{\mathscr{R}}(v_i)\right)}{\sum_{v_i \in \mathscr{H}} \left(0.5\pi^t_{\mathscr{R}}(v_i) + \wp^t_{\mathscr{R}}(v_i) \vee I^t_{\mathscr{R}}(v_i) \vee \mathscr{G}^t_{\mathscr{R}}(v_i)\right)}$;

8. $E_8(\mathscr{R}) = \frac{1}{|\mathscr{H}|} \sum_{v_i \in \mathscr{H}} \frac{0.5\pi^t_{\mathscr{R}}(v_i) + \wp^t_{\mathscr{R}}(v_i) \wedge I^t_{\mathscr{R}}(v_i) \wedge \mathscr{G}^t_{\mathscr{R}}(v_i)}{0.5\pi^t_{\mathscr{R}}(v_i) + \wp^t_{\mathscr{R}}(v_i) \vee I^t_{\mathscr{R}}(v_i) \vee \mathscr{G}^t_{\mathscr{R}}(v_i)}$;

9. $E_9(\mathscr{R}) = \frac{1}{\left(\sqrt{2}-1\right) v_i |\mathscr{H}|} \sum_{v_i \in \mathscr{H}} \left(\sin \frac{1 + \wp^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{R}}(v_i)}{4}\pi\right) + \left(\sin \frac{11\wp^t_{\mathscr{R}}(v_i) + I^t_{\mathscr{R}}(v_i) + \mathscr{G}^t_{\mathscr{R}}(v_i)}{4}\pi - 1\right)$;

10. $E_{10}(\mathscr{R}, \mathscr{U}) = \frac{1}{\left(\sqrt{2}-1\right) v_i |\mathscr{H}|} \sum_{v_i \in \mathscr{H}} \left(\cos \frac{1 + \wp^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{R}}(v_i)}{4}\pi\right) + \left(\cos \frac{11\wp^t_{\mathscr{R}}(v_i) + I^t_{\mathscr{R}}(v_i) + \mathscr{G}^t_{\mathscr{R}}(v_i)}{4}\pi - 1\right)$;

11. $E_{11}(\mathscr{R}, \mathscr{U}) = \frac{1}{|\mathscr{H}|} \sum_{v_i \in \mathscr{H}} \cot\left(\frac{1}{4}\pi + \frac{|\wp^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{R}}(v_i)|}{4\left(1 + \pi^t_{\mathscr{R}}(v_i)\right)}\pi\right)$;

12. $E_{12}(\mathscr{R}, \mathscr{U}) = \frac{1}{|\mathscr{H}|} \sum_{v_i \in \mathscr{H}} \tan\left(\frac{1}{4}\pi - \frac{|\wp^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{R}}(v_i)|}{4\left(1 + \pi^t_{\mathscr{R}}(v_i)\right)}\pi\right)$.

3.4 Inclusion Measure for *T*-SFSs

Let $\mathscr{R}$, $\mathscr{U}$, and $\mathscr{W}$ be three T-SFSs on $\mathscr{H}$. An inclusion measure $I(\mathscr{R}, \mathscr{U})$ is a function I: T-SFS($\mathscr{H}$) × T-SFS($\mathscr{H}$) → [0, 1], carrying the following features:

1. $0 \leq I(\mathscr{R}, \mathscr{U}) \leq 1$;
2. $I(\mathscr{R}, \mathscr{U}) = 1$ iff $\mathscr{R} \subseteq \mathscr{U}$;
3. $I(\mathscr{R}, \mathscr{U}) = 0$ iff $\mathscr{R} = \Phi$ and $\mathscr{U} = \varnothing$;
4. If $\mathscr{R} \subseteq \mathscr{U} \subseteq \mathscr{W}$, then $I(\mathscr{R}, \mathscr{U}) \leq I(\mathscr{R}, \mathscr{W})$ and $I(\mathscr{U}, \mathscr{W}) \leq I(\mathscr{R}, \mathscr{W})$.

Theorem 6 *Let $\mathscr{R}$ and $\mathscr{U}$ be two T-SFSs. Then $I_i(\mathscr{R}, \mathscr{U})$ $(i = 1, 2, \ldots, 7)$ are inclusion measures:*

1. $I_1(\mathscr{R}, \mathscr{U}) =$

$$1 - \frac{1}{2|\mathscr{H}|} \sum_{v_i \in \mathscr{H}} \begin{pmatrix} |\wp^t_{\mathscr{R}}(v_i) - \wp^t_{\mathscr{R}}(v_i) \wedge \wp^t_{\mathscr{U}}(v_i)| + |I^t_{\mathscr{R}}(v_i) \\ -I^t_{\mathscr{R}}(v_i) \wedge I^t_{\mathscr{U}}(v_i)| + |\mathscr{G}^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{R}}(v_i)| \end{pmatrix};$$

2. $I_2(\mathscr{R},\ \mathscr{U}) =$

$$\begin{cases} 1, & \mathscr{R} = \varnothing \\ \dfrac{\sum_{v_i \in \mathscr{H}} \left(1 + \wp^t_{\mathscr{R}}(v_i) \wedge \wp^t_{\mathscr{U}}(v_i) - I^t_{\mathscr{R}}(v_i) \vee I^t_{\mathscr{U}}(v_i) - \mathscr{G}^t_{\mathscr{R}}(v_i) \vee \mathscr{G}^t_{\mathscr{U}}(v_i)\right)}{\sum_{v_i \in \mathscr{H}} \left(1 + \wp^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{R}}(v_i)\right)}, & \mathscr{R} \neq \varnothing \end{cases};$$

3. $I_3(\mathscr{R},\ \mathscr{U}) =$

$$\begin{cases} 1, & \mathscr{R} = \mathscr{U} = \varnothing \\ \dfrac{\sum_{v_i \in \mathscr{H}} \left(1 + \wp^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{R}}(v_i)\right)}{\sum_{v_i \in \mathscr{H}} \left(1 + \wp^t_{\mathscr{R}}(v_i) \vee \wp^t_{\mathscr{U}}(v_i) - I^t_{\mathscr{R}}(v_i) \wedge I^t_{\mathscr{U}}(v_i) - \mathscr{G}^t_{\mathscr{R}}(v_i) \wedge \mathscr{G}^t_{\mathscr{U}}(v_i)\right)}, & others \end{cases};$$

4. $I_4(\mathscr{R},\ \mathscr{U}) =$

$$\begin{cases} 1, & \mathscr{R} = \mathscr{U} = \varnothing \\ \dfrac{\sum_{v_i \in \mathscr{H}} \left(1 + \wp^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{R}}(v_i)\right)}{\sum_{v_i \in \mathscr{H}} \left(1 + \mathscr{G}^t_{\mathscr{R}}(v_i) \wedge \mathscr{G}^t_{\mathscr{U}}(v_i) - I^t_{\mathscr{R}}(v_i) \wedge I^t_{\mathscr{U}}(v_i) - \wp^t_{\mathscr{R}}(v_i) \vee \wp^t_{\mathscr{U}}(v_i)\right)}, & others \end{cases};$$

5. $I_5(\mathscr{R},\ \mathscr{U}) =$

$$\begin{cases} 1, & \mathscr{R} = \mathscr{U} = \varnothing \\ \dfrac{1}{|\mathscr{H}|} \sum_{v_i \in \mathscr{H}} \dfrac{\left(1 + \wp^t_{\mathscr{R}}(v_i) \wedge \wp^t_{\mathscr{U}}(v_i) - I^t_{\mathscr{R}}(v_i) \vee I^t_{\mathscr{U}}(v_i) - \mathscr{G}^t_{\mathscr{R}}(v_i) \vee \mathscr{G}^t_{\mathscr{U}}(v_i)\right)}{\left(1 + \wp^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{R}}(v_i)\right)}, & others \end{cases};$$

6. $I_6(\mathscr{R},\ \mathscr{U}) =$

$$\begin{cases} 1, & \mathscr{R} = \mathscr{U} = \varnothing \\ \dfrac{1}{|\mathscr{H}|} \sum_{v_i \in \mathscr{H}} \dfrac{\left(1 + \wp^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{R}}(v_i)\right)}{\left(1 + \wp^t_{\mathscr{R}}(v_i) \vee \wp^t_{\mathscr{U}}(v_i) - I^t_{\mathscr{R}}(v_i) \wedge I^t_{\mathscr{U}}(v_i) - \mathscr{G}^t_{\mathscr{R}}(v_i) \wedge \mathscr{G}^t_{\mathscr{U}}(v_i)\right)}, & others \end{cases};$$

7. $I_7(\mathscr{R},\ \mathscr{U}) =$

$$\begin{cases} 1, & \mathscr{R} = \mathscr{U} = \varnothing \\ \dfrac{1}{|\mathscr{H}|} \sum_{v_i \in \mathscr{H}} \dfrac{\left(1 + \wp^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{R}}(v_i)\right)}{\left(1 + \mathscr{G}^t_{\mathscr{R}}(v_i) \wedge \mathscr{G}^t_{\mathscr{U}}(v_i) - I^t_{\mathscr{R}}(v_i) \wedge I^t_{\mathscr{U}}(v_i) - \wp^t_{\mathscr{R}}(v_i) \vee \wp^t_{\mathscr{U}}(v_i)\right)}, & others \end{cases}.$$

3.5 Information Measure Transformation Connections for T-SFSs

Theorem 7 *Let $\wp$ be the T-spherical fuzzy distance measure for $\mathscr{R}, \mathscr{U} \in T{-}SFSs$. Then $S(\mathscr{R}, \mathscr{U}) = 1 - \wp(\mathscr{R}, \mathscr{U})$the similarity measurement is of T-SFSs $\mathscr{R}$ and $\mathscr{U}$.*

Proof The proof is straightforward.

Theorem 8 *For $\mathscr{R}, \mathscr{U} \in$ T-SFSs. Then the following hold:*
$\wp_1(\mathscr{R}, \mathscr{U}) = \dfrac{1}{|\mathscr{H}|} \sum_{v_i \in \mathscr{H}} \left(\begin{array}{c} |\wp^t_{\mathscr{R}}(v_i) - \wp^t_{\mathscr{U}}(v_i)| + |I^t_{\mathscr{R}}(v_i) - I^t_{\mathscr{U}}(v_i)| \\ +|\mathscr{G}^t_{\mathscr{R}}(v_i) - \mathscr{G}^t_{\mathscr{U}}(v_i)| + |\pi^t_{\mathscr{R}}(v_i) - \pi^t_{\mathscr{U}}(v_i)| \end{array} \right)$ *then we have*

$S(\mathscr{R},\mathscr{U}) = 1-\frac{1}{2|\mathscr{H}|}\sum_{v_i\in\mathscr{H}}\begin{pmatrix}|\wp^t_{\mathscr{R}}(v_i)-\wp^t_{\mathscr{U}}(v_i)|+|I^t_{\mathscr{R}}(v_i)-I^t_{\mathscr{U}}(v_i)|\\+|\mathscr{G}^t_{\mathscr{R}}(v_i)-\mathscr{G}^t_{\mathscr{U}}(v_i)|+|\pi^t_{\mathscr{R}}(v_i)-\pi^t_{\mathscr{U}}(v_i)|\end{pmatrix} =$ $S_1(\mathscr{R},\mathscr{U})$. *Also* $S_i(\mathscr{R},\mathscr{U}) = 1-\wp_i(\mathscr{R},\mathscr{U})(i=1,2,\ldots 13)$.

Theorem 9 *Let* $\wp$ *and* S *be the measure of distance and measure of similarity of* T*-SFSs, for* $\mathscr{R}\in T$*-SFSs. Then*

$$\aleph(\mathscr{R}) = 1-S(\mathscr{R},\mathscr{R}^c) = 1-S(\mathscr{R},\mathscr{R}^c)$$

is the entropy of T*-SFSs.*

Proof $(\aleph_1)$ Obvious.
$(\aleph_2)$ If $\mathscr{R}$ is a crisp set, then $\mathscr{R}=\varnothing$ or $\mathscr{R}=\Phi$, we have $S(\mathscr{R},\mathscr{R}^c)=0$. Therefore, $E(\mathscr{R})=0$.
$(\aleph_3)$ $E(\mathscr{R})=1 \Leftrightarrow S(\mathscr{R},\mathscr{R}^c)=S(\mathscr{R}^c,\mathscr{R}) \Leftrightarrow \wp^t_{\mathscr{R}}(\mathrm{v}_i)=I^t_{\mathscr{R}}(\mathrm{v}_i)=\mathscr{G}^t_{\mathscr{R}}(\mathrm{v}_i)$ for $\mathrm{v}_i\in\mathscr{H}$.
$(\aleph_4)$ $\aleph(\mathscr{R})=S(\mathscr{R},\mathscr{R}^c)=S(\mathscr{R}^c,\mathscr{R})=\aleph(\mathscr{R}^c)$.
$(\aleph_5)$ Since $\wp^t_{\mathscr{R}}(v_i)\le\wp^t_{\mathscr{U}}(v_i)\le I^t_{\mathscr{U}}(v_i)\le I^t_{\mathscr{R}}(v_i)\le\mathscr{G}^t_{\mathscr{U}}(v_i)\le\mathscr{G}^t_{\mathscr{R}}(v_i)$ implies $\mathscr{R}\subseteq\mathscr{U}\subseteq\mathscr{U}^c\subseteq\mathscr{R}^c$. Therefore, according to the definition of similarity measure of $T-SFSs$, we have $S(\mathscr{R},\mathscr{R}^c)\le S(\mathscr{U},\mathscr{R}^c)\le\aleph(\mathscr{R},\mathscr{R}^c)$, that is, $\aleph(\mathscr{R})\le\aleph(\mathscr{U})$. Similarly, if $\mathscr{G}^t_{\mathscr{R}}(v_i)\le\mathscr{G}^t_{\mathscr{U}}(v_i)\le I^t_{\mathscr{R}}(v_i)I^t_{\mathscr{U}}(v_i)\le\wp^t_{\mathscr{U}}(v_i)\le\wp^t_{\mathscr{R}}(v_i)$, the we have $S(\mathscr{R},\mathscr{R}^c)\le(\mathscr{U},\mathscr{R}^c)\le S(\mathscr{U},\mathscr{U}^c), thatis\aleph(\mathscr{R})\le\aleph(\mathscr{U})$. This completes the proof.

Theorem 10 *For* $\Re,\mathscr{U}\in T$*-SFSs. Then the following hold:*

1. $S(\Re,\mathscr{U}) = S_1(\Re,\mathscr{U}) = 1-\frac{1}{2|\mathscr{H}|}$ $\sum_{v_i\in\mathscr{H}}\begin{pmatrix}|\wp^t_{\Re}(v_i)-\wp^t_{\mathscr{U}}(v_i)|+|I^t_{\Re}(v_i)-I^t_{\mathscr{U}}(v_i)|\\+|\mathscr{G}^t_{\mathscr{R}}(v_i)-\mathscr{G}^t_{\mathscr{U}}(v_i)|+|\pi^t_{\mathscr{R}}(v_i)-\pi^t_{\mathscr{U}}(v_i)|\end{pmatrix}$*, then we have*
2. $\aleph(\Re) = S_1(\Re,\Re^c) = 1-\frac{1}{|\mathscr{H}|}\sum_{v_i\in\mathscr{H}}|\wp^t_{\Re}(\mathrm{v}_i)-I^t_{\Re}(\mathrm{v}_i)-\mathscr{G}^t_{\Re}(\mathrm{v}_i)| = \aleph_4(\Re)$. *Also*
3. $\aleph_4(\Re) = S_2(\Re,\Re^c) = S_3(\Re,\Re^c) = S_4(\Re,\Re^c) = S_{13}(\Re,\Re^c)$,
4. $\aleph_5(\Re) = S_7(\Re,\Re^c) = S_{10}(\Re,\Re^c)$,
5. $\aleph_6(\Re) = S_8(\Re,\Re^c) = S_9(\Re,\Re^c)$,
6. $\aleph_7(\Re) = S_{12}(\Re,\Re^c)$,
7. $\aleph_8(\Re) = S_{11}(\Re,\Re^c)$.

Definition 8 Let $\Re$ be a T-SFS, $m(\Re), n(\Re)\in \mathrm{T}-SFSs$, $\forall\,\mathrm{v}_i\in\mathscr{H}$, $m(\Re)(\mathrm{v}_i) = (\wp_m(\Re)(v_i), I_m(\Re)(v_i), \mathscr{G}_m(\Re)(v_i))$, $n(\Re)(\mathrm{v}_i) = (\wp_n(\Re)(v_i), I_n(\Re)(v_i), \mathscr{G}_n(\Re)(v_i))$; their membership and non-membership functions are defined as follows:

- $\wp_m(\Re)(\mathrm{v}_i) = \sqrt{1+\left(\wp^t_{\Re}(v_i)-I^t_{\Re}(v_i)-\mathscr{G}^t_{\Re}(v_i)\right)^t}$,

- $\mathcal{G}_m\,(\Re)\,(\mathrm{v}_i) = \sqrt{1 - |\wp^t_{\Re}(v_i) - I^t_{\Re}(v_i) - \mathcal{G}^t_{\Re}(v_i)|}$,
- $\wp_n\,(\Re)\,(\mathrm{v}_i) = \sqrt{1 - \left(\wp^t_{\Re}(v_i) - I^t_{\Re}(v_i) - \mathcal{G}^t_{\Re}(v_i)\right)^t}$,
- $\mathcal{G}_n\,(\Re)\,(\mathrm{v}_i) = \sqrt{1 + |\wp^t_{\Re}(v_i) - I^t_{\Re}(v_i) - \mathcal{G}^t_{\Re}(v_i)|}$.

Theorem 11 *Let D be the measure of distance and S be the similarity measure of T-SFSs, for $\Re \in T$-SFSs. Then*

$$\aleph(\Re) = S(m(\Re), n(\Re)) = 1 - D(\Re, \Re^c) = 1 - S(m(\Re), n(\Re))$$

is the entropy of T-SFS $\Re$.

Proof ($\aleph_1$) Obvious.
($\aleph_2$) If $\Re$ is a crisp set, then $\forall\ \mathrm{v}_i \in \mathscr{H}$, we have $\wp_{\Re}(\mathrm{v}_i) = 1$, $\mathcal{G}_{\Re}(\mathrm{v}_i) = 0$ or $\wp_{\Re}(\mathrm{v}_i) = 0, \mathcal{G}_{\Re}(\mathrm{v}_i) = 1$. Therefore, we can achieve

$$\wp_m\,(\Re)\,(v_i) = 1, \mathcal{G}_m\,(\Re)\,(v_i) = 0, \wp_m\,(\Re)\,(v_i) = 0, \mathcal{G}_m\,(\Re)\,(v_i) = 1.$$

This implies that $m(\Re) = \Phi$, $n(\Re) = \varnothing$, consequently, $S(m(\Re), n(\Re)) = 0$.
($\aleph_3$)

$$\aleph(\Re) = 1 \Leftrightarrow S(m(\Re), n(\Re)) = 1 \Leftrightarrow m(\Re) = n(\Re) \Leftrightarrow \wp_m\,(\Re)\,(v_i)$$
$$= \wp_n\,(\Re)\,(v_i),\ \mathcal{G}_m\,(\Re)\,(v_i) = \mathcal{G}_n\,(\Re)\,(v_i).$$

($\aleph_4$) Using the definitions of $m(\Re)$ and $n(\Re)$, we have $m(\Re) = m(\Re^c), n(\Re) = n(\Re^c)$, hence $S(m(\Re), n(\Re)) = S(m(\Re^c), n(\Re^c))$.
($\aleph_5$) Since $\wp_{\Re}(\mathrm{v}_i) \leq \wp_{\mathcal{U}}(\mathrm{v}_i) \leq I_{\Re}(\mathrm{v}_i) \leq I_{\mathcal{U}}(\mathrm{v}_i) \leq \mathcal{G}_{\Re}(\mathrm{v}_i) \leq \mathcal{G}_{\mathcal{U}}(\mathrm{v}_i)$ implies $\Re \subseteq \mathcal{U} \subseteq \mathcal{U}^c \subseteq \Re^c$. Therefore, we have $|\wp^t_{\Re}(\mathrm{v}_i) - I^t_{\Re}(\mathrm{v}_i) - \mathcal{G}^t_{\Re}(\mathrm{v}_i)| \geq |\left(\wp^t_{\mathcal{U}}(v_i) - I^t_{\mathcal{U}}(v_i) - \mathcal{G}^t_{\mathcal{U}}(v_i)\right)|$. It means that $n(A) \leq n(\mathcal{U}) \leq m(\mathcal{U}) \leq m(\Re)$, so we have $S(m(\Re), n(\Re)) \leq S\,(m(\mathcal{U}), n(\Re)) \leq S\,(m(\mathcal{U}), n(\mathcal{U}))$, that is, $\aleph(\Re) \leq \aleph(\mathcal{U})$.

Similarly, if $\mathcal{G}_{\Re}(\mathrm{v}_i) \leq \mathcal{G}_{\mathcal{U}}(\mathrm{v}_i) \leq I_{\mathcal{U}}(\mathrm{v}_i) \leq I_{\Re}(\mathrm{v}_i) \leq \wp_{\mathcal{U}}(\mathrm{v}_i) \leq \wp_{\Re}(\mathrm{v}_i)$, then we have $S(m(\Re), n(\Re)) \leq S\,(m(\Re), n(\mathcal{U})) \leq S\,(m(\mathcal{U}), n(\mathcal{U}))$, that is, $\aleph(\Re) \leq \aleph(\mathcal{U})$. This completes the proof.

Proposition 2 *Assume D and S are the measure of distance and measures of its similarity of T-SFSs, respectively, for $\Re, \mathcal{U} \in T$-SFSs. Then*

$$I(\Re, \mathcal{U}) = S(\Re, \Re \cap \mathcal{U}) = 1 - D(\Re, \Re \cap \mathcal{U})$$

is the measure of inclusion for T-SFSs $\Re$ and $\mathcal{U}$.

Proof (I_1) Obvious.
(I_2) If $\Re \subseteq \mathcal{U}$, then $S(\Re, \Re \cap \mathcal{U}) = S(\Re, \Re) = 1 = S(\Re, \mathcal{U})$.
(I_3) $I(\Re, \mathcal{U}) = 0 \Leftrightarrow S(\Re, \Re \cap \mathcal{U}) = 0 \Leftrightarrow \Re = \Phi, \mathcal{U} = \varnothing$.

(I_4) If $\Re \subseteq \mathscr{U} \subseteq O$, then $I(O, \Re) = S(O, O \cap \Re) = S(O, \Re)$ and $I(\mathscr{U}, \Re) = S(\mathscr{U}, \mathscr{U} \cap \Re) = S(\mathscr{U}, \Re)$. Known by the measure of similarity of T-SFSs, we have $I(O, M) \leq I(\mathscr{U}, \Re)$. Similarly, $I(O, \Re) \leq I(O, \mathscr{U})$. This completes the proof.

Proposition 3 *Assume D and S are the measure of distance and measures of similarities T-SFSs, respectively, for* $\Re$, $\mathscr{U} \in T$*-SFSs. Then*

$$I(\Re, \mathscr{U}) = S(\mathscr{U}, \Re \cup \mathscr{U}) = 1 - D(\mathscr{U}, \Re \cup \mathscr{U})$$

is the measure of inclusion T-SFSs $\Re$ *and* $\mathscr{U}$*.*

Definition 9 Let $\Re$ and $\mathscr{U}$ be two T-SFSs. Then, we described $g(A, B) \in$T$-SFSs$, $\forall \mathrm{v}_i \in \mathscr{H}$,

$\wp_g(\Re, \mathscr{U})(\mathrm{v}_i) =$

$$\sqrt{\frac{1 + \min\left\{|\wp^t_\Re(v_i) - \wp^t_\mathscr{U}(v_i)|, |I^t_\Re(v_i) - I^t_\mathscr{U}(v_i)|, |\mathscr{G}^t_\Re(v_i) - \mathscr{G}^t_\mathscr{U}(v_i)|\right\}}{2}},$$

$\mathscr{G}_g(\Re, \mathscr{U})(\mathrm{v}_i) =$

$$\sqrt{\frac{1 - \max\left\{|\wp^t_\Re(v_i) - \wp^t_\mathscr{U}(v_i)|, |I^t_\Re(v_i) - I^t_\mathscr{U}(v_i)|, |\mathscr{G}^t_\Re(v_i) - \mathscr{G}^t_\mathscr{U}(v_i)|\right\}}{2}}.$$

Proposition 4 *Assume* $\aleph$ *is the entropy measure of T-SFSs, for* $\Re$, $\mathscr{U} \in T$*-SFSs. Then* $\aleph(g(\Re, \mathscr{U}))$ *is the measure of similarity T-SFSs* $\Re$ *and* $\mathscr{U}$*.*

Proof (S_1)$-$(S_2) are clear-cut.
(S_3) known from the entropy definition of T-SFSs, $\aleph(g(\Re, \mathscr{U})) = 1 \Leftrightarrow \wp_g(\Re, \mathscr{U})(\mathrm{v}_i) = I_g(\Re, \mathscr{U})(\mathrm{v}_i) = \mathscr{G}_g(\Re, \mathscr{U})(\mathrm{v}_i) \Leftrightarrow |\wp^t_\Re(\mathrm{v}_i) - \wp^t_\mathscr{U}(\mathrm{v}_i)| = 0, |I^t_\Re(\mathrm{v}_i) - I^t_\mathscr{U}(\mathrm{v}_i)| = 0, |\mathscr{G}^t_\Re(\mathrm{v}_i) - \mathscr{G}^t_\mathscr{U}(\mathrm{v}_i)| = 0 \Leftrightarrow \wp_\Re(\mathrm{v}_i) = \wp_\mathscr{U}(\mathrm{v}_i)$, $I_\Re(\mathrm{v}_i) = I_\mathscr{U}(\mathrm{v}_i), \mathscr{G}_\Re(\mathrm{v}_i) = \mathscr{G}_\mathscr{U}(\mathrm{v}_i) \Leftrightarrow \Re = \mathscr{U}$.
(S_4) If $\Re$ is a crisp set, then $\wp_\Re(\mathrm{v}_i) = 1$, $I_\Re(\mathrm{v}_i) = 0$, $\mathscr{G}_\Re(\mathrm{v}_i) = 0$ or $\wp_\Re(\mathrm{v}_i) = 0$, $I_\Re(\mathrm{v}_i) = 0$, $\mathscr{G}_\Re(\mathrm{v}_i) = 1$. Hence, $\wp_g(\Re, \Re^c)(\mathrm{v}_i) = 1$, $I_\mathscr{R}(\Re, \mathscr{U}^c)(\mathrm{v}_i) = 0, \mathscr{G}_\Re(\Re, \mathscr{U}^c)(\mathrm{v}_i) = 0$, it implies $g(\Re, \mathscr{U}^c)$=Φ, so $\aleph(g(\Re, \mathscr{U}^c)) = 0$.
(S_5) Since $\Re \subseteq \mathscr{U} \subseteq O$, then $\forall$ $\mathrm{v}_i \in \mathscr{H}$, we have $\wp_\Re(\mathrm{v}_i) \leq \wp_\mathscr{U}(\mathrm{v}_i) \leq \wp_O(\mathrm{v}_i)$, $I_O(\mathrm{v}_i) \leq I_\mathscr{U}(\mathrm{v}_i) \leq I_\Re(\mathrm{v}_i)$, $\mathscr{G}_O(\mathrm{v}_i) \leq \mathscr{G}_\mathscr{U}(\mathrm{v}_i) \leq \mathscr{G}_\Re(\mathrm{v}_i)$. Therefore, we have $|\wp^t_\Re(\mathrm{v}_i) - \wp^t_O(\mathrm{v}_i)| \geq |\wp^t_\Re(\mathrm{v}_i) - \wp^t_\mathscr{U}(\mathrm{v}_i)|$, $|I^t_\Re(\mathrm{v}_i) - I^t_O(\mathrm{v}_i)| \geq |I^t_\Re(\mathrm{v}_i) - I^t_\mathscr{U}(\mathrm{v}_i)|$ and $|\mathscr{G}^t_\Re(\mathrm{v}_i) - \mathscr{G}^t_O(\mathrm{v}_i)| \geq |\mathscr{G}^t_\Re(\mathrm{v}_i) - \mathscr{G}^t_\mathscr{U}(\mathrm{v}_i)|$.

Further, we have

$$\begin{aligned} &min\left\{|\wp^t_\Re(v_i) - \wp^t_O(v_i)|, |I^t_\Re(v_i) - I^t_O(v_i)|, |\mathscr{G}^t_\Re(v_i) - \mathscr{G}^t_O(v_i)|\right\} \\ \geq\ &min\left\{|\wp^t_\Re(v_i) - \wp^t_\mathscr{U}(v_i)|, |I^t_\Re(v_i) - I^t_\mathscr{U}(v_i)|, |\mathscr{G}^t_\Re(v_i) - \mathscr{G}^t_\mathscr{U}(v_i)|\right\} \end{aligned}$$

$$\begin{aligned} &\max\left\{|\wp^t_\Re(v_i) - \wp^t_O(v_i)|, |I^t_\Re(v_i) - I^t_O(v_i)|, |\mathscr{G}^t_\Re(v_i) - \mathscr{G}^t_O(v_i)|\right\} \\ \geq\ &\max\left\{|\wp^t_\Re(v_i) - \wp^t_\mathscr{U}(v_i)|, |I^t_\Re(v_i) - I^t_\mathscr{U}(v_i)|, |\mathscr{G}^t_\Re(v_i) - \mathscr{G}^t_\mathscr{U}(v_i)|\right\} \end{aligned}$$

Also, we can know

$$\wp_g(\Re, O)(v_i) \geq \wp_g(\Re, \mathscr{U})(v_i),\ I_g(\Re, O)(v_i) \leq I_g(\Re, \mathscr{U})(v_i)$$

and

$$\begin{aligned}\mathscr{G}_{\Re}(\Re, O)(v_i) \leq \mathscr{G}_{\Re}(\Re, \mathscr{U})(v_i),\ \wp_g(\Re, \mathscr{U})(v_i) \geq I_g(\Re, \mathscr{U})(v_i)\\ \geq \mathscr{G}_{\Re}(\Re, \mathscr{U})(v_i)\end{aligned}$$

that is,

$$\begin{aligned}&\mathscr{G}_g(\Re, O)(v_i) \leq \mathscr{G}_g(\Re, \mathscr{U})(v_i) \leq I_g(\Re, O)(v_i) \leq\\ &I_g(\Re, \mathscr{U})(v_i) \leq \wp_g(\Re, \mathscr{U})(v_i) \leq \wp_g(\Re, O)(v_i)\end{aligned}$$

and understood from the definition, $\aleph(g(\Re, O)) \leq \aleph(g(\Re, \mathscr{U}))$.

Similarly, $\aleph(g(\Re, O)) \leq \aleph(g(\mathscr{U}, O))$.

Proposition 5 *Assume I is the inclusion measure of T-SFSs, for $\Re \in T$-SFSs. Then $\aleph(\Re) = I(\Re \cup \Re^c, \Re \cap \Re^c)$ is the entropy of T-SFSs $\Re$.*

Proof ($\aleph_1$) Obvious.
($\aleph_2$) If $\Re$ is a crisp set, then $\Re = \Phi$ or $\Re = \varnothing$, we have $I(\Re \cup \Re^c, \Re \cap \Re^c) = I(\Phi, \varnothing) = 0$. Therefore, $\aleph(\Re) = 0$.
($\aleph_3$) $\aleph(\Re) = 1 \Leftrightarrow I(\Re \cup \Re^c, \Re \cap \Re^c) = 1 \Leftrightarrow \Re \cup \Re^c \subseteq \Re \cap \Re^c \Leftrightarrow \Re \cup \Re^c = \Re \cap \Re^c \Leftrightarrow \wp_{\Re}(\mathrm{v}_i) = I_{\Re}(\mathrm{v}_i) = \mathscr{G}_{\Re}(\mathrm{v}_i)$.
($\aleph_4$) $\aleph(\Re) = I(\Re \cup \Re^c, \Re \cap \Re^c) = I(\Re^c \cup \Re, \Re^c \cap \Re) = \aleph(\Re^c)$.
($\aleph_5$) Since $\wp_{\Re}(\mathrm{v}_i) \leq \wp_{\mathscr{U}}(\mathrm{v}_i) \leq I_{\mathscr{U}}(\mathrm{v}_i) \leq I_{\Re}(\mathrm{v}_i) \leq \mathscr{G}_{\mathscr{U}}(\mathrm{v}_i) \leq \mathscr{G}_{\Re}(\mathrm{v}_i)$ implies $\Re \subseteq \mathscr{U} \subseteq \mathscr{U}^c \subseteq \Re^c$. Further, $\Re \cap \Re^c \subseteq \mathscr{U} \cap \mathscr{U}^c \subseteq \mathscr{U} \cup \mathscr{U}^c \subseteq \Re \cup \Re^c$.

By definition of inclusion measure, it follows that $I(\Re \cup \Re^c, \Re \cap \Re^c) \leq I(\Re \cup \Re^c, \mathscr{U} \cap \mathscr{U}^c) \leq I(\mathscr{U} \cup \mathscr{U}^c, \mathscr{U} \cap \mathscr{U}^c)$, so $\aleph(\Re) \leq E(\mathscr{U})$.

Similarly, if $\mathscr{G}_{\Re}(\mathrm{v}_i) \leq \mathscr{G}_{\mathscr{U}}(\mathrm{v}_i) \leq I_{\Re}(\mathrm{v}_i) \leq I_{\mathscr{U}}(\mathrm{v}_i) \leq \wp_{\mathscr{U}}(\mathrm{v}_i) \leq \wp_{\Re}(\mathrm{v}_i)$, then we can have $I(\Re \cup \Re^c, \Re \cap \Re^c) \leq I(\mathscr{U} \cup \mathscr{U}^c, \Re \cap \Re^c) \leq I(\mathscr{U} \cup \mathscr{U}^c, \mathscr{U} \cap \mathscr{U}^c)$, that is, $\aleph(A) \leq \aleph(\mathscr{U})$.

Example 1 For $\Re, \mathscr{U} \in T$-SFS*s*.

$$\begin{aligned}I(\Re, \mathscr{U}) &= I_1(\Re, \mathscr{U})\\ &= 1 - \frac{1}{2|\mathscr{H}|}\sum_{v_i \in \mathscr{H}}\begin{pmatrix}|\wp^t_{\Re}(v_i) - \wp^t_{\Re}(v_i) \wedge \wp^t_{\mathscr{U}}(v_i)| + |I^t_{\Re}(v_i)\\ -I^t_{\Re}(v_i) \wedge I^t_{\mathscr{U}}(v_i)| + |\mathscr{G}^t_{\Re}(v_i) - \mathscr{G}^t_{\Re}(v_i)|\end{pmatrix},\end{aligned}$$

then we have $\aleph(\Re) = I_1(\Re \cup \Re^c, \Re \cap \Re^c) = 1 - \frac{1}{2|\mathscr{H}|}\sum_{v_i \in \mathscr{H}}\left(|\wp^t_{\Re}(v_i) - I^t_{\Re}(v_i) - \mathscr{G}^t_{\Re}(v_i)|\right) = \aleph_4(\Re)$.

Theorem 12 $S(\wp_{\Re}(v_i), I_{\Re}(v_i), \mathscr{G}_{\Re}(v_i))$ *is the entropy of T-SFS* $\Re$.

Proof ($\aleph_1$) Obvious.
($\aleph_2$) is the set of crisp $\Re$, then $\wp_{\Re}(v_i) = \Phi$, $I_{\Re}(v_i) = \varnothing$ and $\mathscr{G}_{\Re}(v_i) = \varnothing$ or $\wp_{\Re}(v_i) = \varnothing$, $I_{\Re}(v_i) = \Phi$ and $\mathscr{G}_{\Re}(v_i) = \Phi$. Therefore, $S(\wp_{\Re}(v_i), I_{\Re}(v_i), \mathscr{G}_{\Re}(v_i) = 0$.
(E_3) measured by the notion of similarity, T-SFSs, we have

$$S(\wp_{\Re}(v_i), I_{\Re}(v_i), \mathscr{G}_{\Re}(v_i)) = 1 \Leftrightarrow \wp_{\Re}(v_i) = I_{\Re}(v_i) = \mathscr{G}_{\Re}(v_i) \Leftrightarrow \aleph(\Re) = 1.$$

($\aleph_4$) $\aleph(\Re) = S(\wp_{\Re}(v_i), I_{\Re}(v_i), \mathscr{G}_{\Re}(v_i)) = S(\mathscr{G}_{\Re}(v_i), I_{\Re}(v_i), \wp_{\Re}(v_i)) = \aleph(\mathscr{R}^c)$.
($\aleph_5$) Since $\wp_{\Re}(v_i) \le \wp_{\mathscr{U}}(v_i) \le I_{\mathscr{U}}(v_i) \le I_{\Re}(v_i) \le \mathscr{G}_{\mathscr{U}}(v_i) \le \mathscr{G}_{\Re}(v_i)$ implies $\Re \subseteq \mathscr{U} \subseteq \mathscr{U}^c \subseteq \Re^c$. Namely, $\wp_{\Re} \subseteq \wp_{\mathscr{U}} \subseteq I_{\mathscr{U}} \subseteq I_{\Re} \subseteq \mathscr{G}_{\mathscr{U}} \subseteq \mathscr{G}_{\Re}$. In accordance with the similarity measure's definition, we can have $S(\wp_{\Re}(v_i), I_{\Re}(v_i), \mathscr{G}_{\Re}(v_i)) \le S(\wp_{\mathscr{U}}(v_i), I_{\mathscr{U}}(v_i), \mathscr{G}_{\mathscr{U}}(v_i))$, that is, $\aleph(\Re) \le \aleph(\mathscr{U})$.

Likewise, if $\mathscr{G}_{\Re}(v_i) \le \mathscr{G}_{\mathscr{U}}(v_i) \le I_{\Re}(v_i) \le I_{\mathscr{U}}(v_i) \le \wp_{\mathscr{U}}(v_i) \le \wp_{\Re}(v_i)$, then $\aleph(\Re) \le \aleph(\mathscr{U})$.The proof is now complete.

4 Numerical Examples of Information Measures

This section introduces axiomatic descriptions for T-SFS information measures. Flow chart of information measures is in Fig. 20.1.

4.1 Application of Distance Measures to Pattern Recognition

Herein, we provide numerous examples to illustrate the use of the established distance measures for specific T-SFSs in pattern recognition.

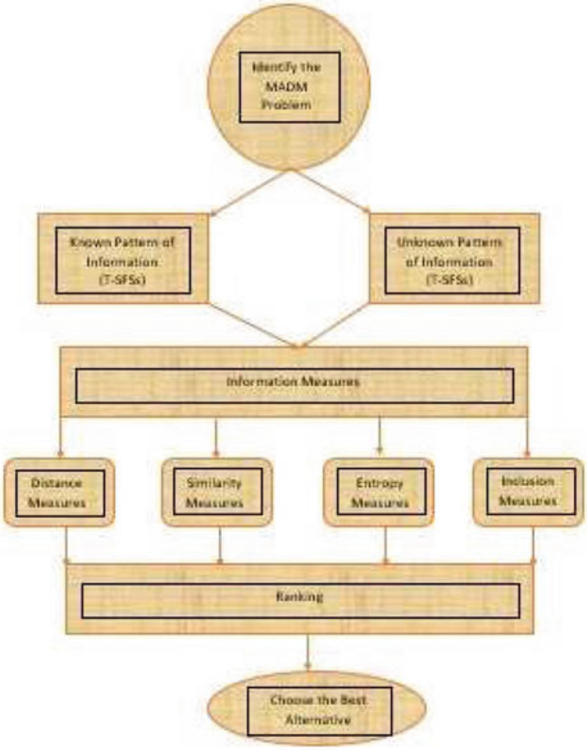
Example 2 Let $\mathscr{W}_1, \mathscr{W}_2, \mathscr{W}_3$, and $\mathscr{W}_4$ be four known patterns which are illustrated by the following T-SFSs in X:

$$\mathscr{W}_1 = \{(v_1, 0.32, 0.14, 0.36), (v_2, 0.41, 0.25, 0.47), (v_3, 0.54, 0.36, 0.48)\},$$
$$\mathscr{W}_2 = \{(v_1, 0.41, 0.34, 0.43), (v_2, 0.52, 0.35, 0.51), (v_3, 0.60, 0.63, 0.32)\},$$
$$\mathscr{W}_3 = \{(v_1, 0.34, 0.47, 0.57), (v_2, 0.56, 0.52, 0.61), (v_3, 0.86, 0.61, 0.81)\},$$
$$\mathscr{W}_4 = \{(v_1, 0.23, 0.21, 0.11), (v_2, 0.31, 0.32, 0.30), (v_3, 0.45, 0.33, 0.35)\}.$$

The following is an unknown pattern $\mathscr{K}$:

$$\mathscr{K} = \{(v_1, 0.25, 0.26, 0.22), (v_2, 0.46, 0.38, 0.34), (v_3, 0.47, 0.49, 0.54)\},$$

Fig. 20.1 Flow chart of information measures



Its aim is to determine the class to which $\mathscr{K}$ belongs. In order to do that, the distance between $\mathscr{K}$ and classes $\mathscr{W}_1$, $\mathscr{W}_2$, $\mathscr{W}_3$, and $\mathscr{W}_4$ is measured, and $\mathscr{K}$ is then allocated to the class $\mathscr{W}_g$ specified by

$$g = arg \max_g \left\{ D\left(\mathscr{W}_g, \mathscr{K}\right) \right\}.$$

For all the newly developed distance measures $(D_1 - D_{13})$ for T-SFS, the distance between $D(\mathscr{W}_1, \mathscr{K})$, $D(\mathscr{W}_2, \mathscr{K})$, $D(\mathscr{W}_3, \mathscr{K})$, and $D(\mathscr{W}_4, \mathscr{K})$ are determined and displayed in Table 20.1. It is observed in Table 20.1 that the pattern which is unknown $\mathscr{K}$ relates to a class $\mathscr{W}_3$ when D_1 to D_{13} are used. It is clear that the cause for this distinction is the first characteristic, i.e., (v_1). The T-SFNs of v_1 are

$$(0.32, 0.14, 0.36), \ (0.41, 0.34, 0.43), \ (0.34, 0.47, 0.57),$$
$$(0.23, 0.21, 0.11), \ (0.25, 0.26, 0.22),$$

for $\mathscr{W}_1$, $\mathscr{W}_2$, $\mathscr{W}_3$, $\mathscr{W}_4$, and $\mathscr{K}$, respectively. As a conclusion, it appears that

$$D(\mathscr{W}_3, \mathscr{K}) > D(\mathscr{W}_2, \mathscr{K}) > D(\mathscr{W}_4, \mathscr{K}) > D(\mathscr{W}_4, \mathscr{K})$$

Table 20.1 Distance measures for Example 2

	$D(\mathscr{W}_1, \mathscr{K})$	$D(\mathscr{W}_2, \mathscr{K})$	$D(\mathscr{W}_3, \mathscr{K})$	$D(\mathscr{W}_4, \mathscr{K})$	Classification results
D_1	0.0956	0.1985	**0.5523**	0.1579	$\mathscr{W}_3$
D_2	0.0382	0.0584	**0.0764**	0.0385	$\mathscr{W}_3$
D_3	0.0669	0.1284	**0.3143**	0.0982	$\mathscr{W}_3$
D_4	0.0572	0.0956	**0.2982**	0.0638	$\mathscr{W}_3$
D_5	0.1051	0.1694	**0.5446**	0.1143	$\mathscr{W}_3$
D_6	0.1082	0.1746	**0.4593**	0.1200	$\mathscr{W}_3$
D_7	0.4656	0.5023	**0.7133**	0.5014	$\mathscr{W}_3$
D_8	0.8219	0.8341	**0.9044**	0.8338	$\mathscr{W}_3$
D_9	0.8125	0.8496	**0.9094**	0.8490	$\mathscr{W}_3$
D_{10}	0.4731	0.5636	**0.7316**	0.5405	$\mathscr{W}_3$
D_{11}	0.6871	0.6973	**0.7564**	0.6850	$\mathscr{W}_3$
D_{12}	0.0613	0.0915	**0.2595**	0.0547	$\mathscr{W}_3$
D_{13}	0.0428	0.0588	**0.0900**	0.0242	$\mathscr{W}_3$

The bold values show the best similarity values according to the given problem

Table 20.2 Symptomatic characteristics of the diagnosis under consideration

	Temperature	Headache	Stomach pain	Cough
Viral fever	(0.2, 0.3, 0.6)	(0.3, 0.4, 0.7)	(0.2, 0.5, 0.6)	(0.1, 0.4, 0.8)
Malaria	(0.3, 0.5, 0.7)	(0.1, 0.3, 0.4)	(0.2, 0.3, 0.4)	(0.1, 0.2, 0.4)
Typhoid	(0.2, 0.3, 0.5)	(0.1, 0.4, 0.5)	(0.1, 0.3, 0.3)	(0.2, 0.2, 0.3)
Chest problem	(0.1, 0.3, 0.4)	(0.2, 0.4, 0.5)	(0.2, 0.3, 0.4)	(0.1, 0.2, 0.3)

Table 20.3 Symptoms and features of the patients under consideration

	Temperature	Headache	Stomach pain	Cough
Al	(0.2, 0.3, 0.4)	(0.1, 0.3, 0.3)	(0.1, 0.3, 0.4)	(0.2, 0.2, 0.3)
Bob	(0.1, 0.4, 0.5)	(0.1, 0.1, 0.4)	(0.2, 0.3, 0.3)	(0.1, 0.1, 0.3)
Joe	(0.1, 0.3, 0.3)	(0.1, 0.4, 0.5)	(0.2, 0.2, 0.3)	(0.2, 0.2, 0.3)
Ted	(0.1, 0.4, 0.4)	(0.2, 0.3, 0.4)	(0.1, 0.3, 0.5)	(0.1, 0.2, 0.3)

is more acceptable. Using standard calculations, we can obtain the noted relation for D_2 to D_{13}.

Example 3 Assume that a doctor would like to diagnose the condition of C = {viral fever, malaria, typhoid, chest problem} for patients P = {Al, Bob, Joe, Ted} with disease symptoms V = {temperature, headache, stomach pain, cough}. The early signs associated with the considered diagnosis are listed in Table 20.2, and the early signs of the disease associated with each patient are listed in Table 20.3. Every table element is represented by a specific T-SFSs. For each patient, a precise diagnosis is necessary.

To determine a condition of the patient, we may assess the distance measure between the symptoms associated with each illness and those associated with the patient. The diagnostic findings are provided in Table 20.4 using the suggested

Table 20.4 The distance between the patient and the set of probable diagnoses by using D_{13}

	Viral fever	Malaria	Typhoid	Cough
Al	**0.1719**	0.0605	0.0324	0.0251
Bob	**0.1654**	0.0527	0.0259	0.0255
Joe	**0.1698**	0.0655	0.0262	0.0230
Ted	**0.1622**	0.0564	0.0352	0.0242

The bold values show the best similarity values according to the given problem

distance measure formula D_{13} and taking $p = 1$, $t_1 = 2$, $t_2 = 2$, $q = 3$. We may conclude from Table 20.4 that all the patients suffer from viral fever.

4.2 Application of the Similarity Measure to Recognition of Pattern

This part, we describe some examples to showing the use of the suggested measure similarity for T-SFS to recognition of pattern.

Let suppose that four classes $\mathscr{W}_1, \mathscr{W}_2, \mathscr{W}_3,$, and $\mathscr{W}_4$ of known construction things and $\mathscr{K}$, an unknown construction things, are defined in the $X = \{v_1, v_2, v_3\}$ showing by T-SFS as given below. Its goal is to ascertain to which class $\mathscr{K}$ belongs to.

$\mathscr{W}_1 = \{(v_1, 0.21, 0.22, 0.33), (v_2, 0.22, 0.42, 0.43), (v_3, 0.32, 0.33, 0.32)\}$,
$\mathscr{W}_2 = \{(v_1, 0.22, 0.32, 0.41), (v_2, 0.32, 0.43, 0.31), (v_3, 0.13, 0.55, 0.34)\}$,
$\mathscr{W}_3 = \{(v_1, 0.11, 0.23, 0.11), (v_2, 0.12, 0.24, 0.21), (v_3, 0.32, 0.21, 0.32)\}$,
$\mathscr{W}_4 = \{(v_1, 0.33, 0.34, 0.11), (v_2, 0.21, 0.12, 0.22), (v_3, 0.23, 0.37, 0.32)\}$,
The following is an known building materials $\mathscr{K}$:
$\mathscr{K} = \{(v_1, 0.25, 0.26, 0.22), (v_2, 0.42, 0.31, 0.34), (v_3, 0.33, 0.23, 0.24)\}$,
Its objective is to determine the class to which $\mathscr{K}$ belongs. To do this, the degrees of similarity between $\mathscr{K}$ and classes $\mathscr{W}_1$, $\mathscr{W}_2$, $\mathscr{W}_3$, and $\mathscr{W}_4$ are measured and $\mathscr{K}$ is then allocated to the class $\mathscr{W}_g$ specified by

$$g = arg \max_g \left\{S\left(\mathscr{W}_g, \mathscr{K}\right)\right\}.$$

For all the established similarity measure $(S_1 - S_{13})$ for T-SFS, the degree of similarity between four classes of known building materials $S(\mathscr{W}_1, \mathscr{K})$, $S(\mathscr{W}_2, \mathscr{K})$, $S(\mathscr{W}_3, \mathscr{K})$, and $S(\mathscr{W}_4, \mathscr{K})$ are determined and displayed in Table 20.5. It is clearly observed in Table 20.5 that the an unknown building material $\mathscr{K}$ belongs to a class $\mathscr{W}_1$ when S_1, S_3, S_7, to S_{10} are used and $\mathscr{K}$ belongs to a class $\mathscr{W}_3$ when S_2, S_4 to S_6 and S_{11} to S_{13} are used. It is obvious that the first characteristic is what has caused this discrepancy, i.e., (v_1). As a conclusion, it appears that $S(\mathscr{W}_1, \mathscr{K}) > S(\mathscr{W}_3, \mathscr{K}) > S(\mathscr{W}_4, \mathscr{K}) > S(\mathscr{W}_2, \mathscr{K})$ is more acceptable. Similarly, we can find the aforementioned relations for S_2 to S_{13}.The graphical representation of similarity measures is given in Fig. 20.2.

Table 20.5 Similarity measure for Example 4.2

	$S(\mathscr{W}_1, \mathscr{K})$	$S(\mathscr{W}_2, \mathscr{K})$	$S(\mathscr{W}_3, \mathscr{K})$	$S(\mathscr{W}_4, \mathscr{K})$	Classification results
S_1	**0.9317**	0.8591	0.8824	0.8681	$\mathscr{W}_1$
S_2	0.9636	0.9378	**0.9923**	0.9838	$\mathscr{W}_3$
S_3	**0.9477**	0.8984	0.9374	0.9260	$\mathscr{W}_1$
S_4	0.9719	0.9345	**0.9736**	0.9687	$\mathscr{W}_3$
S_5	0.1635	0.1592	**0.1637**	0.1631	$\mathscr{W}_3$
S_6	0.4904	0.4772	**0.4910**	0.4894	$\mathscr{W}_3$
S_7	**0.4194**	0.3192	0.3792	0.2682	$\mathscr{W}_1$
S_8	**0.1398**	0.1064	0.1264	0.0894	$\mathscr{W}_1$
S_9	**0.1303**	0.0890	0.1004	0.0840	$\mathscr{W}_1$
S_{10}	**0.4211**	0.2889	0.3565	0.2707	$\mathscr{W}_1$
S_{11}	0.3204	0.3115	**0.3238**	0.3194	$\mathscr{W}_3$
S_{12}	0.9613	0.9347	**0.9716**	0.9583	$\mathscr{W}_3$
S_{13}	0.9778	0.9650	**0.9946**	0.9894	$\mathscr{W}_3$

The bold values show the best similarity values according to the given problem

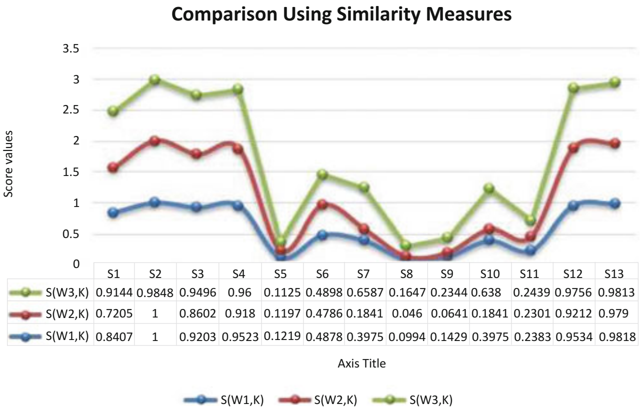


Fig. 20.2 Comparative study

5 Comparative Analysis

To demonstrate how well the proposed similarity measurements work for specific T-SFSs in pattern recognition, we present some examples and compare the novel findings to those reported in the literature.

Example 4 Comparison analysis of similarity measure for three patterns $\mathscr{W}_1, \mathscr{W}_2,$, and $\mathscr{W}_3$ which are presented by the following T-SFSs in $X = \{v_1, v_2, v_3, v_4\}$:
$\mathscr{W}_1 = \{(v_1, 0.3, 0.0, 0.3), (v_2, 0.4, 0.0, 0.4), (v_3, 0.4, 0.0, 0.4), (v_4, 0.4, 0.0, 0.4)\}$,
$\mathscr{W}_2 = \{(v_1, 0.5, 0.0, 0.5), (v_2, 0.1, 0.0, 0.1), (v_3, 0.5, 0.0, 0.5), (v_4, 0.1, 0.0, 0.1)\}$,
$\mathscr{W}_3 = \{(v_1, 0.5, 0.0, 0.4), (v_2, 0.4, 0.0, 0.5), (v_3, 0.3, 0.0, 0.3), (v_4, 0.2, 0.0, 0.2)\}$.

Table 20.6 Comparison for similarity measures for Example 4

	$S(\mathscr{W}_1, \mathscr{K})$	$S(\mathscr{W}_2, \mathscr{K})$	$S(\mathscr{W}_3, \mathscr{K})$	Classification results
S [27]	0.8677	0.7261	**0.9134**	$\mathscr{W}_3$
S [10]	1.0000	1.0000	0.9750	Not classified
S [11]	0.8679	0.7425	**0.8923**	$\mathscr{W}_3$
S [21]	0.8750	0.7500	**0.9000**	$\mathscr{W}_3$
S [21]	0.8141	0.6501	**0.8495**	$\mathscr{W}_3$
S [21]	0.7778	0.6000	**0.8182**	$\mathscr{W}_3$
S [23]	0.8750	0.7500	**0.9250**	$\mathscr{W}_3$
S [17]	1.0000	1.0000	0.9750	Not classified
S [29]	0.9375	0.8750	**0.9500**	$\mathscr{W}_3$
S [30]	0.8750	0.7500	**0.9250**	$\mathscr{W}_3$
S [30]	0.9375	0.8750	**0.9500**	$\mathscr{W}_3$
S [30]	0.9167	0.8333	**0.9417**	$\mathscr{W}_3$
S [33]	0.8750	0.7500	**0.9250**	$\mathscr{W}_3$
S [61]	1.0000	1.0000	0.9969	Not classified
S [52]	1.0000	1.0000	0.9884	Not classified
S [66]	0.5000	0.5000	0.5000	Not classified
S [37]	1.0000	1.0000	0.9775	Not classified
S [37]	0.5038	0.2378	**0.6223**	$\mathscr{W}_3$
S [37]	0.8785	0.7912	**0.9205**	$\mathscr{W}_3$
S [8]	0.9583	0.9167	0.9583	Not classified
S [38]	**0.9841**	0.9727	0.9816	$\mathscr{W}_1$

The bold values show the best similarity values according to the given problem

The following is an unknown pattern $\mathscr{K}$:
$\mathscr{K} = \{(v_1, 0.4, 0.0, 0.4), (v_2, 0.5, 0.0, 0.5), (v_3, 0.2, 0.0, 0.2), (v_4, 0.3, 0.0, 0.3)\}$, Our objective is to ascertain the class to which $\mathscr{K}$ belongs. The results of the suggested similarity measures (S_1-S_{13}), displayed in Table 20.6, are in contrast to the classification result of the existing measure similarity (S[27]–S[38]). From Table 20.7, we observed that the developed similarity measures ($S_1, S_3 - S_{13}$) address the shortcomings of conventional similarity measures S [17], S [61], S [52], S [66], S [37], and S [8].

The graphical representation of comparison using similarity measures is given in the Fig. 20.3:

Comparison analysis of similarity measure for three known patterns $\mathscr{W}_1$, $\mathscr{W}_2$, and $\mathscr{W}_3$ that are presented by the following T-SFSs in $X = \{v_1, v_2, v_3, v_4\}$
$\mathscr{W}_1 = \{(v_1, 0.3, 0.0, 0.3), (v_2, 0.6, 0.0, 0.1), (v_3, 0.2, 0.0, 0.6), (v_4, 0.7, 0.0, 0.3)\}$,
$\mathscr{W}_2 = \{(v_1, 0.5, 0.0, 0.3), (v_2, 0.8, 0.0, 0.1), (v_3, 0.2, 0.0, 0.6), (v_4, 0.7, 0.0, 0.3)\}$,
$\mathscr{W}_3 = \{(v_1, 0.5, 0.0, 0.3), (v_2, 0.6, 0.0, 0.1), (v_3, 0.2, 0.0, 0.6), (v_4, 0.7, 0.0, 0.3)\}$.
The following is an unknown pattern $\mathscr{K}$:
$\mathscr{K} = \{(v_1, 0.4, 0.0, 0.3), (v_2, 0.7, 0.0, 0.1), (v_3, 0.3, 0.0, 0.6), (v_4, 0.7, 0.0, 0.3)\}$,

Table 20.7 Comparison for the proposed similarity measure for Example 4

	$S(\mathscr{W}_1, \mathscr{K})$	$S(\mathscr{W}_2, \mathscr{K})$	$S(\mathscr{W}_3, \mathscr{K})$	classification result
S_1	0.8407	0.7205	**0.9144**	$\mathscr{W}_3$
S_2	**1.0000**	**1.0000**	0.9848	Not classified
S_3	0.9203	0.8602	**0.9496**	$\mathscr{W}_3$
S_4	0.9523	0.9180	**0.9600**	$\mathscr{W}_3$
S_5	0.1219	0.1197	**0.1225**	$\mathscr{W}_3$
S_6	0.4878	0.4786	**0.4898**	$\mathscr{W}_3$
S_7	0.3975	0.1841	**0.6587**	$\mathscr{W}_3$
S_8	0.0994	0.0460	**0.1647**	$\mathscr{W}_3$
S_9	0.1429	0.0641	**0.2344**	$\mathscr{W}_3$
S_{10}	0.3975	0.1841	**0.6380**	$\mathscr{W}_3$
S_{11}	0.2383	0.2301	**0.2439**	$\mathscr{W}_3$
S_{12}	0.9534	0.9212	**0.9756**	$\mathscr{W}_3$
S_{13}	**0.9818**	0.9790	0.9813	$\mathscr{W}_1$

The bold values show the best similarity values according to the given problem

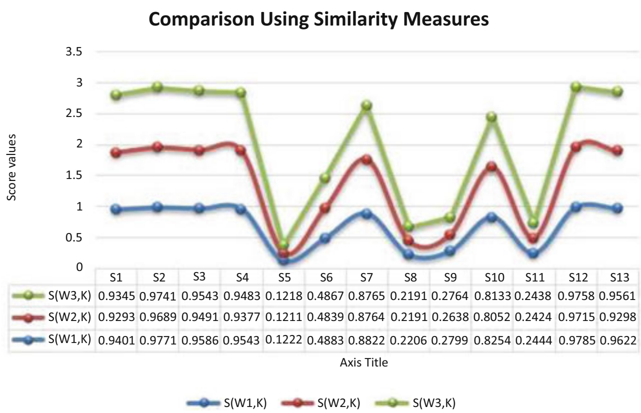


Fig. 20.3 Comparative study

Our objective is to ascertain the class to which $\mathscr{K}$ belongs. Table 20.9 shows a comparison of the classification result of the established similarity measures ($S_1 - S_{13}$) with existing similarity measures (S[27]-S[38]) shown in Table 20.8. Hence, the unknown pattern for the proposed similarity measures ($S_1 - S_{13}$) is classified in the pattern $\mathscr{W}_1$. From Table 20.2, it could be clearly observed that the novel similarity measures ($S_1 - S_{13}$) can address the drawbacks of conventional similarity measures S [17], S [10], S [11], S [21], S [23], S [17] S [29], S [30], S [33], S [61], S [52], and S [8]. The graphical representation of comparison using similarity measures is given in Fig. 20.4 (Table 20.9).

Table 20.8 Comparison for similarity measures for Example 5

	$S(\mathscr{W}_1, \mathscr{K})$	$S(\mathscr{W}_2, \mathscr{K})$	$S(\mathscr{W}_3, \mathscr{K})$	Classification results
S [27]	0.9388	0.9388	0.9388	Not classified
S [10]	0.9625	0.9625	0.9625	Not classified
S [11]	0.8880	0.8902	0.8902	Not classified
S [21]	0.9250	0.9250	0.9250	Not classified
S [21]	0.8857	0.8857	0.8857	Not classified
S [21]	0.8605	0.8605	0.8605	Not classified
S [23]	0.9625	0.9625	0.9625	Not classified
S [17]	0.9625	0.9625	0.9625	Not classified
S [29]	0.9625	0.9625	0.9625	Not classified
S [30]	0.9625	0.9625	0.9625	Not classified
S [30]	0.9625	0.9625	0.9625	Not classified
S [30]	0.9625	0.9625	0.9625	Not classified
S [33]	0.9625	0.9625	0.9625	Not classified
S [61]	0.9949	0.9961	0.9961	Not classified
S [52]	0.9885	0.9943	0.9943	Not classified
S [66]	0.7879	0.8281	0.8229	$\mathscr{W}_2$
S [37]	0.9688	0.9638	0.9663	$\mathscr{W}_1$
S [37]	0.8372	0.8484	0.8410	$\mathscr{W}_2$
S [37]	0.9446	0.9405	0.9415	$\mathscr{W}_1$
S [8]	0.9625	0.9625	0.9625	Not classified
S [38]	0.9771	0.9689	0.9741	$\mathscr{W}_1$

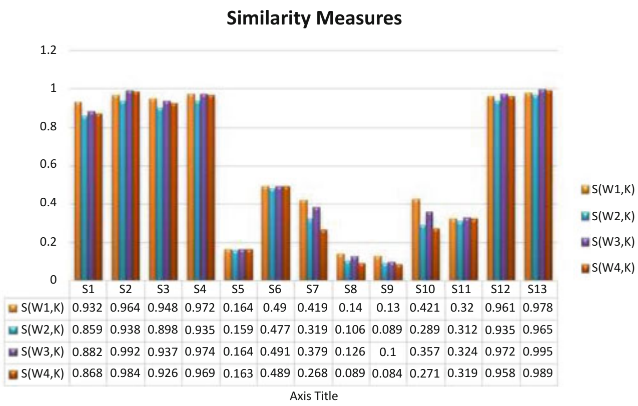


Fig. 20.4 Comparative study

Example 5 Suppose a doctor would likely to diagnose the condition of C = {viral fever,malaria, typhoid} for a patients set P = {Al, Bob, Joe Ted} having symptoms V = {temperature, headache, and cough}. The symptoms associated with the considered diagnosis are listed in Table 20.10, and the symptoms associated

Table 20.9 Comparison for the proposed similarity measures for Example 5

	$S(\mathscr{W}_1, \mathscr{K})$	$S(\mathscr{W}_2, \mathscr{K})$	$S(\mathscr{W}_3, \mathscr{K})$	Classification result
S_1	**0.9401**	0.9293	0.9345	$\mathscr{W}_1$
S_2	**0.9771**	0.9689	0.9741	$\mathscr{W}_1$
S_3	**0.9586**	0.9491	0.9543	$\mathscr{W}_1$
S_4	**0.9543**	0.9377	0.9483	$\mathscr{W}_1$
S_5	**0.1222**	0.1211	0.1218	$\mathscr{W}_1$
S_6	**0.4883**	0.4839	0.4867	$\mathscr{W}_1$
S_7	**0.8822**	0.8764	0.8765	$\mathscr{W}_1$
S_8	**0.2206**	0.2191	0.2191	$\mathscr{W}_1$
S_9	**0.2799**	0.2638	0.2764	$\mathscr{W}_1$
S_{10}	**0.8254**	0.8052	0.8133	$\mathscr{W}_1$
S_{11}	**0.2444**	0.2424	0.2438	$\mathscr{W}_1$
S_{12}	**0.9785**	0.9715	0.9758	$\mathscr{W}_1$
S_{13}	**0.9622**	0.9298	0.9561	$\mathscr{W}_1$

The bold values show the best similarity values according to the given problem

Table 20.10 Symptom features for the diagnosis under consideration

	Temperature	Headache	Stomach pain	Cough	Chest pain
Viral fever	(0.4, 0.0, 0.0)	(0.3, 0.0, 0.5)	(0.1, 0.0, 0.7)	(0.4, 0.0, 0.3)	0.1, 0.0, 0.7
Malaria	(0.7, 0.0, 0.0)	(0.2, 0.0, 0.6)	(0.0, 0.0, 0.9)	(0.7, 0.0, 0.0)	0.1, 0.0, 0.8
Typhoid	(0.3, 0.0, 0.3)	(0.6, 0.0, 0.1)	(0.2, 0.0, 0.7)	(0.2, 0.0, 0.6)	0.1, 0.0, 0.9
Stomach problem	0.1, 0.0, 0.7	0.2, 0.0, 0.4	0.8, 0.0, 0.0	0.2, , 0.0, 0.7	0.2, 0.0, 0.7
Chest problem	(0.1, 0.0, 0.8)	(0.0, 0.0, 0.8)	(0.2, 0.0, 0.8)	(0.2, 0.0, 0.8)	0.8, 0.0, 0.1

Table 20.11 Symptoms of the patients sunder examination

	Temperature	Headache	Stomach pain	Cough	Chest pain
Al	(0.8, 0.0, 0.1)	(0.6, 0.0, 0.1)	(0.2, 0.0, 0.8)	(0.6, 0.0, 0.1)	0.1, 0.0, 0.6
Bob	(0.0, 0.0, 0.8)	(0.4, 0.0, 0.4)	(0.6, 0.0, 0.1)	(0.1, 0.0, 0.7)	0.1, 0.0, 0.8
Joe	(0.8, 0.0, 0.1)	(0.8, 0.0, 0.1)	(0.0, 0.0, 0.6)	(0.2, 0.0, 0.7)	0.0, 0.0, 0.5
Ted	(0.6, 0.0, 0.1)	(0.5, 0.0, 0.4)	(0.3, 0.0, 0.4)	(0.7, 0.0, 0.2)	0.3, 0.0, 0.4

with every patient are listed in Table 20.11. Each table element is represented by a specific T-SFSs. Each patient requires proper diagnosis, which needs to be assessed. Diagnosis will identify a for each patient based on the similar between the symptoms associated with each diagnosis and those associated with the patient. The diagnostic observations are described in Tables 20.12, 20.13, 20.14, and 20.15 for Al, Bob, Joe, and Ted, respectively, using the novel similarity measures formulae ($S_1 - S_{13}$). The patient Al is diagnosed with malaria (Mal.) in 10 of the 13 of the proposed approaches; the remaining approach indicates that Al is diagnosed with viral fever (VF) as presented in Table 20.12. It is obvious that Bob has a stomach problem (SP), since all of the measures yield the same findings, shown in Table 20.13. Joe is diagnosed with typhoid in 12 of the 13 methods; the other

Table 20.12 Symptoms features of the patient Al

	VF	Mal.	TYP	SP	Chest problem
S_1	0.7690	**0.8039**	0.7248	0.6423	0.5910
S_2	**0.8773**	0.8760	0.8384	0.7190	0.6806
S_3	0.8231	**0.8399**	0.7816	0.6807	0.6358
S_4	0.7830	**0.7952**	0.7236	0.6600	0.5912
S_5	0.0883	**0.0887**	0.0854	0.0806	0.0763
S_6	0.4391	**0.4430**	0.4198	0.3976	0.3715
S_7	0.3606	**0.5128**	0.3256	0.0771	0.1201
S_8	0.0860	**0.1150**	0.0746	0.0188	0.0297
S_9	0.0754	**0.1081**	0.0613	0.0082	0.0193
S_{10}	0.3661	**0.5108**	0.3372	0.0798	0.1432
S_{11}	0.1752	**0.1771**	0.1690	0.1473	0.1384
S_{12}	**0.8804**	0.8800	0.8445	0.7489	0.7083
S_{13}	**0.8917**	0.8349	0.8318	0.7275	0.6581

The bold values show the best similarity values according to the given problem

Table 20.13 Symptoms features of the patient Bob

	VF	Mal.	TYP	SP	Chest problem
S_1	0.7054	0.6636	0.7365	**0.8571**	0.6725
S_2	0.8221	0.7308	0.8372	**0.9294**	0.7584
S_3	0.7637	0.6972	0.7869	**0.8933**	0.7154
S_4	0.7200	0.6530	0.7354	**0.8606**	0.6720
S_5	0.0847	0.0819	0.0854	**0.0928**	0.0822
S_6	0.4186	0.3950	0.4238	**0.4625**	0.4019
S_7	0.1552	0.1300	0.2692	**0.5722**	0.1861
S_8	0.0370	0.0322	0.0633	**0.1335**	0.0462
S_9	0.0372	0.0234	0.0730	**0.1272**	0.0399
S_{10}	0.2072	0.1790	0.3379	**0.6514**	0.2770
S_{11}	0.1634	0.1447	0.1659	**0.1845**	0.1458
S_{12}	0.8216	0.7409	0.8324	**0.9257**	0.7540
S_{13}	0.8446	0.7148	0.8285	**0.9136**	0.7170

The bold values show the best similarity values according to the given problem

approach represented that Joe is diagnosed with VF shown in Table 20.14. Similarly, 9 of the 13 measures indicated that Ted has VF, whereas the remaining methods imply that Ted has Mal. presented in Table 20.15. For patient Al, it could be observed from Tables 20.12 and 20.16 that the established similarity measures ($S_1, S_3 - S_{11}$) yield the same findings as those in References [11, 18, 21, 30, 33, 35, 37, 54, 66], and the measures (S_2, S_{12}, S_{13}) provide the same results as in References [8, 10, 17, 23, 27, 30, 35, 37, 38, 46, 47, 50, 52, 61]. For patient Bob the novel similarity measures provide the same results as in literature presented in Table 20.16. Similarly, for patient Joe the measures of similarity provide the same result as in literature shown in Table 20.16 except S_4. For patient Ted, the suggested

Table 20.14 Symptoms features of the patient Joe

	VF	Mal.	TYP	SP	Chest problem
S_1	0.6549	0.6204	**0.6592**	0.6348	0.5922
S_2	0.8229	0.7536	**0.8344**	0.7642	0.6862
S_3	0.7389	0.6870	**0.7468**	0.6995	0.6392
S_4	**0.6812**	0.6168	0.6722	0.6510	0.6000
S_5	0.0818	0.0770	**0.0819**	0.0807	0.0760
S_6	0.4052	0.3815	**0.4020**	0.3943	0.3750
S_7	0.2052	0.2262	**0.3143**	0.1850	0.1322
S_8	0.0490	0.0500	**0.0725**	0.0460	0.0329
S_9	0.0340	0.0398	**0.0558**	0.0189	0.0196
S_{10}	0.2085	0.2212	**0.3261**	0.1702	0.1533
S_{11}	0.1655	0.1559	**0.1678**	0.1539	0.1393
S_{12}	0.8343	0.7762	**0.8403**	0.7858	0.7130
S_{13}	0.8355	0.7431	**0.8397**	0.7608	0.6629

The bold values show the best similarity values according to the given problem

Table 20.15 Symptoms features of the patient Ted

	VF	Mal.	TYP	SP	Chest problem
S_1	**0.7508**	0.7230	0.6767	0.6639	0.5265
S_2	**0.8782**	0.8429	0.8099	0.7809	0.6847
S_3	**0.8145**	0.7830	0.7433	0.7224	0.6056
S_4	**0.7826**	0.7200	0.6882	0.6884	0.5208
S_5	**0.0880**	0.0860	0.0831	0.0821	0.0685
S_6	**0.4390**	0.4186	0.4077	0.4077	0.3425
S_7	0.2256	**0.3938**	0.1520	0.0805	0.0507
S_8	0.0511	**0.0820**	0.0329	0.0191	0.0118
S_9	0.0428	**0.0496**	0.0219	0.0105	0.0097
S_{10}	0.2264	**0.3260**	0.1390	0.0794	0.0546
S_{11}	**0.1771**	0.1724	0.1649	0.1596	0.1431
S_{12}	**0.8842**	0.8528	0.8222	0.8024	0.7156
S_{13}	**0.8790**	0.7989	0.7969	0.7739	0.6617

The bold values show the best similarity values according to the given problem

similarity measures ($S_1 - S_6$, $S_{11} - S_{13}$) yield the same findings as in References [8, 10, 17, 21, 23, 27, 29, 30, 33, 35, 37, 38, 47, 50, 52, 54, 61], and the measures S7-S10 provide the results as in References [11, 18, 37, 46, 66].

5.1 Application of the Inclusion Measures to Pattern Recognition

This section illustrates the applicability of the suggested T-SFS inclusion measures to pattern recognition.

Table 20.16 The summary of existing similarity measures in medical diagnosis

	Al	Bob	Joe	Ted
S [27]	VF	SP	TYP	VF
S [10]	VF	SP	TYP	VF
S [11]	Mal	SP	TYP	Mal
S [21]	Mal	SP	TYP	VF
S [21]	Mal	SP	TYP	VF
S [21]	Mal	SP	TYP	VF
S [23]	VF	SP	TYP	VF
S [17]	VF	SP	TYP	VF
S [29]	VF/Mal	SP	TYP	VF
S [30]	Mal	SP	TYP	VF
S [30]	VF	SP	TYP	VF
S [30]	Mal	SP	TYP	VF
S [33]	Mal	SP	TYP	VF
S [61]	VF	SP	TYP	VF
S [52]	VF	SP	TYP	VF
S [66]	Mal	SP	TYP	Mal
S [37]	Mal	SP	TYP	VF
S [37]	Mal	SP	TYP	Mal
S [37]	VF	SP	TYP	VF
S [8]	VF	SP	TYP	VF
S [38]	VF	SP	TYP	VF
S [54]	Mal	SP	TYP	VF
S [46]	VF	SP	TYP	Mal
S [47]	VF	SP	TYP	VF
S [35]	VF	SP	TYP	VF
S [35]	VF	SP	TYP	VF
S [35]	Mal	SP	SP	VF
S [18]	Mal	SP	TYP	Mal
S [50]	VF	SP	TYP	VF

Example 6 Let, $\mathscr{W}_1, \mathscr{W}_2, \mathscr{W}_3$, and $\mathscr{W}_4$ are the known patterns illustrated by T-SFSs in $X = \{v_1, v_2, v_3\}$ described as follows:
$\mathscr{W}_1 = \{(v_1, 0.21, 0.11, 0.19), (v_2, 0.22, 0.12, 0.23), (v_3, 0.32, 0.14, 0.27)\}$,
$\mathscr{W}_2 = \{(v_1, 0.22, 0.11, 0.12), (v_2, 0.24, 0.12, 0.21), (v_3, 0.27, 0.17, 0.25)\}$,
$\mathscr{W}_3 = \{(v_1, 0.12, 0.11, 0.21), (v_2, 0.14, 0.12, 0.30), (v_3, 0.21, 0.13, 0.32)\}$,
$\mathscr{W}_4 = \{(v_1, 0.10, 0.12, 0.22), (v_2, 0.11, 0.21, 0.31), (v_3, 0.13, 0.22, 0.33)\}$.

The following is an unknown pattern $\mathscr{K}$:
$\mathscr{K} = \{(v_1, 0.15, 0.12, 0.21), (v_2, 0.19, 0.21, 0.27), (v_3, 0.32, 0.23, 0.28)\}$,
Its purpose is to determine the class $\mathscr{K}$. To do this, the inclusion degrees between $\mathscr{K}$ and classes $\mathscr{W}_1, \mathscr{W}_2, \mathscr{W}_3$, and $\mathscr{W}_4$ are measured, and $\mathscr{K}$ is then allocated to the class $\mathscr{W}_g$ specified by

$$g = arg \max_g \left\{ I\left(\mathscr{W}_g, \mathscr{K}\right)\right\}.$$

Table 20.17 Inclusion measure for Example 6

	$I(\mathscr{W}_1, \mathscr{K})$	$I(\mathscr{W}_2, \mathscr{K})$	$I(\mathscr{W}_3, \mathscr{K})$	$I(\mathscr{W}_4, \mathscr{K})$	Classification results
I_1	0.9935	0.9910	0.9970	**0.9997**	$\mathscr{W}_4$
I_2	0.9870	0.9822	0.9939	**0.9995**	$\mathscr{W}_4$
I_3	0.9870	0.9822	0.9940	**0.9995**	$\mathscr{W}_4$
I_4	0.9919	**0.9974**	0.9906	0.9852	$\mathscr{W}_2$
I_5	0.3290	0.3274	0.3313	**0.3332**	$\mathscr{W}_4$
I_6	0.3290	0.3274	0.3313	**0.3332**	$\mathscr{W}_4$
I_7	0.3323	**0.3334**	0.3331	0.3321	$\mathscr{W}_2$

The bold values show the best similarity values according to the given problem

For all the established inclusion measures ($I_1 - I_7$) for T-SFS, the degree of inclusion between $I(\mathscr{W}_1, \mathscr{K})$, $I(\mathscr{W}_2, \mathscr{K})$, $I(\mathscr{W}_3, \mathscr{K})$, and $I(\mathscr{W}_4, \mathscr{K})$ are determined and displayed in Table 20.17. It is shown in Table 20.17 that pattern which is unknown $\mathscr{K}$ lies to a class $\mathscr{W}_4$ when I_1, I_2, I_3, I_5, and I_6 are used and $\mathscr{K}$ belongs to a class $\mathscr{W}_2$ when I_4 and I_7 are used. It is evident that the primary attribute is what caused this difference, i.e., (v_1). T-SFNs of v_1 are $(0.21, 0.11, 0.19)$, $(0.22, 0.11, 0.12)$, $(0.12, 0.11, 0.21)$, $(0.10, 0.12, 0.22)$, and $(0.15, 0.12, 0.21)$ for $\mathscr{W}_1, \mathscr{W}_2, \mathscr{W}_3, \mathscr{W}_4$, and $\mathscr{K}$, respectively. As a conclusion, it appears that $I(\mathscr{W}_1, \mathscr{K}) > I(\mathscr{W}_3, \mathscr{K}) > I(\mathscr{W}_4, \mathscr{K}) > I(\mathscr{W}_2, \mathscr{K})$ is more acceptable. In a similar way, we can find the mentioned above relations for I_2 to I_7.

6 Applications of the Inclusion Measures to Bacteria Recognition

This section illustrates the applicability of the suggested T-SFS inclusion measures to Bacteria recognition.

Example 7 Identification and characterization of microorganisms is an essential part of microbiology. On the basis of classifying bacteria is to examine their response to the Gram's stain test. When stain reacts with the bacteria present, the germs become purple or pink in color. Stay purple if they are Gram-positive. They are Gram-negative if they are pink. Consider the following bacterial collection: Salmonella, Escherichia coli, and Shigella. All gut bacteria are Gram-negative and belong to the bacilli family. The major characteristics of these microorganisms are summarized by

$$\mathscr{W} = \{v_1(\text{size}), v_2(\text{flagellum}), v_3(\text{colony size})\}$$

A team of microbiologists is assumed to evaluate the existence of the aforementioned characteristics in the four bacteria $\mathscr{W}_1, \mathscr{W}_2, \mathscr{W}_3,$ and $\mathscr{W}_4$ and provides an assessment illustrated by T-SFSs in $X = \{v_1, v_2, v_3\}$ described as follows:
$\mathscr{W}_1 = \{(v_1, 0.21, 0.11, 0.19), (v_2, 0.22, 0.12, 0.23), (v_3, 0.32, 0.14, 0.27)\},$
$\mathscr{W}_2 = \{(v_1, 0.22, 0.11, 0.12), (v_2, 0.24, 0.12, 0.21), (v_3, 0.27, 0.17, 0.25)\},$
$\mathscr{W}_3 = \{(v_1, 0.12, 0.11, 0.21), (v_2, 0.14, 0.12, 0.30), (v_3, 0.21, 0.13, 0.32)\},$
$\mathscr{W}_4 = \{(v_1, 0.10, 0.12, 0.22), (v_2, 0.11, 0.21, 0.31), (v_3, 0.13, 0.22, 0.33)\}.$

The main purpose of molecular biologist team is to recognize an unknown bacteria $\mathscr{K}$ which is given as follows:
$\mathscr{K} = \{(v_1, 0.15, 0.12, 0.21), (v_2, 0.19, 0.21, 0.27), (v_3, 0.32, 0.23, 0.28)\},$
Its purpose is to calculate which class $\mathscr{K}$ belongs to. To do this, the inclusion degrees between $\mathscr{K}$ and classes $\mathscr{W}_1$, $\mathscr{W}_2$, $\mathscr{W}_3$, and $\mathscr{W}_4$ are measured, and $\mathscr{K}$ is then allocated to the class $\mathscr{W}_g$ specified by

$$g = arg \max_{g} \left\{ I \left(\mathscr{W}_g, \mathscr{K} \right) \right\}.$$

For all the established inclusion measures $(I_1 - I_7)$ for T-SFS, the degree of inclusion between $I(\mathscr{W}_1, \mathscr{K})$, $I(\mathscr{W}_2, \mathscr{K})$, $I(\mathscr{W}_3, \mathscr{K})$, and $I(\mathscr{W}_4, \mathscr{K})$ are determined and displayed in Table 20.17. It is observed in Table 20.17 that an unknown bacteria $\mathscr{K}$ belongs to a class $\mathscr{W}_4$ when $I_1$to I_3 and I_5 to I_6 are used and $\mathscr{K}$ belongs to a class $\mathscr{W}_2$ when I_4 and I_7 are used. It is obvious that the first characteristic is what caused this discrepancy, i.e., (v_1). The t-SFNs of v_1 are $(0.21, 0.11, 0.19)$, $(0.22, 0.11, 0.12)$, $(0.12, 0.11, 0.21)$, $(0.10, 0.12, 0.22)$, and $(0.15, 0.12, 0.21)$ for $\mathscr{W}_1$, $\mathscr{W}_2$, $\mathscr{W}_3, \mathscr{W}_4$, and $\mathscr{K}$, respectively. It is concluded that the inclusion degree between $(0.10, 0.12, 0.22)$ and $(0.15, 0.12, 0.21)$ is larger than the inclusion degree between $(0.12, 0.11, 0.21)$ and $(0.15, 0.12, 0.21)$; is larger than the inclusion degree between $(0.21, 0.11, 0.19)$ and $(0.15, 0.12, 0.21)$; and is larger than $(0.22, 0.11, 0.12)$ and $(0.15, 0.12, 0.21)$. As a conclusion, it appears that $I(\mathscr{W}_1, \mathscr{K}) > I(\mathscr{W}_3, \mathscr{K}) > I(\mathscr{W}_4, \mathscr{K}) > I(\mathscr{W}_2, \mathscr{K})$ is more acceptable. In a similar way, we can find the mentioned above relations for I_2 to I_7. Table 20.17 summarizes the outcomes for I_1 to I_7. Obviously, the unknown bacteria belong to $\mathscr{W}_4, \mathscr{W}_2$ for $I_1 - I_3$, $I_5 - I_6$ and I_4, I_7, respectively. Interestingly, the findings of all seven inclusion measures are comparable and consistent.

7 Conclusions

The T-SFSs present an excellent framework for expressing imprecise and dubious information owing to their flexible space eminent features and general structure. Further, the distance, the entropy measure, and similarity play pivotal role to boost the caliber and authenticity of decision-making approaches. However, how to precisely measure the similarity, entropy, and inclusion between two T-SFSs remains an open issue. Therefore, this article has keenly developed novel T-SF along with the thorough analysis of the associated transformation relationships. Further, the competency of the proposed distance measures has been unfolded by highlighting their application in recognition of pattern and diagnosis of medical things. To strengthen the arguments in favor of the proposed similarity measures, several counterintuitive examples of existing similarity measures have been presented. Furthermore, these similarity measures have been employed to target the applications of pattern recognition, construction materials, and medical diagnostics to obtain more authentic outcomes. The comparative study has been included in this study to exhibit the validity and competency of similarity measure. Moreover, the presented similarity measures, defined for the outstanding model of T-SFSs, are acknowledged to run over the shortcomings of existing similarity measure which fail to impart the reasonable outcomes in some particular condition. Additionally, we have spotlighted applicability of the suggested T-SFS inclusion measures to pattern recognition with an explanatory example to show their edge in comparison to the existing measures. In a nutshell, the experimental findings demonstrate that the suggested measures achieve more precise classification results. In the future, the similarity measure may be widely explored in scientific investigations for decision-making, recognition of pattern, linguistic summarization, and the mining of data. You most likely resolved your research issue within the contexts of a particular context, location, and culture. As a result, you can propose future studies that tackle the very same research issue in a different sense, placement, and culture. Additionally, we may immerse them in a variety of fuzzy environments.

Conflict of Interest The authors declare no conflict of interest.

Compliance with Ethical Standards

Ethical Approval This article does not contain any studies with human participants or animals performed by the author.

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Chapter 21
New Methods of Computing Correlation Coefficient Based on Pythagorean Fuzzy Information and Their Applications in Disaster Control and Diagnostic Analysis

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1 Introduction

Decision-making is the procedure of identifying and choosing varieties using preferences of the decision-makers (DMs) managed by germane assessment criteria. DMs often come across imprecise information, which tends to obstruct the decision-making process. The conception of fuzzy sets [45], which has the ability to operate in the presence of vague information, becomes appropriate to confront decision-making in uncertain sphere. The quest for a better tool for decision-making under fuzzy setting led to the conception of intuitionistic fuzzy sets (IFSs) [1]. Against the construct of fuzzy set, which features only membership grade (MG) defined within the interval, [0, 1], IFS features MG, nonmembership grade (NMG), and the grade of hesitation (GH), which aggregated to one. Because of the appropriateness of IFS, it has been applied to recognition of patterns [3, 18, 26], diagnosis [6, 16, 38], and decision-making [9, 31].

The concept of intuitionistic fuzzy correlation coefficient (IFCC) has been used to discuss sundry application of IFSs. Correlation coefficient measures the similarity and interrelationship between data, and it has been extended to fuzzy sphere to measure fuzzy data [4, 7]. Subsequently, correlation coefficient was equipped to handle intuitionistic fuzzy data [25]. IFCC had also been studied from statistical

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C. Jana et al. (eds.), *Fuzzy Optimization, Decision-making and Operations Research*, https://doi.org/10.1007/978-3-031-35668-1_21

view point [15, 28, 32, 33, 39]. Zeng and Li [46] extended the method in [25] by including GH for better result. Garg and Kumar [23] developed a method of IFCCusing set pair analysis, and discussed its application in decision-making.

Though model formulation based on IFSs is appropriate for determining decision-making under fuzzy setting, DMs often run into problems when the aggregate of MG and NMG is greater than one, which is a likely possibility. Due to this drawback, the concept of IFS of the second type was conceived [2], which is mostly referred to as the Pythagorean fuzzy set (PFS) [41]. PFS enlarges the scope of IFS so that the aggregate of MG and NMG could also be greater than one. Honestly, every single IFS is a PFS, but every single PFS is not an IFS. Many applications of PFSs have been discussed based on myriad information measures in sundry real-world problems [8, 13, 17, 21, 42–44, 47–49]. Other variants of the fuzzy sets have been studied with applications such as Fermatean fuzzy sets [14], q-rung orthopair fuzzy sets [11], cubic m-polar fuzzy sets [24], linear Diophantine fuzzy soft sets [29], cubic bipolar fuzzy sets [34, 35], and m-polar spherical fuzzy sets [36].

The construct of correlation coefficient has been stretched to accommodate Pythagorean fuzzy data. The studies on Pythagorean fuzzy correlation coefficient (PFCC) were initiated in [22] and applied to determine multiple criteria decision-making (MCDM). Lin et al. [30] extended the methods in [22] to develop new methods of PFCC using unconventional parameters of PFSs. Chen [5] developed a method of PFCC using the approach of Pearson-like correlation coefficient, and discussed its place in MCDM. In ref. [37], method of PFCC was developed by extending the work in [27] to Pythagorean fuzzy setting, and applied in real-life issues. Thao [40] developed a method of computing PFCC from statistical perspective, and used the approach to address MCDM problems. In Ref. [10], a PFCC method, which modified an approach in [22], was developed and applied to real-world problems. Some methods of PFCC based on statistical approach were developed and used to resolve pattern recognition problems and the case of disease diagnosis [12, 19, 20]. The motivation for this research includes the following:

- The method of PFCC in [5] violates the condition of correlation coefficient.
- The methods of PFCC in [10, 22] are defective in the sense that they cannot measure correlation of comparable patterns.
- In ref. [37], the grade of hesitation is not included in the method of PFCC, and so the approach cannot be reliable.
- In ref. [40], the method of PFCC violates the condition of correlation coefficient, and equally does not consider the grade of hesitation.

Due to these lapses, this chapter develops two methods for calculating PFCC to resolve the problems in the existing methods. The new developed methods are discussed and applied to real-world problems like disaster control and diagnosis analysis. The objectives of this chapter include the following:

- Characterizations of the existing methods of PFCC to identify their setbacks.

- Development of two methods of PFCC with better prospects compared to the existing methods of PFCC.
- Validation of the new methods of PFCC with theoretic results to show their agreement with classical correlation operator.
- Demonstration of the applications of the new PFCC methods in real-world problems like disaster control and disease analysis.
- Comparison of the new PFCC methods with other PFCC methods to justify the new developed methods.

What follows is the outline of the remaining parts of the chapter for easy reference: Sect. 2 discusses the concept of PFSs and their correlation operators including some existing methods of PFCC. Section 3 presents the new developed methods of PFCC, discusses their properties, and computation example. Section 4 discusses the numerical applications of the new methods and presents the comparative studies. Section 5 sums up the conclusion and suggests some valid recommendations.

2 Preliminaries

In this section, certain basic notions of PFSs are revised and some methods of calculating PFCC are addressed.

2.1 *Pythagorean Fuzzy Sets*

Some basic definitions of PFSs and operations on them are examined in this section before constructing the approaches of calculating correlation operator. By the inclusion of a nonmembership degree, the idea of fuzzy sets was extended to intuitionistic fuzzy sets. We denote nonempty set by U in the chapter.

Definition 2.1 [1] An IFS $\tilde{N}$ in U is given by

$$\tilde{N} = \left\{ \left\langle u, \xi_{\underset{N}{\sim}}(u), \eta_{\underset{N}{\sim}}(u) \right\rangle | u \in U \right\}, \tag{21.1}$$

where $\xi_{\underset{N}{\sim}}, \eta_{\underset{N}{\sim}} : U \to [0, 1]$ denote the grades of membership and nonmembership for $u \in U$ to the set $\tilde{N}$, with the property

$$0 \leq \xi_{\underset{N}{\sim}}(u) + \eta_{\underset{N}{\sim}}(u) \leq 1. \tag{21.2}$$

The grade of indeterminacy for IFS $\widetilde{N}$ is presented by $\lambda_{\widetilde{N}}(u) = 1 - \xi_{\widetilde{N}}(u) - \eta_{\widetilde{N}}(u)$. For easiness, $\left(\xi_{\widetilde{N}}(u), \eta_{\widetilde{N}}(u)\right)$ is taken as intuitionistic fuzzy number (IFN) and it is denoted by $\widetilde{N} = (\xi, \eta)$.

By extending IFS, Yager [41] presented a new set called PFS as defined in the following manner.

Definition 2.2 [41] A PFS, $\widetilde{\wp}$in U is presented by

$$\widetilde{\wp} = \left\{\left\langle u, \xi_{\widetilde{\wp}}(u), \eta_{\widetilde{\wp}}(u)\right\rangle | u \in U\right\}, \tag{21.3}$$

where $\xi_{\widetilde{\wp}}(u), \eta_{\widetilde{\wp}}(u) \in [0, 1]$ denote the grades of membership and nonmembership for $u \in U$ to the set $\widetilde{\wp}$, with the property

$$0 \leq \xi^2_{\widetilde{\wp}}(u) + \eta^2_{\widetilde{\wp}}(u) \leq 1. \tag{21.4}$$

The grade of indeterminacy for PFS $\widetilde{\wp}$ is given by $\lambda_{\widetilde{\wp}}(u) = \left[1 - \xi^2_{\widetilde{\wp}}(u) - \eta^2_{\widetilde{\wp}}(u)\right]^{\frac{1}{2}}$. With the introduction of PFS, it is clearly realized that the spaces of the membership and nonmembership values have been enlarged to model many real-world problems.

For simplicity, $\left(\xi_{\widetilde{\mathcal{F}}}(u), \eta_{\widetilde{\mathcal{F}}}(u)\right)$ is called a Pythagorean fuzzy number (PFN) and is symbolized by $\widetilde{\wp} = (\xi, \eta)$.

Here, definitions that explain the notion of correlation coefficient under PFSs within $[0, 1]$ and $[-1, 1]$ are provided.

Definition 2.4 [22] Suppose $\widetilde{\wp}_1$ and $\widetilde{\wp}_2$ are PFSs in $U = \{u_1, u_2, \ldots, u_n\}$, then the correlation coefficient under PFSs $\widetilde{\wp}_1$ and $\widetilde{\wp}_2$, denoted by $\Omega\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right)$, is a function, $\Omega : \widetilde{\wp}_1 \times \widetilde{\wp}_2 \to [0, 1]$, which satisfies

(i) $\Omega\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) \in [0, 1]$
(ii) $\Omega\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) = \Omega\left(\widetilde{\wp}_2, \widetilde{\wp}_1\right)$
(iii) $\Omega\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) = 1$if and only if $\widetilde{\wp}_1 = \widetilde{\wp}_2$

As $\Omega\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right)$ moves closer to 1, it shows that the correlation is strong. On the other hand, as $\Omega\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right)$ moves closer to 0, it shows that the correlation is very weak. On the other hand, $\Omega\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) = 1$ and $\Omega\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) = 0$ indicate a perfect correlation and no correlation, respectively.

Definition 2.5 [12] Suppose $\widetilde{\wp}_1$ and $\widetilde{\wp}_2$ are PFSs in $U = \{u_1, u_2, \ldots, u_n\}$, then the correlation coefficient based on statistical view under PFSs $\widetilde{\wp}_1$ and $\widetilde{\wp}_2$ denoted by $\Omega_*\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right)$ is a function, $\Omega_* : \widetilde{\wp}_1 \times \widetilde{\wp}_2 \rightarrow [-1, 1]$, which satisfies

(i) $\Omega_*\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) \in [-1, 1]$

(ii) $\Omega_*\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) = \Omega_*\left(\widetilde{\wp}_2, \widetilde{\wp}_1\right)$

(iii) $\Omega_*\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) = 1$if and only if $\widetilde{\wp}_1 = \widetilde{\wp}_2$

As $\Omega_*\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right)$ moves closer to 1, it shows that there is a strong positive correlation. On the other hand, as $\Omega_*\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right)$ moves closer to-1, it shows that there is a weak negative correlation. Whereas, $\Omega_*\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) = 1$ and $\Omega_*\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) = 0$ indicate a perfect positive correlation and perfect negative correlation, respectively.

2.2 *Problems with Existing Methods for Computing Correlation Coefficient for Pythagorean Fuzzy Sets*

Some existing methods of computing the correlation coefficient of PFSs are enumerated before pinpointing their limitations. Suppose $\widetilde{\wp}_1$ and $\widetilde{\wp}_2$ are two arbitrary PFSs in $U = \{u_1, u_2, \ldots, u_n\}$, then the existing methods for computing correlation coefficient for $\widetilde{\wp}_1$and $\widetilde{\wp}_2$ are as follows:

- **Garg's methods [22]**

$$\Omega_1\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) = \frac{\sum_{j=1}^{n}\left(\xi^2_{\widetilde{\wp}_1}(u_j)\,\xi^2_{\widetilde{\wp}_2}(u_j) + \eta^2_{\widetilde{\wp}_1}(u_j)\,\eta^2_{\widetilde{\wp}_2}(u_j) + \lambda^2_{\widetilde{\wp}_1}(u_j)\,\lambda^2_{\widetilde{\wp}_2}(u_j)\right)}{\sqrt{\sum_{j=1}^{n}\left(\xi^4_{\widetilde{\wp}_1}(u_j) + \eta^4_{\widetilde{\wp}_1}(u_j) + \lambda^4_{\widetilde{\wp}_1}(u_j)\right)\sum_{j=1}^{n}\left(\xi^4_{\widetilde{\wp}_2}(u_j) + \eta^4_{\widetilde{\wp}_2}(u_j) + \lambda^4_{\widetilde{\wp}_2}(u_j)\right)}} \tag{21.5}$$

The method is not reliable for computing correlation coefficient between PFSs. Suppose

$$\widetilde{\wp}_1 = \left(\frac{2}{3}, \frac{2}{3}\right), \widetilde{\wp}_2 = \left(\frac{1}{3}, \frac{2}{3}\right) \text{ and } \widetilde{\wp}_3 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \text{ are PFSs in } U = \{u\}.$$

The hesitation margins are

$$\lambda_{\tilde{\wp}_1}(u) = \frac{1}{3}, \lambda_{\tilde{\wp}_2}(u) = \frac{2}{3}, \lambda_{\tilde{\wp}_3}(u) = \frac{1}{\sqrt{3}}.$$

Applying Eq. (21.5), we get

$$\Omega_1\left(\tilde{\wp}_1, \tilde{\wp}_3\right) = \frac{\left(\frac{2}{3}\times\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{2}{3}\times\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{3}\times\frac{1}{\sqrt{3}}\right)^2}{\sqrt{\left(\frac{2}{3}\right)^4 + \left(\frac{2}{3}\right)^4 + \left(\frac{1}{3}\right)^4}\sqrt{\left(\frac{1}{\sqrt{3}}\right)^4 + \left(\frac{1}{\sqrt{3}}\right)^4 + \left(\frac{1}{\sqrt{3}}\right)^4}}$$

$$= 0.9045 \text{ and}$$

$$\Omega_1\left(\tilde{\wp}_2, \tilde{\wp}_3\right) = \frac{\left(\frac{1}{3}\times\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{2}{3}\times\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{2}{3}\times\frac{1}{\sqrt{3}}\right)^2}{\sqrt{\left(\frac{1}{3}\right)^4 + \left(\frac{2}{3}\right)^4 + \left(\frac{2}{3}\right)^4}\sqrt{\left(\frac{1}{\sqrt{3}}\right)^4 + \left(\frac{1}{\sqrt{3}}\right)^4 + \left(\frac{1}{\sqrt{3}}\right)^4}} = 0.9045$$

We see that, $\Omega_1\left(\tilde{\wp}_1, \tilde{\wp}_3\right) = \Omega_1\left(\tilde{\wp}_2, \tilde{\wp}_3\right)$ although $\tilde{\wp}_1 \neq \tilde{\wp}_2$, which proves that Eq. (21.5) is not reliable. The second method in [22] is

$$\Omega_2\left(\tilde{\wp}_1, \tilde{\wp}_2\right) = \frac{\sum_{j=1}^{n}\left(\xi^2_{\tilde{\wp}_1}(u_j)\,\xi^2_{\tilde{\wp}_2}(u_j) + \eta^2_{\tilde{\wp}_1}(u_j)\,\eta^2_{\tilde{\wp}_2}(u_j) + \lambda^2_{\tilde{\wp}_1}(u_j)\,\lambda^2_{\tilde{\wp}_2}(u_j)\right)}{\max\left\{\begin{array}{c}\sum_{j=1}^{n}\left(\xi^4_{\tilde{\wp}_1}(u_j) + \eta^4_{\tilde{\wp}_1}(u_j) + \lambda^4_{\tilde{\wp}_1}(u_j)\right), \\ \sum_{j=1}^{n}\left(\xi^4_{\tilde{\wp}_2}(u_j) + \eta^4_{\tilde{\wp}_2}(u_j) + \lambda^4_{\tilde{\wp}_2}(u_j)\right)\end{array}\right\}}. \tag{21.6}$$

Again, take

$$\tilde{\wp}_1 = \left(\frac{2}{3}, \frac{2}{3}\right), \tilde{\wp}_2 = \left(\frac{1}{3}, \frac{2}{3}\right) \text{ and } \tilde{\wp}_3 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \text{ as PFSs in } U = \{u\}.$$

Applying Eq. (21.6), we get

$$\Omega_2\left(\tilde{\wp}_1, \tilde{\wp}_3\right) = \frac{\left(\frac{2}{3}\times\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{2}{3}\times\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{3}\times\frac{1}{\sqrt{3}}\right)^2}{\max\left\{\left(\left(\frac{2}{3}\right)^4 + \left(\frac{2}{3}\right)^4 + \left(\frac{1}{3}\right)^4\right), \left(\left(\frac{1}{\sqrt{3}}\right)^4 + \left(\frac{1}{\sqrt{3}}\right)^4 + \left(\frac{1}{\sqrt{3}}\right)^4\right)\right\}}$$

$$= 0.8182 \text{ and}$$

$$\Omega_2\left(\widetilde{\wp}_2,\widetilde{\wp}_3\right)=\frac{\left(\frac{1}{3}\times\frac{1}{\sqrt{3}}\right)^2+\left(\frac{2}{3}\times\frac{1}{\sqrt{3}}\right)^2+\left(\frac{2}{3}\times\frac{1}{\sqrt{3}}\right)^2}{\max\left\{\left(\left(\frac{1}{3}\right)^4+\left(\frac{2}{3}\right)^4+\left(\frac{2}{3}\right)^4\right),\left(\left(\frac{1}{\sqrt{3}}\right)^4+\left(\frac{1}{\sqrt{3}}\right)^4+\left(\frac{1}{\sqrt{3}}\right)^4\right)\right\}}$$
$$=0.8182$$

We see that $\Omega_2\left(\widetilde{\wp}_1,\widetilde{\wp}_3\right)=\Omega_2\left(\widetilde{\wp}_2,\widetilde{\wp}_3\right)$ although $\widetilde{\wp}_1\neq\widetilde{\wp}_2$, which proves that Eq. (21.6) is not reliable.

- **Chen's method [5]**

$$\Omega_3\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)=\frac{1}{3}\left(k_\xi\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)+k_\eta\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)+k_\lambda\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)\right) \tag{21.7}$$

where

$$k_\xi\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)=\frac{\sum_{j=1}^{n}\left[\left(\xi^2_{\widetilde{\wp}_1}(u_j)-\overline{\xi}^2_{\widetilde{\wp}_1}\right)\left(\xi^2_{\widetilde{\wp}_2}(u_j)-\overline{\xi}^2_{\widetilde{\wp}_2}\right)\right]}{\left[\sqrt{\sum_{j=1}^{n}\left(\xi^2_{\widetilde{\wp}_1}(u_j)-\overline{\xi}^2_{\widetilde{\wp}_1}\right)^2}\sqrt{\sum_{j=1}^{n}\left(\xi^2_{\widetilde{\wp}_2}(u_j)-\overline{\xi}^2_{\widetilde{\wp}_2}\right)^2}\right]}$$

$$k_\eta\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)=\frac{\sum_{j=1}^{n}\left[\left(\eta^2_{\widetilde{\wp}_1}(u_j)-\overline{\eta}^2_{\widetilde{\wp}_1}\right)\left(\eta^2_{\widetilde{\wp}_2}(u_j)-\overline{\eta}^2_{\widetilde{\wp}_2}\right)\right]}{\left[\sqrt{\sum_{j=1}^{n}\left(\eta^2_{\widetilde{\wp}_1}(u_j)-\overline{\eta}^2_{\widetilde{\wp}_1}\right)^2}\sqrt{\sum_{j=1}^{n}\left(\eta^2_{\widetilde{\wp}_2}(u_j)-\overline{\eta}^2_{\widetilde{\wp}_2}\right)^2}\right]}$$

$$k_\lambda\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)=\frac{\sum_{j=1}^{n}\left[\left(\lambda^2_{\widetilde{\wp}_1}(u_j)-\overline{\lambda}^2_{\widetilde{\wp}_1}\right)\left(\lambda^2_{\widetilde{\wp}_2}(u_j)-\overline{\lambda}^2_{\widetilde{\wp}_2}\right)\right]}{\left[\sqrt{\sum_{j=1}^{n}\left(\lambda^2_{\widetilde{\wp}_1}(u_j)-\overline{\lambda}^2_{\widetilde{\wp}_1}\right)^2}\sqrt{\sum_{j=1}^{n}\left(\lambda^2_{\widetilde{\wp}_2}(u_j)-\overline{\lambda}^2_{\widetilde{\wp}_2}\right)^2}\right]}$$

$$\overline{\xi}_{\widetilde{\wp}_1}=\frac{\sum_{j=1}^{n}\xi_{\widetilde{\wp}_1}(u_j)}{n},\ \overline{\eta}_{\widetilde{\wp}_1}=\frac{\sum_{j=1}^{n}\eta_{\widetilde{\wp}_1}(u_j)}{n},\ \overline{\lambda}_{\widetilde{\wp}_1}=\frac{\sum_{j=1}^{n}\lambda_{\widetilde{\wp}_1}(u_j)}{n}$$

$$\overline{\xi}_{\widetilde{\wp}_2}=\frac{\sum_{j=1}^{n}\xi_{\widetilde{\wp}_2}(u_j)}{n},\ \overline{\eta}_{\widetilde{\wp}_2}=\frac{\sum_{j=1}^{n}\eta_{\widetilde{\wp}_2}(u_j)}{n},\ \overline{\lambda}_{\widetilde{\wp}_2}=\frac{\sum_{j=1}^{n}\lambda_{\widetilde{\wp}_2}(u_j)}{n}$$

We want to show that Eq. (21.7) is unrealistic. Suppose

$\widetilde{\wp}_1 = \left(\frac{2}{3}, \frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)$ and $\widetilde{\wp}_2 = \left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{2}{3}\right)$ are PFSs in $U = \{u_1, u_2\}$.

The hesitation margins are

$$\lambda_{\widetilde{\wp}_1}(u_1) = \frac{1}{3}, \lambda_{\widetilde{\wp}_1}(u_2) = \frac{2}{3}, \lambda_{\widetilde{\wp}_2}(u_1) = \frac{2}{3}, \lambda_{\widetilde{\wp}_2}(u_2) = \frac{1}{3}$$

The mean values are:

$$\overline{\xi}_{\widetilde{\wp}_1} = \frac{1}{2}, \overline{\eta}_{\widetilde{\wp}_1} = \frac{2}{3}, \overline{\lambda}_{\widetilde{\wp}_1} = \frac{1}{2}$$

$$\overline{\xi}_{\widetilde{\wp}_2} = \frac{2}{3}, \overline{\eta}_{\widetilde{\wp}_2} = \frac{1}{2}, \overline{\lambda}_{\widetilde{\wp}_2} = \frac{1}{2}$$

Applying Eq. (21.7), we get

$$k_\xi\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) = \frac{\left(\frac{2}{3}^2 - \frac{1}{2}^2\right)\left(\frac{2}{3}^2 - \frac{2}{2}^2\right) + \left(\frac{1}{3}^2 - \frac{1}{2}^2\right)\left(\frac{2}{3}^2 - \frac{2}{3}^2\right)}{\left[\sqrt{\left(\frac{2}{3}^2 - \frac{1}{2}^2\right)^2 + \left(\frac{1}{3}^2 - \frac{1}{2}^2\right)^2}\sqrt{\left(\frac{2}{3}^2 - \frac{2}{3}^2\right)^2 + \left(\frac{2}{3}^2 - \frac{2}{3}^2\right)^2}\right]}$$
$$= \infty$$

$$k_\xi\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) = \frac{\left(\frac{2}{3}^2 - \frac{2}{3}^2\right)\left(\frac{1}{3}^2 - \frac{1}{2}^2\right) + \left(\frac{2}{3}^2 - \frac{2}{3}^2\right)\left(\frac{2}{3}^2 - \frac{1}{2}^2\right)}{\left[\sqrt{\left(\frac{2}{3}^2 - \frac{2}{3}^2\right)^2 + \left(\frac{2}{3}^2 - \frac{2}{3}^2\right)^2}\sqrt{\left(\frac{1}{3}^2 - \frac{1}{2}^2\right)^2 + \left(\frac{2}{3}^2 - \frac{1}{2}^2\right)^2}\right]}$$
$$= \infty$$

$$k_\lambda\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) = \frac{\left(\frac{1}{3}^2 - \frac{1}{2}^2\right)\left(\frac{2}{3}^2 - \frac{1}{2}^2\right) + \left(\frac{2}{3}^2 - \frac{1}{2}^2\right)\left(\frac{1}{3}^2 - \frac{1}{2}^2\right)}{\left[\sqrt{\left(\frac{1}{3}^2 - \frac{1}{2}^2\right)^2 + \left(\frac{2}{3}^2 - \frac{1}{2}^2\right)^2}\sqrt{\left(\frac{2}{3}^2 - \frac{1}{2}^2\right)^2 + \left(\frac{1}{3}^2 - \frac{1}{2}^2\right)^2}\right]}$$
$$= -0.9457$$

The fact that $k_\xi\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) = k_\xi\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) = \infty$, then $\Omega_3\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) = \infty$, which is unrealistic, and so is not a reliable approach.

- **Singh and Ganie's method [37]**

$$\Omega_4\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)=\frac{1}{2n}\sum\nolimits_{j=1}^{n}\left(\mu_j\left(1-\Delta\xi_j\right)+\nu_j\left(1-\Delta\eta_j\right)\right)\tag{21.8}$$

where

$$\mu_j=\frac{c-\Delta\xi_j-\Delta\xi_{\max}}{c-\Delta\xi_{\min}-\Delta\xi_{\max}},\ \nu_j=\frac{c-\Delta\eta_j-\Delta\eta_{\max}}{c-\Delta\eta_{\min}-\Delta\eta_{\max}},c>2$$

$$\Delta\xi_{\min}=\min_j\left\{\left|\xi^2_{\widetilde{\wp}_1}\left(u_j\right)-\xi^2_{\widetilde{\wp}_2}\left(u_j\right)\right|\right\},\ \Delta\eta_{\min}=\min_j\left\{\left|\eta^2_{\widetilde{\wp}_1}\left(u_j\right)-\eta^2_{\widetilde{\wp}_2}\left(u_j\right)\right|\right\}$$

$$\Delta\xi_{\max}=\max_j\left\{\left|\xi^2_{\widetilde{\wp}_1}\left(u_j\right)-\xi^2_{\widetilde{\wp}_2}\left(u_j\right)\right|\right\},\ \Delta\eta_{\max}=\max_j\left\{\left|\eta^2_{\widetilde{\wp}_1}\left(u_j\right)-\eta^2_{\widetilde{\wp}_2}\left(u_j\right)\right|\right\}$$

$$\Delta\xi_j=\left|\xi^2_{\widetilde{\wp}_1}\left(u_j\right)-\xi^2_{\widetilde{\wp}_2}\left(u_j\right)\right|,\ \Delta\eta_j=\left|\eta^2_{\widetilde{\wp}_1}\left(u_j\right)-\eta^2_{\widetilde{\wp}_2}\left(u_j\right)\right|$$

The limitation of this method is that it does not consider the hesitation margin, and so, its output is not reliable.

- **Thao's method [40]**

$$\Omega_5\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)=\frac{\sum_j^n\left[\begin{array}{c}\left(\left(\xi^2_{\widetilde{\wp}_1}\left(u_j\right)-\overline{\xi}^2_{\widetilde{\wp}_1}\right)-\left(\eta^2_{\widetilde{\wp}_1}\left(u_j\right)-\overline{\eta}^2_{\widetilde{\wp}_1}\right)\right)\\ \left(\left(\xi^2_{\widetilde{\wp}_2}\left(u_j\right)-\overline{\xi}^2_{\widetilde{\wp}_2}\right)-\left(\eta^2_{\widetilde{\wp}_2}\left(u_j\right)-\overline{\eta}^2_{\widetilde{\wp}_2}\right)\right)\end{array}\right]}{\sqrt{\sum_{j=1}^n\left(\left(\xi^2_{\widetilde{\wp}_1}\left(u_j\right)-\overline{\xi}^2_{\widetilde{\wp}_1}\right)-\left(\eta^2_{\widetilde{\wp}_1}\left(u_j\right)-\overline{\eta}^2_{\widetilde{\wp}_1}\right)\right)^2}\sqrt{\sum_{j=1}^n\left(\left(\xi^2_{\widetilde{\wp}_2}\left(u_j\right)-\overline{\xi}^2_{\widetilde{\wp}_2}\right)-\left(\eta^2_{\widetilde{\wp}_2}\left(u_j\right)-\overline{\eta}^2_{\widetilde{\wp}_2}\right)\right)^2}}\tag{21.9}$$

where

$$\overline{\xi}_{\widetilde{\wp}_1}=\frac{\sum_{j=1}^n\xi_{\widetilde{\wp}_1}\left(u_j\right)}{n},\ \overline{\eta}_{\widetilde{\wp}_1}=\frac{\sum_{j=1}^n\eta_{\widetilde{\wp}_1}\left(u_j\right)}{n}$$

$$\overline{\xi}_{\tilde{\wp}_2} = \frac{\sum_{j=1}^{n} \xi_{\tilde{\wp}_2}(u_j)}{n}, \overline{\eta}_{\tilde{\wp}_2} = \frac{\sum_{j=1}^{n} \eta_{\tilde{\wp}_2}(u_j)}{n}$$

We now show whether this method is reliable. Suppose

$$\tilde{\wp}_1 = \left(\frac{2}{3}, \frac{2}{3}\right) \text{ and } \tilde{\wp}_2 = \left(\frac{2}{5}, \frac{1}{5}\right) \text{ are PFSs in } U = \{u\}.$$

The mean values are

$$\overline{\xi}_{\tilde{\wp}_1} = \frac{2}{3}, \overline{\eta}_{\tilde{\wp}_1} = \frac{2}{3}, \overline{\xi}_{\tilde{\wp}_2} = \frac{2}{5}, \overline{\eta}_{\tilde{\wp}_2} = \frac{1}{5}$$

Thus, $\Omega_5\left(\tilde{\wp}_1, \tilde{\wp}_2\right) = \frac{\left(\left(\frac{2}{3}^2-\frac{2}{3}^2\right)-\left(\frac{2}{3}^2-\frac{2}{3}^2\right)\right)\left(\left(\frac{2}{5}^2-\frac{2}{5}^2\right)-\left(\frac{1}{5}^2-\frac{1}{5}^2\right)\right)}{\sqrt{\left(\frac{2}{3}^2-\frac{2}{3}^2\right)-\left(\frac{2}{3}^2-\frac{2}{3}^2\right)^2}\sqrt{\left(\frac{2}{5}^2-\frac{2}{5}^2\right)-\left(\frac{1}{5}^2-\frac{1}{5}^2\right)^2}} = \infty$, which is unrealistic. In addition, the method does not consider the hesitation margin, and so, its output is not reliable.

- **Ejegwa and Awolola's method [10]**

$$\Omega_6\left(\tilde{\wp}_1, \tilde{\wp}_2\right) = \frac{\sum_{j=1}^{n}\left(\xi^2_{\tilde{\wp}_1}(u_j)\,\xi^2_{\tilde{\wp}_2}(u_j) + \eta^2_{\tilde{\wp}_1}(u_j)\,\eta^2_{\tilde{\wp}_2}(u_j) + \lambda^2_{\tilde{\wp}_1}(u_j)\,\lambda^2_{\tilde{\wp}_2}(u_j)\right)}{\text{Aver}\left\{\begin{array}{c}\sum_{j=1}^{n}\left(\xi^4_{\tilde{\wp}_1}(u_j) + \eta^4_{\tilde{\wp}_1}(u_j) + \lambda^4_{\tilde{\wp}_1}(u_j)\right), \\ \sum_{j=1}^{n}\left(\xi^4_{\tilde{\wp}_2}(u_j) + \eta^4_{\tilde{\wp}_2}(u_j) + \lambda^4_{\tilde{\wp}_2}(u_j)\right)\end{array}\right\}} \quad (21.10)$$

To show that this method is not reliable, take

$$\tilde{\wp}_1 = \left(\frac{2}{3}, \frac{2}{3}\right), \tilde{\wp}_2 = \left(\frac{1}{3}, \frac{2}{3}\right) \text{ and } \tilde{\wp}_3 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \text{ to be PFSs in } U = \{u\}.$$

The hesitation margins are

$$\lambda_{\tilde{\wp}_1}(u) = \frac{1}{3}, \lambda_{\tilde{\wp}_2}(u) = \frac{2}{3}, \lambda_{\tilde{\wp}_3}(u) = \frac{1}{\sqrt{3}}$$

Applying Eq. (21.10), we get

$$\Omega_6\left(\widetilde{\wp}_1, \widetilde{\wp}_3\right) = \frac{\left(\frac{2}{3}\times\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{2}{3}\times\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{3}\times\frac{1}{\sqrt{3}}\right)^2}{\text{Aver}\left\{\left(\left(\frac{2}{3}\right)^4 + \left(\frac{2}{3}\right)^4 + \left(\frac{1}{3}\right)^4\right), \left(\left(\frac{1}{\sqrt{3}}\right)^4 + \left(\frac{1}{\sqrt{3}}\right)^4 + \left(\frac{1}{\sqrt{3}}\right)^4\right)\right\}}$$
$$= 0.8998 \text{ and}$$

$$\Omega_6\left(\widetilde{\wp}_2, \widetilde{\wp}_3\right) = \frac{\left(\frac{1}{3}\times\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{2}{3}\times\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{2}{3}\times\frac{1}{\sqrt{3}}\right)^2}{\text{Aver}\left\{\left(\left(\frac{1}{3}\right)^4 + \left(\frac{2}{3}\right)^4 + \left(\frac{2}{3}\right)^4\right), \left(\left(\frac{1}{\sqrt{3}}\right)^4 + \left(\frac{1}{\sqrt{3}}\right)^4 + \left(\frac{1}{\sqrt{3}}\right)^4\right)\right\}}$$
$$= 0.8998$$

We see that $\Omega_6\left(\widetilde{\wp}_1, \widetilde{\wp}_3\right) = \Omega_6\left(\widetilde{\wp}_2, \widetilde{\wp}_3\right)$ although $\widetilde{\wp}_1 \neq \widetilde{\wp}_2$, which proves that Eq. (21.10) is not reliable.

3 New Methods for Computing Correlation Coefficient for Pythagorean Fuzzy Sets

Having dissected the existing methods of computing of correlation coefficient for PFSs, we present two new methods of calculating correlation coefficient for PFSs, which improve the methods in [22, 37] to enhance reliability.

Let $\widetilde{\wp}_1$ and $\widetilde{\wp}_2$ be any two arbitrary PFSs in $U = \{u_1, u_2, \ldots, u_n\}$, then the new methods for estimating the correlation coefficient between $\widetilde{\wp}_1$ and $\widetilde{\wp}_2$are in Eqs. (21.11) and (21.12), respectively.

$$\widetilde{\Omega}_1\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) = \frac{1}{3n}\sum_{j=1}^{n}\left(\mu_j\left(1-\Delta\xi_j\right) + \nu_j\left(1-\Delta\eta_j\right) + \varphi_j\left(1-\Delta\lambda_j\right)\right) \tag{21.11}$$

where

$$\mu_j = \frac{c - \Delta\xi_j - \Delta\xi_{\max}}{c - \Delta\xi_{\min} - \Delta\xi_{\max}}, \quad \nu_j = \frac{c - \Delta\eta_j - \Delta\eta_{\max}}{c - \Delta\eta_{\min} - \Delta\eta_{\max}},$$

$$\varphi_j = \frac{c - \Delta\lambda_j - \Delta\lambda_{\max}}{c - \Delta\lambda_{\min} - \Delta\lambda_{\max}}, c > 2$$

$$\Delta\xi_{\min} = \min_j\left\{\left|\xi^2_{\widetilde{\wp}_1}\left(u_j\right) - \xi^2_{\widetilde{\wp}_2}\left(u_j\right)\right|\right\}, \Delta\eta_{\min} = \min_j\left\{\left|\eta^2_{\widetilde{\wp}_1}\left(u_j\right) - \eta^2_{\widetilde{\wp}_2}\left(u_j\right)\right|\right\}$$

$$\Delta\lambda_{\min} = \min_j \left\{ \left| \lambda^2_{\underset{\sim}{\wp}_1}(u_j) - \lambda^2_{\underset{\sim}{\wp}_2}(u_j) \right| \right\}$$

$$\Delta\xi_{\max} = \max_j \left\{ \left| \xi^2_{\underset{\sim}{\wp}_1}(u_j) - \xi^2_{\underset{\sim}{\wp}_2}(u_j) \right| \right\}, \Delta\eta_{\max} = \max_j \left\{ \left| \eta^2_{\underset{\sim}{\wp}_1}(u_j) - \eta^2_{\underset{\sim}{\wp}_2}(u_j) \right| \right\}$$

$$\Delta\lambda_{\max} = \max_j \left\{ \left| \lambda^2_{\underset{\sim}{\wp}_1}(u_j) - \lambda^2_{\underset{\sim}{\wp}_2}(u_j) \right| \right\}$$

$$\Delta\xi_j = \left| \xi^2_{\underset{\sim}{\wp}_1}(u_j) - \xi^2_{\underset{\sim}{\wp}_2}(u_j) \right|, \Delta\eta_j = \left| \eta^2_{\underset{\sim}{\wp}_1}(u_j) - \eta^2_{\underset{\sim}{\wp}_2}(u_j) \right|$$

$$\Delta\lambda_j = \left| \lambda^2_{\underset{\sim}{\wp}_1}(u_j) - \lambda^2_{\underset{\sim}{\wp}_2}(u_j) \right|$$

$$\widetilde{\Omega}_2\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) = \frac{\sqrt{\sum_{j=1}^n \left(\xi^2_{\underset{\sim}{\wp}_1}(u_j)\,\xi^2_{\underset{\sim}{\wp}_2}(u_j) + \eta^2_{\underset{\sim}{\wp}_1}(u_j)\,\eta^2_{\underset{\sim}{\wp}_2}(u_j) + \lambda^2_{\underset{\sim}{\wp}_1}(u_j)\,\lambda^2_{\underset{\sim}{\wp}_2}(u_j) \right)}}{\sqrt{\sqrt{\sum_{j=1}^n \left(\xi^4_{\underset{\sim}{\wp}_1}(u_j) + \eta^4_{\underset{\sim}{\wp}_1}(u_j) + \lambda^4_{\underset{\sim}{\wp}_1}(u_j) \right)}\sqrt{\sum_{j=1}^n \left(\xi^4_{\underset{\sim}{\wp}_2}(u_j) + \eta^4_{\underset{\sim}{\wp}_2}(u_j) + \lambda^4_{\underset{\sim}{\wp}_2}(u_j) \right)}}} \tag{21.12}$$

$$= \sqrt[4]{\frac{\left(\sum_{j=1}^n \left(\xi^2_{\underset{\sim}{\wp}_1}(u_j)\,\xi^2_{\underset{\sim}{\wp}_2}(u_j) + \eta^2_{\underset{\sim}{\wp}_1}(u_j)\,\eta^2_{\underset{\sim}{\wp}_2}(u_j) + \lambda^2_{\underset{\sim}{\wp}_1}(u_j)\,\lambda^2_{\underset{\sim}{\wp}_2}(u_j) \right) \right)^2}{\sum_{j=1}^n \left(\xi^4_{\underset{\sim}{\wp}_1}(u_j) + \eta^4_{\underset{\sim}{\wp}_1}(u_j) + \lambda^4_{\underset{\sim}{\wp}_1}(u_j) \right) \sum_{j=1}^n \left(\xi^4_{\underset{\sim}{\wp}_2}(u_j) + \eta^4_{\underset{\sim}{\wp}_2}(u_j) + \lambda^4_{\underset{\sim}{\wp}_2}(u_j) \right)}}$$

Now, we present the computation example of the new method.

3.1 Computation Example to Show Validity and Superiority

Suppose that

$\underset{\sim}{\wp}_1 = \left(\frac{2}{3}, \frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)$ and $\underset{\sim}{\wp}_2 = \left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{2}{3}\right)$ are PFSs in $U = \{u_1, u_2\}$.

The hesitation margins are $\lambda_{\underset{\sim}{\wp}_1}(u_1) = \frac{1}{3}$, $\lambda_{\underset{\sim}{\wp}_1}(u_2) = \frac{2}{3}$, $\lambda_{\underset{\sim}{\wp}_2}(u_1) = \frac{2}{3}$, and $\lambda_{\underset{\sim}{\wp}_2}(u_2) = \frac{1}{3}$.

By applying Eq. (21.11), we obtain the following information:

$$\Delta\xi_1 = 0,\ \Delta\eta_1 = 0.3333,\ \Delta\lambda_1 = -0.3333,$$

$$\Delta\xi_2 = -0.3333,\ \Delta\eta_2 = 0,\ \Delta\lambda_1 = 0.3333$$

$$\Delta\xi_{\min} = -0.3333,\ \Delta\eta_{\min} = 0,\ \Delta\lambda_{\min} = -0.3333$$

$$\Delta\xi_{\max} = 0,\ \Delta\eta_{\max} = 0.3333,\ \Delta\lambda_{\max} = 0.3333$$

$$\mu_1 = 0.9,\ \nu_1 = 0.875,\ \varphi_1 = 1,\ \mu_2 = 1,\ \nu_1 = 1,\ \varphi_1 = 0.7778, \text{ for } c = 3$$

Then, for $n = 2$, we get

$$\begin{aligned}\underset{\sim}{\Omega}_1\left(\underset{\sim}{\wp}_1, \underset{\sim}{\wp}_2\right) &= \frac{1}{6}[(0.9 \times 1) + (0.875 \times 0.6667) + (1 \times 1.333) \\ &\quad + (1 \times 1.3333) + (1 \times 1) + (0.7778 \times 0.6667)] = 0.9448\end{aligned}$$

Using Eq. (21.12), we have

$$\sum_{j=1}^{n}\left(\xi^2_{\underset{\sim}{\wp}_1}(u_j)\,\xi^2_{\underset{\sim}{\wp}_2}(u_j) + \eta^2_{\underset{\sim}{\wp}_1}(u_j)\,\eta^2_{\underset{\sim}{\wp}_2}(u_j) + \lambda^2_{\underset{\sim}{\wp}_1}(u_j)\,\lambda^2_{\underset{\sim}{\wp}_2}(u_j)\right) = 0.5926$$

$$\begin{aligned}&\sum_{j=1}^{n}\left(\xi^4_{\underset{\sim}{\wp}_1}(u_j) + \eta^4_{\underset{\sim}{\wp}_1}(u_j) + \lambda^4_{\underset{\sim}{\wp}_1}(u_j)\right) \\ &= \sum_{j=1}^{n}\left(\xi^4_{\underset{\sim}{\wp}_2}(u_j) + \eta^4_{\underset{\sim}{\wp}_2}(u_j) + \lambda^4_{\underset{\sim}{\wp}_2}(u_j)\right) = 0.8148\end{aligned}$$

and so,

$$\widetilde{\Omega}_2\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)=\sqrt[4]{\frac{(0.5926)^2}{0.8148\times 0.8148}}=0.8528$$

The correlation coefficient values show that $\widetilde{\wp}_1$ and $\widetilde{\wp}_2$ are quite related. By using the other methods, we obtain the following results:

$$\Omega_1\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)=\Omega_2\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)=0.7273,\ \Omega_3\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)=\infty$$

$$\Omega_4\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)=0.9542,\ \Omega_5\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)=-1,\ \Omega_6\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)=0.7273$$

From the computation example, we notice the following:

- $\Omega_1\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)=\Omega_2\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)=\Omega_6\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)$ whenever $\sum_{j=1}^{n}\left(\xi^4_{\widetilde{\wp}_1}(u_j)+\eta^4_{\widetilde{\wp}_1}(u_j)+\lambda^4_{\widetilde{\wp}_1}(u_j)\right)$ and $\sum_{j=1}^{n}\left(\xi^4_{\widetilde{\wp}_2}(u_j)+\eta^4_{\widetilde{\wp}_2}(u_j)+\lambda^4_{\widetilde{\wp}_2}(u_j)\right)$ are equal.
- $\Omega_3\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)$and $\Omega_5\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)$ yield misleading results.
- Although $\Omega_4\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)$, which we modified as $\widetilde{\Omega}_1\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)$, yields a better result at this time, it cannot be trusted, because the hesitation margin parameter was excluded from it.
- Our new methods, $\widetilde{\Omega}_1\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)$ and $\widetilde{\Omega}_2\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)$ are the most reliable correlation coefficient measures, because they give the precise interpretation of the correlation between the considered PFSs.

3.2 Theoretical Results

Some results that validate the new methods of calculating correlation coefficient under PFSs are presented in the following section.

Proposition 3.1 Suppose $\widetilde{\wp}_1$and $\widetilde{\wp}_2$ are PFSs in$U$, then $\widetilde{\Omega}_1\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)$ satisfies the following:

(i) $\widetilde{\Omega}_1\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)=\widetilde{\Omega}_1\left(\widetilde{\wp}_2,\widetilde{\wp}_1\right)$

(ii) $\widetilde{\Omega}_1\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)=1$iff $\widetilde{\wp}_1=\widetilde{\wp}_2$

Proof Recall that

$$\widetilde{\Omega}_1\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) = \frac{1}{3n}\sum_{j=1}^{n}\left(\mu_j\left(1-\Delta\xi_j\right)+\nu_j\left(1-\Delta\eta_j\right)+\varphi_j\left(1-\Delta\lambda_j\right)\right)$$

So, we have

$$\begin{aligned}\widetilde{\Omega}_1\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) = \frac{1}{3n}\sum_{j=1}^{n}&\left(\mu_j\left(1-\left|\xi^2_{\widetilde{\wp}_1}\left(u_j\right)-\xi^2_{\widetilde{\wp}_2}\left(u_j\right)\right|\right)\right.\\ &+\nu_j\left(1-\left|\eta^2_{\widetilde{\wp}_1}\left(u_j\right)-\eta^2_{\widetilde{\wp}_2}\left(u_j\right)\right|\right)\\ &\left.+\varphi_j\left(1-\left|\lambda^2_{\widetilde{\wp}_1}\left(u_j\right)-\lambda^2_{\widetilde{\wp}_2}\left(u_j\right)\right|\right)\right)\end{aligned}$$

$$\begin{aligned}= \frac{1}{3n}\sum_{j=1}^{n}&\left(\mu_j\left(1-\left|\xi^2_{\widetilde{\wp}_2}\left(u_j\right)-\xi^2_{\widetilde{\wp}_1}\left(u_j\right)\right|\right)\right.\\ &+\nu_j\left(1-\left|\eta^2_{\widetilde{\wp}_2}\left(u_j\right)-\eta^2_{\widetilde{\wp}_1}\left(u_j\right)\right|\right)\\ &\left.+\varphi_j\left(1-\left|\lambda^2_{\widetilde{\wp}_2}\left(u_j\right)-\lambda^2_1\left(u_j\right)\right|\right)\right)\end{aligned}$$

$$= \widetilde{\Omega}_1\left(\widetilde{\wp}_2, \widetilde{\wp}_1\right)$$

which verifies (i).

Again, assume $\widetilde{\wp}_1 = \widetilde{\wp}_2$. Then

$$\left|\xi^2_{\widetilde{\wp}_1}\left(u_j\right)-\xi^2_{\widetilde{\wp}_2}\left(u_j\right)\right| = 0, \quad \left|\eta^2_{\widetilde{\wp}_1}\left(u_j\right)-\eta^2_{\widetilde{\wp}_2}\left(u_j\right)\right| = 0$$

$$\left|\lambda^2_{\widetilde{\wp}_1}\left(u_j\right)-\lambda^2_{\widetilde{\wp}_2}\left(u_j\right)\right| = 0$$

Subsequently, $\Delta\xi_{\max} = \Delta\eta_{\max} = \Delta\lambda_{\max} = 0$, $\Delta\xi_j = \Delta\eta_j = \Delta\lambda_j = 0$, $\Delta\xi_{\min} = \Delta\eta_{\min} = \Delta\lambda_{\min} = 0$, thus $\mu_j = \nu_j = \varphi_j = 1$. Hence, $\widetilde{\Omega}_1\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) = 1$.

Conversely, assume that $\widetilde{\Omega}_1\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) = 1$, then $\widetilde{\wp}_1 = \widetilde{\wp}_2$ is straightforward. Hence, (ii) is proved.

Proposition 3.2 Let $\widetilde{\wp}_1$ and $\widetilde{\wp}_2$ be PFSs in U, then $\widetilde{\Omega}_2\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right)$ satisfies the following:

(i) $\widetilde{\Omega}_2\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) = \widetilde{\Omega}_2\left(\widetilde{\wp}_2, \widetilde{\wp}_1\right)$
(ii) $\widetilde{\Omega}_2\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) = 1$ iff $\widetilde{\wp}_1 = \widetilde{\wp}_2$

Proof Recall that

$$\widetilde{\Omega}_2\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) = \frac{\sqrt{\sum_{j=1}^{n}\left(\xi^2_{\widetilde{\wp}_1}(u_j)\,\xi^2_{\widetilde{\wp}_2}(u_j) + \eta^2_{\widetilde{\wp}_1}(u_j)\,\eta^2_{\widetilde{\wp}_2}(u_j) + \lambda^2_{\widetilde{\wp}_1}(u_j)\,\lambda^2_{\widetilde{\wp}_2}(u_j)\right)}}{\sqrt{\sqrt{\sum_{j=1}^{n}\left(\xi^4_{\widetilde{\wp}_1}(u_j) + \eta^4_{\widetilde{\wp}_1}(u_j) + \lambda^4_{\widetilde{\wp}_1}(u_j)\right)}\sqrt{\sum_{j=1}^{n}\left(\xi^4_{\widetilde{\wp}_2}(u_j) + \eta^4_{\widetilde{\wp}_2}(u_j) + \lambda^4_{\widetilde{\wp}_2}(u_j)\right)}}}$$

Then it follows:

$$\widetilde{\Omega}_2\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) = \frac{\sqrt{\sum_{j=1}^{n}\left(\xi^2_{\widetilde{\wp}_1}(u_j)\,\xi^2_{\widetilde{\wp}_2}(u_j) + \eta^2_{\widetilde{\wp}_1}(u_j)\,\eta^2_{\widetilde{\wp}_2}(u_j) + \lambda^2_{\widetilde{\wp}_1}(u_j)\,\lambda^2_{\widetilde{\wp}_2}(u_j)\right)}}{\sqrt{\sqrt{\sum_{j=1}^{n}\left(\xi^4_{\widetilde{\wp}_1}(u_j) + \eta^4_{\widetilde{\wp}_1}(u_j) + \lambda^4_{\widetilde{\wp}_1}(u_j)\right)}\sqrt{\sum_{j=1}^{n}\left(\xi^4_{\widetilde{\wp}_2}(u_j) + \eta^4_{\widetilde{\wp}_2}(u_j) + \lambda^4_{\widetilde{\wp}_2}(u_j)\right)}}}$$

$$= \frac{\sqrt{\sum_{j=1}^{n}\left(\xi^2_{\widetilde{\wp}_2}(u_j)\,\xi^2_{\widetilde{\wp}_1}(u_j) + \eta^2_{\widetilde{\wp}_2}(u_j)\,\eta^2_{\widetilde{\wp}_1}(u_j) + \lambda^2_{\widetilde{\wp}_2}(u_j)\,\lambda^2_{\widetilde{\wp}_1}(u_j)\right)}}{\sqrt{\sqrt{\sum_{j=1}^{n}\left(\xi^4_{\widetilde{\wp}_2}(u_j) + \eta^4_{\widetilde{\wp}_2}(u_j) + \lambda^4_{\widetilde{\wp}_2}(u_j)\right)}\sqrt{\sum_{j=1}^{n}\left(\xi^4_{\widetilde{\wp}_1}(u_j) + \eta^4_{\widetilde{\wp}_1}(u_j) + \lambda^4_{\widetilde{\wp}_1}(u_j)\right)}}}$$

$$= \underset{\sim}{\Omega}_2\left(\underset{\sim}{\wp}_2, \underset{\sim}{\wp}_1\right),$$

which proves (i).

Suppose $\underset{\sim}{\wp}_1 = \underset{\sim}{\wp}_2$, then we have

$$\underset{\sim}{\Omega}_2\left(\underset{\sim}{\wp}_1, \underset{\sim}{\wp}_2\right) = \frac{\sqrt{\sum_{j=1}^{n}\left(\xi^4_{\underset{\sim}{\wp}_1}(u_j) + \eta^4_{\underset{\sim}{\wp}_1}(u_j) + \lambda^4_{\underset{\sim}{\wp}_1}(u_j)\right)}}{\sqrt{\sum_{j=1}^{n}\left(\xi^4_{\underset{\sim}{\wp}_1}(u_j) + \eta^4_{\underset{\sim}{\wp}_1}(u_j) + \lambda^4_{\underset{\sim}{\wp}_1}(u_j)\right)}}$$

$$= 1.$$

On the contrary, assume $\underset{\sim}{\Omega}_2\left(\underset{\sim}{\wp}_1, \underset{\sim}{\wp}_2\right) = 1$, then

$$\sqrt{\sqrt{\sum_{j=1}^{n}\left(\xi^4_{\underset{\sim}{\wp}_1}(u_j) + \eta^4_{\underset{\sim}{\wp}_1}(u_j) + \lambda^4_{\underset{\sim}{\wp}_1}(u_j)\right)}\sqrt{\sum_{j=1}^{n}\left(\xi^4_{\underset{\sim}{\wp}_2}(u_j) + \eta^4_{\underset{\sim}{\wp}_2}(u_j) + \lambda^4_{\underset{\sim}{\wp}_2}(u_j)\right)}}$$

$$= \sqrt{\sum_{j=1}^{n}\left(\xi^2_{\underset{\sim}{\wp}_1}(u_j)\,\xi^2_{\underset{\sim}{\wp}_2}(u_j) + \eta^2_{\underset{\sim}{\wp}_1}(u_j)\,\eta^2_{\underset{\sim}{\wp}_2}(u_j) + \lambda^2_{\underset{\sim}{\wp}_1}(u_j)\,\lambda^2_{\underset{\sim}{\wp}_2}(u_j)\right)} \text{ implies that}$$

$$\sqrt{\sum_{j=1}^{n}\left(\xi^4_{\underset{\sim}{\wp}_1}(u_j) + \eta^4_{\underset{\sim}{\wp}_1}(u_j) + \lambda^4_{\underset{\sim}{\wp}_1}(u_j)\right)}\sqrt{\sum_{j=1}^{n}\left(\xi^4_{\underset{\sim}{\wp}_2}(u_j) + \eta^4_{\underset{\sim}{\wp}_2}(u_j) + \lambda^4_{\underset{\sim}{\wp}_2}(u_j)\right)}$$

$$= \sum_{j=1}^{n}\left(\xi^2_{\underset{\sim}{\wp}_1}(u_j)\,\xi^2_{\underset{\sim}{\wp}_2}(u_j) + \eta^2_{\underset{\sim}{\wp}_1}(u_j)\,\eta^2_{\underset{\sim}{\wp}_2}(u_j) + \lambda^2_{\underset{\sim}{\wp}_1}(u_j)\,\lambda^2_{\underset{\sim}{\wp}_2}(u_j)\right) \text{ implies that}$$

$$\sum_{j=1}^{n}\left(\xi^4_{\widetilde{\wp}_1}(u_j)+\eta^4_{\widetilde{\wp}_1}(u_j)+\lambda^4_{\widetilde{\wp}_1}(u_j)\right)\sum_{j=1}^{n}\left(\xi^4_{\widetilde{\wp}_2}(u_j)+\eta^4_{\widetilde{\wp}_2}(u_j)+\lambda^4_{\widetilde{\wp}_2}(u_j)\right)$$
$$=\left(\sum_{j=1}^{n}\left(\xi^2_{\widetilde{\wp}_1}(u_j)\,\xi^2_{\widetilde{\wp}_2}(u_j)+\eta^2_{\widetilde{\wp}_1}(u_j)\,\eta^2_{\widetilde{\wp}_2}(u_j)+\lambda^2_{\widetilde{\wp}_1}(u_j)\,\lambda^2_{\widetilde{\wp}_2}(u_j)\right)\right)^2$$

impliesthat

$$\sum_{j=1}^{n}\left(\xi^4_{\widetilde{\wp}_1}(u_j)+\eta^4_{\widetilde{\wp}_1}(u_j)+\lambda^4_{\widetilde{\wp}_1}(u_j)\right)\left(\xi^4_{\widetilde{\wp}_2}(u_j)+\eta^4_{\widetilde{\wp}_2}(u_j)+\lambda^4_{\widetilde{\wp}_2}(u_j)\right)$$

$$=\sum_{j=1}^{n}\left(\xi^2_{\widetilde{\wp}_1}(u_j)\,\xi^2_{\widetilde{\wp}_2}(u_j)+\eta^2_{\widetilde{\wp}_1}(u_j)\,\eta^2_{\widetilde{\wp}_2}(u_j)+\lambda^2_{\widetilde{\wp}_1}(u_j)\,\lambda^2_{\widetilde{\wp}_2}(u_j)\right)^2,$$

which implies that $\widetilde{\wp}_1=\widetilde{\wp}_2$. Hence, (ii) is proved.

Theorem 3.3 Suppose $\widetilde{\Omega}_1\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)$ and $\widetilde{\Omega}_2\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)$ are correlation coefficients of PFSs $\widetilde{\wp}_1$ and $\widetilde{\wp}_2$ in U, then $\widetilde{\Omega}_1\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)$, $\widetilde{\Omega}_2\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)\in[0,1]$.

Proof We need to show that $0\le\widetilde{\Omega}_1\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)\le 1$and $0\le\widetilde{\Omega}_2\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)\le 1$, that is,

(i) $\widetilde{\Omega}_1\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)\ge 0$and $\widetilde{\Omega}_1\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)\le 1$

(ii) $\widetilde{\Omega}_2\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)\ge 0$and $\widetilde{\Omega}_2\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)\le 1$

Certainly, $\widetilde{\Omega}_1\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)\ge 0$ and $\widetilde{\Omega}_2\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)\ge 0$. Now, we show that $\widetilde{\Omega}_1\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)\le 1$ as well as $\widetilde{\Omega}_2\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)\le 1$. To establish $\widetilde{\Omega}_1\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)\le 1$, let us assume that

$$\sum_{j=1}^{n}\mu_j\left(1-\Delta\xi_j\right)=\Gamma,\ \sum_{j=1}^{n}\nu_j\left(1-\Delta\eta_j\right)=\Psi,\ \sum_{j=1}^{n}\varphi_j\left(1-\Delta\lambda_j\right)=\Omega.$$

Then

$$\widetilde{\Omega}_1\left(\widetilde{\wp}_1,\widetilde{\wp}_2\right)=\frac{1}{3n}\sum\nolimits_{j=1}^{n}\left(\mu_j\left(1-\Delta\xi_j\right)+\nu_j\left(1-\Delta\eta_j\right)+\varphi_j\left(1-\Delta\lambda_j\right)\right)$$

$$= \frac{\sum_{j=1}^{n} \mu_j \left(1 - \Delta\xi_j\right) + \sum_{j=1}^{n} \nu_j \left(1 - \Delta\eta_j\right) + \sum_{j=1}^{n} \varphi_j \left(1 - \Delta\lambda_j\right)}{3n}$$

$$= \frac{\Gamma + \Psi + \Omega}{3\text{n}}$$

Thus,

$$\widetilde{\Omega}_1\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) - 1 = \frac{\Gamma + \Psi + \Omega}{3n} - 1$$

$$= \frac{\Gamma + \Psi + \Omega - 3n}{3n}$$

$$= -\frac{(3n - \Gamma - \Psi - \Omega)}{3n} \leq 0,$$

Which proves $\widetilde{\Omega}_1\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) \leq 1$, and so , $\widetilde{\Omega}_1\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) \in [0, 1]$.

Now, we establish that $\widetilde{\Omega}_1\left(\widetilde{\wp}_1, \widetilde{\wp}_2\right) \leq 1$. Let us assume that

$$\sum_{j=1}^{n} \xi^2_{\widetilde{\wp}_1}\left(u_j\right) = \Gamma_1, \sum_{j=1}^{n} \xi^2_{\widetilde{\wp}_2}\left(u_j\right) = \Gamma_2$$

$$\sum_{j=1}^{n} \eta^2_{\widetilde{\wp}_1}\left(u_j\right) = \Psi_1, \sum_{j=1}^{n} \eta^2_{\widetilde{\wp}_2}\left(u_j\right) = \Psi_2$$

$$\sum_{j=1}^{n} \lambda^2_{\widetilde{\wp}_1}\left(u_j\right) = \Omega_1, \sum_{j=1}^{n} \lambda^2_{\widetilde{\wp}_2}\left(u_j\right) = \Omega_2$$

Then

$$\tilde{\Omega}_2\left(\tilde{\wp}_1,\tilde{\wp}_2\right)=\frac{\sqrt{\begin{array}{l}\sum_{j=1}^{n}\left(\xi^2_{\tilde{\wp}_1}\left(u_j\right)\xi^2_{\tilde{\wp}_2}\left(u_j\right)+\eta^2_{\tilde{\wp}_1}\left(u_j\right)\eta^2_{\tilde{\wp}_2}\left(u_j\right)\right.\\ \quad\left.+\lambda^2_{\tilde{\wp}_1}\left(u_j\right)\lambda^2_{\tilde{\wp}_2}\left(u_j\right)\right)\end{array}}}{\sqrt{\begin{array}{l}\sqrt{\sum_{j=1}^{n}\left(\xi^4_{\tilde{\wp}_1}\left(u_j\right)+\eta^4_{\tilde{\wp}_1}\left(u_j\right)+\lambda^4_{\tilde{\wp}_1}\left(u_j\right)\right)}\\ \quad\sqrt{\sum_{j=1}^{n}\left(\xi^4_{\tilde{\wp}_2}\left(u_j\right)+\eta^4_{\tilde{\wp}_2}\left(u_j\right)+\lambda^4_{\tilde{\wp}_2}\left(u_j\right)\right)}\end{array}}}$$

$$=\frac{\sqrt{\Gamma_1\Gamma_2+\Psi_1\Psi_2+\Omega_1\Omega_2}}{\sqrt{\sqrt{\Gamma_1^2+\Psi_1^2+\Omega_1^2}\sqrt{\Gamma_2^2+\Psi_2^2+\Omega_2^2}}}$$

$\tilde{\Omega}_2^2\left(\tilde{\wp}_1,\tilde{\wp}_2\right)=\frac{\Gamma_1\Gamma_2+\Psi_1\Psi_2+\Omega_1\Omega_2}{\sqrt{\Gamma_1^2+\Psi_1^2+\Omega_1^2}\sqrt{\Gamma_2^2+\Psi_2^2+\Omega_2^2}}$, and so

$$\tilde{\Omega}_2^2\left(\tilde{\wp}_1,\tilde{\wp}_2\right)-1=\frac{\Gamma_1\Gamma_2+\Psi_1\Psi_2+\Omega_1\Omega_2}{\sqrt{\Gamma_1^2+\Psi_1^2+\Omega_1^2}\sqrt{\Gamma_2^2+\Psi_2^2+\Omega_2^2}}-1$$

$$=\frac{\Gamma_1\Gamma_2+\Psi_1\Psi_2+\Omega_1\Omega_2-\left(\sqrt{\Gamma_1^2+\Psi_1^2+\Omega_1^2}\sqrt{\Gamma_2^2+\Psi_2^2+\Omega_2^2}\right)}{\sqrt{\Gamma_1^2+\Psi_1^2+\Omega_1^2}\sqrt{\Gamma_2^2+\Psi_2^2+\Omega_2^2}}$$

$$=-\frac{\sqrt{\Gamma_1^2+\Psi_1^2+\Omega_1^2}\sqrt{\Gamma_2^2+\Psi_2^2+\Omega_2^2}-(\Gamma_1\Gamma_2+\Psi_1\Psi_2+\Omega_1\Omega_2)}{\sqrt{\Gamma_1^2+\Psi_1^2+\Omega_1^2}\sqrt{\Gamma_2^2+\Psi_2^2+\Omega_2^2}}$$

$$\leq 0$$

Thus, $\tilde{\Omega}_2^2\left(\tilde{\wp}_1,\tilde{\wp}_2\right)-1\leq 0$, and so $\tilde{\Omega}_2\left(\tilde{\wp}_1,\tilde{\wp}_2\right)\leq 1$. Hence, $\tilde{\Omega}_2\left(\tilde{\wp}_1,\tilde{\wp}_2\right)\in[0,1]$.

4 Numerical Applications

The section discusses decision-making process based on the two Pythagorean fuzzy correlation coefficient methods and juxtaposes their effectiveness with the existing Pythagorean fuzzy correlation coefficient methods using the principle of recognition. The applications are in the areas of disaster control and medical diagnosis, respectively.

4.1 Application in Disaster Control

Disaster management is the art of managing resources and responsibilities to deal with disasters by preventing or reducing the destructive effects of disasters. Disaster management consists of preparation, reaction, mitigation, preventive control, and rescue. In short, disaster management is aimed at preventing and reducing the damaging effects of disasters. Most often, the occurrence of disaster is indeterminate, and as such, adopting indeterminate methodologies to check-mate disaster is necessary. We deployed Pythagorean fuzzy correlation coefficient to mitigate disaster, because PFS is one of the verified concepts for curbing uncertainties. To determine the control, an assumption that an emergency agent made emergency rescue planning to control disasters is presented whose data are encapsulated in Pythagorean fuzzy context. The rescue emergency staff is obliged to obtain the situation data after a disaster occurred, and afterward compare with the existing Pythagorean fuzzy disaster data to identify the suitably match with the current disaster. The matching process is carried out by deploying the Pythagorean fuzzy correlation coefficient methods to estimate the matching of the current disaster with the Pythagorean fuzzy data of the disaster rescue planning.

Suppose there are three hypothetical situations of disaster rescue planning represented by Pythagorean fuzzy data D_1, D_2, and D_3 described by means of the set$U = \{$rescue difficulty, scale of people affected, traffic conditions, and emergency supplies$\}$. The current disaster, which is to be controlled, is denoted by nD and described by Pythagorean fuzzy data defined by the elements of U. The Pythagorean fuzzy disaster data can be seen in Table 21.1.

We now find which of the existing disasters can be suitably matched with the current disaster by computing their correlation coefficients based on the existing

Table 21.1 Pythagorean fuzzy disaster information

Disasters	Rescue difficulty	Scale of people affected	Traffic conditions	Emergency supplies
D_1	$(0.8, 0.55)$	$(0.7, 0.0)$	$(0.7, 0.2)$	$(0.7, 0.2)$
D_2	$(0.4, 0.45)$	$(0.5, 0.3)$	$(0.4, 0.5)$	$(0.5, 0.4)$
D_3	$(0.6, 0.3)$	$(0.6, 0.2)$	$(0.7, 0.2)$	$(0.8, 0.0)$
nD	$(0.5, 0.3)$	$(0.6, 0.2)$	$(0.5, 0.4)$	$(0.7, 0.0)$

methods and the new developed methods. The correlation coefficient values can be seen in Table 21.2.

From the correlation coefficient values, the current disaster can be matched with the disasters D_2 and D_3, respectively. Hence, the current disaster can be managed and controlled with the same approaches used to control disasters $D_2$and D_3. With this principle of recognition, it is certain that the current disaster has no similarity with disaster D_1, and so, the approaches used to manage D_1 cannot be adopted for the current disaster nD.

The methods in [10, 22] and the new developed methods are the only methods that incorporate the complete parameters of PFSs, and so, their outputs are more reliable when compared to the methods given in [5, 37, 40], which do not incorporate the complete parameters of PFSs. Besides, the newly developed methods, especially the second method, $\widetilde{\Omega}_2$, produce the best correlation coefficient.

4.2 Application in Medical Diagnosis

Suppose a patient P is sick and is showing some symptoms like high temperature, headache, stomach pain, cough, and chest pain. From the symptoms, it is suspected that the patient is infected by at least one of the diseases in the set $D = \{$viral fever, malaria fever, typhoid fever, peptic ulcer, and chest problem$\}$. The medical information of the patient and the suspected diseases are captured by PFSs, which we called Pythagorean fuzzy medical information data. Table 21.3 contains the Pythagorean fuzzy medical information data for the patient and diseases defined by the symptoms.

Now, we spot which of the diseases has the greatest correlation with the patient to determine the medical diagnosis. In doing this, the Pythagorean fuzzy correlation coefficient methods are deployed and the results are given in Table 21.4.

From the results given in Table 21.4, the patient P is mainly diagnosed with malaria fever, viral fever, and typhoid fever in that order. The diagnosis shows that malaria fever, viral fever, and typhoid fever are related diseases in agreement to real-life medical practice. Again, the newly developed methods take the three parameters

Table 21.2 Correlation coefficient values for the disasters

Methods	(D_1, nD)	(D_2, nD)	(D_3, nD)
Ω_1 [22]	0.8249	0.9618	0.9791
Ω_2 [22]	0.8241	0.9412	0.8931
Ω_3 [5]	−0.3062	0.7539	0.6849
Ω_4 [37]	0.8185	0.8678	0.8994
Ω_5 [40]	0.3851	0.7951	0.6174
Ω_6 [10]	0.8249	0.9616	0.9750
$\widetilde{\Omega}_1$	0.7960	0.8940	0.8919
$\widetilde{\Omega}_2$	0.9083	0.9807	0.9895

Table 21.3 Pythagorean fuzzy medical information data

Patient/diseases	Temperature	Headache	Stomach pain	Cough	Chest pain
Viral fever	(0.4, 0.0)	(0.3, 0.5)	(0.1, 0.7)	(0.4, 0.3)	(0.1, 0.7)
Malaria fever	(0.7, 0.0)	(0.2, 0.6)	(0.0, 0.9)	(0.7, 0.0)	(0.1, 0.8)
Typhoid fever	(0.3, 0.3)	(0.6, 0.1)	(0.2, 0.7)	(0.2, 0.6)	(0.1, 0.9)
Peptic ulcer	(0.1, 0.7)	(0.2, 0.4)	(0.8, 0.0)	(0.2, 0.7)	(0.2, 0.7)
Chest problem	(0.1, 0.8)	(0.0, 0.8)	(0.2, 0.8)	(0.2, 0.8)	(0.8, 0.1)
Patient	(0.8, 0.1)	(0.6, 0.1)	(0.2, 0.8)	(0.6, 0.1)	(0.1, 0.6)

Table 21.4 Correlation coefficient values for diagnostic analysis

Methods	(P, Viral fever)	(P, Malaria fever)	(P, Typhoid fever)	(P, Peptic ulcer)	(P, Chest problem)
Ω_1 [22]	0.8622	0.9047	0.7808	0.6233	0.5080
Ω_2 [22]	0.8328	0.8895	07485	0.6229	0.5075
Ω_3 [5]	0.5773	0.7433	0.2418	−0.0989	−0.1211
Ω_4 [37]	0.9261	0.9300	0.9579	0.7401	0.6869
Ω_5 [40]	0.9361	0.8792	0.7101	−0.6615	−0.4716
Ω_6 [10]	0.8617	0.9046	0.7801	0.6233	0.5080
$\widetilde{\Omega}_1$	0.9316	0.9221	0.9368	0.7853	0.7527
$\widetilde{\Omega}_2$	0.9285	0.9512	0.8836	0.7895	0.7127

of PFSs into account, and so give the most accurate results when compared to the existing methods that also take into account the three parameters of PFSs.

5 Conclusion

In this chapter, we have studied correlation coefficient under PFSs because the concept of correlation coefficient is quite applicable in real-world decision-making. Many authors have presented some methods of correlation coefficient under PFSs, despite some in-built limitations, such as accuracy and reliability. Based on these setbacks, we have presented two methods for calculating correlation coefficient under PFSs, and explicated their properties in consonant to the attributes of the classical correlation coefficient. In addition, the applications of the new methods in real-world problems like disaster control and medical diagnosis were discussed using Pythagorean fuzzy data. Finally, the justification of the new methods was portrayed in comparative analysis involving other methods of correlation coefficient under PFSs, and it was shown that the new developed methods are more reliable when compared to the other methods of calculating correlation coefficient under PFSs. These methods can be applied to study MCDM problems in future under Fermatean fuzzy sets, q-rung orthopair fuzzy sets, cubic m-polar fuzzy sets, linear Diophantine fuzzy soft sets, cubic bipolar fuzzy sets, and m-polar spherical fuzzy sets.

Acknowledgments We express our sincere gratitude to the anonymous reviewers who took time to improve the technical quality of this chapter.

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Chapter 22
Multi-Criteria Group Decision-Making q-Rung Neutrosophic Interval-Valued Soft Set TOPSIS Aggregating Operator for the Selection of Diagnostic Health Imaging

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1 Introduction

Ambiguity is present in nearly all real-world problems. Zadeh [31] has proposed the fuzzy set (FS) as a way of dealing with uncertainties, Atanassov [4] has proposed the intuitionistic FS (IFS), Yager [29] has proposed Pythagorean FS (PFS), and Smarandache [27] proposed the neutrosophic set (NSS). There is an FS in which each element in the set has a membership value corresponding to its level of belongingness, with grades corresponding to these levels. Natural language processing, artificial intelligence, handwriting recognition, and speech recognition are all examples of applications that benefit from using this gradation method. Similar logic was later proposed by Atanassov [4], in which the total MD and NMD values should equal ≤ 1, this as an IFS logic. In the case of MD and NMD sums that exceed 1, we might have difficulty decision-making (DM). For generalizing IFS, Yager [29] has proposed PFS logic, which requires the square total of its MD and NMD to be equal to ≤ 1. Several applications based on PyFS are discussed by Akram et al. [1–3]. Rahman et al. [24] discuss the use of an IVPFS for geometric AOs within the context of a group DM. According to Peng et al. [22], a PyFS based on AO with interval values is recommended. Rahman et al. [25] propose a few MCGDM methods using interval-valued Pythagorean fuzzy Einstein AOs. Multi-

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C. Jana et al. (eds.), *Fuzzy Optimization, Decision-making and Operations Research*, https://doi.org/10.1007/978-3-031-35668-1_22

attribute decision-making (MADM) with IVPyFS was developed by Yang et al. [30]. The square root FS (SRFS) and its weighted AOs were explored by Shami et al. in DM. The q-Rung orthopair FS (q-Rung OFS) were developed by Yager [28] through an expansion of PyFSs. The result of the qth power of MD and qth power of NMD lies between 0 and 1. Since q=1, q-Rung OFS transforms into IFSs, and since q=2, q-Rung OFS transforms into PyFSs; q-Rung OFSs are extensions of IFSs and PyFSs. q-Rung IVFSs have been discussed by Bhagawati [5] et al. Yager [28] introduced the notion of generalized OFSs.

Recent research by Smarandache led to the development of the NSS [27]. This neutrality is referred to as neurosophy, and it is this neutrality that distinguishes FS from IFS. According to the truth degree (TD), the indeterminacy degree (ID), and the false degree (FD). NSS has levels of TD, ID, and FD for every component of the universe that are between 0 and 1. Historically, a classical set, a functional set, an integral set, etc. are generalized by an NSS. Pythagorean NSIV set (PyNSIVS) was introduced for the first time by Smarandache et al. [9]. Medical diagnostics and context analysis are applied to the single-valued NSS [26]. According to Ejegwa [6], distance measures for IFSs are hamming distances (HDs), Euclidean distances (EDs), and normalized Euclidean distances (NEDs), which are common in PyFSs and are examined for use in MCDM and MADM. There are a number of distance functions for PyNSNIVSs presented by Palanikumar et al. [13]. Yang et al. [32] advocated generalizing PyFS against TOPSIS by including MCDM, whereas Peng et al. [23] discussed neutrosophic MADM under MABAC and TOPSIS. In recent year, an approach to Dombi aggregation mappings based on MCDM has been presented by Jana et al. [8]. In several studies [14–21], Palanikumar et al. studied many algebraic structures and its applications.

As a result of Molodtsov's work [12], the theory of soft sets (SS) was developed. A soft set represents real-world DM more accurately than other uncertain theories in terms of objectivity and complexity. In addition, SSs can be integrated with other mathematical models as a valuable research topic. Typically, these concepts are identified as fuzzy soft sets (FSS) [10] and intuitionistic fuzzy soft sets (IFSS) [11]. There are a number of DM problems that can be addressed with these two theories. Adeel et al. discussed fuzzy linguistic TOPSIS using m-polar attributes in 2019, while Eraslan et al. discussed TOPSIS-based GDMs [7]. The concept of single-valued neutrosophic MADM based on MABAC and TOPSIS was proposed by Zhang et al. [32] and Peng et al. [23]. According to Zulqarnain et al. in 2021, TOPSIS can be extended to interval-valued IFSS (IVIFSS). Using TOPSIS, distances to positive ideal solutions (PIS) and negative ideal solutions (NIS) are calculated, and a preference order is found based on the relative closeness of the two distance measures.

Consequently, this work makes the following contributions:

1. TOPSIS q-Rung NSIVSS introduced a new ED measure.
2. An application of the proposed definition for NSIVSS with q-Rungs.
3. According to the q-Rung NSIVSS, PIS and NISs are determined.
4. A decision is made based on q for arriving at a result.

The organization of this article extends the concept of TOPSIS-q-Rung NSIVSS for MCGDM methods. Five sections are presented in the paper. The introduction is found in Sect. 1. An overview of q-Rung FS and q-Rung IVFS is provided in Sect. 2. Section 3 discusses about MCGDM using q-Rung NSIVSS algorithm. Section 4 talks MCGDM based on q-Rung NSIVSS-TOPSIS aggregating operator. Section 5 discusses about the comparison for the q-Rung NSIVSS-TOPSIS and existing approach. Lastly, Sect. 6 provides a conclusion.

2 Preliminary

The purpose of this part is to review some ideas related to NSS and q-Rung FS literary concepts.

Definition 1 An NSS H in the universe $\mathscr{U}$ is $H = \{\mho, \eth^{T}_{H}(\mho), \eth^{I}_{H}(\mho), \eth^{F}_{H}(\mho) | \mho \in \mathscr{U}\}$, where $\eth^{T}_{H}(\mho)$, $\eth^{I}_{H}(\mho)$, and $\eth^{F}_{H}(\mho)$ represent the TD, ID, and FD of H, respectively. Then $\eth^{T}_{H}, \eth^{I}_{H}, \eth^{F}_{H} : \mathscr{U} \to [0, 1]$ and $0 \preceq \sup \eth^{T}_{H}(\mho) + \sup \eth^{I}_{H}(\mho) + \sup \eth^{F}_{H}(\mho) \preceq 3$.

Definition 2 ([28]) The q-Rung FS H in $\mathscr{U}$ is $H = \left\{\mho, \langle \eth^{\mathscr{T}}_{H}(\mho), \eth^{\mathscr{F}}_{H}(\mho)\rangle \middle| \mho \in \mathscr{U}\right\}$, $\eth^{\mathscr{T}}_{H} : \mathscr{U} \to [0, 1]$ and $\eth^{\mathscr{F}}_{H} : \mathscr{U} \to [0, 1]$ denote the MD and NMD of $\mho \in \mathscr{U}$ to H, respectively, and $0 \preceq (\eth^{\mathscr{T}}_{H}(\mho))^{q} + (\eth^{\mathscr{F}}_{H}(\mho))^{q} \preceq 1$, where $q \succeq 1$. The degree of indeterminacy is $\pi(\mho) = \left((\eth^{\mathscr{T}}_{H}(\mho))^{q} + (\eth^{\mathscr{F}}_{H}(\mho))^{q} - (\eth^{\mathscr{T}}_{H}(\mho))^{q}(\eth^{\mathscr{F}}_{H}(\mho))^{q}\right)^{1/q}$. $H = \langle \eth^{\mathscr{T}}_{H}, \eth^{\mathscr{F}}_{H}\rangle$ is called a q-Rung FN.

Definition 3 ([5]) The q-Rung IVFS H in $\mathscr{U}$ is $H = \left\{\mho, \left\langle \widetilde{\eth^{\mathscr{T}}_{H}}(\mho), \widetilde{\eth^{\mathscr{F}}_{H}}(\mho)\right\rangle \middle| \mho \in \mathscr{U}\right\}$, where $\widetilde{\eth^{T}_{H}} : \mathscr{U} \to Int([0, 1])$ and $\widetilde{\eth^{F}_{H}} : \mathscr{U} \to Int([0, 1])$ denote the MD and NMD of $\mho \in \mathscr{U}$ to H, respectively, and $0 \preceq (\eth^{\mathscr{T}+}_{H}(\mho))^{q} + (\eth^{\mathscr{F}+}_{H}(\mho))^{q} \preceq 1$. For convenience, $H = \left\langle \left[\eth^{\mathscr{T}-}_{H}, \eth^{\mathscr{T}+}_{H}\right], \left[\eth^{\mathscr{F}-}_{H}, \eth^{\mathscr{F}+}_{H}\right]\right\rangle$ is called a q-Rung IVFN.

Definition 4 ([9]) A PyNSS H in $\mathscr{U}$ is $H = \{\mho, \eth^{T}_{H}(\mho), \eth^{I}_{H}(\mho), \eth^{F}_{H}(\mho) | \mho \in \mathscr{U}\}$, where $\eth^{T}_{H}(\mho)$, $\eth^{I}_{H}(\mho)$, and $\eth^{F}_{H}(\mho)$ represent the TD, ID, and FD of H, respectively. The mapping $\eth^{T}_{H}, \eth^{I}_{H}, \eth^{F}_{H} : \mathscr{U} \to [0, 1]$ and $0 \preceq (\eth^{T}_{H}(\mho))^{2} + (\eth^{I}_{H}(\mho))^{2} + (\eth^{F}_{H}(\mho))^{2} \preceq 2$. Since $H = \langle \eth^{T}_{H}, \eth^{I}_{H}, \eth^{F}_{H}\rangle$ is called a Pythagorean neutrosophic number(PyNSN).

Definition 5 The PyIVFS $H = \left\{\mho, \left\langle \widetilde{\eth^{T}_{H}}(\mho), \widetilde{\eth^{F}_{H}}(\mho)\right\rangle \middle| \mho \in \mathscr{U}\right\}$, where $\widetilde{\eth^{T}_{H}}(\mho) = \left[\eth^{TL}_{H}(\mho), \eth^{TU}_{H}(\mho)\right]$ and $\widetilde{\eth^{F}_{H}}(\mho) = \left[\eth^{FL}_{H}(\mho), \eth^{FU}_{H}(\mho)\right]$ denote the MD and NMD of H, respectively. Here, $\widetilde{\eth^{T}_{H}}$ and $\widetilde{\eth^{F}_{H}}$ are function from $\mathscr{U}$ into $\mathscr{D}[0, 1]$ and

$0 \le (\widetilde{\eth^{T}_{H}}(\mho))^2 + (\widetilde{\eth^{F}_{H}}(\mho))^2 \le 1$, and it is observed that $0 \le (\eth^{TU}_{H}(\mho))^2 + (\eth^{FU}_{H}(\mho))^2 \le 1$.

Definition 6 The IVNSS $H = \left\{\mho, \left(\widetilde{\eth^{T}_{H}}(\mho), \widetilde{\eth^{I}_{H}}(\mho), \widetilde{\eth^{F}_{H}}(\mho)\right) | \mho \in \mathscr{U}\right\}$, where $\widetilde{\eth^{T}_{H}}(\mho) = \left[\eth^{TL}_{H}(\mho), \eth^{TU}_{H}(\mho)\right]$, $\widetilde{\eth^{I}_{H}}(\mho) = \left[\eth^{IL}_{H}(\mho), \eth^{IU}_{H}(\mho)\right]$, and $\widetilde{\eth^{F}_{H}}(\mho) = \left[\eth^{FL}_{H}(\mho), \eth^{FU}_{H}(\mho)\right]$ represent the TD, ID, and FD of H, respectively. Then $\widetilde{\eth^{T}_{H}} : \mathscr{U} \to D[0,1]$, $\widetilde{\eth^{I}_{H}} : \mathscr{U} \to D[0,1]$, $\widetilde{\eth^{F}_{H}} : \mathscr{U} \to D[0,1]$, and $0 \le (\widetilde{\eth^{T}_{H}}(\mho))^2 + (\widetilde{\eth^{I}_{H}}(\mho))^2 + (\widetilde{\eth^{F}_{H}}(\mho))^2 \le 2$ mean $0 \le (\eth^{TU}_{H}(\mho))^2 + (\eth^{IU}_{H}(\mho))^2 + (\eth^{FU}_{H}(\mho))^2 \le 2$.

Here, $\widetilde{H} = \left(\left[\eth^{TL}_{H}, \eth^{TU}_{H}\right], \left[\eth^{IL}_{H}, \eth^{IU}_{H}\right], \left[\eth^{FL}_{H}, \eth^{FU}_{H}\right]\right)$ is called a neutrosophic interval-valued number (NSIVN).

Definition 7 Let E be the set of parameter. The $\widetilde{(\Gamma, H)}$ or $\widetilde{\Gamma_H}$ is called an NSIVS on $\mathscr{U}$ if $H \sqsubseteq E$ and $\Gamma : H \to NSIV^{\mathscr{U}}$, where $NSIV^{\mathscr{U}}$ denotes the set of all neutrosophic interval-valued subsets of $\mathscr{U}$. That is, $\widetilde{\Gamma_H} = \left\{\left(e, \left\{\frac{\mho}{\left(\left[\eth^{TL}_{\Gamma_H}(\mho), \eth^{TU}_{\Gamma_H}(\mho)\right], \left[\eth^{IL}_{\Gamma_H}(\mho), \eth^{IU}_{\Gamma_H}(\mho)\right], \left[\eth^{FL}_{\Gamma_H}(\mho), \eth^{FU}_{\Gamma_H}(\mho)\right]\right)}\right\}\right) : e \in H, \mho \in \mathscr{U}\right\}$.

Remark 1 If we write $\widetilde{\alpha_{ij}} = \widetilde{\eth^{T}_{\Gamma_H}}(e_j)(\mho_i)$ and $\widetilde{\beta_{ij}} = \widetilde{\eth^{I}_{\Gamma_H}}(e_j)(\mho_i)$ and $\widetilde{\gamma_{ij}} = \widetilde{\eth^{F}_{\Gamma_H}}(e_j)(\mho_i)$, where $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$, then the NSIV set $\widetilde{\Gamma_H}$ defined in matrix form is

$$\begin{aligned}\widetilde{\Gamma_H} &= [(\widetilde{\alpha_{ij}}, \widetilde{\beta_{ij}}, \widetilde{\gamma_{ij}})]_{m\times n} \\ &= \begin{bmatrix} (\widetilde{\alpha_{11}}, \widetilde{\beta_{11}}, \widetilde{\gamma_{11}}) & (\widetilde{\alpha_{12}}, \widetilde{\beta_{12}}, \widetilde{\gamma_{12}}) & \dots & (\widetilde{\alpha_{1n}}, \widetilde{\beta_{1n}}, \widetilde{\gamma_{1n}}) \\ (\widetilde{\alpha_{21}}, \widetilde{\beta_{21}}, \widetilde{\gamma_{21}}) & (\widetilde{\alpha_{22}}, \widetilde{\beta_{22}}, \widetilde{\gamma_{22}}) & \dots & (\widetilde{\alpha_{2n}}, \widetilde{\beta_{2n}}, \widetilde{\gamma_{2n}}) \\ \vdots & \vdots & \ddots & \vdots \\ (\widetilde{\alpha_{m1}}, \widetilde{\beta_{m1}}, \widetilde{\gamma_{m1}}) & (\widetilde{\alpha_{m2}}, \widetilde{\beta_{m2}}, \widetilde{\gamma_{m2}}) & \dots & (\widetilde{\alpha_{mn}}, \widetilde{\beta_{mn}}, \widetilde{\gamma_{mn}}) \end{bmatrix}\end{aligned}$$

This matrix is called neutrosophic interval-valued soft matrix (NSIVSM).

3 MCGDM Based on q-Rung NSIVSS-TOPSIS Aggregating Operator

Definition 8 The cardinal set of the q-Rung NSIVSS $\widetilde{\Gamma_X}$ is denoted by $c\widetilde{\Gamma_X}$ and is defined as

$\widetilde{c\Gamma_X} = \left\{ \frac{e}{\left(\left[\eth^{TL}_{c\theta_X}(e), \eth^{TU}_{c\theta_X}(e)\right], \left[\eth^{IL}_{c\xi_X}(e), \eth^{IU}_{c\xi_X}(e)\right], \left[\eth^{FL}_{c\varphi_X}(e), \eth^{FU}_{c\varphi_X}(e)\right]\right)} : e \in E \right\} = \left\{ \frac{e}{\left(\widetilde{\eth^T}_{c\theta_X}(e), \widetilde{\eth^I}_{c\xi_X}(e), \widetilde{\eth^F}_{c\varphi_X}(e)\right)} : e \in E \right\}$, where $\widetilde{\eth^T}_{c\theta_X}$, $\widetilde{\eth^I}_{c\xi_X}$, and $\widetilde{\eth^F}_{c\varphi_X}$: $E \rightarrow D[0, 1]$ are mapping, respectively; where $\widetilde{\eth^T}_{c\theta_X}(e) = \frac{|\widetilde{\theta}_X(e)|}{|\mathscr{U}|}$, $\widetilde{\eth^I}_{c\xi_X}(e) = \frac{|\widetilde{\xi}_X(e)|}{|\mathscr{U}|}$, and $\widetilde{\eth^F}_{c\varphi_X}(e) = \frac{|\widetilde{\varphi}_X(e)|}{|\mathscr{U}|}$; and where $|\widetilde{\theta}_X(e)|$, $|\widetilde{\xi}_X(e)|$ and $|\widetilde{\varphi}_X(e)|$ denote the scalar cardinalities of the q-Rung NSIVSS $\widetilde{\theta}_X(e)$, $\widetilde{\xi}_X(e)$ and $\widetilde{\varphi}_X(e)$, respectively and $|\mathscr{U}|$ represents cardinality of the universe $\mathscr{U}$. The collection of all cardinal sets of q-Rung NSIVSS of $\mathscr{U}$ is represented as $c\text{q}-Rung\ NSIV^{\mathscr{U}}$. If $X \subseteq E = \{e_i : i = 1, 2, ..., n\}$, then $\widetilde{c\Gamma_X} \in c\text{q}-Rung\ NSIVS^{\mathscr{U}}$ may be represented in a matrix form as $\left[\left([\alpha^L_{1j}, \alpha^U_{1j}], [\beta^L_{1j}, \beta^U_{1j}], [\gamma^L_{1j}, \gamma^U_{1j}]\right)\right]_{1\times n} = \left[\left([\alpha^L_{11}, \alpha^U_{11}], [\beta^L_{11}, \beta^U_{11}], [\gamma^L_{11}, \gamma^U_{11}]\right),\right.$ $\left([\alpha^L_{12}, \alpha^U_{12}], [\beta^L_{12}, \beta^U_{12}], [\gamma^L_{12}, \gamma^U_{12}]\right), \ldots, \left.\left([\alpha^L_{1n}, \alpha^U_{1n}], [\beta^L_{1n}, \beta^U_{1n}], [\gamma^L_{1n}, \gamma^U_{1n}]\right)\right]$, where $\left([\alpha^L_{1j}, \alpha^U_{1j}], [\beta^L_{1j}, \beta^U_{1j}], [\gamma^L_{1j}, \gamma^U_{1j}]\right) = \left[\mu^L_{c\Gamma_X}(e_j), \mu^U_{c\Gamma_X}(e_j)\right]$, for $j = 1, 2, \ldots, n$. For our convenience, consider matrix form as $[(\widetilde{\alpha}_{1j}, \widetilde{\beta}_{1j}, \widetilde{\gamma}_{1j})]_{1\times n} = [(\widetilde{\alpha}_{11}, \widetilde{\beta}_{11}, \widetilde{\gamma}_{11}), (\widetilde{\alpha}_{12}, \widetilde{\beta}_{12}, \widetilde{\gamma}_{12}), \ldots, (\widetilde{\alpha}_{1n}, \widetilde{\beta}_{1n}, \widetilde{\gamma}_{1n})]$, where $(\widetilde{\alpha}_{1j}, \widetilde{\beta}_{1j}, \widetilde{\gamma}_{1j}) = \widetilde{\mu}_{c\Gamma_X}(e_j)$, for $j = 1, 2, ..., n$. Hence, this matrix is called a cardinal matrix of $\widetilde{c\Gamma_X}$ of E.

Definition 9 Let $\widetilde{\Gamma_X} \in q$-Rung NSIVSS$(\mathscr{U})$ and $\widetilde{c\Gamma_X} \in cq$-Rung NSIVSS$(\mathscr{U})$. The q-Rung NSIVSS AO $q - Rung\ NSIVSS_{agg} : cq - Rung\ NSIVSS^{\mathscr{U}} \times q - RungNSIVSS(\mathscr{U}) \rightarrow q - RungNSIVSS(\mathscr{U}, E)$ is defined as $q\text{-}Rung\ NSIVSS_{agg}(\widetilde{c\Gamma_X}, \widetilde{\Gamma_X}) = \left\{ \frac{\mho}{\widetilde{\mu}_{\Gamma^*_X}(\mho)} : \mho \in \mathscr{U} \right\} = \left\{ \frac{\mho}{\left(\widetilde{\eth^T}_{\theta^*_X}(\mho), \widetilde{\eth^I}_{\xi^*_X}(\mho), \widetilde{\eth^F}_{\varphi^*_X}(\mho)\right)} : \mho \in \mathscr{U} \right\}$.

This collection is called $q - RungNSIVSSset\widetilde{\Gamma_X}$.

The TD $\widetilde{\eth^T}_{\theta^*_X}(\mho)$:$\mathscr{U} \rightarrow D[0, 1]$ by $\widetilde{\eth^T}_{\theta^*_X}(\mho) = \frac{1}{|E|}\sum_{e\in E}\left(\widetilde{\eth^T}_{c\theta_X}(e), \widetilde{\eth^T}_{\theta_X}(e)\right)(\mho)$, ID $\widetilde{\eth^I}_{\xi^*_X}(\mho) : \mathscr{U} \rightarrow D[0, 1]$ by $\widetilde{\eth^I}_{\xi^*_X}(\mho) = \frac{1}{|E|}\sum_{e\in E}\left(\widetilde{\eth^I}_{c\xi_X}(e), \widetilde{\eth^I}_{\xi_X}(e)\right)(\mho)$, and FD $\widetilde{\eth^F}_{\varphi^*_X}(\mho) : \mathscr{U} \rightarrow D[0, 1]$ by $\widetilde{\eth^F}_{\varphi^*_X}(\mho) = \frac{1}{|E|}\sum_{e\in E}\left(\widetilde{\eth^F}_{c\varphi_X}(e), \widetilde{\eth^F}_{\varphi_X}(e)\right)(\mho)$. The set $q - RungNSIVSS_{agg}(\widetilde{c\Gamma_X}, \widetilde{\Gamma_X})$ is expressed in a matrix form as

$$\left[\left([\alpha^L_{i1}, \alpha^U_{i1}], [\beta^L_{i1}, \beta^U_{i1}], [\gamma^L_{i1}, \gamma^U_{i1}]\right)\right]_{m\times 1} = \begin{bmatrix} \left([\alpha^L_{11}, \alpha^U_{11}], [\beta^L_{11}, \beta^U_{11}], [\gamma^L_{11}, \gamma^U_{11}]\right) \\ \left([\alpha^L_{21}, \alpha^U_{21}], [\beta^L_{21}, \beta^U_{21}], [\gamma^L_{21}, \gamma^U_{21}]\right) \\ \vdots \\ \left([\alpha^L_{m1}, \alpha^U_{m1}], [\beta^L_{m1}, \beta^U_{m1}], [\gamma^L_{m1}, \gamma^U_{m1}]\right) \end{bmatrix}$$

where $\left[\left([\alpha_{i1}^{L}, \alpha_{i1}^{U}], [\beta_{i1}^{L}, \beta_{i1}^{U}], [\gamma_{i1}^{L}, \gamma_{i1}^{U}]\right)\right] = \left[\mu_{\Gamma_X^*}^{L}(\mho_i), \mu_{\Gamma_X^*}^{U}(\mho_i)\right]$, for $i = 1, 2, ..., m$. This matrix is called q-Rung NSIVSS aggregate matrix of $q-RungNSIVSS_{agg}(\widetilde{c\Gamma_X}, \widetilde{\Gamma_X})$ over $\mathscr{U}$.

Algorithm-IV (q-Rung NSIVSS-TOPSIS)

Step 1: Suppose that $\mathscr{D} = \{\mathscr{D}_i : i \in \mathbb{N}\}$ is a set of decision-makers, $y = \{y_i : i \in \mathbb{N}\}$ is a set of alternatives, and $D = \{e_i : i \in \mathbb{N}\}$ is a set of parameters.

Step 2: Determine the weighted parameter by $\mathscr{P} = \left[w_{ij}^{L}, w_{ij}^{U}\right]_{n\times m}$, where $\left[w_{ij}^{L}, w_{ij}^{U}\right]$ be the weight by the makers $\mathscr{D}_i$ to e_j.

Step 3: Determine the weighted normalized decision by $\widehat{\mathscr{N}} = \left[\widehat{n}_{ij}^{L}, \widehat{n}_{ij}^{U}\right]_{n\times m}$, where $\left[\widehat{n}_{ij}^{L}, \widehat{n}_{ij}^{U}\right] = \left[\frac{w_{ij}^{L}}{\sqrt[q]{\sum_{i=1}^{n} w_{ij}^{qU}}}, \frac{w_{ij}^{U}}{\sqrt[q]{\sum_{i=1}^{n} w_{ij}^{qL}}}\right]$ is the normalized parameter and finding the weighted vector is $\mathscr{W} = \left([m_1^{L}, m_1^{U}], [m_2^{L}, m_2^{U}] ..., [m_m^{L}, m_m^{U}]\right)$, where $[m_i^{L}, m_i^{U}] = \left[\frac{w_i^{L}}{\sqrt[q]{\sum_{l=1}^{n} w_{li}^{U}}}, \frac{w_i^{U}}{\sqrt[q]{\sum_{l=1}^{n} w_{li}^{L}}}\right]$ be the weight of the jth parameter and $[w_j^{L}, w_j^{U}] = \left[\frac{\sum_{i=1}^{n} \widehat{n}_{ij}^{L}}{n}, \frac{\sum_{i=1}^{n} \widehat{n}_{ij}^{U}}{n}\right]$.

Step 4: Form the q-Rung NSIVSS decision by $\mathscr{D}_i = \left[c_{jk}^{Li}, c_{jk}^{Ui}\right]_{l\times m}$. Here, $\left[c_{jk}^{Li}, c_{jk}^{Ui}\right]$ is a ith decision-maker, i.e., $[\mathscr{D}_i^{L}, \mathscr{D}_i^{U}]$ for each i. Calculating the aggregating matrix by

$\left[\mathscr{X}^{L}, \mathscr{X}^{U}\right] = \frac{\left[\mathscr{D}_1^{L}, \mathscr{D}_1^{U}\right] + \left[\mathscr{D}_2^{L}, \mathscr{D}_2^{U}\right] + ... + \left[\mathscr{D}_n^{L}, \mathscr{D}_n^{U}\right]}{n} = \left[x_{jk}^{L}, x_{jk}^{U}\right]_{l\times m}$.

Step 5: Find the weighted q-Rung NSIVSS decision matrix by $[\mathscr{Y}^{L}, \mathscr{Y}^{U}] = \left[y_{jk}^{L}, y_{jk}^{U}\right]_{l\times m}$,

where $\left[y_{jk}^{L}, y_{jk}^{U}\right] = \left[m_k^{L} \times x_{jk}^{L}, m_k^{U} \times x_{jk}^{U}\right]$.

Step 6: Calculate the values for q-Rung NSIVSS-PIS and q-Rung NSIVSS-NIS. Now,

q-Rung NSIVSS-PIS = $\left(\left[y_1^{L+}, y_1^{U+}\right], \left[y_2^{L+}, y_2^{U+}\right] ..., \left[y_l^{L+}, y_l^{U+}\right]\right)$

$= \left\{\left(\vee_k\left[y_{jk}^{L}, y_{jk}^{U}\right], \wedge_k\left[y_{jk}^{L}, y_{jk}^{U}\right], \wedge_k\left[y_{jk}^{L}, y_{jk}^{U}\right]\right) : k = 1, 2, ..., m\right\}$ and

q-Rung NSIVSS-PIS = $\left(\left[y_1^{L-}, y_1^{U-}\right], \left[y_2^{L-}, y_2^{U-}\right] ..., \left[y_l^{L-}, y_l^{U-}\right]\right)$

$= \left\{\left(\wedge_k\left[y_{jk}^{L}, y_{jk}^{U}\right], \vee_k\left[y_{jk}^{L}, y_{jk}^{U}\right], \vee_k\left[y_{jk}^{L}, y_{jk}^{U}\right]\right) : k = 1, 2, ..., m\right\}$.

Here, q-Rung NSIVSS union $\vee$ and q-Rung NSIVSS intersection $\wedge$.

Step 7: Obtain q-Rung NSIVSS-Euclidean distances from q-Rung NSIVSS-PIS and q-Rung NSIVSS-NIS. Now

$$\left[d_j^{L+}, d_j^{U+}\right] = \left[\sqrt{\sum_{k=1}^{m}\left\{\left(\eth_{jk}^{TL} - \eth_j^{TL+}\right)^2 + \left(\eth_{jk}^{IL} - \eth_j^{IL+}\right)^2 + \left(\eth_{jk}^{FL} - \eth_j^{FL+}\right)^2\right\}},\right.$$
$$\left.\sqrt{\sum_{k=1}^{m}\left\{\left(\eth_{jk}^{TU} - \eth_j^{TU+}\right)^2 + \left(\eth_{jk}^{IU} - \eth_j^{IU+}\right)^2 + \left(\eth_{jk}^{FU} - \eth_j^{FU+}\right)^2\right\}}\right]$$

and $\left[d_j^{L-}, d_j^{U-}\right] = \left[\sqrt{\sum_{k=1}^{m}\left\{\left(\eth_{jk}^{TL}-\eth_j^{TL-}\right)^2+\left(\eth_{jk}^{IL}-\eth_j^{IL-}\right)^2+\left(\eth_{jk}^{FL}-\eth_j^{FL-}\right)^2\right\}},\right.$
$\left.\sqrt{\sum_{k=1}^{m}\left\{\left(\eth_{jk}^{TU}-\eth_j^{TU-}\right)^2+\left(\eth_{jk}^{IU}-\eth_j^{IU-}\right)^2+\left(\eth_{jk}^{FU}-\eth_j^{FU-}\right)^2\right\}}\right]$,
where $j = 1, 2, ..., n$.

Step 8: Calculate the relative closeness for the ideal solution by
$\left[C^{L*}(y_j), C^{U*}(y_j)\right] = \left[\frac{d_j^{L-}}{d_j^{U+}+d_j^{U-}}, \frac{d_j^{U-}}{d_j^{L+}+d_j^{L-}}\right]$; hence, $C^*(y_j) = \frac{C^{L*}(y_j)+C^{U*}(y_j)}{2} \in [0, 1]$.

Step 9: Finding the rank of alternatives by using decreasing or increasing order of their relative closeness coefficients. The bigger $C^*(y_j)$, the more desirable alternative y_j.

4 Selection Process Based on Diagnostic Health Imaging

A range of methods for looking inside the body to obtain a diagnosis and identify the cause of a disease or injury are referred to as "diagnostic health imaging." Analysis of diagnostic medical imaging is crucial in contemporary medicine. Due to the difficulty of analyzing and diagnosing from a single image, computer-aided diagnostic tools have been utilized to shed light on potential illness mechanisms. Providing a forum for the dissemination of new research findings in the field of medical and biological image analysis, diagnostic health image analysis focuses on efforts related to the applications of computer vision, virtual reality, and robotics to biomedical imaging challenges. Professionals can see into your body using diagnostic imaging to assist them find any signs of a problem. There are numerous simple, painless, and noninvasive imaging procedures. Some, meanwhile, may demand you to remain still inside the machine for a protracted amount of time, which could be uncomfortable. The test volunteers in certain trials receive a little dose of radiation. Diagnostic health imaging technology has transformed healthcare by enabling early disease diagnosis, reducing the need for unnecessary invasive exploratory procedures, and improving patient outcomes. Due to its higher accuracy, repeatability, and objectivity when compared to conventional diagnosis in many scenarios, computer-assisted automatic processing and analysis of medical images is in high demand. Our research focuses on creating machine learning-based image analysis algorithms and systems to address a variety of crucial yet difficult medical image analysis issues. When used to identify, track, or treat medical conditions, various technologies are referred to as "medical imaging":

1. CT (Computed Tomography) Scan: (y_1)
   A noninvasive medical test or process known as computed tomography (CT), sometimes known as "computerized tomography" or "computed axial tomog-

raphy" (CAT), creates cross-sectional images of the body using specialist X-ray equipment. Similar to the slices in a loaf of bread, each cross-sectional image shows a "slice" of the individual being photographed. There are numerous diagnostic and therapeutic uses for these cross-sectional pictures. Every part of the body can get a CT scan for a variety of purposes (e.g., diagnostic, treatment planning, interventional, or screening). CT is a useful medical tool that can assist a doctor in diagnosing illness, injury, or abnormality, planning and directing interventional or therapeutic operations, and assessing the efficacy of therapy (e.g., cancer treatment).

2. MRI (Magnetic Resonance Imaging) Scan: (y_2)
   MRIs use a powerful magnet rather than radiation to create an image of the patient's body. It gives you an incredibly detailed look inside your body and can be used to check for anomalies in the spine, brain, cysts, tumors, and other areas of your body in addition to breast tissue cancer. An average MRI examination takes between 30 and 60 minutes to complete. The doctor may administer contrast fluid to you in order to improve the clarity of specific components in the ensuing photographs. The development of the MRI scan represents a crucial turning point in medical history. Careful screening of individuals and objects entering the MR environment is crucial to ensure that nothing pierces the magnet.
3. X-Ray: (y_3)
   One of the most popular and well-known diagnostic imaging procedures is the use of X-rays. They are used by doctors to observe the interior of the body. X-ray machines provide a high-energy beam that can pass through other parts of the body but cannot penetrate dense tissue or bones. In the same way, that visible light is an electromagnetic radiation as well as X-rays. A high-energy X-ray can penetrate most materials, including the human body, due to its high energy. X-rays are used to create images of internal structures and tissues. When the device is turned on, X-rays enter the body and are absorbed in different amounts by different tissues depending on their radiological density.
4. Ultrasound: (y_4)
   High-frequency sound waves are used in sonography, another term for ultrasound imaging, to observe within the body. With real-time ultrasound imaging, it is possible to observe the motion of the internal organs of the body as well as the flow of blood via the arteries. A transducer (probe) is inserted into a bodily orifice or on the skin during an ultrasonic examination. In order to allow the ultrasonic waves from the transducer to pass through the skin and into the body, a small coating of gel is applied to the skin. Images of internal organs are produced when waves reflect off them. To create an image, the power (amplitude) of the sound signal and the length of the wave's journey through the body are used. A clinician can analyze, identify, and treat medical diseases using ultrasonic imaging. Since ultrasound imaging has been in use for more than 20 years, it has a stellar safety record.

5. Nuclear Medicine:(y_5)
   Nuclear medicine works well. It can aid in the early detection of a wide range of illnesses, including infection, stress fractures, cancer, heart disease, blood clots, and cancer. Nuclear medicine scans have been used for about 50 years and are mostly safe. Since the radiation dose is so minimal, there are no significant dangers. After the patient receives a radioactive tracer, nuclear medicine imaging is a technique for producing images by detecting radiation from various physiological locations. A nuclear medicine physician uses the images, which are produced digitally on a computer, to diagnose patients. Injections of radioactive tracers into the veins are commonly used in nuclear medicine. A patient normally experiences relatively minimal radiation exposure during a traditional nuclear medicine scan. Nuclear imaging is a common method for identifying or treating ailments.
6. Radiology:(y_6)
   The field of medicine known as radiology uses imaging techniques to diagnose and treat illness. To determine whether a medical condition is present or not, it could be used as a diagnostic tool (such as finding a lung cancer). Radiology has developed methods during the past century for a wide range of disease diagnosis and a variety of treatments, many of which are less invasive than surgery, for a range of medical problems. Its job is to take pictures of the body's interior. It is used to determine certain conditions.
7. MRA (Magnetic Resonance Angiogram) Scan:(y_7)
   A special type of MRI that examines the body's blood arteries is known as magnetic resonance angiography, sometimes known as a magnetic resonance angiogram (MRA). Magnetic resonance angiography is a significantly less intrusive and uncomfortable examination than a standard angiogram, which involves placing a catheter within the body. Doctor may recommend magnetic resonance angiography if they suspect that you have a blocked or narrowed blood artery elsewhere in your body. The images from the magnetic resonance angiography will be analyzed by healthcare provider. Depending on the exact issue found, your healthcare practitioner may recommend additional exams or treatments.
8. PET (Positron Emission Tomography):(y_8)
   Radiation-based imaging shows cellular activity within the body. To perform the scan, radioactive tracers are added to a special dye. In order to examine an area of your body, you either breathe in the tracers, inject them into a vein in your arm, or consume them. Your doctor can evaluate your organs and tissues by detecting the tracers using a PET scanner. There are a number of biological tissues and diseases that are more chemically active than others, so the tracer will collect in these areas. Bright spots will appear on the PET scan when these diseased areas are scanned. The scan can evaluate a variety of things, including blood flow, oxygen utilization, and also how body processes sugar. PET scans reveal cellular-level metabolic alterations in an organ or tissue. This is significant because cellular processes frequently mark the start of illnesses. PET scans are able to identify very subtle cell changes.

9. Mammogram: (y_9)
   Mammography, also referred to as mammograms, is a method used by doctors to find breast cancer early on, before symptoms show. A screening mammography is what it is known as. Most mammography results show benign or noncancerous tissue. Less than one in ten people who require further testing after a mammogram actually have cancer. Mammograms are performed using a special X-ray machine designed to examine breast tissue. The apparatus exposes to less X-ray radiation than is necessary to inspect the bones. During a mammography, the breasts are exposed to very low doses of radiation, but the benefits far outweigh any possible hazards.
10. Fluoroscopy: (y_{10})
   Fluoroscopy, a medical imaging procedure, uses a sequence of X-ray beam pulses to show the movement of internal organs and tissues in real time on a computer screen. Traditional X-rays are like images, whereas fluoroscopy is like a video. Diagnostic imaging and interventional guidance, which use fluoroscopy to guide particular therapeutic procedures like surgeries and catheter insertions, are the two main uses of fluoroscopy in healthcare. Because they can be used to detect a large range of illnesses and to guide a wide range of operations, fluoroscopy imaging examinations are quite common.

Using professional evaluations, we want to select the best option from a large number of options. I have provided the following DM information:

1. Image Restoration
   An image can be improved by image restoration. It is usually objective because it is based on a mathematical or probabilistic model when restoring damaged images. An image can be restored to its original state once noise and blur have been eliminated. In a variety of situations, such as photography, radar imaging, and reducing motion blur brought on by camera shaking, it might be difficult to get rid of image blur. Models that are reliable for individual medical imaging devices are typically much harder to obtain than universal models. Methods of image restoration are crucial for enhancing the usability of medical images and broadening their scope of use. With a linear system model, the process of ultrasound image restoration may be explained. Pictures taken using transmission X-rays were restored using an iterative approach.
2. Image Segmentation
   The process of segmenting an image involves classifying its voxels into a variety of different categories. The core problem with medical image analysis is medical image segmentation, a contentious and challenging field of study. An important first step in image analysis is segmenting CT images. It is a required step in order to provide a trustworthy CT image analysis. Both diagnosis and detection need the use of image segmentation. The purpose of X-ray segmentation is to divide the image into distinct parts so that clinicians can utilize them to examine the bone's structure, identify bone fractures, or choose the best course of action before surgery. Prostate-specific ultrasound images are created for both diagnostic and therapeutic purposes. Nuclear medicine image analysis has been

identified as the core problem due to image segmentation, which remains a contentious and challenging area.

3. Image Registration
   In image registration, two or more images are lined up by fusion, matching, or warping. Image registration in ultrasonography aligns two or more picture files taken at different times, by different cameras, or by different sensors in order to fuse the images from the two sources. The input images in CT imaging modalities must be aligned before fusion can occur. Images can be aligned and stacked in an X-ray scan through image registration. The results of these scans are more accurately interpreted by medical experts when they have these scans. A radioactive tracer can be administered to a patient in order to detect radiation emitted from different physiological regions. Diagnoses are made using digital images made on a computer by nuclear medicine physicians. In order to properly diagnose and treat patients, data from several images taken using diverse modalities must be integrated. The technique in question is image registration.
4. Image Acquisition
   Image acquisition is the action of obtaining an image for subsequent processing from an external source. Since no operation can be started without first obtaining an image, it is always the first stage in the workflow. In a CT scan, picture capture follows a helical or circular path around the patient's body, while X-rays are emitted in a collimated fan beam. When data are collected by the set of detectors at various source locations, a set of projections is produced. Image capture magnetic resonance imaging (MRI or MR), which moves the subject into the center of the tube for imaging, resembles a computed tomography machine in terms of look. X-ray images are produced when an image receptor cassette is exposed to an active X-ray beam. A radiographic film is housed in the image receptor cassette between two intensifying screens. During the ultrasonic picture acquisition process, time, frequency, and velocity are estimated. Nuclear medicine uses positron emission tomography and single-photon emission tomography for image acquisition.
5. Image Enhancement
   Image enhancement techniques allow for an increase in signal-to-noise ratio and emphasize image details by altering the colors or intensities of a photograph. It also has automatic deblurring, linear and nonlinear filtering, and contrast enhancement. Contrast enhancement is used in both MRI and CT scan results. Due to contrast picture enhancement, the radiologist can more easily discern between normal and diseased areas. It is simpler to distinguish between normal and pathological conditions because of the contrast picture enhancement employed in CT scans. X-ray images are dimly lit and have poor contrast. The quality of the X-ray image and consequently the evaluation can be improved by applying image enhancement. In ultrasonic image enhancement, depending on the demands of the user, some features of an image are emphasized, while others are weakened or removed. Following the standard preprocessing and postprocessing carried out by a gamma camera system, improving image quality is offered as a solution.

Suppose that ten types of diagnostic health imaging (alternatives) such as $y = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}\}$, where CT scan ($y_1$), MRI scan ($y_2$), X-ray ($y_3$), ultrasound ($y_4$), nuclear medicine ($y_5$), radiology ($y_6$), MRA scan ($y_7$), PET ($y_8$), mammogram ($y_9$), and fluoroscopy ($y_{10}$):

Step 1: Suppose that $[\mathscr{D}^L, \mathscr{D}^U] = \left\{[\mathscr{D}_i^L, \mathscr{D}_i^U] : i = 1, 2, 3, 4, 5\right\}$ is a finite set of decision-makers, $y = \{y_i : i = 1, 2, ..., 10\}$ is the collection of diagnostic health imaging/alternatives, and $D = \{e_i : i = 1, 2, ..., 5\}$ is a finite set family of parameters, where $e_1 =$ Image Restoration $e_2 =$ image segmentation, $e_3 =$ image registration, $e_4 =$ image acquisition, and $e_5 =$ image enhancement.

Step 2: Linguistic variables in the form as very good report presentation (VGRP) = [0.9, 0.95], good report presentation (GRP) = [0.8, 0.9], average report presentation (ARP) = [0.65, 0.8], poor report presentation (PRP) = [0.5, 0.65], and very poor report presentation (VPRP) = [0.35, 0.5].

Form the weighted parameter matrix given as

$$
\begin{aligned}
\mathscr{P} &= [w_{ij}^L, w_{ij}^U]_{5\times 5} \\
&= \begin{bmatrix}
ARP & GRP & VGRP & VPRP & PRP \\
VPRP & VGRP & GRP & PRP & ARP \\
VGRP & PRP & VPRP & ARP & GRP \\
ARP & VPRP & PRP & VGRP & VPRP \\
PRP & ARP & VGRP & GRP & PRP
\end{bmatrix} \\
&= \begin{bmatrix}
[0.65, 0.80] & [0.80, 0.90] & [0.90, 0.95] & [0.35, 0.50] & [0.50, 0.65] \\
[0.35, 0.50] & [0.90, 0.95] & [0.80, 0.90] & [0.50, 0.65] & [0.65, 0.80] \\
[0.90, 0.95] & [0.50, 0.65] & [0.35, 0.50] & [0.65, 0.80] & [0.80, 0.90] \\
[0.65, 0.80] & [0.35, 0.50] & [0.50, 0.65] & [0.90, 0.95] & [0.35, 0.50] \\
[0.50, 0.65] & [0.65, 0.80] & [0.90, 0.95] & [0.80, 0.90] & [0.50, 0.65]
\end{bmatrix}
\end{aligned}
$$

where $[w_{ij}^L, w_{ij}^U]$ be the weight provided by the specialist $[\mathscr{D}_i^L, \mathscr{D}_i^U]$ to each parameter e_j.

Step 3: Form the normalized weighted decision matrix as follows:

$$
\begin{aligned}
\widehat{\mathscr{N}} &= [\widehat{n}_{ij}^L, \widehat{n}_{ij}^U]_{5\times 5} \\
&= \begin{bmatrix}
[0.3847, 0.4735] & [0.4600, 0.5175] & [0.4964, 0.5240] & [0.2012, 0.2875] & [0.3134, 0.4074] \\
[0.2071, 0.2959] & [0.5175, 0.5462] & [0.4412, 0.4964] & [0.2875, 0.3737] & [0.4074, 0.5015] \\
[0.5326, 0.5622] & [0.2875, 0.3737] & [0.1930, 0.2758] & [0.3737, 0.4600] & [0.5015, 0.5642] \\
[0.3847, 0.4735] & [0.2012, 0.2875] & [0.2758, 0.3585] & [0.5175, 0.5462] & [0.2194, 0.3134] \\
[0.2959, 0.3847] & [0.3737, 0.4600] & [0.4964, 0.5240] & [0.4600, 0.5175] & [0.3134, 0.4074]
\end{bmatrix}.
\end{aligned}
$$

weighted vector

$$
\begin{aligned}
\mathscr{W} = (&[0.058, 0.085], [0.0557, 0.0785], [0.0684, 0.0697], \\
&[0.0557, 0.0785], [0.0629, 0.0982]) .
\end{aligned}
$$

Step 4: The aggregated decision matrix $\left[\mathscr{X}^L, \mathscr{X}^U\right] = \dfrac{\left[\mathscr{D}_1^L, \mathscr{D}_1^U\right] + \left[\mathscr{D}_2^L, \mathscr{D}_2^U\right] +\ldots + \left[\mathscr{D}_5^L, \mathscr{D}_5^U\right]}{5}$

=

([0.55, 0.65], [0.40, 0.50], [0.30, 0.40])	([0.07, 0.45], [0.55, 0.65], [0.46, 0.70])	([0.38, 0.45], [0.40, 0.50], [0.51, 0.60])
([0.45, 0.65], [0.50, 0.60], [0.30, 0.60])	([0.60, 0.60], [0.65, 0.65], [0.10, 0.25])	([0.65, 0.75], [0.25, 0.35], [0.30, 0.40])
([0.55, 0.65], [0.60, 0.67], [0.60, 0.66])	([0.50, 0.60], [0.55, 0.65], [0.15, 0.25])	([0.54, 0.60], [0.23, 0.30], [0.34, 0.40])
([0.50, 0.63], [0.40, 0.50], [0.40, 0.50])	([0.44, 0.55], [0.35, 0.50], [0.47, 0.60])	([0.60, 0.70], [0.50, 0.60], [0.31, 0.40])
([0.55, 0.65], [0.45, 0.53], [0.15, 0.30])	([0.25, 0.35], [0.46, 0.55], [0.65, 0.75])	([0.50, 0.60], [0.40, 0.50], [0.30, 0.55])
([0.62, 0.70], [0.65, 0.73], [0.25, 0.35])	([0.44, 0.51], [0.25, 0.35], [0.55, 0.65])	([0.30, 0.50], [0.45, 0.55, [0.60, 0.70])
([0.40, 0.55], [0.55, 0.62], [0.15, 0.25])	([0.30, 0.42], [0.29, 0.40], [0.60, 0.70])	([0.65, 0.75], [0.20, 0.30], [0.55, 0.75])
([0.45, 0.45], [0.60, 0.74], [0.50, 0.60])	([0.30, 0.40], [0.25, 0.65], [0.65, 0.75])	([0.28, 0.35], [0.40, 0.50], [0.50, 0.60])
([0.45, 0.65], [0.50, 0.73], [0.40, 0.60])	([0.55, 0.65], [0.44, 0.50], [0.55, 0.65])	([0.35, 0.50], [0.34, 0.40], [0.55, 0.62])
([0.35, 0.55], [0.65, 0.80], [0.15, 0.30])	([0.55, 0.65], [0.44, 0.50], [0.55, 0.65])	([0.35, 0.50], [0.34, 0.40], [0.55, 0.62])

([0.45, 0.55], [0.55, 0.65], [0.39, 0.50])	([0.50, 0.60], [0.60, 0.70], [0.50, 0.60])
([0.25, 0.33], [0.60, 0.70], [0.30, 0.60])	([0.50, 0.60], [0.20, 0.30], [0.60, 0.70])
([0.23, 0.30], [0.62, 0.70], [0.20, 0.30])	([0.52, 0.60], [0.22, 0.55], [0.50, 0.73])
([0.60, 0.70], [0.45, 0.55], [0.45, 0.55])	([0.35, 0.50], [0.30, 0.55], [0.55, 0.65])
([0.40, 0.50], [0.55, 0.65], [0.35, 0.50])	([0.44, 0.70], [0.40, 0.55], [0.30, 0.70])
([0.50, 0.60], [0.60, 0.70], [0.40, 0.55])	([0.40, 0.49], [0.38, 0.45], [0.40, 0.50])
([0.49, 0.55], [0.60, 0.75], [0.25, 0.35])	([0.50, 0.60], [0.60, 0.70], [0.20, 0.50])
([0.50, 0.60], [0.70, 0.80], [0.15, 0.35])	([0.36, 0.45], [0.70, 0.78], [0.10, 0.25])
([0.44, 0.60], [0.35, 0.50], [0.40, 0.50])	([0.25, 0.45], [0.65, 0.75], [0.40, 0.55])
([0.29, 0.45], [0.65, 0.75], [0.50, 0.60])	([0.40, 0.50], [0.55, 0.63], [0.60, 0.60])

$= \left[x_{jk}^L, x_{jk}^U\right]_{10\times 5}$

Step 5: The weighted q-Rung NSIVSS decision matrix $\left[\mathscr{Y}^L, \mathscr{Y}^U\right] = \left[m_k^L \times x_{jk}^L, m_k^U \times x_{jk}^U\right]$

=

([0.0537, 0.0933], [0.0390, 0.0718], [0.0293, 0.0574])	([0.0068, 0.0614], [0.0533, 0.0888], [0.0445, 0.0956])
([0.0439, 0.0933], [0.0488, 0.0862], [0.0293, 0.0862])	([0.0581, 0.0956], [0.0629, 0.1024], [0.0097, 0.0341])
([0.0537, 0.0933], [0.0585, 0.0962], [0.0585, 0.0948])	([0.0484, 0.0819], [0.0533, 0.0888], [0.0145, 0.0341])
([0.0488, 0.0905], [0.0390, 0.0718], [0.0390, 0.0718])	([0.0426, 0.0751], [0.0339, 0.0683], [0.0455, 0.0819])
([0.0537, 0.0933], [0.0439, 0.0761], [0.0146, 0.0431])	([0.0242, 0.0478], [0.0445, 0.0751], [0.0629, 0.1024])
([0.0605, 0.1005], [0.0634, 0.1048], [0.0244, 0.0505])	([0.0426, 0.0696], [0.0242, 0.0478], [0.0533, 0.0888])
([0.0390, 0.0790], [0.0537, 0.0890], [0.0146, 0.0359])	([0.0291, 0.0574], [0.0281, 0.0546], [0.0581, 0.0956])
([0.0439, 0.0646], [0.0585, 0.1063], [0.0488, 0.0862])	([0.0291, 0.0546], [0.0242, 0.0888], [0.0629, 0.1024])
([0.0439, 0.0933], [0.0488, 0.1048], [0.0390, 0.0862])	([0.0397, 0.0683], [0.0533, 0.0888], [0.0484, 0.0819])
([0.0342, 0.0790], [0.0634, 0.1149], [0.0146, 0.0431])	([0.0533, 0.0888], [0.0426, 0.0683], [0.0533, 0.0888])

([0.0366, 0.0568], [0.0385, 0.0631], [0.0491, 0.0758])	([0.0436, 0.0751], [0.0533, 0.0888], [0.0378, 0.0683])
([0.0626, 0.0947], [0.0241, 0.0442], [0.0289, 0.0505])	([0.0242, 0.0451], [0.0581, 0.0956], [0.0291, 0.0819])
([0.0520, 0.0758], [0.0222, 0.0379], [0.0328, 0.0505])	([0.0223, 0.0410], [0.0600, 0.0956], [0.0194, 0.0410])
([0.0578, 0.0884], [0.0482, 0.0758], [0.0299, 0.0505])	([0.0581, 0.0956], [0.0436, 0.0751], [0.0436, 0.0751])
([0.0482, 0.0758], [0.0385, 0.0631], [0.0289, 0.0695])	([0.0387, 0.0683], [0.0533, 0.0888], [0.0339, 0.0683])
([0.0289, 0.0631], [0.0434, 0.0695], [0.0578, 0.0884])	([0.0484, 0.0819], [0.0581, 0.0956], [0.0387, 0.0751])
([0.0626, 0.0947], [0.0193, 0.0379], [0.0530, 0.0947])	([0.0474, 0.0751], [0.0581, 0.1024], [0.0242, 0.0478])
([0.0270, 0.0442], [0.0385, 0.0631], [0.0482, 0.0758])	([0.0484, 0.0819], [0.0678, 0.1092], [0.0145, 0.0478])
([0.0434, 0.0695], [0.0241, 0.0442], [0.0626, 0.0947])	([0.0426, 0.0819], [0.0339, 0.0683], [0.0387, 0.0683])
([0.0337, 0.0631], [0.0328, 0.0505], [0.0530, 0.0783])	([0.0281, 0.0614], [0.0629, 0.1024], [0.0484, 0.0819])

([0.0501, 0.0940], [0.0602, 0.1097], [0.0501, 0.0940])
([0.0501, 0.0940], [0.0201, 0.047*o*], [0.0602, 0.1097])
([0.0522, 0.0940], [0.0221, 0.0862], [0.0501, 0.1144])
([0.0351, 0.0784], [0.0301, 0.0862], [0.0552, 0.1019])
([0.0522, 0.1097], [0.0401, 0.0862], [0.0301, 0.1097])
([0.0401, 0.0768], [0.0381, 0.0705], [0.0401, 0.0784])
([0.0501, 0.0940], [0.0602, 0.1097], [0.0201, 0.0784])
([0.0361, 0.0705], [0.0702, 0.1222], [0.0100, 0.0392])
([0.0251, 0.0705], [0.0652, 0.1175], [0.0401, 0.0862])
([0.0401, 0.0784], [0.0552, 0.0987], [0.0602, 0.0940])

$= \left[y_{jk}^L, y_{jk}^U\right]_{10\times 5}$

Step 6: Determine the values for q-Rung NSIVSS-PIS and q-Rung NSIVSS-NIS. Now,

$$\begin{bmatrix} y^{L+}, y^{U+} & q - RungNSIVSS - PIS \\ y_1^{L+}, y_1^{U+} & ([0.0537, 0.0940], [0.0385, 0.0631], [0.0293, 0.0574]) \\ y_2^{L+}, y_2^{U+} & ([0.0626, 0.0956], [0.0201, 0.0442], [0.0097, 0.0341]) \\ y_3^{L+}, y_3^{U+} & ([0.0537, 0.0940], [0.0221, 0.0379], [0.0145, 0.0341]) \\ y_4^{L+}, y_4^{U+} & ([0.0581, 0.0956], [0301, 0.0683], [0.0299, 0.0505]) \\ y_5^{L+}, y_5^{U+} & ([0.0552, 0.1097], [0.0385, 0.0631], [0.0146, 0.0431]) \\ y_6^{L+}, y_6^{U+} & ([0.0605, 0.1005], [0.0242, 0.0478], [0.0244, 0.0503]) \\ y_7^{L+}, y_7^{U+} & ([0.0626, 0.0947], [0.0193, 0.0379], [0.0146, 0.0359]) \\ y_8^{L+}, y_8^{U+} & ([0.0484, 0.0819], [0.0242, 0.0631], [0.0100, 0.0392]) \\ y_9^{L+}, y_9^{U+} & ([0.0439, 0.0933], [0.0241, 0.0442], [0.0387, 0.0683]) \\ y_{10}^{L+}, y_{10}^{U+} & ([0.0533, 0.0888], [0.0328, 0.0505], [0.0146, 0.0431]) \\ y^{L-}, y^{U-} & q - RungNSIVSS - NIS \\ y_1^{L-}, y_1^{U-} & ([0.0068, 0.0568], [0.0602, 0.1097], [0.0501, 0.0956]) \\ y_2^{L-}, y_2^{U-} & ([0.0242, 0.0451], [0.0629, 0.1024], [0.0602, 0.1097]) \\ y_3^{L-}, y_3^{U-} & ([0.0223, 0.0410], [0.0600, 0.0962], [0.0585, 0.1144]) \\ y_4^{L-}, y_4^{U-} & ([0.0351, 0.0751], [0.0482, 0.0862], [0.0552, 0.1019]) \\ y_5^{L-}, y_5^{U-} & ([0.0242, 0.0478], [0.0533, 0.0888], [0.0629, 0.1097]) \\ y_6^{L-}, y_6^{U-} & ([0.0289, 0.0631], [0.0634, 0.1048], [0.0578, 0.0888]) \\ y_7^{L-}, y_7^{U-} & ([0.0291, 0.0574], [0.0602, 0.1097], [0.0581, 0.0956]) \\ y_8^{L-}, y_8^{U-} & ([0.0270, 0.0442], [0.0702, 0.1222], [0.0629, 0.1024]) \\ y_9^{L-}, y_9^{U-} & ([0.0251, 0.0683], [0.0652, 0.1175], [0.0626, 0.0947]) \\ y_{10}^{L-}, y_{10}^{U-} & ([0.0281, 0.0614], [0.0634, 0.1149], [0.0602, 0.094]) \end{bmatrix}$$

Step 7: The q-Rung NSIVSS EDs from q-Rung NSIVSS-PIS and q-Rung NSIVSS-NIS:

$\left[(y_i^L), (y_i^U)\right]$	$\left[d_i^{L+}, d_i^{U+}\right]$	$\left[d_i^{L-}, d_i^{U-}\right]$
$\left[(y_1^L), (y_1^U)\right]$	[0.0681, 0.0980]	[0.0892, 0.1007]
$\left[(y_2^L), (y_2^U)\right]$	[0.0990, 0.0146]	[0.1127, 0.1645]
$\left[(y_3^L), (y_3^U)\right]$	[0.0912, 0.1593]	[0.1029, 0.1682]
$\left[(y_4^L), (y_4^U)\right]$	[0.0511, 0.0769]	[0.0552, 0.0791]
$\left[(y_5^L), (y_5^U)\right]$	[0.0686, 0.1332]	[0.0926, 0.1263]
$\left[(y_6^L), (y_6^U)\right]$	[0.0868, 0.1188]	[0.0782, 0.0976]
$\left[(y_7^L), (y_7^U)\right]$	[0.1000, 0.1524]	[0.0963, 0.1354]
$\left[(y_8^L), (y_8^U)\right]$	[0.1104, 0.1355]	[0.0983, 0.1250]
$\left[(y_9^L), (y_9^U)\right]$	[0.0655, 0.1224]	[0.0783, 0.1031]
$\left[(y_{10}^L), (y_{10}^U)\right]$	[0.1009, 0.1361]	[0.0678, 0.1053]

Step 8: Calculate closeness coefficients using q-Rung NSIVSS-PIS and q-Rung NSIVSS-NIS for every alternatives:
$\left[C_1^{L*}, C_1^{U*}\right] = [0.4488, 0.6403]$, $\left[C_2^{L*}, C_2^{U*}\right] = [0.3631, 0.7767]$, $\left[C_3^{L*}, C_3^{U*}\right] = [0.3140, 0.8667]$, $\left[C_4^{L*}, C_4^{U*}\right] = [0.3541, 0.7435]$, $\left[C_5^{L*}, C_5^{U*}\right] = [0.3569, 0.7830]$, $\left[C_6^{L*}, C_6^{U*}\right] = [0.3613, 0.5914]$, $\left[C_7^{L*}, C_7^{U*}\right] = [0.3346, 0.6900]$, $\left[C_8^{L*}, C_8^{U*}\right] = [0.3772, 0.5993]$, $\left[C_9^{L*}, C_9^{U*}\right] = [0.3471, 0.7170]$, $\left[C_{10}^{L*}, C_{10}^{U*}\right] = [0.2810, 0.6240]$.
The C^* values are $C_1^* = 0.5446$, $C_2^* = 0.5699$, $C_3^* = 0.5904$, $C_4^* = 0.5488$, $C_5^* = 0.5700$, $C_6^* = 0.4764$, $C_7^* = 0.5123$, $C_8^* = 0.4883$, $C_9^* = 0.5320$, $C_{10}^* = 0.4525$.
Step 9: The order of the diagnostic health imaging C_i^* is $y_3 \succeq y_5 \succeq y_1 \succeq y_4 \succeq y_2 \succeq y_9 \succeq y_7 \succeq y_8 \succeq y_{10} \succeq y_6$.

5 Comparison Between the Suggested and the Existing Approach

Here, we compared some existing models with the suggested models. Thus, its benefits and usefulness are demonstrated. A q-rung NSIVSS approach is used with an ED. Here are the various distances:

$q = 1$	q-Rung NSIVSS
$TOPSIS-\ Proposed$	$y_3 \succeq y_5 \succeq y_1 \succeq y_4 \succeq y_2 \succeq y_9 \succeq y_7 \succeq y_8 \succeq y_{10} \succeq y_6$
$TOPSIS-$ [13]	$y_3 \succeq y_5 \succeq y_1 \succeq y_4 \succeq y_2 \succeq y_9 \succeq y_7 \succeq y_8 \succeq y_{10} \succeq y_6$

5.1 Sensitivity Analysis

The reliability of the circumstances for the MCGDM approach with the alternatives. Below are the values and rankings of closeness. The different "q" values are acquired using the q-Rung NSIVSS technique. Using the q-Rung NSIVSS method, if $q = 1$, then ranking of alternative is $y_3 \succeq y_5 \succeq y_1 \succeq y_4 \succeq y_2 \succeq y_9 \succeq y_7 \succeq y_8 \succeq y_{10} \succeq y_6$. If $2 \leq q \leq 10$, then new ranking of alternative is $y_3 \succeq y_5 \succeq y_2 \succeq y_4 \succeq y_1 \succeq y_9 \succeq y_7 \succeq y_8 \succeq y_{10} \succeq y_6$. If $q = 11$, then ranking of new order is $y_3 \succeq y_5 \succeq y_2 \succeq y_1 \succeq y_4 \succeq y_9 \succeq y_7 \succeq y_8 \succeq y_6 \succeq y_{10}$. Thus, the imaging "$y_4$" becomes the imaging "$y_1$" as the best alternative. The TOPSIS approaches q values should be changed. The following relative closeness values are

Relative closeness values										
$q - values$	$\mathscr{D}_1^*$	$\mathscr{D}_2^*$	$\mathscr{D}_3^*$	$\mathscr{D}_4^*$	$\mathscr{D}_5^*$	$\mathscr{D}_6^*$	$\mathscr{D}_7^*$	$\mathscr{D}_8^*$	$\mathscr{D}_9^*$	$\mathscr{D}_{10}^*$
$q = 2$	0.5446	0.5699	0.5904	0.5488	0.5700	0.4764	0.5123	0.4883	0.5320	0.4525
$q = 3$	0.5449	0.5704	0.591	0.5503	0.5705	0.4769	0.5129	0.4887	0.5323	0.4533
$q = 4$	0.5450	0.5705	0.5911	0.5506	0.5706	0.4771	0.513	0.4888	0.5323	0.4536
$q = 5$	0.5449	0.5703	0.5909	0.5502	0.5705	0.4771	0.5129	0.4887	0.5322	0.4535
$q = 6$	0.5447	0.5701	0.5905	0.5492	0.5702	0.4769	0.5126	0.4884	0.5319	0.4532
$q = 7$	0.5444	0.5697	0.59	0.548	0.5698	0.4766	0.5121	0.4880	0.5316	0.4526
$q = 8$	0.5441	0.5692	0.5894	0.5466	0.5694	0.4762	0.5116	0.4876	0.5313	0.4520
$q = 9$	0.5438	0.5688	0.5888	0.5454	0.5689	0.4757	0.5111	0.4872	0.531	0.4511
$q = 10$	0.5434	0.5684	0.5883	0.5440	0.5684	0.4753	0.5106	0.4867	0.5309	0.4505
$q = 11$	0.5432	0.5679	0.5878	0.5427	0.568	0.4751	0.5101	0.4864	0.5305	0.4498

Figure 22.1 shows the graphical representation consists of MCGDM based on TOPSIS.

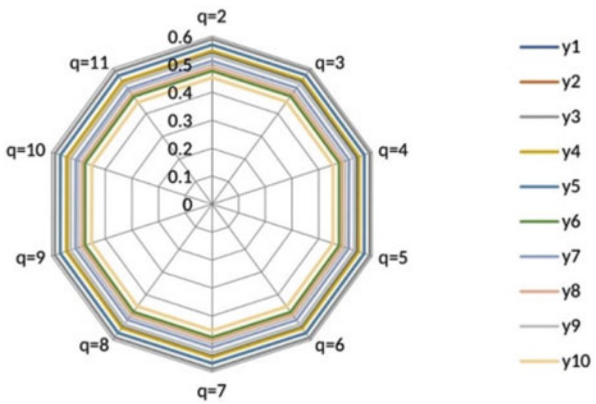

Fig. 22.1 Graphical representation using MCGDM based on TOPSIS

5.2 Advantages

According to the earlier study provided, the technique recommended in the following paragraphs has advantages. By combining the ideas of q-Rung FS and IVNSS, it suggests the concept of q-Rung NSIVSS. In situations where the sum of TD, ID, and FD is higher than two but less than the square total of its TD, ID, and FD, the q-Rung NSIVSS analyzes human behavior and natural phenomena that take place in real life and clarifies ambiguous information. The decision-maker has the freedom to select the outcome based on their individual tastes and q. The flexible accomplishment of varied ranking results of alternatives is made possible by operators like the q-Rung NSIVSS.

6 Conclusion

This article introduced ED measurements for q-Rung NSIVSSs, whose mathematical simplicity is an added benefit. The superiority of ED metrics is illustrated by practical numerical examples. An application case is given to show the usefulness of ED measurements. The q-Rung NSIVSS based on MADM challenges, which arise in diverse DM, is the main subject of this research work. In our discussion of the q-Rung NSIVSS of numerous AOs, we came to a number of conclusions that were relevant to their q-Rung NSIVSS. People can select the best course of action from the available options in settings with ambiguous and conflicting information by using the q-Rung NSIVSS based on MCGDM method. On the basis of q, the operators for the q-Rung NSIVSS have been applied to the MCGDM issue. The MCGDM problem were addressed using the q-Rung NSIVSS. A q-Rung NSIVSS can be used to determine the different rankings of alternatives. A generalized value of q has the greatest impact on ranking alternatives, according to research. By setting q values according to the actual scenario, decision-makers can arrive at the most reasonable ranking. In order to determine the result, the decision-maker must know q. Despite the fact that this field of study is still in its infancy, the author believes that the material in this paper will prove useful to future academics who are interested in it. Numerous types of statistical charts have also been included to visualize the rankings of the alternatives.

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Chapter 23
Cosine Neutrosophic Normal Interval-Valued Aggregation Operators to the Selection of Robotic Engineering

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1 Introduction

The complexity of real-world systems makes it challenging for decision-makers to select the most appropriate option among the various options available. Even though condensing is difficult, it is not impossible. It proves difficult for many businesses to set goals, incentive systems, and viewpoint restrictions. In decision-making (DM), therefore, multiple objectives must be considered at the same time. MADM is the process of choosing the most appropriate option from many possibilities. Our MADM team deals with a wide range of MADM problems on a daily basis. As a result, we all need to improve our DM skills. DM problems are of interest to a number of researchers. Most real problems are characterized by uncertainty. In order to cope with these uncertainties, a number of uncertain theories have been proposed, including fuzzy sets (FSs) [26], intuitionistic FS (IFSs) [4], Pythagorean FSs (PFSs) [24], and neutrosophic sets (NSs) [21]. There is a grade or degree of belonging to an element in a FS that lies between zero and one. Grades such as these are referred to as membership values. Clustering procedures are available for FS, as well as regression predictions for fuzzy time series [23] and fuzzy c-numbers [25]. Natural language processing, artificial intelligence, and handwriting recognition are applications that are well-suited to this gradation concept. According to Atanassov [4], IFS logic can be classified based on its membership degree (MD)

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C. Jana et al. (eds.), *Fuzzy Optimization, Decision-making and Operations Research*, https://doi.org/10.1007/978-3-031-35668-1_23

and non-membership degree (NMD) being equal to or less than one. A problem in decision-making (DM) can arise when MD and NMD are greater than one. A generalization of IFS, PFS is defined as the square sum of MD and NMD is less than or equal to 1, as presented by Yager [24]. PFS can be applied to various applications, as discussed by Akram et al. [1–3]. It has been demonstrated that the geometric aggregation operator (AO) can be applied to group DM in an IVPFS scheme by Rahman et al. [17]. According to Peng et al. [15], a Pythagorean fuzzy AO can be derived from interval values. Rahman et al. [18] have proposed several approaches to MADM using an induced interval-valued Pythagorean fuzzy Einstein AO. In Khan et al. [11], fuzzy Einstein and Choquet operators are applied based on Pythagorean theory. In the DM approach, fuzzy AO was introduced by Liu et al. [12].

The novel theory of NSs was introduced recently. The FS and IFS differ from neurosophy due to their knowledge of neutral thought. According to Smarandache [21], NS is based on philosophical reasoning. According to this logic, each proposition is assigned a degree of truth (TD), indeterminacy (ID), or falsity (FD). As part of NS, every element in the universe receives a degree from 0 to 1 of TD, ID, and FD. A philosophical approach to NSs generalizes classical sets, FSs, and IVFSs. In 1996, Smarandache et al. introduced the PNIVS [10]. In context analysis [20] and medical diagnosis [19], single-valued NSs are applied. Hamming distance (HD) and Euclidean distance (ED), normalized HD and ED and their similarity to MCDM and MADM models in Ejegwa [5]. PNSNIVS distance functions are generally generalized from PNSIVS. Zhang et al. [27] discussed PFS can be generalized by MCDM based on TOPSIS. A discussion of MADM under the MABAC and TOPSIS scheme was then presented by Peng et al. [16]. Various real-life situations have been addressed by Hwang et al. [6]. Jana et al. studied a generalization of bipolar fuzzy soft set (BFSS) with applications [7]. Jana et al. [8] introduced Pythagorean fuzzy Dombi AOs. The measurement of distance in pattern recognition is addressed by Ullah et al. [22]. CT-SVTrN Dombi AOs were identified by Jana et al. [9]. The idea of MADM for PNNIV with AOs is discussed by Palanikumar et al. [13]. Palanikumar et al. discussed spherical vague normal AOs are used for selecting farmers [14]. The following contributions are made by this work:

1. The introduction of a new CTri-NNIVS distance measure based on Euclidean and Hamming distances.
2. For CTri-NNIVN in MADM, we will examine the applicable definition, AOs, and robotic examples.
3. Using CTri-NNIVWA, CTri-NNIVWG, CTri-GNNIVWA, and CTri-GNNIVWG, determine the ideal values.
4. A decision based on Θ is made in order to achieve a specific result.

In this chapter, the new CTri-NNIVS concept is discussed. As a starting point, let us look at the CTri-NNIVS aggregation operators. A ranking will be established using these operators and applied to DM problems by way of an example. According to the table of contents, the chapter is divided into eight sections. In Sect. 1, you will find an introduction. Section 2 refers to some concepts that are explained. Section 3 describes some operations of the MADM based on CTri-NNIVN. The distance between CTri-NNIVNs is discussed in Sect. 4. For CTri-NNIVN, Sect. 5 discusses MADM using cosine AOs. As part of Sect. 6, MADM is compared with proposed and existing models using CTri-NNIV data, an algorithm with the most appropriate selection of robotic data. Section 7 contains the conclusion.

2 Preliminary

We will cover some basic definitions in this section so that we can conduct additional research.

Definition 1 ([24]) Let $\mathbb{Z}$ be the universal set. The PFS L in $\mathbb{Z}$ is $L = \left\{u, \langle \eth_L^{\mathbb{T}}(\varepsilon), \eth_L^{\mathbb{F}}(\varepsilon)\rangle \middle| \varepsilon \in \mathbb{Z}\right\}$, where $\eth_L^{\mathbb{T}} : \mathbb{Z} \to [0, 1]$ and $\eth_L^{\mathbb{F}} : \mathbb{Z} \to [0, 1]$ denotes MD and NMD of $\varepsilon \in \mathbb{Z}$ to L, respectively, and $0 \le (\eth_L^{\mathbb{T}}(\varepsilon))^2 + (\eth_L^{\mathbb{F}}(\varepsilon))^2 \le 1$. For comfortable, $L = \langle \eth_L^{\mathbb{T}}, \eth_L^{\mathbb{F}}\rangle$ is said to be a Pythagorean fuzzy number (PFN).

Definition 2 ([15]) The PIVFS L in $\mathbb{Z}$ is given by $L = \left\{u, \left\langle \eth_L^{\mathbb{T}}(\varepsilon), \eth_L^{\mathbb{F}}(\varepsilon)\right\rangle \middle| \varepsilon \in \mathbb{Z}\right\}$, where the functions $\eth_L^{\mathbb{T}} : \mathbb{Z} \to Int([0, 1])$ and $\eth_L^{\mathbb{F}} : \mathbb{Z} \to Int([0, 1])$ denote MD and NMD of $\varepsilon \in \mathbb{Z}$ to L, $0 \le (\eth_L^{\mathbb{T}+}(\varepsilon))^2 + (\eth_L^{\mathbb{F}+}(\varepsilon))^2 \le 1$. For comfortable, $L = \left\langle \left[\eth_L^{\mathbb{T}-}, \eth_L^{\mathbb{T}+}\right], \left[\eth_L^{\mathbb{F}-}, \eth_L^{\mathbb{F}+}\right]\right\rangle$ is said to be a Pythagorean interval-valued fuzzy number (PIVFN).

Definition 3 ([21]) The NS L in $\mathbb{Z}$ is $L = \left\{u, \langle \eth_L^{\mathbb{T}}(\varepsilon), \eth_L^{\mathbb{I}}(\varepsilon), \eth_L^{\mathbb{F}}(\varepsilon)\rangle \middle| \varepsilon \in \mathbb{Z}\right\}$, where $\eth_L^{\mathbb{T}} : \mathbb{Z} \to [0, 1]$, $\eth_L^{\mathbb{I}} : \mathbb{Z} \to [0, 1]$, and $\eth_L^{\mathbb{F}} : \mathbb{Z} \to [0, 1]$ denote TD, ID, and FD of $\varepsilon \in \mathbb{Z}$ to L, respectively, and $0 \le \eth_L^{\mathbb{T}}(\varepsilon) + \eth_L^{\mathbb{I}}(\varepsilon) + \eth_L^{\mathbb{F}}(\varepsilon) \le 3$. For comfortable, $L = \langle \eth_L^{\mathbb{T}}, \eth_L^{\mathbb{I}}, \eth_L^{\mathbb{F}}\rangle$ is said to be a neutrosophic number (NSN).

Definition 4 ([10]) The PNSS L in $\mathbb{Z}$ is $L = \left\{u, \langle \eth_L^{\mathbb{T}}(\varepsilon), \eth_L^{\mathbb{I}}(\varepsilon), \eth_L^{\mathbb{F}}(\varepsilon)\rangle \middle| \varepsilon \in \mathbb{Z}\right\}$, where $\eth_L^{\mathbb{T}} : \mathbb{Z} \to [0, 1]$, $\eth_L^{\mathbb{I}} : \mathbb{Z} \to [0, 1]$, and $\eth_L^{\mathbb{F}} : \mathbb{Z} \to [0, 1]$ denote TD, ID, and FD of $\varepsilon \in \mathbb{Z}$ to L, respectively, and $0 \le (\eth_L^{\mathbb{T}}(\varepsilon))^2 + (\eth_L^{\mathbb{I}}(\varepsilon))^2 + (\eth_L^{\mathbb{F}}(\varepsilon))^2 \le 2$. For comfortable, $L = \langle \eth_L^{\mathbb{T}}, \eth_L^{\mathbb{I}}, \eth_L^{\mathbb{F}}\rangle$ is said to be a Pythagorean neutrosophic number (PNSN).

Definition 5 ([15]) Let $L = \left\langle [\eth^{\mathbb{T}-}, \eth^{\mathbb{T}+}], [\eth^{\mathbb{F}-}, \eth^{\mathbb{F}+}]\right\rangle$, $L_1 = \left\langle [\eth_1^{\mathbb{T}-}, \eth_1^{\mathbb{T}+}], [\eth_1^{\mathbb{F}-}, \eth_1^{\mathbb{F}+}]\right\rangle$, and

$L_2 = \left\langle [\eth_2^{\mathbb{T}-}, \eth_2^{\mathbb{T}+}], [\eth_2^{\mathbb{F}-}, \eth_2^{\mathbb{F}+}]\right\rangle$ be PIVFNs, and $\Theta > 0$. Then:

1. $L_1 \boxplus L_2 = \begin{bmatrix} \left[\sqrt{(\eth_1^{\mathbb{T}-})^2 + (\eth_2^{\mathbb{T}-})^2 - (\eth_1^{\mathbb{T}-})^2 \cdot (\eth_2^{\mathbb{T}-})^2}, \sqrt{(\eth_1^{\mathbb{T}+})^2 + (\eth_2^{\mathbb{T}+})^2 - (\eth_1^{\mathbb{T}+})^2 \cdot (\eth_2^{\mathbb{T}+})^2}\right], \\ \left[\eth_1^{\mathbb{F}-} \cdot \eth_2^{\mathbb{F}-}, \eth_1^{\mathbb{F}+} \cdot \eth_2^{\mathbb{F}+}\right] \end{bmatrix}$.

2. $L_1 \bigcirc L_2 = \left[\begin{array}{c}\left[\eth_1^{\mathbb{T}-} \cdot \eth_2^{\mathbb{T}-}, \eth_1^{\mathbb{T}+} \cdot \eth_2^{\mathbb{T}+}\right], \\ \left[\sqrt{(\eth_1^{\mathbb{F}-})^2 + (\eth_2^{\mathbb{F}-})^2 - (\eth_1^{\mathbb{F}-})^2 \cdot (\eth_2^{\mathbb{F}-})^2}, \sqrt{(\eth_1^{\mathbb{F}+})^2 + (\eth_2^{\mathbb{F}+})^2 - (\eth_1^{\mathbb{F}+})^2 \cdot (\eth_2^{\mathbb{F}+})^2}\right]\end{array}\right].$

3. $\Theta \cdot L = \left[\left[\sqrt{1 - \left(1 - (\eth^{\mathbb{T}-})^2\right)^{\Theta}}, \sqrt{1 - \left(1 - (\eth^{\mathbb{T}+})^2\right)^{\Theta}}\right], \left[(\eth^{\mathbb{F}-})^{\Theta}, (\eth^{\mathbb{F}+})^{\Theta}\right]\right].$

4. $L^{\Theta} = \left[\left[(\eth^{\mathbb{T}-})^{\Theta}, (\eth^{\mathbb{T}+})^{\Theta}\right], \left[\sqrt{1 - \left(1 - (\eth^{\mathbb{F}-})^2\right)^{\Theta}}, \sqrt{1 - \left(1 - (\eth^{\mathbb{F}+})^2\right)^{\Theta}}\right]\right].$

Definition 6 ([15]) Let $L = \left\langle [\eth^{\mathbb{T}-}, \eth^{\mathbb{T}+}], [\eth^{\mathbb{F}-}, \eth^{\mathbb{F}+}] \right\rangle$ be the PIVFN:

(i) The score function is

$$S(L) = \frac{1}{2}\left((\eth^{\mathbb{T}-})^2 + (\eth^{\mathbb{T}+})^2 - (\eth^{\mathbb{F}-})^2 - (\eth^{\mathbb{F}+})^2\right), \ \textit{where } S(L) \in [-1, 1].$$

(ii) The accuracy function of L is

$$H(L) = \frac{1}{2}\left((\eth^{\mathbb{T}-})^2 + (\eth^{\mathbb{T}+})^2 + (\eth^{\mathbb{F}-})^2 + (\eth^{\mathbb{F}+})^2\right), \ \textit{where } H(L) \in [0, 1].$$

Definition 7 ([25]) The fuzzy number $M(\varepsilon) = e^{-\left(\frac{x-\chi}{\psi}\right)^2}$, $(\psi > 0)$ and $\varepsilon \in R$, is referred to as a normal fuzzy number (NFN) $M = (\chi, \psi)$, N is a set of normal fuzzy numbers (NFNs), and $R \in [-\infty, \infty]$.

Definition 8 ([23]) Let $L = (\chi_1, \psi_1) \in N$and $M = (\chi_2, \psi_2) \in N$, where $(\psi_1, \psi_2 > 0)$. Then $\mathbb{D}(L, M) = \sqrt{(\chi_1 - \chi_2)^2 + \frac{1}{2}(\psi_1 - \psi_2)^2}$.

3 Some Basic Operation Based on CTri-NNIVN

Define cosine trigonometric neutrosophic interval-valued numbers (CTri-NIVN) and neutrosophic interval-valued numbers (NFN). This section develops a new notion of CTri-NIVN with normal (CTri-NNIVN) and its fundamental operations.

Definition 9 Let $(\chi, \psi) \in N$, $L = \langle (\chi, \psi); [\eth^{\mathbb{T}-}, \eth^{\mathbb{T}+}], [\eth^{\mathbb{I}-}, \eth^{\mathbb{I}+}], [\eth^{\mathbb{F}-}, \eth^{\mathbb{F}+}] \rangle$ be NSNIVN. Then we form a cosine trigonometric NNIVN (CTri-NNIVN) set as
$\cos L = \{[\cos\left(\pi/2 \cdot (\eth_L^{\mathbb{T}-}(\varepsilon))\right), \cos\left(\pi/2 \cdot (\eth_L^{\mathbb{T}+}(\varepsilon))\right)],$
$[1 - \cos\left(\pi/2 \cdot 1 - (\eth_L^{\mathbb{I}-}(\varepsilon))\right), 1 - \cos\left(\pi/2 \cdot 1 - (\eth_L^{\mathbb{I}+}(\varepsilon))\right)], [1 - \cos(\pi/2 \cdot$
$(1 - \eth_L^{\mathbb{F}-}(\varepsilon))\right),$
$1 - \cos\left(\pi/2 \cdot (1 - \eth_L^{\mathbb{F}+}(\varepsilon))\right)]\}$. Thus, $\cos L$ is a CTri-NNIVN and satisfied the condition that

$\cos\left(\pi/2 \cdot \partial_L^{\mathbb{T}+}(\varepsilon)\right) \in [0,1]$, $\cos\left(\pi/2 \cdot \partial_L^{\mathbb{I}+}(\varepsilon)\right) \in [0,1]$ and $1 - \cos(\pi/2 \cdot \left(1 - \partial_L^{\mathbb{F}+}(\varepsilon)\right)) \in [0,1]$.

Hence, $\cos L = \left\{\left[\cos\left(\pi/2 \cdot (\partial_L^{\mathbb{T}-}(\varepsilon))\right), \cos\left(\pi/2 \cdot (\partial_L^{\mathbb{T}+}(\varepsilon))\right)\right], 1 - \left[\cos\left(\pi/2 \cdot (1 - (\partial_L^{\mathbb{I}-}(\varepsilon)))\right),\right.\right.$
$\left.\left. 1 - \cos\left(\pi/2 \cdot (1 - (\partial_L^{\mathbb{I}+}(\varepsilon)))\right)\right], \left[1 - \cos\left(\pi/2 \cdot ((1 - \partial_L^{\mathbb{F}-}(\varepsilon)))\right), 1 - \cos(\pi/2 \cdot ((1 - \partial_L^{\mathbb{F}+}(\varepsilon)))\right)\right]\right\}$ is a CTri-NNIVN, put $\left[\partial_L^{\mathbb{T}-}, \partial_L^{\mathbb{T}+}\right] = \left[\partial_L^{\mathbb{T}-} e^{-\left(\frac{y-\chi}{\psi}\right)^2}, \partial_L^{\mathbb{T}+} e^{-\left(\frac{y-\chi}{\psi}\right)^2}\right]$ and $\left[\partial_L^{\mathbb{I}-}, \partial_L^{\mathbb{I}+}\right] = \left[\partial_L^{\mathbb{I}-} e^{-\left(\frac{y-\chi}{\psi}\right)^2}, \partial_L^{\mathbb{I}+} e^{-\left(\frac{y-\chi}{\psi}\right)^2}\right]$ and $\left[\partial_L^{\mathbb{F}-}, \partial_L^{\mathbb{F}+}\right] = \left[\partial_L^{\mathbb{F}-} e^{-\left(\frac{y-\chi}{\psi}\right)^2}, \partial_L^{\mathbb{F}+} e^{-\left(\frac{y-\chi}{\psi}\right)^2}\right]$, $y \in Y$. Here, Y is a non-empty set.

Definition 10 Let $L = \left\langle(\chi, \psi); [\partial^{\mathbb{T}-}, \partial^{\mathbb{T}+}], [\partial^{\mathbb{I}-}, \partial^{\mathbb{I}+}], [\partial^{\mathbb{F}-}, \partial^{\mathbb{F}+}]\right\rangle$ be the CTri-NIVN. Then the score function of L (where $S(L)$ lies between -1 and 1) is founded by

$$S(L) = \frac{\chi}{2}\left(\frac{\begin{array}{c} 2 + (\cos^2(\pi/2 \cdot \partial^{\mathbb{T}-})) + (\cos^2(\pi/2 \cdot \partial^{\mathbb{T}+})) - (\cos^2(\pi/2 \cdot \partial^{\mathbb{I}-})) \\ -(\cos^2(\pi/2 \cdot \partial^{\mathbb{I}+})) - (\cos^2(\pi/2 \cdot \partial^{\mathbb{F}-})) - (\cos^2(\pi/2 \cdot \partial^{\mathbb{F}+})) \end{array}}{2}\right).$$

Definition 11 Let $L = \left\langle(\chi, \psi); [\partial^{\mathbb{T}-}, \partial^{\mathbb{T}+}], [\partial^{\mathbb{I}-}, \partial^{\mathbb{I}+}], [\partial^{\mathbb{F}-}, \partial^{\mathbb{F}+}]\right\rangle$, $L_1 = \left\langle(\chi_1, \psi_1); [\partial_1^{\mathbb{T}-}, \partial_1^{\mathbb{T}+}], [\partial_1^{\mathbb{I}-}, \partial_1^{\mathbb{I}+}], [\partial_1^{\mathbb{F}-}, \partial_1^{\mathbb{F}+}]\right\rangle$, and $L_2 = \left\langle(\chi_2, \psi_2); [\partial_2^{\mathbb{T}-}, \partial_2^{\mathbb{T}+}], [\partial_2^{\mathbb{I}-}, \partial_2^{\mathbb{I}+}], [\partial_2^{\mathbb{F}-}, \partial_2^{\mathbb{F}+}]\right\rangle$ be CTri-NNIVNs, and $\Theta > 0$. Now, we defined the basic operations as follows:

1.

$$\cos L_1 \boxplus \cos L_2 = \left[\begin{array}{c} (\chi_1 \boxplus \chi_2, \psi_1 \boxplus \psi_2); \\ \left[\begin{array}{c} (\cos^2(\pi/2 \cdot \partial_1^{\mathbb{T}-}))^{\Theta} + (\cos^2(\pi/2 \cdot \partial_2^{\mathbb{T}-}))^{\Theta} \\ -(\cos^2(\pi/2 \cdot \partial_1^{\mathbb{T}-}))^{\Theta} \cdot (\cos^2(\pi/2 \cdot \partial_2^{\mathbb{T}-}))^{\Theta}, \\ (\cos^2(\pi/2 \cdot \partial_1^{\mathbb{T}+}))^{\Theta} + (\cos^2(\pi/2 \cdot \partial_2^{\mathbb{T}+}))^{\Theta} \\ -(\cos^2(\pi/2 \cdot \partial_1^{\mathbb{T}+}))^{\Theta} \cdot (\cos^2(\pi/2 \cdot \partial_2^{\mathbb{T}+}))^{\Theta} \end{array}\right], \\ \left[\cos^2(\pi/2 \cdot \partial_1^{\mathbb{I}-}) \cdot \cos^2(\pi/2 \cdot \partial_2^{\mathbb{I}-}), \cos^2(\pi/2 \cdot \partial_1^{\mathbb{I}+}) \cdot \cos^2(\pi/2 \cdot \partial_2^{\mathbb{I}+})\right], \\ \left[\cos^2(\pi/2 \cdot \partial_1^{\mathbb{F}-}) \cdot \cos^2(\pi/2 \cdot \partial_2^{\mathbb{F}-}), \cos^2(\pi/2 \cdot \partial_1^{\mathbb{F}+}) \cdot \cos^2(\pi/2 \cdot \partial_2^{\mathbb{F}+})\right] \end{array}\right].$$

2.

$$\cos L_1 \bigcirc \cos L_2 = \begin{bmatrix} (\chi_1 \bigcirc \chi_2, \psi_1 \bigcirc \psi_2); \\ \left[\cos^2(\pi/2 \cdot \partial_1^{\mathbb{T}-}) \cdot \cos^2(\pi/2 \cdot \partial_2^{\mathbb{T}-}), \cos^2(\pi/2 \cdot \partial_1^{\mathbb{T}+}) \cdot \cos^2(\pi/2 \cdot \partial_2^{\mathbb{T}+})\right], \\ \begin{bmatrix} (\cos^2(\pi/2 \cdot \partial_1^{\mathbb{I}-}))^{\Theta} + (\cos^2(\pi/2 \cdot \partial_2^{\mathbb{I}-}))^{\Theta} \\ -(\cos^2(\pi/2 \cdot \partial_1^{\mathbb{I}-}))^{\Theta} \cdot (\cos^2(\pi/2 \cdot \partial_2^{\mathbb{I}-}))^{\Theta}, \\ (\cos^2(\pi/2 \cdot \partial_1^{\mathbb{I}+}))^{\Theta} + (\cos^2(\pi/2 \cdot \partial_2^{\mathbb{I}+}))^{\Theta} \\ -(\cos^2(\pi/2 \cdot \partial_1^{\mathbb{I}+}))^{\Theta} \cdot (\cos^2(\pi/2 \cdot \partial_2^{\mathbb{I}+}))^{\Theta} \end{bmatrix}, \\ \begin{bmatrix} (\cos^2(\pi/2 \cdot \partial_1^{\mathbb{F}-}))^{\Theta} + (\cos^2(\pi/2 \cdot \partial_2^{\mathbb{F}-}))^{\Theta} \\ -(\cos^2(\pi/2 \cdot \partial_1^{\mathbb{F}-}))^{\Theta} \cdot (\cos^2(\pi/2 \cdot \partial_2^{\mathbb{F}-}))^{\Theta}, \\ (\cos^2(\pi/2 \cdot \partial_1^{\mathbb{F}+}))^{\Theta} + (\cos^2(\pi/2 \cdot \partial_2^{\mathbb{F}+}))^{\Theta} \\ -(\cos^2(\pi/2 \cdot \partial_1^{\mathbb{F}+}))^{\Theta} \cdot (\cos^2(\pi/2 \cdot \partial_2^{\mathbb{F}+}))^{\Theta} \end{bmatrix} \end{bmatrix}.$$

3. $\Theta \cdot \cos L$

$$= \begin{bmatrix} (\Theta \cdot \chi, \Theta \cdot \psi); \\ \left[1 - \left(1 - (\cos^2(\pi/2 \cdot \partial^{\mathbb{T}-}))^{\Theta}\right)^{\Theta}, 1 - \left(1 - (\cos^2(\pi/2 \cdot \partial^{\mathbb{T}+}))^{\Theta}\right)^{\Theta}\right], \\ \left[(\cos^2(\pi/2 \cdot \partial^{\mathbb{I}-}))^{\Theta}, (\cos^2(\pi/2 \cdot \partial^{\mathbb{I}+}))^{\Theta}\right], \\ \left[(\cos^2(\pi/2 \cdot \partial^{\mathbb{F}-}))^{\Theta}, (\cos^2(\pi/2 \cdot \partial^{\mathbb{F}+}))^{\Theta}\right] \end{bmatrix}.$$

4. $(\cos L)^{\Theta}$

$$= \begin{bmatrix} (\chi^{\Theta}, \psi^{\Theta}); \left[(\cos^2(\pi/2 \cdot \partial^{\mathbb{T}-}))^{\Theta}, (\cos^2(\pi/2 \cdot \partial^{\mathbb{T}+}))^{\Theta}\right], \\ \left[1 - \left(1 - (\cos^2(\pi/2 \cdot \partial^{\mathbb{I}-}))^{\Theta}\right)^{\Theta}, 1 - \left(1 - (\cos^2(\pi/2 \cdot \partial^{\mathbb{I}+}))^{\Theta}\right)^{\Theta}\right], \\ \left[1 - \left(1 - (\cos^2(\pi/2 \cdot \partial^{\mathbb{F}-}))^{\Theta}\right)^{\Theta}, 1 - \left(1 - (\cos^2(\pi/2 \cdot \partial^{\mathbb{F}+}))^{\Theta}\right)^{\Theta}\right] \end{bmatrix}.$$

4 Distance for CTri-NNIVNs

The Euclidean and Hamming distances for CTri-NNIVNs are introduced, and their basic mathematical properties are discussed.

Definition 12 Let $L_1 = \left\langle(\chi_1, \psi_1); [\partial_1^{\mathbb{T}-}, \partial_1^{\mathbb{T}+}], [\partial_1^{\mathbb{I}-}, \partial_1^{\mathbb{I}+}], [\partial_1^{\mathbb{F}-}, \partial_1^{\mathbb{F}+}]\right\rangle$ and $L_2 = \left\langle(\chi_2, \psi_2); [\partial_2^{\mathbb{T}-}, \partial_2^{\mathbb{T}+}], [\partial_2^{\mathbb{I}-}, \partial_2^{\mathbb{I}+}], [\partial_2^{\mathbb{F}-}, \partial_2^{\mathbb{F}+}]\right\rangle$ be CTri-NNIVNs. The Euclidean distance and Hamming distance are defined as follows:

$$\mathbb{D}_E\Big(L_1, L_2\Big) = \frac{1}{2}\sqrt{\begin{matrix} \left[\frac{1+I_1+1+I_2}{2}\chi_1 - \frac{1+I_3+1+I_4}{2}\chi_2\right]^2 \\ +\frac{1}{2}\left[\frac{1+I_1+1+I_2}{2}\psi_1 - \frac{1+I_3+1+I_4}{2}\psi_2\right]^2 \end{matrix}}$$

$$\mathbb{D}_H\Big(L_1, L_2\Big) = \frac{1}{2}\left[\begin{array}{c} \left|\frac{1+I_1+1+I_2}{2}\chi_1 - \frac{1+I_3+1+I_4}{2}\chi_2\right| \\ +\frac{1}{2}\left|\frac{1+I_1+1+I_2}{2}\psi_1 - \frac{1+I_3+1+I_4}{2}\psi_2\right| \end{array}\right]$$

$$I_1 = \cos^2(\pi/2 \cdot \eth_1^{\mathbb{T}-}) - \cos^2(\pi/2 \cdot \eth_1^{\mathbb{I}-}) - \cos^2(\pi/2 \cdot \eth_1^{\mathbb{F}-})$$

$$I_2 = \cos^2(\pi/2 \cdot \eth_1^{\mathbb{T}+}) - \cos^2(\pi/2 \cdot \eth_1^{\mathbb{I}+}) - \cos^2(\pi/2 \cdot \eth_1^{\mathbb{F}+})$$

$$I_3 = \cos^2(\pi/2 \cdot \eth_2^{\mathbb{T}-}) - \cos^2(\pi/2 \cdot \eth_2^{\mathbb{I}-}) - \cos^2(\pi/2 \cdot \eth_2^{\mathbb{F}-})$$

$$I_4 = \cos^2(\pi/2 \cdot \eth_2^{\mathbb{T}+}) - \cos^2(\pi/2 \cdot \eth_2^{\mathbb{I}+}) - \cos^2(\pi/2 \cdot \eth_2^{\mathbb{F}+}).$$

Theorem 1 *Let* $L_1 = \Big\langle(\chi_1, \psi_1); [\eth_1^{\mathbb{T}-}, \eth_1^{\mathbb{T}+}], [\eth_1^{\mathbb{I}-}, \eth_1^{\mathbb{I}+}], [\eth_1^{\mathbb{F}-}, \eth_1^{\mathbb{F}+}]\Big\rangle$, $L_2 = \Big\langle(\chi_2, \psi_2); [\eth_2^{\mathbb{T}-}, \eth_2^{\mathbb{T}+}], [\eth_2^{\mathbb{I}-}, \eth_2^{\mathbb{I}+}], [\eth_2^{\mathbb{F}-}, \eth_2^{\mathbb{F}+}]\Big\rangle$, $L_3 = \Big\langle(\chi_3, \psi_3); [\eth_3^{\mathbb{T}-}, \eth_3^{\mathbb{T}+}], [\eth_3^{\mathbb{I}-}, \eth_3^{\mathbb{I}+}], [\eth_3^{\mathbb{F}-}, \eth_3^{\mathbb{F}+}]\Big\rangle$ *be CTri-NNIVNs. Then* $\mathbb{D}_E(L_1, L_2)$ *satisfies the conditions:*

1. $\mathbb{D}_E(L_1, L_2) = 0$, *if and only if* $L_1 = L_2$.
2. $\mathbb{D}_E(L_1, L_2) = \mathbb{D}_E(L_2, L_1)$.
3. $\mathbb{D}_E(L_1, L_3) \leq \mathbb{D}_E(L_1, L_2) + \mathbb{D}_E(L_2, L_3)$.

Proof Proofs of (1) and (2) that are easiest way. It is remaining to Then (3) as below.

Now, $\Big(\mathbb{D}_E(L_1, L_2) + \mathbb{D}_E(L_2, L_3)\Big)^2 = \left[\begin{array}{c} \frac{1}{2}\Big((\Gamma_1\chi_1 - \Gamma_2\chi_2)^2 + \frac{1}{2}(\Gamma_1\psi_1 - \Gamma_2\psi_2)^2\Big)^{1/2} \\ +\frac{1}{2}\Big((\Gamma_2\chi_2 - \Gamma_3\chi_3)^2 + \frac{1}{2}(\Gamma_2\psi_2 - \Gamma_3\psi_3)^2\Big)^{1/2} \end{array}\right]^2$

implies

$$\begin{aligned} &\frac{1}{4}\Big((\Gamma_1\chi_1 - \Gamma_2\chi_2)^2 + \frac{1}{2}(\Gamma_1\psi_1 - \Gamma_2\psi_2)^2\Big) \\ &+ \frac{1}{4}\Big((\Gamma_2\chi_2 - \Gamma_3\chi_3)^2 + \frac{1}{2}(\Gamma_2\psi_2 - \Gamma_3\psi_3)^2\Big) \\ &+ \frac{1}{2}\Bigg(\sqrt{(\Gamma_1\chi_1 - \Gamma_2\chi_2)^2 + \frac{1}{2}(\Gamma_1\psi_1 - \Gamma_2\psi_2)^2} \\ &\times \sqrt{(\Gamma_2\chi_2 - \Gamma_3\chi_3)^2 + \frac{1}{2}(\Gamma_2\psi_2 - \Gamma_3\psi_3)^2}\ \Bigg). \end{aligned}$$

Put $\varGamma_1 = \frac{1+I_1+1+I_2}{2}$, $\varGamma_2 = \frac{1+I_3+1+I_4}{2}$, and $\varGamma_3 = \frac{1+I_5+1+I_6}{2}$.

$$I_1 = \cos^2(\pi/2 \cdot \eth_1^{\mathbb{T}-}) - \cos^2(\pi/2 \cdot \eth_1^{\mathbb{I}-}) - \cos^2(\pi/2 \cdot \eth_1^{\mathbb{F}-}),$$

$$I_2 = \cos^2(\pi/2 \cdot \eth_1^{\mathbb{T}+}) - \cos^2(\pi/2 \cdot \eth_1^{\mathbb{I}+}) - \cos^2(\pi/2 \cdot \eth_1^{\mathbb{F}+}),$$

$$I_3 = \cos^2(\pi/2 \cdot \eth_2^{\mathbb{T}-}) - \cos^2(\pi/2 \cdot \eth_2^{\mathbb{I}-}) - \cos^2(\pi/2 \cdot \eth_2^{\mathbb{F}-}),$$

$$I_4 = \cos^2(\pi/2 \cdot \eth_2^{\mathbb{T}+}) - \cos^2(\pi/2 \cdot \eth_2^{\mathbb{I}+}) - \cos^2(\pi/2 \cdot \eth_2^{\mathbb{F}+}),$$

$$I_5 = \cos^2(\pi/2 \cdot \eth_3^{\mathbb{T}-}) - \cos^2(\pi/2 \cdot \eth_3^{\mathbb{I}-}) - \cos^2(\pi/2 \cdot \eth_3^{\mathbb{F}-}),$$

$$I_6 = \cos^2(\pi/2 \cdot \eth_3^{\mathbb{T}+}) - \cos^2(\pi/2 \cdot \eth_3^{\mathbb{I}+}) - \cos^2(\pi/2 \cdot \eth_3^{\mathbb{F}+}).$$

Hence, $\Big(\mathbb{D}_E(L_1, L_2) + \mathbb{D}_E(L_2, L_3)\Big)^2$

$$\begin{aligned}
&\geq \frac{1}{4}\Big((\varGamma_1\chi_1 - \varGamma_2\chi_2)^2 + \frac{1}{2}(\varGamma_1\psi_1 - \varGamma_2\psi_2)^2\Big) \\
&+ \frac{1}{4}\Big((\varGamma_2\chi_2 - \varGamma_3\chi_3)^2 + \frac{1}{2}(\varGamma_2\psi_2 - \varGamma_3\psi_3)^2\Big) \\
&\quad + \frac{1}{2}\Big((\varGamma_1\chi_1 - \varGamma_2\chi_2) \times (\varGamma_2\chi_2 - \varGamma_3\chi_3) + \frac{1}{2}(\varGamma_1\psi_1 - \varGamma_2\psi_2) \times (\varGamma_2\psi_2 - \varGamma_3\psi_3)\Big) \\
&= \frac{1}{4}(\varGamma_1\chi_1 - \varGamma_2\chi_2 + \varGamma_2\chi_2 - \varGamma_3\chi_3)^2 + \frac{1}{8}(\varGamma_1\psi_1 - \varGamma_2\psi_2 + \varGamma_2\psi_2 - \varGamma_3\psi_3)^2 \\
&= \frac{1}{4}(\varGamma_1\chi_1 - \varGamma_3\chi_3)^2 + \frac{1}{8}(\varGamma_1\psi_1 - \varGamma_3\psi_3)^2 \\
&= \mathbb{D}_E(L_1, L_3).
\end{aligned}$$

Corollary 1 *Let* $L_1 = \Big\langle(\chi_1, \psi_1); [\eth_1^{\mathbb{T}-}, \eth_1^{\mathbb{T}+}], [\eth_1^{\mathbb{I}-}, \eth_1^{\mathbb{I}+}], [\eth_1^{\mathbb{F}-}, \eth_1^{\mathbb{F}+}]\Big\rangle$, $L_2 = \Big\langle(\chi_2, \psi_2); [\eth_2^{\mathbb{T}-}, \eth_2^{\mathbb{T}+}], [\eth_2^{\mathbb{I}-}, \eth_2^{\mathbb{I}+}], [\eth_2^{\mathbb{F}-}, \eth_2^{\mathbb{F}+}]\Big\rangle$, $L_3 = \Big\langle(\chi_3, \psi_3); [\eth_3^{\mathbb{T}-}, \eth_3^{\mathbb{T}+}], [\eth_3^{\mathbb{I}-}, \eth_3^{\mathbb{I}+}], [\eth_3^{\mathbb{F}-}, \eth_3^{\mathbb{F}+}]\Big\rangle$ *be the CTri-NNIVNs. Then* $\mathbb{D}_H(L_1, L_2)$ *satisfies the conditions:*

1. $\mathbb{D}_H(L_1, L_2)=0$, *if and only if* $L_1 = L_2$.
2. $\mathbb{D}_H(L_1, L_2) = \mathbb{D}_H(L_2, L_1)$.
3. $\mathbb{D}_H(L_1, L_3) \leq \mathbb{D}_H(L_1, L_2) + \mathbb{D}_H(L_2, L_3)$.

Proof The proof is devoted to Theorem 1.

5 Aggregation Operators for CTri-NNIVNs

This section introduces the new operators based on CTri-NNIVWA, CTri-NNIVWG, CTri-GNNIVWA, and CTri-GNNIVWG.

5.1 CTri-NNIV Weighted Averaging (CTri-NNIVWA)

Definition 13 Let $L_i = \left\langle (\chi_i, \psi_i); [\eth_i^{\mathbb{T}-}, \eth_i^{\mathbb{T}+}], [\eth_i^{\mathbb{I}-}, \eth_i^{\mathbb{I}+}], [\eth_i^{\mathbb{F}-}, \eth_i^{\mathbb{F}+}] \right\rangle$ be a finite collection of CTri-NNIVNs, and $W = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ be the weight of L_i, $\sigma_i \geq 0$, and $\boxplus_{i=1}^{n} \sigma_i = 1$. Then CTri-NNIVWA $(L_1, L_2, \ldots, L_n) = \boxplus_{i=1}^{n} \sigma_i L_i,\ i = 1, 2, \ldots, n$.

Theorem 2 *Let* $L_i = \left\langle (\chi_i, \psi_i); [\eth_i^{\mathbb{T}-}, \eth_i^{\mathbb{T}+}], [\eth_i^{\mathbb{I}-}, \eth_i^{\mathbb{I}+}], [\eth_i^{\mathbb{F}-}, \eth_i^{\mathbb{F}+}] \right\rangle$ *be a finite collection of CTri-NNIVNs. Then* $CTri-NNIVWA(L_1, L_2, \ldots, L_n)$=

$$\begin{bmatrix} \left(\boxplus_{i=1}^{n} \sigma_i \chi_i, \boxplus_{i=1}^{n} \sigma_i \psi_i \right); \\ \left[1 - \bigcirc_{i=1}^{n} \left(1 - (\cos^2(\pi/2 \cdot \eth_i^{\mathbb{T}-}))^{\Theta} \right)^{\sigma_i}, \right. \\ \left. 1 - \bigcirc_{i=1}^{n} \left(1 - (\cos^2(\pi/2 \cdot \eth_i^{\mathbb{T}+}))^{\Theta} \right)^{\sigma_i} \right], \\ \left[\bigcirc_{i=1}^{n} (\cos^2(\pi/2 \cdot \eth_i^{\mathbb{I}-}))^{\sigma_i}, \bigcirc_{i=1}^{n} (\cos^2(\pi/2 \cdot \eth_i^{\mathbb{I}+}))^{\sigma_i} \right], \\ \left[\bigcirc_{i=1}^{n} (\cos^2(\pi/2 \cdot \eth_i^{\mathbb{F}-}))^{\sigma_i}, \bigcirc_{i=1}^{n} (\cos^2(\pi/2 \cdot \eth_i^{\mathbb{F}+}))^{\sigma_i} \right] \end{bmatrix}.$$

Proof The proof of Theorem 2 follows from the mathematical induction approach. If $n = 2$, then CTri-NNIVWA $(L_1, L_2) = \sigma_1 \cos L_1 \boxplus \sigma_2 \cos L_2$; put

$$\sigma_1 \cos L_1 = \begin{bmatrix} \left(\sigma_1 \chi_1, \sigma_1 \psi_1 \right); \\ \left[1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_1^{\mathbb{T}-}))^{\Theta} \right)^{\sigma_1}, \right. \\ \left. 1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_1^{\mathbb{T}+}))^{\Theta} \right)^{\sigma_1} \right], \\ \left[(\cos^2(\pi/2 \cdot \eth_1^{\mathbb{I}-}))^{\sigma_1}, (\cos^2(\pi/2 \cdot \eth_1^{\mathbb{I}+}))^{\sigma_1} \right], \\ \left[(\cos^2(\pi/2 \cdot \eth_1^{\mathbb{F}-}))^{\sigma_1}, (\cos^2(\pi/2 \cdot \eth_1^{\mathbb{F}+}))^{\sigma_1} \right] \end{bmatrix}$$

and

$$\sigma_2 \cos L_2 = \begin{bmatrix} \Big(\sigma_2\chi_2, \sigma_2\psi_2\Big); \\ \Big[1-\Big(1-(\cos^2(\pi/2\cdot \eth_2^{\mathbb{T}-}))^{\Theta}\Big)^{\sigma_2}, \\ 1-\Big(1-(\cos^2(\pi/2\cdot \eth_2^{\mathbb{T}+}))^{\Theta}\Big)^{\sigma_2}\Big], \\ \Big[(\cos^2(\pi/2\cdot \eth_2^{\mathbb{I}-}))^{\sigma_2}, (\cos^2(\pi/2\cdot \eth_2^{\mathbb{I}+}))^{\sigma_2}\Big], \\ \Big[(\cos^2(\pi/2\cdot \eth_2^{\mathbb{F}-}))^{\sigma_2}, (\cos^2(\pi/2\cdot \eth_2^{\mathbb{F}+}))^{\sigma_2}\Big] \end{bmatrix}.$$

We apply to Definition 11, $\sigma_1 \cos L_1 \boxplus \sigma_2 \cos L_2$

$$= \begin{bmatrix} \Big(\sigma_1\chi_1 \boxplus \sigma_2\chi_2, \sigma_1\psi_1 \boxplus \sigma_2\psi_2\Big); \\ \begin{bmatrix} \Big(1-\Big(1-(\cos^2(\pi/2\cdot \eth_1^{\mathbb{T}-}))^{\Theta}\Big)^{\sigma_1}\Big) \\ +\Big(1-\Big(1-(\cos^2(\pi/2\cdot \eth_2^{\mathbb{T}-}))^{\Theta}\Big)^{\sigma_2}\Big) \\ -\Big(1-\Big(1-(\cos^2(\pi/2\cdot \eth_1^{\mathbb{T}-}))^{\Theta}\Big)^{\sigma_1}\Big) \\ \cdot\Big(1-\Big(1-(\cos^2(\pi/2\cdot \eth_2^{\mathbb{T}-}))^{\Theta}\Big)^{\sigma_2}\Big), \\ \Big(1-\Big(1-(\cos^2(\pi/2\cdot \eth_1^{\mathbb{T}+}))^{\Theta}\Big)^{\sigma_1}\Big) \\ +\Big(1-\Big(1-(\cos^2(\pi/2\cdot \eth_2^{\mathbb{T}+}))^{\Theta}\Big)^{\sigma_2}\Big) \\ -\Big(1-\Big(1-(\cos^2(\pi/2\cdot \eth_1^{\mathbb{T}+}))^{\Theta}\Big)^{\sigma_1}\Big) \\ \cdot\Big(1-\Big(1-(\cos^2(\pi/2\cdot \eth_2^{\mathbb{T}+}))^{\Theta}\Big)^{\sigma_2}\Big) \end{bmatrix} \\ \Big[(\cos^2(\pi/2\cdot \eth_1^{\mathbb{I}-}))^{\sigma_1}\cdot(\cos^2(\pi/2\cdot \eth_2^{\mathbb{I}-}))^{\sigma_2}, \\ (\cos^2(\pi/2\cdot \eth_1^{\mathbb{I}+}))^{\sigma_1}(\cos^2(\pi/2\cdot \eth_2^{\mathbb{I}+}))^{\sigma_2}\Big], \\ \Big[(\cos^2(\pi/2\cdot \eth_1^{\mathbb{F}-}))^{\sigma_1}\cdot(\cos^2(\pi/2\cdot \eth_2^{\mathbb{F}-}))^{\sigma_2}, \\ (\cos^2(\pi/2\cdot \eth_1^{\mathbb{F}+}))^{\sigma_1}(\cos^2(\pi/2\cdot \eth_2^{\mathbb{F}+}))^{\sigma_2}\Big] \end{bmatrix},$$

$$= \begin{bmatrix} \left(\sigma_1\chi_1 \boxplus \sigma_2\chi_2, \sigma_1\psi_1 \boxplus \sigma_2\psi_2\right); \\ \begin{bmatrix} 1-\left(1-(\cos^2(\pi/2\cdot\eth_1^{\mathbb{T}-}))^{\Theta}\right)^{\sigma_1} \\ \cdot\left(1-(\cos^2(\pi/2\cdot\eth_2^{\mathbb{T}-}))^{\Theta}\right)^{\sigma_2}, \\ 1-\left(1-(\cos^2(\pi/2\cdot\eth_1^{\mathbb{T}+}))^{\Theta}\right)^{\sigma_1} \\ \cdot\left(1-(\cos^2(\pi/2\cdot\eth_2^{\mathbb{T}+}))^{\Theta}\right)^{\sigma_2} \end{bmatrix}, \\ \Big[(\cos^2(\pi/2\cdot\eth_1^{\mathbb{I}-}))^{\sigma_1}\cdot(\cos^2(\pi/2\cdot\eth_2^{\mathbb{I}-}))^{\sigma_2}, \\ (\cos^2(\pi/2\cdot\eth_1^{\mathbb{I}+}))^{\sigma_1}\cdot(\cos^2(\pi/2\cdot\eth_2^{\mathbb{I}+}))^{\sigma_2}\Big], \\ \Big[(\cos^2(\pi/2\cdot\eth_1^{\mathbb{F}-}))^{\sigma_1}\cdot(\cos^2(\pi/2\cdot\eth_2^{\mathbb{F}-}))^{\sigma_2}, \\ (\cos^2(\pi/2\cdot\eth_1^{\mathbb{F}+}))^{\sigma_1}\cdot(\cos^2(\pi/2\cdot\eth_2^{\mathbb{F}+}))^{\sigma_2}\Big] \end{bmatrix}.$$

Hence, $CTri-NNIVWA(L_1, L_2)$

$$= \begin{bmatrix} \left(\boxplus_{i=1}^2 \sigma_i\chi_i, \boxplus_{i=1}^2\sigma_i\psi_i\right); \\ \Big[1-\bigcirc_{i=1}^2\left(1-(\cos^2(\pi/2\cdot\eth_i^{\mathbb{T}-}))^{\Theta}\right)^{\sigma_i}, \\ 1-\bigcirc_{i=1}^2\left(1-(\cos^2(\pi/2\cdot\eth_i^{\mathbb{T}+}))^{\Theta}\right)^{\sigma_i}\Big], \\ \Big[\bigcirc_{i=1}^2 (\cos^2(\pi/2\cdot\eth_i^{\mathbb{I}-}))^{\sigma_i}, \bigcirc_{i=1}^2(\cos^2(\pi/2\cdot\eth_i^{\mathbb{I}+}))^{\sigma_i}\Big], \\ \Big[\bigcirc_{i=1}^2 (\cos^2(\pi/2\cdot\eth_i^{\mathbb{F}-}))^{\sigma_i}, \bigcirc_{i=1}^2(\cos^2(\pi/2\cdot\eth_i^{\mathbb{F}+}))^{\sigma_i}\Big] \end{bmatrix}.$$

Also, it is valid for $n \geq 3$. Similarly, $CTri-NNIVWA(L_1, L_2, \ldots, L_l) =$

$$\begin{bmatrix} \left(\boxplus_{i=1}^l \sigma_i\chi_i, \boxplus_{i=1}^l\sigma_i\psi_i\right); \\ \Big[1-\bigcirc_{i=1}^l\left(1-(\cos^2(\pi/2\cdot\eth_i^{\mathbb{T}-}))^{\Theta}\right)^{\sigma_i}, \\ 1-\bigcirc_{i=1}^l\left(1-(\cos^2(\pi/2\cdot\eth_i^{\mathbb{T}+}))^{\Theta}\right)^{\sigma_i}\Big], \\ \Big[\bigcirc_{i=1}^l (\cos^2(\pi/2\cdot\eth_i^{\mathbb{I}-}))^{\sigma_i}, \bigcirc_{i=1}^l(\cos^2(\pi/2\cdot\eth_i^{\mathbb{I}+}))^{\sigma_i}\Big], \\ \Big[\bigcirc_{i=1}^l (\cos^2(\pi/2\cdot\eth_i^{\mathbb{F}-}))^{\sigma_i}, \bigcirc_{i=1}^l(\cos^2(\pi/2\cdot\eth_i^{\mathbb{F}+}))^{\sigma_i}\Big] \end{bmatrix}.$$

If $n = l + 1$, then CTri-NNIVWA $(L_1, L_2, \ldots, L_l, L_{l+1})$

$$= \left[\begin{array}{c} \left(\boxplus_{i=1}^{l}\sigma_i\chi_i \boxplus \sigma_{l+1}\chi_{l+1}, \boxplus_{i=1}^{l}\sigma_i\psi_i \boxplus \sigma_{l+1}\psi_{l+1}\right); \\ \left[\begin{array}{c} \boxplus_{i=1}^{l}\left(1-\left(1-(\cos^2(\pi/2\cdot \eth_i^{\mathbb{T}-}))^{\Theta}\right)^{\sigma_i}\right) \\ +\left(1-\left(1-(\cos^2(\pi/2\cdot \eth_{l+1}^{\mathbb{T}-}))^{\Theta}\right)^{\sigma_{l+1}}\right) \\ -\bigcirc_{i=1}^{l}\left(1-\left(1-(\cos^2(\pi/2\cdot \eth_i^{\mathbb{T}-}))^{\Theta}\right)^{\sigma_i}\right) \\ \cdot\left(1-\left(1-(\cos^2(\pi/2\cdot \eth_{l+1}^{\mathbb{T}-}))^{\Theta}\right)^{\sigma_{l+1}}\right), \\ \boxplus_{i=1}^{l}\left(1-\left(1-(\cos^2(\pi/2\cdot \eth_i^{\mathbb{T}+}))^{\Theta}\right)^{\sigma_i}\right) \\ +\left(1-\left(1-(\cos^2(\pi/2\cdot \eth_{l+1}^{\mathbb{T}+}))^{\Theta}\right)^{\sigma_{l+1}}\right) \\ -\bigcirc_{i=1}^{l}\left(1-\left(1-(\cos^2(\pi/2\cdot \eth_i^{\mathbb{T}+}))^{\Theta}\right)^{\sigma_i}\right) \\ \cdot\left(1-\left(1-(\cos^2(\pi/2\cdot \eth_{l+1}^{\mathbb{T}+}))^{\Theta}\right)^{\sigma_{l+1}}\right) \end{array}\right], \\ \left[\begin{array}{c} \bigcirc_{i=1}^{l}(\cos^2(\pi/2\cdot \eth_i^{\mathbb{I}-}))^{\sigma_i}\cdot(\cos^2(\pi/2\cdot \eth_{l+1}^{\mathbb{I}-}))^{\sigma_{l+1}}, \\ \bigcirc_{i=1}^{l}(\cos^2(\pi/2\cdot \eth_i^{\mathbb{I}+}))^{\sigma_i}\cdot(\cos^2(\pi/2\cdot \eth_{l+1}^{\mathbb{I}+}))^{\sigma_{l+1}} \end{array}\right], \\ \left[\begin{array}{c} \bigcirc_{i=1}^{l}(\cos^2(\pi/2\cdot \eth_i^{\mathbb{F}-}))^{\sigma_i}\cdot(\cos^2(\pi/2\cdot \eth_{l+1}^{\mathbb{F}-}))^{\sigma_{l+1}}, \\ \bigcirc_{i=1}^{l}(\cos^2(\pi/2\cdot \eth_i^{\mathbb{F}+}))^{\sigma_i}\cdot(\cos^2(\pi/2\cdot \eth_{l+1}^{\mathbb{F}+}))^{\sigma_{l+1}} \end{array}\right] \end{array}\right]$$

$$= \left[\begin{array}{c} \left(\boxplus_{i=1}^{l+1}\sigma_i\chi_i, \boxplus_{i=1}^{l+1}\sigma_i\psi_i\right); \\ \left[\begin{array}{c} 1-\bigcirc_{i=1}^{l+1}\left(1-(\cos^2(\pi/2\cdot \eth_i^{\mathbb{T}-}))^{\Theta}\right)^{\sigma_i}, \\ 1-\bigcirc_{i=1}^{l+1}\left(1-(\cos^2(\pi/2\cdot \eth_i^{\mathbb{T}+}))^{\Theta}\right)^{\sigma_i} \end{array}\right], \\ \left[\bigcirc_{i=1}^{l+1}(\cos^2(\pi/2\cdot \eth_i^{\mathbb{I}-}))^{\sigma_i}, \bigcirc_{i=1}^{l+1}(\cos^2(\pi/2\cdot \eth_i^{\mathbb{I}+}))^{\sigma_i}\right], \\ \left[\bigcirc_{i=1}^{l+1}(\cos^2(\pi/2\cdot \eth_i^{\mathbb{F}-}))^{\sigma_i}, \bigcirc_{i=1}^{l+1}(\cos^2(\pi/2\cdot \eth_i^{\mathbb{F}+}))^{\sigma_i}\right] \end{array}\right].$$

Theorem 3 *If all* $L_i = \left\langle(\chi_i, \psi_i); \left[(\cos(\pi/2\cdot\eth_i^{\mathbb{T}-})), (\cos(\pi/2\cdot\eth_i^{\mathbb{T}+}))\right], \left[(\cos(\pi/2\cdot\eth_i^{\mathbb{I}-})), (\cos(\pi/2\cdot\eth_i^{\mathbb{I}+}))\right], \left[(\cos(\pi/2\cdot\eth_i^{\mathbb{F}-})), (\cos(\pi/2\cdot\eth_i^{\mathbb{F}+}))\right]\right\rangle$ *are equal and* $L_i = L$ *with* $\Theta = 1$*, then* CTri-NNIVWA$(L_1, L_2, \ldots, L_n) = L$.

Proof Since $(\chi_i, \psi_i) = (\chi, \psi)$, $[\cos(\pi/2\cdot\eth_i^{\mathbb{T}-}), \cos(\pi/2\cdot\eth_i^{\mathbb{T}+})] = [\cos(\pi/2\cdot\eth^{\mathbb{T}-}), \cos(\pi/2\cdot\eth^{\mathbb{T}+})]$, $[\cos(\pi/2\cdot\eth_i^{\mathbb{I}-}), \cos(\pi/2\cdot\eth_i^{\mathbb{I}+})] = [\cos(\pi/2\cdot\eth^{\mathbb{I}-}), \cos(\pi/2\cdot\eth^{\mathbb{I}+})]$, and $[\cos(\pi/2\cdot\eth_i^{\mathbb{F}-}), \cos(\pi/2\cdot\eth_i^{\mathbb{F}+})] = [\cos(\pi/2\cdot\eth^{\mathbb{F}-}), \cos(\pi/2\cdot\eth^{\mathbb{F}+})]$, for $i = 1, 2, \ldots, n$ and $\boxplus_{i=1}^{n}\sigma_i = 1$. Now, $CTri - NNIVWA(L_1, L_2, \ldots, L_n)$

$$= \begin{bmatrix} \left(\boxplus_{i=1}^{n} \sigma_i \chi_i, \boxplus_{i=1}^{n} \sigma_i \psi_i\right); \\ \begin{bmatrix} 1 - \bigcirc_{i=1}^{n}\left(1 - (\cos(\pi/2 \cdot \eth_i^{\mathbb{T}-}))^{\Theta}\right)^{\sigma_i}, \\ 1 - \bigcirc_{i=1}^{n}\left(1 - (\cos(\pi/2 \cdot \eth_i^{\mathbb{T}+}))^{\Theta}\right)^{\sigma_i} \end{bmatrix}, \\ \left[\bigcirc_{i=1}^{n} (\cos(\pi/2 \cdot \eth_i^{\mathbb{I}-}))^{\sigma_i}, \bigcirc_{i=1}^{n} (\cos(\pi/2 \cdot \eth_i^{\mathbb{I}+}))^{\sigma_i}\right], \\ \left[\bigcirc_{i=1}^{n} (\cos(\pi/2 \cdot \eth_i^{\mathbb{F}-}))^{\sigma_i}, \bigcirc_{i=1}^{n} (\cos(\pi/2 \cdot \eth_i^{\mathbb{F}+}))^{\sigma_i}\right] \end{bmatrix}$$

$$= \begin{bmatrix} \left(\chi \boxplus_{i=1}^{n} \sigma_i, \psi \boxplus_{i=1}^{n} \sigma_i\right); \\ \begin{bmatrix} 1 - \left(1 - (\cos(\pi/2 \cdot \eth^{\mathbb{T}-}))^{\Theta}\right)^{\boxplus_{i=1}^{n}\sigma_i}, \\ 1\left(1 - (\cos(\pi/2 \cdot \eth^{\mathbb{T}+}))^{\Theta}\right)^{\boxplus_{i=1}^{n}\sigma_i} \end{bmatrix}, \\ \left[(\cos(\pi/2 \cdot \eth^{\mathbb{I}-}))^{\boxplus_{i=1}^{n}\sigma_i}, (\cos(\pi/2 \cdot \eth^{\mathbb{I}+}))^{\boxplus_{i=1}^{n}\sigma_i}\right], \\ \left[(\cos(\pi/2 \cdot \eth^{\mathbb{F}-}))^{\boxplus_{i=1}^{n}\sigma_i}, (\cos(\pi/2 \cdot \eth^{\mathbb{F}+}))^{\boxplus_{i=1}^{n}\sigma_i}\right] \end{bmatrix}$$

$$= \begin{bmatrix} (\chi, \psi); \begin{bmatrix} 1 - \left(1 - (\cos(\pi/2 \cdot \eth^{\mathbb{T}-}))^{\Theta}\right), \\ 1 - \left(1 - (\cos(\pi/2 \cdot \eth^{\mathbb{T}+}))^{\Theta}\right) \end{bmatrix}, \\ \left[(\cos(\pi/2 \cdot \eth^{\mathbb{I}-})), (\cos(\pi/2 \cdot \eth^{\mathbb{I}+}))\right], \\ \left[(\cos(\pi/2 \cdot \eth^{\mathbb{F}-})), (\cos(\pi/2 \cdot \eth^{\mathbb{F}+}))\right] \end{bmatrix}$$

$$= L.$$

Theorem 4 *Let* $L_i = \left\langle (\chi_{ij}, \psi_{ij}); [\eth_{ij}^{\mathbb{T}-}, \eth_{ij}^{\mathbb{T}+}], [\eth_{ij}^{\mathbb{I}-}, \eth_{ij}^{\mathbb{I}+}][\eth_{ij}^{\mathbb{F}-}, \eth_{ij}^{\mathbb{F}+}] \right\rangle$ *be a collection of CTri-NNIVWA, where* $(i = 1, 2, \ldots, n)$; $(j = 1, 2, \ldots, i_j)$, $\overleftarrow{\chi} = \inf \chi_{ij}$, $\overrightarrow{\chi} = \sup \chi_{ij}$, $\overleftarrow{\psi} = \sup \psi_{ij}$, $\overrightarrow{\psi} = \inf \psi_{ij}$, $\overleftarrow{\eth^{\mathbb{T}-}} = \inf \eth_{ij}^{\mathbb{T}^-}$, $\overrightarrow{\eth^{\mathbb{T}-}} = \sup \eth_{ij}^{\mathbb{T}^-}$, $\overleftarrow{\eth^{\mathbb{T}+}} = \inf \eth_{ij}^{\mathbb{T}^+}$, $\overrightarrow{\eth^{\mathbb{T}+}} = \sup \eth_{ij}^{\mathbb{T}^+}$, $\overleftarrow{\eth^{\mathbb{I}-}} = \inf \eth_{ij}^{\mathbb{I}^-}$, $\overrightarrow{\eth^{\mathbb{I}-}} = \sup \eth_{ij}^{\mathbb{I}^-}$, $\overleftarrow{\eth^{\mathbb{I}+}} = \inf \eth_{ij}^{\mathbb{I}^+}$, $\overrightarrow{\eth^{\mathbb{I}+}} = \sup \eth_{ij}^{\mathbb{I}^+}$, $\overleftarrow{\eth^{\mathbb{F}-}} = \inf \eth_{ij}^{\mathbb{F}^-}$, $\overrightarrow{\eth^{\mathbb{F}-}} = \sup \eth_{ij}^{\mathbb{F}^-}$, $\overleftarrow{\eth^{\mathbb{F}+}} = \inf \eth_{ij}^{\mathbb{F}^+}$, *and* $\overrightarrow{\eth^{\mathbb{F}+}} = \sup \eth_{ij}^{\mathbb{F}^+}$.

Then, $\left\langle (\overleftarrow{\chi}, \overleftarrow{\psi}); [\overleftarrow{\eth^{\mathbb{T}-}}, \overleftarrow{\eth^{\mathbb{T}+}}], [\overrightarrow{\eth^{\mathbb{I}-}}, \overrightarrow{\eth^{\mathbb{I}+}}], [\overrightarrow{\eth^{\mathbb{F}-}}, \overrightarrow{\eth^{\mathbb{F}+}}] \right\rangle \leq CTri - NNIVWA$ $(L_1, L_2, \ldots, L_n)$

$$\leq \left\langle (\overrightarrow{\chi}, \overrightarrow{\psi}); [\overrightarrow{\eth^{\mathbb{T}-}}, \overrightarrow{\eth^{\mathbb{T}+}}], [\overleftarrow{\eth^{\mathbb{I}-}}, \overleftarrow{\eth^{\mathbb{I}+}}], [\overleftarrow{\eth^{\mathbb{F}-}}, \overleftarrow{\eth^{\mathbb{F}+}}] \right\rangle.$$

Proof Since, $\overleftarrow{\eth^{\mathbb{T}-}} = \inf \eth_{ij}^{\mathbb{T}^-}$, $\overrightarrow{\eth^{\mathbb{T}-}} = \sup \eth_{ij}^{\mathbb{T}^-}$

$\overleftarrow{\eth^{\mathbb{T}+}} = \inf \eth_{ij}^{\mathbb{T}^+}$, $\overrightarrow{\eth^{\mathbb{T}+}} = \sup \eth_{ij}^{\mathbb{T}^+}$ and $\overleftarrow{\eth^{\mathbb{T}-}} \leq \eth_{ij}^{\mathbb{T}^-} \leq \overrightarrow{\eth^{\mathbb{T}-}}$ and $\overleftarrow{\eth^{\mathbb{T}+}} \leq \eth_{ij}^{\mathbb{T}^+} \leq \overrightarrow{\eth^{\mathbb{T}+}}$.
We have $\overleftarrow{\cos^2(\pi/2 \cdot \eth^{\mathbb{T}-})} + \overleftarrow{\cos^2(\pi/2 \cdot \eth^{\mathbb{T}+})}$

$$
\begin{aligned}
&= 1 - \bigcirc_{i=1}^{n} \left(1 - (\overleftarrow{\cos^2(\pi/2 \cdot \eth^{\mathbb{T}-})})^{\Theta}\right)^{\sigma_i} + 1 - \bigcirc_{i=1}^{n} \left(1 - (\overleftarrow{\cos^2(\pi/2 \cdot \eth^{\mathbb{T}+})})^{\Theta}\right)^{\sigma_i} \\
&\leq 1 - \bigcirc_{i=1}^{n} \left(1 - (\cos^2 \cdot \pi/2 \cdot \eth_{ij}^{\mathbb{T}^-})^{\Theta}\right)^{\sigma_i} + 1 - \bigcirc_{i=1}^{n} \left(1 - (\cos^2 \cdot \pi/2 \cdot \eth_{ij}^{\mathbb{T}^+})^{\Theta}\right)^{\sigma_i} \\
&\leq 1 - \bigcirc_{i=1}^{n} \left(1 - (\overrightarrow{\cos^2(\pi/2 \cdot \eth^{\mathbb{T}-})})^{\Theta}\right)^{\sigma_i} + 1 - \bigcirc_{i=1}^{n} \left(1 - (\overrightarrow{\cos^2(\pi/2 \cdot \eth^{\mathbb{T}+})})^{\Theta}\right)^{\sigma_i} \\
&= \overrightarrow{\cos^2(\pi/2 \cdot \eth^{\mathbb{T}-})} + \overrightarrow{\cos^2(\pi/2 \cdot \eth^{\mathbb{T}+})}.
\end{aligned}
$$

Since, $\overleftarrow{\eth^{\mathbb{I}-}} = \inf \eth_{ij}^{\mathbb{I}^-}$, $\overrightarrow{\eth^{\mathbb{I}-}} = \sup \eth_{ij}^{\mathbb{I}^-}$
$\overleftarrow{\eth^{\mathbb{I}+}} = \inf \eth_{ij}^{\mathbb{I}^+}$, $\overrightarrow{\eth^{\mathbb{I}+}} = \sup \eth_{ij}^{\mathbb{I}^+}$ and $\overleftarrow{\eth^{\mathbb{I}-}} \leq \eth_{ij}^{\mathbb{I}^-} \leq \overrightarrow{\eth^{\mathbb{I}-}}$ and $\overleftarrow{\eth^{\mathbb{I}+}} \leq \eth_{ij}^{\mathbb{I}^+} \leq \overrightarrow{\eth^{\mathbb{I}+}}$. We have

$$
\begin{aligned}
&\overleftarrow{\cos^2(\pi/2 \cdot \eth^{\mathbb{I}-})} + \overleftarrow{\cos^2(\pi/2 \cdot \eth^{\mathbb{I}+})} \\
&\quad = \bigcirc_{i=1}^{n} (\overleftarrow{\cos^2(\pi/2 \cdot \eth^{\mathbb{I}-})})^{\sigma_i} + \bigcirc_{i=1}^{n} (\overleftarrow{\cos^2(\pi/2 \cdot \eth^{\mathbb{I}+})})^{\sigma_i} \\
&\quad \leq \bigcirc_{i=1}^{n} (\cos^2 \cdot \pi/2 \cdot \eth_{ij}^{\mathbb{I}^-})^{\sigma_i} + \bigcirc_{i=1}^{n} (\cos^2 \cdot \pi/2 \cdot \eth_{ij}^{\mathbb{I}^+})^{\sigma_i} \\
&\quad \leq \bigcirc_{i=1}^{n} (\overrightarrow{\cos^2(\pi/2 \cdot \eth^{\mathbb{I}-})})^{\sigma_i} + \bigcirc_{i=1}^{n} (\overrightarrow{\cos^2(\pi/2 \cdot \eth^{\mathbb{I}+})})^{\sigma_i} \\
&\quad = \overrightarrow{\cos^2(\pi/2 \cdot \eth^{\mathbb{I}-})} + \overrightarrow{\cos^2(\pi/2 \cdot \eth^{\mathbb{I}+})}.
\end{aligned}
$$

Since, $\overleftarrow{\eth^{\mathbb{F}-}} = \inf \eth_{ij}^{\mathbb{F}^-}$, $\overrightarrow{\eth^{\mathbb{F}-}} = \sup \eth_{ij}^{\mathbb{F}^-}$
$\overleftarrow{\eth^{\mathbb{F}+}} = \inf \eth_{ij}^{\mathbb{F}^+}$, $\overrightarrow{\eth^{\mathbb{F}+}} = \sup \eth_{ij}^{\mathbb{F}^+}$ and $\overleftarrow{\eth^{\mathbb{F}-}} \leq \eth_{ij}^{\mathbb{F}^-} \leq \overrightarrow{\eth^{\mathbb{F}-}}$ and $\overleftarrow{\eth^{\mathbb{F}+}} \leq \eth_{ij}^{\mathbb{F}^+} \leq \overrightarrow{\eth^{\mathbb{F}+}}$.
We have

$$
\begin{aligned}
&\overleftarrow{\cos^2(\pi/2 \cdot \eth^{\mathbb{F}-})} + \overleftarrow{\cos^2(\pi/2 \cdot \eth^{\mathbb{F}+})} \\
&\quad = \bigcirc_{i=1}^{n} (\overleftarrow{\cos^2(\pi/2 \cdot \eth^{\mathbb{F}-})})^{\sigma_i} + \bigcirc_{i=1}^{n} (\overleftarrow{\cos^2(\pi/2 \cdot \eth^{\mathbb{F}+})})^{\sigma_i} \\
&\quad \leq \bigcirc_{i=1}^{n} (\cos^2 \cdot \pi/2 \cdot \eth_{ij}^{\mathbb{F}^-})^{\sigma_i} + \bigcirc_{i=1}^{n} (\cos^2 \cdot \pi/2 \cdot \eth_{ij}^{\mathbb{F}^+})^{\sigma_i} \\
&\quad \leq \bigcirc_{i=1}^{n} (\overrightarrow{\cos^2(\pi/2 \cdot \eth^{\mathbb{F}-})})^{\sigma_i} + \bigcirc_{i=1}^{n} (\overrightarrow{\cos^2(\pi/2 \cdot \eth^{\mathbb{F}+})})^{\sigma_i} \\
&\quad = \overrightarrow{\cos^2(\pi/2 \cdot \eth^{\mathbb{F}-})} + \overrightarrow{\cos^2(\pi/2 \cdot \eth^{\mathbb{F}+})}.
\end{aligned}
$$

Since, $\overleftarrow{\chi} = \inf \chi_{ij}$, $\overrightarrow{\chi} = \sup \chi_{ij}$, $\overleftarrow{\psi} = \sup \psi_{ij}$, $\overrightarrow{\psi} = \inf \psi_{ij}$ and $\overleftarrow{\chi} \leq \chi_{ij} \leq \overrightarrow{\chi}$ and $\overrightarrow{\psi} \leq \psi_{ij} \leq \overleftarrow{\psi}$.
Hence, $\boxplus_{i=1}^{n}\sigma_i \overleftarrow{\chi} \leq \boxplus_{i=1}^{n}\sigma_i \chi_{ij} \leq \boxplus_{i=1}^{n}\sigma_i \overrightarrow{\chi}$ and $\boxplus_{i=1}^{n}\sigma_i \overrightarrow{\psi} \leq \boxplus_{i=1}^{n}\sigma_i \psi_{ij} \leq \boxplus_{i=1}^{n}\sigma_i \overleftarrow{\psi}$.
Therefore,

$$\frac{\boxplus_{i=1}^{n}\sigma_i \overleftarrow{\chi}}{2} \times \left[\frac{2+\left(1-\bigcirc_{i=1}^{n}\left(1-(\overleftarrow{\cos^2(\pi/2\cdot\eth^{\mathbb{T}-})})^{\Theta}\right)^{\sigma_i}\right)+\left(1-\bigcirc_{i=1}^{n}\left(1-(\overleftarrow{\cos^2(\pi/2\cdot\eth^{\mathbb{T}+})})^{\Theta}\right)^{\sigma_i}\right)}{2} - \frac{\left(\bigcirc_{i=1}^{n}(\overrightarrow{\cos^2(\pi/2\cdot\eth^{\mathbb{I}-})})^{\sigma_i}\right)+\left(\bigcirc_{i=1}^{n}(\overrightarrow{\cos^2(\pi/2\cdot\eth^{\mathbb{I}+})})^{\sigma_i}\right)}{2} - \frac{\left(\bigcirc_{i=1}^{n}(\overrightarrow{\cos^2(\pi/2\cdot\eth^{\mathbb{F}-})})^{\sigma_i}\right)+\left(\bigcirc_{i=1}^{n}(\overrightarrow{\cos^2(\pi/2\cdot\eth^{\mathbb{F}+})})^{\sigma_i}\right)}{2}\right]$$

$$\leq \frac{\boxplus_{i=1}^{n}\sigma_i \chi_{ij}}{2} \times \left[\frac{2+\left(1-\bigcirc_{i=1}^{n}\left(1-(\cos^2\cdot\pi/2\cdot\eth_{ij}^{\mathbb{T}-})^{\Theta}\right)^{\sigma_i}\right)+\left(1-\bigcirc_{i=1}^{n}\left(1-(\cos^2\cdot\pi/2\cdot\eth_{ij}^{\mathbb{T}+})^{\Theta}\right)^{\sigma_i}\right)}{2} - \frac{\left(\bigcirc_{i=1}^{n}(\cos^2\cdot\pi/2\cdot\eth_{ij}^{\mathbb{I}-})^{\sigma_i}\right)+\left(\bigcirc_{i=1}^{n}(\cos^2\cdot\pi/2\cdot\eth_{ij}^{\mathbb{I}+})^{\sigma_i}\right)}{2} - \frac{\left(\bigcirc_{i=1}^{n}(\cos^2\cdot\pi/2\cdot\eth_{ij}^{\mathbb{F}-})^{\sigma_i}\right)+\left(\bigcirc_{i=1}^{n}(\cos^2\cdot\pi/2\cdot\eth_{ij}^{\mathbb{F}+})^{\sigma_i}\right)}{2}\right]$$

$$\leq \frac{\boxplus_{i=1}^{n}\sigma_i \overrightarrow{\chi}}{2} \times \left[\frac{2+\left(1-\bigcirc_{i=1}^{n}\left(1-(\overrightarrow{\cos^2(\pi/2\cdot\eth^{\mathbb{T}-})})^{\Theta}\right)^{\sigma_i}\right)+\left(1-\bigcirc_{i=1}^{n}\left(1-(\overrightarrow{\cos^2(\pi/2\cdot\eth^{\mathbb{T}+})})^{\Theta}\right)^{\sigma_i}\right)}{2} - \frac{\left(\bigcirc_{i=1}^{n}(\overleftarrow{\cos^2(\pi/2\cdot\eth^{\mathbb{I}-})})^{\sigma_i}\right)+\left(\bigcirc_{i=1}^{n}(\overleftarrow{\cos^2(\pi/2\cdot\eth^{\mathbb{I}+})})^{\sigma_i}\right)}{2} - \frac{\left(\bigcirc_{i=1}^{n}(\overleftarrow{\cos^2(\pi/2\cdot\eth^{\mathbb{F}-})})^{\sigma_i}\right)+\left(\bigcirc_{i=1}^{n}(\overleftarrow{\cos^2(\pi/2\cdot\eth^{\mathbb{F}+})})^{\sigma_i}\right)}{2}\right].$$

Hence, $\Big\langle(\overleftarrow{\chi}, \overleftarrow{\psi}); [\overleftarrow{\cos^2(\pi/2\cdot\eth^{\mathbb{T}-})}, \overleftarrow{\cos^2(\pi/2\cdot\eth^{\mathbb{T}+})}], [\overrightarrow{\cos^2(\pi/2\cdot\eth^{\mathbb{I}-})}, \overrightarrow{\cos^2(\pi/2\cdot\eth^{\mathbb{I}+})}],$

$\overrightarrow{[\cos^2(\pi/2 \cdot \eth^{\mathbb{F}-})}, \overrightarrow{\cos^2(\pi/2 \cdot \eth^{\mathbb{F}+})]}\Big\rangle \leq CTri - NNIVWA(L_1, L_2, \ldots, L_n) \leq \Big\langle (\overrightarrow{\chi}, \overrightarrow{\psi}); \overrightarrow{[\cos^2(\pi/2 \cdot \eth^{\mathbb{T}-})}, \overrightarrow{\cos^2(\pi/2 \cdot \eth^{\mathbb{T}+})]}, \overleftarrow{[\cos^2(\pi/2 \cdot \eth^{\mathbb{I}-})}, \overleftarrow{\cos^2(\pi/2 \cdot \eth^{\mathbb{I}+})]}, \overleftarrow{[\cos^2(\pi/2 \cdot \eth^{\mathbb{F}-})}, \overleftarrow{\cos^2(\pi/2 \cdot \eth^{\mathbb{F}+})]}\Big\rangle$.

Theorem 5 *Let* $L_i = \Big\langle (\chi_{t_{ij}}, \psi_{t_{ij}}); [\eth^{\mathbb{T}-}_{t_{ij}}), \eth^{\mathbb{T}+}_{t_{ij}})], [\eth^{\mathbb{I}-}_{t_{ij}}), \eth^{\mathbb{I}+}_{t_{ij}})], [\eth^{\mathbb{F}-}_{t_{ij}}), \eth^{\mathbb{F}+}_{t_{ij}})]\Big\rangle$ *and* $W_i = \Big\langle (\chi_{h_{ij}}, \psi_{h_{ij}}); [\eth^{\mathbb{T}-}_{h_{ij}}), \eth^{\mathbb{T}+}_{h_{ij}})], [\eth^{\mathbb{I}-}_{h_{ij}}), \eth^{\mathbb{I}+}_{h_{ij}})], [\eth^{\mathbb{F}-}_{h_{ij}}), \eth^{\mathbb{F}+}_{h_{ij}})]\Big\rangle (i = 1, 2, \ldots, n)$; $(j = 1, 2, \ldots, i_j)$ *be the two families of CTri-NNIVWAs. For any* i*, if there is* $\chi_{t_{ij}} \leq \psi_{h_{ij}}$, $\Big(\cos^2(\pi/2 \cdot \eth^{\mathbb{T}-}_{t_{ij}})\Big) + \Big(\cos^2(\pi/2 \cdot \eth^{\mathbb{T}+}_{t_{ij}})\Big) \leq \Big(\cos^2(\pi/2 \cdot \eth^{\mathbb{T}-}_{h_{ij}})\Big) + \Big(\cos^2(\pi/2 \cdot \eth^{\mathbb{T}+}_{h_{ij}})\Big)$ *and* $\Big(\cos^2(\pi/2 \cdot \eth^{\mathbb{I}-}_{t_{ij}})\Big) + \Big(\cos^2(\pi/2 \cdot \eth^{\mathbb{I}+}_{t_{ij}})\Big) \geq \Big(\cos^2(\pi/2 \cdot \eth^{\mathbb{I}-}_{h_{ij}})\Big) + \Big(\cos^2(\pi/2 \cdot \eth^{\mathbb{I}+}_{h_{ij}})\Big)$ *and* $\Big(\cos^2(\pi/2 \cdot \eth^{\mathbb{F}-}_{t_{ij}})\Big) + \Big(\cos^2(\pi/2 \cdot \eth^{\mathbb{F}+}_{t_{ij}})\Big) \geq \Big(\cos^2(\pi/2 \cdot \eth^{\mathbb{F}-}_{h_{ij}})\Big) + \Big(\cos^2(\pi/2 \cdot \eth^{\mathbb{F}+}_{h_{ij}})\Big)$ *or* $L_i \leq W_i$*, then* $CTri - NNIVWA\,(L_1, L_2, \ldots, L_n) \leq CTri - NNIVWA\,(W_1, W_2, \ldots, W_n)$.

Proof For every i, $\chi_{t_{ij}} \leq \psi_{h_{ij}}$. Thus, $\boxplus_{i=1}^n \chi_{t_{ij}} \leq \boxplus_{i=1}^n \psi_{h_{ij}}$.

For any i, $\Big(\cos^2(\pi/2 \cdot \eth^{\mathbb{T}-}_{t_{ij}})\Big) + \Big(\cos^2(\pi/2 \cdot \eth^{\mathbb{T}+}_{t_{ij}})\Big) \leq \Big(\cos^2(\pi/2 \cdot \eth^{\mathbb{T}-}_{h_{ij}})\Big) + \Big(\cos^2(\pi/2 \cdot \eth^{\mathbb{T}+}_{h_{ij}})\Big)$.

Therefore, $1 - \Big(\cos^2(\pi/2 \cdot \eth^{\mathbb{T}-}_{t_i})\Big) + 1 - \Big(\cos^2(\pi/2 \cdot \eth^{\mathbb{T}+}_{t_i})\Big) \geq 1 - \Big(\cos^2(\pi/2 \cdot \eth^{\mathbb{T}-}_{h_i})\Big) + 1 - \Big(\cos^2(\pi/2 \cdot \eth^{\mathbb{T}+}_{h_i})\Big)$.

Hence, $\bigcirc_{i=1}^n \Big(1 - \Big(\cos^2(\pi/2 \cdot \eth^{\mathbb{T}-}_{t_i})\Big)\Big)^{\sigma_i} + \bigcirc_{i=1}^n \Big(1 - \Big(\cos^2(\pi/2 \cdot \eth^{\mathbb{T}+}_{t_i})\Big)\Big)^{\sigma_i} \geq \bigcirc_{i=1}^n \Big(1 - \Big(\cos^2(\pi/2 \cdot \eth^{\mathbb{T}-}_{h_i})\Big)\Big)^{\sigma_i} + \bigcirc_{i=1}^n \Big(1 - \Big(\cos^2(\pi/2 \cdot \eth^{\mathbb{T}+}_{h_i})\Big)\Big)^{\sigma_i}$ and $\Big(1 - \bigcirc_{i=1}^n \Big(1 - \Big(\cos^2(\pi/2 \cdot \eth^{\mathbb{T}-}_{t_i})\Big)^{\Theta}\Big)^{\sigma_i}\Big) + \Big(1 - \bigcirc_{i=1}^n \Big(1 - \Big(\cos^2(\pi/2 \cdot \eth^{\mathbb{T}+}_{t_i})\Big)^{\Theta}\Big)^{\sigma_i}\Big) \leq \Big(1 - \bigcirc_{i=1}^n \Big(1 - \Big(\cos^2(\pi/2 \cdot \eth^{\mathbb{T}-}_{h_i})\Big)^{\Theta}\Big)^{\sigma_i}\Big) + \Big(1 - \bigcirc_{i=1}^n \Big(1 - \Big(\cos^2(\pi/2 \cdot \eth^{\mathbb{T}+}_{h_i})\Big)^{\Theta}\Big)^{\sigma_i}\Big)$

and
$$\frac{2 + \Big(1 - \bigcirc_{i=1}^n \Big(1 - \Big(\cos^2(\pi/2 \cdot \eth^{\mathbb{T}-}_{t_i})\Big)^{\Theta}\Big)^{\sigma_i}\Big) + \Big(1 - \bigcirc_{i=1}^n \Big(1 - \Big(\cos^2(\pi/2 \cdot \eth^{\mathbb{T}+}_{t_i})\Big)^{\Theta}\Big)^{\sigma_i}\Big)}{2} \leq \frac{2 + \Big(1 - \bigcirc_{i=1}^n \Big(1 - \Big(\cos^2(\pi/2 \cdot \eth^{\mathbb{T}-}_{h_i})\Big)^{\Theta}\Big)^{\sigma_i}\Big) + \Big(1 - \bigcirc_{i=1}^n \Big(1 - \Big(\cos^2(\pi/2 \cdot \eth^{\mathbb{T}+}_{h_i})\Big)^{\Theta}\Big)^{\sigma_i}\Big)}{2}.$$

For every i,

$\left(\cos^2(\pi/2 \cdot \eth^{\mathbb{I}-}_{t_{ij}})\right)^{\Theta} + \left(\cos^2(\pi/2 \cdot \eth^{\mathbb{I}+}_{t_{ij}})\right)^{\Theta} \geq \left(\cos^2(\pi/2 \cdot \eth^{\mathbb{I}-}_{h_{ij}})\right)^{\Theta} + \left(\cos^2(\pi/2 \cdot \eth^{\mathbb{I}+}_{h_{ij}})\right)^{\Theta}$.

Therefore,

$$-\frac{\left(\bigcirc_{i=1}^{n} \cos^2(\pi/2 \cdot \eth^{\mathbb{I}-}_{t_{ij}})\right) + \left(\bigcirc_{i=1}^{n} \cos^2(\pi/2 \cdot \eth^{\mathbb{I}+}_{t_{ij}})\right)}{2} \leq -\frac{\left(\bigcirc_{i=1}^{n} \cos^2(\pi/2 \cdot \eth^{\mathbb{I}-}_{h_{ij}})\right) + \left(\bigcirc_{i=1}^{n} \cos^2(\pi/2 \cdot \eth^{\mathbb{I}+}_{h_{ij}})\right)}{2}.$$

For any i, $\left(\cos^2(\pi/2 \cdot \eth^{\mathbb{F}-}_{t_{ij}})\right) + \left(\cos^2(\pi/2 \cdot \eth^{\mathbb{F}+}_{t_{ij}})\right) \geq \left(\cos^2(\pi/2 \cdot \eth^{\mathbb{F}-}_{h_{ij}})\right) + \left(\cos^2(\pi/2 \cdot \eth^{\mathbb{F}+}_{h_{ij}})\right)$.

Therefore,

$$-\frac{\left(\bigcirc_{i=1}^{n} \cos^2(\pi/2 \cdot \eth^{\mathbb{F}-}_{t_{ij}})\right) + \left(\bigcirc_{i=1}^{n} \cos^2(\pi/2 \cdot \eth^{\mathbb{F}+}_{t_{ij}})\right)}{2} \leq -\frac{\left(\bigcirc_{i=1}^{n} \cos^2(\pi/2 \cdot \eth^{\mathbb{F}-}_{h_{ij}})\right) + \left(\bigcirc_{i=1}^{n} \cos^2(\pi/2 \cdot \eth^{\mathbb{F}+}_{h_{ij}})\right)}{2}.$$

$$\frac{\boxplus_{i=1}^{n} \chi_{tij}}{2} \times \left[\begin{array}{c} \dfrac{2 + \left(1 - \bigcirc_{i=1}^{n}\left(1 - (\cos^2 \cdot \pi/2 \cdot \eth^{\mathbb{T}^-}_{ti})^{\Theta}\right)^{\sigma_i}\right) + \left(1 - \bigcirc_{i=1}^{n}\left(1 - (\cos^2 \cdot \pi/2 \cdot \eth^{\mathbb{T}^+}_{ti})^{\Theta}\right)^{\sigma_i}\right)}{2} \\ -\dfrac{\left(\bigcirc_{i=1}^{n}(\cos^2 \cdot \pi/2 \cdot \eth^{\mathbb{I}^-}_{tij})\right) + \left(\bigcirc_{i=1}^{n}(\cos^2 \cdot \pi/2 \cdot \eth^{\mathbb{I}^+}_{tij})\right)}{2} \\ -\dfrac{\left(\bigcirc_{i=1}^{n}(\cos^2 \cdot \pi/2 \cdot \eth^{\mathbb{F}^-}_{tij})\right) + \left(\bigcirc_{i=1}^{n}(\cos^2 \cdot \pi/2 \cdot \eth^{\mathbb{F}^+}_{tij})\right)}{2} \end{array}\right].$$

$$\leq \frac{\boxplus_{i=1}^{n} \chi_{hij}}{2} \times \left[\begin{array}{c} \dfrac{2 + \left(1 - \bigcirc_{i=1}^{n}\left(1 - (\cos^2 \cdot \pi/2 \cdot \eth^{\mathbb{T}^-}_{hi})^{\Theta}\right)^{\sigma_i}\right) + \left(1 - \bigcirc_{i=1}^{n}\left(1 - (\cos^2 \cdot \pi/2 \cdot \eth^{\mathbb{T}^+}_{hi})^{\Theta}\right)^{\sigma_i}\right)}{2} \\ -\dfrac{\left(\bigcirc_{i=1}^{n}(\cos^2 \cdot \pi/2 \cdot \eth^{\mathbb{I}^-}_{hij})\right) + \left(\bigcirc_{i=1}^{n}(\cos^2 \cdot \pi/2 \cdot \eth^{\mathbb{I}^+}_{hij})\right)}{2} \\ -\dfrac{\left(\bigcirc_{i=1}^{n}(\cos^2 \cdot \pi/2 \cdot \eth^{\mathbb{F}^-}_{hij})\right) + \left(\bigcirc_{i=1}^{n}(\cos^2 \cdot \pi/2 \cdot \eth^{\mathbb{F}^+}_{hij})\right)}{2} \end{array}\right].$$

Hence, $CTri-NNIVWA\,(L_1, L_2, \ldots, L_n) \leq CTri-NNIVWA\,(W_1, W_2, \ldots, W_n)$.

5.2 Generalized CTri-NNIVWA (CTri-GNNIVWA)

Definition 14 Let $L_i = \left\langle(\chi_i, \psi_i); [\eth^{\mathbb{T}-}_i, \eth^{\mathbb{T}+}_i], [\eth^{\mathbb{I}-}_i, \eth^{\mathbb{I}+}_i], [\eth^{\mathbb{F}-}_i, \eth^{\mathbb{F}+}_i]\right\rangle$ be the finite collection of CTri-NNIVN. Then CTri-GNNIVWA $(L_1, L_2, \ldots, L_n) = \left(\boxplus_{i=1}^{n} \sigma_i (\cos L_i)^{\Theta}\right)^{1/\Theta}$.

Theorem 6 *Let* $L_i = \left\langle(\chi_i, \psi_i); [\eth_i^{\mathbb{T}-}, \eth_i^{\mathbb{T}+}], [\eth_i^{\mathbb{I}-}, \eth_i^{\mathbb{I}+}], [\eth_i^{\mathbb{F}-}, \eth_i^{\mathbb{F}+}]\right\rangle$ *be the finite collection of CTri-NNIVNs. Then CTri-GNNIVWA* $(L_1, L_2, \ldots, L_n) =$

$$\left[\begin{array}{c} \left(\left(\boxplus_{i=1}^n \sigma_i \chi_i^{\Theta}\right)^{1/\Theta}, \left(\boxplus_{i=1}^n \sigma_i \psi_i^{\Theta}\right)^{1/\Theta}\right); \\ \left[\begin{array}{c} \left(1 - \bigcirc_{i=1}^n \left(1 - \left((\cos^2(\pi/2 \cdot \eth_i^{\mathbb{T}-}))^{\Theta}\right)^{\Theta}\right)^{\sigma_i}\right)^{1/\Theta}, \\ \left(1 - \bigcirc_{i=1}^n \left(1 - \left((\cos^2(\pi/2 \cdot \eth_i^{\mathbb{T}+}))^{\Theta}\right)^{\Theta}\right)^{\sigma_i}\right)^{1/\Theta} \end{array}\right], \\ \left[\begin{array}{c} 1 - \left(1 - \left(\bigcirc_{i=1}^n \left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_i^{\mathbb{I}-}))^{\Theta}\right)^{\Theta}\right)^{\sigma_i}\right)^{\Theta}\right)^{1/\Theta}, \\ 1 - \left(1 - \left(\bigcirc_{i=1}^n \left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_i^{\mathbb{I}+}))^{\Theta}\right)^{\Theta}\right)^{\sigma_i}\right)^{\Theta}\right)^{1/\Theta} \end{array}\right], \\ \left[\begin{array}{c} 1 - \left(1 - \left(\bigcirc_{i=1}^n \left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_i^{\mathbb{F}-}))^{\Theta}\right)^{\Theta}\right)^{\sigma_i}\right)^{\Theta}\right)^{1/\Theta}, \\ 1 - \left(1 - \left(\bigcirc_{i=1}^n \left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_i^{\mathbb{F}+}))^{\Theta}\right)^{\Theta}\right)^{\sigma_i}\right)^{\Theta}\right)^{1/\Theta} \end{array}\right] \end{array}\right].$$

Proof It is compulsory to prove that $\boxplus_{i=1}^n \sigma_i (\cos L_i)^{\Theta} =$

$$\left[\begin{array}{c} \left(\left(\boxplus_{i=1}^n \sigma_i \chi_i^{\Theta}\right), \left(\boxplus_{i=1}^n \sigma_i \psi_i^{\Theta}\right)\right); \\ \left[\begin{array}{c} 1 - \bigcirc_{i=1}^n \left(1 - \left((\cos^2(\pi/2 \cdot \eth_i^{\mathbb{T}-}))^{\Theta}\right)^{\Theta}\right)^{\sigma_i}, \\ 1 - \bigcirc_{i=1}^n \left(1 - \left((\cos^2(\pi/2 \cdot \eth_i^{\mathbb{T}+}))^{\Theta}\right)^{\Theta}\right)^{\sigma_i} \end{array}\right], \\ \left[\begin{array}{c} \bigcirc_{i=1}^n \left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_i^{\mathbb{I}-}))^{\Theta}\right)^{\Theta}\right)^{\sigma_i}, \\ \bigcirc_{i=1}^n \left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_i^{\mathbb{I}+}))^{\Theta}\right)^{\Theta}\right)^{\sigma_i} \end{array}\right], \\ \left[\begin{array}{c} \bigcirc_{i=1}^n \left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_i^{\mathbb{F}-}))^{\Theta}\right)^{\Theta}\right)^{\sigma_i}, \\ \bigcirc_{i=1}^n \left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_i^{\mathbb{F}+}))^{\Theta}\right)^{\Theta}\right)^{\sigma_i} \end{array}\right] \end{array}\right].$$

The proof uses a mathematical induction approach. If $n = 2$, then $\sigma_1(\cos L_1)^{\Theta} \boxplus \sigma_2(\cos L_2)^{\Theta}$

$$
= \left[\begin{array}{c}
\left(\sigma_1 \chi_1^{\Theta} \boxplus \sigma_2 \chi_2^{\Theta}, \sigma_1 \psi_1^{\Theta} \boxplus \sigma_2 \psi_2^{\Theta} \right); \\
\left[\begin{array}{c}
\left(1 - \left(1 - \left((\cos^2(\pi/2 \cdot \eth_1^{\mathbb{T}-}))^{\Theta} \right)^{\Theta} \right)^{\sigma_1} \right)^{\Theta} \\
+ \left(1 - \left(1 - \left((\cos^2(\pi/2 \cdot \eth_2^{\mathbb{T}-}))^{\Theta} \right)^{\Theta} \right)^{\sigma_1} \right)^{\Theta}, \\
- \left(1 - \left(1 - \left((\cos^2(\pi/2 \cdot \eth_1^{\mathbb{T}-}))^{\Theta} \right)^{\Theta} \right)^{\sigma_1} \right)^{\Theta} \\
\cdot \left(1 - \left(1 - \left((\cos^2(\pi/2 \cdot \eth_2^{\mathbb{T}-}))^{\Theta} \right)^{\Theta} \right)^{\sigma_1} \right)^{\Theta} \\
\left(1 - \left(1 - \left((\cos^2(\pi/2 \cdot \eth_1^{\mathbb{T}+}))^{\Theta} \right)^{\Theta} \right)^{\sigma_1} \right)^{\Theta} \\
+ \left(1 - \left(1 - \left((\cos^2(\pi/2 \cdot \eth_2^{\mathbb{T}+}))^{\Theta} \right)^{\Theta} \right)^{\sigma_1} \right)^{\Theta} \\
- \left(1 - \left(1 - \left((\cos^2(\pi/2 \cdot \eth_1^{\mathbb{T}+}))^{\Theta} \right)^{\Theta} \right)^{\sigma_1} \right)^{\Theta} \\
\cdot \left(1 - \left(1 - \left((\cos^2(\pi/2 \cdot \eth_2^{\mathbb{T}+}))^{\Theta} \right)^{\Theta} \right)^{\sigma_1} \right)^{\Theta}
\end{array} \right], \\
\left[\begin{array}{c}
\left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_1^{\mathbb{I}-}))^{\Theta} \right)^{\Theta} \right)^{\sigma_1} \\
\cdot \left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_2^{\mathbb{I}-}))^{\Theta} \right)^{\Theta} \right)^{\sigma_1}, \\
\left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_1^{\mathbb{I}+}))^{\Theta} \right)^{\Theta} \right)^{\sigma_1} \\
\cdot \left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_2^{\mathbb{I}+}))^{\Theta} \right)^{\Theta} \right)^{\sigma_1}
\end{array} \right], \\
\left[\begin{array}{c}
\left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_1^{\mathbb{F}-}))^{\Theta} \right)^{\Theta} \right)^{\sigma_1} \\
\cdot \left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_2^{\mathbb{F}-}))^{\Theta} \right)^{\Theta} \right)^{\sigma_1}, \\
\left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_1^{\mathbb{F}+}))^{\Theta} \right)^{\Theta} \right)^{\sigma_1} \\
\cdot \left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_2^{\mathbb{F}+}))^{\Theta} \right)^{\Theta} \right)^{\sigma_1}
\end{array} \right]
\end{array} \right]
$$

$$= \begin{bmatrix} \left(\boxplus_{i=1}^{2} \sigma_i \chi_i^{\Theta}, \boxplus_{i=1}^{2} \sigma_i \psi_i^{\Theta} \right); \\ \begin{bmatrix} 1 - \bigcirc_{i=1}^{2} \left(1 - \left((\cos^2(\pi/2 \cdot \eth_1^{\mathbb{T}-}))^{\Theta} \right)^{\Theta} \right)^{\sigma_i}, \\ 1 - \bigcirc_{i=1}^{2} \left(1 - \left((\cos^2(\pi/2 \cdot \eth_1^{\mathbb{T}+}))^{\Theta} \right)^{\Theta} \right)^{\sigma_i} \end{bmatrix}, \\ \begin{bmatrix} \bigcirc_{i=1}^{2} \left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_i^{\mathbb{I}-}))^{\Theta} \right)^{\Theta} \right)^{\sigma_i}, \\ \bigcirc_{i=1}^{2} \left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_i^{\mathbb{I}+}))^{\Theta} \right)^{\Theta} \right)^{\sigma_i} \end{bmatrix}, \\ \begin{bmatrix} \bigcirc_{i=1}^{2} \left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_i^{\mathbb{F}-}))^{\Theta} \right)^{\Theta} \right)^{\sigma_i}, \\ \bigcirc_{i=1}^{2} \left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_i^{\mathbb{F}+}))^{\Theta} \right)^{\Theta} \right)^{\sigma_i} \end{bmatrix} \end{bmatrix}.$$

In general, the form $\boxplus_{i=1}^{l} \sigma_i (\cos L_i)^{\Theta} =$

$$\begin{bmatrix} \left(\boxplus_{i=1}^{l} \sigma_i \chi_i^{\Theta}, \boxplus_{i=1}^{l} \sigma_i \psi_i^{\Theta} \right); \\ \begin{bmatrix} 1 - \bigcirc_{i=1}^{l} \left(1 - \left((\cos^2(\pi/2 \cdot \eth_1^{\mathbb{T}-}))^{\Theta} \right)^{\Theta} \right)^{\sigma_i}, \\ 1 - \bigcirc_{i=1}^{l} \left(1 - \left((\cos^2(\pi/2 \cdot \eth_1^{\mathbb{T}+}))^{\Theta} \right)^{\Theta} \right)^{\sigma_i} \end{bmatrix}, \\ \begin{bmatrix} \bigcirc_{i=1}^{l} \left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_i^{\mathbb{I}-}))^{\Theta} \right)^{\Theta} \right)^{\sigma_i}, \\ \bigcirc_{i=1}^{l} \left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_i^{\mathbb{I}-}))^{\Theta} \right)^{\Theta} \right)^{\sigma_i} \end{bmatrix}, \\ \begin{bmatrix} \bigcirc_{i=1}^{l} \left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_i^{\mathbb{F}-}))^{\Theta} \right)^{\Theta} \right)^{\sigma_i}, \\ \bigcirc_{i=1}^{l} \left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_i^{\mathbb{F}-}))^{\Theta} \right)^{\Theta} \right)^{\sigma_i} \end{bmatrix} \end{bmatrix}.$$

If $n = l + 1$, then $\boxplus_{i=1}^{l} \sigma_i (\cos L_i)^{\Theta} \boxplus \sigma_{l+1} (\cos L_{l+1})^{\Theta} = \boxplus_{i=1}^{l+1} \sigma_i (\cos L_i)^{\Theta}$.
Now, $\boxplus_{i=1}^{l} \sigma_i (\cos L_i)^{\Theta} \boxplus \sigma_{l+1} (\cos L_{l+1})^{\Theta} = \sigma_1 (\cos L_1)^{\Theta} \boxplus \sigma_2 (\cos L_2)^{\Theta} \boxplus \ldots \boxplus \sigma_l (\cos L_l)^{\Theta} \boxplus \sigma_{l+1} (\cos L_{l+1})^{\Theta}$

$$
= \left[\begin{array}{c}
\left(\boxplus_{i=1}^{l} \sigma_i \chi_i^{\Theta} \boxplus \sigma_{l+1} \chi_{l+1}^{\Theta}, \boxplus_{i=1}^{l} \sigma_i \psi_i^{\Theta} \boxplus \sigma_{l+1} \psi_{l+1}^{\Theta} \right); \\
\left[\begin{array}{c}
\left(1 - \bigcirc_{i=1}^{l} \left(1 - \left((\cos^2(\pi/2 \cdot \eth_i^{\mathbb{T}-}))^{\Theta} \right)^{\Theta} \right)^{\sigma_i} \right)^{\Theta} \\
+ \left(1 - \left(1 - \left((\cos^2(\pi/2 \cdot \eth_{l+1}^{\mathbb{T}-}))^{\Theta} \right)^{\Theta} \right)^{\sigma_1} \right)^{\Theta}, \\
- \left(1 - \bigcirc_{i=1}^{l} \left(1 - \left((\cos^2(\pi/2 \cdot \eth_i^{\mathbb{T}-}))^{\Theta} \right)^{\Theta} \right)^{\sigma_i} \right)^{\Theta} \\
\cdot \left(1 - \left(1 - \left((\cos^2(\pi/2 \cdot \eth_{l+1}^{\mathbb{T}-}))^{\Theta} \right)^{\Theta} \right)^{\sigma_1} \right)^{\Theta} \\
\left(1 - \bigcirc_{i=1}^{l} \left(1 - \left((\cos^2(\pi/2 \cdot \eth_i^{\mathbb{T}+}))^{\Theta} \right)^{\Theta} \right)^{\sigma_i} \right)^{\Theta} \\
+ \left(1 - \left(1 - \left((\cos^2(\pi/2 \cdot \eth_{l+1}^{\mathbb{T}+}))^{\Theta} \right)^{\Theta} \right)^{\sigma_1} \right)^{\Theta} \\
- \left(1 - \bigcirc_{i=1}^{l} \left(1 - \left((\cos^2(\pi/2 \cdot \eth_i^{\mathbb{T}+}))^{\Theta} \right)^{\Theta} \right)^{\sigma_i} \right)^{\Theta} \\
\cdot \left(1 - \left(1 - \left((\cos^2(\pi/2 \cdot \eth_{l+1}^{\mathbb{T}+}))^{\Theta} \right)^{\Theta} \right)^{\sigma_1} \right)^{\Theta}
\end{array} \right], \\
\left[\begin{array}{c}
\bigcirc_{i=1}^{l} \left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_i^{\mathbb{I}-}))^{\Theta} \right)^{\Theta} \right)^{\sigma_i} \\
\cdot \left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_{l+1}^{\mathbb{I}-}))^{\Theta} \right)^{\Theta} \right)^{\sigma_1}, \\
\bigcirc_{i=1}^{l} \left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_i^{\mathbb{I}+}))^{\Theta} \right)^{\Theta} \right)^{\sigma_i} \\
\cdot \left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_{l+1}^{\mathbb{I}+}))^{\Theta} \right)^{\Theta} \right)^{\sigma_1}
\end{array} \right], \\
\left[\begin{array}{c}
\bigcirc_{i=1}^{l} \left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_i^{\mathbb{F}-}))^{\Theta} \right)^{\Theta} \right)^{\sigma_i} \\
\cdot \left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_{l+1}^{\mathbb{F}-}))^{\Theta} \right)^{\Theta} \right)^{\sigma_1}, \\
\bigcirc_{i=1}^{l} \left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_i^{\mathbb{F}+}))^{\Theta} \right)^{\Theta} \right)^{\sigma_i} \\
\cdot \left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_{l+1}^{\mathbb{F}+}))^{\Theta} \right)^{\Theta} \right)^{\sigma_1}.
\end{array} \right]
\end{array} \right]
$$

Thus, $\boxplus_{i=1}^{l+1}\sigma_i(\cos L_i)^{\Theta} =$

$$\left[\begin{array}{c} \left(\boxplus_{i=1}^{l+1}\sigma_i\chi_i^{\Theta}, \boxplus_{i=1}^{l+1}\sigma_i\psi_i^{\Theta}\right); \\ \left[\begin{array}{c} 1-\bigcirc_{i=1}^{l+1}\left(1-\left((\cos^2(\pi/2\cdot\eth_1^{\mathbb{T}-}))^{\Theta}\right)^{\Theta}\right)^{\sigma_i}, \\ 1-\bigcirc_{i=1}^{l+1}\left(1-\left((\cos^2(\pi/2\cdot\eth_1^{\mathbb{T}+}))^{\Theta}\right)^{\Theta}\right)^{\sigma_i} \end{array}\right], \\ \left[\begin{array}{c} \bigcirc_{i=1}^{l+1}\left(1-\left(1-(\cos^2(\pi/2\cdot\eth_i^{\mathbb{I}-}))^{\Theta}\right)^{\Theta}\right)^{\sigma_i}, \\ \bigcirc_{i=1}^{l+1}\left(1-\left(1-(\cos^2(\pi/2\cdot\eth_i^{\mathbb{I}-}))^{\Theta}\right)^{\Theta}\right)^{\sigma_i} \end{array}\right], \\ \left[\begin{array}{c} \bigcirc_{i=1}^{l+1}\left(1-\left(1-(\cos^2(\pi/2\cdot\eth_i^{\mathbb{F}-}))^{\Theta}\right)^{\Theta}\right)^{\sigma_i}, \\ \bigcirc_{i=1}^{l+1}\left(1-\left(1-(\cos^2(\pi/2\cdot\eth_i^{\mathbb{F}-}))^{\Theta}\right)^{\Theta}\right)^{\sigma_i} \end{array}\right] \end{array}\right].$$

Hence, $\left(\boxplus_{i=1}^{l+1}\sigma_i(\cos L_i)^{\Theta}\right)^{1/\Theta}$

$$\left[\begin{array}{c} \left(\left(\boxplus_{i=1}^{l+1}\sigma_i\chi_i^{\Theta}\right)^{1/\Theta}, \left(\boxplus_{i=1}^{l+1}\sigma_i\psi_i^{\Theta}\right)^{1/\Theta}\right); \\ \left[\begin{array}{c} \left(1-\bigcirc_{i=1}^{l+1}\left(1-\left((\cos^2(\pi/2\cdot\eth_i^{\mathbb{T}-}))^{\Theta}\right)^{\Theta}\right)^{\sigma_i}\right)^{1/\Theta}, \\ \left(1-\bigcirc_{i=1}^{l+1}\left(1-\left((\cos^2(\pi/2\cdot\eth_i^{\mathbb{T}+}))^{\Theta}\right)^{\Theta}\right)^{\sigma_i}\right)^{1/\Theta} \end{array}\right], \\ \left[\begin{array}{c} 1-\left(1-\left(\bigcirc_{i=1}^{l+1}\left(1-\left(1-(\cos^2(\pi/2\cdot\eth_i^{\mathbb{I}-}))^{\Theta}\right)^{\Theta}\right)^{\sigma_i}\right)\right)^{1/\Theta}, \\ 1-\left(1-\left(\bigcirc_{i=1}^{l+1}\left(1-\left(1-(\cos^2(\pi/2\cdot\eth_i^{\mathbb{I}+}))^{\Theta}\right)^{\Theta}\right)^{\sigma_i}\right)\right)^{1/\Theta} \end{array}\right], \\ \left[\begin{array}{c} 1-\left(1-\left(\bigcirc_{i=1}^{l+1}\left(1-\left(1-(\cos^2(\pi/2\cdot\eth_i^{\mathbb{F}-}))^{\Theta}\right)^{\Theta}\right)^{\sigma_i}\right)\right)^{1/\Theta}, \\ 1-\left(1-\left(\bigcirc_{i=1}^{l+1}\left(1-\left(1-(\cos^2(\pi/2\cdot\eth_i^{\mathbb{F}+}))^{\Theta}\right)^{\Theta}\right)^{\sigma_i}\right)\right)^{1/\Theta} \end{array}\right] \end{array}\right].$$

It is true for any l.

In the case of $\Theta = 1$, the CTri-GNNIVWA becomes the CTri-NNIVWA.

Theorem 7 *If all* $L_i = \left\langle(\chi_i, \psi_i); \left[(\cos(\pi/2\cdot\eth_i^{\mathbb{T}-})), (\cos(\pi/2\cdot\eth_i^{\mathbb{T}+}))\right], \left[(\cos(\pi/2\cdot\eth_i^{\mathbb{I}-})), (\cos(\pi/2\cdot\eth_i^{\mathbb{I}+}))\right],\right.$
$\left.\left[(\cos(\pi/2\cdot\eth_i^{\mathbb{F}-})), (\cos(\pi/2\cdot\eth_i^{\mathbb{F}+}))\right]\right\rangle$ *are equal and* $L_i = L$ *with* $\Theta = 1$*, then* *CTri-GNNIVWA*$(L_1, L_2, \ldots, L_n) = L$.

Proof The proof of Theorem 7 is based on Theorem 3.

CTri-GNNIVWA is valid for Theorem 4 and Theorem 5 satisfy the boundedness and monotonicity properties.

5.3 Generalized CTri-NNIVWG (CTri-GNNIVWG)

Definition 15 Let $L_i = \left\langle (\chi_i, \psi_i); [\eth_i^{\mathbb{T}-}, \eth_i^{\mathbb{T}+}], [\eth_i^{\mathbb{I}-}, \eth_i^{\mathbb{I}+}], [\eth_i^{\mathbb{F}-}, \eth_i^{\mathbb{F}+}] \right\rangle$ be the finite collection of CTri-NNIVNs. Then CTri-GNNIVWG $(L_1, L_2, \ldots, L_n) = \frac{1}{\Theta}\left(\bigcirc_{i=1}^{n} (\Theta \cos L_i)^{\sigma_i} \right)$, $(i = 1, 2, \ldots, n)$.

Theorem 8 *Let* $L_i = \left\langle (\chi_i, \psi_i); [\eth_i^{\mathbb{T}-}, \eth_i^{\mathbb{T}+}], [\eth_i^{\mathbb{I}-}, \eth_i^{\mathbb{I}+}], [\eth_i^{\mathbb{F}-}, \eth_i^{\mathbb{F}+}] \right\rangle$ *be the finite collection of CTri-NNIVNs. Then CTri-GNNIVWG*$(L_1, L_2, \ldots, L_n) =$

$$\left[\begin{array}{c} \left(\frac{1}{\Theta} \bigcirc_{i=1}^{n} (\Theta \chi_i)^{\sigma_i}, \frac{1}{\Theta} \bigcirc_{i=1}^{n} (\Theta \psi_i)^{\sigma_i} \right); \\ \left[\begin{array}{l} 1 - \left(1 - \left(\bigcirc_{i=1}^{n} \left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_i^{\mathbb{T}-}))^{\Theta} \right)^{\Theta} \right)^{\sigma_i} \right)^{\Theta} \right)^{1/\Theta}, \\ 1 - \left(1 - \left(\bigcirc_{i=1}^{n} \left(1 - \left(1 - (\cos^2(\pi/2 \cdot \eth_i^{\mathbb{T}+}))^{\Theta} \right)^{\Theta} \right)^{\sigma_i} \right)^{\Theta} \right)^{1/\Theta} \end{array} \right], \\ \left[\begin{array}{l} \left(1 - \bigcirc_{i=1}^{n} \left(1 - \left((\cos^2(\pi/2 \cdot \eth_i^{\mathbb{I}-}))^{\Theta} \right)^{\Theta} \right)^{\sigma_i} \right)^{1/\Theta}, \\ \left(1 - \bigcirc_{i=1}^{n} \left(1 - \left((\cos^2(\pi/2 \cdot \eth_i^{\mathbb{I}+}))^{\Theta} \right)^{\Theta} \right)^{\sigma_i} \right)^{1/\Theta} \end{array} \right], \\ \left[\begin{array}{l} \left(1 - \bigcirc_{i=1}^{n} \left(1 - \left((\cos^2(\pi/2 \cdot \eth_i^{\mathbb{F}-}))^{\Theta} \right)^{\Theta} \right)^{\sigma_i} \right)^{1/\Theta}, \\ \left(1 - \bigcirc_{i=1}^{n} \left(1 - \left((\cos^2(\pi/2 \cdot \eth_i^{\mathbb{F}+}))^{\Theta} \right)^{\Theta} \right)^{\sigma_i} \right)^{1/\Theta} \end{array} \right] \end{array} \right].$$

Proof The proof of Theorem 8 is devoted to Theorem 6.

With $\Theta = 1$, the CTri-GNNIVWG becomes the CTri-NNIVWG.

According to Theorem 4 and Theorem 5, the CTri-GNNIVWG operator fulfills the boundedness and monotonicity properties.

Theorem 9 *If all* $L_i = \left\langle (\chi_i, \psi_i); \left[(\cos(\pi/2 \cdot \eth_i^{\mathbb{T}-})), (\cos(\pi/2 \cdot \eth_i^{\mathbb{T}+})) \right], \left[(\cos(\pi/2 \cdot \eth_i^{\mathbb{I}-})), (\cos(\pi/2 \cdot \eth_i^{\mathbb{I}+})) \right], \left[(\cos(\pi/2 \cdot \eth_i^{\mathbb{F}-})), (\cos(\pi/2 \cdot \eth_i^{\mathbb{F}+})) \right] \right\rangle$ *are equal and* $L_i = L$ *with* $\Theta = 1$*, then CTri-GNNIVWG*$(L_1, L_2, \ldots, L_n) = L$.

6 MADM Concept Using CTri-NNIV Approach

Let $L = \{L_1, L_2, \ldots, L_n\}$ be the set of n-alternatives and $C = \{M_1, M_2, \ldots, M_m\}$ be the set of m-attributes and their corresponding weights of attributes are $w = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$, where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$ $L_{ij} = \Big\langle (\chi_{ij}, \psi_{ij}); [\cos^2(\pi/2 \cdot \eth_{ij}^{\mathbb{T}-}), \cos^2(\pi/2 \cdot \eth_{ij}^{\mathbb{T}+})], [\cos^2(\pi/2 \cdot \eth_{ij}^{\mathbb{I}-}), \cos^2(\pi/2 \cdot \eth_{ij}^{\mathbb{I}+})][\cos^2(\pi/2 \cdot \eth_{ij}^{\mathbb{F}-}), \cos^2(\pi/2 \cdot \eth_{ij}^{\mathbb{F}+})]\Big\rangle$ denote CTri-NNIVN of L_i in M_j. Thus, $\Big[\cos^2(\pi/2 \cdot \eth_{ij}^{\mathbb{T}-}), \cos^2(\pi/2 \cdot \eth_{ij}^{\mathbb{T}+})\Big], \Big[\cos^2(\pi/2 \cdot \eth_{ij}^{\mathbb{I}-}), \cos^2(\pi/2 \cdot \eth_{ij}^{\mathbb{I}+})\Big], \Big[\cos^2(\pi/2 \cdot \eth_{ij}^{\mathbb{F}-}), \cos^2(\pi/2 \cdot \eth_{ij}^{\mathbb{F}+})\Big] \in [0, 1]$ and the values of $\cos^2(\pi/2 \cdot \eth_{ij}^{\mathbb{T}+})(z) + \cos^2(\pi/2 \cdot \eth_{ij}^{\mathbb{I}+})(z) + \cos^2(\pi/2 \cdot \eth_{ij}^{\mathbb{F}+})(z)$ lie between 0 and 2. The $n \times m$ decision matrix is $\mathbb{D} = (L_{ij})_{n\times m}$, where n-alternative sets and m-attribute sets.

6.1 Algorithm

Step 1 The decision values should be entered based on the CTri-NNIV model.

Step 2 Find normalized decision values. The decision matrix $\mathbb{D} = (L_{ij})_{n\times m}$ is to normalized decision matrix $\overline{\mathbb{D}} = (\widehat{L_{ij}})_{n\times m}$. Since,

$$\widehat{L_{ij}} = \Big\langle (\overline{\chi_{ij}}, \overline{\psi_{ij}}); [\overline{\eth_{ij}^{\mathbb{T}-}}, \overline{\eth_{ij}^{\mathbb{T}+}}], [\overline{\eth_{ij}^{\mathbb{I}-}}, \overline{\eth_{ij}^{\mathbb{I}+}}], [\overline{\eth_{ij}^{\mathbb{F}-}}, \overline{\eth_{ij}^{\mathbb{F}+}}]\Big\rangle$$

and

$$\overline{\chi_{ij}} = \frac{\chi_{ij}}{\sup_i(\chi_{ij})}, \overline{\psi_{ij}} = \frac{\psi_{ij}}{\sup_i(\psi_{ij})} \cdot \frac{\psi_{ij}}{\chi_{ij}}, \ \overline{\eth_{ij}^{\mathbb{T}-}} = \eth_{ij}^{\mathbb{T}-}, \overline{\eth_{ij}^{\mathbb{T}+}} = \eth_{ij}^{\mathbb{T}+}.$$

Step 3 Find the aggregated values for every alternative. CTri-NNIV is used to represent the attribute M_j in L_i, and

$$\widehat{L_{ij}} = \Big\langle (\overline{\chi_{ij}}, \overline{\psi_{ij}}); [\overline{\eth_{ij}^{\mathbb{T}-}}, \overline{\eth^{\mathbb{T}+})_{ij}}], [\overline{\eth_{ij}^{\mathbb{I}-}}, \overline{\eth_{ij}^{\mathbb{I}+}}], [\overline{\eth_{ij}^{\mathbb{F}-}}, \overline{\eth_{ij}^{\mathbb{F}+}}]\Big\rangle$$

is aggregated into

$$\widehat{L_i} = \Big\langle (\overline{\chi_i}, \overline{\psi_i}); [\overline{\eth_i^{\mathbb{T}-}}, \overline{\eth_i^{\mathbb{T}+}}], [\overline{\eth_i^{\mathbb{I}-}}, \overline{\eth_i^{\mathbb{I}+}}], [\overline{\eth_i^{\mathbb{F}-}}, \overline{\eth_i^{\mathbb{F}+}}]\Big\rangle.$$

Step 4 Compute the ideal values of each alternative:

$$\widehat{L}^+ = \Big\langle \Big(\sup_{1\le i\le n} \overline{\chi_{ij}}, \inf_{1\le i\le n} \overline{\psi_{ij}} \Big); [1, 1], [0, 0], [0, 0] \Big\rangle$$

and

$$\widehat{L}^{-} = \left\langle \left(\inf_{1 \le i \le n} \overline{\chi_{ij}}, \sup_{1 \le i \le n} \overline{\psi_{ij}} \right); [0, 0], [1, 1], [1, 1] \right\rangle.$$

Step 5 To illustrate, EDs between each alternative include ideal values:

$$\mathbb{D}_i^{+} = \mathbb{D}_E\left(\widehat{L}_i, \widehat{L}^{+}\right); \ \mathbb{D}_i^{-} = \mathbb{D}_E\left(\widehat{L}_i, \widehat{L}^{-}\right).$$

Step 6 To illustrate, the relative nearness values are

$$\mathbb{D}_i^{*} = \frac{\mathbb{D}_i^{-}}{\mathbb{D}_i^{+} + \mathbb{D}_i^{-}}.$$

Step 7 The output is the greatest value of $\mathbb{D}_i^{*}$, and thus the optimal solution is to be chosen as the most appropriate solution to our problem.

6.2 Robotic Engineering Real-Life Example

In computing, it is a branch of interdisciplinary study. Robotics is the design, construction, operation, and use of robots. According to an agreement, robots are primarily used to replace humans. Software can be used in a wide range of fields, such as mechanics, electronics, computer science, graphics, and software engineering. There are many ways in which it can replace humans. **Aircraft Manufacturing:** A scientist reported that robots are used to manufacture most aircraft. Artificial intelligence is the fastest-growing area in robotics. The majority of the work is done without human assistance. First and foremost, the ultimate robot is used in aircraft manufacturing. **Cafes and Hotels:** Nowadays, most hotels in China and other countries serve as servers. What makes it so interesting is that it has the most majestic function of speaking to people. The etymology of robot was introduced by Kern Capek. Scientists conduct extensive research in high-demand fields, such as hotels. **Army:** Robots are developed in such a way that robots are used in the army. They are considered soldiers as they are used in many mis-functional areas in critical situations. Their usage of robots can save a human life that is in the army. Most of the technical and emotional fireworks and bullets for humans can be avoided in robots. They are really very useful in the army. **Agriculture:** Robots are now being utilized to cultivate crops in agriculture. In the agriculture field, robots are having a great time. Some of the robots are used in Merlin robots, milkers, Rospheres, orange harvesters, and Lethice robots. Robotics are used widely in farming. A particularly suitable example is the milk bot. It is the first and foremost agricultural bot, Dubble Viro, that is capable of picking tomatoes without causing any damage. There is no doubt that Cambridge University has been instrumental in the development of robotics. It is a Vegebot. **Humanoid bot:** The first humanoid bot is WABLT-1, the world's first full-scale humanoid intelligent robot. It was created at Waseda University in 1967 and 1972. It is known as the WABLT project. It is a full-scale humanoid intelligence. The other bot, known as General Motors, was the first robot to work in Factor in 1961. If five different uses of robotics (alternatives) such as Aircraft Manufacturing (L_1), Cafes and Hotels (L_2), Army (L_3), Agriculture (L_4), Humanoid bot (L_5) with four attributes are considered as tasks (M_1), precision (M_2), speed (M_3), and completion of work (M_4) and their weights are $w = \{0.4, 0.3, 0.2, 0.1\}$. The best robotics are those that perform well in the real world. In making decisions, we consider the following factors:

	M_1	M_2	M_3	M_4
L_1	$\langle(0.85, 0.6); [0.15, 0.2], [0.1, 0.5], [0.3, 0.5]\rangle$	$\langle(0.55, 0.3); [0.15, 0.28], [0.3, 0.52], [0.1, 0.52]\rangle$	$\langle(0.5, 0.35); [0.15, 0.7], [0.52, 0.59], [0.3, 0.64]\rangle$	$\langle(0.75, 0.55); [0.13, 0.2], [0.3, 0.45], [0.45, 0.52]\rangle$
L_2	$\langle(0.95, 0.65); [0.22, 0.28], [0.3, 0.45], [0.31, 0.52]\rangle$	$\langle(0.4, 0.25); [0.22, 0.48], [0.3, 0.59], [0.52, 0.59]\rangle$	$\langle(0.55, 0.3); [0.22, 0.56], [0.3, 0.45], [0.3, 0.36]\rangle$	$\langle(0.9, 0.65); [0.15, 0.56], [0.45, 0.5], [0.45, 0.5]\rangle$
L_3	$\langle(0.9, 0.7); [0.22, 0.56], [0.3, 0.59], [0.3, 0.45]\rangle$	$\langle(0.6, 0.35); [0.48, 0.7], [0.3, 0.64], [0.45, 0.59]\rangle$	$\langle(0.5, 0.25); [0.15, 0.48], [0.45, 0.64], [0.31, 0.36]\rangle$	$\langle(0.75, 0.55); [0.27, 0.56], [0.23, 0.59], [0.52, 0.59]\rangle$
L_4	$\langle(0.75, 0.45); [0.2, 0.48], [0.23, 0.59], [0.59, 0.64]\rangle$	$\langle(0.45, 0.2); [0.22, 0.7], [0.23, 0.59], [0.3, 0.45]\rangle$	$\langle(0.65, 0.5); [0.2, 0.28], [0.3, 0.59], [0.3, 0.64]\rangle$	$\langle(0.85, 0.6); [0.2, 0.42], [0.23, 0.5], [0.3, 0.31]\rangle$
L_5	$\langle(0.85, 0.55); [0.22, 0.56], [0.3, 0.5], [0.1, 0.52]\rangle$	$\langle(0.55, 0.3); [0.2, 0.56], [0.52, 0.59], [0.31, 0.52]\rangle$	$\langle(0.6, 0.4); [0.15, 0.56], [0.52, 0.59], [0.45, 0.59]\rangle$	$\langle(0.8, 0.65); [0.15, 0.27], [0.3, 0.59], [0.52, 0.71]\rangle$

A normalized decision matrix can be illustrated as below:

	M_1	M_2	M_3
L_1	$\langle(0.8947, 0.605); [0.15, 0.2], [0.1, 0.5], [0.3, 0.5]\rangle$	$\langle(0.9167, 0.4675); [0.15, 0.28], [0.3, 0.52], [0.1, 0.52]\rangle$	$\langle(0.7692, 0.49); [0.15, 0.7], [0.52, 0.59], [0.3, 0.64]\rangle$
L_2	$\langle(1, 0.6353); [0.22, 0.28], [0.3, 0.45], [0.31, 0.52]\rangle$	$\langle(0.6667, 0.4464); [0.22, 0.48], [0.3, 0.59], [0.52, 0.59]\rangle$	$\langle(0.8462, 0.3273); [0.22, 0.56], [0.3, 0.45], [0.3, 0.36]\rangle$
L_3	$\langle(0.9474, 0.7778); [0.22, 0.56], [0.3, 0.59], [0.3, 0.45]\rangle$	$\langle(1, 0.5833); [0.48, 0.7], [0.3, 0.64], [0.45, 0.59]\rangle$	$\langle(0.7692, 0.25); [0.15, 0.48], [0.45, 0.64], [0.31, 0.36]\rangle$
L_4	$\langle(0.7895, 0.3857); [0.2, 0.48], [0.23, 0.59], [0.59, 0.64]\rangle$	$\langle(0.75, 0.254); [0.22, 0.7], [0.23, 0.59], [0.3, 0.45]\rangle$	$\langle(1, 0.7692); [0.2, 0.28], [0.3, 0.59], [0.3, 0.64]\rangle$
L_5	$\langle(0.8947, 0.5084); [0.22, 0.56], [0.3, 0.5], [0.1, 0.52]\rangle$	$\langle(0.9167, 0.4675); [0.2, 0.56], [0.52, 0.59], [0.31, 0.52]\rangle$	$\langle(0.9231, 0.5333); [0.15, 0.56], [0.52, 0.59], [0.45, 0.59]\rangle$

	M_4
L_1	$\langle(0.8333, 0.6205); [0.13, 0.2], [0.3, 0.45], [0.45, 0.52]\rangle$
L_2	$\langle(1, 0.7222); [0.15, 0.56], [0.45, 0.5], [0.45, 0.5]\rangle$
L_3	$\langle(0.8333, 0.6205); [0.27, 0.56], [0.23, 0.59], [0.52, 0.59]\rangle$
L_4	$\langle(0.9444, 0.6516); [0.2, 0.42], [0.23, 0.5], [0.3, 0.31]\rangle$
L_5	$\langle(0.8889, 0.8125); [0.15, 0.27], [0.3, 0.59], [0.52, 0.71]\rangle$

We can find aggregate values based on the CTri-NNIVWA operator for every alternative that are as below:

	$CTri - NNIVWA\ operator\ (\Theta = 1)$
$\widehat{L_1}$	$\langle(0.8701, 0.5423); [0.3608, 0.9273], [0.3395, 0.5610], [0.2135, 0.4326]\rangle$
$\widehat{L_2}$	$\langle(0.8692, 0.5257); [0.3396, 0.972], [0.1079, 0.7957], [0.5524, 0.6462]\rangle$
$\widehat{L_3}$	$\langle(0.9161, 0.5982), [0.4069, 0.9601], [0.1348, 0.4732], [0.3467, 0.7246]\rangle$
$\widehat{L_4}$	$\langle(0.8352, 0.4495), [0.4082, 0.9445], [0.0779, 0.8066], [0.2198, 0.4105]\rangle$
$\widehat{L_5}$	$\langle(0.9064, 0.5315), [0.3736, 0.9747], [0.2769, 0.5641], [0.8591, 0.7866]\rangle$

Illustrate that the ideal values of every alternative are below:

$\widehat{L}^+$
$\langle(0.9161, 0.4495), [1, 1], [0, 0], [0, 0]\rangle$

$\widehat{L}^-$
$\langle(0.8352, 0.5982), [0, 0], [1, 1], [1, 1]\rangle$

The Hamming distance between every alternative with ideal values is below:

$\mathbb{D}_1^+$	$\mathbb{D}_2^+$	$\mathbb{D}_3^+$	$\mathbb{D}_4^+$	$\mathbb{D}_5^+$
0.4349	0.7170	0.4234	0.7107	0.8799

$\mathbb{D}_1^-$	$\mathbb{D}_2^-$	$\mathbb{D}_3^-$	$\mathbb{D}_4^-$	$\mathbb{D}_5^-$
0.9276	0.6455	0.9391	0.6517	0.4826

It illustrates the relative nearness values are $\mathbb{D}_1^* = 0.6808$, $\mathbb{D}_2^* = 0.4738$, $\mathbb{D}_3^* = 0.6892$, $\mathbb{D}_4^* = 0.4783$, and $\mathbb{D}_5^* = 0.3542$. The ranking of alternatives is $L_3 \geq L_1 \geq L_4 \geq L_2 \geq L_5$. For real-world performance, L_3 provides the most suitable solution.

6.3 Comparison for Proposed Approach with Existing Approach

Based on the above news, we handover to the CTri-NNIVWG, CTri-GNNIVWA, and CTri-GNNIVWG operators. As a result, the following tabulation was made:

$\Theta = 1$	$CTriNNIVWA$	$CTriNNIVWG$	$GCTriNNIVWA$	$GCTriNNIVWG$
$TOPSIS -\ Hamming$	$L_3 \geq L_1 \geq L_4$	$L_3 \geq L_1 \geq L_4$	$L_3 \geq L_1 \geq L_4$	$L_3 \geq L_1 \geq L_4$
$distance\,(proposed)$	$L_2 \geq L_5$	$L_5 \geq L_2$	$L_2 \geq L_5$	$L_5 \geq L_2$
$Hamming\,distance$ [13]	$L_3 \geq L_5 \geq L_1$	$L_3 \geq L_5 \geq L_2$	$L_3 \geq L_5 \geq L_1$	$L_3 \geq L_5 \geq L_2$
	$L_2 \geq L_4$	$L_1 \geq L_4$	$L_2 \geq L_4$	$L_1 \geq L_4$

Replace the values of Θ from the CTri-NNIVWA approach. Then the relative nearness values and their orders are in Figs. 23.1 and 23.2:

In the above discussion, the alternative ranking is determined in the case of the CTri-NNIVWA operator. If $\Theta = 1$, we can find the ranking of alternative is $L_3 \geq L_1 \geq L_4 \geq L_2 \geq L_5$; if $\Theta = 2$, we can find the ranking of alternative is $L_3 \geq L_2 \geq L_1 \geq L_4 \geq L_5$; if $\Theta = 3$, then finding a new order is $L_5 \geq L_2 \geq L_1 \geq L_3 \geq L_4$. Hence, the optimal L_3 is converted into L_5. Similarly, we can handover to the alternative rankings that are founded using CTri-NNIVWG, CTri-GNNIVWA, and CTri-GNNIVWG operators based on Θ.

7 Conclusion

In this chapter, we introduce ED and HD measures for CTri-NNIVSs, whose mathematical simplicity makes them even more appealing. Through appropriate

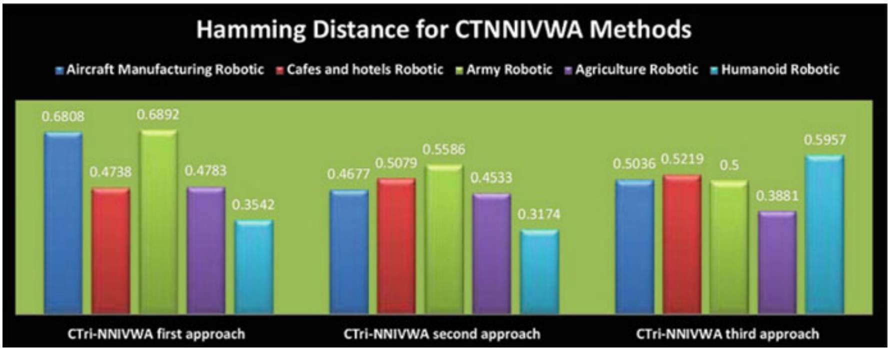

Fig. 23.1 Graphical representation based on Hamming distance for CTri-NNIVWA

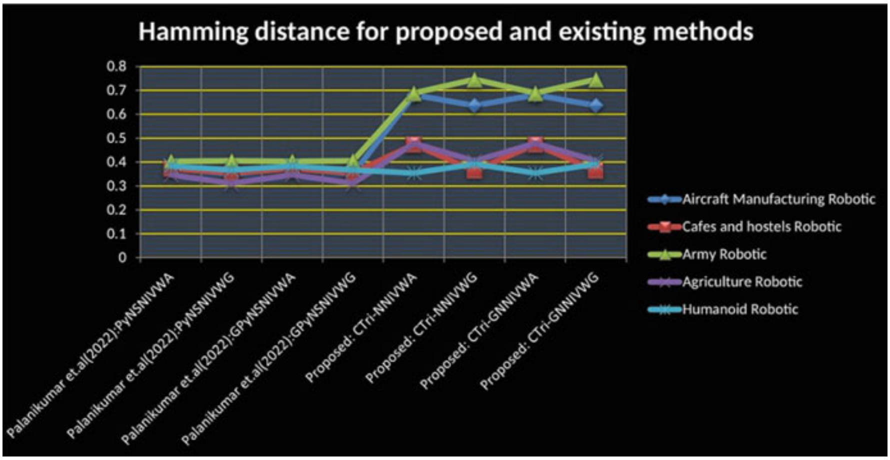

Fig. 23.2 Graphical representation based on Hamming distance for the existing and proposed models

numerical examples, the ED and HD measures are shown to be superior. In real-life examples, the ED and HD measures are demonstrated to be applicable. Our proposal improves the AO rules for CTri-NNIVWA, CTri-NNIVWG, CTri-GNNIVWA, and CTri-GNNIVWG. A few examples were provided of how these AOs can be developed, as well as some properties. Using the CTri-NNIV MADM approach can help people make the right decision out of available alternatives in indeterminate and inconsistent information environments. To solve MADM problems under Θ, we applied the CTri-NNIVWA, CTri-NNIVWG, CTri-GNNIVWA, and CTri-GNNIVWG operators. According to Θ, the CTri-NNIVWA, CTri-NNIVWG, CTri-GNNIVWA, and CTri-GNNIVWG operators rank alternatives differently. As a result of the above analysis, Θ appears to have the greatest impact on ranking. By adjusting Θ based on the actual situation, decision-makers can determine an appropriate and reasonable ranking. Using Θ values, the decision-maker can arrive at a conclusion. An analysis of data is involved in a number of real-life applications of ED and HD measures of NSs. The topic of this chapter will be helpful to future researchers interested in this emerging field.

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Chapter 24
An Integrated Weighted Distance-Based Approximation Method for Interval-Valued Spherical Fuzzy MAGDM

Utpal Mandal and Mijanur Rahaman Seikh

1 Introduction

An organization identifies, evaluates, and contracts with suppliers during the selection of suppliers. For any organization to succeed, it is crucial. When suppliers are selected incorrectly, suppliers may perform poorly, supply may disrupt, and business processes may be inadequate. Most organizations lack a well-defined supplier selection process, which makes reducing excess spending difficult. Now, selecting the most suitable supplier for an organization consists of multiple conflicting attributes such as quality of the product, price of the product, past performance record, financial stability, the technical ability of the supplier, etc. As a result, it can be considered a multi-attribute decision-making (MADM) problem. Again, no single person can assess the significance of each attribute in a MADM process. So, we need to consult with several experts from different domains to gather the required data. Thus, the problem is similar to a MAGDM problem.

Now, expressing information about the attributes subject to a specific alternative is a crucial component of MAGDM problems. These kinds of issues frequently arise in our daily lives. Because, in the real world, due to insufficient information and the complexity of MAGDM, it is more appropriate to convey attribute values using fuzzy sets rather than crisp values. However, the fuzzy set is inappropriate for making several real-life decisions since it is limited to membership degree (MD) only. As an extension of the fuzzy set, Atanassov [2] presented the notion of the intuitionistic fuzzy set (IFS). The IFS is associated with the MD and the nonmembership degree (NMD), whose sum cannot be more than one. Several researchers developed the MADM method under the intuitionistic fuzzy environment [8, 30, 31]. Later, Yager [29] enlarged the range of IFS and proposed Pythagorean fuzzy sets (PFS). The

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C. Jana et al. (eds.), *Fuzzy Optimization, Decision-making and Operations Research*, https://doi.org/10.1007/978-3-031-35668-1_24

sum of the squares of MD and NMD in PFS is restricted to 1. There have been many significant contributions to the PFS [14–16, 32].

Gundogdu and Kahraman [11] developed the notion of a spherical fuzzy set (SFS) as an extension of PFS. With the SFSs, decision experts can define a membership function on a spherical surface and independently assign the parameters to a larger domain using the membership function. Several researchers are investigating the properties of SFSs and how they can be applied in MADM scenarios [1, 19]. Mathew et al. [21] combined the analytical hierarchy process (AHP) and TOPSIS under SFS and applied them to the selection of an advanced manufacturing system. Mahmood et al. [20] defined some operational laws of SFSs and applied them to solve medical diagnosis-based problems.

Currently, Gundogdu and Kahraman [12] combined SFS and interval-valued fuzzy set [9] and introduced the notion of IVSF set. Several researchers utilized the IVSF set to deal with uncertainty and develop a decision-making model. Gul and Ak [10] extended the TOPSIS model under the IVSF environment and used it to select a marble manufacturing facility. Gundogdu and Kahraman [13] developed the AHP method using IVSF sets and utilized it for hospital performance assessment. Erdogan [7] combined SWARA and MAIRCA methods under the IVSF environment to assess farmers' attitudes toward Agriculture 4.0 technologies. Duleba et al. [6] developed the AHP method for assessing public transportation problems using IVSF data. Aydogdu and Gul [4] combined the entropy method and ARAS method under the IVSF environment.

In the following, we review existing fuzzy extensions to the WDBA method [23]. According to the WDBA method, the best alternative should be closest to the ideal solution and farthest away from the anti-ideal solution. Dorfeshan et al. [5] developed the WDBA method with interval-valued fuzzy data. Peng et al. [22] extended the WDBA method under q-rung orthopair fuzzy environment and applied it in emergency decision-making. In light of the above literature review, we can conclude that WDBA can be used effectively to address real-world decision-making scenarios with conflicting attributes. But the WDBA method using IVSF sets has not existed yet. Thus, WDBA should be extended in the IVSF environment.

Now, the weight of the decision experts appointed to solve MAGDM problems cannot be equalized. The importance of attributes may be judged differently by different decision experts. Therefore, in this chapter, we calculate the experts' weights utilizing the AHP. Again, not all attributes are equally important in decision-making. In most cases, decision-makers choose attribute weights at random. However, this is not realistic. To avoid the influence of decision experts in determining attribute weights, we enhance the entropy method under the IVSF environment and use it to determine attribute weights in this chapter.

The motivation for this study can be summarized as follows:

- The IVSF sets are a new concept in the literature. A wide range of applications enables the IVSF environment-based decision-making model to be applied more flexibly and in greater detail to reveal obscure information. In this context, it is essential to focus more on the decision-making model within IVSF.

- The existing score function for IVSF sets has limitations. The existing score functions cannot distinguish IVSF sets properly. Thus, it is important to develop a new score function that is more accurate for IVSF sets.
- In the MAGDM process, attribute weights play a vital role. For determining the attribute weights, this study developed an IVSF environment-based entropy model intending to avoid undesirable effects of decision-makers on the outcome of decision-making.
- The WDBA method has emerged as an efficient and straightforward decision-making tool. Thus, the WDBA method, when combined with the entropy method and IVSF sets in a decision-making environment, will have a more robust decision-making structure than the existing methods.

This chapter makes the following contributions:

- Introduced an improved score function and Euclidean distance measure under an IVSF environment.
- Developed entropy method using the proposed score function to determine the attribute weights.
- Developed a new approach called IVSF-AHP-entropy-WDBA method utilizing AHP, WDBA, and entropy method under IVSF environment. Here, the AHP, entropy, and WDBA methods are employed to determine the decision experts' weights, attribute weights, and ranking outcomes, respectively.
- Presented a numerical example of selecting a supplier for a textile manufacturing company within the IVSF context to demonstrate the practicality and efficiency of the developed IVSF-AHP-WDBA-entropy methodology.

This chapter is arranged in the following manner: In Sect. 2, some basic preliminaries are incorporated. In Sect. 3, we propose an improved score function to distinguish IVSF sets. An entropy method and Euclidean distance measure are also introduced in this section. The AHP-WDBA-entropy methodology for MAGDM under the IVSF environment is presented in Sect. 4. Here, the decision experts' weights are obtained by using the AHP method and attribute weights are calculated using the proposed entropy method. Section 5 presents a supplier selection problem in a textile manufacturing company to illustrate the efficiency of the developed IVSF-AHP-WDBA-entropy methodology. This section also compares the results to various existing methods to demonstrate the feasibility of our model. Finally, we concluded in Sect. 6.

2 Preliminaries

Here, we review some fundamental preliminaries. The set Y is regarded as a universal set all through the paper.

Definition 1 ([11]) The SFS $\tilde{S}$ over Y is expressed as

$$\tilde{S} = \{\langle g, \eta_S(g), \theta_S(g), \pi_S(g)\rangle | g \in Y\},$$

where η_S, θ_S, and π_S are the functions from Y to $[0, 1]$ and they, respectively, represent the MD, NMD, and hesitancy degree (HD). The MD, NMD, and HD satisfy the condition $(\eta_S(g))^2 + (\theta_S(g))^2 + (\pi_S(g))^2 \leq 1$ for each $g \in Y$.

Definition 2 ([12]) The IVSF set S over Y is expressed as

$$S = \{\langle g, ([\eta_S^-(g), \eta_S^+(g)], [\theta_S^-(g), \theta_S^+(g)], [\pi_S^-(g), \pi_S^+(g)])\rangle | g \in Y\},$$

where $0 \leq \eta_S^-(g) \leq \eta_S^+(g) \leq 1$, $0 \leq \theta_S^-(g) \leq \theta_S^+(g) \leq 1$, and $0 \leq (\eta_S^+(g))^2 + (\theta_S^+(g))^2 + (\pi_S^+(g))^2 \leq 1$. For every $g \in Y$, $\eta_S^+, \theta_S^+,$ and π_S^+, respectively, represent the upper MD, NMD, and HD of g to the set S. For simplicity, the triplet $s = \langle[\eta^-, \eta^+], [\theta^-, \theta^+], [\pi^-, \pi^+]\rangle$ where all of $[\eta^-, \eta^+], [\theta^-, \theta^+],$ and $[\pi^-, \pi^+]$ are the subsets of $[0, 1]$ represents as IVSF number (IVSFN).

Note that if $\eta^- = \eta^+, \theta^- = \theta^+,$ and $\pi^- = \pi^+$, then IVSF set converted into SFS.

Definition 3 ([12]) Let $s = \langle[\eta^-, \eta^+], [\theta^-, \theta^+], [\pi^-, \pi^+]\rangle$, $s_1 = \langle[\eta_1^-, \eta_1^+], [\theta_1^-, \theta_1^+], [\pi_1^-, \pi_1^+]\rangle$, and $s_2 = \langle[\eta_2^-, \eta_2^+], [\theta_2^-, \theta_2^+], [\pi_2^-, \pi_2^+]\rangle$ be three IVSFNs and $\lambda > 0$ be a real number. Then, some IVSFN operations are listed below.

1. $s_1 \oplus s_2 = \langle[\sqrt{(\eta_1^-)^2 + (\eta_2^-)^2 - (\eta_1^-)^2(\eta_2^-)^2}, \sqrt{(\eta_1^+)^2 + (\eta_2^+)^2 - (\eta_1^+)^2(\eta_2^+)^2}], [\theta_1^-\theta_2^-, \theta_1^+\theta_2^+],$ $[\sqrt{(1-(\eta_2^-)^2)(\pi_1^-)^2 + (1-(\eta_1^-)^2)(\pi_2^-)^2 - (\pi_1^-)^2(\pi_2^-)^2},$ $\sqrt{(1-(\eta_2^+)^2)(\pi_1^+)^2 + (1-(\eta_1^+)^2)(\pi_2^+)^2 - (\pi_1^+)^2(\pi_2^+)^2}]\rangle.$
2. $s_1 \otimes s_2 = \langle[\eta_1^-\eta_2^-, \eta_1^+\eta_2^+], [\sqrt{(\theta_1^-)^2 + (\theta_2^-)^2 - (\theta_1^-)^2(\theta_2^-)^2}, \sqrt{(\theta_1^+)^2 + (\theta_2^+)^2 - (\theta_1^+)^2(\theta_2^+)^2}],$ $[\sqrt{(1-(\theta_2^-)^2)(\pi_1^-)^2 + (1-(\theta_1^-)^2)(\pi_2^-)^2 - (\pi_1^-)^2(\pi_2^-)^2},$ $\sqrt{(1-(\theta_2^+)^2)(\pi_1^+)^2 + (1-(\theta_1^+)^2)(\pi_2^+)^2 - (\pi_1^+)^2(\pi_2^+)^2}]\rangle.$
3. $\lambda s = \langle[\sqrt{1-(1-(\eta^-)^2)^\lambda}, \sqrt{1-(1-(\eta^+)^2)^\lambda}], [(\theta^-)^\lambda, (\theta^+)^\lambda],$ $[\sqrt{(1-(\eta^-)^2)^\lambda - (1-(\eta^-)^2-(\pi^-)^2)^\lambda}, \sqrt{(1-(\eta^+)^2)^\lambda - (1-(\eta^+)^2-(\pi^+)^2)^\lambda}]\rangle.$
4. $s^\lambda = \langle[(\eta^-)^\lambda, (\eta^+)^\lambda], [\sqrt{1-(1-(\theta^-)^2)^\lambda}, \sqrt{1-(1-(\theta^+)^2)^\lambda}],$ $[\sqrt{(1-(\theta^-)^2)^\lambda - (1-(\theta^-)^2-(\pi^-)^2)^\lambda}, \sqrt{(1-(\theta^+)^2)^\lambda - (1-(\theta^+)^2-(\pi^+)^2)^\lambda}]\rangle.$

Definition 4 ([12]) Let $s_x = \langle[\eta_x^-, \eta_x^+], [\theta_x^-, \theta_x^+], [\pi_x^-, \pi_x^+]\rangle, x = 1, 2, \ldots, n$ be a collection of IVSFNs; then the aggregated value using interval-valued spherical weighted arithmetic mean (IVSWAM) operator is given by

$$IVSWAM(s_1, s_2, \ldots, s_n) = \left\langle \left[\sqrt{1 - \prod_{x=1}^{n}(1-(\eta_x^-)^2)^{\varpi_x}}, \sqrt{1 - \prod_{x=1}^{n}(1-(\eta_x^+)^2)^{\varpi_x}}\right],\right.$$
$$\left[\prod_{x=1}^{n}(\theta_x^-)^{\varpi_x}, \prod_{x=1}^{n}(\theta_x^+)^{\varpi_x}\right], \left[\sqrt{\prod_{x=1}^{n}(1-(\eta_x^-)^2)^{\varpi_x} - \prod_{x=1}^{n}(1-(\eta_x^-)^2-(\pi_x^-)^2)^{\varpi_x}},\right.$$
$$\left.\left.\sqrt{\prod_{x=1}^{n}(1-(\eta_x^+)^2)^{\varpi_x} - \prod_{x=1}^{n}(1-(\eta_x^+)^2-(\pi_x^+)^2)^{\varpi_x}}\right]\right\rangle$$

where ϖ_x is the weight of s_x with $\varpi_x > 0$, $\sum_{x=1}^{n} \varpi_x = 1$.

Definition 5 ([12]) The score and the accuracy functions of the IVSFN $s = \langle[\eta^-, \eta^+], [\theta^-, \theta^+], [\pi^-, \pi^+]\rangle$ are defined by

$$\Phi_1(s) = \frac{(\eta^-)^2 + (\eta^+)^2 - (\theta^-)^2 - (\theta^+)^2 - (\frac{\pi^-}{2})^2 - (\frac{\pi^+}{2})^2}{2} \quad (24.1)$$

and

$$\Psi(\sigma) = \frac{(\eta^-)^2 + (\eta^+)^2 + (\theta^-)^2 + (\theta^+)^2 + (\pi^-)^2 + (\pi^+)^2}{2} \quad (24.2)$$

respectively. Here $\Phi(\sigma) \in [-1, 1]$, and $\Psi(\sigma) \in [0, 1]$.

Definition 6 ([12]) According to Definition 5, the comparison between two IVSFNs $\Phi(s_1)$ and $\Phi(s_2)$ is defined as follows:

1. If $\Phi(s_1) > \Phi(s_2)$, then $s_1 \succ s_2$.
2. If $\Phi(s_1) = \Phi(s_2)$, and
   - If $\Psi(s_1) > \Psi(s_2)$, then $s_1 \succ s_2$.
   - If $\Psi(s_1) = \Psi(s_1)$, then $s_1 \sim s_2$.

2.1 Limitations of the Existing Score Functions

The score functions are an important factor in the MADM process for ranking the various options. In the following example, we show that the current score function on the IVSF set cannot distinguish between the alternatives adequately.

Example 1 Let $s_1 = \langle[0, 0.5], [0.1, 0.7], [0, 0.1]\rangle$ and $s_2 = \langle[0.3, 0.4], [0.5, 0.5], [0, 0.1]\rangle$ be two IVSFNs. Now, using Eq. 24.1 we have $\Phi_1(s_1) = -0.1275$ and $\Phi_1(s_2) = -0.1275$. Hence, we cannot distinguish between s_1 and s_2 using the existing score function Φ_1. Therefore, the score function Φ_1 fails to rank the IVSFNs s_1 and s_2.

Hence, according to Definition 6, we calculate the accuracy values of s_1 and s_2. By using Eq. 24.2, we obtain $\Psi(s_1) = 0.38$ and $\Psi(s_2) = 0.38$.

Therefore, based on Definition 6, $s_1 \sim s_2$. But we can easily observe that $s_1 \neq s_2$. Thus, it is not possible to distinguish between IVSFNs using the existing score and accuracy functions. Hence, it is necessary to develop an improved score function.

3 Improved Score Function and Distance Measure

Example 1 shows that the score function Φ_1 is not sufficient to distinguish IVSFNs. If the score value of two distinct IVSFNs is the same, then it is necessary to implement an accuracy function to rank them. Therefore, we aim to develop an improved score function to make it easier to distinguish IVSFNs despite this complexity of calculation.

3.1 Improved Score Function

Definition 7 The score function for the IVSFN $s = \langle[\eta^-, \eta^+], [\theta^-, \theta^+], [\pi^-, \pi^+]\rangle$ is defined by

$$\Phi(s) = (\eta^-)^2[1 + \sqrt{(\eta^-)^2 + (\theta^-)^2 + (\pi^-)^2}] + (\eta^+)^2[1 + \sqrt{(\eta^+)^2 + (\theta^+)^2 + (\pi^+)^2}] \quad (24.3)$$

Definition 8 The IVSFNs are compared according to the following rule: Suppose s_1 and s_2 are two IVSFNs; then

- $s_1 \succ s_2$ if $\Phi(s_1) > \Phi(s_2)$
- $s_1 = s_2$ if $\Phi(s_1) = \Phi(s_2)$

Now, we consider the following example to show how well the suggested score function works.

Example 2 Let $s_1 = \langle[0, 0.5], [0.1, 0.7], [0, 0.1]\rangle$ and $s_2 = \langle[0.3, 0.4], [0.5, 0.5], [0, 0.1]\rangle$ be two IVSFNs. Now, using Eq. (24.3) we have $\Phi(s_1) = 0.1875$ and $\Phi(s_2) = 0.2224$. Hence, according to Definition 8, we have $s_2 \succ s_1$, i.e., s_2 is better than s_1.

Here, we consider the identical IVSFNs s_1 and s_2 in Examples 1 and 2. However, the suggested score function easily distinguishes between s_1 and s_2, whereas the existing score function is unable to do so. As a result, the improved score function is superior to the existing score function. Also, it can reduce computational complexity during the decision-making process.

Now, we will define some fundamental properties of the improved score function.

Property 1 (Zero Property) If s is the smallest IVSFN, i.e., if $s = \langle[0, 0], [1, 1], [0, 0]\rangle$, then $\Phi(s) = 0$.

Property 2 (One Property) If s is the largest IVSFN, i.e., if $s = \langle[1, 1], [0, 0], [0, 0]\rangle$, then $\Phi(s) = 1$.

3.2 Distance Measure

Definition 9 The Euclidean distance between two IVSFNs $s_1 = \langle[\eta_1^-, \eta_1^+], [\theta_1^-, \theta_1^+], [\pi_1^-, \pi_1^+]\rangle$ and $s_2 = \langle[\eta_2^-, \eta_2^+], [\theta_2^-, \theta_2^+], [\pi_2^-, \pi_2^+]\rangle$ is defined by

$$\Theta(s_1, s_2) = \sqrt{\frac{1}{6}((\eta_1^- - \eta_2^-)^2 + (\eta_1^+ - \eta_2^+)^2 + (\theta_1^- - \theta_2^-)^2 + (\theta_1^+ - \theta_2^+)^2 + (\pi_1^- - \pi_2^-)^2 + (\pi_1^+ - \pi_2^+)^2)}. \tag{24.4}$$

Example 3 The Euclidean distance between the largest and smallest IVSFNs $\langle[1, 1], [0, 0], [0, 0]\rangle$ and $\langle[0, 0], [1, 1], [0, 0]\rangle$ using Definition 9 is 1.

Remark 1 If the lower and upper limits of MD, NMD, and hesitancy degree in the IVSF set are the same, the proposed distance measure converts into the Euclidean distance measure of the SFS. Therefore, the Euclidean distance between two SFSs $\tilde{s}_1 = \langle\eta_1, \theta_1, \pi_1\rangle$ and $\tilde{s}_2 = \langle\eta_2, \theta_2, \pi_2\rangle$ is given by $\Theta'(\tilde{s}_1, \tilde{s}_2) = \sqrt{\frac{1}{3}((\eta_1 - \eta_2)^2 + (\theta_1 - \theta_2)^2 + (\pi_1 - \pi_2)^2)}$.

4 Weighted Distance-Based Approximation Method for MAGDM

Here, we develop an integrated IVSF-AHP-entropy-WDBA model by combining AHP, entropy, and WDBA methods within IVSF sets.

Our proposed model consists of three stages. In the first stage, we estimate the DE's weight using the AHP method. The attribute weight is estimated in the second stage using the entropy method. Finally, we used the WDBA method to derive the ranking results of the various options in the corresponding MAGDM problem.

The framework of the integrated IVSF-AHP-entropy-WDBA approach is presented in Fig. 24.1 and described in the steps below.

Stage 1: Here, we estimate the DE's weight using the AHP method. The procedure is given in the following steps:

Step 1.1. Selection of alternatives and attributes using DE's preference. Let $A = \{A_1, A_2, \ldots, A_m\}$, $B = \{B_1, B_2, \ldots, B_n\}$, and $DE = \{DE_1, DE_2, \ldots, DE_k\}$ be the sets of m alternatives, n attributes, and k decision experts, respectively.

Step 1.2. Constitute pairwise comparison matrix $M = (a_{ij})_{k\times k}$ among the decision experts.

$$M = \begin{pmatrix} 1 & a_{12} & a_{13} & \ldots & a_{1k} \\ a_{21} & 1 & a_{23} & \ldots & a_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & r_{k3} & \ldots & 1 \end{pmatrix}$$

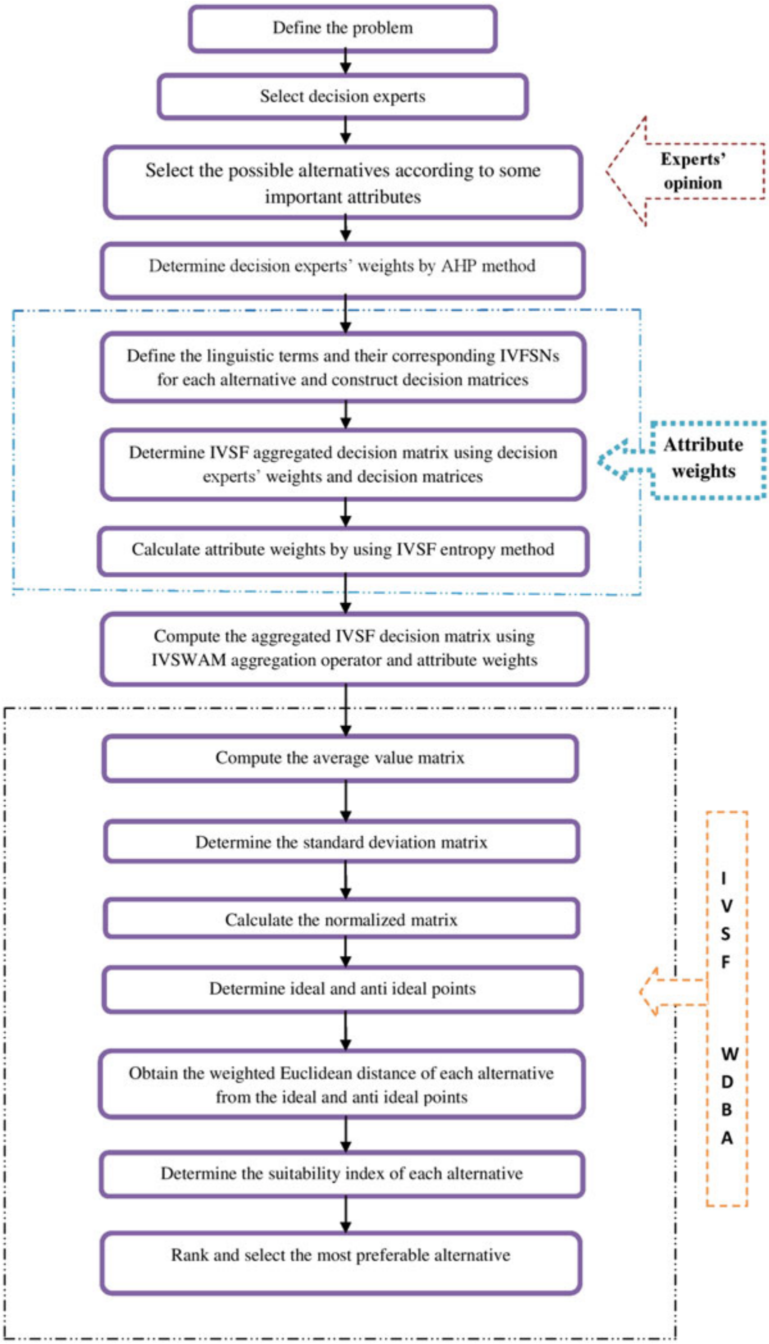

Fig. 24.1 Framework of the IVSF-AHP-entropy-WDBA approach

Table 24.1 Scale of importance

Importance level	Definition
1	Equal importance
3	Medium importance
5	High importance
7	Very high importance
9	Extreme importance
2,4,6,8	Intermediate values
$\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}$	Value for inverse comparison

Here, each entry a_{ij} represents the significance of the i^{th} DE compared to the j^{th} DE, and each a_{ij} is measured on a scale from 1 to 9, as shown in Table 24.1.

Step 1.3. Determine the normalized pairwise comparison matrix $\tilde{M} = (\tilde{a}_{ij})_{k\times k}$, where $\tilde{a}_{ij}$ is calculated by using Eq. (24.5)

$$\tilde{a}_{ij} = \frac{a_{ij}}{\sum\limits_{i=1}^{k} a_{ij}}. \tag{24.5}$$

Step 1.4. Weights $\varpi_i (i = 1, 2, \ldots, k)$ of the decision experts are calculated, utilizing Equation (24.6)

$$\varpi_i = \frac{\sum\limits_{j=1}^{k} \tilde{a}_{ij}}{k}. \tag{24.6}$$

Stage 2: At this stage, we combine the decision matrices obtained from the decision experts. Also, this stage develops the entropy method using the proposed score function, which is used to compute the attribute weights.

Step 2.1. Identify the linguistic terms and construct the decision matrices.
The DEs express their viewpoints using the following linguistic variables (LVs): extremely low (EL), very low (VL), low (L), moderate (M), high (H), very high (VH), and extremely high (EH). The IVSFNs are used to model the LVs to portray the qualitative data in a better way. Table 24.2 shows the relationships between LVs and IVSFNs.
Then, we construct the decision matrices using Table 24.2. Let $D^t = (r^t_{xy})_{m\times n}$ be decision matrix provided by DE_t. Here, $r^t_{xy} = \langle[\eta^{-t}_{xy}, \eta^{+t}_{xy}], [\theta^{-t}_{xy}, \theta^{+t}_{xy}], [\pi^{-t}_{xy}, \pi^{+t}_{xy}]\rangle$ is an IVSFN that reflects the assessed value of the alternative A_x for the attribute B_y given by DE_t.

Step 2.2. Compute IVSF aggregated decision matrix.

Table 24.2 Relationship between LVs and IVSFNs

LVs	IVSFNs
EH	⟨[0.84, 0.91], [0.14, 0.21], [0.07, 0.14]⟩
VH	⟨[0.77, 0.84], [0.21, 0.28], [0.14, 0.21]⟩
H	⟨[0.70, 0.77], [0.28, 0.35], [0.21, 0.28]⟩
M	⟨[0.49, 0.56], [0.42, 0.49], [0.35, 0.42]⟩
L	⟨[0.28, 0.35], [0.70, 0.77], [0.21, 0.28]⟩
VL	⟨[0.21, 0.28], [0.77, 0.84], [0.14, 0.21]⟩
EL	⟨[0.14, 0.21], [0.84, 0.91], [0.07, 0.14]⟩

Here, we utilize the weights of the DEs $(\varpi_1, \varpi_2, \ldots, \varpi_k)$ from Stage 1 and IVSWAM operator to aggregate the decision matrices $D^t, t = 1, 2, \ldots, k$. Let $\tilde{D} = ((\alpha_{xy}))_{m\times n}$ be the aggregated decision matrix where

$$\alpha_{xy} = \Big\langle \Big[\sqrt{1 - \prod_{t=1}^{k}(1-(\eta^{-t}_{xy})^2)^{\varpi_t}}, \sqrt{1 - \prod_{t=1}^{k}(1-(\eta^{+t}_{xy})^2)^{\varpi_t}}\Big], \Big[\prod_{t=1}^{k}(\theta^{-t}_{xy})^{\varpi_t}, \prod_{t=1}^{k}(\theta^{+t}_{xy})^{\varpi_t}\Big],$$
$$\Big[\sqrt{\prod_{t=1}^{k}(1-(\eta^{-t}_{xy})^2)^{\varpi_t} - \prod_{t=1}^{k}(1-(\eta^{-t}_{xy})^2-(\pi^{-t}_{xy})^2)^{\varpi_t}},$$
$$\sqrt{\prod_{t=1}^{k}(1-(\eta^{+t}_{xy})^2)^{\varpi_t} - \prod_{t=1}^{k}(1-(\eta^{+t}_{xy})^2-(\pi^{+t}_{xy})^2)^{\varpi_t}}\Big]\Big\rangle.$$

Step 2.3. Calculate the attribute weights.
Here, we extend the entropy method [28] under IVSF environment using the proposed score function. The proposed entropy method for determining the attribute weights is exhibited in Eq. (24.7).

$$\gamma_y = \frac{\frac{1}{m}\sum_{x=1}^{m}\Phi(\alpha_{xy})}{\sum_{y=1}^{n}[\frac{1}{m}\sum_{x=1}^{m}\Phi(\alpha_{xy})]}. \tag{24.7}$$

Stage 3: This stage derives the ranking results of the various options in a MAGDM problem using the IVSF-WDBA model. The following steps outline the IVSF-WDBA model:

Step 3.1. Here, the attribute weights, aggregated decision matrix $\tilde{D}$ which is obtained from Stage 2, and Definition 3 are utilized to obtain the weighted aggregated decision matrix. Let $\tilde{D}' = (\beta_{xy})_{m\times n}$ be the weighted aggregated decision matrix.

Step 3.2. Compute the average value matrix $E_y(y = 1(1)n)$ by using Eq. (24.8).

$$E_y = \frac{1}{m}\sum_{x=1}^{m}\Phi(\beta_{xy}). \tag{24.8}$$

Step 3.3. Determine the standard deviation matrix $F_y, y = 1(1)n$ by using Eq. (24.9).

$$F_y = \sqrt{\frac{1}{m}\sum_{x=1}^{m}(\Phi(\beta_{xy}) - E_y)^2}. \tag{24.9}$$

Step 3.4. Calculate the normalized matrix $(G_{xy})_{m\times n}$ by utilizing Equation (24.10).

$$G_{xy} = \frac{\Phi(\beta_{xy}) - E_y}{F_y}. \tag{24.10}$$

Step 3.5. Determine ideal points G^+ and anti-ideal points G^- by Eq. (24.11).

$$\begin{aligned} G^+ &= \{\max_x\{G_{x1}\}, \max_x\{G_{x2}\}, \ldots, \max_x\{G_{xn}\}\} \\ G^- &= \{\min_x\{G_{x1}\}, \min_x\{G_{x2}\}, \ldots, \min_x\{G_{xn}\}\}. \end{aligned} \tag{24.11}$$

Step 3.6. Obtain the weighted Euclidean distance (WED) of each alternative A_x from the ideal points G^+ and anti-ideal points G^-. The distance from G^+ and G^- are, respectively, denoted by $\Theta(A_x, G^+)$ and $\Theta(A_x, G^-)$. The distances are given in Eq. (24.12).

$$\begin{aligned} \Theta(A_x, G^+) &= \sqrt{\sum_{y=1}^{n}\gamma_y(G_{xy} - G^+)^2} \\ \Theta(A_x, G^-) &= \sqrt{\sum_{y=1}^{n}\gamma_y(G_{xy} - G^-)^2}. \end{aligned} \tag{24.12}$$

Step 3.7. Using Eq. (24.13), calculate the suitability index value $\Upsilon(A_x)$ for every alternative.

$$\Upsilon(A_i) = \frac{\Theta(A_x, G^-)}{\Theta_{\max}(A_x, G^-)} - \frac{\Theta(A_x, G^-)}{\wp_{\min}(A_x, G^-)}. \tag{24.13}$$

The suitability index is used for measuring the extent to which the alternative A_x is close to the ideal points and far from anti-ideal points [22]. The best alternative should have the highest closeness coefficient and vice versa.

5 Supplier Selection Problem

Here, we apply the proposed IVSF-AHP-entropy-WDBA methodology to select the best supplier in a textile manufacturing company.

Suppose X is a renowned textile manufacturing company. To produce a large number of textiles, X needs a large amount of fabric. Thus, the company must choose a fabric supplier who meets its requirements. Now, choosing the most suitable supplier for the company is not an easy task due to the presence of apparently contradictory attributes. For this, the managing board has formed a team of three experienced DEs (DE_1, DE_2, DE_3) from different fields to choose the best supplier for the company. These experts have decision-making abilities and over 8 years of field experience. Each expert can independently analyze the criteria and alternatives within the evaluation process. The following attributes are taken into consideration according to the experts' expertise and experience.

- **Quality of the product** (B_1): In today's life, everyone wants to buy a product of the best quality. Therefore, delivering quality products to the customer is the key to capturing the market and gaining customer confidence. The target of quality products depends on the quality of the material supplied by the supplier. So, the company must choose a supplier who distributes better-quality materials.
- **Past performance record** (B_2): The organization must carefully examine the past performance record of the supplier. It is a good investment to do business with a supplier that has maintained good relationships with clients and maintained quality levels over the years.
- **Price of the product** (B_3): The price of the material is the most significant factor that affects the supplier selection process. It is not the amount of money that matters but its value. There is, however, a possibility that a cheap supplier may sell low-quality material and a costly supplier may sell good-quality material. Therefore, a company should choose a supplier who offers good-quality products at a reasonable cost.
- **Financially strong** (B_4): If the supplier is not financially strong, it cannot be a reliable source for the organization, because supplier also needs to acquire or prepare material and run their day-to-day operations.
- **Delivery capability** (B_5): Every company is aware of whether suppliers can deliver the right amount of product at the right time. In a just-in-time environment, companies depend on suppliers to deliver small quantities of products quickly. Suppliers' locations may also affect the delivery time.
- **Technical ability** (B_6) : A supplier's technical ability refers to their ability to acquire new technologies and technical resources.

Based on the above attributes, the decision experts have selected five distinct suppliers, namely, A_1, A_2, A_3, A_4, and A_5.

In Fig. 24.2, we present a hierarchical framework of supplier selection in the textile industry.

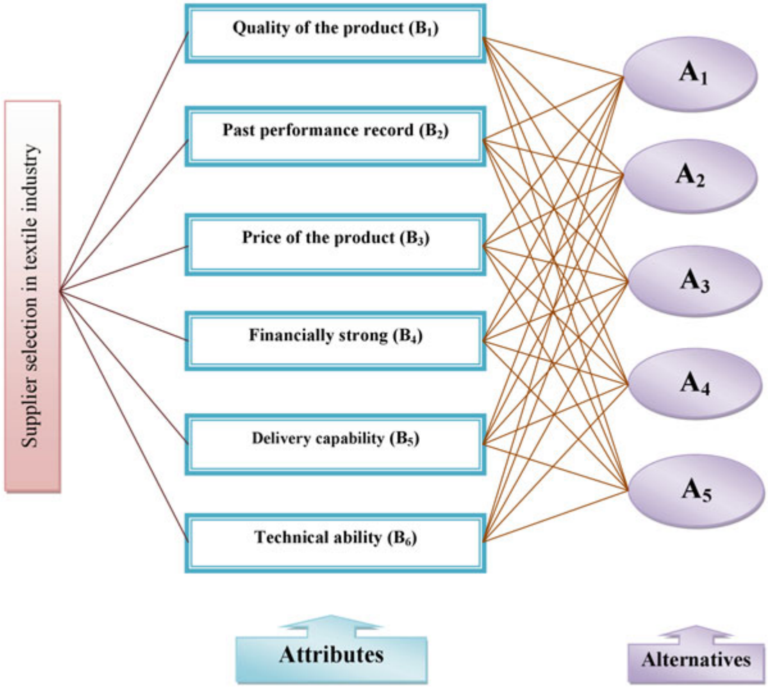

Fig. 24.2 Hierarchical framework of supplier selection process

5.1 Decision-Making Process

Here, we exploit the proposed IVSF-entropy-WDBA approach to solve the above supplier selection problem.

Stage 1: In this stage, we compute the weights of the decision experts using the AHP method.

Step 1.1. We take the opinion of three decision experts $DE_k (k = 1, 2, 3)$ and they select five alternative suppliers and six attributes.

Step 1.2. Based on the scale of importance of decision experts' from Table 24.1, construct the pairwise comparison matrix M.

$$M = \begin{array}{c} \\ DE_1 \\ DE_2 \\ DE_3 \end{array} \begin{array}{c} \begin{array}{ccc} DE_1 & DE_2 & DE_3 \end{array} \\ \begin{bmatrix} 1 & 5 & \frac{1}{3} \\ \frac{1}{5} & 1 & 7 \\ 3 & \frac{1}{7} & 1 \end{bmatrix} \end{array}$$

Step 1.3. Determine the normalized pairwise matrix $\tilde{M}$ using Eq. (24.5).

$$\tilde{M} = \begin{array}{c} \\ DE_1 \\ DE_2 \\ DE_3 \end{array} \begin{array}{c} \begin{array}{ccc} DE_1 & DE_2 & DE_3 \end{array} \\ \begin{bmatrix} 0.238 & 0.813 & 0 \\ 0.047 & 0.162 & 0.833 \\ 0.714 & 0.023 & 0.125 \end{bmatrix} \end{array}$$

Step 1.4. Using Eq. (24.6) we obtain the DE's weights as (0.364, 0.350, 0.286).

Stage 2: Here, we combine the decision matrices obtained from three DEs. Also, we determine the attribute weights.

Step 2.1. Construct the decision matrices in Table 24.3 using the LVs presented in Table 24.2.

Step 2.2. We utilize the IVSWAM aggregation operator and decision experts' weights which are obtained from Stage 1 to combine the decision experts' opinions exhibited in Table 24.3. Let $\tilde{D}$ be the aggregated decision matrix $\tilde{D}$. Then

Table 24.3 Decision matrices

Decision-maker	Alternatives	B_1	B_2	B_3	B_4	B_5	B_6
DE_1	A_1	EH	VH	M	VH	M	H
	A_2	VH	H	L	H	M	M
	A_3	M	H	H	M	L	L
	A_4	H	M	H	L	H	H
	A_5	M	H	H	M	M	L
DE_2	A_1	VH	EH	H	M	H	VH
	A_2	M	VH	M	H	M	VH
	A_3	H	VH	H	M	L	H
	A_4	H	M	M	M	H	H
	A_5	M	H	VH	VL	H	L
DE_3	A_1	H	VH	H	VH	M	H
	A_2	VH	H	H	VH	M	M
	A_3	M	M	M	H	VL	VH
	A_4	M	M	M	H	L	EH
	A_5	H	L	H	L	VL	EL

$\tilde{D}=$

	B_1	B_2	B_3	B_4	B_5	B_6
A_1	⟨[0.783, 0.857], [0.196, 0.268], [0.136, 0.203]⟩	⟨[0.797, 0.869], [0.182, 0.253], [0.116, 0.184]⟩	⟨[0.640, 0.712], [0.324, 0.395], [0.259, 0.328]⟩	⟨[0.702, 0.777], [0.267, 0.340], [0.212, 0.279]⟩	⟨[0.582, 0.654], [0.364, 0.435], [0.299, 0.368]⟩	⟨[0.727, 0.797], [0.253, 0.323], [0.185, 0.255]⟩
A_2	⟨[0.747, 0.818], [0.232, 0.302], [0.164, 0.234]⟩	⟨[0.727, 0.797], [0.253, 0.323], [0.185, 0.255]⟩	⟨[0.522, 0.594], [0.450, 0.524], [0.269, 0.339]⟩	⟨[0.722, 0.793], [0.257, 0.328], [0.190, 0.259]⟩	⟨[0.490, 0.560], [0.420, 0.490], [0.350, 0.420]⟩	⟨[0.623, 0.699], [0.253, 0.323], [0.274, 0.341]⟩
A_3	⟨[0.582, 0.654], [0.364, 0.435], [0.299, 0.368]⟩	⟨[0.686, 0.760], [0.284, 0.356], [0.224, 0.292]⟩	⟨[0.654, 0.726], [0.314, 0.385], [0.249, 0.318]⟩	⟨[0.567, 0.639], [0.374, 0.445], [0.308, 0.377]⟩	⟨[0.262, 0.331], [0.719, 0.789], [0.193, 0.262]⟩	⟨[0.637, 0.713], [0.359, 0.437], [0.191, 0.262]⟩
A_4	⟨[0.654, 0.726], [0.314, 0.385], [0.249, 0.318]⟩	⟨[0.490, 0.560], [0.420, 0.490], [0.350, 0.420]⟩	⟨[0.585, 0.657], [0.362, 0.433], [0.297, 0.366]⟩	⟨[0.522, 0.594], [0.450, 0.524], [0.269, 0.339]⟩	⟨[0.629, 0.702], [0.363, 0.438], [0.213, 0.285]⟩	⟨[0.751, 0.825], [0.229, 0.302], [0.170, 0.237]⟩
A_5	⟨[0.567, 0.639], [0.374, 0.445], [0.308, 0.377]⟩	⟨[0.629, 0.702], [0.363, 0.438], [0.213, 0.285]⟩	⟨[0.797, 0.797], [0.253, 0.323], [0.185, 0.255]⟩	⟨[0.360, 0.428], [0.600, 0.673], [0.266, 0.335]⟩	⟨[0.542, 0.614], [0.433, 0.508], [0.261, 0.330]⟩	⟨[0.248, 0.317], [0.737, 0.807], [0.183, 0.250]⟩

Step 2.3. Compute γ_y using Eq. (24.7). The attributes along with their weights (γ_y) are exhibited in Fig. 24.3.

Stage 3: Here, we utilize the WDBA method to get ranking of the selected suppliers.

Step 3.1. Here, we utilize Equation (3) and the attribute weights obtained from Stage 2 to get the IVSF weighted aggregated decision matrix $\tilde{D}'$.

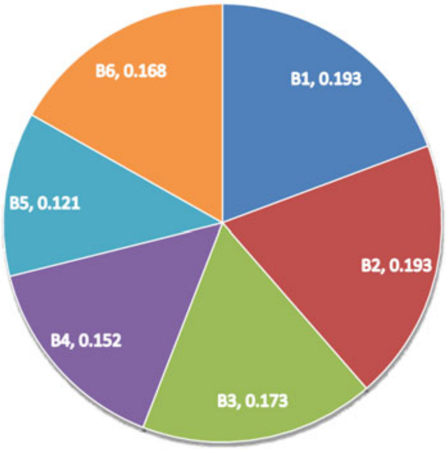

Fig. 24.3 Weights of the attributes

$$\tilde{D}' = \begin{array}{c} \\ A_1 \\ \\ A_2 \\ \\ A_3 \\ \\ A_4 \\ \\ A_5 \end{array} \begin{array}{cccccc} B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \\ \left[\begin{array}{c} \langle[0.409, 0.475], \\ [0.730, 0.775], \\ [0.088, 0.157]\rangle \\ \langle[0.381, 0.438], \\ [0.754, 0.793], \\ [0.101, 0.166]\rangle \\ \langle[0.276, 0.319], \\ [0.822, 0.851], \\ [0.159, 0.213]\rangle \\ \langle[0.319, 0.366], \\ [0.799, 0.831], \\ [0.140, 0.198]\rangle \\ \langle[0.268, 0.310], \\ [0.827, 0.855], \\ [0.163, 0.216]\rangle \end{array}\right. & \begin{array}{c} \langle[0.420, 0.487], \\ [0.719, 0.767], \\ [0.077, 0.146]\rangle \\ \langle[0.367, 0.420], \\ [0.767, 0.804], \\ [0.111, 0.174]\rangle \\ \langle[0.339, 0.391], \\ [0.784, 0.819], \\ [0.129, 0.189]\rangle \\ \langle[0.227, 0.264], \\ [0.845, 0.871], \\ [0.177, 0.227]\rangle \\ \langle[0.304, 0.350], \\ [0.822, 0.852], \\ [0.116, 0.170]\rangle \end{array} & \begin{array}{c} \langle[0.295, 0.339], \\ [0.822, 0.851], \\ [0.137, 0.192]\rangle \\ \langle[0.231, 0.269], \\ [0.870, 0.894], \\ [0.130, 0.175]\rangle \\ \langle[0.303, 0.348], \\ [0.818, 0.847], \\ [0.133, 0.189]\rangle \\ \langle[0.264, 0.305], \\ [0.838, 0.865], \\ [0.151, 0.202]\rangle \\ \langle[0.349, 0.400], \\ [0.788, 0.822], \\ [0.106, 0.167]\rangle \end{array} & \begin{array}{c} \langle[0.313, 0.362], \\ [0.818, 0.848], \\ [0.112, 0.168]\rangle \\ \langle[0.325, 0.373], \\ [0.813, 0.844], \\ [0.102, 0.160]\rangle \\ \langle[0.239, 0.276], \\ [0.861, 0.884], \\ [0.146, 0.194]\rangle \\ \langle[0.217, 0.252], \\ [0.885, 0.906], \\ [0.122, 0.165]\rangle \\ \langle[0.144, 0.170], \\ [0.600, 0.673], \\ [0.111, 0.146]\rangle \end{array} & \begin{array}{c} \langle[0.220, 0.255], \\ [0.884, 0.904], \\ [0.128, 0.173]\rangle \\ \langle[0.180, 0.210], \\ [0.900, 0.917], \\ [0.142, 0.183]\rangle \\ \langle[0.092, 0.118], \\ [0.960, 0.971], \\ [0.069, 0.097]\rangle \\ \langle[0.243, 0.280], \\ [0.884, 0.904], \\ [0.094, 0.138]\rangle \\ \langle[0.203, 0.235], \\ [0.925, 0.941], \\ [0.108, 0.147]\rangle \end{array} & \left.\begin{array}{c} \langle[0.344, 0.394], \\ [0.793, 0.827], \\ [0.105, 0.165]\rangle \\ \langle[0.281, 0.326], \\ [0.829, 0.858], \\ [0.141, 0.194]\rangle \\ \langle[0.289, 0.335], \\ [0.841, 0.870], \\ [0.098, 0.148]\rangle \\ \langle[0.360, 0.417], \\ [0.780, 0.817], \\ [0.099, 0.162]\rangle \\ \langle[0.102, 0.132], \\ [0.950, 0.964], \\ [0.077, 0.108]\rangle \end{array}\right] \end{array}$$

Step 3.2. By utilizing Equation (24.8), we obtain the average value matrix E_y as

$$E_y = (0.376, 0.381, 0.276, 0.212, 0.118, 0.277).$$

Step 3.3. We obtain the standard deviation matrix F_y using Eq. (24.9) as

$$F_y = (0.132, 0.145, 0.077, 0.105, 0.053, 0.138).$$

Step 3.4. By utilizing Equation (24.10), we get the normalized matrix $(G)_{m\times n}$ as

$$G = \begin{pmatrix} 1.473 & 1.539 & 0.093 & 1.035 & 0.648 & 0.814 \\ 0.855 & 0.468 & -1.366 & 1.261 & -0.304 & -0.129 \\ -0.944 & 0.020 & 0.305 & -0.277 & -1.713 & -0.047 \\ -0.303 & -1.462 & -0.643 & -0.604 & 1.236 & 1.128 \\ -1.059 & -0.549 & 1.628 & -1.394 & 0.167 & -1.755 \end{pmatrix}$$

Step 3.5. We obtain ideal and anti-ideal points as follows from Eq. (24.11):
$G^+ = \{1.473, 1.539, 1.628, 1.261, 1.236, 1.128\}$
$G^- = \{-1.059, -1.462, -1.366, -1.394, -1.713, -1.755\}$.

Step 3.6. Utilizing Equation (24.12), we get the distances between the alternative A_x and the ideal points G^+ as well as the anti-ideal points G^-, and the outcomes are recorded in Table 24.4.

Step 3.7. By using Eq. (24.13), we get the suitability index of each alternatives. The ranking outcomes are exhibited in Table 24.4 and Fig. 24.4.

According to Fig. 24.4 and Table 24.4, we get the ranking of five available suppliers as $A_1 \succ A_2 \succ A_4 \succ A_5 \succ A_3$. Therefore, from Table 24.4 and Fig. 24.4, it can be concluded that supplier A_1 is the most suitable for the textile manufacturing company.

Table 24.4 Decision obtained by IVSF-WDBA method

Alternative	$\Theta(A_x, G^+)$	$\Theta(A_x, G^-)$	$\Upsilon(A_x)$	Ranking
A_1	0.281	1.001	0.781	1
A_2	0.632	0.728	0.535	2
A_3	0.765	0.514	0.401	5
A_4	0.792	0.676	0.460	3
A_5	0.597	1.623	0.402	4

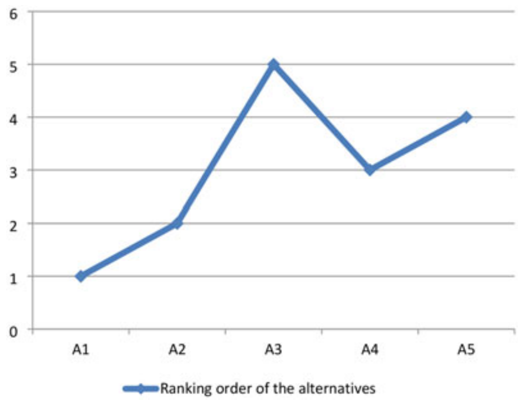

Fig. 24.4 Ranking of the alternatives

5.2 Comparison Analysis

Here, we compare the proposed approach with some existing approaches, such as the TOPSIS method [12] and MULTIMOORA method (fuzzy multiplicative and ratio system approach) [3] under the IVSF environment. To accomplish this, we employ the TOPSIS method [12] and the MULTIMOORA method [3] to solve the supplier selection problem given in Sect. 5. Table 24.5 and Fig. 24.5 display the outcomes for supplier options using various methods. We demonstrate, using Table 24.5 and Fig. 24.5, that our method does not differ significantly from the other methods for selecting the best supplier. The supplier A_1 is the best supplier obtained using all the methods. However, the ranking of the other suppliers slightly differs. As a result of the comparison analysis, the proposed method produces highly reliable results.

In comparison to the existing model, the proposed model has the following advantages:

- The proposed model uses the AHP method to select each DE weight, whereas, in the existing models, the DE weights are selected arbitrarily.
- The proposed model investigates the MAGDM using unknown attribute weights. To estimate these unknown weights, we employ the entropy method. But, in the existing models, attribute weights are chosen arbitrarily. Since decision-makers have limited abilities and lack sufficient data and knowledge, they must be unsure about the appropriate attribute weights. So, calculating attribute weights

Table 24.5 Ranking of the suppliers using various approaches

Methods	Ranking of the alternatives
TOPSIS [12]	$A_1 \succ A_2 \succ A_4 \succ A_5 \succ A_3$
Ratio system approach-based MULTIMOORA [3]	$A_1 \succ A_2 \succ A_4 \succ A_3 \succ A_5$
Fuzzy multiplicative approach-based MULTIMOORA [3]	$A_1 \succ A_2 \succ A_4 \succ A_3 \succ A_5$
WDBA (proposed)	$A_1 \succ A_2 \succ A_4 \succ A_5 \succ A_3$

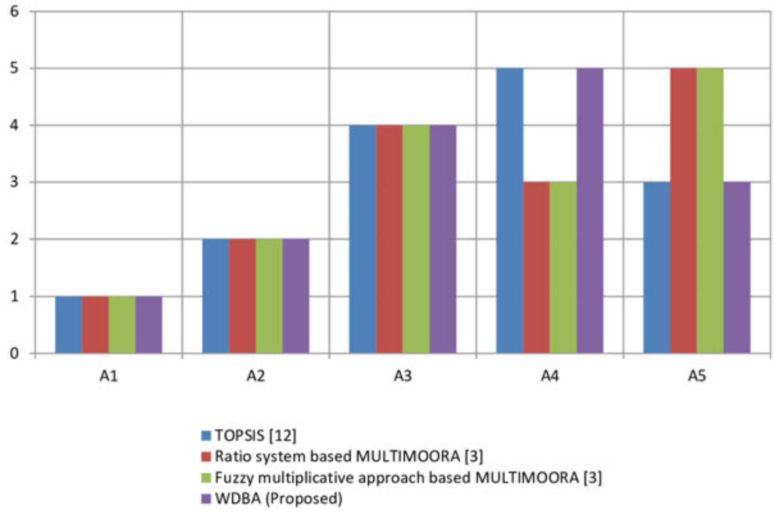

Fig. 24.5 Outcomes for supplier options using various methods

by applying a suitable method is more reasonable. In conclusion, the proposed model will provide a more robust solution.

6 Conclusion

Due to insufficient information, experts cannot precisely quantify their judgment in many real-life situations. In such a situation, decision-makers should provide their judgments through the IVSFS due to their broader space. This chapter develops an integrated AHP and entropy-based WDBA method using IVSF data. At first, we have defined an improved score function for IVSF sets. Then, for IVSF sets, we defined Euclidean distance measures and investigated their significant characteristics. Next, we have developed an entropy method by utilizing the proposed score function. Then, to solve MAGDM problems, we developed the WDBA method within the IVSF context, utilizing the proposed Euclidean distance measure. The

AHP method and the entropy method are used in the developed method to determine the DE's weights and the unknown attributed weights, respectively. Then, we solved a supplier selection problem for a textile manufacturing company to illustrate the practicality and efficiency of the proposed model. The result shows that A_1 is the best supplier for the company. Finally, we compared the results to several existing methods to demonstrate the feasibility of our model.

There are some limitations to the present study. Future research will address the limitations of the present study. This study was conducted with data from three decision-makers regarding five alternative suppliers. It is possible to confirm the results of this study by collecting data from multiple suppliers in the future. Furthermore, the MAGDM model is only used for selecting suppliers. But it has the potential to address various other decision-making issues, such as assistant professor selection problem [18], job selection problem [24], company investment problem [25], etc. Furthermore, other fuzzy set extensions, such as quasirung fuzzy sets [26, 27], Pythagorean cubic fuzzy sets [17], etc., could be used to construct fuzzy WDBA.

Acknowledgments The comments and suggestions of the anonymous referee are also greatly appreciated. We sincerely thank the editors of the book for providing all supports in preparing the manuscript.

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Chapter 25
Investigating Some Parameters of Cubic Fuzzy Graphs and an Application in Decision-Making Problem

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Mathematics Subject Classification 05C99, 03E72

1 Introduction

Graphs are usually used to model relationships between objects. They have many applications in different fields including networks, computers, systems analysis, operations research, transportation, and economics. In recent years, various parameters have been studied in graphs. During these studies, domination and covering have been of great importance due to the concrete role they have had in the practical and real issues of human life.

Among many issues in the world today, there are many variables for which no exact value can be considered. These values, which are categorized in the field of ambiguous and uncertain concepts, have long occupied the minds of researchers. Until Zadeh [1], for the first time, by introducing a fuzzy set (FS), was able to convert uncertain variables into intelligible numerical values by assigning a value between 0 and 1. Ten years later, Rosenfeld [2] proposed the fuzzy graph (FG) concept using FS. Next, the researchers investigated the FG types and their features. Mordeson [3] provided some operations definitions on FGs. In 2013, Akram [4] broadened the bipolar FGs concept and explored some of their properties. Considering the FS non-membership values, Atanasov [5] extended it to an intuitionistic fuzzy set (IFS). Rashmanlou et al. [6, 7] examined some properties of an intuitionistic fuzzy graph (IFG) and highly irregular bipolar fuzzy graphs. Borzooei et al. [8] conducted many studies on vague graph (VG). Talebi et al. [9, 10]

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C. Jana et al. (eds.), *Fuzzy Optimization, Decision-making and Operations Research*, https://doi.org/10.1007/978-3-031-35668-1_25

revised a selection of graph parameters on the IVIFG. Some features of VG are investigated in the research of Kosari et al. [11–13].

The crucial grounds on dominating set (DS) was put forward by T.W. Haynes et al. [14]. Somasundaram and Somasundaram [15, 16] introduced the domination in FG. Gani and Chandrasekaran [17] presented domination in FGs by means of strong edges which was also examined by Mohideen and Ismayil [18]. Several domination concepts in FGs were presented and its various types were examined in FGs by researchers. Borzooei et al. [19] mentioned several concepts on VGs. The DS in an IFG was proposed by Parvathi et al. [20].

A vertex covering (VC) in a graph acts as an array of vertices where each edge has at least one endpoint. Manjusha and Sunitha [21] established the paired DS, VC, and matching in FGs applying strong edges. The VC and paired DS in IFGs was analyzed by Sahoo et al. [22].

Jun et al. [23] described the cubic fuzzy set (CFS) idea as an FS and an IVFS. Jun et al. [24] brought together the neutrosophic complex with CFS and organized the concept of neutrosophic CFS. Jun et al., also, evaluated some CFS-based algebraic features incorporating cubic IVIFSs [25], cubic structures [26], cubic sets in semigroups [27], cubic soft sets [28], and cubic intuitionistic structures [29]. Mohiuddin et al. [30] presented a new definition of a CFG. Rashmanlou et al. [31–33] illustrated some CFG concepts. Khan et al. [34] put forward some graphical structures of cubic IFG. Gulistan et al. [35] conducted a study on neutrosophic CFGs regarding real-life applications in industries. Jan et al. [36] defined some cubic bipolar fuzzy graphs concepts.

In this chapter, we introduce two parameters related to the vertices of a graph, i.e., VC and DS in a CFG. The inadequacies of previous information in this field have led us to redefine the DS and VC in a CFG. Certain concepts related to DS and VC in a CFG were investigated. Some features of DS and VC have been studied in complete CFG. An application of the DS in a decision-making problem in the field of power transmission is presented at the end of the chapter.

2 Preliminaries

Some of the concepts needed to be mentally prepared to enter the main topic would be reviewed in this section.

Let U be a non-empty set. Then, $\vartheta : U \to [0, 1]$ is referred to as a fuzzy set (FS) on U. An FG is a pair $\digamma = (\vartheta, \varrho)$, where ϑ is an FS on U and ϱ is an FS on $U \times U$, therefore, $\varrho(ab) \leq \vartheta(a) \wedge \vartheta(b)$, for all $a, b \in U$. Here, ϱ is considered a reflective and symmetric fuzzy relation in ϑ.

The underlying graph $\digamma^* = (\vartheta^*, \varrho^*)$ is referred to as the crisp graph of $\digamma$ whenever $\vartheta^* = \{a \in U | \vartheta(a) > 0\}$ and $\varrho^* = \{ab \in U \times U | \varrho(ab) > 0\}$. The FG $H = (\varphi, \psi)$ is named the partial fuzzy subgraph of $\digamma = (\vartheta, \varrho)$ providing that $\varphi \subseteq \vartheta$ and $\psi \subseteq \varrho$. Two vertices a and b in $\digamma$ are adjacent to each other if

$\varrho(ab) > 0$. An FG $F = (\vartheta, \varrho)$ is termed complete FG if $\varrho(ab) = \vartheta(a) \wedge \vartheta(b)$, for all $a, b \in U$. [37]

Definition 2.1 ([23]) A cubic fuzzy set (CFS) on U is outlined as

$$\mathfrak{X} = \{\langle[\iota(a), \zeta(a)], \eta(a)\rangle | a \in U\},$$

where $[\iota(a), \zeta(a)]$ is referred to as the interval-valued fuzzy membership value and $\eta(a)$ is signified as the fuzzy membership value of a, where $\iota, \zeta, \eta : U \to [0, 1]$.

Definition 2.2 ([30]) A cubic fuzzy graph (CFG) on U is a pair $\mathfrak{S} = (\mathfrak{X}, \mathfrak{Y})$ where $\mathfrak{X}$ is a CFS in U and $\mathfrak{Y}$ is a CFS in $U \times U$, so that for all $ab \in \mathfrak{Y}$

$$\iota_{\mathfrak{Y}}(ab) \leq \iota_{\mathfrak{X}}(a) \wedge \iota_{\mathfrak{X}}(b),$$
$$\zeta_{\mathfrak{Y}}(ab) \leq \zeta_{\mathfrak{X}}(a) \wedge \zeta_{\mathfrak{X}}(b),$$
$$\eta_{\mathfrak{Y}}(ab) \leq \eta_{\mathfrak{X}}(a) \wedge \eta_{\mathfrak{X}}(b).$$

Definition 2.3 ([30]) Suppose that $\mathfrak{S} = (\mathfrak{X}, \mathfrak{Y})$ be a CFG on U. The strength of a path P is defined as $S(P) = \langle[S_\iota(P), S_\zeta(P)], S_\eta(P)\rangle$ where

$$S(P) = \left\langle \left[\bigwedge_{i=1}^{k} \iota_{\mathfrak{Y}}(z_{i-1}z_i), \bigwedge_{i=1}^{k} \zeta_{\mathfrak{Y}}(z_{i-1}z_i) \right], \bigwedge_{i=1}^{k} \eta_{\mathfrak{Y}}(z_{i-1}z_i) \right\rangle.$$

Definition 2.4 ([30]) An edge zw in a CFG $\mathfrak{S}$ is named CSE whenever

$$\iota_{\mathfrak{Y}}(ab) \geq \max S\iota_(P_i), \quad \zeta_{\mathfrak{Y}}(ab) \geq \max S_\zeta(P_i),$$
$$\eta_{\mathfrak{Y}}(ab) \geq \max S_\eta(P_i), \text{ for } i = 1, 2, \ldots$$

Definition 2.5 A CFG $\mathfrak{S} = (\mathfrak{X}, \mathfrak{Y})$ is signified as the complete CFG whenever $a, b \in U$

$$\iota_{\mathfrak{Y}}(ab) = \iota_{\mathfrak{X}}(a) \wedge \iota_{\mathfrak{X}}(b),$$
$$\zeta_{\mathfrak{Y}}(ab) = \zeta_{\mathfrak{X}}(a) \wedge \zeta_{\mathfrak{X}}(b),$$
$$\eta_{\mathfrak{Y}}(ab) = \eta_{\mathfrak{X}}(a) \wedge \eta_{\mathfrak{X}}(b).$$

Definition 2.6 Let $\mathfrak{S} = (\mathfrak{X}, \mathfrak{Y})$ be a CFG. Then, the vertex and edge cardinalities of $\mathfrak{S}$ are expressed as follows:

$$p = \sum_{a \in U} \left(\frac{1 - \iota_{\mathfrak{X}}(a) + \zeta_{\mathfrak{X}}(a) + \eta_{\mathfrak{X}}(a)}{3} \right)$$
$$q = \sum_{ab \in E} \left(\frac{1 - \iota_{\mathfrak{Y}}(ab) + \zeta_{\mathfrak{Y}}(ab) + \eta_{\mathfrak{Y}}(ab)}{3} \right).$$

Table 25.1 Abbreviations

Notation	Meaning
FS	Fuzzy set
FG	Fuzzy graph
DS	Dominating Set
VC	Vertex covering
IFS	Intuitionistic fuzzy set
IVFS	Interval-valued fuzzy set
IVFG	Interval-valued fuzzy graph
IVIFS	Interval-valued intuitionistic fuzzy set
IVIFG	Interval-valued intuitionistic fuzzy graph
Min-Car	Minimum cardinality
Max-Car	Maximum cardinality
CFS	Cubic fuzzy set
CFG	Cubic fuzzy graph
CSE	Cubic strong edge
CSN	Cubic strong neighborhood
SVCS	Strong vertex covering set
SVIS	Strong vertex independent set

Definition 2.7 Suppose that $\mathfrak{S} = (\mathfrak{X}, \mathfrak{Y})$ be a CFG and $S \subseteq U$. The cardinality of S is outlined by

$$|S| = \sum_{a \in S} \left(\frac{1 - \iota_{\mathfrak{X}}(a) + \zeta_{\mathfrak{X}}(a) + \eta_{\mathfrak{X}}(a)}{3} \right).$$

Some abbreviations in the article are listed in Table 25.1.

3 The Dominating Set in Cubic Fuzzy Graphs

In this section, we describe DS in a CFG and study its related associated properties. This concept is one of the characteristics attributed to the vertices of a graph that the set containing those vertices dominates the other vertices. This means that there is at least one edge between the vertices of this set and the outside vertices. The following definition introduces the concept with the help of CSEs.

Definition 3.1 Suppose that $\mathfrak{S} = (\mathfrak{X}, \mathfrak{Y})$ be a CFG on U. $S \subseteq U$ is referred to as a dominating set (DS) in $\mathfrak{S}$ whenever for every $z \notin S$, there exists $w \in S$ so that w dominates z. In other words, there exists a CSE between w and z. The Min-Car of all DSs in $\mathfrak{S}$ is referred to as the domination number of $\mathfrak{S}$ signified by $\gamma(\mathfrak{S})$ or γ.

The above definition is applied in the following example.

Example 3.2 As given in Fig. 25.1, a CFG of $\mathfrak{S} = (\mathfrak{X}, \mathfrak{Y})$ is taken into account.

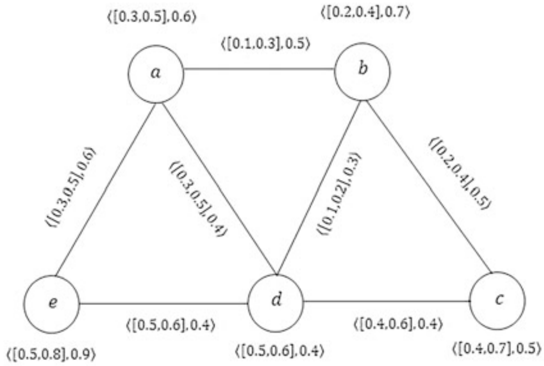

Fig. 25.1 The CFG $\mathfrak{S} = (\mathfrak{X}, \mathfrak{Y})$

Here, ab and bd are CSE. The DSs are as follows:

$$S_1 = \{a, c\}, \quad S_2 = \{b, e\}, \quad S_3 = \{c, e\},$$
$$S_4 = \{a, b\}, \quad S_5 = \{b, d\}.$$

Upon measuring the cardinality of S_1, S_2, S_3, S_4, S_5, we obtain

$$|S_1| = 1.2, \quad |S_2| = 1.36, \quad |S_3| = 1.33,$$
$$|S_4| = 1.23, \quad |S_5| = 1.13.$$

Thus, $\gamma = 1.13$.

In the following definitions, the neighborhood of a vertex and its neighborhood degree are discussed.

Definition 3.3 Suppose that $\mathfrak{S} = (\mathfrak{X}, \mathfrak{Y})$ be a CFG. A CSN of z is specified as

$$\mathcal{N}(z) = \{w \in U | wz \; is \; a \; CSE\}.$$

Furthermore, $\mathcal{N}[z] = \mathcal{N}(z) \cup \{z\}$ is referred to as a closed CSN of z.

Definition 3.4 The CSN degree of vertex z is classified as

$$\mathcal{D}\mathcal{N}(z) = \left\langle \left[\sum_{w \in \mathcal{N}(z)} \iota(w), \sum_{w \in \mathcal{N}(z)} \zeta(w) \right], \sum_{w \in \mathcal{N}(z)} \eta(w) \right\rangle,$$

and the minimum and Max-Car CSN of $\mathfrak{S}$ are represented by $\delta_{\mathcal{N}}$ and $\Delta_{\mathcal{N}}$, respectively.

The above definitions are used in the following example.

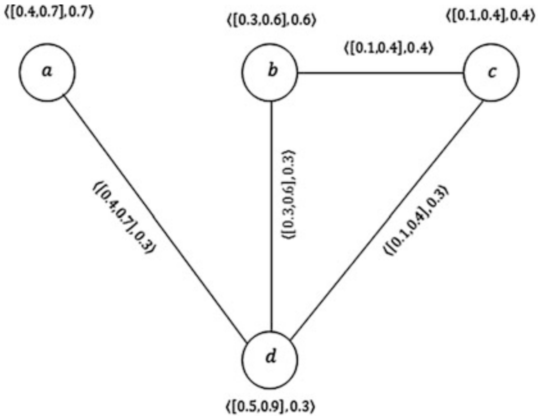

Fig. 25.2 A strong CFG

Example 3.5 Suppose that $\mathfrak{S} = (\mathfrak{X}, \mathfrak{Y})$ be a CFG as depicted in Fig. 25.2. All edges are strong. We have

$$\mathcal{N}(a) = \{d\}, \quad \mathcal{N}(b) = \{c, d\}, \quad \mathcal{N}(c) = \{b, d\},$$
$$\mathcal{N}(d) = \{a, b, c\}.$$

Therefore,

$$\mathcal{D}\mathcal{N}(a) = \langle [0.5, 0.9], 0.3\rangle, \quad \mathcal{D}\mathcal{N}(b) = \langle [0.6, 1.3], 0.7\rangle,$$
$$\mathcal{D}\mathcal{N}(c) = \langle [0.8, 1.5], 0.9\rangle, \quad \mathcal{D}\mathcal{N}(d) = \langle [0.8, 1.7], 1.7\rangle.$$

Then, $\delta_{\mathcal{N}} = |\langle [0.5, 0.9], 0.3\rangle| = 0.57$, and $\Delta_{\mathcal{N}} = |\langle [0.8, 1.7], 1.7\rangle| = 1.2$.

Definition 3.6 In a CFG $\mathfrak{S} = (\mathfrak{X}, \mathfrak{Y})$, a vertex $z \in U$ is referred to as an isolated vertex if $\mathcal{N}(z) = \emptyset$, i.e., for any $w \in U$ where $z \neq w$, wz is not a CSE.

The following theorem shows the relation between γ and neighborhood degree.

Theorem 3.7 *If $\mathfrak{S} = (\mathfrak{X}, \mathfrak{Y})$ is a CFG without an isolated vertex, then*

$$\gamma \leq p - \Delta_{\mathcal{N}} \ \ \textit{and} \ \ \gamma \leq p - \delta_{\mathcal{N}}.$$

Proof Suppose that z be a vertex in CFG $\mathfrak{S} = (\mathfrak{X}, \mathfrak{Y})$. Let $\mathcal{D}\mathcal{N}(z) = \Delta_{\mathcal{N}}$ and $U - \mathcal{N}(z)$ be a DS of $\mathfrak{S}$ so that

$$\gamma \leq |U - \mathcal{N}(z)| = p - \Delta_{\mathcal{N}}.$$

Since $\delta_{\mathcal{N}} \leq \Delta_{\mathcal{N}}$, then, $\gamma \leq p - \delta_{\mathcal{N}}$.

4 The Vertex Covering in Cubic Fuzzy Graphs

This section discusses the VC in CFGs comparing some of their properties. This concept is one of the other parameters attributed to the vertices of a graph whose members cover all the edges of the graph. With the definition given in this chapter, this vertices covering is done by CSEs.

Definition 4.1 Suppose that $\mathfrak{S} = (\mathfrak{X}, \mathfrak{Y})$ be a CFG. An SVCS in $\mathfrak{S}$ is a set $\mathcal{C}$ of vertices in such a way that each CSE in $\mathfrak{S}$ is adjacent to at least one vertex in $\mathcal{C}$.

The set $\mathcal{C}$ is named the minimal SVCS of the CFG $\mathfrak{S}$ if $\mathcal{C} \setminus \{z\}$ is not an SVCS, for all $z \in \mathcal{C}$.

The SVCS number of $\mathfrak{S}$ is the Min-Car among all the minimal SVCSs of $\mathfrak{S}$, and it is shown by $\eth_s(\mathfrak{S})$ or $\eth_s$. In that case, it is labeled as $\eth_s$-set.

Example 4.2 Consider the CFG $\mathfrak{S}$ as shown in Fig. 25.3. Here, ab, cd, and ef are CSEs. The minimal SVCSs in Fig. 25.3 are as follows:

$$\mathcal{C}_1 = \{a, c, e\}, \quad \mathcal{C}_2 = \{a, d, e\}, \quad \mathcal{C}_3 = \{a, c, f\}, \quad \mathcal{C}_4 = \{a, d, f\},$$
$$\mathcal{C}_5 = \{b, d, f\}, \quad \mathcal{C}_6 = \{b, c, f\}, \quad \mathcal{C}_7 = \{b, d, e\}, \quad \mathcal{C}_8 = \{b, c, e\}.$$

From the cardinality measurement of the above SVCSs, we have

$$|\mathcal{C}_1| = 1.59, \quad |\mathcal{C}_2| = 1.75, \quad |\mathcal{C}_3| = 1.46, \quad |\mathcal{C}_4| = 1.62,$$
$$|\mathcal{C}_5| = 1.72, \quad |\mathcal{C}_6| = 1.56, \quad |\mathcal{C}_7| = 1.85, \quad |\mathcal{C}_8| = 1.69.$$

It is observed that $\mathcal{C}_3$ comprises the Min-Car among other SVCSs. Hence, $\eth_s(\mathfrak{S}) = 1.46$ and $\mathcal{C}_3$ is the $\eth_s$-set of $\mathfrak{S}$.

The following theorem proves that by removing a vertex from a CFG, $\eth_s(\mathfrak{S})$ can be reduced.

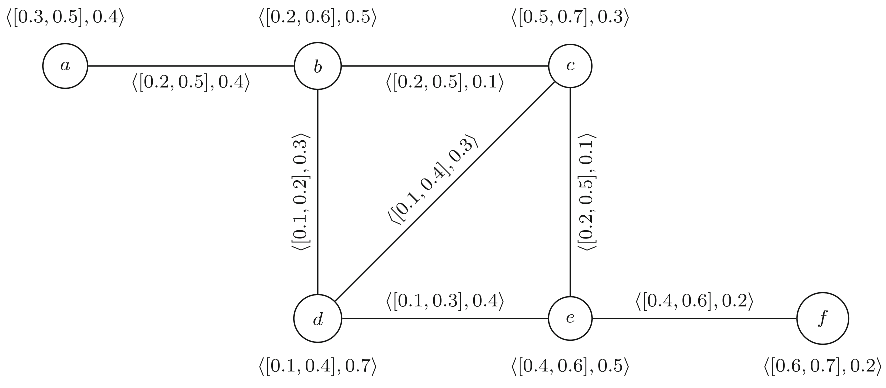

Fig. 25.3 The CFG $\mathfrak{S}$

Theorem 4.3 *If $\mathfrak{S}$ is a CFG and $z \in U$, then, $\eth_s(\mathfrak{S} - z) \le \eth_s(\mathfrak{S})$.*

Proof Consider $\mathcal{C}$ to be a minimum SVCS of $\mathfrak{S}$.
Case i. Assume $z \notin \mathcal{C}$.

Let u and v be two adjacent vertices of $\mathfrak{S} - z$. Therefore, they are also adjacent in $\mathfrak{S}$. Since $\mathcal{C}$ is an SVCS, then, $u \in \mathcal{C}$ or $v \in \mathcal{C}$. Thus, $\mathcal{C}$ is an SVCS of $\mathfrak{S} - z$. Hence, $\eth_s(\mathfrak{S} - z) \le \eth_s(\mathfrak{S})$.
Case ii. Assume $z \in \mathcal{C}$.

Consider the set $\mathcal{C}_1 = \mathcal{C} \setminus \{z\}$. Suppose that u and v as two nodes of $\mathfrak{S} - z$ which are adjacent in $\mathfrak{S} - z$. Hereupon, $u \in \mathcal{C}$ or $v \in \mathcal{C}$.

Since $u \neq z$ and $v \neq z$, $u \in \mathcal{C}_1$ or $v \in \mathcal{C}_1$. Hence, $\mathcal{C}_1$ is an SVCS of $\mathfrak{S} - z$. Then,

$$\eth_s(\mathfrak{S} - z) \le |\mathcal{C}_1| < |\mathcal{C}| = \eth_s(\mathfrak{S}).$$

Therefore, $\eth_s(\mathfrak{S} - z) \le \eth_s(\mathfrak{S})$.

Theorem 4.4 *If $\mathfrak{S} = (\mathfrak{X}, \mathfrak{Y})$ is a CFG without an isolated vertex, then,* $\eth_s(\mathfrak{S}) \le \frac{p}{2}$.

Proof Suppose that $\mathfrak{S} = (\mathfrak{X}, \mathfrak{Y})$ be a CFG without an isolated vertex and the set $\mathcal{C}$ be an SVCS of $\mathfrak{S}$. Therefore, $U \setminus \mathcal{C}$ is also an SVCS of $\mathfrak{S}$. Thus,

$$\eth_s(\mathfrak{S}) = \min\{|\mathcal{C}|, |U \setminus \mathcal{C}|\},$$
$$\eth_s(\mathfrak{S}) \le \frac{p}{2}.$$

Whenever we talk about an SVCS, the independent SVCS (SVIS) is also mentioned. Next, this concept is discussed in CFG.

Definition 4.5 Suppose that $\mathfrak{S}$ be a CFG. A set $\mathcal{F} \subseteq U$ is referred to as the SVIS of $\mathfrak{S}$ if the connecting edge between both vertices of $\mathcal{F}$ is not CSE. An SVIS is named maximal SVIS if no set larger than $\mathcal{F}$ is an SVIS. The Max-Car of SVISs in $\mathfrak{S}$ is termed as the SVIS number, and it is signified by $ß_s(\mathfrak{S})$ or $ß_s$.

Example 4.6 In Fig. 25.3, only $\mathcal{C}_7$ is a $ß_s$-set of $\mathfrak{S}$ with the Max-Car, so $ß_s(\mathfrak{S}) = |\mathcal{C}_7| = 1.85$.

The following theorem expresses the correlation between two SVCS and SVIS.

Theorem 4.7 *In CFG $\mathfrak{S}$, $\mathcal{F}$ is an SVIS if and only if $V^* \setminus F$ is an SVCS.*

Proof The proof is clear because every CSE has at least one endpoint in $\mathcal{F}$ and $U \setminus \mathcal{F}$.

In the next theorem, the relationship between SVCS and SVIS is presented in the CFG.

Theorem 4.8 *If $\mathfrak{S}$ is a CFG of order p without an isolated vertex, then,* $\eth_s(\mathfrak{S}) + ß_s(\mathfrak{S}) = p$.

Proof Suppose that $\mathcal{C}$ and $\mathcal{F}$ be two $\eth_s$-sets of $\mathfrak{S}$, respectively. Thus, $U \setminus \mathcal{C}$ is an SVCS, and $U \setminus \mathcal{F}$ is an SVIS. Hence,

$$p - \eth_s(\mathfrak{S}) = |U \setminus \mathcal{C}| \leq ß_s(\mathfrak{S}), \qquad p - ß_s(\mathfrak{S}) = |U \setminus \mathcal{F}| \geq \eth_s(\mathfrak{S}).$$

The last two inequalities show that $\eth_s(\mathfrak{S}) + ß_s(\mathfrak{S}) = p$.

5 Application

Today, the increase in urban population, the restructuring of the electricity industry, and the move toward privatization are among the main concerns of electricity companies. In this situation, one of the serious needs of the country's electricity industry in the distribution sector is to establish an automation system for these networks, so that in a reasonable period of time, the reliability of the system can be significantly improved. The use of remote control keys as a suitable method to achieve the desired goals can cause fault detection, isolation, and load transfer in order to maintain the continuity and reliability of the network.

Automation of distribution systems plays an important role in reducing the load recovery program execution time by using remote control keys. The use of an efficient methodology to determine the location of remote control switches is very important for power companies because this process can improve network recovery time as well as reliability indicators. Determining the appropriate location of these switches is one of the important priorities of power companies, considering the multiple network maneuvering points and economic constraints. The control keys should be installed in such a way that it dominates the surrounding points. Figure 25.4 shows the single-line diagram of the regional medium pressure feeder.

Loads in the distribution network are considered as the residential, commercial, and industrial ones. The criteria related to the number of subscribers and their level of sensitivity are considered in combination. All the information required for the purpose of checking and analyzing the network, including the total load capacity and the distance of the points from the beginning of the feeder, which are extracted

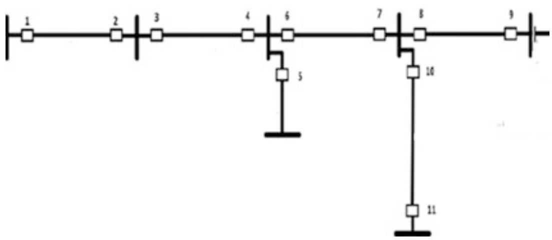

Fig. 25.4 Single-line diagram of a power distribution feeder

based on the data available in the geographic information system (GIS) as well as comprehensive exploitation software (ENOX), are listed in Table 25.2.

If we consider the single-line diagram in Fig. 25.4 as a CFG, then by considering the distance as an interval-valued fuzzy number and the total capacity as a fuzzy number, the cubic fuzzy values of each point are determined according to Table 25.3. Fuzzy values are obtained by dividing each number by the maximum numbers.

The location of points is such that all edges can be assumed to be strong. The minimum DSs are as follows:

$$S_1 = \{1, 4, 8, 10\}, \quad |S_1| = 1.98$$

$$S_2 = \{1, 4, 9, 10\}, \quad |S_2| = 2.01$$

$$S_3 = \{1, 4, 8, 11\}, \quad |S_3| = 1.89$$

$$S_4 = \{1, 4, 9, 11\}, \quad |S_4| = 1.92$$

$$S_5 = \{2, 4, 8, 10\}, \quad |S_5| = 2.01$$

$$S_6 = \{2, 4, 8, 11\}, \quad |S_6| = 1.92$$

Table 25.2 Medium pressure feeder information

Points	Distance from the feeder	Total load capacity
1	100	85
2	435	155
3	475	315
4	710	215
5	775	130
6	765	250
7	1055	510
8	1070	350
9	1540	420
10	1085	715
11	1800	525

Table 25.3 The cubic fuzzy values of each of the points

Points	Cubic fuzzy values
1	$\langle[0.04, 0.06], 0.11\rangle$
2	$\langle[0.23, 0.25], 0.21\rangle$
3	$\langle[0.25, 0.27], 0.44\rangle$
4	$\langle[0.38, 0.39], 0.30\rangle$
5	$\langle[0.42, 0.44], 0.18\rangle$
6	$\langle[0.42, 0.44], 0.34\rangle$
7	$\langle[0.57, 0.59], 0.71\rangle$
8	$\langle[0.58, 0.60], 0.48\rangle$
9	$\langle[0.84, 0.86], 0.58\rangle$
10	$\langle[0.59, 0.61], 1\rangle$
11	$\langle[0.99, 1], 0.73\rangle$

$$S_7 = \{2, 4, 9, 10\}, \quad |S_7| = 2.04$$

$$S_8 = \{2, 4, 9, 11\}, \quad |S_8| = 1.95.$$

Therefore, S_3 is a γ-set. So, points 1, 4, 8, and 11 are the best places to install remote control keys. It should be noted that according to the mentioned process, two or more adjacent points may be given the high priority for installing the equipment; in which case the experts of the region should choose one of them as the best point. The need to pay attention to the geographic and environmental conditions and location of the studied feeders, the possible limitations of the real network, and the existence of unique characteristics of each of the medium pressure feeders cause that all the components related to the network are carefully examined in choosing the location of these switches.

6 Conclusions

The CFGs have better flexibility in modeling uncertain phenomena since they support both fuzzy membership and interval-valued membership. In this chapter, due to the importance of the concepts of DS and VC in graph theory, we introduced these concepts in CFG and examined some of its properties. In this context, attention has been paid to the definition of these concepts in some types of CFGs. The results show a significant relationship between these concepts in FGs. In this study, the parameters values are shown as a real number to make it easy to compare between different sets. In addition, the obtained values depend on the upper and lower bounds of the membership intervals. One of the limitations of this method is the loss of some data. In future works, the authors try to study the connectivity index and the Wiener index in a CFG.

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Chapter 26
Imperfect Production Inventory System Considering Effects of Production Reliability

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1 Introduction

To attain maximum profit, each production company wishes to manufacture only perfect quality products. Due to various factors like as labor problems, machinery breakdowns, etc., malfunctioning of the manufacturing process increases as production time increases and hence manufacturing unit out lets partly perfect and partly defective/imperfect quality items. By the implication of development cost, the production of defective units can be reduced. Still now, several works on EPQ models by several researchers have been developed. On imperfect productions, several research works have been developed by Panja and Mondal [37], Dolai and Mondal [35], Dolai et al. [36], Nandra et al. [38], Seung et al. [15, 29], Nandra et al. [41], Tayyab et al. [39], Goyal et al. [7], Sana et al. [24], Sarkar et al. [40], Dolai [26], Yadav et al. [42], and Khouja and Mehrez [12]. With the incorporation of learning effect in random planning horizon, Kar et al. [10] have assumed a production model with permissible delay due to stimulate consumers. In connection with the EPQ/EOQ model with imperfect products, Salameh and Jaber [23] worked on inventory model.

Integrated production inventory approach deteriorating as a result of pricing policy, Chung and Wee [4] formulated a model in which warranty period and inspection planning have been assumed. In an imperfect production process, Cardenas-Barron [2] formulated an EPQ model in which manufactured imperfect items are repaired within the same cycle of the production due to ready the lot size of produced items as well as backorders' size. Under consideration of imperfect inspection and sales return, Yoo et al. [34] developed an EPQ model.

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C. Jana et al. (eds.), *Fuzzy Optimization, Decision-making and Operations Research*, https://doi.org/10.1007/978-3-031-35668-1_26

Moreover, reliability of a produced item mostly depends on the production system as well as production rate, and hence production cost has an impact due to reliability of a produced item. In this regard, so many research works have been developed. Sana [25] developed a production inventory model in which production cost depends on production rate as well as product reliability parameter. Under fuzziness with multiple constraints, Gupta et al. [8] presented optimization problems in perspective of reliability. To deal with the discrepancy and deficiency, Saxena and Sarkar [43] studied production process reliability with random misplacement.

In general, for deteriorating items, deterioration increases as time passes. In this connection, Singh et al. [31] developed time-dependent linear demand rate along with time-proportional deterioration rate in an optimal policy for deteriorating items. In the sense of probabilistic deterioration, Sarkar and Sarkar [28] formulated an EMQ model in a production system. With multiple production setups for deteriorating items, Uthayakumar and Tharani [33] have presented an EPQ with time-dependent demand. For instantaneous deteriorating items with backlogging as well as trade credit under inflation, Rajan and Uthayakumar [22] considered replenishment policies with optimal pricing.

At present, publicity has a strong effect to rise the demand of a commodity. In this regard, in an imperfect production inventory model with advertisement-dependent demand, Manna et al. [16] developed a production inventory model along with production rate-dependent defective rate. Using branch and bound technique with advertisement-dependent demand, Khara et al. [11] developed an integrated imperfect production model.

At present, the economic condition changes rapidly in the financial market, due to high inflation rate. In this regard, several researchers have developed different types of model considering the effect of inflation and also time value of money. Under the effect of inflation in an imperfect production process, Sarkar [27] presented an inventory model with reliability.

2 Research Gaps

Under the environment of fuzzy rough, a supply chain model containing three layers was presented by Manna et al. [14]. Paul et al. [20] studied an imperfect production model under managing disruption. The reliability-dependent development cost is done by Mettas [17] and Sana [25]. In real scenario, it has been noticed that every production system produces almost all good items within a fractional part of the production time as all components are newly set up and going start to run at the beginning of production and produces a mixture of imperfect and perfect items as malfunctioning increases when system runs through long duration. Again, when time passes, the development cost increases. Therefore, in this chapter the total production time has been divided into two parts (i) a time during which all produced items are perfect and (ii) a time within which imperfect and perfect items are produced. Also we have considered the development cost depends on time and reliability connected to the system.

It is observed that various investigator have considered various patterns of demand structure. Baker and Urban [1] studied a model with stock-dependent demands. Hariga [9] developed time variable demand. Mandal and Phaujdar [13] and Ray and Chaudhuri [21] have assumed stock-dependent demand. Datta and Pal [5] and Teng and Chang [32] investigated an inventory system with stock- and price-dependent demand. Ghosh and Chaudhuri [6] considered an EOQ model, where demand nature is in quadratic form. Mondal et al. [18], Panda and Maiti [19], and Chen et al. [3] studied a demand function connected with price of the item. Hence, from the seeing of the literature, it appears no one has previously taken into account the selling value, publicity cost, as well as reliability connected with the product.

3 Novelty and Contribution

In this work, a production model has been studied where production system manufactures only perfect quality of products within a fractional part of the total production duration at the initial stage of production because of all components of the production system are new in this initial stage and produces perfect and imperfect items within the remaining part of the production duration as malfunctioning of the production system increases due to various kinds of problems like labor, machinery, and technology whenever production run time increases. Every time a person goes to a supermarket or shopping mall to purchase a product, he or she faces an important question. First one is: How long does it last with its specific operating hours? That is, how much is it reliable? Second things is what is its selling price? Under this aspect requirement of the item has been adopted as a function of product reliability, selling value, and advertisement. To reduce imperfect production as well as to regulate the reliability of the machinery system, time-dependent development cost has been introduced in this production inventory model because of imperfect increases as time passes. In this work, three variables which are reliability connected with the product, manufacturing system, and the production time have been adopted that maximizes the item's profit. The gist of connected literature from which novelty of this chapter can be understood easily has been given in Table 26.1.

The chapter is arranged as follows: In Sect. 4, we have discussed the inventory management in operations research (OR). Some basic concept and terminologies have been given in Sect. 5. In Sect. 6, notations and model assumptions are given. In Sect. 7, the model formulation is illustrated. In Sect. 8, the numerical results have been discussed. In Sect. 9, sensitivity analysis and in Sect. 10 managerial insight are drawn. Finally, the paper is concluded including the future research in Sect. 11.

Table 26.1 Gist of connected literature

Author(s)	EOQ /EPQ	Development cost depends on	Cost of materials depends on	Demand depends on	Produced items within production time
Manna et al. [14]	EPQ	Constant(labor)	Constant	Stock-dependent	Imperfect and perfect items
Sana [25]	EPQ	reliability of System	Constant	Time	Imperfect and perfect items
Sarkar [27]	EPQ	reliability of System	System reliability	Selling price and advertisement cost	Imperfect and perfect items
Sana and Goyal & Chaudhuri [24]	EPQ	labor	Constant	constant	Imperfect and perfect items
Sarkar and Sarkar [28]	EPQ	reliability of System	System reliability	Time	Imperfect and perfect items
Manna et al. [16]	EPQ	Constant(labor)	Constant	Time	Imperfect and perfect items
Shah and Shah [30]	EPQ	reliability of System	Constant	Time	Imperfect and perfect items
Present paper	EPQ	System reliability and time	Reliability of product	Selling price, advertisement and product reliability	Imperfect and Perfect items

4 Inventory Management in Operations Research

To solve a problem as well as for making a decision, operations research (OR) is a useful analytical method which is applied in the management of organizations. In this consequence, a specific problem is divided into some components, and next it is solved by several steps into mathematical analysis. Mathematical logic, network analysis, simulation, game theory, and queuing theory are well-known analytical methods used in operations research. At the time of the Second World War, the mystical term operations research (OR) was originated whenever a group of scientists were called by the British military management to perform a scientific approach in the study of military operations to win the battle. The main purpose was to assign limited amount of resources in an appropriate technique to several military operations and activities within every operation. In England, the activity of this group of researchers was called "operational research" because of dealing with research on (military) operations. Operations research means the method of the scientific application, tools, as well as techniques to the problems of decision-making involving the system operations such that to apply these in the monitoring of the operations with optimum results.

After completion of the war, industrial managers focused their attention on the advancement of the military teams to solve their complex executive-type problems. In 1947, a mathematical technique was developed, which is related to this field (known as the simplex method in the problem of linear programming). After that, in both academic institution and industry, these new techniques along with applications have been revealed with effort and cooperations of sensible individuals. At present, in many areas the impact of operations research has been realized. Besides military and business applications, in so many sections such as transportation, financial institution, hospitals, city planning, and libraries, the activities of OR have been implemented. In this connection, in a real-life situation, for an example, it has been observed that a retailer roughly places an order to his/her supplier according to his/her customer's demand during a period of month or a week to satisfy his/her customer's demand. This is not the fact for a manager of a large departmental store or a large retailer, as the stocking in these cases depends on different factors, e.g., demand, time of ordering, time lag between the orders and actual receipt, amelioration, deterioration, time value of money, inflation, etc., and the impreciseness of these factors. So the problem for managers/retailers is to have a compromise between over-stocking and under-stocking. The study of such type of problem is known by the term "inventory control."

In general, inventory is nothing but the stock of the raw materials. For an enterprise, it is also known as the idle resource. Furthermore, the items which are either to be hold for sale or are kept for the process of manufacturing or are stocked in the form of raw materials are also defined as inventory. It has been observed that the time interval between the receiving of the purchased raw material and converting them into finished items varies from factory to factory depending on the production cycle time. Therefore, it is most essential to hold inventories of several types to perform as a buffer between supply and demand for efficient operation of the system. Thus, for smooth running of the production cycle with least interruptions, it is must needed to control inventory. In broad sense, inventory is defined as an idle resource of a company/manufacturing firm. It can be defined as a stock of physical goods, commodities, or other economic resources which are used to satisfy the customer's demand or requirement of production. This means that the inventory acts as a buffer stock between a supplier and a customer. In any one of the following categories, inventory of commodity can be stored:

- Raw materials: The collected products or extracted raw materials which are shifted to products.
- Components: Several parts that are used to make a final product.
- Work-in-process (WIP): An improved item which belongs to some stage of completion in a production system.
- Finished goods: Any completed item produced by the manufacturing system which is ready to deliver for the consumers.
- Maintenance, repair, and operational (MRO) inventory (often called supplies): some specific part that are used in production process but is not any part of the finished product.

In inventory control, materials are systematically moved from suppliers to production facilities, and then products are moved to consumers via distribution centers. Materials and final products are planned, acquired, stored, moved, and controlled by this department. Its main objective is to obtain the right goods at the right time with the right price to continue desired service level at minimum cost. Basically, it focuses with mainly two problems: (i) At what time an order should be placed? (ii) How much quantity should be ordered to satisfy market demand?

Using inventory models, answers are given. The scientific inventory control system balances between the loss due to non-availability of an item and carrying cost due stock of the product. The main goal of the scientific inventory control is to maintain the stock of required goods at optimum level at minimum cost in favor of the company. Therefore, the basic objectives of inventory control are:

- To assure sufficient supply of items to the customer and prevent shortages as far as feasible.
- To ensure minimum financial investment for inventories.
- For materials procurement, keeping, use for purpose, and accounting are important objectives.
- Timely initiative for replenishment is needed.
- To continue timely account of stocks of products and to keep the stock within a desired range.
- To give in hand stock for different lead times of supply of materials.
- To provide a technical concept for both short-term and long-term ideas of materials.

For the same reasons, stocks of other departments are used. Some costs like holding, replenishment/set-up, material costs, purchasing, shortage, etc. are involved in an inventory system. Although inventories constitute an idle resource which incurs holding costs, they are supported by the result protecting in shortage, replenishment, and procurement costs. The problem of inventory control is primarily concerned with the following fundamental questions:

- Is there a need to produce which items?
- How much of each of these products should be manufactured?
- When to produce? or When should an order be placed?
- What type of stock of the system should be used?

According to the above questions, the inventory problem plays a vital role in determining optimal decisions. Moreover, an inventory model reflects the decisions which enhance the favor of the organization, consumer service, etc. and optimize either the profit function or cost function of the inventory model. Basically, it is a tremendous job to decide an appropriate stocking management that deals with the mentioned questions in a practical problem. To solve inventory control problems, OR techniques are used to develop a mathematical model. Basically, the model represents the problems in a simplified way by considering only the most significant characteristics. Then, most beneficial or an optimal solution is acquired. In this study, we are developing some mathematical models and solving them using different mathematical methodologies for various realistic inventory control systems.

5 Basic Concepts and Terminologies

Inventory: Generally, inventory is defined as greatest resource of a company/enterprise/manufacturing firm. Normally, stock of commodities and physical goods which are supplied to satisfy the customers' demand can also be denoted as inventory. Also it reveals that it is considered as a buffer stock between a customer and a supplier. The inventories are categorized into the following forms:

I. Direct Inventories: Products that take part as an important preface in the production system and become an important portion of the finished products are familiar as direct inventories, which has been divided into four major groups:

   (a) Raw materials inventory: It consists of the procure products or extracted items which are shifted into products or components. It is provided for cheap (i) to make larger purchasing, (ii) to allow to change production rate, (iii) to enable buffer stock of production in regarding lateness transportation, and also (iv) for seasonal variations.

(b) Inventory for work-in-process (WIP): In the manufacturing process, this type of inventory consists of completion of some stages for an item. It is provided (i) to allow for economical lot size of production, (ii) to continue to produce the varieties of items, (iii) for replacing of dissipation, and (iv) to continue consistency in production though amount of sales may change.
(c) Inventory of completed products: This type of inventory contains accomplished items which will be delivered to consumers. It is given (i) to enable sales of promotion, (ii) for keeping off-self delivery, and (iii) to maintain stability of the level of the production.
(d) Elements of the inventory: It includes of some portion that are used to form the complete item.

II. Indirect Inventories: Some components like petrol, grease, oil, lubricants, office materials, maintenance materials, etc. that are needed to produce an item but are not the component of finished production known as indirect inventories.

Demand: In a given period of time, it is the number of consumers who are willing and able to buy products at different prices. The consumer's willingness and ability to pay for the commodity determines demand for any commodity.

Production: The resources and the products determine the production system. A product is designed to meet market demand using these resources.

Imperfect Production: It is seen that in reality in any manufacturing process, the production of imperfect quality products is a natural phenomenon which occur due to different causes such as inadequate working instructions, quality of raw materials, long-term production process, machinery faults, labor issues, etc.

Inspection/Screening Process: In any manufacturing process, it is observed that the system produces both imperfect and perfect quality items due to the various kinds of issues connected with the manufacturing process. So, the screening process is necessary to identify whether a manufactured product is good or not. After screening and shorting, the manufacturer satisfies the retailer's/customer's demand.

Rework Policy: In an imperfect production systems, it has been seen that a portion of the produced items are imperfect in quality which may be reworked to become as good as new one and can be sold further. It is important to eliminate waste and reduce manufacturing costs as a result of rework.

Reliability: In production inventory system, reliability means the ability of a device or a manufacturing process to execute its function subject to stated conditions for a specified length of time. It is measured by the ratio of the number of survival units to the total number of operating units inside a specified time interval. In a manufacturing system, the ratio of perfect number of items to the total number of produced items within an operating hour is known as the reliability of the production system.

Credit Period Policy/Delay in Payment: It is a time period that a customer is allowed to wait before paying an invoice. In present competitive situation, the credit period has special significance to reduce demand error for the manufacturer.

Advance Payment Policy: An advance payment means a payment that is made prior to replenish the order. Frequently, retailers try to increase the demand through the motivation by price discount offer. Further, if a customer pays an advance payment, then he may get some price discount on the ordering quantity. It has been observed that, in presence of several brands available in the market, demand rate of a product is decreased. So, to ensure the demand, the manufacturer/retailer frequently asks their retailer/customer to pay a certain percentage of the procurement cost of the replenishment quantity prior to delivery of the products in order to lessen the chance of cancellations of orders and invites to enjoy some other facilities offered by them.

Discount Policy: In order to employ more transection, the manufacturer frequently offers a reduced price if ordered quantity is greater than some specified minimum quantity. As a result, if you place a larger order, you will pay a lower price per unit. In general, two kinds of quantity price structure are assumed: (a) all unit discount and (b) incremental quantity discount. Furthermore, several types of discount are also offered under certain conditions.

Supply Chain Model (SCM): In present competitive market, supply chain management(SCM) takes part a significant role in economy as it provides an integrated networking system among supplier, manufacturer, retailer, and customer and tells how to survive in the present competitive market situation through the cooperation among supplier, manufacturer, retailer, and customer. To obtain tensionless stable sources of supply and demand of reliable products and to achieve optimum profit, it is very important to attain a long-term cooperation among manufacturer, supplier, and retailer which balances a series of inter-connected business procedure on account of:

(i) Optimal purchasing quantity of raw materials delivered by the supplier;
(ii) Shifting the raw materials into storehouse;
(iii) Manufacture of items in the manufacturing center and transportation of finished product to retailer to satisfy customer's required demand.

Recycling: Recycling is the procedure of converting waste substance into new materials and articles. It saves natural resources and helps to reduce greenhouse gas emissions. At present, in connection with the crisis of conventional resource of energy as well as environment pollution due to increase of greenhouse gasses, recycling of used product shows a vital role to save conventional resource of energy and to rescue us from an environmental catastrophe. Because much less energy is required for recycling of used product, huge energy is needed to produce an item from virgin raw materials, and consequently recycling saves our natural resources. So, it not only saves natural resource but it also saves energy.

Closed-Loop Supply Chain (CLSC): Closed-loop supply chains (CLSC) indicates a network that include the returns processes, and the manufacturer has the intent of capturing additional value and further integrating all supply chain activities. Closed-loop supply chain management(CLSCM) includes all forward logistics and also the reverse logistics to procure and process returned products and/or some portion of the products due to assurance of an ecologically as well as socioeconomically sustainable fulfillment. The main objective of the conventional supply chain is to reduce the cost and increase the efficiency of supply chain company so that as to maximum economic benefits can be achieved. Furthermore, CLSCM try to maximize economic profits, to reduce the consumption of natural resources and energy, and also to reduce the emissions of pollutants.

Cost of Purchase: It expenses to purchasing material and making the material in sellable condition.

Inventory Cost: It is the cost of keeping and holding down the inventory over a specified time period. There are several costs associated with inventory, for example, costs of production, purchasing costs, screening costs, reworking costs, setup costs, holding costs, idle costs, advertisement costs, warranty costs, stock-out costs, disposal worth, etc.

Reworking Cost: This is a cost required to make new products from defective items to restrain the waste of functional portion of an imperfect product, also to make less the use of virgin raw materials, to decrease energy utilization and environmental pollution in manufacturing center. Therefore, for this reason, the imperfect products are remanufactured at a cost to make a perfect one product.

Warranty Cost: The cost that a business expects to or has already incurred for the repair or replacement of goods that it has sold. The total amount of warranty expense is limited by the warranty period that a business typically allows.

Setup Cost: It includes packaging, delivery, shipping, and handling costs associated with actually ordering the inventory.

Shortage or Stock-Out Cost: This is the penalty imposed when there is not enough stock to meet customer demand. Inventory that has not been delivered to the customer determines this cost parameter, not where stock is replenished.

Disposal Cost: If an excess of some units remain at the end of an inventory cycle and are sold at a less price, and at the next time to get certain benefits, such as finishing the stock, closing the business, etc., the disposal cost is the revenue earned as a result of such a process.

6 Assumption and Notation

Following are the assumptions and notations used to develop the imperfect production model.

6.1 Notations

- $q(t)$: Perfect items inventory at time $t \geq 0$
- $q_d(t)$: Faulty items inventory at $t \geq 0$
- p : Fixed production value per item.
- λ : Reliability parameter of the manufacturing system, decision variable, which belongs in $[\lambda_{min}, \lambda_{max}]$
- r : Reliability of the product.
- r_{max} : Highest value of r
- r_{min} : Lowest value of r
- v : Number of advertisement, $v \in [0,1]$
- c_v : Advertisement cost per unit time per unit advertisement. It is the cost allotted to advertise for an item in popular media such as TV, radio, newspaper, magazine, etc. and also with the sales representative to increase the sale of that item.
- $V(\lambda, t)$: Development cost.
- A : Energy and labor cost which is not dependent on λ.
- B : The cost of resource, design complexity, and technology for manufacturing. when $\lambda=\lambda_{max}$ and $t=(\tau+1)$.
- k : The difficulties in increasing reliability of the manufacturing system.
- s : Perfect item selling price per unit.
- s_1 : Imperfect item selling price per unit.
- $D(r, s, v)$: Demand function on r, s and v.
- d_0 : Constant demand due to reliability of the product.
- d_1 : Demand constant due to selling price.
- d_2 : Demand constant due to advertisement.
- D_3 : Demand of defective items.
- $C(\lambda, r, t)$: Production cost per product.
- $M(r)$: Material cost depending on r.
- M_0 : Constant ingredient cost.
- M_1 : The reliability of the item to be produced increases by this ingredient cost.
- C_1 : Per unit item holding cost per unit time. It is the cost to keep the items store house of the inventory until its utilization or selling.
- C_2 : Screening cost per unit item.
- t_1 : Duration of production, dependent decision variable.
- τ : A fractional part of the production time t_1
- T : Finite time duration. Depending upon the characteristic of the stock system, it can be finite or infinite.
- α : Tool/die cost variance constant.

6.2 Assumptions

On the basis of following assumptions, this model of the inventory management system has been developed which are adopted by the authors only. So, there is no reference against each assumption:

(i) In an imperfect production process, this model is formulated for a single product.

(ii) It is obvious that the production system is under control in the initial stage of production run time as every machinery components of the system is new initially. As a result, it produces only perfect quality items within this initial stage of production. But after a fractional part of the production time (i.e., after this initial stage of production time), it produces a mixture of imperfect and perfect quality items because it is shifted to out of control state due to malfunctioning of the machinery parts, machinery breakdown, labor, etc. as production run time increases. In this connection, it has been assumed that the system manufactures only fair items within a period of time length $(0,\tau)$ when the manufacturing system is under control state and after time τ the production process outlets a mixture of both imperfect and perfect items.

(iii) In this model, $e^{-\lambda(t-\tau)}$ is assumed as the reliability of the production system where λ is the parameter due to reliability is explained as

$$\lambda=\frac{\text{number of imperfect products}}{\text{whole number of manufactured product within a specified interval of time}}$$

which reflects that reliability of the system increases with the deceasing value of λ.

(iv) In perspective of our assumption (iii), it is obvious that, whenever time passes, production system reliability is reduced. Again, following the assumption (ii), the manufacturing process is under control state within a duration of time lying between 0 and τ, and system reaches out of control situation after τ. So, it is appropriate to consider a constant maintenance cost within a period of time interval $(0,\tau)$ to run the system and a time-dependent development cost after time τ to obtain a constant reliability connected with the manufacturing process within production system. Moreover, the cost for development should be increased to increase the reliability of the system. Hence, this cost $V(\lambda,t)$ should be related to λ, and t is assumed as follows:

$$V(\lambda,t)=\begin{cases} A & if\, 0<t\leq\tau \\ A+B(t-\tau)e^{k\left(\frac{\lambda_{max}-\lambda}{\lambda-\lambda_{min}}\right)} & if\, \tau\leq t\leq t_1 \end{cases}$$

where A is a constant maintenance cost due to energy and labor which independent of λ and the cost of resource, technology, as well as design complexity is B, whenever t=$(1+\tau)$ and λ=λ_{max}.

(v) Generally, product reliability r depends on the raw material as well as on the manufacturing procedure. Again, reliability r connected with the product increases with increase in quality of the raw material, but this rate is not uniform. This rate of increase is decreased. Incorporating this fact, the ingredient cost $M(r)$ have been formulated as below

$$M(r) = M_0 - M_1 e^{-\mu r}$$

where $M_0 > 0$, $M_1 > 0$, with $M_0 > M_1$ and $\mu > 0$.

(vi) In this model, the production cost per product $C(\lambda, r, t)$ assumed as a function of development cost as well as material cost of the following form:

$$C(\lambda, r, t) = M(r) + \frac{V(\lambda, t)}{p} + \alpha p$$

where α is fixed cost depending on manufactured items p in the production system.

(vii) In present situation, it is noted that the selling of an article relies on the product promotion in connection with the common life. So, increasing the demand of an article, the advertisement impacts positively to motivate the consumers to purchase the product. Again, it is noted that the customers pay attention on the reliability connected with the product r and also on the selling price s of an article. In this regard, the demand $D(r, s, v)$ of an article has been assumed as the following form:

$$D(r, s, v) = d_0 \frac{(r - r_{min})}{(r_{max} - r)} + d_1 s^{-b} + d_2 v \tag{26.1}$$

where $0 < v < 1$ and here the demand decreases with selling price (s) and increases with the reliability (r) and the advertisement (v).

(viii) In this model, the finite time horizon is assumed.

(ix) The terminal and the initial inventory levels are considered as zero.

(x) As all produced items are considered as perfect within time τ, the screening is considered for the items produced after time (τ) to identify whether it is perfect or imperfect.

(xi) As τ a fractional part of the production time t_1, in this model we have considered $\tau = xt_1$ where $0 < x < 1$.

Lemma 1 *The ingredient cost* $M(r) = M_0 - M_1 e^{-\mu r}$ *increases with product reliability* (r) *and increases from* $M_0 - M_1$ *at a decreasing rate.*

Proof Here, $M(r) = M_0 - M_1 e^{-\mu r}$ and for zero reliability, we have $M(0) = M_0 - M_1$ is the constant material cost for production. Since $M(0) > 0$, so $M_0 > M_1$.

Therefore, $\frac{dM(r)}{dr} = M_0 \mu e^{-\mu r} > 0$ which reflects that $M(r)$ is an rising function. Again, $\frac{d^2 M(r)}{dr^2} = -M_0 \mu^2 e^{-\mu r} < 0$ shows that $\frac{dM(r)}{dr}$ decreases with r. Hence, $M(r)$ is an increasing function of r with a decreasing rate.

7 Model Formulation

In real scenario, it has seen that there is no manufacturing system which can produce 100% perfect items when it continues through long time. It is true that it produces a mixture of both imperfect and perfect quality items due to various kinds of issue like machinery breakdown, labor, technology, etc. when it process through long duration. In our model, we have considered that all produced items are perfect within time τ and after time τ when manufacturing system becomes out of control state due to several kinds of problem like labor, machinery breakdown, technology, etc. producing a mixture of both imperfect and perfect items. Items in perfect condition are available for sale, while defective items are on sale for a reduced cost. In order to reduce these defective items, the cost of development $V(\lambda, t)$ is introduced, which is enhanced with the time t. The schematic diagram of this model has been given in Fig. 26.1.

Here, p is the constant production rate, and the reliability of the manufacturing system is $e^{-\lambda(t-\tau)}$, number of perfect items produced by the system is $pe^{-\lambda(t-\tau)}$, and the number of imperfect items is $p(1 - e^{-\lambda(t-\tau)})$. Therefore, the differential equation of the inventory $q(t)$ is given by

$$\frac{dq(t)}{dt} = \begin{cases} p - D & if\, 0 \le t \le \tau \\ pe^{-\lambda(t-\tau)} - D & if\, \tau \le t \le t_1 \\ -D & if\, t_1 \le t \le T \end{cases}$$

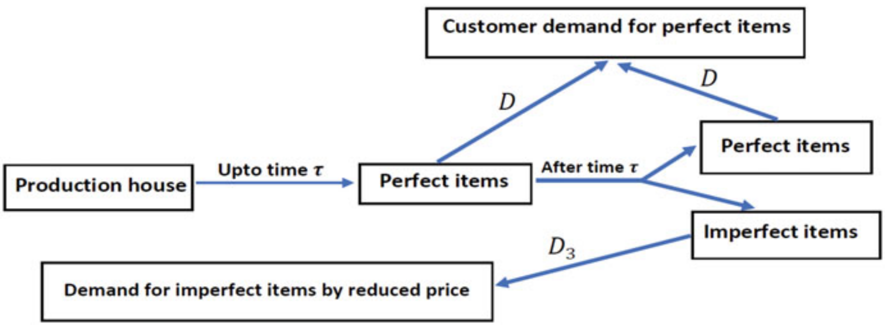

Fig. 26.1 The schematic diagram of imperfect production inventory system

with boundary conditions $q(0) = q(T) = 0$.

By solving the abovementioned differential equation with the boundary conditions, the stocking level $q(t)$ of perfect items at any time t is given by

$$q(t) = \begin{cases} (p - D)t & if\ 0 \le t \le \tau \\ \frac{p}{\lambda}\{1 - e^{-\lambda(t-\tau)}\} - Dt + p\tau & if\ \tau \le t \le t_1 \\ \frac{p}{\lambda}\{1 - e^{-\lambda(t_1-\tau)}\} - Dt + p\tau & if\ t_1 \le t \le T \end{cases}$$

The graph inventory for perfect item for Example 2 is shown in Fig. 26.2 and that of for imperfect item is depicted in Fig. 26.3.

Lemma 2 *The variables r, λ, and t_1 are connected as follows:*

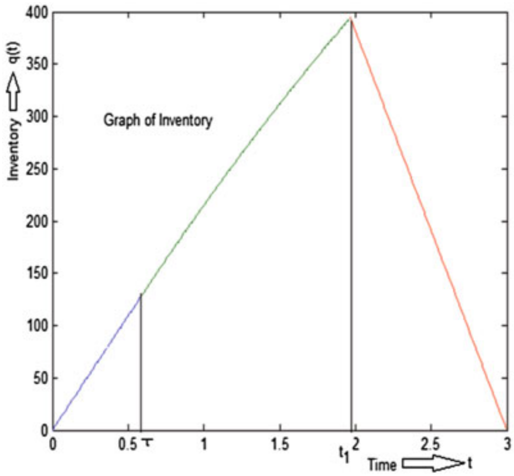

Fig. 26.2 Graph of inventory $q(t)$ of perfect items based on Example 2

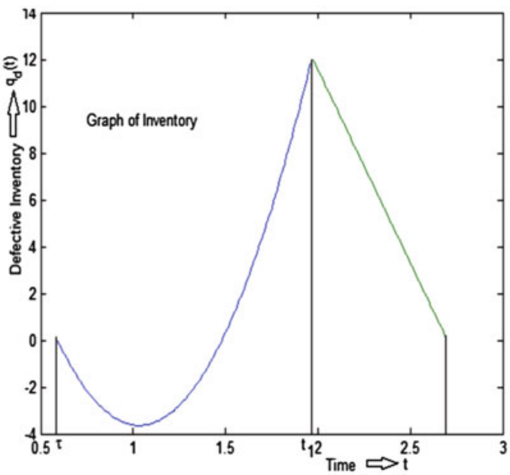

Fig. 26.3 Graph of inventory $q_d(t)$ of imperfect items based on Example 2

$$\frac{1}{\lambda}p\{1-e^{-\lambda(t_1-\tau)}\}=DT-p\tau$$

where $D=d_0\frac{(r-r_{min})}{(r_{max}-r)}+d_1s^{-b}+d_2v$

Proof Since, the stocking level $q(t)$ is continuous at time $t=\tau$ and at $t=t_1$, hence we have

$$\frac{1}{\lambda}p\{1-e^{-\lambda(t_1-\tau)}\}-Dt_1+p\tau=D(T-t_1)$$

$$\text{i.e.,}\frac{1}{\lambda}p\{1-e^{-\lambda(t_1-\tau)}\}+p\tau=DT \tag{26.2}$$

Again, the differential equation of the stock of the imperfect products $q_d(t)$ is as follows:

$$\frac{dq_d(t)}{dt}=\begin{cases} p\{1-e^{-\lambda(t-\tau)}\}-D_3 & if\ \tau\le t\le t_1 \\ -D_3 & if\ t_1\le t\le t_3 \end{cases}$$

with boundary conditions $q_d(\tau)=0=q_d(t_3)$ and $D_3=d_3s_1^{-w}$ has been considered as the demand of defective items.

By solving the abovementioned differential equation with the initial conditions, the stocking level $q_d(t)$ at any time t is given by

$$q_d(t)=\begin{cases} p[t-\tau-\frac{1}{\lambda}\{1-e^{-\lambda(t-\tau)}\}]-(t-\tau)D_3 & if\ \tau\le t\le t_1 \\ p[t_1-\tau-\frac{1}{\lambda}\{1-e^{-\lambda(t_1-\tau)}\}]-(t-\tau)D_3 & if\ t_1\le t\le t_3 \end{cases}$$

The graph inventory for perfect item for Example 3 is shown in Fig. 26.4 and that of for imperfect item is depicted in Fig. 26.5.

Lemma 3 *The decision variables* λ *and* t_1 *are connected as follows:*

$$p[t_1-\tau-\frac{1}{\lambda}\{1-e^{-\lambda(t_1-\tau)}\}]=(t_3-\tau)D_3 \tag{26.3}$$

Proof The stocking level $q_d(t)$ of the imperfect items at time $t=t_3$ vanishes; therefore, $q_d(t_3)=0$ implies

$$p[t_1-\tau-\frac{1}{\lambda}\{1-e^{-\lambda(t_1-\tau)}\}]-(t_3-\tau)D_3=0$$

$$\text{i.e.,}p[t_1-\tau-\frac{1}{\lambda}\{1-e^{-\lambda(t_1-\tau)}\}]=(t_3-\tau)D_3 \tag{26.4}$$

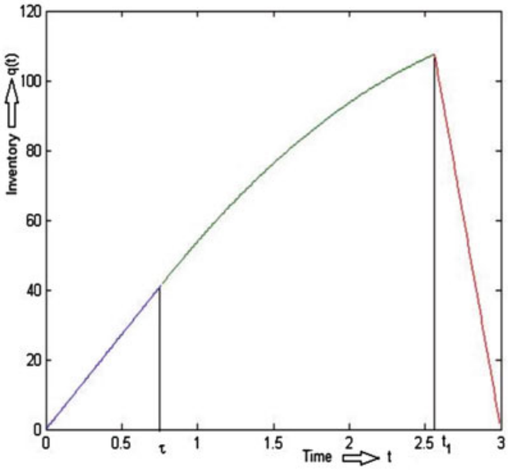

Fig. 26.4 Graph of inventory $q(t)$ of perfect items based on Example 3

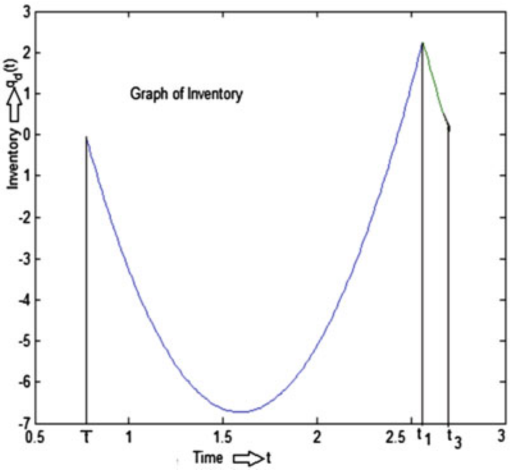

Fig. 26.5 Graph of inventory $q_d(t)$ of imperfect items based on Example 3

Lemma 4 *The decision variables such as the product reliability r and the production time t_1 are related as follows $t_1 = E + GD$ where $E = \frac{D_3 t_3}{p+D_3 x}$, $G = \frac{T}{p+D_3 x}$ and $\tau = x t_1$*

Proof From Lemmas 2 and 3: using two constraint Equations, we attained as follows:

$$p[t_1 - \tau - \frac{1}{\lambda}\{1 - e^{-\lambda(t_1-\tau)}\}] - (t_3 - \tau)D_3 = 0$$

$$\text{i.e.,} p(t_1 - \tau) - (DT - p\tau) = (t_3 - \tau)D_3$$

$$\text{i.e.,} pt_1 - DT = D_3t_3 - D_3xt_1$$
$$\text{i.e.,} t_1 = \frac{D_3t_3}{p + D_3x} + \frac{T}{p + D_3x}D$$
$$\text{i.e.,} t_1 = E + GD \tag{26.5}$$

The revenue for the perfect items is given by

$$S_{rev} = s\int_0^{\tau} pdt + s\int_{\tau}^{t_1} pe^{-\lambda(t-\tau)}dt = sp[\tau + \frac{1}{\lambda}\{1 - e^{-\lambda(t_1-\tau)}\}] \tag{26.6}$$

The revenue for the imperfect items with reduced selling price is given by

$$S_{revd} = s_1\int_{\tau}^{t_1} p\{1 - e^{-\lambda(t-\tau)}\}dt = s_1p[t_1 - \tau - \frac{1}{\lambda}\{1 - e^{-\lambda(t_1-\tau)}\}] \tag{26.7}$$

The total cost C_p for production is given by

$$C_p = p\int_0^{\tau} C(\lambda, r, t)dt + p\int_{\tau}^{t_1} C(\lambda, r, t)dt$$
$$= p\int_0^{\tau}\{M(r) + \frac{A}{p} + \alpha p\}dt + p\int_{\tau}^{t_1}\{M(r) + \frac{V(\lambda, t)}{p} + \alpha p\}dt$$
$$= [p\{M_0 - M_1e^{-\mu r}\} + A + \alpha p^2]t_1 + \frac{B}{2}(t_1 - \tau)^2e^{k(\frac{\lambda max - \lambda}{\lambda - \lambda_{min}})} \tag{26.8}$$

The cost C_{hp} for holding the perfect items is obtained as

$$C_{hp} = c_1[\int_0^{\tau} q(t) + \int_{\tau}^{t_1} q(t) + \int_{t_1}^{T} q(t)]dt$$
$$= c_1\frac{p}{\lambda}[t_1 - \tau - \frac{1}{\lambda}\{1 - e^{-\lambda(t_1-\tau)}\}] + c_1pT\tau + c_1(T - t_1)$$
$$\frac{p}{\lambda}\{1 - e^{-\lambda(t_1-\tau)}\} \tag{26.9}$$
$$- \frac{c_1}{2}(DT^2 + p\tau^2) \tag{26.10}$$

The holding cost C_{hd} of imperfect items is obtained as

$$C_{hd} = c_1[\int_{\tau}^{t_1} q_d(t) + \int_{t_1}^{t_3} q_d(t)]dt$$
$$= c_1\frac{p}{2}(t_1-\tau)^2 - c_1\frac{D_3}{2}(t_3-\tau)^2 + p(t_3 - t_1 - \frac{1}{\lambda})[t_1 - \tau$$
$$-\frac{1}{\lambda}\{1 - e^{-\lambda(t_1-\tau)}\}] \quad (26.11)$$

The total cost for publicity $C_{adv} = c_v \int_0^{yT} v dt = c_v v y T$

The cost for screening the products whether it is perfect or defective is $C_{scr} = c_2 \int_{\tau}^{t_1} p dt = c_2 p(t_1 - \tau)$

Therefore, in this model the average profit using Lemmas 1 and 2: is obtained as follows:

$$F(\lambda, r, t_1) = \frac{1}{T}[S_{rev} + S_{revd} - C_p - C_{hp} - C_{hd} - C_{adv} - C_{scr}]$$
$$= \frac{1}{T}[(sT - \frac{1}{2}c_1T^2 + c_1Tt_1)\{d_0\frac{(r - r_{min})}{(r_{max} - r)} + d_1 s^{-b} + d_2 v\}$$
$$-\{s_1 D_3 x + p(M_0 - M_1 e^{-\mu r}) + \alpha p^2 + A - c_1 D_3 t_3 + c_2 p(1-x)\}t_1$$
$$+\{\frac{1}{2}c_1 D_3 x^2 - \frac{1}{2}B(1-x)^2 e^{k(\frac{\lambda_{max}-\lambda}{\lambda-\lambda_{min}})} - \frac{1}{2}c_1 p - c_1 D_3 x\}t_1^2$$
$$-\frac{1}{2}c_1 D_3 t_3^2 - c_v v y T + s_1 D_3 t_3] \quad (26.12)$$

Here three variables λ, r, and t_1 are connected through the average profit $F(\lambda, r, t_1)$. As there are two constraint equations in Lemmas 2 and 3 connected with three variables λ, r, and t_1, the objective function $F(\lambda, r, t_1)$ finally reduces to a one independent decision variable. As profit function is not linear and complicated form, it is difficult to reduce as a function of a single independent variable. After substitution of the value of t_1 from Lemma 4, the function $F(\lambda, r, t_1)$ transforms to $F(\lambda)$ as follows:

$$F(\lambda) = \frac{1}{T}[\mu_1 D - \{p(M_0 - M_1 e^{-\mu r}) + \mu_2\}(E + GD)$$
$$+ \{\mu_3 - \frac{1}{2}B(1-x)^2 e^{k(\frac{\lambda_{max}-\lambda}{\lambda-\lambda_{min}})}\}(E+GD)^2 + c_1 T D(E+GD) + \mu_4]$$
$$= \frac{1}{T}[\{\mu_5 - Gp(M_0 - M_1 e^{-\mu r}) - EGB(1-x)^2 e^{k(\frac{\lambda_{max}-\lambda}{\lambda-\lambda_{min}})}\}D$$

$$+ \{\mu_6 - \frac{1}{2}G^2B(1-x)^2e^{k(\frac{\lambda max-\lambda}{\lambda-\lambda_{min}})}\}D^2 - Ep(M_0 - M_1e^{-\mu r})$$

$$- \frac{1}{2}E^2B(1-x)^2e^{k(\frac{\lambda max-\lambda}{\lambda-\lambda_{min}})} + \mu_7] \quad (26.13)$$

where $D = \{d_0\frac{(r-r_{min})}{(r_{max}-r)} + d_1s^{-b} + d_2v\}$
$\mu_1 = st - (1/2)c_1T^2$, $\mu_2 = s_1D_3x + \alpha p^2 + A - c_1D_3t_3 + c_2p(1-x)$
$\mu_3 = (1/2)c_1D_3x^2 - (1/2)c_1p - c_1D_3x$, $\mu_4 = c_1D_3t_3 - (1/2)c_1D_3t_3^2 - c_vvyt$
$\mu_5 = \mu_1 - G\mu_2 + 2EG\mu_3 + c_1TE$, $\mu_6 = G^2\mu_3 + c_(1)TG$ and $\mu_7 = \mu_4 - E\mu_2 + E^2\mu_3$
and r is considered as a function of λ, suppose $r = \phi(\lambda)$ is obtained from the following equation where r and λ are related implicitly

$$\frac{p}{\lambda}\{1 - e^{-\lambda(t_1-\tau)}\} = DT - p\tau$$

$$\text{i.e.,} \frac{p}{\lambda}\{1 - e^{-\lambda(1-x)t_1}\} = DT - pxt_1$$

$$\text{i.e.,} \frac{p}{\lambda}\{1 - e^{-\lambda(1-x)(E+GD)}\} = DT - px(E+GD)$$

$$\text{i.e.,} \frac{p}{\lambda}\{1 - e^{-\lambda(1-x)(E+GD)}\} = (T - pxG)D - pxE \quad (26.14)$$

So, the main objective is to maximize $F(\lambda) = F(\lambda, r)$ subject to $r = \phi(\lambda)$ (say) obtained from Eq. (26.14).

Therefore, by the chain rule from differential calculus, $\frac{dF(\lambda)}{d\lambda} = \frac{\partial F(\lambda,r)}{\partial\lambda} + \frac{\partial F(\lambda,r)}{\partial r}\frac{dr}{d\lambda}$ and $\frac{d^2F}{d\lambda^2} = \frac{\partial^2F(\lambda,r)}{\partial\lambda^2} + 2\frac{\partial^2F(\lambda,r)}{\partial\lambda\partial r}\frac{dr}{d\lambda} + \frac{\partial F(\lambda,r)}{\partial r}\frac{d^2r}{d\lambda^2} + \frac{\partial^2F(\lambda,r)}{\partial\lambda^2}(\frac{dr}{d\lambda})^2$ (see Appendix).

Lemma 5 *Average profit F is finally a function of* λ. *Here, parametric values are associated with the equation* $\frac{dF(\lambda)}{d\lambda} = 0$. *There exists at least one value of* $\lambda = \lambda^*$ *at which* $\frac{d^2F}{d\lambda^2} < 0$; *then we get the optimum value of the profit at* $\lambda = \lambda^*$.

8 Numerical Illustration

To get the optimum value of the model, the numerical description has been done taking the following examples. It is noted that due to unavailability of the real data, all data in these examples have been considered hypothetically but similar to the real.

Example 1 Suppose a production house manufactures an article with a fixed rate of production monthly with ten units to fulfill the requirement of the consumer during the business cycle of 1 month due to assumption (vi). Here fixed rate of demand is given by $d_0 = 2$ units, $d_1 = 8$ units, and $d_2 = 2$ units, per unit selling price for perfect quality of items, $s = \$90$, and other parameters $b = 2$ and amount of advertisement $v = 0.7$. To regulate the reliability connected with the manufacturing process, the manufacturer has assumed a time-dependent manufacturing development cost based on assumption (iii) with $A = \$1$, $k = 0.1$, $\lambda_{min} = 0.01$, $B = \$0.5$, and $\lambda_{max} = 0.9$. In this system, the cost for material (assumption (iv)) is defined where $M_1 = \$30$, $M_0 = \$60$, $\mu = 0.5$ and production cost per unit has been considered based on assumption (v) where $\alpha = 0.01$. Here, to sell the produced imperfect quality of items, the company has considered reduced selling price $s_1 = \$25$ per imperfect product. In this system, the cost for holding single produced item per unit time is $c_1 = \$0.4$, unit advertisement cost $c_v = \$20$. To obtain the maximum profit, find out the optimum time of production due to the optimum reliability connected with the manufacturing process. Hence, we have to obtain the optimum reliability.

Solution In Table 26.2, the hypothetic values of the parameters for this example are assumed.

It is very complicated to optimize the objective function analytically, due to nonlinear of the profit function of this current model. To obtain the outcomes of the model, we have used LINGO 12 software. With respect to the system reliability parameter λ, Fig. 26.6 reveals that the profit function $F(\lambda)$ is concave and due to the reliability r. Also, Fig. 26.7 shows that the profit function $F(\lambda)$ is concave in nature, which indicates that the profit function is globally optimum. Hence, due to these hypothetic values of all parameters, the model is optimized numerically which gives the following results:

Optimum values of the profit, λ, duration of production, and reliability of the product are given by $F^* = \$382.92$, $\lambda^* = 0.0253446$, $t_1^* = 0.6720670$, and $r^* = 0.6558947$, respectively.

Now, here optimal condition of the objective function are verified numerically as follows:

At $\lambda^* = 0.0253446$, the first derivative $\frac{dF(\lambda)}{d\lambda} = 0$ and the second derivative $\frac{d^2F}{d\lambda^2} = -36950150$ which indicates the optimal point at $\lambda^* = 0.02534456$. Henceforth, it confirms the existence of the maximum value.

Table 26.2 Parametric values

Parameter	Value	Parameter	Value	Parameter	Value	Parameter	Value
s	90	M_0	60	p	10	t_3	0.9
s_1	25	T	1	A	1	B	0.5
μ	0.5	y	0.7	λ_{max}	0.9	λ_{min}	0.01
v	0.7	α	0.01	d_0	2	c_1	0.4
M_1	30	k	0.1	d_1	8	c_2	0.2
x	0.3	b	2	d_2	2	c_v	20
w	1	y_3	0.9	d_3	1	τ	0.2016201

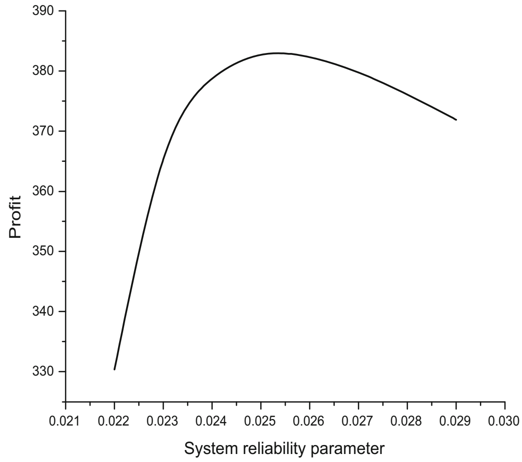

Fig. 26.6 Concave nature of $F(\lambda)$ based on Example 1

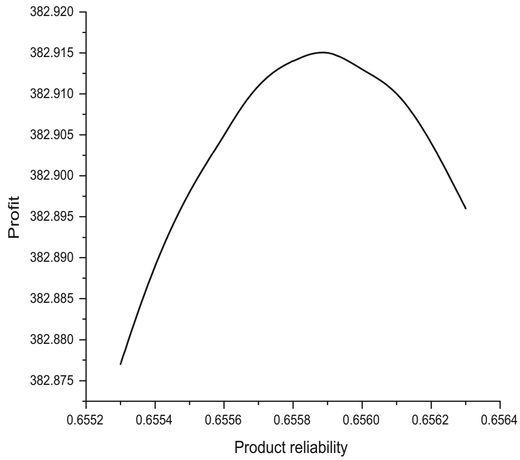

Fig. 26.7 Concave of the profit function $F(r)$ based on Example 1

Example 2 The same type of problem is considered in this example with some different input parametric values shown in Table 26.3.

Solution In Table 26.3, the parametric values are given.

In the same way as in Example 1, Fig. 26.8 shows that the profit function $F(\lambda)$ is concave in nature w.r.t system reliability parameter λ and Fig. 26.9 shows that $F(\lambda)$ is concave in nature with respect to the product reliability r which indicates global maximum value of the function. Hence, the optimal solution is given by:

Optimal value of λ, profit, duration of production and reliability of the product are as $\lambda^* = 0.0630804$, $F^* = \$19397.85$, $t_1^* = 1.967591$, and $r^* = 0.8549242$ respectively.

Now, here optimal condition of the objective function are verified numerically as follows:

Table 26.3 parametric values

Parameter	Value	Parameter	Value	Parameter	Value	Parameter	Value
s	100	M_0	60	p	600	t_3	2.7
s_1	45	T	3	A	1	B	1
μ	0.5	y	0.7	λ_{max}	0.9	λ_{min}	0.01
v	0.7	α	0.01	d_0	20	c_1	0.4
M_1	30	k	0.5	d_1	8	c_2	0.4
x	0.3	b	2	d_2	10	c_v	20
w	0.05	y_3	0.9	d_3	20	τ	0.5902774

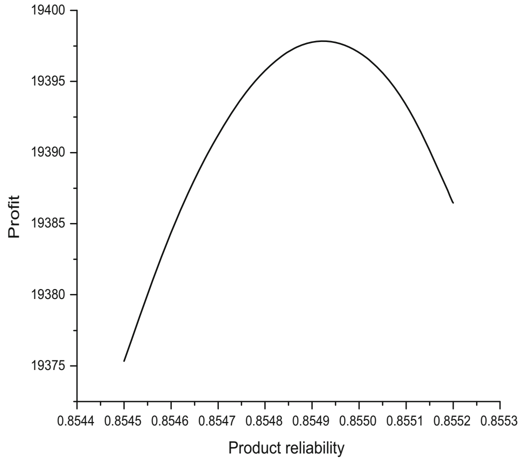

Fig. 26.8 Concave nature of $F(\lambda)$ based on Example 2

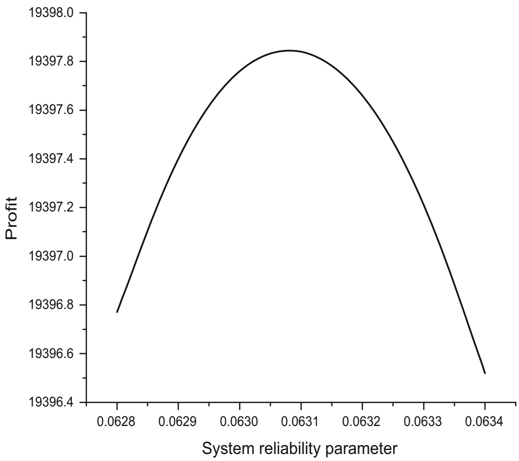

Fig. 26.9 Graph of profit F vs parameter λ based on Example 2

Table 26.4 Parametric values

Parameter	Value	Parameter	Value	Parameter	Value	Parameter	Value
s	100	M_0	60	p	300	t_3	2.7
s_1	45	T	3	A	1	B	1
μ	0.5	y	0.7	λ_{max}	0.9	λ_{min}	0.01
v	0.7	α	0.01	d_0	20	c_1	0.4
M_1	30	k	0.5	d_1	8	c_2	0.4
x	0.3	b	2	d_2	10	c_v	20
w	0.05	y_3	0.9	d_3	20	τ	0.7686275

At $\lambda^* = 0.0630804$, the first derivative $\frac{dF(\lambda)}{d\lambda} = 0$ and the second derivative $\frac{d^2F}{d\lambda^2} = -0.3039915E + 08$ which indicates the optimal point at $\lambda^* = 0.0630804$. Henceforth, it confirms the existence of the maximum value.

Example 3 In Table 26.4, the corresponding input several parameters are given as in Examples 1 and 2.

In the same way of Examples 1 and 2, in this example the optimal solutions are given by

Optimum profit, values of λ, duration of production, and reliability of the product are as $F^* = \$13243.07$, $\lambda^* = 0.0689405$, $t_1^* = 2.562092$, and $r^* = 0.8549242$, respectively.

9 Sensitivity Analysis

From the previous Example 1, we have done sensitivity analysis of the proposed model to variations in some parameters in the following section.

Here, we assume, $\triangle\lambda = (\lambda' - \lambda)/\lambda \times 100\%$, $\triangle x = (x' - x)/x \times 100\%$, $\triangle p = (p' - p)/p \times 100\%$, $\triangle t_1^* = (t_1^{*'} - t_1^*)/t_1^* \times 100\%$, $\triangle F^* = (F^{*'} - F^*)/F^* \times 100\%$, $\triangle r^* = (r^{*'} - r^*)/r^* \times 100\%$, where λ, p, t_1^*, F^*, r^* be the exact values and the corresponding estimated values are λ', p', $t_1^{*'}$, $F^{*'}$, $r^{*'}$. We have shown the sensitivity analysis by fluctuating the parameters p and x by different percentage(%), where one or more are taken at a time with keeping the rest at their exact values. The results are as follows (Tables 26.5 and 26.6).

It is seen from Table 26.7 that the reliability connected with the system λ increases (i.e., the reliability connected with the production system decreases) as the parameter k increases. For this reason, Table 26.7 depicts that the production time t_1, profit F, and product reliability r are decreased as the parameter k increases.

Table 26.8 shows that the system reliability parameter λ decreases with the increasing of selling price s which explores that for more selling price s, the demand

Table 26.5 Sensitivity w.r.t production rate p based on Example 1

Change of p (%) ($\triangle p$)	Change of λ (%) ($\triangle\lambda^*$)	Change of t_1 (%) ($\triangle t_1^*$)	Change of r (%) ($\triangle r^*$)	Change of F (%) ($\triangle F^*$)
−30	+4.67	+14.40	−8.36	−19.92
−20	+2.86	+8.85	−5.03	−12.93
−10	+1.32	+4.11	−2.29	−6.30
+10	−1.15	−3.61	+1.96	+6.03
+20	−2.17	−6.82	+3.65	+11.82
+30	−3.08	−9.70	+5.14	+17.40

Table 26.6 Sensitivity analysis w.r.t. the parameter x

Change of x (%) ($\triangle x$)	Change of λ (%) ($\triangle\lambda^*$)	Change of t_1 (%) ($\triangle t_1^*$)	Change of r (%) ($\triangle r^*$)	Change of F (%) ($\triangle F^*$)
−30	+1.01	−7.45	−2.76	−7.75
−20	+0.68	−5.07	−1.83	−5.28
−10	+0.35	−2.58	−0.91	−2.69
+10	−0.35	+2.68	+0.90	+2.81
+20	−0.72	+5.47	+1.80	+5.73
+30	−1.09	+8.35	+2.67	+8.77

is decreased. Again, demand rate decreases imply production rate decreases for which the reliability connected with the system is increased.

10 Managerial Implication

Using the numerical results and sensitivity analysis, the observed managerial implications are as follows:

- Figures 26.6 and 26.9 reveal that the objective function profit is concave in nature w.r.t λ. Here, the system reliability decreases as production time increases and to maintain the same level of production reliability throughout the process, the production cost must be increased and hence profit decreases after getting its optimum value.
- From Figs. 26.7 and 26.8, one can see that w.r.t. product reliability r, the profit is concave in nature. It is obvious because more production cost is needed for more product reliability r which decreases the profit.
- In this current model, it is taken in consideration that all manufacturing products are perfect up to time τ in the initial stage of production, so screening is not needed up to time τ for which total screening cost can be reduced.

Table 26.7 The change of $F(\lambda)$ w.r.t. k

k	λ	t_1	r	F
0.08	0.0225648	0.7068327	0.6670160	403.63
0.09	0.0239620	0.6886712	0.6613357	392.78
0.1	0.0253446	0.6720670	0.6558947	382.91
0.2	0.0385805	0.5581755	0.6106469	316.44
0.3	0.0510626	0.4919235	0.5756759	278.56
0.4	0.0629947	0.4469979	0.5467233	253.15
0.5	0.0744734	0.4138940	0.5218484	234.53
0.6	0.0855573	0.3881774	0.4999669	220.14
0.7	0.0962875	0.3674514	0.4803988	208.57
0.8	0.1066943	0.3502885	0.4626845	199.02
0.9	0.1168021	0.3357764	0.4464960	190.95
1.0	0.1266310	0.3233007	0.4315896	184.03

Table 26.8 The variation of λ, t_1, r, and profit F with respect to the selling price s based on Example 2

s	λ	t_1	r	F
85	0.0649297	1.943189	0.8543554	13704.19
90	0.0642287	1.952326	0.8545700	15597.75
95	0.0636188	1.960386	0.8547577	17491.96
100	0.0630804	1.967591	0.8549242	19397.85
105	0.0625992	1.974102	0.8550736	21310.64
110	0.0621650	1.980037	0.8552089	23229.69
115	0.0617699	1.985486	0.8553324	25154.48

11 Conclusion

In this chapter, a production inventory model with defective items has been taken in which the production run time is broken into two parts: (i) a fractional part τ of the production time t_1 in the initial stage of production where all produced items are good because of in-control-state of the machinery system and (ii) second the remaining part $t_1 - \tau$ of the production time t_1 where imperfect and perfect items are manufactured because of faulty system as due to long run process. Here demand depends on selling price, advertisement, and the product reliability. Basically, this type of work is related to the items of saree (cloths), the other goods made by brass, etc. In our model, the time-dependent development cost has been considered, so less production time implies less development cost when system reliability is fixed. Further, by motivating the customers to purchase product, the reliability can be regulated by regulating the production rate and also the development cost. The proposed model is further illustrated with numerical illustrations and sensitivity analysis to demonstrate the impact of changing the various parameters involved. Evaluating this work, we have the following conclusions:

(i) The product reliability has an influence to rise the consumer demand and hence to increase the profit.

(ii) In this model, it is taken in consideration that all yielding products are perfect up to time τ in the initial stage of production. Thus, we can conclude that the screening is not needed up to time τ for which total screening cost can be reduced.

Some possible future works are as follows: (i) to investigate the EPQ model with multi-item, where demand depends on different reliable product as well as selling price, (ii) and to consider the warranty period of the produced items and (iii) to introduce the demand depending on product reliability, stock of the product, and selling price and (iv) uncertainty in price.

This current model is limited with a single-level inventory model, which may be further directed toward an integrated inventory model. An inventory model that incorporates imperfect items may contribute to reducing carbon emissions in future work. In this model, we can add the advertisement effects of the product to the customer's demand.

Appendix

Here, the profit function $F(\lambda, r)$ is formed as a function of r and λ. Again r and λ are related implicitly by the equation $r = \phi(\lambda)$, so finally $F(\lambda)$ is formed by λ only. Therefore, by chain rule from differential calculus, we have $\frac{dF(\lambda)}{d\lambda} = \frac{\partial F(\lambda,r)}{\partial\lambda} + \frac{\partial F(\lambda,r)}{\partial r}\frac{dr}{d\lambda}$. Differentiating again with respect to λ, we obtain

$$
\begin{aligned}
\frac{d^2F}{d\lambda^2} &= \frac{d}{d\lambda}(\frac{\partial F}{\partial\lambda}) + \frac{d}{d\lambda}(\frac{\partial F}{\partial r}\frac{dr}{d\lambda}) = \frac{\partial}{\partial\lambda}(\frac{\partial F}{\partial\lambda}) + \frac{\partial}{\partial r}(\frac{\partial F}{\partial\lambda})\frac{dr}{d\lambda} + \frac{d}{d\lambda}(\frac{\partial F}{\partial r})\frac{dr}{d\lambda} \\
&\quad + \frac{\partial F}{\partial r}\frac{d}{d\lambda}(\frac{dr}{d\lambda}) \\
&= \frac{\partial^2 F}{\partial\lambda^2} + \frac{\partial^2 F}{\partial r\partial\lambda}\frac{dr}{d\lambda} + \{\frac{\partial^2 F}{\partial\lambda\partial r} + \frac{\partial^2 F}{\partial r^2}\frac{dr}{d\lambda}\}\frac{dr}{d\lambda} + \frac{\partial F}{\partial r}\frac{d^2r}{d\lambda^2} \\
&= \frac{\partial^2 F}{\partial\lambda^2} + 2\frac{\partial^2 F}{\partial r\partial\lambda}\frac{dr}{d\lambda} + \frac{\partial F}{\partial r}\frac{d^2r}{d\lambda^2} + \frac{\partial^2 F}{\partial r^2}(\frac{dr}{d\lambda})^2 \qquad (26.15)
\end{aligned}
$$

where the values of $\frac{\partial F}{\partial\lambda}, \frac{\partial F}{\partial r}, \frac{dr}{d\lambda}, \frac{\partial^2 F}{\partial\lambda^2}, \frac{\partial^2 F}{\partial\lambda\partial r}, \frac{d^2r}{d\lambda^2}, \frac{\partial^2 F}{\partial r^2}$ are given as follows:

$\frac{\partial F}{\partial\lambda} = \frac{1}{T}[Bk(1-x)^2(GED + \frac{1}{2}G^2D^2 + \frac{1}{2}E^2)\frac{(\lambda_{max}-\lambda_{min})}{(\lambda-\lambda_{min})^2}e^{k(\frac{\lambda_{max}-\lambda}{\lambda-\lambda_{min}})}]$

$$
\begin{aligned}
\frac{\partial F}{\partial r} &= \frac{1}{T}[-(E+GD)pM_1\mu e^{-\mu r} + \{\mu_5 - Gp(M_0 - M_1e^{-\mu r}) \\
&\quad - GEB(1-x)^2e^{k(\frac{\lambda_{max}-\lambda}{\lambda-\lambda_{min}})}\}d_0\frac{(\lambda_{max}-\lambda_{min})}{(\lambda-\lambda_{min})^2}
\end{aligned}
$$

$$+ 2D\{\mu_6 - \frac{1}{2}G^2B(1-x)^2 e^{k(\frac{\lambda_{max}-\lambda}{\lambda-\lambda_{min}})}\}d_0\frac{(\lambda_{max}-\lambda_{min})}{(\lambda-\lambda_{min})^2}] \quad (26.16)$$

Again from Eq. (26.14) differentiating both sides with respect to λ, we obtain

$$\frac{dD}{d\lambda} = \frac{\frac{p}{\lambda^2} - \frac{p}{\lambda}\{\frac{1}{\lambda} + (E+GD)(1-x)\}e^{-\lambda(1-x)(E+GD)}}{pG(1-x)e^{-\lambda(1-x)(E+GD)} - T + pxG} = \frac{U_0}{V_0}, say \quad (26.17)$$

Again differentiating with respect to λ

$$\frac{d^2D}{d\lambda^2} = \frac{V_0\frac{dU_0}{d\lambda} - U_0\frac{dV_0}{d\lambda}}{V_0^2} \quad (26.18)$$

where $\frac{dU_0}{d\lambda}$ and $\frac{dV_0}{d\lambda}$ are as follows:

$$\frac{dU_0}{d\lambda} = -\frac{2p}{\lambda^3} + [\frac{2p}{\lambda^3} + \frac{p}{\lambda^2}(1-x)(E+GD) - \frac{p}{\lambda}(1-x)G\frac{dD}{d\lambda}$$
$$+ (1-x)(E+GD+\lambda G\frac{dD}{d\lambda})\{\frac{p}{\lambda^2} + \frac{p}{\lambda}(1-x)(E+GD)\}]e^{-\lambda(1-x)(E+GD)}$$
$$\frac{dV_0}{d\lambda} = -pG(1-x)^2(E+GD+\lambda G\frac{dD}{d\lambda})e^{-\lambda(1-x)(E+GD)} \quad (26.19)$$

Again differentiating $D = \{d_0\frac{(r-r_{min})}{(r_{max}-r)} + d_1 s^{-b} + d_2 v\}$ with respect to λ, we obtain

$$\frac{dD}{d\lambda} = \frac{d}{d\lambda}\{d_0\frac{(r-r_{min})}{(r_{max}-r)} + d_1 s^{-b} + d_2 v\} = d_0\frac{(r_{max}-r_{min})}{(r_{max}-r)^2}\frac{dr}{d\lambda} \quad (26.20)$$

So

$$\frac{dr}{d\lambda} = \frac{(r_{max}-r)^2}{d_0(r_{max}-r_{min})}\frac{dD}{d\lambda} \quad (26.21)$$

Again differentiating equation (27) with respect to λ we obtain

$$\frac{d^2r}{d\lambda^2} = \frac{1}{d_0(r_{max}-r_{min})}\{(r_{max}-r)^2\frac{d^2D}{d\lambda^2} - 2\frac{(r_{max}-r)^3}{d_0(r_{max}-r_{min})}(\frac{dD}{d\lambda})^2\} \quad (26.22)$$

$$\frac{\partial^2 F}{\partial\lambda^2} = -Bk(1-x)^2(GED + \frac{1}{2}G^2D^2 + \frac{1}{2}E^2)\{k\frac{(\lambda_{max}-\lambda_{min})^2}{(\lambda-\lambda_{min})^4}$$

$$+ 2\frac{(\lambda_{max} - \lambda_{min})}{(\lambda - \lambda_{min})^3}\}e^{k(\frac{\lambda_{max}-\lambda}{\lambda-\lambda_{min}})}$$

$$\frac{\partial^2 F}{\partial\lambda\partial r} = BkG(1-x)^2\frac{(\lambda_{max} - \lambda_{min})}{(\lambda - \lambda_{min})}d_0\frac{(r_{max} - r_{min})}{(r_{max} - r)^2}(E + GD)e^{k(\frac{\lambda_{max}-\lambda}{\lambda-\lambda_{min}})}$$

$$\frac{\partial^2 F}{\partial r^2} = DGpM_1\mu^2 e^{-\mu r} - 2GpM_1\mu e^{-\mu r}\frac{dD}{dr} + \{\mu_5 - Gp(M_0 - M_1 e^{-\mu r})$$
$$- GEB(1-x)^2 e^{k(\frac{\lambda_{max}-\lambda}{\lambda-\lambda_{min}})}\}\frac{d^2D}{dr^2} + 2\{\mu_6 - \frac{1}{2}G^2B(1-x)^2 e^{k(\frac{\lambda_{max}-\lambda}{\lambda-\lambda_{min}})}\}$$
$$\times\{(\frac{dD}{dr})^2 + \frac{d^2D}{dr^2}\} \quad (26.23)$$

where $\frac{dD}{dr} = d_0\frac{(r_{max}-r_{min})}{(r_{max}-r)^2}$ and $\frac{d^2D}{dr^2} = 2d_0\frac{(r_{max}-r_{min})}{(r_{max}-r)^3}$

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Chapter 27
A Fuzzy EOQ Model with Exponential Demand and Deterioration with Preservation Technology

Ganesh Kumar and Sunita

1 Introduction

A deterministic inventory system was developed considering dual storage house allowing different item depreciation levels in warehouses by Bhunia and Maiti [2]. Kar et al. [14] devised an inventory model for an individual entity comprising two distinct storage facilities and an arithmetic progression of the lengths of the subsequent replenishment cycles. An inventory system for ameliorating commodities demonstrated for the prescribed period by Mondal et al. [23]. When dealing with a supplier's trade credit plans and price promotions on purchasing merchandise, a strategy for increased revenue was proposed by Sana and Chaudhuri [31] for retailers and inventory system was devised for a product whose units were not in pristine condition by selling through two distinct outlets: a primary shop and a secondary shop. A system of unified management was developed by Singh et al. [37]. A study was conducted by Sana [30] on a demand that is impacted by price and a degradation rate that changes with time.

An approach to managing an inventory system for goods that deteriorate over time, taking into account the possibility of controlling the degradation process through preservation methods conducted by Mishra [22]. A mathematical model was devised by Mishra [21] to analyze a unique inventory system for goods that are simultaneously deteriorating, with a focus on the idea that the degradation process can be controlled through the implementation of preservation methods. This model takes into consideration both the constant demand rate and a deterioration rate that changes over time, utilizing the Weibull distribution to come to its conclusions. "Preserving Perishables: Unraveling the Secrets of Seasonal Product Longevity" is a study that delves into the potential of industries to elongate the lifespan of

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C. Jana et al. (eds.), *Fuzzy Optimization, Decision-making and Operations Research*, https://doi.org/10.1007/978-3-031-35668-1_27

seasonal products through strategic preservation investments by He and Huang [8]. The impact of investing in preservation techniques on inventory decisions was broadly described by Dye [6] when they took into account an inventory system with a non-instantaneously degrading commodity. Zhang et al. [43] unveiled the mystery of striking a balance between pricing and inventory management for products in decline, an intricate challenge that was once thought to be insurmountable. The examination of consumers' reactions to expiration dates for perishable items in grocery stores was conducted by Tayal et al. [39]. Mishra [20] crafted a strategy for determining the optimal replenishment cycle and total expenditure, with preservation technology cost as a defining factor in the convex cost function. An innovative solution was proposed by Singh et al. [36], where merchants invest in preservation technologies to decelerate the depreciation of goods that gradually lose value. This inventory policy ensures that demand for these goods is based on available stock levels, resulting in a balanced approach to inventory management. Shah et al. [33] crafted a narrative that delves into the retailer's calculated decision to fortify their commodities by investing in preservation techniques, leading to a slowing down of their degradation rate. Roy et al. [29] developed a strategic framework for managing stock of perishable goods, incorporating the likelihood of demand and the rate of product decay to prevent stock-outs and optimize inventory. Pervin et al. [25] created a revolutionary supplier-customer model for slowly losing value goods, a novel approach to attract more patrons.

Ullah et al. [40] developed a sophisticated interplay of production and inventory management to mimic a dual-layer supply network, where the decline of goods was meticulously regulated by chance. Maity et al. [16] created an innovative approach to tackle the challenge of managing the inventory of products that lose value over time and a complex, nonlinear demand pattern by creating a system that incorporated the use of nonlinear heptagonal fuzzy demand. Chakraborty et al. [3] described hexagonal fuzzy numbers, which are useful in fuzzy inventory systems. De et al. [5] put forth a forward-thinking production system for perishable goods that takes into account the realities of delayed payments and prioritizes environmentally conscious carbon emission practices. Maity et al. [17] implemented a cutting-edge inventory model that leverages cloud-based intuitionistic demand to manage backlogged commodities with maximum efficiency. The innovative approach takes into consideration the delicate interplay of supply, demand, and depreciation. Maity et al. [19] illustrated a unique and innovative EOQ model, incorporating parabolic demand amidst a hazy and uncertain atmosphere. Maity et al. [18] unraveled a fascinating EOQ approach in a nebulous atmosphere, examining the inventory management of wavering commodities that were facing declining consumer demand. Jaggi et al. [12] created a novel dual-warehouse solution for perishable items facing backlogs, incorporating flexible credit terms and adjusting for economic inflation. Taleizadeh et al. [38] proposed a cost-effective solution for inventory management in the presence of time and credit-period-sensitive demand for items with imperfections. Shah et al. [32] explored and evaluated the intricate workings of the supply chain mechanism to understand its impact on inventory decisions. Pervin et al. [27] unveiled a novel strategy of mitigating product depreciation by incorporating preservation techniques within the vendor-buyer system. This innovative approach aimed to strike a balance between controlling the pace of product degradation and meeting customer demand in a production environment with perishable components. Pervin et al. [26] adopted preservation technology. Jaggi and Singh [11] optimized resource allocation by devising a strategic plan for efficient management of declining resources in a disaster response scenario. Iqbal and Sarkar [10] developed a revolutionary solution to ensure the longevity of goods by incorporating preservatives, addressing public health concerns with a model that promises an endless shelf life.

Akbar et al. [34] examined the intricate interplay of inflation, production reliability, price dependence, and supply network degradation in devising a comprehensive strategy for inventory management of perishing commodities with a partial trade credit plan. Rana et al. [28] made efforts to reduce the emission in their model. Barman et al. [1] made a unique production system for goods that were facing declining demand and backlogs, taking into account the effect of time on consumer requests within a hazy and uncertain environment. Zadeh [42] depicted the problems in real-world situations due to unclear data and the use of fuzzy sets in data management and problem-solving. Further, expanded use of fuzzy theory has been incorporated into various areas of mathematics, with inventory management being one such field. Chen and Hsieh [4] proposed two streamlined fuzzy inventory models utilizing generalized trapezoidal fuzzy numbers to illustrate fuzzy parameters and variables. Hsieh [9] proposed the utilization of either crisp or fuzzy quantities in inventory management systems and relying on the cost of each unit and ordering cycle for optimization considered by Wang et al. [41]. Utilizing two innovative methods that incorporate either crisp or fuzzy production quantities as fuzzy parameters, thoroughly evaluating the cost of each unit quantity and the order cost of each cycle, Mahata and Goswami [15] used trapezoidal and triangular fuzzy numbers. Shekarian et al. [35] used and described a sample of 210 papers to discover similar model traits. Fathalizadeh et al. [7] presented two distinct inventory strategies for products undergoing depreciation, featuring constant demand and steady degradation rates, in order to determine the most cost-effective ordering plan in the context of inflation and partial backlog using two different modeling techniques. Kalaiarasi and Gopinath [13] sought to outline the EOQ model and its underlying stochastic processes, streamlining operations and impacting inventory costs positively. Nagamani [24] designed a novel dual-warehouse EOQ framework, encompassing uncertain and faulty merchandise within a fuzzy atmosphere.

The following highlights the innovative features of the current study:

- This contribution involves constant and instantaneous deterioration in which demand is taken exponentially, and shortages are not allowed with costs as trapezoidal fuzzy numbers.
- Both in crisp and fuzzy scenarios, preservation technology is employed to mitigate the degradation.
- To determine the system's least overall cost in both a crisp and fuzzy sense, mathematical formulation is created and solved using differentiation tools.

- In order to explicate and validate system in both scenarios, there are numerical examples.
- Changing one parameter while leaving the others at their default levels allows for managerial insights and parameter observations.
- In two scenarios, the impact of changing the input parameters on the overall cost, the economic order quantity, and the optimal time is investigated.

Different steps follow the construction of this chapter: The preliminaries related to this system are offered in Sect. 2. Section 3 specifies the notations and assumptions. Section 4 outlines the model's mathematical formulation in both a crisp and fuzzy perspective. We have demonstrated the solution procedure in Sect. 5. A numerical example of this inventory system in both a crisp and fuzzy circumstances is performed in Sect. 6. Section 7 presents a sensitivity investigation of the developed model by assigning values to the parameters. Graphical representation is portrayed in Sect. 8. Observations and managerial insights are reported in Sect. 9. The paper's conclusion is discussed in Sect. 10.

2 Preliminaries

2.1 Fuzzy Set

The elements of a specified universal set X, a crisp set, are mapped into the real numbers in the range [0, 1] by each membership function of a fuzzy set $\tilde{B}$.

So $\mu_{\tilde{B}}(x) : X \to [0, 1]$ is defined as the membership value of $x \in X$.

Hence, we can defined a fuzzy set $\tilde{B}$ in the following manner:

$\tilde{B} = \{(x, \mu_{\tilde{B}}(x) : x \in X\}$

2.2 Fuzzy Number

If the membership function of a fuzzy set $\tilde{B}$, specified on set of all real numbers $\mathbb{R}$, meets the criteria listed below, the set is said to be fuzzy number.

1. $\mu_{\tilde{B}} : \mathbb{R} \to [0, 1]$ is continuous.
2. $\mu_{\tilde{B}}(x) = 0$ for all $x \in (-\infty, a] \cup [d, \infty)$
3. $\mu_{\tilde{B}}(x)$ strictly increasing on $[a, b]$ and strictly decreasing on $[c, d]$.
4. $\mu_{\tilde{B}}(x) = 1$ for all $x \in [b, c]$, where $a < b < c < d$.

2.3 α-cut of a Fuzzy Number

The α-cut, $\alpha \in (0, 1]$ of a fuzzy number $\tilde{B}$ is a crisp set designated as $B(\alpha) = \{x \in \mathbb{R} : B(x) \geq \alpha\}$. Every B_α is a closed interval of the form $[B_L(\alpha), B_U(\alpha)]$.

2.4 Trapezoidal Fuzzy Number

A trapezoidal fuzzy number denoted by $\tilde{B}$ is defined as (l, m, n, u) where the membership function is given by $\mu_{\tilde{B}}(x) = \begin{cases} 0, & x \leq l \\ \frac{x-l}{m-l}, & l \leq x \leq m \\ 1, & m \leq x \leq n \\ \frac{u-x}{u-n}, & n \leq x \leq u \\ 0, & x \geq u \end{cases}$

Figures 27.1 and 27.2 are geometrical forms of α-cut and trapezoidal fuzzy numbers.

2.5 Defuzzification Method

The value of a fuzzy number $\tilde{B}$ is denoted and defined as
$val(B) = \int_0^1 \alpha \{B_U(\alpha) + B_L(\alpha)\} \, d\alpha$

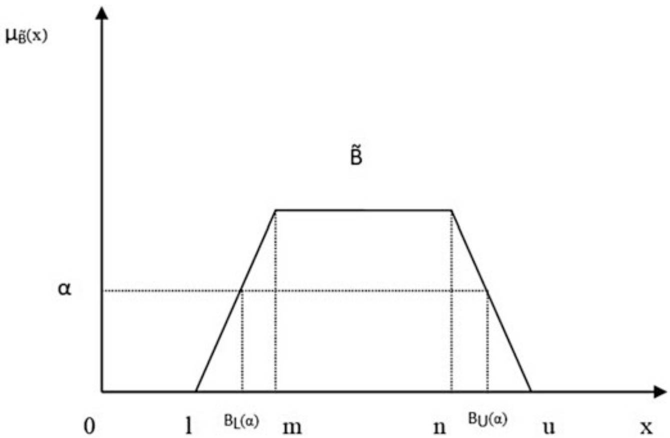

Fig. 27.1 α-cut of a trapezoidal fuzzy number

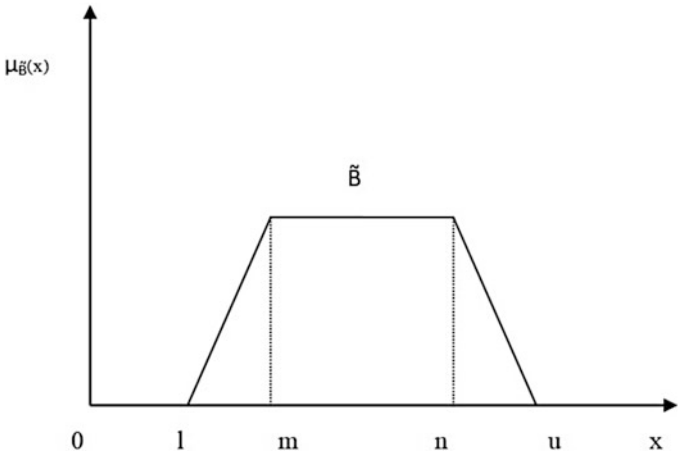

Fig. 27.2 Trapezoidal fuzzy number

Proposition 1 *The value of a trapezoidal fuzzy number* $\tilde{B} = (a, b, c, d)$ *is demonstrated by* $val(B) = \frac{a}{6} + \frac{b}{3} + \frac{c}{3} + \frac{d}{6}$.

Proof The α-cut of the trapezoidal fuzzy number $\tilde{B} = (a, b, c, d)$ is stated by

$$B(\alpha) = [B_L(\alpha), B_U(\alpha)] = [a + \alpha(b - a), d - \alpha(d - c)].$$

Then value of the fuzzy number $\tilde{B} = (a, b, c, d)$ is denoted and specified by

$$\begin{aligned} val(B) &= \int_0^1 \alpha \{B_U(\alpha) + B_L(\alpha)\} d\alpha \\ &= \int_0^1 [a + \alpha(b - a), d - \alpha(d - c)] d\alpha \\ &= \frac{a}{6} + \frac{b}{3} + \frac{c}{3} + \frac{d}{6} \\ &= \frac{a + 2b + 2c + d}{6}. \end{aligned}$$

3 Assumptions and Notations

To explain this concept, we have shown the following presumptions and notations.

Table 27.1 Notations

Notation	Description
θ	Deterioration rate, where $0 < \theta < 1$
T	Length of cycle time (years)
Q	Economic order quantity (units)
h	Holding cost ($/unit/unit time)
$\tilde{h}$	Fuzzy holding cost ($/unit/unit time)
A	Setup cost ($/order)
$\tilde{A}$	Fuzzy setup cost ($/order)
x	Preservation technology investment per unit time to reduce the deterioration rate
$\tilde{x}$	Fuzzy preservation technology investment per unit time to reduce the deterioration rate
q	Unit transportation cost of a shipment from supplier to the retailer($/unit/unit time)
$\tilde{q}$	Fuzzy unit transportation cost of a shipment from supplier to the retailer($/unit/unit time)
$I(t)$	Inventory level with respect to time t
TC	Total cost ($/cycle)
$\widetilde{TC}$	Fuzzy total cost ($/cycle)

3.1 Notations

Table 27.1 in the current model employs the following notations.

3.2 Assumptions

1. Time influences the demand rate and expressed as $D(t) = ae^{bt}$ i.e. demand is exponential.
2. The proportion of diminished depreciation rate is represented by the function $f(x) = 1 - e^{-x}$, where $0 \leq f(x) \leq 1$, if $x \geq 0$.
3. Shortages are not allowed.
4. Deterioration is instantaneous.
5. Planning horizon is infinite.
6. Lead time is ignorable.

4 Formulation of Mathematical Model

Our inventory control system is set up so that at time $t = 0$, the initial inventory level is Q. The stock supply gradually depletes as a consequence of the ongoing rise in demand and level of degradation. The stock level drops to zero, and the cycle starts afresh once time $t = T$ has passed. We have displayed our inventory system in Fig. 27.3.

4.1 Crisp Model

To demonstrate this inventory system, the governing differential equation are as follows:

$$\frac{dI_1(t)}{dt} + \{\theta - f(x)\}I(t) = -D(t)$$

$$\Longrightarrow \frac{dI_1(t)}{dt} + \{\theta - f(x)\}I(t) = -ae^{bt},\ 0 \le t \le T. \tag{27.1}$$

Coupled with boundary conditions,

$$I(0) = Q \quad and \quad I(T) = 0 \tag{27.2}$$

The boundary conditions (27.1) are used to effectively resolve the differential equation (27.2) in the manner shown below:

$$I(t) = \frac{a}{b + \theta - f(x)} \left[e^{bt} - e^{bT} e^{(\theta - f(x))(T-t)} \right]. \tag{27.3}$$

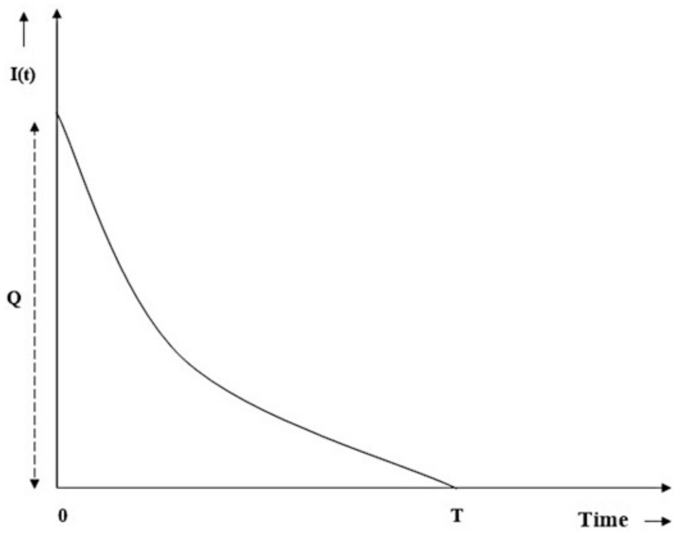

Fig. 27.3 Visualization of present model

Maximum inventory is determined by utilizing the boundary condition $I(0) = Q$:

$$Q = \frac{a}{b + \theta - f(x)} \left[1 - e^{(b+\theta-f(x))T}\right]. \tag{27.4}$$

Annual holding cost is given by

$$\begin{aligned} HC &= h\left[\int_0^T I(t)dt\right] \\ \implies HC &= \frac{ha}{b+\theta-f(x)}\left[\frac{e^{bt}}{b} + \frac{e^{bT}}{\theta - f(x)}\left\{1 - e^{(\theta-f(x))T}\right\}\right]. \end{aligned} \tag{27.5}$$

Annual preservation technology cost is given by

$$PC = \frac{xaT}{b + \theta - f(x)} \left[1 - e^{(b+\theta-f(x))T}\right]. \tag{27.6}$$

Annual transportation cost is given by

$$TrC = \frac{qa}{b + \theta - f(x)} \left[1 - e^{(b+\theta-f(x))T}\right]. \tag{27.7}$$

Annual setup cost is given by

$$SC = A \tag{27.8}$$

Total cost per cycle,

$$\begin{aligned} TC &= \frac{1}{T}[SC + HC + PC + TrC] \\ \implies TC &= \frac{A}{T} + \frac{ha}{T\{b+\theta-f(x)\}}\left[\frac{e^{bt}}{b} + \frac{e^{bT}}{\theta - f(x)}\left\{1 - e^{(\theta-f(x))T}\right\}\right] \\ &+ \frac{xa}{\{b+\theta-f(x)\}}\left[1 - e^{(b+\theta-f(x))T}\right] \\ &+ \frac{qa}{T\{b+\theta-f(x)\}}\left[1 - e^{(b+\theta-f(x))T}\right]. \end{aligned} \tag{27.9}$$

4.2 *Fuzzy Model*

To face the uncertainty in inventory problems, consider the costs h, A, q, and x as trapezoidal fuzzy numbers. So taking

$\tilde{h} = (h_1, h_2, h_3, h_4)$,

$\tilde{A} = (A_1, A_2, A_3, A_4),$
$\tilde{q} = (q_1, q_2, q_3, q_4),$
$\tilde{x} = (x_1, x_2, x_3, x_4).$

The total cost function becomes

$$\widetilde{TC} = \frac{\tilde{A}}{T} + \frac{\tilde{h}a}{T\{b+\theta-f(\tilde{x})\}}\left[\frac{e^{bt}}{b} + \frac{e^{bT}}{\theta - f(\tilde{x})}\left\{1 - e^{(\theta - f(\tilde{x}))T}\right\}\right]$$
$$+ \frac{\tilde{x}a}{\{b+\theta-f(\tilde{x})\}}\left[1 - e^{(b+\theta-f(\tilde{x}))T}\right]$$
$$+ \frac{\tilde{q}a}{T\{b+\theta-f(\tilde{x})\}}\left[1 - e^{(b+\theta-f(\tilde{x}))T}\right].$$

where $\widetilde{TC} = (TC_1, TC_2, TC_3, TC_4)$ and $R(\widetilde{TC}) = \frac{1}{6}(TC_1 + 2TC_2 + 2TC_3 + TC_4)$, where

$$TC_1 = \frac{A_1}{T} + \frac{h_1 a}{T\{b+\theta-f(x_1)\}}\left[\frac{e^{bt}}{b} + \frac{e^{bT}}{\theta - f(x_1)}\left\{1 - e^{(\theta - f(x_1))T}\right\}\right]$$
$$+ \frac{x_1 a}{\{b+\theta-f(x_1)\}}\left[1 - e^{(b+\theta-f(x_1))T}\right]$$
$$+ \frac{q_1 a}{T\{b+\theta-f(x_1)\}}\left[1 - e^{(b+\theta-f(x_1))T}\right]$$

$$TC_2 = \frac{A_2}{T} + \frac{h_2 a}{T\{b+\theta-f(x_2)\}}\left[\frac{e^{bt}}{b} + \frac{e^{bT}}{\theta - f(x_2)}\left\{1 - e^{(\theta - f(x_2))T}\right\}\right]$$
$$+ \frac{x_2 a}{\{b+\theta-f(x_2)\}}\left[1 - e^{(b+\theta-f(x_2))T}\right]$$
$$+ \frac{q_2 a}{T\{b+\theta-f(x_2)\}}\left[1 - e^{(b+\theta-f(x_2))T}\right]$$

$$TC_3 = \frac{A_3}{T} + \frac{h_3 a}{T\{b+\theta-f(x_3)\}}\left[\frac{e^{bt}}{b} + \frac{e^{bT}}{\theta - f(x_3)}\left\{1 - e^{(\theta - f(x_3))T}\right\}\right]$$
$$+ \frac{x_3 a}{\{b+\theta-f(x_3)\}}\left[1 - e^{(b+\theta-f(x_3))T}\right]$$
$$+ \frac{q_3 a}{T\{b+\theta-f(x_3)\}}\left[1 - e^{(b+\theta-f(x_3))T}\right]$$

$$TC_4 = \frac{A_4}{T} + \frac{h_4 a}{T\{b+\theta-f(x_4)\}}\left[\frac{e^{bt}}{b} + \frac{e^{bT}}{\theta - f(x_4)}\left\{1 - e^{(\theta - f(x_4))T}\right\}\right]$$
$$+ \frac{x_4 a}{\{b+\theta-f(x_4)\}}\left[1 - e^{(b+\theta-f(x_4))T}\right]$$
$$+ \frac{q_4 a}{T\{b+\theta-f(x_4)\}}\left[1 - e^{(b+\theta-f(x_4))T}\right]$$

5 Solution Procedure

To find the best answer to the issue (27.9), we'll follow the steps below:

Step-1: Initialize the total cost function by entering the parameter values.
Step-2: Determine the total cost function's first- and second-order partial derivatives in relation to cycle length T.
Step-3: With the aid of MATLAB, we obtain the stationary value of T by equating the first-order partial derivative of the total cost function relative to T equals zero.
Step-4: With the support of MATLAB, we obtain the second-order partial derivative of the total cost function in relation to T greater than zero.
Step-5: Using optimal value T^* in $TC(T)$ function, we get the minimum cost.

6 Numerical Examples

6.1 Numerical Example in Crisp Environment

Let us consider $A = 1000$, $h = 6$, $q = 3$, $x = 6$, $T = 3$, $a = 5$, $b = 0.5$, and $\theta = 0.5$. Then we get the following optimum values with the help of the MATLAB software: $t^* = 0.2301$, $Q^* = 1551.9$, and $TC^* = 14827$.

6.2 Numerical Example in Fuzzy Environment

(i) Let $A_1 = 600$, $h_1 = 3$, $q_1 = 1$, $x_1 = 4$, $T = 3$, $a = 5$, $b = 0.5$, and $\theta = 0.5$. Then we get the following optimum values with the help of the MATLAB software: $t^* = 0.6999$, $Q^* = 78.3961$, and $TC^* = 2109.9$.
(ii) Let $A_2 = 800$, $h_2 = 5$, $q_2 = 2$, $x_2 = 5$, $T = 3$, $a = 5$, $b = 0.5$, and $\theta = 0.5$. Then we get the following optimum values with the help of the MATLAB software: $t^* = 0.3668$, $Q^* = 468.0031$, and $TC^* = 5877.6$.

(iii) Let $A_3 = 1000$, $h_3 = 7$, $q_3 = 3$, $x_3 = 6$, $T = 3$, $a = 5$, $b = 0.5$, and $\theta = 0.5$. Then we get the following optimum values with the help of the MATLAB software: $t^* = 0.2131$, $Q^* = 1586.3$, and $TC^* = 15414$.

(iv) Let $A_4 = 1200$, $h_4 = 9$, $q_4 = 4$, $x_4 = 7$, $T = 3$, $a = 5$, $b = 0.5$, and $\theta = 0.5$. Then we get the following optimum values with the help of the MATLAB software: $t^* = 0.1268$, $Q^* = 4787.2$, and $TC^* = 40779$.

So we have,
$\tilde{A} = (600, 800, 1000, 1200)$, $\tilde{h} = (3, 5, 7, 9)$, $\tilde{q} = (1, 2, 3, 4)$, $\tilde{x} = (4, 5, 6, 7)$, $T = 3$, $a = 5$, $b = 0.5$, and $\theta = 0.5$. Then we get the following optimum values using MATLAB software: $t^* = 0.3311$, $Q^* = 1495.7$, and $TC^* = 14245$.

7 Sensitivity Analysis

Each and every parameter of the inventory is subjected to the sensitivity analysis. One approach to assessing the sensitivity of inventory parameters entails altering one parameter by -20%, -10%, 0%, $+10\%$, and $+20\%$ while maintaining the same values for the other variables.

7.1 Case: I Sensitivity Analysis in Crisp Environment

7.2 Case: II Sensitivity Analysis in Fuzzy Environment

8 Graphical Representations

9 Observations and Managerial Insights

9.1 Observations

We have observed the following situations by studying the above tables and with the help of the above figures:

1. When the value of the initial demand parameter a is changed from -20 to $+20$ percentage, Table 27.2, Figs. 27.4 and 27.5 show that the optimal time decreases, the lot size increases and the overall cost rises for the crisp example, which leads us to propose reducing the value of the parameter a to reduce costs and increase revenue simultaneously.
2. An organization should avoid adjusting the value of parameter b in this range (-20 to $+20$ percentage of the original value) because no closed-form determination can be established for it. The same holds for our attempt to find a monotonic

Table 27.2 Change in t, Q, and TC w.r.t. input parameters

Parameter		Percentage change				
Input parameters	Output parameters	−20%	−10%	0%	+10%	+20%
a	t	0.2585	0.2431	0.2301	0.2189	0.2092
	Q	1195.5	1373	1551.9	1731.9	1912.8
	TC	12680	13767	14827	15864	16882
b	t	3	3	0.2301	1.0709	1.3318
	Q	88.2480	195.7788	1551.9	−12.2481	−23.2022
	TC	−2058.5	−5000.3	14827	1969.1	1331.9
θ	t	3	3	0.2301	1.0107	1.2233
	Q	88.2480	195.7788	1551.9	−6.2081	−17.3376
	TC	−2071.7	−5012.4	14827	2078.8	1475.9

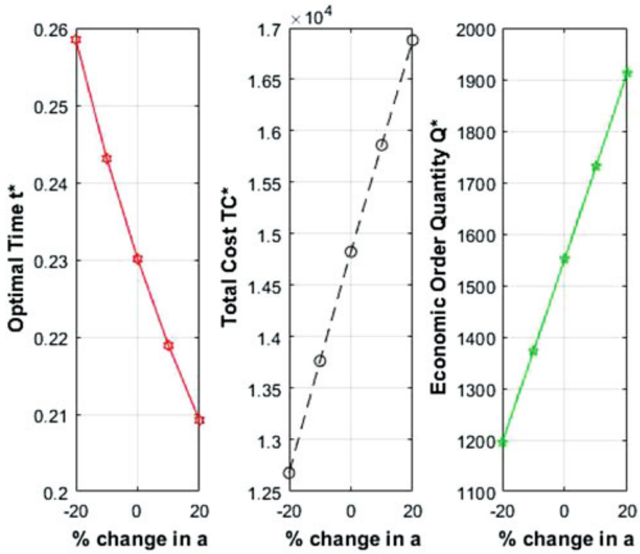

Fig. 27.4 Change in total cost, optimal time, and EOQ related to parameter a in crisp case

representation of the data for the parameter θ. It is only sometimes possible to locate the parameters in the crisp form in real-world situations (Fig. 27.6).

3. As a result of some ambiguity and imprecise information, we require a new theory of fuzziness to accommodate these factors. Table 27.3 and Figs. 27.7, 27.8, and 27.9 display the results of the fuzzy inventory model. The graph of output parameter variation versus input parameter value is similar to that of the crisp model. However, the fuzzy inventory model reduces the overall system cost. The crisp inventory model has been considered less significant, and the fuzzy model is recommended as an alternative.

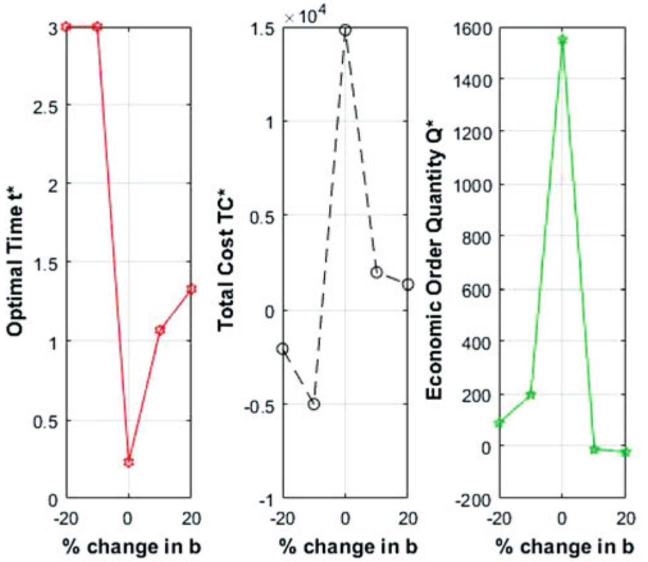

Fig. 27.5 Change in total cost, optimal time, and EOQ related to parameter b in crisp case

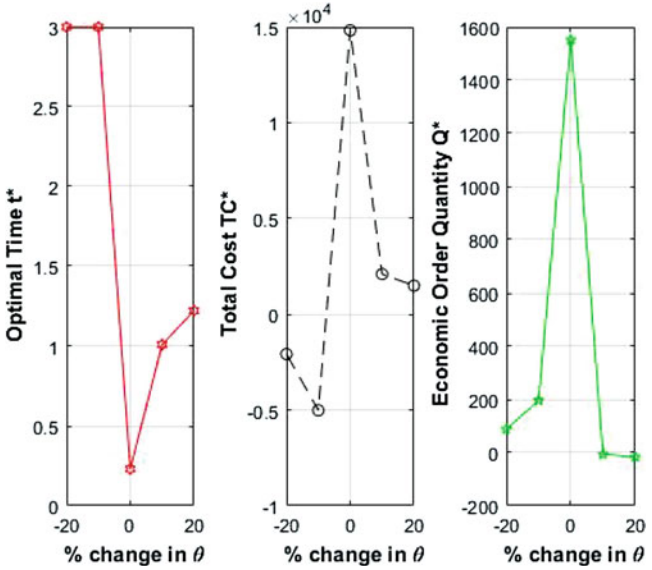

Fig. 27.6 Change in total cost, optimal time, and EOQ related to parameter θ in crisp case

Table 27.3 Change in t, Q, and TC w.r.t. input parameters in fuzzy case

Parameter		Percentage change				
Input parameters	Output parameters	−20%	−10%	0%	+10%	+20%
a	t	0.3734	0.3505	0.3311	0.3145	0.3000
	Q	1159	1326.8	1495.7	1665.5	1836.3
	TC	12152	13211	14245	15259	16255
b	t	3	3	0.3311	1.0439	1.2884
	Q	92.8253	219.0670	1495.7	−8.1166	−20.0972
	TC	−2329.4	−5735.4	14245	1833	1259.8
θ	t	3	3	0.3311	0.9872	1.1824
	Q	92.8253	219.0670	1495.7	−2.8058	−14.5476
	TC	−2340.7	−5744.9	14245	1929.6	1388.5

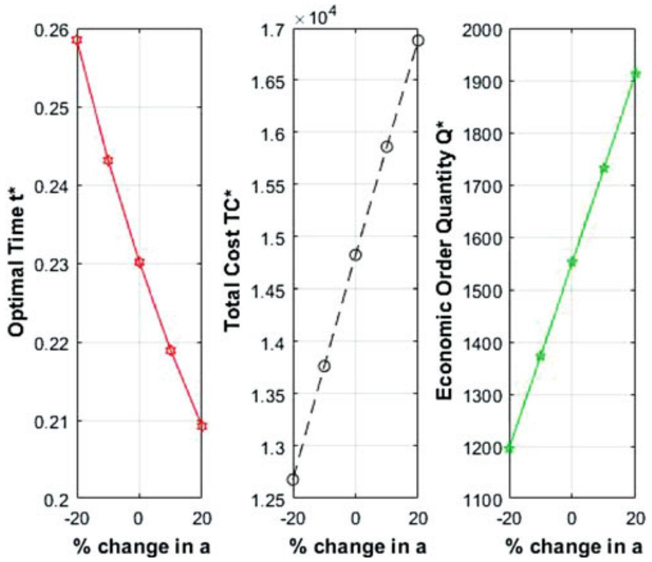

Fig. 27.7 Change in total cost, optimal time, and EOQ related to parameter a in fuzzy case

9.2 *Managerial Insights*

A business manager can apply the model using the following points:

1. This is a very simple and effective approach to reducing the total cost in the management of any inventory system that even a less qualified business manager can understand easily.
2. A business manager can lower the total cost by using the fuzzy approach compared to the crisp model.

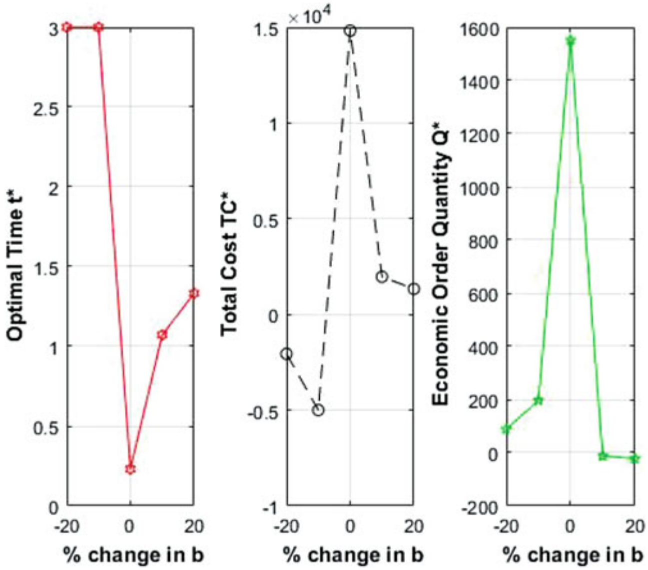

Fig. 27.8 Change in total cost, optimal time, and EOQ related to parameter *b* in fuzzy case

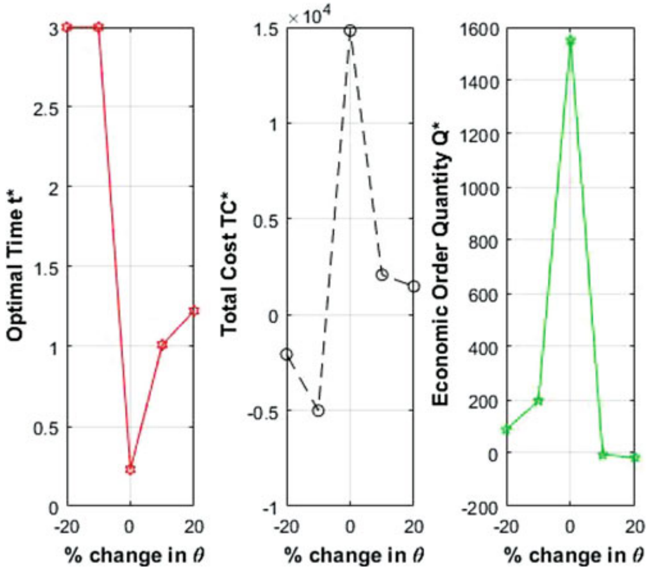

Fig. 27.9 Change in total cost, optimal time, and EOQ related to parameter θ in fuzzy case

3. In a fuzzy environment as opposed to crisp conditions, a business manager will order fewer units in order to reduce the overall cost while maintaining all the parameters at their best levels.
4. They can increase the cycle length by using our model in a fuzzy environment so that stock lasts longer than the usual model in a crisp sense.

9.3 Discussion

The fuzzy approach in inventory management is a simple and effective method to reduce the total cost, making it accessible and understandable even for less qualified business managers. This approach is known to lower the total cost when compared to a crisp model. The advantage of using a fuzzy approach lies in the fact that it allows business managers to order fewer units while still maintaining optimal parameters, ultimately reducing the overall cost. Additionally, by using the fuzzy approach, the cycle length can be increased, which allows the stock to last longer than it would in a crisp environment.

10 Conclusion

Stock management is the most crucial factor to consider in the business of perishable commodities, such as medications, eatables, and blood. It is essential to use preservation technologies to ensure they remain useful for an extended period. This article is a contribution that examines continuous and instantaneous degradation under the assumption that demand is exponential and that shortages are not allowed. By using preservation strategies, we can regulate the degradation process. In general, it is optional to determine the various expenses associated with the model precisely. Therefore, to estimate the outcomes, we utilized a fuzzy approach and then employed a method to defuzzify it to be more precise. A mathematical formulation is developed in both crisp and fuzzy cases and then solved with the help of a differentiation tool to determine the procedure that achieves the result at the lowest total cost. A numerical illustration is provided as a means of both explaining and justifying the model. Keeping one parameter in its original state while making changes to the others allows for generating managerial viewpoints and observations of the parameters. The findings from the observations in both the crisp and fuzzy cases suggest that we can reduce costs in an environment with more uncertainty. In addition, they can be developed further in neutrosophic and intuitionist cases.

Acknowledgments The authors express their gratitude toward the editor and reviewers for offering valuable insights and recommendations that have elevated the standard of this manuscript.

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Chapter 28
An EOQ Model with Price and Stock-Dependent Demand Including Trade Credit Using De-intuitification Technique Under Triangular Intuitionistic Fuzzy Environment

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1 Introduction

Impreciseness concept plays a crucial role in several fields of advance mathematical modeling. Recently, researchers from several sectors like E-commerce, social science, accounting, finance, medical etc., have linked the theory of ambiguity in their applicable arenas. The classical EOQ model considers constant demand with no shortages and deterioration. But as the time elapsed, the restriction on the above parameters is relaxed and the researcher tries to develop the model that fits with the real-life scenario.

Thus, in this chapter, we focused on various defuzzification techniques of triangular intuitionistic fuzzy number with application in inventory management. We focused on price and stock-dependent demand where shortages are backordered and there are two options of trade credit under inflation. It further used defuzzification techniques of triangular intuitionistic fuzzy number and optimized the inventory model.

1.1 Literature Review

In 1965, Professor Zadeh [1] first invented the general idea of fuzzy set (FS) and its mathematical notations. Since then, a large number of works have been established in this fuzzy zone research arena [2–6] and applied in various sectors

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C. Jana et al. (eds.), *Fuzzy Optimization, Decision-making and Operations Research*, https://doi.org/10.1007/978-3-031-35668-1_28

of technical fields. But fuzzy set mainly deals with the membership functions, whereas in many cases, we need to focus on both truth and false components of the uncertain number. In 1986, Atanassov [7] stretched and explained the perception of FS into intuitionistic FS (IFS) by introducing the idea of both truth and false components of a fuzzy uncertain parameter. After the invention of IFS, Zhang et al. [8] portrayed an idea of interval-valued IFS and discussed its utility in detail. Also, a few researchers focused on the structural development of IFS [9–11] to capture, tackle, and discover several complex features in various real-life problems. Further, several works based on arithmetic operations [12], assignment difficulty [13], and similarity measure [14] in interval-valued IFS were developed in research zone and people also utilized it to solve several decision-making problems. A few important numbers of arithmetic operator-based research work like aggregation operator [15], exponential operator [16] and important MCDM process has been established based on few crucial measures like similarity [17, 18], inclusion [19], entropy [20, 21], cross-entropy [22], and distance [23] in IFS field to resolve some useful problems in engineering domain. Also, researchers have applied the IFS in inventory management problem: Chakraborty et al. [24] utilized IFS in optimization skill for Pareto optimal solution of industrial inventory models having presence of shortages; De and Sana [25] used it in periodic production model; De et al. [26] incorporated the idea of time sensitivity backlogging in EOQ model; multiobjective stochastic inventory model has been portrayed by Banerjee and Roy [27]; backlogging EOQ model related with time-dependent effect has been manifested by Das et al. [28]; generalized IFS based inventory model highlighted by Garai et al. [29]; IFS-related backlogging EOQ model ignited by De and Sana [30]; fractional inventory model manifested by Ali et al. [31] and Kaur and Deb [32] focused on an IFS tactic IN A without shortage problem in an inventory model; Chakraborty et al. [33] focused on pentagonal intuitionistic number and applied it in inventory model; Maity et al. [34] incorporated the idea of fuzzy-related demand rate-based inventory model.

Sana and Chaudhuri [35] projected research on numerous natures of demand rates having trade credit policies along with the effect of price discounting. A deterministic model in inventory field having delay effect in payment and backorder price discount effect has been introduced by Pal and Chandra [36] and Jaggi et al. [37], respectively. Pal et al. [38, 39] worked on ramp type demand with and without shortages in their different papers under fuzzy condition. Chakraborty et al. [40] have developed Hexagonal fuzzy number with ranking and defuzzification technique with its application in inventory management; Maity et al. [41] extended the concept to nonlinear heptagonal dense fuzzy environment for a backlogging EOQ model; Maity et al. [42] also worked on EOQ model of discounted items under cloudy fuzzy demand rate; Mondal et al. [43] focused on EOQ model by introducing the advertisement effect; Mahapatra et al. [44] introduced time-dependent deterioration model under uncertainty; Bhuniya et al. [45] manifested supply chain model in uncertain environment; Sarkar et al. [46] focused on marketing-based inventory problem under trade credit policy; Kumar et al. [47] incorporated the logistic model under fuzzy arena by considering the carbon emission factor; Choi et al. [48] introduced online to offline supply chain model under demand variability; Gupta et al. [49] projected multiobjective supply chain model under intuitionistic arena; Maity et al. [50] established the backorder EOQ model under dense intuitionistic environment.

1.2 Motivation

The idea of uncertainty theory plays a useful role in creation of mathematical model, soft computing, engineering structural problems, medical diagnoses, etc. Generally, a few questions will arise in mind that if uncertainty (Specifically intuitionistic fuzzy scenario) is present in inventory model, then how can we tackle it? How can we convert an intuitionistic parameter into crisp one? How it affects the inventory model system? What is the difference between crisp results and intuitionistic fuzzy result? Researchers have already applied triangular fuzzy number in inventory model to different situations, but if the model parameter acts like triangular intuitionistic fuzzy type (it means both membership and non-membership functions are present), then how we shall tackle it? After, arising these questions in our mind we started to build up this article to find out the reliable solution of the system.

1.3 Novelty

In this chapter, we have mainly focused on price and stock-dependent demand where shortages are backordered. There are two options of trade credit with inflation under crisp and triangular IFN. Thus the main contribution of the paper is given below:

(i) We have introduced an innovative de-intuitification skill of triangular intuitionistic fuzzy number here.
(ii) We have developed the EOQ model with inflation under crisp and triangular intuitionistic fuzzy environment.
(iii) We have showed the reliability of this model by performing sensitivity analysis.
(iv) A comparative study has been made under both crisp and triangular intuitionistic fuzzy domain. Also, the result is analyzed with a numerical example.

1.4 Structure of the Chapter

The research chapter is structured as follows:

In Sect. 2, mathematical preliminaries have been addressed; while Sect. 3 shows de-intuitification technique of triangular IFN and its theoretical computation; Sect. 4 focuses on the application part of this chapter and developed the EOQ model under uncertain environment; in Sect. 5, we have performed the numerical simulations to understand the effect of the various parameters; in Sect. 6, we have performed the sensitivity analysis of the model and finally, in Sect. 7, we have drawn the conclusion part of this chapter.

2 Mathematical Preliminaries

Definition 28.1: Intuitionistic Fuzzy Set A set $\tilde{I}$ in the universal discourse X is said to be an intuitionistic fuzzy set if $\tilde{I} = \left\{\left\langle x; \left[\tau_{\tilde{I}}(x), \varphi_{\tilde{I}}(x)\right]\right\rangle \vdots x \in X\right\}$, where $\tau_{\tilde{I}}(x) : X \to [0, 1]$ is the truth membership function, $\varphi_{\tilde{I}}(x) : X \to [0, 1]$ is the falsity membership function, which satisfies the following relation:

$$0 \leq \tau_{\tilde{I}}(x) + \varphi_{\tilde{I}}(x) \leq 1.$$

Definition 28.2: Triangular Intuitionistic Fuzzy Number A triangular intuitionistic fuzzy number is defined as $\tilde{\boldsymbol{A}}_{\boldsymbol{I}} = (\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}; \boldsymbol{b_1}, \boldsymbol{b_2}, \boldsymbol{b_3})$ whose truth membership and falsity membership are defined as follows:

$$T_{\tilde{A}_I}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & a_1 \leq x < a_2 \\ 1 & x = a_2 \\ \frac{a_3-x}{a_3-a_2} & a_2 < x \leq a_3 \\ 0 & otherwise \end{cases} \quad ; \quad F_{\tilde{A}_I}(x) = \begin{cases} \frac{b_2-x}{b_2-b_1}, & b_1 \leq x < b_2 \\ 0 & x = b_2 \\ \frac{x-b_2}{b_3-b_2} & b_2 < x \leq b_3 \\ 1 & otherwise. \end{cases}$$

where $0 \leq T_{\tilde{A}_I}(x) + F_{\tilde{A}_I}(x) \leq 1, x \in \tilde{A}_I$

The parametric form of the above type number is

$$\left(\tilde{A}_I\right)_{\alpha,\beta} = [T_I(\alpha), T_I(\alpha); F_I(\beta), F_I(\beta)]$$

where

$T_I(\alpha) = a_1 + \alpha(a_2 - a_1)$
$T_I(\alpha) = a_3 - \alpha(a_3 - a_2)$
$F_I(\beta) = b_2 - \beta(b_2 - b_1)$
$F_I(\beta) = b_2 + \beta(b_3 - b_2)$

Here, $0 \leq \alpha \leq 1, 0 \leq \beta \leq 1$, and $0 \leq \alpha + \beta + \gamma \leq 1$

3 De-intuitification of Triangular Intuitionistic Fuzzy Number

3.1 De-intuitification Skill Using Removal Area Method

Let us consider a triangular intuitionistic fuzzy number as $\tilde{A}_{Ne}= (P_1, P_2, P_3; S_1, S_2, S_3)$ whose Pictorial representation of De-intuitification skill is shown as below:

We consider a number $s \in R$ and an uncertain number $\check{A}$ for the lower triangle of the intuitionistic number, then left side removal of $\check{A}$ with respect to k is $R_l\left(\check{A}, s\right)$,which one is called the left spade of $\check{A}$ with respect to lower part. Similarly, the right side removal of $\check{A}$ with respect to k is $R_r\left(\check{A}, s\right)$. Further, consider a number $s \in R$ and an uncertain number $\check{B}$ for the left most upper triangle, then left side removal of $\check{B}$ with respect to k is $R_l\left(\check{B}, s\right)$,which is called the left spade of $\check{A}$ with respect to upper part. Similarly, the right side removal of $\check{B}$ with respect to k is $R_r\left(\check{B}, s\right)$.

Mean is defined as $R\left(\check{A}, s\right) = \frac{R_l\left(\check{A},\ S\right)+R_r\left(\check{A},S\right)}{2}$, $R\left(\check{B}, s\right) = \frac{R_l\left(\check{B},S\right)+R_r\left(\check{B},S\right)}{2}$

Then, we defined the De-intuitification of triangular intuitionistic fuzzy number as follows:

$$R\left(\check{D}, s\right) = \frac{R\left(\check{A}, S\right) + R\left(\check{B}, S\right)}{2}$$

For $S = 0$, $R\left(\check{A}, 0\right) = \frac{R_l\left(\check{A},0\right)+R_r\left(\check{A},0\right)}{2}$, $R\left(\check{B}, 0\right) = \frac{R_l\left(\check{B},0\right)+R_r\left(\check{B},0\right)}{2}$

So, $R\left(\check{D}, 0\right) = \frac{R\left(\check{A},0\right)+R\left(\check{B},0\right)}{2}$

$$R_l\left(\check{A}, 0\right) = \text{Area of trapezium of Fig. 28.1} = \frac{(P_1 + S_1)}{2}.1$$

$$R_r\left(\check{A}, 0\right) = \text{Area of trapezium of Fig. 28.2} = \frac{(P_3 + S_2)}{2}..1$$

$$R_l\left(\check{B}, 0\right) = \text{Area of trapezium of Fig. 28.3} = \frac{(P_2 + S_1)}{2}.1$$

$$R_r\left(\check{B}, 0\right) = \text{Area of trapezium of Fig. 28.4} = \frac{(P_2 + S_3)}{2}.1$$

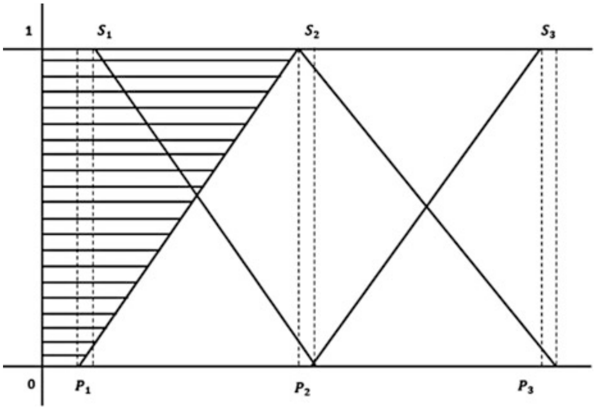

Fig. 28.1 Area Computation (Step-1)

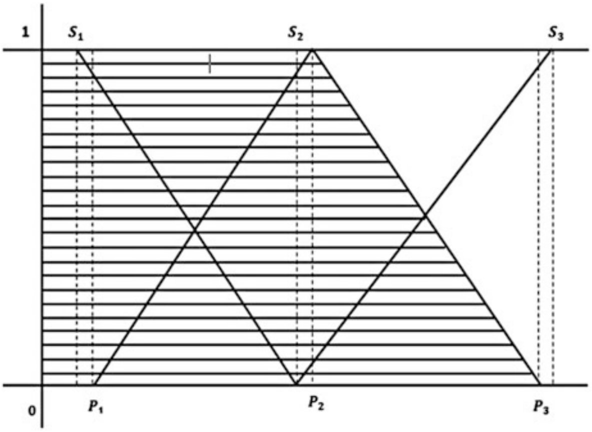

Fig. 28.2 Area Computation (Step-2)

Hence, $R\left(\check{A},0\right)=\frac{(a+2b+c)}{4}$, $R\left(\check{B},0\right)=\frac{(d+2e+f)}{4}$, $R\left(\check{C},0\right)=\frac{(g+2h+k)}{4}$

$$\text{So, } R\left(\check{D},0\right)=\frac{(P_1+2P_2+P_3+S_1+2S_2+S_3)}{8} \tag{28.1}$$

4 Application of Triangular Intuitionistic Fuzzy Number in EOQ Model

In this chapter, we formulate an EOQ model for deteriorating items with price and stock-dependent demand under the effect of shortage, inflation, and delay in

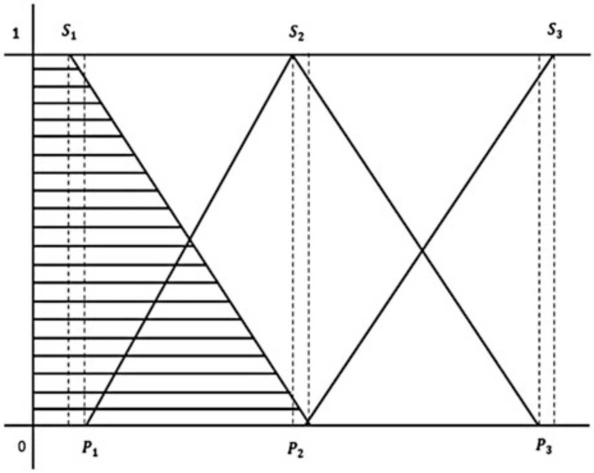

Fig. 28.3 Area Computation (Step-3)

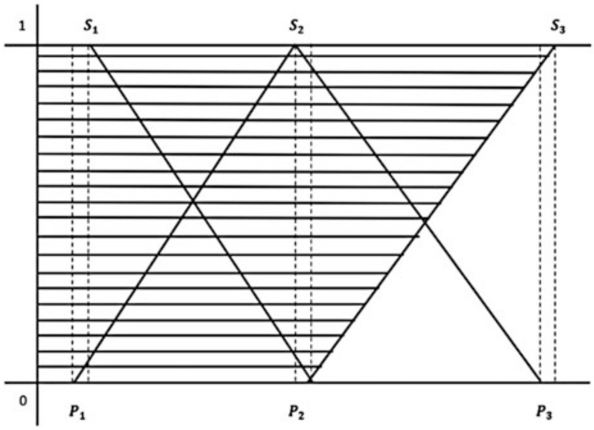

Fig. 28.4 Area Computation (Step-4)

payment. Here we also observe the effect of triangular intuitionistic fuzzy number in the model and observe a comparative study under crisp and intuitionistic fuzzy environment. In this case study, the effect of trade credit is considered and two cases arise under these circumstances. Either, the supplier collects the money before the inventory ends, in other words, credit period end before or at the end of the cycle time, or, the supplier collects the money after the inventory ends. Finally, the total cost is minimized.

4.1 Notations

The following notations are used to represent the variables in this chapter:

D(Q(t), s)	Demand rate at time t
Q(t)	Inventory amount at time t
s	Selling price of the stock
γ	Constant rate of deterioration
k	Inflation rate
t_1	Time when the stock finishes
T	Replenishment time
c_1	Holding cost per unit item
c_2	Purchase cost per unit item
c_3	Shortage cost per unit item
I_e	Rate of interest earned
I_p	Rate of interest payable
R	Credit period

4.2 Assumptions

The assumptions of the model are as follows:

1. Demand rate $D(Q(t), s) = p(s)[a + bQ(t)]$ where $p(s) = \alpha e^{-s\beta}$ is the price feature where α, $\beta > 0$ are the parameters, b is stock-dependent parameter $0 \leq b \leq 1$.
2. Shortages are fully backlogged.
3. Inflation rate (k) is considered constant, which will affect the future value of the inventory cost.
4. Deterioration rate (γ) is constant and no replenishment is done inside the cycle.
5. If the vendor pays back within the credit period R, then no interest is chargeable to the former; otherwise, the former has to pay interest at the rate I_p to the supplier.
6. The vendor earned interest at the rate of I_e and paid charges to the supplier for delay, that is, I_p is rate of interest that is payable by the retailer to supplier for delay in payment.

4.3 Prototypical Design of Inventory Model

In this chapter, we have considered an EOQ model with price and stock-dependent demand rate with constant deteriorations. Q_{max} quantity of items are ordered at the beginning of the inventory and it depletes with time till $t = t_1$. In this model, we have considered a delay in payment, that is, there is an option for the supplier if he arrives before or after the stock finishes, which we have incorporated in two different cases as shown in Fig. 28.5. After the stock finishes still there is demand, which leads to shortage and thus there is backorder till t = T.

Therefore, Fig. 28.5 shows the model, and the mathematical formulation of the model is as follows:

$$\frac{dQ(t)}{dt} + \gamma Q(t) = -p(s)\left[a + bQ(t)\right], \quad 0 \le t \le t_1$$

$$\frac{dQ(t)}{dt} = -p(s)a, \qquad t_1 \le t \le T \tag{28.2}$$

Where the conditions are $Q(0) = Q_{max}$ *and* $Q(t_1) = 0$

$$\text{Thus, solving (28.2), } Q(t) = \begin{cases} \frac{e^{-\vartheta t}\left(e^{\vartheta t_1} - e^{\vartheta t}\right)p(s)a}{\vartheta}, & if\ 0 \le t \le t_1 \\ p(s)a\,(t_1 - t)\,, & if\ t_1 \le t \le T \end{cases} \tag{28.3}$$

where $\vartheta = \gamma + br(s)$

The maximum amount of inventory obtained after ordering at the beginning of the cycle t = 0 is $Q_{max} = Q(0)$ and is given as follows:

$$Q_{max} = Q(0) = \frac{ap(s)}{\vartheta}\left(e^{\vartheta t_1} - 1\right) \tag{28.4}$$

The inventory finishes at time $t = t_1$ and thus, the shortages begin from that time. The maximum shortage is at t = T and is given by

$$Q_s = -p(s)a\,(t_1 - T) = p(s)a\,(T - t_1)$$

Now the current value of inventory carrying cost is HC where its per unit cost is c_1

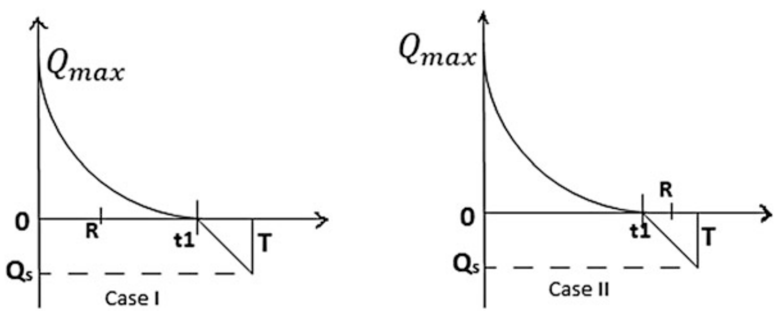

Fig. 28.5 On hand inventory

$$HC = c_1 \int_0^{t_1} \left(\frac{p\,(s)\,ae^{-at}\left(e^{\vartheta t_1} - e^{\vartheta t}\right)}{\vartheta} \right) e^{-kt} dt$$

$$= \left(\frac{c_1 p(s)\vartheta}{\vartheta k\,(\vartheta + k)} \right) \left(\vartheta e^{-kt_1} + k e^{\vartheta t_1} - (\vartheta + k) \right)$$

The present value of purchase cost is PC, where per unit cost is c_2

$$\begin{aligned} PC &= c_2 Q_{max} + c_2 e^{-kt} \textstyle\int_0^{T-t_1} p(s)adt \\ &= c_2 Q_{max} + c_2 e^{-kt} a\,(T - t_1) \end{aligned}$$

The present value of shortage cost is SC, where per unit cost is c_3

$$SC = c_3 \int_{t_1}^{T} Q(t) e^{-kt} dt$$

$$= c_3 \left(\frac{e^{-k(T+t_1)} c_2 p(s) a}{k^2} \right) \left(e^{kT} - e^{kt_1}\,(1 + k\,(T - t_1)) \right)$$

The present value of deterioration cost is DC, where per unit cost is c_4

$$DC = c_2 \Big(Q_{max} - \int_0^{T} \mathrm{D}\,(\mathrm{Q}\,(\mathrm{t})\,,\,\mathrm{s})\,dt$$

$$= c_2 \left(Q_{max} - \left(\frac{p(s)a}{\vartheta^2} \right) \left(\vartheta^2 t^1 + \left(e^{\vartheta t_1} - 1 \right) p(s) b - \vartheta r(s) t_1 b \right) \right)$$

The ordered cost OC is O_c.

Case I: Supplier Arrives Before the Completion of Inventory ($R \le T_1$)

As the supplier arrives before the stock ends so the retailer has to give money prior his/her income. So retailer can't receive the interest of the money, which she or he could have received by selling the items if the supplier doesn't arrive. Thus, the interest payable to the supplier by the retailer is given by IP_1 and the amount is given by

$$IP_1 = c_2 I_p \int_R^T Q(t) dt$$

$$= c_2 I_p \left(\left(\frac{p(s)a\left(e^{\vartheta(t_1-R)}-1\right)-1-\vartheta\left(R-t_1\right)}{\vartheta^2} \right) - \left(\frac{1}{2}\right) p(s)a(T-t_1)^2 \right)$$

Interest earned by the retailer due to sale up tot_1 is given by IE_1

$$IE_1 = c_2 I_e \int_0^{t_1} t\mathrm{D}\left(\mathrm{Q}\left(\mathrm{t}\right), \mathrm{s}\right) dt$$

$$= \left(\frac{c_2 I_e p(s)a}{2\vartheta^3}\right)\left(2\left(e^{\vartheta T}-1\right) r(s)b + \vartheta t_1 \left(\vartheta^2 t_1 - r(s)\left(2+\vartheta t_1\right) b\right)\right)$$

Thus, total cost is $TC_1 = \left(\frac{1}{T}\right)\left(OC + HC + SC + DC + PC + IP_1 - IE_1\right)$

$$= \frac{1}{T}\left[O_c + \left(\frac{c_1 p(s)\vartheta}{\vartheta k\left(\vartheta+k\right)}\right)\left(\vartheta e^{-kt_1} + ke^{\vartheta t_1} - \left(\vartheta+k\right)\right) + c_2 Q_{max} \right.$$
$$+ c_2 e^{-kt} a\left(T-t_1\right) + c_3\left(\frac{e^{-k(T+t_1)} c_2 p(s)a}{k^2}\right)\left(e^{kT} - e^{kt_1}\left(1+k\left(T-t_1\right)\right)\right)$$
$$\left. + c_2\left(Q_{max} - \left(\frac{p(s)a}{\vartheta^2}\right)\left(\vartheta^2 t^1 + \left(e^{\vartheta t_1}-1\right) r(s)b - \vartheta p(s)t_1 b\right)\right)\right]$$
$$+ c_2 I_p \left(\left(\frac{p(s)a\left(e^{\vartheta(t_1-R)}-1\right)-1-\vartheta\left(R-t_1\right)}{\vartheta^2} \right) - \left(\tfrac{1}{2}\right) p(s)a(T-t_1)^2 \right)$$
$$- \left(\tfrac{c_2 I_e p(s)a}{2\vartheta^3}\right)\left(2\left(e^{\vartheta T}-1\right) p(s)b + \vartheta t_1\left(\vartheta^2 t_1 - p(s)\left(2+\vartheta t_1\right) b\right)\right)$$

Case II: Supplier Arrives After the Stocks Ends ($t_1 \leq R$)

As the supplier arrives after the stock ends, the retailer will earn the capital ($c_2 Q_{max}$) and also the interest (IE_2) from that capital. Due to this the retailer won't have to pay the interest to the supplier, that is, $IP_2 = 0$.

Thus, the interest earned in this case is

$$IE_2 = c_2 I_e \left[\int_0^{t_1} tD\left(Q(t), s\right) dt + \left(R-t_1\right)\int_0^{t_1} D\left(Q(t), s\right) dt\right]$$

$$= \frac{c_2 I_e p(s)a}{2\vartheta^3}\left[\left(2\left(e^{\vartheta T}-1\right) p(s)b + \vartheta t_1\left(\vartheta^2 t_1 - p(s)\left(2+\vartheta t_1\right) b\right)\right)\right.$$
$$\left. + \left(\frac{\left(R-t_1\right) a}{\vartheta^2}\right)\left(\vartheta^2 t_1 + \left(e^{\vartheta t_1}-1\right) p(s)b - \vartheta p(s)t_1 b\right)\right]$$

Thus, total cost is $TC_2 = \left(\frac{1}{T}\right)\left(OC + HC + SC + DC + PC + IP_2 - IE_2\right)$

$$= \frac{1}{T}\left[O_c + \left(\frac{c_1 p(s)\vartheta}{\vartheta k(\vartheta+k)}\right)\left(\vartheta e^{-kt_1} + ke^{\vartheta t_1} - (\vartheta + k)\right) + c_2 Q_{max} + c_2 e^{-kt} a\,(T - t_1)\right.$$
$$+ c_3\left(\frac{e^{-k(T+t_1)} c_2 p(s) a}{k^2}\right)\left(e^{kT} - e^{kt_1}\,(1 + k\,(T - t_1))\right)$$
$$\left. + c_2\left(Q_{max} - \left(\frac{p(s)a}{\vartheta^2}\right)\left(\vartheta^2 t^1 + \left(e^{\vartheta t_1} - 1\right) p(s)b - \vartheta p(s) t_1 b\right)\right)\right]$$
$$- \frac{c_2 I_e p(s) a}{2\vartheta^3}\left[\left(2\left(e^{\vartheta T} - 1\right) p(s)b + \vartheta t_1\left(\vartheta^2 t_1 - p(s)\,(2 + \vartheta t_1)\,b\right)\right)\right.$$
$$\left. + \left(\frac{(R - t_1)\,a}{\vartheta^2}\right)\left(\vartheta^2 t_1 + \left(e^{\vartheta t_1} - 1\right) p(s)b - \vartheta p(s) t_1 b\right)\right]$$

Now the minimize of total cost with respect to time is obtained if $\frac{dTC_1}{dt_1}$ $and \frac{dTC_2}{dt_1}$ exist for then the necessary condition to minimize TC_1 and TC_2 for a specified value of M are $\frac{dTC_1}{dt_1} = 0$ $and \frac{dTC_2}{dt_1} = 0$ and we get the extreme point of TC_1 and TC_2. Again, the sufficient condition to minimize TC_1 and TC_2 are $\frac{d^2TC_1}{dt_1^2} > 0$ $and \frac{d^2TC_2}{dt_1^2} > 0$. Since the total cost is intricated function, it is tough to show the analytic validation of the above sufficient conditions in both cases. Thus, the sufficient condition is assessed numerically.

Effect of Triangular Intuitionistic Fuzzy Number in the Proposed Model

IFN is a new mathematical number to handle impreciseness properly. Normally it is used when there is both the chance of truthiness or falseness of a parameter. Thus, in this chapter, we have considered selling price (s) and inflation (k) as intuitionistic fuzzy parameter. Thus, we have modified the model $TC_1\left(\tilde{s}, \tilde{k}\right)$ and $TC_2\left(\tilde{s}, \tilde{k}\right)$. Now applying Eq. 28.1 to defuzzify the parameter ($\widetilde{s_{dint}}$, $\widetilde{k_{dint}}$) the model we obtained is given below

$$\widetilde{TC_1} = \frac{1}{T}\left[O_c + \left(\frac{c_1 p\left(\widetilde{s_{dint}}\right)\vartheta}{\vartheta k\left(\vartheta + \widetilde{k_{dint}}\right)}\right)\left(\vartheta e^{-\widetilde{k_{dint}} t_1} + \widetilde{k_{dint}} e^{\vartheta t_1} - \left(\vartheta + \widetilde{k_{dint}}\right)\right)\right.$$
$$+ c_2 Q_{max} + c_2 e^{-\widetilde{k_{dint}} t} a\,(T - t_1)$$
$$+ c_3\left(\frac{e^{-\widetilde{k_{dint}}(T+t_1)} c_2 p\left(\widetilde{s_{dint}}\right) a}{\tilde{k}^2}\right)\left(e^{\widetilde{k_{dint}} T} - e^{\widetilde{k_{dint}} t_1}\left(1 + \widetilde{k_{dint}}\,(T - t_1)\right)\right)$$
$$\left. + c_2\left(Q_{max} - \left(\frac{p\left(\widetilde{s_{dint}}\right) a}{\vartheta^2}\right)\left(\vartheta^2 t_1 + \left(e^{\vartheta t_1} - 1\right) p\left(\widetilde{s_{dint}}\right) b - \vartheta p\left(\widetilde{s_{dint}}\right) t_1 b\right)\right)\right]$$
$$+ c_2 I_p\left(\left(\frac{p\left(\widetilde{s_{dint}}\right) a\left(e^{\vartheta(t_1 - R)} - 1\right) - 1 - \vartheta(R - t_1)}{\vartheta^2}\right) - \left(\frac{1}{2}\right) p\left(\widetilde{s_{dint}}\right) a(T - t_1)^2\right)$$
$$- \left(\frac{c_2 I_e p\left(\widetilde{s_{dint}}\right) a}{2\vartheta^3}\right)\left(2\left(e^{\vartheta T} - 1\right) p\left(\widetilde{s_{dint}}\right) b + \vartheta t_1\left(\vartheta^2 t_1 - p\left(\widetilde{s_{dint}}\right)(2 + \vartheta t_1)\,b\right)\right)$$

$$\widetilde{TC_2} = \frac{1}{T}\left[O_c + \left(\frac{c_1 p(\widetilde{s_{dint}})\vartheta}{\vartheta k(\vartheta +)}\right)\left(\vartheta e^{-\widetilde{k_{dint}}t_1} + \widetilde{k_{dint}}e^{\vartheta t_1} - \left(\vartheta + \widetilde{k_{dint}}\right)\right)\right.$$

$$+c_2 Q_{max} + c_2 e^{-\widetilde{k_{dint}}t} a\left(T - t_1\right) + c_3\left(\frac{e^{-\widetilde{k_{dint}}(T+t_1)} c_2 p(\widetilde{s_{dint}})a}{\tilde{k}^2}\right)$$

$$\times\left(e^{\widetilde{k_{dint}}T} - e^{\widetilde{k_{dint}}t_1}\left(1 + \widetilde{k_{dint}}\left(T - t_1\right)\right)\right)$$

$$\left.+c_2\left(Q_{max} - \left(\frac{p(\widetilde{s_{dint}})a}{\vartheta^2}\right)\left(\vartheta^2 t_1 + \left(e^{\vartheta t_1} - 1\right)p\left(\widetilde{s_{dint}}\right)b - \vartheta p\left(\widetilde{s_{dint}}\right)t_1 b\right)\right)\right]$$

$$-\frac{c_2 I_e p(\widetilde{s_{dint}})a}{2\vartheta^3}\left[\begin{array}{l}\left(2\left(e^{\vartheta T} - 1\right)p\left(\widetilde{s_{dint}}\right)b + \vartheta t_1\left(\vartheta^2 t_1 - p\left(\widetilde{s_{dint}}\right)\left(2 + \vartheta t_1\right)b\right)\right)\\ +\left(\frac{(R-t_1)a}{\vartheta^2}\right)\left(\vartheta^2 t_1 + \left(e^{\vartheta t_1} - 1\right)p\left(\widetilde{s_{dint}}\right)b - \vartheta p\left(\widetilde{s_{dint}}\right)t_1 b\right)\end{array}\right]$$

5 Numerical Illustration

The existing model is demonstrated with the following case. A supermarket ABC has demand rate dependent on the stock present and price of the item, that is, demand rate is *D(Q(t), s)*, here $\alpha = 200,\ \beta = 1.3,\ a = 500\ units,\ b = 0.15$. Also, 10% of the total stock deteriorates, costing *$* 2 per item. Let the purchasing price of each item is $ 3, retailing price is $ 6 per item, and to carry the item, it entails $ 0.6 per unit. They spend $ 250 for setup of the inventory. The inflation rate in the market is 12% and let the vendor earns 15% of interest and pays 20% interest for full year. If the supplier comes (i) Two Monthly (ii) Four months after (iii) Six months (iv) Eight months. Now we minimize the total cost per unit item per unit time for the above situations.

$$\begin{aligned}\mathrm{D}\left(\mathrm{Q}\left(\mathrm{t}\right),\mathrm{s}\right) = \mathrm{r}\left(\mathrm{s}\right)\left[\mathrm{a+bQ}\left(\mathrm{t}\right)\right] = 200\mathrm{e}^{-6*1.3}\left[500+0.15Q(t)\right], s = 6, \gamma = 0.1, O_c\\ = 250 \text{ per order},\quad \mathrm{c}_1 = 0.6 \text{ per year, k} = 0.12, c_2 = 3 \text{ per year, c}_3\\ = 2 \text{ per year, I}_e = 0.15 \text{ per year, I}_p = 0.2 \text{ per year},\quad \mathrm{T} = 1 \text{ year.}\end{aligned}$$

Inflation and price of the items are also considered as triangular intuitionistic fuzzy number. Here k = (0.1,0.13, 0.14, 0.11, 0.12, 0.15), s = (5,6,7,5.5,6.5,7.5)

The results of the inventory model for four diverse situations are represented in Table 28.1.

Thus, from Tables 28.1 and 28.2, it is seen that supplier arriving in 2nd month (R = 0.1667) it contradicts case 2 and case 1 hold with minimum total cost TC_1 under the intuitionistic scenario. If the supplier arrives at 4th month, (R = 0.333) then both case 1 and case 2 are satisfied and $TC_1 < TC_2$ under both scenarios. If the supplier arrives at 6th month (R = 0.5), then it contradicts case 1, while case 2 holds with minimum total cost TC_2 under the intuitionistic scenario. If the supplier arrives at 8th month (R = 0.667) that contradicts case 1 while case 2 hold with minimum total cost TC_2 under the intuitionistic scenario. Thus, from Table 28.2 it

Table 28.1 Supplier arrives before inventory finishes (Case 1)

Delay in payment	t_1^* (in year) Crisps case	TC_1^* ($) Crisps case	t_1^* (in year) Intuitionistic case	$\widetilde{TC}_1^*$ ($) Intuitionistic case
2 months	0.373	376.42	0.374	341.32
4 months	0.41	375.83	0.412	340.89
6 months	0.447	375.78	0.449	340.85
8 months	0.484	376.26	0.486	341.12

Table 28.2 Supplier arrives after inventory finishes (Case 2)

Delay in payment	t_1^* (in year) Crisps case	TC_2^* ($) Crisps case	t_1^* (in year) Intuitionistic case	$\widetilde{TC}_2^*$ ($) Intuitionistic case
2 months	0.178	387.31	0.156	349.19
4 months	0.244	379.41	0.226	342.05
6 months	0.311	368.96	0.297	332.21
8 months	0.378	355.98	0.368	319.73

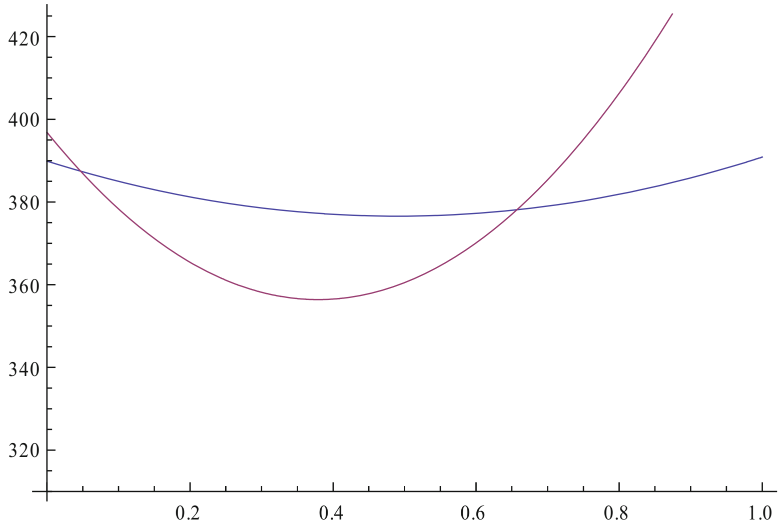

Fig. 28.6 Case 1 and Case 2 in crisps scenario

is observed that the total cost is minimum is under Intuitionistic scenario when the supplier arrives at 8th month.

Figures 28.6 and 28.7 show that the total cost is minimum under intuitionistic scenario, which is useful from managerial point of view.

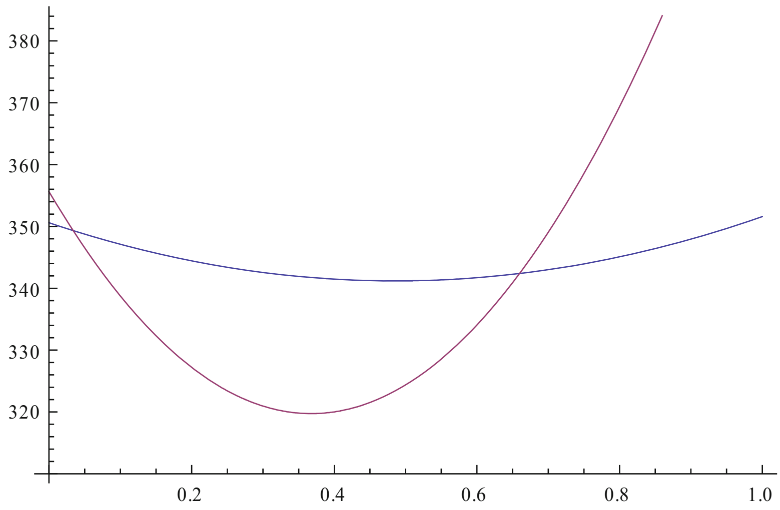

Fig. 28.7 Case 1 and case 2 in Intuitionistic Scenario

6 Sensitivity Analysis

In order to understand the effect of change of various parameter, we take one parameter at a time and by changing −20%, −10%, 10%, 20% for each parameter and keeping other unchanged. Here we have considered M = 8 month as the total costing is minimum in this situation.

Thus, from Table 28.3, it is seen that price of the items is highly sensitive, replacement time is moderately sensitive, while rest of the parameters are less sensitive.

6.1 Managerial Insight and Limitation of Work

In this chapter, we observed that our model fits better under case 2. That is, the supplier comes after the stock finishes. This is realistic in managerial point of view, because if the supplier comes after the stock finishes, then the retailer earns the money of selling the entire stock as well as she or he earns the interest until the supplier arrives.

The limitation of the works is, here we have mainly focused at the optimum time when the stock finishes. We could have extended the research work by focusing on how much the selling price will be in order to have our optimum total cost. Also, we could have extended the model by considering finite time horizon under different scenarios of intuitionistic fuzzy number.

Table 28.3 Sensitivity analysis of parameters

Parameter	Change (%)	Case 1		Case 2		Remarks	Change (%)
		t_1*	TC_1*	t_1*	TC_1*		
p	−20	0.463	855.66	0.444	848.445	Case 2 holds	138.34
	−10	0.481	526.63	0.412	511.026	Case 2 holds	43.55
	10	0.494	307.98	0.357	285.34	Case 2 holds	−19.84
	20	0.496	276.57	0.345	252.69	Case 2 holds	−29.02
k	−20	0.519	378.09	0.385	358.32	Case 2 holds	0.66
	−10	0.504	377.35	0.382	357.36	Case 2 holds	0.39
	10	0.475	375.81	0.376	355.46	Case 2 holds	−0.15
	20	0.46	375.02	0.373	354.52	Case 2 holds	−0.41
I_e	−20	0.474	377.014	0.385	361.58	Case 2 holds	1.57
	−10	0.482	376.803	0.382	358.997	Case 2 holds	0.85
	10	0.498	376.359	0.377	353.81	Case 2 holds	−0.61
	20	0.507	376.126	0.375	351.22	Case 2 holds	−1.34
I_p	−20	0.505	377.14	0.378	355.98	Case 2 holds	0.00
	−10	0.497	376.86	0.378	355.98	Case 2 holds	0.00
	10	0.482	376.3	0.378	355.98	Case 2 holds	0.00
	20	0.475	376	0.378	355.98	Case 2 holds	0.00
T	−20	0.425	439.2	0.358	409.92	Case 2 holds	15.15
	−10	0.458	404.37	0.368	379.98	Case 2 holds	6.74
	10	0.521	353.92	0.389	337.44	Case 2 holds	−5.21
	20	0.55	335.08	0.4	321.93	Case 2 holds	−9.57

7 Conclusion

In order to deal with the real-life scenario, it is observed that considering the parameters as constant is not realistic. Few parameters always have vagueness in it and thus to represent the vagueness in the parameter it is better to consider triangular intuitionistic fuzzy number. In this chapter, an EOQ model is considered under shortages and delay in payment. After studying the model numerically, it is seen that under different situations, the model works differently and depending on delay in payment period, the different cases are satisfied. Under all scenarios, it is seen that the model works better if we consider inflation and price of the items as triangular intuitionistic fuzzy number. It is seen that overall total cost is minimum when the supplier arrives at 8th month and this means total cost is minimum at case 2, that is, the supplier arrives once the stock finishes, which is realistic for retailer point of view. This model can be extended by considering price as decision variable, or considering shortage as lost in sales, etc.

In future, the model can be further extending by incorporating it in supply chain model or by collaborating with interdisciplinary ideas of control theory. The uncertainty of the model can also be represented by using stochastic process.

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Chapter 29
A Study of an EOQ Model Under Triangular Cloudy Fuzzy Neutrosophic Demand Rate

Sujit Kumar De and Sanchita Mahato

1 Introduction

It is very crucial to have genuine crisp data for any decision-making problem. But in modern competing scenario, it is very difficult to have adequate crisp data, because many data are hidden for governmental and political issues. The reliable data are not actually acceptable because of its vagueness and piecewise untruth nature. Zadeh [24] first studied Fuzzy set theory but few decennary later a new concept on hesitant fuzzy set has been developed. Torra [20] developed the concept on hesitant fuzzy set. Indeed, in intuitionistic fuzzy environment, eminent practitioner studied numerous research articles. Atannosov [1, 2] and Dubois et al. [10] independently developed the concept on intuitionistic fuzzy set (IFS). Between its process, Wang et al. [21, 22], Pei and Zheng [19] discussed over new concepts on evidence-based IFS and a unique perspective for decision-making respectively.

A study with four-valued logic, namely, Truth (T), false (F), Unknown (U), and Contradiction (C), was performed by [3]. Considering these four components as interconnected in nature, a bilattice structure was used by him. Smarandache [19] developed four valued logics: the Neutrosophic set (NS), Neutrosophic logic, Neutrosophic probability, and Neutrosophic statistics. Numerous ranking methods for NS have been discussed by several researchers like [23], Biswas et al. [5, 6], [16], etc. Recently, Peng et al. [18] developed the multivalued power operator in NS.

EOQ models under different types of dense fuzzy environment were uniquely studied by Maity et al. [12–14]. Some inventory models for defective or imperfect quality items have been generously established by De and Mahata [8]. Maity et al. [15] made a great study of a Backorder EOQ model for cloud-type intuitionistic

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C. Jana et al. (eds.), *Fuzzy Optimization, Decision-making and Operations Research*, https://doi.org/10.1007/978-3-031-35668-1_29

dense fuzzy demand rate. Several researchers like Karmakar et al. [11] and De et al. [7] developed a production inventory supply chain model with partial back ordering and disruption under triangular linguistic dense fuzzy lock set approach greatly. A pollution-sensitive fuzzy EPQ model with endogenous reliability and product deterioration based on lock fuzzy game theoretic approach has been evolved by Bhattacharya et al. [4].

Generally, a decision maker finds suitable membership grade for various characteristics of data. But real situation is different as predicted data of previous day is inconvenient for tomorrow due to the reason of changing frequency of various ownership enterprises. Time gaping is also responsible for this type of problematic situation. So, it is inconvenient to discover the raw data, as most of the genuine data are concealed under national or international rules and regulations. For instance, due to several unreliability in the merchandise, the demand rate actually varies from one cycle time to another. Moreover, for developing inventory model, a major challenge faced by a DM is to prognosticate the annual demand whose characteristics are found to be standard and nonstandard fuzzy flexibilities.

To model this situation in this chapter, a classical EOQ model is considered in TCFN demand rate environment. For numerical illustration, a solution algorithm has been constructed. After comparison of numerical result with other connected models, the uniqueness of this new approach is proved. Graphical illustration and sensitivity analysis are also made for proper justification of this model.

This chapter is organized as follows: In Sect. 2, we have described some preliminary definitions and membership grades related to different NS. In this section, we also have expressed about score value of an NS. In Sect. 3, we have developed crisp mathematical EOQ model. In the next part, we have developed crisp equivalent of TCFN model. In Sect. 4, we have solved a numerical problem through crisp, general fuzzy and TCFN model and compare the results in a table. A table for different submodels related to TCFN model also has been created in this section. In Sect. 5, a table for sensitivity analysis of different parameters has been developed. In Sect. 6, we have illustrated different graphs like optimum inventory cost and order quantity under various methods and graph for sensitivity analysis of different parameters. Finally, conclusion has been given in Sect. 7.

2 Basic Concepts

This section contains some basic definitions and concepts of NS.

Definition 1 (Biswas et al. 2014a) Let y be the element of a space point $\mathcal{X}$. Then an NS $\mathcal{A}$ in $\mathcal{X}$ is characterized by a truth grade α, an indeterminacy grade β, and a falsity grade γ, respectively. Thus, $N_S = \ <\alpha, \beta, \gamma>$ with the functions α, β, γ are of real standard or non-standard subsets of $]0^-, 1^+[$. That is:$Y \to\ ]0^-, 1^+[$, $\beta : Y \to\ ]0^-, 1^+[$ and $\gamma : Y \to\ ]0^-, 1^+[$ satisfying the relation $0^- \leq \sup \alpha(y) + \sup \beta(y) + \sup \gamma(y) \leq 3^+$

2.1 Triangular Cloudy Fuzzy System

Cloudy fuzzy environment in different cases of NS has been discussed in this section.

Definition 2 Let a fuzzy set $\tilde{A}$ have components of time τ with the membership grade satisfying the functional relation $\mu(y, \tau) \to 1$. Now as $\tau \to \infty$ if $\mu(y, \tau) \to 1$ for some $y \in \mathcal{R}$, then the set $\tilde{A}$ is a cloudy fuzzy set. If $\tilde{A}$ is triangular, then it is called "Triangular cloudy Fuzzy Set" or TCFS. Now, if for some τ, $\mu(y, \tau)$ attains 1, then it is called "Normalized Triangular Cloudy Fuzzy Set" or NTCFS (shown in Fig. 29.1).

Example 1 Following definitions (2) we assume the NTCFS as follows:

$$\tilde{A} =< p_2\left(1 - \frac{\eta_1}{(1+\tau)}\right), p_2, p_2\left(1 + \frac{\delta_1}{1+\tau}\right) > \tag{29.1}$$

Having membership function

$$\gamma_\alpha(y, \tau) = \begin{cases} 0 \quad if \quad y < p_2\left(1 - \frac{\eta_1}{1+\tau}\right) \;\; and \;\; y > p_2\left(1 + \frac{\delta_1}{1+\tau}\right) \\ \dfrac{y - p_2\left(1 - \frac{\eta_1}{1+\tau}\right)}{\frac{\rho_1 a_2}{1+t}} \quad if \;\; p_2\left(1 - \frac{\eta_1}{1+\tau}\right) \le y \le p_2 \\ \dfrac{p_2\left(1 + \frac{\delta_1}{1+\tau}\right) - y}{\frac{\delta_1 p_2}{1+\tau}} \quad if \;\; p_2 \le y \le p_2\left(1 + \frac{\delta_1}{1+\tau}\right) \end{cases} \tag{29.2}$$

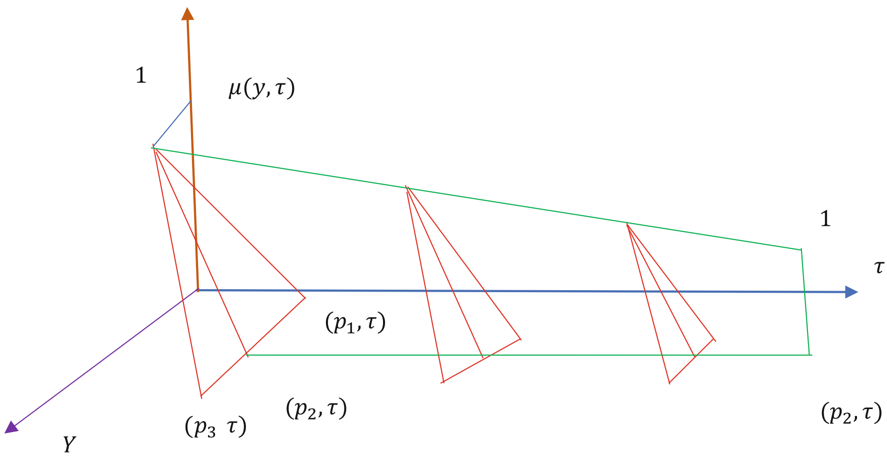

Fig. 29.1 Truth function of NTCFS

Similarly, here also we consider the membership functions of falsehood and indeterminacy, respectively, described as follows:

$$\gamma_\beta(y,\tau) = \begin{cases} 0 \quad if\ y < q_2\left(1-\frac{\eta_2}{1+\tau}\right) \quad and\ y > q_2\left(1+\frac{\delta_2}{1+\tau}\right) \\ \frac{q_2-y}{\frac{\eta_2 q_2}{(1+\tau)}} \quad if\ q_2\left(1-\frac{\eta_2}{1+\tau}\right) \le y \le q_2 \\ \frac{y-q_2}{\frac{\delta_2 q_2}{1+\tau}} \quad if\ q_2 \le y \le q_2\left(1+\frac{\delta_2}{1+\tau}\right) \end{cases} \tag{29.3}$$

$$\gamma_\gamma(y,\tau) = \begin{cases} 0 \quad if\ y < \gamma_2\left(1-\frac{\eta_3}{1+\tau}\right) \quad and\ y > r_2\left(1+\frac{\delta_3}{1+\tau}\right) \\ \frac{r_2-y}{\frac{\eta_3 r_2}{(1+\tau)}} \quad if\ r_2\left(1-\frac{\eta_3}{1+\tau}\right) \le y \le r_2 \\ \frac{y-r_2}{\frac{\eta_3 r_2}{1+\tau}} \quad if\ r_2 \le y \le r_2\left(1+\frac{\delta_3}{1+\tau}\right) \end{cases} \tag{29.4}$$

Definition 3 TCFS based on nonmembership and indeterminacy function [10] Let $\tilde{A}$ be the fuzzy number whose components are the function of time. Now as $\tau \to \infty$ if $\nu(y,\tau) \to 0$ for some $y \in \mathcal{R}$, then we call the set $\tilde{A}$ as cloudy fuzzy set. If we consider the fuzzy number $\tilde{A}$ of the form $\tilde{A} =< p_1, p_2, p_3 >$, then we call it "Triangular cloudy Fuzzy Set" or TCFS. Now, if $\tau = 0$ in T and $\nu(y,\tau)$ takes 1, then we call it as "Normalized Triangular Cloudy Fuzzy Set" or NTCFS (shown in Fig. 29.2).

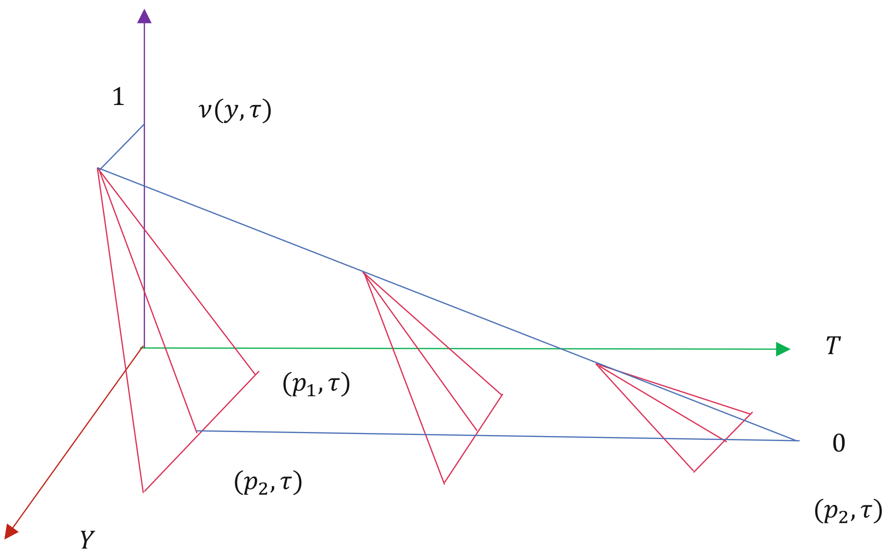

Fig. 29.2 Falsity function of NTCFS

Example 2 Let the falsity set is given by

$$\widetilde{B} =< q_2 (1 - \eta_2) e^{-\tau}, q_2 e^{-\tau}, q_2 (1 + \delta_2) e^{-\tau} > \quad for\ 0 < \eta_2, \delta_2 < 1 \tag{29.5}$$

and its nonmembership function for $\tau \geq 0$ is defined by

$$\nu (y, \tau) = \begin{cases} 0 \quad if\ y < q_2 (1 - \eta_2) e^{-\tau} \quad and\ y > q_2 (1 + \delta_2) e^{-\tau} \\ \frac{q_2 e^{-\tau} - y}{\eta_2 q_2 e^{-\tau}} \quad if\ q_2 (1 - \eta_2) e^{-\tau} \leq y \leq q_2 e^{-\tau} \\ \frac{y - q_2 e^{-\tau}}{\delta_2 q_2 e^{-\tau}} \quad if\ q_2 e^{-\tau} \leq y \leq q_2 (1 + \delta_2) e^{-\tau} \end{cases} \tag{29.6}$$

And that for the indeterminacy dense fuzzy set $\widetilde{C}$ be of the form

$$\widetilde{C} =< r_2 (1 - \eta_3) e^{-\tau}, r_2 e^{-\tau}, r_2 (1 + \delta_3) e^{-\tau} > \quad for\ 0 < \eta_3, \delta_3 < 1 \tag{29.7}$$

representing the membership function

$$\pi (y, \tau) = \begin{cases} 0 \quad if\ y < r_2 (1 - \eta_3) e^{-\tau} \quad and\ y > r_2 (1 + \delta_3) e^{-\tau} \\ \frac{r_2 e^{-\tau} - y}{\eta_3 r_2 e^{-\tau}} \quad if\ r_2 (1 - \eta_3) e^{-\tau} \leq y \leq r_2 e^{-\tau} \\ \frac{y - r_2 e^{-\tau}}{\delta_3 r_2 e^{-\tau}} \quad if\ r_2 e^{-\tau} \leq y \leq r_2 (1 + \delta_3) e^{-\tau} \end{cases} \tag{29.8}$$

respectively.

Remark 1 The NS for dependency components [9] Here a plain Venn diagram for NS dependency components has been drawn out for realization of aggregate analysis of fuzzy components. As per theorem of probability, the net score for NS-dependent components was obtained from the Fig. 29.3. Moreover, the concepts of Figs. 29.1 and 29.2 have been utilized to draw Fig. 29.3. Note that, if $p_2 = q_2 = r_2$ hold in the relations (29.1), (29.5), and (29.7), then we reach at a crisp set.

2.2 *Score Value of an NS [9]*

Here we shall discuss over the aggregated score or expected index value under the standard fuzzy set of the proposed NS $N_s = <\alpha, \beta, \gamma>$ where all the NS components are assumed to be dependent. Then the score $s(y)$ for standard neutrosophic set is given by

$$\begin{aligned} s(y) = (&\alpha(y) + \beta(y) + \gamma(y) - \alpha(y)\beta(y) - \beta(y)\gamma(y) \\ &- \alpha(y)\gamma(y) + \alpha(y)\beta(y)\gamma(y))^{\frac{1}{3}} \end{aligned} \tag{29.9}$$

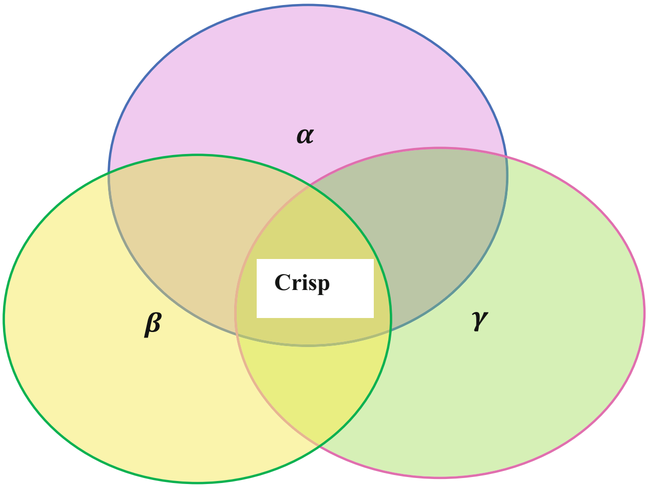

Fig. 29.3 Venn diagram of general NS

And that for nonstandard NS we have

$$I(S) = [I(\alpha) + I(\beta) + I(\gamma) - I(\alpha)I(\beta) - I(\beta)I(\gamma) - I(\alpha)I(\gamma) + I(\alpha)I(\beta)I(\gamma)] \tag{29.10}$$

3 Considerations and Symbols

The following notations and assumptions are used to develop the model.

Considerations

1. Orders are placed on demand.
2. The cycle time is infinite (weeks).
3. Backlogging is absent.
4. Lead time is zero

Symbols

q The stock per cycle (a variable)
D Week basis Demand (units)
K Cycle wise Set-up cost ($)
h Inventory holding cost per unit quantity per cycle ($)
c Purchasing price of unit item ($)
T_0 Threshold inventory run time (years)
T_1 Final inventory run time (years) (Decision variable)
$T_1 - T_0$ Cycle time (years)
Z Total average cost of the model (a variable) ($)

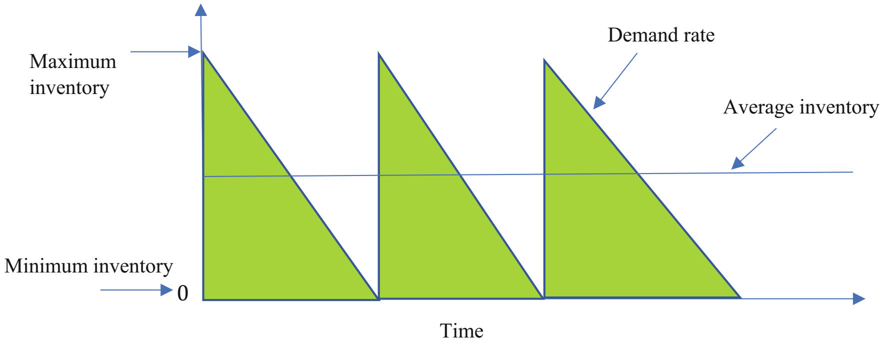

Fig. 29.4 Schematic view of the EOQ process

3.1 Crisp Model Formulation

Considering the initial inventory starts with order quantity q and the constant consumed demand rate is D (shown in Fig. 29.4). Then at the end of the cycle time $T_1 - T_0$, the inventory becomes zero. The costs related with it are unit purchasing price c, unit holding cost h, and set up cost K only. Therefore, the proposed inventory problem for minimization of overall total cost is given as

$$\begin{cases} Total\ Average\ inventory\ cost \\ \quad = purchasing\ cost + set - up\ cost + holding\ cost \\ \qquad\qquad = cD + \frac{k}{T_1 - T_0} + \frac{hD(T_1 - T_0)}{2} \\ \qquad Subject\ to \quad q = D\,(T_1 - T_0) \end{cases}$$

So, the proposed EOQ model becomes

$$\begin{cases} Find\ X = (q, T)\ so\ that \\ \text{Minimize } Z = cD + \frac{k}{T_1 - T_0} + \frac{hD(T_1 - T_0)}{2} \\ subject\ to \qquad q = D\,(T_1 - T_0) \end{cases} \tag{29.11}$$

3.2 Neutrosophic Fuzzy Mathematical Model

Let us assume that the inventory process is demand sensitive by means of NS. Thus, taking the demand rate D as a TCFN

$$\widetilde{D} = \left\{ < p_2\left(1 - \frac{\eta_1}{1+t}\right), p_2, p_2\left(1 + \frac{\delta_1}{1+t}\right) > \quad for\ 0 < \eta_1, \delta_1 < 1\right.$$

Now the membership function for truth component of Neutrosophic fuzzy demand is given by

$$\mu_{\alpha D}(y,\tau)=\begin{cases} 0 & if y < p_2\left(1-\frac{\eta_1}{1+\tau}\right) and\ y > p\left(1+\frac{\delta_1}{1+\tau}\right) \\ \dfrac{y-p_2\left(1-\frac{\eta_1}{1+\tau}\right)}{\frac{\eta_1 p_2}{1+\tau}} & if\ p_2\left(1-\frac{\eta_1}{1+\tau}\right)\le y\le p_2 \\ \dfrac{p_2\left(1+\frac{\delta_1}{1+\tau}\right)-y}{\frac{\delta_1 p_2}{1+\tau}} & if\ p_2\le y\le p_2\left(1+\frac{\delta_1}{1+\tau}\right) \end{cases} \tag{29.12}$$

Similarly, membership function for falsehood of Neutrosophic fuzzy demand is given by

$$\mu_{\beta D}(y,\tau)=\begin{cases} 0 & if\ y < q_2\left(1-\frac{\eta_2}{1+\tau}\right)\ \ and\ y > q_2\left(1+\frac{\delta_2}{1+\tau}\right) \\ \dfrac{q_2-y}{\frac{\eta_2 q_2}{(1+\tau)}} & if\ q_2\left(1-\frac{\eta_2}{1+\tau}\right)\le y\le q_2 \\ \dfrac{y-q_2}{\frac{\delta_2 q_2}{1+\tau}} & if\ q_2\le y\le q_2\left(1+\frac{\delta_2}{1+\tau}\right) \end{cases} \tag{29.13}$$

and that of the membership function for indeterminacy is given by

$$\mu_{\gamma D}(y,\tau)=\begin{cases} 0 & if\ y < r_2\left(1-\frac{\eta_3}{1+\tau}\right)\ \ and\ y > r_2\left(1+\frac{\delta_3}{1+\tau}\right) \\ \dfrac{r_2-y}{\frac{\eta_3 r_2}{(1+\tau)}} & if\ r_2\left(1-\frac{\eta_3}{1+\tau}\right)\le y\le r_2 \\ \dfrac{y-r_2}{\frac{\delta_3 r_2}{1+\tau}} & if\ r_2\le y\le r_2\left(1+\frac{\delta_3}{1+\tau}\right) \end{cases} \tag{29.14}$$

Now we consider the TCFN demand set as

$$N_s = < \mu_{\alpha D}(y,\tau), \mu_{\beta D}(y,\tau), \mu_{\gamma D}(y,\tau) > = < \alpha_d, \beta_d, \gamma_d > \text{ say,}$$

where α_d is the truth membership function, β_d is the falsehood membership function, and γ_d is the indeterminacy function, respectively. If these three components are dependent and keep positive sign then using inclusion-exclusion principle and De and Beg [9], the ultimate score be, the case of standard NS

$$\begin{aligned} s(x) = (&\alpha(y) + \beta(y) + \gamma(y) - \alpha(y)\beta(y) - \beta(y)\gamma(y) \\ &-\alpha(y)\gamma(y) + \alpha(y)\beta(y)\gamma(y))^{\frac{1}{3}} \end{aligned} \tag{29.15}$$

Now utilizing De and Beg [9], the defuzzied value of NS can be obtained with the help of the formula

$$\begin{aligned} I\left(\tilde{A}\right) &= \frac{1}{2T_1}\iint_{\theta=0,t=T_0}^{\theta=1,t=T_1} \{L(\theta,t) + R(\theta,t)\}\, d\theta dt \\ &= p_2\left\{1 + \frac{\delta - \eta}{4T_1} Log\,(1 + T_1)\right\} \end{aligned} \tag{29.16}$$

Therefore, the index values for T_d, I_d, F_d are given by

$$\begin{cases} I_\alpha\left(\tilde{D}\right) = p_2\left\{1 + \frac{\delta_1 - \eta_1}{4T_1} Log\,(1 + T_1)\right\} \\ I_\beta\left(\tilde{D}\right) = q_2\left\{1 + \frac{\delta_2 - \eta_2}{4T_1} Log\,(1 + T_1)\right\} \\ I_\gamma\left(\tilde{D}\right) = r_2\left\{1 + \frac{\delta_3 - \eta_3}{4T_1} Log\,(1 + T_1)\right\} \end{cases} \tag{29.17}$$

So, the aggregated index value for Neutrosophic fuzzy demand D can be found with the help of (29.15)

$$\begin{aligned} I\left(\tilde{D}\right) = \Big(&I_\alpha\left(\tilde{D}\right) + I_\beta\left(\tilde{D}\right) + I_\gamma\left(\tilde{D}\right) - I_\alpha\left(\tilde{D}\right) I_\beta\left(\tilde{D}\right) \\ &-I_\alpha\left(\tilde{D}\right) I_\gamma\left(\tilde{D}\right) - I_\beta\left(\tilde{D}\right) I_\gamma\left(\tilde{D}\right) + I_\alpha\left(\tilde{D}\right) I_\beta\left(\tilde{D}\right) I_\gamma\left(\tilde{D}\right)\Big)^{\frac{1}{3}} \end{aligned} \tag{29.18}$$

Again considering (29.10), the truth membership function for Neutrosophic fuzzy objective is given by

$$\mu_{\alpha Z}(y,\tau)=\begin{cases}0 & if \quad Z<p_2\left(1-\frac{\eta_1}{1+\tau}\right)\left(c+\frac{h\tau}{2}\right)+\frac{K}{\tau} \\ & and\ Z>p_2\left(1+\frac{\delta_1}{1+\tau}\right)\left(c+\frac{h\tau}{2}\right)+\frac{K}{\tau} \\ \left\{\frac{\frac{2(Z\tau-K)}{\left(2c\tau+h\tau^2\right)}-p_2\left(1-\frac{\eta_1}{1+\eta}\right)}{\frac{\eta_1 p_2}{1+\tau}}\right\} & if\ p_2\left(1-\frac{\eta_1}{1+\tau}\right)\left(c+\frac{h\tau}{2}\right)+\frac{K}{\tau} \\ & \leq Z\leq p_2\left(c+\frac{h\tau}{2}\right)+\frac{K}{\tau} \\ \left\{\frac{p_2\left(1+\frac{\delta_1}{1+\tau}\right)-\frac{2(Z\tau-K)}{2c\tau+h\tau^2}}{\frac{\delta_1 p_2}{1+\tau}}\right\} & if\ p_2\left(c+\frac{h\tau}{2}\right)+\frac{K}{\tau}\leq Z \\ & \leq p_2\left(1+\frac{\delta_1}{1+\tau}\right)\left(\mathrm{c}+\frac{\mathrm{h}\tau}{2}\right)+\frac{\mathrm{K}}{\tau}\end{cases} \tag{29.19}$$

Similarly, the membership function for falsity and indeterminacy of the Neutrosophic fuzzy objective is as follows:

$$\mu_{\beta Z}(y,\tau)=\begin{cases}0 & if\ Z<q_2\left(1-\frac{\eta_2}{1+\tau}\right)\left(c+\frac{h\tau}{2}\right)+\frac{K}{\tau} \\ & and\ Z>q_2\left(1+\frac{\delta_2}{1+\tau}\right)\left(c+\frac{h\tau}{2}\right)+\frac{K}{\tau} \\ \left\{\frac{q_2-\frac{2(Z\tau-K)}{\left(2c\tau+h\tau^2\right)}}{\frac{\eta_2 q_2}{(1+\tau)}}\right\} & if\ q_2\left(1-\frac{\eta_2}{1+\tau}\right)\left(c+\frac{h\tau}{2}\right)+\frac{K}{\tau} \\ & \leq Z\leq q_2\left(c+\frac{h\tau}{2}\right)+\frac{K}{\tau} \\ \left\{\frac{\frac{2(Z\tau-K)}{\left(2c\tau+h\tau^2\right)}-\ q_2}{\frac{\delta_2 q_2}{1+\tau}}\right\} & if\ q_2\left(c+\frac{h\tau}{2}\right)+\frac{K}{\tau}\leq Z \\ & \leq q_2\left(1+\frac{\delta_2}{1+\tau}\right)\left(c+\frac{h\tau}{2}\right)+\frac{K}{\tau}\end{cases} \tag{29.20}$$

$$\mu_{\gamma Z}(y,\tau)=\begin{cases}0 & if\ Z<r_2\left(1-\frac{\eta_3}{1+\tau}\right)\left(c+\frac{h\tau}{2}\right)+\frac{K}{\tau} \\ & and\ Z>r_2\left(1+\frac{\delta_3}{1+\tau}\right)\left(c+\frac{h\tau}{2}\right)+\frac{K}{\tau} \\ \left\{\frac{r_2-\frac{2(Z\tau-K)}{\left(2c\tau+h\tau^2\right)}}{\frac{\eta_3 r_2}{(1+\tau)}}\right\} & if\ r_2\left(1-\frac{\eta_3}{1+\tau}\right)\left(c+\frac{h\tau}{2}\right)+\frac{K}{\tau}\leq Z \\ & \leq r_2\left(c+\frac{h\tau}{2}\right)+\frac{K}{\tau} \\ \left\{\frac{\frac{2(Z\tau-K)}{\left(2c\tau+h\tau^2\right)}\ -\ r_2}{\frac{\delta_3 r_2}{1+\tau}}\right\} & if\ r_2\left(c+\frac{h\tau}{2}\right)+\frac{K}{\tau}\leq Z \\ & \leq r_2\left(1+\frac{\delta_3}{1+\tau}\right)\left(c+\frac{h\tau}{2}\right)+\frac{K}{\tau}\end{cases} \tag{29.21}$$

Now from the formula stated in (29.17), we find the index values of truth, falsity, and indeterminacy components of fuzzy objective variable, which are given below:

$$\begin{cases} I_\alpha\left(\tilde{Z}\right) = \frac{K Log\left(\frac{T1}{T0}\right)}{T_1} + p_2\left(c + \frac{hT_1}{4}\right) + \frac{p_2}{4T_1}c\left(\delta_1 - \eta_1\right) Log\left(1 + T_1\right) \\ \quad + \frac{a_2}{8T1}h\left(\delta_1 - \eta_1\right)\left(T_1 - Log\left(1 + T_1\right)\right) \\ I_\beta\left(\tilde{Z}\right) = \frac{K Log\left(\frac{T1}{T0}\right)}{T_1} + q_2\left(c + \frac{hT_1}{4}\right) + \frac{q_2}{4T_1}c\left(\delta_2 - \eta_2\right) Log\left(1 + T_1\right) \\ \quad + \frac{b_2}{8T1}h\left(\delta_2 - \eta_2\right)\left(T_1 - Log\left(1 + T_1\right)\right) \\ I_\gamma\left(\tilde{Z}\right) = \frac{K Log\left(\frac{T1}{T0}\right)}{T_1} + r_2\left(c + \frac{hT_1}{4}\right) + \frac{r_2}{4T_1}c\left(\delta_3 - \eta_3\right) Log\left(1 + T_1\right) \\ \quad + \frac{c_2}{8T1}h\left(\delta_3 - \eta_3\right)\left(T_1 - Log\left(1 + T_1\right)\right) \end{cases} \tag{29.22}$$

Now the aggregated index value for objective function Z becomes

$$I\left(\tilde{Z}\right) = \left(I_\alpha\left(\tilde{Z}\right) + I_\beta\left(\tilde{Z}\right) + I_\gamma\left(\tilde{Z}\right) - I_\alpha\left(\tilde{Z}\right) I_\beta\left(\tilde{Z}\right) - I_\alpha\left(\tilde{Z}\right) I_\gamma\left(\tilde{Z}\right) - I_\beta\left(\tilde{Z}\right) I_\gamma\left(\tilde{Z}\right) + I_\alpha\left(\tilde{Z}\right) I_\beta\left(\tilde{Z}\right) I_\gamma\left(\tilde{Z}\right)\right)^{\frac{1}{3}} \tag{29.23}$$

However, the truth membership function for fuzzy order quantity q is

$$\mu_{\alpha Q}\left(y,\tau\right) = \begin{cases} 0 & if\ \ Q < p_2 t\left(1 - \frac{\eta_1}{1+\tau}\right) and\ Q > p_2\tau\left(1 + \frac{\delta_1}{1+\tau}\right) \\ \frac{Q - p_2\tau\left(1 - \frac{\eta_1}{1+\tau}\right)}{\frac{\eta_1 p_2\tau}{1+\tau}} & if\ \ p_2\tau\left(1 - \frac{\eta_1}{1+\backslash tau}\right) \le Q \le p_2\tau \\ \frac{p_2\tau\left(1 + \frac{\delta_1}{1+\tau}\right) - Q}{\frac{\delta_1 p_2\tau}{1+\tau}} & if\ \ p_2\tau \le Q \le p_2\tau\left(1 + \frac{\delta_1}{1+\tau}\right) \end{cases} \tag{29.24}$$

Similarly, the membership function for falsity and indeterminacy of fuzzy order quantity q is

$$\mu_{\beta Q}\left(y,\tau\right) = \begin{cases} 0 & if\ Q < q_2\tau\left(1 - \frac{\eta_2}{1+\tau}\right) and\ Q > q_2\tau\left(1 + \frac{\delta_2}{1+\tau}\right) \\ \frac{q_2\tau - Q}{\frac{\eta_2 q_2\tau}{(1+\tau)}} & if\ q_2\tau\left(1 - \frac{\eta_2}{1+\tau}\right) \le Q \le q_2\tau \\ \frac{Q - q_2\tau}{\frac{\delta_2 q_2\tau}{1+\tau}} & if\ q_2\tau \le Q \le q_2\tau\left(1 + \frac{\delta_2}{1+\tau}\right) \end{cases} \tag{29.25}$$

$$\mu_{\gamma Q}(y,\tau)=\begin{cases}0 & if\, Q<r_2\tau\left(1-\frac{\eta_3}{1+\tau}\right) and\ Q>r_2\tau\left(1+\frac{\delta_3}{1+\tau}\right)\\ \dfrac{r_2\tau-Q}{\frac{\eta_3 r_2\tau}{(1+\tau)}} & if\ r_2\tau\left(1-\frac{\eta}{1+\tau}\right)\le Q\le r_2\tau\\ \dfrac{\frac{Q}{\tau}-r_2\tau}{\frac{\delta_3 r_2\tau}{1+\tau}} & if\ r_2\tau\le Q\le r_2\tau\left(1+\frac{\delta_3}{1+\tau}\right)\end{cases} \tag{29.26}$$

Now from the formula stated in (29.24), we find the index values of truth, falsity, and indeterminacy components of fuzzy order quantity, which are given below:

$$\begin{cases} I_\alpha\left(\tilde{Q}\right)=\frac{p_2T_1}{2}+\frac{p_2}{4T_1}(\delta_1-\eta_1)(T_1-Log(1+T_1))\\ I_\beta\left(\tilde{Q}\right)=\frac{q_2T_1}{2}+\frac{q_2}{4T_1}(\delta_2-\eta_2)(T_1-Log(1+T_1))\\ I_\gamma\left(\tilde{Q}\right)=\frac{r_2T_1}{2}+\frac{r_2}{4T_1}(\delta_3-\eta_3)(T_1-Log(1+T_1))\end{cases} \tag{29.27}$$

Now the total index value for fuzzy order quantity q becomes

$$I\left(\tilde{Q}\right)=\Big(I_\alpha\left(\tilde{Q}\right)+I_\beta\left(\tilde{Q}\right)+I_\gamma\left(\tilde{Q}\right)-I_\alpha\left(\tilde{Q}\right)I_\beta\left(\tilde{Q}\right)-I_\alpha\left(\tilde{Q}\right)I_\gamma\left(\tilde{Q}\right)$$
$$-I_\beta\left(\tilde{Q}\right)I_\gamma\left(\tilde{Q}\right)+I_\alpha\left(\tilde{Q}\right)I_\beta\left(\tilde{Q}\right)I_\gamma\left(\tilde{Q}\right)\Big)^{\frac{1}{3}} \tag{29.28}$$

So, the crisp equivalent of the TDFN Model becomes

$$\begin{cases}\text{Min } I\left(\tilde{Z}\right)=\Big(I_\alpha\left(\tilde{Z}\right)+I_\beta\left(\tilde{Z}\right)+I_\gamma\left(\tilde{Z}\right)-I_\alpha\left(\tilde{Z}\right)I_\beta\left(\tilde{Z}\right)-I_\alpha\left(\tilde{Z}\right)\\ \quad\times I_\gamma\left(\tilde{Z}\right)-I_\beta\left(\tilde{Z}\right)I_\gamma\left(\tilde{Z}\right)+I_\alpha\left(\tilde{Z}\right)I_\beta\left(\tilde{Z}\right)I_\gamma\left(\tilde{Z}\right)\Big)^{\frac{1}{3}}\\ I\left(\tilde{D}\right)=\Big(I_\alpha\left(\tilde{D}\right)+I_\beta\left(\tilde{D}\right)+I_\gamma\left(\tilde{D}\right)-I_\alpha\left(\tilde{D}\right)I_\beta\left(\tilde{D}\right)-I_\alpha\left(\tilde{D}\right)\\ \quad\times I_\gamma\left(\tilde{D}\right)-I_\beta\left(\tilde{D}\right)I_\gamma\left(\tilde{D}\right)+I_\alpha\left(\tilde{D}\right)I_\beta\left(\tilde{D}\right)I_\gamma\left(\tilde{D}\right)\Big)^{\frac{1}{3}}\\ S.t.I\left(\tilde{Q}\right)=\Big(I_\alpha\left(\tilde{Q}\right)+I_\beta\left(\tilde{Q}\right)+I_\gamma\left(\tilde{Q}\right)-I_\alpha\left(\tilde{Q}\right)I_\beta\left(\tilde{Q}\right)-I_\alpha\left(\tilde{Q}\right)\\ \quad\times I_\gamma\left(\tilde{Q}\right)-I_\beta\left(\tilde{Q}\right)I_\gamma(Q)+I_\alpha\left(\tilde{Q}\right)I_\beta\left(\tilde{Q}\right)I_\gamma\left(\tilde{Q}\right)\Big)^{\frac{1}{3}}\end{cases}$$

Subject to

$$\begin{cases} I_\alpha\left(\tilde{Z}\right) = \frac{K\,Log\left(\frac{T1}{T0}\right)}{T_1} + p_2\left(c + \frac{hT_1}{4}\right) + \frac{p_2}{4T_1}c\left(\delta_1 - \eta_1\right)Log\left(1 + T_1\right) \\ \quad + \frac{p_2}{8T1}h\left(\delta_1 - \eta_1\right)\left(T_1 - Log\left(1 + T_1\right)\right. \\ I_\beta\left(\tilde{Z}\right) = \frac{K\,Log\left(\frac{T1}{T0}\right)}{T_1} + q_2\left(c + \frac{hT_1}{4}\right) + \frac{q_2}{4T_1}c\left(\delta_2 - \eta_2\right)Log\left(1 + T_1\right) \\ \quad + \frac{q_2}{8T1}h\left(\delta_2 - \eta_2\right)\left(T_1 - Log\left(1 + T_1\right)\right. \\ I_\gamma\left(\tilde{Z}\right) = \frac{K\,Log\left(\frac{T1}{T0}\right)}{T_1} + r_2\left(c + \frac{hT_1}{4}\right) + \frac{r_2}{4T_1}c\left(\delta_3 - \eta_3\right)Log\left(1 + T_1\right) \\ \quad + \frac{r_2}{8T1}h\left(\delta_3 - \eta_3\right)\left(T_1 - Log\left(1 + T_1\right)\right. \\ I_\alpha\left(\tilde{D}\right) = p_2\left\{1 + \frac{\delta_1 - \eta_1}{4T_1}Log\left(1 + T_1\right)\right\} \\ I_\beta\left(\tilde{D}\right) = q_2\left\{1 + \frac{\delta_2 - \eta_2}{4T_1}Log\left(1 + T_1\right)\right\} \\ I_\gamma\left(\tilde{D}\right) = r_2\left\{1 + \frac{\delta_3 - \eta_3}{4T_1}Log\left(1 + T_1\right)\right\} \\ I_\alpha\left(\tilde{Q}\right) = \frac{p_2T_1}{2} + \frac{p_2}{4T_1}\left(\delta_1 - \eta_1\right)\left(T_1 - Log\left(1 + T_1\right)\right) \\ I_\beta\left(\tilde{Q}\right) = \frac{q_2T_1}{2} + \frac{q_2}{4T_1}\left(\delta_2 - \eta_2\right)\left(T_1 - Log\left(1 + T_1\right)\right) \\ I_\gamma\left(\tilde{Q}\right) = \frac{r_2T_1}{2} + \frac{r_2}{4T_1}\left(\delta_3 - \eta_3\right)\left(T_1 - Log\left(1 + T_1\right)\right) \end{cases} \tag{29.29}$$

Similarly, we shall use the formula of nonstandard fuzzy set to get another set of results.

4 Numerical Experiment

Let us take the values of model parameters $c = \$60$, $D = 50$ *units*, $K = \$800$, $h = \$3.5$ and the fuzzy deviation parameters $\delta_1 = .1$, $\eta_1 = .15$, $\delta_2 = .2$, $\eta_2 = .25$, $\delta_3 = .3$, $\eta_3 = .35$.Then we get the optimal solution of the proposed model given in Table 29.1. Taking fuzzy demand rate as $\tilde{D} =< 46, 50, 52 >$ and that for the TCFN model and the outputs are shown in this table. For TCFN model, we take $p_2 = 50$, $q_2 = 50$, $r_2 = 50$ and get the result

From Table 29.1, it is seen that the average inventory cost under different standard fuzzy environments ranges from \$3299.668 to \$ 3528.150. For the nonstandard fuzzy environment, the cost value hikes to 7979790.87 with respect to stock quantity 47,662, which is the 315.25 multiple of that of crisp result. The cost value gives better benefit to the decision maker in standard Neutrosophic fuzzy environment for cycle time 1.534 years and stock quantity 74.51 units, which gives

Table 29.1 Optimum solution of the proposed process

Model	Time cycle $(T_1 - T_0)^*$ (years)	Stock quantity q^* (units)	Average System cost Z^*	RC (%)
Crisp	3.0323	151.1858	3528.150	
General fuzzy (Triangular)	3.038	150.4280	3496.498	−0.89
TCFN (standard)	1.534	74.51	3299.668	−6.47
TCFN (nonstandard)	1.5	47662	28153.9×10^6	+797979010.87

Note: $RC = \frac{Z^* - Z^* crisp}{Z^* crisp} \times 100\%$

6.47% cost reduction with respect to crisp model. Although, in the fuzzy model, the cost benefit assumes 0.89% with respect to that of crisp model. The order quantities of the other models, assumes values near 150 units explicitly.

5 Sensitivity Analysis

Here we perform a sensitivity analysis for the TCFN (standard) model for c, h, K, a_2, b_2, c_2, which are associated in the EOQ model. Considering the parametric changes from −50 % to 50% for each of the parameters and the optimum results are denoted by (*) and they are shown in Table 29.2.

From Table 29.2, we see that the purchasing cost parameter c is highly sensitive with its changes from −50% to +50% and the change of inventory cost is from −48.7% to 35.8 % . The holding cost h and set up cost k are almost insensitive whenever they are changed from −50% to +50%. For Neutrosophic truth parameter a_2 we get value of optimum inventory cost and order quantity for 50%change and −30 % change. In other cases, we get infeasible solution. For Neutrosophic falsity parameter we get optimal solution for −30% changes and other cases give infeasible solution. For Neutrosophic indeterminacy parameter c_2 we observe mild changes for 50% and 30% changes. For parameter a_2 there is small changes for −30% and 50% changes. Parameter c_2 has very mild sensitivity for 50% and 30% changes.

6 Graphical Illustration

Taking data from Table 29.1 we illustrate the following graphs below.

From Fig. 29.5, we see that the inventory cost for crisp model is \$3528.150, for general fuzzy model, it is 3496.488, and for TCFN (standard) model, it is \$3299.668. So, we see that inventory cost for TCFN (standard) model gives lesser value than crisp and general fuzzy model.

Table 29.2 Sensitivity analysis of TCFN (standard) model

Parameters	Changes	q^*	$(T_1 - T_0)^*$	Z^*	RC
$c = 60$	50%	74.519	1.5343	4791.047	35.8%
	30%	74.524	1.5345	4209.424	19.3%
	−30%	74.482	1.5328	2404.733	−31.8%
	−50%	74.467	1.5322	1808.139	−48.7%
$h = 3.5$	50%	74.474	1.5324	3365.576	−4.6%
	30%	74.462	1.532	3339.121	−5.35%
	−30%	74.509	1.5339	3273.212	−7.2%
	−50%	74.597	1.5347	3237.111	−8.2%
$K = 800$	50%	74.5298	1.5347	3392.575	−3.8%
	30%	74.4974	1.5334	3355.321	−4.9%
	−30%	74.477	1.5326	3243.835	−8%
	−50%	74.597	1.5318	3206.647	−9.1%
$p_2 = 50$	50%	85.310	1.5341	3748.695	6.2%
	30%	Infeasible	Infeasible	Infeasible	–
	−30%	66.050	1.5348	2953.071	−16.2%
	−50%	Infeasible	Infeasible	Infeasible	–
$q_2 = 50$	50%	Infeasible	Infeasible	Infeasible	–
	30%	Infeasible	Infeasible	Infeasible	–
	−30%	65.995	1.5323	2952.903	−16.3%
	−50%	Infeasible	Infeasible	Infeasible	Infeasible
$r_2 = 50$	50%	85.373	1.5323	3753.373	6.4%
	30%	81.397	1.5337	3585.612	1.6%
	−30%	Infeasible	Infeasible	Infeasible	–
	−50%	Infeasible	Infeasible	Infeasible	–

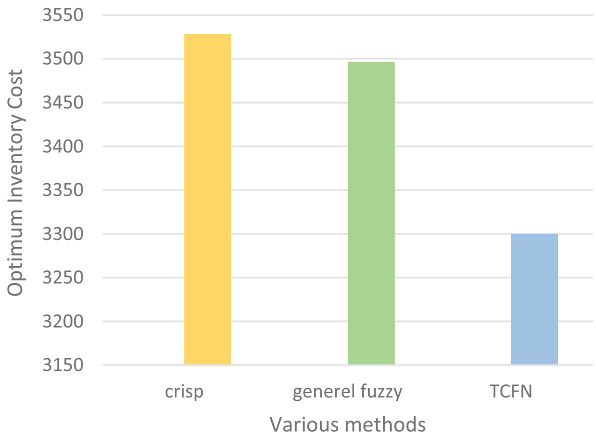

Fig. 29.5 Inventory cost under various methods

Figure 29.6 shows that the order quantity for crisp model is 151.1858 units, for general fuzzy model is 150.4280 units and that for TCFN (standard) model is 74.51

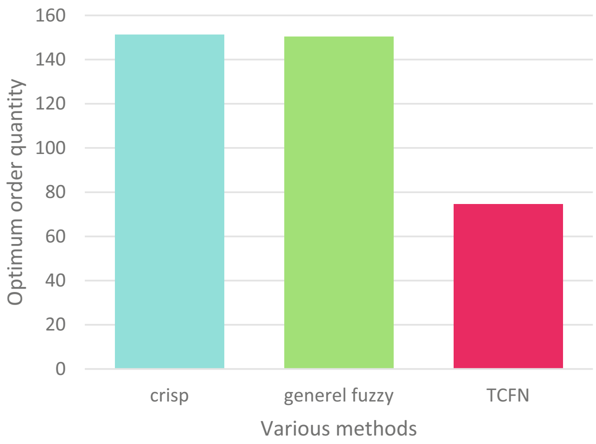

Fig. 29.6 Order quantity under various methods

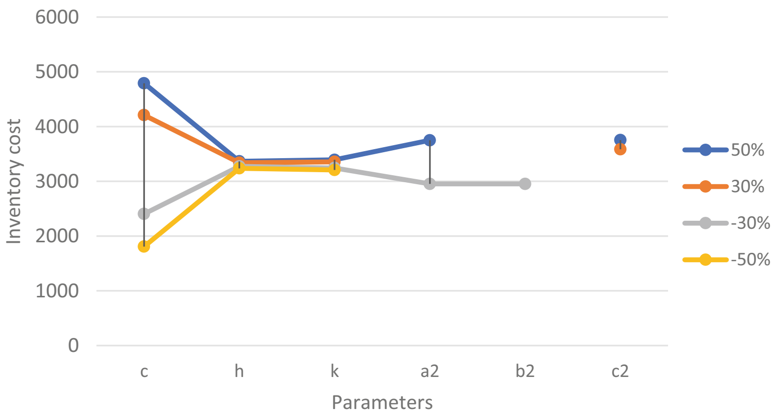

Fig. 29.7 Inventory cost under parametric sensitivity

units. So, it is obvious that optimum order quantity for TCFN (standard) model is lesser than crisp and general fuzzy model.

Taking data from Table 29.2, we illustrated the following graphs below.

Figure 29.7 shows that the purchasing cost parameter c is highly sensitive. The changes of inventory cost for this parameter range from \$1808.139 to \$4791.047. The h and K are almost insensitive and the deviation of inventory cost for these parameters is from $\$3206.647\ to\ \$3392.575$ exclusively. For the Neutrosophic truth, falsity, indeterminacy parameters, the changes of inventory cost range from $\$2750.018\ to\ \$3753.373$.Also this Figure shows that maximum and minimum inventory cost is \$4791.047 and \$1808.139 for $+50\ \%$ and $-\ 50\%$ changes of purchasing cost parameter c.

6.1 Managerial Insights

This study reveals an EOQ inventory decision under standard and nonstandard neutrosophic fuzzy flexibility of the demand parameter. In fact, in cloudy fuzzy environment, all fuzzy parameters are assumed to be changing with learning experiences gained by the decision maker from one cycle to another. The basic aim of cloudy fuzzy situation is to avail minimum cost of every modeling. Here, the following managerial insights can be achieved.

A. Standard neutrosophic fuzzy environment is more comfortable for the DM because of its cost benefit up to 6.47% with respect to the crisp decision.
B. In standard NS, the order quantity almost gets half of that of crisp and general fuzzy model.
C. The entire cycle time also becomes minimum (1.5 years) than that of crisp model (3.0323 years).
D. The time complexity is minimum all the time.
E. The only limitation is that if the DM wishes to order more and more quantities, then the cost will get an unexpected value within the same / specific cycle time period.

7 Conclusion

In this study, we have explained the existing EOQ model under TCFN demand rate. As in contemporary circumstances, predicted data is always swapping for change of learning experience of human being. Time gap is also accountable for exchanges of crisp data. So, lack of genuine data causes substantial issue to DM for making decision in inventory problem. To clarify this problem, we defuzzify classical EOQ Model under TCFN environment with standard and nonstandard situations. After analyzing a real-life data under this model, we came to the decision that this model gives better result than other existing models. The newness's of this model are

1. The fuzzy demands, which are obtained directly from data can be used easily in this model to get optimum value.
2. Changes of fuzzy flexibility with time elapsed and interactions covered will not affect in optimum result.
3. An actual real world inventory problem can be easily analyzed rather than a hypothetical world problem.
4. Learning experiences measured by the cloudy fuzzy set can get the model global minimum all the time.

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Chapter 30
An Application of Intuitionistic Fuzzy Differential Equation to the Inventory Model

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1 Introduction

When the system's knowledge is insufficient to define a fuzzy set or to describe the ambiguity (vagueness) around a declaration of assertions about the financial world in the collections themselves, the idea of IFS can be seen as an alternate option. Since human skill and knowledge are legitimate and dependable, it is envisaged that IFS may be utilized to imitate human decision-making processes and activities. Here, rejection and satisfaction levels are considered so that the total of both values is never greater than one. In this chapter, the traditional EOQ model in uncertain intuitionistic fuzzy phenomena will be covered. The following subsections describe the motivation, objective, research gaps, contribution, and orientation of this chapter.

1.1 Motivation and Objectives

The idea of fuzzy set theory is expanded more comprehensively by the intuitionistic fuzzy set theory, which considers belongingness and nonbelongingness (rejection). A phenomenon that makes decisions could experience an ambiguous scenario with

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C. Jana et al. (eds.), *Fuzzy Optimization, Decision-making and Operations Research*, https://doi.org/10.1007/978-3-031-35668-1_30

the choice between acceptance and rejection. A manager may share an acceptance-rejection dilemma when using an economic order quantity (EOQ) model with the set aim of reducing costs as much as feasible. As a result, an EOQ model may be considered in a fuzzy intuitionistic uncertain environment. It is preferable to use intuitionistic fuzzy calculus to explain the uncertain model when the parameters and decision variables are of the imprecise intuitionistic fuzzy kind.

1.2 Research Gaps

After reviewing existing literature, we have seen the following points.

(i) Few papers [21–26] describe inventory models in fuzzy environment using fuzzy differential equation approach.
(ii) Few papers [27–33] exist in the existing literature where intuitionistic fuzzy numbers are applied in a differential equation.
(iii) Some recent studies [34–39] discussed inventory control problems in the intuitionistic fuzzy environment.

But, to our knowledge, there is nothing in the existing literature describing the inventory control problem under the fuzzy differential equation approach. Table 30.1 describes the research gaps in the existing literature and contribution of this chapter.

1.3 Contribution

To fill the gaps, the current chapter is going to explore the following novel ideas:

(i) An EOQ model is considered in the intuitionistic fuzzy environment. Here, the intuitionistic fuzzy differential equation is incorporated in this current study to describe the fuzzy model.
(ii) The optimal solutions obtained in terms of (α, β)-level are defuzzified by a new defuzzification approach.

1.4 Orientation of the Manuscript

The rest of this chapter is organized into the following sections: Sect. 2 describes shortly the preliminaries section regarding intuitionistic fuzzy numbers. Section 3 stands for describing the notations and assumptions on which the model is constructed and developed. Section 4 describes the mathematical formulation of the proposed model and described it by intuitionistic fuzzy differential equations.

Table 30.1 Comparison of contributions in recent literatures and present chapter

Article details	Uncertain differential equation approach	Inventory (EOQ/EPQ)	Uncertain environment	Main contribution
Das et al. [21]	Fuzzy differential equation	EOQ	Fuzzy	Solution of an EOQ model using fast and elitist multi-objective genetic algorithm (MOGA) and interactive fuzzy decision making
Guchhait et al. [22]	Fuzzy differential equation	EPQ	Fuzzy	Solution of an EPQ model using interval compared genetic algorithm
Mondal et al. [23]	Fuzzy differential equation	EPQ	Fuzzy	Solution of an EPQ model using modified graded mean integration value (MGMIV) and fuzzy preference ordering of interval (FPOI)
Majumder et al. [24]	Fuzzy differential equation	EPQ	Fuzzy	Solution of an EPQ model using generalized Hukuhara derivative approach
Mondal [25]	Fuzzy differential equation	EOQ	Fuzzy	Solution of an EOQ model using fuzzy differential and interval differential approach
Debnath et al. [26]	Fuzzy differential equation	EOQ	Fuzzy	Solution of an EOQ model using generalized Hukuhara derivative approach.
Lata and Kumar [27]	Intuitionistic fuzzy differential equation	–	Intuitionistic fuzzy	A novel approach is suggested for resolving these nth-order time-dependent intuitionistic fuzzy linear differential equations.
Melliani and Chadli [28]	Intuitionistic fuzzy differential equation	–	Intuitionistic fuzzy	Solving partial differential equation in intuitionistic fuzzy phenomena
Abbasbandy and Allahviranloo [29]	Intuitionistic fuzzy differential equation	–	Intuitionistic fuzzy	Numerical solution approach of intuitionistic fuzzy differential equation by Runge-Kutta method
Melliani and Chadli [30]	Intuitionistic fuzzy differential equation	–	Intuitionistic fuzzy	Concept of solving initial valued differential equation in intuitionistic fuzzy phenomena

(continued)

Table 30.1 (continued)

Article details	Uncertain differential equation approach	Inventory (EOQ/EPQ)	Uncertain environment	Main contribution
Mondal and Roy [31]	Intuitionistic fuzzy differential equation	–	Intuitionistic fuzzy	Triangular intuitionistic fuzzy number as the initial value of a system of differential equations and its application
Mondal and Roy [32]	Intuitionistic fuzzy differential equation	–	Intuitionistic fuzzy	The first-order homogeneous ordinary differential equation with a triangular fuzzy beginning value
De and Sana [33]	–	EOQ	Intuitionistic fuzzy	The essay's subject is the backorder EOQ (economic order quantity) model with the promotional index for fuzzy decision variables
De and Sana [34]	–	EOQ	Intuitionistic fuzzy	Utilizing the score functions for the member and nonmembership functions, an intuitionistic fuzzy economic order quantity (EOQ) inventory model with the backlog is examined
De et al. [35]	–	EOQ	Intuitionistic fuzzy	Consumers' demand fluctuates with the selling price and promotional activity, and this chapter describes the EOQ crisp model with backlogged orders in an intuitionistic fuzzy scenario
De and Sana [36]	–	EPQ	Intuitionistic fuzzy	The relevant linear programming problem (L.P.P.) with various constraints has been first presented in a crisp model. The cost function is then adjusted to account for three different fuzzy and intuitionistic fuzzy assumptions

(continued)

Table 30.1 (continued)

Article details	Uncertain differential equation approach	Inventory (EOQ/EPQ)	Uncertain environment	Main contribution
Das et al. [37]	–	EOQ	Intuitionistic fuzzy	The current study examines a backorder economic order quantity (EOQ) model for a natural leisure/closing time system. The overall scarcity period and the seasonal effect influence the demand rate
This paper	Intuitionistic fuzzy differential equation	EOQ	Intuitionistic fuzzy	An EOQ model is considered in the intuitionistic fuzzy environment and it is solved using intuitionistic fuzzy differential equation approach

Numerical illustration in different scenarios is presented in Sect. 5. Finally, the conclusion is given in Sect. 6.

2 Literature Review

The literature review was done based on the keywords, which can be categorized into following three subsections.

2.1 Fuzzy Set Theory

Zadeh [1] was the first to introduce the concept of fuzzy uncertainty, concentrating on the degree of exclusion and inclusion of a point in a given set. However, the degree of hesitation in terms of the membership and nonmembership functions was incorporated into the notion of intuitionistic fuzzy uncertainty by Atanassov [2, 3]. Several investigations have established many arithmetic operations of the intuitionistic fuzzy sets [4–6]. The uncertainty due to the intuitionistic fuzzy sense is improved and popularly used for decision-making in different domains of science and technology [7–13].

2.2 Fuzzy Differential Equation Approach in Inventory Models

The fuzzy differential equation is an essential tool for describing many dynamical models. In this context, Kandel and Byatt first introduced the term "fuzzy differential equation" [14]. To construct the fuzzy differential equation, definitions of fuzzy derivatives were introduced [15]. Several investigations were carried out aiming at the analytical solution and proper interpretation of the first-order linear fuzzy differential equations [16–20]. When applied to inventory management issues, the inclusion of fuzzy demand rates results in FDE for the current state of the inventory level. Until recently, creating and using the fuzzy differential equation to solve the various fuzzy inventory models was challenging. The usage of two methods for solving an initial valued first-order FDE on a fuzzy EOQ model was discussed by Das et al. [21] using the fuzzy extension idea and the centroid formula for defuzzification. Guchhait et al. [22] used fuzzy differential equations and an interval-valued genetic algorithm approach to create a production inventory model with fuzzy demand and production rate in an imperfect manufacturing process. A production recycling model was developed and solved by Mondal et al. [23]. Majumber et al. [24] examined an application of the generalized Hukuhara derivative technique of FDE using an economic production quantity (EPQ) model with a partial trade credit policy in a fuzzy environment. Mondal [25] explained how to solve the fundamental inventory model using FDE and an IDE technique in an imprecise and interval environment. Using the extended Hukuhara derivative of FDE, Debnath et al. [26] proposed a sustainable fuzzy economic production quantity model with the demand expressed as a type-2 fuzzy number.

2.3 Inventory Problems in Intuitionistic Fuzzy Environment

In the current literature, intuitionistic fuzzy numbers have been used in a few articles [27–32] to solve differential equations. Recent research has examined inventory control issues in an intuitionistic and fuzzy environment. De and Sana [33] published the Classical EOQ model for promotional effort-sensitive demand, which was seen as an intuitionistic fuzzy variable. De and Sana [34] devised the EOQ model with back ordering for marketing initiatives and selling price-sensitive demand using an intuitionistic fuzzy technique. For the EOQ model with time-sensitive backlogging, De et al. [35] recently created an intuitionistic fuzzy technique-based decision-making phenomenon. In an intuitionistic fuzzy environment, De and Sana [36] created an optimal global solution for the multiperiod production—inventory model with capacity restrictions for many producers. Das et al. [37] investigated a backorder EOQ model in a natural leisure/closing time system where the demand rate depends on the full scarcity length owing to the seasonal influence.

3 Preliminaries

Definition 2.1 [31, 32] An intuitionistic fuzzy set $\tilde{A}$ in the universe of discourse X is given by the triplet $\tilde{A} = \{< x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x) >: x \in X\}$, where $\mu_{\tilde{A}}, \nu_{\tilde{A}} : X \to [0, 1]$ are two functions such that $0 \leq \mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) \leq 1$. Here, $\mu_{\tilde{A}}(x)$ is called the membership function and $\nu_{\tilde{A}}(x)$ is called the nonmembership function.

Definition 2.2 [31, 32] A triangular intuitionistic fuzzy number (TIFN) $\tilde{A} = (a_1, a_2, a_3; a_1{}', a_2, a_3{}')$ is given by the membership and nonmembership functions as the following:

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, \text{ where } a_1 \leq x \leq a_2 \\ \frac{a_3-x}{a_3-a_2}, \text{ where } a_2 \leq x \leq a_3 \\ 0, \text{ otherwise} \end{cases} \text{ and } \nu_{\tilde{A}}(x) = \begin{cases} \frac{a_2-x}{a_2-a_1{}'}, \text{ where } a_1{}' \leq x \leq a_2 \\ \frac{x-a_2}{a_3{}'-a_2}, \text{ where } a_2 \leq x \leq a_3{}' \\ 1, \text{ otherwise} \end{cases}$$

Definition 2.3 [31, 32] The (α, β)-level of the fuzzy set $\tilde{A}$ of X is given by $A_{(\alpha,\beta)} = \{x : \mu_{\tilde{A}}(x) \geq \alpha \ , \nu_{\tilde{A}}(x) < \beta : x \in X, \quad \alpha, \beta \in [0, 1], \alpha + \beta \leq 1\}$.

In parametric form the (α, β)-level of the fuzzy set $\tilde{A}$ is given by $A_{(\alpha,\beta)} = \{[A_1(\alpha), A_2(\alpha)]; [A_1{}'(\beta), A_2{}'(\beta)]\}$, where $\frac{dA_1(\alpha)}{d\alpha} > 0$, $\frac{dA_2(\alpha)}{d\alpha} < 0$, $A_1(1) \leq A_2(1)$ and $\frac{dA_1{}'(\beta)}{d\beta} < 0$, $\frac{dA_2{}'(\beta)}{d\beta} > 0$, $A_1{}'(0) \leq A_2{}'(0)$ for all $\alpha, \beta \in [0, 1], \alpha + \beta \leq 1$.

The (α, β)-level of the TIFN $\tilde{A} = (a_1, a_2, a_3; a_1{}', a_2, a_3{}')$ix given by$A_{(\alpha,\beta)} = \{[A_1(\alpha), A_2(\alpha)]; [A_1{}'(\beta), A_2{}'(\beta)]\}$, where

$$\begin{cases} A_1(\alpha) = a_1 + \alpha(a_2 - a_1) \\ A_2(\alpha) = a_3 - \alpha(a_3 - a_2) \\ A_1{}'(\beta) = a_2 - \beta(a_2 - a_1{}') \\ A_2{}'(\beta) = a_2 + \beta(a_3{}' - a_2) \end{cases}$$

Definition 2.4 [31, 32] Let f be an intuitionistic fuzzy value function defined on R. Suppose the (α, β)-level of f be given by $[f(t)]_{(\alpha,\beta)} = \{[f_1(t, \alpha), f_2(t, \alpha)]; [f_1{}'(t, \beta), f_2{}'(t, \beta)]\}$ for all $t \in R$. Also let, $f_1(t, \alpha), f_2(t, \alpha), f_1{}'(t, \beta)$ and $f_2{}'(t, \beta)$ are differentiable functions. Then,

(a) $f$is (i)-differentiable when $\frac{df(t)}{dt} = \left\{\left[\frac{df_1(t,\alpha)}{dt}, \frac{df_2(t,\alpha)}{dt}\right]; \left[\frac{df_1{}'(t,\beta)}{dt}, \frac{df_2{}'(t,\beta)}{dt}\right]\right\}$

(b) f is (ii)-differentiable when $\frac{df(t)}{dt} = \left\{\left[\frac{df_2(t,\alpha)}{dt}, \frac{df_1(t,\alpha)}{dt}\right]; \left[\frac{df_2{}'(t,\beta)}{dt}, \frac{df_1{}'(t,\beta)}{dt}\right]\right\}$.

4 Notations and Assumptions to Define Proposed EOQ Model

To describe our proposed problem, we use the following notation with certain units and description:

Notations	Units	Descriptions
Crisp model		
h	\$/unit	Holding cost per unit time
oc	\$/unit	Ordering cost per unit time
D	Units	Demand rate per cycle
T	Year	Total time cycle
Q	Units	Lot size
TAP	\$/Year	Total average profit
Decision variable for crisp model		
T Q	Year Unit	Total time cycle Lot size
Objective function for crisp model		
TAP	\$/Year	Total average profit

For the fuzzy models, the above notations h, oc, D, Q,and TAP are replaced by $\tilde{h}$, $\tilde{oc}$, $\tilde{D}$, $\tilde{Q}$, and $T\tilde{A}P$, respectively.

This chapter revisited the EOQ model given by Harris [38] in intuitionistic fuzzy environment. Harris [38]'s work contains the following assumptions.

1. Demand is constant.
2. No shortage is allowed.
3. Replenishment rate is infinite, but size is finite.
4. The horizon is finite.
5. Lead time is zero.

We assume that the demand, holding cost, and ordering cost to be uncertain in nature and given in terms of triangular intuitionistic fuzzy numbers.

5 Formulation of Mathematical Model

5.1 Crisp Model

Initially, an inventory system starts with the maximum stock Q. The stock gradually decreases meeting up the demand of the customers and reaches to zero level completing the cycle length T.

Then the classical EOQ model is governed by the differential equation

$$\begin{cases} \frac{dq(t)}{dt} = -D, \; for \; 0 \le t \le T \\ where \; q(0) = Q \\ and \; q(T) = 0 \end{cases} \tag{30.1}$$

Solving the Eq. (30.1), we get

$$q(t) = D(T - t), \text{ for } 0 \le t \le T \tag{30.2}$$

And the lot size is given by

$$Q = DT \tag{30.3}$$

Now, we calculate the costs.

Holding Cost The holding cost per unit is h and it is constant. Then, the total holding cost (*HC*) for the whole cycle is given by

$$HC = h \int_0^T D(T - t)\, dt = \frac{hDT^2}{2} \tag{30.4}$$

Ordering Cost One time ordering cost per cycle is *oc* and it is constant.

Total Average Cost The total cost is $\left(oc + \frac{hDT^2}{2}\right)$ in the whole cycle. Therefore, total average cost (*TAC*) is given by

$$TAC = \frac{oc}{T} + \frac{hDT}{2} \tag{30.5}$$

So, the optimization problem is

$$\begin{cases} \text{Min } TAC \\ TAC = \frac{oc}{T} + \frac{hDT}{2} \\ Q = DT \\ Subject \; to \; T > 0 \end{cases} \tag{30.6}$$

5.2 *Intuitionistic Fuzzy Differential Equation Approach*

Let $\tilde{Q}$ and $\tilde{D}$ be the TIFNs given by $\tilde{Q} = \left(Q_1, Q_2, Q_2; Q_1{}', Q_2, Q_2{}'\right)$ and $\tilde{D} = \left(d_1, d_2, d_3; d_1{}', d_2, d_3{}'\right)$.

Then the fuzzy differential equation is of the form

$$\begin{cases} \frac{d\tilde{q}(t)}{dt} = -\tilde{D},\ for\ 0 \le t \le T \\ where\ \tilde{q}(0) = \tilde{Q} \\ and\ \tilde{q}(T) = \tilde{0} \end{cases} \tag{30.7}$$

The (α,β)-level of $\tilde{D}$ is $D_{(\alpha,\beta)} = \{[D_1(\alpha), D_2(\alpha)]; [D_1{}'(\beta), D_2{}'(\beta)]\}$, where,

$$\begin{cases} D_1(\alpha) = d_1 + \alpha(d_2 - d_1) \\ D_2(\alpha) = d_3 - \alpha(d_3 - d_2) \\ D_1{}'(\beta) = d_2 - \beta\left(d_2 - d_1{}'\right) \\ D_2{}'(\beta) = d_2 + \beta\left(d_3{}' - d_2\right) \end{cases} \tag{30.8}$$

The (α, β)-level of $\tilde{Q}$ is $Q_{(\alpha,\beta)} = \{[Q_1(\alpha), Q_2(\alpha)]; [Q_1{}'(\beta), Q_2{}'(\beta)]\}$ where

$$\begin{cases} Q_1(\alpha) = Q_1 + \alpha(Q_2 - Q_1) \\ Q_2(\alpha) = Q_3 - \alpha(Q_3 - Q_2) \\ Q_1{}'(\beta) = Q_2 - \beta\left(Q_2 - Q_1{}'\right) \\ Q_2{}'(\beta) = Q_2 + \beta\left(Q_3{}' - Q_2\right) \end{cases} \tag{30.9}$$

Let, the on-hand inventory level $\tilde{q}(t)$ at any time t has the (α, β)-level representation $\{[q_1(t,\alpha), q_2(t,\alpha)], [q_1{}'(t,\beta), q_2{}'(t,\beta)]\}$. Now, the following two cases are considered:

Case-1 when $\tilde{q}(t)$ is (i)-differentiable

Then, from (30.7), we get

$$\left\{\left[\frac{dq_1(t,\alpha)}{dt}, \frac{dq_2(t,\alpha)}{dt}\right], \left[\frac{dq_1{}'(t,\beta)}{dt}, \frac{dq_2{}'(t,\beta)}{dt}\right]\right\} = -\left\{[D_1(\alpha), D_2(\alpha)]; \left[D_1{}'(\beta), D_2{}'(\beta)\right]\right\}$$

This gives the following system of differential equations

$$\begin{cases} \frac{dq_1(t,\alpha)}{dt} = -D_2(\alpha) \\ \frac{dq_2(t,\alpha)}{dt} = -D_1(\alpha) \\ \frac{dq_1{}'(t,\beta)}{dt} = -D_2{}'(\beta) \\ \frac{dq_2{}'(t,\beta)}{dt} = -D_1{}'(\beta) \end{cases} \tag{30.10}$$

The initial and terminal values in the (α, β)-level representation provide the following conditions for the system (30.10):

$$\begin{cases} q_1(0,\alpha) = Q_1(\alpha) \\ q_2(0,\alpha) = Q_2(\alpha) \\ q_1{}'(0,\beta) = Q_1{}'(\beta) \\ q_2{}'(0,\beta) = Q_2{}'(\beta) \end{cases} \quad (30.11)$$

And

$$\begin{cases} q_1(T,\alpha) = 0 \\ q_2(T,\alpha) = 0 \\ q_1{}'(T,\beta) = 0 \\ q_2{}'(T,\beta) = 0 \end{cases} \quad (30.12)$$

Then, solving (30.10) and using conditions given by (30.11) and (30.12),

$$\begin{cases} q_1(t,\alpha) = D_2(\alpha)(T-t) \\ q_2(t,\alpha) = D_1(\alpha)(T-t) \\ q_1{}'(t,\beta) = D_2{}'(\beta)(T-t) \\ q_2{}'(t,\beta) = D_1{}'(\beta)(T-t) \end{cases} \quad (30.13)$$

Also, the (α,β)-level of the lot size $\tilde{Q}$ in terms of demand is given by

$$\begin{cases} Q_1(\alpha) = D_2(\alpha)T \\ Q_2(\alpha) = D_1(\alpha)T \\ Q_1{}'(\beta) = D_2{}'(\beta)T \\ Q_2{}'(\beta) = D_1{}'(\beta)T \end{cases} \quad (30.14)$$

Now, we calculate the intuitionistic fuzzy costs.

Holding Cost The intuitionistic fuzzy holding cost per unit is $\tilde{h}$ and let, the (α,β)-level of the holding cost per unit quantity per unit time are given by $\{[h_1(\alpha), h_2(\alpha)]; [h_1{}^{'}(\beta), h_2{}^{'}(\beta)]\}$. Then, the (α,β)-level of the total holding cost $(\widetilde{HC})$ for the whole cycle is given by $\{[HC_1(T,\alpha), HC_2(T,\alpha)]; [HC_1{}^{'}(T,\beta), HC_2{}^{'}(T,\beta)]\}$, where

$$\begin{cases} HC_1(T,\alpha) = h_1(\alpha)\int_0^T D_2(\alpha)(T-t)\,dt = \frac{h_1(\alpha)D_2(\alpha)T^2}{2} \\ HC_2(T,\alpha) = h_2(\alpha)\int_0^T D_1(\alpha)(T-t)\,dt = \frac{h_2(\alpha)D_1(\alpha)T^2}{2} \\ HC_1{}'(T,\beta) = h_1{}'(\beta)\int_0^T D_2{}'(\beta)(T-t)\,dt = \frac{h_1{}'(\beta)D_2{}'(\beta)T^2}{2} \\ HC_2{}'(T,\beta) = h_2{}'(\beta)\int_0^T D_1{}'(\beta)(T-t)\,dt = \frac{h_2{}'(\beta)D_1{}'(\beta)T^2}{2} \end{cases} \quad (30.15)$$

Ordering Cost One-time intuitionistic fuzzy ordering cost per cycle is $\tilde{oc}$ and let the (α,β)-level of the ordering cost is given by$\{[c_1(\alpha), c_2(\alpha)]; [c_1{}^{'}(\beta), c_2{}^{'}(\beta)]\}$.

Total Average Cost Then, the (α, β)-level of the total average cost $(\widetilde{TAC})$ for the whole cycle is given by $\{[TAC_1(T, \alpha), TAC_2(T, \alpha)]; [TAC_1{}'(T, \beta), TAC_2{}'(T, \beta)]\}$, where

$$\begin{cases} TAC_1(T, \alpha) = \frac{c_1(\alpha)}{T} + \frac{h_1(\alpha)D_2(\alpha)T}{2} \\ TAC_2(T, \alpha) = \frac{c_2(\alpha)}{T} + \frac{h_2(\alpha)D_1(\alpha)T}{2} \\ TAC_1{}'(T, \beta) = \frac{c_1{}'(\beta)}{T} + \frac{h_1{}'(\beta)D_2{}'(\beta)T}{2} \\ TAC_2{}'(T, \beta) = \frac{c_2{}'(\beta)}{T} + \frac{h_2{}'(\beta)D_1{}'(\beta)T}{2} \end{cases} \tag{30.16}$$

Then, the optimization problem is given by

$$\begin{cases} \text{Maximize } \alpha \\ \text{Minimize } \beta \\ TAC_{(\alpha,\beta)} = \{[TAC_1(\alpha), TAC_2(\alpha)]; [TAC_1{}'(\beta), TAC_2{}'(\beta)]\} \\ Subject\ to\ TAC_1(\alpha) \le TAC_2(\alpha)\ and\ TAC_1{}'(\beta) \le TAC_2{}'(\beta)\ and\ values\ are\ given\ by\ (30.16) \\ Q_{(\alpha,\beta)} = \{[Q_1(\alpha), Q_2(\alpha)]; [Q_1{}'(\beta), Q_2{}'(\beta)]\}\ and\ values\ are\ given\ by\ (30.14) \\ 0 \le \alpha \le 1, 0 \le \beta \le 1\ and\ \alpha + \beta \le 1 \\ T > 0 \end{cases}$$

Equivalently the optimization problem can be given by

$$\begin{cases} \text{Maximize } (\alpha - \beta) \\ \alpha > \beta \\ TAC_{(\alpha,\beta)} = \{[TAC_1(\alpha), TAC_2(\alpha)]; [TAC_1{}'(\beta), TAC_2{}'(\beta)]\} \\ Subject\ to\ TAC_1(\alpha) \le TAC_2(\alpha)\ and\ TAC_1{}'(\beta) \le TAC_2{}'(\beta)\ and\ values\ are\ given\ by\ (30.16) \\ Q_{(\alpha,\beta)} = \{[Q_1(\alpha), Q_2(\alpha)]; [Q_1{}'(\beta), Q_2{}'(\beta)]\}\ and\ values\ are\ given\ by\ (30.14) \\ 0 \le \alpha \le 1, 0 \le \beta \le 1\ and\ \alpha + \beta \le 1 \\ T > 0 \end{cases}$$

Case-2 when $\tilde{q}(t)$ is (ii)-differentiable

Then, from (30.7), we get

$$\left\{\left[\frac{dq_2(t, \alpha)}{dt}, \frac{dq_1(t, \alpha)}{dt}\right], \left[\frac{dq_2{}'(t, \beta)}{dt}, \frac{dq_1{}'(t, \beta)}{dt}\right]\right\} = -\{[D_1(\alpha), D_2(\alpha)]; [D_1{}'(\beta), D_2{}'(\beta)]\}$$

This gives the following system of differential equations

$$\begin{cases} \frac{dq_1(t,\alpha)}{dt} = -D_1(\alpha) \\ \frac{dq_2(t,\alpha)}{dt} = -D_2(\alpha) \\ \frac{dq_1{}'(t,\beta)}{dt} = -D_1{}'(\beta) \\ \frac{dq_2{}'(t,\beta)}{dt} = -D_2{}'(\beta) \end{cases} \tag{30.17}$$

Then, solving (30.17) and using conditions given by (30.11) and (30.12),

$$\begin{cases} q_1(t,\alpha) = D_1(\alpha)(T-t) \\ q_2(t,\alpha) = D_2(\alpha)(T-t) \\ q_1{}'(t,\beta) = D_1{}'(\beta)(T-t) \\ q_2{}'(t,\beta) = D_2{}'(\beta)(T-t) \end{cases} \tag{30.18}$$

Also, the (α, β)-level of the lot size $\tilde{Q}$ in terms of demand is given by

$$\begin{cases} Q_1(\alpha) = D_1(\alpha)T \\ Q_2(\alpha) = D_2(\alpha)T \\ Q_1{}'(\beta) = D_1{}'(\beta)T \\ Q_2{}'(\beta) = D_2{}'(\beta)T \end{cases} \tag{30.19}$$

Now, we calculate the intuitionistic fuzzy costs.

Holding Cost The intuitionistic fuzzy holding cost per unit is $\tilde{h}$ and let the (α, β)-level of the holding cost per unit quantity per unit time be given by $\{[h_1(\alpha), h_2(\alpha)]; [h_1{}'(\beta), h_2{}'(\beta)]\}$. Then, the (α, β)-level of the total holding cost $(\widetilde{HC})$ for the whole cycle is given by $\{[HC_1(T,\alpha), HC_2(T,\alpha)]; [HC_1{}'(T,\beta), HC_2{}'(T,\beta)]\}$, where

$$\begin{cases} HC_1(T,\alpha) = h_1(\alpha)\int_0^T D_1(\alpha)(T-t)\,dt = \frac{h_1(\alpha)D_1(\alpha)T^2}{2} \\ HC_2(T,\alpha) = h_2(\alpha)\int_0^T D_2(\alpha)(T-t)\,dt = \frac{h_2(\alpha)D_2(\alpha)T^2}{2} \\ HC_1{}'(T,\beta) = h_1{}'(\beta)\int_0^T D_1{}'(\beta)(T-t)\,dt = \frac{h_1{}'(\beta)D_1{}'(\beta)T^2}{2} \\ HC_2{}'(T,\beta) = h_2{}'(\beta)\int_0^T D_2{}'(\beta)(T-t)\,dt = \frac{h_2{}'(\beta)D_2{}'(\beta)T^2}{2} \end{cases} \tag{30.20}$$

Ordering Cost One-time intuitionistic fuzzy ordering cost per cycle is $\tilde{oc}$ and let the (α, β)-level of the ordering cost be given by $\{[c_1(\alpha), c_2(\alpha)]; [c_1{}'(\beta), c_2{}'(\beta)]\}$.

Total Average Cost Then, the (α, β)-level of the total average cost $(\widetilde{TAC})$ for the whole cycle is given by $\{[TAC_1(T,\alpha), TAC_2(T,\alpha)]; [TAC_1{}'(T,\beta), TAC_2{}'(T,\beta)]\}$, where

$$\begin{cases} TAC_1(T,\alpha) = \frac{c_1(\alpha)}{T} + \frac{h_1(\alpha)D_1(\alpha)T}{2} \\ TAC_2(T,\alpha) = \frac{c_2(\alpha)}{T} + \frac{h_2(\alpha)D_2(\alpha)T}{2} \\ TAC_1{}'(T,\beta) = \frac{c_1{}'(\beta)}{T} + \frac{h_1{}'(\beta)D_1{}'(\beta)T}{2} \\ TAC_2{}'(T,\beta) = \frac{c_2{}'(\beta)}{T} + \frac{h_2{}'(\beta)D_2{}'(\beta)T}{2} \end{cases} \tag{30.21}$$

Then the optimization problem will be

$$\begin{cases} \text{Maximize } (\alpha - \beta) \\ \alpha > \beta \\ TAC_{(\alpha,\beta)} = \{[TAC_1(\alpha), TAC_2(\alpha)]; [TAC_1'(\beta), TAC_2'(\beta)]\} \\ \textit{Subject to } TAC_1(\alpha) \le TAC_2(\alpha) \textit{ and } TAC_1'(\beta) \le TAC_2'(\beta) \textit{ and values are given by } (30.19) \\ Q_{(\alpha,\beta)} = \{[Q_1(\alpha), Q_2(\alpha)]; [Q_1'(\beta), Q_2'(\beta)]\} \textit{ and values are given by } (30.21) \\ 0 \le \alpha \le 1, 0 \le \beta \le 1 \textit{ and } \alpha + \beta \le 1 \\ T > 0 \end{cases}$$

5.3 Defuzzification of Total Average Cost and Lot Size

The defuzzified value of $TAC_{(\alpha,\beta)}$ is given by $TAC_{i\to c} = TAC_{crisp} + (TAC_\alpha - TAC_\beta)$, $TAC_\alpha = \frac{\sum_{i=1}^{m}\{TAC_1(\alpha_i)+TAC_2(\alpha_i)\}}{2m}$ and $TAC_\beta = \frac{\sum_{j=1}^{n} TAC_1'(\beta_j)+TAC_1'(\beta_j)}{2n}$, where i and j are not distinct.

The defuzzified value of $Q_{(\alpha,\beta)}$ is given by $Q_{i\to c} = Q_{crisp} + (Q_\alpha - Q_\beta)$, $Q_\alpha = \frac{\sum_{i=1}^{m}\{Q_1(\alpha_i)+Q_2(\alpha_i)\}}{2m}$ and $Q_\beta = \frac{\sum_{j=1}^{n} Q_1'(\beta_j)+Q_1'(\beta_j)}{2n}$, where i and j not are distinct.

6 Numerical Illustration

For numerical simulation, we used the hypothetical data. Then, the crisp and fuzzy models after defuzzification as described in Sect. 5.3 were optimized. We used LINGO 18.0 software for numerical solution. The machine was 64 bits. The following three subsections describe the numerical simulation in details.

6.1 Crisp and Fuzzy Solutions

(a) For crisp model, let $oc = 550$, $h = 3$, and $D = 350$. Then, the optimal values of the objective function and the decisions variables are given by
$TAC = 1500$, $T = 0.432$, and $Q = 151.19$.

(b) Let us consider the values of the fuzzy parameters as TIFNs:

$\tilde{oc} = (500, 550, 600; 490, 550, 610)$, $\tilde{h} = (2.5, 3, 3.5; 2.4, 3, 3.6)$ and $\tilde{D} = (300, 350, 400; 290, 350, 410)$. Then, the solutions of T, $\underset{\sim}{Q}$, and $T\tilde{A}C$ for the case of (i) and (ii)-differentiability are given in Tables 30.2 and 30.3, respectively.

Following Table 30.2, the components of total average cost in the (α, β)-level against the total time cycle and the acceptance and rejection level are plotted three-dimensionally and given in Figs. 30.1, 30.2, 30.3, and 30.4.

Table 30.2 Values of the objective function and decision variables (for the case of (i)-differentiability)

α	β	T	$Q_1(\alpha)$	$Q_2(\alpha)$	$Q_1{}'(\beta)$	$Q_2{}'(\beta)$	$TAC_1(\alpha)$	$TAC_2(\alpha)$	$TAC_1{}'(\beta)$	$TAC_1{}'(\beta)$
1	0	1.307	457.57	457.57	457.57	457.57	1107.05	1107.05	1107.05	1107.05
0.9	0	0.144	51.12	49.68	50.40	50.40	3860.21	3930.01	3895.13	3895.13
0.9	0.1	0.682	242.09	235.27	242.77	234.59	1156.27	1172.64	1154.60	1174.24
0.8	0	0.753	272.28	257.16	264.72	264.72	1108.77	1139.00	1124.26	1124.26
0.8	0.1	0.137	49.31	46.57	48.76	47.11	4014.27	4160.98	4043.65	4131.68
0.8	0.2	0.437	157.25	148.51	158.12	147.64	1464.29	1515.26	1459.39	1516.96
0.7	0	0.251	91.70	84.16	87.93	87.93	2260.15	2381.44	2321.08	2321.08
0.7	0.1	0.149	54.53	50.05	53.19	51.39	3658.78	3860.70	3719.49	3800.27
0.7	0.2	0.140	51.24	47.03	50.82	47.45	3883.66	4098.40	3905.19	4076.98
0.7	0.3	0.231	84.39	77.45	85.08	76.76	2434.18	2565.67	2420.92	2578.70
0.6	0	0.133	49.18	43.87	46.52	46.52	4056.06	4358.31	4207.46	4207.46
0.6	0.1	0.159	58.93	52.56	56.70	54.79	3410.12	3662.85	3498.87	3574.69
0.6	0.2	0.142	52.67	46.98	51.53	48.12	3796.68	4079.08	3853.35	4022.79
0.6	0.3	0.130	48.28	43.06	48.02	43.32	4129.55	4437.42	4145.00	4422.08
0.6	0.4	0.566	209.39	186.75	211.65	184.49	1229.68	1306.02	1221.55	1313.16
0.5	0	0.163	61.12	52.97	57.04	57.04	3305.20	3614.01	3460.12	3460.12
0.5	0.1	0.522	195.63	169.55	185.72	179.46	1275.35	1377.71	1315.78	1340.35
0.5	0.2	0.238	89.25	77.35	86.16	80.45	2328.56	2541.61	2384.52	2486.79
0.5	0.3	0.981	367.77	318.73	360.90	325.60	1041.00	1104.25	1051.33	1096.87
0.5	0.4	0.615	230.80	200.028	230.19	200.64	1170.36	1259.29	1172.29	1257.66
0.4	0	0.132	50.16	42.24	46.20	46.20	4006.97	4463.48	4235.82	4235.82
0.4	0.1	0.275	104.54	88.03	97.94	94.63	2031.34	2253.56	2121.42	2165.86
0.4	0.2	0.534	202.93	170.89	193.32	180.50	1247.69	1368.06	1285.82	1333.96
0.4	0.3	0.526	199.85	168.30	193.54	174.61	1258.54	1380.51	1284.44	1357.63
0.3	0	0.677	260.82	213.40	237.11	237.11	1105.78	1220.96	1167.52	1167.52
0.3	0.1	0.588	226.30	185.16	209.25	202.20	1176.01	1305.38	1233.10	1255.28
0.3	0.2	0.163	62.56	51.19	58.83	54.93	3252.05	3685.65	3395.39	3544.06
0.2	0	0.157	61.15	48.61	54.88	54.88	3332.24	3845.61	3590.18	3590.18
0.2	0.1	0.455	177.49	141.08	162.01	156.55	1351.38	1536.27	1433.52	1461.25
0.1	0	0.666	262.93	203.02	232.98	232.98	1093.89	1244.08	1175.72	1175.72

Table 30.3 Values of the objective function and decision variables (for the case of (ii)-differentiability)

α	β	T	$Q_1(\alpha)$	$Q_2(\alpha)$	$Q_1^{'}(\beta)$	$Q_2^{'}(\beta)$	$TAC_1(\alpha)$	$TAC_2(\alpha)$	$TAC_1^{'}(\beta)$	$TAC_1^{'}(\beta)$
1	0	0.02	0.65	0.65	0.65	0.65	296078.6	296078.6	296078.6	296078.6
0.9	0	0.140	48.16	49.56	48.86	48.86	3975.11	4051.28	4013.17	4013.17
0.9	0.1	0.599	206.62	212.60	206.02	213.20	1214.78	1250.94	1211.20	1254.59
0.8	0	0.900	305.99	323.99	314.99	314.99	1043.70	1124.42	1083.61	1083.61
0.8	0.1	0.554	188.28	199.36	190.50	197.14	1248.13	1320.24	1262.38	1305.65
0.8	0.2	0.580	197.15	208.74	195.99	209.90	1217.15	1289.33	1210.06	1296.68
0.7	0	0.696	233.17	254.06	243.62	243.62	1100.90	1211.87	1155.60	1155.60
0.7	0.1	0.551	184.51	201.04	189.47	196.08	1234.28	1342.45	1266.21	1309.47
0.7	0.2	0.664	222.43	242.35	224.42	240.36	1122.72	1232.64	1133.44	1221.38
0.7	0.3	0.677	226.85	247.16	224.82	249.19	1113.32	1223.65	1102.63	1235.02
0.6	0	0.704	232.41	260.58	246.49	246.49	1077.92	1226.28	1150.69	1150.69
0.6	0.1	0.603	199.11	223.24	207.56	214.80	1157.17	1301.90	1206.73	1250.15
0.6	0.2	0.797	263.11	295.01	269.49	288.63	1033.09	1186.91	1062.83	1155.13
0.6	0.3	0.536	176.85	198.29	177.93	197.22	1236.54	1380.85	1243.55	1373.43
0.6	0.4	0.592	195.22	218.88	192.85	221.25	1169.22	1313.74	1155.29	1328.72
0.5	0	0.169	54.88	63.32	59.10	59.10	3184.48	3508.02	3345.72	3345.72
0.5	0.1	0.562	182.60	210.69	193.28	200.02	1185.49	1365.78	1252.35	1295.62
0.5	0.2	0.602	195.64	225.73	203.46	217.91	1141.15	1322.04	1186.74	1273.56
0.5	0.3	0.675	219.35	253.10	224.08	248.38	1079.46	1263.22	1104.17	1236.48
0.5	0.4	0.147	47.86	55.22	48.01	55.08	3630.90	3994.37	3638.14	3987.06
0.4	0	0.302	96.51	114.61	105.56	105.56	1854.46	2112.21	1981.98	1981.98
0.4	0.1	0.218	69.91	83.01	75.15	77.77	2474.67	2791.92	2600.63	2664.08
0.4	0.2	0.435	139.17	165.27	147.00	157.44	1383.52	1606.29	1448.71	1537.81
0.4	0.3	0.134	42.79	50.81	44.39	49.21	3946.50	4421.27	4041.07	4325.93
0.3	0	0.670	211.06	257.97	234.52	234.52	1048.26	1305.17	1172.61	1172.61
0.3	0.1	0.566	178.24	217.84	194.64	201.43	1146.34	1398.77	1247.55	1290.83
0.3	0.2	0.798	251.24	307.07	269.58	288.72	978.59	1247.81	1062.74	1155.04
0.2	0	0.515	159.66	200.86	180.26	180.26	1197.80	1487.04	1338.30	1338.30
0.2	0.1	0.480	148.95	187.38	165.28	171.05	1255.09	1546.51	1375.19	1418.90
0.1	0	0.150	45.61	59.07	52.34	52.34	3435.15	4080.73	3756.42	3756.42

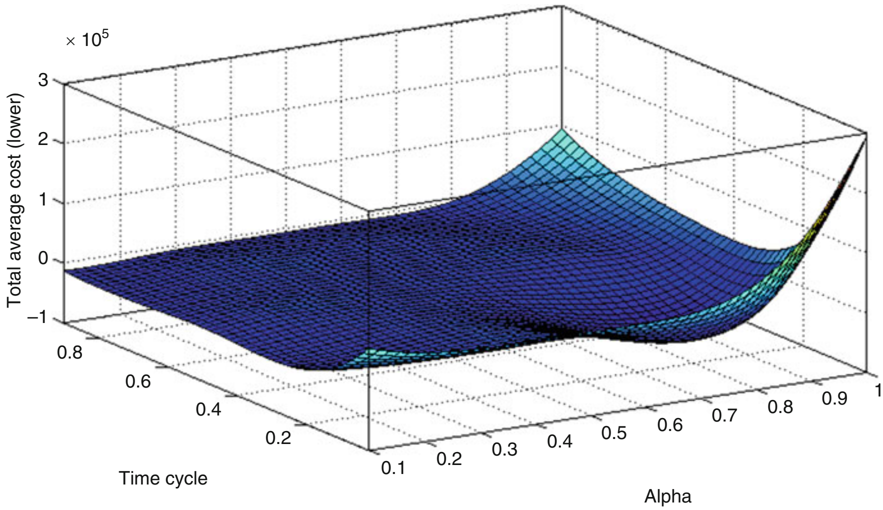

Fig. 30.1 Inter-dependency among total average cost (lower), time cycle, and alpha for the case of (i)-differentiability

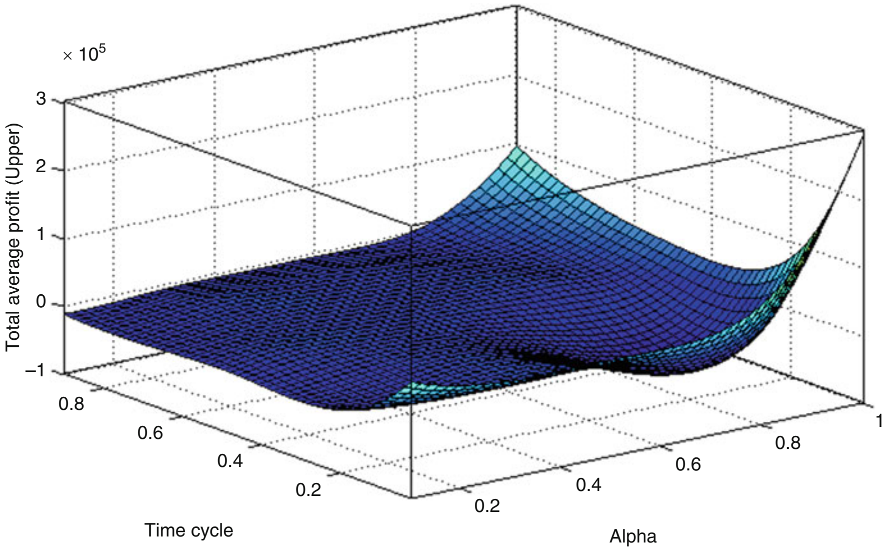

Fig. 30.2 Inter-dependency among total average cost (upper), time cycle, and alpha for the case of (i)-differentiability

Following Table 30.3, the components of total average cost in the (α, β)- level against the total time cycle and the acceptance and rejection level are plotted three-dimensionally and given in Figs. 30.5, 30.6, 30.7, and 30.8.

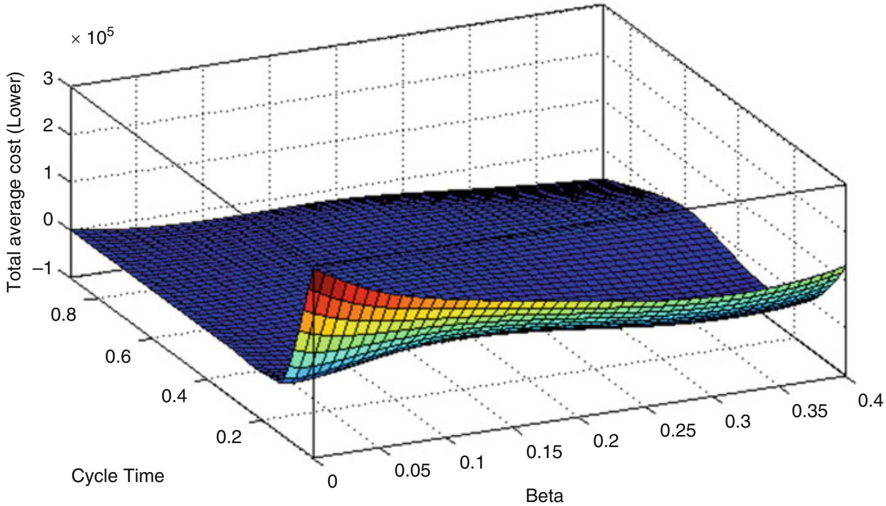

Fig. 30.3 Inter-dependency among total average cost (lower), time cycle, and beta for the case of (i)-differentiability

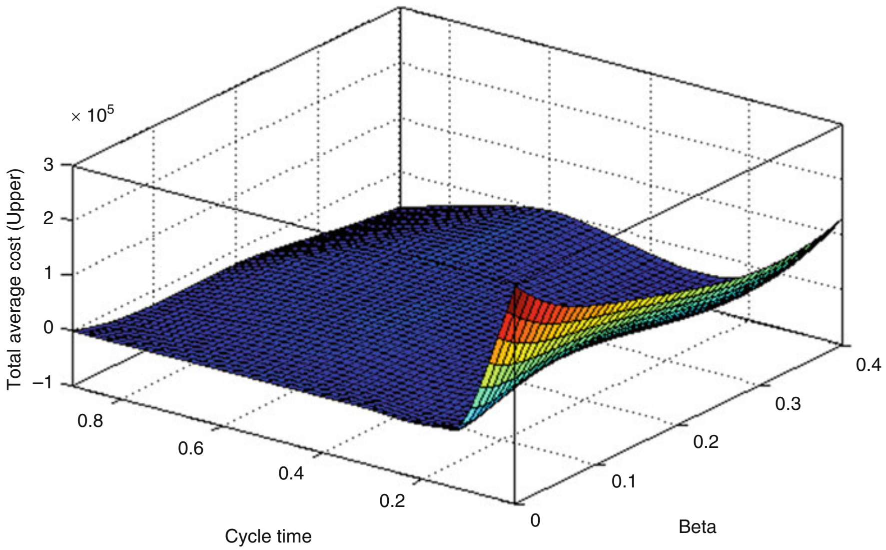

Fig. 30.4 Inter-dependency among total average cost (Upper), time cycle, and beta for the case of (i)-differentiability

6.2 Comparison Among Three Cases

When (i)-differentiability is considered, then from Table 30.2, we get

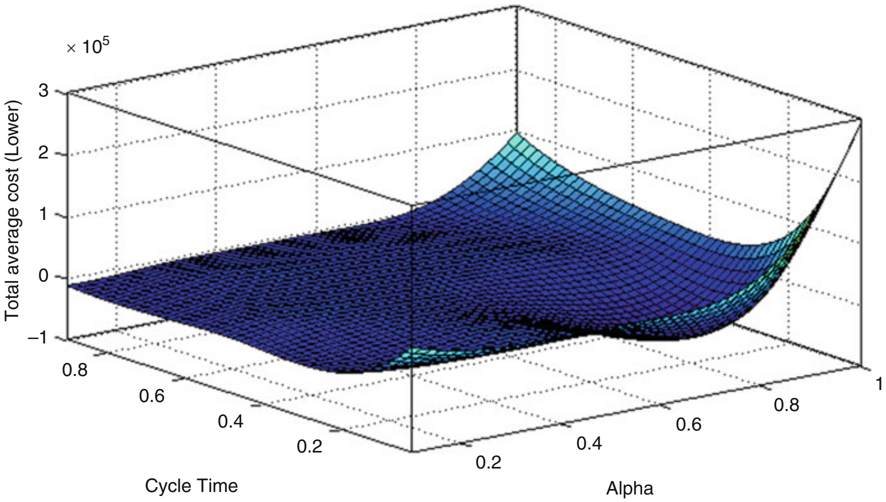

Fig. 30.5 Inter-dependency among total average cost (lower), time cycle, and alpha for the case of (ii)-differentiability

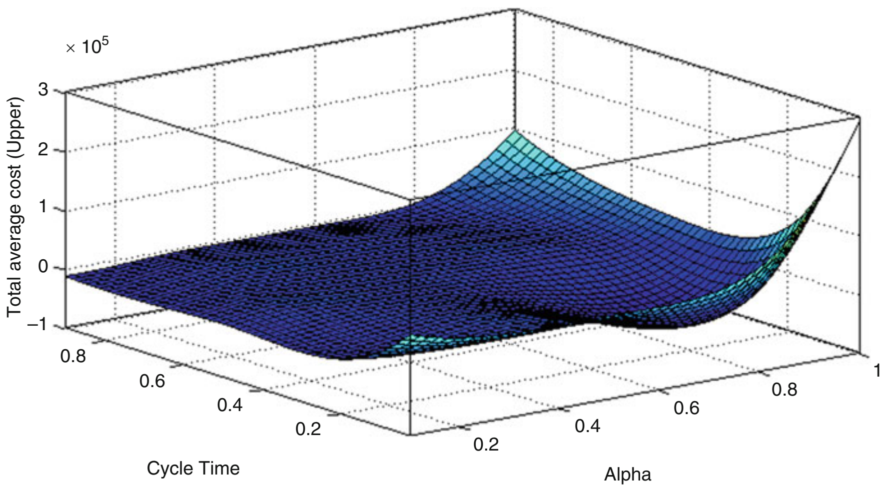

Fig. 30.6 Inter-dependency among total average cost (upper), time cycle, and alpha for the case of (ii)-differentiability

$TAC_\alpha = 2442.02$, $TAC_\beta = 2442.99$, and $Q_\alpha = 140.67(approx.) = Q_\beta$
Then, $TAC_{\mathrm{i} \to \mathrm{c}} = 1499.03$ and $Q_{\mathrm{i} \to \mathrm{c}} = 151.19$
Again, when (ii)-differentiability is considered, then from Table 30.3, we get
$TAC_\alpha = 11565.84$, $TAC_\beta = 11564.67$, and $Q_\alpha = 175.19(approx.) = Q_\beta$
Then, $TAC_{\mathrm{i} \to \mathrm{c}} = 1501.17$ and $Q_{\mathrm{i} \to \mathrm{c}} = 151.19$

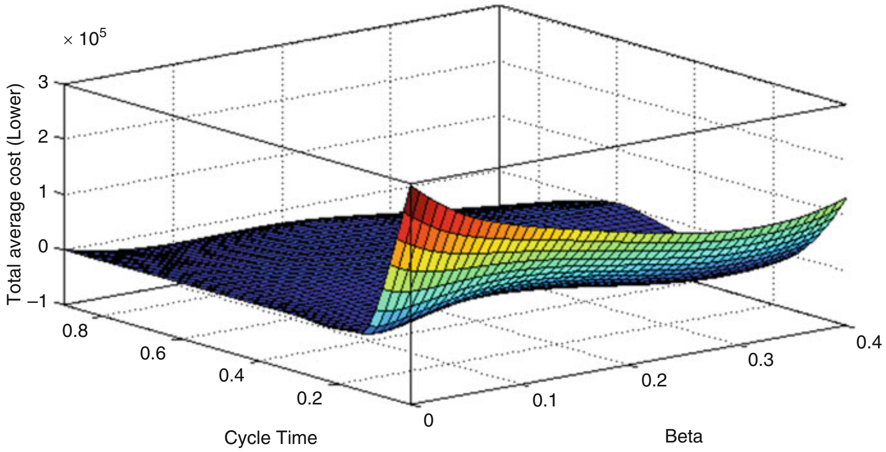

Fig. 30.7 Inter-dependency among total average cost (lower), time cycle, and beta for the case of (ii)-differentiability

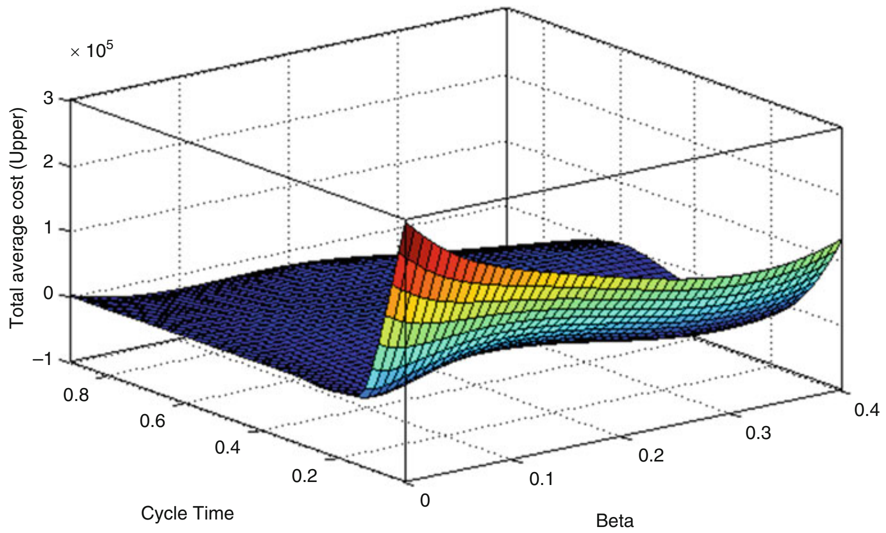

Fig. 30.8 Inter-dependency among total average cost (Upper), time cycle, and beta for the case of (ii)-differentiability

The defuzzified optimal values of total average cost and lot size along with the crisp result are represented in Table 30.4.

The lot size in the three different situations remains the same. But the total average cost varies less significantly. The comparison of the values of the total average cost is presented in Fig. 30.9.

Table 30.4 Values of total average cost and lot size in different scenarios

Scenario	Lot size	Total average cost
Crisp	151.19	1500
Intuitionistic fuzzy[(i)-differentiability]	151.19	1499.03
Intuitionistic fuzzy[(ii)-differentiability]	151.19	1501.17

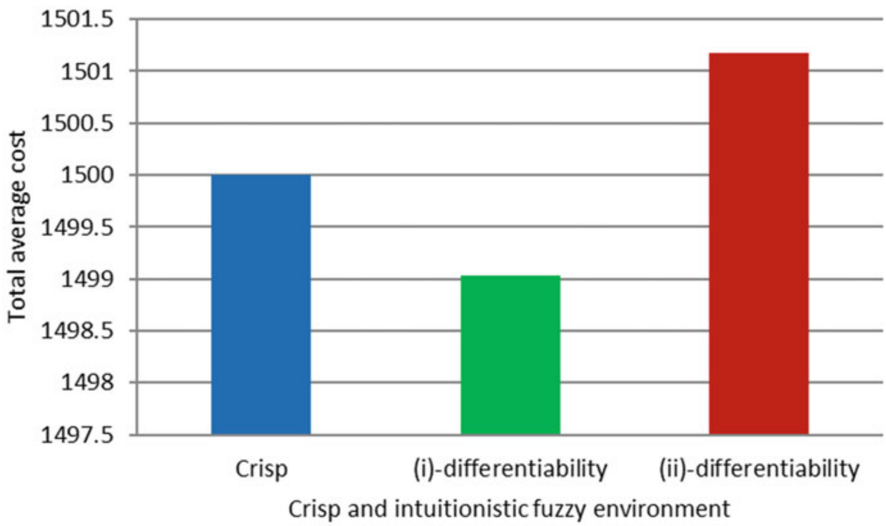

Fig. 30.9 Comparison of crisp total average cost, defuzzified total average costs for the cases of (i) and (ii)-differentiability

Though the variation of Total average cost in the three different considerations with the present data set is almost negligible, from the bar diagram given in Fig. 30.9, our basic observation is the following:

(i) The case regarding (i)-differentiability of $q(t)$ gives the best result in the cost minimization objective. Even the result, in this case, is better than the crisp result.
(ii) The case of (ii)-differentiability of $q(t)$ is not suitable for this cost minimization perspective as it gives a worse result than the crisp result.

6.3 Managerial Insights

The choice between acceptance and rejection might present as a phenomenon that makes decisions. When utilizing an economic order quantity (EOQ) model to lower costs as much as possible, a manager may experience an acceptance-rejection dilemma. An EOQ model can therefore be taken into account in a fuzzy intuitionistic uncertain environment. In cases when the decision variables and parameters are of the imprecise intuitionistic fuzzy kind, it is better to describe the uncertain model using intuitionistic fuzzy calculus. This chapter suggests an approach to deal with the dilemma of acceptance-rejection associated with impreciseness for wise managerial decisions. This is the central executive insight of this chapter.

7 Conclusion

This chapter has attempted to describe an inventory control problem under the intuitionistic fuzzy differential equation approach. In such an initial study, we consider the classical EOQ model to analyze in a new direction. A defuzzification technique is also developed anew to compare with the crisp model. From the numerical analysis, it is perceived that the intuitionistic fuzzy environment with (i)-differentiability of $q(t)$provides the best result in favor of the cost minimization goal among the three discussed approaches, while the intuitionistic fuzzy environment with (ii)-differentiability of $q(t)$shows the worst one. As per our knowledge, this will be a novel approach in the literature to discuss the inventory model in a fuzzy situation. In the end, we admit our limitations in discussing this chapter. We redefined the classical EOQ model in an intuitionistic fuzzy environment and discussed the model in an intuitionistic fuzzy differential equation approach. Though the proposed method is very new and logical, we discussed the classical EOQ model, which was a straightforward model and may not catch complex retailing scenarios properly. Further, we explained the model based on the hypothetical data. In the future, the intuitionistic fuzzy differential equation approach can be applied to discuss the more complicated and accurate marketing-retailing-based model. Instead of the theoretical data, the raw data collected from the real-world business field can validate the proposed approach in the upcoming days.

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Chapter 31
Solution of the Second-Order Linear Intuitionistic Fuzzy Difference Equation by Extension Principle Scheme

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1 Introduction

The notion based on fuzzy numbers and fuzzy arithmetic was initially developed by Zadeh [1], Dubois, and Parade [2]. The theory of generalized fuzzy sets [1] is regarded as one of the intuitionistic fuzzy sets (IFS). Later, Atanassov created the notion of an intuitionistic fuzzy set and expanded the fuzzy set concept [3–5].We consider a second-order linear nonhomogeneous difference equation with its initial information as follows:

$$\begin{cases} a_0 u_{n+2} + a_1 u_{n+1} + a_2 u_n = f(n) \\ u_{n=0} = u_0, u_{n=1} = u_1 \end{cases} \tag{31.1}$$

In Eq. (31.1), $a_0 \neq 0$, a_1 and a_2 are coefficients of the linear difference equation, and $f(n)$ is the nonhomogeneous part of the above difference equation, which is a function of n alone. Equation (31.1) can be viewed as an intuitionistic fuzzy difference equation, taking one or more of the initial information u_0 and u_1 and the coefficients $a_0 \neq 0$, a_1 and a_2 as intuitionistic fuzzy number(s). The following subsections describe the motivation, objectives, research gaps, and contribution of this chapter.

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C. Jana et al. (eds.), *Fuzzy Optimization, Decision-making and Operations Research*, https://doi.org/10.1007/978-3-031-35668-1_31

1.1 Motivation and Objectives

The discrete calculus and fuzzy decision-making methods used in the fuzzy difference equation investigate the computational framework's theoretical underpinnings that depict the uncertainty inherent in the modeling and the underflowing discrete behavior. The nonlinear fuzzy difference equations with various beginning conditions and coefficients represented as fuzzy numbers allow for the identification of the model. The fuzzy difference equation efficiently establishes the mathematical correspondence of a discrete dynamical system with modeling-related parameter uncertainty. Numerous authors consider intuitionistic fuzzy numbers in various works and use them in multiple contexts. However, only a few scholars have solved the difference equations accompanied by intuitionistic fuzzy numbers or functions. But what mathematical tool will represent the phenomenon if a physical situation passes through a discrete dynamical system and has uncertainty with an acceptance-rejection sense? We look for the best work on this topic based on this motive.

1.2 Research Gaps

An extensive literature survey based on the keywords of this chapter, which are summarized in the next section, reveals the lacuna in literature as follows:

(i) The theory of fuzzy differential equations is not discussed too much compared to fuzzy difference equations. However, there are causes to develop the approach of fuzzy difference equation to describe the discrete dynamical system under uncertainty.
(ii) Majority of the fuzzy difference equations are discussed in a general fuzzy environment. Few (e.g., Mondal et al. [6]) used intuitionistic fuzzy phenomena.
(iii) Up to the authors' knowledge, no such literature has discussed difference equations of order two in an intuitionistic fuzzy environment.

1.3 Contribution

The contributions of this present chapter can be summarized as follows:

(i) The second-order difference equation is analyzed using an intuitionistic fuzzy frame.
(ii) The problem is addressed extensively using the extension principal approach.
(iii) An appropriate application and numerical examples are provided to illustrate the proposed theory.

Table 31.1 Comparison of contributions in recent literatures and this chapter

Article details	System taken (discrete/ continuous)	Linear/nonlinear	Environment	Main contribution
Mondal and Roy [7]	Continuous	Linear	Intuitionistic fuzzy	Discussion of first-order linear homogenous differential equation in intuitionistic fuzzy environment
Mondal and Roy [8]	Continuous	Linear	Intuitionistic fuzzy	Discussion of first-order linear nonhomogenous differential equation in intuitionistic fuzzy environment
Mondal and Roy [9]	Continuous	Linear	Intuitionistic fuzzy	Solving the system of differential equations in intuitionistic fuzzy environment
Mondal [10]	Continuous	Linear	Interval valued intuitionistic fuzzy	Discussion of first-order linear differential equation in interval valued intuitionistic fuzzy environment
Alamin et al. [11]	Discrete	Linear	General fuzzy	The solution techniques of nonhomogeneous fuzzy linear difference equation in fuzzy environment
Alamin et al. [12]	Discrete	Linear	Neutrosophic	The solution of the homogeneous difference equation with initial information, coefficient and both as neutrosophic numbers
Rahaman et al. [13]	Discrete	Linear	Gaussian fuzzy	Properties of Gaussian fuzzy numbers and their applications to describe fuzzy difference equations
Rahaman et al. [14]	Discrete	Linear	Interval number	Solution of linear difference equations in interval environment
Mondal et al. [6]	Discrete	Linear	Intuitionistic fuzzy	Solution of linear difference equations in intuitionistic fuzzy environment
This paper	Discrete	Quadratic	Intuitionistic fuzzy	Solution of second-order difference equations in intuitionistic fuzzy environment

The contributions of the recent literature and this chapter are summarized in Table 31.1.

1.4 Organization of This Chapter

The rest of this chapter is divided into the following sections. Some suitable mathematical premises related to the proposed theory are presented in Sect. 3. The fundamental contribution of this chapter is detailed in Sect. 4, which introduces the notion of an intuitionistic fuzzy difference equation. The numerical examples in Sect. 5 clarify the theory. Section 6 suggests a suitable application. Final remarks are provided in Sect. 7.

2 Literature Review

This section consists of three subsections related to the keywords as follows:

2.1 Intuitionistic Fuzzy Number and Its Application

The fuzzy set merely takes into account the level of belongingness and nonbelongingness. The degree of hesitation is not taken into account by fuzzy set theory. Atanassov [4] examined the idea of fuzzy set theory through intuitionistic fuzzy set (IFS) theory to deal with such circumstances. Only the degree of acceptance in fuzzy sets is relevant; otherwise, IFS is defined by a membership function and a nonmembership function, the sum of which is smaller than one [5]. More IFS theory developments, like intuitionistic fuzzy generalized nets, intuitionistic fuzzy logic, intuitionistic fuzzy topology, and an intuitionistic fuzzy approach to artificial intelligence, may all be found in [15]. IFSs have numerous valuable applications in a variety of fields, including pattern classification [16], clinical issues [17], and drug screening [18]. As a result, they are an essential and effective tool for modeling imprecision.

2.2 Fuzzy Differential Equation in Intuitionistic Environment

Interest in fuzzy differential equations (FDEs) has exploded recently. The first-order system is the most significant among all fuzzy differential equations. Only a few studies use intuitionistic fuzzy numbers for differential equations. Melliani and Chadli [19] described fuzzy differential equation of first order in intuitionistic fuzzy environment. They also discussed partial differential equation in the same environment [20]. In this environment, the fuzzy differential equations of first-order, homogenous and nonhomogenous types were discussed in a series of research works by Mondal and Roy [7–10, 21, 22].

2.3 Fuzzy Difference Equation

The discrete calculus and fuzzy decision-making methods used in the fuzzy difference equation, respectively, investigate the computational framework's theoretical underpinnings that depict the uncertainty inherent in the modeling and the underflowing discrete behavior. The nonlinear fuzzy difference equations with various beginning conditions and coefficients represented as fuzzy numbers allow for the identification of the model. The fuzzy difference equation efficiently establishes the mathematical correspondence of a discrete dynamical system with modeling-related parameter uncertainty. Deeba et al. [23, 24] accounted for the initial ideas of fuzzy difference equations and potential applications. Lakshmikantham and Vatsala [25] also investigated the foundational theory of fuzzy difference equations. Many academics [11–14, 26–28] followed in their footsteps and examined many kinds of fuzzy difference equations from their unique points of view.

3 Preliminaries

Definition 3.1 [3] A triangular intuitionistic fuzzy number $\tilde{A}^i_{TIFN}$ is defined as an ordered triplet $\left(x, \mu_{\tilde{A}^i}(x), \vartheta_{\tilde{A}^i}(x)\right)$, where x belongs to some universal set X and the membership function $\mu_{\tilde{A}^i}(x)$ and nonmembership function $\vartheta_{\tilde{A}^i}(x)$ of x in the set A satisfies the relation $0 \le \mu_{\tilde{A}^i}(x) + \vartheta_{\tilde{A}^i}(x) \le 1$ where $\mu_{\tilde{A}^i}(x)$ and $\vartheta_{\tilde{A}^i}(x)$ are defined as

$$\mu_{\tilde{A}^i}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1} & for\ a_1 \le x \le a_2 \\ \dfrac{a_3 - x}{a_3 - a_2} & for\ a_2 \le x \le a_3 \\ 0 & otherwise \end{cases}$$

$$\text{and } \vartheta_{\tilde{A}^i}(x) = \begin{cases} \dfrac{a_2 - x}{a_2 - a_1'} & for\ a_1' \le x \le a_2 \\ \dfrac{x - a_2}{a_3' - a_2} & for\ a_2 \le x \le a_3' \\ 1 & otherwise. \end{cases}$$

Here, $a_1' \le a_1 \le a_2 \le a_3 \le a_3'$ and TIFN are denoted by $\tilde{A}^i_{TIFN} = \left(a_1, a_2, a_3; a_1', a_2, a_3'\right)$

Definition 3.2 [3] Let the real valued intuitionistic fuzzy number is denoted by the set $\mathcal{F}_1 = \left\{\langle \zeta, \eta \rangle \mid \mathbb{R} \to [0, 1]^2, \forall x \in \mathbb{R}; 0 \le \zeta(x) + \eta(x) \le 1\right\}$. An element $\langle \zeta, \eta \rangle$ in $\mathcal{F}_1$ to be an intuitionistic fuzzy number if the following conditions are maintained

$\langle \zeta, \eta \rangle$ is normal, that is, for some t_0 and $t_1 \in \mathbb{R}$, $\zeta(t_0) = 1$ and $\eta(t_1) = 1$.

(i) The membership functions of ζ and η are fuzzy convex and fuzzy concave respectively.
(ii) ζ is a lower semicontinuous and η is a n upper semicontinuous function.
(iii) $Supp\langle\zeta, \eta\rangle$ is bounded.

Definition 3.3 [3, 29] If α, $\beta \in [0, 1]$ and $\alpha + \beta = 1$, the parametric form of the α, β- cut of the intuitionistic fuzzy number $\langle\zeta, \eta\rangle$ is of the form $\langle\zeta, \eta\rangle_{(\alpha,\beta)} = \left[\zeta_L^1(\alpha), \zeta_R^1(\alpha); \eta_L^2(\beta), \eta_R^2(\beta)\right]$ where $\left[\zeta_L^1(\alpha), \zeta_R^1(\alpha)\right]$ and $\left[\eta_L^2(\beta), \eta_R^2(\beta)\right]$ are the parametric form of the α-cut of ζ and β-cut of η, respectively.

Definition 3.4 *Zadeh's extension principle* [30]: Let U be a crisp set and $\tilde{A}$ be a fuzzy set in $U. f : U \to V$ be a function defined by $v = f(u)$ then the extension principle introduces a fuzzy set $\underset{\sim}{B}$ in V as $\underset{\sim}{B} = \left\{\left(v, \mu_{\underset{\sim}{B}}(v) | v = f(u),\ \ u \in U\right)\right.$ where $\mu_{\underset{\sim}{B}}(v) = \begin{cases} \sup_{u\in f^{-1}(v)} \left(\mu_{\tilde{A}}(u)\right), iff\ f^{-1}(v) \neq \phi \\ 0 \quad otherwise \end{cases}$

Example 3.1 Let A_f be a fuzzy set given by the membership function as follows:

$$\mu_{A_f}(x) = \begin{cases} 0\ if\ x \leq 3 \\ x - 3\ if\ 3 \leq x < 4 \\ 1\ if\ x = 4 \\ \dfrac{6 - x}{2}\ if\ 4 < x \leq 6 \\ 0\ if\ x \geq 6 \end{cases}$$

Let us choose a function $F(x) = 2x + 3$. Using the concept of Zadeh's extension principle, another fuzzy set $F(A_f)$ can be determined. The membership function of $F(A_f)$ is obtained as follows:

$$\mu_{F(A_f)}(y) = \begin{cases} 0\ if\ x \leq 9 \\ \dfrac{y - 9}{2}\ if\ 9 \leq y < 11 \\ 1\ if\ y = 11 \\ \dfrac{15 - y}{4}\ if\ 11 < y \leq 15 \\ 0\ if\ y \geq 15 \end{cases}$$

Definition 3.5 *Extension principle on Intuitionistic fuzzy sets* [31]: Let us take some usual set R_R and let us choose some fuzzy set $A_{if} \in IFS(X_R)$.The extension principle for fuzzy sets states that if $G(A_{if}) \in FS(Y_R)$ such that $y \in Y_R$,

$$\mu_{G(A_{if})}(y) = \begin{cases} \sup\left\{\mu_{A_{if}}(x) : x \in G^{-1}(y)\right\}, if\ y \in Range(G) \\ 0, \qquad if\ y \notin Range(G) \end{cases}$$

$$\vartheta_{G(A_{if})}(y) = \begin{cases} \inf\left\{\vartheta_{A_{if}}(x) : x \in G^{-1}(y)\right\}, if\ y \in Range(G) \\ 1,\quad otherwise \end{cases}$$

And for every $B_{if} \in IFS(Y_R)$, $G^{-1}(B_{if})$ is defined in the following way

$$\mu_{G^{-1}(B_{if})}(x) = \mu_{B_{if}}(G(x))$$

$$\vartheta_{G^{-1}(B_{if})}(x) = \vartheta_{B_{if}}(G(x)), \quad \text{for every } x \in X_R.$$

Example 3.2 Let A_{if} be an Intuitionistic fuzzy set whose membership and non-membership functions are given as follows:

$$\mu_{A_{if}}(x) = \begin{cases} 0\ if\ x \leq 0 \\ x\ if\ 0 \leq x < 1 \\ 1\ if\ x = 2 \\ \frac{4-x}{2}\ if\ 2 < x \leq 4 \\ 0\ if\ x \geq 4 \end{cases}$$

$$\vartheta_{A_{if}}(x) = \begin{cases} 1\ if\ x \leq -1 \\ \frac{2-x}{3}\ if -1 < x \leq 2 \\ 0\ if\ x = 2 \\ \frac{x-2}{3}\ if\ 2 \leq x < 5 \\ 1\ if\ x \geq 5 \end{cases}$$

Let us choose a function $G(x) = x + 5$. Using the concept of Zadeh's extension principle, another fuzzy set $G(A_{if})$ can be determined. The membership and nonmembership functions of $G(A_{if})$ are obtained as follows:

$$\mu_{G(A_{if})}(y) = \begin{cases} 0\ if\ y \leq 5 \\ y-5\ if\ 5 \leq y < 6 \\ 1\ if y = 7 \\ \frac{9-y}{2}\ if\ 7 < y \leq 9 \\ 0\ if\ y \geq 9 \end{cases}$$

$$\vartheta_{G(A_{if})}(y) = \begin{cases} 1\ if\ y \leq 4 \\ \frac{7-y}{3}\ if\ 4 < x \leq 7 \\ 0\ \ if\ y = 7 \\ \frac{y-7}{3}\ if\ 7 < y \leq 10 \\ 1\ if\ y \geq 10 \end{cases}$$

Theorem 3.1 Let $F : R^m \rightarrow R^n$ is continuous function. Then, the Zadeh's extension function $\tilde{F} : (R_f)^m - (R_f)^n$ is well defined, continuous and $\left[\tilde{F}(u)\right]_\alpha = F\left(\left[\tilde{u}\right]_\alpha\right), \forall\alpha \in [0, 1]$.

Note 3.1 The above theorem is also valid for $F : U \rightarrow R^n$, where U is an open subset in R^n.

4 Second-Order Linear Difference Equation in Intuitionistic Fuzzy Environment

Consider the second-order linear nonhomogeneous crisp difference equation as

$$\begin{cases} a_0 u_{n+2} + a_1 u_{n+1} + a_2 u_n = f(n) \\ u_{n=0} = u_0, u_{n=1} = u_1 \end{cases} \tag{31.2}$$

With the initial condition $u_{n=0} = u_0$ and $u_{n=1} = u_1$

where $a_0 \neq 0$, a_1 and a_2 are the coefficient of the linear difference Eq. (31.2) and $f(n)$ is the nonhomogeneous part of the above difference equation, which is either constant or function of n alone. In the case of linear difference equation, the coefficients are independent of u_n. The solution of Eq. (31.2) includes two parts: one is complementary and other is particular solution.

The upstairs Eq. (31.2) is called intuitionistic fuzzy difference equation if:

(i) The initial information is of intuitionistic fuzzy valued number.
(ii) The coefficient or coefficients are intuitionistic fuzzy valued number.
(iii) The initial information and coefficient or coefficients are intuitionistic fuzzy valued numbers.

5 Extension Principle on Intuitionistic Fuzzy Second-Order Difference Equation

Consider a second-order homogeneous difference equation

$$a_0 u_{n+2} + a_1 u_{n+1} + a_2 u_n = 0 \tag{31.3}$$

With the fuzzy intuitionistic initial conditions $u_{n=0} = \tilde{u}_0$ and $u_{n=1} = \tilde{u}_1$ whereas $a_0 \neq 0$, a_1 and a_2 are the coefficient of the linear homogeneous difference equation.

The auxiliary equation of the difference Eq. (31.3) is

$$a_0 m^2 + a_1 m + a_2 = 0 \tag{31.4}$$

Let the intuitionistic(α, β)-cut of the initial conditions $u_{n=0} = \tilde{u}_0$ and $u_{n=1} = \tilde{u}_1$ are

$$\begin{cases} [\tilde{u}_0]_{(\alpha,\beta)} = \left[u^1_{L,0}(\alpha), u^1_{R,0}(\alpha); u^2_{L,0}(\beta), u^2_{R,0}(\beta)\right] \\ [\tilde{u}_1]_{(\alpha,\beta)} = \left[u^1_{L,1}(\alpha), u^1_{R,1}(\alpha); u^2_{L,1}(\beta), u^2_{R,1}(\beta)\right] \end{cases} \tag{31.5}$$

Now, depending on the roots of the auxiliary Eq. (31.4), the following cases arise.

Case I The roots m_1 and m_2 are real and distinct.

Suppose m_1 and m_2 are two real roots of the auxiliary Eq. (31.4) with $\mathrm{m}_1 \neq \mathrm{m}_2$. The general solution of (31.2) in crisp sense is

$$u_n = k_1 (m_1)^n + k_2 (m_2)^n + G(n) \tag{31.6}$$

Here $m_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}{2a_0}$
In Eq. (31.6), $G(n)$ is the particular solution of Eq. (31.2).
Using the initial conditions, $u_{n=0} = u_0$ and $u_{n=1} = u_1$, we have

$$u_n = \frac{u_0 m_2 - u_1}{m_2 - m_1}(m_1)^n + \frac{u_0 m_1 - u_1}{m_1 - m_2}(m_2)^n + \varnothing(n) \tag{31.7}$$

In Eq. (31.7), $\varnothing(n)$ is given as follows:

$$\varnothing(n) = \left(\frac{G(1) - m_2 G(0)}{m_2 - m_1}\right)(m_1)^n + \left(\frac{m_1 G(0) - G(1)}{m_2 - m_1}\right)(m_2)^n + G(n) \tag{31.8}$$

Now the sequence of intuitionistic fuzzy solution $\tilde{u}_n^i$ is obtained by substituting the intuitionistic fuzzy initial conditions (31.5) in place of u_0 and u_1 respectively in Eq. (31.6).

Let, $\tilde{u}_n^{i_e}$ be the sequence of intuitionistic fuzzy solutions obtained from Eq. (31.6) after applying Zadeh's extension principle and setting the values of coefficients $(m_1)^n$ and $(m_2)^n$ as $f_1(u_0, u_1)$ and $f_2(u_0, u_1)$ respectively. Then, the(α, β)-cut of $\tilde{u}_n^{i_e}$ is given by

$$\left[\tilde{u}_n^{i_e}\right]_{(\alpha,\beta)} = \left[u^{i_{e_1}}_{L,n}(\alpha), u^{i_{e_1}}_{R,n}(\alpha); u^{i_{e_2}}_{L,n}(\beta), u^{i_{e_2}}_{R,n}(\beta)\right].$$

The components of the (α, β)-cut of $\tilde{u}_n^{i_e}$ are obtained as

$$\begin{cases} u_{L,n}^{i_{e_1}}(\alpha)=\min\{f_1(u_0,u_1)(m_1)^n+f_2(u_0,u_1)(m_2)^n:u_0\epsilon\ [\tilde{u}_0]_\alpha,\ \ u_1\epsilon\ [\tilde{u}_1]_\alpha,\alpha\ \epsilon[0,1]\}+\varnothing(n) \\ u_{R,n}^{i_{e_1}}(\alpha)=\max\{f_1(u_0,u_1)(m_1)^n+f_2(u_0,u_1)(m_2)^n:u_0\epsilon\ [\tilde{u}_0]_\alpha,\ \ u_1\epsilon\ [\tilde{u}_1]_\alpha,\alpha\ \epsilon[0,1]\}+\varnothing(n) \\ u_{L,n}^{i_{e_2}}(\beta)=\min\left\{f_1(u_0,u_1)(m_1)^n+f_2(u_0,u_1)(m_2)^n:u_0\epsilon\ [\tilde{u}_0]_\beta,\ \ u_1\epsilon\ [\tilde{u}_1]_\beta,\beta\ \epsilon[0,1]\right\}+\varnothing(n) \\ u_{R,n}^{i_{e_2}}(\beta)=\max\left\{f_1(u_0,u_1)(m_1)^n+f_2(u_0,u_1)(m_2)^n:u_0\epsilon\ [\tilde{u}_0]_\beta,\ \ u_1\epsilon\ [\tilde{u}_1]_\beta,\beta\ \epsilon[0,1]\right\}+\varnothing(n) \end{cases} \tag{31.9}$$

The function $\varnothing(n)$ is obtained from Eq. (31.8).

After computing $u_{L,n}^{i_{e_1}}(\alpha)$, $u_{R,n}^{i_{e_1}}(\alpha)$; $u_{L,n}^{i_{e_2}}(\beta)$ and $u_{R,n}^{i_{e_2}}(\beta)$; $u_n^{i_e}$ be the solution of Eq. (31.2),if all the α-components and β-components satisfy intutionistic the difference Eq. (31.2) and the corresponding initial conditions.

The following condition is required for $u_n^{i_e}$ to solve the difference equation, it is easy to see $u_n^{i_e}$ satisfy the initial condition, so we only need to consider $n \geq 2$.

For each $\alpha \in [0,1]$ and $\beta \in [0,1]$, there exists u_{01} and $u_{02} \in [\tilde{u}_0]_\alpha$; u_{01}^* and $u_{02}^* \in [\tilde{u}_0]_\beta$; $u_{11}, u_{12} \in [\tilde{u}_1]_\alpha$; u_{11}^* and $u_{12}^* \in [\tilde{u}_1]_\beta$, then

$$\begin{cases} u_{L,n}^{i_{e_1}}(\alpha) = f_1(u_{01}, u_{11})(m_1)^n + f_2(u_{01}, u_{11})(m_2)^n + \varnothing(n) \\ u_{L,n}^{i_{e_2}}(\beta) = f_1\left(u_{01}^*, u_{11}^*\right)(m_1)^n + f_2\left(u_{01}^*, u_{11}^*\right)(m_2)^n + \varnothing(n) \end{cases} \tag{31.10}$$

And for all $n \geq 2$

$$\begin{cases} u_{R,n}^{i_{e_1}}(\alpha) = f_1(u_{02}, u_{12})(m_1)^n + f_2(u_{02}, u_{12})(m_2)^n + \varnothing(n) \\ u_{R,n}^{i_{e_2}}(\beta) = f_1\left(u_{02}^*, u_{12}^*\right)(m_1)^n + f_2\left(u_{02}^*, u_{12}^*\right)(m_2)^n + \varnothing(n) \end{cases} \tag{31.11}$$

To check the increasing and decreasing nature of the solution u_n from Eq. (31.6) we have

$$\begin{cases} \frac{\partial u_n}{\partial u_0} = h_1(n) = \frac{m_2(m_1)^n - m_1(m_2)^n}{m_2 - m_1} \\ \frac{\partial u_n}{\partial u_1} = h_2(n) = \frac{(m_2)^n - (m_1)^n}{m_2 - m_1} \end{cases} \tag{31.12}$$

Case II The roots m_1 and m_2 are real and equal

Suppose, $m_1 = m_2 = m$ are two real roots of the auxiliary Eq. (31.4)**.** The general solution of the Eq. (31.2) is of the form

$$u_n = k_1 m^n + k_2 n(m)^n + G(n) \tag{31.13}$$

$$m = \frac{-a_1}{2a_o}.$$

Using the initial condition, the unique solution is obtained as

$$u_n = u_0 m^n + \frac{u_1 - u_0 m}{m} n m^n + \varphi(n) \tag{31.14}$$

$$\varphi(n) = -G(0)m^n + \frac{mG(0) - G(1)}{m} nm^n + G(n) \tag{31.15}$$

To check the increasing and decreasing nature of the solution u_n, we calculate the following

$$\begin{cases} \frac{\partial u_n}{\partial u_0} = (1-n)\, m^n \\ \frac{\partial u_n}{\partial u_1} = nm^{n-1} \end{cases} \tag{31.16}$$

Case III The roots m_1 and m_2 are complex conjugate

Let two complex conjugate roots are given by $m_1 = \alpha - i\beta,\ m_2 = \alpha + i\beta; \alpha, \beta,$ $\alpha = \frac{-a_1}{2a_0}, \beta = \frac{\sqrt{4a_0a_2 - a_1^2}}{2a_0}$ are real with $\beta > 0$. Then the general solution of the Eq. (31.2) is

$$u_n = p^n\, (k_1 \cos{(n\theta)} + k_2 \sin{(n\theta)}) + G(n) \tag{31.17}$$

In Eq. (31.17), values of p and θ are obtained as

$$\begin{cases} p = \sqrt{\alpha^2 + \beta^2} \\ \theta = \tan^{-1}\left({}^{\beta}\!\big/_{\alpha} \right) \end{cases} \tag{31.18}$$

Using the initial conditions, we find the unique solution as

$$u_n = p^n \left(u_0 \cos{(n\theta)} + \frac{u_1 - pu_0 \cos\theta}{\sin\theta} \sin{(n\theta)} \right) + \chi(n) \tag{31.19}$$

In Eq. (31.19), $\chi(n)$ is obtained as

$$\chi(n) = p^n \left(-G(0) \cos{(n\theta)} + \frac{pG(0) \cos\theta - G(1)}{\sin\theta} \sin{(n\theta)} \right) + G(n) \tag{31.20}$$

6 Numerical Illustration

Example 6.1 Consider the second-order difference equation $y_{n+2} + y_{n+1} - 6y_n = 0$ with initial intuitionistic conditions $y_0 = (2, 3, 4; 1, 3, 5)$, $y_1 = (15, 17, 19; 14, 17, 20)$.

Solution: First we solve the difference equation $y_{n+2} + y_{n+1} - 6y_n = 0$,

$$y_n = c_1(-3)^n + c_2(2)^n \tag{31.21}$$

Using the initial conditions in terms of y_0 and y_1, the solution of the difference equation takes the form

$$y_n = \left(\frac{2y_0 - y_1}{5}\right)(-3)^n + \left(\frac{3y_0 + y_1}{5}\right)(2)^n \tag{31.22}$$

We check the increasing and decreasing nature of the Eq. (31.22)

$$\begin{cases} \frac{\partial y_n}{\partial y_0} = \frac{2}{5}(-3)^n + \frac{3}{5}(2)^n = g_1(n) \\ \frac{\partial y_n}{\partial y_1} = -\frac{1}{5}(-3)^n + \frac{1}{5}(2)^n = g_2(n) \end{cases} \tag{31.23}$$

The(α, β)-cuts of the intuitionistic fuzzy initial conditions y_0 and y_1 are

$$\left[\tilde{y}_0\right]_{(\alpha,\beta)} = [2+\alpha, 4-\alpha; 3-2\beta, 3+2\beta],$$

$$\left[\tilde{y}_1\right]_{(\alpha,\beta)} = [15+2\alpha, 19-2\alpha; 17-3\beta, 17+3\beta]$$

Using the initial conditions, we find the general intuitionistic fuzzy solution $y_n^{i_e}$ of the second-order intuitionistic fuzzy difference equation, given by example 6.1 and it's α, β-cuts are as follows:

Case I $n \in \mathbb{N}$ *is an even number*

When $\in \mathbb{N}$, an even number as a power of the auxiliary roots of the difference equation in example 6.1, then the α, β-cuts of the intuitionistic fuzzy solution $y_n^{i_e}$ are given by

$$\begin{cases} y_{L,n}^{i_{e_1}}(\alpha) = \left(\frac{-15+4\alpha}{5}\right)(-3)^n + \left(\frac{5\alpha+21}{5}\right)(2)^n \\ y_{R,n}^{i_{e_1}}(\alpha) = \left(\frac{-7-4\alpha}{5}\right)(-3)^n + \left(\frac{31-5\alpha}{5}\right)(2)^n \\ y_{L,n}^{i_{e_2}}(\beta) = \left(\frac{-11-7\beta}{5}\right)(-3)^n + \left(\frac{26-9\beta}{5}\right)(2)^n \\ y_{R,n}^{i_{e_2}}(\beta) = \left(\frac{-11+7\beta}{5}\right)(-3)^n + \left(\frac{26+9\beta}{5}\right)(2)^n \end{cases} \tag{31.24}$$

The subfigures in Fig. 31.1 describe the Case I of the Example 6.1 in different situations.

Case II $n \in \mathbb{N}$ *is an odd number*

When $n \in \mathbb{N}$, an odd number as a power of the auxiliary roots of the difference equation in example 6.1, then the α, β-cuts of the intuitionistic fuzzy solution $y_n^{i_e}$ are given by

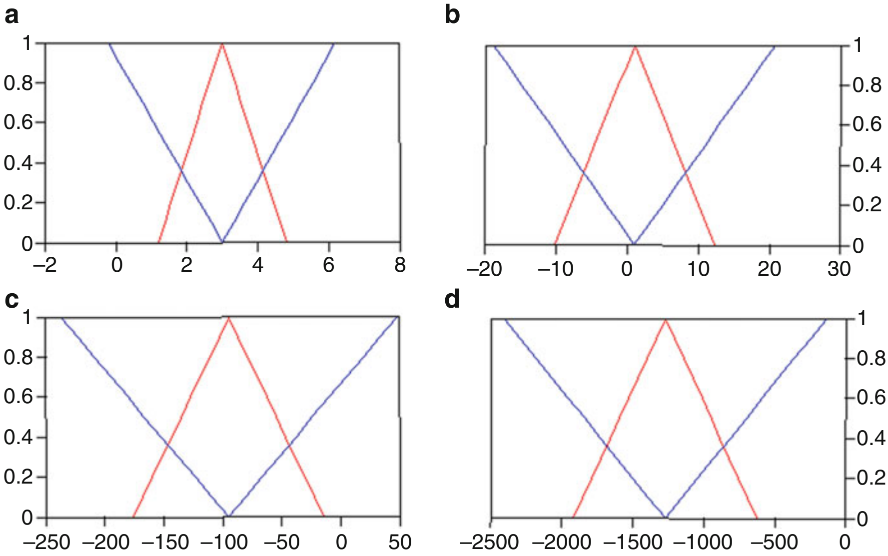

Fig. 31.1 Solution of the second-order difference equation $y_{n+2} + y_{n+1} - 6y_n = 0$ with initial intuitionistic conditions $y_0 = (2, 3, 4; 1, 3, 5)$, $y_1 = (15, 17, 19; 14, 17, 20)$ for n as even numbers. In Fig. 31.1., subfigures (**a**), (**b**), (**c**), and (**d**) represent solutions in terms of intuitionistic fuzzy numbers for $n = 0, 2, 4, 6$ respectively. The red and blue colors represent the membership and nonmembership functions respectively in all the subfigures

$$\begin{cases} y_{L,n}^{i_{e_1}}(\alpha) = \left(\frac{-7-4\alpha}{5}\right)(-3)^n + \left(\frac{5\alpha+21}{5}\right)(2)^n \\ y_{R,n}^{i_{e_1}}(\alpha) = \left(\frac{-15+4\alpha}{5}\right)(-3)^n + \left(\frac{31-5\alpha}{5}\right)(2)^n \\ y_{L,n}^{i_{e_2}}(\beta) = \left(\frac{-11+7\beta}{5}\right)(-3)^n + \left(\frac{26-9\beta}{5}\right)(2)^n \\ y_{R,n}^{i_{e_2}}(\beta) = \left(\frac{-11-7\beta}{5}\right)(-3)^n + \left(\frac{26+9\beta}{5}\right)(2)^n \end{cases} \tag{31.25}$$

The subfigures in Fig. 31.2 describe the Case II of the Example 6.1 in different situations.

Example 6.2 Consider the second-order difference equation $y_{n+2} - 16y_{n+1} + 64y_n = 0$ with initial intuitionistic conditions, $y_0 = (4, 5, 6; 3, 5, 7)$ and $y_1 = (9, 10, 11; 8, 10, 12)$.

Solution First we solve the difference equation of the example 6.2, we have

$$y_n = (c_1 + nc_2)\, 8^n \tag{31.26}$$

where c_1 and c_2 are arbitrary constants to be obtained from Eq. (31.26) using the initial conditions specified in the example. Therefore,

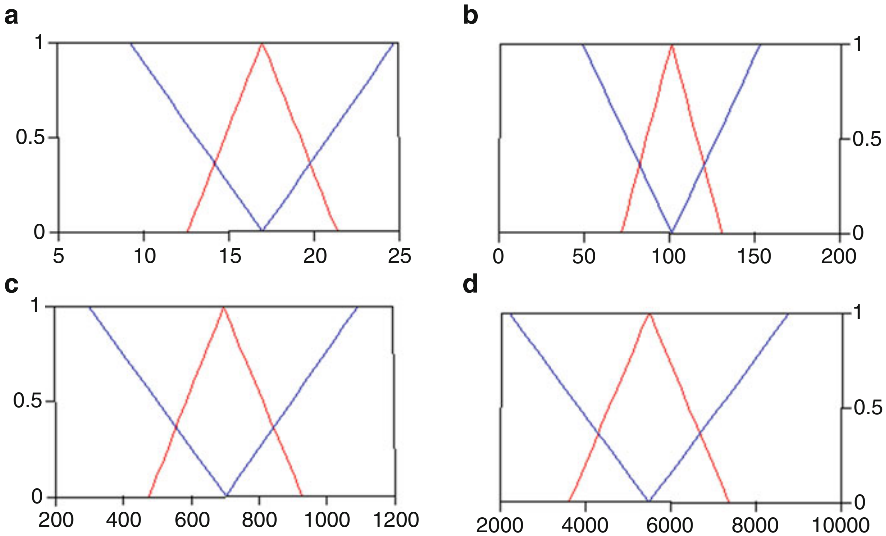

Fig. 31.2 Solution of the second-order difference equation $y_{n+2} + y_{n+1} - 6y_n = 0$ with initial intuitionistic conditions $y_0 = (2, 3, 4; 1, 3, 5)$, $y_1 = (15, 17, 19; 14, 17, 20)$ for n as odd numbers. In Fig.31.2., subfigures (**a**), (**b**), (**c**), and (**d**) represent solutions in terms of intuitionistic fuzzy numbers for $n = 1, 3, 5, 7$ respectively. The red and blue colors represent the membership and nonmembership functions respectively in all the subfigures

$$y_n = \{ny_1 - 8(n-1)y_0\}(8)^{n-1} \tag{31.27}$$

We check the increasing and decreasing properties of the solution y_n in Eq. (31.27),

$$\begin{cases} \frac{\partial y_n}{\partial y_0} = (8)^n - n(8)^n = g_1(n) \\ \frac{\partial y_n}{\partial y_1} = n(8)^{n-1} = g_2(n) \end{cases} \tag{31.28}$$

By extension principle

Using initial condition, we find the general solution $y_n^{i_e}$ of second-order intuitionistic fuzzy difference equation as its α, β-cuts are given by

$$\begin{cases} y_{L,n}^{i_{e_1}}(\alpha) = \{n(9+\alpha) - 8(n-1)(6-\alpha)\}(8)^{n-1} \\ y_{R,n}^{i_{e_1}}(\alpha) = \{n(11-\alpha) - 8(n-1)(4+\alpha)\}(8)^{n-1} \\ y_{L,n}^{i_{e_2}}(\beta) = \{n(10-2\beta) - 8(n-1)(5+2\beta)\}(8)^{n-1} \\ y_{R,n}^{i_{e_2}}(\beta) = \{n(10+2\beta) - 8(n-1)(5-2\beta)\}(8)^{n-1} \end{cases} \tag{31.29}$$

Example 6.3 Let us consider second-order difference equation $y_{n+2} + 16y_n = 0$ with initial intuitionistic condition $y_0 = (1.5, 3, 4.5; 1, 3, 5)$, $y_1 = (12, 13, 14; 11, 13, 15)$

Solution: The α, β-cut of the initial condition $\left[\tilde{y}_0\right]_{(\alpha,\beta)} = [1.5 + 1.5\alpha, 4.5 - 1.5\alpha; 3 - 2\beta, 3 + 2\beta]$

$$\left[\tilde{y}_1\right]_{(\alpha,\beta)} = [12 + \alpha, 14 - \alpha; 13 - 2\beta, 13 + 2\beta]$$

Using initial conditions, we find the general solution of second-order intuitionistic fuzzy difference equation as its α, β-cut:

(a) when $n = 4k$, $k \in \mathfrak{Z}^+$

$$\begin{cases} y_{L,n}^{i_{e_1}}(\alpha) = (4)^{4k}(1.5 + 1.5\alpha) \\ y_{R,n}^{i_{e_1}}(\alpha) = (4)^{4k}(4.5 - 1.5\alpha) \\ y_{L,n}^{i_{e_2}}(\beta) = (4)^{4k}(3 - 2\beta) \\ y_{R,n}^{i_{e_2}}(\beta) = (4)^{4k}(3 + 2\beta) \end{cases} \tag{31.30}$$

(b) when $n = 4k + 1$, $k \in \mathfrak{Z}^+$

$$\begin{cases} y_{L,n}^{i_{e_1}}(\alpha) = (4)^{4k+1}(12 + \alpha) \\ y_{R,n}^{i_{e_1}}(\alpha) = (4)^{4k+1}(14 - \alpha) \\ y_{L,n}^{i_{e_2}}(\beta) = (4)^{4k+1}(13 - 2\beta) \\ y_{R,n}^{i_{e_2}}(\beta) = (4)^{4k+1}(13 + 2\beta) \end{cases} \tag{31.31}$$

(c) when $n = 4k + 2$, $k \in \mathfrak{Z}^+$

$$\begin{cases} y_{L,n}^{i_{e_1}}(\alpha) = (4)^{4k+2}(1.5\alpha - 4.5) \\ y_{R,n}^{i_{e_1}}(\alpha) = (4)^{4k+2}(-1.5\alpha - 1.5) \\ y_{L,n}^{i_{e_2}}(\beta) = (4)^{4k+2}(-3 - 2\beta) \\ y_{R,n}^{i_{e_2}}(\beta) = (4)^{4k+2}(-3 + 2\beta) \end{cases} \tag{31.32}$$

(d) when $n = 4k + 3$, $k \in \mathfrak{Z}^+$

$$\begin{cases} y_{L,n}^{i_{e_1}}(\alpha) = (4)^{4k+3}(\alpha - 14) \\ y_{R,n}^{i_{e_1}}(\alpha) = (4)^{4k+3}(-\alpha - 12) \\ y_{L,n}^{i_{e_2}}(\beta) = (4)^{4k+3}(-13 - 2\beta) \\ y_{R,n}^{i_{e_2}}(\beta) = (4)^{4k+3}(-13 + 2\beta) \end{cases} \tag{31.33}$$

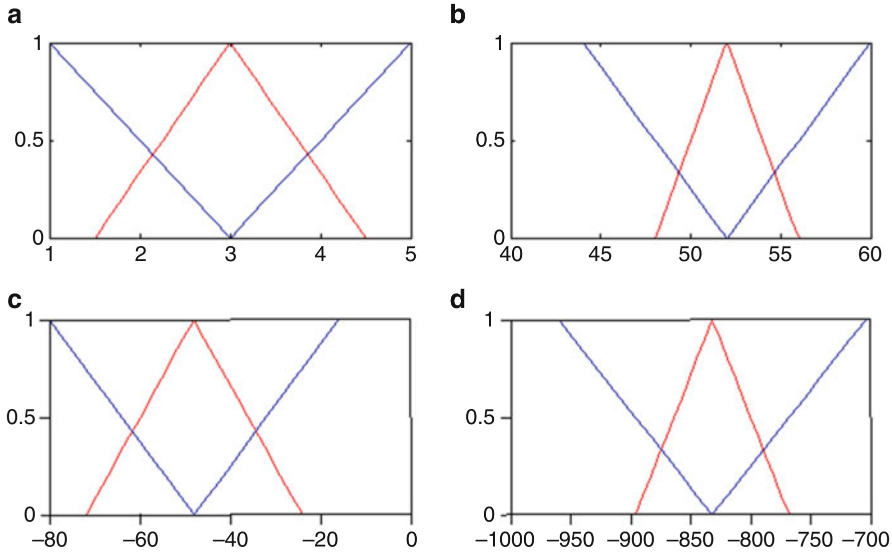

Fig. 31.3 Solution of the second-order difference equation $y_{n+2} + 16y_n = 0$ with initial intuitionistic conditions $y_0 = (1.5, 3, 4.5; 1, 3, 5)$, $y_1 = (12, 13, 14; 11, 13, 15)$. In Fig. 31.3, subfigures (**a**), (**b**), (**c**), and (**d**) represent solutions in terms of intuitionistic fuzzy numbers for $n = 0, 1, 2, 3$ respectively. The red and blue colors represent the membership and nonmembership functions respectively in all the subfigures

The subfigures in Fig. 31.3 describe all the cases of the Example 6.3 in different situations.

Example 6.4 Consider the second-order difference equation $y_{n+2} - 5y_{n+1} + 6y_n = 4n$ with initial intuitionistic condition, $y_0 = (1.5, 2, 2.5; 1, 2, 3)$, $y_1 = (4.5, 5, 5.5; 4, 5, 6)$

Solution First we solve the difference equation of example (31.23)

$$y_n = c_1(2)^n + c_2(3)^n + 2n + 3 \tag{31.34}$$

Using the values of initial conditions in Eq. (31.34), we obtain the values of arbitrary constants c_1 and c_2 in terms of y_0 and y_1. Then the Eq. (31.34) reduces to the form

$$y_n = (3y_0 - y_1 - 4)\,(2)^n + (y_1 - 2y_0 + 1)\,(3)^n + 2n + 3 \tag{31.35}$$

The α, β-cut of the initial condition $\left[\tilde{y}_0\right]_{(\alpha,\beta)} = [1.5 + 0.5\alpha, 2.5 - 0.5\alpha; 2 - \beta, 2 + \beta]$

$$\left[\tilde{y}_1\right]_{(\alpha,\beta)} = [4.5 + 0.5\alpha, 5.5 - 0.5\alpha; 5 - \beta, 5 + \beta]$$

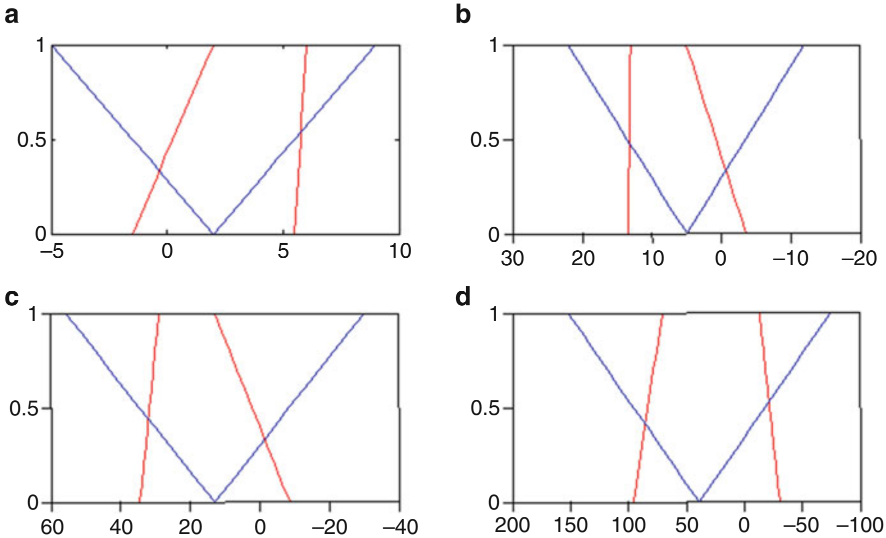

Fig. 31.4 Solution of the second-order difference equation $y_{n+2} - 5y_{n+1} + 6y_n = 4n$ with initial intuitionistic conditions $y_0 = (1.5, 2, 2.5; 1, 2, 3)$, $y_1 = (4.5, 5, 5.5; 4, 5, 6)$. In Fig. 31.4, subfigures (**a**), (**b**), (**c**), and (**d**) represent solutions in terms of intuitionistic fuzzy numbers for $n = 0, 1, 2, 3$ respectively. The red and blue colors represent the membership and nonmembership functions respectively in all the subfigures

Using initial conditions, we find the general solution of second-order intuitionistic fuzzy difference equation of example 6.3 as its α, β-cut is given by

$$\begin{cases} y_{L,n}^{i_{e_1}}(\alpha) = (-5 + 2\alpha)(2)^n + (0.5 + 1.5\alpha)(3)^n + 2n + 3 \\ y_{R,n}^{i_{e_1}}(\alpha) = (-1 - 2\alpha)(2)^n + (3.5 - 1.5\alpha)(3)^n + 2n + 3 \\ y_{L,n}^{i_{e_2}}(\beta) = (-3 - 4\beta)(2)^n + (2 - 3\beta)(3)^n + 2n + 3 \\ y_{R,n}^{i_{e_2}}(\beta) = (-3 + 4\beta)(2)^n + (2 + 3\beta)(2)^n + 2n + 3 \end{cases}$$

The subfigures in Fig. 31.4 describe all the cases of the Example 6.4 in different situations.

7 Application

Considering an experiment in which the pressure of gas in a cylinder is measured in each second and the pressure (in standard units) in n seconds is denoted by p_n. The measurements satisfy the difference equation

$$p_{n+2} = \frac{1}{2}\left(p_{n+1} + p_n\right) \tag{31.36}$$

with the initial intuitionistic condition are $p_0 = (7, 8, 9; 6, 8, 10;), p_1 = (12, 13, 14; 11, 13, 15) (n \geq 0)$. Find an explicit formula for p_n in terms of n, and state the value to which the pressure settles down in the long term.

Solution The general solution of the difference Eq. (31.36) in this application is as follows:

$$p_n = c_1 + c_2\left(-\frac{1}{2}\right)^n \tag{31.37}$$

Using the initial conditions in terms of p_0 and p_1, the solution of the difference Eq. (31.36) takes the form

$$p_n = \frac{1}{3}\left(p_0 + 2p_1\right) + \frac{2}{3}\left(p_0 - p_1\right)\left(\frac{-1}{2}\right)^n \tag{31.38}$$

We check the increasing and decreasing of p_n in Eq. (31.38),

$$\begin{cases} \frac{\partial p_n}{\partial p_0} = \frac{1}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^n = g_1(n) \\ \frac{\partial p_n}{\partial p_1} = \frac{2}{3} - \frac{2}{3}\left(-\frac{1}{2}\right)^n = g_2(n) \end{cases} \tag{31.39}$$

The intuitionistic fuzzy initial conditions are

$$\left[\tilde{p}_0\right]_{(\alpha,\beta)} = [7 + \alpha, 9 - \alpha; 8 - 2\beta, 8 + 2\beta] \tag{31.40}$$

and

$$\left[\tilde{p}_1\right]_{(\alpha,\beta)} = [12 + \alpha, 14 - \alpha; 13 - 2\beta, 13 + 2\beta] \tag{31.41}$$

By extension principle, using the intuitionistic α, β-cut of the initial conditions, we find the general intuitionistic solution $p_n^{i_e}$ of second-order intuitionistic fuzzy difference equation $p_{n+2} = \frac{1}{2}\left(p_{n+1} + p_n\right)$ as its α, β-cut is given by the following cases:

Case I when $n \in \mathbb{N}$ is an even number

When $n \in \mathbb{N}$, an even number as a power of the auxiliary roots of the difference equation in this application, then the α, β-cuts of the intuitionistic fuzzy solution $p_n^{i_e}$ are given by

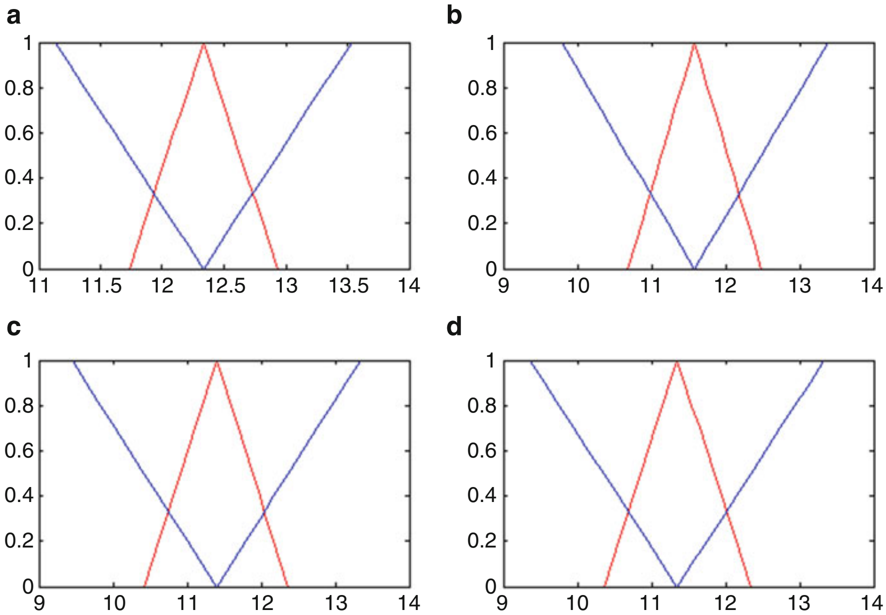

Fig. 31.5 Pressure of gas in a cylinder with initial pressure as intuitionistic conditions $p_0 = (7, 8, 9; 6, 8, 10;)$, $p_1 = (12, 13, 14; 11, 13, 15)$. In Fig. 31.5, subfigures (**a**), (**b**), (**c**), and (**d**) represent solutions in terms of intuitionistic fuzzy numbers for $n = 0, 2, 4, 6$ respectively. The red and blue colors represent the membership and nonmembership functions respectively in all the subfigures

$$\begin{cases} p_{L,n}^{i_{e_1}}(\alpha) = \frac{1}{3}(31 + 3\alpha) + \frac{2}{3}(-7 + 2\alpha)\left(\frac{-1}{2}\right)^n \\ p_{R,n}^{i_{e_1}}(\alpha) = \frac{1}{3}(37 - 3\alpha) + \frac{2}{3}(-3 - 2\alpha)\left(\frac{-1}{2}\right)^n \\ p_{L,n}^{i_{e_2}}(\beta) = \frac{1}{3}(34 - 6\beta) + \frac{2}{3}(-5 - 4\beta)\left(\frac{-1}{2}\right)^n \\ p_{R,n}^{i_{e_2}}(\beta) = \frac{1}{3}(34 + 6\beta) + \frac{2}{3}(-5 + 4\beta)\left(\frac{-1}{2}\right)^n \end{cases} \tag{31.42}$$

The subfigures in Fig. 31.5 describe the Case I of the application in different situations.

Case II when $n \in \mathbb{N}$ is an odd number

When $\epsilon\mathbb{N}$, an odd number as a power of the auxiliary roots of the difference equation in this application, then the α, β-cuts of the intuitionistic fuzzy solution $y_n^{i_e}$ are given by

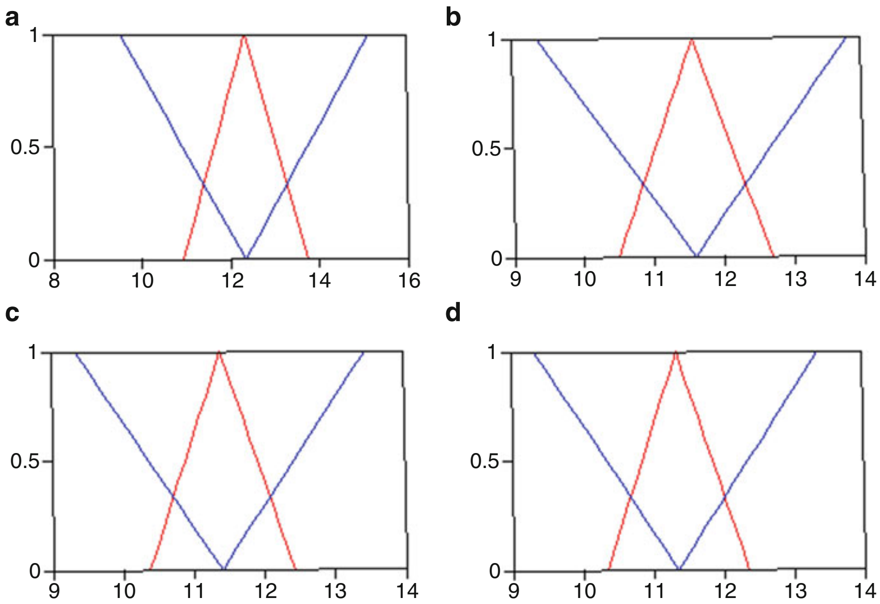

Fig. 31.6 Pressure of gas in a cylinder with initial pressure as intuitionistic conditions $p_0 = (7, 8, 9; 6, 8, 10;)$, $p_1 = (12, 13, 14; 11, 13, 15)$. In Fig. 31.6, subfigures (**a**), (**b**), (**c**), and (**d**) represent solutions in terms of intuitionistic fuzzy numbers for $n = 1, 3, 5, 7$ respectively. The red and blue colors represent the membership and nonmembership functions respectively in all the subfigures

$$\begin{cases} p_{L,n}^{i_{e_1}}(\alpha) = \frac{1}{3}(31 + 3\alpha) + \frac{2}{3}(-3 - 2\alpha)\left(\frac{-1}{2}\right)^n \\ p_{R,n}^{i_{e_1}}(\alpha) = \frac{1}{3}(37 - 3\alpha) + \frac{2}{3}(-7 + 2\alpha)\left(\frac{-1}{2}\right)^n \\ p_{L,n}^{i_{e_2}}(\beta) = \frac{1}{3}(34 - 6\beta) + \frac{2}{3}(-5 + 4\beta)\left(\frac{-1}{2}\right)^n \\ p_{R,n}^{i_{e_2}}(\beta) = \frac{1}{3}(34 + 6\beta) + \frac{2}{3}(-5 - 4\beta)\left(\frac{-1}{2}\right)^n \end{cases} \tag{31.43}$$

The subfigures in Fig. 31.5 describe the Case II of the application in different situations.

8 Conclusion

The difference equations are significant for modeling numerous difficulties in diverse fields in the discrete system. It is imperative if it is learning in an intuitionistic fuzzy setting. When it can be taught in an intuitionistic but imprecise environment, it behaves differently, and the solution process becomes more complicated than in fuzzy instances. The intuitionistic extension principle approach, which is an extension of extension principle in fuzzy set, is used in this study to solve the second-order linear intuitionistic fuzzy difference equation. If the beginning data or (and) the coefficients are intuitionistic fuzzy valued numbers, the difference equation is referred to as an intuitionistic fuzzy difference equation. We discussed the situation using intuitionistic fuzzy valued integers as the initial information. This work's numerical examples and applications illustrate the simplest method of solving an unknown difference equation.

Suppose we are looking for a mathematical tool where all changes are discrete and the uncertain phenomena have a sense of acceptance-rejection measure. In that scenario, the proposed chapter can fulfill the purpose, and this is the most significant managerial benefit of the discussion of this chapter. Ultimately, we acknowledge our limitation in this chapter, that we manifested the second-order difference equation in an intuitionistic environment. The whole argument is theoretical, and the data used here are all artificial. It is better if the data from natural phenomena validate the hypothetical results. In the future, the proposed theory can be extended for nonlinear difference equations with the uncertainty of different kinds. Also, the investigation for aptly fitted applications of the proposed idea in discrete dynamics under uncertainty may bring worthy outcomes in the future.

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Chapter 32
The Probabilistic Games and the Shapley Function

Surajit Borkotokey, Sujata Goala, and Rajnish Kumar

1 Introduction

A cooperative game with transferable utilities (TU games in short) deals with situations where players make coalitions to generate worth. It is assumed that all the players join together to form the grand coalition and a suitable allocation rule determines the share of each player in the grand coalitional worth. The Shapley value [19] is perhaps the most popular allocation rule in the literature of TU games. In many social situations, even though players tend to cooperate among themselves for generating worth, their complex and difficult to predict interactions, emotional traits and preferences may raise competency issues in such cooperative behaviour [17, 21]. Thus, while forming a coalition, the players must first agree on its sub-coalitions. Take, for example, if player 1, 2 and 3 want to form a coalition $\{1, 2, 3\}$, first, there should be agreements between players 1 & 2, players 2 & 3 and players 3 & 1. Because of the uncertainties involved in the negotiations among these three players in pairs, we can never be fully sure to have the coalition $\{1, 2, 3\}$. Instead, we need to assign some probabilities to each of these sub-coalitions, and the worth of the coalition is the expectation with respect to the probabilities over all these sub-coalitions. However, in the deterministic framework, in order to form coalition $\{1, 2, 3\}$, for example, no such prior interactions among the players in the smaller coalitions is considered. Thus, in the deterministic case, we have that in particular, the probability of formation of $\{1, 2, 3\}$ is 1 while the probabilities of all its sub-coalitions that lead to the formation of $\{1, 2, 3\}$ are 0.

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C. Jana et al. (eds.), *Fuzzy Optimization, Decision-making and Operations Research*, https://doi.org/10.1007/978-3-031-35668-1_32

In [15], several instances are reported where coalitions among agents from different spatial and cultural backgrounds fail to sustain even after a few successful interactions due to a lack of mutual appreciation among the agents that should develop over a period of time. Considering this fact, in this paper, we propose a probabilistic model of worth generation through cooperative activities which we call a probabilistic game, and the corresponding Shapley function is proposed as a suitable mean to allocate the worth so generated among the players. A special class of probabilistic game is studied, and the corresponding Shapley function is characterized using some intuitive axioms.

Few essential recent studies in this direction are by Dehez and Ferey [8], Ferey and Dehez [9], Hou et al. [11], Kamionko and Marakulin [12], and Koster et al. [13]; however, the references are indicative of an extensive literature only. In [12] a TU game with probabilistic endogenous coalitions is proposed. The authors assume that different agents have different incentives of cooperation with other agents in the coalition formation process: the outcome of these incentives is represented by a probability function. The prediction value is the difference between the conditional expectations of the worths of a coalition when an agent cooperates with a given probability distribution introduced in [13]. In [11] the allocation problem of the collective probability of success in cooperation is studied where different agents have a common target, and the probability of success with which each agent succeeds in attaining the target is a common prior. In [8], the joint liabilities of a group of agents are allocated according to the Shapley value under a probabilistic setup. In [9], the appointment of damage resulting from the action of several tortfeasors is studied. However, in either of these works, the worth is again a probability. This probability is shared among the agents using an allocation scheme. This is why in all these studies, the aggregate of two probabilistic games is not a probabilistic game again. On the other hand, in our model, the worth being a non-negative real number can represent any divisible goods, and the class of probabilistic games in our model is a vector space under the standard addition and scalar multiplication. The interested reader is referred to [5–7, 14, 18] for some alternative models that describe similar situations.

There is another group of researchers who discuss the uncertainty of the coalition formation process using fuzzy set theory [1–4, 16, 20]. In these models, the worth of coalitions depends on the degree of membership of each player in this coalition. In [1, 3, 4], TU games with fuzzy coalitions are defined as a weighted aggregate of the worths on the level subsets over the membership values. In [16], the notion of multilinear extension is used to define a fuzzy cooperative game, and in [20], the idea of Choquet integral is used to define a fuzzy cooperative game. In [2], both coalitions and the worths generated by the coalitions are taken as fuzzy quantities. However, none of these models consider the competency possibilities among the players in small groups while forming a coalition. We deviate from most of the existing literature in two respects: first, we extend the class of TU games to its probabilistic counterpart by incorporating probabilities of coalition formation which is given by a probability measure, and second, the allocation schemes are also specific to this particular probability measure.

The rest of the paper proceeds as follows. In Sect. 2, we briefly mention the preliminary ideas related to the development of our model. Section 3 studies the class of probabilistic games and the corresponding Shapley function. In Sect. 4 we propose a special class of probabilistic games and obtain its Shapley function. We also provide a characterization of this Shapley function. Finally Sect. 5 concludes.

2 Preliminary

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}^n$ denote the Euclidian space of typical vectors $(x_i)_{i=1}^n$ with $x_i \in \mathbb{R}$. Let $\aleph$ denote the set of all non-empty, finite subsets of a countably infinite set; we call this the universe of all players. For each $N \in \aleph$, let 2^N denote the power set of N. The members of 2^N are called coalitions, and N, the largest among them, is called the grand coalition. To simplify the notations, we use $S \cup i$, $S \setminus i$, etc., instead of $S \cup \{i\}$, $S \setminus \{i\}$, etc. Also let small letters s, t, etc. denote the size of coalitions S, T, etc. A cooperative game with transferable utilities (TU game) is a pair (N, v) with $N \in \aleph$ and a characteristic function $v : 2^N \mapsto \mathbb{R}$, such that $v(\emptyset) = 0$. The real number $v(S)$ represents the worth of the coalition S. The set of all TU games with variable players set $N \in \aleph$, (fixed player set N) is denoted by G, $(G(N))$. If there is no ambiguity with the choice of N, we denote the TU game (N, v) only by v. The restriction of (N, v) to a player set $S \subseteq N$ is denoted by (S, v). The identity game $e_T \in G(N)$ is defined as

$$e_T(S) = \begin{cases} 1 \text{ if } S = T, \\ 0 \text{ if } S \neq T. \end{cases} \tag{32.1}$$

and the unanimity game $u_T \in G(N)$ is defined as

$$u_T(S) = \begin{cases} 1 \text{ if } T \subseteq S, \\ 0 \text{ otherwise}. \end{cases} \tag{32.2}$$

The set of unanimity games $\{u_T : T \subseteq N,\ T \neq \emptyset\}$ and the class of identity games $\{e_T : T \subseteq N,\ T \neq \emptyset\}$ are bases for the game space $G(N)$. The null game (N, v_0) is given by $v_0(S) = 0\ \forall\ S \subseteq N$.

Since $G(N)$ is a linear space and $\{u_T : T \subseteq N,\ T \neq \emptyset\}$ is a basis for $G(N)$, every $v \in G(N)$ can be expressed uniquely as a linear combination of these basis vectors as follows:

$$v = \sum_{\emptyset \neq S \subseteq N} \Delta_v(S) u_S, \tag{32.3}$$

where the term $\Delta_v(S)$ is called the Harsanyi dividend [10] and is given for all $S \subseteq N$ by

$$\Delta_v(S) = \begin{cases} 0 & \text{for } S = \emptyset, \\ v(S) - \sum_{R \subsetneq S} \Delta_v(R) & \text{otherwise.} \end{cases} \quad (32.4)$$

An alternative expression of the Harsanyi dividend is given by the following formula.

$$\Delta_v(S) = \sum_{T \subseteq S} (-1)^{s-t} v(T). \quad (32.5)$$

A solution to the class $G(N)$ of TU games is a function defined on $G(N)$ that assigns each TU game an n-tuple of real numbers. An intuitive assumption in this framework is the formation of grand coalition, and the n-tuple given by a solution is usually a distribution of the worth of this grand coalition. The single point solutions are called values. The most popular and transverse value is the Shapley value [19], defined in terms of Harsanyi dividends[10] as follows:

$$\Phi_i^{sh}(N, v) = \sum_{S \subseteq N : i \in S} \frac{\Delta_v(S)}{s}, \quad \forall\, i \in N. \quad (32.6)$$

The Shapley value is characterized, among others by the following axioms; see [19]. Before proceeding to these axioms, we give some definitions.

Definition 1 (Symmetric Players) Players $i, j \in N$ are symmetric in $(N, v) \in G(N)$ if $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$.

Definition 2 (Null Player) Player $i \in N$ is a null player in $(N, v) \in G(N)$ if for all $S \subseteq N \setminus i$, $v(S \cup i) = v(S)$.

Definition 3 (Carrier) A coalition C is said to be a carrier of $(N, v) \in G(N)$ if $v(S \cap C) = v(S)\ \forall S \subseteq N$.

It follows that if C is a carrier and $i \in N \setminus C$, then for any $S \subseteq N \setminus i$, $v(S \cup i) = v(S \cup i \cap C) = v(S \cap C) = v(S)$. Thus if C is a carrier of (N, v), all players $j \in N \setminus C$ are null players in (N, v).

Axiom 1 (Carrier) A value Φ on $G(N)$ satisfies Carrier, i.e. for any game (N, v) and any carrier C in (N, v), $\sum_{i \in C} \Phi_i(N, v) = v(C) = v(N)$ and $\Phi_j(N, v) = 0$ for all $j \in N \setminus C$.

Note that Carrier axiom is equivalent to efficiency that implies that the total payoff to all the members of the grand coalition is equal to the worth of the grand coalition; hence there is neither deficit nor saving over the total worth $v(N)$.

Axiom 2 (Symmetry) A value Φ on G satisfies Symmetry if $\Phi_i(N, v) = \Phi_j(N, v)$ for each pair of symmetric players $i, j \in N$.

The last axiom is that of Linearity (Additivity).

Axiom 3 (Linearity (Additivity)) A value Φ on G is Linear if for arbitrary (N, v), & $(N, w) \in G$, and $\alpha, \beta \in \mathbb{R}$, one must have

$$\Phi(N, \alpha v + \beta w) = \alpha\Phi(N, v) + \beta\Phi(N, w).$$

Φ is Additive if the above conditions holds only for $\alpha = \beta = 1$.

Shapley's[19] characterization theorem goes as follows.

Theorem 1 *There exists exactly one function $\Phi : G(N) \mapsto \mathbb{R}^n$ that satisfies axioms Carrier, Symmetry and Linearity, and it is given by Eq. (32.6).*

3 A Class of Probabilistic Games and Its Shapley Function

Following our discussion in Sect. 1, now we assume that the formation of the coalitions is realized with an endogenous probability distribution over the set 2^N of coalitions of N. Our second assumption is that the probability of formation of $S \in 2^N$ depends on the probabilities of formation of all its sub-coalitions given by this probability distribution. Thus, for each $S \subseteq N$, we associate a probability measure $p^N : 2^N \mapsto [0, 1]$ such that the set $\{p^N(S) | S \subseteq N, \ \sum_{S \subseteq N} p^N(S) = 1\}$ forms a probability distribution over 2^N. With an abuse of notation, we call the probability measure p^N a probability distribution. Denote the class of all such distributions by $\mathbb{P}^N$. Thus formally, we have

$$\mathbb{P}^N = \{p^N : 2^N \mapsto [0, 1] | \sum_{S \subseteq N} p^N(S) = 1\}.$$

Let $\mathbb{P} = \{\mathbb{P}^N : N \in \aleph\}$ denote the class of all probability distributions over a variable player set $N \in \aleph$.

Definition 4 An ordered pair $(p^N, S) \in \mathbb{P}^N \times 2^N$ consisting of a probability distribution over N and a coalition $S \subseteq N$ is called a probabilistic coalition. With an abuse of notations, we denote by $i \in (p^N, S)$ to represent that i is an element of the probabilistic coalition (p^N, S); however, the notion $i \in (p^N, S)$ differs from $i \in S$ in the sense that, here, player i joins coalition S as prescribed by the probability distribution p^N over N. Further, for a fixed $p^N \in \mathbb{P}^N$, we define a partial order between two probabilistic coalitions (p^N, S) and (p^N, T) by $(p^N, S) \subseteq (p^N, T)$ if and only if $S \subseteq T$.

It follows that when a probabilistic coalition (p^N, S) is such that $p^N(S) = 1$ and $p^N(T) = 0, \ \forall \, T \neq S$, then we can say that coalition S forms with certainty. In other words, every coalition $S \in 2^N$ corresponds to a probabilistic coalition (p^N, S) such that $p^N(S) = 1$ and $p^N(T) = 0 \, \forall \, T \neq S$.

Definition 5 A probabilistic game with transferable utilities is a pair (N, w) with $N \in \aleph$ and a function $w : \mathbb{P}^N \times 2^N \to \mathbb{R}$: call it the probabilistic coalition function that satisfies $w(p^N, \emptyset) = 0$, for all $p^N \in \mathbb{P}^N$.

Let $G(\mathbb{P}^N)$ denote the set of the probabilistic games (N, w) over a fixed player set N and $G(\mathbb{P})$ denote the set of the probabilistic games (N, w) for all $N \in \aleph$. The probabilistic game $(N, w) \in G(\mathbb{P}^N)$ is called the probabilistic null game if $w(p^N, S) = 0$, for all $(p^N, S) \in \mathbb{P}^N \times 2^N$.

Definition 6 A probabilistic coalition $(p^N, S) \in \mathbb{P}^N \times 2^N$ is called a carrier in (N, w) if

$$w(p^N, S \cap T) = w(p^N, T) \ \ \forall (p^N, T) \in \mathbb{P}^N \times 2^N.$$

Let $C(N, w)$ denote the class of all carriers in (N, w). In the following, we prove that intersection of the carriers in a probabilistic non-null game (N, w) is non-empty and is again a carrier.

Proposition 1 Let $(p^N, S_1),\ (p^N, S_2) \in C(N, w)$. Then $(p^N, S_1 \cap S_2) \in C(N, w)$ for each probabilistic non-null game $(N, w) \in G(\mathbb{P}^N)$.

Proof We prove our proposition in two steps.

Let (p^N, S_1) and (p^N, S_2) be two carriers in $(N, w) \in G(\mathbb{P}^N)$. As a deduction, we have $w(p^N, S_1 \cap T) = w(p^N, T)$ and $w(p^N, S_2 \cap T) = w(p^N, T)$, $\forall\ (p^N, T) \in \mathbb{P}^N \times 2^N$. In particular taking $T = S_1$ and $T = S_2$, we get

$$w(p^N, S_1 \cap S_2) = w(p^N, S_1) = w(p^N, S_2) \tag{32.7}$$

Let $S_1 \cap S_2 = P$, then using Eq. (32.7), for all $(p^N, T) \in \mathbb{P}^N \times 2^N$, $w(p^N, P \cap T) = w(p^N, S_1 \cap (S_2 \cap T)) = w(p^N, S_2 \cap T) = w(p^N, T)$. Hence P is also a carrier.

Our next assertion is that the intersection of two carriers cannot be empty in a probabilistic non-null game. Since otherwise, by the first part, $(p^N, \emptyset)$ is also a carrier which implies that (N, w) is the probabilistic null game. Hence for a probabilistic non-null game, intersection of two carriers cannot be empty.

Similar to its counterpart in classical TU games, a solution to the class $G(\mathbb{P}^N)$ can be defined as a function $f : G(\mathbb{P}^N) \mapsto (\mathbb{R}^n)^{\mathbb{P}^N}$ such that for each $(N, w) \in G(\mathbb{P}^N)$, and $p^N \in \mathbb{P}^N$, $f(N, w)(p^N)$ is an n-vector of real numbers representing the payoffs to the players. Next, we define the Shapley function on the class $G(\mathbb{P}^N)$ based on a set of properties specific to the set of probabilistic games.

Definition 7 Given $(N, w) \in G(\mathbb{P}^N)$, and $p^N \in \mathbb{P}^N$, a probabilistic coalition (p^N, M_w) is called a maximal probabilistic coalition of (N, w) with respect to p^N if $w(p^N, S) = w(p^N, S \cap M_w)$ for all $(p^N, S) \nsubseteq (p^N, M_w)$ and $w(p^N, M_w) \neq 0$.

It is easy to verify that M_w is the intersection of all carriers of the probabilistic TU game (N, w). In view of proposition 1, therefore, M_w is also a carrier. Based on this, we define the following axiom, namely, the "Maximal Coalitional Efficiency".

Axiom 4 (Maximal Coalitional Efficiency) A function $\Phi : G(\mathbb{P}^N) \mapsto (\mathbb{R}^n)^{\mathbb{P}^N}$ satisfies Maximal Coalitional Efficiency, that is, for each $(N, w) \in G(\mathbb{P}^N)$ and $p^N \in \mathbb{P}^N$,

$$\sum_{i \in (p^N, M_w)} \Phi_i(N, w)(p^N) = w(p^N, M_w)$$

$$\Phi_i(N, w)(p^N) = 0 \quad \text{whenever } i \notin (p^N, M_w).$$

The "Maximal Coalitional Efficiency" indicates that given a probability distribution $p^N \in \mathbb{P}^N$ and $(N, w) \in G(\mathbb{P}^N)$, the worth $w(p^N, M_w)$ so obtained should be completely exhausted by distributing it among all the members of the maximal coalition. The next axiom is also a slight deviation from the symmetry axiom of the classical Shapley function. Here, we define a property called p^N-Compatibility that looks into how two players can contribute to the worth generation through their probabilities of joining a coalition.

Definition 8 Given $(N, w) \in G(\mathbb{P}^N)$ and $p^N \in \mathbb{P}^N$, players $i, j \in N$ are called equally compatible with respect to p^N in (N, w) if

$$w(p^N, S \cup i) = w(p^N, S \cup j), \text{ for all } S \subseteq N \setminus \{i, j\}.$$

The corresponding axiom, which we call the p^N-Compatibility axiom goes as follows.

Axiom 5 (p^N-Compatibility) The function $\Phi : G(\mathbb{P}^N) \mapsto (\mathbb{R}^n)^{\mathbb{P}^N}$ satisfies p^N-Compatibility, namely,

$$\Phi_i(N, w)(p^N) = \Phi_j(N, w)(p^N)$$

for each pair i, j of equally compatible players with respect to p^N in (N, w).

Axiom p^N-Compatibility states that players i and j are rewarded with equal payoffs if they are equally compatible in forming coalitions and generating worth. The next axiom is Additivity which is same as the standard Additivity axiom for classical TU games.

Axiom 6 (Additivity) The function Φ on $G(\mathbb{P}^N)$ is said to satisfy Additivity, i.e. given $(N, w), (N, w') \in G(\mathbb{P}^N)$,

$$\Phi(N, w + w') = \Phi(N, w) + \Phi(N, w').$$

Based on these axioms, we define the Shapley function for the class of probabilistic games as follows.

Definition 9 A function $\Phi : G(\mathbb{P}^N) \to (\mathbb{R}^n)^{\mathbb{P}^N}$ is said to be a Shapley function on the class $G(\mathbb{P}^N)$ if it satisfies Maximal Coalitional Efficiency, p^N-Compatibility, and Additivity. We denote the Shapley value on the class $G(\mathbb{P}^N)$ by Ψ.

4 A Special Collection of Probabilistic Games

In this section, we define a special class of probabilistic games. Recall that for any probability distribution $p^N \in \mathbb{P}^N$ and $S \subseteq N$, we denote by $p^N(S)$ the probability of formation of the coalition $S \subseteq N$. We assume that in order to from a probabilistic coalition (p^N, S), all the players in S must first form the sub-coalitions of S according to the probability distribution p^N. Thus the worth of the probabilistic coalition (p^N, S) in this framework is the expectation of the worths of all the sub-coalitions of S with respect to the probabilities $p^N(T)$ where $T \subseteq S$. For this, we associate a TU game to obtain the worths of the sub-coalitions. Formally we define a probabilistic game as follows.

Definition 10 Given the player set N and a TU game (N, v), a probabilistic game $(N, w_{(N,v)})$ is the pair where the function $w_{(N,v)} : \mathbb{P}^N \times 2^N \mapsto \mathbb{R}$ is defined as

$$w_{(N,v)}(p^N, S) = \sum_{T \subseteq S} v(T) p^N(T) \text{ for each pair } (p^N, S) \in \mathbb{P}^N \times 2^N. \tag{32.8}$$

The TU game (N, v) is called the associate TU game of $(N, w_{(N,v)})$.

It follows from Definition 10 that, for each pair $(p^N, S) \in \mathbb{P}^N \times 2^N$, the value $w_{(N,v)}(p^N, S)$ represents the expected worth of the coalition S with respect to the probability distribution p^N over the worths of all its sub-coalitions. Let $G_0(\mathbb{P}^N)$ denote this particular class of probabilistic games over player set N. Clearly $G_0(\mathbb{P}^N) \subseteq G(\mathbb{P}^N)$. In the following, we obtain the probabilistic Shapley value for the class $G_0(\mathbb{P}^N)$. Given $(N, w_{(N,v)}) \in G_0(\mathbb{P}^N)$ with the associate TU game $(N, v) \in G(N)$ and a probability distribution $p^N \in \mathbb{P}^N$, define a TU game $v^{p^N} : 2^N \mapsto \mathbb{R}$ by

$$v^{p^N}(S) = w_{(N,v)}(p^N, S) \;\; \forall S \subseteq N. \tag{32.9}$$

The Shapley function of (N, v^{p^N}) is given by

$$\Phi_i^{sh}(N, v^{p^N}) = \sum_{S \subseteq N : i \in S} \frac{\Delta_{(N,v^{p^N})}(S)}{s}, \text{ where } \Delta_{(N,v^{p^N})}(S) \text{ is given by Eq. (32.5)}. \tag{32.10}$$

On simplification of the expression in Eq. (32.10), we obtain

$$\Phi_i^{sh}(N, v^{p^N}) = \sum_{S \subseteq N: i \in S} \frac{\sum_{T \subseteq S} (-1)^{s-t} v^{p^N}(T)}{s}$$
$$= \sum_{S \subseteq N: i \in S} \frac{1}{s} \Big(\sum_{T \subseteq S} (-1)^{s-t} \sum_{K \subseteq T} v(K) p^N(K) \Big)$$
$$= \sum_{S \subseteq N: i \in S} \frac{v(S) p^N(S)}{s}. \tag{32.11}$$

Now, define a function $\Psi : G_0(\mathbb{P}^N) \mapsto (\mathbb{R}^n)^{\mathbb{P}^N}$ by

$$\Psi(N, w_{(N,v)})(p^N) = \Phi^{sh}(N, v^{p^N}) \text{ for each } p^N \in \mathbb{P}^N. \tag{32.12}$$

Following theorem states that Ψ so defined is the probabilistic Shapley function on $G_0(\mathbb{P}^N)$.

Theorem 2 The function Ψ defined by Eq. (32.12) is the unique Shapley function for the class $G_0(\mathbb{P}^N)$.

Proof We first show that Ψ given by Eq. (32.12) satisfies Maximal Coalitional Efficiency, p^N-Compatibility, and Additivity.

(a) **Maximal Coalitional Efficiency**: Given $(N, w_{(N,v)}) \in G_0(\mathbb{P}^N)$ and $p^N \in \mathbb{P}^N$, let (p^N, M_w) be a maximal coalition of $(N, w_{(N,v)})$. Since $w_{(N,v)}(p^N, T) = w_{(N,v)}(p^N, T \cap M_w)$, therefore $v(S) = 0$, for all $S \not\subset M_w$, we have

$$\sum_{i \in (p^N, M_w)} \Psi_i(N, w_{(N,v)})(p^N) = \sum_{i \in M_w} \Phi^{sh}(N, v^{p^N})$$
$$= \sum_{i \in M_w} \left\{ \sum_{S \subseteq M_w : i \in S} \frac{v(S)}{s} p^N(S) \right\}$$
$$= \sum_{S \subseteq M_w} s \frac{v(S)}{s} p^N(S)$$
$$= \sum_{S \subseteq M_w} p^N(S) v(S)$$
$$= w_{(N,v)}(p^N, M_w)$$

Also,

$$\Psi_i(N, w_{(N,v)})(p^N) = \sum_{S \subsetneq M_w : i \in S} \frac{v(S)}{s} p^N(S) = 0, \quad \text{for all } i \notin M_w.$$

Hence Ψ satisfies Maximal Coalitional Efficiency.

(b) **p^N-Compatibility:**
Given $(N, v) \in G(N)$, $(N, w_{(N,v)}) \in G_0(\mathbb{P}^N)$ and $p^N \in \mathbb{P}^N$, let $i, j \in N$, be equally compatible players in $(N, w_{(N,v)})$ with respect to p^N, i.e.

$$w_{(N,v)}(p^N, S \cup i) = w_{(N,v)}(p^N, S \cup j), \text{ for all } S \subseteq N \setminus \{i, j\}.$$

In view of Eq. (32.8), with some simplifications, we obtain

$$v(S \cup i)p^N(S \cup i) = v(S \cup j)p^N(S \cup j), \text{ for all } S \subseteq N \setminus \{i, j\}$$

It follows from Eqs. (32.11) and (32.12) that

$$\begin{aligned}\Psi_i(N, w_{(N,v)})(p^N) &= \sum_{S \subseteq N : i \notin S} \frac{v(S \cup i)}{s+1} p^N(S \cup i) \\ &= \sum_{S \subseteq N : j \notin S} \frac{v(S \cup j)}{s+1} p^N(S \cup j) \\ &= \Psi_j(N, w_{(N,v)})(p^N).\end{aligned}$$

(c) **Additivity:** For any $(N, v), (N, v') \in G(N)$, $p^N \in \mathbb{P}^N$ and corresponding $(N, w_{(N,v)}), (N, w_{(N,v')}) \in G(\mathbb{P}^N)$, we have from Eqs. (32.11) and (32.12)

$$\begin{aligned}\Psi_i(N, w_{(N,v)} + w_{(N,v')})(p^N) &= \sum_{S \subseteq N : i \in S} \frac{(v + v')(S) p^N(S)}{s} \\ &= \Psi_i(w_{(N,v)})(p^N) + \Psi_i(w_{(N,v')})(p^N).\end{aligned}$$

For the uniqueness part, let Φ be a probabilistic value satisfying Maximal Coalitional Efficiency, p^N-Compatibility, and Additivity. We show that $\Phi = \Psi$. Let $(N, v) \in G(N)$. Then, there exist unique real numbers $c_T \in \mathbb{R}$, $T \neq \emptyset$ such that $v = \sum_{T \subseteq N, T \neq \emptyset} c_T u_T$. Then by Eq. (32.8), we have

$$\begin{aligned}w_{(N,v)}(p^N, S) &= \sum_{T \subseteq S} v(T) p^N(T) \\ &= \sum_{T \subseteq S} \Big(\sum_{K \subseteq N} c_K u_K(T) \Big) p^N(T) \\ &= \sum_{K \subseteq N} c_K \Big(\sum_{T \subseteq S} u_K(T) p^N(T) \Big) \\ &= \sum_{K \subseteq N} c_K w_{(N,u_K)}(p^N, S) \;\; \forall \, (p^N, S) \in \mathbb{P}^N \times 2^N.\end{aligned}$$

Thus, $w_{(N,v)} = \sum_{K \subseteq N} c_K w_{(N,u_K)}$. Using Additivity of Ψ and Φ, it follows that

$$\Phi(N, w_{(N,v)}) = \sum_{T \subseteq N, T \neq \emptyset} \Phi(N, c_T w_{(N,u_T)}) \tag{32.13}$$

$$\Psi(N, w_{(N,v)}) = \sum_{T \subseteq N, T \neq \emptyset} \Psi(N, c_T w_{(N,u_T)}) \tag{32.14}$$

Therefore, it is enough to show that $\Phi(N, cw_{(N,u_T)}) = \Psi(N, cw_{(N,u_T)}),\ \forall\, T \subseteq N,\ T \neq \emptyset$ and $c \in \mathbb{R}$.

In view of Eqs. (32.1) and (32.2), $u_T = \sum_{T \subseteq K \subseteq N} e_K$ and $M_{e_K} = K$. Therefore, using Additivity of Φ, we have

$$\Phi_i(N, cw_{(N,u_T)}) = \sum_{T \subseteq K \subseteq N} \Phi_i(N, cw_{(N,e_K)})$$

Here, $M_{e_K} = K$ and any pair of $i, j \in K$ are equally compatible with respect to p^N in $w_{(N,e_K)}$. Therefore, using Maximal Coalitional Efficiency and p^N-Compatibility,

$$\begin{aligned}
&\sum_{i \in M_{e_K}} \Phi_i(N, cw_{(N,e_K)})(p^N) = \sum_{S \subseteq K} c\ p^N(S) e_K(S) \\
\Rightarrow \quad & k\Phi_i(N, cw_{(N,e_K)})(p^N) = c\ p^N(K) \\
\Rightarrow \quad & \Phi_i(N, cw_{(N,e_K)})(p^N) = \frac{cp^N(K)}{k} \\
\Rightarrow \quad & \Phi_i(N, cw_{(N,e_K)})(p^N) = \Psi_i(N, cw_{(N,e_K)})(p^N) \ \text{ for all } i \in K.
\end{aligned}$$

Also, $\Phi_i(N, cw_{(N,e_K)})(p^N) = 0 = \Psi_i(N, cw_{(N,e_K)})(p^N)$, for all $i \notin K$.

Therefore, $\Psi_i(N, cw_{(N,u_T)}) = \sum_{T \subseteq K : i \in K} \frac{cp^N(K)}{k} = \Phi_i(N, cw_{(N,u_T)}),\ \forall\, i \in N, T \subseteq N$.

This completes the proof.

Remark 1 The logical independence of the three axioms in theorem (2) are shown below:

1. The value $\Phi_i(N, w_{(N,v)})(p^N) = \frac{\Psi_i(N, w_{(N,v)})(p^N)}{2^n - 1}$ satisfies p^N–Compatibility and Additivity, but not Maximal Coalitional Efficiency.
2. The value $\Phi_i(N, w_{(N,v)})(p^N) = \Psi_i(N, w_{(N,v)})(p^N) + \frac{w_{(N,v)}(p^N, N)}{[\frac{n}{2}]}$ for first $[\frac{n}{2}]$ number of players and

$$\Phi_i(N, w_{(N,v)})(p^N) = \Psi_i(N, w_{(N,v)})(p^N) - \frac{w_{(N,v)}(p^N, N)}{[\frac{n}{2}]}$$

for the next $[\frac{n}{2}]$ number of players and $\Phi_i(N, w_{(N,v)})(p^N) = \Psi_i(N, w_{(N,v)})(p^N)$ for the remaining players (if exist) satisfies Maximal Coalitional Efficiency and Additivity, but not p^N−Compatibility.

3. The value $\Phi_i(N, w_{(N,v)})(p^N) = \Psi_i(N, w_{(N,v)})(p^N)$ for $w_{(N,v)}(p^N, N) \geq K$, and for all $w_{(N,v)}(p^N, N) < K$, $\Phi_i(N, w_{(N,v)})(p^N) = \frac{w_{(N,v)}(p^N,N)}{n}$ where $K \in \mathbb{R}$ satisfies p^N-Compatibility and Maximal Coalitional Efficiency, but not Additivity.

Remark 2 An alternative expression for $\Psi : G_0(\mathbb{P}^N) \mapsto (\mathbb{R}^n)^{\mathbb{P}^N}$ can be obtained from the standard simplifications used in classical TU games as follows:

$$\Psi(N, w_{(N,v)})(p^N) = \sum_{S \subseteq N \setminus i} \frac{(n-s-1)!s!}{n!} \{\sum_{T \subseteq S} v(T \cup i) p^N(T \cup i)\} \qquad (32.15)$$

From now onward, we call Ψ the probabilistic Shapley value on $G_0(\mathbb{P}^N)$. In what follows next, we present a characterization of the probabilistic Shapley value specific to the class $G_0(\mathbb{P}^N)$. In the following, we present a numerical example to illustrate our proposed model.

Example 1 Consider the player set $N = \{1, 2, 3, 4, 5\}$, and the TU game (N, v) is defined as follows:
$v(S) = 0$ for all $S \subseteq N$ such that $5 \in S$; $v(\{1\}) = v(\{2\}) = 2$; $v(\{3\}) = v(\{4\}) = 3$;$v(\{1, 3\}) = v(\{2, 3\}) = 5$; $v(\{1, 4\}) = v(\{2, 4\}) = 4$; $v(\{1, 2\}) = 3$, $v(\{3, 4\}) = 2$; $v(\{1, 3, 4\}) = v(\{2, 3, 4\}) = 5$; $v(\{1, 2, 3\}) = v(\{1, 2, 4\}) = 6$; $v(\{1, 2, 3, 4\}) = 5$.
It is easy to show that in the probabilistic game $(N, w_{(N,v)})$, the maximal probabilistic coalition is given by (p^N, M_w), where $M_w = \{1, 2, 3, 4\}$. Assume that the endogenous probability distribution p^N is given by the following:
$p^N(\emptyset) = \frac{1}{6}$; $p^N(\{i\}) = \frac{1}{30}$; $p^N(\{i, j\}) = \frac{1}{60}$; $p^N(\{i, j, k\}) = \frac{1}{60}$; $p^N(\{i, j, k, l\}) = \frac{1}{30}$; $p^N(N) = \frac{1}{30}$, for all $i, j, k, l \in N$, i.e.

$$p^N(S) = \begin{cases} \frac{1}{6} & \text{if } S = \emptyset \text{ or } S = N \\ \frac{1}{6 \times \binom{4}{s}} & \text{otherwise} \end{cases} \quad \text{and } \; p^N(S) = 0 \; \text{ for all } \; S \subseteq N : 4 \in S. \qquad (32.16)$$

Now we have

$$\Psi_i(N, w_{(N,v)})(p^N) = \begin{cases} \sum_{S \subseteq M_w : i \in S} \frac{v(S)}{s} p^N(S), & \text{for all } i \in M_w. \\ 0, & \text{for all } i \notin M_w \end{cases}$$

Thus the payoff for each $i \in N$ in the game $w_{(N,v)}$ corresponding to the probability distribution p^N given by Eq. (32.16) is given by $\Psi(N, w_{(N,v)})(p^N) = (\frac{20}{72}, \frac{20}{72}, \frac{25}{72}, 0, 0)$. Note that, since $p^N(S) = 0$ for all $S \subseteq N : 4 \in S$, therefore, we have $\Psi_4(N, w_{(N,v)})(p^N) = 0$. Moreover, player 1 and 2 satisfy p^N compatibility therefore, $\Psi_1(N, w_{(N,v)})(p^N) = \frac{20}{72} = \Psi_2(N, w_{(N,v)})(p^N)$. Finally, due to the Maximum Coalitional Efficiency $\Psi_5(N, w_{(N,v)})(p^N) = 0$ and $\Psi_3(N, w_{(N,v)})(p^N) = \frac{25}{72}$.

4.1 Characterization on $G_0(\mathbb{P}^N)$

In this section, we present a characterization of the probabilistic Shapley value on $G_0(\mathbb{P}^N)$ along the line of Young's characterization of the Shapley value for TU games; see [22]. Let $w_{(N,v)}$ be the probabilistic null game with associate game $(N, v) \in G(N)$. That is, $w_{(N,v)}(p^N, S) = 0$ for each $(p^N, S) \in \mathbb{P}^N \times 2^N$. Note that a sufficient condition for $w_{(N,v)}$ to be null is that (N, v) is the null game in $G(N)$, that is, $v(S) = 0$ for all $S \subseteq N$; however, this condition is not necessary. Given a probability distribution $p^N \in \mathbb{P}^N$, let $\mathbb{V}_0 \subseteq G(N)$ be the class of TU games $(N, v) \in G(N)$ for which $(N, w_{(N,v)})$ is the probabilistic null game. Let us denote a probabilistic null game by $(N, O_{(N,v)})$ with $(N, v) \in \mathbb{V}_0$. We have the next axiom as follows.

Axiom 7 (Probabilistic Null Game Property) A function $\Phi : G_0(\mathbb{P}^N) \mapsto (\mathbb{R}^n)^{\mathbb{P}^N}$ satisfies probabilistic null game property if for each $p^N \in \mathbb{P}^N$

$$\Phi_i(N, O_{(N,v)})(p^N) = 0, \quad \forall\, i \in N, \; (N, v) \in \mathbb{V}_0.$$

Following similar formulations as in their counterparts in classical TU games, for a given $(N, v) \in G(N)$, $(p^N, S) \in \mathbb{P}^N \times 2^N$ and a player $i \in N$, let us call the term $w_{(N,v)}(p^N, S \cup i) - w_{(N,v)}(p^N, S)$ the marginal contribution of player i from $S \subseteq N \backslash i$ with respect to the probabilistic game $(N, w_{(N,v)})$. The next axiom is strong p^N-Monotonicity which suggests that if the marginal contribution of a player with respect to a probabilistic game is not less than the marginal contribution with respect to another probabilistic game, then the payoff to this player by the former game cannot be less than the latter.

Axiom 8 (Strong p^N-Monotonicity) Given $i \in N$, if $(N, v), (N, v') \in G$ and the corresponding $(N, w_{(N,v)}), (N, w_{(N,v')}) \in G(\mathbb{P})$ are such that $w_{(N,v)}(p^N, S \cup i) - w_{(N,v)}(p^N, S) \geq w_{(N,v')}(p^N, S \cup i) - w_{(N,v')}(p^N, S)$ for all $p^N \in \mathbb{P}^N$ and all

$S \in 2^N$ with $i \notin S$, then the probabilistic value Φ satisfies Strong p^N-Monotonicity, namely,

$$\Phi_i(N, w_{(N,v)})(p^N) \geq \Phi_i(N, w_{(N,v')})(p^N).$$

Definition 11 A player $i \in N$ is called the probabilistic null player with respect to (N, v) and p^N if $w_{(N,v)}(p^N, T) - w_{(N,v)}(p^N, T \backslash i) = 0,\ \forall\ T$ such that $i \in T \subseteq N$.

Axiom 9 (Probabilistic Null Player Property) A probabilistic value Φ satisfies probabilistic null player property if $\Phi_i(N, w_{(N,v)})(p^N) = 0$ for a probabilistic null player i with respect to (N, v) and p^N.

Theorem 3 *A probabilistic value Φ satisfies Maximal Coalitional Efficiency, Monotonicity, and Strong p^N-Monotonicity if and only if $\Phi(N, w_v) = \Psi(N, w_v)$.*

Proof In view of Theorem 2, the probabilistic Shapley value Ψ satisfies Maximal Coalitional Efficiency and p^N-Compatibility. Also, it is easy to show that Ψ satisfies Strong p^N-Monotonicity. For the converse part, let Φ be a function satisfying Maximal Coalitional Efficiency, p^N-Compatibility, and Strong p^N-Monotonicity.

We first claim that if a probabilistic value satisfies Maximal Coalitional Efficiency and p^N-Compatibility, then it also satisfies the probabilistic null game property. Note that, in the probabilistic null game $(N, O_{(N,v)})$, all the players are equally compatible. Because $O_{(N,v)}(p^N, S) = 0,\ \forall\ S \subseteq N \Rightarrow v(S)p^N(S) = 0,\ \forall\ S \subseteq N$, therefore for arbitrary pair i, j and $\forall\ S \subseteq N \setminus \{i, j\}$, we have $v(S \cup i)p^N(S \cup i) = 0 = v(S \cup j)p^N(S \cup j)$ where $(N, v) \in \mathbb{V}_0$.

Therefore following Maximal Coalitional Efficiency and p^N-Compatibility, i.e.

$$\Phi_i(N, O_{(N,v)})(p^N) = \Phi_j(N, O_{(N,v)})(p^N).$$

For any pair of compatible players i, j, we have

$$\sum_{i \in M_O} \Phi_i(N, O_{(N,v)})(p^N) = 0 \quad \text{and} \quad \Phi_i(N, O_{(N,v)})(p^N) = 0\ \forall\ i \notin M_O$$

$$\Rightarrow \Phi_i(N, O_{(N,v)})(p^N) = 0,\ \ \forall\ \ i \in N \ \text{ and } (N, v) \in \mathbb{V}_0.$$

Next, we show that a value satisfying probabilistic null game property and Strong p^N—Monotonicity also satisfies Probabilistic Null player property.

Let $i \in N$ be a Probabilistic Null player in $w_{(N,v)}$ hence, $w_{(N,v)}(p^N, T) = w_{(N,v)}(p^N, T \setminus i)\ \forall\ T$ such that $i \in T$. Moreover, $O_{(N,v)}(T) = 0\ \forall\ T \subseteq N$ i.e. $O_{(N,v)}(p^N, T) = 0 = O_{(N,v)}(p^N, T \setminus i)$. Therefore, following Strong p^N-Monotonicity, $\Phi_i(N, w_{(N,v)})(p^N) = \Phi_i(O_{(N,v)})(p^N) = 0$. This proves our assertion.

Since $w_{e_T}(p^N, S) = w_{e_T}(p^N, S \setminus i)$, for all $S \subseteq N \setminus i$, therefore for the class of game w_{e_T}, all $i \notin T$ are Probabilistic Null players. Similarly since $e_T(S \cup i)p^N(S \cup$

$i) = 0.p^N(S \cup i) = 0 \cdot\ p^N(S \cup j) = e_T(S \cup j)p^N(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$, therefore all $i, j \in T$ are compatible players with respect to e_T and p^N.

Therefore, using Maximal Coalitional Efficiency, p^N-Compatibility, and Probabilistic Null player property,

$$\Phi_i(N, w_{(N,e_K)})(p^N) = \Psi_i(N, w_{e_K})(p^N) = 0,\ \forall\, i \notin K$$

$$\Phi_i(N, w_{(N,e_K)})(p^N) = \Psi_i(N, w_{e_K})(p^N) = \frac{p^N(K)}{k},\ \forall\, i \in K$$

We prove the uniqueness using induction on the size of number of non-zero $\Delta_{w_{(N,v)}(p^N,T)}$s. Note that $\Delta_{w_{(N,v)}(p^N,T)} = p^N(T)v(T) \neq 0$ implies $p^N(T) \neq 0$ and $v(T) \neq 0$. Therefore, for any $p^N \in \mathbb{P}^N$, it is enough to apply induction on the size of $\Delta_{w_{(N,v)}(p^N,T)}$.

Let $\Im_{w_{(N,v)}} = \{T : \Delta_{w_{(N,v)}(p^N,T)} = v(T).p^N(T) \neq 0,\ T \subseteq N\}$, i.e. $p^N(T) \neq 0$ as well as $v(T) \neq 0$. We apply induction on the size of $\Im_{w_{(N,v)}}$. For $|\Im_{w_{(N,v)}}| = 0 \Rightarrow v = v_0$.

Therefore, $\Phi_i(N, w_{(N,v)})(p^N) = \Phi_i(w_{(N,v_0)})(p^N) = 0 = \Psi_i(w_{(N,v_0)})(p^N) = \Psi_i(N, w_{(N,v)})(p^N)$.

For $|\Im_{w_{(N,v)}}| = 1$, there exists $T \subseteq N$ with $v(T) \neq 0,\ p^N(T) \neq 0$ and hence using Maximal Coalitional Efficiency:

$$\Phi_i(N, w_{(N,v)})(p^N) = \frac{v(T)p^N(T)}{t} = \Psi_i(N, w_{(N,v)})(p^N)$$

Let $\Phi_i(N, w_{v'})(p^N) = \Psi_i(N, w_{v'})(p^N),\ \forall\, (N, v') \in G(N)$ with $|\Im_{w_{v'}}| < k,\ k \geq 2$. Suppose $(N, v) \in G(N)$ be a game with $|\Im_{w_{(N,v)}}| = k$.

There exist T_r, $r \in \{1, 2, \ldots, k\}$ such that $v(T_r) \neq 0$, $p^N(T_r) \neq 0$ and hence $w_{(N,v)}(p^N, N) = \sum_{r=1}^{K} v(T_r)p^N(T_r)$. Consider $D = \cap_{r=1}^{K} T_r$.

For $i \in N \backslash D$, we define $w_{v^i}(p^N, S) = \sum_{i \in T_r} v(T_r)w_{(N,e_{T_r})}(p^N, S)$ for $S \subseteq N$. Because $|\Im_{w_{v_i}}| < k$, the induction hypothesis implies, $\Phi_i(N, w_{(N,v^i)}) = \Psi_i(N, w_{(N,v^i)})$.

Note that, for $i \notin D$, but $i \in T_r$ for some r, we have $w_{(N,v)}(p^N, N) = \sum_{i \in T_r} v(T_r)p^N(T_r) = w_{v^i}(p^N, N)$. Following Strong p^N—Monotonicity,

$\Phi_i(N, w_{(N,v)})(p^N) = \Phi_i(N, w_{(N,v^i)})(p^N) = \Psi_i(N, w_{(N,v^i)})(p^N) = \Psi_i(N, w_{(N,v)})(p^N)$. On the other hand if $i \notin T_r$ for any T_r then $i \notin M_w$ and hence $\Phi_i(N, w_{(N,v)})(p^N) = 0 = \Psi_i(w_{v^i})(p^N)$ and consequently

$$\Phi_i(N, w_{(N,v)})(p^N) = \Psi_i(N, w_{(N,v^i)})(p^N),\ \text{for all } i \in N \setminus D. \tag{32.17}$$

By Maximal Coalitional Efficiency,

$$\sum_{i \in M_w} \phi_i(N, w_{(N,v)}(p^N) = \sum_{i \in M_w} \psi_i(N, w_{(N,v)})(p^N)$$

Moreover, for $i \notin D$, using Eq. (4.1)

$$\sum_{i \in M_w} \Phi_i(N, w_{(N,v)})(p^N) = \sum_{i \in M_w} \Psi_i(N, w_{(N,v)})(p^N)$$

$$\Rightarrow \sum_{i \in D} \Phi_i(N, w_{(N,v)})(p^N) + \sum_{i \in M_w \setminus D} \Phi_i(N, w_{(N,v)})(p^N) = \sum_{i \in D} \Psi_i(N, w_{(N,v)})(p^N) + \sum_{i \in M_w \setminus D} \Psi_i(N, w_{(N,v)})(p^N)$$

$$\Rightarrow \sum_{i \in D} \Phi_i(N, w_{(N,v)})(p^N) + \sum_{i \in M_w \setminus D} \Psi_i(N, w_{(N,v)})(p^N) = \sum_{i \in D} \Psi_i(N, w_{(N,v)})(p^N) + \sum_{i \in M_w \setminus D} \Psi_i(N, w_{(N,v)})(p^N)$$

$$\Rightarrow \sum_{i \in D} \Phi_i(N, w_{(N,v)})(p^N) = \sum_{i \in D} \Psi_i(N, w_{(N,v)})(p^N).$$

Note that for all $i, j \in D \Rightarrow i, j \in T_r$ for some r. For all $i, j \in T_r$, $v(S \cup i)p^N(S \cup i) = 0 = v(S \cup j)p^N(S \cup j)$, since $S \cup j \neq T_r$ and $S \cup i \neq T_r$ for any r and $S \subseteq N \setminus \{i, j\}$; therefore, any pair $i, j \in D$ are compatible. Hence, using p^N-Compatibility we obtain, $\Psi_i(N, w_{(N,v)}) = \Psi_j(N, w_{(N,v)})$, for all $i, j \in D$. Again for any $D \subseteq N$,

$$\sum_{i \in D} \Phi_i(N, w_{(N,v)})(p^N) = \sum_{i \in D} \Psi_i(N, w_{(N,v)})(p^N)$$

$$\Rightarrow d.\Phi_i(N, w_{(N,v)})(p^N) = d.\Psi_i(N, w_{(N,v)})(p^N)$$

$$\Rightarrow \Phi_i(N, w_{(N,v)})(p^N) = \Psi_i(N, w_{(N,v)})(p^N) \text{ for all } i \in D.$$

This completes the proof.

Remark 3 The logical independence of each of the axioms in theorem (3) are shown below:

1. The value $\Phi_i(N, w_{(N,v)})(p^N) = \frac{w_{(N,v)}(p^N, N)}{n}$ satisfies p^N-Compatibility and Maximal Coalitional Efficiency but not Strong p^N—Monotonicity.
2. The value $\Phi_i(N, w_{(N,v)})(p^N) = \Psi_i(N, w_{(N,v)})(p^N) + 1$ for first $[\frac{n}{2}]$ number of player and $\Phi_i(N, w_{(N,v)})(p^N) = \Psi_i(N, w_{(N,v)})(p^N) - 1$ for next $[\frac{n}{2}]$ number of player $\Phi_i(N, w_{(N,v)})(p^N) = \Psi_i(N, w_{(N,v)})(p^N)$ for the rest player (if exist), satisfies Maximal Coalitional Efficiency and Strong p^N—Monotonicity, but not p^N-Compatibility.
3. The value $\Phi_i(N, w_{(N,v)})(p^N) = \frac{\Psi_i(N, w_{(N,v)})(p^N)}{2^n - 1}$ satisfies p^N-Compatibility and Strong p^N—Monotonicity, but not Maximal Coalitional Efficiency.

5 Conclusions

In this work, we proposed the probabilistic extension of the classical TU games. First, we defined a general class of probabilistic games over the set of probabilistic coalitions. Then we introduced a special subclass of this general class having some interesting properties. Our assumption is that the formation of a coalition depends on how the players in it interact with each other in the smaller coalitions. Thus we can assign a probability distribution over all possible coalitions, and the worth of any coalition under this setup is the expectation of the worths at sub-coalitions with respect to this probability distribution. The Shapley value is defined for the general class first, and based on that we obtained the corresponding Shapley value for this special subclass. We proposed a characterization of the Shapley value of the special subclass. Various properties of the classical TU games can be extended to the probabilistic framework. However, one drawback of our proposal is that we are able to discuss only a very primitive model involving prior probabilities of forming of a coalition. Updating these probabilities on the basis of players' successive interactions among themselves would provide more credibility to the model. The probabilistic approach of TU games has a natural extension to network games. However, due to their complex nature, a number of problems in this area are still open. Strategic nature of players in forming networks in absence of prior knowledge is, therefore, an interesting area to work on. This we keep for our future studies.

Acknowledgments The first author gratefully acknowledges the financial support from ASTEC Grant No. ASTEC/S&T/192(171)/2019- 20/2762. The suggestions and opinions of the anonymous referee are also greatly appreciated. We sincerely thank the editors of the book for providing all supports in preparing the manuscript.

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C. Jana et al. (eds.), *Fuzzy Optimization, Decision-making and Operations Research*, https://doi.org/10.1007/978-3-031-35668-1