Chapter 9
A Framework for Creative Insights Within Internalization of Mathematics

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9.1 Introduction

One explanation for the dominant role of constructivism in mathematics education is its clearly defined methodology of research in which the researcher acts as a teacher and embodies discovery learning, i.e., provides minimal verbal explanations to the student. In constructivist theory, understanding a problem situation begins with an attempt to assimilate its information into existing schemes. Acts of accommodation, viewed as based upon creative moments of insight, may occur when a problem cannot readily be assimilated into an existing scheme. Piaget describes the mechanism of accommodation, which he terms reflective abstraction, as occurring through reflection upon an existing scheme or solution activity in a problem-solving environment. The primary objective of this article is to use the theoretical framework of Koestler (1964) integrated with the work of constructivist theory, based upon Piaget, to analyze moments of insight within social discourse, i.e., the internalization of Vygotsky (1978).

In previous work, the bisociative framework for creativity Koestler (1964) has been translated into or integrated with that developed by Piaget and Garcia (1989) to analyze acts of accommodation (Baker, 2016). This earlier translation work (the integrated frame) was used to analyze individual moments of creative insight leading to the formation of new schemes. The integrated frame was extended to include the foundational type of reflective abstraction known as interiorization (Baker, 2021a; Czarnocha & Mason, 2021).

The first objective of this article is to review and modify the integrated frame to analyze students' moments of insight, leading to concept development, and accommodation, what Koestler refers to as growth in understanding, within social discourse. Learners struggling to internalize new content may not rate highly according

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to the measures of assessment associated with creative and gifted individuals, such as flexibility, fluency, proficiency, or in terms of productive-divergent thinking (Leikin & Pitta-Pantazi, 2013). That said, for such students, every moment of insight may represent an important step. In Vygotsky's framework, an activity that is initially viewed as externally directed becomes internal through a process of conscious imitation. Thus, the child first imitates or performs an activity with a teacher's assistance and ultimately acts independently. This framework is ideally situated for analyzing classroom instruction, especially teacher-led discourse. However, its drawback is that the process of internalization is not well understood. "As yet, the barest minimum of this process is known" (Vygotsky, 1978, p.57).

A second objective of this article is to clarify the notion of constructivist pedagogy to support moments of insight within social discourse. Constructionism clearly has a major influence on mathematics pedagogy, yet its implementation has been difficult on the institutional level. One problematic issue is that the role of the teacher managing discovery learning is unclear, or ill- defined (Baker, 2021b). One common explanation is that professional development efforts have not been sufficient to impart the essence of discovery learning. Another factor is that the strong focus of constructivist research and pedagogy on individual reflection and abstraction simply does not resonate with the practice of teaching and learning within social discourse. In this line of thought, the strict focus on an individual's subjective understanding is simply not well equipped to provide insight into the teacher's role in a social learning community. Thus, the second objective is to highlight the role of the teacher in guiding students to moments of insight within the process of internalization. That is in bringing the spirit of discovery embodied in constructivist methodology into the daily lesson.

9.2 Theoretical Foundations

9.2.1 Matrices Codes, Schemes

Koestler's (1964) notion of "bisociation" is used to describe moments of insight within a diverse range of fields including humor, art, literature, math, and science. The term bisociation is used to distinguish between associative routine thought – exercise of understanding (assimilation), as opposed to bisociative thought existing simultaneously in two previously unrelated frames of reference or progress in understanding (accommodation). Koestler develops his theory of creative insight in terms of matrices or frames of reference and the code or rules of the game that govern each matrix.

> The matrix is the pattern before you, representing the ensemble of permissible moves. The code which governs the matrix can be put into simple mathematical equations…or it can be expressed in words. The code is the fixed invariable factor in a skill or habit, the matrix is the variable part. The two words do not refer to different entities, they refer to different aspects of the same activity. (p.40)

In the integrated frame, matrices correspond to schemes, which are viewed as matrices whose codes can be expressed in a quasi-mathematical language. In constructivist learning, schemes represent activity that has been generalized into stable, predictable, and repeatable activity (Confrey, 1994a, p.4). Schemes, described in detail presently, are important in analyzing the cognitive effects of moments of insight in the individual's learning process. In the integrated frame, the term "code" translates into the constructivist notion of an "invariant relationship," which may be understood as the conceptual reasoning that underlies solution activity. Thus, the formation of schemes and development of concept are intertwined.

9.2.2 Blocked Non-assimilatory Situation and Discovery of Hidden Analogy

For Koestler (1964), creative insight takes place in a blocked situation, in which the subject does not initially have an appropriate matrix.

> [The situation] still resembles in some respect other situations encountered in the past yet contains new features or complexities which make it impossible to solve by the rules of the game that were used in those past situations. (p.119)

In constructivist theory, this is referred to as a non-assimilatory situation, in which the subject understands the goal in a situation but does not have an appropriate scheme to assimilate the situation and thus complete the goal. Koestler describes selective attention on previously unimportant aspects of a situation, alongside disregard for other details as the foundation for moments of creative insight to resolve a blocked situation, "selective emphasis on the relevant factors and omission of the rest" Koestler (1964, p.72). In creativity theory more generally, selective attention can be understood as three processes: selective encoding, in which the subject applies consciousness attention to a previously overlooked aspects of the situation; selective comparison, in which the subject compares previous knowledge with the given situation; or selective combination of existing but previously unrelated aspects within the situation (Perking, 2000; Davidson, 2002).

Koestler (1964) refers to the resolution of a blocked situation, as the result of selective attention as the discovery of a hidden analogy: "[T]he displacement of attention to something not previously noted, which was irrelevant in the old and is relevant in the new context; the discovery of hidden analogies…" (p.120). Koestler (1964) describes the search through one's collection of possibly related matrices leading to the discovery of a hidden analogy that will remove the block as looking for something one does not know:

> We have seen that one of the basic mechanisms of the Eureka process is the discovery of a hidden analogy…Yet the word 'search,' so often used in the context of problem-solving, is apt to create confusion because it implies that I knew what I am searching for, whereas in fact I do not…[T]he subject looks for a clue, the nature of which he does not know except that it should be a 'clue'…a link to a type of problem familiar to him. Instead of looking through a given filter-frame for an object which matches the filter, he must try out one form

> after another to match the object under his nose, until he finds the frame which it fits, i.e., until the problem presents some familiar aspect- which is then perceived as an analogy with past experience and allows him to come to grips with it. (pp.653–654)

In creativity research, like problem-solving research, the subject often does not have a matrix-scheme to assimilate the situation and thus searches (selective comparison) to find a hidden analogy. In constructivist theory, this search leads to a situation-activity link, which is the first component of an action scheme. The second type of search (selective encoding) occurs after a relevant matrix has been found, yet the subject has not learned how to use it to obtain the desired effect: "The problem in problem solving consists firstly in discovering which routine is appropriate to the problem-what type of game is to be played, and secondly, how to play it-i.e., which strategy to follow, which members of the flexible matrix are to be brought into play according to the lie of the land" (Koestler, 1964, p. 638). In constructivist theory, this leads to an activity-effect link.

9.2.3 Bisociation

Koestler (1964) describes the mechanism of creative insight, "bisociation" as the result of an idea, event, or concept existing or vibrating on two previous unrelated matrices:

> [T]he perceiving of a situation or idea L, in two self-consistent but habitually incompatible frames of reference M1 and M2. The event L in which the two intersect is made to vibrate simultaneously on two different wavelengths, as it were. While this unusual situation lasts, L is not merely linked top one associative context, but bisociated with two. (p.35)

The effect of the bisociative vibration of concepts on two previously unrelated matrices, during the discovery of a hidden analogy, is the transfer of thought from one matrix to another: "The sudden transfer of a mental event with two habitually incompatible matrices results in an abrupt transfer of the train of thought from one associative context to another" (p.59). Thus, for Koestler (1964), the transfer of analogic reasoning during the discovery of a hidden analogy results is the result of a bisociative synthesis of existing matrices:

> The creative act is not an act of creation in the sense of the Old Testament. It does not create something out of nothing; it uncovers, selects, re-shuffles, combines, synthesizes already existing facts, ideas facilities, skills. The more familiar the parts, the more striking the new whole. (p.120)

9.2.4 Understanding and Discovery

Skemp (1987) suggests that "To understand something is to assimilate it into an appropriate schema" (p.29). However, in a blocked or non-assimilatory situation, there is potential for growth in understanding. Sierpinska (1994) comments on the relationship between discovery and understanding when she states "While any invention assures understanding, the latter does not necessarily imply the former…" (p.69). We take the distinction to be that, while understanding thought assimilation (exercise in understanding) occurs without discovery, the process of discovery (rediscovery) or creative insight always results in what Koestler refers to as growth in understanding. Piaget (1977) expresses the view that growth in understanding is based upon moments of insight:

> To understand is to discover, or reconstruct by rediscovery, and such conditions must be complied with if in the future individuals are to be formed who are capable of production and creativity and not simply repetition. (p.20)

Here we take Piaget's use of the phrase "to understand" as referring to accommodation of new information through scheme development, i.e., growth in understanding.

9.2.5 Integrated Frame: Bisociation-Koestler

In the integrated frame, based upon the work of Koestler (1964), moments of insight occur when an individual finds themselves in a blocked situation. Their understanding of the blocked situation, and search to resolve it, is referred to as their search matrix or M1. M1 may be a scheme or some part of a scheme that has been applied in similar situations but is inadequate to assimilate the situation. This creates a tension between their M1 understanding of the situation and their motive to resolve it; constructivists often refer to this as "perturbation." The subjects' search to understand and resolve the situation may, in a moment of insight, lead them to discover a relevant M2 matrix previously unrelated to their M1 (discovery of a hidden analogy-selective comparison) or it may lead them to uncover a feature of their M1, previously not considered relevant (selective encoding). The uncovered or hidden features of M1 are often properties of the objects being acted upon, that when realized direct solution activity. These uncovered features can be understood as a matrix M2 that emerges from the M1 matrix.

In the moment of insight, the discovered analogic matrix M2 undergoes bisociative synthesis with M1; during this process, analogical M2 reasoning transfers to M1 directing solution activity.

During selective encoding, the uncovered matrix M2 has its genesis within aspects of M1; in this situation, the uncovered concepts exist organically within M1and M2 (bisociation). The conceptual relationships established during the bisociation of M1 and M2 represents the birth of what Koestler (1964) would refer to as a "pseudo-code," i.e., the building block for a new matrix- code, that has typically not yet been "reified" into a code (p.639). Thus, the end-product of a moment of insight is novel activity based upon a newly formed conceptual relationship or developing code. Note, such moments of insight take place within problem-solving but are not the result of deductive reasoning, as the actual moment of insight is intuitive in nature insight and has been described by Koestler (1964) as being induced through, "putting thinking aside" (p.182).

Thus, moments of insight in the integrated frame have three defining characteristics: the search process, the connection between M1 and M2 that leads to a transfer of analogic thought, and finally a novel activity that is directed by the conceptual relationship established.

9.2.6 Reflective Abstraction and the Formation of the Action Scheme

Koestler's frame has several weaknesses for analyzing insights leading to the development of mathematical structure; the first is that because it is used so broadly, it is vague, and a second related issue is the relationships between concepts, matrices, and solution activity are not described in depth.

Piaget's notion of reflective abstraction expresses acts of accommodation as containing two parts. The first part is the projection of an existing relevant scheme into a blocked or non-assimilatory situation. The projected scheme, and the subject's sense of why it is lacking in the situation their M1. The projected scheme cannot assimilate the situation hence, it requires coordination with new aspects of the problem, selective attention to these new aspects results in concept formation and/or discovery of new scheme (M2) that when coordinated with M1, what Piaget and Garcia (1989) refer to as "constructive generalization," and Koestler as bisociation, resolves the situation.

Simon et al. (2004) consider that the two-stage notion of reflective abstraction, as presented by Piaget, is simply not well understood and, thus, not useful in guiding pedagogy in mathematics educational research. "It is our contention that the mechanism itself is underspecified for guiding the design of instructional interventions intended to address challenging learning problems in mathematics" (p.313). These authors re-interpret Piaget's notion of reflective abstraction in terms of Von Glasersfeld's (1995) action-scheme, which contains three parts: a situation, activity, and its effect. The action-scheme or scheme can be expressed as two connections or relationships, the first is the situation-activity relationship (S-A), and the second is the activity-effect relationship . Simon et al. (2004) provide a plausible and readily understandable metaphorical descriptions of the search process leading to an A-E relationship, built upon the abstraction of invariant relationships:

> We offer the following physical metaphor to promote an image of the records of experience and how they are used in reflective abstraction. Each record of experience can be thought of as being stored in a jar. Inside of each jar is a particular instance of the activity and the effect of that activity. Each jar is labeled as to whether the record of experience inside was associated with a positive result or a negative result. In the first phase of Piaget's reflective abstraction, the projection phase, jars are sorted according to their labels (i.e., learners mentally-though not necessarily consciously--compare /sort records based on the results).

> In the second phase, the reflection phase, the contents of the jars that have been grouped together are compared and patterns observed. Thus, within each subset of the records of experience (positive versus negative results), the learner's mental comparison of the records allows for recognition of patterns, that is, abstraction of the relationship between activity and effect. (p.319)

Thus, in the first, projection phase, the subject attempts solution activity based upon a relevant but insufficient scheme M1. In the second reflection phase, the subject abstracts the invariant relationship or conceptual relationship from his previous attempts, i.e., the A-E-dyads. This abstraction is the result of coordination between the features of the new situation and properties of the objects being acted upon highlighted by the inadequate effects of this solution activity.

The formation of an A-E-dyad within the transition from intuitive, or externally directed thought to independent, "interior thought is described in Steffe (1991). His work is summarized below and is taken as the blueprint for the constructivist notion of "interiorization." It is meant to understand how human cognition, historically and individually, develops new conscious structure from our empirical reasoning ability, i.e., our common-sense reasoning based upon our perception of the world.

9.2.7 Interiorization

The term interiorization is often used by constructivists to describe the genesis, or birth of a process (interior scheme). While there is general agreement that interiorization is the result of reflection upon and abstraction of solution activity, succinct descriptions are harder to find. Sfard (1991) credits Piaget for the following characterization of an interiorized process as one that could be "carried out through [mental] representations, and in order to be considered, analyzed and compared, it needs no longer be actually performed" (Sfard, 1991, p.18). That is what is meant when they say the A-E relationship has been abstracted or made interior. In Czarnocha et al. (1999), activity before interiorization is referred to as taking place within an action conception, where an action is defined as a transformation that "is a reaction to stimuli that the subject perceives to be external" and hence cannot be carried out independently (p.98). These authors consider an interiorized scheme to be a "process" which is distinguished from an action conception by the conscious control the individual has over it. Arnon et al. (2014) suggest that repetition and reflection are central to promoting interiorization, "As actions are repeated and reflected upon, the individual moves from relying on external cues to having internal control over them. This is characterized by the ability to image carrying out the steps without necessarily having to perform each one explicitly and by being able to skip steps as well as reverse them" (p.20). These statements provide insight into the nature of an interiorized process. However, they do not describe the actual process of interiorization.

Steffe (1991) refers to an internal process as one that proceeds by making a mental reference to physical objects. In this situation, the child can use his intuitive M1 scheme to complete count-up activity and has internalized the process, by making mental references to physical objects even when they are not present. However, when the count-up value increased beyond the child's spontaneous control (from 3 to 5), he lacked the cognitive ability to coordinate his mental reference of objects with this count-up process, and thus he did not know when to stop counting. Applying selective attention to the five objects he needed to count-up, the child abstracted a numerical conception for 5 objects (M2). He could coordinate this conception with his count-up process and thus successfully stopped at the correct result. This novel activity is said to be an interiorized process because it is directed by the subject's conscious reasoning without the need for mental references to physical objects (internal process). Thus, the birth of the interior counting-up process occurred simultaneously with his conception of the numerical value 5. In the integrated frame, his internal process of counting-up is his M1; his newly developed number conception is his M2, which can be said to emerge from M1 through selective encoding. This conception underlies his new count-up activity, i.e., the pseudocode for his novel count-up scheme. This process is referred to as interiorization, and it presents what is presumably a fair description of the historical development of how humankind learned the count-up process.

9.2.8 Participatory and Anticipatory Schemes

Tzur (2007), Simon et al. (2016), and Tzur (2021) refer to a subject's need for assistance or use of situation-dependent schemes as being a participatory scheme. They describe the so-called "next day effect" in which a subject can use a scheme in class but cannot act appropriately in the same situation the next day as evidence of participatory schemes. With a participatory scheme, the A-E relationship is developing, and the S-A relationship requires prompts or external assistance:

> The participatory stage is characterized by a provisional, prompt-dependent access to a newly forming scheme…at this stage the learner has abstracted a new anticipation, that is, an activity-effect [A-E] dyad…this is yet to be linked with a situation/goal part of a new scheme. (p.333)

These authors refer to a participatory scheme that has become interiorized, as an anticipatory scheme. Hackenberg (2010) characterizes an anticipatory scheme as one in which the subject understands the conceptual reasoning for the activity and thus can anticipate the results, "anticipation involves the attempt to attain the results of prior experience by generating the cause of them" (p.387). She notes that, with an anticipatory scheme, because the subject knows the cause, it may be used for planning and reflection.

9.2.9 Moments of Insight and Internalization-Interiorization

Tzur (2021) suggests that moments of insight occur within both participatory and anticipatory schemes. Those moments of insight that occur across A-E dyads lead to an understanding of the invariant nature of the A-E relationships, what he refers to as the "logical necessity," i.e., the cause or pseudo-code of a newly formed A-E dyad. In the work of Hackenberg (2010), the term conceptual reasoning is used to signify when the subject understands the cause/logical necessity or invariant nature of the A-E relationships. Thus, a conceptual understanding of the cause allows the subject to abstract the A-E from the situation, allowing it to be used as input in another situation, reversed, and combined or synthesized with other processes. In short, understanding the conceptual reason for a previously participatory scheme is the hallmark of an "interior" scheme.

In this article, participatory schemes reflect the process of internalization, i.e., the subject has a need for assistance from the learning community. Elucidating the relationship between internal and interior process is a central research issue. Tzur's (2021) statement highlights the research question of interest: "what is the nature of moments of insights within a participatory scheme, i.e., during internalization?" It also raises an important pedagogical issue, what markers can a teacher use as evidence of internalization.

9.2.10 Internalization: Vygotsky

Vygotsky (1978) posits that learning, in its early stages, takes place through communication with adults who model meaningful activity:

> [w]hen the child imitates the way adults use tools and objects, she masters the very principle involved in a particular activity…repeated actions pile up, one upon another…the common traits become clear, and the differences become blurred. (p.22)

The notion of internalization as developed by Vygotsky (1978) represents the transition from interpersonal (external and socially directed) activity to intrapersonal (activity under one's internal control), through a process of conscious imitation:

> We call the internal reconstruction of an exterior operation, internalization…The process of internalization consists of a series of transformations: (a) An operation that initially represents an external activity is reconstructed and begins to occur internally… (b) An interpersonal process is transformed into an intrapersonal one. Every function in the child's cultural development appears twice: first, on the social level, and later on the individual level…(c) The transformation of interpersonal to intrapersonal one is the result of a long series of developmental events.

Vygotsky illustrates this transformation with a child's attempt to grasp an object, and how a parent gives meaning to this grasping process, and transitions it into pointing "The child's unsuccessful attempt engenders a reaction not from the object but from another person…consequently the primary meaning of unsuccessful grasping movement is established by others" (Vygotsky, 1978, p.56).

Internalization, especially during the transition from arithmetical to algebra thought, is brought about through socially mediated instruction as the subject reflects upon their intuitive-spontaneous concepts to build organized structures, i.e., scientific concepts Vygotsky (1997). The internalization process begins with recall and like "interiorization" ends with understanding the cause or logical necessity of the activity. "For the young child, to think means to recall; but for the adolescent, to recall means to think. Her memory is so 'logicalized' that remembering is reduced to establishing and finding logical relations; recognizing consists in discovering that element which the task indicates has to be found…At the transitional age all ideas and concepts, all mental structures, cease to be organized according to family types and become organized as abstract concepts" (1978, p.51). Vygotsky (1978) describes this transition from spontaneous to abstract concepts as passing through a phase of linking similar examples (schemes) together in series, as one develops a sense of "family type." "Children's concepts relate to a series of examples and are constructed in a manner similar to the way we represent family names" (1978, p.50).

9.2.11 Appropriation and Internalization

Vygotsky's notion of an internal scheme, which is used in this treatise, refers to the process of learning within mentor-led discourse, leading to independent activity through the subjects' understanding of the "logical" or conceptual relationships that underlie the activity. Both characteristics are used by constructivists as markers for interiorization. The need to verify independent activity is embedded in the first internalize and then verify-test methodology of most math classrooms.

For constructivists, who study an individual's solution activity, evidence of the conceptual relationships that underlies an interior process or scheme is provided by the ability to understand when the scheme/process is relevant in a new situation, to reverse the process, and/or to coordinate it with other schemes, e.g., use it as input into another scheme. For social constructivists, who focus on social discourse, understanding conceptual relationships is evidenced by the ability to communicate such knowledge to other members in the learning community. The term appropriation is often used to analyze learning within social discourse; it has its origins in the work of the Russian psychologist M.M. Bahktin to understand how children learn language.

> The word in language is half someone else's. It becomes "one's own" only when the speaker populates it with his own intention, his own accent, when he appropriates the work, adapting it to his own semantic and expressive intention. (Bahktin, 1994, p.293)

This explanation suggests there are two phases to appropriation; the first is grasping and interpreting another's communication and the second is making it your own. The goal of the first is to assimilate another's communication at least partially. Bahktin describes this phenomenon as being half yours and half someone else's. The goal of the second phase is to transform this partially assimilated understanding into one's own internal structure, adding it to one's repertoire or toolbox of concepts, words, and activities.

The use of the term appropriation to describe the internalization of an activity through communication requires some caveats in the translation from language to mathematics. In language learning, the second phase of appropriation is often embodied in one's ability to use a word in a productive manner or perhaps to explain what the word means. However, as pointed out in Confrey (p.43, 1994b), the ability to explain the definition of a math concept (definition of a function) does not imply the "ability to act accordingly." In Baker (2021b) a similar situation was highlighted, in which a subject could explain or interpret the work of another; however, they were unable to engage in independent solution activity. Thus, independent activity remains a cornerstone for evaluation of internalization and learning in general.

The painful reality for teachers of the 'next day' effect, and the difficulty translating students' ability to talk or communicate during today's classroom with the ability to act independently on tomorrow's exam, suggests that a more significant analysis of appropriation is needed. Although this is beyond the scope of this work, it is important to note that, when looking for markers of appropriation in student communication, the quality of the student's cognition must be a deciding criterion. In other words, having a sense of another's communication (first phase) does not mean you have made it your own (second phase). Thus, the ability to memorize a definition, does not imply one has any idea how to use it. The ability to interpret another's activity does not necessarily mean you understand why they employed it, how it works, or engage in such activity yourself.

In this study, beyond the criteria of independent activity, evidence for the completion of the second phase of the appropriation process (making it one's own) is taken as the communication of the conceptual relationships that underlie one's solution activity, the abstraction of solution activity into family types, and the abstraction of the objects acted upon, into relevant categories. Sfard (2020) introduces the term "commognition" to refer to the quality of the cognition, specifically, the level of abstraction, communicated in dialogue. The notion of commognition includes three processes, "Saming, Encapsulating, and Reifying." Saming involves relating previously unrelated aspects of a situation. Encapsulation involves understanding that what was previously viewed as separate objects are one entity. Reifying involves giving an A-E that is at least partially related to a situation, a name, thus indicating it has reached a noun status or object level. Appropriation in this context, alongside the ability for independent activity, will be taken as evidence of the completion of an internalization as well as interiorization.

9.3 Pedagogy: Constructivism and Vygotsky

There is a lack of certainty about what exactly is "constructivist pedagogy" and heated debate about the causes of its failure when applied system-wide in the mathematics classroom (Baker, 2021b). That said, constructivist pedagogy is associated with instruction founded upon an individual's reflection on available schemes. In particular, the second tenet of constructivism states instruction should not begin with unfamiliar definitions or activity beyond student experience. The role of the instructor is to guide students with minimal instruction to reflect upon their schemes:

> If the pedagogue is a Piagetian constructivist, he/she will refrain from verbal explanations because he or she believes the source of understanding is in the individual's actions of physical or mental objects. (Sierpinska, p.39, 1998)

Thus, a central feature of the role of an instructor in constructivist pedagogy can be understood as to present helpful problems that assist in reflection upon available schemes during solution activity that leads to accommodation. This methodology requires the use of well thought-out, innovative examples that are designed to "(a) Bring forth learning relevant activity-effect anticipations and (b) Bring forth noticing of intended effects" Tzur (2021, p.329). In constructivist pedagogy, the desired effect is to induce interiorization, understood as the foundation for growth in process-object duality (Dubinsky, 1991; Czarnocha et al., 1999; Sfard, 1991).

In contrast, Vygotsky's notion of internalization involves a subject's understanding of an externally modeled activity, one that may not be an available scheme. Ideally, in Vygotsky's social constructivist methodology, instruction is within the upper level of the student's ZPD, i.e., a bit above, yet relatable to their available schemes. In this methodology, higher level activity, notation, and content are to be introduced, and the instructor's role is to assist the student make a connection to their existing schemes that will provide meaning to the higher-level activity-notation-content (Baker, 2021b).

Berger (2005) argues that the focus of constructivism on transforming activity on existing schemes into interior processes does not account for how students learn math symbols and terminology within social discourse:

> But much of this process−object theory does not resonate with a great deal of what I see in my mathematics classroom. For example, it does not help me explain or describe what is happening when a learner fumbles around with 'new' mathematical signs making what appear to be arbitrary connections between these new signs and other apparently unrelated signs. Similarly, it does not explain how these incoherent−seeming activities can lead to usages of mathematical signs that are both acceptable to professional members of the mathematical world and that are personally meaningful to the learner. I suggest that the central drawback of these neo−Piagetian theories is that they are rooted in a framework in which conceptual understanding is regarded as deriving largely from interiorized actions; the crucial role of language (or signs) and the role of social regulation and the social constitution of the body of mathematical knowledge is not integrated into the theoretical framework. (pp.154)

Berger does not appear to completely accept the notion, embedded in constructivist pedagogy, that new notation and schemes should not be presented, until all perquisite schemes and notation have been understood. Indeed, as the instructor, she expects students to "fumble around" and make "arbitrary connections" as the new notation is at the upper level of the student's ZPD. This raises the question, "what type of activity is required to induce a moment of insight leading to concept formation when the subject does not have a suitable scheme?" If direct definitions are of no avail, can the mentor actively guide, present, or explain to the mentee suitable examples and to what extent can listening to the explanations of a mentor result in concept development for a mentee?

As noted, unlike constructivists, Vygotsky believes that meaning for activity comes the development of scientific concepts within instruction. "The scientific concepts evolve under the conditions of systematic cooperation between the child and the teacher. Developmental and maturation of the child's higher functions are products of this cooperation" (Vygotsky, 1997, p.148). From a pedagogical viewpoint, the keyword here is cooperation, and this does not imply direct instruction:

> A concept is more than the sum of certain associative bonds formed by memory…it is a complex and genuine act of thought that cannot be taught by drilling…Practical experience also shows that direct teaching of concepts is impossible and fruitless. A teacher who does this usually accomplishes noting but empty verbalism. (Vygotsky, 1997, pp.149–159)

Koestler, whose bisociation theory is the basis for the integrated frame, believed that spontaneous acts of creativity occur throughout the learning process, but only in untutored learning. "Minor bisociative processes do occur on all levels and are the main vehicles of untutored learning" (Koestler, 1964, p.658). Thus, like most creativity theory, the focus is on the individual's search process, and not instruction, or assistance in guiding this search process.

Indeed, Koestler devotes an entire section of his book to the "boredom of science" in which he argues that formalistic discourse has reduced math and science to a boredom of definitions, ruining the spirit of intuitive discovery required to appreciate the beauty of these subjects:

> The same inhuman, in fact anti-human-trend pervades the climate in which science is taught, the classrooms, and the textbooks. To derive pleasure from the art of discovery, as from other acts, the consumer-in the case, the student- must be made to re-live, to some extent, the creative process. In other words, he must be induced, with proper aid and guidance, to make some of the fundamental discoveries of science by himself, to experience in his own mind some of those flashes of insight which have lightened its path. (Koestler, 1964, p.265–266)

As a teacher researcher, this work is founded on the belief that students at all levels of mathematics must experience moments of insight to relieve the boredom of mathematics and to grow in understanding. Following Vygotsky, the focus of this study is internalization within social discourse, and the pedagogical issue is to highlight the role of the teacher in inducing such moments of insight.

9.4 Research

The goal of this study is to provide supporting evidence that bisociative moments of insight occur and are the mechanism for acts of accommodation during internalization, i.e., when the subject requires mentor assistance, thus extending Koestler's view that such moments of insight occur primarily in "untutored learning." To accomplish this goal, the primary objective is to review the integrated frame used to analyze an individual's moment of insight leading to acts of accommodation, developed previously, and then extend this frame to include internalization. The first research question is to describe the genesis of concepts during internalization and compare this to genesis of concepts embedded in developing schemes within the process of interiorization, as presented by Steffe (1991). A second related research question is to investigate how matrices or schemes emerge and are grouped together into conceptual structures, or toolboxes of schemes, during internalization, essentially what Vygotsky (1978) refers to as a "family type." In this second research question, the constructivist notions that are useful to analyze these moments of insight to build up a toolbox during internalization include Piaget's notion of "constructive generalization" and the notion of invariant relationships.

Pedagogically, the research objective is to highlight how teacher-mentors can support moments of insight within the internalization process, i.e., social discourse. The empirical evidence collected will then be used to briefly reflect upon the role of the teacher within the context of constructivist and social constructivist pedagogy.

9.5 Stages of Internalization: Empirical Examples

Vygotsky's conception of internalization suggests many levels of development the child may experience as they internalize mathematical content. In our analysis, we consider two overarching categories; one is characterized by passive reception or processing of teacher-mentor- directed activity (external), and the other is marked by an active effort to provide meaning for such activity. Within the active category of internalization, we consider three stages. These stages loosely correspond to Berger's (2004a, b) use of Vygotsky's three stages of concept development: heap, complex, and pseudo-concept/concept.

9.5.1 Interpersonal Learning Social Participatory Stage

For Vygotsky, internalization involves conscious imitation of adult behavior; as such it begins with watching and listening to modeled solution activity, which is perceived as externally driven. During the social participatory stage, internalization begins as the learner (albeit passively) struggles to make sense out of modeled solution activity, new definitions, or symbols.

Example 1: The Elephant Aha Moment (Kadej (1999), Czarnocha and Baker (2021, pp.95–98)).

The discovery of a hidden analogy to abstract a concept within direct instructional methodology.

Two children were trying to solve the algebraic equation $x + (x + 12) = 76$, one child-mentor readily understood the task and was attempting to explain it to his peer-mentee who did not understand the concept of the variable "x" as representing an unknown numerical value. Following the textbook, the mentor-child points out that the "x" term can be viewed as a blank square, into which one inputs any numerical value. This simply confuses the peer-mentee, who is thinking along a concrete line of reasoning and, thus, rejects the obvious inconsistency that "x" is a blank square. Then, the mentor child describes the blank square as a window while simultaneously simplifying the problem as two windows (blank squares) that are equal to 64, while asking "what is each window?". However, the mentee child continues to engage in concrete reasoning rejecting the comparison of a blank square to a window as more confusion. Finally, the mentor child attempts an analogy using elephants, asking "two elephants are 64, what is each elephant?". The mentee child ponders and then proclaims one elephant is 32. I understand now!

The mentor reframes the situation to make sure the mentee understands, asking "if two elephants are 60, how much is each?". Immediately the mentee-child provides the correct response.

9.5.2 Discussion: Example 1

Initially, the mentee-child grasps very little of the variable concept; thus, the M1 search matrix does not contain any scheme to reflect upon; instead, it consists of a nonsensical symbol "x." The mentor employs a common approach of introducing different analogies to guide the mentee towards this concept. Radford (2003) describes such a situation as the tension between the individual and the learning community leading to "transitional language" that results in the individual giving meaning to symbols using "metaphor." Initially, the subject does not relate to these transitional terms – they are simply not part of his toolbox. The discovery of the elephant metaphor can be seen as what Matsushima (2020) refers to as dynamic composition within appropriation, i.e., it exists in the mind of the mentor first and is borrowed by the mentee. The moment of insight occurs when the mentee grasps the uncovered previously hidden analogy between taking half of two elephants-M2 and the symbol-variable "x"-M1. During this bisociative moment of realization, the concept of an unknown exists in both matrices, and the intuitive M2 matrix gives meaning to the mentee-child's previously non-existent M1 solution activity. Thus, the bisociative connection between elephants and the symbol x results in the internalization of the variable concept, a novel concept, i.e., growth in understanding.

In Steffe's (1991) example of interiorization, M1 is an intuitive count-up scheme, while M2 is the abstraction of the number concept required to end the count-up process. In this case, M1 is a search matrix with no scheme; indeed, the variable "*x*" appears to have little meaning to the subject. The M2 analogy that gives meaning to the variable *x* or M1 is an intuitive scheme the child can grasp. Pedagogically, this is a good example of Vygotsky's educational approach, in which a mentor is ahead of a child's developmental level, yet within their ZPD.

9.5.3 Active Internalization: Interpersonal to Intrapersonal – Heap Stage

Vygotsky (1978) notes that for children to think means to recall, and at this stage understanding consists of the recall of activities viewed as "isolated instances" with little to no logical structure or conceptual understanding. At the early stages of internalization, the link between situation and activity is essentially founded upon imitation of previous examples; thus, if the teacher mixes up examples or reintroduces after a pause, a similar example, students at this level may be confused and resort to guessing what activity is appropriate or choose a non-relevant activity based upon a superficial understanding of the new situation.

Berger (2004a) suggests that in heap thinking, "the person links ideas or objects together as a result of an idiosyncratic association" (p.3). She elaborates on the quality of the link at this stage, noting that "objects are linked by chance in the child's perception" (p.5). Berger refers to such links as surface associations, which result when the solver reads and interprets problem information in a superficial manner. Berger provides examples in which a solver attempts to employ symbols or phrases (keywords) in a problem situation without reasoning based upon any real understanding. Their selection of symbols and keywords is designed to simplify their cognitive load. Hence, the resulting activity is often incorrect.

Solvers at this stage, lacking logical structure, or what Vygotsky refers to as a notion of "family type," frequently attempt to direct their activity through imitation of modeled activity. Thus, learning often consists of reviewing, class notes, textbook examples, or videos of similar exercises to assist in problem solving. At this stage, the S-G and A-E relationships are weak based upon superficial associations, and the so-called "next-day effect" where solvers perform with some degree of proficiency in class, but not independently, i.e., the next day.

9.5.4 Example 2: T-INTERVAL

This example is taken from work with students in an online (zoom) math class during the pandemic. It represents an example of students cooperating, which was not a common event as most students were very passive while participating using online platforms.

Previous class time had been spent on using calculator commands for hypothesis testing and confidence intervals. Wendy internalized that a collection-toolbox of commands were available using the STAT/TEST menu. She also understood that for a normal distribution, one typically used either the Z-Test or the Z-Interval command in this menu or toolbox of commands. However, this problem required her to understand a new concept, T-distribution. Indeed, students were now being asked to choose between two types of commands involving either Z or T distributions, further complicating the issue that each type could require one of the two categories, either internal confidence or testing commands. There was a third involving sample proportions, but it will be simpler for the reader if left out of the discussion. Thus, she needed to generalize the two commands learned previously into two categories, and two types, or family names.

Students traditionally experience a high degree of uncertainty and frustration trying to understand which of these commands is appropriate. The mentee student (Wendy) is in the process of consciously imitating modeled problem-solving behavior; she knows to use the STATS/TESTS toolbox collection of commands on her calculator. However, she is not sure which one to use. At this point, Wendy brings the exercise to class and asks what to do! (Interpersonal process). Wendy states that it appeared to involve a T command instead of Z command; she was unsure why and was not clear about whether it involved a test or interval command, and her voice suggested she was overwhelmed.

The instructor, who had modeled the variations of problems require several times, realizes that Wendy needs another voice (preferably a peer, not an authority) to explain, and asks if any other student can help her. A mentor student (Marisol) volunteers to explain. Marisol first, explains that the question asks for an interval, and thus you need to use an Interval command under the TESTS options. This explanation helps Wendy formulate the concept-categories of interval versus testing commands. It also reduces the search to discriminating between the two family types, Z- Interval or T-Interval. Wendy listens attentively and appears to understand. Second, Marisol points out that there is a hint which explicitly states that, because there is a sample standard deviation, the problem requires a T-distribution, and thus, she concludes the T-Interval command is appropriate. At this point, Wendy is silently processing what Marisol has said, then indicates she now understands, and thanks Marisol.

9.5.5 Discussion

Wendy's initial understanding of the situation (M1) included a sense of what to do, based upon her appropriation of previous examples (her hybrid or partial conception). However, her partial conception was inadequate to support independent activity. Her statement that it was probably a T-distribution (not sure why) suggests a faltering ability to express this appropriated conception. Thus, she needed confirmation to enter confidently into the second phase of appropriation, in part because the need to navigate four variations of commands was overwhelming. Instead, her search process, classic to internalization, was to ask for assistance and then try to understand what she was told. Kosko (2014) refers to this as active listening. Thompson et al. (2004) suggest that such listening is the foundation of critical thinking and involves receiving knowledge, comparing received to previous knowledge leading to comprehension and finally to evaluation.

The mentor, Marisol, guided Wendy's attention to coordinate the previously unclear problem information, with the required commands (M2). First, Marisol guided by pointing out there were two overarching categories, testing – find the percent or interval problems. Wendy listens intently and spontaneously understands the need to discriminate between Test and Interval commands. Although online learning makes it difficult to observe the learners affect, it was clear that while listening to Marisol, Wendy received or appropriated the knowledge imparted by Marisol, and as she reframed from further questions and could act independently afterword this suggests she comprehended it, i.e., her growth in understanding allowed her to direct future solution activity – M2. This suggests that, as in the first example, a moment of insight or transfer from Marisol to Wendy occurred during active learning, providing another example of "dynamic composition" Matsushima (2020). Marisol's second explanation served to highlight uncovered problem information (the hint), which provided a positive evaluation confirmation for Wendy that her tentative appropriation knowledge (T-distribution) was correct.

Although, it was not possible to observe her affect (it was an online session, and her camera was off) it was clear Wendy's understanding of Marisol's comments was real and immediate, as she thanked Marisol. Furthermore, it became clear that Wendy had developed a new scheme, as she independently completed the related homework, and received a 100% on the next exam. Previously she was a B+ student. Wendy also demonstrated motivation and did well on the final and included several such (Test/Interval) problems with six different related commands required, indicating she had developed an organized toolbox for these six related commands or A-E-dyads under the STAT/TEST menu. Finally, it is worth noting that, in this example, the active listening by Wendy was essentially a reflection upon Marisol's guidance through a solution activity Wendy had previously attempted. Thus, although not exactly reflection upon her own solution activity, it contained elements of reflective abstraction, as well as reflection during active listening-appropriation.

9.5.6 Internalization Complex Stage

The complex stage marks the transformation of the operation into an intrapersonal process. This occurs with the development of a "family name" for the activity that is a sense of "problem type" obtained not necessarily by an understanding of the underlying structure rather through the linking of different but similar examples together one at a time. Berger (2004b) describes this stage in terms of developing a nucleus built up by relating similar examples and then linking them together initially by superficial but ultimately by their invariant relationships. Thus, it is in this stage that moments of insight occur as students bring their conscious awareness to their own independent activity comparing A-E-dyads for similar exercises to develop an invariant relationship and observe other's activity that somehow relate to their available schemes. Tzur (2021, pp.336) refers to such comparing as awareness of "across activity-effect instances."

Berger (2004b) suggests that the common pedagogical technique of teaching by examples often results in students (Complex Stage) incorrectly linking the observed activity in superficially similar but cognitively different problem examples. She illustrates this, with the example of students who understand that one multiplies the rate with the time to find the distance. However they continue with this multiplication-activity when the problem gives distance, time and asks for the rate, or gives distance, the rate and asks for the time. Another case where learning by examples yielding incorrect links is provided by Berger (2004b) when students who have learned that $f(x) = |x|$ is everywhere continuous but not differentiable apply this model to conclude the same is true for all absolute value functions even $g(x) = |x^2|$.

9.5.7 Example 3: The Domain Aha Moment (Czarnocha & Baker, 2021, p. 99–101)

A student understands the domain of the proto-type example ($f(x) = \sqrt{x}$) as being the values $x \geq 0$. She incorrectly uses this template to conclude that the domain of the similar function $f(x) = \sqrt{x+3}$ is also $x \geq 0$. At the direction of the instructor, she checks several negative integer values for x, including -3, and the student begins to realize something is wrong; when the instructor asks whether $x = -2$ works, the student ponders before declaring. "Those x's which are smaller than -3 can't be used here!". As noted in Czarnocha and Baker (2021, p.99–101), this marks the student's first realization of a new guiding principle. Thus, this realization is the result of selective encoding, in which the new action scheme M2 emerges from conscious attention to her available scheme M1. Selective combination-coordination is also involved as she coordinates her instructor-led activity with her initial understanding of the domain. The instructor, to determine whether an invariant relationship has been established, asks the student for the domain of $g(x) = \sqrt{x-1}$, and after pondering for a minute, she provides the correct response.

In this realization, the student's M1 was her previous understanding of the domain of the absolute value function $f(x) = |x|$, $x \geq 0$; after her work following instructor guidance, she realizes that values such as $x = -1, -2, -3$ are all exceptions, i.e., they are part of the domain for $f(x) = \sqrt{x+3}$; these exceptions to her M1 rule or code are the basis of a new conception or M2, and as she coordinates these values with her previous understanding, she generalizes the code from $x \geq 0$ to $x + 3 \geq 0$ ($x \geq -3$). Thus, she forms new S-A and A-E relationships evidenced in her understanding of the domain of the similar function $g(x) = \sqrt{x-1}$ ($x \geq 0.1$). With these moments of realization, she is developing a new code or invariant relationship for linear factors under a radical, based upon connections between similar examples or family type. This can be seen as an example of a realization leading to an invariant relationship across A-E dyads (Tzur, 2021, pp.336). It can also be understood as an example of the commognition Sfard (2020) refers to as "Saming."

9.5.8 *Internalization: Pseudo-Concept Stage*

At this stage, the subject shows signs of conceptual reasoning, i.e., activity based upon concepts and their relationship to the situation. Thus, they are developing an abstracted A-E relationship based upon conceptual reasoning that underlies connections. In Koestler terminology, we say that a code is being abstracted.

In the preceding example the subject has successfully provided the domain for $f(x)=\sqrt{x+3}$ and $g(x)=\sqrt{x-1}$ based upon the coordination of her initial scheme (domain of the radical x function) with her reasoning that the linear expression under the radical must be greater than or equal to zero (the invariant relationship). Next, the instructor asks for the domain of $h(x)=\sqrt{x-a}$ after pondering she has a second realization and provides the correct answer $x \geq a$. At this point, the student has abstracted the invariant relationship or guiding principle into an algebraic expression or code that works for all such problems (linear factors beneath a radical). Her ability to express the domain using symbolic notation can be viewed as a communication of her conceptual reasoning and hence as an example of appropriation, one of the characteristics for the completion of internalization. Thus, this realization involves an abstraction of the invariant relationship across A-E-dyads (Tzur, 2021, pp.336) or the abstraction of a new code.

9.5.9 *Discussion*

In this example, the constructivist notion of reflective abstraction begins to merge with Vygotsky's notion of internalization. Thus, reflection upon another's solution activity, i.e., appropriation demonstrated in previous examples is replaced with reflection upon one's own solution activity, albeit activity guided by the instructor, i.e., reflective abstraction. Analyzed using the integrated frame, the subject has an initial understanding of the situation M1 that directs her activity. However, it is based upon a template example, which she does not generalize correctly. Under the instructor's tutelage, she engages in solution activity with different values of x, until she experiences an "Uphs effect" or the realization that her initial M1 scheme is inappropriate.

In the initial moments of insight, the subject uses her innate ability to compare different concrete experiences of substitution to abstract the A-E dyad for $\sqrt{x+3}$, and the A-E-dyad $\sqrt{x-1}$ in this process, M1 was her understanding of the domain of $\sqrt{x}$ and M2 was her understanding of solving linear equations such as $x + 3 \geq 0$, $x{-}1 \geq 0$. Thus, the instructor-guided solution activity resulted in selective coordination-combination, or bisociation of her M1 and M2 schemes, that resulted in a new albeit local code for each A-E-dyad. This new code is the logical necessity or the "reason why," i.e., the cause that transfers from M2 to M1 for each dyad.

The second moment of insight occurs as she reflects across her newly constructed A-E- dyads for radical $(x + 3)$ and radical $(x - 1)$, which now represent her M1, and her insight can be viewed as the result of her innate ability for pattern recognition as she coordinates the problem, (finding the domain of the radical $(x + a)$ function) with these M1.

In this case, the subject recognizes and expresses in symbolic language the invariant conceptual relationship that underlies the linking of her A-E-dyads. This communication represents an act of appropriation (second phase) or the type of commognition Sfard (2020) refers to as "Reifying" as she is communicating her understanding of this invariant relationship with an algebraic formula.

9.6 Concluding Remarks

The goal of this article is to highlight moments of insight, as students struggle to internalize mathematics in a classroom situation. The primary research objective is to extend the integrated frame to internalization, and in so doing to highlight the relationship between this process, and the constructivist notion of interiorization.

The first research question is to describe concept development within the internalization process and compare this to concept development within interiorization. In the first example, the child is struggling to appropriate the concept of an unknown as expressed by the symbol "x," i.e., he is struggling to give meaning to a new object or sign. His motive is to understand what is being presented, and his M1 understanding is very limited. As the mentee presents different M2 analogies, the subject finally realizes (in a flash) the connection between taking half the given weight of two unknown (twin) elephants to find the weight of each one. This guided discovery of a previously hidden analogies allows meaning to transfer from his intuitive scheme of taking half of elephants to the required symbol manipulation of symbol "x." As the concept of an unknown emerges from his spontaneous or intuitive taking half scheme, it represents both the genesis of a concept as well as of a scheme. This act of appropriating another's understanding of a symbol, sign, or word name is essentially what Sfard (2020) describes as "reifying," i.e., the giving of a name (object level status) to a process. In the process of interiorization as described by Steffe (1991), the number concept (five) also emerges from an intuitive scheme and, thus, represents both the genesis of a concept and a process (count-up).

During interiorization, the M1 is the intuitive scheme, and the M2 is a concept that emerges through reflection upon one's M1 solution activity; hence interiorization is a foundational process of reflective abstraction. As this reflection uncovers what was previously known, albeit only intuitively, hence it can be viewed as selective encoding. In contrast, during internalization, the M1 is essentially not functional, and meaning or growth in structural understanding is obtained during active listening. This is an example of reflection during social communication, a process Vygotsky (1997) describes as synonymous with reflective consciousness. As the search to appropriate meaning of the symbol “x” involved active listening, the moment of insight can be understood as the guided realization of a previously hidden analogy, i.e., selective comparison without synthesis of schemes.

Internalization as presented by Vygotsky often involves the recall of externally modeled activity. Whereas, in the first example, the subject (heap stage) is learning a completely new concept embodied within a symbol, and M1 is essentially non-functional, in the second example (complex stage), the subject has an M1 scheme, based upon recall, and is in the process of building up a toolbox of schemes or a collection of schemes within a family type.

In this example, her M1 search matrix includes a vague recall of how to proceed, i.e., where to go on the calculator to find the appropriate toolbox of commands. However, she lacks the ability to select the relevant problem information, and coordinate it, with her solution activity. The instructor’s methodology was to ask other students to explain. The mentor-student did not attempt to explain the concepts needed; instead, they focused on recognizing these concepts in the problem situation and how these concepts connect to and thus direct (novel) solution activity. The growth in understanding was to first discriminate between two types (categories) or problems, confidence interval and hypothesis testing, and second, between two types of tools or schemes (Z versus T distributions). Thus, through active listening, the subject learned that there is a new tool to be used in two types of problems, and she learned to recognize when this new tool was required. This growth in understanding can be understood as the result of a projection of an existing M1 into a novel situation, and the resulting coordination (bisociative synthesis) between M1 and M2 features required to resolve the problem, while leading to a new code. Hence, it is like Piaget’s notion of “constructive generalization.” This distinction being that it involves active listening as opposed to reflection upon one’s own solution activity.

In the third example, although directed by the teacher, the subject’s reflection is completely upon her own solution activity, not upon communicated knowledge; thus, her moments of insight can be analyzed within the integrated frame as reflective abstraction. As the instructor skillfully directs the student to reflect upon patterns of substitution, she is guided to an “Upps effect” and ultimately the synthesis of her M2 conceptual understanding of the (more-than /less-than zero) solution of a linear inequality with her initial M1 understanding of the domain of a radical, as being greater than or equal to zero. Thus, her moment of insight can be understood as “constructive generalization”; in that, she has an existing M1 scheme that is first projected into the new situation and then coordinated (bisociative synthesis) with features of this situation that were previously overlooked to begin the formation of a new concept and code.

In the constructivist frame, one can also understand her moments of insight as interiorization. In this view, her initial M1 scheme was dependent upon the situation (participatory schemes). The coordination (bisociative synthesis) of her initial M1 domain scheme with her M2 linear inequality scheme provides a good example of constructive generalization after the instructor recognition of the invariant relationships embedded in the patterns of substitution (discovery of a hidden analogy-M2). Her communication of these domains to the instructor represents the second phase of appropriation and indicates a new code is developing.

In the fourth example, the subject abstracted the invariant relationship of her previous work. Thus, she understands them as reflecting one code, encapsulation (Sfard, 2020), which she gives a name in symbolic form, reifying (Sfard, 2020). These latter examples highlight aspects of the second research question. In the second, active listening, on communication of content analogous to an existing scheme, leads to construction of similar "family-type" schemes. In the latter two examples, pattern recognition and abstraction of an invariant relationship lead to encapsulation and reifying of solution activity (processes) into a code expressed in symbolic form. This demonstrates her object level understanding of these processes as different schemes that can be organized as one, named entity.

Pedagogically, these examples demonstrate the effectiveness of guiding students to "Upps effects" through pattern recognition as opposed to direct instruction. They also show that peer-peer dialogue involving direct analogies can be very effective in leading to moments of insight.

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