Chapter 16
Mathematical Modelling: A Philosophy of Science Perspective

Uwe Schürmann

16.1 Introduction

The (analytical) separation between mathematics and reality can be found in numerous publications on mathematical modelling. For instance, PISA, the Programme for International Student Assessment (OECD, 2009), uses the following diagram in its mathematical framework (Fig. 16.1), where mathematics and the real world are considered to be separate domains.

Also, the introduction of the 14th study of the International Commission on Mathematical Instruction (ICMI) on modelling and applications (Blum et al., 2007) shows a modelling cycle distinguishing between mathematics and an extra-mathematical world. Additionally, this separation is also postulated in various contributions to the volumes of the International Community of Teachers of Mathematical Modelling and Application (ICTMA).

Figure 16.2 presents a modelling cycle by Blum and Leiß, which is frequently cited in German-language literature on mathematical modelling and is used (sometimes modified or extended) in various works (cf. Greefrath, 2011; Ludwig & Reit, 2013). Borromeo Ferri (2006) offers a carefully elaborated overview of many of these modelling cycles. It is clear from this overview that the (analytical) separation between mathematics and reality is omnipresent in the reconstruction of modelling processes.

In contrast, only a few publications are questioning this separation. For instance, Biehler et al. (2015) analyse modelling processes in engineering classes and conclude from their analysis that it is rather inadequate to separate mathematics and the "rest of the world" as well as to divide modelling processes into certain distinct phases. From

U. Schürmann (✉)
Primary School Institute, University of Applied Sciences and Arts Northwestern Switzerland, Muttenz, Switzerland
e-mail: uwe.schuermann@fhnw.ch

M. A. V. Bicudo et al. (eds.), *Ongoing Advancements in Philosophy of Mathematics Education*, https://doi.org/10.1007/978-3-031-35209-6_16

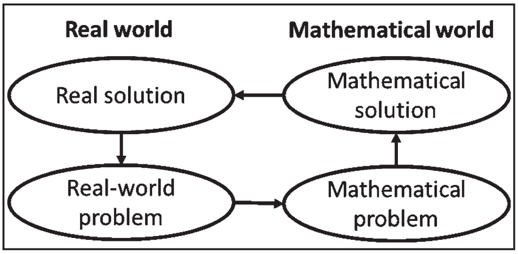

Fig. 16.1 Modelling cycle in PISA's theoretical framework (OECD, 2009, p. 105)

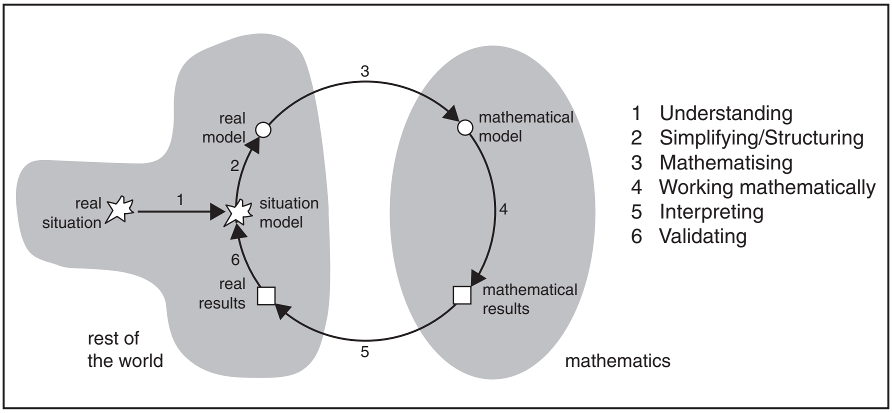

Fig. 16.2 Modelling cycle by Blum and Leiß (2006)

their point of view, mathematical aspects must be considered during the step of simplification (part of the "rest of the world" in most of the modelling cycles), already. This theoretical insight is supported by a subsequent empirical investigation by the authors. Furthermore, Voigt (2011, p. 868) identifies the analytical separation between mathematics and reality as a problem that can only be solved if we take a close look at the area between the "rest of the world" and "mathematics". Consequently, he considers this "intermediate realm" as substantial. Voigt strongly advocates examining not the separation but the connection between these two spheres.

This builds the starting point: In the following, the relationship between mathematics and reality will be explored in more detail. In this way, the question is posed whether and to what extent the analytical separation of mathematics and reality can be justified or whether it should be supplemented or even replaced by an alternative interpretation of the relationship between mathematics and reality.

16.1.1 Orientation

The separation between mathematics and reality, as found in many modelling cycles, can be interpreted in at least three different ways.

1. As an ontological separation according to which mathematics by its very nature would have to be distinguished from reality, the real world, or the rest of the world
2. As an analytical separation primarily serving to describe modelling activities adequately, i.e. to be able to empirically research them
3. As a separation that makes sense from a constructive point of view and serves to support learners while working on modelling tasks

The three interpretations mentioned are neither mutually exclusive nor mutually dependent. Nevertheless, the author hypothesises that when mathematics education research considers mathematics and reality as two distinct realms promoting a constructive point of view becomes more likely. Each of the three interpretations mentioned is problematised in the literature against the background of different perspectives. For instance, Voigt (2011, p. 869) asks whether in placing the "real situation" at the beginning of the modelling process—far from mathematics—the ideal of an everyday life orientation is expressed, under which one imagines that mathematics develops out of an everyday life untainted by any mathematics. Such notions are undermined in various contributions to mathematics education research. Niss (1994, p. 371), for example, mentions that mathematics is confronted with a "relevance paradox". On the one hand, mathematics is becoming more and more relevant and, at the same time, more and more irrelevant, since mathematics plays a pivotal role in the development of technical devices, yet the operation of these technical devices no longer requires mathematical literacy. Keitel's (1989) pair of terms de-/mathematisation points in the same direction. However, these terms emphasise the social significance of mathematics more strongly and problematise the use of supposedly realistic mathematics tasks in the classroom. Keitel introduces the pair of terms de-/mathematisation to describe those processes leading to mathematics—in terms of mechanisation and automation— increasingly determining our living environment (mathematisation). At the same time, mathematics increasingly disappears from everyday life (demathematisation) since the skills that were previously required are henceforth taken from humans by a technical device. Skovsmose and Borba (2004) critically examine the ideological effect of mathematics and its teaching within social contexts. They argue that if mathematics is considered a perfect system and an infallible tool for solving real problems, political control is in favour.

So, the separation between mathematics and reality cannot be understood as a fixed boundary, at least not within social contexts. A domain that is part of the "rest of the world" can be mathematised very soon. Since students gain experience in their mathematised environment way before mathematical concept formation processes take place in the classroom, the everyday life orientation of mathematics education, as outlined by Voigt with critical intent, should rather be rejected.

Another problematising perspective on the relationship between mathematics and reality is offered by those historical-philosophical approaches that are usually assigned to postmodernism. These approaches explicate the historical contexts from which a specific, prevailing image of mathematics has emerged. Deleuze (Deleuze, 1994; Deleuze & Guattari, 1987), for instance, sees a problematising side of mathematics alongside the prevailing axiomatising formalisation of mathematics.

By using historical examples— first and foremost the development of calculus by Leibniz—he elaborates on the possibility of dynamic mathematics emerging from concrete problems (cf. de Freitas, 2013; Smith, 2006). Châtelet (2000) highlights the representational side of mathematics by using historical examples to illustrate several ways in which mathematics' innovations and concepts are strongly dependent on the mathematical tools and forms of graphic representation used at a given time. In doing so, he interprets diagrams as a section of a sequence of physical gestures and thus relates the formal side of mathematics to its material and, above all, physical basis.

De Freitas and Sinclair (2014) take up this idea when they map out their didactics of mathematical concepts. They emphasise the material and ontological side of mathematics in addition to the logical and epistemological. Schürmann (2018a) points in the same direction as he attempts to show that mathematical models, in particular, do not merely serve knowledge, but should also be understood as entities, i.e. in addition to their epistemological function, their ontological side needs consideration, too. Furthermore, Schürmann (2018b) deals with the origin of historical knowledge formations that may have contributed to the separation of mathematics and reality. Using Frege's logicism and Hilbert's formalist programs as paradigms (Frege, 1884, 1892; Hilbert, 1903) against the background of what Foucault calls the episteme of modernity (Foucault, 1996) this separation is understood as a reaction to the relativisation of mathematical truth claims within the nineteenth century.

The literature cited here clarifies that the boundary between mathematics and reality is historically conditioned. A further problematisation of the separation of mathematics and reality emerges from those empirical studies focusing on individual modelling processes. Regarding these studies, students already consider relationships between the mathematical content and parts of the real world long before setting up a mathematical model. Biehler et al. (2015) and Meyer and Voigt (2010) give a critique of the analytical separation of mathematics and reality based on this empirical finding.

16.1.2 Focus

Since mathematics education research on modelling is largely detached from the philosophical discussion on models, which goes on for more than 100 years,[1] this chapter elucidates the separation of mathematics and reality against the background of the philosophy of science on models.

Here, the philosophy of science is understood as a subdomain of philosophy in which the validity claims of empirical sciences and mathematics are scrutinised, for

[1] In order to prove this thesis, the author has reviewed the bibliographies of all contributions in the ICTMA volumes published so far. It turns out that none of these contributions refer to relevant works from the philosophy of science.

instance, by reconstructing scientific theories. However, the philosophy of science concerning the humanities (e.g. hermeneutics) is excluded, even though such an approach may be of interest under certain conditions.[2]

Additionally, the following is mainly about the relationship between mathematics and reality in the context of theories and models. An epistemological question of the perception of reality is not raised here, although it is not intended to deny the importance of fundamental epistemological questions for the understanding of mathematical modelling.

The approach is to take up considerations from the philosophy of science on the relationship between theories, models, and reality and apply them to mathematics education research. For this purpose, two central views within analytical philosophy, the syntactic and the semantic view, are juxtaposed and related to mathematical modelling in the classroom. This selection is not intended to question divergent approaches, such as the pragmatic view on models (cf. Gelfert, 2017; Winther, 2016). The restriction to the two views mentioned above is merely for pragmatic reasons. Even these two views can only be outlined here. However, their discussion provides valuable information for answering the following questions:

1. *Epistemological question:* Is the analytical separation between mathematics and reality, often found in mathematics education research on modelling, tenable as such against the background of analytical philosophy, or does it need to be revised or at least relativised?
2. *Methodological question:* Does the discussion on the syntactic and semantic view on models and theories offer new insights into the description of mathematical modelling in the classroom? In particular, can methodological tools be derived that describe modelling processes more appropriately and accurately?

A third, rather constructive question, arising from an assumed separation between mathematics and reality, is excluded here. It is not asked whether the separation between mathematics and reality supports the learners in the processing of modelling tasks.

16.2 Analysis

Large parts of the philosophy of science's discussion on models have their origins in model theory, a subdomain of mathematical logic. To also grasp scientific models and theories, mathematical logic's angle, formerly focused on formal languages, was widened. From now on, natural and scientific languages are considered as well, i.e. formal languages are understood as subsets of natural languages.

[2] Frigg and Salis (2019), for example, compare models with (literary) fiction.

The syntax of a language L consists of its vocabulary and the rules for forming well-defined expressions in L. The semantics of L allows the interpretation of well-defined expressions by mapping them to another relational structure R. Thus, on the one hand, well-defined expressions from L are made comprehensible, and, on the other hand, these expressions can be examined within L for their validity. Then, the distinction between syntax and semantics initially leads to two opposing (but related) views on models and theories, the syntactic and the semantic view. The syntactic view on scientific theories was developed primarily by representatives of the Vienna Circle. Due to this, this view on theories and models is closely connected to logical positivism or logical empiricism,[3] which had a huge impact on the philosophy of science in the twentieth century until the 1960s (cf. Gelfert, 2017). Very likely, the achievements of the natural sciences in conjunction with the rapidly developing axiomatic-formal mathematics at the beginning of the twentieth century were decisive for the increasing influence of logical positivism.

The semantic view on theories and models has emerged largely in response to the syntactic view and its associated obstacles (some of them will be discussed below). The main difference between the two may be that the syntactic view attempts to describe theory building in an idealised form, while the semantic approach tries to outline theory building in terms of scientific practice. Due to the large amount of literature, it is necessary to select among the authors referred to in this chapter. From the syntactic view, the oeuvre of Rudolf Carnap is considered paradigmatic (Carnap, 1939, 1956, 1958, 1969). The analysis of the semantic view is based on the works of Patrick Suppes (1957, 1960, 1962, 1967).

16.2.1 Carnap's Syntactic View on Models

From Carnap's (1969, pp. 255 ff., 1958; see also Suppe, 1971) syntactic point of view, theories can be reconstructed based on propositions. A theory is formulated in a language L that consists of two sub-languages, the theoretical language L_T and the observational language L_O. The descriptive constants of L_T are named theoretical terms or *t*-terms. Those of L_O are called "observable" (Carnap, 1969, p. 225), observational terms or *o*-terms (Carnap, 1969, p. 255). *O*-terms denote observable objects or processes and the relations between them, e.g. "Zurich", "cold" and "heavy". *T*-terms are those that cannot be explicitly defined by *o*-terms, i.e. they cannot be derived from perception. Carnap's given examples are fundamental terms of theoretical physics such as "mass" or "temperature" (Carnap, 1958, p. 237). This distinction leads to three different types of propositions:

[3] Even though the Vienna Circle's members did not use the term "logical positivism" for themselves, this chapter does not distinguish between logical positivism and logical empiricism. Creath (2017) points out that a distinction between the two terms along theoretical assumptions and sociological viewpoints cannot be made meaningfully anyway.

1. Observational propositions containing *o*-terms but no *t*-terms
2. Mixed propositions containing *t*-terms and *o*-terms
3. Theoretical propositions containing *t*-terms but no *o*-terms

According to this approach, a theory in language *L* is based on two types of postulates: the theoretical or *t*-postulates and the correspondence or *c*-postulates, also called correspondence rules (Carnap, 1969) or protocol theorems (Carnap, 1932). *T*- postulates are pure *t*-propositions, i.e. they belong to type (3) of the three types of propositions listed above. *T*-postulates comprise all fundamental laws of a theory. For instance, these can be the fundamental laws of classical mechanics or the main laws of thermodynamics. *T*-postulates are therefore the axioms of a theory. They are taken for granted. All statements *s* that can be derived purely syntactically from the *t*-postulates also belong to L_T. The derivation of such statements is based on syntactic rules, which can contain further rules of formation in addition to mathematical rules. L_T in itself has no (empirical) meaning. The meaning of *t*-terms is only given indirectly using L_O. Carnap assumes that *o*-terms refer to directly observable or at least almost directly observable physical objects or processes and relations between them (Carnap, 1969, pp. 225 ff.). In the following, this direct interpretation will be called *d*-interpretation. Thus, the semantics of *o*-terms is directly given. It is not possible to derive empirical statements from theoretical statements, i.e. from propositions of type (3), it is not possible to conclude propositions of type (1) without further ado. Rules are needed, the so-called *c*-postulates, to connect *t*-terms with *o*-terms. For instance, Carnap (1969, p. 233) mentions the measurement of electromagnetic oscillations of a certain frequency, which is made visible by the display of a certain colour. *C*-postulates thus connect something visible with something invisible. Nevertheless, they do not thereby make the invisible itself visible.

The *t*-term to be interpreted remains theoretical. This kind of interpretation has therefore to be distinguished from the *d*-interpretation of the *o*-term. Moreover, the interpretation remains incomplete since it is always possible to establish further rules to connect *t*-terms with *o*-terms. Since the interpretation of *t*-terms using *c*-postulates is partial, it is called *p*-interpretation in the following.

To Carnap, it is important not to confuse *c*-postulates with definitions (Carnap, 1956, p. 48). The definition of *t*-terms itself is theoretical and can only be given adequately within L_T. A *t*-term is interpreted logically within L_T, which is why this kind of interpretation is called *l*-interpretation in the following. It is not possible to define a *t*- term completely by relating it to *o*-terms via *c*-postulates. Carnap gives us the following explanation: The terms of geometry as defined by Hilbert are entirely theoretical. However, if they are used within an empirical theory, their empirical use would have to be introduced with the help of *c*-postulates. However, no geometric *o*-term, such as “ray of light” or “taut string”, corresponds to the theoretical properties of the *t*-term straight line (Carnap, 1969, p. 236).

Equipped with this repertoire of concepts, Carnap’s understanding of empirical theories can be defined.

> A theory is a proposition. This proposition is the conjunction of the two propositions *T* and *C*, where *T* is the conjunction of all *t*-postulates and *C* is the conjunction of all *c*-postulates. (Carnap, 1969, p. 266, translation by the author)

To emphasise this connection, Carnap uses the abbreviation *TC* for theories. Now that we have a clear and distinctive definition of what Carnap calls a theory, we go on to explicate Carnap's view on models. Carnap distinguishes descriptive models of physics, which are built from real objects like a model ship, from scientific models in a contemporary sense. As in mathematics and logic, a model in the natural sciences in the twentieth century was understood to be an "abstract, conceptual structure". In this sense, a model is a simplified description of a (physical, economic, sociological, or other) structure in which abstract concepts are mathematically connected (Carnap, 1969, pp. 174–175).

By highlighting the importance of non-Euclidean geometry for physics, especially for the development of the theory of relativity, Carnap infers that it is not disadvantageous for theories if they cannot be visualised without difficulty. In this way, he opposes the idea that models are a sort of visualisation. For Carnap, the visualising character of models is only a makeshift or a didactic aid that merely brings the benefit of being able to think about theories in vivid pictures (Carnap, 1939, p. 210). According to Carnap, models only play a significant role in the development of empirical theories if they establish a connection between L_T and L_O. These "constructing models" (Carnap, 1959, p. 204) serve the *p*- interpretation of *t*-terms and, in this sense, are nothing else than *c*-postulates.

16.2.2 Suppes' Semantic View on Models

The objections to the syntactic view are numerous (cf. Achinstein, 1963, 1965; Suppe, 1971, 1989, 2000; Suppes, 1967; van Fraassen, 1980; see also Liu, 1997; Winther, 2016). Some of these objections are:

1. The formalisation of theories as linguistic entities is inadequate and obscures the underlying structures of theories.
2. Theory testing is oversimplified in the syntactic view since it is assumed that propositions from L_O can be directly linked to phenomena.
3. The pure distinction between *o*- and *t*-terms is not tenable if the characterisation of *o*-terms or *t*-terms is insufficient.
4. *P*-interpretation remains undefined and all possible ways to define *p*-interpretation lead to inconsistencies in the syntactic view.

The semantic view on theories and models can essentially be understood as a reaction to the shortcomings of the syntactic view outlined here (Gelfert, 2017; Portides, 2017). Thus, the meta-mathematical description of theories through formal languages is (largely) rejected in the semantic view. While the syntactic view tries to describe scientific theories in logical languages, the semantic approach asks what kind of mathematical models are used in the sciences (Winther, 2016). Mathematical tools are available for the direct analysis of such structures. In contrast, a reformulation of a theory in a specific formal language tends to be impractical, especially for those theories with rather complicated structures (Suppes, 1957, pp. 248–249).

Moreover, a direct description of mathematical structures may be independent of a particular language. From Suppes' semantic point of view, a theory is composed of a set of set-theoretic structures satisfying the different linguistic formulations of a theory. Worth mentioning that besides this conception of the semantic view, at least one differing semantic approach—the so-called state-space approach—exists (e.g. van Fraassen, 1980), which describes physical systems by vectors. In the semantic view, a model of a theory is a structure and should not be confused with the linguistic description of that structure. Propositions of a theory, expressed in a particular linguistic formulation, are merely interpreted within that structure.

> [A] model of a theory may be defined as a possible realization in which all valid sentences of the theory are satisfied, and a possible realization of the theory is an entity of the appropriate set-theoretical structure. (Suppes, 1962; see also Suppes, 1957, 1960, p. 253)

This emphasises the importance of models for theory building, and along with it the importance of nonlinguistic structures overall. Furthermore, Suppes points out that theories cannot be related directly to experimental data. Accordingly, the *d*-interpretation of *o*-terms in experimental settings is dismissed. Rather, this connection is only established indirectly via what Suppes calls models of data (Suppes, 1962). While models of a theory are possible realisations of a theory, models of data are possible realisations of experimental data. By this conception, Suppes circumvents objection (2), as listed above. In addition, even if a hierarchy between these different types of models is assumed, they are nevertheless connected by an isomorphism between the two types of models (for a critique of this connection by isomorphism, see Suárez, 2003). Objections (3) and (4) are discussed in more detail in the following sections "Theoretical and Empirical Concepts" and "Correspondence Rules and Partial Interpretation".

16.2.3 Theoretical and Empirical Concepts

The separation between *o*- and *t*-terms is challenged from different perspectives. Putnam (1962), for instance, indicates the possibility of formulating theories that do not contain any *t*-terms. He quotes Darwin's theory of evolution as an example. He thus questions whether the separation of L_O and L_T is at all necessary. Consequently, theories that manage without *t*-terms could also not be reconstructed as the proposition *TC* in Carnap's sense.

Putnam then goes on to say that the mere distinction between *o*- and *t*-terms is not sufficient at all. He points out that terms that do not belong to L_O cannot be considered *t*- terms automatically.

Moreover, it is unclear which criterion separates L_T from L_O. Carnap assumes that from a pragmatic point of view, a clear distinction can usually be made between the two (Carnap, 1969, p. 255). It is only decisive whether a term designates a directly or at least indirectly observable entity. Otherwise, it is a *t*-term. According to Achinstein (1965), this criterion is not exhaustive. For instance, an electron, usually a non-observable term, can be considered observable in certain contexts and under certain conditions. He concludes that the term electron cannot be unambiguously assigned to either L_T or L_O. Rather, the conditions for *o*-terms must be made explicit in more detail.

Therefore, Achinstein discusses another criterion that could justify the separation into *o*- and *t*-terms. *T*-terms could be distinguished from *o*-terms based on their theoretical character (cf. Hanson, 1958). According to this distinction, a term would be theory-laden and thus a *t*-term if it cannot be understood without its theoretical background. To Achinstein, even this distinction is not sufficient to divide *o*- and *t*-terms more clearly. A term can be essential in the context of a certain theory, while in another corresponding theory, it is rather independent. Thus, for each term, it must be made clear which theory in particular forms the background. Putnam (1962) also argues that there are no terms that belong exclusively to L_O. For instance, the colour red, which is considered an *o*-term in everyday language, is a *t*-term (red corpuscles) in Newton's corpuscular theory of light. So, the question is posed how to define *t*-terms more precisely.

Another criticism of the syntactic view deals with the possibility to make a theory-free perception at all. This focuses upon the syntactic view's assumption that *o*-terms can be interpreted by direct or at least indirect observation of real phenomena. Seen from the syntactic perspective, *o*-terms must be interpreted with direct reference to real phenomena, since indirect observation by instruments already implies *l*-interpretation.

To perceive objects without recourse to a theoretical background is questioned by other authors. Can there be such a thing as mere observation or does observation always require interpretation of sensory impressions? Hanson (1958, pp. 5–13) gives us various examples here: two biologists looking at an amoeba may see different things because of their different theoretical backgrounds, Tycho Brahe who would not recognise the telescope in a cylinder, as Kepler presumably would, etc. Hanson goes on by describing optical perceptions. He explains that seeing as a mere perception on the retina is always already an interpretation as soon as it enters consciousness. This also illustrates that observational concepts cannot be related to objects directly.

16.2.4 Correspondence Rules and Partial Interpretation

According to the syntactic view, *o*-terms are connected to *t*-terms by correspondence rules (Carnap's *c*-postulates). The assumption is that correspondence rules are the *p*-interpretation of a *t*-term. However, not all *t*-terms of a theory have to be partially interpretable. While an *o*-term must always be directly interpretable, *t*-terms may exist without *c*-postulate partially interpreting them. Such *t*-terms are only indirectly connected with L_O by being connected in L_T with other *t*-terms that can be partially interpreted. For instance, the square root of 2 is unobservable. Nevertheless, it can be indirectly connected with *o*-terms via an *l*-interpretation if it is interpreted as the side length of the square with the area 2. For the *c*-postulates of a theory, Carnap (1956) formulates the following rules.

1. The set of *c*-postulates of a theory must be finite.
2. All *c*-postulates must be logically compatible with the *t*-postulates.
3. The *c*-postulates do not contain terms neither belonging to L_T nor to L_O.
4. Each *c*-postulate must contain at least one *t*-term and *o*-term.

However, apart from the explanation by examples and these rules for *c*-postulates, Carnap does not define more clearly what is meant by *p*-interpretation. This lack of clarification is criticised by various authors (cf. Achinstein, 1963, 1965; Putnam, 1962). Hence, Putnam discusses three ways to define *p*-interpretation:

1. [T]o 'partially interpret' a theory is to specify a non-empty class of intended models. If the specified class has one member, the interpretation is complete; if more than one, properly partial.
2. To partially interpret a term P could mean […] to specify a verification-refutation procedure.
3. Most simply, one might say that to partially interpret a formal language is to interpret part of the language (e.g. to provide translations into common language for some terms and leave the others mere dummy symbols). (Putnam, 1962)

Definition 1 Putnam objects to the first definition. To define a class of models similar in structure to the theory in parts, (a) mathematical concepts, theoretical by definition, are required, and the argument would become circular. Furthermore, he points out (b) that models require certain broad-spectrum terms (e.g. physical object or physical quantity). Such terms cannot be defined a priori, as Quine (1957) illustrates by the meta-concept "science". Accordingly, it is possible that such terms do not acquire their meaning through *p*-interpretation in a particular model, but within a theoretical framework based on the conventions of a research community. Consequently, logical positivists like Carnap must reject such concepts as metaphysical. Ultimately, it refers (c) to the problem that a theory with an empty class of models can no longer be called false, but merely meaningless.

Definition 2 According to Putnam, the second understanding of *p*-interpretation also proves to be unsustainable. If for every concept or proposition a procedure for its confirmation or its refutation is specified, this would lead to curious statements against the background of the philosophical position of verificationism as advocated by Carnap. According to verificationism, only those (synthetic) statements may be true that can be empirically verified. Using the example of the sun and the helium it contains, Putnam draws attention to the following problem. Although it is possible to prove that the sun contains helium, no procedure can be used to prove that helium exists in every part of the sun. If this confirming or refuting procedure is missing, the truth value is indeterminate. In consequence, one would have to claim that the sun contains helium, whereas it cannot be said for parts of the sun, whether there is helium or not.

Definition 3 The third and last possible definition of *p*-interpretation, that L_T is only interpreted in parts, is rejected by Putnam in just one sentence. Such a view would lead to certain theoretical terms ultimately having no meaning at all. A part of L_T would be interpreted into everyday language, for example, and the remaining part of the *t*-terms would merely consist of dummy terms.

16.3 Modelling in Mathematics Classroom from a Syntactic Point of View

In the following, mathematical modelling in the classroom is interpreted against the background of Carnap's syntactic view, while bearing in mind criticism from a semantic point of view. For that, the posed epistemological and methodological questions are focused. Since Carnap's syntactic view first and foremost describes an ideal of empirical sciences, modifications must be made to transfer this to modelling in mathematics education. Axiomatised mathematics cannot be assumed for mathematics teaching, but mathematics in L_T that students master. Furthermore, it is not assumed that an understanding of mathematics in Carnap's formal sense prevails among the students. To describe a modelling process, it is sufficient to reformulate students' usage of terms in Carnap's sense. In this context, mathematical terms used by students in theoretical regards are classified as *t*-terms. Those that refer to observable objects are classified as *o*- terms.

The problem of theoretical terms is serious. Nevertheless, when it comes to mathematical modelling, most of the terms used are mathematical terms and therefore of theoretical nature. Thus, mathematical concepts in school also have a certain theoretical character if students can *l*-interpretate them to a certain extent. Likewise, students can understand that mathematical concepts are in principle unobservable, even if they can be illustrated. However, Achinstein's (1963, 1965) and Putnam's (1962) objections to the separation of theoretical and empirical terms remain considered insofar that the *t*-terms used in the context of mathematical modelling are always *t*-theoretical. In means of students' modelling processes, this implies that *t*-terms are dependent on the mathematics available to students.

Even when transferring Carnap's syntactic view to the description of learners' mathematical modelling, Putnam's objections (1962, p. 245) to different definitions of *p*- interpretation are still considered. If *p*-interpretation of mathematical terms is considered as building a set of intended models, theoretical terms are required indeed. However, this science-theoretical problem concerns the consistency of the syntactic view of theory building. This problem may be less important when it comes to *p*-interpretation within modelling processes taking place in the mathematics classroom. In fact, trying to provide an appropriate procedure for confirming or refuting each *t*-term can lead to some odd statements. For modelling problems in the classroom, however, this can also be a rather subordinate problem. Mathematics lessons usually consider those parts of reality for which such confirmation or refutation procedures exist. Furthermore, the third definition of *p*-interpretation, interpreting parts of L_T and leaving the remaining terms aside, is rather a duty for mathematics teaching than a real objection. Every *t*-term of mathematics should be made semantically accessible to students. Here, the psychological argument is that interpretation of mathematical content through its application leads to an improved and deeper understanding of such content (cf. Blum, 1996, p. 21–22).

16.3.1 Epistemological Question

The purpose of this chapter is to prove if the separation between mathematics and reality, often found in mathematics education research on modelling, is tenable as such against the background of analytical philosophy. By discussing Carnap's syntactic view of theories and models, it becomes clear that this separation needs to be revised.

Carnap's syntactic view captures more precisely the connection between mathematics and reality. In contrast to the dichotomous separation between mathematics and reality in many modelling cycles, there is at least a twofold gradation from mathematics in L_T, via empirical-mathematical concepts in L_O, to real-world phenomena. In a modelling process, (school) mathematics is to be understood as the theoretical (part of a) language with which students can proceed syntactically. Reality, or the "rest of the world", is henceforth divided into an observational language, which itself is not yet a reality, and a part that is identified with real-world phenomena (Fig. 16.3).

Bearing this picture in mind, the criticism of many so-called modelling tasks in mathematics textbooks can be justified by the fact that no real problem is actually solved by the students in the context of a modelling process. Most likely, those tasks take place only in the sphere of theoretical and observational language. While the translation process between these two parts of the language is crucial for making sense of pure mathematical concepts, it does not involve any connection to real-world phenomena. This insight is probably obscured by an overly simplistic juxtaposition of reality and mathematics in many modelling cycles.

Moreover, Carnap's interpretation of models in science as *c*-postulates, marking the area between theoretical and observational language, and Suppes' objection that models of data, marking the area between observational language and real-world phenomena, have to be considered as well. While three models appear in the modelling cycle proposed by Blum and Leiß ("situation model" and "real model" in the realm of reality, and "mathematical model" in the realm of mathematics, Fig. 16.2), we can now capture more accurately the nature of models in mathematical modelling processes. Models are translation rules both for the translation between

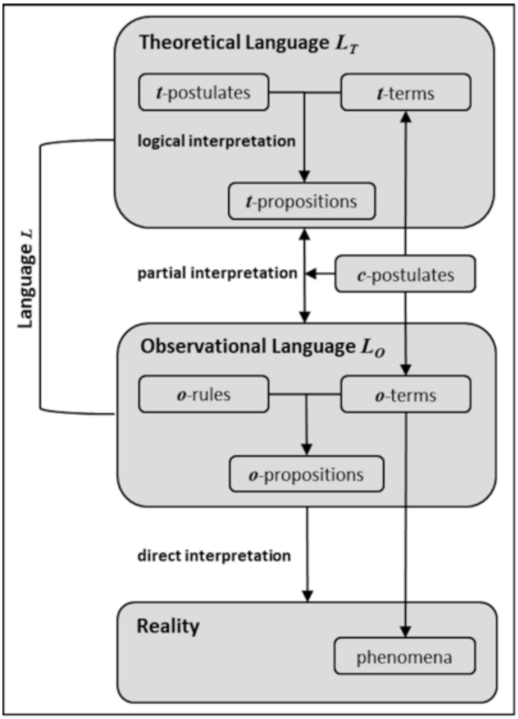

Fig. 16.3 Carnap's syntactic view on theories visualised

theoretical and observational language and for the translation between real-world phenomena and observational language.

Here, the crucial point is that the connection between the two domains of the language at issue is given by assumed rules, not by nature. This finding circumvents lots of epistemological obstacles (e.g. questions about the nature of mathematical terms and their possible empirical origin do not need to be answered for the syntactic view to work). Finally, the competencies described in the modelling cycle can now be interpreted against the background of the previous discussion of theories and models. Working mathematically (step 4 in the modelling cycle according to Blum & Leiß, 2006) can be identified with the *l*-interpretation, mathematising as a transition from the "rest of the world" to "mathematics" (step 3, ibid.) and interpreting as a transition into the opposite direction (step 5, ibid.) is associated with Carnap's *p*-interpretation. The decisive difference is that *p*-interpretation does not indicate the transition from mathematics to reality and vice versa but only a transition between two parts of a language. The *d*- interpretation is pivotal in making the transition from L_O to real-world phenomena (step 1, ibid.). Here, models of data are crucial.

It becomes obvious why, as Meyer and Voigt (2010) note, connections from mathematics need to be considered already in the step of simplification. Learners work with *o*-terms in the step of simplification. However, these must be connected, even implicitly, with *t*-terms. They form what Voigt (2011) calls the "intermediate area".

16.3.2 Methodological Question

The methodological question of whether the discussion on the syntactic and semantic views on models and theories offers new insights for the description of mathematical modelling in the classroom can now be answered against the background of the previous discussion. The goal is to derive methodological tools that describe modelling processes more appropriately and accurately compared to standard modelling cycles. To this end, Carnap's syntactic view and Suppes' criticism of it are considered.

One of the main objections to the syntactic view is that the formalisation of theories as linguistic entities tends to be inadequate because it obscures the underlying structures of theories. While this objection may be crucial for discussion in the philosophy of science, the attempt to focus on the underlying (mathematical) structures tends to be a hindrance when it comes to empirical research in mathematics education. Students' utterances (written, spoken, or expressed by gestures) can be directly observed, whereas the underlying mathematical structures can only be conjectured. With its distinction between theoretical and observational terms, the syntactic view provides a tool for a more detailed analysis of students' utterances. For instance, when a student uses the word "triangle", it is decisive whether the word is used in a theoretical way, for example, in a mathematical theorem, or whether it is used in a sentence to describe real-world phenomena. At this point, Suppes' objection to the theoretical-observational distinction must be considered. The discussion in the section on "Theoretical and Empirical Concepts" shows very briefly that it cannot be said that a concept, by its nature, belongs to either L_T or L_O. Nevertheless, the distinction holds when the theoretical or observational character of a term is considered against the background of the theory T in question. Stegmüller's (1970) solution to this problem is that a term can be called T-theoretical (or T-observable) in the case that T is the theory under consideration. The theoretical character of a term depends on the theory we are talking about. Carnap's definition of a theory (TC is the conjunction of T and C, while T is the conjunction of all t-postulates and C is the conjunction of all c-postulates) reminds us that a clear description of the theoretical background taught to students is necessary when mathematical modelling processes are captured empirically. The question is what theoretical tools (i.e. mathematical theorems, procedures, etc.) are on the theoretical side and what correspondence rules for translation between L_T and L_O are accessible to students.

In order to take a closer look at students' modelling processes, it is necessary to reconsider Carnaps' notion of the connection between L_T and L_O given by c-postulates. Although for Carnap, an ideal theory only includes c-postulates, i.e. axioms that translate between L_T and L_O, he points out that it is not essential for this connection that correspondence rules have the character of an axiom.

> The particular form chosen for the rules C is not essential. They might be formulated as rules of inference or as postulates. (Carnap, 1956, p. 47)

We can thus distinguish between the individual mental models expressed in students' utterances and the more normative models thought in class to make sense of pure mathematical concepts and to give students the ability to solve real-world problems.

From a normative point of view, it is necessary to describe rather abstract models that fit a wide range of situations. This involves the four rules for the formation of *c*- postulates (see section "Correspondence Rules and Partial Interpretation"). Under these conditions, the goal is to formulate as many (but still independent) c-postulates as possible, so that as many situations as possible that fit a notion of pure mathematics are covered by a certain set of *c*-postulates. Putnam's main objections to this understanding of partial interpretation (specifying a class of intended models) are that (a) pure mathematical terms (e.g. set) are needed and the procedure would become circular, and that (b) broad-spectrum terms are needed (e.g. physical object, physical quantity, etc.) that cannot be defined a priori and whose meaning cannot be given by partial interpretation via correspondence rules. While these objections can seriously affect the syntactic view when the focus is on the normative description of an ideal theory, they tend not to negatively affect the goal of describing students' modelling processes. Rather, these objections remind us that every description of individual modelling processes and even the establishment of normative models are limited by the framing through inherently broad-spectrum terms and (meta-) mathematical terms in use.

Bearing in mind, that correspondence rules not necessarily need to be formulated in a set of axioms, this offers an opportunity to analyse students' (implicit) use of correspondence rules in modelling processes. As we will see, this provides a methodological tool that leads to different results than the analyses that depend on standard modelling cycles and their inherent epistemological assumptions. Let us take a look at the mathematics task from a textbook for fifth and sixth graders:

> The African grey parrot can grow up to 40 cm long; a flamingo of about 200 cm. How many times bigger is the flamingo compared to the grey parrot? (Prediger, 2009, p. 6, translation by the author)

From a normative point of view, the area of mathematics addressed in this task can be narrowed down to the structure of natural numbers in connection with multiplication $(N, \cdot)$. On the observational side, questions can be formulated such as "how often does one length fit into another?" or "how many times larger is this length compared to another?". The connection between L_T and L_O is then given by a *p*-interpretation containing at least two *c*-rules. Thus, *c*-rule c_1 connects—for instance—the number 1 with the observable length of 1 cm, while *c*-rule c_2 connects multiplication with a temporal- successive action (e.g. "an empirical length is juxtaposed until the length used for comparison is reached"). The model at issue here is the description of the structure $(N, \cdot)$ using the linguistic means from L_O, given by c_1 and c_2. If T is the conjunction of all true propositions in L_T about the structure $(N, \cdot)$ and C is the conjunction of c_1 and c_2, the theoretical background of the task is given by the conjunction TC. This interpretation of $(N, \cdot)$ by c_1 and c_2 remains partial. In contrast, an *l*-interpretation of $(N, \cdot)$ within L_T, (e.g. as the addition of equal addends) is complete. Again, it should be mentioned that c_1 and c_2 do not connect mathematics with reality, but theoretical terms of mathematics (*t*-terms) with empirical terms (*o*-terms). Hence, it is a purely linguistic connection. With this revision of the task in mind, we can now analyse individual utterances of students confronted with this task. To do this, individual modelling is analysed by reconstructing the reasoning within the modelling according to the Toulmin scheme (Toulmin, 1996). Thereby, the use of *c*-rules—whether implicit or explicit—has to be taken into account.

Due to the limitation of a book chapter, the focus is on a single case study, the student Anton. Anton is interviewed while solving the task (cf. Prediger, 2009). At the beginning of the interview, Anton soon says, "The flamingo is 160 cm taller". The genesis of this statement can be reconstructed with the help of the Toulmin scheme as follows (Fig. 16.4).

Against the background of common modelling cycles, Anton's statement must be interpreted as an individual construction of whether a situation model, a real model, or a mathematical model. However, a rational reconstruction shows that none of Anton's possible considerations can be the mere result of modelling processes taking place exclusively in the "rest of the world". Anton's statement cannot be interpreted without (implicit) translations between L_T and L_O.

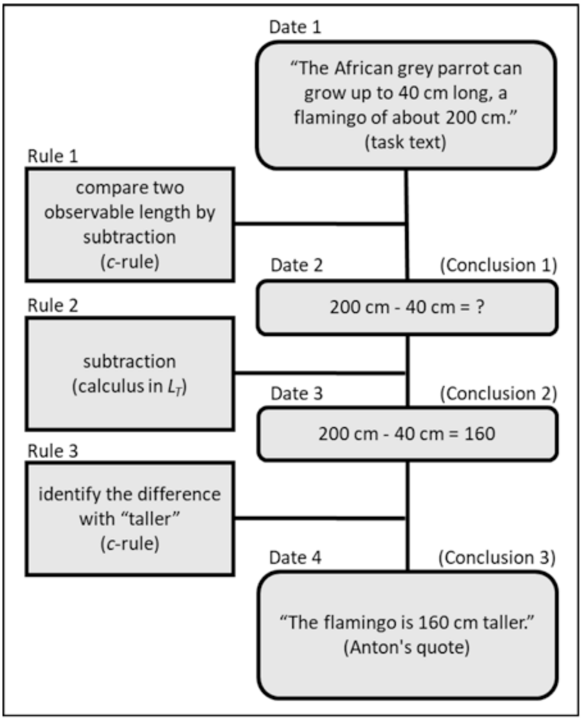

Fig. 16.4 Anton's statement rationally reconstructed

16.4 Conclusion and Outlook

Against the background of the syntactic view on theories and models and its critique by the adherents of the semantic view of theory building, mainstream modelling cycles and their inherent epistemological assumptions about the relation between mathematics and reality have been problematised. The goal of the chapter is to show that the description of students' modelling processes cannot rely on a simple separation between mathematics and reality. The syntactic view, as offered by Carnap, indicates that distinguishing between the theoretical and observational side of a language can be helpful in capturing the translation processes of students that take place when mental models are used to interpret pure mathematical terms, and vice versa, to interpret the empirical part of a language through the means of mathematics.

Based on a single case study, it was shown that the twofold separation between L_T and L_O and the connection via *c*-rules—in combination with Toulmin's scheme—provides a methodological tool to investigate students' translations between mathematics understood as a theoretical language and everyday language and the empirical use of mathematical terms contained therein. In detail, this attempt allows us to reconstruct also those more implicit translation steps that are necessary to explain subsequent explicit utterances and that would remain hidden against the background of mainstream modelling cycles.

To give an outlook: While this brief chapter has paid attention to the multiple translations between the theoretical and empirical sides of a language used in modelling processes, the connection of *o*-terms with real-world phenomena was omitted to a large extent. In order to get a comprehensive picture of all the translations taking place in modelling processes, this connection needs to be described and problematised in more detail. Follow-up questions arise when not only the epistemological and ontological aspects but also constructive aspects of mathematical modelling are considered. Here, questions may arise concerning the design of textbook tasks to promote students' modelling skills, the teaching of adequate models for proper connection between pure mathematical terms and everyday language, and whether and what meta-knowledge about mathematical modelling should be taught in the class.

References

Achinstein, P. (1963). Theoretical terms and partial interpretation. *British Journal for the Philosophy of Science, 14*(54), 89–10.

Achinstein, P. (1965). The problem of theoretical terms. *American Philosophical Quarterly, 2*(3), 193–203.

Biehler, R., Kortemeyer, J., & Schaper, N. (2015). Conceptualizing and studying students' processes of solving typical problems in introductory engineering courses requiring mathematical competences. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the ninth congress of the European Society for Research in mathematics education* (pp. 2060–2066). Charles University, Faculty of Education.

Blum, W. (1996). Anwendungsbezüge im Mathematikunterricht - Trends und Perspektiven. In G. Kadunz et al. (Eds.), *Trends und Perspektiven. Schriftenreihe der Didaktik der Mathematik* (Vol. 23, pp. 15–38). Hölder-Pichler-Temsky.

Blum, W., & Leiß, D. (2006). "Filling up" – The problem of Independence-preserving teacher interventions in lessons with demanding modelling tasks. In M. Bosch (Ed.), *Proceedings of the fourth conference of the European Society for Research in mathematics education* (pp. 1623–1633). FUNDEMI IQS – Universitat Ramon Llull and ERME.

Blum, W., Galbraith, P., Henn, H. W., et al. (2007). Introduction. In W. Blum, P. Galbraith, H. W. Henn, et al. (Eds.), *Modelling and applications in mathematics education – The 14th ICMI study* (pp. 3–32). Springer.

Borromeo Ferri, R. (2006). Theoretical and empirical differentiations of phases in the modelling process. *Zentralblatt für Didaktik der Mathematik, 38*(2), 86–95.

Carnap, R. (1932). Über Protokollsätze. *Erkenntnis, 3*(1), 215–228.

Carnap, R. (1939). *Foundations of logic and mathematics*. University of Chicago Press.

Carnap, R. (1956). The methological character of theoretical concepts. In H. Feigl & M. Scriven (Eds.), *Minnesota studies in the philosophy of science* (pp. 38–76). University of Minnesota Press.

Carnap, R. (1958). Beobachtungssprache und theoretische Sprache. *Dialectica, 12*, 236–248.

Carnap, R. (1959). *Introduction to semantics and formalization of logic*. Harvard University Press.

Carnap, R. (1969). *Einführung in die Philosophie der Naturwissenschaft*. Nymphenburger Verlagshandlung.

Châtelet, G. (2000). *Figuring space – Philosophy, mathematics, and physics*. Kluwer Academic Publishers.

Creath, R. (2017). Logical empiricism. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy*. https://plato.stanford.edu/archives/fall2017/entries/logical- empiricism/. Accessed 28 Feb 2022

de Freitas, E. (2013). The mathematical event: Mapping the axiomatic and the problematic in school mathematics. *Studies in Philosophy and Education, 32*(6), 581–599.

de Freitas, E., & Sinclair, N. (2014). *Mathematics and the body: Material entanglements in the classroom*. Cambridge University Press.

Deleuze, G. (1994). *Difference and repetition*. (trans: Patton P). Columbia University Press.

Deleuze, G., & Guattari, F. (1987). *A thousand plateaus: Capitalism and schizophrenia*. (trans: Massumo B). University of Minnesota Press.

Foucault, M. (1996). *Die Ordnung der Dinge – Eine Archäologie der Humanwissenschaften*. Suhrkamp.

Frege, G. (1884). *The foundations of arithmetic: A logico-mathematical enquiry into the concept of number* (Vol. 1960). Harper.

Frege, G. (1892). *Über Sinn und Bedeutung. Zeitschrift für Philosophie und philosophische Kritik NF* 100:25–50.

Frigg, R., & Salis, F. (2019). Of rabbits and men: Fiction and scientific modelling. In B. Armour-Garb & F. Kroon (Eds.), *Fictionalism in philosophy* (pp. 187–206). Oxford University Press.

Gelfert, A. (2017). The ontology of models. In L. Magnani & T. Bertolotti (Eds.), *Springer handbook of model-based science* (pp. 5–23). Springer.

Greefrath, G. (2011). Using technologies: New possibilities of teaching and learning modelling – Overview. In G. Kaiser, W. Blum, R. Borromeo Ferri, et al. (Eds.), *Trends in teaching and learning of mathematical modelling – ICTMA 14* (pp. 301–304). Springer.

Hanson, N. R. (1958). *Patterns of discovery – An inquiry into the conceptual foundations of science*. Cambridge University Press.

Hilbert, D. (1903). *Grundlagen der Geometrie*. Teubner.

Keitel, C. (1989). Mathematics education and technology. *For the Learning of Mathematics, 9*(1), 7–13.

Liu, C. (1997). Models and theories I – The semantic view revisited. *International Studies in the Philosophy of Science, 11*(2), 147–164.

Ludwig, M., & Reit, X. R. (2013). A cross-sectional study about modelling competency in secondary school. In G. Stillman, G. Kaiser, W. Blum, et al. (Eds.), *Teaching mathematical modelling: Connecting to research and practice – ICTMA 15* (pp. 327–337). Springer.

Meyer, M., & Voigt, J. (2010). Rationale Modellierungsprozesse. In B. Brandt, M. Fetzer, & M. Schütte (Eds.), *Auf den Spuren interpretativer Unterrichtsforschung in der Mathematikdidaktik* (pp. 117–148). Waxmann.

Niss, M. (1994). Mathematics in society. In R. Biehler, R. W. Scholz, R. Sträßer, et al. (Eds.), *Didactics of mathematics as a scientific discipline* (pp. 367–378). Kluwer Academic Publishers.

OECD. (2009). *PISA 2009 assessment framework – key competencies in reading, mathematics and science*. http://www.oecd.org/dataoecd/11/40/44455820.pdf. Accessed 28 Feb 2022.

Portides, D. (2017). Models and theories. In L. Magnani & T. Bertolotti (Eds.), *Springer handbook of model-based science* (pp. 25–48). Springer.

Prediger, S. (2009). „Aber wie sag ich es mathematisch?" – Empirische Befunde und Konsequenzen zum Lernen von Mathematik als Mittel zur Beschreibung von Welt. In Höttecke D (Ed.), *Entwicklung naturwissenschaftlichen Denkens zwischen Phänomen und Systematik* (pp. 6–20). LIT-Verlag.

Putnam, H. (1962). What theories are not. In E. Nagel, P. Suppes, & A. Tarski (Eds.), *Logic, methodology and philosophy of science* (pp. 240–251). Stanford University Press.

Quine, W. V. O. (1957). The scope and language of science. *British Journal for the Philosophy of Science, 8*, 1–17.

Schürmann, U. (2018a). Towards an alternative approach to modelling in school mathematics. *The Mathematics Enthusiast, 15*(1), 228–250.

Schürmann, U. (2018b). The separation of mathematics from reality in scientific and educational discourse. In P. Ernest (Ed.), *The philosophy of mathematics education today – ICME-13 monographs* (pp. 241–251). Springer.

Skovsmose, O., & Borba, M. (2004). Issues of power in theory and methodology. In P. Valero & R. Zevenbergen (Eds.), *Researching the socio-political dimensions of mathematics education* (Mathematics education library) (Vol. 35, pp. 207–226). Kluwer Academic Publishers.

Smith, D. W. (2006). Axiomatics and problematics as two modes of formalisation – Deleuze's epistemology of mathematics. In S. Duffy (Ed.), *Virtual mathematics – The logic of difference* (pp. 145–168). Clinamen.

Stegmüller, W. (1970). *Probleme und Resultate der Wissenschaftstheorie und Analytischen Philosophie*, Bd. 2 Theorie und Erfahrung: Studienausgabe, Teil C Beobachtungssprache, theoretische Sprache und die partielle Deutung von Theorien. Springer.

Suárez, M. (2003). Scientific representation – Against similarity and isomorphism. *International Studies in the Philosophy of Science, 17*(3), 225–244.

Suppe, F. (1971). On partial interpretation. *The Journal of Philosophy, 68*(3), 57–76.

Suppe, F. (1989). *The semantic conception of theories and scientific realism*. Illinois University Press.

Suppe, F. (2000). Understanding scientific theories: An assessment of developments, 1969– 1989. *Philosophy in Science, 67*, S102–S115.

Suppes, P. (1957). *Introduction to logic*. Van Nostrand.

Suppes, P. (1960). A comparison of the meaning and uses of models in mathematics and the empirical sciences. *Synthese, 12*(2/3), 287–301.

Suppes, P. (1962). Models of data. In E. Nagel, P. Suppes, & A. Tarski (Eds.), *Logic, methodology and philosophy of science – Proceedings of the 1960 international congress* (pp. 252–261). Stanford University Press.

Suppes, P. (1967). What is a scientific theory? In S. Morgenbesser (Ed.), *Philosophy of science today* (pp. 55–67). Basic Books.

Toulmin, S. E. (1996). *Der Gebrauch von Argumenten*. Beltz.

van Fraassen, B. C. (1980). *The scientific image*. Oxford University Press.

Voigt, J. (2011). Rationale Modellierungsprozesse. In GDM (Ed.), *Beiträge zum Mathematikunterricht – 45. Jahrestagung der Gesellschaft für Didaktik der Mathematik* (pp. 867–870). WTM.

Winther, R. G. (2016). The structure of scientific theories. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy*. https://plato.stanford.edu/archives/win2016/entries/structure-scientific-theories/. Accessed 28 Feb 2022.