Chapter 1
The Ontological Problems of Mathematics and Mathematics Education

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1.1 Introduction

Ontology is the branch of philosophy that studies the fundamental nature of reality, the first principles of being, identity and changes to being, that is, becoming. In this chapter, I want to explore being, existence and identity as they concern mathematics and mathematics education. In particular, I want to address the ontological problems of mathematics and mathematics education. The ontological problem of mathematics is that of accounting for the nature of mathematical objects and their relationships.[1] What are mathematical objects? Of what 'stuff' are they made and do they consist?

The ontological problem of mathematics education concerns persons. What is the nature and being of persons, including both children and adults? In the context of this chapter, I will restrict my attention to human mathematical identities, that part of being which pertains to mathematics, namely the mathematical identity of mathematicians and the developing mathematical identities of students. What are these mathematical identities and how are they constituted? Human beings are located in, and constituted through the cultures they inhabit, so my answer will encompass how these contribute to mathematical identities, as well.

The twin ontological problems of mathematics and mathematics education concern the chief entities in the two domains. These are mathematical objects first, and second, persons, restricted to their mathematical identities. The structural similarity

[1] I use the term nature without presuming essentialism or assuming 'natural' states of being. I shall answer the question of how the properties and characteristics of mathematical objects and human beings as mathematical subjects are inscribed within them as a process of becoming without the presuppositions of essentialism.

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does not end with these parallel twin focuses of inquiry. In each of these two domains, there are dominant myths that must be critiqued or cut down before their respective problems can be addressed adequately. In mathematics, there is the myth of Platonism, namely that mathematical objects exist in some eternal, superhuman realm. According to this view, mathematical objects were there before we came along, and they will still exist after we are all gone.

In mathematics education, there is the rather more hidden problem of individualism. This is the view that persons are all existentially separated creatures whose actions, learning and even whose being take place in hermetically sealed and separated personal domains.

However, there is also dissimilarity in the treatments I can hope to offer. While I can aspire to giving an account of the nature of mathematical objects, I cannot hope to treat the nature and being of persons except in a very partial way. As I have indicated, I restrict my inquiry to those aspects of human being that pertain to learning and doing mathematics, and their personal foundations.

1.2 Mathematical Objects

In this part, in a number of linked sections, I offer an attempt to giving an account of the nature of mathematical objects. I start by trying to clear some of the obstructive conceptual undergrowth that stands in the way of my account. In the exposition that follows on, all the elements that make up the social constructionist account of the ontology of mathematical objects are introduced and then summarized in Sect. 1.2.8.

1.2.1 Critique of Platonism

According to Platonism, mathematics comprises an objective, timeless and superhuman realm populated by the objects of mathematics. These objects are pure abstractions, and they exist in an unchanging ideal realm quite distinct from the empirical world of our day-to-day living. Plato's doctrine of Platonism locates other abstract ideals beyond mathematics such as Justice, Beauty and Truth, in this realm. Not surprisingly, the nature, status and location of abstract ideas has been a matter of debate at least since the time of Plato. The medievalists divided into the camps of nominalists (abstract objects are primarily linguistic names), conceptualists (abstract objects are ideas in our minds), and realists (abstract objects are real entities that are located in some platonic-like realm). All of these positions have their problems.

Many of the greatest philosophers and mathematicians have subscribed to the doctrine of Platonism in the subsequent two plus millennia since Plato's time. In the modern era, the view has been endorsed by many leading thinkers including Frege (1884, 1892), Gödel (1964), and Penrose (2004).

Although I shall reject it, quite a lot is gained by this view. First of all, mathematicians and philosophers have a strong belief in the absolute certainty of mathematical truth and in the objective existence of mathematical objects, and a belief in Platonism is consonant with this and even validates this view. Platonism posits a quasi-mystical realm into which only the select few – initiates into the arcane practices of mathematics – are permitted to gaze, and within there – to discern mathematical objects and mathematical truth.

Second, this view is a concomitant of, and validates purism, the ideology that mathematics is value-free and ethics-free. Human values are excluded by definition, for they cannot seep into or taint the hermetically sealed superhuman platonic realm, since it exists in another dimension. As I have recounted elsewhere, purism is an ideology that was strong in Plato's time and then again in the nineteenth- and twentieth-century mathematics (Ernest, 2021a).

Third, Platonism supplies mathematics with a theory of meaning. According to this theory, mathematical signs and terms refer to objects in this ideal realm. Likewise, mathematical sentences, claims, and theorems refer to true, or otherwise, according to their status, states of affairs and relationships between the constituent objects that hold in the Platonic realm.

However, a distinction should be made between Platonism and mathematical realism. According to the latter, mathematical objects are real; they are something verifiably shared amongst many people. However, they do not necessarily exist in a superhuman and supraphysical realm. For example, as I shall argue, they can be social objects. However, developing a social theory of mathematical objects is more complex than simply positing a Platonic realm, which can be conjured, ready-made out of a hat. Explaining and validating a social constructionist theory of mathematical objects requires the development of conceptual machinery and to a certain degree, a suspension of disbelief, because Platonism has penetrated so deeply into our understanding of mathematics and universals.

Platonism is not without its problems. Two major problems concern access and causality. How can mathematicians access the Platonic realm? With what faculties can they peer into it to discern its objects and truths? No such sixth sense is known unless one strays into the realms of the mystic and shaman. And even if one did stray there, and could discern mathematical objects and truths directly, what justifications could be given to others for the existence of the objects and the validity of the truths discerned? To say I saw it with my mind's eye is not enough. Mathematical objects need to be accurately defined to be communicated, and mathematical truths need to be convincingly proven in public texts to be acceptable. So, their means of validation are just those that one would need even if there were no superhuman Platonic realm into which one could peer (Benacerraf, 1973).

In terms of causality, there are problems both ways (Linnebo, 2018). How are newly defined concepts and newly proven results inserted into the Platonic realm? What is there about our inventions and discoveries that cause them to appear there? Plato argued these 'new' objects were there all along and we can only discern them when we have recreated them for ourselves. This is surely an unsatisfactory *ad hoc* answer. If we can only discern what we have recreated, why not dispense with the mystification and acknowledge we *created* them, in the first place? In the reverse direction, how can the truths of the Platonic realm causally determine outcomes in the material world? Why is pure mathematics so unreasonably effective in the real world? How and why are mathematical truths so real and so persuasive to children, students, and adults prior to demonstrations? I suppose that if mathematical truths hold in all possible worlds, then those found in the Platonic realm must hold in the material world. But this is not a causal argument implying that mathematical truths from the Platonic realm force their applications to hold in the physical world. Once again it leaves the Platonic realm superfluous.

Although positing the Platonic realm as the home of mathematical objects and as a source for mathematical truths opens a number of serious problems, it remains a widespread, legitimate, and irrefutable view. Like many ideologies that posit other realms full of celestial beings it remains a matter of choice and belief. I choose to use Occam's razor, the principle of ontological parsimony, that entities should not be multiplied beyond necessity. Extending this to the multiplication of ontological realms, I regard the expansion of mathematical ontology through the addition of the superhuman Platonic realm to be unnecessary. It creates new problems of access and causality. It represents a succumbing to the historical vice of Idealism. My claim is that socially based mathematical realism can accommodate many of the benefits of Platonism without all these extra costs. So, I reject Platonism while embracing mathematical realism.

1.2.2 Meaning Theory

Above I acknowledge that Platonism supplies mathematics with a theory of meaning. According to this theory, mathematical signs and terms refer to objects and their manifested relationships in the ideal Platonic realm. Most simply, this is a referential or picture theory of meaning. Ernest (2018a) shows some of the inadequacies of this theory, which is also widely criticized elsewhere (e.g. Rorty, 1979). But if one is to reject this theory what is to stand in its place? If mathematical signs and words are not simply the names of objects in a Platonic realm how else can they signify? How can we offer a way to understand their meanings? In my view, Wittgenstein's (1953) theory of meaning, according to which much of the meaning of words and other signs is given by their use, offers the best solution.

With regard to meaning, Wittgenstein says that much of meaning is given by use: "for a large class of cases – though not for all – in which we employ the word 'meaning' it can be defined thus: the meaning of a word is its use in the language" (Wittgenstein, 1953, I, sec. 43). He allows for three other sources of meaning – custom, rule-following, and physiognomic meaning (Finch, 1995; Cunliffe, 2006). Focusing on meaning as use, it is important to hedge this in the way that Wittgenstein does. Namely that the use of words or signs is always located within language games situated within forms of life. Thus, according to this theory, the meanings of words and signs are the roles they play within conversations located in social forms of life. But these are not free-floating conversations, they are conversations centered on, and intrinsically a part of, shared activities with a goal or object in mind. In one extreme case this might be conversing after dinner with friends with combined aims of sharing information (or gossip), consolidating relationships or just for the intrinsic joy of relaxing with friends and family. Such discussion, although perhaps capturing the popular meaning of the term 'conversation', is trivial and fails to reflect the central importance of conversation.

Conversations are not just trivial decorations but an integral part of social activities. The function of conversations is to facilitate important joint and productive activities through directions, confirmations, and other means. The meanings of the terms and signs employed are their functions within these activities. Joint action within a form of life is usually directed and punctuated by discourse. In other words, language in conversation is a tool employed to further a joint activity and take it towards its goal. Indeed, the language used in productive material activities is as often imperative or interrogative as it is declarative. Such as in the kitchen: 'stir this', 'is there enough salt in the sauce?', and 'this is tonight's meal'. Where conversation is lacking in a joint activity, often custom and rules have been laid down conversationally in earlier manifestations of the form of life rendering repeated conversations and directions superfluous, so the joint activity can progress without verbal instructions or elaboration.

Wittgenstein makes it clear that meanings depend on the language games in which they are used, and 'When language-games change, then there is a change in concepts, and with the concepts the meanings of words change' (Wittgenstein, 1969, sect. 65).

Two other dimensions of Wittgenstein's ideas of meaning, custom and rule-following, are also important. Cunliffe (2006: p. 65) points out that there are deontic dimensions of meaning entangled with the other uses, with widespread imperatives imposing or requiring rule-following, the meeting of obligations, lawfulness, and respect for customary usage. Language use is far from limited to the alethic mode – meaning that it encompasses epistemic, factual, and truth-orientated functions. It also commonly employs the imperative mode. This is very important when understanding the meaning of mathematical texts, where the imperative mode far outweighs the declarative or indicative modes, as an analysis of verb usage in the corpus of mathematical texts reveals (Ernest, 1998, 2018a; Rotman, 1993). I shall argue that the institutions of mathematics are held up by tacit or explicit rule-following and custom, so this dimension of meaning is very significant. Indeed, I hope to show that the very objects of mathematics are created and maintained by tacit agreements, rule-following, and embedded customs inscribed within the objects themselves. For example, to count you must follow a string of rules, but as counting skills develop, and numbers come into being as self-subsistent mathematical entities, then the rules and norms appear to dissolve or disappear into the perceived nature of the numbers themselves. But I get ahead of myself.

1.2.3 What Are the Objects of Mathematics?

Platonism and realism offer answers as to where mathematical objects are to be found (Skovsmose & Ravn, 2019). But apart from the fact that they are universals and abstractions, these ontologies do not tell us what the objects of mathematics are; they do not answer the question of what is the stuff of which they are made?

Unfortunately, traditional ontology is not a lot of help here. It seems to be satisfied with a category of being, rather than a deeper inquiry into the very stuff or substance of the existents. What is needed is a multi-disciplinary approach that combines insights from philosophy of mathematics, mathematics itself, semiotics, cognitive science, psychology, sociology, linguistics, and mathematics education into the nature of mathematical objects. No one of these disciplines is sufficient of itself, as I intend to show, to satisfactorily answer the question: Of what stuff are mathematical objects made?

There is another obstacle in the way of a naturalistic account of the ontology of mathematical objects, namely, the ideologies of essentialism and presentism (Irvine, 2020). In this context, essentialism presupposes that mathematical objects are made of some fixed stuff, analogous to diamonds or other precious stones, but existing in an eternal realm, where they are found to be permanent and unchanging.

The ideology of presentism searches for answers to all questions in a timeless present, where there is no change, development, or becoming. In my view, ignoring Plato's admission of 'becoming' into ontology, presentism underpins much of modern Anglo philosophy. There logical arguments are timelessly valid, and concepts presumed fixed and permanent. Where such properties are attributed to mathematical concepts and objects, they are presumed completed and there is no need to discuss how they came to be and how this shapes what they are. Becoming is ignored and disallowed. I think that to fully understand mathematical concepts and objects you need to know how they became as they are. Especially, as being abstractions, they have been abstracted from lower orders of abstractions or actions. Mathematical objects do not have a fixed essence, for they change over time, and they have different meanings in different contexts.

Let me illustrate this with arguably the simplest of all mathematical concepts, the number one. This first appears in human history and in child development as the first word in a spoken count ('one', 'uno', 'yek', 'tik') or as the first tally in symbolically recorded counting. The number, or rather numeral, 'one' is the first ordinal in a counting sequence (meaning 'first'). The early use of this numeral is enactive, with the action of pointing or making a mark accompanying its utterance. When the last ordinal number in counting out a set becomes defined as its cardinal number, the word 'one' or sign '1' gains its cardinal meaning as the number one. Counting out a triplet with the ordinals 1, 2, 3 ends with 3 which by definition is its cardinality. Counting a singleton – ordinal one – results in cardinal one. This marks the arrival of the concept of one in its first rudimentary but complete form, the cardinal number one. At this stage in its development, number one is understood to be on a par with the other natural numbers 2, 3, 4, and so forth.

It is important to notice that the concept of one is doubly abstracted. First from the physical action of counting, from tallying, or from just saying the number names out loud prior to counting. Second, once ordinal counting is mastered for small values, the cardinal number one is abstracted from the ordinal one, as comes to represent the value of a completed count.

In tallying, strokes or marks are used to represent the outcomes of counts (e.g. '///' representing the count of three. Each stroke is itself a part of a unit action and ultimately each of these units 'one more' is identified with 'one'. The tally '///' represents one (more) and one more and one more which is a compound way of representing three (via the ordinal 'third').

In number systems, the numeral '1' resembles a tally stroke. In several number systems, such as those of the Ancient Egyptians, Sumerians, and Romans, numerals made up of one, two, and three strokes (I, II, III, respectively) are used for the first few numbers representing both repeated ones and unit strokes in a tally. Studies of proto-language suggest that the early, possibly prehistoric name for one was 'tik', also meaning digit or finger (Lambek, 1996). Use of fingers for counting evidently goes back a long way, and this word for one also stands for a single digit (as finger). Indeed, the modern use of the word digit retains this ambiguity, standing both for individual numerical signs (1 to 9) as well as for any single finger.

This is just the beginning of the development of the concept of one. The use of the sign '1' becomes more elaborate within systems of numeration, calculation, and measurement. The numeral '1' represents a unit in an abacus or place value system in compound numbers (one ten, one hundred, etc. indicated in decimal place value as 10, 100, respectively, with '1' as an atomic component in a molecular sign). Subsequently, with the introduction of multiplication, one serves as the multiplicative identity element.

As number systems and structures are extended, '1' has different meanings and properties, across N, Z, Q, R, C. In Q, the numeral '1' is used both in numerators and denominators. In R (and Q), '1' is used in extended place value notation as a fraction of denomination ten to a negative power (e.g. $0.001 = 10^{-3}$). '1' and other numerals are used in algebra, length measures (and indeed all measures), in fractions, extended place value notations, vectors, matrices, probability theory (representing certainty), Boolean algebra (representing truth). The property of 'one' as the multiplicative identity in Q, R, and C is generalized throughout algebraic structures such as groups, rings, fields (together with the more basic additive identity 0).

In each of these different roles, '1' has different uses and meanings, so its meaning can never be said to be fixed, but is always dependent on the context of use, on the background theory. Thus, the number one cannot be claimed to be a single fixed mathematical object or concept. However, what we can say is that the number one, like all Natural numbers, in its first emergence, is an abstraction of an action using signs. An instance of a counting action can be physical, such as touching individual members a set of objects (already conceptualized as countable units for the purposes of counting, Ernest, 2021b), while uttering the sequence of ordinal names. Or it can be conceptual where the units are counted without physical contact or movement. Either way the act of counting is an instance, a token of counting, corresponding to an abstracted type, the count. This count has an endpoint, a designated sign that represents the ordinal position of the last counted unit, which is abstracted as the number, the cardinality of the set counted. Thus, the resultant number, the cardinality is doubly abstracted from the instance of counting. First, as the type or class of the designated count. Second, as the cardinal number abstracted from the derived ordinal number.

This simplest of all the numbers, 'little' number one, serves to show both how complex and multiply meaningful mathematical concepts are, as well as of what they are formed. 'One' begins as an action associated with a sign, which is then abstracted. The process is reified into an object. Virtually all named mathematical objects consist of abstracted operations or actions on simpler mathematical objects or actions.[2] To enable a differentiation of levels into simpler/more complicated, one can posit a hierarchy through which the relation of 'simpler than' can be defined. The lowest level of mathematical actions (level 0) is made up of those that have a physical correlate, like counting or drawing a line. The lowest levels of mathematical objects (level 1) are abstractions of, or from, mathematical actions of level 0. A mathematical action of level $n + 1$ operates on actions and objects that include at the highest level those of level n. Likewise, a mathematical object of level $n + 1$ abstracts actions and objects that include those up to and including level n.

What I have only exemplified in the case of 'one' is that there is a sign associated with every (named) mathematical action or object. Frequently, there are several signs. So, with one there is the spoken verbal name or word (in almost every language), a written verbal name ('one'), and a mathematical sign ('1'). In various arithmetics, there are in fact an infinite number of expressions with the numerical value of 1 that could also be called names for 1 (e.g. 22–21, 0 + 1). The signs of mathematics are of paramount importance. The signs not only help to create the objects of mathematics, but they are also entangled with them. Mathematical actions are typically actions on or with mathematical signs.

1.2.4 Mathematical Signs and Their Performativity

In order to fully engage with the role of signs in mathematics, with the semiotics of mathematics, it is necessary to understand the performativity of mathematical signs. As syntactical objects, mathematical signs are both the objects acted upon and the crystallized residue of acts in themselves. In the first instance, all the 'atoms', the ur-elements of mathematical signing are performed actions. Thus, as we have seen, 3 represents the product of tallying III which is itself the residual mark of the repetitive physical act of counting one, two, three. In this way, counting employs

[2] Various scholars in mathematics education research make this point (Sfard, 1994; Tall, 2013; Dubinsky, 1991).

indexical signing because each stroke or physical tally movement corresponds to a counted entity by proximity in space, time, or thought.

Within semantics, the performativity of mathematical signs is ontological. The signs create their own meanings; the abstract objects of mathematics that they denote. What we have is a 'a set of repeated acts within a highly rigid regulatory frame that congeal over time to produce the appearance of substance, of a natural sort of being' (Butler, 1999, p. 43–44). Although Butler is referring to another domain, the process is identical for the construction of mathematical objects. Thus, numerals and number words 'do not refer to numbers, they *serve as* numbers' (Wiese, 2003, p. 5, original emphasis). This is an important point that contradicts any referential theory of meaning, including both the picture theory of meaning and Platonism. Numerals, number word terms, and by extension all mathematical signs need not indicate or refer beyond themselves to other objects as their meaning, let alone to a supraphysical and ideal realm of existence. They themselves serve as their own objects of meaning, coupled with the actions that they embody (and their inferential antecedents and consequents).[3] Mathematical language is thus performative, for mathematical terms create, over time, the objects to which they refer. As I have argued, counting via abstraction is the basis for the creation of numbers, and likewise operations create mathematical functions. In the first instance, these are inscribed numerals and enacted operations. Repeated usage reifies and solidifies them into abstract mathematical objects.[4] Furthermore, their currency of use serves as a social warrant for them, verifying their legitimacy and existence.[5]

Elsewhere I argue that mathematical signs are performative in two ways, which I term inner and outer. What I describe above is part of the inner performativity, whereby mathematical sign usage creates mathematical objects. The outer performativity of mathematics is the way it formats the way we experience and interact with the material world (Skovsmose, 2019, 2020; Ernest, 2019). I will not discuss this outer performativity further here (but see O'Halloran, 2005 and Ernest, 2018b).

[3] This has been used as an explicit strategy within mathematics. Henkin (1949) defines the reference of each sign within the system to be itself, in his classic proof of the completeness of the first-order functional calculus.

[4] This is supported both within philosophy (for example, Machover, 1983) and empirically by research into the psychology of learning mathematics (Ernest, 2006; Tall, 2013).

[5] In writing that signs create their own meanings, it is taken for granted and unwritten here that it is signs-in-use by persons that perform actions, for it is persons that use signs and create and comprehend meanings.

1.2.5 The Constitutional Role of Social Agreement for Mathematical Objects

Social agreements play a decisive role in the constitution of mathematical objects and the validation of mathematical knowledge. Such agreements may be tacit and are introduced both implicitly and explicitly through mathematical practices and everyday games and practices with young children. Before counting can even begin, the idea of separate objects or units needs to be introduced into a learner or child's worldview. This is the idea that some of the features of the environment can be construed as self-contained objects. Take for example, collections of toys, pebbles, or sweets. Each item in the collection can be treated as a separate independent object. In addition, steps in climbing a stairway, walking along the ground step by step, or other sequences of actions can also be seen in this way as series of discrete actions. Such a way of viewing a domain, although concrete and limited in extension at first, prepares it to be viewed as countable (Ernest, 2021b).

Once the experienced world is thus construed into repeated units, the foundations of counting can be laid. A further prerequisite to be learned and tacitly agreed is a list of word numerals that must be used in a repeatable order. This list must be both stable, that is invariant, and as at least as long as the number of items to be counted. This is the first of Gelman and Gallistel's (1978) five counting principles, (1) The stable-order principle.

The other principles are as follows.

2. The one-one principle – this requires the assignment of one, and only one, distinct counting word to each of the items to be counted.
3. The cardinal principle – this states that, on condition that the one-one and stable-order principles have been followed, the number name allocated to the final object in a collection represents the number of items in that collection.
4. The abstraction principle – this allows that the preceding principles can be applied to any collection of objects, whether tangible or not.
5. The order-irrelevance principle – this involves the knowledge that the order in which items are counted is irrelevant.

These principles are something that a child must learn in their schooling or early home life. But although often viewed as knowledge, they are deontic social agreements. A quasi-counting activity must conform to them, or it is not socially acceptable. Entering a game or any social practice requires conforming to the rules and regulations of that activity as a participant. The rules are compulsory. They are expressed in the deontic modality that indicates how behaviours must be, to accord with the relevant norms.

The five counting principles listed here are part of the social agreements about what constitutes counting and ultimately regulates what numbers are. All mathematicians will adhere to such agreements, but they are so basic, so deeply entrenched, that with familiarity they seem obvious, unnecessary, and not needing to be articulated. Rules and agreements like these become subsumed into the perceived essence of counting actions and number objects. Starting as necessary features of counting they become seen as features of numbers, intrinsic properties of the reified mathematical objects themselves. Only when someone like Cantor introduces his theory of transfinite numbers is the one-one principle made explicit, jolted back into focus, and considered in the light of the new problematic theoretical context, the equipollence of infinite sets. Otherwise, the one-one principle in counting is seen as intrinsic and definitional, rather than a norm that is (must be) followed.

One of the most valuable and remarkable features of mathematics is how the rich, deep, and complex concepts and objects come into being from simpler objects and actions. This allows the dizzy heights of abstraction to be scaled and objects to be created that exceed by so much what we perceive and experience in the material world, such as the concept of infinite sets. However, one cost of such repeated objectification and abstraction processes is that the rules and social agreements that determine the nature and limits of lower-level objects, concepts, and actions become perceived as essential characteristics of the more abstract objects created from them. The social agreements that shape and constitute arithmetic, for example, become hidden, forgotten, and indeed eventually denied as being the social agreements underpinning number. It is not that their observance is breached, but that they are seem as so essential that they become regarded as intrinsic to the constitution of the objects. Many mathematicians and philosophers state that the natural numbers are something given to humankind by nature (Penrose, 2004). The relationships, extrinsic constraints, and norms that govern their proper and permissible usages become seen as intrinsic properties of the objects in themselves. The social agreements that give shape to objects of mathematics become seen as inscribed in the essence and very being of the objects. The intellectual struggles of humankind over millennia to create counting and numeration systems are no longer seen as processes that through their notational inventions, their actions and conceptions, created what are now seen as the independent objects, the natural numbers. Even their name suggests that these numbers are natural, that is, given by nature, rather than the outcomes of processes of social construction based on imposed rules and norms.

My claim is that in this way social agreements play a constitutional role in mathematical objects. Cole (2009, p. 9) proposes 'The thesis that mathematical entities—specifically mathematical domains—are pure constitutive social constructs constituted by mathematical practices, i.e. the rationally constrained social activities of mathematicians'. In other words, mathematical objects are social constructs, built up from the socially enacted and socially warranted actions described above, and founded on the social agreements of the community of mathematicians. These agreements are expressions of the deontic nature of mathematical practices and are manifested in conforming to their rules and norms. Many, if not most, of these agreements are tacit, agreements in forms of life, as in mutually aligned mathematical practices, not as explicit verbal agreements.

1.2.6 Signs as Constitutive of Mathematical Objects

I have argued that the signs of mathematics play a constitutive role in the formation of mathematical objects. Actions on signs and objects become the next level of abstract objects, themselves depicted by signs. However, it should be made clear that not all mathematical objects are named by signs. Sometimes abstractions create whole classes of mathematical objects. For example, abstracting the set of Natural numbers 1, 2, 3, 4, 5, 6, … into a completed whole named 'N' does not result in an infinite number of names for all the members of N. We have a procedure for naming arbitrarily large natural numbers, but we can never name more than the members of a finite subset of N. Likewise mathematical abstraction creates many sets and classes of mathematical objects which can never all be named. Only a finite number of these mathematical objects can be named, even when the set to which they all belong is named. Thus abstraction, generalization and, in particular, the idealized completion of sets, sequences, and series cannot name all their members when they are infinite.[6]

Quine (1969) argues that the ontological commitment of any theory, mathematical or scientific, is to the domains of objects over which its variables range. Thus, Peano arithmetic, a scientific canonization of the rules of arithmetic, is ontologically committed to that which the variable n ranges over. This domain is N, the set of all Natural numbers, and so Peano's theory is committed to the existence of all of the Natural numbers. Zermelo–Fraenkel set theory is committed to the existence of all of the sets its variable x ranges over. This class of sets is very large and contains sets of several orders of infinite magnitude. In both of these examples, our ontological commitments, the classes of mathematical objects that the theories incorporate or bring into being includes many, many objects that cannot ever be all named. In both the cases of N and V,[7] the universe of sets created by Zermelo–Fraenkel (ZFC) set theory, the global mathematical object constructed is a mathematical domain, a space containing many mathematical objects. These are themselves mathematical objects that encapsulate the endless processes of generating their members, each becoming a single entity within the space of mathematics.

1.2.7 The Human Construction of Mathematical Objects

I have argued that mathematical objects are formed through actions on mathematical objects and signs which are then abstracted and reified into higher level mathematical concepts and objects. Notice that the verbs involved are all active: 'to abstract', 'to reify', 'to act', which all represent the action of a subject on an object.

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[6] This answers the criticism of Cole (2008) that names cannot be constituent of mathematical objects because there are too many objects to be named.

[7] V is the von Neumann class of hereditary well-founded sets.

The subjects in question are irrefutably humans. It is their (our) activities that create mathematical objects, for mathematical objects do not create themselves. Their performativity lies in the capacity of mathematical objects to engender actions and changes through the humans that use them; they are not possessed of any intrinsic or self-subsistent agency. It is humans who perform all these actions, and it is humans that abstract from these actions to create new mathematical objects.

Although mathematical objects are real, their reality is part of cultural activity and its products. Like all of culture, from money, clothing and cookery to languages, movies and ideas, mathematical objects are cultural objects. They are created in mathematical activities, which to a large extent can be represented as language games that take place within mathematical forms of life (Wittgenstein, 1953). Humans acting socially, within mathematical forms of life or mathematical practices, over time, are what create, enact, develop, and sustain mathematical processes, concepts, and objects. This is why the assumptions of presentism are problematic. They deny or disregard the passage of time which is ineliminable in the emergence and being of mathematical objects. Thus, for example, Endress (2016, p. 130) critiques John Searle's (1995) account of social construction because 'his entire work fails to answer or even discuss the question of how the status of "something," as well as its "functions," socially emerge'. This may not be Searle's focus but his analyses do partially indicate how the physical comes to have social function and so be socially constructed. However, unless one understands their becoming, the transitions in the formation of any social constructed entity, including mathematical objects, with its shift from process to structural object, one cannot fully understand what they are. Transitions and shifts occur over time, and these affect the constitution of the emergent mathematical objects, so time and becoming cannot be dispensed with.

Time is implicated in mathematics in three ways. First, there is historical time over which the mathematics in cultures comes to be and develops. I have considered in passing how counting and numeration systems have developed from oral counting, tally marks, and then written numerals of increasing complexity and sophistication. Second, there is personal time in which a person's knowledge of mathematics and grasp of its objects develops. I will say more about this in the next section, but development over time in this domain is undeniable.

Third, there is the foundational analogue of time, the logical development, over the course of which, starting with primitive notions, the theoretical framework of a mathematical theory develops. This last is not real time, but a strong analogue because of the logical before and after relations.[8] Concepts, definitions, results, and proofs are built up in a logical sequence when the later elements depend logically on the former ones.[9]

[8] Lakatos (1976) points out that the 'logical time' of justification often subverts the 'chronological time' of discovery, when the presentation of a completed proof inverts the order in which it was created.

[9] I reject a possible fourth aspect of time. This is the consideration of mathematics as having universal validity across time and space, as this contradicts the sequential emergence of mathematics, as well as the social constructionist assumptions of this chapter.

This is similar to Lakoff and Nunez's (2000) conceptual metaphor that maps from an image schema for temporal succession into the more abstract domain of logic. They call this the *logical consequence is temporal succession* metaphor.

Consider how Peano arithmetic begins with two primitive notions, a starting number, 0 say (historically Peano started with 1), and the successor function denoted by S such that S(n) is the successor of n (Peano used '+1'). It also includes a number of axioms. These make the following five assertions. Zero is a natural number. Every natural number has a successor in the natural numbers. Zero is not the successor of any natural number. If the successor of two natural numbers is the same, then the two original numbers are the same. Lastly, there is the induction axiom.[10] If 0 has a certain property, and whenever n has that property, so does $n + 1$, then all of the natural numbers have that property. There are also the three standard identity axioms (reflexivity, symmetry, and transitivity) specifying the properties of the equality relation (=).

On the basis of these small beginnings, the operations of addition, multiplication, and exponentiation can be defined as well as subtraction and division in the limited ways they apply to natural numbers. From this modest foundation, number theory can now be built up with increasingly complex concepts, functions, and theorems. Ontologically, it can be said that the initial axioms bring the natural numbers into being, within the formal theory, but as the theory progresses, new objects corresponding to the subsequently defined concepts and functions are also brought into being.

A realist, either a Platonist or another kind of realist, can respond to these assertions with the answer that Peano arithmetic does not create the natural numbers but merely provides an elegant and minimal axiomatization of the properties and assumptions underpinning the already existent natural numbers. Over history and in personal development, this is true. The historical growth in the formulation of the natural numbers and number theory does precede Peano's axiomatization. Foundationally this is not true, the simpler parts of the theory logically precede the more complex and dependent later parts. There is an irreversible flow, if not of time, of its logical analogue from the simpler to the more complex later parts of the theory.

My argument is that we also need to allow for time in philosophy, and in particular, in ontology. Both the objects of mathematics and the mathematical identities of persons, that I consider in the next section, grow and change their characters over time, they are subject to processes of becoming. Ontology needs to permit emergence and change in the entities for which it accounts. A static snapshot of being will not suffice to explain of what it is formed.

[10] If P is a subset of N, and 0 belongs to P, and if n belonging to P implies *S(n)* also belongs to P, then P=N.

1.2.8 The Ontology of Mathematical Objects

At this point in the exposition, I have now introduced all the elements that make up the social constructionist account of the ontology of mathematical objects.

What constitutes mathematical objects? Mathematical objects are abstract objects in shared cultural space (the space of mathematics) constituted by rules and agreements established in and by the community of mathematicians, many of which are also sustained and upheld in wider society. These rules and agreements include the tacit rules and conventions into which mathematicians are socialized as they participate in the shared social practices of mathematics. In Wittgensteinian (1953) terms, these include agreement in forms of life in which the actions and practices of participants are aligned, that is, run in the same direction, often without this direction ever being explicitly articulated. Mathematical objects gain their legitimacy through usage, for every instance of use confirms their validity, both among mathematicians and in wider society. In addition, the patterns of, and connections within, their usage also gives them their meanings. Mathematical domains are also objects of mathematics, even if they are populated with an infinite number of mathematical objects which cannot all be named.

What 'stuff' are mathematical objects made from? Mathematical objects are reifications built from abstracted actions on simpler mathematical objects and actions. Humans have a capacity for and a tendency towards nominalization. Just as nouns are created in the nominalization of verbs describing actions, so too mathematical objects are created from the nominalization of mathematical actions. This is a process of reification, encapsulation, and transformation in which actions become structural objects (Sfard, 1994; Dubinsky, 1991). Furthermore, this process is cumulative with increasing levels of abstraction, as actions on simpler objects become more complex objects in themselves.

Where are mathematical objects to be found? Mathematical objects exist in the cultural space of mathematics, a shared domain of signs and operations, whose rule-governed uses provide their meanings. This domain is primarily added and used by mathematicians, but also widely accessed by the public for simple constructs like numbers, whose constitution links them to actions in the empirical world.

Why are mathematical objects objective? Mathematical objects are objective because at any given time they appear 'solid' (inflexible and invariant) founded on mathematicians' agreements and fixed, publicly shared uses in the domain of mathematics and beyond. Their uses are rule governed and there is widespread agreement without ambiguity as to correct usage. Once created mathematical objects 'detach from their originator' (Hersh, 1997, p. 16) becoming independent and self-subsistent entities within a shared domain, the cultural space of mathematics. However, if mathematical practices shift over time, so too may the rules and objects of mathematics themselves, reflecting such cultural shifts.

Why are mathematical objects and their relationships viewed as necessary? The necessity arises from the deontic nature of the rules of mathematics. Mathematicians' agreements are often tacit, being obligations assumed with participation in mathematical practices, and these determine what 'must be so'. The rules are imperatives, analogous to those that must be followed in order to play chess. To engage in mathematical activity, you *must* use the objects of mathematics in the prescribed ways. Mathematics and mathematical entities are non-contingent because they necessarily conform to and obey the rules, customs, and conventions of mathematics. Furthermore, mathematical results and theorems necessarily follow by logic from the axioms and assumptions laid down in mathematical theories.[11] Logic also rests on deontic necessity, for it follows laid down and inflexible tracks of reasoning that, it is accepted, 'must be so' (Ernest, In preparation).

Why do mathematical rules have the modal status of necessity? Mathematical rules and customs make up the institution of mathematics. The institutionalization of social processes such as mathematical practices grows out of the habitualization and customs, gained through mutual observation with subsequent mutual tacit agreement on the 'way of doing things' in these practices. Thus, to engage in a mathematical practice is to be habituated into the norms, customs, and uses of the rules and to follow and apply them unquestioningly as imperatives. Associated with institutions such as mathematics are a set of beliefs that 'everybody knows' (e.g. 'there is a set of natural numbers {1, 2, 3, 4, …}', 1 + 1 = 2, 50 + 50 = 100, 9 > 8, and so on). These beliefs make the institutionalized structure plausible and acceptable, thus providing legitimation for the necessity of the institution of mathematics (Berger & Luckmann, 1966). Much of the language of mathematical texts is imperative, in the deontic modality, as engaging in the practice ('playing the game') necessitates following the tacit and explicit rules and norms embedded in, and constituting the institution of, mathematics (Ernest, In preparation).

1.3 Human Subjects and Mathematical Identities

1.3.1 Being in Terms of Mathematical Identity

In this section, I aim to address and tentatively answer the ontological problem of mathematics education mentioned above. This problem concerns persons, for in mathematics education, the primary concern is with human beings, both learners and teachers. What is the nature of a living, thinking human being? We know it is (we are all) a biological animal but are there any special features of a human being that pertain to mathematics and its teaching and learning mathematics?

Here, my concern is what I term mathematical identity. By this I mean those acquired capacities in the child and adult that enable participation in mathematical activities. This could be termed a person's mathematical power or capability. I do

[11] Note that some of the axioms and assumptions that underpin mathematics can be contingent, as they may follow from mathematicians' choices, albeit constrained choices. The same holds for mathematical logic.

not mean the person's self-image, social image, or sense of belonging to a group, be it as a mathematician, mathematics teacher, or mathematics student. These are the sociological senses of the term mathematical identity widely used in mathematics education, such as in Owens (2008).

In considering the nature of individual human beings from the perspective of their capabilities, one must not overlook the social dimension: social practices, social groups, and even social constructions and structures. To think that individual persons exhaust all of social being is to fall into the reductionist traps of individualism. An active social group is more than a set of individuals. It includes a history of interactions with other individuals expressing themselves through actions and speech and reactions to involvement in such activities. The impact of the activities of the groups will be to change the individuals involved to greater or lesser extent. This is the fundamental principle of education, organizing social activities intended to help human beings to develop and become something different.

In this section, I want to consider both a fully grown person, an adult, and a developing person, a child. By looking at these two aspects of humanity, time has already been admitted, because a child develops over time into an adult. I also signalled here and above, in the introduction, that in considering the nature of human being, there is the problem of the ideology of individualism.

1.3.2 The Ideology of Individualism

The ideology of individualism is a perspective that puts individuals first. Not only is it a social theory favouring freedom of action for individuals over collective action, social responsibilities, or state control. It also positions the individual as ontologically prior to the social. Individualism may be related to the top-down position of the modernist metanarrative in which the 'gaze' of a reasoning Cartesian subject with its legitimating rational discourse is assumed to precede all knowledge and philosophy. The rational knower comes first and is a universal intelligence that is embodied to a greater or lesser extent in individual humans (Scheman, 1983). Modern individualism acknowledges that human beings are embodied, and we are more than just knowers for we also have drives. Our primary motivations are to seek our own survival and the satisfaction of our own, individual desires. Individualism validates this ethical self-centeredness.

According to the individualistic view, humans are entirely separate and independently living creatures (Rand, 1961). Although it is conceded that our independence is not wholly complete, because we do depend on each other for help in survival, nevertheless individualism emphasizes that we are autonomous, self-motivated, agentic creatures who have a great deal of freedom in choosing how we act and behave. Our capacities for understanding, knowing, thinking, and feeling belong to ourselves as individuals and to ourselves alone. Our consciousness is independent, unique, and unconnected with that of other people (Soares, 2018).

Individualism underpins various modern theories such as Piaget's genetic epistemology. According to Piaget, children develop individually following a number of inscribed stages in their growing understanding and capacities. There is an inbuilt logic to cognitive development, perhaps analogous with how a living organism grows, directed by its internal genetic programme. Thus, persons are all existentially separated creatures whose actions, learning, and even whose being take place in hermetically sealed personal domains. The social and physical environment may help or hinder a person's development, just like water, nutrients, and being located in a sunny place will help a plant to grow. But the endpoint or goal of growth is internally encoded and driven.

My criticism of this perspective is that it radically underestimates our ontological dependence on other fellow human beings (Lukes, 1968). First of all, we originate inside another human's body, our mother's, and cannot survive physically without close proximity to and regular attention from a primary caregiver including, but not limited to, feeding. Beyond physical survival our mental, emotional, and personality development requires caring attention for the first decade or two of the years of our lives (Lewis et al., 2000). That attention includes many thousands of hours of involvement with others in social activities through which we acquire spoken language, or an equivalent, and other aspects of cultural knowledge. The mechanism by means of which we make our needs known and receive assurances is conversation, understood broadly. This includes the pre-verbal enacted forms of conversation involving touching, holding, crying, pre-verbal vocalization, facial expressions, gestures, and other embodied actions. It is through such means that we learn the use of words and language. We also develop our identities as persons and our emotional being by these means. Thus, my objection to individualism is that although as primates we are separate animals, as humans we are socially constituted beings. Our very formation and becoming human depends essentially on the social experiences that shape us. We would lack our special human characteristics of shared languages, shared cultures and shared modes of thinking and being, were we hived off from each other in the way that individualism supposes. Our identities are socially constructed, and we could not be the full human beings that we are if we were not socialized and enculturated.

1.3.3 Conversation and the Social Construction of Persons

Ontologically, I want to distinguish between the biological genesis of the human animal and the cultural genesis of the human being as a person. Obviously, the animal provides the material and biological basis of being human, but my claim is that building on that basis, the human being needs to be socially constructed. At the heart of social constructionism lies the dialogical pattern of interactions and knowledge growth and warranting. The unit of analysis, the fundamental atom upon which social constructionism is built, is that of conversation. In its minimal manifestation, this occurs between two persons, who are communicating as participants in a jointly shared social activity, in a social context. There is a continuum of contexts in which conversations take place from face-to face preverbal and verbal interactions all the way to the mediated conversations using letters, emails, and other forms of media over extended distances and timespans.[12]

Conversation and dialogue are widely occurring and utilized notions across philosophy and the social sciences. For the philosopher Mead (1934), conversation is central to human being, mind and thinking. Rorty (1979) uses the concept of conversation as a basis for his epistemology. Wittgenstein's (1953) key idea of language games situated in forms of life is evidently conversational, and I draw on this heavily. Many other philosophers and theorists could be cited, including Gadamer, Habermas, Buber, Bakhtin, Volosinov, Vygotsky, Berger, and Luckmann, and more generally, social constructionists.

> Central to the social constructionist ontology is the view (shared with Gergen and Harré) that the primary human reality is conversation. (Shotter, 1993, pp. 13)

Because of the evidently interpersonal nature of teaching, references to conversation and dialogue are very widespread in the mathematics education literature. However, concerning the philosophy and foundations of mathematics, the references are more limited. But there is growing attention to conversational, dialogical, and dialectical interpretations and philosophies of mathematics (Ernest, 1994, 1998; Dutilh Novaes, 2021; Larvor, 2001).

The original form of conversation is evidently interpersonal dialogue, which consists of persons exchanging speech, or other constellations of signs generated or uttered during the period of contact, based on shared experiences, understandings, interests, values, respect, activities, demands, orders, etc. Thus, in Wittgensteinian terms, it is comprised of language games situated in human forms of life. 'One may view the individual's everyday life in terms of the working away of a conversational apparatus that ongoingly maintains, modifies and reconstructs his subjective reality' (Berger & Luckmann, 1966 p.152).

Two secondary forms of conversation are derived from this most immediate and primary form. First, there is intrapersonal conversation, that is thought as constituted and formed by conversation. According to this view, (verbal) thinking is an originally internalized conversation with an imagined other (Vygotsky, 1978; Mead, 1934). Intrapersonal conversation becomes much more than 'words in the mind', and the conversational roles of proponent and critic discussed below are internalized, becoming part of one's mental functions (Ernest & Sfard, 2018).

Second, there is cultural conversation, which is an extended variant, consisting of the creation and exchange of texts at a distance in embodied material form. I am thinking primarily of chains of correspondence be they made up of letters, papers, email messages, transmitted diagrams, and so forth, exchanged between persons. Such conversations can be extended over years, lifetimes even. It can be argued that

[12] In this chapter I use dialogue and conversation interchangeably. Some authors use 'dialogue' to mean a democratic and ethically more valuable type of conversation. Here I am using these terms descriptively without prescriptive attribution of greater ethical value to one over the other.

they extend beyond a single person's lifetime, if new persons join in and maintain and extend the conversation. Indeed, human culture made up of the ideas, texts, and artefacts made and shared and exchanged by people over millennia has been termed the 'great conversation' (Hutchins, 1959; Oakshott, 1967).

These three forms of conversation are all social. They are either social in their manifestation, in the case of interpersonal and cultural conversations, or social in nature and origin, as is the case with intrapersonal conversation. In the latter, thinking is a conversation one has with oneself, based on one's experience of, and participation in, interpersonal conversations (Sfard, 2008). In all manifestations, stemming from its interpersonal origins, conversation has an underlying dialogical form of ebb and flow, comprised of the alternation of voices in one register followed by another in the same register or of assertion and counter assertion. Conversations result in affirmation and bonding, unless the responses are in the less common forms of negation, refutation, rejection, or the silencing of a speaker. Just cooperation in the form of keeping the channel open provides feelings of enhancement for the speakers. More generally, a fully extended concept of interpersonal conversation including non-verbal communication, mimesis and touch encompasses all of human interaction and is the basis for all social cohesion, identity formation, and culture.

In addition, I wish to claim that all human knowledge and knowing are conversational, including mathematics. Elsewhere I have described the specific features of mathematics that support this analysis, namely that many mathematical concepts are at base conversational, as are the processes of discovery and justification of mathematical knowledge (Ernest, 1994, 1998). However, a word of caution is needed before I further develop conversational theory. Although mathematics is at its root conversational, it is also the discipline *par excellence* which hides its dialogical nature under its monological appearance. Research mathematics texts expunge all traces of multiple voices, and human authorship is concealed behind a rhetoric of objectivity and impersonality. This is why the claim that mathematics is conversational might seem so surprising. It is well hidden, and it subverts the traditional view of mathematics as disembodied, superhuman, monolithic, certain, and eternally true.

1.3.4 The Critical Roles of Proponent and Responder in Conversation

Conversation is the basis of all feedback, whether it be in the form of acceptance, elaboration, reaction, asking for reasons, correction, and criticism. Such feedback is in fact essential for all human knowledge growth and learning. In performing such functions, the different conversational roles include the two main forms of proponent and critic, which occur in all of the modes (inter, intra, cultural), but originate in the interpersonal.

The role of the proponent lies in initiation, reaching out, putting forward an idea or emergent sequence of ideas, a line of thinking, a narrative, a thought experiment, or a reasoned argument. The aim is to share feelings, make demands, communicate an idea, build understanding, or convince the listener (Peirce, 1931-58; Rotman, 1993). Elsewhere I have described how this is the mechanism underpinning the construction of new mathematical knowledge (Ernest, 1994, 1998).

In contrast, there is the role of responder, including critic, in which an utterance or communicative act is responded to in terms of acknowledgement of its comprehensibility, acting in response to a request, actively demonstrating a shared understanding, providing an elaboration of the content, or in other ways. In the role of the critic, the action or utterance may be responded to in terms indicating weaknesses in its understandability and meaning, its weaknesses as a proposal, its syntactic flaws, how it transgresses shared rules, and so on. Critical responses need not be negative, and the role of the responder includes that of friendly listener following a line of thinking, narrative, or a thought experiment sympathetically in order to understand and appreciate it, and perhaps offer suggestions for its extension variation or improvement.

In conversation, ideally the voices or inputs of the proponent and critic alternate in a dialectical see-saw or waltz pattern. In its most rational or mature discursive mode, the proponent puts forward a thesis. The critic responds with a critical antithesis. Third, the proponent, prompted by the critic, modifies the thesis and puts forward a synthesis, a correction, or replacement that is the new thesis in the next iteration of the cycle. Thus, we have a dialectical process approximating the thesis-antithesis-synthesis pattern. In this cycle, the speed of the iterations can vary greatly. In a face-to-face conversation about a mathematical problem at a whiteboard, there can be many mathematical back-and-forth contributions in the space of an hour. But in submitting a mathematics paper to a teacher or journal, it may be that weeks or months pass before critical feedback is received.

From the outset, or nearly so, persons will adopt both the positions of proponent and critic, sometimes within the same conversation. This can also be the case with intrapersonal conversations in which, say, someone thinking about anything, such as a mathematical problem, having internalized these roles alternates between proponent and self-critic.

These two roles are widely present in teaching (teacher/expositor vs. examiner/assessor) and learning (listener/engagement with learning tasks vs. responder/reviser following formative assessment). Indeed, my claim is that these two roles reappear throughout all human social interactions in the form of communicator and responder, although not all elements of conversation need necessarily fall into these two categories.

1.3.5 The Significance of Conversation in Social Activity

Wittgenstein (1953) interprets human living in terms of his fundamental concepts, language games, and forms of life. Language games can be understood as conversations, and these are embedded in human forms of life, that is as social activities. Every social activity has a purpose, a goal, and language and conversation are communicative techniques for working together towards that goal. Examples include mothering an infant with the goal of the infant flourishing (holding, feeding, responding, etc.), working together in a carpentry workshop with the aim of building furniture, working mathematical problems in a classroom with the goal of learning mathematics, and so on.[13] In all such activities, the goal of the activity comes first, and the ways of working, the conversational communications are all about furthering the goal. In such activities, both roles of conversation are important. Conversation can help to focus attention, bond the participants, and direct activities.[14] In this context, the role of responder or critic is vital. When a colleague or more expert participant demonstrates or suggests a way of working or guides the other utterances of the sort 'like this, not like that' embody the role of the critic. This can take the form of Show, Copy, Guide (correction). The teacher shows an action, the learner copies the action, and the teacher guides and corrects the action. By teacher, I mean anybody in the role of guiding partner and correcting responder, whether they be a parent, peer, schoolteacher, workmate, or trainer; in short, the more knowledgeable the other within the learner's Zone of Proximal Development (Vygotsky, 1978). Given that meaning is largely given by use, following Wittgenstein (1953), through guiding and correcting use, the more knowledgeable other is shaping the associated meanings for the learner.

I want to stress that this process, this mode of interaction, is vital in learning how to conduct any practice. This is not only true in concrete production practices, such as carpentry, baking, building brick walls, and so on. It is the mechanism by mean of which all social rules are communicated. These rules include the correct use of spoken language, modes of acceptable behaviour in public, how to treat people, animals and things (ethics in practice), mathematical activities, and so on. This conversational mechanism is how the rules and agreements that make up social institutions are communicated and maintained. Some rules and agreements may have explicit linguistic formulations, like laws of the land, or mathematical axioms, but by far the majority of socially accepted rules and agreements are implicit and are learned through copying others' performances and the novice's own corrected usage.

[13] This is not to deny that persons working together in a shared practice can have different goals, such as reluctant student not fully participating in the classroom practice that the teacher is directing.

[14] Conversation can also be used to further separate the interlocutors by asserting and reinforcing power and status differences, such as a teacher imposing order on an unruly class.

1.3.6 The Realities We Inhabit

I want to start with the assumption that all humans share some indescribable underlying realities. My claims about this shared reality are ontological not epistemological, we can never fully know these realities, but we learn how to operate in them. In some unchallengeable pre-scientific and pre-philosophical sense, human beings all have the experience of living together on the Earth. As a common species, we have comparable bodily functions and experiences that make our sense of being who we are and of daily life commensurable. In virtually all cases, these shared realities are in fact the social activities in which we participate.

Heidegger's (1962) view is that we all have a given, 'thrown' preconceptualized experience of being an embodied person living in some sort of society. He celebrates authenticity, our 'being-here-now' existence (*Dasein*), an attitude that acknowledges our multiple existence in the linked but disparate worlds of our experience: the bodily, mundane, discursive, political, professional, institutional, and cultural realms. Our experience in these social and worldly forms-of-life is taken for granted. It provides the grounds on which all knowing and philosophy begins, although no essential knowledge or interpretation of the basal lived reality is either assumed or possible. This is a bottom-up perspective that contrasts with the top-down position of modernist metanarratives in which a legitimating rational discourse and the 'gaze' of a reasoning Cartesian subject is assumed to precede all knowledge and philosophy. The rational knower does not come first, he (and I use the masculine deliberately) is not a universal disembodied intelligence, but a construction with historically shaped sensibilities. Once again, this illustrates the need to accommodate growth, development, emergence, and becoming in ontology.

Virtually all of our capacities are shaped by the social practices in which we participate. We could not learn, understand, use, or make mathematics unless we were educated, language and sign using social beings with personal histories and mathematical learning trajectories. Like every (academic or school) subject, mathematical knowledge requires an already present knower, and this must be a fleshy, embodied human being with both developmental history (including an educational history) and a social presence and location. Obvious as these statements are, they have been ruled irrelevant and inadmissible by generations of philosophers and mathematicians that subscribe to an absolutist or Platonist philosophies of mathematics.

Paramount amongst the realities we inhabit are the social institutions of which we are a part. Our understandings, as evidenced by the ways in which we participate, are shaped by long histories of conversational exchanges, situated in various social practices. These formative conversations have not only inducted us as participating entrants to, and members of, the institutional practices and domains. They have also shaped our actions so as to be maintainers and onwards developers of the social institutions. Social institutions and social realities are kept alive and afloat (metaphors for continuing and enduring existence) by the myriad actions of the participants in reaffirming the conventions and rules intrinsic to the institutions.

These reaffirming actions are directed through conversations with both insiders, that is participant members, and outsiders, demarcating the rules, norms, conventions, and boundaries that define and constitute the institutions and entry to the associated social practices.

1.3.7 Conversation and the Genesis of Thinking

An important part of conversation theory concerns how it is implicated in the genesis of thinking and constitutive of thought. To sketch the genesis of thinking, we start with a human baby with its sense impressions of its world of experience, probably beginning in a rudimentary way during its period of gestation. Some of these sense data originate from outside its own body, such as from light (impacting via seeing), from sound (via hearing), from touch (via skin pressure nerve arousal). Some of these experiences originate from within the baby's own body, such as hunger, bodily discomfort, and what we might see as spontaneous emotions. These two sources of experience are deeply interwoven. The distinction is far from absolute since sensory inputs must be interpreted in both cases. I believe that the baby notices invariants and starts to impose some order, structure, or pattern on its experiences giving rise to what Vygotsky (1978) calls spontaneous concepts. What such concepts are I cannot say precisely but they may not only include regularities in sensations but also regularities in responses such as movements, vocalizations, etc. Undoubtedly, these concepts vary and grow over time; they are not static and need not be constant. The baby is not isolated in this world of experiences, actions, and concepts because the baby is involved in preverbal dialogues, comprising reciprocal actions and what we might call signalling with others, most notably the mother or primary caregiver.

At this stage, it is hard to know what the baby's thinking is like. Presumably there will be 'inner' sensations and experiences such as pleasure and discomfort or displeasure, recognition of familiar persons, objects and experiences, desires, associated emotions and feelings, sensory images recalled from memory. There may well be reactions to familiar and non-familiar persons, objects, and experiences, accompanied by emotions such as interest, curiosity, desire. The baby will also experience negative emotions including anger, anxiety, or fear, in response to such experiences as being startled by sudden loud noises. What the flow of ideas and experiences brought into consciousness is like I cannot say, but I expect it will be led by sensory stimuli, whether external or internal in origin.

Now we move to the next stage, although of course this overlaps with the preverbal phase, and ultimately engulfs it, as I shall argue. Other persons, such as the mother, will start to use words with the baby, beginning a verbal dialogue, accompanying embodied exchanges such as looking, touching, holding, rocking, and so on. There are intermediary phases in the development of language such as the baby babbling in what we can interpret as pretend speech in the 'game of talking'. After some exposure to adult speech, the baby will start to use words back, mummy, daddy, ball, dog, or whatever. The baby starts to use these words in a regular and recognizable way. At this stage, the baby/young child is starting to develop what Vygotsky (1962) terms scientific concepts, which would be better termed social or cultural concepts. The use and mastery of language takes quite a long time and during this time the child develops and uses a growing set of linguistic capabilities. Of course, this development is triggered by engagement in a growing range of activities with accompanying dialogues in different contexts, with different purposes, and with different but overlapping vocabularies.

Somewhat later during childhood, after the acquisition of spoken language, most children also start to learn to read and write, and these encounters with written language may also feed into the development of their thinking. This includes written arithmetic and other parts of mathematics. However, I won't speculate on the impact of reading and writing on thinking beyond its role as an add-on and expansion of spoken language.

A second strand of development concerns attention, which is part of human agency. A baby turns to look at objects or people that interest it or that move and draw their attention. Part of this is following their mother's or caregiver's gaze (Deák, 2015). Of course, other sensory stimuli also capture its attention, sounds, touch, smells, tastes, pain, and so on. As the child develops, its power of self-directed attention grows and becomes increasingly volitional. In addition to choosing what to attend to in its experiential (perceptual) world, the child can choose and initiate its own activities. It can direct its attention to different activities including toys, games, video, TV programmes, touch screens, nature, animals, and other things. One of the most important things that a child attends to is other humans and dialogue. The child attends to many utterances from others and participates in dialogues.

So now the stage is set for me to propose what thinking is or at least might be. According to Vygotsky (1962), the child's spontaneous and scientific (that is, linguistically acquired) concepts meld or at least start to interact and form one inner system of concepts from quite early on. Words and linguistic utterances have been experienced in various contexts and the uses they are put to and the activities they are a part constitute their initial meanings. Young children will have spoken dialogues with themselves in which they may instruct themselves mimicking what they have experienced with more capable, older speakers. After a while, these self-directed conversations become silent, internalized but perhaps visible through sub-vocal lip movements.

On this basis, children's private thinking consists of an inner dialogue the person has with themselves. This is learned from participation in conversations and discourse with others. But this inner dialogue is not just made up of words – it is supplemented by and may even have elements replaced by visual imagery, memory episodes, feelings (emotions, etc.) within the experienced stream of ideas. An associational logic is at play so perceived external persons, objects, or events may trigger associations that become contents in the inner dialogue. Thus thinking, the inner dialogue, may be a string or cluster of meanings, concepts, or reasonings. This may be prompted by external stimuli, such as conversations/speech from someone else, experiences or events in the world, or may be internally generated, such as when I solve a mathematical problem mentally. The stream of ideas, etc., that I experience in thought is multimodal and can involve words and associated concepts, imagery both real and imagined, smell and touch impressions, or memories of them, etc. We also have some control over this internal dialogue, we can choose to remember something, direct our attention to some idea, memory, problem, etc. Of course, things also come unbidden to our thought, either because of some deep unconscious trigger or an association that draws our attention aside or onwards.

Although our thought originates in interpersonal conversation, in becoming intrapersonal conversation, it differs from public speech. For as conversation is internalized, it combines with our preverbal thought, sensory perceptions, visual images, emotions to become a richer multimodal conversation we have with ourselves. All these aspects as well as personal meanings are attached to the words and signs we use. Thus, we can think spatially as well as verbally. Vygotsky (1962) argues that the contents of our mind are not structured the way our speech is. When we engage in social, interpersonal conversations we also communicate multimodally using gestures, expressions, tone of voice, objects, and other props, as well as our oral linguistic utterances.

Our thinking, this internal stream of ideas and thoughts, is a dialogue in three ways. First, learning to speak is by means of participation in dialogue and conversation. So languaging is a process driven by public speech, that is words and speech. These evoke meaningful concepts and reasoning responses in us – their content and form are irrevocably tied in with their origin, that is spoken dialogue or conversation. Vygotsky is often interpreted as saying that speech and dialogue become internalized. This is of course a metaphorical rather than a literal description. Children learn to imitate phrases. I expect they can also imagine the sounds of these utterances subvocally, that is solely in the mind. So, exposure to speech leads to something like speech in the mind.

Second, our streams of thought come in segments. How these are demarcated or segmented varies, but each segment will have a coherent meaning. Each of these thought segments evokes an association or follow on, a response or reply. Thus, we follow each thought by its echo or answer, like question and answer, thus exhibiting the dialogue form. Just as in a spoken dialogue, we have choices as to how to choose/make our replies thus steering the conversation. Likewise in our internal dialogue, we can choose how to follow on a line of thought. Of course, some people with compulsions find it difficult to steer away from a recurrent pattern of thought. Indeed, this can happen to any of us if we are stressed by a difficult situation or conflicting or difficult demands or an unsatisfactory ending to a previous conversation. So, all my general claims must be hedged with caveats because less typical events and cases can always occur.

Third, our thinking is dialogical when we are reacting to an artefact – a piece of writing, painting, a performance, or even someone talking including a lecture. The attended-to part of the artefact is one voice in the conversation and our reactive or reflective thoughts constitute the second voice, which we may or may not utter in public.

Our internal dialogue can have a variety of functions. It might involve planning some action, a solution to a problem, a plan for making something, the development of ideas. This is imagination at work in thought. This may involve all sorts of meanings including concepts, word meanings (associations), visual imagery, practical sequences of actions. However, such planning or creative imagination need not anticipate or take place separately from our activities. For often we can be involved in making something, such as me writing these observations, and not know beyond a hazy idea, if one has that, where our stream of ideas or words – our internal dialogue – is going to lead. Often our next step in the creative process is enacted as the moment arises. It is a choice, often what feels like the right choice, possibly the necessary choice, but made in the moment.

1.3.8 Extending the Meaning as Use Theory

Following Wittgenstein (1953), I have adopted his operationalization that the meaning of a word is in many cases given by its use. However, this needs disambiguation, for 'use' has multiple meanings. The particular use which I make when I utter the word 'red' or 'addition', say, at a specific event within a particular form of life is one such enactment of meaning. But the *system* of use or usage has another meaning. This includes a systematic grammatical theory of usage that describes past correct usages and potentially includes future correct uses or at least the rules that will guide them. De Saussure made this point when distinguishing *Parole* (utterances of spoken language) from *Langue* (the system of language). Specific uses are one thing but systematic patterns of use which entail imperatives about future specific uses are another.

What one can say is that the spoken utterance meaning of use comes first. Use in the systematic, theoretical sense is secondary to specific instances of participation in conversations and making or hearing utterances. (Parole precedes Langue.) There is a history (we all have histories) of language uses, and we all have a set of memories of instances of language uses – our own and others. In addition, these memories will include the corrections we have received, observed, or given, via conversations, which have shaped our capacities for spoken language. In fact, we may not remember many such corrections, but our linguistic know-how will have been shaped by such instances of correction and correct usage, from childhood on. Many persons will not have explicit or full theories of word use, but have the capacity to make and understand meanings from word utterances based on their implicit know-how.

In the present context, the significance is that the meaning of a word as given by a specific utterance or instance of use is only partial. In the broader sense, meaning as use depends on a whole pattern of usage to give a better indication of meaning. This pattern might only be encapsulated in a tacit set of guidelines or intuitions whose function is, in effect, to regulate which uses are correct and which are not.

At this point, Robert Brandom's (2000) inferentialist account of meaning is helpful. For Brandom, the meaning of words and sentences is largely given by their use in language, but it is a central aspect of use, namely the nexus of inferential connections with other words and sentences. For Brandom, the inferentialist meaning of a word or sentence S is its connections through reasoning with antecedents (reasonings leading to) S and its consequences (reasonings that follow from S). These uses are shown through enacted utterances, but meaning reflects past uttered links and is always open towards the future. So, the current meaning of a word or sentence, at any time, is partial and never final, for further patterns of use will supplement the meaning. As Wittgenstein (1978) says, a new proof of a proposition, changes the meaning of the proposition. Adding logical antecedents or consequents to a sentence changes its meaning.

1.3.9 Dialogic Space

I have considered how children acquire language and the ability to communicate meanings. In addition, I have described how children internalize conversation as a basis for their thinking. On this basis, I can now offer an account of the zone in which meanings are communicated and shared, termed dialogic space (Wegerif, 2013; Lambirth, 2015). This is the virtual space in which words, gestures, and signs are uttered, perceived, and responded to. Dialogic space or spaces are both public and private. A conversation between persons has 'visible' multimodal utterances which are public, but also runs through our private spaces of understanding where we attend to the dialogue and create or conjure up associations, narratives, imagery, emotional responses in our reception of the dialogue. We may engage in an intrapersonal dialogue in response.

Figure 1.1 represents some of the basic elements of dialogic space and its participants. I emphasized the key actions with of italics. As participants, through *listening* we pay *attention* to what is being said, understanding it in terms of building the *meaning* links to what we know (the network of words and concepts to which we have personal access). Through understanding we take personal *ownership* of the meaning links to antecedent and consequent expressions in our network of reasoning relations. When we have an *expressive* impulse, we loosely assemble the *idea* or remark and express it as our *chosen* supplement to the dialogue that we *utter*. (Note that our remark is not usually created in private and then uttered. It normally comes into being as it is uttered.) Every participant in the dialogue does this. Participants also own a set of *rules* about how the dialogue should be conducted in terms of participative membership, the appropriate form of contributions, and the conceptual content of contributions. This overall process is illustrated in Fig. 1.1.

Thus, in addition to exploring and developing the ideas under discussion (the content of the dialogue) participants' contributions can also be utterances that are about regulating or policing the dialogue based on rules that should reflect shared values and democratic principles. For example, in a dialogue between friends and colleagues, one or more contributors may intervene about imbalances in contributions, such as some participant speaking too much or another being

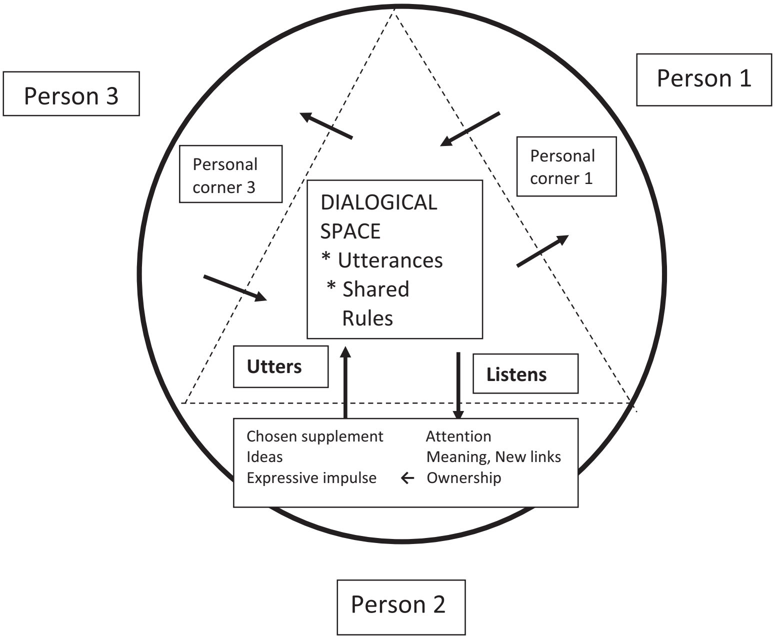

Fig. 1.1 Dialogic space with its personal corners

encouraged to contribute and be attended to. There can also be rules-based utterances on the content of the dialogue, which may be commenting on, redirecting, or curtailing some contribution to steer the direction or thrust of the dialogue in terms of the content and concepts discussed. But the most important part is the pattern of utterances that extend and develop the subject matter, the joint understanding of a topic, the solution of a shared problem, or a creative ensemble made by the group. Not mentioned but underpinning the dialogue is participation in a shared activity (a form of life) which may be purely conversational or may be making or doing something, accompanying the conversation.

1.3.10 Roles and Power Differentials in Conversation

In any dialogue, persons as active agents in that dialogue take on a variety of roles. Two of the most important are speaker and listener. Speaking can involve offering new links that are responses to the previous utterances. Such responses can build on, extend what was previously said. Or they can interrogate and question what was said. Listening can be actively following the narrative and making sense of it through linking utterances with our own concepts and meaning associations. We can follow the flow of a narrative adding our own associations and responses, which we may (or may not) utter audibly or publicly. We can listen critically whereby we interrogate, question, or challenge the narrative as we are hearing it. We can make these reactions public or keep the thoughts to ourselves. And we should never forget that we are embodied, not just passive in listening or active in speaking – we are all the while engaged in bodily activities beyond the actions of communicating (vocalizing, facial expressions, arm, and bodily movements) for we can also be drinking coffee, walking along a road, or even building a model or material artefact together in some shared activity in our joint form of life.

In addition, there are power differentials between contributors in most dialogues, based on personal force or institutional authorization. Table 1.1 lists some sample types of conversation with the relative power of the participants indicated.

Table 1.1 exemplifies the more powerful within institutionalized groups as those, not only with knowledge of the rules (for progressing towards the group goal) but, most importantly, being institutionally authorized to impose the rules in regulating the activity. In informal groups, power is softer and may shift among participants to those with better knowledge of the rules, but without institutional authorization, they may be challenged and have to try to demonstrate the validity of the rules they are suggesting.

Table 1.1 Types of conversation and the relative power of participants

Type of conversation	More powerful participants (MPP)	Less powerful participants (LPP)
Family – Parenting	Parents – more knowledgeable and laying down behavioural rules	Children
Working in learner's zone of proximal development	More knowledgeable parent, teacher, or peer demonstrating rules, etc.	Learner
Friends in discussion	Power may move around group	Power may move around group
Collaborative work on school mathematics problem	Asserter of mathematical rules or moves is MPP at the time	Proposer of next step needing to be regulated (LPP at the time)
Collaborative research project	Power moves around group	Working researchers less powerful if there is principal researcher
Informal conversation between colleagues	Power moves around group unless power hierarchy has been established	Power moves around group unless power hierarchy has been established
School maths class	Teacher directs teaching and the learning activities	Student follows teacher instructions and rules for participation
School maths examination	Examiners	Students (examinees)
University seminar	Visiting lecturer	Audience – but audience can take some power in the questions slot
Journal editorial board	Editor, referees	Author

1.3.11 Mathematical Enculturation

Mathematical enculturation takes place over the course of development from childhood to adulthood. Prior to elementary schooling commencing at 5 to 7 years of age, the child will typically gain a growing mastery of spoken language and very likely engage in simple number and shape games. Typically, these will include learning and using the names of simple geometric shapes (square, circle, triangle, ball, etc.) and spoken number names (one, two, three, four, five, etc.) as well as the correct order of these first few names. There will also very likely be some learning of the single digit numerals (1, 2, 3, 4, 5, etc.).

Such activities will continue in kindergarten and early elementary school plus the introduction of elementary operations, most notably addition and the addition sign '+'. Mathematical activity for learners typically shifts from being wholly spoken, to spoken and textual, with a shift towards the dominance of text for children's activities. Children very likely will engage in enactive activities (counting objects such as buttons), activities presented in iconic forms (working simple tasks mostly shown with repeated pictures, such as simple flower pictures), moving on to symbolic work with texts using words and mathematical symbols.

Perhaps the most central activity in the mathematics classroom is the imposition of mathematics learning tasks on students (Ernest, 2018a). These will be orally or textually presented and may be enacted in a variety of media. But over the years of schooling, throughout elementary and secondary (high school), these will become almost exclusively presented via written texts. A mathematical learning task:

1. Is an activity that is externally imposed or directed by a person or persons in power representing and on behalf of a social institution (e.g. teacher).
2. Is subject to the judgement of the persons in power as to when and whether it is successfully completed.
3. Is a purposeful and directional activity that requires human actions and work in the striving to achieve its goal.
4. Requires learner acceptance of the imposed goal, explicitly or tacitly, in order for the learner to consciously work towards achieving it[15].
5. Requires and consists of working with texts: both reading and writing texts in attempting to achieve the task goal.
6. A mathematical task begins with a mathematical representation (text) and requires the application of mathematical rules to transform the representation, in a series of steps, to a required end form (e.g. in a calculation, the numerical answer).

Power is at work in a mathematical task at two levels. First, at the social level, the teacher imposes the task and requires that it be attempted by the learner. Second, within the task itself, power is at work through the permitted rules and transformations

[15] Gerofsky (1996) adds that tasks, especially 'word problems', also bring with them a set of assumptions about what to attend to and what to ignore among the available meanings.

of the text. In other words, the apprentice mathematician must act as a conduit through which the imperatives of mathematics work. They must follow certain prescribed actions in the correct sequence. As the tasks become more complex, the apprentice mathematician will have some choices as to which rules to apply in constructing the sequence of actions or operations towards the solution, but otherwise all is imperative driven. Mathematics is a rule-driven game, and the rules are a major part of the institution of mathematics.

Later in the process of mathematical enculturation, the institutional rule-based nature of mathematics is internalized, and apprentice mathematicians adopt a more general concept of mathematical task that includes self-imposed tasks that are not externally imposed and not driven by direct power relationships.[16] However, in research mathematicians' work, although tasks may not be individually subject to power relations, particular self-selected and self-imposed tasks may be undertaken within a culture of performativity that requires measurable outputs. So, power relations are at play at a level above that of individual tasks. Even where there is no external pressure to perform, the accomplishment of a self-imposed task requires the internationalization and tacit understanding of the concept of task. Such an understanding includes the roles of assessor and critic, based on the experience of social power relations. This faculty provides the basis for an individual's own judgement as to when a task is successfully completed. Within institutional rule-based mathematics imperatives are at work, the dominant actions (rules) inscribed within the texts themselves. The role of the critic is to judge that the institutional rules of mathematics are applied appropriately and followed faithfully.

Mathematical learning tasks are important because they introduce the learner to the rules of mathematics and its textual imperatives. For this reason, such tasks make up the bulk of school activity in the teaching and learning of mathematics. During most of their mathematics learning careers, which in Britain continues from 5 to 16 years and beyond, students mostly work on textually presented tasks. I estimate that an average British child works on 10,000 to 200,000 tasks during the course of their statutory mathematics education. This estimate is based on the not unrealistic assumptions that children each attempt 5 to 50 tasks per day, and that they have a mathematics class every day of their school career (estimated as 200 days per annum).[17]

A typical school mathematics task concerns the rule-based transformation of text. Such tasks consist of a textual starting point, the task statement. These texts can be presented multimodally, with the inscribed starting point expressed in written language or symbolic form, possibly with illustrative iconic representations or

[16] There are also more open mathematical tasks such as problem solving (choose your own methods) and investigational work (pose your own questions) in school but these are not frequently encountered.

[17] Much of the mathematics education literature concerns optimal teaching approaches intended to enhance cognitive, affective, critical reasoning or social justice gains (prescriptive). Here my concern is just with teaching as a process that enables students to learn mathematics, without problematizing the teaching itself, that is, purely descriptive.

figures. In the classroom, these are typically accompanied by a metatext, spoken instructions from the teacher. Learners carry out such set tasks by writing a sequence of texts, including figures, literal and symbolic inscriptions, ultimately arriving, when successful, at a terminal text which is the required 'answer'. Sometimes this sequence of actions involves the serial inscription of distinct texts. For example, in the case of the addition of two fraction numerals with distinct denominators or the solution of an equation in linear algebra. Sometimes this involves the elaboration or superinscription of a single piece of text, such as the carrying out of 3-digit column addition or the construction of a geometric figure. It can also combine both types of inscriptions. In each of these cases, there is a common structure. The learner is set a task, central to which is an initial text, the specification or starting point of the task. The learner is then required to apply a series of transformations to this text and its derived products, thus generating a finite sequence of texts terminating, when successful, in a final text, the 'answer'. This answer text represents the goal state of the task, which the transformation of signs is intended to attain.[18] In some solution sequences, new texts will be freshly introduced, such as axioms, lemmas, or methods, and therefore are not strictly transformations of the preceding text but play an integral part in the overall sequence.

Formally, a successfully completed mathematical task is a sequential transformation of, say, n texts or signs ('$\boldsymbol{S}_i$') written or otherwise inscribed by the learner, with each text implicitly derived by n-1 rule based transformations ('$\Rightarrow i$').[19] This can be shown as the sequence:

$$S_1^{\Rightarrow_1} S_2^{\Rightarrow_2} S_3^{\Rightarrow_3} \ldots^{\Rightarrow_n -1} S_n$$

$\boldsymbol{S}_1$ is a representation of the task as initially inscribed or recorded by the learner. This may be the text presented in the original task specification. However, the initial given text presenting the task may have been curtailed, or may be represented in some other mode than that given, such as a figure, when first inscribed by the learner. $\boldsymbol{S}_n$ is a representation of the final text, intended to satisfy the goal requirements as interpreted by the learner. The rhetorical requirements and other rules at play within the social context and following mathematical imperatives (the mathematical rules) determine which sign representations $\boldsymbol{S}_k$ and which steps, $\Rightarrow_k$ for $k < n$, are acceptable. Indeed, the selection of mathematical rules applied, and the transformed representations inscribed by the learner, up to and including the final goal representation

[18] I use the word text broadly to include whatever multimodal representations are required in the task including writing, symbolism, diagrams and even 3-D models.

[19] Normally learners of school mathematics are not expected to specify the transformations used. Rather they are implicitly evidenced in the difference between the antecedent and the subsequent text in any adjacent (i.e., transformed) pair of texts in the sequence. In some forms of proof, including some versions of Euclidean geometry not generally included in modern school curricula, a proof requires a double sequence. The first is a standard deductive proof and the second a parallel sequence providing justifications for each step, that is specifications for each deductive rule application. Only in cases like this are the transformations specified explicitly.

($\boldsymbol{S}_n$), are the major focus for negotiation and correction between learner and teacher, both during production and after the completion of the transformational sequence. This focus will be determined according to whether in the given classroom context, the learner is required only to display the terminal text (the answer) or a sequence of transformed texts representing its derivation, whether calculation, problem solution, proof, or application of mathematics (Ernest, 2018a).

The extended apprenticeship in completing many thousands of mathematical tasks over the years of schooling, where successful, represents an enculturation into the social practice of mathematics. For persons going on to use mathematics professionally in their careers, or going on to be professional mathematicians, this apprenticeship is extended and intensified with introduction to more abstract and complex mathematical topics with a greater range of more sophisticated rules, as well as more demanding mathematical tasks. This occurs over the years of college or university specialization in mathematics. The extensive involvement and engagement with mathematical practices and activities or tasks result in a deeper engagement with mathematical rules. Some of these rules become automatic and are identified with the nature of mathematical objects. Imperatives become inscribed in mathematical objects so they cannot be seen as existing without what are deontic rules prescribing their possible uses. The journeyman mathematician accesses a cultural realm of mathematical objects that have all the appearances of solid real objects (within their own realm) whose nature necessitates a limited and prescribed range of properties and powers. These are embodiments of the rules that brought the objects of mathematics into being and limit their possible uses.

At this stage, the student or apprentice mathematician has developed into a practicing mathematician, and signs, symbols, and concepts of mathematics correspond to full independent mathematical objects embodying the restrictions and rules of their possible uses and the community-wide agreements as to their constitutions and operability.

During the pursuit of mathematical activities, mathematicians and others engage in extended work with texts and symbols in dialogic space. This crystallizes into 'math-worlds' where the objects of mathematics have a speaker-independent existence and reality. It is not only these objects whose independent being is confirmed and strengthened. It is also persons' identities as mathematicians that is validated by their access to dialogic space and its population of mathematical objects. However, this description is deceptive, for it is the submission to internalization and absorption of the many rules, norms, and tacit agreements through which mathematical activities and objects are constituted, that makes a mathematician. Just as these institutions bring forth the objects of mathematics, so too they bring forth the special powers and capabilities of the mathematician qua mathematician. Namely, a person empowered mathematically through obedience to the deontics of mathematics. Of course, a mathematician is not beaten down and cowed through this submission. The many thousand-fold experiences of successful pursuit of the goals of mathematical tasks have shaped, sharpened, and directed the desire of the mathematician to answer the questions, solve the problems, pursue the holy grail of proving new theorems. A chess master internalizes the rules of chess and turns them into intuitions of desirable outcomes many moves ahead. Similarly, the mathematician's rule-shaped intuition suggests where actions and processes on mathematical objects in dialogic space may lead.

In this account, conversation provides the epistemic and ontic basis of mathematical knowledge and object existence. It grounds them in physically embodied, socially situated acts of human knowing, communication, and agreement. Because of the tight rules, norms, and conventions, mathematical conversation has minimal of ambiguity compared to every other domain of discourse. Nevertheless, the philosophical bases of mathematics are in the final analysis deontic, resting on the shared explicit rules and hidden norms of mathematical practice, as communicated via conversation. Conversation includes the roles of proponent and critic, and both of these roles are necessary in fruitful conversation, at any level. Their existence is the reason why a mathematician stranded alone on a desert island for 20 years proving theorems is still engaging in a social practice.

1.4 Conclusion

This concludes my treatment of the ontological problems of mathematics and mathematics education. I have argued that mathematical objects are formed out of actions on simpler objects, which are abstracted and reified into self-subsistent objects. All the actions involved in this process are heavily constrained by the rules of mathematics which are entangled with and woven into the objects. The norms and constraints that make mathematical objects possible are necessary elements of their existence. Because of these definitionally necessary limits, the objects are necessary objects. Their necessity is the product of the deontic modality, which in describing mathematical objects, indicates how their world ought to or must be according to the norms and expectations of mathematical culture. Contrary to the traditional view that accounts of mathematical objects are in epistemic or alethic modality expressing possibility, prediction, and truth, the deontic modality of mathematical language indicates an obligation that becomes a necessity. If mathematical objects exist, and they are present to all mathematicians and students of mathematics to a varying degree, then 'this' is how they must be. Here, 'this' refers to the necessary character of mathematical objects as the conventions and rules require them to be.

The ontological problem of mathematics education concerns the nature of mathematicians and students of mathematics. I have argued that the formation of their mathematical identities, which are perhaps only a small part of their overall beings as persons, develop through mathematical enculturation. The key element of this process is subjection to rules, conventions, orders, instructions that must be obeyed, at three levels, during engagement with mathematical activities.

First, there is the social, interpersonal level. In schooling, the teacher sets the tasks and their goals. However, they may be hedged, the teacher issues orders to the children that requires that they engage in the set mathematical activities or tasks.

The teacher also demonstrates and reinforces the rules and solution processes that the learners must use to attempt to achieve these goals. There may be a limited degree of flexibility as in some tasks the learner can select their preferred method of solution from amongst the approved methods or their variations. But overall, this is the level made up of the imperatives issued directly by the teacher in social or interpersonal space.

The second level of necessity is that inscribed within the texts of the tasks. The most common verb forms in mathematics, both in school and research texts, are imperatives requiring the reader to complete the activity in prescribed ways (Ernest, 2018a; Rotman, 1993). Such prescriptions may be tacit, but there is a repertoire of agreed rules and methods to be employed. Here, the key characteristic is that the imperatives are in the text themselves.

Third, there are the tacit and explicit rules and conventions of mathematics that delimit the permitted actions and textual transformations. These are part of the culture of mathematics and a key element of what students and practitioners pick up and internalize as a residue of the myriad conversational exchanges in the dialogic space of mathematics. These make up much of what is termed the knowledge of mathematics, that which is learned through mathematics education. It is these rules that must be selected from and utilized in the performance of mathematical activities and tasks by students of mathematics and mathematicians.

Thus, my two big ontological problems, the nature of mathematical objects and the nature of mathematical identities, with their associated powers, converge. It is the rules and conventions of mathematical culture that help build up and constitute both of these types of entity. The objects of mathematics are abstracted actions encapsulating these rules. Mathematical identities are shaped, constituted, and constrained through the internalizations of these rules.

This convergence in explanations is why an interdisciplinary approach to these problems is necessary. I am tempted to claim that only such a multidisciplinary analysis, drawing on philosophy, linguistics, mathematics education, and other disciplines, can address both of these two problems together. Furthermore, the solutions offered are interdependent and co-constituting. Interacting with mathematical objects is an essential dimension in the construction of mathematical identities. Coming to, becoming and being a mathematician depends essentially on engagement with and using mathematical objects. Conversely, the formation and maintenance of mathematical objects depends on human capacities to actively maintain cultures through extended conversational interactions, including the capacities for abstraction and rule formation. Only through the induction of persons into the culture of mathematics are mathematical identities formed and the culture of mathematics, which is the location of mathematical objects, maintained and extended.

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