Maria Aparecida Viggiani Bicudo · Bronislaw Czarnocha · Maurício Rosa · Małgorzata Marciniak *Editors*

Ongoing Advancements in Philosophy of Mathematics Education

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Preface

Ongoing Advancements in Philosophy of Mathematics Education approaches the philosophy of mathematics education in a forward movement, analyzing, reflecting, and proposing significant contemporary themes in the field of mathematics education. It furthers the proposal of *The Philosophy of Mathematics Education Today*, edited by Paul Ernest, published by Springer, in 2018. The book contains many articles presented and discussed at ICME 13, within TSG 53, Philosophy of Mathematics Education, which took place in Hamburg, Germany, in 2016. Besides those articles, the editor, Paul Ernest, invited other authors to contribute relevant work in the field.

This book, Ongoing Advancements in Philosophy of Mathematics Education contains work presented and discussed during the ICME 14, which took place in 2021 in Shanghai, China, within TSG 56, whose main objective was to focus on the relationship between the philosophy of mathematics and mathematics education. Its goal was to characterize the interaction and dialogue between these areas, including what can be highlighted when one uses the methodology of philosophical research to question the ontology, epistemology, or ethics of mathematics regarding mathematics education, or conversely when one unveils the philosophical outreach of mathematical ideas, concepts, or methodologies, especially in an educational context where mathematical practices may be worked through teaching and learning processes.

ICME 14 was a hybrid event, that is, some people participated in person, others remotely. The event should have taken place in 2020, with all participants on site. However, that became impossible in view of the pandemic that plagued humanity between 2019 and 2020. The original date was thus postponed to July 2021. During the closing section of the activities of TSG 56, the participants proposed that a book be published in order to compile the articles discussed during the activities of TSG. As a TGS 56 Team Member, I have taken on the task of organizing such book in collaboration with Bronisław Czarnocha, Maurício Rosa, and Małgorzata Marciniak. It is important to point out that, as ICME 14 had been scheduled to take place in 2020, the Team Members of TSG 56 were different from those who took over the organization of ICME 2021. For several reasons, all of the members of that team were unable take on TSG 56 in 2021. I, also for health reasons arising from having contracted Coronavirus, also could not continue as chair.

Bronisław Czarnocha graciously accepted the position, and then I acted as co-chair to help him with the articles derived from work conducted previously. With this new structure, the work of TSG 56 was successfully conducted.

The chapters comprising this volume present studies conducted by the authors who participated in the Topic Study Group that focused the theme "Philosophy of Mathematics and Mathematics Education," one of the activities that took place during the International Congress of Mathematics Education – ICME 14, 2021, in Shanghai, China. Also, in this volume, there are chapters written by guest authors and relevant researchers who study this subject on the international scene.

The theme that gives life to the book is philosophy of mathematics education understood as arising from the intertwining between philosophy of mathematics and philosophy of education which, through constant analytical and reflective work regarding teaching and learning practices in mathematics, is materialized in its own discipline, philosophy of mathematics education. This is the field of investigation of the chapters in the book.

The book aims to present to teachers and scholars of the philosophy of mathematics and mathematics education current investigations and didactic proposals by authors who conduct their activities in different countries.

As editor and co-editors, we have managed all the organization, sent invitations, and established schedules for the work necessary so that the book could materialize. We collaborated with the authors as follows: each of the co-editors conducted the revision of a number of chapters; each of the authors revised one of the chapters; Reviews 1 and 2 (R1 and R2) were sent to each respective author, keeping in mind that after considering the observations they could accept or reject them. Each one of the co-editors then conducted a third review of the chapters and implemented the necessary adjustments in cooperation with the authors. The principle that guided the inclusion of chapters was the book's own proposal, previously known by authors. Through our reading and analyses, we focused on the philosophical and educational discussion present in the texts and respective foundations, as well as the internal coherence and logical clarity of the articulations made. We do not take a position on the ideas presented and articulated in each chapter, as we understand that their maintenance is the responsibility of the respective authors.

The different chapters are organized into four parts, which deal important themes concerning the philosophy of mathematical education: Part I – A Broad View of the Philosophy of Mathematics Education; Part II – Philosophy of Mathematics Education: Creativity and Educational Perspectives; Part III – Philosophy of Critical Mathematics Education, Modelling and Education for Sustainable; Part IV – Philosophy of Mathematics Education in Diverse Perspectives, Cultures, and Environments. Those parts are introduced through an exposition of the understanding of the themes treated, written by one of the editors of the book.

Each part is presented below with the sequence of chapters it comprises, the respective author(s), and a brief summary through which the ideas considered are articulated.

Part I – A Broad View of the Philosophy of Mathematics Education

1 *Paul Ernest*

"The Ontological Problems of Mathematics and Mathematics Education" represents a movement in the process of understanding the constitution of mathematical objects and of the mathematician himself, moving beyond the epistemological aspects present in this constitution and focusing on the deontic modality. According to this author, contrary to the traditional view that accounts of mathematical objects are in epistemic or alethic modality expressing possibility, prediction, and truth, the deontic modality of mathematical language indicates an obligation that becomes a necessity. The chapter presents a synthesis of the author's understanding of Mathematics Education, of Mathematics, of Education, pointing out the great disciplinary fields that are interwoven in the constitution of Mathematics Education and presenting its readers with his comprehensions of important and diverse subjects, object of specific disciplines, such as Psychology, Linguistics, Sociology. The authors he mentions are important in the scenario of Philosophy and Mathematics Education.

2 *Michael Otte and Mircea Radu*

Otte and Radu, in "Scientific Revolutions: From Popper to Heisenberg," compare several seminal interpretations of Thomas Kuhn's theory of scientific revolutions such as those of Karl Popper (1902-1994) and Werner Heisenberg (1901-1976), emphasizing two theses of particular importance for the analysis they present. The first states that theories are always underdetermined by the data they are supposed to represent, so that theories appear as realities of their own. The authors explore how this thesis can be upheld and proved fruitful while avoiding a nominalist interpretation. The second thesis concerns a favorite commonplace in philosophical and historical discussions of Renaissance. It states that the key element of the Renaissance was recognizing the individual human subject as the central agency of cultural, social, and economic progress. While discussing Kuhn's conception and its reception, they explore some of the implications of these theses for a better understanding of the debates concerning the development of science, mathematics, society, and indeed education to this day. In this context, it is explained how aesthetic experience emerges as a fundamental ingredient capable of bridging the gap between theory and practice, nature and culture, between individual and social generality, and even between the two theses proposed above. The discussion developed by them proceeds in terms of Peircean semiotics.

3 Maria Aparecida Viggiani Bicudo

Bicudo, in "Questions That Are at the Core of a Mathematics Education Project," presents an essay focused on those questions. The author emphasizes that the project of mathematics education needs to work by realizing philosophical thinking. This chapter outlines a way of understanding the production of mathematics, as understood by the western civilization, and its role in the constitution of scientific and technological thinking present in the world we live in today; it points out the urgency of not succumbing to the loss of meaning of life and of the world, as we are immersed in a sea of explanations and predictions issued and supported by the scientific and technological apparatuses; it is evidenced that mathematics education can contribute to the accomplishment of this task in a unique manner, which is critical and urgent for humanity. In this chapter, a list of themes defined as important and worthy of study and practice is not pointed out. Rather, the arguments intertwined conduct an analytical and reflexive way to exercise and point out understandings regarding the characteristics of scientific and technological work, which is supported by mathematics as understood by the western civilization, defending the premise that within the scope of mathematics education it is necessary to comprehend such characteristics and integrate them into educational practices with ways other cultures work mathematically.

4 Thomas Hausberger and Frédéric Patras

In chapter "Networking Phenomenology and Didactics: Horizons of Didactical Milieus with a Focus on Abstract Algebra," the authors work with Husserlian horizons intertwined with notions from Brousseau's Theory of Didactical Situations (TDS). They present these as tools to analyze the shifts of attention and interconnectedness of knowledge in learners attending to an abstract structure. Then, they go forward in order to encompass a larger spectrum of horizons and methods in a pioneering application in the context of university mathematics education, allowing for a fine-grain analysis of the work of learners engaged in the elaboration of a structuralist mathematical theory around the given structure. At the theoretical level of frameworks, they contribute by combining/coordinating notions from TDS with the perspective of phenomenology, in the spirit of networking. They believe that such a dual framework may be applied in a large variety of contexts and educational levels.

5 Jonh Mason

"Specifying, Defining, Generalizing and Abstracting Mathematically All Seen as Subtly Different Shifts of Attention" by John Mason focusses on mathematical abstraction as a process, in relation to specifying, defining, and generalizing. In the wake of his earlier investigations concerning symbols (signs), when he suggested in 1980 that these entities could be experienced initially as abstract in the sense of being unconnected to other experiences, but over time could become perfectly confidently manipulable, as they were concrete entities, and later, in 1989, when he suggested that abstracting mathematically involves a "delicate" shift, not so much in what is attended to, but in how it is attended to, he has led to point out implications for choices of pedagogical actions to initiate when working with learners. In this chapter, he moves on and proposed that the acts of specifying, defining, and generalizing also involve delicate shifts of attention, subtly different from each other and from abstracting. He argues that through the use of multiple examples, readers are invited to refine the distinctions they make in the form of their own attention, so as to work more effectively with learner attention.

6 Steven Watson

"Toward a Systems Theory Approach to Mathematics Education" by Steven Watson presents a systems approach to thinking about mathematics education. It is an important text where the author aims to introduce mathematics education as a social system to contemporary systems theory. As he states, he outlines some features of the theory itself and the directions and themes which he takes up in his preliminary inquiry into mathematics education. According to him, systems theory facilitates the understanding of the social and cognitive dimensions within a theory of society and, from this, a theory of mathematics education, as a social system of communication.

7 Gerald A. Goldin

Goldin in chapter "On Mathematical Validity and Its Human Origins" suggests the desirability of a fully integrated philosophy of mathematics education that builds on several distinct, mutually compatible foundational pillars. These pillars have their intellectual bases in different philosophical school of thought. The discussion focuses on one aspect of those epistemological foundations – the interplay between the human origins and uses of mathematics, and its objective truth and validity. Various philosophical trends in education have centralized just one of these aspects, often to the extent of denying or dismissing the other. He argues for their compatibility, maintaining that objective mathematical truth and the fact of culturally situated human mathematical invention should both be guiding teaching and learning, with neither diminished in importance. Several meanings given to the "why" of mathematics are discussed – logical and empirical reasons that underlie mathematical truths and relationships, sociocultural and contextual reasons for developing and teaching mathematics, and the in-the-moment experiences that afforded students to motivate their study. Some sources of cultural relativism, historical change, and "fallibility" in mathematics are identified, and the value of "mistakes" in powerful mathematical problem solving is highlighted. His goal is to argue that an intellectually sound philosophy of mathematics education must incorporate all of the aforementioned features of mathematics and its practice, dismissing none.

Part II – Philosophy of Mathematics Education: Creativity and Educational Perspectives

8 Bronisław Czarnocha

"Towards a philosophy of creativity in mathematics education," by Bronisław Czarnocha, addresses a rarely explored area, philosophy of creativity and its relevance to mathematics education. The central question of this chapter – "What can the practice of and research in creativity of mathematics education contribute to the philosophy of creativity in mathematics education and possibly to the philosophy of creativity in general?" – guided the author's presentation and argumentation. The innovative contributions made in the chapter are pertinent to the discussion about creativity, how to define it, how to measure it, how to work with it in mathematics classes.

9 William Baker

In the chapter "A Framework for Creative Insights within Internalization of Mathematics," William Baker analyzes cognitive changes during moments of insight realized within a math classroom. He makes his argument explicit by pointing to previous studies by authors significant to the subject, such as Piaget and Vygotsky. He poses that constructivism is arguably the prevailing theory of mathematics educational research; based upon the work of Piaget, it posits that human knowledge is built up through an individual's reflection and abstraction upon their solution activity. He continues saying that social constructivists, in contrast, frequently use the work of Vygotsky focused on the internalization of knowledge within social discourse. In this chapter, Baker takes up these approaches and integrates them into a framework based upon the work of Koestler to study and analyze cognitive changes during moments of insight realized within a math classroom. Two dominant themes underlying his attempt are: first, classroom discourse is the unit of analysis for both student development and reflection upon pedagogy, and second, Vygotsky's framework while ideally situated for analysis of classroom instruction is lacking in detail of the actual process of internalization.

10 Mitsuru Matsushima

Mitsuru Matsushima's chapter "A Reconsideration of Appropriation from a Sociocultural Perspective" pursues the following questions: Why does interaction in the learning community deepen mathematics learning? How does individual learning contribute to the learning community through dialogue and deepen mathematics learning? He works out with these questions from a sociocultural perspective, considering them jointly. In this chapter, he offers a reconsideration from a sociocultural perspective, basing his discussion on two appropriation features of previous studies, dynamic composition and mutual composition, and an extended sign appropriation and use model. He deepens his investigation about the concept of internalization and articulates ideas in order to answer the questions posed by him: "Why does dialogue deepen mathematics learning?" and "Will mathematics learning deepen without dialogue?".

11 Regina D. Möller and Peter Collignon

"Towards a Philosophy of Algorithms as an Element of Mathematics Education" is the theme focused by the authors in this chapter. They consider that nowadays the concept of algorithms is used in a rather broad range, since many subject matters refer to this notion, and they shape it along the respective desired usefulness or requirements. They argue that algorithms are one of several fundamental mathematical ideas, and they structure the content of math classes throughout the school years, that is, the primary and the secondary levels. They ponder that their roles and their importance for mathematics education have undergone substantial changes especially during the last 30 years. They understand that these changes give reason to investigate and reflect upon this emerging phenomenon and ask for analyzing the contemporary need in actual math classes as a response to everyday life experiences related to algorithms often hidden in technical devices. They go further in the line of their investigations and in this chapter, from a philosophical point of view, they pursue new questions to be considered within the framework of (post-)modernism and within a constructivist approach.

12 Małgorzata Marciniak

Marciniak in "The times of transitions in the modern education" points toward expanded professional development for teacher education, in that she understood that living and working during these times of transitions in the modern education may be extremely confusing for teachers of all levels since the education they received and were taught to provide is not what they are required to perform. As she says, this was particularly exposed during the pandemic when thousands of teachers worldwide were forced to teach remotely regardless of their digital skills. She considers the questions of the character and shape of this development remains open, and so in this work she tries to analyze a few pivoting moments in the history of education to follow up on the ideas of Thomas Kuhn as presented in his book *The Structure of Scientific Revolutions*. Here the discussion is applied to the structure of the revolutions of education with the pandemic being one of them. The question: what will be the long-term influences of the pandemic on the teaching and learning, remains open and fully credible answers can be provided only with time. She will try to answer this question based on short-term recent experiences and observations. Weaving articulations between Kuhn's thinking about scientific revolutions and educational theory is an important point in this chapter.

13 Yenealem Ayalew

"Some Examples of Mathematical Paradoxes with Implications for the Professional Development of Teachers" addresses some clever mathematical paradoxes that challenge or trouble traditional interpretations of mathematical results; it uses examples as evidence for the underlying argument. For instance, it elaborates the sum of one and one with possible results 0,1,2,3,10, or ∞. The discussions were made based on an eclectic position of the philosophies of mathematics. A question on the possibility of bringing qualitative mathematical relations into classroom context is posed in the middle of the text. Generally, the chapter deals on what creative imagination looks like, at the level of mathematics, mathematics teaching, mathematics teacher education, etc. Thus, it appears to be a theoretical exploration of the subjectivity of mathematical development and the professional development of mathematics educators. A multi-stage collaborative work model is also forwarded.

Part III – Philosophy of Critical Mathematics Education, Modelling and Education for Sustainable

14 Ole Skovsmose

Skovsmose's chapter, "A Performative Interpretation of Mathematics," suggests a performative interpretation of mathematics, inspired by a performative interpretation of language. The author states that according to this interpretation, any form of mathematics is intrinsically linked to potential or actual actions. His interpretation, as he says, is elaborated upon with respect to both advanced mathematics and school mathematics. He highlights that any kind of mathematics exercises a symbolic power, which brings to the forefront the ethical dimension of a philosophy for mathematics. Any kind of action is in need of ethical reflections, and so specifically are mathematics-based actions. He points out that ethical reflections concern the possible impact of mathematics, the different groups of people that might be affected by such actions, the possible acting subjects that might be hidden behind the curtain of mathematics, the possible intentions behind the action, and the ethical reflections themselves. He concludes that, taken together, ethical reflections concern how symbolic acts rooted in mathematics might form our life-worlds.

15 Nadia Stoyanova Kennedy

The chapter "Reflective Knowing in the Mathematics Classroom: The Potential of Philosophical Inquiry for Critical Mathematics Education," by Nadia Stoyanova Kennedy, brings philosophizing to the very activity of teaching and learning mathematics. The text explains how to understand the process of philosophizing, which includes analytical and reflective thinking about issues concerning the way of producing mathematical knowledge, advancing to questions of ethical and epistemological nature. The author proposes ways for the teacher to work in the classroom with her students, keeping the dialogue alive in a collaborative way. The advance goes beyond the clear explicitness of the understanding of philosophizing, as it goes into the details of how and what can be worked out in the classroom.

16 Uwe Schürmann

Schürmann, in chapter "Mathematical Modelling: A Philosophy of Science Perspective," questions the separation between mathematics and reality or the "rest of the world," which is often found in mathematics education research on modelling, against the background of the syntactic and the semantic view of models and theories. He argues that the syntactic view more accurately captures the connection between mathematics and reality by distinguishing between theoretical and observational terms of a theory. Moreover, as he points out, the distinction provides a tool to analyze students' utterances more precisely. He links (analytic) philosophy of science and mathematics education research on mathematical modelling in the classroom. He does so because as this subject matter is not directly related to philosophical findings on models in sciences, it is not directly concerned with questions of a philosophical nature. Even so, it always deals with reality and so it carries underlying ontological questions.

17 Hui Chuan Li

In the chapter "Education for Sustainable Development (ESD) in Mathematics Education: Reconfiguring and Rethinking the Philosophy of Mathematics for the 21st Century," Hui Chuan Li discusses the disparity between the current trends in mathematics education and Education for Sustainable Development (ESD) approaches. The author argues toward a call for reconfiguring and rethinking the philosophy of mathematics for twenty-first-century learning priorities, as he detected a stalemate between ESD educational propositions and the growing trend in mathematics education toward teaching to the tests, which has become an increasingly common phenomenon in many education systems across the world.

Part IV – Philosophy of Mathematics Education in Diverse Perspectives, Cultures, and Environments

18 Antonio Miguel, Elizabeth Gomes Souza and Carolina Tamayo

"Asé o'u toryba 'ara îabi'õnduara!". In this chapter, the authors aim to problematize the alleged uniqueness and universality of Western logical-formal mathematics and the philosophies that support it with the purpose of deconstructing this belief as a problem, and not exactly defending a new philosophy of mathematics or mathematics education, but a therapeutic-decolonial way of educating and of educating oneself mathematically through the non-disciplinary problematization of normative cultural practices. They argue that any practice aimed at fulfilling normative social purposes in a way of life can be seen as a mathematical language game in Wittgenstein's sense. And that, by extension, the mathematical practices, and the ways in which they affect the different forms of life in the contemporary world, must be the focus of the therapeutic problematization of a mathematics education that intends to be decolonial. Six persons participate in this therapeutic debate: Oiepé, Mokoi, Mosapyr, Irundyk, Mbó, and Opá kó mbó. Their names correspond, respectively, to the numerals one, two, three, four, five, and ten in the ancient language Tupi, today considered one of the two most important linguistic branches of the hundreds of different languages currently spoken by indigenous communities in Brazil. Other remote interlocutors are invited to participate in the debate.

19 Maurício Rosa

In the chapter "Mathematics Education and Ubuntu Philosophy: The Analysis of Antiracist Mathematics Activity With Digital Technologies," Maurício Rosa investigates how mathematics education can encourage/provoke the understanding/constitution of social responsibility of mathematics teachers and students, specifically, in relation to social issues such as structural racism that inhabits our reality, including educational ones. His research question is: How does one discuss racism in a mathematics class with Digital Technologies in a way that mathematical concepts support the discussion? This investigation he conducts is guided by his understanding that currently, from the polarization of worldviews, it is important that education with mathematics teachers highlights the political and social dimensions of this form/a(c)tion, so that mathematics is a reflective resource, language or field of study articulated with Digital Technologies (DT) and with questions related to these dimensions. Given these considerations and their questioning, he analyzes a mathematical activity with Digital Technologies that discusses colorism, and uses the African philosophy called Ubuntu as an analytical resource, which does not conceive the existence of a being independent of the other, but of a "being" who thinks, acts and lives with others, be-being, that is, a becoming-being that promotes a transformation in reality from its agency with others, with nature, with life. As he points out, he found interesting pedagogical possibilities, which under a decolonial perspective demarcate equity and social justice arising from a class that takes DT and mathematics as a basis.

20 Min Bahadur Shrestha

"Philosophy, Rigor, and Axiomatics in Mathematics: Imposed or Intimately Related?". In this chapter, Min Bahadur Shrestha shows his movement to examine how philosophy, rigor, and axiomatics are related, as he was motivated by the two main tendencies of mathematical development in the nineteenth and twentieth centuries, which he understands as having to do with rigor and formalization. He argues that rigor and formalization took place on axiomatic basis leading to more abstraction and that Euclidean type of an axiomatic model became a model of mathematics even for constructively developed analysis. He goes on to point out his reasoning, arguing that although rigor and axiomatic method differ, and rigor does not need to be based on axiomatic method, in practice, that basis has been required for mathematical validity. He observes that among different interpretations made about it, some have explained it as a mathematical necessity, while others have attributed it to the philosophical underpinnings of formalism and foundationalism. From these arguments, he goes on to clearly state his argument by pointing out that, for him, it seems that philosophy has distant but decisive impression on the nature of mathematical knowledge, whereas rigor and axiomatic seems to be relatively internal to mathematics. However, because such trends have mostly been associated with European tradition, he argues, they need to be examined in the light of non-European traditions, including Hindu mathematical traditions, which have made significant contributions to mathematics without any axiomatic proof or philosophical presumption of absolute certainty.

21 Karla Sepúlveda Obreque and Javier Lezama Andalon

In the chapter "Idealism and materialism in mathematics teaching, an analysis from the socio epistemological theory," the authors aim to reflect on the influence of idealism and materialism, as philosophical currents, on the school curriculum and the work of teaching mathematics. They take social epistemological theory as the theoretical framework to guide their reflection, as well as to analyze the information obtained through direct observation and unstructured interviews with Chilean teachers.

22 Thomas E. Ricks

In the chapter "Cognitive and Neurological Evidence of Non-Human Animal Mathematics, and Implications for Mathematics Education," Thomas Rick focuses on a challenging issue for the mathematics educator Community. He reviews recent scientific evidence legitimizing animal mathematics. He states that, in particular, numerous cognitive and neurological studies suggest that animals mathematize like humans. Such findings run counter belief held by many in mathematics education that mathematics is a uniquely human enterprise. He concludes by suggesting possible benefits animal-mathematics studies may hold for the work of mathematics education.

Part V – Concluding

23 Bronisław Czarnocha and Małgorzata Marciniak

“Living in the Ongoing Moment” is a summarizing chapter. Czarnocha and Marciniak, co-editors of this book, wrote it as a moment of reflection on the topics brought by the authors in the current, just assembled book and a sudden, yet somehow expected, appearance of an advanced AI chatbot. Having no answers, they keep asking questions hoping that they will serve as an inspiration for more philosophy so much needed in mathematics education today.

São Paulo, Brazil
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Maria Aparecida Viggiani Bicudo

Introduction

Ongoing Advancements in Philosophy of Mathematics Education approaches the philosophy of mathematics education in a forward movement, analyzing, reflecting, and proposing significant contemporary themes in the field of mathematics education. The four parts that comprise the book share this view of the themes that interpret the investigative procedures and studies, as well as the didactic-pedagogical practices conducted in the area. Although the four parts are dedicated to specific subjects, the themes treated are intertwined and often displayed as oppositions, feeding the dialectic of thought that questions and seeks analyzes which significantly support arguments that advance, either through clarification, or proposals for new topics to be researched, or educational proposals.

We are a group of four researchers who dedicated ourselves to organizing this book, since ICME 14, which took place in 2021 in Shanghai, China. Four different people with different views. However, the four remain united by the common goal of bringing to the community of mathematics educators the diversity and strength of the ways of understanding mathematics and mathematics education, through critical and reflective analyses, present in the different papers discussed by TSG 56.

Thus, we believe that it is difficult or even impossible to amalgamate these four voices. Therefore, we decided that each of us, in their own way of thinking and explaining their understandings, would be responsible for presenting the introduction to one of the parts with which they were most familiarized, due to their own investigations and areas of interest. These presentations materialized as a brief text called "Introducing the theme of Part X." Maria Aparecida Viggiani Bicudo wrote the introduction to Part I; Bronisław Czarnocha wrote the introduction to Part II; Małgorzata Marciniak wrote the introduction to Part III; and Maurício Rosa the introduction to Part IV. The parts are presented below as well as the respective introduction of the themes covered, in order to expose the reader the logic of the organization of such parts which evidences our way of seeing the content of different chapters.

Introducing the Theme of Part I

In this first part of the book, we take a comprehensive look at mathematics education, resulting from philosophical thinking regarding what is understood by the authors of the different chapters in this book, which are proposed for analysis and reflection by those who operate such education.

The topic addressed in this part focuses on essays developed by their authors based on extensive investigations and debates that have taken place throughout their lives, while they immerse themselves in the philosophy of mathematics education. Proposals are presented regarding possible ways of understanding mathematics education, based on work already published but considered for reflection within the horizon of mathematics education. While reading them, I feel they go deep into topics already clarified and reviewed, advancing toward contributing with views that are articulated in organic units and involve more complex levels, seen from the perspective of a theorization work.

It is a movement that demonstrates the commitment to lead mathematics education beyond the aspects concerning themes regarding teaching and learning mathematics, when, for example, learning theories and proposals for ways of teaching are brought up, as well as topics related to the history of mathematics, philosophy, education, and other significant issues for this area, seeking to penetrate the intricacies of the ontological, epistemological, axiological aspects lying at the core of philosophy. The discipline imposed by the rigor of this search contributes to an attitude of constantly questioning the sense and meaning of what is said, both in one's authorial productions and those published in this area. Questions are asked, such as: What does this text say about education? About mathematics education? About mathematics? About teaching? About knowledge? About scientific knowledge and natural knowledge? About ways to critically understand scientific production? About ethical and aesthetic aspects that are shown between the lines of texts, research, and practices experienced among mathematics teachers and researchers? Why teach mathematics to all people, in the manner it is inserted in the school curricula of the Western world societies in recent centuries? In what direction should one go when performing mathematics education, that is, from a teleological point of view, what are the purposes assumed as valid, in order to guide the realization of this education?

Questions of this kind underlie the texts presented in this Part I and are treated in a unique way by different authors. The diversity of views and authors mentioned is evident in the chapters presented. Likewise, there is diversity of paths traveled toward the theorization pointed out above. This shows the strength of the area that is revealed from multiple perspectives. It is important to point out that all authors perform a rigorous exercise committed to issues which are central to philosophy, whether naming them explicitly or not.

The logic of the sequence of chapters goes from broader questions placed through reflections of the author's own work, moving on to texts that intertwine, in an articulated way, different philosophies of important authors such as Thomas Kuhn, Karl

Popper, and Werner Heisenberg, to authors who assume specific philosophical views, such as phenomenological and hermeneutic views, understood as critical for the movement of philosophically thinking mathematics education; as well as texts that focus on actions specific to human knowledge, with emphasis on mathematical knowledge, emphasizing abstraction as a process, and explaining actions that perform this process, such as specifying, defining, generalizing, abstracting mathematically; including texts that highlight the importance of analyzing and understanding mathematical education from the perspective of a system; and also a text that argues that an intellectually sound philosophy of mathematics education must incorporate various philosophical trends in education without centralizing just one of their aspects while denying or dismissing others.

These chapters challenge readers and invite them to follow the paths of the thought exposed, certainly raising other questions, and thinking from the perspective of their own pursuits and concerns.

Introducing the Theme of Part II

Part II develops beginnings of the philosophy of creativity in mathematics education. It contains three chapters which concern creativity directly and three chapters which touch upon it tangentially. All of them contribute to our understanding of this fundamental yet elusive phenomenon of creativity. In this short introductory section, we explicitly focus on that common thread. At the same time, we discover interesting connections between the chapters which suggests new questions to the emerging Philosophy of Creativity in Mathematics Education (PCME).

PCME is the emerging subdomain of consideration within our profession, which we hope will provide us with hints when, where, and how we can facilitate students' creativity and through its expression let them experience its power both along the cognitive and affective dimensions. Creativity and innovation have become the buzz words within the professional business circles. Against that background, Bronislaw Czarnocha's chapter "Towards a Philosophy of Creativity in Mathematics Education" addresses rarely explored area, philosophy of creativity and its relevance to mathematics education. The central question of this chapter – "What can the practice of and research in creativity of mathematics education contribute to the philosophy of creativity in mathematics education and possibly to the philosophy of creativity in general?" – guided the author's presentation and argumentation. Through the bisociation that is the theory of Aha! Moment, seen here as the act of creation following Koestler (1964), the author arrives at the conclusion that creativity should be at the basis of mathematics curriculum design. Regina Möller and Peter Collignon's chapter "Towards a Philosophy of Algorithms as an Element of Mathematics Education" arrive at a similar assertion but from the point of view of the role of algorithms in mathematics education. The authors point to the dramatic change in that role occurring within last several decades, both in our profession as well as in the world. As an introduction to the subject, the authors provide an interesting historical sketch, which includes the description of the recent changes, which employed algorithms is daily life, in particular in electronic calculators and computers. The contemporary user however, doesn't know about the existence of the algorithms at all. The main issue in question is about the impact of algorithms on math classes and phenomena that were completely unknown until recently. Since those new possibilities involve fascinating opportunities and enormous threats at the same time, the authors assert that the concept of algorithms should be in the center of didactical considerations. And that brings philosophical/didactical question: Both proposals, to organize the curriculum with creativity at its base and with the algorithm at the center of didactic attention, when implemented together lead us to the philosophical question: What is the relationship between creativity and an algorithm, or creativity and a procedure? Following on its heels comes the didactic question: How do we organize classroom teaching based on these two, ultimately antithetical concepts?

Equally interesting relationship exists between William Baker's chapter "A Framework for Creative Insights Within Internalization of Mathematics" and Mitsuru Matsushima's chapter "A Reconsideration of Appropriation from a Sociocultural Perspective" both of which include a similar goal but approach it from a different, one could even say, opposite points of view. The goal is to identify creative process, creativity within sociocultural approach.

Baker supports himself by a recently created concept of integrated bisociative frame grounded in Piagetian constructivism and Koestler's bisociation, and he shows the existence of such frames within internalization. That clarifies where and how creativity is found within internalization. Bisociative frame is one of the central concepts of the creativity theory of Aha! Moment presented in the Czarnocha chapter; it plays the role of the "discoverer" of creativity. Baker's efforts in the chapter are the continuation of Baker (2021), where the coordination of bisociation with Piagetian constructive generalization led to the formulation of the integrated bisociative frame – a new tool to identify moments of creative insight within student learning. Identification of such integrated bisociative frames within both Piagetian and Vygotskian approaches is very important. It suggests that a bisociative creativity underlies both of the approaches and can serve as their unifying principle. Moreover, the discussion of the close relationship between interiorization – the characteristics of Piagetian approaches – and internalization – the characteristic of Vygotskian approach – culminates the chapter.

Matsushima's chapter "A Reconsideration of Appropriation from a Sociocultural Perspective" pursues the following questions: Why does interaction in the learning community deepen mathematics learning? He addresses two fundamental questions within the sociocultural approach: Why does interaction in the learning community deepen mathematics learning of its members? How does individual learning contribute to the learning community through dialogue and deepen mathematics learning?

The chapter approaches the common goal of identifying creativity through the sociocultural vantage point of appropriation. He uses the newly introduced concepts of dynamic composition and mutual composition during the process of appropriation between an individual learner and the community of learners to identify three problems within internalization showing at the same time how these new concepts can deal with the described difficulties. And creativity is found possible during the appropriation as a deviation from the original concept to be appropriated. Such deviations are a manifestation of gaps between the concepts of the learners interacting with each other and the concepts of the learning community as seen by each learner. That means that bisociative frame is created by two different frames of reference of individuals between whom the deviation takes place.

Creative imagination of our profession is the common aspect of Ayalev and Marciniak chapters, especially seen at their sociocultural background. While for Ayalev it is the principle of the proposed professional development for teachers, for Marciniak it is the reality of the rapid transformation from face-to-face to online mode. She points out to several published reports from the time of pandemic which emphasized the critical role of creative imagination in the rapid transformation for which majority of teachers and students were not prepared. The chapter of Małgorzata Marciniak "The Times of Transitions in the Modern Education" addresses very deep revolutionary changes in education that are taking place at present due to the impact of pandemic as well as due to the growth of the role of Internet. The author recalls the incredibly high speed with which both teachers/faculty and students had to change the mode of teaching and learning from face-to-face to online Internet platforms. The process of change included significant difficulties for students, parents, and teachers leading ultimately to lowering students' passing rates, yet at the same time, it offered significant creative possibilities. Creativity involved in solving presented problems, when integrated with the creative possibilities of Internet, may signal that creativity itself is becoming the central underlying feature of contemporary pivotal point in education. The author places that present moment as the most recent pivotal point within a sequence of similar pivotal revolutionary changes in education's history: introduction of compulsory education, transformation from religious to secular schooling, and creation of public education. She points to the fundamental role of education of females brought by the paradigm of public education. Yenealem Ayalev's chapter "Some Examples of Mathematical Paradoxes with Implications for the Professional Development of Teachers" investigates the relationship between mathematics, teacher, and education as the basis for the professional education of teachers. He emphasizes that vital component of that relationship is creative imagination. He advocates the socio/cultural approach to creative imagination, and especially in the context of mathematical paradoxes, whose both posing and solving invokes large dose of creativity. The author finds the essence of mathematics as the science of computation and operation, as the provision of skills to learn and create shared meanings by way of socializing the field of mathematics. An excellent example of that multi-meaning which can be attributed to one mathematical concept leads the author to the analysis of known mathematical paradoxes, which through history have led to many creative ideas and discoveries. One of them has been the 5^{th} postulate of Euclid whose many possible meanings have led to the creation of non-Euclidean geometries. Another one is the collection of different interpretations and solutions to "1 + 1 = ?", which illuminate the role of interpretation as well as existence of multiple truths in mathematics.

Introducing the Theme of Part III

Young children often ask for the purpose of live, but when discouraged, they stop and engage in other activities. Similarly, young researchers frequently inquire about the meaning of mathematics and its place in education. But when the answer is not found, they tend to shift to other, more rewarding topics. In this part of the book, the authors stubbornly search for the answers of revised and reformulated questions about the meaning of mathematics, its place in education and reality. Unfortunately, mathematics is nowadays taught as the universal and unquestionable truth not responding to the needs inquisitive and future oriented minds. Even college students of engineering programs frequently express their surprise when they find out that their teachers do research in mathematics beyond what is already known.

In modern mathematics, classroom logicism, formalism, and intuitionism are already well-established dimensions. In his chapter, Ole Skovsmose finds ways for expanding these dimensions and introduces performative interpretation of mathematics as a niche in mathematical ethics. He justifies the needs for adding this new dimension to the contemporary mathematics as a crucial factor in creation of the life-world. In his works, performative interpretation based on mathematical results plays a key role in creating reality. At the same time, Skovsmose points out the new roles of mathematical algorithms in political, social, and financial decision making on a large scale and emphasizes the urgent needs for ethical discussions about such uses.

It is worthwhile to mention that the aspects of ethics in terms of performative interpretation go far beyond well-known mathematical literacy (sometimes called numeracy) denoted by further authors as mathemacy. Ole gives an example of how formalism (language) affects the discussion using mathematical modelling. In his view, a choice for a particular model shapes the discussion, for example about climate change and affects the conclusions. No doubts, the way the data is cleansed, analyzed, and used is important for the future shape of our global society. The fact that the language influences the content of conversation is well known to bilingual people. While it may be intuitively clear that the choice of language shapes spoken or written exchange of ideas, this may be less obvious for pure wordless thinking. Especially when we would like to see thinking as a process is entirely owned by us. But surprisingly so, some, thoughts, including mathematical thoughts are highly dependent on the language.

Practical classroom applications of performative interpretation in mathematics classroom are described in the chapter written by Nadia Stoyanova-Kennedy. She suggests that inquiry joined with collaborative group work may be the best way for creating a suitable environment for critical and creative thought. Stoyanova-Kennedy points out that the role of math in the society has been unclear. On the one hand, math skills are valued and used in many aspects of human life (banking, decision making, cryptography, coding, etc.), but on the other hand, students of all ages and nationalities fear math and often sincerely dislike it. This leaves math educators in a century-lasting dilemma of how to connect mathematics with happy feelings and natural interactions of humans with the reality. Here mathemacy comes as a rescue as it is designed to build interactions between the mind and the reality. At the same time, Stoyanova-Kennedy warns that mathemacy, if engaged only for technical or mathematical understanding, will not contribute to the growth and development of modern society. Thus, she calls for well-framed and properly facilitated classroom discussions that aim for building ethical and social awareness of students of all ages. Stoyanova-Kennedy gives an example of discussion about mathematical modelling which could contain questions of the type: Should a model describe reality with a prefect precision? Why mathematical models are need? Can they describe everything? How do we know how much to trust them?

Uwe Schürmann notices that even in courses about mathematical modeling, calculations are heavily overrepresented not leaving any time and attention for such fundamental epistemological and ontological questions crucial for the role of models in creating reality. He points serious shortages in such courses:

- Lack of empirical experience of mathematical modeling, i.e., collecting the data, finding the parameters of the models, comparing the results from the models to the actual outcomes, and explaining the obvious existence of discrepancies
- Social separation: Who builds the model and for what purpose, how are the results used and with what intentions?
- Ontological separation: How does the model contribute to existing view of nature, for example, one differential equation tends to appear in many different areas and describe unrelated phenomena, pendulum equation describes not only vibration but level of sugar and insulin, what does it mean to science? How does this discovery contribute to our view of reality?

Education with such separation will not create responsible citizens prepared for challenges of the future. Thus, mathematics education urgently needs a revision in its paradigms. Hui-Chuan Li brings as an option UNESCO-supported ideas of Education for Sustainable Development (ESD) which has three aspects: economic growth, social inclusion, environmental protection. He emphasizes that in mathematics curriculum, these aspects may not be actual topics. But they should influence the way of implementation of math topics in the class activities. Li gives an example of own research topics related to ESD introduced as workshops for his students in Scotland. Topics include biodiversity, climate change, and sustainable energy (wind turbines and biodiesel) and the workshops contain non-traditional activities such as: respectful debates and discussions, reflections, sharing opinions, and points of view of the value of mathematics and its limitations. At this point, Li rightfully rises another issue for discussion related to testing in math classes. As he points out, standardized testing influenced teachers to focus on teaching just for these tests supporting the opinion that the style and content of testing affects the way of teaching. Opposite holds as well, i.e., the style of testing should mirror the teaching. Thus, bringing discussions and debates to the curriculum should influence the way of testing. In my own teaching experience, written reflections on mathematical results rescued validity of online testing during the pandemic. This experience significantly changed my view on tests and testing in math courses.

As a conclusion, STEM education has been drawing increasing attention as it became more politically relevant due to its role in technology and decision making needed for the future workforce. Mathematics curriculum makes attempts to accommodate such needs by introducing problem solving, which, in Li's opinion, is insufficient for preparing students to become critical citizens of future reality. Thus, at the end of his chapter, Li calls for culturally and ethnically sensitive mathematical teaching.

Introducing the Theme of Part IV

Part IV of this book brings chapters that deal with the philosophy of mathematics education in different perspectives, cultures, and environments. This means that the chapters that make up this part undertake to discuss, to reflect, to reflect on, for example, colonial mathematics problematizing the supposed singularity and universality of this western logical-formal mathematics and these philosophies that sustain it for the purpose of deconstructing the belief of a single and universal mathematics as a problem. Therefore, highlighting the ways of doing mathematics of different peoples, valuing it, is a movement that arises as a potential restorative of the historical invisibility of different cultures.

In this perspective, the conception of culture is understood as a whole way of life, its common meanings, to designate the arts, learning, the processes of discovery, and the creative effort (Williams, 2015). Also, according to Eagleton (2003), based on the etymological meaning of the word that comes from the Latin colere, the term culture is used to designate distinct things such as habitation (in the words "colony" and "colonist") and religious worship ("worship"). It is also used to manual labor. The original meaning of word "culture" is "crop" or "agricultural cultivation" and, thus, what previously designated a particular material activity, takes an abstract meaning, which is imposed as the general cultivation of the intellect, in the individual sense and also in the collective. Thus, this part of the book highlights the ideas of freedom and determinism, activity and resistance, change and identity, what is given and what is created (Eagleton, 2003) in the mathematical production of different peoples and environments and what can be taught to empower mathematical doing and equity of human beings. Therefore, the philosophy of mathematics education in various cultural perspectives, for example, seeks the individual and collective growth of what has already been cultivated or that will be, either by itself or by the group to which one belongs, in a dialectical movement between what is said natural and what is said artificial, according to rules agreed by the group itself, and between what we do to the world and what the world does to us in an inseparable flow, which presents itself in proper act of educating by mathematics.

This part of the book assumes possibilities of defense in a decolonial way of educating mathematically and educating by mathematics through the non-disciplinary problematization of normative cultural practices. The discussion of how mathematics education can stimulate/provoke the understanding/constitution of the social responsibility of teachers and mathematics students is raised. Specifically, in relation to social issues such as structural racism that inhabits our reality, including educational, there is a movement that highlights the African philosophy Ubuntu, evidencing ways in which mathematics can sustain the understanding of respect for the collective. The chapters promote reflection on decolonial ways of educating by mathematics. Thus, some interesting mathematical practices are pointed out, which from a decolonial perspective demarcation equity and social justice and can be performed in recurrent educational spaces.

Nevertheless, the understanding that Hindu mathematical traditions, for example, have made significant contributions to European mathematics even without any axiomatic proof (of the way these tests are (im)put by the coloniality of knowledge) or philosophical presumption of absolute certainty (likewise), can raise empowerment flights to various groups of students from minorities (black) in mathematical educational spaces.

In Latin American terms, there is a movement around social epistemology as a theory of educational mathematics. This epistemology studies didactic phenomena related to mathematical knowledge, assuming the legitimacy of all forms of knowledge, whether popular, technical, or formal, because it considers that they, as a whole, constitute human wisdom.

Also, as an innovative perspective, this part of the book presents a discussion on recent scientific evidence that legitimizes the mathematics of animals. Cognitive and neurological studies are indicated that suggest that animals mathematize like humans and this destabilizes the common belief that mathematics is an exclusively human enterprise.

However, the disarticulations of universal truths are present in this part of the book, because, although the meanings we attribute to a given situation or object are individual and subjective, they have their origins and their importance in the culture in which they are created. Therefore, in this cultural locus is the guarantee of negotiation and communication of these senses in order to characterize meanings ("agreed" or "(im)posed"). What matters, in cognitive terms is that meanings become the basis for cultural exchange. In the symbolic systems of culture, there is a possible realization of what is understood together and what can be communicated in terms of eurocentrical ideas. Thus, this part of the book seeks to debate issues that do not fit, at first, into a formal, academic, white, and Eurocentric mathematics. In other words, there is no framing of diverse mathematical thoughts in an order of hegemonic and European structure, because these different mathematical forms of thinking are outside the characteristics of rigor and axiomatic proof, previously conceived in history by the dominant culture. However, the proper mathematical way of thinking of the dominant culture is philosophically questioned and other forms are discussed in the subsequent chapters of this part of the book. These debates, in turn, open different possibilities from those commonly defended and potentiate knowledge that can often be considered as decolonial ways of thinking.

At the end, concluding the treatise on the different parts of the book, there is Chap. 23, understood as a reflection on the topics brought up by the authors. It is an essay in which two of the co-editors, Dr. Czarnocha and Dr. Marciniak, carry out a review of the book, highlighting its unity understood as based on philosophical thinking and the way of proceeding of the Philosophy of Mathematics Education, projecting inquiries visualized in the horizon opened by the AI chatbot.

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Porto Alegre, Rio Grande do Sul, Brazil — Maurício Rosa
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Part I
A Broad View of the Philosophy of Mathematics Education

Chapter 1
The Ontological Problems of Mathematics and Mathematics Education

Paul Ernest

1.1 Introduction

Ontology is the branch of philosophy that studies the fundamental nature of reality, the first principles of being, identity and changes to being, that is, becoming. In this chapter, I want to explore being, existence and identity as they concern mathematics and mathematics education. In particular, I want to address the ontological problems of mathematics and mathematics education. The ontological problem of mathematics is that of accounting for the nature of mathematical objects and their relationships.[1] What are mathematical objects? Of what 'stuff' are they made and do they consist?

The ontological problem of mathematics education concerns persons. What is the nature and being of persons, including both children and adults? In the context of this chapter, I will restrict my attention to human mathematical identities, that part of being which pertains to mathematics, namely the mathematical identity of mathematicians and the developing mathematical identities of students. What are these mathematical identities and how are they constituted? Human beings are located in, and constituted through the cultures they inhabit, so my answer will encompass how these contribute to mathematical identities, as well.

The twin ontological problems of mathematics and mathematics education concern the chief entities in the two domains. These are mathematical objects first, and second, persons, restricted to their mathematical identities. The structural similarity

[1] I use the term nature without presuming essentialism or assuming 'natural' states of being. I shall answer the question of how the properties and characteristics of mathematical objects and human beings as mathematical subjects are inscribed within them as a process of becoming without the presuppositions of essentialism.

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does not end with these parallel twin focuses of inquiry. In each of these two domains, there are dominant myths that must be critiqued or cut down before their respective problems can be addressed adequately. In mathematics, there is the myth of Platonism, namely that mathematical objects exist in some eternal, superhuman realm. According to this view, mathematical objects were there before we came along, and they will still exist after we are all gone.

In mathematics education, there is the rather more hidden problem of individualism. This is the view that persons are all existentially separated creatures whose actions, learning and even whose being take place in hermetically sealed and separated personal domains.

However, there is also dissimilarity in the treatments I can hope to offer. While I can aspire to giving an account of the nature of mathematical objects, I cannot hope to treat the nature and being of persons except in a very partial way. As I have indicated, I restrict my inquiry to those aspects of human being that pertain to learning and doing mathematics, and their personal foundations.

1.2 Mathematical Objects

In this part, in a number of linked sections, I offer an attempt to giving an account of the nature of mathematical objects. I start by trying to clear some of the obstructive conceptual undergrowth that stands in the way of my account. In the exposition that follows on, all the elements that make up the social constructionist account of the ontology of mathematical objects are introduced and then summarized in Sect. 1.2.8.

1.2.1 Critique of Platonism

According to Platonism, mathematics comprises an objective, timeless and superhuman realm populated by the objects of mathematics. These objects are pure abstractions, and they exist in an unchanging ideal realm quite distinct from the empirical world of our day-to-day living. Plato's doctrine of Platonism locates other abstract ideals beyond mathematics such as Justice, Beauty and Truth, in this realm. Not surprisingly, the nature, status and location of abstract ideas has been a matter of debate at least since the time of Plato. The medievalists divided into the camps of nominalists (abstract objects are primarily linguistic names), conceptualists (abstract objects are ideas in our minds), and realists (abstract objects are real entities that are located in some platonic-like realm). All of these positions have their problems.

Many of the greatest philosophers and mathematicians have subscribed to the doctrine of Platonism in the subsequent two plus millennia since Plato's time. In the modern era, the view has been endorsed by many leading thinkers including Frege (1884, 1892), Gödel (1964), and Penrose (2004).

Although I shall reject it, quite a lot is gained by this view. First of all, mathematicians and philosophers have a strong belief in the absolute certainty of mathematical truth and in the objective existence of mathematical objects, and a belief in Platonism is consonant with this and even validates this view. Platonism posits a quasi-mystical realm into which only the select few – initiates into the arcane practices of mathematics – are permitted to gaze, and within there – to discern mathematical objects and mathematical truth.

Second, this view is a concomitant of, and validates purism, the ideology that mathematics is value-free and ethics-free. Human values are excluded by definition, for they cannot seep into or taint the hermetically sealed superhuman platonic realm, since it exists in another dimension. As I have recounted elsewhere, purism is an ideology that was strong in Plato's time and then again in the nineteenth- and twentieth-century mathematics (Ernest, 2021a).

Third, Platonism supplies mathematics with a theory of meaning. According to this theory, mathematical signs and terms refer to objects in this ideal realm. Likewise, mathematical sentences, claims, and theorems refer to true, or otherwise, according to their status, states of affairs and relationships between the constituent objects that hold in the Platonic realm.

However, a distinction should be made between Platonism and mathematical realism. According to the latter, mathematical objects are real; they are something verifiably shared amongst many people. However, they do not necessarily exist in a superhuman and supraphysical realm. For example, as I shall argue, they can be social objects. However, developing a social theory of mathematical objects is more complex than simply positing a Platonic realm, which can be conjured, ready-made out of a hat. Explaining and validating a social constructionist theory of mathematical objects requires the development of conceptual machinery and to a certain degree, a suspension of disbelief, because Platonism has penetrated so deeply into our understanding of mathematics and universals.

Platonism is not without its problems. Two major problems concern access and causality. How can mathematicians access the Platonic realm? With what faculties can they peer into it to discern its objects and truths? No such sixth sense is known unless one strays into the realms of the mystic and shaman. And even if one did stray there, and could discern mathematical objects and truths directly, what justifications could be given to others for the existence of the objects and the validity of the truths discerned? To say I saw it with my mind's eye is not enough. Mathematical objects need to be accurately defined to be communicated, and mathematical truths need to be convincingly proven in public texts to be acceptable. So, their means of validation are just those that one would need even if there were no superhuman Platonic realm into which one could peer (Benacerraf, 1973).

In terms of causality, there are problems both ways (Linnebo, 2018). How are newly defined concepts and newly proven results inserted into the Platonic realm? What is there about our inventions and discoveries that cause them to appear there? Plato argued these 'new' objects were there all along and we can only discern them when we have recreated them for ourselves. This is surely an unsatisfactory *ad hoc* answer. If we can only discern what we have recreated, why not dispense with the mystification and acknowledge we *created* them, in the first place? In the reverse direction, how can the truths of the Platonic realm causally determine outcomes in the material world? Why is pure mathematics so unreasonably effective in the real world? How and why are mathematical truths so real and so persuasive to children, students, and adults prior to demonstrations? I suppose that if mathematical truths hold in all possible worlds, then those found in the Platonic realm must hold in the material world. But this is not a causal argument implying that mathematical truths from the Platonic realm force their applications to hold in the physical world. Once again it leaves the Platonic realm superfluous.

Although positing the Platonic realm as the home of mathematical objects and as a source for mathematical truths opens a number of serious problems, it remains a widespread, legitimate, and irrefutable view. Like many ideologies that posit other realms full of celestial beings it remains a matter of choice and belief. I choose to use Occam's razor, the principle of ontological parsimony, that entities should not be multiplied beyond necessity. Extending this to the multiplication of ontological realms, I regard the expansion of mathematical ontology through the addition of the superhuman Platonic realm to be unnecessary. It creates new problems of access and causality. It represents a succumbing to the historical vice of Idealism. My claim is that socially based mathematical realism can accommodate many of the benefits of Platonism without all these extra costs. So, I reject Platonism while embracing mathematical realism.

1.2.2 Meaning Theory

Above I acknowledge that Platonism supplies mathematics with a theory of meaning. According to this theory, mathematical signs and terms refer to objects and their manifested relationships in the ideal Platonic realm. Most simply, this is a referential or picture theory of meaning. Ernest (2018a) shows some of the inadequacies of this theory, which is also widely criticized elsewhere (e.g. Rorty, 1979). But if one is to reject this theory what is to stand in its place? If mathematical signs and words are not simply the names of objects in a Platonic realm how else can they signify? How can we offer a way to understand their meanings? In my view, Wittgenstein's (1953) theory of meaning, according to which much of the meaning of words and other signs is given by their use, offers the best solution.

With regard to meaning, Wittgenstein says that much of meaning is given by use: "for a large class of cases – though not for all – in which we employ the word 'meaning' it can be defined thus: the meaning of a word is its use in the language" (Wittgenstein, 1953, I, sec. 43). He allows for three other sources of meaning – custom, rule-following, and physiognomic meaning (Finch, 1995; Cunliffe, 2006). Focusing on meaning as use, it is important to hedge this in the way that Wittgenstein does. Namely that the use of words or signs is always located within language games situated within forms of life. Thus, according to this theory, the meanings of words and signs are the roles they play within conversations located in social forms of life. But these are not free-floating conversations, they are conversations centered on, and intrinsically a part of, shared activities with a goal or object in mind. In one extreme case this might be conversing after dinner with friends with combined aims of sharing information (or gossip), consolidating relationships or just for the intrinsic joy of relaxing with friends and family. Such discussion, although perhaps capturing the popular meaning of the term 'conversation', is trivial and fails to reflect the central importance of conversation.

Conversations are not just trivial decorations but an integral part of social activities. The function of conversations is to facilitate important joint and productive activities through directions, confirmations, and other means. The meanings of the terms and signs employed are their functions within these activities. Joint action within a form of life is usually directed and punctuated by discourse. In other words, language in conversation is a tool employed to further a joint activity and take it towards its goal. Indeed, the language used in productive material activities is as often imperative or interrogative as it is declarative. Such as in the kitchen: 'stir this', 'is there enough salt in the sauce?', and 'this is tonight's meal'. Where conversation is lacking in a joint activity, often custom and rules have been laid down conversationally in earlier manifestations of the form of life rendering repeated conversations and directions superfluous, so the joint activity can progress without verbal instructions or elaboration.

Wittgenstein makes it clear that meanings depend on the language games in which they are used, and 'When language-games change, then there is a change in concepts, and with the concepts the meanings of words change' (Wittgenstein, 1969, sect. 65).

Two other dimensions of Wittgenstein's ideas of meaning, custom and rule-following, are also important. Cunliffe (2006: p. 65) points out that there are deontic dimensions of meaning entangled with the other uses, with widespread imperatives imposing or requiring rule-following, the meeting of obligations, lawfulness, and respect for customary usage. Language use is far from limited to the alethic mode – meaning that it encompasses epistemic, factual, and truth-orientated functions. It also commonly employs the imperative mode. This is very important when understanding the meaning of mathematical texts, where the imperative mode far outweighs the declarative or indicative modes, as an analysis of verb usage in the corpus of mathematical texts reveals (Ernest, 1998, 2018a; Rotman, 1993). I shall argue that the institutions of mathematics are held up by tacit or explicit rule-following and custom, so this dimension of meaning is very significant. Indeed, I hope to show that the very objects of mathematics are created and maintained by tacit agreements, rule-following, and embedded customs inscribed within the objects themselves. For example, to count you must follow a string of rules, but as counting skills develop, and numbers come into being as self-subsistent mathematical entities, then the rules and norms appear to dissolve or disappear into the perceived nature of the numbers themselves. But I get ahead of myself.

1.2.3 What Are the Objects of Mathematics?

Platonism and realism offer answers as to where mathematical objects are to be found (Skovsmose & Ravn, 2019). But apart from the fact that they are universals and abstractions, these ontologies do not tell us what the objects of mathematics are; they do not answer the question of what is the stuff of which they are made?

Unfortunately, traditional ontology is not a lot of help here. It seems to be satisfied with a category of being, rather than a deeper inquiry into the very stuff or substance of the existents. What is needed is a multi-disciplinary approach that combines insights from philosophy of mathematics, mathematics itself, semiotics, cognitive science, psychology, sociology, linguistics, and mathematics education into the nature of mathematical objects. No one of these disciplines is sufficient of itself, as I intend to show, to satisfactorily answer the question: Of what stuff are mathematical objects made?

There is another obstacle in the way of a naturalistic account of the ontology of mathematical objects, namely, the ideologies of essentialism and presentism (Irvine, 2020). In this context, essentialism presupposes that mathematical objects are made of some fixed stuff, analogous to diamonds or other precious stones, but existing in an eternal realm, where they are found to be permanent and unchanging.

The ideology of presentism searches for answers to all questions in a timeless present, where there is no change, development, or becoming. In my view, ignoring Plato's admission of 'becoming' into ontology, presentism underpins much of modern Anglo philosophy. There logical arguments are timelessly valid, and concepts presumed fixed and permanent. Where such properties are attributed to mathematical concepts and objects, they are presumed completed and there is no need to discuss how they came to be and how this shapes what they are. Becoming is ignored and disallowed. I think that to fully understand mathematical concepts and objects you need to know how they became as they are. Especially, as being abstractions, they have been abstracted from lower orders of abstractions or actions. Mathematical objects do not have a fixed essence, for they change over time, and they have different meanings in different contexts.

Let me illustrate this with arguably the simplest of all mathematical concepts, the number one. This first appears in human history and in child development as the first word in a spoken count ('one', 'uno', 'yek', 'tik') or as the first tally in symbolically recorded counting. The number, or rather numeral, 'one' is the first ordinal in a counting sequence (meaning 'first'). The early use of this numeral is enactive, with the action of pointing or making a mark accompanying its utterance. When the last ordinal number in counting out a set becomes defined as its cardinal number, the word 'one' or sign '1' gains its cardinal meaning as the number one. Counting out a triplet with the ordinals 1, 2, 3 ends with 3 which by definition is its cardinality. Counting a singleton – ordinal one – results in cardinal one. This marks the arrival of the concept of one in its first rudimentary but complete form, the cardinal number one. At this stage in its development, number one is understood to be on a par with the other natural numbers 2, 3, 4, and so forth.

It is important to notice that the concept of one is doubly abstracted. First from the physical action of counting, from tallying, or from just saying the number names out loud prior to counting. Second, once ordinal counting is mastered for small values, the cardinal number one is abstracted from the ordinal one, as comes to represent the value of a completed count.

In tallying, strokes or marks are used to represent the outcomes of counts (e.g. '///' representing the count of three. Each stroke is itself a part of a unit action and ultimately each of these units 'one more' is identified with 'one'. The tally '///' represents one (more) and one more and one more which is a compound way of representing three (via the ordinal 'third').

In number systems, the numeral '1' resembles a tally stroke. In several number systems, such as those of the Ancient Egyptians, Sumerians, and Romans, numerals made up of one, two, and three strokes (I, II, III, respectively) are used for the first few numbers representing both repeated ones and unit strokes in a tally. Studies of proto-language suggest that the early, possibly prehistoric name for one was 'tik', also meaning digit or finger (Lambek, 1996). Use of fingers for counting evidently goes back a long way, and this word for one also stands for a single digit (as finger). Indeed, the modern use of the word digit retains this ambiguity, standing both for individual numerical signs (1 to 9) as well as for any single finger.

This is just the beginning of the development of the concept of one. The use of the sign '1' becomes more elaborate within systems of numeration, calculation, and measurement. The numeral '1' represents a unit in an abacus or place value system in compound numbers (one ten, one hundred, etc. indicated in decimal place value as 10, 100, respectively, with '1' as an atomic component in a molecular sign). Subsequently, with the introduction of multiplication, one serves as the multiplicative identity element.

As number systems and structures are extended, '1' has different meanings and properties, across N, Z, Q, R, C. In Q, the numeral '1' is used both in numerators and denominators. In R (and Q), '1' is used in extended place value notation as a fraction of denomination ten to a negative power (e.g. $0.001 = 10^{-3}$). '1' and other numerals are used in algebra, length measures (and indeed all measures), in fractions, extended place value notations, vectors, matrices, probability theory (representing certainty), Boolean algebra (representing truth). The property of 'one' as the multiplicative identity in Q, R, and C is generalized throughout algebraic structures such as groups, rings, fields (together with the more basic additive identity 0).

In each of these different roles, '1' has different uses and meanings, so its meaning can never be said to be fixed, but is always dependent on the context of use, on the background theory. Thus, the number one cannot be claimed to be a single fixed mathematical object or concept. However, what we can say is that the number one, like all Natural numbers, in its first emergence, is an abstraction of an action using signs. An instance of a counting action can be physical, such as touching individual members a set of objects (already conceptualized as countable units for the purposes of counting, Ernest, 2021b), while uttering the sequence of ordinal names. Or it can be conceptual where the units are counted without physical contact or movement. Either way the act of counting is an instance, a token of counting, corresponding to an abstracted type, the count. This count has an endpoint, a designated sign that represents the ordinal position of the last counted unit, which is abstracted as the number, the cardinality of the set counted. Thus, the resultant number, the cardinality is doubly abstracted from the instance of counting. First, as the type or class of the designated count. Second, as the cardinal number abstracted from the derived ordinal number.

This simplest of all the numbers, 'little' number one, serves to show both how complex and multiply meaningful mathematical concepts are, as well as of what they are formed. 'One' begins as an action associated with a sign, which is then abstracted. The process is reified into an object. Virtually all named mathematical objects consist of abstracted operations or actions on simpler mathematical objects or actions.[2] To enable a differentiation of levels into simpler/more complicated, one can posit a hierarchy through which the relation of 'simpler than' can be defined. The lowest level of mathematical actions (level 0) is made up of those that have a physical correlate, like counting or drawing a line. The lowest levels of mathematical objects (level 1) are abstractions of, or from, mathematical actions of level 0. A mathematical action of level $n + 1$ operates on actions and objects that include at the highest level those of level n. Likewise, a mathematical object of level $n + 1$ abstracts actions and objects that include those up to and including level n.

What I have only exemplified in the case of 'one' is that there is a sign associated with every (named) mathematical action or object. Frequently, there are several signs. So, with one there is the spoken verbal name or word (in almost every language), a written verbal name ('one'), and a mathematical sign ('1'). In various arithmetics, there are in fact an infinite number of expressions with the numerical value of 1 that could also be called names for 1 (e.g. 22–21, 0 + 1). The signs of mathematics are of paramount importance. The signs not only help to create the objects of mathematics, but they are also entangled with them. Mathematical actions are typically actions on or with mathematical signs.

1.2.4 Mathematical Signs and Their Performativity

In order to fully engage with the role of signs in mathematics, with the semiotics of mathematics, it is necessary to understand the performativity of mathematical signs. As syntactical objects, mathematical signs are both the objects acted upon and the crystallized residue of acts in themselves. In the first instance, all the 'atoms', the ur-elements of mathematical signing are performed actions. Thus, as we have seen, 3 represents the product of tallying III which is itself the residual mark of the repetitive physical act of counting one, two, three. In this way, counting employs

[2] Various scholars in mathematics education research make this point (Sfard, 1994; Tall, 2013; Dubinsky, 1991).

indexical signing because each stroke or physical tally movement corresponds to a counted entity by proximity in space, time, or thought.

Within semantics, the performativity of mathematical signs is ontological. The signs create their own meanings; the abstract objects of mathematics that they denote. What we have is a 'a set of repeated acts within a highly rigid regulatory frame that congeal over time to produce the appearance of substance, of a natural sort of being' (Butler, 1999, p. 43–44). Although Butler is referring to another domain, the process is identical for the construction of mathematical objects. Thus, numerals and number words 'do not refer to numbers, they *serve as* numbers' (Wiese, 2003, p. 5, original emphasis). This is an important point that contradicts any referential theory of meaning, including both the picture theory of meaning and Platonism. Numerals, number word terms, and by extension all mathematical signs need not indicate or refer beyond themselves to other objects as their meaning, let alone to a supraphysical and ideal realm of existence. They themselves serve as their own objects of meaning, coupled with the actions that they embody (and their inferential antecedents and consequents).[3] Mathematical language is thus performative, for mathematical terms create, over time, the objects to which they refer. As I have argued, counting via abstraction is the basis for the creation of numbers, and likewise operations create mathematical functions. In the first instance, these are inscribed numerals and enacted operations. Repeated usage reifies and solidifies them into abstract mathematical objects.[4] Furthermore, their currency of use serves as a social warrant for them, verifying their legitimacy and existence.[5]

Elsewhere I argue that mathematical signs are performative in two ways, which I term inner and outer. What I describe above is part of the inner performativity, whereby mathematical sign usage creates mathematical objects. The outer performativity of mathematics is the way it formats the way we experience and interact with the material world (Skovsmose, 2019, 2020; Ernest, 2019). I will not discuss this outer performativity further here (but see O'Halloran, 2005 and Ernest, 2018b).

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[3] This has been used as an explicit strategy within mathematics. Henkin (1949) defines the reference of each sign within the system to be itself, in his classic proof of the completeness of the first-order functional calculus.

[4] This is supported both within philosophy (for example, Machover, 1983) and empirically by research into the psychology of learning mathematics (Ernest, 2006; Tall, 2013).

[5] In writing that signs create their own meanings, it is taken for granted and unwritten here that it is signs-in-use by persons that perform actions, for it is persons that use signs and create and comprehend meanings.

1.2.5 The Constitutional Role of Social Agreement for Mathematical Objects

Social agreements play a decisive role in the constitution of mathematical objects and the validation of mathematical knowledge. Such agreements may be tacit and are introduced both implicitly and explicitly through mathematical practices and everyday games and practices with young children. Before counting can even begin, the idea of separate objects or units needs to be introduced into a learner or child's worldview. This is the idea that some of the features of the environment can be construed as self-contained objects. Take for example, collections of toys, pebbles, or sweets. Each item in the collection can be treated as a separate independent object. In addition, steps in climbing a stairway, walking along the ground step by step, or other sequences of actions can also be seen in this way as series of discrete actions. Such a way of viewing a domain, although concrete and limited in extension at first, prepares it to be viewed as countable (Ernest, 2021b).

Once the experienced world is thus construed into repeated units, the foundations of counting can be laid. A further prerequisite to be learned and tacitly agreed is a list of word numerals that must be used in a repeatable order. This list must be both stable, that is invariant, and as at least as long as the number of items to be counted. This is the first of Gelman and Gallistel's (1978) five counting principles, (1) The stable-order principle.

The other principles are as follows.

2. The one-one principle – this requires the assignment of one, and only one, distinct counting word to each of the items to be counted.
3. The cardinal principle – this states that, on condition that the one-one and stable-order principles have been followed, the number name allocated to the final object in a collection represents the number of items in that collection.
4. The abstraction principle – this allows that the preceding principles can be applied to any collection of objects, whether tangible or not.
5. The order-irrelevance principle – this involves the knowledge that the order in which items are counted is irrelevant.

These principles are something that a child must learn in their schooling or early home life. But although often viewed as knowledge, they are deontic social agreements. A quasi-counting activity must conform to them, or it is not socially acceptable. Entering a game or any social practice requires conforming to the rules and regulations of that activity as a participant. The rules are compulsory. They are expressed in the deontic modality that indicates how behaviours must be, to accord with the relevant norms.

The five counting principles listed here are part of the social agreements about what constitutes counting and ultimately regulates what numbers are. All mathematicians will adhere to such agreements, but they are so basic, so deeply entrenched, that with familiarity they seem obvious, unnecessary, and not needing to be articulated. Rules and agreements like these become subsumed into the perceived essence of counting actions and number objects. Starting as necessary features of counting they become seen as features of numbers, intrinsic properties of the reified mathematical objects themselves. Only when someone like Cantor introduces his theory of transfinite numbers is the one-one principle made explicit, jolted back into focus, and considered in the light of the new problematic theoretical context, the equipollence of infinite sets. Otherwise, the one-one principle in counting is seen as intrinsic and definitional, rather than a norm that is (must be) followed.

One of the most valuable and remarkable features of mathematics is how the rich, deep, and complex concepts and objects come into being from simpler objects and actions. This allows the dizzy heights of abstraction to be scaled and objects to be created that exceed by so much what we perceive and experience in the material world, such as the concept of infinite sets. However, one cost of such repeated objectification and abstraction processes is that the rules and social agreements that determine the nature and limits of lower-level objects, concepts, and actions become perceived as essential characteristics of the more abstract objects created from them. The social agreements that shape and constitute arithmetic, for example, become hidden, forgotten, and indeed eventually denied as being the social agreements underpinning number. It is not that their observance is breached, but that they are seem as so essential that they become regarded as intrinsic to the constitution of the objects. Many mathematicians and philosophers state that the natural numbers are something given to humankind by nature (Penrose, 2004). The relationships, extrinsic constraints, and norms that govern their proper and permissible usages become seen as intrinsic properties of the objects in themselves. The social agreements that give shape to objects of mathematics become seen as inscribed in the essence and very being of the objects. The intellectual struggles of humankind over millennia to create counting and numeration systems are no longer seen as processes that through their notational inventions, their actions and conceptions, created what are now seen as the independent objects, the natural numbers. Even their name suggests that these numbers are natural, that is, given by nature, rather than the outcomes of processes of social construction based on imposed rules and norms.

My claim is that in this way social agreements play a constitutional role in mathematical objects. Cole (2009, p. 9) proposes 'The thesis that mathematical entities—specifically mathematical domains—are pure constitutive social constructs constituted by mathematical practices, i.e. the rationally constrained social activities of mathematicians'. In other words, mathematical objects are social constructs, built up from the socially enacted and socially warranted actions described above, and founded on the social agreements of the community of mathematicians. These agreements are expressions of the deontic nature of mathematical practices and are manifested in conforming to their rules and norms. Many, if not most, of these agreements are tacit, agreements in forms of life, as in mutually aligned mathematical practices, not as explicit verbal agreements.

1.2.6 Signs as Constitutive of Mathematical Objects

I have argued that the signs of mathematics play a constitutive role in the formation of mathematical objects. Actions on signs and objects become the next level of abstract objects, themselves depicted by signs. However, it should be made clear that not all mathematical objects are named by signs. Sometimes abstractions create whole classes of mathematical objects. For example, abstracting the set of Natural numbers 1, 2, 3, 4, 5, 6, … into a completed whole named 'N' does not result in an infinite number of names for all the members of N. We have a procedure for naming arbitrarily large natural numbers, but we can never name more than the members of a finite subset of N. Likewise mathematical abstraction creates many sets and classes of mathematical objects which can never all be named. Only a finite number of these mathematical objects can be named, even when the set to which they all belong is named. Thus abstraction, generalization and, in particular, the idealized completion of sets, sequences, and series cannot name all their members when they are infinite.[6]

Quine (1969) argues that the ontological commitment of any theory, mathematical or scientific, is to the domains of objects over which its variables range. Thus, Peano arithmetic, a scientific canonization of the rules of arithmetic, is ontologically committed to that which the variable n ranges over. This domain is N, the set of all Natural numbers, and so Peano's theory is committed to the existence of all of the Natural numbers. Zermelo–Fraenkel set theory is committed to the existence of all of the sets its variable x ranges over. This class of sets is very large and contains sets of several orders of infinite magnitude. In both of these examples, our ontological commitments, the classes of mathematical objects that the theories incorporate or bring into being includes many, many objects that cannot ever be all named. In both the cases of N and V,[7] the universe of sets created by Zermelo–Fraenkel (ZFC) set theory, the global mathematical object constructed is a mathematical domain, a space containing many mathematical objects. These are themselves mathematical objects that encapsulate the endless processes of generating their members, each becoming a single entity within the space of mathematics.

1.2.7 The Human Construction of Mathematical Objects

I have argued that mathematical objects are formed through actions on mathematical objects and signs which are then abstracted and reified into higher level mathematical concepts and objects. Notice that the verbs involved are all active: 'to abstract', 'to reify', 'to act', which all represent the action of a subject on an object.

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[6] This answers the criticism of Cole (2008) that names cannot be constituent of mathematical objects because there are too many objects to be named.

[7] V is the von Neumann class of hereditary well-founded sets.

The subjects in question are irrefutably humans. It is their (our) activities that create mathematical objects, for mathematical objects do not create themselves. Their performativity lies in the capacity of mathematical objects to engender actions and changes through the humans that use them; they are not possessed of any intrinsic or self-subsistent agency. It is humans who perform all these actions, and it is humans that abstract from these actions to create new mathematical objects.

Although mathematical objects are real, their reality is part of cultural activity and its products. Like all of culture, from money, clothing and cookery to languages, movies and ideas, mathematical objects are cultural objects. They are created in mathematical activities, which to a large extent can be represented as language games that take place within mathematical forms of life (Wittgenstein, 1953). Humans acting socially, within mathematical forms of life or mathematical practices, over time, are what create, enact, develop, and sustain mathematical processes, concepts, and objects. This is why the assumptions of presentism are problematic. They deny or disregard the passage of time which is ineliminable in the emergence and being of mathematical objects. Thus, for example, Endress (2016, p. 130) critiques John Searle's (1995) account of social construction because 'his entire work fails to answer or even discuss the question of how the status of "something," as well as its "functions," socially emerge'. This may not be Searle's focus but his analyses do partially indicate how the physical comes to have social function and so be socially constructed. However, unless one understands their becoming, the transitions in the formation of any social constructed entity, including mathematical objects, with its shift from process to structural object, one cannot fully understand what they are. Transitions and shifts occur over time, and these affect the constitution of the emergent mathematical objects, so time and becoming cannot be dispensed with.

Time is implicated in mathematics in three ways. First, there is historical time over which the mathematics in cultures comes to be and develops. I have considered in passing how counting and numeration systems have developed from oral counting, tally marks, and then written numerals of increasing complexity and sophistication. Second, there is personal time in which a person's knowledge of mathematics and grasp of its objects develops. I will say more about this in the next section, but development over time in this domain is undeniable.

Third, there is the foundational analogue of time, the logical development, over the course of which, starting with primitive notions, the theoretical framework of a mathematical theory develops. This last is not real time, but a strong analogue because of the logical before and after relations.[8] Concepts, definitions, results, and proofs are built up in a logical sequence when the later elements depend logically on the former ones.[9]

[8] Lakatos (1976) points out that the 'logical time' of justification often subverts the 'chronological time' of discovery, when the presentation of a completed proof inverts the order in which it was created.

[9] I reject a possible fourth aspect of time. This is the consideration of mathematics as having universal validity across time and space, as this contradicts the sequential emergence of mathematics, as well as the social constructionist assumptions of this chapter.

This is similar to Lakoff and Nunez's (2000) conceptual metaphor that maps from an image schema for temporal succession into the more abstract domain of logic. They call this the *logical consequence is temporal succession* metaphor.

Consider how Peano arithmetic begins with two primitive notions, a starting number, 0 say (historically Peano started with 1), and the successor function denoted by S such that S(*n*) is the successor of *n* (Peano used '+1'). It also includes a number of axioms. These make the following five assertions. Zero is a natural number. Every natural number has a successor in the natural numbers. Zero is not the successor of any natural number. If the successor of two natural numbers is the same, then the two original numbers are the same. Lastly, there is the induction axiom.[10] If 0 has a certain property, and whenever n has that property, so does $n + 1$, then all of the natural numbers have that property. There are also the three standard identity axioms (reflexivity, symmetry, and transitivity) specifying the properties of the equality relation (=).

On the basis of these small beginnings, the operations of addition, multiplication, and exponentiation can be defined as well as subtraction and division in the limited ways they apply to natural numbers. From this modest foundation, number theory can now be built up with increasingly complex concepts, functions, and theorems. Ontologically, it can be said that the initial axioms bring the natural numbers into being, within the formal theory, but as the theory progresses, new objects corresponding to the subsequently defined concepts and functions are also brought into being.

A realist, either a Platonist or another kind of realist, can respond to these assertions with the answer that Peano arithmetic does not create the natural numbers but merely provides an elegant and minimal axiomatization of the properties and assumptions underpinning the already existent natural numbers. Over history and in personal development, this is true. The historical growth in the formulation of the natural numbers and number theory does precede Peano's axiomatization. Foundationally this is not true, the simpler parts of the theory logically precede the more complex and dependent later parts. There is an irreversible flow, if not of time, of its logical analogue from the simpler to the more complex later parts of the theory.

My argument is that we also need to allow for time in philosophy, and in particular, in ontology. Both the objects of mathematics and the mathematical identities of persons, that I consider in the next section, grow and change their characters over time, they are subject to processes of becoming. Ontology needs to permit emergence and change in the entities for which it accounts. A static snapshot of being will not suffice to explain of what it is formed.

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[10] If P is a subset of N, and 0 belongs to P, and if *n* belonging to P implies *S(n)* also belongs to P, then P=N.

1.2.8 The Ontology of Mathematical Objects

At this point in the exposition, I have now introduced all the elements that make up the social constructionist account of the ontology of mathematical objects.

What constitutes mathematical objects? Mathematical objects are abstract objects in shared cultural space (the space of mathematics) constituted by rules and agreements established in and by the community of mathematicians, many of which are also sustained and upheld in wider society. These rules and agreements include the tacit rules and conventions into which mathematicians are socialized as they participate in the shared social practices of mathematics. In Wittgensteinian (1953) terms, these include agreement in forms of life in which the actions and practices of participants are aligned, that is, run in the same direction, often without this direction ever being explicitly articulated. Mathematical objects gain their legitimacy through usage, for every instance of use confirms their validity, both among mathematicians and in wider society. In addition, the patterns of, and connections within, their usage also gives them their meanings. Mathematical domains are also objects of mathematics, even if they are populated with an infinite number of mathematical objects which cannot all be named.

What 'stuff' are mathematical objects made from? Mathematical objects are reifications built from abstracted actions on simpler mathematical objects and actions. Humans have a capacity for and a tendency towards nominalization. Just as nouns are created in the nominalization of verbs describing actions, so too mathematical objects are created from the nominalization of mathematical actions. This is a process of reification, encapsulation, and transformation in which actions become structural objects (Sfard, 1994; Dubinsky, 1991). Furthermore, this process is cumulative with increasing levels of abstraction, as actions on simpler objects become more complex objects in themselves.

Where are mathematical objects to be found? Mathematical objects exist in the cultural space of mathematics, a shared domain of signs and operations, whose rule-governed uses provide their meanings. This domain is primarily added and used by mathematicians, but also widely accessed by the public for simple constructs like numbers, whose constitution links them to actions in the empirical world.

Why are mathematical objects objective? Mathematical objects are objective because at any given time they appear 'solid' (inflexible and invariant) founded on mathematicians' agreements and fixed, publicly shared uses in the domain of mathematics and beyond. Their uses are rule governed and there is widespread agreement without ambiguity as to correct usage. Once created mathematical objects 'detach from their originator' (Hersh, 1997, p. 16) becoming independent and self-subsistent entities within a shared domain, the cultural space of mathematics. However, if mathematical practices shift over time, so too may the rules and objects of mathematics themselves, reflecting such cultural shifts.

Why are mathematical objects and their relationships viewed as necessary? The necessity arises from the deontic nature of the rules of mathematics. Mathematicians' agreements are often tacit, being obligations assumed with participation in mathematical practices, and these determine what 'must be so'. The rules are imperatives, analogous to those that must be followed in order to play chess. To engage in mathematical activity, you *must* use the objects of mathematics in the prescribed ways. Mathematics and mathematical entities are non-contingent because they necessarily conform to and obey the rules, customs, and conventions of mathematics. Furthermore, mathematical results and theorems necessarily follow by logic from the axioms and assumptions laid down in mathematical theories.[11] Logic also rests on deontic necessity, for it follows laid down and inflexible tracks of reasoning that, it is accepted, 'must be so' (Ernest, In preparation).

Why do mathematical rules have the modal status of necessity? Mathematical rules and customs make up the institution of mathematics. The institutionalization of social processes such as mathematical practices grows out of the habitualization and customs, gained through mutual observation with subsequent mutual tacit agreement on the 'way of doing things' in these practices. Thus, to engage in a mathematical practice is to be habituated into the norms, customs, and uses of the rules and to follow and apply them unquestioningly as imperatives. Associated with institutions such as mathematics are a set of beliefs that 'everybody knows' (e.g. 'there is a set of natural numbers {1, 2, 3, 4, …}', 1 + 1 = 2, 50 + 50 = 100, 9 > 8, and so on). These beliefs make the institutionalized structure plausible and acceptable, thus providing legitimation for the necessity of the institution of mathematics (Berger & Luckmann, 1966). Much of the language of mathematical texts is imperative, in the deontic modality, as engaging in the practice ('playing the game') necessitates following the tacit and explicit rules and norms embedded in, and constituting the institution of, mathematics (Ernest, In preparation).

1.3 Human Subjects and Mathematical Identities

1.3.1 Being in Terms of Mathematical Identity

In this section, I aim to address and tentatively answer the ontological problem of mathematics education mentioned above. This problem concerns persons, for in mathematics education, the primary concern is with human beings, both learners and teachers. What is the nature of a living, thinking human being? We know it is (we are all) a biological animal but are there any special features of a human being that pertain to mathematics and its teaching and learning mathematics?

Here, my concern is what I term mathematical identity. By this I mean those acquired capacities in the child and adult that enable participation in mathematical activities. This could be termed a person's mathematical power or capability. I do

[11] Note that some of the axioms and assumptions that underpin mathematics can be contingent, as they may follow from mathematicians' choices, albeit constrained choices. The same holds for mathematical logic.

not mean the person's self-image, social image, or sense of belonging to a group, be it as a mathematician, mathematics teacher, or mathematics student. These are the sociological senses of the term mathematical identity widely used in mathematics education, such as in Owens (2008).

In considering the nature of individual human beings from the perspective of their capabilities, one must not overlook the social dimension: social practices, social groups, and even social constructions and structures. To think that individual persons exhaust all of social being is to fall into the reductionist traps of individualism. An active social group is more than a set of individuals. It includes a history of interactions with other individuals expressing themselves through actions and speech and reactions to involvement in such activities. The impact of the activities of the groups will be to change the individuals involved to greater or lesser extent. This is the fundamental principle of education, organizing social activities intended to help human beings to develop and become something different.

In this section, I want to consider both a fully grown person, an adult, and a developing person, a child. By looking at these two aspects of humanity, time has already been admitted, because a child develops over time into an adult. I also signalled here and above, in the introduction, that in considering the nature of human being, there is the problem of the ideology of individualism.

1.3.2 The Ideology of Individualism

The ideology of individualism is a perspective that puts individuals first. Not only is it a social theory favouring freedom of action for individuals over collective action, social responsibilities, or state control. It also positions the individual as ontologically prior to the social. Individualism may be related to the top-down position of the modernist metanarrative in which the 'gaze' of a reasoning Cartesian subject with its legitimating rational discourse is assumed to precede all knowledge and philosophy. The rational knower comes first and is a universal intelligence that is embodied to a greater or lesser extent in individual humans (Scheman, 1983). Modern individualism acknowledges that human beings are embodied, and we are more than just knowers for we also have drives. Our primary motivations are to seek our own survival and the satisfaction of our own, individual desires. Individualism validates this ethical self-centeredness.

According to the individualistic view, humans are entirely separate and independently living creatures (Rand, 1961). Although it is conceded that our independence is not wholly complete, because we do depend on each other for help in survival, nevertheless individualism emphasizes that we are autonomous, self-motivated, agentic creatures who have a great deal of freedom in choosing how we act and behave. Our capacities for understanding, knowing, thinking, and feeling belong to ourselves as individuals and to ourselves alone. Our consciousness is independent, unique, and unconnected with that of other people (Soares, 2018).

Individualism underpins various modern theories such as Piaget's genetic epistemology. According to Piaget, children develop individually following a number of inscribed stages in their growing understanding and capacities. There is an inbuilt logic to cognitive development, perhaps analogous with how a living organism grows, directed by its internal genetic programme. Thus, persons are all existentially separated creatures whose actions, learning, and even whose being take place in hermetically sealed personal domains. The social and physical environment may help or hinder a person's development, just like water, nutrients, and being located in a sunny place will help a plant to grow. But the endpoint or goal of growth is internally encoded and driven.

My criticism of this perspective is that it radically underestimates our ontological dependence on other fellow human beings (Lukes, 1968). First of all, we originate inside another human's body, our mother's, and cannot survive physically without close proximity to and regular attention from a primary caregiver including, but not limited to, feeding. Beyond physical survival our mental, emotional, and personality development requires caring attention for the first decade or two of the years of our lives (Lewis et al., 2000). That attention includes many thousands of hours of involvement with others in social activities through which we acquire spoken language, or an equivalent, and other aspects of cultural knowledge. The mechanism by means of which we make our needs known and receive assurances is conversation, understood broadly. This includes the pre-verbal enacted forms of conversation involving touching, holding, crying, pre-verbal vocalization, facial expressions, gestures, and other embodied actions. It is through such means that we learn the use of words and language. We also develop our identities as persons and our emotional being by these means. Thus, my objection to individualism is that although as primates we are separate animals, as humans we are socially constituted beings. Our very formation and becoming human depends essentially on the social experiences that shape us. We would lack our special human characteristics of shared languages, shared cultures and shared modes of thinking and being, were we hived off from each other in the way that individualism supposes. Our identities are socially constructed, and we could not be the full human beings that we are if we were not socialized and enculturated.

1.3.3 Conversation and the Social Construction of Persons

Ontologically, I want to distinguish between the biological genesis of the human animal and the cultural genesis of the human being as a person. Obviously, the animal provides the material and biological basis of being human, but my claim is that building on that basis, the human being needs to be socially constructed. At the heart of social constructionism lies the dialogical pattern of interactions and knowledge growth and warranting. The unit of analysis, the fundamental atom upon which social constructionism is built, is that of conversation. In its minimal manifestation, this occurs between two persons, who are communicating as participants in a jointly shared social activity, in a social context. There is a continuum of contexts in which conversations take place from face-to face preverbal and verbal interactions all the way to the mediated conversations using letters, emails, and other forms of media over extended distances and timespans.[12]

Conversation and dialogue are widely occurring and utilized notions across philosophy and the social sciences. For the philosopher Mead (1934), conversation is central to human being, mind and thinking. Rorty (1979) uses the concept of conversation as a basis for his epistemology. Wittgenstein's (1953) key idea of language games situated in forms of life is evidently conversational, and I draw on this heavily. Many other philosophers and theorists could be cited, including Gadamer, Habermas, Buber, Bakhtin, Volosinov, Vygotsky, Berger, and Luckmann, and more generally, social constructionists.

> Central to the social constructionist ontology is the view (shared with Gergen and Harré) that the primary human reality is conversation. (Shotter, 1993, pp. 13)

Because of the evidently interpersonal nature of teaching, references to conversation and dialogue are very widespread in the mathematics education literature. However, concerning the philosophy and foundations of mathematics, the references are more limited. But there is growing attention to conversational, dialogical, and dialectical interpretations and philosophies of mathematics (Ernest, 1994, 1998; Dutilh Novaes, 2021; Larvor, 2001).

The original form of conversation is evidently interpersonal dialogue, which consists of persons exchanging speech, or other constellations of signs generated or uttered during the period of contact, based on shared experiences, understandings, interests, values, respect, activities, demands, orders, etc. Thus, in Wittgensteinian terms, it is comprised of language games situated in human forms of life. 'One may view the individual's everyday life in terms of the working away of a conversational apparatus that ongoingly maintains, modifies and reconstructs his subjective reality' (Berger & Luckmann, 1966 p.152).

Two secondary forms of conversation are derived from this most immediate and primary form. First, there is intrapersonal conversation, that is thought as constituted and formed by conversation. According to this view, (verbal) thinking is an originally internalized conversation with an imagined other (Vygotsky, 1978; Mead, 1934). Intrapersonal conversation becomes much more than 'words in the mind', and the conversational roles of proponent and critic discussed below are internalized, becoming part of one's mental functions (Ernest & Sfard, 2018).

Second, there is cultural conversation, which is an extended variant, consisting of the creation and exchange of texts at a distance in embodied material form. I am thinking primarily of chains of correspondence be they made up of letters, papers, email messages, transmitted diagrams, and so forth, exchanged between persons. Such conversations can be extended over years, lifetimes even. It can be argued that

[12] In this chapter I use dialogue and conversation interchangeably. Some authors use 'dialogue' to mean a democratic and ethically more valuable type of conversation. Here I am using these terms descriptively without prescriptive attribution of greater ethical value to one over the other.

they extend beyond a single person's lifetime, if new persons join in and maintain and extend the conversation. Indeed, human culture made up of the ideas, texts, and artefacts made and shared and exchanged by people over millennia has been termed the 'great conversation' (Hutchins, 1959; Oakshott, 1967).

These three forms of conversation are all social. They are either social in their manifestation, in the case of interpersonal and cultural conversations, or social in nature and origin, as is the case with intrapersonal conversation. In the latter, thinking is a conversation one has with oneself, based on one's experience of, and participation in, interpersonal conversations (Sfard, 2008). In all manifestations, stemming from its interpersonal origins, conversation has an underlying dialogical form of ebb and flow, comprised of the alternation of voices in one register followed by another in the same register or of assertion and counter assertion. Conversations result in affirmation and bonding, unless the responses are in the less common forms of negation, refutation, rejection, or the silencing of a speaker. Just cooperation in the form of keeping the channel open provides feelings of enhancement for the speakers. More generally, a fully extended concept of interpersonal conversation including non-verbal communication, mimesis and touch encompasses all of human interaction and is the basis for all social cohesion, identity formation, and culture.

In addition, I wish to claim that all human knowledge and knowing are conversational, including mathematics. Elsewhere I have described the specific features of mathematics that support this analysis, namely that many mathematical concepts are at base conversational, as are the processes of discovery and justification of mathematical knowledge (Ernest, 1994, 1998). However, a word of caution is needed before I further develop conversational theory. Although mathematics is at its root conversational, it is also the discipline *par excellence* which hides its dialogical nature under its monological appearance. Research mathematics texts expunge all traces of multiple voices, and human authorship is concealed behind a rhetoric of objectivity and impersonality. This is why the claim that mathematics is conversational might seem so surprising. It is well hidden, and it subverts the traditional view of mathematics as disembodied, superhuman, monolithic, certain, and eternally true.

1.3.4 The Critical Roles of Proponent and Responder in Conversation

Conversation is the basis of all feedback, whether it be in the form of acceptance, elaboration, reaction, asking for reasons, correction, and criticism. Such feedback is in fact essential for all human knowledge growth and learning. In performing such functions, the different conversational roles include the two main forms of proponent and critic, which occur in all of the modes (inter, intra, cultural), but originate in the interpersonal.

The role of the proponent lies in initiation, reaching out, putting forward an idea or emergent sequence of ideas, a line of thinking, a narrative, a thought experiment, or a reasoned argument. The aim is to share feelings, make demands, communicate an idea, build understanding, or convince the listener (Peirce, 1931-58; Rotman, 1993). Elsewhere I have described how this is the mechanism underpinning the construction of new mathematical knowledge (Ernest, 1994, 1998).

In contrast, there is the role of responder, including critic, in which an utterance or communicative act is responded to in terms of acknowledgement of its comprehensibility, acting in response to a request, actively demonstrating a shared understanding, providing an elaboration of the content, or in other ways. In the role of the critic, the action or utterance may be responded to in terms indicating weaknesses in its understandability and meaning, its weaknesses as a proposal, its syntactic flaws, how it transgresses shared rules, and so on. Critical responses need not be negative, and the role of the responder includes that of friendly listener following a line of thinking, narrative, or a thought experiment sympathetically in order to understand and appreciate it, and perhaps offer suggestions for its extension variation or improvement.

In conversation, ideally the voices or inputs of the proponent and critic alternate in a dialectical see-saw or waltz pattern. In its most rational or mature discursive mode, the proponent puts forward a thesis. The critic responds with a critical antithesis. Third, the proponent, prompted by the critic, modifies the thesis and puts forward a synthesis, a correction, or replacement that is the new thesis in the next iteration of the cycle. Thus, we have a dialectical process approximating the thesis-antithesis-synthesis pattern. In this cycle, the speed of the iterations can vary greatly. In a face-to-face conversation about a mathematical problem at a whiteboard, there can be many mathematical back-and-forth contributions in the space of an hour. But in submitting a mathematics paper to a teacher or journal, it may be that weeks or months pass before critical feedback is received.

From the outset, or nearly so, persons will adopt both the positions of proponent and critic, sometimes within the same conversation. This can also be the case with intrapersonal conversations in which, say, someone thinking about anything, such as a mathematical problem, having internalized these roles alternates between proponent and self-critic.

These two roles are widely present in teaching (teacher/expositor vs. examiner/assessor) and learning (listener/engagement with learning tasks vs. responder/reviser following formative assessment). Indeed, my claim is that these two roles reappear throughout all human social interactions in the form of communicator and responder, although not all elements of conversation need necessarily fall into these two categories.

1.3.5 The Significance of Conversation in Social Activity

Wittgenstein (1953) interprets human living in terms of his fundamental concepts, language games, and forms of life. Language games can be understood as conversations, and these are embedded in human forms of life, that is as social activities. Every social activity has a purpose, a goal, and language and conversation are communicative techniques for working together towards that goal. Examples include mothering an infant with the goal of the infant flourishing (holding, feeding, responding, etc.), working together in a carpentry workshop with the aim of building furniture, working mathematical problems in a classroom with the goal of learning mathematics, and so on.[13] In all such activities, the goal of the activity comes first, and the ways of working, the conversational communications are all about furthering the goal. In such activities, both roles of conversation are important. Conversation can help to focus attention, bond the participants, and direct activities.[14] In this context, the role of responder or critic is vital. When a colleague or more expert participant demonstrates or suggests a way of working or guides the other utterances of the sort 'like this, not like that' embody the role of the critic. This can take the form of Show, Copy, Guide (correction). The teacher shows an action, the learner copies the action, and the teacher guides and corrects the action. By teacher, I mean anybody in the role of guiding partner and correcting responder, whether they be a parent, peer, schoolteacher, workmate, or trainer; in short, the more knowledgeable the other within the learner's Zone of Proximal Development (Vygotsky, 1978). Given that meaning is largely given by use, following Wittgenstein (1953), through guiding and correcting use, the more knowledgeable other is shaping the associated meanings for the learner.

I want to stress that this process, this mode of interaction, is vital in learning how to conduct any practice. This is not only true in concrete production practices, such as carpentry, baking, building brick walls, and so on. It is the mechanism by mean of which all social rules are communicated. These rules include the correct use of spoken language, modes of acceptable behaviour in public, how to treat people, animals and things (ethics in practice), mathematical activities, and so on. This conversational mechanism is how the rules and agreements that make up social institutions are communicated and maintained. Some rules and agreements may have explicit linguistic formulations, like laws of the land, or mathematical axioms, but by far the majority of socially accepted rules and agreements are implicit and are learned through copying others' performances and the novice's own corrected usage.

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[13] This is not to deny that persons working together in a shared practice can have different goals, such as reluctant student not fully participating in the classroom practice that the teacher is directing.

[14] Conversation can also be used to further separate the interlocutors by asserting and reinforcing power and status differences, such as a teacher imposing order on an unruly class.

1.3.6 The Realities We Inhabit

I want to start with the assumption that all humans share some indescribable underlying realities. My claims about this shared reality are ontological not epistemological, we can never fully know these realities, but we learn how to operate in them. In some unchallengeable pre-scientific and pre-philosophical sense, human beings all have the experience of living together on the Earth. As a common species, we have comparable bodily functions and experiences that make our sense of being who we are and of daily life commensurable. In virtually all cases, these shared realities are in fact the social activities in which we participate.

Heidegger's (1962) view is that we all have a given, 'thrown' preconceptualized experience of being an embodied person living in some sort of society. He celebrates authenticity, our 'being-here-now' existence (*Dasein*), an attitude that acknowledges our multiple existence in the linked but disparate worlds of our experience: the bodily, mundane, discursive, political, professional, institutional, and cultural realms. Our experience in these social and worldly forms-of-life is taken for granted. It provides the grounds on which all knowing and philosophy begins, although no essential knowledge or interpretation of the basal lived reality is either assumed or possible. This is a bottom-up perspective that contrasts with the top-down position of modernist metanarratives in which a legitimating rational discourse and the 'gaze' of a reasoning Cartesian subject is assumed to precede all knowledge and philosophy. The rational knower does not come first, he (and I use the masculine deliberately) is not a universal disembodied intelligence, but a construction with historically shaped sensibilities. Once again, this illustrates the need to accommodate growth, development, emergence, and becoming in ontology.

Virtually all of our capacities are shaped by the social practices in which we participate. We could not learn, understand, use, or make mathematics unless we were educated, language and sign using social beings with personal histories and mathematical learning trajectories. Like every (academic or school) subject, mathematical knowledge requires an already present knower, and this must be a fleshy, embodied human being with both developmental history (including an educational history) and a social presence and location. Obvious as these statements are, they have been ruled irrelevant and inadmissible by generations of philosophers and mathematicians that subscribe to an absolutist or Platonist philosophies of mathematics.

Paramount amongst the realities we inhabit are the social institutions of which we are a part. Our understandings, as evidenced by the ways in which we participate, are shaped by long histories of conversational exchanges, situated in various social practices. These formative conversations have not only inducted us as participating entrants to, and members of, the institutional practices and domains. They have also shaped our actions so as to be maintainers and onwards developers of the social institutions. Social institutions and social realities are kept alive and afloat (metaphors for continuing and enduring existence) by the myriad actions of the participants in reaffirming the conventions and rules intrinsic to the institutions.

These reaffirming actions are directed through conversations with both insiders, that is participant members, and outsiders, demarcating the rules, norms, conventions, and boundaries that define and constitute the institutions and entry to the associated social practices.

1.3.7 Conversation and the Genesis of Thinking

An important part of conversation theory concerns how it is implicated in the genesis of thinking and constitutive of thought. To sketch the genesis of thinking, we start with a human baby with its sense impressions of its world of experience, probably beginning in a rudimentary way during its period of gestation. Some of these sense data originate from outside its own body, such as from light (impacting via seeing), from sound (via hearing), from touch (via skin pressure nerve arousal). Some of these experiences originate from within the baby's own body, such as hunger, bodily discomfort, and what we might see as spontaneous emotions. These two sources of experience are deeply interwoven. The distinction is far from absolute since sensory inputs must be interpreted in both cases. I believe that the baby notices invariants and starts to impose some order, structure, or pattern on its experiences giving rise to what Vygotsky (1978) calls spontaneous concepts. What such concepts are I cannot say precisely but they may not only include regularities in sensations but also regularities in responses such as movements, vocalizations, etc. Undoubtedly, these concepts vary and grow over time; they are not static and need not be constant. The baby is not isolated in this world of experiences, actions, and concepts because the baby is involved in preverbal dialogues, comprising reciprocal actions and what we might call signalling with others, most notably the mother or primary caregiver.

At this stage, it is hard to know what the baby's thinking is like. Presumably there will be 'inner' sensations and experiences such as pleasure and discomfort or displeasure, recognition of familiar persons, objects and experiences, desires, associated emotions and feelings, sensory images recalled from memory. There may well be reactions to familiar and non-familiar persons, objects, and experiences, accompanied by emotions such as interest, curiosity, desire. The baby will also experience negative emotions including anger, anxiety, or fear, in response to such experiences as being startled by sudden loud noises. What the flow of ideas and experiences brought into consciousness is like I cannot say, but I expect it will be led by sensory stimuli, whether external or internal in origin.

Now we move to the next stage, although of course this overlaps with the preverbal phase, and ultimately engulfs it, as I shall argue. Other persons, such as the mother, will start to use words with the baby, beginning a verbal dialogue, accompanying embodied exchanges such as looking, touching, holding, rocking, and so on. There are intermediary phases in the development of language such as the baby babbling in what we can interpret as pretend speech in the 'game of talking'. After some exposure to adult speech, the baby will start to use words back, mummy, daddy, ball, dog, or whatever. The baby starts to use these words in a regular and recognizable way. At this stage, the baby/young child is starting to develop what Vygotsky (1962) terms scientific concepts, which would be better termed social or cultural concepts. The use and mastery of language takes quite a long time and during this time the child develops and uses a growing set of linguistic capabilities. Of course, this development is triggered by engagement in a growing range of activities with accompanying dialogues in different contexts, with different purposes, and with different but overlapping vocabularies.

Somewhat later during childhood, after the acquisition of spoken language, most children also start to learn to read and write, and these encounters with written language may also feed into the development of their thinking. This includes written arithmetic and other parts of mathematics. However, I won't speculate on the impact of reading and writing on thinking beyond its role as an add-on and expansion of spoken language.

A second strand of development concerns attention, which is part of human agency. A baby turns to look at objects or people that interest it or that move and draw their attention. Part of this is following their mother's or caregiver's gaze (Deák, 2015). Of course, other sensory stimuli also capture its attention, sounds, touch, smells, tastes, pain, and so on. As the child develops, its power of self-directed attention grows and becomes increasingly volitional. In addition to choosing what to attend to in its experiential (perceptual) world, the child can choose and initiate its own activities. It can direct its attention to different activities including toys, games, video, TV programmes, touch screens, nature, animals, and other things. One of the most important things that a child attends to is other humans and dialogue. The child attends to many utterances from others and participates in dialogues.

So now the stage is set for me to propose what thinking is or at least might be. According to Vygotsky (1962), the child's spontaneous and scientific (that is, linguistically acquired) concepts meld or at least start to interact and form one inner system of concepts from quite early on. Words and linguistic utterances have been experienced in various contexts and the uses they are put to and the activities they are a part constitute their initial meanings. Young children will have spoken dialogues with themselves in which they may instruct themselves mimicking what they have experienced with more capable, older speakers. After a while, these self-directed conversations become silent, internalized but perhaps visible through sub-vocal lip movements.

On this basis, children's private thinking consists of an inner dialogue the person has with themselves. This is learned from participation in conversations and discourse with others. But this inner dialogue is not just made up of words – it is supplemented by and may even have elements replaced by visual imagery, memory episodes, feelings (emotions, etc.) within the experienced stream of ideas. An associational logic is at play so perceived external persons, objects, or events may trigger associations that become contents in the inner dialogue. Thus thinking, the inner dialogue, may be a string or cluster of meanings, concepts, or reasonings. This may be prompted by external stimuli, such as conversations/speech from someone else, experiences or events in the world, or may be internally generated, such as when I solve a mathematical problem mentally. The stream of ideas, etc., that I experience in thought is multimodal and can involve words and associated concepts, imagery both real and imagined, smell and touch impressions, or memories of them, etc. We also have some control over this internal dialogue, we can choose to remember something, direct our attention to some idea, memory, problem, etc. Of course, things also come unbidden to our thought, either because of some deep unconscious trigger or an association that draws our attention aside or onwards.

Although our thought originates in interpersonal conversation, in becoming intrapersonal conversation, it differs from public speech. For as conversation is internalized, it combines with our preverbal thought, sensory perceptions, visual images, emotions to become a richer multimodal conversation we have with ourselves. All these aspects as well as personal meanings are attached to the words and signs we use. Thus, we can think spatially as well as verbally. Vygotsky (1962) argues that the contents of our mind are not structured the way our speech is. When we engage in social, interpersonal conversations we also communicate multimodally using gestures, expressions, tone of voice, objects, and other props, as well as our oral linguistic utterances.

Our thinking, this internal stream of ideas and thoughts, is a dialogue in three ways. First, learning to speak is by means of participation in dialogue and conversation. So languaging is a process driven by public speech, that is words and speech. These evoke meaningful concepts and reasoning responses in us – their content and form are irrevocably tied in with their origin, that is spoken dialogue or conversation. Vygotsky is often interpreted as saying that speech and dialogue become internalized. This is of course a metaphorical rather than a literal description. Children learn to imitate phrases. I expect they can also imagine the sounds of these utterances subvocally, that is solely in the mind. So, exposure to speech leads to something like speech in the mind.

Second, our streams of thought come in segments. How these are demarcated or segmented varies, but each segment will have a coherent meaning. Each of these thought segments evokes an association or follow on, a response or reply. Thus, we follow each thought by its echo or answer, like question and answer, thus exhibiting the dialogue form. Just as in a spoken dialogue, we have choices as to how to choose/make our replies thus steering the conversation. Likewise in our internal dialogue, we can choose how to follow on a line of thought. Of course, some people with compulsions find it difficult to steer away from a recurrent pattern of thought. Indeed, this can happen to any of us if we are stressed by a difficult situation or conflicting or difficult demands or an unsatisfactory ending to a previous conversation. So, all my general claims must be hedged with caveats because less typical events and cases can always occur.

Third, our thinking is dialogical when we are reacting to an artefact – a piece of writing, painting, a performance, or even someone talking including a lecture. The attended-to part of the artefact is one voice in the conversation and our reactive or reflective thoughts constitute the second voice, which we may or may not utter in public.

Our internal dialogue can have a variety of functions. It might involve planning some action, a solution to a problem, a plan for making something, the development of ideas. This is imagination at work in thought. This may involve all sorts of meanings including concepts, word meanings (associations), visual imagery, practical sequences of actions. However, such planning or creative imagination need not anticipate or take place separately from our activities. For often we can be involved in making something, such as me writing these observations, and not know beyond a hazy idea, if one has that, where our stream of ideas or words – our internal dialogue – is going to lead. Often our next step in the creative process is enacted as the moment arises. It is a choice, often what feels like the right choice, possibly the necessary choice, but made in the moment.

1.3.8 Extending the Meaning as Use Theory

Following Wittgenstein (1953), I have adopted his operationalization that the meaning of a word is in many cases given by its use. However, this needs disambiguation, for 'use' has multiple meanings. The particular use which I make when I utter the word 'red' or 'addition', say, at a specific event within a particular form of life is one such enactment of meaning. But the *system* of use or usage has another meaning. This includes a systematic grammatical theory of usage that describes past correct usages and potentially includes future correct uses or at least the rules that will guide them. De Saussure made this point when distinguishing *Parole* (utterances of spoken language) from *Langue* (the system of language). Specific uses are one thing but systematic patterns of use which entail imperatives about future specific uses are another.

What one can say is that the spoken utterance meaning of use comes first. Use in the systematic, theoretical sense is secondary to specific instances of participation in conversations and making or hearing utterances. (Parole precedes Langue.) There is a history (we all have histories) of language uses, and we all have a set of memories of instances of language uses – our own and others. In addition, these memories will include the corrections we have received, observed, or given, via conversations, which have shaped our capacities for spoken language. In fact, we may not remember many such corrections, but our linguistic know-how will have been shaped by such instances of correction and correct usage, from childhood on. Many persons will not have explicit or full theories of word use, but have the capacity to make and understand meanings from word utterances based on their implicit know-how.

In the present context, the significance is that the meaning of a word as given by a specific utterance or instance of use is only partial. In the broader sense, meaning as use depends on a whole pattern of usage to give a better indication of meaning. This pattern might only be encapsulated in a tacit set of guidelines or intuitions whose function is, in effect, to regulate which uses are correct and which are not.

At this point, Robert Brandom's (2000) inferentialist account of meaning is helpful. For Brandom, the meaning of words and sentences is largely given by their use in language, but it is a central aspect of use, namely the nexus of inferential connections with other words and sentences. For Brandom, the inferentialist meaning of a word or sentence S is its connections through reasoning with antecedents (reasonings leading to) S and its consequences (reasonings that follow from S). These uses are shown through enacted utterances, but meaning reflects past uttered links and is always open towards the future. So, the current meaning of a word or sentence, at any time, is partial and never final, for further patterns of use will supplement the meaning. As Wittgenstein (1978) says, a new proof of a proposition, changes the meaning of the proposition. Adding logical antecedents or consequents to a sentence changes its meaning.

1.3.9 Dialogic Space

I have considered how children acquire language and the ability to communicate meanings. In addition, I have described how children internalize conversation as a basis for their thinking. On this basis, I can now offer an account of the zone in which meanings are communicated and shared, termed dialogic space (Wegerif, 2013; Lambirth, 2015). This is the virtual space in which words, gestures, and signs are uttered, perceived, and responded to. Dialogic space or spaces are both public and private. A conversation between persons has 'visible' multimodal utterances which are public, but also runs through our private spaces of understanding where we attend to the dialogue and create or conjure up associations, narratives, imagery, emotional responses in our reception of the dialogue. We may engage in an intrapersonal dialogue in response.

Figure 1.1 represents some of the basic elements of dialogic space and its participants. I emphasized the key actions with of italics. As participants, through *listening* we pay *attention* to what is being said, understanding it in terms of building the *meaning* links to what we know (the network of words and concepts to which we have personal access). Through understanding we take personal *ownership* of the meaning links to antecedent and consequent expressions in our network of reasoning relations. When we have an *expressive* impulse, we loosely assemble the *idea* or remark and express it as our *chosen* supplement to the dialogue that we *utter*. (Note that our remark is not usually created in private and then uttered. It normally comes into being as it is uttered.) Every participant in the dialogue does this. Participants also own a set of *rules* about how the dialogue should be conducted in terms of participative membership, the appropriate form of contributions, and the conceptual content of contributions. This overall process is illustrated in Fig. 1.1.

Thus, in addition to exploring and developing the ideas under discussion (the content of the dialogue) participants' contributions can also be utterances that are about regulating or policing the dialogue based on rules that should reflect shared values and democratic principles. For example, in a dialogue between friends and colleagues, one or more contributors may intervene about imbalances in contributions, such as some participant speaking too much or another being

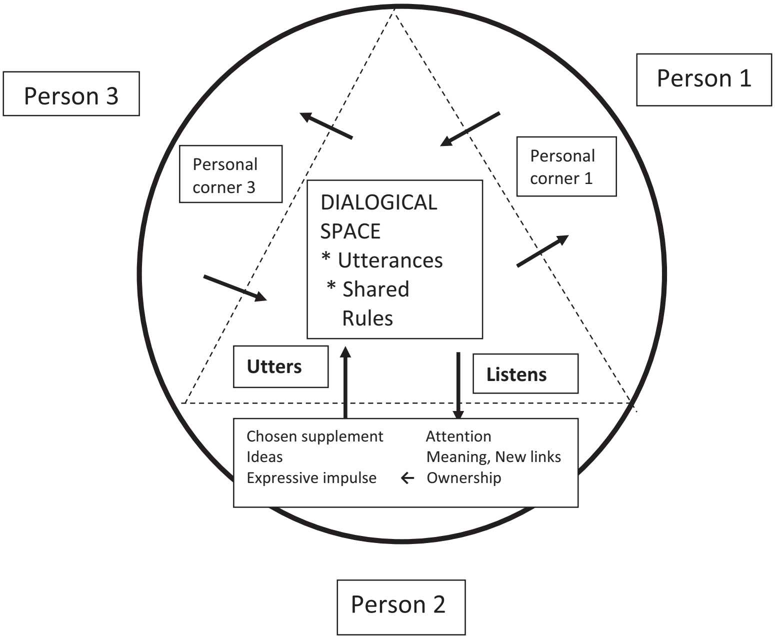

Fig. 1.1 Dialogic space with its personal corners

encouraged to contribute and be attended to. There can also be rules-based utterances on the content of the dialogue, which may be commenting on, redirecting, or curtailing some contribution to steer the direction or thrust of the dialogue in terms of the content and concepts discussed. But the most important part is the pattern of utterances that extend and develop the subject matter, the joint understanding of a topic, the solution of a shared problem, or a creative ensemble made by the group. Not mentioned but underpinning the dialogue is participation in a shared activity (a form of life) which may be purely conversational or may be making or doing something, accompanying the conversation.

1.3.10 Roles and Power Differentials in Conversation

In any dialogue, persons as active agents in that dialogue take on a variety of roles. Two of the most important are speaker and listener. Speaking can involve offering new links that are responses to the previous utterances. Such responses can build on, extend what was previously said. Or they can interrogate and question what was said. Listening can be actively following the narrative and making sense of it through linking utterances with our own concepts and meaning associations. We can follow the flow of a narrative adding our own associations and responses, which we may (or may not) utter audibly or publicly. We can listen critically whereby we interrogate, question, or challenge the narrative as we are hearing it. We can make these reactions public or keep the thoughts to ourselves. And we should never forget that we are embodied, not just passive in listening or active in speaking – we are all the while engaged in bodily activities beyond the actions of communicating (vocalizing, facial expressions, arm, and bodily movements) for we can also be drinking coffee, walking along a road, or even building a model or material artefact together in some shared activity in our joint form of life.

In addition, there are power differentials between contributors in most dialogues, based on personal force or institutional authorization. Table 1.1 lists some sample types of conversation with the relative power of the participants indicated.

Table 1.1 exemplifies the more powerful within institutionalized groups as those, not only with knowledge of the rules (for progressing towards the group goal) but, most importantly, being institutionally authorized to impose the rules in regulating the activity. In informal groups, power is softer and may shift among participants to those with better knowledge of the rules, but without institutional authorization, they may be challenged and have to try to demonstrate the validity of the rules they are suggesting.

Table 1.1 Types of conversation and the relative power of participants

Type of conversation	More powerful participants (MPP)	Less powerful participants (LPP)
Family – Parenting	Parents – more knowledgeable and laying down behavioural rules	Children
Working in learner's zone of proximal development	More knowledgeable parent, teacher, or peer demonstrating rules, etc.	Learner
Friends in discussion	Power may move around group	Power may move around group
Collaborative work on school mathematics problem	Asserter of mathematical rules or moves is MPP at the time	Proposer of next step needing to be regulated (LPP at the time)
Collaborative research project	Power moves around group	Working researchers less powerful if there is principal researcher
Informal conversation between colleagues	Power moves around group unless power hierarchy has been established	Power moves around group unless power hierarchy has been established
School maths class	Teacher directs teaching and the learning activities	Student follows teacher instructions and rules for participation
School maths examination	Examiners	Students (examinees)
University seminar	Visiting lecturer	Audience – but audience can take some power in the questions slot
Journal editorial board	Editor, referees	Author

1.3.11 Mathematical Enculturation

Mathematical enculturation takes place over the course of development from childhood to adulthood. Prior to elementary schooling commencing at 5 to 7 years of age, the child will typically gain a growing mastery of spoken language and very likely engage in simple number and shape games. Typically, these will include learning and using the names of simple geometric shapes (square, circle, triangle, ball, etc.) and spoken number names (one, two, three, four, five, etc.) as well as the correct order of these first few names. There will also very likely be some learning of the single digit numerals (1, 2, 3, 4, 5, etc.).

Such activities will continue in kindergarten and early elementary school plus the introduction of elementary operations, most notably addition and the addition sign '+'. Mathematical activity for learners typically shifts from being wholly spoken, to spoken and textual, with a shift towards the dominance of text for children's activities. Children very likely will engage in enactive activities (counting objects such as buttons), activities presented in iconic forms (working simple tasks mostly shown with repeated pictures, such as simple flower pictures), moving on to symbolic work with texts using words and mathematical symbols.

Perhaps the most central activity in the mathematics classroom is the imposition of mathematics learning tasks on students (Ernest, 2018a). These will be orally or textually presented and may be enacted in a variety of media. But over the years of schooling, throughout elementary and secondary (high school), these will become almost exclusively presented via written texts. A mathematical learning task:

1. Is an activity that is externally imposed or directed by a person or persons in power representing and on behalf of a social institution (e.g. teacher).
2. Is subject to the judgement of the persons in power as to when and whether it is successfully completed.
3. Is a purposeful and directional activity that requires human actions and work in the striving to achieve its goal.
4. Requires learner acceptance of the imposed goal, explicitly or tacitly, in order for the learner to consciously work towards achieving it[15].
5. Requires and consists of working with texts: both reading and writing texts in attempting to achieve the task goal.
6. A mathematical task begins with a mathematical representation (text) and requires the application of mathematical rules to transform the representation, in a series of steps, to a required end form (e.g. in a calculation, the numerical answer).

Power is at work in a mathematical task at two levels. First, at the social level, the teacher imposes the task and requires that it be attempted by the learner. Second, within the task itself, power is at work through the permitted rules and transformations

[15] Gerofsky (1996) adds that tasks, especially 'word problems', also bring with them a set of assumptions about what to attend to and what to ignore among the available meanings.

of the text. In other words, the apprentice mathematician must act as a conduit through which the imperatives of mathematics work. They must follow certain prescribed actions in the correct sequence. As the tasks become more complex, the apprentice mathematician will have some choices as to which rules to apply in constructing the sequence of actions or operations towards the solution, but otherwise all is imperative driven. Mathematics is a rule-driven game, and the rules are a major part of the institution of mathematics.

Later in the process of mathematical enculturation, the institutional rule-based nature of mathematics is internalized, and apprentice mathematicians adopt a more general concept of mathematical task that includes self-imposed tasks that are not externally imposed and not driven by direct power relationships.[16] However, in research mathematicians' work, although tasks may not be individually subject to power relations, particular self-selected and self-imposed tasks may be undertaken within a culture of performativity that requires measurable outputs. So, power relations are at play at a level above that of individual tasks. Even where there is no external pressure to perform, the accomplishment of a self-imposed task requires the internationalization and tacit understanding of the concept of task. Such an understanding includes the roles of assessor and critic, based on the experience of social power relations. This faculty provides the basis for an individual's own judgement as to when a task is successfully completed. Within institutional rule-based mathematics imperatives are at work, the dominant actions (rules) inscribed within the texts themselves. The role of the critic is to judge that the institutional rules of mathematics are applied appropriately and followed faithfully.

Mathematical learning tasks are important because they introduce the learner to the rules of mathematics and its textual imperatives. For this reason, such tasks make up the bulk of school activity in the teaching and learning of mathematics. During most of their mathematics learning careers, which in Britain continues from 5 to 16 years and beyond, students mostly work on textually presented tasks. I estimate that an average British child works on 10,000 to 200,000 tasks during the course of their statutory mathematics education. This estimate is based on the not unrealistic assumptions that children each attempt 5 to 50 tasks per day, and that they have a mathematics class every day of their school career (estimated as 200 days per annum).[17]

A typical school mathematics task concerns the rule-based transformation of text. Such tasks consist of a textual starting point, the task statement. These texts can be presented multimodally, with the inscribed starting point expressed in written language or symbolic form, possibly with illustrative iconic representations or

[16] There are also more open mathematical tasks such as problem solving (choose your own methods) and investigational work (pose your own questions) in school but these are not frequently encountered.

[17] Much of the mathematics education literature concerns optimal teaching approaches intended to enhance cognitive, affective, critical reasoning or social justice gains (prescriptive). Here my concern is just with teaching as a process that enables students to learn mathematics, without problematizing the teaching itself, that is, purely descriptive.

figures. In the classroom, these are typically accompanied by a metatext, spoken instructions from the teacher. Learners carry out such set tasks by writing a sequence of texts, including figures, literal and symbolic inscriptions, ultimately arriving, when successful, at a terminal text which is the required 'answer'. Sometimes this sequence of actions involves the serial inscription of distinct texts. For example, in the case of the addition of two fraction numerals with distinct denominators or the solution of an equation in linear algebra. Sometimes this involves the elaboration or superinscription of a single piece of text, such as the carrying out of 3-digit column addition or the construction of a geometric figure. It can also combine both types of inscriptions. In each of these cases, there is a common structure. The learner is set a task, central to which is an initial text, the specification or starting point of the task. The learner is then required to apply a series of transformations to this text and its derived products, thus generating a finite sequence of texts terminating, when successful, in a final text, the 'answer'. This answer text represents the goal state of the task, which the transformation of signs is intended to attain.[18] In some solution sequences, new texts will be freshly introduced, such as axioms, lemmas, or methods, and therefore are not strictly transformations of the preceding text but play an integral part in the overall sequence.

Formally, a successfully completed mathematical task is a sequential transformation of, say, n texts or signs ('$\boldsymbol{S}_i$') written or otherwise inscribed by the learner, with each text implicitly derived by $n\text{-}1$ rule based transformations ('$\Rightarrow i$').[19] This can be shown as the sequence:

$$S_1^{\Rightarrow_1} S_2^{\Rightarrow_2} S_3^{\Rightarrow_3} \ldots^{\Rightarrow_n -1} S_n$$

$\boldsymbol{S}_1$ is a representation of the task as initially inscribed or recorded by the learner. This may be the text presented in the original task specification. However, the initial given text presenting the task may have been curtailed, or may be represented in some other mode than that given, such as a figure, when first inscribed by the learner. $\boldsymbol{S}_n$ is a representation of the final text, intended to satisfy the goal requirements as interpreted by the learner. The rhetorical requirements and other rules at play within the social context and following mathematical imperatives (the mathematical rules) determine which sign representations $\boldsymbol{S}_k$ and which steps, $\Rightarrow_k$ for $k < n$, are acceptable. Indeed, the selection of mathematical rules applied, and the transformed representations inscribed by the learner, up to and including the final goal representation

[18] I use the word text broadly to include whatever multimodal representations are required in the task including writing, symbolism, diagrams and even 3-D models.

[19] Normally learners of school mathematics are not expected to specify the transformations used. Rather they are implicitly evidenced in the difference between the antecedent and the subsequent text in any adjacent (i.e., transformed) pair of texts in the sequence. In some forms of proof, including some versions of Euclidean geometry not generally included in modern school curricula, a proof requires a double sequence. The first is a standard deductive proof and the second a parallel sequence providing justifications for each step, that is specifications for each deductive rule application. Only in cases like this are the transformations specified explicitly.

($\boldsymbol{S}_n$), are the major focus for negotiation and correction between learner and teacher, both during production and after the completion of the transformational sequence. This focus will be determined according to whether in the given classroom context, the learner is required only to display the terminal text (the answer) or a sequence of transformed texts representing its derivation, whether calculation, problem solution, proof, or application of mathematics (Ernest, 2018a).

The extended apprenticeship in completing many thousands of mathematical tasks over the years of schooling, where successful, represents an enculturation into the social practice of mathematics. For persons going on to use mathematics professionally in their careers, or going on to be professional mathematicians, this apprenticeship is extended and intensified with introduction to more abstract and complex mathematical topics with a greater range of more sophisticated rules, as well as more demanding mathematical tasks. This occurs over the years of college or university specialization in mathematics. The extensive involvement and engagement with mathematical practices and activities or tasks result in a deeper engagement with mathematical rules. Some of these rules become automatic and are identified with the nature of mathematical objects. Imperatives become inscribed in mathematical objects so they cannot be seen as existing without what are deontic rules prescribing their possible uses. The journeyman mathematician accesses a cultural realm of mathematical objects that have all the appearances of solid real objects (within their own realm) whose nature necessitates a limited and prescribed range of properties and powers. These are embodiments of the rules that brought the objects of mathematics into being and limit their possible uses.

At this stage, the student or apprentice mathematician has developed into a practicing mathematician, and signs, symbols, and concepts of mathematics correspond to full independent mathematical objects embodying the restrictions and rules of their possible uses and the community-wide agreements as to their constitutions and operability.

During the pursuit of mathematical activities, mathematicians and others engage in extended work with texts and symbols in dialogic space. This crystallizes into 'math-worlds' where the objects of mathematics have a speaker-independent existence and reality. It is not only these objects whose independent being is confirmed and strengthened. It is also persons' identities as mathematicians that is validated by their access to dialogic space and its population of mathematical objects. However, this description is deceptive, for it is the submission to internalization and absorption of the many rules, norms, and tacit agreements through which mathematical activities and objects are constituted, that makes a mathematician. Just as these institutions bring forth the objects of mathematics, so too they bring forth the special powers and capabilities of the mathematician qua mathematician. Namely, a person empowered mathematically through obedience to the deontics of mathematics. Of course, a mathematician is not beaten down and cowed through this submission. The many thousand-fold experiences of successful pursuit of the goals of mathematical tasks have shaped, sharpened, and directed the desire of the mathematician to answer the questions, solve the problems, pursue the holy grail of proving new theorems. A chess master internalizes the rules of chess and turns them into intuitions of desirable outcomes many moves ahead. Similarly, the mathematician's rule-shaped intuition suggests where actions and processes on mathematical objects in dialogic space may lead.

In this account, conversation provides the epistemic and ontic basis of mathematical knowledge and object existence. It grounds them in physically embodied, socially situated acts of human knowing, communication, and agreement. Because of the tight rules, norms, and conventions, mathematical conversation has minimal of ambiguity compared to every other domain of discourse. Nevertheless, the philosophical bases of mathematics are in the final analysis deontic, resting on the shared explicit rules and hidden norms of mathematical practice, as communicated via conversation. Conversation includes the roles of proponent and critic, and both of these roles are necessary in fruitful conversation, at any level. Their existence is the reason why a mathematician stranded alone on a desert island for 20 years proving theorems is still engaging in a social practice.

1.4 Conclusion

This concludes my treatment of the ontological problems of mathematics and mathematics education. I have argued that mathematical objects are formed out of actions on simpler objects, which are abstracted and reified into self-subsistent objects. All the actions involved in this process are heavily constrained by the rules of mathematics which are entangled with and woven into the objects. The norms and constraints that make mathematical objects possible are necessary elements of their existence. Because of these definitionally necessary limits, the objects are necessary objects. Their necessity is the product of the deontic modality, which in describing mathematical objects, indicates how their world ought to or must be according to the norms and expectations of mathematical culture. Contrary to the traditional view that accounts of mathematical objects are in epistemic or alethic modality expressing possibility, prediction, and truth, the deontic modality of mathematical language indicates an obligation that becomes a necessity. If mathematical objects exist, and they are present to all mathematicians and students of mathematics to a varying degree, then 'this' is how they must be. Here, 'this' refers to the necessary character of mathematical objects as the conventions and rules require them to be.

The ontological problem of mathematics education concerns the nature of mathematicians and students of mathematics. I have argued that the formation of their mathematical identities, which are perhaps only a small part of their overall beings as persons, develop through mathematical enculturation. The key element of this process is subjection to rules, conventions, orders, instructions that must be obeyed, at three levels, during engagement with mathematical activities.

First, there is the social, interpersonal level. In schooling, the teacher sets the tasks and their goals. However, they may be hedged, the teacher issues orders to the children that requires that they engage in the set mathematical activities or tasks.

The teacher also demonstrates and reinforces the rules and solution processes that the learners must use to attempt to achieve these goals. There may be a limited degree of flexibility as in some tasks the learner can select their preferred method of solution from amongst the approved methods or their variations. But overall, this is the level made up of the imperatives issued directly by the teacher in social or interpersonal space.

The second level of necessity is that inscribed within the texts of the tasks. The most common verb forms in mathematics, both in school and research texts, are imperatives requiring the reader to complete the activity in prescribed ways (Ernest, 2018a; Rotman, 1993). Such prescriptions may be tacit, but there is a repertoire of agreed rules and methods to be employed. Here, the key characteristic is that the imperatives are in the text themselves.

Third, there are the tacit and explicit rules and conventions of mathematics that delimit the permitted actions and textual transformations. These are part of the culture of mathematics and a key element of what students and practitioners pick up and internalize as a residue of the myriad conversational exchanges in the dialogic space of mathematics. These make up much of what is termed the knowledge of mathematics, that which is learned through mathematics education. It is these rules that must be selected from and utilized in the performance of mathematical activities and tasks by students of mathematics and mathematicians.

Thus, my two big ontological problems, the nature of mathematical objects and the nature of mathematical identities, with their associated powers, converge. It is the rules and conventions of mathematical culture that help build up and constitute both of these types of entity. The objects of mathematics are abstracted actions encapsulating these rules. Mathematical identities are shaped, constituted, and constrained through the internalizations of these rules.

This convergence in explanations is why an interdisciplinary approach to these problems is necessary. I am tempted to claim that only such a multidisciplinary analysis, drawing on philosophy, linguistics, mathematics education, and other disciplines, can address both of these two problems together. Furthermore, the solutions offered are interdependent and co-constituting. Interacting with mathematical objects is an essential dimension in the construction of mathematical identities. Coming to, becoming and being a mathematician depends essentially on engagement with and using mathematical objects. Conversely, the formation and maintenance of mathematical objects depends on human capacities to actively maintain cultures through extended conversational interactions, including the capacities for abstraction and rule formation. Only through the induction of persons into the culture of mathematics are mathematical identities formed and the culture of mathematics, which is the location of mathematical objects, maintained and extended.

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Chapter 2
Scientific Revolutions: From Popper to Heisenberg

Michael Otte and Mircea Radu

2.1 Introduction

Following Charles S. Peirce we assume that all thinking takes place in signs. Thinking is representation, i.e., is semiotic in nature. Each representation is a generalization. And any generalization is accompanied by a sense of liberation that can be as deceptive as it is empowering.

For example, to justify statements like 2 + 2 = 4 or 7 + 5 = 12 (to take Kant's examples), one first argues, as in discourse on ordinary knowledge, that the propositions express matters of fact reacted by "synthetic construction in pure intuition" (Kant). The justification process however does not stop here. In time additional reflection work about the matter continues in search of a self-contained unifying theoretical *explanation* of the isolated facts in case. The arithmetical axioms as put forward by people like Hermann Grassmann, Giuseppe Peano, or Richard Dedekind are theoretical explanations of this kind. Once this point is reached, the quest for the nature of numbers is answered by the formal-axiomatic approach. From an axiomatic viewpoint, however, numbers can be nearly anything. Logicians like Frege and Russell saw this consequence of the axiomatic explanation of the number concept as a problematic form of reductionism. They attempted to provide alternative explanations of the number concept, based on logic and conceptual thinking rather than generalization as practiced in terms of formal theories. Paul Mouy echoes these issues by proposing a social and cultural genealogy of the

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necessity concept. In our history, necessity initially manifests itself as blind fate. It is initially rooted in

> (…) circumstance and leads men to their destruction, which perfidiously drove Oedipus to incest and patricide. A primitive idea. Thanks to mathematical proof, this concept passes without change of name from the exterior to the interior, from things to spirit, from the domain of mysticism to the domain of reason. (…) It becomes what man by force of reason compels himself to obey. It constitutes an obligation on the part of the mind, an intellectual value. (Mouy, 1971, vol.II, p. 48–57, p. 49)

In a sense, from the perspective of Frege and Russell, axiomatics appears a return to the type of formal "exterior" necessity criticized by Mouy. The perpetual quest for this type of shift from the "exterior" to the "interior", which Mouy ultimately links to mathematics and to mathematical proof, constitutes the fundamental theme of many ongoing debates in mathematics education. Frege and Russell see logic as the means of achieving such a shift. Logic and Ethics are understood by them as normative sciences. According to Peirce, himself a logician, the situation looks different:

> (…) it is generally said that the three normative sciences are logic, ethics, and esthetics, being the three doctrines that distinguish good and bad; Logic in regard to representations of truth, Ethics in regard to efforts of will, and *Esthetics in objects considered simply in their presentation.* Now that third Normative science, (…) is evidently the basic normative science upon which as a foundation, the doctrine of ethics must be reared to be surmounted in its turn by the doctrine of logic. (Peirce, CP 5.36)

For Hegel, Esthetics has the special meaning that in it the essence of that which appears to us is adequately presented. Esthetic perception is most likely to do justice to the essence of appearance (Hegel, 1950, *Ästhetik*, vol.2, p. 606).

In a sense, our chapter is an attempt to clarify the meaning of the latter claim, by exploring Thomas Kuhn's concept of *scientific revolution* and some of the debates triggered by it. This leads us to a reexamination of a series of episodes in the history of science and art. We begin by highlighting some aspects of Karl Popper's understanding of the logic of research.

2.2 A Reassuring Fact?

In the foreword to his well-known book *Vermutungen und Widerlegungen* (Conjectures and Refutations), Karl Popper informs the reader that his book consists of variations on the comforting fact "that we are able to learn from our errors, from the mistakes that we have made." According to Popper, this represents the foundation of scientific progress (Popper, 2009, XIII). But how helpful is this statement in explaining the evolution of scientific theories?

Popper's view of science is based on the following criterion: Only beliefs that are susceptible of being falsified count as scientific ones. This criterion, of course, opens a fundamental gap between science and mathematics. Consider, for example, the hypothesis that we can reach all natural numbers by counting. We can neither prove nor falsify this hypothesis, but have to treat it as an axiom. Poppers criterion excludes mathematics as well as philosophy from the realm of the sciences. Is this a productive philosophical approach?

At the beginning of the nineteenth century, Goethe, the great poet, indicated that as a separate discipline *Philosophy* seemed superfluous to him. As Goethe recollects in his autobiography: "*I maintained a separate philosophy was not necessary*, as the whole of it was already contained in religion and poetry" (Goethe, 1998, 200). If the philosophy of science can be truly reduced to falsificationism and its consequences, does this not then mean that all this conception is telling us about natural science and mathematics is already contained in these disciplines?

Popper's falsificationism seems to imply that we all have always been scientists, since the days of the Stone Age, and that science, therefore, is a continuous and natural affair, indistinguishable from our everyday experiences. On the other hand, however, never before has there been such a large gulf between common and specialized knowledge as it exists today. And never before have such rapid changes in the philosophy of science taken place.

Poppers views were frequently used as arguments against Thomas Kuhn's conception of scientific progress as it is presented in his *The Structure of Scientific Revolutions*. During a conference hosted by the *Bedford College* in 1965 (both Kuhn and Popper attended the conference), Kuhn responded to this criticism by pointing out that Popper's conception cannot actually explain the evolution of scientific theories. Kuhn said:

> Compare the situation of the astronomer and the astrologer. If an astronomer's prediction failed (…), he could hope to set the situation right. Perhaps the data were fault (…). Or perhaps theory needed adjustment, either by the manipulations of epicycles (…) or by more fundamental reforms of astronomical technique. For more than a millennium these were the theoretical and mathematical puzzles around which together with their instrumental counterparts the astronomer's research tradition was constituted. The astrologer, by contrast, had no such puzzles. The occurrence of failures could be explained, but particular failures did not give rise to research puzzles, for no man, however skilled, could make use of them in a constructive attempt to revise the astrological tradition. There were too many possible sources of difficulty. (Kuhn, 1970, 9)

Widespread and diverse changes were necessary in order to transform alchemy or astrology into natural sciences. Think of the plethora of ill-founded rules, requirements, and recipes that accompanied the activity of the alchemist. One had to pay attention to the purity of the soul, the position of the stars had to be considered, the avoidance of the full moon and of the influence of black cats was necessary, and many other things seemed essential in alchemy, especially when one wanted to made gold from clay.

Johann Friedrich Böttger (1682–1719), condemned by the Duke of Saxony to make gold out of clay, could have avoided all sinful thoughts and all black cats and still could not have made gold. Alchemy does not become scientific chemistry by such advice as Popper provides. Popper's strategies and maxims concern the political application of science and its role in society, rather than science itself.

Stefan Amsterdamski responds to Popper's problems by suggesting a functional demarcation of the sciences, rather than a descriptive one:

> It seems that one of the functions science performs permanently in human culture consists in unifying into a coherent system practical skills and cosmological beliefs, the episteme and the techne. It is, of course, hard to pinpoint the place and the exact moment when this requirement was accepted for the first time. (Amsterdamski, 1975, 43)

Amsterdamski's ideas make the situation better and worse too. According to them, alchemy is undoubtedly a science like chemistry, in fact it appears as more science-like, because the heyday of hermetic emblematic coincided with the decline of classical alchemy, which was still capable to merge technological skills and practical experience with spiritual components (Roob, 2019, 18). On the other hand, it becomes clear that the evolution of science is a cultural and socio-historical problem with unclear boundaries. But in terms of methodology, it was underdeveloped.

One of the most important concepts of Kuhn's theory is the notion of *paradigm-change*. Kuhn's definition of the concept of paradigm is rather vague. How far can and should the researcher go in restricting his search space, in order to identify or replace a doubtful hypothesis?

Any non-trivial problem that we manage to identify and solve is part of a more general unsolved problem. The problem-solving process creates its own solution-space, and this leads to a sort of co-evolution of the problem and of the methods used. This process essentially requires complexity reduction, which, in the end, leads to narrowing the thinking process down to constructing, calculating, and measuring, on the one hand, and making uncertain choices between alternative goals and objects, on the other hand.

If you look into books of mathematics or theoretical natural sciences, you will discover many equations and formulas containing symbols which denote purely functional quantities. Quantities that serve the functional interpretation of the equations. In mathematics, the imaginary numbers, the additional constructions in geometry provide simple examples. To put it bluntly, functionality emerges through mathematics. Mathematics, therefore, is not a language, but rather an instrument of reasoning.

Purely descriptive representations of natural or technical conditions are much more difficult to understand or remain incomprehensible altogether. Bare words or characters are replaced with new words as in a dictionary, as if one were simply explaining *John* means *Johann*. Newton's mathematized science has often been criticized as explaining nothing. According to legend, Newton replied: "It tells you how the Earth moves."

The equations and formulas of natural science do something like this too. But the difficulty— in comparison with pure mathematics— is that the physical elements are supposed to represent something real, something given in experiene. In physics, for example, "constants" like the speed of light, Planck's constant or the electric elementary charge, etc., are such empirical constraints. Quite a number of the symbols are supposed to have real counterparts in the empirical sense, but is this really the case? The Nobel Prize laureate Richard Feynman is right when saying that "nobody understands quantum mechanics" (Feynman, 1965, p. 129). Nineteenth-century debates over such questions have led to two very different conceptions of what logic actually is: a language or a calculus (Heijenoort, 1967).

Doesn't an ordinary declarative sentence simply have the function, logically speaking, of attributing a property to an object? And is it not the use of the knife that tells us whether the wood we are carving is hard or soft, smooth or fibrous, etc. Charles S. Peirce speaks similarly of the hardness of the diamond (see: Peirce, CP. 7340). The basis of the operation of abstraction is that it very often starts from one's own actions and from constructed schemata and representations and not from given objects, as is the case with empirical abstraction.

And the use we make of our tools makes us aware of ourselves too. The hands must think by themselves. "The analysis of a skillful feat in terms of its constituent motions remains always incomplete" (Polanyi, 1961, 460). During the *Industrial Revolution,* the hands became substituted by working mechanisms and machinery. *A machine to spin without hands*. "This was the specification of the Jacquard loom in the patent document of John Wyatt (1700–1766) of 1735" (Essinger, 2004, 37).

Conclusion: Everywhere, from quantum theory to everyday experience and to technical production, we learn that the actual object of our cognition is not something purely objective, but that we are always dealing with our own interactions with reality such that the epistemic subject and the object become impossible to be completely distinguished.

2.3 What About the Object of Cognition

At times the object of our interest appears as mere desire or motive. As pointed out above, Böttger wanted to turn clay into gold, without, however, having any clue whatsoever as to the practical path that might lead to this goal, ending up with porcelain. Similarly, the finding that certain bodies increase in weight as they burn was known as early as the seventeenth century, but it was not until the eighteenth century that Lavoisier made this fact the subject of scientific inquiry and explanation. One of the reasons for this delay, as argued by Thomas Kuhn, was that the "gradual assimilation of Newton's gravitational theory led chemists to insist that gain in weight must mean gain in quantity of matter" (Kuhn, 1962, 71).

Theories of scientific discovery are normally of two types: those which rely on mentalist notions (Popper) and those which employ a conception of social or cultural determination (Kuhn). The analysis and clarification of the conflict between Popper and Kuhn, therefore, begins with the insight that all thinking relates to something that can be called the *motif* or *intended objective* of cognition, something that, at the beginning of a process, may be fairly vague and ill defined, yet becoming more and more effective and precise in the course of the activity. In order to be able to think about something, one must first represent it one way or the other. The world is therefore always conceptualized through the angle of some possible representation and design. In this respect, Max Bense noticed that "the perspective representation of the world in painting corresponds to the epistemological view of the world in philosophy" (Bense, 1949, 79, our translation).

In art, the subject itself became an object like any other, losing its absolute position it had detained in religion and philosophy. Architects from Filippo Brunelleschi (1377–1446) to Girard Desargues (1591–1661) and artists like Piero del la Francesca (1412–1492), Leonardo da Vinci (1452–1519), and Albrecht Dürer (1471–1528) made space an object of something to which they themselves belonged.

> The scientific interest that accompanied the artistic activity is inspired by the belief that the evidence of the outward shape is also the warrant of its internal truth. The intense systematic study of anatomy and geometry culminating in the investigations of Leonardo da Vinci shows clearly that the artistic naturalism of those generations did not derive from the interests and mental habits of merchants and bookkeepers, as is sometimes represented, but was rather the expression of a general spiritual desire for order, balance, clarity and truth. (Olschki, 1950, 295f)

However, while the painters formulated the problem of perspective as a relation between the picture and the reality of an individual observer, Desargues (1591–1661) formulated it as a problem of the relation between two such pictures. In this way, he led mathematics on the path to projective geometry. In Desargues' geometry of "perspective", a circle and an ellipse are considered to be the same mathematical object, since a circle can become an ellipse when the point of view changes. The individual perspective, which became so important for epistemology, did not exist in the Middle Ages (Eco, 2002, Chap. 5.2). According to Olschki, during the Renaissance, even scientific spirits were attracted "by aesthetic, rather than by purely scientific questions."

The same development toward an individual perspective based on personal interest became also evident in scientific and technical contexts. For example, if you want to survey something, a building plot, or maybe the territory of Denmark, then you have to ram three stakes into the ground at suitably chosen points, which together form a triangle. Everything else can then be located and represented in the form of arithmetical coordinates relative to these basic points. An essential feature of measurement is the combination between the determination of an object by individual specification and the determination of the same object by some conceptual means. The latter is only possible relative to objects which must be given directly. Students learn this today under the name of *analytic geometry*.

Agostino Ramelli (1531–1610) was an engineer best known for writing and illustrating a splendid book *Le diverse et artificiose machine del Capitano Agostino Ramelli* (publ. 1588), which contains more than 100 extremely elaborated engravings. The book was considered "a landmark work in the depiction of the inventing range of technological imagination" (Rothenberg, 1993, IX) and its production must have been so expensive that an officer like Ramelli, who must have looked at it as an important means of self-advertisement, certainly could not have financed it by himself.

This shows that the link between artistic and scientific skill or genius (e.g., Leonardo da Vinci) was stronger in the days of the Renaissance than it is today.

> Actually, the belief in the divine origin of beauty and the conviction that truth is embodied in the perfection of form have been basic principles of Italian civilization ever since the throngs of the faithful were enraptured by a Madonna of Giotto (…) not as in earlier days

> because of their miraculous power but because of the supernatural charm emanating from those beautiful human images of a perfect divine creature. (…) The general feeling that truth and beauty are equal attributes of the divinity led those generations to contemplate the external world with religious eyes and to express religious emotions in concrete realistic forms. (Olschki, 1950, 251)

Leonardo da Vinci even saw the science of perspective together with mechanics as the most important ones, the absence of which would make one look like the captain of a ship without a compass. In his diaries he writes: "La meccanica è il paradiso delle scienzie matematiche, perche si viene al frutto matematico" (Leonardo da Vinci, *Philosophische Tagebücher*). Martin Kemp seems right when he explains that

> The ambition to invent a machine or device for the perfect imitation of nature appears to have been virtually limited to Renaissance and Post-Renaissance Western Art. (…) I do not think it is coincidental that this was also the period in which the technologies of scientific and utilitarian devices came to occupy a central place in European man's striving for intellectual and material progress. Indeed, the whole notion of progress in this phase of Western thought is deeply shared by science, technology and naturalistic art. (Kemp, 1990, 167)

2.4 Some Remarks on Semiotics, Logic, and Epistemology

A photograph gives a snapshot of an object or a person. The best example of this kind is perhaps a biometric photograph or a radiography. In a painted portrait, one looks for something else, namely for a glimpse at the character and essence of a living person. Anton van Dyck (1599–1641) revolutionized the genre and became one of the greatest and most important portrait painters of all times. The people depicted in his paintings engage the viewer in a characteristic way, entering, as it were, in a direct communication with the viewer. Generality and movement characteristic for painting are missing from the products of a data-giving apparatus, because life demands continuity, and continuity is indefinite and open.

This kind of combination of the individual/singular and the general, which is so crucial in aesthetics and philosophy is usually viewed as a disadvantage in technical applications, in science or in logic. Leibniz and later Frege and Russell struggled with the fact that any descriptive language is unable to fully determine an individual object thus forcing us to rely on contextual indexical signs such as names or the infamous x of algebra for clarification. As Russell points out:

> *I met Jones* and *I met a man* would count traditionally as propositions of the same form, but in actual fact they are of quite different forms: the first names an actual person, Jones; while the second involves a propositional function, and becomes, when made explicit: *The function 'I met x and x is human' is sometimes true.* (Russell, 1919/1998, 167f)

The arts—painting as well as writing and language—present the specific as a general as a continuum. Logic and mathematics, in contrast, depend on the principle of logical consistency and because of this they must break this continuity of space or time, because *continuity is generality* and the principle of consistency does not apply on continuous realities. For example, when we speak of the *Londoner* in general, we cannot so easily say: the Londoner is blond or funny or friendly, etc. The Londoner can be this and that and all together. The logical principle of the excluded middle does not apply in this case. The same situation arises when the geometer wants to speak of the *general* triangle. In 1710, Berkeley had asked the readers of Locke's *Essay concerning Human Understanding* to try and find out whether they could possibly have

> an idea that shall correspond with the description here given of the general idea of a triangle, which is neither oblique, nor rectangle, equilateral, equicrural, nor scalenon, but all and none of these at once. (Berkeley, 1975, 70)

And Peirce says in his Lectures on Pragmatism of 1903 writes:

> The old definition of a general (…) recognizes that the general is essentially ***predicative***. (…) In another respect, however, the definition represents a very degenerate sort of generality. None of the scholastic logics fails to explain that ***sol*** is a general term; because although there happens to be but one sun yet the term ***sol aptum natum est dici de multis.*** But (…) if ***sol*** is apt to be predicated of ***many,*** it is apt to be predicated of any multitude however great (…) In short, the idea of a general involves the idea of possible variations which no multitude of existent things could exhaust. (Peirce CP 5.103)

We therefore possess two forms of the general: predicative generality in language and the continuum in space and geometry. Since they depend on the principle of logical consistency, logic and mathematics invented a way of avoiding both.

Gotthold Lessing (1729–1781) had interpreted in his treatise Laocoon of 1766, a famous sculpture of Hellenistic times, which can be admired in the museums of the Vatican. Lessing describes how the artist found the "fertile moment," in which a whole story, that of the priest Laocoon and his sons, is summarized and compressed in one moment in a particularly meaningful way. Normally poetry and painting, according to Lessing, are different because they make use of entirely different means—the first namely, of form and color in space, the second of articulated sounds in time. Lessing writes:

> Painting and poetry should be like two just and friendly neighbors, (…) Physical beauty results from the harmonious action of various parts which can be taken in at a glance. (…) The poet, who must necessarily detail in succession the elements of beauty, (…) must feel that these elements arranged in a series cannot possibly produce the same effect as in juxtaposition. (…) When a poet personifies abstractions he sufficiently indicates their character by their name and employment. These means are wanting to the artist, who must therefore give to his personified abstractions certain symbols by which they may be recognized. These symbols, because they are something else and mean something else, constitute them allegorical figures. (Lessing, 1776/2005, chapter XVIII)

The most important aspect of poetical language is, in fact, the combination of metaphor and structuralist innovations. Nowadays, this is frequently referred to as "the *spatiality of literature.*" The notion of spatial form in modern literature was introduced about 1945 by Joseph Frank. Frank argues that modern aesthetic theory has evolved not from a set of fixed norms or categories imposed on the work of art or literature, but from the relation between the work and the conditions of human perception and activity.

Frank reports that he carefully studied Lessing's *Laocoon* and following Lessing, he came to see a poem more as a spatial shape or structure than as an event in time. Frank believed that the structure of modern works took on aspects that required them to be apprehended spatially instead of according to the natural temporal order of language. Smitten and Daghistany write in their presentation of Frank's work:

> The concept of spatial form in literature was introduced by Joseph Frank after having been stimulated by the observation that modern poetry – that of Eliot and Pound, for example - often breaks or undermines the normal consecutiveness of language forcing the reader to perceive the elements of the poem not as unrolling in time, but as juxtaposed in space. This undermining is accomplished primarily by the suppression of causal/temporal connectives, those words and word groups by which a literary work is tied to external reality and to the tradition of mimesis. This suppression of connectives alters the whole character of the literary work and forces the reader to perceive it in a new unconventional way. (Smitten and Daghistany, 1981, 17f)

A frequently presented example of structural poetry is T H. Eliot's great poem: "*The Waste Land,*" where syntactical sequence is abandoned and replaced by a structure depending on the perception of relationships between disconnected word groups. Joseph Frank comments:

> Aesthetic form in modern poetry, then, is based on a space-logic that demands a complete reorientation in the reader's attitude toward language. Since the primary reference of any word-group is to something inside the poem itself, language in modern poetry is really reflexive. The meaning-relationship is completed only by the simultaneous perception in space of word-groups that have no comprehensible relation to each other when read consecutively in time. (Frank, 1991, 14f)

The previous examples should suffice to illustrate the thesis, that artistic expression is characterized by the complementarity of time and space, of conceptual expression and spatial representation as invoked by Peirce in his praise of Esthetics as the fundamental normative science. Can science and mathematics be reduced to consistency or can we find similar aesthetic moments there as well?

2.5 Heisenberg's Ambivalent Praise of Kuhn: Some Implications

Werner Heisenberg (1901–1976), Nobel Prize winner and one of the fathers of Quantum Theory did not feel satisfied with Niels Bohr's presentation of the theory in terms of familiar macro-physical language. He therefore renounced using such linguistic descriptions and designed a non-commutative algebraic structure that would be better suited to directly express available physical data. In his autobiography, Heisenberg describes how his exploratory attempts in mathematization led him to the new foundations of quantum mechanics:

> It had become clear to me what precisely had to take the place of the Bohr-Sommerfeld quantum conditions in an atomic physics working with none but observable magnitudes. (…) Then I noticed that there was no guarantee that the new mathematical scheme could be

> put into operation without contradictions. In particular, it was completely uncertain whether the principle of the conservation of energy would still apply, and I knew only too well that my scheme stood or fell by that principle. (…) Other than that, however, several calculations showed that the scheme seemed quite self-consistent. Hence I concentrated on demonstrating that the conservation law held (…). (Heisenberg, 1971, 61)

Many years after Heisenberg's great contributions to quantum mechanics, Carl von Weizsäcker (1912–2007) physicist, philosopher, and Heisenberg's student recalls having once drawn Heisenberg's attention to Kuhn's picture of the history of science as a succession of paradigm shifts and scientific revolutions. Contrary to Weizsäcker's expectations, Heisenberg was disappointed by Kuhn's book:

> Historically he's right. But he spoils the punch line. What he calls paradigms are actually closed theories. They must follow each other discontinuously because they are simple. The real philosophical problem is why can there be simple theories that are true? (...) That is the key to the history of science. One has not understood anything about the possibility of science as long as one has not understood it. (Weizsäcker, 1992, 799)

Heisenberg rejects Kuhn's conception of scientific revolutions, even though he admits that Kuhn is moving in the right direction, because Kuhn does not go far enough. What is then the true merit of Kuhn's conception?

We will deal with this issue below. In the meantime, it seems worthwhile taking a brief look at another example taken from physics and at the problem of the existence of paradigms in mathematics.

The example we have in mind is provided by the history of classical Newtonian mechanics and its generalization by Einstein. Thomas Kuhn indicates that the term *Mass* has different meanings in classical Newtonian Mechanics and in Einstein's Special Theory of Relativity: the Newtonian mass is stable, independent of velocity, whereas the Einsteinian one depends on the velocity:

$$m = m_0 / \left(1\text{-}v^2 / c^2\right)^{1/2}$$

When we assume that the velocity of light c passes to infinity, we get $m = m_0$ and we see a continuity between the old and the new theory. However, this *passing to infinity* is explicitly forbidden, because the constancy of the velocity of light is essential to Einstein's theory of relativity. The Michelson–Morley experiment showed that the velocity of light is constant and independent of the relative position and movement of the source.

In the theory only the concept, that is, the sense or meaning appears, while in experiment and technology, the references to the corresponding objects are established. When we consider successive formal structures, then the continuity and correspondence is clear. The situation changes when we pass from the syntactical approach to semantics. Semiotically speaking, it is the complementarity between sense and reference that counts and makes the real difference. Nobel Prize winner, Richard Feynman, compares the different formulations of classical mechanics as given by Newton, Lagrange, and Hamilton, respectively, showing this fact implicitly:

> Mathematically each of the three different formulations, Newton's law, the local field method and the minimum principle, gives exactly the same consequences. What do we do then? You will read in all the books that we cannot decide scientifically on one or the other. That is true. (...) But psychologically they are different because they are completely unequivalent when you are trying to guess new laws. As long as physics is incomplete, and we are trying to understand the other laws, then the different possible formulations may give clues about what might happen in other circumstances. (Feynman, 1965, 53)

For a further clarification, it seems helpful to briefly explore the applicability of these reflections to a formal science like mathematics and to ask whether there can be revolutions in mathematics. In the past, this issue has indeed been the subject to heated debates. Michael Crowe, for instance, maintained that "revolutions never occur in mathematics" (Crowe, 1975, 165), because in mathematics, the development of new theories does not lead to older theories being irrevocably discarded. In contrast, Caroline Dunmore argued that revolutions do occur in mathematics,

> (…) but are confined entirely to the meta-level component of the mathematical world. (...) Consider what a major revolution in thought was entailed in the acceptance of non-Euclidean geometry. (…) Although Euclidean geometry itself was retained the belief that it was the only kind of geometry there could possibly be was discarded (…). The evolution of mathematics is conservative on the object-level, but revolutionary on the meta-level. (Dunmore, 1992, 212)

However, this position is not entirely accurate either, because first of all, mathematics and meta-mathematics became indistinguishable insofar as axioms in the sense of Grassmann, Peano or Hilbert are meta-mathematical systems and not systems of immediate mathematical truths. Until about 1800, the terms axiom and hypothesis were opposites, after which they became synonyms.

Moreover, revolutions in the meta-mathematical domain also result in revolutions in the object domain. Quite simply because everything that satisfies the axioms of a formal mathematical theory must be included in its object domain. For example, Peano's axioms do not answer questions, like: "What is the number 1, or 2"? Numbers could be anything, like Conway- Numbers, Vectors, or Hackenbusch-Games. This formal generality enlarges the usefulness of the mathematical number concept, rather than restricting it. A theory becomes a pair consisting of a formal axiomatical structure and some intended applications or models.

Let us present an argument linked to the problem of determining the angle-sum theorem for triangles as an illustration. Suppose we pass along the periphery of a triangle. By how many degrees have we turned after having reached our starting position again? Simple answer: 360°, because our input direction coincides with the end position. This response, although intuitively convincing, is based on the assumption that it amounts to the very same thing, to turn around on the spot to a full angle of 360°, on the one hand, or alternatively, do the same thing by travelling along a closed line, the periphery of an arbitrarily large triangle, for example, on the other hand.

One case, however, is based on *local* characteristics of space, the other is not, at least not if the triangle may be assumed as arbitrarily large! For arbitrary triangles, our conclusion is only valid in the Euclidean plane, but is invalid on the surface of the sphere, for example. Spherical geometry, like the geometries of Lobatchevsky and of Riemann are generalizations of Euclidean geometry. The latter is a limit case of them when the curvature radius K goes to infinity. Non-Euclidean geometries became accepted only after Beltrami had proved their consistency in 1868. The ordinary sphere was adopted as a model of a space with positive curvature.

Descartes invented co-ordinate geometry by assigning number pairs to the points of plane Euclidean geometry, thus presenting a certain meta-perspective on classical geometry. Axiomatic linear algebra provides a meta-perspective on Cartesian classical analytical geometry.

The strength of algebra and of formal axiomatics lies in the possibility of turning even unknown objects - objects the existence of which is not guaranteed in advance - into objects of investigation and of mathematical operation. And this becomes an important moment in the series of events that led to the emergence of mathematics as a linguistic tool of the new sciences. Lavoisier, the Newton of chemistry, writes at the beginning of his *Traité Elémentaire de Chimie*:

> While engaged in this employment, I perceived, better than I had ever done before, the justice of the following maxims of the Abbé de Condillac, in his *System of Logic*, and some other of his works: "Algebra, which is adapted to its purpose in every species of expression, in the most simple, most exact, and best manner possible, is at the same time a language and an analytical method." (Lavoisier, 1952, 1)

Therefore, if we understand algebra as a meta-theory of chemistry, then we only need to know whether two things of our interest are the same or are different from the given perspective. The similarities and differences refer only to what can be expressed mathematically, length, width, weight, temperature, etc. Lavoisier continues:

> Thus, while I thought myself employed only in forming a nomenclature and while I proposed myself nothing more than to improve the chemical language, my work transformed itself by degrees (...) into a treatise upon the elements of chemistry. (Lavoisier, 1952, 1)

And this becomes an important moment in the series of events that led to the emergence of mathematics as a tool of the new sciences. The reference to the physical weight, the consideration of which Lavoisier led to his theory of combustion, points to a general characteristic of research. We usually do not know what is behind the things; we do not know their genus from the outset, nor their specificities, or their essence. But we have to be able to see whether two structures or two objects are the same or are different. Lavoisier writes:

> I have been obliged to depart from the usual order of courses (…) which always assume the first principles of science as known and begin by treating the elements of matter and by explaining the tables of affinities without considering that in so doing they must bring the principal phenomena of chemistry into view at the very outset: they make use of terms which have not been defined and suppose the science to be understood at the beginning. (Lavoisier, 1952, 2)

It's not just about the new mathematical perspective, but about establishing a whole new experimental practice that makes Nature appear in a new light:

> When in the early 1780s Lavoisier and Laplace invented the device that they called a machine for measuring heat, but that soon became the calorimeter, they designed it as an analogue of that epitome of simple machines, the balance. (…) Despite their collaboration, however, Lavoisier and Laplace recognized somewhat different balances in the calorimeter. With his primary interest in chemistry, Lavoisier saw a balance of chemical substances. (…) Laplace saw a balance of forces. (…) The calorimeter mediated between theories and things. It exchanges theoretical entities for concrete realities. (Wise, 2010, 208ff)

2.6 Aesthetics and Scientific Theory Building: Heisenberg and Kepler

In addition to the objective generality, something like an individual generality of an aesthetic nature appears in modern thinking. Werner Heisenberg, for example, believed—like Kepler—that mathematical beauty has a grasp on truth: "If nature leads us to mathematical forms of great simplicity and beauty, we cannot help thinking that they are 'true,' that they reveal a genuine feature of nature."

In retrospect, he traces this conviction back to his school time where atoms were displayed with "hooks and eyes." This irritated the young Heisenberg, since atoms of such complexity could never be basic building blocks of matter. At this point, Heisenberg followed Plato believing that "the ultimate root of appearances is therefore not matter but mathematical law, symmetry, mathematical form." (Heisenberg, 1974, p. 10) This conviction accompanied him throughout his entire life.

On occasion of one of Niels Bohr's visits at Göttingen, Heisenberg had asked him whether he was indeed convinced the classical concepts were sufficient for dealing with quantum physics. Bohr had replied, as Heisenberg remembered, that classical concepts were adequate also in the sub-atomic domain of Rutherford's model of the atom. Heisenberg rejected this. As an alternative, Heisenberg constructed, as said, a non-commutative algebraic structure better suited to match the available physical data and certain theoretical principles. Thereby, he made use of a new mathematical calculus, namely matrix algebra, something Paul Jordan had pointed out to him (de Toledo Piza, 2003, p. 90ff).

In his autobiography, Heisenberg describes how his exploratory attempts in mathematization provided him access to the foundations of quantum mechanics:

> I noticed that there was no guarantee that the new mathematical scheme could be put into operation without contradictions. In particular, it was completely uncertain whether the principle of the conservation of energy would still apply, and I knew only too well that my scheme stood or fell by that principle. (Heisenberg, 1971, 61)

As soon as he had success in showing that his framework of matrix algebra would confirm the energy theorem in all parts, he wrote that

> [I] could no longer doubt the mathematical consistency and coherence of the kind of quantum mechanics to which my calculations pointed. (…) [And I] felt almost giddy at the thought that I now had to probe this wealth of mathematical structures nature had so generously spread out before me. (ibid.)

Heisenberg's work shows that theories are realities *sui generis* in distanced relation to concrete reality. Theories and works of art are both build on aesthetic integration. Sabine Hossenfelder commenting on the (aesthetic) disagreements between Heisenberg and Schrödinger writes: "The advent of quantum mechanics wasn't the only beauty fail in physics" (Hossenfelder, 2018, p. 20). Hossenfelder examines the mathematical representation of the system of planetary orbits given by Johannes Kepler (1571—1630) in terms of the relations between the Platonic solids and shows how this led to an example "aesthetically motivated failure." Of course, one can simply say, with Hossenfelder, that Kepler's model was wrong (Hossenfelder, 2018, 18). And in fact, Kepler later convinced himself that his first model did not apply, and he concluded that the planets move in ellipses, not circles, around the Sun.

However, it seems of some importance to understand that the intention to search for the exact form of the planetary orbits, i.e. the search for general laws, and not being content with any more or less plausible interpolation of the measurement data, was stimulated and guided by aesthetic desire and aesthetic imagination in the first place. The measurement data alone never determine the natural laws of which they are supposed to be the expression of. John Banville has written a very interesting and ingenious novel about Kepler. Banville explains that after thorough investigations, Kepler finally

> (…) made the discovery. He realized that it was not so much in what he had done that Copernicus had erred: his sin had been one of omission. The great man, Kepler now understood, had been concerned only to see the nature of things demonstrated, not explained. (…) Copernicus had devised a better system which yet (...) was intended only to *save the phenomena*, to set up a model which need not be empirically true, but only plausible according to the observations. (Banville, 1981, 25)

Astronomers like Copernicus and Kepler could not undertake laboratory experiments and had experimented with theories instead. They had to imagine different models that did not come into conflict with available data in order "to save the phenomena." From mere formal hypotheses, no true and at the same time necessary conclusions can be obtained, an insight that was the key source of inspiration for Kant's epistemology, even if Kant's explanation turned out to be too rigid. The hypotheses themselves must be *experienced* as true.

Kepler's decision to view the heliocentric system as a fact of physics rather than just as a convenient instrument for calculations and predictions led to strong opposition not just from the Catholics Church but also by the Protestant movement. The link between aesthetic and scientific ideas in Kepler's work is concentrated in the concept of "harmony":

> Firsts and centrally, he means he has reached a new conception of causality, that is, he thinks of the underlying mathematical harmony discoverable in the observed facts as the *cause* of the latter, the reason, as he usually puts it, why they are as they are (…) He was antecedently convinced that genuine causes must always be in the nature of underlying mathematical harmonies. (…) Causality, to repeat, becomes reinterpreted in terms of mathematical simplicity and harmony (…) A true hypothesis for Kepler must be a statement in the mathematical harmony discoverable in the effects. (Burtt, 2003, 64f)

In the light of the things outlined above, the fallibility of a theory, therefore, appears not simply as a matter of falsifying one set of assertions of the theory or the other but rather as a fundamentally determined, among other things, by the esthetic choices made while the theories were developed. In our view, an aesthetic choice of this kind is not simply a lucky but arbitrary decision on the part of the creator of a theory, but rather one that *emerges* during the process of making sense of available data and puzzles. This leaves no room for nominalist language-games, and is – that, at least, is our interpretation of Heisenberg's criticism of Kuhn-, something that occurs objectively. This, of course, as said, does not exclude failure, but it allows a different understanding of what the falsification of a theory requires.

Galileo Galilei seems to never have adopted Kepler's discovery of the elliptical orbits of the planets. He considered the whole matter from a strictly technical point of view and always looked at it from the perspective of classic geometric ideas and processes (yet another aesthetic decision!). In his extensive biography of Galileo, J. Heilbron writes, among other things: "Nowhere in his later work is there any acknowledgment that Kepler motion in an ellipse is a substantial technical improvement over equant motion in an eccentric circle" (Heilbron, 2010, 72).

Cardinal Bellarmino (1542–1621), Grand Inquisitor and Galileo's principal adversary, in 1615 notified Galileo of a forthcoming decree of the Church, condemning the Copernican doctrine of heliocentrism and ordered him to abandon it as an explanation of the world. He argued that mathematicians always used to speak hypothetically or "*ex suppositione"* only (Bellarmino, Letter to Father Foscarini of April 1615). Galileo agreed and disagreed.

In his "Assayer" (Il Saggiatore) of 1623, Galileo compared God's Word in the *Bible*, which is adapted to the frame and imagination of the people, on the one hand, and the *Great Book of Nature*, on the other hand, which presents the realities of Nature in geometrical figures. But it seems unclear, whether he simply wanted to say that the Book of Nature cannot be read by everybody or whether he wanted to contradict the Bible. Galileo, to his credit, also rejected the idea that mathematicians might employ convenient hypotheses, because of technical reasons alone. However, when three comets appeared in the sky in 1618, Galileo held on (for technical and aesthetic reasons of his own) to the traditional thesis of the circularity of the orbits of the stars, ignoring Kepler's "bold break with this tradition" to classify the appearance of the comets "as atmospheric phenomena in the sense of the ancient *Meteora*" (Blumenberg, 1981, 73, our translation).

2.7 Aesthetic Desire and Experience

We may and indeed must emphasize the importance of the creative and artists of the *Renaissance* because their contributions contained the germs of the Scientific Revolution of the modern age.

An aesthetic experience is something extremely personal, something difficult to share with others. You cannot make somebody to see or feel something that he/she does not see. It is only indirectly manifest in the products of art and science, just as the laws of nature are only, as said before, indirectly given in the morphology and functions of a mechanical machine. Aesthetic experience is something that we would like to call the *individual general*. It is a union between general assumptions, material marks, and emotions concentrated in one individual artwork. The work of art, therefore, represents a reality sui generis (linked to the aesthetic activity that shaped it) and at the same time it is a sign that reaches further sometimes enabling the emergence of further new and unexpected reactions and developments.

Theory is not reality, the map is not the territory, the menu is not a substitute for the meal, and a drawing of the dead *Marat* in the bathtub consists of nothing but a few lines of graphite on white paper, impressive as it may be. When we approach a theory or a drawing, we can engage in two types of effort comprehending it. One may proceed from the whole toward the identification of its parts, or, conversely, from the recognition of a group of presumed parts toward grasping their mutual relations in the whole.

And we can further understand a piece of science or of the arts as representative for the whole of a certain culture at a certain time in history. Johan Huizinga writes at the beginning of his Preface to his monumental *The Waning of the Middle-Ages*:

> The present work deals with the history of the fourteenth and fifteenth centuries regarded as a period of termination, as the close of the Middle Ages. Such a view of them presented itself to the author of this volume, whilst endeavoring to arrive at a genuine understanding of the art of the brothers Van Eyck and their contemporaries, that is to say, to grasp its meaning by seeing it in connection with the entire life of their times (Huizinga, 1999).

Other philosophers and historians have argued in a similar way. Think of Ernst Cassirer or Agnes Heller. Cassirer, for example, wrote that the individual

> (…) is form only by giving himself his form, and therefore we must not see in this form merely a limit, but we must recognize it as a genuine and original force. The general that reveals itself to us in the sphere of culture, in language, in art, in religion, in philosophy is therefore always individual and universal at the same time. (quoted from Schwemmer, 1997, 145)

And in Agnes Heller's book, *Der Mensch der Renaissance* (Renaissance Man), we find the following passage:

> It's a popular truism to say that in the Renaissance Man became the center of interest. (…) But this is not the problem, the question rather is, how to interpret the relationship between Man (society) and Nature. (…) Nature appears as an object governed by its own laws (…). And the knowledge of Nature becomes a continuous task of the human mind. Spirit and continuity are equally important here. Spirit means that success in no way depends on ethical behavior, while continuity separates knowledge from logic. (Heller, 1988, 18f.)

As we have seen, foundation and development of knowledge can only be discussed in relation to one another. Which of the three normative sciences named by Peircean and mentioned in the beginning of this paper are better suited for giving a more adequate account of this evolutionary interdependence, is it Aesthetics, is it Ethics, or is it Logic? Since Heller excludes Ethics and Logic regarding the Renaissance and the Scientific Revolution, we are left with Aesthetics as the only remaining guiding thread. This was new!

Traditionally, from Plato and through the Middle Ages, some *principle of sufficient reason*, occasionally rooted in the belief in God's wisdom, dominated most attempts to explain the world and the human destination within that world. At times, even Descartes appears to endorse the *principle of sufficient reason*. For example, he argues for the existence of God in the third *Meditation* on the basis of the principle that there must be at least as much reality in the cause as in the effect.

> Car d'où est-se que l'effet peut tirer sa réalité sinon de sa cause? Et de là il suit, non seulement que le néant ne saurait produire aucune chose, mais aussi que ce qui est plus parfait, c'est-à-dire que contient en soi plus réalité, ne peut être une suite et une dépendance du moins parfait. (Descartes, 1953, 289)

And he justifies this causal principle by claiming that "Nothing comes from nothing."

Most important, however, remain, as said, principles of aesthetic integration and individual or exemplary generality. The emergence of individualism has for the first time in history led to a theory of excellence and of human genius. No such theory was available before the *Renaissance*. During the *Renaissance,* this theory of geniality was linked to the ideal of universal accessibility of art and not simply, as in the case of Bacon and Galileo, with science as such. In this sense, therefore, art involved early on an element of democratic creative education (compare Heller, 423ff). In science and in mathematics education, the opposition between conceptual consistency and continuous experience still pose a serious challenge to educational efforts to this day.

Although reflections on art and beauty have, of course existed since Plato, it is only since about the middle of the eighteenth century in Europe that a distinct discipline called "aesthetics" develops as a consequence of what had occurred earlier.

> Nevertheless, the artist is the paradigmatic personality of the Renaissance. The imagination of the genius is the actual creation of this great epoch, even the man of action has an aesthetic character in it. But behind the concept of the artist hides that of the creative person in general. From now on this concept plays a previously unknown role in the intellectual life of the West. (Baeumler, 1923, 2)

And curiously enough, the situation repeated itself during the Romantic Movement, in the last decades of the 18th and the first decades of the 19th century, particularly in the context of the more abstract empirical sciences of electricity and thermodynamics and at the same time in the Humboldtian conception of unity of research and teaching in Germany. As S. Turner describes the situation:

> In Germany, unlike other European nations, the universities have traditionally been the major centers for the creation of academic knowledge as well as for its transmission. This was especially true in the first two thirds of the nineteenth century (…). The ultimate cause of this burgeoning of German scholarship was the new and at that point uniquely German conviction that the professor's responsibility is not only to transmit academic learning but also to expand it, through criticism and research. This ideal of the professor's proper function can be called the *research imperative.* (Turner, 1981, 109ff)

The role of the aesthetic and the importance of the arts lies in the fact that art can deal more freely with the fundamental complementarity of meaning and information.

2.8 Conclusions

In his *Critique of Pure Reason,* Kant had said: "*Philosophical cognition regards the particular in the general, mathematical the general in the particular, nay, in the individual*" (Kant, B742). For Kant, this meant that in mathematics the general becames, so to speak, alive in the particular construction.

The latter is also true with respect to the arts. Mathematics has always had some affinity with the arts. And we find, in fact, that it was aesthetics—in addition to technical utility—that had stimulated the mathematization of the sciences. Pure mathematics had always had an aesthetic appeal to people, and it was sometimes even regarded as the highest philosophy. A mathematician, says G. Hardy, the most important British mathematician up to the Second World War

> is like a painter or a poet a maker of patterns (...). The mathematician's patterns like the painters of the poets must be beautiful, the ideas like the colors or the words must fit together in a harmonious way. There is no place in the world for ugly mathematics. (Hardy, 2012, 84)

Analytic mathematical thinking, such as traditional twentieth-century analytic philosophy, is stuck in a computer-induced impasse which may perhaps grow even deeper, than it is today. In teaching contexts algorithmic reductionism presents itself as a convenient simplification. The procedures we teach in our schools are simple and, if not, they can be easily performed by technical devices such as calculators and computers. If practiced unilaterally—as this is often the case, due *not least* to the social systemic constraints upon the educational process —this approach loses sight of the aesthetic component of mathematical and indeed scientific practice. Analytic mathematics, if it is to work for the student, must be experienced, and a "technical" approach is only one side of the equation. Structures must be contemplated in action and experienced, they cannot be directly communicated. This, on the other hand, cannot be achieved by simply delegating the task to supposedly familiar semantic environments either.

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Chapter 3
Questions That Are at the Core of a Mathematics Education "Project"

Maria Aparecida Viggiani Bicudo

3.1 Introduction

It is assumed that mathematics, as a science and the way it exists, is at the core of the issues concerning mathematics education, as well as those concerning the formation of individuals and citizens. This core is surrounded and expanded by issues concerning the complexity of the world in which we live and requires that socio-historical-cultural contexts be considered. From this core and its surroundings, arise diverse themes of research centered on lines of investigation and in sub-areas of mathematics education. To elucidate this statement, I list the following works which focus on many of such themes. One that has achieved prominence in recent years is coloniality, about which many articles have been written, mainly in Brazil. The objective of such works is to highlight the supremacy of knowledge brought about by Western mathematics, which has been imposed as a parameter of truth and accuracy for the knowledge produced in other cultures. Fernandes (2021) exposes thoughts about relations between Mathematics and Western modernity in the configuration of the colonial matrix of power. Tamayo (2017), Tuchapesk da Silva, and

Project is being understood as "*projection*," as Heidegger exposes its meaning in Being and Time (1962, p. 185), that is, as an idea launched to the possibilities of happening, that is, of its becoming. He says "Dasein is thrown into the kind of Being" which we call "projecting." Projecting has nothing to do with comporting oneself toward a plan that was conceived and in accordance with which Dasein arranges its Being (Heidegger, 1962, p. 185). The objective of bringing this meaning to mathematics education, as set out in the title of this chapter, is to affirm that at the core of mathematics education there are issues considered important that are actualized in the very movement of becoming in which mathematics education is cast into the mundanity of the world actualizing its possibilities.

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Tamayo (2022) have authored work highlighting this idea. Ethnomathematics has been seen as a precursor of studies focused on sociocultural issues within mathematics education. Tamayo et al. (2018) stated that ethnomathematics studies in mathematics education

> [...] triggered "a shift in the ways of thinking about mathematical knowledge. As a result of such movements, initiated by ethnomathematics, which we understand as counter- conduct, new branches, and possibilities, for thinking about mathematics education through different perspectives blossomed." (Tamayo et al., 2018, p. 588)

By assuming social justice as a germane value in educational practice, critical mathematics has also highlighted the mathematics affected within the context of daily practices. Gutestein (2018) explained this view clearly by stating

> While I acknowledge the multiple meanings of these terms, for me, they all mean the same: to learn and use mathematics to study social reality, as a way to deepen learners' understanding of the roots of injustice and prepare them to change the world, as they see fit, in both the present and future. (Gutestein, 2018, p. 133)

Another theme that has been brought to the foreground is that of *gender and mathematics education*. For instance, Souza and Fonseca, assume in their article that

> [...] the emergence of the concept of gender in education showing different nuances and proposing its incorporation as a category of analysis in the field of mathematics education, in which discussions on gender are rarely detected, especially when we analyze scientific research in Brazil. Using women scholars in the field of gender studies as references, we have reflected on the need to incorporate this concept into the investigation of the processes of teaching and learning mathematics, the subjects in pedagogical relations, and the cultural mode of conceiving, using, and evaluating mathematical knowledge. Such incorporation would imply, however, a disruption in the ways which we have thought about concepts related to female, male and mathematics." (Souza & Fonseca, 2009, p. 29)

This questioning regarding mathematics and reality leads some researchers to investigate "The Separation of Mathematics from Reality in Scientific and Educational Discourse" (Schürmann, 2018). The article aimed to illustrate that the separation between mathematics and reality is an outcome of several shifts in historic mathematical discourse.

This is a broad spectrum of issues. How can this diversity be articulated? How can we articulate relevant criticisms of the primacy of European science,[1] prevalent in Western civilization, in relation to the ways other cultures tackle knowledge and the understanding of mathematics, which underpins scientific and technological

[1] "European sciences" is a term that refers to the logic of science that was built in Western civilization and that expanded throughout the world, as a support for scientific investigations and their applications and as a tool for technology. "The term 'science' was used here in a precise sense: that which refers to the disciplines, theories, investigations that were configured in western thought in two fundamental stages: Greek speculation for Geometry and modern speculation for Mathematics and Physics" (Ales Bello, 1986, p.10) (Il termine "scienza" stato usato in una precisa accezione: quella che si referisce alle discipline, teorie indagine che sono configurate nel pensiero occidentale nelle due tappe fondamentalle: le speculzione greca per la geometria e la speculazione moderna per la mathematica e física).

practices, which have become common both in Western and Easter cultures? Notably, there is a certain ambiguity in the way mathematics is treated, according to the discourse and the emphasis attributed either to the ways of routinely dealing with it, or the scope and positivity of scientific knowledge, present in the areas of health, engineering, technology. Depending on the perspective, the so-called European science is viewed as important and crucial for *progress* and the possibility of improving life and, at the same time, it is rejected, for bringing too many generalizations which are imposed on particular worldviews, especially those of different cultures. How can we account for such distinct perspectives, when working with mathematics education in institutions dedicated to the formation of individuals and citizens?

Kennedy (2018) had already shown concern regarding the need to create space in the curriculum for the inclusion of philosophical thought within mathematics education. In "Towards a Wider Perspective: Opening a Philosophical Space in the Mathematics Curriculum" she stated that

> philosophical inquiry may aid in the opening of a "wider horizon of interpretations" that includes a critical dimension. Such an opening represents a potential expansion of students' mathematical experience, and promises to provide bridges for establishing richer, critical, and more meaningful connections and interactions between students' personal experience and the broader culture. (Kennedy, 2018, p. 309)

I agree with Kennedy that it is necessary to bring philosophical thought to the mathematics education curriculum. To point to possibilities I chose to write this chapter, as an essay, albeit short.

The objective of this chapter is *not* to produce a list of themes defined as important and worthy of study and practice. Rather, it is to conduct an analytical and reflexive exercise and point out understandings regarding the characteristics of scientific work, which is supported by mathematics, defending the premise that within the scope of mathematics education it is necessary to comprehend such characteristics and integrate them into educational practices.

Husserl (1970a, b, c, 2006) conducted an important study regarding the crisis engulfing the European community in the 1920s and 1930s. He emphasized the role Western science plays in that crisis when it makes absolute and imposes its way of dogmatizing the world and its reality, even covering ethical issues. While seeking to understand the manner through which this science was produced, he presented an enlightening study of pre-categorial and categorial knowledge. They lie at the base of Euclidian geometry, which, in turn, laid out the manner mathematics is used, assumed by Galileo's physics, as well as the ramifications of the advancement of science and its application to natural and spiritual sciences, due to the success of their application and research. He emphasized the logic of such production, which encompasses the work with idealities of mathematical objects, with the exactness based on such idealities and, as a result, points to the mathematization of nature.

The strength of his thinking brings possibilities for conducting an analytical-reflective exercise, bringing his considerations to mathematics education. Significant work has been conducted to highlight the articulations of ideas with mathematics education, explained, and understood by that philosopher. Bicudo (1991, 2018, 2020), Garnica (1992), Hausberger and Patras (2019), Hausberberger (2020), Rosa and Pinheiro (2020), Paulo and Ferreira (2020), Batistela (2017), Kluth (2005) are some of the many authors who have committed to studying that philosopher's work and who, through exhaustive hermeneutic reading of his texts, opened themselves to the understanding of what he expressed. While reflecting their comprehension, prominent issues in the field of mathematics education are intertwined with their thoughts. They endeavor to highlighting possible articulations both in the scope of the philosophy of mathematics education, as well as in the practices which foster teaching and learning. Bicudo and Rosa, for instance, sought out to understand the *Lebenswelt* in which we are immersed nowadays, in which science and computer technology are present, constituting cybernetic space. They asked: How does one get around to teaching and learning mathematics within this reality? Hausberger and Patras (2019) dedicated their efforts to understanding the notions of horizon, the ways by which hermeneutic studies of texts are conducted, including in mathematics, articulating their understandings with the didactics of mathematics. Bicudo (1991) authored a text on different ways of understanding hermeneutics and advocated the importance of bringing it into the didactic-pedagogical activities conducted by mathematics teachers. Garnica (1992) also dedicated himself to studying hermeneutics, investigating how mathematics texts could be hermeneutically read during an activity conducted by teachers with students, describing the meanings that words opened in the readers' horizon of interpretation. Kluth (2005) investigated the algebraic structure, focusing on the way through which its logical-mathematical determinations and language are presented throughout the historicity of the formalizations effected within the discipline. Batistela (2017) endeavored to study and understand Gödel's Incompleteness Theorem, advancing into the field of mathematics teaching within mathematics teachers' education courses. Since the 1990s, Paulo has studied topics relevant to comprehension and perception, articulating them to the understanding of the ways through which children and also university students constitute and develop knowledge of geometry.

This chapter focused on the studies of Husserl about the specificities of mathematics, Sciences of Nature, and Sciences of Spirit,[2] and technology. While highlighting them, this author wonders what horizons of philosophical, thus critical and reflexive thought, are opened regarding their importance, and what they say about the world, despite understanding that they do *not* dictate, or are not the guardians of the truth, much less establish the parameters that should guide human knowledge and the ethics of personal and social interrelations.

In this chapter, the goal is to make these ideas explicit. Therefore, the meaning of the mathematization of nature, the way to achieve accuracy at the level of categorial thinking, the way through which mathematical objects become idealities, enabling

[2] Sciences of Spirit are sometimes called cultural, humanistic, or moral and political (Mora, 2000, p. 465). They are also generally treated as Humanities. When understood as Humanities, in the Western world, in the modern era, in the present, they can be investigated through the logic of modern science, as developed since Galileo.

advances in the field of mathematics and other sciences that stream toward the liberty of procedures, free from the empirical experience performed in the concreteness of the natural world.

It is assumed that Mathematics education deals with mathematics, thus with the logic underlying it. But also deals with issues concerning human sciences. It promotes critical thinking about society and ideological-political issues that populate everyday life. It is complex work that requires a vision that encompasses both the logic of the production of mathematics itself and its presence in other sciences and technology, as philosophical thinking, thus critical and reflective, regarding the same sciences and how they+ came to prevail, dictating "truths" about the reality of the world. This work must be the goal of mathematics education when mathematically educating citizens and forming individuals.

The abovementioned goal of mathematics education is a broad endeavor to be thought about and assumed by those who care about educating mathematically and who seek to accomplish that. This text focuses on a *mode* of understanding the production of mathematical idealities and the mathematization of sciences—natural and human—that entail the imposition of a scientific truth above the reality experienced by people in their daily lives. This imposition creates obstacles to the understanding of the articulation between scientific theory and reality, as well as opens paths for separate cultures, groups, and worldviews.

Due to the importance of Edmund Husserl's work, the rigor present in his investigations and the pertinence of the themes he dealt with, which are still current today, this text presents the manner through which he understands the production of mathematics, sciences of nature and sciences of spirit, and his view on the imposition of "scientific truth" as dominating in relation to other possibilities for human beings to explore and understand the world. At the same time, articulated considerations are presented based on the enlightening arguments regarding mathematics education. To do that, the following are addressed: the production of mathematics, highlighting the logic present in pre-categorial knowledge; the change of target that takes place in the production of geometric knowledge; the extent of this way of knowing which is established as structuring of scientific knowledge in the Modern Age, to the present day with Galileo, underlying the positivist view of science. The treatment of such themes is articulated by contemplating *mathematics education: the necessary critical-reflexive view*.

3.2 The Production of Mathematics

This production results from a historical-cultural movement established and which endures in the very manner through which human beings cope with the world[3] in their daily dealings. However, it is not a mere representation of what is seen,

[3] *World is used here in its original meaning.*

understood, and performed in those dealings. Nonetheless, in the process of its constitution, there is a change in the view of the objects given in their physical concreteness and space-time position and dealing with them under a perspective that is different from that which prevails in the natural world. Focusing on this change and seeking to understand the threads that are intertwined in it is an arduous task that has been the objective of many scholars, from different areas, for centuries. To this end, a selection is made in order to look at what has been accepted as being the cradle of science in Western civilization, ancient Greek culture. I will take the work of Edmund Husserl, specially *The Crisis of European Sciences* (1970a),[4] to elucidate that movement.

In that work, Husserl produced an interpretative study about the movement that, according to his understanding, structures the transformation of pre-categorial knowledge, present in day-to- day life in the world, into scientific knowledge, which he calls categorial. Pre-categorial knowledge is produced in the world of common experience which we navigate in space-time, making predictions and acting according to expectations based on them. Categorial is knowledge which is already based on the theoretic view of reality. It is law-oriented knowledge. According to Ales Bello (1986), those laws can be understood in the 1927 lectures, when Husserl wonders about the *Gesetzeswissenschaft* law. According to such law, he reports as follows:

> (a) the law that applies to the world surrounding us, according to which the world is homogeneous; it represents the generality of all that is "regulated"; therefore, we are talking about a regulated "event", of a regulated "*becoming*", of a regulated "causality";
>
> (b) the "exact" laws that contrast with the first regularity that can be defined as purely "typical". Opposition is justified as accuracy, which consists of presuming the identity of singularity in a priori sciences, and can assume two meanings, depending on whether identity is understood as "ideally-exact" or as "typical" (this is the difference between mathematization and generalization) (Ales Bello, 1986, p. 152).[5]

[4] It is important to state that the *crisis* of contemporary European sciences is a theme that had already concerned Husserl. In 1922, he addressed this theme in an article for Kaizo Magazine (Husserl, 2006). Underlaying this concern is his view of philosophy which, in a continuum, from the beginning of his studies, shows that it must "[...] 'teach us to carry out the eternal work of humanity'. It must not only enlighten man about actual states of affairs but also to give leadership in ethical and religious matters" (De Boer, 1978, p. 497).

[5] *Nelle lezioni del 1927 (Manoscritto F 1 32) Husserl si domanda in che cosa consista la legge e quindi che cosa sia la scienza della legge (Gesetzeswissenschaft). Egli risponde distinguendo: (a) la legge che vale nel nostro mondo circostante secondo la quale il mondo si presenta come omogeneo; essa rappresenta la generalità di tutto ciò che è "regolato", per cui si parla di un "accadere" regolato, di un "divenire" regolato, di una "causalità" regolata; (b) le leggi "esatte" che si contrappongono alla prima regolarità che può esser definita come puramente "tipica". La contrapposizione si giustifica in quanto l'esattezza, che consiste nel presupporre l'identità della singolarità nelle scienza* a priori*, può assumere due significati a secondo che l'identità sia intesa come "'esatta-ideale'" o come "tipica" (questa è la differenza fra matematizzazione e generalizzazione).*

3.2.1 *Logical Aspects of Pre-Categorial Knowledge and What They Mean for Mathematics Education*

Phenomenologically, Husserl worked to delimit an *invariance* in relation to the natural world, to expose the characteristics of the way of knowing conveyed in it. He explained that what is characteristic of the pre-scientific knowledge of nature, present in Greek culture, lies in the belief in a *undivided-world* and in the connections established that are perceived as empirically definite, because, in the practicality of everyday life they are valid, and because they make sense within the naïvely assumed context of vision, and are supported by the uniqueness of the intuitable world. The basis of such pre-scientific knowledge lies in its concreteness, given in daily empirical intuition. "If we take the intuitable world as whole, in the flowing present in which it is straightforwardly there for us, it has even as a whole its 'habit,' i.e., that of continuing habitually as it has up to now" (Husserl, 1970a, p. 31). As a whole, the world allows us to presume that, if it has been that way so far, it will continue to be so. "In the natural spiritual attitude, a world that exists is before our eyes, a world that extends infinitely in space, that is, that was and that will be; this consists of an inexhaustible multiplicity of things that sometimes persist in their state, sometimes change, that become interlocked with each other, and then break apart, that carry out reciprocal actions and, reciprocally are the object of such actions" (Husserl, 2009, p. 4–5 author's translation).[6]

The knowledge of this world is founded on the *certainty* that the world is based on the understanding of the perceived physicality of things, of the possible explanations about how they are arranged among themselves, in space and time, and on the fact that these relationships remain valid in the empiricism of daily experiences. This is a natural concept of the world which is imposed onto everyday life. "The world understood as the unity of all constitutive experiences of associative formations, founded on *Glauben* (intuition) is empirically cognoscible, according to the *logical principle* of *induction* which encompasses the same breadth of objectifying experience" (Ales Bello, 1986, p. 148, author's translation).[7] The invariants pointed out in the analyses performed reveal a structure of science examined here in the soil in which it takes root, that is, the natural world. This does not import asserting that science is absolute, but that in science invariant aspects are found which reveal themselves as inherent to its structure. This idea is found in the *Introduction to Logic* and the *Theory of Knowledge*, when Husserl, in § 2, declares that the idea of science presents that which is logical as the essence of science in general. He

[6] Nell'atteggiamento spirituale naturale ci sta davanti agli occhi um mondo che esiste, um mondo che si estende infinitamente nello spazio, che è, che è stato e che sarà in futuro; esso consiste di uma inesauribile molteplicità di cose che ora persistono nel loro stato ed. ora si modificano, che si entrecciano l'um l'altra per poi separarsi, che esercitano azioni reciproche, e che reciprocamente le subiscono.

[7] Il mondo inteso come l'unità di tutte le esperienze, come formazione costitutiva delle formazioni associative fondate sul Glauben, è conoscibile empiricamente secondo il principio logico dell'induzione e questo ha la stessa estensione dell'esperienza obiettivante.

affirms: "[...] it is of course that the character of this which is logical is constitutive of science" (Husserl, 2019, p. 29. author's translation).[8]

The following invariants are highlighted: pre-categorical knowledge is structured by a *logic*, in which induction, causality, measurement, hypothesis, confirmation of the hypothesis are present. *Logic* is founded on a conviction based on a straight, intuitive view of what is immediately perceived in an environment that surrounds us and is close to us, coexisting with us. It is maintained by our confrontation between being seen and seeing, touching, and being touched. This type of coexistence incorporates conductive threads that guide us from perception to perception, so that the surrounding space is revealed as real, as well as time in which perceptions flow as near and far occurrences, which bring meanings that lead us to *infer* those that may or may not occur. *Inference* is inherent to the style of the experience of the world, which is taken as an *a priori* that manifests itself in the total course of experience. The validity of *inference* is confirmed in empirical practice, through reasoning based on approximations that, in an imaginative variation, can be thought of as: happened fully as anticipated; almost happened, etc. *Certainty*, or confidence is established when expectation, which takes on the role of *hypothesis,* that the next event will be similar, based on previous events, in similar circumstances, is confirmed. The calculation of the event, or its absence, of what is expected is approximate. If it occurs as expected, this event fills the void of the wait for the hypothesis to be confirmed and, based on that *causality,* it is shaped. It is established with the successful *repetition* of what *happened* and what was expected to happen in a certain way. "This universal causal style of the intuitively given surrounding world makes possible hypotheses, inductions, predictions about the unknowns of its present, its past and its future. In the life of the prescientific knowing we remain, however, in the sphere of the [merely] approximative" (Husserl, 1970a, p. 31).

According to Husserl (1970a, b), the logic introduced as structuring such knowledge is present in the natural world of Greek civilization and is evidenced as the soil in which Euclidean thinking flourished. However, it is important to emphasize that it is also present in the way we deal with the world in which we live, in our daily lives. It is prior knowledge, as Heidegger (1962, p. 182) refers to it in the "Being-there as understanding" which brings previous sight, understanding, and interpreting. We conjecture about possibilities, bet on those whose occurrence seems more possible, making decisions; we calculate distances, even if approximately; we conceive forms and, through actions that in practical life are successful, based on those forms, use them to build utensils.

In daily school life, pre-categorial knowledge supports the actions, interpretations, and evaluations performed. The educational work carried out in schools is based on the intertwining of that knowledge and categorial knowledge, characteristic of science, as conceived and dealt with in western civilization. In the case of mathematics teaching, it is imperative to point out this intertwining between the activities proposed and developed, with emphasis on the modification of the

[8] "[...] è ovvio allora che il carattere di ciò che è logico è constitutivo per il carattere dela scienza".

perspective, that is, the difference in the aspects of the objects with which individuals are working and the respective ways of treating them. For instance, in the natural world, counting and measuring are carried out in an approximate way. It is possible to count stones, group them into small piles, etc. However, numbering stones is an action that involves aspects present in the constitution and production of *idealities,*[9] such as how to work with the essence of numbers, the way of positioning it, denominating it, etc. Measurements can be performed in an approximate way, through steps, size of hands, for instance. Nonetheless, measuring requires a standard, which is taken and accepted as accurate. The acceptance of the standard can be based on social consensus. However, this criterion does not satisfy those of a broader generality, which transcend a specific social group or culture. What is necessary to establish such a criterion? How should exactness be established? The categorial knowledge method makes a difference.[10]

Educational action is responsible for highlighting such differences, clarifying instances when a way of dealing with the knowledge of the world and its practices, and the decisions resulting from it are acceptable and valid, and others in which other tools are needed, for instance, scientific tools.[11] The formation of individuals and citizens calls for the discussion of the non-supremacy of one knowledge over another. Moreover, it requires people to ponder, individually and with peers, in different instances of social organizations in which they operate, about the bases on which decisions will be made and taken responsibly.

3.2.2 Logical Aspects of Categorial Knowledge and Its Implications to Mathematics Education

Changes occur within categorial knowledge, in comparison with pre-categorial logic explained above. In the studies mentioned herein, Husserl conducts analyses that show the profound difference between the gnoseological process, which is the base of Euclidean Geometry, and the ontological view that underlies the pre-predicative science of nature. Euclidean Geometry seeks *exactness*. Moreover, along with this search, there is a process of constituting a *method*, which will prove powerful in advancing scientific and technological thinking.

As an *a priori* to Euclidean work, that is, the foundation in which it is based and generated, is the natural and pre-scientific knowledge of the world. However, it goes

[9] Will be explained in the next section.

[10] Will be explained in the next section.

[11] This article is focused on pre-categorial and categorial knowledge, as my aim is to deal with the characteristics of mathematics and the mathematization of sciences of nature and, by extension, of the conception of the world. However, the complexity of life is such that there are situations in which religious knowledge, or other ways of tackling reality, are more consistent with the issue to be decided. Examples are the ethical question of turning off equipment that keeps organisms alive, when there are no longer signs of survival; organ donation; civil disobedience, etc.

beyond that, it is dissimilar to nature, through the establishment of a working method that becomes invariant in the logic of Western science. In *The Elements* (Bicudo, 2009),[12] Euclid contemplates the method that sustains the transformation of the primitive empiric mathematical knowledge of Egyptians and Babylonians into deductive, systematic, and based on definitions and axioms Greek mathematical sciences. This is a change in the way to produce knowledge, notably founded on experience (*empiria*), for one which is based on statements expressed in a formalized way.

The transformation of the knowledge labeled above as primitive is not the work of a single individual, in the case considered herein, Euclid. It takes place in the becoming *(devir)*of the historical-social movement underway in Greek culture which, in the III A.C., when Euclid lived, already relied on the philosophy of the pre-Socratics, Parmenides, Heraclides, Democritus, and the philosophers who followed them chronologically, Sophists, Socrates, Plato, and Aristotle, in whose thinking the supremacy of *episteme* in relation to *doxa* is emphasized, with emphasis on *logos*, to sustain true knowledge, which is the subject of their investigations. Aristotelian syllogistics[13] brings a systematization of this logic. It works with predicative judgments, as expressed in predicative language, proceeding to a formalization of predicative forms, which support connections of logical reasons, also formalized.

Aristotle was the first propose the idea of form which was to determine the fundamental meaning of "formal logic" [...] Aristotle was the first, we may say, to execute in the apophantic sphere—the sphere of assertive statements ("judgments" in the sense expressed by the word in traditional logic) —that "formalization" or algebraization [...] (Husserl, 1978, p. 42).

In the demonstrations presented by Euclid, we find the idea of formalization, as well as forms of progression between the prepositions, generating demonstrative connections. However, the novelty regarding the works of philosophers are the space-time figures of the surrounding world with which Euclid worked. There is a crucial modification that remains at the structural basis of Western science: these figures are not mere representations of the forms of objects as seen in direct experience with the world, but conceived and dealt with as ideals, that is, as eidetic.[14] Such space- time figures are *idealities*, resulting from idealization which is constituted

[12] This reference is from the first translation of *The Elements*, from Greek to Portuguese. Translation made by Irineu Bicudo, a mathematician and Greek scholar. In the "Introduction" which the translator wrote and is 79 pages long, his erudition and care for the philosophical and historical bases, thus evidencing his concern with the important work of "translating" such significant work, both from a theoretical and historical point of view, and whose manuscripts have faded away with the passage of time.

[13] Aristotle, 384 B.C.–Athens, 322 B.C.

[14] In Husserlian phenomenology, *eidos* or *essence* derives from the *noesis-noema* movement, that is, of bringing, by intentionality, what is perceived in the action of perceiving. The *idea* is that what is perceived, apprehended at the moment of the act. It is with the idea that consciousness operates, articulating meanings, given in the experiences of the living-body, and the meanings presents the language that conveys understanding and interpretations.

and produced in the dimension of subjectivity-intersubjectivity, in which the meanings made within the living-body of each individual, and the meanings conveyed by the language in the socio-historical-cultural intersubjective sphere are present. Thus, they become mathematical *objectualities*. They display a generic universality, not related to a specific figure.

By following Husserl (1970a, b, c), through imaginative variation reasoning, one can comprehend how that happens. He exposes his reasoning as follows: space-time figures are perceived in the common movement of the world, in which the perfection of the forms is pursued, moving further to imaginary improvements in an open horizon. Such movement leads to the possibility of seeing *limit-shapes* that are shown as convergent poles of improvement. The perfected forms of *limit-shapes* lead to pure-thinking, without the contents perceived in the concreteness of the empiricism. Pure thinking, devoid of perceived contents, is now in the domain of these figures. "In place of real praxis – that of action or that of considering empirical possibilities having to do with actual and really [i.e., physically] possible empirical bodies – we now have an ideal praxis of 'pure thinking' which remains exclusively within the realm of pure limit-shapes" (Husserl, 1970a, p. 26). What is at work here is "[...] a method of idealization and construction which historically has long since been worked out and can be practiced intersubjectively in a community [...]" (Husserl, 1970a, p. 26).

This method of idealization and intersubjective construction creates ideal objectualities that become the soil in which *exactness* is obtained. With them, it is possible to determine absolute identities that can be known in a uniquely identical way, as well as it is possible to generate a systematic and aprioristic method, of maximum scope. This method points to the methodology of exact measurement.

A separation of empirical practicality is established in relation to ideal practice. Working in the dimension of ideal practice and with limit-shapes becomes the methodical practice of mathematicians. They deal with ideal objectivity and not with empirical content. In the ideal dimension, it is possible to obtain *exactness*, which is not achievable in empirical practice. This is because limit-shapes, in their ideality, enable the determination of an *absolute identity*. Through this methodical way of proceeding, it is possible to work with that which is absolutely identical and methodically univocal. Singular configurations, such as sections of lines, triangles, or circles, can be highlighted.

Geometry is structured through in this practice. Given this mode of operation, by virtue of its generating method, it is possible to construct other figures, which are determined in a unique way. "For in the end the possibility emerges of producing constructively and univocally, through an *a priori*, all-encompassing systematic method, all possibly conceivable ideal shapes" (Husserl, 1970a, p. 27). With the support of Aristotelian logic, the field for the formalization of geometry is open and can be realized. The connection between logic and science is thus established, in this specific case, geometry. The connections of inferences are systematized and explained at the dimension of the idealities of the geometric bodies and logical laws with which geometry operates.

The novelty of Euclidean geometry, in relation to pre-categorial science, lies in exactness, operated by an indirect mathematization. Indirect because it does not work with the given object in its empirical concreteness, but with the *idea* and the idealities—perfect objects, as the methodof obtaining *exactness* enables the production of perfection.

Euclidean Geometry's implications to mathematics educations. It has been a strong component of the curricula that prevail in Western civilization schools. Its logic underpins the very structure of traditional curricula; from mode of inference laws ranging from the simplest to the most complex topics, even imposing prerequisites for content to be addressed. Moreover, the chronological time that prevails in this structure is idealized, not considering what is experienced in the movement of teaching and learning of the people involved in such actions. With regard to the mathematics curriculum, geometry is always present as content to be worked on and studies reveal that it supported the teaching of Mathematics. Imenes (1989) explains that the teaching of mathematics follows *The Elements* of Euclid, in terms of the content presented in that work, but mainly through the didactic model derived from it. He states that "Although it was not written for didactic purposes, for many centuries, that work was used as a reference for the teaching of Euclidean Geometry, as a true textbook" (Imenes 1989, p. 193).

Given the intertwining between the geometric figures that "can be visualized" in the natural world and the limit-shapes with which Geometry works, within the didactic work of school routines, it is easy to naively overlap them, without due care, regarding the change of perspective in that view. Formal science, which works with idealities, and empirical factual science have been regarded by many mathematicians and mathematics teachers as equivalent. The gap between the work at the level of formalized geometry and the work at the level of pre-categorial knowledge has been increasingly established with greater vigor. The difficulties in understanding what scientific "truths" say about the world multiply. They grow even larger with Galileo and the "transposition" of Euclidian geometry into physics, extending toward the sciences of nature and the spirit.

3.2.3 The Loss of Meaning Implicit in the Transposition of Geometric Logic into Physics and What it Means for Mathematics Education

This transposition occurs within the framework built by Galileo[15] and other scholars who came after him in the following centuries, assuming his vision, corroborating it with that of science which prevails today.

In order to understand what was said regarding the loss of meaning in the title of this subitem, it is important to always bring the meanings opened by the word

[15] Galileo Galilei; Pisa, February 15, 1564 — Florence, January 8, 1642.

meaning in Husserlian phenomenological philosophy. Meaning is made for the subject at the moment he/she receives, through sense organs (touch, smell, vision, taste, hearing, and kinesis), the aspects of the thing shown to them in the dialectics of noesis-noema,[16] which now reveals itself as a phenomenon. It is effected in the articulation between the different sensations performed by the living body, in the stream of consciousness, considered in the conception of transcendental[17] phenomenology. Therefore, the loss of meaning occurs when there is a distancing from such experiences and the concept expressed and defined in language. Therefore, it is not a social, historical, and cultural transformation of the concept, but a dichotomy between what is perceived and lived, and what is expressed either in pre-categorial or categorial language.

Therefore, the loss of meaning mentioned in the title of this sub-item refers to the distancing between the experience felt by the individual in the empiricality of the world in which they live and the ideality of the objects constituted and produced in the dimension of subjectivity and intersubjectivity. It is noteworthy that the intuition of the individual about what they see and experience occurs in the flow of experience and in the carnality of the living body[18] (Bicudo, 2020). This intuition engenders acts of transcendental consciousness and transcends subjectivity, through language-mediated expression, launching itself into the intersubjective sphere, intertwined with history and culture, in which the individual is with the other (whoever they may be) as a living body.

With the work of Galileo, this distancing is strengthened, as the theory that is under construction moves away from the vision of the natural world, which sustains Euclidean work, and begins to work with the idealities made explicit by concepts and formulas. There is an epistemological cut and a change in the view of reality. With Euclid, reality is what he experiences and where he acts in the natural world, also understood from the perspective of the logic of the knowledge thus produced. With Galileo and his science that is always under construction; the reality in which the thinker-scientist moves is that which springs from theoretical truths. An epistemological cut, as, in the first case, with Euclid, meaning is effected within the realization of experience, and the explanations are structured according to a non-propositional, exact logic. In the second case, with Galileo, it is necessary to

[16] The structure of intentionality, as exposed in Husserl's writings, can be analyzed through two components: the object as intended, that is, as something to which consciousness is directed, the *noema*, and the conscious act that intends the object, the act of *noesis*. The noema secures the side of the object in the intentional relationship and the noesis highlights the side of the individual, intending the ways in which it is given to consciousness.

[17] The living-body, as explained by Husserl's (2002) phenomenological investigation, is shown as the center of orientation to the world, the other and the self. It is the core of distinctive and diffuse sensations, of the constitution of mental and spiritual life. The living-body, unlike other physical bodies, moves by itself, showing autonomy and freedom, to come and go, to direct itself in a targeted manner.

[18] Transcendental consciousness is understood as not regional, fluid, moving, originator of meaning; therefore, it is not separated from the body, even though it transcends and goes beyond the body, through its characteristic intentionality.

seek links between conceptual language, the logic that structures the theory that is already sustained in the logical system, engendered by Aristotle and the world experienced, in which experiences are expressed in a common language.

Apparently, this difference between pre-categorial and categorial logic could be understood as that between spontaneous and scientific language, which are concepts exposed by Vygotsky. However, it is not just about using signs (psychological tools) as language, before understanding their function or meaning. That author distinguishes between two types of concepts, namely scientific and spontaneous, each encompassing different genetic origins and histories. Scientific concepts evolve through instruction; spontaneous concepts emerge from everyday experiences (Vygotsky, 1986). According to Vygotsky, the development of scientific and spontaneous concepts follows different paths. The former moves from abstract to concrete whereas the latter moves to a greater abstraction, starting from concreteness. However, when Husserl mentions the distancing that initially occurs with Euclid's work and what is established beginning with Galileo's work, this does not refer solely to ways of expressing different concepts, but to the vision of reality assumed and the logic that structures the synthesis of ideas that a concept expresses. This is the shift from pre-categorial to categorial knowledge.[19]

Despite the loss of meaning pointed out, from the first half of the sixteenth century, science continued on its path of development. It is based on the logic of Euclidean Geometry that opens possibilities for all ideal figures, generally imaginable, to be constructively and univocally generated, by a systematic aprioristic method of maximum extension.

Back to Galileo. The work of Galileo mathematizes the physics then known. Thus, there is a great change in the view derived from the way Galileo works with physical bodies and their movements in space, as idealities, thus achieving accuracy. The work is supported by the logic of Euclidean Geometry by the possibility of generating constructively and univocally, through a systematic aprioristic method of maximum extension, all ideal figures *in general imaginable*, as stated above.

The work of Galileo leads to a *mathematization* of the physics then known. The great modification of perspective that occurs comes from Galileo working with physical bodies, and their movements in space, as idealities, thus obtaining

[19] I believe that working with language as a mediator between subjective dimensions (psychological and all others the living-body encompasses) and intersubjective, thus also referring to social, historical, and cultural contexts, evidences two aspects that must be assumed. One concerns the view of reality seen as primarily social, in which language is a given, as well as the relationships among individuals, and the positions they occupy in the social structure. Presuming that aspect takes as given (takes for granted) that language is first; with immediate concreteness. Another aspect evidenced is not questioning the constitution and production of language and social organization. This does not at all mean to affirm that man comes first and creates social structure. It means to affirm that, in the movement of life experienced by the individual, in the world where there are others, the individual-word-other; intertwiningly, social, historical, and cultural reality are constituted and produced. As I see it, Husserlian phenomenology strives to comprehend that movement.

accuracy. To do so, he took Euclidean Geometry and its method as given, that is, as being there in the world to be put into operation and expanded.

The change from one paradigm to another is intensified and expanded in its strength with the operationalization of accuracy, sustained by measurement. It is conducted through the *art of measurement*.

> The art of measuring discovers *practically* the possibility of picking up as [standard] measures certain empirical basic shapes, concretely fixed on empirical rigid bodies which are in fact generally available; and by means of the relations which obtain (or can be discovered) between these and other body-shapes it can determines the latter intersubjectively and in practice univocally – at first within narrow spheres (as in the art of surveying land), then in new spheres where shape is involved. (Husserl, 1970a)

Galileo had before him Euclidean geometry, contained in a well-articulated theory, according to the laws of Aristotelian logic, which brought within its structure the idealizing method and the possibility of new and accurately measurable creations. This procedure leads to a rupture in the conceptions assumed between men of science and commoners. It establishes a separation between those visions, intensified by the interest in the pure application of geometry to physics, without questioning followed by enlightening understandings of the basis of scientific knowledge rooted in the logic of the knowledge of the natural world. According to Husserl (1970a), the origin of theory was hidden, and this concealment generated, within the scope of modern science in the West, a crisis seen as insufficient understanding of the world.

In an imaginative variation, the author explains:

> Galileo said to himself: Wherever such a methodology is developed, there we have also overcome the relativity of subjective interpretations which is, after all, essential to the empirically intuited world. For in this manner, we attain an identical, nonrelative truth of which everyone who can understand and use this method can convince himself. Here, then, we recognize something that truly is — though only in the form of a constantly increasing approximation, beginning with what is empirically given, to the geometrical ideal shape which functions as a guiding pole. (Husserl, 1970a)

The intention of getting to know the world in a scientific way engenders the need to *discover* a method to systematically construct, previously, the world and its causalities and *confirm* this construction in a safe manner. Given the impossibility of conducting direct mathematization to fulfill this intention, indirect mathematization is performed. It should be emphasized that in the knowledge of the pre-categorial world, a universal concrete causality prevails, anticipating that the intuitable world can only be intuited as an open world, on an infinitely open horizon.

> To be sure, this inductivity was not understood by Galileo as a hypothesis. For him, a physics was immediately almost as certain as the previous pure and applied mathematics. The hypothesis also immediately traced also for him (its own) path of realization (a realization whose success necessarily has the sense in our eyes, of a *verification of the hypothesis* – this by no means obvious hypothesis related to [previously] inaccessible factual structure of the concrete world). (Husserl, 1970a, p. 39)

In the scientific knowledge of physics, which is emerging, it is necessary to seek far- reaching methods; measure speeds; accurately apprehend universal causality.[20] It is necessary to determine, in a positive way, the knowledge generated based on proof of hypotheses. In the quest to account for his task, Galileo used practical action, obtaining accurate measurements and determinations. And the meaning of the *logic of the art of measurement*, whose crucial characteristic is to always be in motion perfecting the measure, toward accuracy remained hidden. The implicit meaning in this logic: improve, *repeatedly and continuously*[21] the method for obtaining exactness.

This goal can be achieved through an *indirect mathematization of the intuitable world* with the application of general numerical formulas that, once found, can serve for its application, to conduct the factual objectification in particular replaceable cases.

> The formulae obviously express general causal interrelations, "laws of nature," laws of real dependencies in the form of "functional" dependencies of numbers. Thus, their true meaning does not lie in the pure interrelations between numbers (as if they were formulae in the purely arithmetical sense); it lies in what Galilean idea of universal physics, which its (as we have seen) highly complicated meaning-content, gave as a task of scientific humanity and in what the process of its fulfillment through successful physics results in – a process of developing particular methods, and mathematical formulae and "theories" shaped by them. (Husserl, 1970a, p. 41)

The very meaning underlying this mathematization and which is *hidden* when scientists and the science teachers turn into technicians are not the connections of numerical values, mathematical relationships, operations, etc., but the Galilean idea of universal physics. The enthusiastic interest of the researcher of nature, that also becomes that of the mathematician, is in *formulas*. The former is aimed at the *natural scientific* method, understood as the method of true knowledge. The latter aims to work with exact formulas.

What does formulation bring*?* Implicitly, its logic results *in loss of meaning*. That is why the formulation provides numbers, in general, based on general propositions that express laws of functional dependencies, moving away from the concreteness of the empirical world and freeing thought to advance toward imagined possibilities, however not in the dimension of fantasy, but delineated in the context of the previously determined prediction. This freedom is expanded with the help of algebra, through the symbols that replace figures and their respective ways of operating with them, always working in a categorial dimension. The effect of this way of thinking seems to be beneficial, on one hand, and sinister, on the other. First of all, algebraic formulations mean "an immense extension of the possibilities of the arithmetic thinking that was handed down in old, primitive forms. It becomes free, systematic, a priori thinking, completely liberated from all intuited actuality, about numbers, numerical relations, numerical laws" (Husserl, 1970a, p. 44).

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[20] Husserl shows how such quest led to analytic geometry.

[21] In mathematics, the idea of *always and continuously* takes on the meaning of *infinitum.* It operates as a center of convergence.

It is necessary to clarify that this broadening had already occurred with arithmetic, which also brought about freedom from empiricism. There is an intersection between arithmetic and algebra which is therefore in the nature of their objects; symbols. However, the former deals with symbols which denote numbers, operations, and relations, such as equality. The latter broadens the symbolic set and the notion of operations beyond usual numbers. In the words of Husserl, "the sign for +, for instance, is not the sign for arithmetic addition, but a general connection, for which law such as 'a+b=b+a' are valid" (Husserl, 1962, p. 94, translated by the author). This means, for instance, that while in arithmetic the equality '2 + 3 = 3 + 2' is valid, in algebra, the law given by 'a+b=b+a' doesn't even require that a and b be numbers but establishes a link for all objects which are compatible with it. In this sense, there is a broadening, a liberation from arithmetic, from all intuitive reality, made possible by algebra. However, following the thoughts of Peacock (1830), generalizing arithmetic is just one of the possible explanations for algebra. From that, in its modern conception, one can deduce what Husserl called "supreme algebraic thought" (Husserl, 2012, p. 35), which enabled a theoretical leap for mathematics and cognitive for human thought, as it deals with "'something in general' constructible in pure thought, in pure-formal generality" (Husserl, 2012, p. 35, author's emphasis). Thus, algebra allowed mathematical thought to be freed from all empiricism, from all factuality, expressed objects defined solely through a determined axiomatic which "in relation to its matter, (...) remain entirely indetermined" (Husserl, 2014, p. 186).

The beneficial aspects brought about by formulation activity show the amplification of what has already been determined. However, these same benefits encompass an emptying of their meaning, as pure intuitions are transformed into pure numerical figures, with algebraic configurations, which, as previously explained, make it possible to express "supreme algebraic thought" (Husserl, 2012, p. 35).

The beneficial aspects, which can be understood, show the expansions of what is already determined. However, these same benefits they also bring an *emptying of their meaning*. Pure intuitions turn into pure numerical figures, with algebraic configurations. In algebraic calculus, the geometric meaning automatically recedes, or is abandoned. One calculates and, only in the end, remembers that the numbers should mean greatness. One calculates with symbols. "This process of method-transformation, carried out instinctively, unreflectively in the praxis of theorizing, begins in the Galilean age and leads, in an incessant forward movement, to the highest stage of, and at the same time, a surmounting of, 'arithmetization;' it leads to a completely universal formalization" (Husserl, 1970a, p. 45). Thus, the distance between the intuitions that occur in the empiricalness of experiences enjoyed in the natural world increases, moreover, there is a rupture caused by the change of point of view and by the non-reflection on what is being done and obtained with this practice.

The emptying of meaning is accentuated with technization. Within the framework of the mathematical disciplines themselves, for example, with operation with letters, by connecting signs (+, x, =, etc.) and according to the connective ordering *rules of the game*. The original reasoning, even if regarding formal truth, is left out of the loop. One operates, obtains correct results, because they are accurate. Thus, "purely geometric thinking is also emptied, as well as its application to factual nature, to scientific-natural thinking". It is important to emphasize that the possibility of getting lost in technization is inherent to the essence of every method. Therefore, natural science undergoes a multiple transformation which also encompasses the concealment of meaning. The natural world becomes a forgotten foundation of meaning in natural science.

It falls onto *mathematics education* to focus on the beneficial sides of generalization and formalization, as well as the loss of meaning, when working with formulas and their importance from the perspective of scientific explanations and, at the same time, highlight what worldly reality they hide. It is always work that develops in a middle line interweaving scientific practice and philosophical thinking regarding the meaning of the world and what science and technology mean in it and enables such meaning to be accomplished.

3.3 Mathematization as Methodological and Ontological Foundation of Reality: What it Means for Mathematics Education

The mathematization initiated with Galileo's work which prevailed throughout Modern Age, with the establishment of Positivism, imposing itself as a way of investigating and obtaining accurate knowledge, which is considered "*true*" knowledge. It advanced into technical and technological practice, throughout the contemporary era, with the advent of mathematical logic, analysis and calculus, algebra, and its possibilities of application to natural sciences, and gradually to spiritual sciences as well. This expansion was possible by replacing the pre-scientific view of nature with an idealized one, which could be the stage for hypothetically correct investigations, because they are accurate. They support explanations and predictions also calculated by general numerical formulas. These lead to the objectification of particular cases, of events that occur at the level of the natural world. This results in the application of the general to the particular, that is, from theoretical laws to particular cases, consummating the application of theory to practice.

At the heart of this conception is the meaning of the hypothesis, always understood as hypothesis, although confirmed by scientific research. The methodical process of establishing, proving, and assuming proof brings in itself the possibility of errors which are calculated and assumed within the process. It is a method that propagates the idea of continuous improvement, understood as progress. "In the total idea of an exact Science, just as in all the individual concepts, propositions, and methods which express an 'exactness' (i.e., ideality) – and in the total idea of physics as well as the idea of pure mathematics – is embedded *infinitum*, the permanent form of that peculiar inductivity which first brought geometry into the historical world" (Husserl, 1970a, p. 42). The idea of *infinitum* encompasses that of increasing refinement that offers a better representation of nature. This view presumes supremacy and underscores explanations and predictions in the diversity of ways that life manifests itself. It fulfills its destiny, that of constant improvement, becoming absolute. It adorns reality with robes woven with symbols, masking experienced reality. It shows a reality seen and understood through the lens of mathematical-scientific idealities.

Thus, there is an inversion in terms of the conception of the reality of the natural world. This reality is affirmed in the positivity of scientific laws and is no longer seen as the soil in which opinion and reason are exercised, explaining a pre-categorial and epistemic knowledge about reality. One enters the path opened by Galileo's physics and ends up pontificating the so-called scientific "truths," which become synonymous with what is, that is, with being. The prevailing view is that of an infinite world obtained by a rational, coherent, and systematic method. With such a view, an infinite horizon opens up for mathematics. It has supported sciences, both through the possibility of applying its theories and by serving them as a methodical and ontological foundation, consummating an ontology of mathematics, which Husserl called the mathematization of nature.

Here, a forcible break is made in the realm of the positivist sciences. Truth is dictated by scientific theory, which postulates mundane reality. It goes beyond lived reality and the experiences of individuals who, in their sensitivity, perceive nuances that are dismissed by scientific theory. This break implies a schizophrenic view, as the individual must deny their sensitivity and perception and impose the "scientific" truth on themselves. There is a distinct separation between pre-categorial and categorial knowledge, and what science states about the world does not make sense to human beings, taken in their subjective individuality.

Husserl argued that the new sciences—those built from Galileo's work—undoubtedly seemed initially successful when they displayed their favorable results through the application of their theories. However, he considered that this initial impulse gave way to a sense of failure. He claimed that this is the beginning of "a long period extending from Hume and Kant to our time, of enthusiastic struggle for a clear, reflecting understanding of the true reasons for this centuries world failure" (Husserl, 1970a, p. 11). He stated that the sciences dissolve internally but fail to understand the meaning of their original foundation when it appears as a branch of philosophy. This means that when sciences are separated from philosophy, they stop thinking philosophically about its meaning, what it says about the world, humanity, and life itself. Thus, a crisis is established in the European community, initially latent, then acute, which signals the meaninglessness of its cultural life, viewed in terms of its total existence. This is the meaning of the *crisis* of European sciences, explained in several works by that author, mainly in *The Crisis of European Sciences* (1970a). For Husserl, the origin of theory was hidden, and this concealment generated a crisis understood as the lack of understanding of the world in modern Western science.

It is this understanding, that is, the search for meaning that the world brings us, that needs to be resumed. It is not a matter of denying logic, practice, and the importance of "European Science," but of understanding it, in its genesis and in the dimension of its own practice and the respective reach of its explanations and forecasts. To this end, philosophical thinking, characterized by being comprehensive, critical, and reflective, becomes necessary. It enables us to go beyond scientific discourse that positions itself as absolute to pontificate about life and the world and always ask: what do these statements mean to me, to the community in which I participate, to society in which I am inserted regarding different issues? It is in the play, between philosophical thinking about science and scientific expertise that the sovereignty of scientific thinking and practice, installed in our world, comes under suspicion, and can be weakened. The imposition of scientific thinking leads individuals to distance themselves from the world of life experience; moreover, it fosters distrust and rejection of original intuition, replaced by the preciseness and certainty of science.

This rupture must be addressed within the realm of *mathematics education;* its goal must be the understanding of the very production of science and what it says about the reality of the world. It is necessary to exert philosophical thinking regarding what scientific knowledge means and what it brings in terms of improvement of living conditions, for individuals, society, and the planet itself. This thinking is critical and reflective, it expands to the manner of knowing of each person individually, and the community in which they live, enabling the comprehension of one's own feelings and perceptions, the ways of expressing them, and those brought about in their daily life in which they interact socially with others.

This alternation, from mathematical and scientific statements to pre-categorial ones and vice-versa, opens possibilities for emphasizing diverse ways of knowing and talking about the world, both in the realm of other concerns, such as, for instance, religious ones, as well as those of cultures and peoples inhabiting other regions. Therefore, horizons of understanding and ways in which the meaning of the world can be reestablished are expanded. It is necessary to foster credence and acceptance of the original intuition and think about their relativity, which can be overcome by confrontation and subsequent social consensus of the group. In this game, intuition always means something to the person, understood in its individuality, and needs to be maintained in the dialogue with others, that is, in the dimension of the intersubjectivity of life with the group, as long as it persists as a clear view, which occurred, for the person, at the moment it happened. Moving toward the realization of critical-reflective thinking, it is necessary to bring statements based on idealities and the formulas built with them, in order to consider what is obtained with this generality and what it means in the activities performed with students. The realization of this thinking requires that individuals, in their subjective dimension, be focused and attentive to the movement of the articulations that are being made for them, as well as to the language that expresses them and supports the communication with other people with whom they are interconnected in the dimension of intersubjectivity. In this dimension, mathematics education is made in its becoming, that is, in its ways of happening.

3.4 Mathematics Education: Necessary Philosophical Thought

This text involves issues to be worked on in mathematics education, outlining a way of understanding the production of mathematics and its role in the constitution of scientific and technological thinking present in the world we live in today, and the urgency of not succumbing to the loss of meaning of life and of the world, as we are immersed in a sea of explanations and predictions issued and supported by the scientific and technological apparatuses. It pointed out that mathematics education can contribute to the accomplishment of this task in a unique manner, which is critical and urgent for humanity. The justifications developed in this chapter and that support this statement are associated to the possibility of mathematics education working with mathematics, as critically and reflexively understood, and produced in Western civilization. This manner is characteristic of philosophical thinking, which does not mean that it is based on works of philosophers and their different visions of mathematics and science, but, only, that it performs an exercise of understanding what is said, so as not to take it as absolute truth and to always and repeatedly wonder if what the affirmation tells about the body of knowledge being worked on, as well as the life of people, the community that they share, the social organization in which they live and worldly reality.

The work of Edmund Husserl, mainly “The Crisis of the European Sciences,” a theme so dear to him, was used to elicit a way to understand the movement of production and establishment of sciences in the world in which we live. These sciences present a universalizing vision of the reality of the word, to the extent that their logic and theoretical and methodological framework are assumed both by Western and eastern civilizations, particularly as far as technology is concerned.

To explain the aforementioned production, the ways of knowledge that are characteristic of a science present in the natural world were brought forward, named by the above-mentioned author as pre-categorial, and the logic underlying them was evidenced.[22] It was explained that the pre- categorial mode of knowledge is the soil in which Euclidean geometry flourished, giving rise to a different mode of knowledge, categorial or scientific. The characteristic of this way of knowing is to work with *idealities* constituted and produced based on a shift of view from the empirically given in the reality of physical and concrete bodies, to the *idea* or *eidos* or *essence*, sustaining the *idea* in language and in procedures that lead to formalization and categorization. The expansion of Euclidean geometry from mathematics to physics, with Galileo, was also explained, evidencing the transformations that are being implemented and maintained throughout the Modern Age, entering the twentieth and twenty-first centuries. These transformations bring mathematics, its theories, and possible applications, as well as its logic, now formal and algebraic. An ontology that speaks about the world and its reality is established, from the perspective of Western science.

[22] Descriptions and analyses regarding this affirmation are presented in §9 of The Crisis of European Sciences (Husserl, 1970a).

The universalizing trend of this view leads mathematics educators to seek gateways by bringing ways to introduce mathematics in other cultures, criticize accuracy, and the respective value judgments that can underlie it, work with approximate knowledge, work with concrete materials, and everyday situations.

This chapter argues that, underlying these different attempts, is the question of the intertwining of pre-categorial and categorial knowledge, the respective perspectives in which they are produced and, specially, the imposition of the mathematical-scientific-technological way of conceiving and dealing with worldly reality. It is argued that it is necessary to know about the specificities of mathematics and science to conduct work that opens horizons for critical and reflective thinking about their importance and what they say about the world, understanding, at the same time, that they *do not* dictate the truth nor are they the guardians of it, much less establish the parameters that should guide human knowledge and the ethics of personal and social interrelations. This chapter underlies the understanding that forming individuals and citizens requires that the knowledge of sciences, as present in the logic of the "European sciences," be the object of teaching and learning in curricular activities of schools, and that, in doing so, the discussion of the non-supremacy of one knowledge over the other must be included in the agenda; moreover, that an exercise of philosophical thinking be carried out, leading people to ponder, individually and with others with whom they live, in different instances of the social organizations in which they work, about the bases on which their decisions will be made and responsibly assumed.

The conduction of such thinking requires that people, in their subjective dimension, be focused and attentive to the movement of the articulations that are being created for them, as well as the language that expresses them and that supports the communication with other people interconnected in the dimension of intersubjectivity. Through this dimension, in the dialogical situation, understood as being both an atmosphere for acceptance of others and for revealing arguments and ways of understanding, mathematics education develops in its becoming (*devir*), i.e., in ways of happening.

Due to the requirement for extending the text, at the end of each item when I focused on what it means to mathematics education there was no dialogue with the texts listed in the references indicated regarding the same themes, in the realm of mathematics education. Such a dialogue would be fruitful. Nonetheless, it would render this chapter too long. Initially, it had 20 pages, as required by the editors of this book. In order to comply with reviewers' requests, it was expanded to 27 pages.

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Chapter 4
Networking Phenomenology and Didactics: Horizon of Didactical Milieus with a Focus on Abstract Algebra

Thomas Hausberger and Frédéric Patras

4.1 Introduction

In his broad overview of the philosophy of mathematics education as a sub-field of mathematics education, Ernest (2018) emphasized the following characterization of philosophy as a discipline: "Philosophy is about systematic analysis and the critical examination of fundamental problems. It involves the exercise of the mind and intellect, including thinking, analysis, enquiry, reasoning and its results: judgements, conclusions, beliefs and knowledge." To wit, philosophy is about knowledge and the mind's access to knowledge and, as a consequence, there are many ways to apply philosophical concepts, results, or methods to mathematics education research (MER). Among them, we feature first that one should adopt a "critical attitude" to claims, theories, methodologies of MER. Second, one should use contributions of philosophical domains (ontology and metaphysics, aesthetics, epistemology, ethics, etc.) and approaches to enhance theoretical development in MER as one cannot disentangle the ambition to offer a secure basis for knowledge from the very analysis of what knowledge is, should be, and how it can be acquired, in mathematical education theories as elsewhere.

Unfortunately, editorial constraints imposed by the main mathematics education journals, notably the standard format of an article, which must include analysis and interpretation of data, rarely allow time for discussion of the foundations of the theories that are applied and for consideration of potential developments. In other words, the vocation of mathematics education to improve teaching and learning

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would pull the field toward a form of pragmatism that has little tolerance for the subtleties of language offered by what some reviewers call ‘philosophical jargon’. Such use must show the full force of the results produced or else the same discourse cannot be held in a more common vocabulary without significant loss of nuance. This creates a strange situation, where the methods used by the science that aims to study the process of knowledge acquisition depart from the way science is usually built. Indeed, science in general does not progress primarily by experiments and data analysis, but by a combination of methods that run from theoretical constructs and research programs to actual experimentation. The interplay of practices, not a dogmatic and uniform approach, is the key to progress, also in MER. From our point of view, MER has a lot to gain from taking more advantage of philosophical writings, which we will try to highlight in this chapter by taking up Husserl’s theoretical developments on the notion of horizon.

The inclusion of Husserlian horizons in MER has already been proposed by Zazkis and Mamolo (2011) as a way to interpret the “knowledge at the mathematical horizon” (KMH; Ball & Bass, 2009). We will begin by briefly presenting this work and show, in the “critical attitude” of Philosophy, how, although relevant in the context of teacher training, it moves away from Husserl’s project. It also presents limitations when it is a question of carrying out a more advanced analysis of the cognitive processes at stake in learning a topic such as Abstract Algebra.

In the following sections of the chapter, we come to the core of our contribution, which is mainly theoretical in nature, and present how we have articulated phenomenology and mathematics education in order to study the manifestation and acquisition of structuralist thinking in groups of university students. We will rather say “didactics of mathematics” since the main theory considered in the sequel, Brousseau’s (1997) theory of didactical situations (TDS), takes its origin in the French tradition of the field (Artigue et al., 2019).

On the methodological level, our work may be described as a form of networking of theoretical frameworks (Bikner-Ahsbahs & Prediger, 2010). The same phenomenon in mathematics education, namely how students solve a given problem in Abstract Algebra, can be analyzed from the perspectives of both Husserlian phenomenology and Brousseau’s theory. The networking of didactical theories is a research practice allowing the combination of complementary insights. It also leads to the linking of theories at different levels and by means of different strategies (by comparison, contrast, synthesis, local integration, or more). In our case, the joint analysis (first stage: comparing/contrasting) of the data led to the identification of common features (second stage: combining/coordinating) between Brousseau’s didactic contract and Jauss’ horizon of expectation in Hermeneutics (Hausberger & Patras, 2019; Hausberger, 2020).

In a third and more advanced stage of networking, *synthetizing* and *integrating locally* are relevant concepts whenever theoretical development is aimed at. This chapter is a first step in this direction as it aims at a local integration of theoretical constructs of phenomenology to supplement TDS. To do so, we will connect the notion of *milieu* in TDS with that of *world* in Phenomenology and draw further connections around the notion of horizon. Although the richness of the notion has not been completely taken advantage of in our previous work, we will not come back here to the horizon of expectation but focus mainly on Husserl.

His philosophy is thus contributing to unraveling the hermeneutical and phenomenological dimensions of the learner's interaction with the milieu. Key is the learner's intentionality. Intentionality, in a phenomenological sense, does not refer here merely to intentions (goals such as acquiring understanding or insights, for example), but in a subtler way to the structures of conscience underlying the relationships between the individual and the world. This idea of intentionality comes from a scholastic notion, which Husserl inherited through Brentano. In medieval philosophy, *intentio* referred to the application of the mind to an object. We owe it to Husserl to have made it a foundation of Phenomenology. Intentionality has multiple forms and accounts for example for both the gaze we direct on the surrounding things and the theoretical gaze we have on mathematical objects. All these dimensions contribute to make it a central idea for the use of phenomenology in didactics; hereafter, we will focus on intentionality in relation to the structures of consciousness underlying the relationships between the learner and the milieu. This philosophical analysis at the level of principles will serve as a background to analyze, in a second step, how these ideas unfold to grasp key aspects of elaborated theoretical knowledge in Abstract Algebra. This is where the key notion of horizon in the sense of Husserl comes into play and supplements the more general one of intentionality, as horizon structures are indeed structures of intentionality.

Our theoretical elaboration will be illustrated in the last section of the chapter through the analysis of excerpts of a dialogue between a pair of advanced students (PhD level and beyond) engaged in solving Abstract Algebra tasks. We will unveil a large spectrum of horizon types, without attempting to be exhaustive, featuring in particular a richness and complexity that depart from the descriptions offered by Zazkis and Mamolo's interpretation of horizons in the teacher education context.

4.2 Horizons in Teacher Education

Zazkis and Mamolo (2011) focus on the "knowledge at the mathematical horizon" (KMH; Ball & Bass, 2009), a component of the subject-matter knowledge, in the classical sense of Schulman, which designates (primary or secondary) teachers' advanced mathematical knowledge (from university or college) that may prove useful in teaching at school. Their interpretation is driven by the metaphor of horizon as a place "where the land appears to meet the sky" and the distinction between inner and outer horizon, after Husserl. Whereas the inner horizon corresponds to "aspects of an object that are not the focus of attention but are also intended," the outer horizon represents the "greater world" in which the object exists. Zazkis and Mamolo connected these two types of horizons to the first two components of KMH, respectively: the surroundings of the current topic under study and "the major disciplinary ideas and structures." Subsequently, Mamolo and Pali (2014) attempt to add in their descriptions, knowledge related to practices and values, in other words to account for the remaining two components of KMH: “key mathematical practices” and “core mathematical values and sensibilities.” But they didn’t draw further connections with phenomenology. Their goal is to study how these horizons may impact teacher’s actions in teaching situations.

As a main case study, Mamolo and Taylor (2018) exemplify connections between Abstract Algebra content (a part of the “blue sky”) and secondary school content. Although numerous examples are provided, by pointing out to studies in the volume contributed to, these examples rely on a similar schema: the object attended to is part of the school curriculum and its outer horizon, inside Abstract Algebra, consists of the “generalities which are exemplified in the particular object” (Zazkis & Mamolo, 2011, p. 10). In Philosophy, this is called a type-token relationship, and it certainly doesn’t exhaust the possible types of relationships, as we will see at the end of this chapter. To summarize, in this approach, the focus is how Abstract Algebra understanding may influence decision-making in teaching situations at school. The analysis of intentionality does not aim at relating abstract structures to lower-level mathematical objects in mathematical practices. Nor does it aim at shedding light on the ways and means a consciousness interacts with abstract mathematical objects (and thus achieves learning in Abstract Algebra). In particular, horizons of objects belonging to Abstract Algebra (horizons *inside* the “blue sky”) are not considered, whereas they will be central in our work that focuses on higher education teaching and learning.

4.3 Modeling Teaching-Learning Phenomena

The Theory of Didactical Situations (TDS; Brousseau, 1997) offers a general model and tools for the analysis of any didactical system: the main point is that a learner interacts with a *milieu* shaped by the teacher, according to a *didactical contract*. Learning is then asserted when the adequate adaptation to the milieu may be observed in the student.

Precisely, the didactical contract designates the “system of reciprocal obligation” that determines “explicitly to some extent, but mainly implicitly – what each partner, the teacher and the student, will have the possibility for managing and, in some way or another, be responsible to the other person for” (Brousseau, 1997, p. 31). It is expected from the milieu to be antagonistic, in the sense that it will provide retroactions (to the students’ attempts to solve the problem) and allow the target knowledge to emerge due to the “internal logic of the situation.” At this stage of the learning process, the milieu is *a-didactical,* in the sense that students shall experience an “absence of [direct] intentional direction” (didactical intention). The new knowledge acquires the status of a piece of the mathematical text at the later didactic phase of institutionalization by the teacher.

In fact, Brousseau distinguishes different patterns of situations, which are usually integrated in a sequence: an action pattern, in which students act on a material milieu; a formulation pattern, which aims to make explicit the students’ “implicit models of action"; and finally a validation pattern, in which a debate is organized to discuss the truth value of the students' findings. Brousseau refers to cognitive psychology when it comes to identifying these implicit models of action and understanding their role in the acquisition of knowledge. He mentions the conceptual field theory (TCF) program of Vergnaud (1990), and others can be cited (Dubinsky's APOS theory, Tall and Vinner's theoretical construct of concept-image, etc.). If such a study falls within the scope of the interactions between psychology and didactics, our aim in this chapter is to show what an interaction between phenomenology and didactics can bring to shed light on the psychogenesis of concepts when a learner is confronted with an a-didactical milieu.

This is where the notion of horizon comes into play. Before presenting Husserlian horizons and their connections to Brousseau's theory, we need to introduce other works that extend TDS on some aspects that may be related to phenomenology. In a pioneering paper, Brousseau and Centeno (1991) investigated how teachers handle the temporary and transient knowledge of pupils to promote learning. They called *didactic memory of the teacher* the knowledge that teachers may evoke on purpose to reactivate and facilitate the transformation of previous knowledge toward the target knowledge. Flückinger (2005) combined the perspective of TDS with TCF to study how students' numerical knowledge on division evolved through the construction of schemes connected to classes of situations partly organized by the teachers and partly emerging as new knowledge in the conceptualization process. She called such a feature the *didactic memory of students* since responsibility for memory processes has been partially devolved to students through a specific didactical contract: for instance, it is the students' responsibility to decide which objects of knowledge are the most pertinent to handle the assigned problems. We argue that the notion of horizon is a tool to capture features of the interaction of the students with a-didactical milieus and will give evidence of its relevance to analyze the evolution of forms of knowledge from implicit models of action to their explicitation (formulation) and then to a path toward a formal proof (validation).

4.4 The Horizon According to Husserl

We assume, then, that the construction of meaning, as we understand it, implies a constant interaction between the student and problem-situations, a dialectical interaction (because the subject anticipates and directs her actions) in which she engages her previous knowings, submits them to revision, modifies them, completes them or rejects them to form new conceptions. The main object of didactique is precisely to study the conditions that the situations or the problems put to the student must fulfill in order to foster the appearance, the working and the rejection of these successive conceptions (Brousseau, 1997, p. 83).

In the global project summarized by Brousseau's quote, two questions will retain our attention and govern our approach to Husserl's ideas in an a-didactical context. First, how to describe the modalities of interactions between the student and the milieu? This can be done at two levels: first, a functional, descriptive level based on experience or observation. For example, a group of students can start playing a game naively to "see how it works" and decide later on a protocol to look for an optimal strategy, or look immediately for a strategy, or mix the two approaches in various ways that the teacher can observe and partially expect. Achieving such a description is important because it can allow the concrete engineering of didactical situations. However, the question can be addressed at the higher level of principles: why is such a thing as the interaction between a student and a milieu possible? What are the available tools to speak of such a thing? How can it be described in a way that will allow didactics to explain and theorize the corresponding processes?

The other question, closely connected in our opinion, as we shall see, is how do previous knowledge play a role in this interaction? Of course, we know practical answers: for example, these knowledges are the tools that will allow her to grasp and analyze the problems. But, once again, at the level of principles, the question is harder to treat: why is it so, for example, that a student will be led to use induction to solve a counting problem and not a direct argument (a bijection with a set of known cardinality, for example)? How is such a thing as a path of successive guesses, modifications, completions of knowledge possible? Or, more precisely, in what space of cognitive actions, theoretical behaviors, does this path live?

Let us consider the milieu from a phenomenological point of view. Recall from Brousseau's Glossary (2010) that "a situation is characterized in an institution by a set of relations and reciprocal roles of one or more subjects (pupil, teacher, etc.) with a milieu, aimed at transforming that milieu according to a project. The milieu consists of objects (physical, cultural, social or human) with which the subject interacts in a situation."

In the phenomenological language, the subject interacts with a world. Most of the time, this world is the natural world, the *Lebenswelt* (the world of life). The key role of the *Lebenswelt* for Phenomenology was emphasized by Husserl in various texts, two of the most relevant for us here being the *Crisis* (*The Crisis of European Sciences and Transcendental Phenomenology*, Husserl, 1954) and his contemporary essay on the *Origin of Geometry.* A key thesis, defended in both texts in different forms, is that modern mathematics ultimately refers to a proto foundation in a system of original evidences whose origin is to be found in our immediate relationship to the world (that is, to the *Lebenswelt*). Mathematical ideas thus have a complex historicity, which is not only the result of their history but also of this necessary reference to fundamental intuitions. These ideas are extremely important and of considerable significance for didactics, but we will not go down that road here: instead, we will emphasize the role of the *Lebenswelt* in the constitution of the horizon of mathematical objects and concepts in the classroom.

Another important observation is that, however important the *Lebenswelt,* it is not the only "world" we can be embedded in or interact with: "I can for example also occupy myself with pure numbers and laws of numbers. The world of numbers is also there for me; it constitutes precisely the field of objects where the activity of the arithmetician takes place. During this activity, she will focus on some numbers or numerical constructions surrounded by an arithmetical horizon, partially determined, partially undetermined" (Husserl, 1913, [51], our translation).

A key step when interacting with a world, whatever it is, is to change attitude: the same person can behave naturally and interact with her *Lebenswelt,* her surrounding world, or switch to a theoretical attitude and behave as an arithmetician, or a student in arithmetics. The corresponding world will then be shaped differently. A milieu in the didactical sense can be thought of as a particular kind of world. It exists as a milieu precisely because the student (or the teacher) adopts toward it the right attitude. For example, scissors, a pen, and a sheet of paper can constitute a milieu suited for elementary Euclidean geometry or be simply the tools given to a kid to play.

In Phenomenology, a world cannot be disentangled from its horizon. The horizon is, roughly stated, the configuration of possibilities, meanings, tools, intentions that shape the world/milieu. The horizon is at the same time what makes the dynamical and constructive interactions between the subject and the world possible and the "place" where they occur. A feature of the a-didactical horizons we will consider is that the didactical memory of students is a key ingredient in their constitution and structuration. When the young student considers numbers and properties of numbers, she may already know that there are operations she can perform: addition, subtraction, multiplication. She also maybe knows that there are more complex operations like division or exponentiation that she remembers only vaguely, and she knows that using them would require some care. Lastly, she maybe has learned more advanced ideas, for example the reasoning by induction, but at the moment does not connect this knowledge to numbers, although she could remember it at some stage of a reasoning. These operations, some clearly determined, some still undetermined or under-determined, are one component of the horizon of numbers. They are also tools that I can use to reshape the current milieu. For example, I can transform the problem of computing (6+7)*2 into the problem of computing 13*2. But in the horizon of possible shaping my interactions with (6+7)*2 other paths of reasoning would be possible, for example its transformation into (6+7)+(6+7). Here, again we can observe constructive interactions between phenomenology and didactics. The notion of didactic memory, together with its theorization and documentation on classroom experiments, can enrich phenomenology by documented examples, where the behavior of students can be analyzed. Conversely, phenomenology enriches the didactical theory with its precise tools of analysis, especially of the theoretical endeavors and ideas formation in the context of interactions between an individual and the world.

Studying the milieu from the phenomenological perspective leads to answers to our initial questions. The horizon is a locus where interesting, dynamical, transforming interactions with the milieu take place. Prior knowledge are some of the components of the milieu, they also contribute to the shaping of the horizon and to the action on the milieu. The horizon and the didactic memory are certainly not exhausting the analysis of the interactions with the milieu, but our thesis is that they are an important constituent that allows us to understand various important didactical phenomena.

4.5 Toward a Typology of Horizons

In cognitive sciences, Jorba (2020) argues that the perceptual intentional horizon in Husserl's phenomenology, besides being a general structure of the experience, extends to a viable notion of cognitive horizon that relates to affordances (possibilities of action present in experience). She proposes "to characterize a specific structure of the cognitive horizon – that which presents possibilities for action – as a cognitive affordance. Cognitive affordances present cognitive elements as opportunities for mental action (i.e., a problem affording trying to solve it, a thought affording calculating, an idea affording reflection)" (p. 847). Following Husserl, she also features various types of horizon structures that we will use later to characterize several structures showing up in (a)-didactical experiments. We detail their content, building on her analysis.

The inner horizon accounts, in the phenomenology of perception, for the various ways in which I can have access to an object: "Every experience has its own horizon... this implies that every experience refers to the possibility... of obtaining, little by little as experience continues, new determinations of the same thing. (...) Thus every experience of a particular thing has its internal horizon" (Husserl 1973, §8: 32, quoted in Jorba, 2020). In didactics, we propose to use the notion as referring to the various access I can have to an object (a notion, a concept...) that are directly contained, either in the object itself (for example as direct consequences of its definition or as properties of its components), or in a given milieu. Here, "given" refers to the components of the milieu that go immediately with the (a-)didactical situation. Notice that this is a subtle notion. Whereas the inner horizon of a spatio-temporal object or being amounts simply to the various experiences I can make—for example by turning around a building, visiting it, seeing its roof from an airplane, etc.—the inner horizon of a theoretical object such as a mathematical one highly depends on the way this object is given. A sphere defined using the classical axioms of Euclidean geometry can be identified with an object in the space $\mathbf{R}^3$ equipped with a positive definite quadratic form, but the (technical, conceptual, methodological) horizons that go together with these two definitions are quite different. In other terms, this notion of inner horizon also depends on the learner's background.

Outer horizons refer instead to the possibility of putting an object (notion, concept,…) in relation to other objects or in another context (Jorba, 2020, p. 849ff). In our previous example, quadratic forms, metrics, scalar products belong to an outer horizon of spheres in naive Euclidean geometry. Phenomenology itself enters the scene by providing theoretical tools to analyze how horizons structure the relationships of a consciousness to its objects, whatever they are.

This phenomenology of horizons is further enriched by two families of relations. In our didactical context, we propose to call *associative* outer horizons those relations based on relating two objects, two notions in a non-straightforward way (once again, what straightforward means will depend on the learner's background). The example of spheres and quadratic forms can be analyzed that way, for example. Instead, *inferential* outer horizons will denote relations acquired through reasoning, provided new elements, notions, ideas, insights result in this process. For example, the late nineteenth-century insight that a finite set is a set that cannot be put in bijection with a proper subset could be analyzed that way: appealing implicitly to the infinite to define finiteness, besides being counter-intuitive, requires upgrading the horizon of finite collections through a process that, at least at the very beginning, relies more on technicality and reason than intuition.

Turning back to Jorba's general program, we agree with her analyses relating cognitive horizons and cognitive affordances and point out that the didactic approach to Jauss' notion of horizon of expectation in (Hausberger & Patras, 2019) goes in the same overall direction. Many features of phenomenological horizons of the *Lebenswelt* actually translate into features meaningful in a didactical context. For example: "Every cogito, an external perception or a remembering, and so on, for example, carries with itself in a detectable manner an immanent potentiality: the one of possible life experiences, linked to the same intentional object, that the self can realize […]. In each cogito, we discover horizons" (Husserl, 1950, p. 18[1]). They induce potentialities, in the natural behavior, for example the possibility to turn my head to the left to discover new components of the countryside, or in arithmetics, the possibility to perform first a sum or a product in a given formula, with some priority constraints on the operations that contribute to shape the horizon of possible arithmetical actions.

An important point that we will start to develop implicitly in this chapter is that the notion of horizon is not a vague concept that would allow to speak of certain phenomena without giving conceptual and methodological tools to investigate their properties and structure: "I can investigate an intentional experience, which means that I can penetrate its horizons, interpret them and, that way, unravel potentialities of my life and, on another side, clarify, at the objective level, the targeted meaning [Ibid]."

Taking again the elementary example of arithmetical operations, the expression (2+3+4)*(2+1) can be transformed into (5+4)*(2+1), (2+7)*(2+1), (2+3+4)*3, and so on. These potentialities are all open and part of the horizon of the expression. They are the beginning of paths that will lead the student (hopefully) to 27. When I analyze the structures underlying these potentialities, key ideas of arithmetics will show up if I push the analysis to its limit. For example, the equivalence of the first two transformations—which is not obvious—points out at the associativity of addition, an highly sophisticated notion that, in its modern, structural interpretation, appeared relatively late in mathematics (Leibniz, Grassmann, …).

The second question we raised in the beginning of section 4 (how does prior knowledge impact the interaction of the student with the milieu?) has started to be addressed by noticing that knowledge is a key component of the structure of the horizon and by pointing out at the relevance of didactic memory in our context. Let us expand briefly on this and make these observations concrete. If I already know

[1] On the phenomenological definition of the horizon and its fundamental properties, see also op. cit., p. 82.

what associativity and distributivity mean, I will be able to devise more complex strategies to solve equations and will be much more confident on their validity. However, the question relates to a very general feature of intentionality that goes beyond the particular case of didactic memory, namely the fact that each life experience has an horizon of anteriority (my past experiences and the memory I have of them). This horizon of anteriority has several components. Short-term memory is important, for example, when solving a problem. The ideas and results I just obtained contribute to shape my current understanding of the problem in its present state. In the French educational system, this phenomenon is illustrated by a marked difference between exercises, usually focusing on a few directly related questions, and problems, much longer and where drawing connections between arguments in different parts of the problem is essential to its solution. In such situations (exercise, problem) where didactical and a-didactical components are mixed (depending on the reliance of the solution on already acquired skills), an horizon is constructed largely internally to the situation—in the sense that it is shaped by previous answers. Long-term memory impacts differently the interactions of the student with the milieu. For example, recognizing certain prototypical features of a question (for example, to perform a computation involving sums and products) will lead her to use the priority rules and distributivity laws for arithmetical operations that she had learned some time ago and had remained before one among the many and largely indistinct components of the horizon of the problem.

In conclusion, horizon, didactic memory, and their constructive interactions can be understood and documented in many ways. We will focus now on a specific example in order to illustrate the fertility of our theoretical ideas on concrete empirical data.

4.6 Application to Abstract Algebra

The purpose of this section is to study, in the spirit of Husserl, the learning processes of advanced students engaged in solving a mathematical problem in Abstract Algebra: the theory of banquets. Cognitive processes will be explored using the lens of phenomenology with the notion of horizon as the main tool: progresses in solving the problem are thus related to changes in the horizon structure which potentially result in new cognitive affordances.

Throughout our analysis, the main questions will therefore be: Which is the main intentional object (or noema) that consciousness is focusing on in crucial moments of the mathematical experience? What is the underlying motivation structuring intentionality and, more generally, what is structuring its noetic moment: the way mathematical conscience is conscience of… intuition of… grasping of…? How are inner and outer horizons of intentional objects structured by the learners' interpretation of the milieu and background knowledge (or didactic memory)? We will rely on language and other semiotic representations produced by learners as warrants for our claims; moreover, the chronology of reasonings makes it possible to detect partly implicit features of cognitive anticipation of the horizon through the evidence of how the horizon unfolds in subsequent cogita.

Let us now present the problem. Mathematical structuralism has had a large impact on contemporary mathematical practices (Patras, 2001) but also on modern didactics of mathematics. Various members of its founding fathers have indeed been strongly influenced by the problems that arose together with the emergence of "modern maths" where abstract axiomatic structures serve, especially in algebra, as organizing principles in the exposition of mathematical theories and as tools to pose and solve mathematical problems. Pre-structuralist theories about numbers, polynomials, and other standard mathematical objects appear as a background to motivate and apply the abstract unifying and generalizing point of view of structures. Structures also give rise to new questions: Which identity principle to adopt (which are the natural morphisms between objects of a given type of structure)? How to classify objects up to isomorphism? Which structuralist theorems govern the decomposition of objects into simpler ones? As a piece of didactic engineering (Artigue, 2014), the theory of banquets (Hausberger, 2020; Hausberger, 2023) has been designed in order to tackle these kinds of questions in the context of an Abstract Algebra course at the transition between undergraduate and graduate studies in pure mathematics. The main prerequisite is a course in Group Theory, so that students have already encountered similar structuralist questions and results that will be thematized in the context of banquets.

A banquet is a set E endowed with a binary relation R which satisfies the following axioms: (i) No element of E satisfies xRx; (ii) If xRy and xRz then $y = z$; (iii) If yRx and zRx then $y = z$; (iv) For all x, there exists at least one y such that xRy.

In part I.1 of the worksheet, students are asked the following questions:

1 a. Coherence: is it a valid (non-contradictory) mathematical theory? In other words, does there exist a model?; b. Independence: is any axiom a logical consequence of others or are all axioms mutually independent?

In part I.2, they are asked to classify banquets of small cardinalities and link banquets of order 4 with their knowledge in Group Theory (in particular with the cyclic group of order 4). The abstract/concrete relationship is reversed in part II of the worksheet, which begins with the empirical definition of a table of cardinal number n as a configuration of n people sitting around a round table. Its aim is to prove that any banquet decomposes as a disjoint union of tables (the "structure theorem"). We won't give more details here since excerpts of students' work that will be analyzed are restricted to part I.1 as we prefer to insist on our method, its significance, and concrete use than on all the conclusions that can be drawn from experiments on the theory of banquets.

The banquet structure possesses a large variety of models since the system of axioms may be interpreted in quite different worlds, beginning with the empirical interpretation of guests sitting around tables (whence its name): xRy if x is sitting on the left (or right) of y. Other domains of interpretation include Set Theory (the binary relation is represented by its graph), Functions ($xRy \Leftrightarrow y = f(x)$ defines a function f according to axioms (ii) and (iv); the other two axioms mean that it is injective without fixed points), Permutation Groups (f is a bijection when E is finite, in other words a permutation without fixed points) or even Matrix Theory and Graph Theory (see Hausberger, 2021, for a full mathematical analysis). The structure theorem of banquets thus corresponds to the well-known theorem of canonical cycle decomposition of a permutation, but the analogy remains hidden since the binary relation of banquets is different from binary operations that define groups. These remarks explain why the theory of banquets is mathematically rich but may not be found in any textbook (it is equivalent, in the finite case, to permutation groups). Moreover, it is a simpler theory (in the sense of mathematical technicality) than Group Theory, and it carries the underlying intuition and mental image of guests sitting around tables (a wedding banquet).

4.7 Horizons of the Abstract Structure of Banquets

Question I.1 of the worksheet (coherence; existence of a model; independence of axioms) may be regarded as a first situation in the sense of Brousseau, dedicated to logical analysis. Its milieu contains the axiomatic definition of the banquet structure, the concept of model of a system of axioms and the language of Set Theory.

The intentional object is, in general, the object (of senses, or abstract, theoretical…) toward which consciousness is directed. In this exercise, the main intentional object is the definition of banquet. At any moment during the solution of the exercise, this consciousness and the attention given to the axiom system is embedded into various horizons. The important point is that these horizons are not fixed: every time consciousness is going to be directed toward a particular feature of the axiom system, new horizons will present themselves as surrounding this state of consciousness. On the other hand, taking into account the presence of these horizons will help students to progress and understand the axioms in different ways, so that consciousness itself will evolve accordingly.

What we claim here is simple, but essential and too often forgotten by authors appealing to Phenomenology as a method of philosophical investigation: one can describe the process of thinking by investigating such phenomena. Comprehension of learning in particular is a topic particularly well-suited to such analyses. Our claim is that they help understand the didactical processes and could also be useful in didactical engineering by providing tools to analyze what steps students are expected to perform to reach a satisfactory construction of knowledge.

In the first part of the exercise, the investigation of the meaning of the definition of banquets goes through the logical investigation of the system of axioms (coherence and independence) using a semantic approach (construction of models). The related work of the two (very advanced) students, called Alice and Bob hereafter, took the form of a dialogue that has been registered and transcribed. It is made of a sequence of 31 speeches. The integrality of the dialogue can be found in Hausberger (2016, annex 4), we use here only some parts to illustrate and support our analysis. Numbers indicated below in front of Alice or Bob statements correspond to the position of the statements in the dialogue: (5) will refer to the fifth speech among 31, and so on, so as to indicate the progression of the argumentation.

Concretely, investigating the meaning of the definition could usually be done in three ways:

- Appeal to prior knowledge (their own didactic memory, entangled with the teacher's didactic memory)
- Try to grasp directly the meaning of the axioms (with some training it is indeed possible to have a purely formal understanding of algebraic axiomatic systems)
- Explore empirically the axioms' content

In general, mathematical thinking is a blend of several such processes. Each approach goes together with distinct intentional modalities. We will try to account for those that appear in the two students' dialogue. It will appear that several successive horizons may be uncovered and disentangled.

1. First horizon (inner): theoretical memory. As a first attempt, students try to use direct knowledge on binary relations (antisymmetry, irreflexivity) to make sense of the axioms. They appeal therefore to didactic memory in one of its simpler forms that we may call formal or theoretical: going back to the known properties of the objects and notions under consideration.

(1) Alice: Classical, we specify the structure through relations, okay.
(2) Bob: Antisymmetry [about axiom (i)].

Our memories shape horizons of possibilities and horizons of understanding. In her statement, Alice explicitly acknowledges the idea that a structure can be defined through relations and that such a fact belongs to classroom knowledge. In Brousseau's language, this idea has been already institutionalized, it is contained in the paramathematical[2] concept of structure and is also based on the notion of a binary relation taught in Set Theory. Recognition of institutionalized knowledge is essential; it provides a ground on which further advances can be made.

2. Second horizon (outer): natural semantics. Interestingly, this first (inner) horizon of the banquet structure soon leaves place to a quite different horizon evoked by the name of banquet:

(3) Alice: there's one guy on the right and one on the left, that's the idea; there's nobody sitting alone at a table.

This second horizon is thus driven by natural semantics, empirical knowledge, and more generally our embedding in a *Lebenswelt*: the mental image of banquets acquired from perceptual experience. The theory is embedded in a wider, extra-mathematical, context. Here, didactical engineering is involved since the name of

[2] Paramathematical concepts are "named objects whose characteristics are studied but which have, for various reasons, not yet been organized and theorized, such as the notion of function in the 19th century, or that of equation in the 16th century, or that of variable in the 20th" (Brousseau, 1997, p. 59). The students have not been taught Category Theory, the mathematical framework that aims at theorizing the notion of structure, but Alice has been introduced to Model Theory in her studies.

the theory is a main didactical variable (in the sense of TDS) of the situation. By the name « theory of banquets », the instructor has chosen to drive the learners toward a certain type of models and intuitions – he enforced the building of a specific outer horizon.

We emphasize that "natural semantics" refers here to the fact that students give a meaning to the theory of banquets by a "fulfilling of intentions of signification," in the language of Husserl. By referring to daily life situations, the theory becomes concrete and can be grasped: an element of the set E is now "a guy at the table." The desire to associate a meaning to the axioms (intention of signification) starts to be fulfilled.

3. Third horizon (outer): associative. The second horizon is subsequently augmented with knowledge from elementary set theory and logic to give rise to a third horizon with powerful cognitive affordances to construct models and check the validity of statements.

   As it is based on relating two horizons—the outer one of natural semantics of banquets and the inner one of set theory—the new horizon is associative:

(5) Alice: To show that it's not contradictory, you can show that there exists a model. I suggest we take one guy. No, one guy doesn't work, 2 guys sitting next to each other. [...]

(7) Alice: Let's take $E=\{a,b\}$ and for the relationship the couples (a,b) and (b,a). So it is indeed a model. [...]

(9) Alice: Yes, a set with 2 elements, they are sitting opposite each other… obviously, there is at most one on the right and one on the left, they are in relation with the one opposite.

Here, Alice makes explicitly a move from natural to formal semantics. She uses theoretical memory to relate the non-contradiction of axioms with the existence of a model but appeals then to the idea of people around a table to build a model. In speech (7), Alice and Bob have obtained a first mathematical statement: the axiom system has a model and is consistent. Speech (9) interestingly confirms the formal, mathematical, sentence by a translation into natural semantics.

4. Fourth (outer) horizon: inferential. The dialogue proceeds with some easy arguments on independence that will be omitted. Later, as they stumble on a difficulty to deny (ii) while keeping other axioms (that is, when trying to prove the independence of axiom (ii) from the others), Alice feels the need to produce another interpretation of banquets:

(15) Alice: So this thing, it's nice… there are some and at most one, so this thing, it's a function. To x we associate the unique y such that xRy. And we have the injectivity a priori.

The relationship with functions is thus the main component of a fourth horizon that may be qualified as both outer and inferential since it involves several concepts not directly related to the axiomatic system (multivalued functions—Bob mentions for example the possibility of two images of an element—injectivity) and results (equivalence of injectivity and bijectivity for functional relations between sets of same finite cardinal number), and leads to a break-through:

(26) Alice: Perhaps an infinite set is needed, it is possible.
(27) Bob: I have the impression that this is not possible.
(28) Alice: It's a bijectivity thing that makes you need an infinite set.

A formal proof of the necessity of infinite cardinality is not produced, but what they achieve is enough for the production of the counterexample they were looking for.

The experiment we have treated allows us to reach several conclusions. The solution to an exercise is a dynamical process. Understanding it requires the understanding of how the students' thoughts evolve and move forth and back from the object under investigation to a series of insights, some of which are given with the problem (inner horizon: the acquired knowledge directly related in that case to relations and axiomatic systems), some others have to be found in relation to outer horizons that unravel progressively.

We feature once again that, in spite of a common reference to Husserl's horizons, our approach is much more general than a mere type-token analysis, as it appeared for example in Zazkis and Mamolo. Indeed, whereas the latter is restricted to understanding the subordination of a given mathematical object or problem to a more advanced and general theory, our use of horizons gets into the very dynamic process of knowledge-building in the classroom. The semantical aspects involved in the idea of banquets are a good illustration of the generality of Husserlian use of intentionality and horizons.

4.8 Conclusion and Perspectives

The main contribution of this chapter is the further development of Husserlian horizons, first introduced in a didactical context by Zazkis and Mamolo (2011), as tools to analyze the shifts of attention and interconnectedness of knowledge in learners attending to an abstract structure. Our extension encompasses a larger spectrum of horizons and methods in a pioneering application in the context of university mathematics education, allowing for a fine-grain analysis of the work of learners engaged in the elaboration of a structuralist mathematical theory around the given structure.

The features of horizons that we managed to identify in relation to the manifestation of structural sense among a pair of advanced students are but a first step in understanding the genesis of structuralist thinking in educational contexts. At the theoretical level of frameworks, we contribute by combining/coordinating notions from TDS with the perspective of phenomenology, in the spirit of networking. We believe that such a dual framework may be applied in a large variety of contexts and educational levels. We also point out the coordination with studies in cognitive sciences. These links should be investigated further in subsequent research.

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Chapter 5
Specifying Defining, Generalising and Abstracting Mathematically All Seen as Subtly Different Shifts of Attention

John Mason

5.1 Introduction

Mathematics is often referred to as 'abstract', usually in reference to the use of multiple unfamiliar symbols. Although it is true that the objects and concepts of mathematics are rarely tangible, using the label *abstract* overlooks and obscures *abstracting* as a process, as a change of relationship between a person and a set of words, and hence a change of relationship with self- constructed 'virtual objects'. My aim here is to examine more closely mathematical abstraction as a process, in relation to specifying, defining and generalising.

5.1.1 Background Frames

Mathematics teaching is taken here to mean the initiation of pedagogic and mathematical actions which tailor learner experiences of encountering mathematical actions, themes, and the use of their own powers (Gattegno, 1981 p6; Mason, 2002a, 2008; Mason & Johnston-Wilder, 2004a, b). In order to inform choices of pedagogical actions it is necessary that aspects of a teaching situation resonate with or trigger relevant actions, and this triggering can be enriched through the use of labels for distinctions which enable discernment of those choices. Labels are gathered together as frameworks, and it is these frameworks of distinctions which

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constitute the theoretical foundations of mathematics education (Love & Mason, 1992, p29–53). As will emerge, this is itself a process of the abstract becoming familiar through abstracting. Herewith, some relevant frameworks.

Brunerian Spirals

Inspired by Jerome Bruner's articulation of three modes of (re)presentation (enactive, iconic, symbolic) (Bruner, 1966), my colleagues and I at the Open University (1982) found it instructive and informative to propose a spiral of developmental experience in which, through manipulating objects which are familiar and confidence inspiring, learners get a sense of some possible relationships, and over time find themselves articulating those relationships more and more succinctly, until the labels/relationships themselves become confidently manipulable, to be used to explore yet more relationships (Floyd et al., 1981; Mason, 2002b; Mason & Johnston-Wilder, 2004a, b) (Fig. 5.1).

Whenever confidence ebbs or confusions arise, the sensible thing to do is to backtrack to a more confidently manipulable situation in order to reinforce and re-substantiate the sense of relationships and to add substance to the articulation. The purpose of manipulating, what George Pólya (1962) and Mason, Burton, & Stacey (1982/2010) referred to as 'specialising', is to develop a sense of possible relationships, and perhaps to articulate these as properties that may hold in more general situations. When confidence ebbs, moving back down the spiral to a domain of greater confidence from which to rebuild 'a sense of' relationships and properties is a useful way to regain confidence and to re-vivify familiarity.

The purpose of specialising is to gain both confidence with and insight into possible generalisations which are conjectured, followed by attempts to justify these. As Whitehead (1911 p4) put it, "To see what is general in what is particular and what is permanent in what is transitory is the aim of scientific thought." This aligns with the ancient Greek meaning of *theorem* as 'a seeing', so that proofs become attempts to get other people to 'see' what I am seeing. These 'seeings' involve subtle shifts in how the person attends to the 'objects' being attended to (Mason 1989), as will emerge shortly.

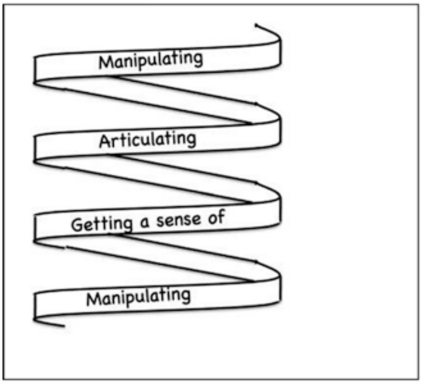


Fig. 5.1 The MGA spiral

Example Spaces

There is a strong link between MGA as a developmental sequence, and the notion of an *accessible example space* (Watson & Mason, 2005), which refers to mathematical objects, construction tools, and relationships, which underpin or constitute appreciating and comprehending. I use ‘appreciating and comprehending’ in place of ‘understanding’, which for me is too highly jargonised to be a useful construct. An example space, accessed from a particular direction according to current context and triggers, constitutes the domain of familiar and confidently manipulable mathematical objects accessible to the learner in any given situation, not simply as a collection, but as a gateway to generality.

Forms of Attention

Over some 70 years of being inducted into and engaging in mathematical thinking, and some 50 years of trying to support and stimulate people in their teaching of mathematics, I have become more and more convinced that attention is a core aspect of human psyche, playing a key role in the success or failure of classroom interactions. Discerning different ways to attend can be used to make sense of many classroom phenomena.

The fundamental conjecture is that when teacher and learners are attending to different things, their attempts to communicate are at best impoverished. But even when attending to the same thing, they may be attending in different ways, and still their communication is likely to be at best incomplete.

Until relatively recently, William James was one of the few philosophers to address questions about attention. Whereas James (1890) felt that attention was not responsible for “discerning, analysing or relating but that the most that can be said is that it is a condition of our doing so” (p426–7), I find it useful to think in terms of different ways of attending, which include the following readily recognisable forms:

Holding Wholes (gazing); Discerning details; Recognising Relationships;
Perceiving Properties (as being instantiated); Reasoning on the Basis of Agreed Properties.

These forms of noticing were derived from juxtaposing personal experience with a neo- Pythagorean perspective on qualities of number (Bennett, 1993). Wholes are of course associated with unity or oneness; distinctions involve ‘this-not-that’, a quality of twoness; relationships involve three elements, two discerned wholes held in relationship and so mediated by a third; properties involve four elements as components of activity; reasoning involves five elements concerned with identity and potential (Bennett, 1993; Shantock Systematics Group, 1975).

Readers familiar with van Hiele levels (van Hiele-Geldof, 1957; Usiskin, 1982; van Hiele, 1986; Burger & Shauenessy 1986), will recognise a close relationship between the ‘levels’ and these five forms of attending, which is interesting because the forms of attention were derived independently, from observing my own experience while informed by the systematics of Bennett (1993; Shantock Systematics Group, 1975). The significant difference is that whereas van Hiele levels apply to developmental stages of young children and their awareness, these forms of attention are states which are far from stable. Indeed transitions or shifts of attention can be very rapid indeed and are rarely progressive in the sense of proceeding from wholeness through distinctions to relationships and beyond, but rather tend to bounce around as new distinctions create new wholes, instantiated properties give rise to fresh relationships and new distinctions, and so on. Readers will be offered contexts in which to experience this for themselves in what follows.

5.1.2 Attention, Awareness, and Consciousness

A much-repeated adage says that if you want to know about water, don't ask a fish. Put another way, "you are (where) your attention is"; what you are attending to is by definition what you are experiencing, what you are aware or conscious of. Experience is informed by attention, and attention is what is experienced. William James (1890 p4202) put it that "*My experience is what I agree to attend to*." Only those items which I *notice* shape my mind; without selective interest, experience is an utter chaos (his italics), and (p424) "each of us literally chooses, by [their] ways of attending to things, what sort of a universe [they] shall appear to themself to inhabit." I use *awareness* in the sense of Gattegno (1970, 1987) to mean having an action become available to be enacted, whether automatically (such as adjusting pulse rate or skin pores), subconsciously (such as breathing or reacting to stimuli), or consciously (choosing to act). Subtle difference in how that attention is structured makes different experiences and different awarenesses available.

William James (*op cit* p91) claimed that "All attention involves excitement from within of the tract concerned in feeling the objects to which attention is given," which aligns with the poetic voice of Mary Oliver (2015) "Attention without feeling, I began to learn, is only a report. An openness – an empathy – was necessary if the attention was to matter." I see the different forms of attending as forms of empathy with mathematics and mathematical thinking. For an overview of contemporary philosophical discussions around attention, awareness, and consciousness, see Watzl (2011a, b).

5.2 Method

My approach is fundamentally phenomenological as I am interested in the lived experience of thinking mathematically rather than collecting observations of other people's behaviour and trying to infer experience. Consequently, my 'method' involves offering readers tasks to undertake during which they may catch a taste of abstraction and other mathematical actions as a process.

The psychological phenomena of interest here, attention, and its co-relates awareness and consciousness, are not amenable to extra-spective research methods, in which indicators of phenomena are chosen and measured under varying conditions. My method involves intra-spection (being careful to avoid retrospective "I must have …" constructions) mixed with inter-spection, in which colleagues offer brief-but-vivid descriptions of experiences, seeking resonance and dissonance with colleagues, and building up a collection of shared experiences and descriptive labels. In order to maintain validity, these must be tested against the experiences of further colleagues, who then continue the same process. This is the essence of the Discipline of Noticing (Mason, 2002c).

5.3 Shifts of Attention

Ways in which mathematical actions labelled by specifying, defining, generalising and abstracting mutually intersect and interact are elaborated below. At first, forms of attending are italicised so as to alert the reader to the possibility of noticing them.

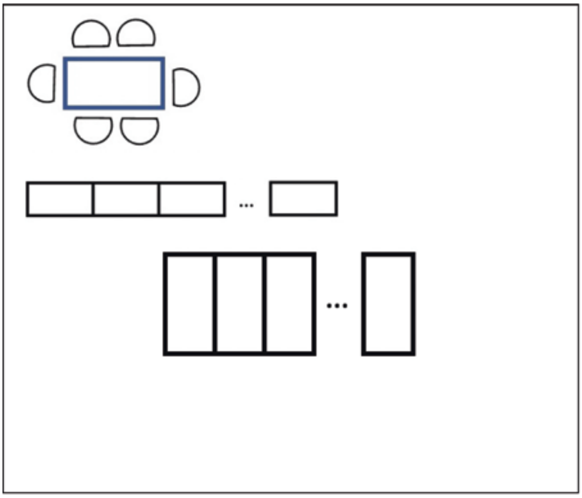
5.4 Generalising and Abstracting

The transition from *recognising a relationship* in a situation, to *perceiving a property as being instantiated*, is characteristic of the mathematical act of generalising. By contrast, but not always a sharp distinction, abstracting mathematically involves the act of defining, which means declaring that 'anything which has these properties is considered to be a (label provided)'. In Mason and Czarnocha (2021), we considered an instance of a student apparently abstracting for themselves, without, of course, producing a label. For mathematicians, the label is the short form used to refer to the effect of the abstracting process that is assumed to have taken place. Such an act releases the possibility of *reasoning with properties* independently of the nature of the objects under consideration. Attention shifts from objects and specific relationships to *properties* of objects.

5.5 Generalising But Not Abstracting

There is a wide range of tasks which invite the expressing of generality, often associated with counting configurations or predicting terms of a sequence (Recorde, 1543; Mason et al., 2005; Küchemann, 2021). For example, in a hall with a large number of tables that seat six people, how many people can be seated if there are a specified number of rows each made up of some specified number of tables placed end to end? Placed side by side? (Fig. 5.2)

Fig. 5.2 Showing chairs at a table, and two ways of forming rows of tables



Notice that for a very short period of time your attention *gazed* at the diagrams in the figure. *Discernment* of two distinct diagrams and of components (the individual tables) then released a sense of *relationship* as described previously in the text, and latterly in the figure caption.

In these sorts of generalisation tasks, learners are invited to shift their attention away from drawing all the tables and then counting, to *discerning details* amongst which they can *recognise a relationship* between the number of chairs added when a new table is adjoined, and expressing this as a *property*. Although learners unfamiliar with using symbols to express generality might experience the effect as ‘abstract’, that is, as unfamiliar, it is not what mathematicians would call the result of abstracting. Notice that the way the task is stated no particular number of desks in a row or number of rows is given, in an attempt to direct attention away from particular numbers and towards a generality (*perceiving a property*). It is anticipated that those who feel the need to specialise using specific numbers will do so or can be reminded to do so if they are stuck. This contrasts with a standard worksheet approach in which specific numbers are used, intended as scaffolding to support generalising, but actually directing attention towards the particular rather than the general.

5.6 Poised Between Generalising and Abstracting

In this section, different ways of attending are highlighted in italics. Consider the following task:

Two people are each thinking of their own number.

They are about to subtract the smaller of their numbers from the larger, when suddenly they are both instructed to add one to their number.
What difference will this make to their subtraction result?

Notice the difference between providing a range of instances of performing this action with the associated subtractions, and this version of a task which relegates the unimportant detail (the specific numbers) to the background. Attention is directed to the effect of the action of adding 1 to both numbers, to *recognising relationships*. The generality implied by not specifying the two starting numbers actually helps thinking, by omitting the distraction caused by specifics, and by drawing attention towards *perceiving properties*.

After discussion amongst learners, it is likely to emerge that the two instances of ‘1’ can be replaced by any number. Performing the same (additive) action on both numbers makes no difference. This too is a generalisation, as well as an example of the mathematical theme of invariance-in-the-midst-of-change. Different learners may require different amounts of time manipulating confidence inspiring objects (numbers) before the realisation (literally, making real through *recognising* and getting-a-sense of *relationships* and *perceiving* these *as instances of properties*). This is why blocking access to overly familiar objects (specific numbers) can sometimes actually assist learners to *recognise relationships* as *instances of propertie*s, which is a core shift in developing mathematical thinking.

The teacher may at some point think it worth using a label such as *compensation* for this overall invariance, which in itself is a pedagogical-abstracting move, for it is providing a label so that reference can be made to instances in the future, supporting and promoting *recognising relationships* as *instances of properties*. Labels for experiences make scientific advance possible, in line with the earlier quote from Whitehead (1911): “To see what is general in what is particular and what is permanent in what is transitory is the aim of scientific thought”.

The label also opens up the possibility of other ways to compensate while leaving the result invariant, or perhaps changed in some predictable way. Similarly, my drawing attention, through labelling, to the mathematical theme is an abstracting move, providing a label for later reference back to this situation.

At another time, various variants can be considered, such as adding different numbers or adding something to the larger and subtracting something else from the smaller before the final subtraction or even replacing the final subtraction by addition. These all focus attention on relationships, and invite thinking in generalities, perceiving relationships as potential properties.

Switching to multiplying and dividing the two numbers by the same thing before dividing, or even before dividing one by the other extends the domain of *compensation* by making similar shifts of attention. Fruitful discussion and realisation can emerge from seeking similarities and differences between the properties associated with subtracting and with dividing, and also with adding and with multiplying. Justification of conjectures involves *reasoning on the basis* of previously articulated *properties*.

5.7 Abstracting as an Action

Breive (2022) reports observing two 5-year-old boys explore the concept of reflection symmetry using a doll's pram. In the activity, the two boys first point to specific familiar features of the pram which are symmetrically placed, then one of the boys' attention gradually moves to the imagined and finally to grasping a general and establishing symmetry as a new point of view. This illustrates the essential role of gestures, bodily actions, and rhythm, in conjunction with spoken words, in the two boys' gradual process of encountering a general, challenging the traditional view of abstraction as decontextualised higher order thinking. Abstraction, according to Brieve, is not a matter of moving from the concrete to the abstract, but rather an emergent and context-bound process, arising naturally from activity with confidence-inspiring objects.

Here is an example of an abstracting action in a specifically mathematical domain.

5.8 Difference Divisible

Honsberger (1970, p.87) drew attention to the sequence 1, 3, 7, 13, 21, 31, 43, 57, 73, 91…, appearing in the middle of the following array (Fig. 5.3):

The sequence goes 1, 3, 7, 13, 21, 31, 43, 57, 73, 91, …, $n^2 - n + 1$, …, based on the triangular layout which generates the terms. This is generalisation, shifting from specific relationships (for example, related to the positions of the square numbers in the array) to a way to express each term. There is a shift to articulating a comprehension of *all* terms being generated by a formula, which marks the shift in how one is attending.

He noted that this sequence has the property that if you start at the second term (3) and count along 3 terms the new term is also divisible by 3, as indeed are the terms found by counting along a further 3, and then a further 3. Furthermore, if you start at the third term (7) and count along 7, the new term is divisible by 7, as is the

Fig. 5.3 An array of numbers

			10	
		5	11	
	2	6	12	
1	3	7	13	. . .
	4	8	14	
		9	15	
			16	

term found by counting along a further 7, and so on. What a world of exploration is encompassed or signalled by those three little words *and so on*!

These are specific relationships. By stressing the relationship rather than the specific numbers, two conjectures emerge from asking whether these relationships 'continue':

For any positive integer n, t_n divides t_{n+t_n}
For any positive integer n and λ, t_n divides $t_{n+\lambda t_n}$

The expression of generality has already converted a few specific relationships into potential or conjectured general properties. There has been a shift in thinking, in both what is being attended to and in how these are being attended to. Note that many colleagues with whom I have used this task have struggled to express the general properties, largely due to the unfamiliarity of using subscripted terms in subscripts!

Shifting from specific relationships to a property of each term, and counting along by multiples of that term, is again a generalisation, marked by the shift to a sense of *all*. The urge to check may be great, including the somewhat empty relationship when n is 1. In order to check some particular cases, so as to get a sense of what the property is claiming, it may be necessary to generate some more terms, perhaps by following the evident pattern of the layout or using the formula. If the formula is itself confidence inspiring, then there may be a shift to expressing a conjecture algebraically, and then verifying the divisibility property. In order to justify the claim that the property always holds, it is necessary to find some way to express the general term of the sequence, and this involves *discerning details, recognising relationships* and treating them as *properties*, together with a shift in the way you are attending, from concentration on specifics to a sense of 'all'.

Having checked that the conjectures hold for this particular sequence, abstracting provides a label for any sequence satisfying the same properties.

A sequence of numbers is said to be *term-wise divisible* if for every positive integer n and every positive integer λ, t_n divides $t_{n+\lambda t_n}$.

A mathematical habit-of-mind (Cuoco et al., 1996) immediately raises the question as to whether there are other sequences with the same property, and indeed there are. For example, any geometric progression clearly satisfies the property. To think of trying geometric progressions arises from familiarity with, and access to, geometric progressions along with arithmetic progressions in one's accessible example space (Watson & Mason, 2005).

Close inspection of the original sequence reveals more: the difference between the third and first terms is divisible by 2 as is $t_4 - t_2$ and $t_5 - t_3$; $t_5 - t_2$ is divisible by 3 as is $t_6 - t_3$ and $t_7 - t_4$; $t_6 - t_2$ is divisible by 4 and so on. Expressing these relationships as instances of a property leads to the conjecture that for any positive integer d, d is a factor of $t_{n+d} - t_n$. This property is readily verified in general for the sequence generated by $n^2 - n + 1$. If, while carrying out the algebra, attention is primarily focused on how the reasoning develops rather than simply on the algebraic manipulation, the possibility arises that the same reasoning will work for any polynomial with integer coefficients. Again this is readily verified (Mason, 1990).

The mathematical abstracting action is to name this property for ease of future reference and to declare it as the defining property of such sequences. Call such sequences *difference- divisible*. The abstracting process brings with it the sense of a collection of sequences all of which satisfy the difference-divisibility property, and so it is natural to ask about actions on this new set of objects which preserve the property. For example, term-wise addition and multiplication, scalar multiplication, and composition of difference-divisible sequences remain difference-divisible, as do first-differences. Such a plethora of properties (preservation of a property under various actions) suggests that characterising all difference-divisible sequences might be possible, and so it proves. Characterisation involves finding properties which, unlike the global property involving divisibility, apply locally, allowing explicit construction.

It turns out that all difference-divisible sequences are generated by polynomials of the form for some d and integers c_k (Mason, 1990).

$$t_n = c_0 + c_1\binom{n}{1} + c_2\mathrm{lcm}(1,2)\binom{n}{2} + c_3\mathrm{lcm}(1,2,3)\binom{n}{3} + \cdots + c_d\mathrm{lcm}(1,2,\ldots,d)\binom{n}{d}$$

5.9 Reflection

Abstracting is the action of isolating a collection of properties, formulating a label, and declaring that any ‘thing’ that satisfies those properties is to be considered an instance of that label. Van Oers & Poland (2007 p13–14) describe abstracting similarly as a shift to “a point of view from which the concrete can be seen as meaningfully related”. Gaining familiarity with a range of examples, developing tools for constructing new examples from old, and characterising such examples using other properties, is how accessible example spaces become enriched and is the stuff of pure mathematics.

5.10 Defining and Specifying

Mathematicians have a habit of using the word *define* in two completely different senses. One form is the familiar formulating of a definition of a concept, such as *function* or *circle*, as has been illustrated as part of the process of abstracting. The other form is actually a specification of a particular. Thus, I might specify *the* function f to be (to denote for the moment; notice the definite article rather than the indefinite article) $f(x) = x^2 + 1$. To use the word *define* in this context, although common, is misleading for novices, because it is not defining the concept of a function, but specifying that for the time being f will denote a particular function.

Notice that comprehending the specification involves discerning and then relating two things, the letter f and the rule for evaluating it as a function. The consequence is that whenever (locally) the letter f is used, it has the property that $f(x) = x^2 + 1$, where x is a placeholder and so can have various things substituted for it, including itself. The experience of 'taking this on board' seems to me to involve a shift in how I am attending to the symbol f, but to be subtly different from the shift in attending which accompanies abstraction, since although the specified value can be used in any context in which the symbol f appears, it is not abstracting a property, but rather denoting it.

5.11 Other Instances of Abstracting in School Mathematics

What follows are some sample situations from school-aged mathematics which involve the action of abstracting. I begin with what might be considered the more obvious examples and proceed towards the more subtle.

5.12 Names

The act of naming something is an act of abstraction. It provides a label for 'things discerned as having the properties I am currently attending to'. What makes mathematical abstracting distinctive is that the properties are made explicit, whereas ordinary naming makes assumptions that everyone is attending to the same things as necessary properties. Thus, I point to a tractor and say "tractor", but my young son appears to be attending to the end of my finger! What is rather amazing is that children acquire grammar and vocabulary, often over-generalising at first, but then narrowing down to common usage (Brown, 1973).

5.13 Geometry

5.13.1 Shape Naming

Much of early years spatial thinking is, unfortunately, taken up with the naming of shapes, rather than actions which can be performed on shapes and which preserve some set of properties.

Furthermore, most of the physical shapes made available to young children are special instances of geometrical shapes that deserve more general mathematical treatment. So when the word *triangle* is used largely in the context of a plastic or wooden equilateral triangle, children are expected to experience what is really an abstracting process of moving from properties to naming and defining. Where the adult draws attention to the three vertices and three edges, properties are being identified, but when the adult is *not* attending to the regularity, children may not be aware of the implicit generality. This aligns with the learning of vocabulary generally, in which new words are used, sometimes with articulation of meaning and sometimes not. But mathematically, it is important that the full range of properties are brought to attention. For example, Sfard (2007 p597) found that her students often distinguished between 'ordinary' triangles and extreme triangles, labelling the latter as 'sticks', rather than 'seeing them as' triangles, and this parallels initial reluctance to accept a square as an instance of a rectangle. In some contexts, naming of sub-categories is taken on board easily, and in other contexts it is often resisted.

The vital but overlooked ingredient of abstracting is the sense of 'all' or 'any' which accompanies it. 'Anything', with the stated properties, qualifies. It is perhaps largely within mathematics and science that it is necessary to be specifically articulate about exactly which properties are required in order to qualify for the label, so that reasoning on the basis of properties can take place.

Perimeter and Area

It is well known (Kouba et al., 1988; Dickson, 1989; Reinke, 2010) that learners sometimes confuse perimeter and area when asked to calculate them. Usually confusion arises because of the procedures invoked for calculating: it is easy to confuse the role of 'counting squares' when figures are presented on squared paper (sometimes you count inner squares, sometimes you count boundary squares, inside, or outside). It is possible that young children have only the haziest notion of what is being attended to by either concept. To make area and perimeter objects of attention and hence to abstract them mathematically requires recognising relationships and perceiving properties that are pretty sophisticated (which may be why they don't show up again until calculus) and probably out of reach for young children. Even so, there is an element of abstracting in directing attention to the distance around and the area encompassed by a shape drawing on intuitions based on specific confidence-inspiring examples. Dealing with area and perimeter separately, at different times may add to student confusion; one way to juxtapose them is through the following task, based on an idea of Dina Tirosh & Pessia Tsamir (Tirosh & Tsamir, 2020).

Figure 5.4 shows a shape in the middle cell. For each of the remaining cells, draw a shape which has the properties indicated by the row and column headings, making as few changes as possible to the central shape. Bring to articulation what you become aware of as principles for tinkering with shapes so as to modify their area and perimeter appropriately. These principles could be the starting point for full abstraction of area and perimeter.

5.14 Constructions in Triangles

Various constructions in triangles (medians, angle-bisectors, altitudes, perpendicular- bisectors, cevians, …), and various special cases of polygons (regular, cyclic, equi-angular, equi- edged, oppositely-parallel-edges, alternately paralleled, …) are

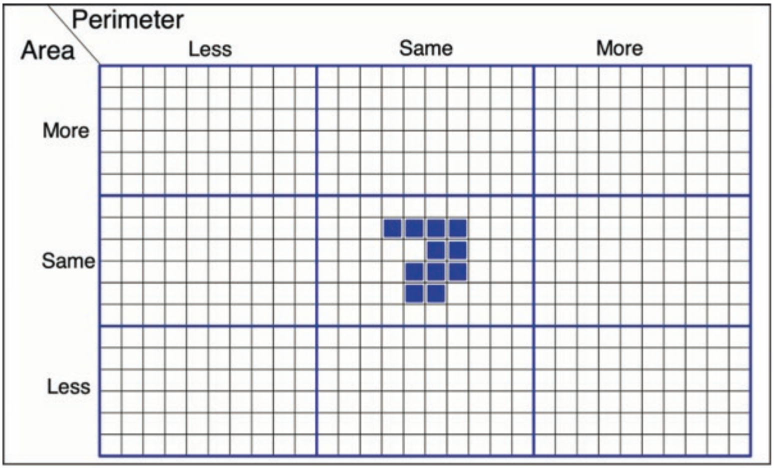


Fig. 5.4 An example of a more-or-less-grid for area and perimeter

labels for properties and so are examples of an abstracting action. Again the critical element is the word *any* or *all* when stating that *any* object satisfying the property is to be so labelled, for it is these properties, and these properties alone which can be used in reasoning concerning, for example, that the three medians, three angle-bisectors, three altitudes, and three perpendicular bisectors, are each coincident or the conditions in which the three cevians are guaranteed to be coincident.

In geometrical reasoning, it is sometimes the case that deductions are made from the property (points on an angle-bisector forming right-triangles with the edges of the angle which are congruent), and sometimes the property is known to hold (a point equidistant from two adjacent edges must lie on the angle-bisector). The fluency and flexibility required in order to reason geometrically depend on the abstracting process having been fully appreciated, especially the sense of 'all' which the abstraction encompasses.

5.15 Pythagoras

Appreciating and comprehending the Pythagorean relationship involves abstracting as a consequence of reasoning rather than defining. Given any three positive numbers a, b, c for which $a^2 + b^2 = c^2$, there is a right-angled triangle with edges a, b and c, and vice versa, given a right-angled triangle, the relationship holds. The adjective *Pythagorean* applies to number triples and involves a further abstracting process. Comprehension is extended when a second 'any' is introduced: placing similar figures on the edges of a (any) right-angled triangle leads to the areas satisfying the Pythagorean relationship.

5.16 Trig Functions

Each of the trigonometric functions is the result of an abstracting process, but it may take time and multiple exposures to appreciate them fully. At first, any right-angled triangle containing the specified angle (between 0° and 90°) provides six measures of that angle, the sine, the cosine and the tangent, secant, cosecant and cotangent, precisely and exclusively because ratios are preserved between similar triangles. It is this invariance which provides the *all*, so that the definition makes sense. Later, other definitions are encountered (power series, solutions of differential equations), which require reasoning to justify the claim that they too classify or capture equivalence and are indeed alternative characterisations.

5.17 Cosine and Sine Rules

These two trigonometric relationships which apply to triangles in Euclidean geometry depend on the presence of *all* or *any*, in that they hold for any planar triangle. Appreciating the scope of this *any* is necessary to comprehend the abstracting that has taken place.

5.18 Numbers

5.18.1 Types of Numbers

Whole numbers, zero, and the integers, positive, negative, and zero are labelled and used as nouns, usually without any reference to properties. It is only when real numbers are being placed on a firmer foundation that it is deemed necessary to return to integers and to isolate and articulate defining properties. The abstracting is somehow deemed to have taken place already. This is the source of some confusion for tertiary students for whom ‘facts’ that they have long been familiar with are ‘proved’ using lists of properties which appear ‘obvious’.

An interesting issue arises with fractions, introduced formally as the basis for rational numbers. A fraction is usually defined as a pair consisting of an integer and a positive integer.

Consequently, neither $\frac{2\pi}{3\pi}$ nor $\frac{2}{-3}$ are fractions, though of course as real numbers they are equivalent to fractions and hence are representatives of rational numbers. Based on learners’ experiences prior to meeting rational numbers, fractions might more usefully be defined as pairs of integers with the second being non-zero, but even so, formal reasoning is required in order to check relations which underpin equivalence of fractions and hence rational numbers are indeed equivalence relations (note the extra abstraction with the label 'equivalence relation'), and this formality can be off putting. The formalities required to check that properties continue under equivalence require fluency and flexibility with the notions of equivalence and with the arithmetic of fractions.

5.19 Symbolised Numbers: π, $\sqrt{2}$ and $\sqrt{-1}$

Interestingly, learners seem to accept the symbol π as standing for some number, even if they are misled into believing it to be 22/7, or more mysteriously, some terminating decimal such as 3.1415927. Mathematically, there are actually two π's which happen to be the same: the ratio of the semi-circumference of a circle to the radius, which is invariant under changes of radius, and the area of a circle divided by the square of the radius, which is also invariant under change of radius. Thus, to appreciate π involves appreciating two invariants, together with reasoning which justifies the claim that the two invariants are actually the same. Of course, a good way of developing comprehension of this is to contrast it with other more easily and confidently manipulable shapes such as equilateral triangles, squares, and rectangles with a specified ratio, because for these shapes the two invariants are different. Might there be a collection of shapes which, like the circle, have the ratio of semi-perimeter to radius and area to radius squared equal?

Learners seem to have slightly more difficulty dealing with numbers denoted by surds, such as $\sqrt{2}$, perhaps because of the need to shift to properties. Surds challenge the certainty-based confidence of whole numbers which are finite in nature and in expression, while most real numbers have unknowable decimal digits. The symbol $\sqrt{2}$ denotes a number with two properties: it is positive and its square is 2. That is all that is known, and all that can be used. Although $\sqrt{2}$ 'knows all of its decimal digits,' it cannot tell us what they all are. We can find out as finitely many as we wish to know, but there will always be ignorance of what follows. Of course, the same is true of π, though this does not seem to agitate learners in the same way.

In Mason (1980), I asked when a symbol is actually experienced as symbolic, drawing attention to the way in which what begins as an unfamiliar sign, such as π or $\sqrt{2}$, gradually over time and with experience, becomes confidence inspiring and manipulable. In terms of attention, the sign is no longer simply something at which to gaze, but is discerned from other signs, its relationships recognised, its properties perceived, and reasoning can take place on the basis of its properties.

Experience with $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{2}$, etc. opens the way for the introduction of $\sqrt{-1}$, whose only property is that its square is -1. Interestingly, it follows (reasoning on the basis of properties) that $-\sqrt{-1}$ has the same property. They are distinguished by placing them on an Argand diagram.

Dedekind noticed that there is a problem when reasoning with surds and negative numbers, because simply using the familiar properties for positive numbers leads to contradiction:

$$1=\sqrt{1}=\sqrt{(-1)(-1)}=\sqrt{-1}\sqrt{-1}=-1$$

Dedekind raised the question of why it is the case that $\sqrt{ab}=\sqrt{a}\sqrt{b}$ when a and b are non- negative. His answer was to define real numbers using what came to be known as Dedekind cuts, which apply to non-negative a and b and leave as problematic manipulations of surds of negative numbers. Eventually, this was resolved by Riemann using what are now known as Riemann surfaces. Dedekind undertook a process of abstraction in order to put decimal numbers on affirm foundation.

5.20 GCD and LCM

Finding the greatest common divisor of two specified numbers involves a sequence of mathematical actions. Shifting to thinking of the greatest common divisor of two as-yet-unspecified numbers requires a subtle shift from recognising relationships (divisibility, maximality) to perceiving properties as being instantiated. In order to proceed mathematically, this act of generalisation is accompanied by formalisation, in order to ascertain the properties that are needed. To reduce the words required, the notation $a|b$ is read as, and means that, the integer a divides into the integer b or in other words that there exists an integer c such that $ac = b$. Then the properties of the gcd of two positive integers are that

gcd(n, m) | n and gcd(n, m) | m, and for any h for which $h|n$ and $h|m$, h | gcd(n, m). gcd(n, n) = n; gcd(1, n) = 1.

These are acts of generalisation, listing properties. Mathematical abstracting means reversing that action, choosing one or more properties as being ‘critical’ and creating a definition:

Suppose $g|n$ and $g|m$, and suppose further that for any h, $h|a$ and $h|b$ implies that h | g. Then g is the gcd(a, b).

Mathematically, one then proves that the other properties are consequences of the chosen properties.

The interested reader may wish to repeat the movement from recognising relationships to perceiving properties to abstracting (ready to reason on the basis of properties) in the case of the lowest common multiple of two positive integers.

Abstracting involves a declaration. Notice that during the abstracting process, the nature of the ground-set (positive integers) was omitted. This enables the properties to be perceived as instantiated in other contexts and then the transfer of all reasoned consequences of properties to the new context.

For example, consider $g_p(n, m)$ defined for any positive integer p as the greatest power of p which divides both n and m. The function g_p also has the properties of a gcd, but only when p is a prime.

But abstracting is also an on-going process. Together, gcd and lcm satisfy the following properties:

gcd(n, m) = gcd (m, n) and lcm(n, m) = lcm(m, n)
gcd(n, n) = n = lcm(n, n)
gcd(n, m)lcm(n, m) = nm
gcd(n, m) | n | lcm(n, m)
gcd(1, n) = 1; lcm(1, n) = n.
for any h for which $h|n$ and $h|m$, h | gcd(n, m)
for any h for which $n|h$ and $m|h$, lcm(n, m) | h.

Abstracting means isolating at least some of these properties and declaring that any pair of function that satisfies those properties are a gcd/lcm pair.

Choosing a minimal set of properties from which the remainder can be deduced is one domain of exploration. It exploits the shift of attention from properties as the focus of attention, to properties as tools for reasoning. Being on the lookout for other pairs of functions satisfying the same properties enriches the sense of what gcd and lcm are about. For example,

Do the functions min(n, m) and max(n, m), defined on the integers, satisfy these properties of gcd and lcm, respectively?
Do the functions $f(n, m)$ = product of each of the primes which divide both n and m, and $g(n, m)$ = product of the primes which divide at least one of n and m satisfy the properties?

Extending functions to three or more variables may involve extra properties. Abstracting further by replacing 'divides' by some more general relationship, perhaps with some specified properties, introduces connections to yet other mathematical domains such as lattices and partially ordered sets.

5.21 Euclidean Algorithm

The Euclidean algorithm is the result of an abstracting process, in which, starting with a pair of positive numbers, repeated actions of subtracting smaller from larger until 0 is achieved, provides an output. Throughout, the current pair of numbers preserve the property of having the same greatest common divisor. So labelling the process, the action involves abstracting over the domain of pairs of positive integers (which can be extended to rationals and to polynomials with integer coefficients). Appreciating the 'all' is what contributes to comprehending the algorithm itself.

5.22 Zero and Infinity

5.22.1 Completed and Unfolding Infinities

One of the awkward features of a concept defined in terms of its negation is that it is hard to escape a concept image of negation. Infinity is just such a concept. The word *infinity* is usually associated with a process that 'goes on forever', and so is by its very nature, uncompleted, incomplete, and incompletable. And yet, it is possible to take a stance that somehow the process has been completed, perhaps by letting go of unfolding-in-time, as in the names of irrationals as mentioned earlier. Different stances are taken by different folk (Pajk, 1983; Hamming, 1989; Sierpinska, 1994; Mamolo, 2010, 2017).

5.22.2 Empty-Sets and Zero

The introduction of zero to denote an empty place in a positional system of numerals seems to have come to the west from India in the eighth or ninth centuries, though it may have been present in Central America as well. But it is an important construct at the intersection of cosmologies, psychology, and philosophy: the notion of the emergence of matter out of apparently nothing (the void), as in the 'big bang', aligns with personality seen as layers protecting people from encountering their essential emptiness, and in mathematics, the empty set.

I suggest that there is a very delicate shift of mathematical attention required (Mason, 1989), in order to move from the sense of something being empty, to that emptiness actually being a something, and hence a potential member of another set. Thus, $\{\}$, which is usually denoted by $\varnothing$, involves discerning an emptiness, treating it as a something, as a whole (potential object of gazing), recognising a relationship between what has been discerned and non-emptiness, providing a label, and then perceiving properties. Thus, $\varnothing$ leads to $\{\{\}\} = \{\varnothing\}$ and so on into the emergence out of the void of something analogous to numbers (Halmos, 1960). Properties include for any set A, $A \cup \varnothing = \varnothing \cup A = A$ and $A \cap \varnothing = \varnothing \cap A = \varnothing$

Abstracting mathematically is the move from treating $\varnothing$ has a thing, to using the properties to say that any set E with the property that for all sets A, $A \cup E = E \cup A = A$ and $A \cap E = E \cap A = E$, is the empty set.

This is actually a theorem to be justified with reasoning:

Suppose for all sets A, $A \cup E = E \cup A = A$ and $A \cap E = E \cap A = E$

Then in particular, these properties must hold when $A = \varnothing$

Consequently, $\varnothing \cup E = E \cup \varnothing = \varnothing$ and so $E = \varnothing$

Alternatively,

$$\varnothing = \varnothing \cap E = E \cap \varnothing = E. //$$

In the face of alternative reasoning, the question arises as to whether both properties are required or whether each can be deduced from the other. In other words, is it possible for a set to satisfy either property but not the other? Mathematical exploration of minimal axioms (defining properties) usually follows.

5.22.3 *Wholeness of One*

Wholeness, or oneness, seems at first to be entirely natural and unproblematic. Yet it involves a choice, a decision to discern difference, perhaps even boundary, in order to include while at the same time excluding. As you read, you (mostly) ignore the ambient temperature, the sensation of your body as you sit or stand, your heartbeat, what you had for breakfast, and so on. What you are attending to has an innate sense of wholeness. It is the whole of your attention.

Being able to gaze, to hold a whole without proceeding to make further distinctions and so engage in an analytic or decomposing programme, is remarkably difficult. The temptation of the human organism to enact any available actions is very great. And yet it is through gazing, through attending to form rather than function, through holding wholes, that insight comes. It is through gazing that access to creative energy is gained. For me, this is the point of linkage between mathematical thinking and other forms of creative arts, including insight that might be classified as spiritual or 'as coming from somewhere else'. It turns out that many people, amongst them Sylvanus Thompson (1917 p106–109) have had similar thoughts.

Contemplation of moments of insight opens up a sense of wholeness, which, through being distinguished from multiplicity, and through recognition of what used to be known as the middle-aged stance that 'everything is connected', engages an ongoing abstracting of ecological-spiritual- mathematical awareness of the unity and interconnectedness of everything, while at the same time providing a setting for discerning detail, recognising other relationships and perceiving properties.

If you have ever had the experience of looking but not seeing, such as looking in the fridge for something but not seeing it as being present, you know the effect of a limited wholeness and how it blocks out other details and relationships. So the analytic mode is capable of contributing to an increasing rich sense of wholeness, as long as it does not become its own exclusive world. Perceiving properties and reasoning on the basis of properties can serve to enrich one's sense of wholeness, one's subsequent gazing, but by becoming a wholeness of its own, it can also block out that richness.

5.23 Conclusions

Discerning different ways of attending to mathematical constructs enriches possibilities for informing both teaching and learning, not simply by refining analysis of experience, but by providing a practical vocabulary for directing attention.

A process of mathematical abstracting takes place throughout mathematics, at all ages. It is not confined to the ‘higher mathematics’ of tertiary institutions. Discerning abstracting as a shift in how someone is attending, discerning what is stressed and what ignored, and emphasising the properties required in order to justify use of a label arising from a definition, could inform teacher practice by providing a useful way of undertaking and displaying the abstracting.

Distinguishing between generalising and abstracting could go a long way to improve communication between teacher and learners, because they involve different shifts not only in what is attended to, but how. Generalising involves a shift of attention from specific relationships in a particular situation, to focusing simply on properties. It seems so natural to those familiar with mathematical thinking but can be mysterious for novices. Abstracting involves a further shift in which attention is not on the objects in a situation, but the general properties which have been isolated and subsumed under some defining label. Again, the expert barely notices the shift, while learners may be mystified when it is done quickly and implicitly.

Mathematical definitions come in two types, the naming of an object as something satisfying a list of properties, and, unhelpfully, temporarily specifying a particular object within a domain. Pedagogically, it makes sense to distinguish these by using *specify* for the act of specifying. Pedagogic support for shifting the way learners are attending to the more mathematically productive generalising and abstracting requires increasingly complex appreciation of the *for all* feature of properties. This means enriching the space of accessible examples and associated construction tools so that *all* (or *any*, *each*, and even *a*) encompass a wide range of instances for learners. Where characterisation is possible, then *all* is actually articulated in terms of the local properties which are equivalent to the original global properties on which the definition of the abstraction is based.

Knowing when something is worth naming, or in other words, when a distinction is worth making, is not trivial. Locally, it may ease articulating reasoning for the writer, but it may also introduce a hurdle for the reader who has not yet internalised the concept being labelled; globally, it may introduce yet more burden on teachers, by, for example, introducing distinctions and associated pedagogical actions which may overcomplicate teacher attention, leading to frustration.

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Chapter 6
Toward a Systems Theory Approach to Mathematics Education

Steven Watson

6.1 Introduction

This chapter presents a systems approach to thinking about mathematics education. Little attention has been paid to systems theory in mathematics education, although there has been some interest in the forebears of contemporary systems theory. Notably, Ernst von Glasersfeld's (1995) radical constructivism draws on earlier renditions of systems theory and second-order cybernetics; Davis and Simmt (2003) draw on complexity theory and present the mathematics classroom as a complex adaptive system, and Proulx and Simmt (2013) relate systems theory to enactivism. However, systems theory has itself developed considerably since Bertalanffy (1968) proposed a general systems theory of open systems. Here, I will present those developments and begin to bring the ideas of systems theory to mathematics education. While this may appear to have commonality with postmodern theory, and the problem of the loss of grand narrative, it is resolutely modern, but as a modern critique of modernity.

I will draw upon the work of the German sociologist Niklas Luhmann (1927–1998) who developed a systems theory of society. Luhmann was a prolific author; he is notable for his public intellectual rivalry with the Frankfurt School philosopher Jurgen Habermas and his difficult and sometimes impenetrable writing. However, he is recognized in many quarters as one of the most significant social theorists of the latter half of the twentieth century but is quite often misunderstood and misrepresented (see Borch, 2011). Even amongst sociologists of the anglophone world, little attention is given to his work, but his influence in northern Europe is considerable and across a range of disciplines. This is often attributed to the fact that Luhmann wrote in German, and there has been something of a lag

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before his work was translated into English. Even translated into English his writing style is difficult, and systems theory itself is challenging as it marks a departure from the traditions of post-Enlightenment and humanist thinking.

In what follows, I will set out some of the key ideas presented in Luhmann's systems theory and then begin to consider mathematics education from this perspective. In the spirit of the title of this book as presenting philosophy of mathematics education *works-in-progress*, this marks the beginning of consideration of a systems perspective on mathematics education.

It is important to address the question, What issues in mathematics education does systems theory address? While this chapter will unfold some propositions as I introduce systems theory, the central issue is the possibility of the observation of mathematics education as an entity. In other words, scholars and practitioners who are active in mathematics education are increasingly asking about the nature of mathematics education as a whole. There is an imperative for self-observation. Within the field of mathematics education, there has been increasing interest in the self-observation of mathematics education, for example, from the perspective of history (see, for example, Howson, 1982; Kilpatrick, 2014; Inglis & Foster, 2018) and also theoretically (Sriraman & English, 2010). Within the 'unity' of mathematics education, there is also much difference: differences between theory and practice as well as approaches to practice and approaches to research, with positions often presented as polarized and intractable, within the field and in the media (Watson, 2020a; Watson & Barnes, 2021). Overall, however, mathematics education takes place in a complex world, where knowledge appears to be increasingly uncertain and is sometimes contradictory; information proliferates within the media and is often conflicting and even 'fake'. I argue systems theory offers an approach to understanding this complexity.

6.2 Systems Theory

Systems, as functional conglomerations of elements, have been conceptualized since at least classic antiquity (Bertalanffy, 1972). However, in the last 200 years, the idea of a system has taken on a new significance. This development has run in parallel with notions of complexity and emergence. Human beings increasingly encounter and also become part of complex and fluid social and technological entities. Deleuze and Guattari described such entities as *assemblages* (Deleuze & Guattari, 1987). Here, I will refer to such emergent and dynamic entities as systems.

Mathematics education, a mathematics education research paradigm, a classroom (see Davis & Simmt, 2003), a teacher or learner's cognition can all be thought of as systems. All are entities consisting of elements and processes and each can distinguish itself from its environments. All subsystems can see themselves as within the totality of mathematics education, yet within that totality, *that unity*, there is difference: paradigmatic differences in scholarship and research, differences between theory and practice and differences in values, for example.

Traditionally, and going back to antiquity, the distinction has been made between 'whole' and 'parts' in the analysis of the world (Luhmann, 1995). Mathematics education in this sense represents an aggregation of, say, policy, policymakers, practice, teachers and learners, and research and researchers, etc. However, the unity of the whole is more than the sum of its parts. In other words, mathematics education, in totality, has overall characteristics that cannot be identified within the elements that contribute to it. The totality is not simply an aggregation of all its elements. To address the limitations of the *whole-and-part* distinction, systems theory makes a distinction between *system and environment* in order to understand the relationship between the unity of a totality, such as mathematics education, and its elements. This apparently simple change also results in some profound changes in epistemology and ontology.

The distinction between system and environment is not concerned with the function of the system alone, but with how the system functions in relation to its environment. It is how that system distinguishes itself from its environment (Luhmann, 1995). It is the system itself that distinguishes itself from its environment through self-reference, but self-reference in relation to its environment. The part/whole characterization is replaced by the idea of system differentiation, systems are differentiated from their environments and within systems, subsystems differentiate themselves, e.g., research and practice. The environment of each subsystem includes other subsystems within that system and the environment beyond that system.

The environment of the mathematics classroom includes education policy making, knowledge generated through mathematics education research, individual learners, the teacher, as well as the mass media and social media. Mathematics education is, through policy, practice and scholarship in a constant process of defining itself in relation to environmental prompts. Its overall purpose, its function, is not controversial, in that mathematics education intends to develop the mathematical knowledge and mathematical capabilities of learners. How it does that is subject to internal (and external) debate, contested approaches, organizational and institutional policy and practices, and appeals to knowledge, values, and morality.

A further distinction is made between open and closed systems. A closed system is a limiting case where notionally no material or energy crosses the boundary between system and environment (Luhmann, 1995). All 'real' systems are open, they are responsive to the environment that they are distinguished from. Real systems also have a boundary with their environment, for example, a cell has a membrane yet remains open. This apparent contradiction between openness and closure in living systems was addressed by Chilean biologist Humberto Maturana and later with Francisco Varela (Maturana & Varela, 1980). They suggested that living systems were operationally closed and that although cell membranes are permeable and therefore open, the cell can be considered to be closed operationally. It responds to its environment through its internal biochemical operations, those operations recursively create the distinction between system and environment (the membrane). They termed this *autopoiesis* as the self-referential operational closure of a system. Self-reference is important since the consequence of autopoiesis is that any experience of the environment is entirely constructed by the system internally. The system iteratively responds to a complex and unknowable world.

Maturana and Varela's work was principally based on living systems, but Luhmann wanted to use autopoiesis in the context of consciousness and society (i.e. in psychic and social systems). To do this, he generalized Maturana and Varela's theory of autopoiesis by de-temporalizing and de-ontologizing the elements of a system. For Maturana and Varela, elements of living systems involve relatively stable chemical molecules which are replaced from time to time. Luhmann argued that (1) the replacement of stable molecules is a momentary event and therefore in de-temporalized terms elements are not dependent on temporal references beyond the system itself and (2) elements are internally differentiated by the system, they are not pre-given and are constituted by the system they are given to. Elements of a system do not have an ontology beyond that designated by the system. The move then is to a more general theory of autopoiesis.

> [I]f we abstract from life and define autopoiesis as a general form of system building using self-referential closure, we would have to admit that there are non-living autopoietic systems, different modes of autopoietic reproduction, and general principles of autopoietic organization which materialize as life, but also in other modes of circularity and self- reproduction. In other words, if we find non-living autopoietic systems in our world, then and only then will we need a truly general theory of autopoiesis which carefully avoids references which hold true only for living systems. (Luhmann, 1986, p. 172)

Luhmann translated the idea of self-referential, operationally closed, autopoietic systems to society and social systems. He distinguishes between machines, organic and neurophysiological systems (cells, nervous systems, immune systems etc.) and systems "constituted by the production and processing of meaning" (Luhmann, 1995, p. 37) i.e. social systems and psychic systems (consciousness) (see Fig. 6.1). In this chapter, I will focus on the social and psychic systems constituted on a phenomenological interpretation of the idea of meaning.

Social and psychic systems can only experience their environment through meaning (see Luhmann, 1995, p. 102). Similarly physical, chemical, and organic use physical forms and processes to experience the environment. This probably explains why neurophysiological approaches appear only to have limited explanatory power in relation to cognition. The individual cognitive system from the perspective of systems theory is a self-referential system constituted on meaning, clearly neurophysiology is necessary for consciousness, but consciousness by its very definition is abstracted from the physical.

What then distinguishes social and psychic systems from other systems is that the medium (the elements of these systems) that constitutes both is meaning, and

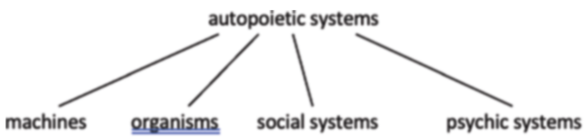

Fig. 6.1 The structure of autopoietic systems. (Adapted from Luhmann, 1995)

this is central to language and communication. The explanation for this lies with a phenomenological rather than the hermeneutic sense of meaning used by Luhmann.

> Luhmann defines meaning as the "horizon" of possibilities that is virtually present in every one of its actualizations. As the difference between the possible and the actual, meaning itself is a category "without difference" (differenzlos), which designates the medium through which social systems process world-complexity. (Foreword by Eva Knodt in Luhmann, 1995, p. xiii)

Meaning, in Luhmann's terms, means consciousness and communication are constituted on the difference between actual and possible. This concretizes the 'actual' in thought and language while permitting a horizon of possibility. This ultimately is a means through which the complexity of the environment is reduced by the operations of consciousness (cognition) and of social systems.

Luhmann, rather controversially, characterizes society as a social system of communication a totality of communication. That is rather than one of human beings and/or their actions. This while the consequence of such theorization is not so – suggests to some, denial of agency or humanity even. Luhmann argues that it is only through communication that society is created, as we shall see that conscious systems constitute the environment of the social system of society and vice versa. Therefore, action perturbates or stimulates conscious systems which in turn stimulate the system of society. Some action is communicative, of course, but many actions are not communicative and are external to society (detailed justification for this can be found in Luhmann, 1995, 2013a, for example).

In the later phases of developing systems theory, Luhmann was influenced by British mathematician George Spencer Brown (Watson, 2020b) and his Laws of Form (Spencer Brown, 1969). This represents a further abstraction of autopoiesis, although Spencer Brown developed Laws of Form sometime before Maturana and Varela. Laws of Form offers a psychological, mathematical, and philosophical development of a distinction as 'the mark' between system and environment, its unity in difference, its paradoxical nature, and self-referentially recursive nature (Watson, 2020b).

The most profound consequence is that the foundation of consciousness, the psychic system, is the capacity to make a distinction, in the abstract, between one thing and another. Complex phenomena emerge from the recursive distinction of different patterns of distinction (think here of the generation of fractal patterns of repetition and emergence in Mandelbrot sets). There is limited space here to go into the recursive basis of complex patterns through the process of re-entry, where a simple distinction, the identification of an 'object', leads to further distinctions within that distinction. The distinction re-enters itself (see Watson, 2020b).

Distinction is also important in understanding the nature of information and communication in systems theory. Luhmann follows Gregory Bateson's (1972) characterization of information, as a difference that makes a difference. What marks something out as information is its difference from other things. Information is in itself a distinction, but is fleeting and momentary, although it can be recorded as an event. Luhmann's theory of communication involves a distinction between information and utterance, in a similar way to recognizing a pattern of sound against a background of noise. Language provides the medium of communication (which is in itself a form in the medium of meaning). The process of communication involves utterance and selection of information, the receiver independently selects information from the utterance, this is less than likely to be the same selection that is made by the sender. It is a corollary of the general theory of autopoietic systems that communication refers only to previous communication.

Again, in a somewhat unorthodox move, Luhmann puts forward the idea that it is not people who communicate, it is communications that communicate (Luhmann, 1995). Again, this can cause a jolt and shudder on the grounds of denial of human agency. However, the fact that humans experience, perceive, make sense and construct meaning, and articulate a selection of that through language, which is itself an evolving recursion, indicates the plausibility of society and social systems as an autopoietic system of communication.

Systems theory draws on evolutionary theory and has an important role in explaining the improbability of system differentiation as a result of variation, selection, and stabilization (Luhmann, 2013b). The implication of both evolutionary theory and systems theory is the proliferation and selection of distinctions, as well as the importance of both the inseparability and distinctness of psychic and social systems of consciousness and communication and language.

From an evolutionary perspective, social systems and psychic systems have developed concurrently, one is inconceivable without the other. They are, in the terminology of Luhmann, *structurally coupled,* a means by which systems regulate relations with their environment (Luhmann, 1995). Structure is employed by an autopoietic system to 'simplify' its relationship with its environment. For example, the evolutionary development of muscles and a skeleton is a structural development to allow movement in the context of gravitational forces. Another example is the way in which the brain is structurally coupled to the environment through the eye and ear; this structural coupling reduces what can be seen and what can be heard. Structural coupling maintains the boundary between system and environment but does not influence the system's autopoiesis.

Psychic and social systems are uniquely constituted in the medium of phenomenological meaning and depend on structural coupling. The structural features of spoken and written language reduce environmental complexity and form the basis of structural coupling between psychic and social systems.

> ... language excludes a lot in order to include very little, and that it can become complex only for this reason. If we begin with spoken language, we see that noises are excluded except for those few highly articulated noises that can function as language. Even small variations and slight shifts, or the replacement of one sound by another make communication impossible and irritate consciousness. [...] And the same is true of writing, only very few standardized symbols are suitable for writing, and everything else that can be seen is simply out of the question. Structural coupling is a highly selective form that uses relatively simple patterns. (Luhmann, 2013a, p. 87)

It is with a limited number of letters, a sophisticated alphabetic phonetic writing system, some standardized pitches and acoustic signs that with which we can recreate complexity, and at the same time as reducing environmental complexity. The implication of this is that society is only coupled with the physical world via consciousness. Luhmann argues "that there is no physical, chemical, or purely biological effects that influence social communication" (Luhmann, 2013a, p. 87). As such, everything passes through the limits of language which forms the basis of structural coupling between psychic systems and the social.

With this brief introduction to systems theory, it is possible to examine mathematics education, if only to a preliminary extent.

6.3 What Kind of System Is Mathematics Education?

I will begin by presenting mathematics education as a social system of society, a system of communication, specifically a functional system. Coupled to mathematics education, as well as the consciousness (i.e., psychic systems) of those involved in mathematics education, are other functional subsystems of society. These include, for example, the political system, the legal system, the economic system, the education system, the system of science (as the general system of knowledge production), and mass and social media.

6.4 The System of Society and Education

The evolution of society is a concomitant feature of the evolution of consciousness, and for human beings, the evolution of language and communication allows the possibility of forms of structural coordination that improve the likelihood of survival of individual 'consciousnesses'. This is an assumption central to a systems theory of societal differentiation (Luhmann, 1995, 2013a, b, c). Coordination takes place through the differentiation of society by recursive distinction. Early societal differentiation involved family, kinship, and tribes, which Luhmann characterizes as *segmentary* society. This allowed the potential for coordination between families who in smaller units were engaged in survival. In parts of the world, tribal structures evolved and with the emergence of eminent families, this *centre-periphery* differentiation gave way to empires and civilizations, through which and with the invention of writing, were to give way to *stratified* differentiation, with the differentiation of the nobility from commoner. In Europe, stratification was the predominant mode of differentiation through the Middle Ages and continued to be the dominant mode of differentiation of society until a *functional differentiation* of society emerged as dominant in Europe from about the eighteenth century (Luhmann, 2013b, c).

Before elaborating on functional differentiation, it is important to provide some caveats and conditions with which Luhmann presents the four modes of differentiation: segmentary, centre- periphery, stratified, functional. (1) They are not indications of society's epochs; (2) societal differentiation is not limited to these forms; these, Luhmann argues, are the key types of differentiation in the history of society, (3) different types of differentiation can exist at the same time, e.g. families continue to exist in contemporary functionally differentiated society, (4) the internal differentiation of society reflects an increasing complexity with which the world can be described and experienced, and (5) societal differentiation follows evolutionary principles of variation and selection. Also, Luhmann stresses the improbability of such structures on the basis of evolution:

> Evolution theory shifts the problem [of the "improbability of structural coordination'] to time and explains how it is possible that ever more demanding and ever more improbable structures develop and function as normal. (Luhmann, 2013b, pp. 251–252)

Functional differentiation developed from stratification with the appearance of specialized functions in society and through the division of labour. There are various aspects that mark the transformation from stratified society to functionally differentiated society from the late Middle Ages and into early modernity. These include the growth of international trade, surplus money, and a debt crisis in the fifteenth and sixteenth centuries. The economy 'learned' to perpetuate itself through prices and became independent of the nobility.

> What is important is that at some point or other, the recursivity of autopoietic reproduction began to take hold and achieved closure, after which only politics counted for politics, only art for art, only aptitude and willingness to learn for education, only capital and profit for the economy, and the corresponding intrasocietal environments—which included stratification—were now seen only as irritating noise, as disturbances or opportunities. (Luhmann, 2013c, p. 66)

I am not going to expound here on the differentiation of each functional system from the Middle Ages, of politics, economy, law, media, religion, art, etc. (for an elaboration, see Luhmann, 2013c, pp. 65–86). Although, it is important to consider the dynamics of this 'outdifferentiation', the process through which functional systems differentiate from stratified society. Politics emerged from political rivalry in stratified societies, which by the mid-seventeenth century, had given way to a nascent form of democracy and then to ideas of the sovereign state, as well as the emergence of a legal system.

> As part and parcel of this movement, education progressively switched from being primarily achieved through mere socialization to becoming more systemically organized and differentiated from the rest of society. What emerged then across Europe was a sort of 'popular religious education' (Luhmann, 2013c, p. 219) whose contents were still aligned with dogmatic teachings (Luhmann & Schorr, 2000, pp. 74–75) and differentiated according to rank and status. The notion that schools and families operate as sites for the systematic, intentional and still very religious education of the population became generalized around the sixteenth century in Europe. (Mangez & Vanden Broeck, 2020, p. 679)

Replacing the 'commoner' of stratified feudal society was an increasingly economic active working class and middle class, which allowed education to escape the authority of religion, i.e. the church. Education differentiates from the system of religion in the eighteenth century (Luhmann & Schorr, 2000). The impact of the differentiation of education is given little attention in educational research. However, Luhmann and Shorr articulate the changes in the organization of society and the consequences of the functional differentiation of society.

> A young person grows up in his own family without problem and has barely any transition when he enters life in the society in the same degree that his radius of action and his circle of friends expand. By being forced to attend school, however, he is confronted for the first time and suddenly with a society that is no longer negotiated by the family. But the school, being a special institution of a function system, is not a representative sample of societal life; it socializes for the school, not the society. The fact that the first contact with society apart from the family takes on precisely this form - one can think of it as a concentration of people of the same age in a relatively big interaction system - rather than another form, must have deep-reaching repercussions on the cognitive and motivational resources of societal life. And it is clearly impossible to level out this socialization-related imbalance through school curricula, social studies, sexual education, etc. (Luhmann & Schorr, 2000, p. 31)

Hannah Arendt made some similar reflections upon education in the United States in her 1954 essay, *The crisis of education*, especially the artificiality of the separation of childhood and youth from the adult world (Arendt, 1977). However, I agree with Luhmann and Schorr that this macroscopic view of education, its role and function in society, and the experience of individuals is given little attention in educational studies. What comes to the fore is the investigation of educational processes, both cognitive and social, within the system of education and the relationship between politics and education (i.e. policy making). A blind spot occurs as a result of seeing 'education' as a seamless component of society rather than its difference in the unity of the communications system of society. Assumptions can be made that if education is 'successful' in its mission, then it must be of benefit to society. This denies the complexity presented by functionally differentiated systems of society and points to some considerable questions for the sociology of education (and sociology of mathematics education). Too often do these fields become entrenched in moral rather than analytic accounts of society. Morality is too contingent to be the basis for any scientific endeavour.

Mathematics education is a subsystem of the social system of education, the overarching theme within education is the construct of the educability of the individual. In mathematics education, we are concerned with a particular set of knowledge, mathematical knowledge. Before concluding this chapter, it is necessary to give some attention to mathematics as a system of communication and which is closely connected to the system of science as the functional system of knowledge production.

6.5 The System of Mathematics

Through practices and programmes of empirical and theoretical work, science is engaged in a process of the production of knowledge – the identification of validity and the rejection of invalid claims. This permits a range of possibilities for science with a plurality of research approaches, extensive communication, and peer evaluation of the knowledge produced. The differentiation of science from society accelerated during the Enlightenment as human reason challenged religiously authorized determinations of what knowledge was. Mathematics has further specificity within a system of society; mathematical knowledge is generated through reasoning processes and proof rather than the more general empiricism of science. It is useful at this point to think about mathematics as a subsystem of science, but it is possible that there is a different systemic relationship with science. It could be argued that mathematics is a distinct system of communication that is structurally coupled with science. Whatever the relationship, I agree with Mehrtens, one of the few systems theorists of mathematics: "'Mathematics' means here the social system of the discipline with its central function of producing and disseminating knowledge of a specific character" (Mehrtens, 1993, p. 220). Mehrtens argues, based on an analysis of mathematics in Germany, that mathematics becomes functionally differentiated in the nineteenth century. He presents mathematics as a knowledge-producing social system like the system of science and that it characterizes itself similarly through the specificities of its knowledge with a core of 'pure' mathematics. Another important feature is the compelling nature of mathematical argument (Mehrtens, 1993).

> Mathematical knowledge is not simply a "parade of syntactic variations," a set of "structural transformations," or "concatenations of pure form." [...] Mathematical forms and objects come to be seen as sensibilities, collective formations, and world views. The foundations of mathematics are not located in logic or systems of axioms but rather in social life. Mathematical forms or objects *embody* math worlds. They are produced *in* and *by* math worlds. It is, in the end, math worlds, not individual mathematicians, that manufacture mathematics. (cf. Becker, 1982) (Restivo, 1993, p. 250)

Like Mehrtens, Restivo is emphasizing the systemic and communicative nature of mathematics as an autopoietic system.

6.6 Toward a System Theory Approach to Mathematics Education

This section poses the question, What kind of system is mathematics education? So far in what I have presented, mathematics education can be seen as an autopoietic social system of communication. Through its own internal operations and programmes, it responds to its environment and maintains its own operations and limits. While it is mathematics education that defines, in an on-going way, what mathematics education is, this is not as autonomous as one might first conclude, the self-referentiality is entirely in response to an environment that the system can only make sense of through its own operations.

Here, like the argument set out for the system of mathematics, I will treat mathematics education as a subsystem of education. As a coda, and before setting out a provisional research agenda for systems theory in and of mathematics education, I want to consider the nature of mathematics education as a functional system of communication in society.

A social system of communication is constituted on a phenomenological interpretation of meaning – meaning as the difference between actual and possible. This permits the actualization of experience but with a horizon of possibility – *what is* does not deny the possibility of it being anything else. The mathematics classroom is an important interactive system in the production of meaning, as it is in the classroom that conscious systems (the individual psychic systems of teacher and student/ pupil/ adult learner) encounter and are compelled to engage with mathematical knowledge. Psychic systems are in a constant process of making sense of environmental stimuli. The social system of mathematics education in the context of classroom interactions prompts communication. The process of experiencing, meaning-making and communication stimulate each other. However, there are particular sets of purposes, assumptions, and roles in the mathematics classroom, these include the purpose of developing learners' mathematical knowledge and capability, there is the assumption of (mathematical) educability and development, and the teacher is tasked with instruction and facilitation and the learner to learn (mathematics).

There is the important task of determining what knowledge is within the classroom context. The curriculum is an articulation of this knowledge. While the mathematics curriculum is an expression of the system of mathematics, it is distinct. Its development and expression reflect a range of influences from within mathematics education but are influenced by the system's environment. Environmental influences might include politics, as the process of policy making, the needs of the businesses and the economic system, as well as science (including educational research that I will discuss shortly) and mathematics. Luhmann and Schorr (2000) argue that the curriculum is strongly influenced by the contingency formula of educability, in other words what kinds of knowledge 'stick'. Contingency formulae are a means by which functional systems like education can deal with environmental complexity, they have their historical origins within religion but have continued as functionally equivalent into modern functionally differentiated system of education.

> An awareness of contingency, which accepts "this or also that" for a given condition is a form of generalization and for that reason necessitates a re-specification; because one cannot live with the notion of the equal possibility of anything and everything. (Luhmann & Schorr, 2000, p. 66)

The potential to be educated or educability is a key contingency formula within education, it deals with uncertainty by offering a process that can be actualized in the current context and allowing a surplus of possibility, albeit with somewhat prescribed possibilities set by historical practices. The mathematics classroom as a system of interaction within education has played a key role in defining and redefining what mathematics education is and what it isn't. Cuban (1993, 2009) has illustrated this point from the perspective of the historical analysis of classrooms in the United States and points to the continuity and contingency that exists over time. He also shows the limitations and frequent failure of attempts at teaching reforms in the light of the continuity of practice. This can be explained also by considering the contingency formula of educability and the historical continuity of educational practice.

While education became functionally differentiated in the eighteenth century, it was not until the twentieth century that there was a developed system of education research or mathematics education research. With the increasing number of participants in state education, the need to develop assessment and evaluate policy prompted (mathematics) education to reflect upon itself (see Howson, 2009; Kilpatrick, 2014; Inglis & Foster, 2018). Mathematics education, increasingly professionally staffed and higher education based, is 'coupled' with science through the use of theory and methodology. It is connected to but distinct from science, what distinguishes and closely integrates mathematics education research with the system of mathematics education is the object of research and that is mathematics education. While traditions of educational assessment and evaluation have been influential on educational research, mathematics education has drawn on psychology and mathematics, now there is a plurality of paradigms and perspectives. The most significant of these is the distinction between mathematics education research that is concerned with cognitive processes and research that is concerned with social perspectives. The former follows traditions of psychological research and in particular experimental approaches, the latter draws on social theory and approaches to sociological research in education more generally. The social turn in mathematics education (Lerman, 2000) employs a range of qualitative methods with theoretical perspectives that might draw on the sociocultural psychology of Vygotsky or in a more macroscopic sociology of mathematics education using Marx's theory of capitalist society.

The plurality within the system of mathematics education not only draws across the division of educational research and educational practice but also within alternative traditions and paradigms of research. Certainly, studies of mathematical cognition and sociocultural studies of mathematics education are distinct, and it seems like there is no language that would allow a synthesis between the two. However, systems theory offers the possibility of a meta-theory of consciousness and communication that although it may not permit an integration it provides an approach for the meta-analysis of these components within a totality of mathematics education and within the totality of society. It would seem likely that this is one possible way forward and a meta- language of systems addresses a fundamental problem in the imperative for society and its systems to be able to observe themselves. As William Rasch explains in the introduction of a collection of essays by Luhmann on theories of distinction, the challenge facing scholarship in general in late modernity:

> … the immanent, partial, and severed world, the posited world gradually achieves autonomy and takes center stage. What was once "the whole" or the nature of "all things" that could be seized in an instant and for all time as a totality now becomes an immanent field of observations, descriptions, and communications, a "totality of facts," as Wittgenstein wrote […] that must contend with the uncomfortable situation that any observation of a fact is itself a fact that can be observed. The whole that is modernity is the whole that strains to see itself and thus a whole that forever divides itself with every observation into more and

> more "facts." The whole we now deal with is a self-referential whole, thus an inescapably paradoxical one. Accordingly, we are no longer in the realm of a foundationalist "first" philosophy but rather in the realm of a "second-order" philosophy of observations of the observations of self and other. (Rasch in the introduction to Luhmann, 2002, p. 3)

While within education research this agenda is being addressed with a focus on, for example, school organization and governance (Andersen & Pors, 2021), policy and the second- order observation of education (Mangez & Vanden Broeck, 2020), inclusion (Qvortrup & Qvortrup, 2018) and curriculum change (Hilt & Riese, 2022), the challenges of the plurality of mathematics education for both research and practice begs a development of this initial systems theory analysis, to understand the plurality of mathematics education and, importantly, its relationship to society.

Systems theory may be useful beyond the sociological and the social psychological. It is possible that the study of mathematical cognition may benefit from a systems perspective, especially in relation to the conscious processes of distinction and recursion as the basis of the construction of mathematical objects and processes. It might be that studies of the learning of early number (see Gilmore et al., 2018) reveal more about the development in the sophistication of distinctions made by psychic systems in response to symbols and communication. Experimental studies of young children and number seem to suggest the observation of distinctions as much as it does as the development of operational processes and symbolic representation.

6.7 Concluding Remarks

Here, I have aimed at introducing (the social system of) mathematics education to contemporary systems theory by outlining some features of the theory itself and the directions and themes which I am taking up in my preliminary inquiry into mathematics education. I began contextualizing this in terms of the paradoxes that contemporary society increasingly makes evident, this is captured in the Luhmannian notion of the paradox of difference in unity. Society represents a difference in unity and so does the system of mathematics education in terms of a systemic treatment. Rather than suppress the paradoxical reality, systems theory begins with an unfolding of paradox, accepting that society and cognition stabilizes self-referentially, autopoietically, and improbably through evolutionary processes of variation and selection and through societal differentiation.

The question arises from this then of what use is this to mathematics education? While my purpose here has been to outline a theory of social systems consistent with volume's intent to present works in progress in relation to the philosophy of mathematics education, a central issue for mathematics education is the improvement of mathematical capabilities of individuals in order that they can sustain themselves and contribute to society. Systems theory is not the first theoretical approach that provides a critique of the instrumental and sometimes reductive accounts of individual mathematical learning. However, it does not reject the cognitive aspects of learning. It does offer a way of understanding the relationship between the social (as communication) and cognitive (individual psychic systems). Again, these distinctions can be integrated within social psychology, socio-cultural, and Vygotskian derivatives such as cultural-historical activity theory or the anthropological theory of didactics. What systems theory facilitates is understanding of the social and cognitive dimensions within a theory of society and from this a theory of mathematics education as a social system of communication. Reflections upon mathematical learning and in relation to human development often take for granted that learning processes, pedagogy, curriculum and that represent foundational substrates. Systems theory treats these aspects as constructs, and while it does not reject them, it asks about the social and contingent basis in which these constructs became enduring systems of meaning. In this way, systems theory provides a more precise way of dealing with the concepts that we use in studying and reflecting on mathematics education.

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Chapter 7
On Mathematical Validity and Its Human Origins

Gerald A. Goldin

7.1 Introduction: Essentials for a Philosophy of Mathematics Education

For a philosophy of mathematics education to have full integrity, it should address several fundamental issues. One of these is the nature of objective mathematical truth and validity, in all its complexity. A second one is the human origin of mathematics: its development and application, its teaching and learning, in reasons for existing – all taking place in environmental, cultural, social, cognitive, and affective contexts. Many eminent philosophers have addressed particular aspects of these two issues in great depth; a review is well beyond the scope of this chapter. Often one of the two is taken as fundamental – explicitly or tacitly – with the consequence of overtly denying or dismissing the other entirely.

Here, I maintain that they are not contradictory. This is why I argue for an integrated philosophy incorporating both, rather than a diversity of mutually opposed philosophies. Both objective mathematical truth and validity, and the view of mathematics as invented and developed by human beings, have philosophically sound underpinnings, and both must emerge as central to the teaching and learning of mathematics. However, the ways they most commonly enter mathematics education now are not necessarily optimal or desirable. This chapter explores some of the interplay between them – the senses in which mathematics is socially and culturally dependent and the senses in which it is true and universal.

A related issue might be termed the "why" of mathematics. We have several important interpretations of the term "why": What does it mean to understand a mathematical concept? Why does a pattern occur, or why does a theorem hold? (e.g., Davis, 1992). Why have we formulated various axiom systems of

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mathematics as we have? Why have we defined various concepts, categories, and objects the way that we have? Why have we named them as we have? Why do we represent mathematical constructs with the conventional symbol systems that we use? What is the wider purpose of a concept? Why should students be required to learn various topics? Moreover, and perhaps most importantly, why might a student want to engage at all with learning or doing mathematics (Middleton & Jansen, 2011)?

I hope to offer some perspectives on the above issues to which I have come over many years, as both a mathematics education researcher and a mathematical physicist. I hold a personal philosophy of mathematics education, reflected in the views I espouse here, but it is not an "ism." Rather, I take a more eclectic perspective, drawing on well-known elements of various fashionable and less-fashionable schools of thought in the philosophical, psychological, and mathematics education literature, while not adopting the limiting assumptions behind any one of them. Sometimes I have found this perspective dismissed categorically by adherents of one or another school – often for a priori philosophical reasons – because adherents value certain features of mathematics learning very strongly, while devaluing others (Goldin, 2003, 2008). However, my purpose here is not to propose one new, eclectic philosophy. Rather, I want to strengthen the case that any intellectually sound philosophy must base itself on foundational pillars that support the centrality of both objective mathematical truth and its psychologically and culturally situated human origins.

Academic philosophizing about mathematics education does not take place in a vacuum. The philosophical pendulum in mathematics education has swung repeatedly in my lifetime: back and forth, between radical positivism and radical constructivism, between behaviorism and radical social constructivism, between powerful absolutist and extreme relativist perspectives. Research and evaluation studies offering justifications or critiques of educational policy and practice are typically situated within one or another fashionable and influential philosophical school of thought, characterized as a "theoretical framework." Some variables and constructs are then centralized, while other important ones can be dismissed outright and disregarded. In my view, the sociology of the academic world in education favors proponents of extreme claims and powerfully dismissive philosophies.

Partly as a consequence, energized by political forces, educational policies and practices have veered rather wildly. In the United States, we moved from "old math" to "new math" to "back to basics" and have oscillated between "traditional" and "reform" curricula and teaching methods. After much advocacy of "alternative assessments," we are presently in an era of mandated standardized testing – and declining scores.

Each of the aforementioned philosophical "isms" is based on some unquestionably valuable idea – some essential feature of mathematics, its teaching and learning, or a related educational research methodology. Many careful and valuable studies have been carried out by researchers energized by their belief in one of the "isms." I propose that we build thoughtfully on such research. Nevertheless, we must also recognize the damage that occurs when a philosophical system claims universality and exclusivity as a founding principle, dismissing concepts that do not fit – based not on evidence of their inapplicability, but on a priori argumentation.

To be specific, behaviorists – with a philosophy rooted ultimately in twentieth-century logical positivism and radical empiricism (Ayer, 1946) – rejected discussions of mathematical understanding as "mentalistic," allowing *only* the meaningfulness of straightforwardly observable outcomes – e.g., defined performance objectives or test scores (Skinner, 1974; Sund & Picard, 1972). Radical constructivists – with a philosophy rooted ultimately in the seventeenth-century idealism of Bishop George Berkeley (Downing, 2021) – dismissed objective truth and validity as admissible topics of discussion, declaring the *only* acceptable focus to be the viability of a concept for an individual or (in the case of radical social constructivism) for a social group. Thus, correctness in mathematical reasoning and the validity of science could all be reduced either to the individual's "experiential reality," or else to social consensus (Confrey, 2000; von Glasersfeld, 1996).

These philosophical belief systems—behaviorism, radical, and social constructivism—only implicitly addressed the foundations of mathematics per se. They pertained most directly to its learning and teaching, especially how we define, assess, or evaluate learning processes and outcomes. I have termed them "dismissive epistemologies." Within each, one finds both centrally important ideas and serious philosophical flaws (e.g., Slezak, 2000; Goldin, 2003). I want to advocate here for a far more inclusive approach to the philosophy of mathematics education. Its epistemological foundations belong in empiricism, rationalism, socio-culturalism, pragmatism, and ethical philosophy, broadly construed. It should be a philosophy addressing human needs through objective mathematical truths and objectively valid applications. It should also be a philosophy taking full account of the human origins of mathematics. It should be able to account for truth and validity through features that distinguish mathematics from other, less objective domains of human inquiry.

I think such a philosophy is not only tenable and defensible, but necessary if we are to achieve excellent and universally accessible learning in mathematics.

At the outset, let me highlight some limitations of this chapter. Most of the ideas presented are well known and have been explored in depth by philosophers, mathematicians, and educators. Consequently, the citations are necessarily far from adequate. It is the integration of these ideas into a comprehensive, practical philosophy for which I argue. We need a philosophy that can drive our research and our educational practices progressively and productively, rather than fostering wild swings of fashion.

In the next section, I discuss sources of validity and truth in mathematics and the meaning of these ideas in the human context of their development. In so doing, I offer responses to certain frequently raised objections to such constructs. I also suggest some roles they can best play in excellent mathematics teaching. The discussion also delves somewhat more deeply into the notion of fallibility in mathematics (e.g., Ernest, 1991).

The third section explores and elaborates on the "why" of mathematics and the importance of many interpretations of this "why" in teaching and learning. Here, I distinguish misconceptions from alternate (valid) conceptions learners may have, and stress the essential role of errors and misconceptions in powerful mathematical problem solving. The concluding section of the chapter relates the study of mathematics directly to the meeting of fundamental human needs, suggesting directions to modify our educational priorities accordingly.

7.2 Sources of Objective Truth and Validity in Mathematics

7.2.1 Social Constructivism

Mathematics as we know it was formulated by human beings, in order to meet human needs. It is likely that mathematics originated in prehistoric times, through its creation for various practical purposes. Counting probably dates to well before recorded history. Measures of length, area, volume, and time served numerous goals that took different forms in different ancient civilizations. Stories and myths surrounded the sun, the moon, the planets and the stars, whose observed regularities formed the basis of human calendars and beliefs. Various increasingly elaborate systems of representation developed differently in different cultures, describing numerical and fractional quantities as well as geometrical constructs.

Human beings noticed patterns in the systems they created and asked questions about such patterns. Do numbers go on forever? Some groupings of object, described by numbers, cannot be arranged in rectangular arrays. Do such "prime" numbers go on forever? Assumptions were systematized and reasoning formalized, always by human beings, and in different ways by different people in different cultures. The questions asked were socially and culturally situated, as they continue to be today. Isn't then mathematics entirely a human, culturally dependent social construct?

Of course, it is, as is the product of every other human activity. To say mathematics is a human construct merely places it alongside all science, all myth, all religion, all beliefs, all language, all literature, all culture, all economics and politics – as well as all farms, all factories, all dwellings, all buildings and bridges, all clothing, all objects created through arts and crafts, etc.

The statement adds little or no actual information. However, it focuses the mind in a certain way. It encourages us to attend closely to important questions that should influence or be explicitly addressed in mathematics education, for example:

- How does and can mathematics meet human needs?
- Through what processes do people create and study mathematical ideas?
- Through what historical processes has mathematics developed?
- How and why do different cultures vary in the mathematics they create and use?
- What are the psychological processes behind mathematical learning, problem solving, inventiveness, and creativity?
- What culturally dependent aesthetic features encourage appreciation of beauty in mathematics?
- How do our policies and practices in mathematics education facilitate or impede mathematical engagement and the learning, appreciation, and effective uses of mathematics by our students?
- How can we best engage in culturally relevant mathematics teaching?
- How do power, politics, racial and gender discrimination, ethnocentricity, and economics shape the development of mathematics as a field, establish hegemony, and limit who has access and whose ideas gain notice?

These are essential questions. To the extent that the statement, "Mathematics is a human social construct" helps us focus on them, it is a helpful foundational pillar for our philosophy.

In some philosophical approaches, however, the statement takes the unfortunate form, "Mathematics is *just* [or *only*] a human social construct." Here, the not-yet-stated implication is that mathematical truth or validity is fundamentally indistinguishable from the truth or validity of any other human belief system – science, religion, myth, superstition, or individual idiosyncrasy of conception or misconception. All are *merely* sociocultural or psychological constructs, to be kept or discarded according to their social, cultural, or personal viability for the group or individual that uses them. An approach that begins by denying the very possibility of characteristics such as truth and validity – features that ultimately *distinguish* mathematics from other forms of human activity – should not be the basis for a widely accepted philosophy of mathematics education. A detailed critique of radical social constructivism is offered by Slezak (2000).

7.2.2 *Objective Truth in Mathematics*

Platonism, logicism, and formalism (Horsten, 2022) Mathematicians and philosophers have taken various approaches to the nature of mathematical truth. Some of these are also tacitly or overtly dismissive. In a Platonist perspective, the objects described by mathematics (number, geometrical objects) exist in a separate, timeless world of ideal forms. Their properties – the truths of mathematics – are eternal and independent of human beings, with the possibility of being (partially) known through processes of thought. The school of thought known as logicism, associated with Gottlob Frege, Bertrand Russell, and Alfred North Whitehead (but with roots going back to Euclid), sought to establish the objects and truths of mathematics, including sets and numbers, as derived purely through logical deduction from initial axioms. In the formalist philosophy advanced by David Hilbert, mathematics consists of symbols and consistent rules for manipulating them – without their holding any *intrinsic* meaning at all. Kurt Gödel succeeded in showing, however, that such formalist or logicist systems, if sufficient to include infinite sets, could never be proven complete or consistent – except by assuming rules of inference that themselves could not be proven complete or consistent (Nagel & Newman, 1958; Hofstadter, 1979). Gödel himself appears to have been a Platonist (Goldstein, 2005).

Intuitionism and Lakatosian philosophy In contrast to these approaches, the intuitionist school of philosophy, credited to Luitzen Egbertus Jan (L. E. J.) Brouwer, begins with mental constructs and the mathematical properties that are evident from those constructs; all the rest of mathematics that is meaningful should be obtained from these intuitions through constructive processes. Nonconstructive mathematical principles are thus inadmissible. Lakatos focused on the actual practices of mathematicians, who arrive at a consensus about mathematical truths (e.g., the correctness of a published proof) through discussion and explanatory discourse, not through strict application of a purely symbolic formalism. One sees in intuitionism and Lakatosian philosophy some roots of radical constructivist and social constructivist philosophy, respectively, that are specific to mathematics and an influence on humanist approaches to its teaching (e.g., Davis & Hersh, 1981).

I maintain that a philosophy of mathematics education does not require exclusive commitment to a specific philosophy of mathematics per se. In fact, to the extent that Platonism, formalism, logicism, intuitionism, Lakatosian philosophy, or other approaches are exclusionary of different perspectives, it is far better to rest the foundations of our philosophy on multiple pillars. When our educational goal is to convey a sense of awesome, timeless beauty, Platonist ideas may help us do so. We can then focus on what is universal in mathematics, the constructs that seem to transcend cultures and eras. When our goal is to explore logical deduction, theorems and proofs, a logicist perspective provides a deeper understanding of the power of mathematics.

When we seek to convey an understanding of non-Euclidean geometry, we may point out that in a formalist perspective, we are free to interpret the terms in Euclid's postulates (such as "point," "straight line," and "distance") as we wish; only the relationships stated in the postulates matter. Thus, the term "point" might be interpreted as a pair of antipodal points on the surface of a sphere; "line" might stand for a great circle; and "distance" might refer to the length of a segment of the great circle. Then a "straight line" is still the shortest "distance" between two "points" – but the parallel postulate no longer holds.

Focusing on mental imagery and constructive processes in the psychology of mathematics learning centralizes an intuitionist perspective, and we have already discussed the opportunities inherent in a social constructivist perspective. Indeed, there is great educational value in attending to all these dimensions of mathematics, without taking any one to be its "real" or only foundation. The source of truth that becomes relevant depends on what question we want to ask or what issue we want to address. The various foundational sources are complementary, not contradictory.

There are two main ways to approach the objective character of mathematics: through the idea of *empirical* truth or that of *rational* or logical truth.

Empirical truth When we create a mathematical system of representation, together with its definitions, rules, and procedures, we often do so to describe a class of situations and to make inferences and predictions about them. If the mathematics we are using is *valid*, these will be borne out by subsequent information and events; otherwise, our predictions will at some point fail. Here, the empirical validity of the mathematics depends on at least two aspects: (a) the applicability of the model used to describe the situation and (b) the internal consistency and correctness of reasoning within the mathematics.

(a) It is not difficult to find examples of (objectively) correct mathematics applied inappropriately to reach (objectively) false conclusions. For example, causation is often inferred falsely from correlation. If the conclusions serve (viable) political or social purposes, as is often the case, they may come to be widely accepted. Such social consensus does not suffice for their truth and should not suffice for us to believe they are true. Nor should we settle for saying they are "true for this group" while "untrue for that group" – a stance wholly lacking in intellectual integrity. Mathematics education should provide students with the *power of knowledge* – the ability to distinguish valid from invalid mathematical applications or models.

(b) A different failure of empirical validity may occur if a mathematical conclusion reached within the mathematical system is itself (objectively) false. An incorrect computation may be due to a clerical error or to a more fundamental misconception. This may lead to a recipe that fails, a loan that is disadvantageous to the borrower, or furniture that fails to fit in the planned space. The prediction fails empirically because the mathematics used to generate it contradicts mathematical truth.

The property of being a human construction offers no protection from objective empirical truth or falsity—a traffic bridge is a human construct, but whether the bridge stands or falls is objective fact.

Rational truth Having created a mathematical system (e.g., a system of counting) with accompanying ways of representing it (e.g., spoken and written numbers), we can proceed to axiomatize it—to identify the rules governing our creation and our reasoning about it. Then truth can be defined entirely in relation to the assumed system of axioms and rules of inference. A vector, for example, is no longer simply a way to represent something in real life, such as a velocity, by "a directed line segment" or as "a quantity with both magnitude and direction" —an interpretation suggesting human intuitions with potential empirical validation of the mathematics of vectors. Instead, a vector becomes "an element of a vector space," where the vector space is a set defined in relation to another set, called a scalar field. Both sets are endowed with operations that obey certain axioms. Now the "truths" about vectors and scalars are the axioms, true by virtue of their assumption, together with theorems provable from those axioms through well-defined, logical reasoning processes. To characterize mathematical truth this way – i.e., through logicism – is to place it in the domain of what philosophers have termed "analytic" truth (e.g. Ayer, 1946) – statements that are true a priori by virtue of meanings or definitions, requiring no empirical verification. It appears that the capability of such reasoning, sometimes called "logical thinking," is common to human beings in every culture.

It is virtually non-existent in mathematics (though the situation is spectacularly different in science) that different social or cultural groups engaged in mathematics come to believe in systems of truths that actually contradict each other.[1]

[1] However, there have been historical periods of disagreement, followed by radical shifts in belief among mathematicians, over some metamathematical issues, for example: whether Euclid's postulates are "self-evident truths,", whether negative numbers "really exist,", or whether mathematics can be formalized in a complete and consistent way (e.g., Kline, 1980).

Does this agreement across cultures alone demonstrate that mathematical truth and validity are indeed universal and eternal? Of course, in a certain sense it does. The convergence of belief, attributable to the objective truth of mathematics, has probably encouraged an "absolutist" philosophical stance to thrive across the millennia. Paradoxically, mathematical consensus seems also to have allowed the more recent opposite conception to thrive, that rational truth in mathematics has no objectivity beyond the fact of a powerful social consensus of mathematicians. Of course, radical social constructivism fails spectacularly in empirical domains when consensus is absent. In the biological sciences today, in this time of pandemic, substantial numbers of powerful people adhere firmly to and promulgate the (objectively false) belief that vaccination is ineffective and dangerous – with deaths occurring as a consequence of the false belief system. A philosophy of mathematics education that has integrity should be wholly compatible with a sound philosophy of science, where mathematics finds both important applications and sources of inspiration. Denial a priori of the very possibility of rational objectivity, distinguishing mathematics from other forms of human activity, should not be a feature of our mathematics education philosophy.

7.2.3 Objectivism

Formalists and logicisits attribute mathematical truth to logic alone. From a purely formalist point of view, the structures of mathematics and the structures of games such as chess or checkers are quite parallel. Once the set of rules is assumed, the question of whether with best play chess will always end in a draw has just one true answer, now and forever. This does not, of course, contradict the human origin of chess – its rules are quite evidently socially constructed, without any necessity whatsoever, and clearly culturally situated.[2]

Platonists, on the other hand, ascribe to abstract mathematical structures a kind of independent, eternal existence in a realm of ideals that we access but partially. In this philosophical viewpoint, perfect circles, squares, and regular polyhedra ("Platonic solids") have always existed, apart from human beings—but then, presumably, so must have far more exotic (and equally beautiful) constructs, such as the Sierpinski triangle, the Mandelbrot set, and the quantum plane.

"Absolutist" philosophical systems – embracing the idea that there is only one mathematics, that it is eternal and universal, transcending not only all cultures, but humanity itself – may encourage us to focus on aspects of mathematics education

[2] Of course, chess is a finite game and thus such a question has a definite answer—though we may not know now what the answer is. In contrast, mathematics involves infinite sets, where the work of Gödel demonstrates that with a given set of rules of inference, not every proposition can be resolved as either true or false. The notion of "truth" extends beyond what is provable with the specified rules of inference.

that are valuable, but different from those suggested by radical constructivism, for example:

- How can we develop students' skills so that they reason validly and solve problems powerfully and correctly?
- How can we foster students' processes of abstraction and their understanding of abstract structures and proof in mathematics?
- What role should axiomatization, logical reasoning from axioms, and theorem proving play to enable students to grasp both the generality and the certainty of mathematical results?
- How can we best assess, objectively, when and in what ways students have acquired valid conceptual understandings and are proficient in performing correct computations in arithmetic, algebra, and calculus?
- How can we incorporate powerful and valid applications of abstract mathematics most effectively in the curriculum?
- Observing that people in all cultures are capable of logical reasoning, can the resulting universality of mathematical truths help to bridge cultural gaps across the world?
- How can we educate students to appreciate the beauty to be found in abstract mathematics, to find fascination in the mathematical objects in a Platonic world of ideal forms?

These are essential questions to ask. Absolutist philosophies are helpful to the extent that they remind us of the importance of correctness, valid conceptions, and the power that comes with understanding mathematical truth and its sources. They lead us to the power of abstraction in mathematics and the idea of a structure that may apply to or describe a variety of very different concrete situations. As mathematics educators, we should want students to understand the concepts of structure, isomorphism, homomorphism, etc., as well as the idea that features of mathematical structures may be established via theorems proved by logical deduction from definitions and axioms. Our goals should include helping students acquire some facility with reasoning and proof – with what is often termed "logical thinking."

Mathematicians tend to favor approaching objective truth rationally and analytically, through theorems and proofs. To create mathematics, researchers may identify a pattern, investigate it, formulate a conjecture, and then try to prove it or to demonstrate its falsity through counterexample or by proving its negation.

Applied mathematicians, physicists, statisticians, and other scientists may approach mathematical truth empirically. To create mathematics, researchers invent a mathematical object or method, motivated by a situation, to model the situation. It may be that the invented object does not even "exist" (yet) mathematically, but using it seems to work. Rules motivated by the situation are applied – and only later is the object or method defined precisely, the rules axiomatized, and theorems proven.

But as we see, formalism and Platonism are also open to some fundamental philosophical objections. The limitations of formalism in fully establishing foundations for mathematics have been quite rigorously and beautifully discussed (Hofstadter, 1979). In mathematics education, formal theorems and proofs alone may convey little or no understanding of the *meanings* or *ideas* behind the formalized mathematics. And the reasons outside of pure mathematics behind the choices of systems to axiomatize, from plane geometry to the natural numbers and onwards, are not addressed.

Alternatively, the Platonist assumption of the independent existence of mathematical truths implicitly accords extra ontological status to *all* possible systems of assumptions and sets of inferencing procedures: not only games like chess and poker, number systems, and non-Euclidean and noncommutative geometries, but also all not-yet-created games and mathematical categories. All must "exist" as ideal forms, independently of their ever having been imagined or invented. While this is a fascinating mental image, taking such an assertion as one's only starting point – like starting with the assertion that mathematics is "only" a culturally situated human construction – focuses attention, but erases distinctions and conveys little or no actual information.

Absolutism becomes damaging when it takes an unfortunately narrow and authoritarian form. The assertion that there is one universally true mathematics is sometimes tacitly assumed to rule out alternate (but valid) conceptions. Algorithms that are nonstandard in one culture but standard in another, as well as potentially fruitful and valid paths of inquiry that do not follow canonical ways of thinking – i.e., *alternate* conceptions, as opposed to misconceptions – can be unfortunately rejected by the teacher who "knows" what mathematics is "supposed" to be.

Authoritarian absolutists may also greatly devalue misconceptions, mistakes, and blind alleys. An exclusive focus on correctness can lead teachers to dismiss or overlook the substantial value to learners that can result from misconceiving something or making a mistake during problem-solving activity. Each occurrence of an error provides a singular opportunity for students to discuss it, to come to understand why the conception fails or why the error occurred – and an opportunity to strengthen problem-solving processes that reveal the error and enable more profound understanding. It is also an opportunity for the teacher to congratulate the student for opening the door to deeper, nonroutine insights for everyone.

7.2.4 Fallibilism

Fallibilist philosophy in mathematics education takes mathematics to be socially constructed but goes further to argue that mathematical truth itself is forever open to question (Ernest, 1991). Indeed, the field of mathematics has not evolved historically through "progress" alone – the definition of new constructs, the formulation of new axiom systems, the proving of new theorems, and the discovery of new applications. Substantial changes in mathematical definitions, notations, concept, proofs, and consensus perspectives on these have all taken place, to the extent that some statements once regarded as true would now be taken as false or ill-defined. Fallibilism has value for mathematics education by focusing special attention on specific processes of change – those that have modified our notions about what is true or certain (e.g., Kline, 1980). It is also suggestive of the necessary, positive role that false starts, blind alleys, and misconceptions play in powerful mathematical problem solving as they occur and are addressed productively.

In its most extreme form, fallibilism has been interpreted to suggest that mathematical truth per se does not exist – that every mathematical assertion is subject to eventual falsification. Thus, not only is authoritarianism rejected – the very basis for authority in mathematics is called into question.

It is helpful in teaching mathematics, and in constructing a philosophy of mathematics education, to explore the social and historical processes that have led to changes in mathematics. In doing so, I think we can identify certain specific, distinct sources of change that have affected, or that hypothetically would affect, the truth of mathematical statements. Knowing these means that we can understand their role – and we need not, and should not, dismiss the very notion of objective truth and validity in mathematics because of them. Instead we can offer students, at appropriate points in their mathematical development, an understanding of how change in mathematics has come about.

Changes in the meaning of truth One kind of change has been in our notion of what meaning we attribute to mathematical "truth." Early Greek geometers regarded Euclid's postulates as "self-evident truths." That is, postulating them was not to merely create a label for them as true, but to state as fact an already-existing truth. Thus, Euclid's "parallel postulate" was understood as an "eternal truth of mathematics," albeit less self-evident than Euclid's other postulates.

With the development of non-Euclidean geometries, we came to a different understanding – that the truth of axioms and postulates, and the resulting theorems, holds only within the system where they are defined. Euclid's parallel postulate is false in Riemannian and Lobachevskian geometries. And the empirical success of general relativity suggests that the geometry of our universe is in fact non-Euclidean.

Another change of meaning has been due to the work of Gödel already mentioned. It was formerly thought that the truth of a mathematical statement could be taken as synonymous with its provability from the assumed axioms using well-defined processes of inference. But this idea did not hold up – in a system that allows for an infinite set, such as the natural numbers, statements must exist that are true but unprovable.

Changes in representational conventions Many "true" statements in mathematics are merely facts about currently agreed-upon systems of representations. For example, we agree conventionally that numbers grow larger toward the right on the number line and as we move upward on a vertical axis. Such a "truth" is quite obviously arbitrary, though we can suggest psychological and socio-cultural reasons for it. We agree conventionally to perform multiplication and division operations before addition and subtraction, so that the "correct" value of an expression such as "8 + 12/4" is 11. In teaching, this rule is sometimes called the "fundamental order of operations," though there is nothing "fundamental" about it. Here, "truth" changes easily. In a different context, where the convention is to perform operations in order from left to right, the "correct" value of the above expression would be 5.

The meanings attributed conventionally to mathematical expressions are malleable. Before the advent of algebra, and in some contexts now, $3x$ might suggest a 2-digit number with "3" in the "tens" place and an uncertain "units" digit, rather than 3 times x. Depending on the context, dy may refer to a constant d multiplying a variable y, to an infinitesimal change in the value of y, or to an element of a cotangent space. The truth of statements relying on such contexts is changeable.

Changes in the meaning of constructs The definitions that come to be accepted in mathematics also change over time. Earlier conceptions of continuous curves would not have accommodated the possibility of a function $f(x)$ continuous at all irrational values of x, but discontinuous at all rational values. In the time of Newton, when calculus was being invented, the conception of continuity most widely held did not accommodate functions that are everywhere continuous but nowhere differentiable. Our concept of continuity now permits such constructs. As meanings change and evolve, new "truths" supplant earlier ones.

As concepts in mathematics are generalized, statements earlier regarded as "true" are true no longer. The common "misconception" in children that "Multiplication always makes numbers larger, never smaller" is true, when "numbers" are interpreted solely as natural numbers 1, 2, 3, … (and the reference is to numbers other than 1). When "numbers" are generalized to include fractions, and multiplication is correspondingly generalized, the statement becomes false. Similar observations apply to "misconceptions" such as "You can't subtract a larger number from a smaller one," and "there is no square root of -1."

Changes in language As concepts evolve, new constructs are introduced, and as new applications of mathematics found, our language changes. New terms are introduced, and old ones change their meanings. New contexts occur, and natural language interpretation depends heavily on context. Consequently, "true" statements with one set of meanings or in one context may, with different meanings or in other contexts, be "untrue." In this sense, *statements* in natural language of mathematical truths can never be absolute or eternal.

Mathematics education and mathematical communication always occur through natural language as well as symbols, and teachers typically evaluate students' understandings based on the perceived correctness of their statements. Here, we must all be sensitive to the possibility of our own "fallibility."

The hypothetical possibility of consistent human error All of us occasionally make mathematical mistakes and oversights. In any given session of mathematical thinking by a particular person or group, the probability of error is nonzero. The probability that the error remains undetected for a period of time is smaller, but still non-zero, as is the probability that it is never detected. Theoretically, then, the probability must be nonzero that all human beings have consistently made the same mathematical error, in computation or in reasoning, in studying a mathematical result and in determining its truth. One may enjoy trying to estimate an order of magnitude for its value, expressed in negative powers of 10. I think the smallness of the estimate leaves us with no practical implications for mathematics education, except perhaps for offering students an amusing exploration in probability and the meaning of "practical certainty."

Radical fallibilism as the foundation for a philosophy of mathematics education has been critiqued in detail elsewhere (e.g., Rowland et al., 2010).

7.3 The "Why" of Mathematics

In a way, "The Why of Mathematics" could have been the title of this article. Ultimately, a philosophy of mathematics education should guide us toward understanding this question of "why" in its many different interpretations.

Patterns A teacher may ask her students to explore why a pattern occurs—e.g., Why does taking the differences between the successive square numbers *1,4,9,16,25,…* result in the sequence of successive odd numbers *3,5,7,9,…*? One possible answer is via an algebraic calculation: $n^2-(n-1)^2 = n^2-(n^2-2n+1) = 2n-1$. Such a computation is at the heart of a formal proof – one possible way to answer the question. But the resulting understanding, essentially formalist, is limited.

A different kind of answer is provided by discovering a way to partition a concrete set of chips arranged in a square (say *5* by *5*). One can form a smaller square (*4* by *4*), together with one smaller row and column (each with *4* chips) and *1* chip sitting in the corner. So we *display* the difference between the two squares (twice *4* plus *1*, or *9*). This can be compared with the algebraic result (for $n = 5$, twice *5* minus *1*). Answering the question "why" by connecting multiple representations is a path to mathematical truth not suggested by formalism but highlighted in constructivism – and it leads to deeper understanding of a "true" reason for the pattern.

Definitions Another teacher may ask his class to explore why something is defined mathematically the way it is. For example, why do we define $n^0 = 1$ (for $n = 1,2,3,\ldots$) after we have defined n^k as the product n times n times n … (k times). Why should multiplying a number by itself *0* times be equal to *1* and not *0*? And in the case of a negative exponent, why should n^{-k} be defined as the reciprocal $1/n^k$?

Here, answers to both questions eventually take the form of our wanting to *extend* certain "laws of exponents" so that they will hold not only for positive exponents, but for all integer exponents—a great idea! Does n^{-k} "really" equal $1/n^k$? Obviously, this definition is a socially agreed-upon convention, and in that sense, it is not “absolute truth”. But just as obviously, there is an important *reason* behind the convention, which is characteristic of the development of mathematics and behind its power to encompass more objective truths. The understanding of the learner who discovers this is enhanced.

Arbitrary conventions Why do we have the arbitrary conventions in mathematics that we do? For example, why is “ten” the base of our system of numeration? Why then does 12 appear importantly in the English system of measurement? What led to the metric system replacing it in many places? Why do both 12 and 60 play such important roles in our measurement of time? Answers to such questions are *both* historical and intrinsically mathematical. I have mentioned earlier conventions about the order of arithmetic operations and about the directionality of numbers on coordinate axes. The “why” behind these also leads to interesting discussions.

Exploring such topics enhances students’ powerful understanding of mathematics, and our philosophy of mathematics education should encourage *every* aspect of their exploration. Thus, it should be possible to identify and distinguish what is true and mathematically relevant (e.g., that 60 is divisible by 2,3,4,5,6 and 12), what is historical (e.g., Babylonian mathematics and Ptolemy’s approach to astronomical measurement), what may be guided by universal human experience (e.g., that we have ten fingers, or that larger quantities form a heap that increases vertically), what is cultural (e.g., that we read Western languages from left to right), and what is simply convenient or arbitrary (e.g., the order of performing arithmetic operations) – and to discuss the interplay among all of these.

Applications and models Why do we need numbers, measures of length, area, and volume, algebraic equations, geometric constructions and theorems, probability and statistics, trigonometry, differential and integral calculus, abstract algebra, etc. at all? What led to their construction? The empirical origins of particular mathematical topics are often mentioned in passing, but deserve to be explored by students in depth. For example, can the idea of a “unit square” be used to define the area of a rectangle? A triangle? A circle? How can we do so, in order that area measures fulfill the applications we have in mind? At a more advanced level, why do the axioms defining a mathematical group include the associative property, but not the commutative property? Consider what axioms allow us to describe the permutations of a set or the symmetry transformations of a 2- or 3-dimensional object. Why might we want to?

Here, exploration can lead to a deeper understanding of both the mathematics and the motivation for the mathematics as representing a situation. If area measure should describe how much paint is needed to cover a floor of irregular shape, we expect that two triangles forming a rectangle should require twice the paint needed for each triangle. If one approximates a circular area of given diameter with triangular wedges, one can discover the area formula $A = \pi r^2$. It is mathematically true in Euclidean geometry, while we have just seen how the very notion of area is socially constructed for many practical purposes. Again, a comprehensive and valuable philosophy should encourage both.

Reasons to learn mathematics Why should students study mathematics at all? Why should they have to learn ratio and proportion, descriptive statistics, the quadratic formula, or trigonometric identities? Students are typically offered a variety of reasons for the importance of their study. These include areas of application in everyday life, the importance of STEM to scientific and technological progress or to national goals, course prerequisites for college admission, the many desirable career opportunities that mathematics may open for them later in life, and our belief that studying mathematics will help them learn to think logically about many other things, too. Some of these answers are sustainable for particular topics, while others are patently untrue. Efforts to modernize the mathematics curriculum typically focus on the immediate practical importance and the long-term utility of the content – important criteria. For example, statistics and probability greatly exceed trigonometry in these respects.

But there are other important criteria too that deserve consideration. Some students may want to become mathematicians or physicists, while most will not. Some students are mathematically talented and others less so – but mathematical talent is not synonymous with mathematical speed. The in-the-moment psychological needs and satisfactions of students are often disregarded or misconceived, resulting in unresolved frustration and the widespread incidence of "math anxiety" among adults. I discuss this further in the next section.

A comprehensive philosophy of mathematics education should provide a foundation for embracing all of these important reasons to study mathematics – remaining open to many diverse sources of motivation, while valuing those that foster engagement and learning.

7.4 Mathematics and Fundamental Human Needs: Conation and Mathematical Engagement

The term *conation* refers to the domain of human psychology pertaining to drives, needs, desires, "will," or choice – the "why" of all that we do (e.g., Snow et al., 1996).

Elsewhere, my colleagues and I have considered diverse in-the-moment desires that motivate students to engage in mathematical activity or to disengage from it (Goldin et al., 2011). The approach that I advocate is to offer students the *experience of fundamental needs being met* throughout their mathematics learning, from kindergarten through higher education (Goldin, 2020).

There are several ways in which mathematics can authentically fulfill fundamental needs "in the moment" without relying on applicability to daily life, extrinsic rewards, or the promise of achieving distant goals. Here, I suggest five: through *aesthetic* experience, the experience of *power*, the gaining of *insight*, the reward of social *connection*, and *self-expression* through creativity. Each of these addresses a domain of basic human need. Our philosophy of mathematics of education should provide a foundation for all of them.

Aesthetics Human beings universally appreciate *beauty*, and students can experience mathematics as beautiful – through patterns in nature that mathematics describes, through a feeling of reverence for its truth and universality, through the elegance of reasoning in certain proofs, and through the sense of wonder that often accompanies understanding – e.g., in an “Aha! experience (Czarnocha & Baker, 2021). Our philosophy of mathematics education should include a basis for knowing and valuing these sources of aesthetic appreciation – i.e., a foundation for empirical and rational truth, as well as for depth of understanding.

Power Adler (1927) writes of the “will to power” as a basic human drive. As students learn, they can experience power in different ways – the power to do things they could not do before, to solve new problems, and to understand aspects of the world about them in new ways, and the power of their own reasoning to gain insight. For such power, mathematical validity is essential, and our philosophy should provide a foundation for it. But this sense of power does not stem from compliance with or submission to authority – it is the opposite. “True” mathematics is not synonymous here with “authority-based” mathematics.

Connection Social connection is likewise a fundamental human need – a sense of belonging and contributing, acknowledgment by others, friendship and partnership, status and respect. Mathematics need not and should not be an activity performed only in isolation. The social processes of doing and using mathematics – exploration and problem-solving problems with a partner or in a group, bouncing around ideas, responding to each other’s thoughts, reaching a consensus of understanding – all provide opportunities for “in the moment” fulfillment through social interactions. Our philosophy of mathematics education should incorporate a foundation for the social process of doing mathematics – mathematics as a human activity, with “human” understood to include the affective as well as the cognitive.

Insight Children exhibit curiosity from an early age – a fundamental human drive to experience and understand the world. A student’s insight can occur dramatically, as in the “Aha!” experience, or bit by bit, as one sees more when one climbs higher on a scenic path. To provide opportunities for insight, the “why” of mathematics in all its interpretations is critical. Our philosophy should provide a foundation for all of them.

Self-expression Maslow (1943) places “self-actualization” at the apex of his pyramid of human needs. Self-expression, a form of self-actualization, is most often thought of in relation to music, literature, or the arts and crafts. But opportunities for acts of creation and invention abound in mathematics. Skilled teachers seek to foster creativity in their students, as learners invent their own rules and patterns, make up original problems, put forth original conjectures, generate original proofs, and find new ways to solve well-known problems. Our philosophy of education can draw on constructivist ideas, highlighting the human invention of mathematics and inviting all to participate in its continuing invention and reinvention.

As students seek to fulfill their motivating desires during mathematical activity, they typically experience frustration with impasses during problem solving, unease when they do not understand, or disappointment when their ideas turn out to be "wrong" or when their suggestions are not accepted by others. The skilled teacher does not seek to remove negative affect, but to enable students to "see it through" – to experience fulfillment of the desire accompanied by elation, satisfaction, a sense of accomplishment, and increased self-efficacy.

7.5 Summary and Conclusion

A philosophy of mathematics education should draw on valid, nondismissive, and noncontradictory elements in various philosophies of what mathematics is. It should not be a theory of just one way to build the foundations of mathematics technically. I hope I have made the case that such a philosophy should provide foundational pillars for all that is involved in the complexities of mathematical learning and teaching. None of the "dismissive epistemologies" can do this – not radical behaviorism, authoritarian absolutism, strict formalism, logicism, or Platonism, radical constructivism, radical social constructivism, or any other epistemology that excludes important features of mathematics a priori: its human origins, its purposes, its contexts, or its truth and validity.

Behaviorists maintain that all anyone can observe in mathematics learning and teaching is behavior and its productions. Radical constructivists argue that an individual has access only to that person's individual world of experience and can have no knowledge beyond that. Radical social constructivists see objectivity as impossible, as all knowledge is constructed socially. Some proponents of such "isms" may argue that the features ruled out can be accounted for within their system – but at best they are devalued, and in the extreme, one is not even allowed to talk about them. Such philosophical approaches should not form the basis for a comprehensive philosophy of mathematics education.

What is needed is a far more eclectic approach, based on several philosophical pillars – one that accounts for, rather than dismisses, truth and validity in mathematics, and accounts for and explores its culturally situated human origins. I do not think we can authentically claim to be mathematics educators while rejecting the importance of objective mathematical truth and correctness, or rejecting discussions of mathematical understanding and mental processes, or disregarding the affective domain as inessential, or dismissing the importance of ethnomathematics.

But we must recognize that each of the aforementioned "isms" has a valid reason for its appeal – an important nugget of real value. Behaviorism reminds us of the importance of observable phenomena, empirical inquiry, and more objective measures of learning. Constructivism focuses attention on the thinking of the individual, particularly on constructive processes that occur during learning. Social constructivism points to the negotiation of meaning, calling attention to classroom cultures and culturally relevant teaching, and welcoming the study of diverse cultural practices and origins of mathematical systems.

A comprehensive philosophy of mathematics education should provide strong foundational pillars for all considerations that have to do with the learning, experience, appreciation, and use of mathematics by students in a variety of cultures. It should strengthen teachers in striving to enable all their students to be successful and highly motivated learners, making use of manifold tools and techniques. I have offered here some reasons as to why such a philosophy is needed and suggested some of its important elements.

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Part II
Philosophy of Mathematics Education: Creativity and Educational Perspectives

Chapter 8
Towards a Philosophy of Creativity in Mathematics Education

Bronisław Czarnocha

8.1 Introduction

This chapter discusses the formulation of a philosophy of creativity in mathematics education as a subdomain of the philosophy of mathematics education. We have observed the recent increase of interest in philosophical issues in our profession (Ernest, 2018); however, creativity is not specifically mentioned in this volume. One reason is that the general philosophy of creativity seems to be *in statu nascendi*. Paul and Kaufman (2013, p.3) tell us that "philosophy of creativity is still a neologism in most quarters."

Two recently published surveys of Philosophy of Creativity by Paul and Kaufman (2013) and Gaut and Kieran (2018) guide our discussions. Classical themes such as ethics and the value of creativity and creativity in the arts and sciences are explored together with creativity in the context of mind, cognitive science, AI, and in nature. With respect to aspects of creativity that are relevant to us but are not "covered" by the surveys of Paul and Kaufman (2013) and Gaut and Kieran (2018), such as creativity and learning or measurability of creativity, we direct interested readers to supplementary material published outside the philosophical mainstream.

Paul and Kaufman (2013) provide us with a methodological tip on how to start the process of formulation of the subdomain. As the researchers note, even though creativity has not occupied a central place in philosophy today, there has been a surge of interest in creativity within psychology and related domains, including research and practice of mathematics education.

At the same time, however, we note that Glaveanu (2014) in his critical reading of the Psychology of Creativity observes that although the field is thriving, the "discipline is in crisis." He attributes the crisis to the large number of divergent ideas in the field and the relatively little constructive accumulation of ideas. This has

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resulted in abandoning big ideas in favor of specialized inquiries. As a consequence, Glaveanu (2014) asserts that this process ends up effectively excluding everyday creative tasks including creativity that takes place in a classroom. In response to the crisis in the psychology of creativity, there has been increased interest in Maslow's humanistic psychology, which views creativity through the lens of individual self-realization posited at the very top of his pyramid of typology of human needs. Maslow's humanistic approach directs attention to everyday tasks called for by Glaveanu (2014): "I learned from her and others like her that a first-rate soup is more creative than a second rate painting" Maslow (2018).

Thus, to make the connection between the manifold of ideas being investigated, among others, within the thriving domain of Philosophy of Creativity and creativity of the classroom, we approach the philosophy of creativity from two directions, using Ernest's (2018) metaphors: bottom-up and top-down. The bottom-up approach uses the teaching-research methodology of starting with the practice of creativity in our classes (Czarnocha et al., 2016) and develops that pathway into a research approach. The top-down approach, by contrast, relies on the general philosophy of creativity included in the two volumes and guides us in exploring appropriate results and reflections of researchers and teachers of mathematics concerned with mathematical creativity in the classroom.

This chapter aims at answering the following question: What can the practice of and research in creativity of mathematics education contribute to the philosophy of creativity in mathematics education and possibly to the philosophy of creativity in general?

Finally, since Ernest (2018, p.15) suggests that the point of departure for such an endeavor should be "the critical examination of its fundamental problems" in mathematics education in the context of the systemic analysis of the domain will we follow that lead. This chapter is the synthesis and amplification of two recent presentations (Czarnocha, 2022a, b).

The last two sections of this chapter provide a summary of the discussion followed by Philosophical Conclusions.

8.2 What Is Creativity?

Nearly all teachers of mathematics have encountered situations in our classes when we have intuitively recognized a student who has come up with an unusually creative response to a problem or asked an insightful question. Without a clear and accepted definition of creativity, however, we do not always understand what was creative in that student's remark or how to reinforce its impact on the other students in the class.

To help us answer this question, Paul and Kaufman (2013) offer the following definition of creativity: "The term "creative" is used to describe three kinds of things: a person, a process or activity, or a product…There is an emerging consensus that a product must meet two conditions to be creative. It must be *new* of course, …, it must also be *of value*." Boden (2004), based on her involvement with AI, defines creativity as the ability to generate creative ideas (artifacts) – where a creative idea is *novel*, *surprising* and *valuable*.

To these definitions, we add another. Moruzzi (2021), who is interested in those aspects of natural and artificial creativity that do not depend on external factors but only on the inner structure of the creative system, bases her analysis on three creative features: problem solving, evaluation, and naivety. Of course, there are many more definitions in the field. Mann (2006) has found over 100 different definitions of creativity in the profession. Investigations of the approach based on the self-awareness of learners as creative individuals have been conducted by Shriki and Lavy (2014), while a new approach via the relationship between imagination, creativity and innovation has been proposed by Karwowski et al. (2017) in the context of conjunctional model of creative imagination.

In mathematics education, two approaches to creativity stand out. The first is the Gestalt approach adopted by Wallas (2014) and Hadamard (1945); the second originated with the work of Guilford (1950). There are interesting basic differences between the two: whereas the Wallas/Hadamard theory describes the creative process, the Guilford/Torrance theory addresses human creative capacities.

The Gestalt approach emphasizes the creative process which has several, more or less consecutive stages of preparation and incubation leading to illumination or insight as the sudden restructuring of a problem's situation. Under Poincare's influence, Wallas added the last stage: verification. Sadler-Smith's (2015) recent examination of Wallas's work suggests a fifth stage of intimation between incubation and illumination. Davidson's (1996) three-process theory helps in identifying different types of insight. Selective combination takes place when someone suddenly puts together elements of the problem situation in a way that previously was not obvious to the individual. Selective encoding occurs when a person suddenly sees one or more features that previously have not been obvious (Davidson, 1996). Selective comparison occurs when a person suddenly discovers a nonobvious relationship between new and old information.

Guilford's approach introduces the concept of divergent thinking as the central feature of creative personality characterized by fluency, that is, by the number of relevant ideas; flexibility, the ability to generate qualitatively different ideas; and originality and elaboration, the ability to develop ideas (Guilford, 1967). Guilford's approach became the basis for the Torrance Test of Creative Thinking (TTCT) (2018), which has had a strong impact on research on creativity from the point of view of its product. Leikin and her research group (Leikin et al., 2009) introduced these ideas into mathematics education research using the criteria of fluency, flexibility, and originality. Fluency refers to the number of solutions to a problem as well as the pace of solving produced by an individual. Flexibility refers to the number of solutions using different methods, while originality is measured by insight-based or unconventionality/conventionality of the solution in relation to the full sample of participating students (Leikin, 2009).

Naturally, each approach determines a different pedagogy and different research techniques. Mann (2006) and Moruzzi (2021) point to the research difficulties arising from the multiple definitions of creativity. Our aim is to identify the concerns arising in classroom teaching choices and pedagogies because of these differences. Teachers' intent on facilitating creativity in their classroom will have to choose whether to facilitate the development of creative originality through the development of fluency and flexibility or through the originality of the structural insight. Depending on the goal, the teacher will choose the definition of creativity and gear the pedagogy together with the types of problems necessitated by that approach. A teacher might question the pedagogical utility of such an arbitrariness. It would make sense if each approach described a different aspect of creativity. In such a case, research is needed to better understand the relationship between these two approaches.

Moreover, it could be argued that different aspects of creativity appear in different student populations. For example, in a standard mathematics classroom in an urban community college neither definition is useful for facilitating creativity or assessing its depth. This is because most first-year students require a great deal of mathematics remediation and hence are neither fluent nor flexible in their mathematical thinking. Yet we, their math instructors, know they are creative.

On the other hand, the Gestalt approach, with its four stages, is hard to control within the classroom curriculum and does not offer much guidance on how to assess the insight of illumination. The comments here lead naturally to the subtlety of the relationship between creativity and learning, given that different student populations might respond better to different manifestations and different aspects of creativity.

8.3 Creativity and Learning

Surprisingly, our examination of the literature yielded few studies on the relationship between creativity and learning within the general philosophy of creativity, indicating that this topic has not yet reached the attention of philosophers. By contrast, creativity in the field of psychology has several pathways in that direction, the most interesting for us being the work of Kaufman and Beghetto (2009). They address common ways of classifying creativity through C-creativity (called also H-creativity, which focuses on creativity that has had an impact on society at large) and c-creativity (called also P-creativity, which impacts primarily an individual thinker but not necessarily the society).

The authors isolate mini-creativity out of c-creativity claiming that creative insights of students as they learn new concepts or make a new metaphor is overlooked in the world of little-c. This new category of mini-creativity "was designed to encompass creativity inherent in the learning process" p (3). The authors give an example of a young child's mini-c creativity, who stated she wants to be a "mushroom princess" when she grows up. The child's insight here was a combination of two things she valued: mushrooms (probably because her parent is a mycologist, or someone who studies mushrooms) and princesses. In the words of Czarnocha and Baker (2021), it is clear that the creative insight of the child took place within the bisociative frame (see below) between the two components mentioned. Thus, mini-c represents for the authors creativity present in the learning process. Kaufman and Beghetto (2009) also isolate also a Pro-c creativity from representing the developmental and effortful progression little-c creativity (that has not yet attained Big-C status).

To clarify the relationship between learning and creativity to the needs of mathematics education, we will start by posing three basic questions: Is there creativity in learning? Is there learning in creativity? Can creativity be taught?

The second question is the easiest to answer in the pure affirmative. By pure affirmative, I mean "nothing but…" namely that creativity of Aha! moment in mathematics is nothing but learning, in fact, conceptual learning.

Yerushalmy (2009) observes that discussions of creativity and curriculum take place independently of each other; quite often, they do not even share the same lexicon. In other words, creativity and curriculum (the organization of learning in mathematics classroom) are two essentially unconnected matrices of thought. The theory of creativity, which we propose in the further reaches of this chapter, looks upon such a pair of matrices as a bisociative frame: a frame within which creative insights have larger chance to emerge. The creative insights may connect elements of both the discourse of creativity and the discussion of learning within the given curriculum.

The central question here is how can we connect such two matrices of thought so that a student's creativity develops simultaneously with learning? That question might be easier to answer if we focus on the creativity of Aha! moment. According to Koestler (1964), this is *a spontaneous leap of insight which connects two or more unconnected matrices of discourse by unearthing hidden analogies*.[1]

Note that if indeed the Aha! moment insight connects unconnected matrices, it participates in the process of a thinking schema construction. In other words, the Aha! moment is the act of thought through which understanding expresses itself by building a new conceptual connection. Consequently, it is the element of conceptual learning, a term Simon et al. (2004) introduced recently. We see that creativity of an Aha! moment is closely related to the development of conceptual understanding. Now, we need to ask an inverse question, what kind of learning can engender a creative Aha! moment? Of course, there is a related question of why we would like to engender such a moment of creativity in the classroom, or alternatively what is its value in the classroom? The answer to this question is discussed later.

The second and third questions above are connected to each other. If creativity can be taught and the object of teaching is learning, then we should be able to find a lot of creativity in learning. But we do not. And that is the second point of contention between creativity and learning: how to recognize the possibility of

[1] Unconnected matrices of discourse called also unconnected frames of reference are two unconnected ways of thinking which get connected through the Aha! moment. Quite known example of such two matrices can be found in Archimedes' Aha! moment "Eureka" where one matrix of discourse was geometry and the other, the discourse connected with taking bath in a bathtub. Both matrices being originally unconnected, became connected through the Aha! Moment which created the new buoyancy law.

creativity at a given moment and how to promote it in the classroom so that the presence of creative processes is maximized. This idea is one of the unresolved issues in our profession despite our increased understanding of the influence of problem solving, inquiry, and discovery methods. While these three approaches have been introduced into mathematics classrooms for at least two or three last decades, the reports of their long-term introduction are still lacking, possibly because of the absence of understanding along the interphase of creativity with curriculum design. The question asked of teachers at that point is how to recognize those areas of the daily mathematics curriculum where a possibility for creative insight may emerge and what to do with it once it emerges.

Nevertheless, one of the most difficult obstacles on the creative pathways along the curriculum is simply habit: our habitual nature that can turn every new skill we learn into an automatic habit. And the more automatic, the better because it can be enacted more precisely and without thinking in a critical situation. The problem is that we tend to transfer its role beyond the authentically critical situations. In that form, the habit becomes the obstacle for the creative act, which is essentially a spontaneous one. Davis (2011) asserts that barriers are blocks, internal or external, that either inhibit creative thinking and inspiration or else prevent innovative ideas from being accepted and implemented.

Most barriers result from learning. They may originate with a person's family, peer group, community, educational environment, or from others in the culture or business organizations. We try to imbue our students with the reasoning skills and the steps of algorithms, often closing at the same time the space available for the creative act.

Habits and their relationship with originality are of utmost importance for Koestler (1964):

> Matrices vary from fully automatized skills to those with a high degree of plasticity; but even the latter are controlled by rules of the game which function below the awareness. These silent codes can be regarded as condensation of learning into habit. Habits are indispensable core of stability and ordered behaviour; they also tend to become mechanized and to reduce man to the status of a conditioned automaton. The creative act, by connecting previously unrelated [possibly habitual] dimensions of experience, enables him [or her] to attain to a higher level of mental evolution. It is an act of liberation – the defeat of habit by originality. (p.96)

That contradiction between habit and originality as expression of spontaneous creativity needs to be understood very clearly and taken into account if we want to focus education on creativity and innovation. To some degree, it is our conviction of and attachment to the fundamental role of reason that is possibly one the strongest forces which inhibit the occurrence of the creative act by placing reasoning necessarily before or during the act of creation. For example, recent changes in the typology of thought from the standard Bloom taxonomy to the revised one create such an obstacle. The pyramid of the old Bloom taxonomy has a synthesis step followed by evaluation as the top stage, whereas the revised Bloom taxonomy has evaluation first followed by creativity (instead of synthesis). In other words, conscious evaluation comes before the creative process in general, which may severely limit the spontaneous aspect of creativity.

On the other hand, the creativity of an Aha! moment depends to some degree on the conscious formation of knowledge before the illumination, in the preparation stage. The fundamental question is this: How much conscious preparation through curriculum design is necessary to facilitate the insight? Moruzzi (2021) reminds us that a creative process is a process of exploration, which does not necessarily demand expertise and self-education but rather, can simply evoke everyday psychological abilities such as capacity of observation and of making analogies (p.7).

Let us remember that aha! moments come not only in problem solving but in any learning that may lead to progress in understanding as opposed to exercising understanding by applying it to similar mathematical situations. A case in point is recurring concerns about teaching to the test, which through its pressure and failings stifle creativity (Lim, 2010).

A similar concern arises during the discussion on the nature of creative insight, the Aha! moment. Whereas some researchers see Aha! moments intrinsically connected with reasoning, others see the insight as, quite possibly, emptied of conscious rational content. A good example of such situation is described in (Czarnocha & Baker, 2021; Chap. 13), when a young student grasps the possible simplification of the division of fractions with the same denominator yet has no idea why it works. It takes several prompts of the researchers to lead her to understanding.

Similarly, Poincare, when describing his famous Aha! moments, tells us he suddenly knew it, but the process of verification (explicit conscious understanding through mathematical reasoning) took him several hours of work after that and only after he returned home. Poincare (1914) asserts,"It never happens that unconscious work supplies a ready-made result of a lengthy calculation in which we have only to apply the fixed rules…All that we can hope from these inspirations, which are the fruits of unconscious work, is to obtain points of departure for such calculations. As for calculations themselves, they must be made in the second period of conscious work, which follows the inspiration and in which the results of inspiration are verified, and the consequences deduced." (p.62–63).

Wallas (2014, p.50) offers the following thought: Many people would agree that any attempt to control the thought process at this point (the moment of illumination and its neighborhood in time) *will always do more harm than good.* We can ask, what is that knowledge we get during the illumination stage when we do not yet understand explicitly how it works? We just know it does. Poincare uses the French word *sensibilite*, which accordingly to Wallas is extremely ambiguous (p.34). Instead, Wallas (p.47) directs our attention to the "'fringe of consciousness' which surrounds our 'focal' consciousness as the Sun's 'corona' surrounds the disk of full luminosity" to which he gives a special name of intimation (p.48). Admittedly, Wallas's main goal in introducing the term is to describe the sensation leading up to the illumination (which in the hands of S.-Smith (2015) became the new stage in the Walla/Hadamard theory of stages). Wallas continues, "This fringe consciousness may last up to the flash instance, may accompany it, and in some cases may continue beyond that." In such a case, intimation can be an excellent point of departure for calculations in the verification stage with certain though unclear anticipation for its results.

These considerations bring us to one of the oldest problems in the philosophy of creativity, the relationship between creativity and rational thinking. Paul and Kaufman (2013) remind us that Plato's Socrates saw great poets as being divinely inspired by the Muses in a state of possession that exhibits kind of madness. Aristotle, in contrast, characterized the work of a poet as a rational, goal-oriented activity of making (poesis) (p.3).

On the other hand, Nietzsche (1999) saw the tragic poetry of ancient Greece as being born out of a rare cooperation between the Dionysian spirit of ecstatic intoxication and the Apollonian spirit of sober restrain, which tempers chaos with order and form (p.4). In light of our previous discussion, intimation and illumination might be that cooperation between the ecstatic intuition and sober reasoning, especially if we consider the emotional impact of the Aha! moment that often accompanies it (Koestler, 1964; Liljedahl, 2004).

8.4 Summary

In presenting this complex relationship between creativity and learning, we can easily see where to find learning in creativity. The bigger challenge of how to induce creativity from learning is more complicated. The discussion has touched on the difficulty of the interphase of creativity and curriculum that led to the habits developed through curriculum and in particular the habit of rational thinking as obstacles for the successful facilitation of creativity. We are led to the age-old question of the relationship between creativity and rationality, the clarification of which is necessary, in our opinion, for its smooth composition in the mathematics curriculum.

8.5 Creativity and Its Value

Let us remember the original description of creativity by Paul and Kaufman (2013), who end their statement by claiming "…it must also be of value." Boden's (2004) characterization of creativity as the "ability to come up with ideas or artefacts which are new, surprising and valuable" reinforces the concept of value, suggesting at the same time the question "of value for whom?"

In response, Gaut (2018) introduces several criteria to get to the concept; he divides it into instrumental and intrinsic value. Intrinsic value, for Gaut, is to be valuable as an end or to be valuable for its own sake. Instrumental value then is the value of the means of creativity. For example, the value of creative cooking is to produce not only unusual but also good food; or the instrumental value of a scientific idea such as quantum mechanical "entangled states" created by Schrodinger is the possibility and emergent reality of teleportation immortalized 60 years ago captain James Kirk in the science fiction Star Trek through his "Beam me up, Scottie!"

What is the value of creativity for us in mathematics education? Again, it depends for whom? For us as teachers to teach a creative person is very pleasant and enlightening. We can establish intellectual contact with the student, experience pleasure when a creative student comes up with a new solution or new understanding. As teachers, we would like to find the means to help the student flourish to the maximum. Here is another site for creativity, that of the teacher who needs to design a problem, a hint (or learning environment) that is adequate for the high degree of student creativity. For a student in mathematics, this provides a very strong influence in several dimensions (Czarnocha & Baker, 2021; Chap. 10). Here we just quote simple expressions of happiness with the moment of creativity, "It felt great," "I was so relieved; I could barely contain my happiness," "That was the best feeling," "I never knew I could feel so good while doing math," "The joy I felt was like no other," and "It made me feel like I could do anything."

Simply, creativity feels good. Is it an intrinsic or extrinsic value? Personally, I would see creativity as an intrinsic value, remembering that it may produce good and bad objects. For example, creativity involved in the understanding the mathematical/physical structure of an atom and energetic possibilities contained there led to nuclear power plants and the nuclear bombs. Here creativity in mathematics and mathematics education encounters ethics. To deal with that problem, Gaut (2018) introduces the distinction between something being good, as distinct from something being good of its kind. In that sense, nuclear bombs are very good as bombs but certainly they are not good, period (p.128). That of course suggests that judging the value of creativity by its product is not very useful as the manifestation of that value. This realization leads to the concept of conditional creativity. Following Kant (1993), we can call things that are valuable only under some circumstances as "conditionally valuable", while those valuable under all circumstances – "unconditionally valuable." This implies that creativity is conditionally valuable since its products may or may not be valuable. Thus, the sense of feeling good as a result of a creative act is an unconditional value, while the creativity involved in understanding the structure of an atom is conditional because it depends on what product is created. The discussion of ethical dimension of creativity in mathematics education encounters closely related discussion of Ernest (2018), who analyzes more general problem of ethics of mathematics, where he had offered "the metaphor that mathematics has two faces, good and bad." Similarly, we have seen that creativity can have two faces and the conditional/unconditional creativity criterion introduced by Gaut (2018) distinguishes between those two faces. Ernest points out to the role of social image of mathematics as experienced by many learners' image of mathematics and its effect on success or failure in the subject. Below we demonstrate the role of creativity of Aha! moment in overcoming the sense of failure through acting upon students' extrinsic motivation.

We see the sense of pride and confidence: In reflecting upon this Aha! moment, I feel the sense of pride that I accomplished this mathematical idea all by myself, "The Aha! moment is inspiring!" It makes students believe that they solved the question through reasoning and deep thought and inspires him or her to seek more of these moments to obtain sort of confidence and further knowledge.

Clearly, creativity as experienced by authors of those two fragments is extrinsic as it reveals products of self-belief and self-confidence within the experiencer psyche or inner life. Are those products valuable unconditionally or conditionally? I think "the sense of pride" in the accomplishment of solving a classroom problem is quite unconditional and I would say intrinsic, while the inspiration to seek more of these moments, might be extrinsic conditional depending on the circumstances within which they will occur and product they will create. According to Ernest (2018), Davis (1988) suggested there should be a Hippocratic Oath for mathematicians. Since doing mathematics to large degree depends on creativity, maybe the discussion of the Hippocratic Oath should take place in the mathematics classrooms together with the emphasis on two faces of creativity.

It is interesting that one creative experience can produce things that are both intrinsic and extrinsic, conditionally and unconditionally valuable. In light of the difficulty of establishing the value of creativity, Hills and Bird (2018) question the presence of "value" in the very definition of creativity. Rather than value, they suggest that we focus deeper attention on originality in the definition.

Coming back to the creativity in our classes of mathematics, we educators feel and recognize the value of our students' as well as our, teachers' creativity. The examples above demonstrate the benefits of creativity on our students. However, to the author of remedial and undergraduate mathematics course, the most precious value of Aha! moment creativity is its influence on the attitudes and beliefs about mathematics. DeBellis and Goldin (2006) have introduced the concept of a student bonding with mathematics while developing a more "intimate" relationship with and knowledge of the subject. Consequently, for our profession, which is constantly mired by the difficulties of attitudes while teaching the subject, that value of bonding with mathematics through creativity is of highest and in my opinion extrinsic degree.

8.6 Is Creativity Measurable?

Measuring the depth of creative experience has not been addressed by the volumes of Philosophy of Creativity discussed here, namely, Paul and Kaufman (2013) and Gaut and Kieran (2018). This is a relatively a new theme, which arises as the result of dissatisfaction with psychometric approaches, notably the ones promoted by Guilford (1967) and others. The new articles appearing in professional journals are only partially related to creativity. Piffer (2000) argues that the difficulties in the measuring of creativity are due to the absence of clarity concerning its definition.

Piffer (2000) formulates a new framework based on novelty, appropriateness, and impact within which creativity is measurable. Basing himself on the new definition, the author argues that "Divergent Thinking, Remote Associates or some personality scales can be considered neither the only components of the creative process/cognition/potential nor 'creativity tests'." Finally, he suggests that the creativity of a person cannot be accessed directly and can only be assessed through self-report questionnaires. Below we discuss how using the progress of understanding as the measure of the change of the depth of knowledge obtained during the Aha! moment can give us a direct assessment tool of creativity in mathematics. Elizabeth Kaplunov (2016), a recent PhD calls upon students in the UK: Surely between all the students in the UK, we can find a new way to harness and assess this brainpower, capacity, imagination, resourcefulness, inspiration creativity (whatever you want to call it). Let us measure creativity in creative ways!

Moruzzi (2021), on the other hand, tells us that "the question of how to measure creativity is arguably under-discussed in the field...creating a serious gap in our capacity to delve into the analysis of creativity depths."

8.7 Why Is It Important to Measure Creativity?

Even the traditional view of creativity as the product of a "genius" or extremely talented people in arts and science involves an intuitive assessment of that work. When we intuitively detect the impact of it in ourselves, this work, this painting, that music affects me deeper than this one, or this solution is more elegant because of its creative component compared to that one. Of course, these assessments are subjective and vague. Nonetheless, their existence reminds us that measuring creativity is present within our inner experience. For Moruzzi (2021), it is important to find a way "to measure the distance that sets performance of machine learning systems apart from human creativity, if there is any." For those of us in mathematics education, measuring creativity is as essential as the clarity of its definition, due primarily to our need to assess the depth of progress of a student's creative thoughts and actions or depth of knowledge (DoK).

It is important for us to know how to translate a student's creative discovery into general classroom knowledge to be shared even though creativity of a person at a given moment might be a fleeing feeling or thought that disappears because of external factors. For this reason, it could be very useful to measure creativity with the help of tools that do not depend on external-to-creativity factors but on the inner structure of creativity itself, which does not undermine creativity or affect it during measurement. The need for such an approach increased significantly once our profession understood that creativity is not limited to the very gifted and talented only but is within the reach of "everyman," using Boden's (2004) characterization. In mathematics education, research and practice of creative teaching attention to that difference is expressed in the division into H-creativity, historical creativity which have influenced society, and P-creativity, personal creativity of an individual. We also use the similar distinction between C-, and c-creativity.

Moruzzi is using three features of creativity to substantiate the approach: problem solving, evaluation, and the feature of naivety. The main problem solving quality taken here into account is the "connection-making" process which "consists in drawing links between apparently disconnected pieces of knowledge and in exploring novel paths towards the successful conclusion of the problem-solving process."

Evaluation feature, what Moruzzi (2021 p.7) sees as the "ability to assess the process and to know 'when to stop'." This feature is especially important during the trail-and-error solving process when the learner or an investigator adjusts her or his thinking, which leads to the solution. That process is of course akin to the divergent thinking of Guilford, (1967). Naivety feature “relates to various aspects that in the literature have a place among the core traits of creativity, unconscious thought processing, challenging domain norms, independence from rigid structures of thought.” Moruzzi points out that from the point of view naivety “A creative process is a process of exploration, which does not necessarily demand expertise and self-education but, rather can simply evoke everyday psychological abilities such as the capacity for of observation and of making analogies.” One of the central qualities of naivety feature is “lack of previous exposure to the situation at hand.” The lack of previous exposure to the situation at hand is one of the central condition in the process of facilitation creativity of Aha! moment.

Moruzzi’s proposal is to see creativity of the individual as proportional to the “sum” of naivety, novelty and the distance of connections, evaluative ability, and efficiency. The proposal emphasizes the analysis of the creative process, in distinction to widely employed Torrance tests approach, which focuses on the product. We have argued that focus on the process is especially useful for regular students for whom fluency and flexibility in mathematics are not fully there, but who nonetheless display creativity.

8.8 New Theory of Creativity in Mathematics Education

Should creativity be facilitated in mathematics classes? Can it be done in all classes or just in some? Can it be done at all?

If the answer to these questions is yes, then as we discussed in Sections I and III, this situation requires a definition, theory, and practice of creativity that suitable is for the task. Prabhu and Czarnocha (2014) have already proposed the definition of the act of creation as formulated by Koestler (1964). To a certain degree, this definition addresses the topics we raised earlier in our discussion. Before we proceed, however, I would like to sketch our motivation for this work. While working at a community college in South Bronx, it has become very clear to me that the most common approaches to creativity in our profession, which are based fully or partly on the Guilford/Torrance approach, do not work well with urban college students.

Most first-year students coming to such colleges require remediation in mathematics. Many, in addition, lack both fluency and flexibility. Yet as we have observed, they can be highly creative in the subject. The question thus arose on how to identify and describe their mathematical creativity. One favorable circumstance for this project has become the relatively recent realization of the distinction between C- and c-creativity in the profession (Kaufman and Beghetto 2009; Boden 2004; Hadamard, 1945). c-creativity is understood as bringing insights that are new and original to the thinker but not necessarily to anyone else. That allowed us to understand that any knowledge obtained or a solution to a problem done by a student or as Koestler would have said “within non-tutored” setting can be a minor creative act. As a result, the investigation of the degree of the progress of understanding created during the insight is a proper measure of creativity in mathematics classroom. That turned our attention to Aha! moments in our classrooms.

The first investigation of their occurrence in our classroom was done by Prabhu (2016), who observed a significant increase in their number as well as the impact they made on students in the class led by her new pedagogy.

Consider Chamberlin (2013), who states the following: Missing is information on what initiatives are in place to develop and facilitate mathematical creativity in underserved and under-identified populations. This type of discussion would be informative to the field of gifted education and counter the criticism that field is not inclusive. (p. 856)

Moreover, Sriraman's assertion that "there is almost little, or no literature related to the synthetic abilities of "ordinary" individuals" (p. 120) is a strong confirmation of the fact that until recently the knowledge about creativity of so-called underserved and underrepresented student population has been nil. Because Aha!moment insights take place among the general population in different circumstances, the theory of creativity of Aha! moment can be seen as the theory of "creativity of and for all" (Prabhu and Czarnocha, 2014; Czarnocha, 2022b). That is why we are aiming at the definition, theory, and practice of creativity that fits the creativity of "everyman," of rank-and-file as well as of gifted students.

8.8.1 Definition

Bisociation (creativity of Aha! moment/Eureka experience) is the spontaneous leap of insight that connects two unconnected frames of reference (matrices of thought) by unearthing a hidden analogy (p. 45).

But what is a theory in mathematics education? According to Redford (2008), such a theory contains the following:

- A system P of principles delineating the frontier of the universe of discourse.
- A methodology M that includes the data interpretation that is supported by P.
- A set Q of paradigmatic research question.

We will discuss these points heuristically. Those details may be found in Czarnocha and Baker (2021; Conclusions). We are now directing our attention to the definition of bisociation. We see that it contains several tightly related concepts: spontaneity, process of connecting and the two unconnected frames of reference that get connected by the insight. The definition also describes the process of connecting and the product of connection – the conceptual bridge between unconnected matrices of thought. We abstract here the bisociative frame, which is the two unconnected frames of reference within which the Aha! moment takes or might take place, as one of the basic principles of the theory.

The universe of the bisociation theory has a several dimensions in mathematics education:

- In its relation with student affect and its relationship with conation (Goldin, 2021).
- In its cognitive dimension, it studies the intrinsic structure of creativity as the process of understanding through the development of the schema of thinking accomplished during the Aha! moment.
- In its learning dimension, it studies the changes of understanding mathematical concepts occasioned by Aha! moments, the relationship between internalization and interiorization (Baker, this volume) and the role of creativity in abstraction.
- In its networking theories' dimension, it investigates the process of integration of different theories of learning through which bisociation expresses itself in terms of the host theory.[2] The notion of a bisociative frame turned out to be a powerful creativity "detector" within different (constructivists) theories of learning. It identifies sites of heightened possibility of creativity within Tzur's (2021) R*AF theory. It also helped formulate dynamics of creativity within different stages of Mason's attention theory (Mason & Czarnocha, 2021), and it coordinated a bisociation process with the reflective abstraction of Piaget's approach of showing that creativity can occur on every level of development of student thinking (Baker, 2016). In this volume, Baker (2023) shows where bisociation can be found within Vygotsky's concept of internalization, providing an extremely important unifying connection between the Piagetian and Vygotskian approaches to learning.
- The connection between different theories of learning analyzed according to networking theory indicates that creativity can be identified in each constructive research and teaching approach, responding this way to a somewhat scattered approach to creativity in learning indicated by Mann (2006) and Moruzzi (2021). Moreover, it formulates new research questions on creativity within the terms and structures of the host theories.
- Directing our attention on the process of schema construction as the product of creativity, we see that the definition suggests a theory of schema development as the approach to understand the depth of inside (DoK) reached during an Aha! moment. We used the well-known Triad of Piaget and Garcia (1987) as a tool of assessment. This tool allows the use of the three stages of the Triad: -intra, -inter, and -trans to relatively precisely gauge the degree of progress of understanding within the insight (Czarnocha & Baker, 2021, Chap. 4). The information to that degree was obtained through students' written descriptions of mathematical concepts entering into insight followed by student interviews. This approach fulfils an important condition stated by Morruzzi of focusing solely on the intrinsic qualities of bisociation without in any way disturbing the creative process.

[2] We discuss here the relationship between the new bisociation theory of creativity proposed and different theories of learning called host theories. This relationship is understood in terms of the networking of theories approach formulated by Prediger and Bickner-Ahsbahs (2014). The host theory, an example of which is Tzur's R*AF theory (2004) mentioned in the next paragraph, is the theory where bisociative frame can be expressed within concepts of that theory itself.

8.9 Summary of Arguments

The central question of this chapter is the following: What can the practice of and research in creativity of mathematics education contribute to the philosophy of creativity in mathematics education and possibly to the philosophy of creativity in general?

We began our discussion with a known problem in our field, namely, the multitude of definitions of creativity. We also pointed out how different definitions focus on different aspects of creativity and how different student population might be more sensitive to a particular definition. That, of course, suggests a research question on the relationship between different definitions. From the point of view of the standard mathematics classroom, neither definition adequately addresses the facilitation of creativity.

We also pointed out that the theme creativity and learning has not yet attracted attention from the general philosophy of creativity; however, it does contain within itself deep and basic issues. Interestingly enough, whereas we did not have much difficulty in identifying learning in the context of creativity, the reverse question of identifying creativity in the learning process revealed serious obstacles leading to the discussion of automatic habits, particularly the habit of overemphasized rationality.

This discussion underscored the need to finding a balance between the standards and habits of learning and the requirements of creative learning environment. Closer analysis of Poincare's comments (1914) suggests that Wallas's intimation discussed by Sadler-Smith as the fifth stage of the Gestalt approach might also carry a substantial value not only before but also after the insight illumination.

Our discussion of the value of creativity was undertaken within the dichotomy of intrinsic and extrinsic value suggested by Gaut (2018). We also incorporated the concept of conditional and nonconditional value suggesting the possibility of integrating the elements of ethics while facilitating creativity in the classroom.

The pedagogical classroom experience suggests that the main value of creative insight is the bonding with mathematics through the positive feelings the insight induces. The last theme undertaken in this chapter is that of measuring creativity or assessing the depth of the progress of understanding reached during the insight. This theme does not yet appear in the general philosophy of creativity; however, forcefully including creativity in mathematics classes suggests the need to measure it. We bring forward Moruzzi's thinking (2021) on the subject who points to the quality of naivety, which is close to the conditions we create in the classroom to facilitate the insight. The discussion of measurement of creativity must consider that different measures might be necessary for different student populations.

In the final pages of this chapter, we discuss the details of the Koestler's (1964) definition of bisociation or creativity of the Aha! moment and Eureka experience as the basis of the new theory of creativity in mathematics education. We characterize the theory with the help of criteria established by Redford (2008) in the context of networking theories approach. A main component of the theory is the bisociative frame, which enables a person to establish a close relationship between creativity and constructivist and socio-cultural learning theories. That coordination can be a significant help in facilitating creativity through different pathways of learning.

8.10 Philosophical Conclusions

What issues and qualities of creativity in mathematics education have been brought forth by discussion of related fundamental problems in mathematics education?

Paul and Kaufman inform us that the adjective "creative" is seen as the description of a process, product, and a person – a creator. We summarize the answer to the question above by taking these three components of the subject as the initial framework underlying at the same time discovered connections between them.

The section "What Is Creativity?" brought forth the many ways through which we approach it; however, instead of seeing it as the obstacle for research and teaching practice we see it as different "cuts" or different components of creativity, each addressing its different aspects. Moreover, each "cut" determines different research questions, different pedagogical approaches in the classroom, and different measurement techniques, each of them providing useful though often different knowledge about creativity. Philosophically that translates into a wide avenue of research investigating the inner structure of creativity.

Such investigations could be conducted using recently formulated networking theories approach (Bikner-Ahsbahs & Prediger, 2014). We see in here an important connection between the ontological question of what is the being called creativity with the epistemological question of how do we gain knowledge of it? One new concept brought forth by our field, Mathematics Education to Philosophy of Creativity, is the question of its measurement. Here we immediately want to assert that although the concept of measurement in psychology very easily associates itself with positivistic philosophy and psychometric methods, which of course are used in the context Guilford/Torrance/Leikin approaches, we also bring here a very different approach whose aim is to assess the depth of creative endeavor (Moruzzi, 2021). We use here Piaget-based techniques of cognitive developmental psychology, which allows us to assess the degree of the depth of knowledge gained by the creativity of Aha! moment. This method allows us to solve Leikin (2016) knowledge-creativity paradox formulated as follows: "Creativity is a necessary condition for knowledge construction, whereas knowledge is a necessary condition for creative processing" (p.19). Knowledge necessary for creative processing is different from knowledge constructed through the act of creation. Knowledge is different in the degree of connectedness and therefore in the degree of conceptual maturity of the creator. (Czarnocha & Baker, 2021; p.281). In other words, considerations within mathematics education are adding a new useful quality to the nature of being called creativity, to the ontology of the concept – its depth.

One of the most significant issues brought forth by several later sections of this chapter is that the nature of creativity in mathematics is the same for the mature inventor and for a student solving a problem in algebra or geometry. As Hadamard (1945) reminds us the difference between them is that of scope, possibly of the depth and not of the creative process. Similarly, more detailed considerations of Boden (2004) or mathematics education researchers (Shriki, 2010) have brought forth the concept of C- and c- creativity or corresponding with it division into H-historical and P- personal creativity that allows us to investigate creativity of "rank-and-file" students or as Boden (2004) would have had it, the creativity of "everyman."

The questions of how to do it, what definition of creativity to use, or what is the relationship between creativity of everyman and that of gifted and talented students needs to be explored. The philosophy of creativity in mathematics education could and should take the lead in this process by pointing out that creativity is the quality of us all. However, the philosophy of who is a creator does not stop at this juncture. Is creativity a solely human endeavor? The discussion of recent and past investigations into animal creativity represented in this volume by Thomas Rick's chapter, extended by Boden (2004) into, on the one hand, biological creativity of life and, on the other hand, to the recently formulated computer creativity suggests a possibility of a different answer. Either creativity is not just a human quality or we have not yet identified that aspect of creativity which is uniquely human.

The section "Creativity and Learning" followed by the last two sections on measurability and new theory hide within themselves epistemological aspects of creativity. How do we know the being called creativity? How do we recognize its process? A central issue is that we cannot fully predict when creative process will commence or how creative its product might be. Thus, the methods of knowing creativity have an irreducible probabilistic base. And yet, we generally know it, at least in mathematics classroom, when something, we call creativity takes place. So we have a certain intuition of creativity that can express itself in many ways; for Poincare it was unconscious aesthetics. And then we choose a certain definition of creativity with the help of how we organize our investigations/exploration into its being. Having understood circumstances when it sporadically occurs, we can reproduce and refine them to increase the probability of its future occurrence. This is one way of knowing creativity, by continuously interacting with it through the cycles of recognizing it intuitively, by seeing and understanding it, and by increasing the probability of its occurrence to investigating the results. Note that the whole process of knowing creativity depends essentially upon the choice of the definitions, which might be quite different, yet each describing certain of its aspects. Consequently, to explore epistemology of creativity in mathematics education we have to make a profound shift of attention from creativity per se to different aspects of creativity, its different "cuts" irreducibly connected to the psychological processes of learning and knowing.

This sense of irreducible connectiveness is reinforced when we reflect upon the Koestler's definition of the act of creation as *the spontaneous leap of insight that connects unconnected matrices of thought* (Koestler, 1964 p.45). We have here psychology of the spontaneous leap of insight and the element of philosophy of being connecting the unconnected matrices of thought. These considerations suggest that the unit of analysis here is not within creativity nor psychology but should be situated at the very act of creation itself which connects both domains.

The presented new theory of creativity in the last section offers an example of the approach to creativity in mathematics education, which addresses several raised issues. Its central philosophical significance is hidden in the concept of the bisociative frame, one of the main concepts of the theory abstracted from the definition of the act of creation as the two unconnected matrices of thought that get (or might get) connected during the insight. It is our epistemological tool, the instrument with the help of which we can identify areas of possible creativity in different theories of learning present in mathematics education. The ability of the bisociative frame to identify creative possibilities within different approaches to learning suggests a general question in front of the philosophy of creativity: should not creativity be taken as the foundation of learning in mathematics classrooms?

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Chapter 9
A Framework for Creative Insights Within Internalization of Mathematics

William Baker

9.1 Introduction

One explanation for the dominant role of constructivism in mathematics education is its clearly defined methodology of research in which the researcher acts as a teacher and embodies discovery learning, i.e., provides minimal verbal explanations to the student. In constructivist theory, understanding a problem situation begins with an attempt to assimilate its information into existing schemes. Acts of accommodation, viewed as based upon creative moments of insight, may occur when a problem cannot readily be assimilated into an existing scheme. Piaget describes the mechanism of accommodation, which he terms reflective abstraction, as occurring through reflection upon an existing scheme or solution activity in a problem-solving environment. The primary objective of this article is to use the theoretical framework of Koestler (1964) integrated with the work of constructivist theory, based upon Piaget, to analyze moments of insight within social discourse, i.e., the internalization of Vygotsky (1978).

In previous work, the bisociative framework for creativity Koestler (1964) has been translated into or integrated with that developed by Piaget and Garcia (1989) to analyze acts of accommodation (Baker, 2016). This earlier translation work (the integrated frame) was used to analyze individual moments of creative insight leading to the formation of new schemes. The integrated frame was extended to include the foundational type of reflective abstraction known as interiorization (Baker, 2021a; Czarnocha & Mason, 2021).

The first objective of this article is to review and modify the integrated frame to analyze students' moments of insight, leading to concept development, and accommodation, what Koestler refers to as growth in understanding, within social discourse. Learners struggling to internalize new content may not rate highly according

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to the measures of assessment associated with creative and gifted individuals, such as flexibility, fluency, proficiency, or in terms of productive-divergent thinking (Leikin & Pitta-Pantazi, 2013). That said, for such students, every moment of insight may represent an important step. In Vygotsky's framework, an activity that is initially viewed as externally directed becomes internal through a process of conscious imitation. Thus, the child first imitates or performs an activity with a teacher's assistance and ultimately acts independently. This framework is ideally situated for analyzing classroom instruction, especially teacher-led discourse. However, its drawback is that the process of internalization is not well understood. "As yet, the barest minimum of this process is known" (Vygotsky, 1978, p.57).

A second objective of this article is to clarify the notion of constructivist pedagogy to support moments of insight within social discourse. Constructionism clearly has a major influence on mathematics pedagogy, yet its implementation has been difficult on the institutional level. One problematic issue is that the role of the teacher managing discovery learning is unclear, or ill- defined (Baker, 2021b). One common explanation is that professional development efforts have not been sufficient to impart the essence of discovery learning. Another factor is that the strong focus of constructivist research and pedagogy on individual reflection and abstraction simply does not resonate with the practice of teaching and learning within social discourse. In this line of thought, the strict focus on an individual's subjective understanding is simply not well equipped to provide insight into the teacher's role in a social learning community. Thus, the second objective is to highlight the role of the teacher in guiding students to moments of insight within the process of internalization. That is in bringing the spirit of discovery embodied in constructivist methodology into the daily lesson.

9.2 Theoretical Foundations

9.2.1 Matrices Codes, Schemes

Koestler's (1964) notion of "bisociation" is used to describe moments of insight within a diverse range of fields including humor, art, literature, math, and science. The term bisociation is used to distinguish between associative routine thought – exercise of understanding (assimilation), as opposed to bisociative thought existing simultaneously in two previously unrelated frames of reference or progress in understanding (accommodation). Koestler develops his theory of creative insight in terms of matrices or frames of reference and the code or rules of the game that govern each matrix.

> The matrix is the pattern before you, representing the ensemble of permissible moves. The code which governs the matrix can be put into simple mathematical equations…or it can be expressed in words. The code is the fixed invariable factor in a skill or habit, the matrix is the variable part. The two words do not refer to different entities, they refer to different aspects of the same activity. (p.40)

In the integrated frame, matrices correspond to schemes, which are viewed as matrices whose codes can be expressed in a quasi-mathematical language. In constructivist learning, schemes represent activity that has been generalized into stable, predictable, and repeatable activity (Confrey, 1994a, p.4). Schemes, described in detail presently, are important in analyzing the cognitive effects of moments of insight in the individual's learning process. In the integrated frame, the term "code" translates into the constructivist notion of an "invariant relationship," which may be understood as the conceptual reasoning that underlies solution activity. Thus, the formation of schemes and development of concept are intertwined.

9.2.2 Blocked Non-assimilatory Situation and Discovery of Hidden Analogy

For Koestler (1964), creative insight takes place in a blocked situation, in which the subject does not initially have an appropriate matrix.

> [The situation] still resembles in some respect other situations encountered in the past yet contains new features or complexities which make it impossible to solve by the rules of the game that were used in those past situations. (p.119)

In constructivist theory, this is referred to as a non-assimilatory situation, in which the subject understands the goal in a situation but does not have an appropriate scheme to assimilate the situation and thus complete the goal. Koestler describes selective attention on previously unimportant aspects of a situation, alongside disregard for other details as the foundation for moments of creative insight to resolve a blocked situation, "selective emphasis on the relevant factors and omission of the rest" Koestler (1964, p.72). In creativity theory more generally, selective attention can be understood as three processes: selective encoding, in which the subject applies consciousness attention to a previously overlooked aspects of the situation; selective comparison, in which the subject compares previous knowledge with the given situation; or selective combination of existing but previously unrelated aspects within the situation (Perking, 2000; Davidson, 2002).

Koestler (1964) refers to the resolution of a blocked situation, as the result of selective attention as the discovery of a hidden analogy: "[T]he displacement of attention to something not previously noted, which was irrelevant in the old and is relevant in the new context; the discovery of hidden analogies…" (p.120). Koestler (1964) describes the search through one's collection of possibly related matrices leading to the discovery of a hidden analogy that will remove the block as looking for something one does not know:

> We have seen that one of the basic mechanisms of the Eureka process is the discovery of a hidden analogy…Yet the word 'search,' so often used in the context of problem-solving, is apt to create confusion because it implies that I knew what I am searching for, whereas in fact I do not…[T]he subject looks for a clue, the nature of which he does not know except that it should be a 'clue'…a link to a type of problem familiar to him. Instead of looking through a given filter-frame for an object which matches the filter, he must try out one form

> after another to match the object under his nose, until he finds the frame which it fits, i.e., until the problem presents some familiar aspect- which is then perceived as an analogy with past experience and allows him to come to grips with it. (pp.653–654)

In creativity research, like problem-solving research, the subject often does not have a matrix-scheme to assimilate the situation and thus searches (selective comparison) to find a hidden analogy. In constructivist theory, this search leads to a situation-activity link, which is the first component of an action scheme. The second type of search (selective encoding) occurs after a relevant matrix has been found, yet the subject has not learned how to use it to obtain the desired effect: "The problem in problem solving consists firstly in discovering which routine is appropriate to the problem-what type of game is to be played, and secondly, how to play it-i.e., which strategy to follow, which members of the flexible matrix are to be brought into play according to the lie of the land" (Koestler, 1964, p. 638). In constructivist theory, this leads to an activity-effect link.

9.2.3 Bisociation

Koestler (1964) describes the mechanism of creative insight, "bisociation" as the result of an idea, event, or concept existing or vibrating on two previous unrelated matrices:

> [T]he perceiving of a situation or idea L, in two self-consistent but habitually incompatible frames of reference M1 and M2. The event L in which the two intersect is made to vibrate simultaneously on two different wavelengths, as it were. While this unusual situation lasts, L is not merely linked top one associative context, but bisociated with two. (p.35)

The effect of the bisociative vibration of concepts on two previously unrelated matrices, during the discovery of a hidden analogy, is the transfer of thought from one matrix to another: "The sudden transfer of a mental event with two habitually incompatible matrices results in an abrupt transfer of the train of thought from one associative context to another" (p.59). Thus, for Koestler (1964), the transfer of analogic reasoning during the discovery of a hidden analogy results is the result of a bisociative synthesis of existing matrices:

> The creative act is not an act of creation in the sense of the Old Testament. It does not create something out of nothing; it uncovers, selects, re-shuffles, combines, synthesizes already existing facts, ideas facilities, skills. The more familiar the parts, the more striking the new whole. (p.120)

9.2.4 Understanding and Discovery

Skemp (1987) suggests that "To understand something is to assimilate it into an appropriate schema" (p.29). However, in a blocked or non-assimilatory situation, there is potential for growth in understanding. Sierpinska (1994) comments on the relationship between discovery and understanding when she states "While any invention assures understanding, the latter does not necessarily imply the former…" (p.69). We take the distinction to be that, while understanding thought assimilation (exercise in understanding) occurs without discovery, the process of discovery (re-discovery) or creative insight always results in what Koestler refers to as growth in understanding. Piaget (1977) expresses the view that growth in understanding is based upon moments of insight:

> To understand is to discover, or reconstruct by rediscovery, and such conditions must be complied with if in the future individuals are to be formed who are capable of production and creativity and not simply repetition. (p.20)

Here we take Piaget's use of the phrase "to understand" as referring to accommodation of new information through scheme development, i.e., growth in understanding.

9.2.5 Integrated Frame: Bisociation-Koestler

In the integrated frame, based upon the work of Koestler (1964), moments of insight occur when an individual finds themselves in a blocked situation. Their understanding of the blocked situation, and search to resolve it, is referred to as their search matrix or M1. M1 may be a scheme or some part of a scheme that has been applied in similar situations but is inadequate to assimilate the situation. This creates a tension between their M1 understanding of the situation and their motive to resolve it; constructivists often refer to this as "perturbation." The subjects' search to understand and resolve the situation may, in a moment of insight, lead them to discover a relevant M2 matrix previously unrelated to their M1 (discovery of a hidden analogy-selective comparison) or it may lead them to uncover a feature of their M1, previously not considered relevant (selective encoding). The uncovered or hidden features of M1 are often properties of the objects being acted upon, that when realized direct solution activity. These uncovered features can be understood as a matrix M2 that emerges from the M1 matrix.

In the moment of insight, the discovered analogic matrix M2 undergoes bisociative synthesis with M1; during this process, analogical M2 reasoning transfers to M1 directing solution activity.

During selective encoding, the uncovered matrix M2 has its genesis within aspects of M1; in this situation, the uncovered concepts exist organically within M1and M2 (bisociation). The conceptual relationships established during the bisociation of M1 and M2 represents the birth of what Koestler (1964) would refer to as a "pseudo-code," i.e., the building block for a new matrix- code, that has typically not yet been "reified" into a code (p.639). Thus, the end-product of a moment of insight is novel activity based upon a newly formed conceptual relationship or developing code. Note, such moments of insight take place within problem-solving but are not the result of deductive reasoning, as the actual moment of insight is intuitive in nature insight and has been described by Koestler (1964) as being induced through, “putting thinking aside” (p.182).

Thus, moments of insight in the integrated frame have three defining characteristics: the search process, the connection between M1 and M2 that leads to a transfer of analogic thought, and finally a novel activity that is directed by the conceptual relationship established.

9.2.6 Reflective Abstraction and the Formation of the Action Scheme

Koestler’s frame has several weaknesses for analyzing insights leading to the development of mathematical structure; the first is that because it is used so broadly, it is vague, and a second related issue is the relationships between concepts, matrices, and solution activity are not described in depth.

Piaget’s notion of reflective abstraction expresses acts of accommodation as containing two parts. The first part is the projection of an existing relevant scheme into a blocked or non-assimilatory situation. The projected scheme, and the subject’s sense of why it is lacking in the situation their M1. The projected scheme cannot assimilate the situation hence, it requires coordination with new aspects of the problem, selective attention to these new aspects results in concept formation and/or discovery of new scheme (M2) that when coordinated with M1, what Piaget and Garcia (1989) refer to as “constructive generalization,” and Koestler as bisociation, resolves the situation.

Simon et al. (2004) consider that the two-stage notion of reflective abstraction, as presented by Piaget, is simply not well understood and, thus, not useful in guiding pedagogy in mathematics educational research. “It is our contention that the mechanism itself is underspecified for guiding the design of instructional interventions intended to address challenging learning problems in mathematics” (p.313). These authors re-interpret Piaget’s notion of reflective abstraction in terms of Von Glasersfeld’s (1995) action-scheme, which contains three parts: a situation, activity, and its effect. The action-scheme or scheme can be expressed as two connections or relationships, the first is the situation-activity relationship (S-A), and the second is the activity-effect relationship . Simon et al. (2004) provide a plausible and readily understandable metaphorical descriptions of the search process leading to an A-E relationship, built upon the abstraction of invariant relationships:

> We offer the following physical metaphor to promote an image of the records of experience and how they are used in reflective abstraction. Each record of experience can be thought of as being stored in a jar. Inside of each jar is a particular instance of the activity and the effect of that activity. Each jar is labeled as to whether the record of experience inside was associated with a positive result or a negative result. In the first phase of Piaget’s reflective abstraction, the projection phase, jars are sorted according to their labels (i.e., learners mentally-though not necessarily consciously--compare /sort records based on the results).

> In the second phase, the reflection phase, the contents of the jars that have been grouped together are compared and patterns observed. Thus, within each subset of the records of experience (positive versus negative results), the learner's mental comparison of the records allows for recognition of patterns, that is, abstraction of the relationship between activity and effect. (p.319)

Thus, in the first, projection phase, the subject attempts solution activity based upon a relevant but insufficient scheme M1. In the second reflection phase, the subject abstracts the invariant relationship or conceptual relationship from his previous attempts, i.e., the A-E-dyads. This abstraction is the result of coordination between the features of the new situation and properties of the objects being acted upon highlighted by the inadequate effects of this solution activity.

The formation of an A-E-dyad within the transition from intuitive, or externally directed thought to independent, "interior thought is described in Steffe (1991). His work is summarized below and is taken as the blueprint for the constructivist notion of "interiorization." It is meant to understand how human cognition, historically and individually, develops new conscious structure from our empirical reasoning ability, i.e., our common-sense reasoning based upon our perception of the world.

9.2.7 Interiorization

The term interiorization is often used by constructivists to describe the genesis, or birth of a process (interior scheme). While there is general agreement that interiorization is the result of reflection upon and abstraction of solution activity, succinct descriptions are harder to find. Sfard (1991) credits Piaget for the following characterization of an interiorized process as one that could be "carried out through [mental] representations, and in order to be considered, analyzed and compared, it needs no longer be actually performed" (Sfard, 1991, p.18). That is what is meant when they say the A-E relationship has been abstracted or made interior. In Czarnocha et al. (1999), activity before interiorization is referred to as taking place within an action conception, where an action is defined as a transformation that "is a reaction to stimuli that the subject perceives to be external" and hence cannot be carried out independently (p.98). These authors consider an interiorized scheme to be a "process" which is distinguished from an action conception by the conscious control the individual has over it. Arnon et al. (2014) suggest that repetition and reflection are central to promoting interiorization, "As actions are repeated and reflected upon, the individual moves from relying on external cues to having internal control over them. This is characterized by the ability to image carrying out the steps without necessarily having to perform each one explicitly and by being able to skip steps as well as reverse them" (p.20). These statements provide insight into the nature of an interiorized process. However, they do not describe the actual process of interiorization.

Steffe (1991) refers to an internal process as one that proceeds by making a mental reference to physical objects. In this situation, the child can use his intuitive M1 scheme to complete count-up activity and has internalized the process, by making mental references to physical objects even when they are not present. However, when the count-up value increased beyond the child's spontaneous control (from 3 to 5), he lacked the cognitive ability to coordinate his mental reference of objects with this count-up process, and thus he did not know when to stop counting. Applying selective attention to the five objects he needed to count-up, the child abstracted a numerical conception for 5 objects (M2). He could coordinate this conception with his count-up process and thus successfully stopped at the correct result. This novel activity is said to be an interiorized process because it is directed by the subject's conscious reasoning without the need for mental references to physical objects (internal process). Thus, the birth of the interior counting-up process occurred simultaneously with his conception of the numerical value 5. In the integrated frame, his internal process of counting-up is his M1; his newly developed number conception is his M2, which can be said to emerge from M1 through selective encoding. This conception underlies his new count-up activity, i.e., the pseudocode for his novel count-up scheme. This process is referred to as interiorization, and it presents what is presumably a fair description of the historical development of how humankind learned the count-up process.

9.2.8 Participatory and Anticipatory Schemes

Tzur (2007), Simon et al. (2016), and Tzur (2021) refer to a subject's need for assistance or use of situation-dependent schemes as being a participatory scheme. They describe the so-called "next day effect" in which a subject can use a scheme in class but cannot act appropriately in the same situation the next day as evidence of participatory schemes. With a participatory scheme, the A-E relationship is developing, and the S-A relationship requires prompts or external assistance:

> The participatory stage is characterized by a provisional, prompt-dependent access to a newly forming scheme…at this stage the learner has abstracted a new anticipation, that is, an activity-effect [A-E] dyad…this is yet to be linked with a situation/goal part of a new scheme. (p.333)

These authors refer to a participatory scheme that has become interiorized, as an anticipatory scheme. Hackenberg (2010) characterizes an anticipatory scheme as one in which the subject understands the conceptual reasoning for the activity and thus can anticipate the results, "anticipation involves the attempt to attain the results of prior experience by generating the cause of them" (p.387). She notes that, with an anticipatory scheme, because the subject knows the cause, it may be used for planning and reflection.

9.2.9 Moments of Insight and Internalization-Interiorization

Tzur (2021) suggests that moments of insight occur within both participatory and anticipatory schemes. Those moments of insight that occur across A-E dyads lead to an understanding of the invariant nature of the A-E relationships, what he refers to as the "logical necessity," i.e., the cause or pseudo-code of a newly formed A-E dyad. In the work of Hackenberg (2010), the term conceptual reasoning is used to signify when the subject understands the cause/logical necessity or invariant nature of the A-E relationships. Thus, a conceptual understanding of the cause allows the subject to abstract the A-E from the situation, allowing it to be used as input in another situation, reversed, and combined or synthesized with other processes. In short, understanding the conceptual reason for a previously participatory scheme is the hallmark of an "interior" scheme.

In this article, participatory schemes reflect the process of internalization, i.e., the subject has a need for assistance from the learning community. Elucidating the relationship between internal and interior process is a central research issue. Tzur's (2021) statement highlights the research question of interest: "what is the nature of moments of insights within a participatory scheme, i.e., during internalization?" It also raises an important pedagogical issue, what markers can a teacher use as evidence of internalization.

9.2.10 Internalization: Vygotsky

Vygotsky (1978) posits that learning, in its early stages, takes place through communication with adults who model meaningful activity:

> [w]hen the child imitates the way adults use tools and objects, she masters the very principle involved in a particular activity…repeated actions pile up, one upon another…the common traits become clear, and the differences become blurred. (p.22)

The notion of internalization as developed by Vygotsky (1978) represents the transition from interpersonal (external and socially directed) activity to intrapersonal (activity under one's internal control), through a process of conscious imitation:

> We call the internal reconstruction of an exterior operation, internalization…The process of internalization consists of a series of transformations: (a) An operation that initially represents an external activity is reconstructed and begins to occur internally… (b) An interpersonal process is transformed into an intrapersonal one. Every function in the child's cultural development appears twice: first, on the social level, and later on the individual level…(c) The transformation of interpersonal to intrapersonal one is the result of a long series of developmental events.

Vygotsky illustrates this transformation with a child's attempt to grasp an object, and how a parent gives meaning to this grasping process, and transitions it into pointing "The child's unsuccessful attempt engenders a reaction not from the object but from another person…consequently the primary meaning of unsuccessful grasping movement is established by others" (Vygotsky, 1978, p.56).

Internalization, especially during the transition from arithmetical to algebra thought, is brought about through socially mediated instruction as the subject reflects upon their intuitive-spontaneous concepts to build organized structures, i.e., scientific concepts Vygotsky (1997). The internalization process begins with recall and like "interiorization" ends with understanding the cause or logical necessity of the activity. "For the young child, to think means to recall; but for the adolescent, to recall means to think. Her memory is so 'logicalized' that remembering is reduced to establishing and finding logical relations; recognizing consists in discovering that element which the task indicates has to be found…At the transitional age all ideas and concepts, all mental structures, cease to be organized according to family types and become organized as abstract concepts" (1978, p.51). Vygotsky (1978) describes this transition from spontaneous to abstract concepts as passing through a phase of linking similar examples (schemes) together in series, as one develops a sense of "family type." "Children's concepts relate to a series of examples and are constructed in a manner similar to the way we represent family names" (1978, p.50).

9.2.11 Appropriation and Internalization

Vygotsky's notion of an internal scheme, which is used in this treatise, refers to the process of learning within mentor-led discourse, leading to independent activity through the subjects' understanding of the "logical" or conceptual relationships that underlie the activity. Both characteristics are used by constructivists as markers for interiorization. The need to verify independent activity is embedded in the first internalize and then verify-test methodology of most math classrooms.

For constructivists, who study an individual's solution activity, evidence of the conceptual relationships that underlies an interior process or scheme is provided by the ability to understand when the scheme/process is relevant in a new situation, to reverse the process, and/or to coordinate it with other schemes, e.g., use it as input into another scheme. For social constructivists, who focus on social discourse, understanding conceptual relationships is evidenced by the ability to communicate such knowledge to other members in the learning community. The term appropriation is often used to analyze learning within social discourse; it has its origins in the work of the Russian psychologist M.M. Bahktin to understand how children learn language.

> The word in language is half someone else's. It becomes "one's own" only when the speaker populates it with his own intention, his own accent, when he appropriates the work, adapting it to his own semantic and expressive intention. (Bahktin, 1994, p.293)

This explanation suggests there are two phases to appropriation; the first is grasping and interpreting another's communication and the second is making it your own. The goal of the first is to assimilate another's communication at least partially. Bahktin describes this phenomenon as being half yours and half someone else's. The goal of the second phase is to transform this partially assimilated understanding into one's own internal structure, adding it to one's repertoire or toolbox of concepts, words, and activities.

The use of the term appropriation to describe the internalization of an activity through communication requires some caveats in the translation from language to mathematics. In language learning, the second phase of appropriation is often embodied in one's ability to use a word in a productive manner or perhaps to explain what the word means. However, as pointed out in Confrey (p.43, 1994b), the ability to explain the definition of a math concept (definition of a function) does not imply the "ability to act accordingly." In Baker (2021b) a similar situation was highlighted, in which a subject could explain or interpret the work of another; however, they were unable to engage in independent solution activity. Thus, independent activity remains a cornerstone for evaluation of internalization and learning in general.

The painful reality for teachers of the 'next day' effect, and the difficulty translating students' ability to talk or communicate during today's classroom with the ability to act independently on tomorrow's exam, suggests that a more significant analysis of appropriation is needed. Although this is beyond the scope of this work, it is important to note that, when looking for markers of appropriation in student communication, the quality of the student's cognition must be a deciding criterion. In other words, having a sense of another's communication (first phase) does not mean you have made it your own (second phase). Thus, the ability to memorize a definition, does not imply one has any idea how to use it. The ability to interpret another's activity does not necessarily mean you understand why they employed it, how it works, or engage in such activity yourself.

In this study, beyond the criteria of independent activity, evidence for the completion of the second phase of the appropriation process (making it one's own) is taken as the communication of the conceptual relationships that underlie one's solution activity, the abstraction of solution activity into family types, and the abstraction of the objects acted upon, into relevant categories. Sfard (2020) introduces the term "commognition" to refer to the quality of the cognition, specifically, the level of abstraction, communicated in dialogue. The notion of commognition includes three processes, "Saming, Encapsulating, and Reifying." Saming involves relating previously unrelated aspects of a situation. Encapsulation involves understanding that what was previously viewed as separate objects are one entity. Reifying involves giving an A-E that is at least partially related to a situation, a name, thus indicating it has reached a noun status or object level. Appropriation in this context, alongside the ability for independent activity, will be taken as evidence of the completion of an internalization as well as interiorization.

9.3 Pedagogy: Constructivism and Vygotsky

There is a lack of certainty about what exactly is "constructivist pedagogy" and heated debate about the causes of its failure when applied system-wide in the mathematics classroom (Baker, 2021b). That said, constructivist pedagogy is associated with instruction founded upon an individual's reflection on available schemes. In particular, the second tenet of constructivism states instruction should not begin with unfamiliar definitions or activity beyond student experience. The role of the instructor is to guide students with minimal instruction to reflect upon their schemes:

> If the pedagogue is a Piagetian constructivist, he/she will refrain from verbal explanations because he or she believes the source of understanding is in the individual's actions of physical or mental objects. (Sierpinska, p.39, 1998)

Thus, a central feature of the role of an instructor in constructivist pedagogy can be understood as to present helpful problems that assist in reflection upon available schemes during solution activity that leads to accommodation. This methodology requires the use of well thought-out, innovative examples that are designed to "(a) Bring forth learning relevant activity-effect anticipations and (b) Bring forth noticing of intended effects" Tzur (2021, p.329). In constructivist pedagogy, the desired effect is to induce interiorization, understood as the foundation for growth in process-object duality (Dubinsky, 1991; Czarnocha et al., 1999; Sfard, 1991).

In contrast, Vygotsky's notion of internalization involves a subject's understanding of an externally modeled activity, one that may not be an available scheme. Ideally, in Vygotsky's social constructivist methodology, instruction is within the upper level of the student's ZPD, i.e., a bit above, yet relatable to their available schemes. In this methodology, higher level activity, notation, and content are to be introduced, and the instructor's role is to assist the student make a connection to their existing schemes that will provide meaning to the higher-level activity-notation-content (Baker, 2021b).

Berger (2005) argues that the focus of constructivism on transforming activity on existing schemes into interior processes does not account for how students learn math symbols and terminology within social discourse:

> But much of this process−object theory does not resonate with a great deal of what I see in my mathematics classroom. For example, it does not help me explain or describe what is happening when a learner fumbles around with 'new' mathematical signs making what appear to be arbitrary connections between these new signs and other apparently unrelated signs. Similarly, it does not explain how these incoherent−seeming activities can lead to usages of mathematical signs that are both acceptable to professional members of the mathematical world and that are personally meaningful to the learner. I suggest that the central drawback of these neo−Piagetian theories is that they are rooted in a framework in which conceptual understanding is regarded as deriving largely from interiorized actions; the crucial role of language (or signs) and the role of social regulation and the social constitution of the body of mathematical knowledge is not integrated into the theoretical framework. (pp.154)

Berger does not appear to completely accept the notion, embedded in constructivist pedagogy, that new notation and schemes should not be presented, until all perquisite schemes and notation have been understood. Indeed, as the instructor, she expects students to "fumble around" and make "arbitrary connections" as the new notation is at the upper level of the student's ZPD. This raises the question, "what type of activity is required to induce a moment of insight leading to concept formation when the subject does not have a suitable scheme?" If direct definitions are of no avail, can the mentor actively guide, present, or explain to the mentee suitable examples and to what extent can listening to the explanations of a mentor result in concept development for a mentee?

As noted, unlike constructivists, Vygotsky believes that meaning for activity comes the development of scientific concepts within instruction. "The scientific concepts evolve under the conditions of systematic cooperation between the child and the teacher. Developmental and maturation of the child's higher functions are products of this cooperation" (Vygotsky, 1997, p.148). From a pedagogical viewpoint, the keyword here is cooperation, and this does not imply direct instruction:

> A concept is more than the sum of certain associative bonds formed by memory…it is a complex and genuine act of thought that cannot be taught by drilling…Practical experience also shows that direct teaching of concepts is impossible and fruitless. A teacher who does this usually accomplishes noting but empty verbalism. (Vygotsky, 1997, pp.149–159)

Koestler, whose bisociation theory is the basis for the integrated frame, believed that spontaneous acts of creativity occur throughout the learning process, but only in untutored learning. "Minor bisociative processes do occur on all levels and are the main vehicles of untutored learning" (Koestler, 1964, p.658). Thus, like most creativity theory, the focus is on the individual's search process, and not instruction, or assistance in guiding this search process.

Indeed, Koestler devotes an entire section of his book to the "boredom of science" in which he argues that formalistic discourse has reduced math and science to a boredom of definitions, ruining the spirit of intuitive discovery required to appreciate the beauty of these subjects:

> The same inhuman, in fact anti-human-trend pervades the climate in which science is taught, the classrooms, and the textbooks. To derive pleasure from the art of discovery, as from other acts, the consumer-in the case, the student- must be made to re-live, to some extent, the creative process. In other words, he must be induced, with proper aid and guidance, to make some of the fundamental discoveries of science by himself, to experience in his own mind some of those flashes of insight which have lightened its path. (Koestler, 1964, p.265–266)

As a teacher researcher, this work is founded on the belief that students at all levels of mathematics must experience moments of insight to relieve the boredom of mathematics and to grow in understanding. Following Vygotsky, the focus of this study is internalization within social discourse, and the pedagogical issue is to highlight the role of the teacher in inducing such moments of insight.

9.4 Research

The goal of this study is to provide supporting evidence that bisociative moments of insight occur and are the mechanism for acts of accommodation during internalization, i.e., when the subject requires mentor assistance, thus extending Koestler's view that such moments of insight occur primarily in "untutored learning." To accomplish this goal, the primary objective is to review the integrated frame used to analyze an individual's moment of insight leading to acts of accommodation, developed previously, and then extend this frame to include internalization. The first research question is to describe the genesis of concepts during internalization and compare this to genesis of concepts embedded in developing schemes within the process of interiorization, as presented by Steffe (1991). A second related research question is to investigate how matrices or schemes emerge and are grouped together into conceptual structures, or toolboxes of schemes, during internalization, essentially what Vygotsky (1978) refers to as a "family type." In this second research question, the constructivist notions that are useful to analyze these moments of insight to build up a toolbox during internalization include Piaget's notion of "constructive generalization" and the notion of invariant relationships.

Pedagogically, the research objective is to highlight how teacher-mentors can support moments of insight within the internalization process, i.e., social discourse. The empirical evidence collected will then be used to briefly reflect upon the role of the teacher within the context of constructivist and social constructivist pedagogy.

9.5 Stages of Internalization: Empirical Examples

Vygotsky's conception of internalization suggests many levels of development the child may experience as they internalize mathematical content. In our analysis, we consider two overarching categories; one is characterized by passive reception or processing of teacher-mentor- directed activity (external), and the other is marked by an active effort to provide meaning for such activity. Within the active category of internalization, we consider three stages. These stages loosely correspond to Berger's (2004a, b) use of Vygotsky's three stages of concept development: heap, complex, and pseudo-concept/concept.

9.5.1 Interpersonal Learning Social Participatory Stage

For Vygotsky, internalization involves conscious imitation of adult behavior; as such it begins with watching and listening to modeled solution activity, which is perceived as externally driven. During the social participatory stage, internalization begins as the learner (albeit passively) struggles to make sense out of modeled solution activity, new definitions, or symbols.

Example 1: The Elephant Aha Moment (Kadej (1999), Czarnocha and Baker (2021, pp.95–98)).

The discovery of a hidden analogy to abstract a concept within direct instructional methodology.

Two children were trying to solve the algebraic equation $x + (x + 12) = 76$, one child-mentor readily understood the task and was attempting to explain it to his peer-mentee who did not understand the concept of the variable "x" as representing an unknown numerical value. Following the textbook, the mentor-child points out that the "x" term can be viewed as a blank square, into which one inputs any numerical value. This simply confuses the peer-mentee, who is thinking along a concrete line of reasoning and, thus, rejects the obvious inconsistency that "x" is a blank square. Then, the mentor child describes the blank square as a window while simultaneously simplifying the problem as two windows (blank squares) that are equal to 64, while asking "what is each window?". However, the mentee child continues to engage in concrete reasoning rejecting the comparison of a blank square to a window as more confusion. Finally, the mentor child attempts an analogy using elephants, asking "two elephants are 64, what is each elephant?". The mentee child ponders and then proclaims one elephant is 32. I understand now!

The mentor reframes the situation to make sure the mentee understands, asking "if two elephants are 60, how much is each?". Immediately the mentee-child provides the correct response.

9.5.2 Discussion: Example 1

Initially, the mentee-child grasps very little of the variable concept; thus, the M1 search matrix does not contain any scheme to reflect upon; instead, it consists of a nonsensical symbol "x." The mentor employs a common approach of introducing different analogies to guide the mentee towards this concept. Radford (2003) describes such a situation as the tension between the individual and the learning community leading to "transitional language" that results in the individual giving meaning to symbols using "metaphor." Initially, the subject does not relate to these transitional terms – they are simply not part of his toolbox. The discovery of the elephant metaphor can be seen as what Matsushima (2020) refers to as dynamic composition within appropriation, i.e., it exists in the mind of the mentor first and is borrowed by the mentee. The moment of insight occurs when the mentee grasps the uncovered previously hidden analogy between taking half of two elephants-M2 and the symbol-variable "x"-M1. During this bisociative moment of realization, the concept of an unknown exists in both matrices, and the intuitive M2 matrix gives meaning to the mentee-child's previously non-existent M1 solution activity. Thus, the bisociative connection between elephants and the symbol x results in the internalization of the variable concept, a novel concept, i.e., growth in understanding.

In Steffe's (1991) example of interiorization, M1 is an intuitive count-up scheme, while M2 is the abstraction of the number concept required to end the count-up process. In this case, M1 is a search matrix with no scheme; indeed, the variable “x” appears to have little meaning to the subject. The M2 analogy that gives meaning to the variable x or M1 is an intuitive scheme the child can grasp. Pedagogically, this is a good example of Vygotsky’s educational approach, in which a mentor is ahead of a child’s developmental level, yet within their ZPD.

9.5.3 Active Internalization: Interpersonal to Intrapersonal – Heap Stage

Vygotsky (1978) notes that for children to think means to recall, and at this stage understanding consists of the recall of activities viewed as “isolated instances” with little to no logical structure or conceptual understanding. At the early stages of internalization, the link between situation and activity is essentially founded upon imitation of previous examples; thus, if the teacher mixes up examples or reintroduces after a pause, a similar example, students at this level may be confused and resort to guessing what activity is appropriate or choose a non-relevant activity based upon a superficial understanding of the new situation.

Berger (2004a) suggests that in heap thinking, “the person links ideas or objects together as a result of an idiosyncratic association” (p.3). She elaborates on the quality of the link at this stage, noting that “objects are linked by chance in the child’s perception” (p.5). Berger refers to such links as surface associations, which result when the solver reads and interprets problem information in a superficial manner. Berger provides examples in which a solver attempts to employ symbols or phrases (keywords) in a problem situation without reasoning based upon any real understanding. Their selection of symbols and keywords is designed to simplify their cognitive load. Hence, the resulting activity is often incorrect.

Solvers at this stage, lacking logical structure, or what Vygotsky refers to as a notion of “family type,” frequently attempt to direct their activity through imitation of modeled activity. Thus, learning often consists of reviewing, class notes, textbook examples, or videos of similar exercises to assist in problem solving. At this stage, the S-G and A-E relationships are weak based upon superficial associations, and the so-called “next-day effect” where solvers perform with some degree of proficiency in class, but not independently, i.e., the next day.

9.5.4 Example 2: T-INTERVAL

This example is taken from work with students in an online (zoom) math class during the pandemic. It represents an example of students cooperating, which was not a common event as most students were very passive while participating using online platforms.

Previous class time had been spent on using calculator commands for hypothesis testing and confidence intervals. Wendy internalized that a collection-toolbox of commands were available using the STAT/TEST menu. She also understood that for a normal distribution, one typically used either the Z-Test or the Z-Interval command in this menu or toolbox of commands. However, this problem required her to understand a new concept, T-distribution. Indeed, students were now being asked to choose between two types of commands involving either Z or T distributions, further complicating the issue that each type could require one of the two categories, either internal confidence or testing commands. There was a third involving sample proportions, but it will be simpler for the reader if left out of the discussion. Thus, she needed to generalize the two commands learned previously into two categories, and two types, or family names.

Students traditionally experience a high degree of uncertainty and frustration trying to understand which of these commands is appropriate. The mentee student (Wendy) is in the process of consciously imitating modeled problem-solving behavior; she knows to use the STATS/TESTS toolbox collection of commands on her calculator. However, she is not sure which one to use. At this point, Wendy brings the exercise to class and asks what to do! (Interpersonal process). Wendy states that it appeared to involve a T command instead of Z command; she was unsure why and was not clear about whether it involved a test or interval command, and her voice suggested she was overwhelmed.

The instructor, who had modeled the variations of problems require several times, realizes that Wendy needs another voice (preferably a peer, not an authority) to explain, and asks if any other student can help her. A mentor student (Marisol) volunteers to explain. Marisol first, explains that the question asks for an interval, and thus you need to use an Interval command under the TESTS options. This explanation helps Wendy formulate the concept-categories of interval versus testing commands. It also reduces the search to discriminating between the two family types, Z- Interval or T-Interval. Wendy listens attentively and appears to understand. Second, Marisol points out that there is a hint which explicitly states that, because there is a sample standard deviation, the problem requires a T-distribution, and thus, she concludes the T-Interval command is appropriate. At this point, Wendy is silently processing what Marisol has said, then indicates she now understands, and thanks Marisol.

9.5.5 Discussion

Wendy's initial understanding of the situation (M1) included a sense of what to do, based upon her appropriation of previous examples (her hybrid or partial conception). However, her partial conception was inadequate to support independent activity. Her statement that it was probably a T-distribution (not sure why) suggests a faltering ability to express this appropriated conception. Thus, she needed confirmation to enter confidently into the second phase of appropriation, in part because the need to navigate four variations of commands was overwhelming. Instead, her search process, classic to internalization, was to ask for assistance and then try to understand what she was told. Kosko (2014) refers to this as active listening. Thompson et al. (2004) suggest that such listening is the foundation of critical thinking and involves receiving knowledge, comparing received to previous knowledge leading to comprehension and finally to evaluation.

The mentor, Marisol, guided Wendy's attention to coordinate the previously unclear problem information, with the required commands (M2). First, Marisol guided by pointing out there were two overarching categories, testing – find the percent or interval problems. Wendy listens intently and spontaneously understands the need to discriminate between Test and Interval commands. Although online learning makes it difficult to observe the learners affect, it was clear that while listening to Marisol, Wendy received or appropriated the knowledge imparted by Marisol, and as she reframed from further questions and could act independently afterword this suggests she comprehended it, i.e., her growth in understanding allowed her to direct future solution activity – M2. This suggests that, as in the first example, a moment of insight or transfer from Marisol to Wendy occurred during active learning, providing another example of "dynamic composition" Matsushima (2020). Marisol's second explanation served to highlight uncovered problem information (the hint), which provided a positive evaluation confirmation for Wendy that her tentative appropriation knowledge (T-distribution) was correct.

Although, it was not possible to observe her affect (it was an online session, and her camera was off) it was clear Wendy's understanding of Marisol's comments was real and immediate, as she thanked Marisol. Furthermore, it became clear that Wendy had developed a new scheme, as she independently completed the related homework, and received a 100% on the next exam. Previously she was a B+ student. Wendy also demonstrated motivation and did well on the final and included several such (Test/Interval) problems with six different related commands required, indicating she had developed an organized toolbox for these six related commands or A-E-dyads under the STAT/TEST menu. Finally, it is worth noting that, in this example, the active listening by Wendy was essentially a reflection upon Marisol's guidance through a solution activity Wendy had previously attempted. Thus, although not exactly reflection upon her own solution activity, it contained elements of reflective abstraction, as well as reflection during active listening-appropriation.

9.5.6 Internalization Complex Stage

The complex stage marks the transformation of the operation into an intrapersonal process. This occurs with the development of a "family name" for the activity that is a sense of "problem type" obtained not necessarily by an understanding of the underlying structure rather through the linking of different but similar examples together one at a time. Berger (2004b) describes this stage in terms of developing a nucleus built up by relating similar examples and then linking them together initially by superficial but ultimately by their invariant relationships. Thus, it is in this stage that moments of insight occur as students bring their conscious awareness to their own independent activity comparing A-E-dyads for similar exercises to develop an invariant relationship and observe other's activity that somehow relate to their available schemes. Tzur (2021, pp.336) refers to such comparing as awareness of "across activity-effect instances."

Berger (2004b) suggests that the common pedagogical technique of teaching by examples often results in students (Complex Stage) incorrectly linking the observed activity in superficially similar but cognitively different problem examples. She illustrates this, with the example of students who understand that one multiplies the rate with the time to find the distance. However they continue with this multiplication-activity when the problem gives distance, time and asks for the rate, or gives distance, the rate and asks for the time. Another case where learning by examples yielding incorrect links is provided by Berger (2004b) when students who have learned that $f(x) = |x|$ is everywhere continuous but not differentiable apply this model to conclude the same is true for all absolute value functions even $g(x) = |x^2|$.

9.5.7 Example 3: The Domain Aha Moment (Czarnocha & Baker, 2021, p. 99–101)

A student understands the domain of the proto-type example ($f(x) = \sqrt{x}$) as being the values $x \geq 0$. She incorrectly uses this template to conclude that the domain of the similar function $f(x) = \sqrt{x+3}$ is also $x \geq 0$. At the direction of the instructor, she checks several negative integer values for x, including −3, and the student begins to realize something is wrong; when the instructor asks whether $x = -2$ works, the student ponders before declaring. "Those x's which are smaller than −3 can't be used here!". As noted in Czarnocha and Baker (2021, p.99–101), this marks the student's first realization of a new guiding principle. Thus, this realization is the result of selective encoding, in which the new action scheme M2 emerges from conscious attention to her available scheme M1. Selective combination-coordination is also involved as she coordinates her instructor-led activity with her initial understanding of the domain. The instructor, to determine whether an invariant relationship has been established, asks the student for the domain of $g(x) = \sqrt{x-1}$, and after pondering for a minute, she provides the correct response.

In this realization, the student's M1 was her previous understanding of the domain of the absolute value function $f(x) = |x|$, $x \geq 0$; after her work following instructor guidance, she realizes that values such as $x = -1, -2, -3$ are all exceptions, i.e., they are part of the domain for $f(x) = \sqrt{x+3}$; these exceptions to her M1 rule or code are the basis of a new conception or M2, and as she coordinates these values with her previous understanding, she generalizes the code from $x \geq 0$ to $x + 3 \geq 0$ ($x \geq -3$). Thus, she forms new S-A and A-E relationships evidenced in her understanding of the domain of the similar function $g(x) = \sqrt{x-1}$ ($x \geq 0.1$). With these moments of realization, she is developing a new code or invariant relationship for linear factors under a radical, based upon connections between similar examples or family type. This can be seen as an example of a realization leading to an invariant relationship across A-E dyads (Tzur, 2021, pp.336). It can also be understood as an example of the commognition Sfard (2020) refers to as "Saming."

9.5.8 Internalization: Pseudo-Concept Stage

At this stage, the subject shows signs of conceptual reasoning, i.e., activity based upon concepts and their relationship to the situation. Thus, they are developing an abstracted A-E relationship based upon conceptual reasoning that underlies connections. In Koestler terminology, we say that a code is being abstracted.

In the preceding example the subject has successfully provided the domain for $f(x)=\sqrt{x+3}$ and $g(x)=\sqrt{x-1}$ based upon the coordination of her initial scheme (domain of the radical x function) with her reasoning that the linear expression under the radical must be greater than or equal to zero (the invariant relationship). Next, the instructor asks for the domain of $h(x)=\sqrt{x-a}$ after pondering she has a second realization and provides the correct answer $x \geq a$. At this point, the student has abstracted the invariant relationship or guiding principle into an algebraic expression or code that works for all such problems (linear factors beneath a radical). Her ability to express the domain using symbolic notation can be viewed as a communication of her conceptual reasoning and hence as an example of appropriation, one of the characteristics for the completion of internalization. Thus, this realization involves an abstraction of the invariant relationship across A-E-dyads (Tzur, 2021, pp.336) or the abstraction of a new code.

9.5.9 Discussion

In this example, the constructivist notion of reflective abstraction begins to merge with Vygotsky's notion of internalization. Thus, reflection upon another's solution activity, i.e., appropriation demonstrated in previous examples is replaced with reflection upon one's own solution activity, albeit activity guided by the instructor, i.e., reflective abstraction. Analyzed using the integrated frame, the subject has an initial understanding of the situation M1 that directs her activity. However, it is based upon a template example, which she does not generalize correctly. Under the instructor's tutelage, she engages in solution activity with different values of x, until she experiences an "Uphs effect" or the realization that her initial M1 scheme is inappropriate.

In the initial moments of insight, the subject uses her innate ability to compare different concrete experiences of substitution to abstract the A-E dyad for $\sqrt{x+3}$, and the A-E-dyad $\sqrt{x-1}$ in this process, M1 was her understanding of the domain of $\sqrt{x}$ and M2 was her understanding of solving linear equations such as $x + 3 \geq 0$, $x-1 \geq 0$. Thus, the instructor-guided solution activity resulted in selective coordination-combination, or bisociation of her M1 and M2 schemes, that resulted in a new albeit local code for each A-E-dyad. This new code is the logical necessity or the "reason why," i.e., the cause that transfers from M2 to M1 for each dyad.

The second moment of insight occurs as she reflects across her newly constructed A-E- dyads for radical $(x + 3)$ and radical $(x - 1)$, which now represent her M1, and her insight can be viewed as the result of her innate ability for pattern recognition as she coordinates the problem, (finding the domain of the radical $(x + a)$ function) with these M1.

In this case, the subject recognizes and expresses in symbolic language the invariant conceptual relationship that underlies the linking of her A-E-dyads. This communication represents an act of appropriation (second phase) or the type of commognition Sfard (2020) refers to as "Reifying" as she is communicating her understanding of this invariant relationship with an algebraic formula.

9.6 Concluding Remarks

The goal of this article is to highlight moments of insight, as students struggle to internalize mathematics in a classroom situation. The primary research objective is to extend the integrated frame to internalization, and in so doing to highlight the relationship between this process, and the constructivist notion of interiorization.

The first research question is to describe concept development within the internalization process and compare this to concept development within interiorization. In the first example, the child is struggling to appropriate the concept of an unknown as expressed by the symbol "x," i.e., he is struggling to give meaning to a new object or sign. His motive is to understand what is being presented, and his M1 understanding is very limited. As the mentee presents different M2 analogies, the subject finally realizes (in a flash) the connection between taking half the given weight of two unknown (twin) elephants to find the weight of each one. This guided discovery of a previously hidden analogies allows meaning to transfer from his intuitive scheme of taking half of elephants to the required symbol manipulation of symbol "x." As the concept of an unknown emerges from his spontaneous or intuitive taking half scheme, it represents both the genesis of a concept as well as of a scheme. This act of appropriating another's understanding of a symbol, sign, or word name is essentially what Sfard (2020) describes as "reifying," i.e., the giving of a name (object level status) to a process. In the process of interiorization as described by Steffe (1991), the number concept (five) also emerges from an intuitive scheme and, thus, represents both the genesis of a concept and a process (count-up).

During interiorization, the M1 is the intuitive scheme, and the M2 is a concept that emerges through reflection upon one's M1 solution activity; hence interiorization is a foundational process of reflective abstraction. As this reflection uncovers what was previously known, albeit only intuitively, hence it can be viewed as selective encoding. In contrast, during internalization, the M1 is essentially not functional, and meaning or growth in structural understanding is obtained during active listening. This is an example of reflection during social communication, a process Vygotsky (1997) describes as synonymous with reflective consciousness. As the search to appropriate meaning of the symbol "x" involved active listening, the moment of insight can be understood as the guided realization of a previously hidden analogy, i.e., selective comparison without synthesis of schemes.

Internalization as presented by Vygotsky often involves the recall of externally modeled activity. Whereas, in the first example, the subject (heap stage) is learning a completely new concept embodied within a symbol, and M1 is essentially nonfunctional, in the second example (complex stage), the subject has an M1 scheme, based upon recall, and is in the process of building up a toolbox of schemes or a collection of schemes within a family type.

In this example, her M1 search matrix includes a vague recall of how to proceed, i.e., where to go on the calculator to find the appropriate toolbox of commands. However, she lacks the ability to select the relevant problem information, and coordinate it, with her solution activity. The instructor's methodology was to ask other students to explain. The mentor-student did not attempt to explain the concepts needed; instead, they focused on recognizing these concepts in the problem situation and how these concepts connect to and thus direct (novel) solution activity. The growth in understanding was to first discriminate between two types (categories) or problems, confidence interval and hypothesis testing, and second, between two types of tools or schemes (Z versus T distributions). Thus, through active listening, the subject learned that there is a new tool to be used in two types of problems, and she learned to recognize when this new tool was required. This growth in understanding can be understood as the result of a projection of an existing M1 into a novel situation, and the resulting coordination (bisociative synthesis) between M1 and M2 features required to resolve the problem, while leading to a new code. Hence, it is like Piaget's notion of "constructive generalization." This distinction being that it involves active listening as opposed to reflection upon one's own solution activity.

In the third example, although directed by the teacher, the subject's reflection is completely upon her own solution activity, not upon communicated knowledge; thus, her moments of insight can be analyzed within the integrated frame as reflective abstraction. As the instructor skillfully directs the student to reflect upon patterns of substitution, she is guided to an "Upps effect" and ultimately the synthesis of her M2 conceptual understanding of the (more-than /less-than zero) solution of a linear inequality with her initial M1 understanding of the domain of a radical, as being greater than or equal to zero. Thus, her moment of insight can be understood as "constructive generalization"; in that, she has an existing M1 scheme that is first projected into the new situation and then coordinated (bisociative synthesis) with features of this situation that were previously overlooked to begin the formation of a new concept and code.

In the constructivist frame, one can also understand her moments of insight as interiorization. In this view, her initial M1 scheme was dependent upon the situation (participatory schemes). The coordination (bisociative synthesis) of her initial M1 domain scheme with her M2 linear inequality scheme provides a good example of constructive generalization after the instructor recognition of the invariant relationships embedded in the patterns of substitution (discovery of a hidden analogy-M2). Her communication of these domains to the instructor represents the second phase of appropriation and indicates a new code is developing.

In the fourth example, the subject abstracted the invariant relationship of her previous work. Thus, she understands them as reflecting one code, encapsulation (Sfard, 2020), which she gives a name in symbolic form, reifying (Sfard, 2020). These latter examples highlight aspects of the second research question. In the second, active listening, on communication of content analogous to an existing scheme, leads to construction of similar "family-type" schemes. In the latter two examples, pattern recognition and abstraction of an invariant relationship lead to encapsulation and reifying of solution activity (processes) into a code expressed in symbolic form. This demonstrates her object level understanding of these processes as different schemes that can be organized as one, named entity.

Pedagogically, these examples demonstrate the effectiveness of guiding students to "Upps effects" through pattern recognition as opposed to direct instruction. They also show that peer-peer dialogue involving direct analogies can be very effective in leading to moments of insight.

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Chapter 10
A Reconsideration of Appropriation from a Sociocultural Perspective

Mitsuru Matsushima

10.1 Introduction

Why does interaction in the learning community deepen mathematics learning? How does individual learning contribute to the learning community through dialogue and deepen mathematics learning? These questions can be answered from not the viewpoint of dualism, which considers the learning of society and the individual separately, but from a sociocultural perspective, which considers these aspects jointly. Sfard (2008), who pioneered the unique concept of "commognition" based on psychology and philosophy, emphasized that communication represents thinking itself, focusing on the connection between individual thinking and community learning from a sociocultural approach.

Ernest (1998, 2010) demonstrated the structural deepening of mathematics learning within an individual from the perspective of social constructivism based on sociocultural perspective. Ernest (2010) showed that the key to facilitating mathematics learning was the publication of individual sign use and appropriation. The concept of appropriation took shape from Bakhtin's linguistic philosophy and Vygotsky's psychology, and it plays an important role in the study of learning from a sociocultural perspective. Appropriation is defined as a "process that has as its end result the individual's reproduction of historically formed human properties, capacities, and modes of behavior" (Leontyev, 1981, p. 422). Following the above studies, Matsushima (2020, 2021) connected individual thinking with learning communities from the perspective of appropriation to present the structure of deepening mathematical learning through dialogue as shown in Fig. 10.1. The two main features of appropriation are given below.

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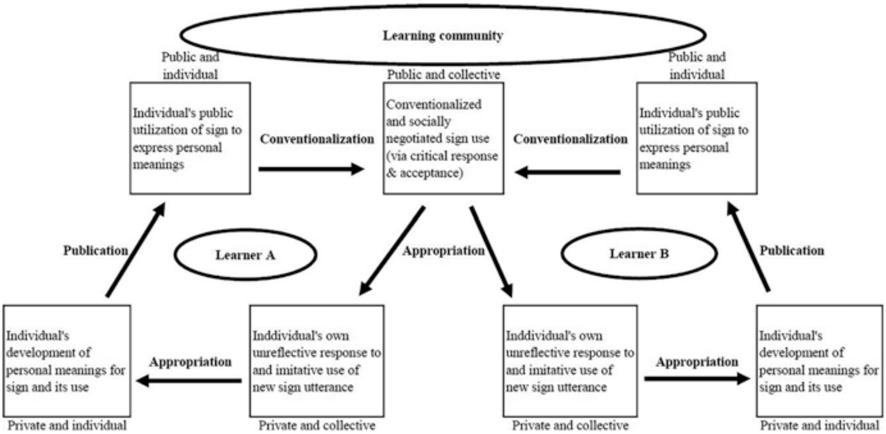

Fig. 10.1 Extended model of sign appropriation and use (Matsushima, 2020, p. 113)

Feature 1 (dynamic composition):

Gradually forming one's concept by speaking, while borrowing the concept of others.

Feature 2 (mutual composition):

The concept of learning community is formed in the process of forming the concept of self. (Matsushima, 2021)

As the above two appropriation features indicate, appropriation is dynamically composed of the concept of learner and learning community and mutually composed of both by bidirectionally influencing the learner's concept and learning community's concept. In other words, appropriation facilitates the concepts of individual learners and learning community to develop interactively (Brown et al., 1993). Conceptual development here refers to the ability to use language appropriately. In mathematics learning, being able to use mathematical words and signs appropriately indicates a deep understanding of mathematics.

> However, only a few studies have so far examined the concept of appropriation, with the concept itself appearing confusing. For example, few bidirectional discussions on appropriation can be found in the literature, with many studies examining the concept within each individual (e.g., Moschkovich, 2004; Solomon et al., 2021). Furthermore, it is difficult to distinguish between appropriation and similar concepts (e.g., Brown et al., 1993). One study points out the problem how to think about the effects of appropriation when the learner does not speak to others (Carlsen, 2010).

To solve these research problems, we need to first clarify the features of appropriation. Therefore, the purpose of this chapter is to reconsider the concept of appropriation and clarify its features.

10.2 Structure of This Chapter

In this chapter, we first review previous research on appropriation and point out three problems with the concept of internalization, which is closely related to the concept of appropriation. To show that the appropriation as a concept that overcomes the three problems of internalization, we first discuss how the learner's concept may be transformed by appropriation based on Figure 10.1. In the discussion, we proceed with the discussion separately for the case where the learner is the speaker of the dialogue and the case where the learner is the listener, and solve the first two problems of internalization. Then we will cite the discussion of intersubjectivity regarding the connections between individual learners and communities from knowledge of developmental psychology to overcome the third problem. Through these discussions, the concept of appropriation is clarified as a concept to overcome the three problems of internalization, and its six characteristics are pointed out. Finally, from the standpoint of a sociocultural approach, we will answer the following questions: "Why does dialogue deepen mathematics learning?" "Will mathematics learning deepen without dialogue?"

10.3 Problems Related to the Concept of Appropriation

As stated in the previous section, the concept of appropriation took shape from Bakhtin's philosophy of language and Vygotsky's psychology. Bakhtin (1981) explains the polyphonic nature of spoken language as follows:

> The word in language is half someone else's. It becomes "one's own" only when the speaker populates it with his own intention, his own accent, when he appropriates the word adapting it to his own semantic and expressive intention. (Bakhtin, 1981, p. 293)

This quotation outlines the concept of appropriation. People discuss a concept at the beginning using the concept of others and gradually form their concept based on it. Note the start point of concept formation here. The first point is borrowing the concept from others. Conventions such as concepts, ideas, and values of others are shared with the community even before we join the community. Individuals in the learning community assimilate conventions individually through appropriation, the start point of concept formation. Discussing and acting on these conventions also affect the concept formation of others and learning community. This bidirectional concept formation chain between individuals and the learning community transforms the conventions of individuals and the learning community. In other words, individuals and the learning community continue to form new conventions (Cazden, 2001; Rogoff, 2003). These features involve both dynamic and mutual composition (Matsushima, 2021). However, a new question arises here. How does one borrow the concept of others? Furthermore, are there any restrictions on borrowing the concept of others? The two features mentioned above do not answer these questions. This study therefore tries to examine whether these two features are valid and answers the above new questions in terms of the two previous studies that clearly state the appropriation features.

The first study describes three characteristics of the appropriation process (Nunokawa & Kuwayama, 2003) based on a case study as follows:

> (a) In the process of appropriation, the student created kinds of hybrids between the old and new ideas.
>
> (b) When the new idea was presented by the others, the student attempted to interpret it in the framework of the old method he had used up to then.
>
> (c) There were long time-lags even before the student began to incorporate some aspects of the new idea into the method he had used up to then. (Nunokawa & Kuwayama, 2003, pp. 303–304)

Characteristic (a) is the same as the dynamic composition of an individual's appropriation but refers to the quality of the object to be composed. The object created through appropriation is not completely new but based on the previous object. New ideas are thus constrained by old ideas. Characteristics (b) and (c) need to be recombined from the old framework to formulate the new object to be created in the new framework, but this takes time. These points relate to appropriation and resistance (Wertsch, 1998). Appropriation is carried out on the basis of existing individual and learning community conventions, but these existing conventions historically and culturally constrain appropriation.

The second study examined the appropriation process (Carlsen, 2010) and clarified the following five appropriation processes in mathematics learning:

1. Be involved in joint activity.
2. Establish a shared focus of attention with others. Students have to develop some kind of working consensus of what to pay attention to in a mathematical task.
3. Develop shared meanings of words and concepts, i.e., meanings in accordance with the mathematics community through participating in joint decision-making processes.
4. Be involved in the activity of transforming, a process where the students appropriate actions and utterances by fellow students in the collaborative problem-solving context and use them in ongoing activities.
5. Attend to the problem of the relationship between sense and meaning by identifying the relations between pre-existing established mathematical knowledge in the classroom and students' joint activity in the small group. (Carlsen, 2010, p.99)

Like these five processes, from the standpoint of a sociocultural approach, a concept is a way of using signs. The use of signs for mathematics as concepts in dialogue with others is appropriated by the acceptance and criticism of the learning community. This is a process of collaborative creation based on existing concepts and experiences, involving sharing with the learning community. At the same time, individual learners themselves try to form a concept. This process is highly consistent with the extended model in Fig. 10.1 and can be considered to have features of both dynamic and mutual composition. In particular, note that descriptions that are conscious of the two-way concept formation can be found between the learner and other learners. Also, note the objectification (e.g., Radford, 2003; Roth & Radford, 2011) of the learning objects in this paper. Objectification in the learning community is "embedded in socio-psycho-semiotic meaning-making processes framed by cultural modes of knowing that encourage and legitimize particular forms of sign and tool use" (Radford, 2003, p.44). Objectification is the process and result of creating a method for using signs as a concept, with focus on the connection between the learning community and individual learners, and it has much in common with appropriation. Carlsen (2010) often mentions objectification in his study. A comparison of these two concepts shows that appropriation is better to express the polyphony of Bakhtin (1981) and objectification is better to overcome dualism by emphasizing the connection with the concept of subjectification. However, the significance of these concepts will be a subject for future research. Moreover, objectification focuses on the reflection/refraction of individual learners in the process (Roth & Radford, 2011). It shows the importance of reflective thinking and the gap in thinking with the learning community when deepening individual thinking based on the connection with the learning community. The reflection/refraction viewpoint is also very important when deepening the concept of appropriation.

> From the previous two research models and a comparison of the two features of the dynamic and mutual composition, we see no reason to deny the two features. And additional factor found is the historical and cultural restrictions of the appropriation as existing conventions interfere with the appropriation. Note that reflective thinking and deviations occur when connecting with the thinking of the learning community.

So far, we considered appropriation based on Bakhtin's philosophy of language. Another source of appropriation is Vygotsky (1978), but this source uses the term internalization rather than appropriation for the internal reconstruction of external operations. Vygotsky's disciple Leont'ev replaced Piaget's notion of assimilation with appropriation (Leontyev, 1981). In assimilation, a learner takes information from the outside world in the framework of the learner's individual knowledge without change. This transformation from assimilation to appropriation can be due to the change in focus from biological ontogeny to a socio-historical perspective. Leont'ev (1974) also emphasized on activity and thought that the mediation of artifacts in activity would connect learners, objects, and others within the learning community. However, some researchers have pointed out that the concept of internalization has the following problems:

> 1: It easily leads to dualism between individual and social. This assumes that internalization occurs solely through personal influence (Wertsch, 1998).
>
> 2: It is misunderstood to be a concept of passively copying information from the outside world to an individual (Cazden, 2001).
>
> 3: The mechanism of internalization is not clear (Brown et al., 1993).

In view of these points, Wertsch (1998) classified internalization into two types, mastery and appropriation. Internalization as mastery allows the use of cultural signs as an intermediary, whereas internalization as appropriation is the process of taking something belonging to others and making it your own (Wertsch, 1998). However, this appropriation, which is as an elaboration of internalization in Wertsch (1998), lacks the viewpoint of mutual composition. As mentioned above, a confusion exists with regard to various other terms on appropriation because of the limited number of studies. In the next section, we show that appropriation can overcome the above- mentioned problems of internalization.

10.4 Appropriation as a Concept to Overcome the Problems of Internalization

In this section, we show that appropriation can overcome the three problems of internalization mentioned in the previous section. First, problems 1 and 2 can be overcome with the features of the dynamic and mutual composition. The sociocultural approach in psychology defines internalization as the reconstruction of an individual's knowledge through interaction with others (Vygotsky, 1978). This internalization was shown as the entire process of reconstructing the knowledge of an individual, triggered by interaction with others in the activity. Later, this was developed as the reconstruction of knowledge in social processes between individuals (Leont'ev, 1974). In both of these processes, the learner's existing knowledge and experience contribute to the reconstruction of new knowledge. To overcome problems 1 and 2 of internalization, we need to emphasize that appropriation is an active concept with an aspect of bidirectional concept formation between individuals and learning communities. Appropriation allows learners of all ages, expertise levels, and interests to return to the learning community the ideas and knowledge they have dedicated to their desires and the zones of proximal development of the learning that they are working on (Brown et al., 1993), with the individuals and learning communities influencing each other. That is, appropriation is a concept with dynamic and mutual composition. Therefore, it can be seen as a concept to overcome problems 1 and 2 of internalization.

Second, we examine whether appropriation can overcome problem 3 of internalization. Both dynamic and mutual composition only outline the mechanism of appropriation. Therefore, we refer to the extended model in Fig. 10.1 and examine the mechanism in detail. Consider the subject in Fig. 10.1. The learning community can be small groups of two to four people or have the size of a whole class. In a learning community of any size, multiple people continue to speak in turn. When learner *A* speaks, *A* is the only one speaking, with the others listening. Next, learner *B* speaks, representing the learning community responding to *A*'s utterance. *B*'s utterance represents that of the learning community, but it corresponds to *A*'s utterance. If we consider a certain utterance as the starting point of a dialogue, we need to note that the individual speakers change one after another, but the moment of dialogue is an individual-to-individual dialogue. Dialogue in a learning community can be considered the accumulation of individual-to-individual dialogue in the learning community. Therefore, basically two learners form the structure of the social interaction of dialogue in the learning community. However, Fig. 10.1 shows the model of three learners as an ellipse. Of the three learners, two are real learners and the third is the learning community as a virtual learner. If we consider the learning community as a virtual learner with some information about the learning target, the model in Fig. 10.1 illustrates a three-party dialogue model. Although we need not increase the elements of consideration when examining the learner's concept formation, we need to include the learning community as an element of consideration from sociocultural perspective, because, from the standpoint of it, learners need to be included in the history and culture of the learning community, implicitly restricting appropriation. Furthermore, the existence of appropriation may become clear when the constraints of the learning community are also considered. This is because the difference in concepts that learners *A* and *B* have appropriated and the difference in concepts that the learning community has as seen by learners *A* and *B*, are clarified. These deviations are unique to the learners because they are constrained by the existing conventions and experiences of learners *A* and *B* (Newman et al., 1989). This difference in concepts of the learning community from the perspectives of learners *A* and *B*, or the difference in the concept that they appropriate, will be useful to interpret the process of deepening mathematics learning and develop lesson designs that would be easy for children to make sense.

Next, we consider the specific appropriation process shown in Fig. 10.1. This is to show how the process of appropriation differs between the appropriation of the speaker and the listener. Here, we use a simplified symbol to clarify the process of concept formation. Let $A(x)$ show that learner *A* is in the state of concept x about the learning object and $B(\alpha)$ show that learner *B* is in the state of concept α about the learning object. The state of concept x shared by the learning community from the perspective of learner *A* is shown as $C_A(x)$, and the partial transformation of the state of learner *A*'s concept x into state x_1 is shown as $A(x_1)$. Here, we assume that learner *A* begins to talk about concept x and learner *B* is just listening.

First, learner *A* (speaker) publishes his/her sign use with regard to concept x for the first time in the public/individual domain. This is publication 1. This utterance gives the learning community's consent 1 or criticism 1 in the public/collective domain. Learner *A*'s first appropriation in response to consent 1 and criticism 1 occurs in the private/collective domain and the private/individual domain. This is appropriation 1. Following appropriation 1, the cycle proceeds to new publication 2. Here, we need to note the content of consent 1 or critique 1. If the learning community agrees to the publication of learner *A*, learner *A*'s concept remains $A(x)$ and the appropriation dynamic composition does not work. Then, the learning community from the viewpoint of learner *A* becomes $C_A(x)$ owing to the appropriation's mutual composition. However, when the use of learner *A*'s sign is criticized, learner *A* would transform the concept into a partially transformed version x_1 or completely different version y, that is, $A(x_1)$ or $A(y)$. This is a transformation in concept due to the appropriation dynamic composition. The concept of learning community from the viewpoint of learner *A* owing to mutual composition is $C_A(x_1)$ or $C_A(y)$. Table 10.1 shows learner *A*'s appropriation 1 process as a speaker. The transformation of learner *A*'s concepts in this way indicates the transformation of learner *A*'s method of using signs.

Table 10.1 Learner *A*'s appropriation 1 process as a speaker

Initial state of concept	Consent or criticism from the learning community	Appropriation 1	Learner *A* concept	Learner *A*'s concept of learning community
$A(x)$	Consent 1	$A(x)$	$A(x)$	$C_A(x)$
	Criticism 1	$A(x_1)$ or $A(y)$	$A(x_1)$ or $A(y)$	$C_A(x_1)$ or $C_A(y)$

Next, we consider learner B's appropriation 1 process as a dialogue listener. Let $B(\alpha)$ be the initial state of learner B's concept as a listener. Publication of $A(x)$ is not the same from the perspective of learner B and A because it is subject to historical and cultural restrictions based on learner B's existing conventions and experiences, interpreted as a partially transformed x_2. This is expressed as $A_B(x_2)$. This $A_B(x_2)$ indicates the state of learner A's concept as interpreted by learner B. To analyze learner B's appropriation process as a listener, we need to classify appropriation situations. First, we have three cases: the learning community agrees with, partially denies, and completely denies $A_B(x_2)$. At the same time, after obtaining consent, partial negation, and complete denial of the learning community with regard to $C_B(x_2)$, learner B needs to agree with his/her own concept α, partially deny it, or completely deny it, representing the three cases. Therefore, this will be divided into nine cases, that is, 3 × 3. The second is the classification based on the relationship between the concept x_2 and the concept α. Here, we have four cases: concept x_2 and concept α are independent, they have partial intersection, concept α contains concept x_2, and concept x_2 contains concept α. Depending on these combinations, the classification results in 9 × 4 = 36 cases. Strict case classification requires more detailed case classification, for example, whether the partial negation part is at the intersection of the two concepts, to result in 36 or more cases. However, the purpose of making this table is to show how appropriation modifies the original concept, rather than analyze its detailed processes. The paper therefore discusses appropriation considering only a part, that is, 24 of the basic 36 cases. Table 10.2 shows the process analysis of learner B's appropriation 1 as a listener using the above simplified symbols and case classification.

The symbols in Table 10.2 are described in a supplementary explanation. In column No. 1 appropriation 1, "$A_B(x_2)$, $B(\alpha)$" indicates that the state of concept of learner B includes two kinds of concepts, $A_B(x_2)$ and $B(\alpha)$. In column No. 2, "$A_B(x_2) + B(\alpha)$" indicates that learner B appropriates the concept combining the two concepts of $A_B(x_2)$ and $B(\alpha)$. Thus, column No. 2 shows that the state of concept of learner B transforms into the $B(\alpha_1)$, which is an extension of $B(\alpha)$. In column No. 3, "$A_B(x_2) \supset B(\alpha)$" indicates that concept x_2 contains concept α. In column No. 6, the partially denied part of learner B's concept α is expressed as concept $B_{sub}(\alpha)$. The concept excluding the partially denied $B_{sub}(\alpha)$ from $B(\alpha)$ is "$B(\alpha) - B_{sub}(\alpha)$". Learner B then appropriates the idea of adding the concept to $A_B(x_2)$ and shows that the concept of learner B has been transformed into $B(\alpha_3)$. In column No. 9 appropriation 1, "$\neg B(\alpha)$" indicates the complete denial of concept α.

From Table 10.1, concept $A(x)$ of learner A may be of three types, $A(x)$, $A(x_1)$, and $A(y)$, through an appropriation. Moreover, from the perspective of learner A, the concept of the learning community may be of three types, $C_A(x)$, $C_A(x_1)$, and $C_A(y)$, through an appropriation.

In Table 10.2, concept $B(\alpha)$ of learner B may become $B(\alpha)$, $B(\alpha_1)$, $B(\alpha_2)$, $B(\alpha_3)$, $B(\alpha_4)$, $B(\alpha_5)$, $B(\alpha_6)$, $B(\alpha_7)$, $B(x_2)$, $B(x_3)$, and $B(x_4)$ through an appropriation. The concept of the learning community can also be of two types, $C_B(x_2)$ and C_B (x_3), from the perspective of learner B. Table 10.2 is an extract of a part of learner B's appropriation 1 process. Therefore, at least 11 types of concept transformations and two

Table 10.2 Learner B's appropriation process as a listener (partial excerpt)

No.	State of the concept of $A_B(x_2)$ and (α)	Relationship between the concepts of $A_B(x_2)$ and $B(\alpha)$	Appropriation 1	Learner B's concept	Learner B's concept of learning community
1	$A_B(x_2)$: Agree $B(\alpha)$: Agree	Independence	$A_B(x_2)$, $B(\alpha)$	$B(x_2)$, $B(\alpha)$	$C_B(x_2)$
2		Have a partial intersection	$A_B(x_2) + B(\alpha)$	$B(\alpha_1)$	$C_B(x_2)$
3		$A_B(x_2)$ contains $B(\alpha)$	$A_B(x_2) \supset B(\alpha)$	$B(x_2)$	$C_B(x_2)$
4		$B(\alpha)$ contains $A_B(x_2)$	$B(\alpha) \supset A_B(x_2)$	$B(\alpha)$	$C_B(x_2)$
5	$A_B(x_2)$: agree $B(\alpha)$: Partial negation	Independence	$A_B(x_2)$, $B(\alpha) - B_{\text{sub}}(\alpha)$	$B(x_2)$, $B(\alpha_2)$	$C_B(x_2)$
6		Have a partial intersection	$A_B(x_2) + \{B(\alpha) - B_{\text{sub}}(\alpha)\}$	$B(\alpha_3)$	$C_B(x_2)$
7		$A_B(x_2)$ contains $B(\alpha)$	None	None	None
8		$B(\alpha)$ contains $A_B(x_2)$	$\{B(\alpha) - B_{\text{sub}}(\alpha)\} \supset A_B(x_2)$	$B(\alpha_4)$	$C_B(x_2)$
9	$A_B(x_2)$: agree $B(\alpha)$: Complete denial	Independence	$A_B(x_2)$, $\neg B(\alpha)$	$B(x_2)$	$C_B(x_2)$
10		Have a partial intersection	None	None	None
11		$A_B(x_2)$ contains $B(\alpha)$	None	None	None
12		$B(\alpha)$ contains $A_B(x_2)$	None	None	None
13	$A_B(x_2)$ Partial negation $B(\alpha)$: Agree	Independence	$A_B(x_2) - A_{B\text{sub}}(x_2)$, $B(\alpha)$	$B(x_3)$, $B(\alpha)$	$C_B(x_3)$
14		Have a partial intersection	$\{A_B(x_2) - A_{B\text{sub}}(x_2)\} + B(\alpha)$	$B(\alpha_5)$	$C_B(x_3)$
15		$A_B(x_2)$ contains $B(\alpha)$	$\{A_B(x_2) - A_{B\text{sub}}(x_2)\} \supset B(\alpha)$	$B(x_3)$	$C_B(x_3)$
16		$B(\alpha)$ contains $A_B(x_2)$	$B(\alpha) \supset \{A_B(x_2) - A_{B\text{sub}}(x_2)\}$	$B(\alpha)$	$C_B(x_3)$
17	$A_B(x_2)$: Partial negation $B(\alpha)$: Partial negation	Independence	$A_B(x_2) - A_{B\text{sub}}(x_2)$, $B(\alpha) - B_{\text{sub}}(\alpha)$	$B(x_3)$, $B(\alpha_6)$	$C_B(x_3)$
18		Have a partial intersection	$\{A_B(x_2) - A_{B\text{sub}}(x_2)\} + \{B(\alpha) - B_{\text{sub}}(\alpha)\}$	$B(\alpha_7)$	$C_B(x_3)$
19		$A_B(x_2)$ contains $B(\alpha)$	$\{A_B(x_2) - A_{B\text{sub}}(x_2)\} \supset B(\alpha)$	$B(x_3)$	$C_B(x_3)$
20		$B(\alpha)$ contains $A_B(x_2)$	$\{B(\alpha) - B_{\text{sub}}(\alpha)\} \supset \{A_B(x_2) - A_{B\text{sub}}(x_2)\}$	$B(\alpha_6)$	$C_B(x_3)$
21	$A_B(x_2)$: Partial negation $B(\alpha)$: Complete denial	Independence	$A_B(x_2) - A_{B\text{sub}}(x_2)$, $\neg B(\alpha)$	$B(x_3)$	$C_B(x_3)$
22		Have a partial intersection	$\{A_B(x_2) - A_{B\text{sub}}(x_2)\} - B(\alpha)$	$B(x_4)$	$C_B(x_3)$
23		$A_B(x_2)$ contains $B(\alpha)$	$\{A_B(x_2) - A_{B\text{sub}}(x_2)\} - B(\alpha)$	$B(x_4)$	$C_B(x_3)$
24		$B(\alpha)$ contains $A_B(x_2)$	None	None	None

types of learning community concepts may occur in an appropriation from the listener learner's perspective. What is important here is that the appropriation as both speaker and listener changes the concept of both the learner and learning community through the appropriation process. The learner and learning community are thus connected.

In addition, note the difference between the speaker's and the listener's appropriation variations. When the learning community agrees on concept x of speaker A, the concept of the speaker becomes $A(x)$, the concept of the learning community from the speaker A's perspective becomes $C_A(x)$, and the concept of the learning community from the listener B's perspective becomes $C_B(x_2)$. The three concepts are fixed but are not exactly equal. However, if $A_B(x_2)$ is agreed upon when the concept of listener B is α, the concept of listener B may become one of six types, $B(x_2)$, $B(\alpha)$, $B(\alpha_1)$, $B(\alpha_2)$, $B(\alpha_3)$, and $B(\alpha_4)$, and not exactly match $A(x)$, $C_A(x)$, and $C_B(x_2)$. In particular, when the listener B's concept is not $B(x_2)$, the variation of difference between learner B and others has many possibilities. This difference is present in almost all cases, whether or not the speaker's concept is partially or completely denied. Even if the speaker and listener participate in the same dialogue, their appropriations may differ. This process is likely to lead to a different concept for the listener rather than speaker. Even in case of slight difference between the two concepts due to appropriation, if the listener only listens to the dialogue continues without speaking, the difference in concept with others may widen as the dialogue progresses.

Whether the learner is a speaker or listener, the appropriation process described above transforms the concept of the individual by triggering the publication of the concept, and the concept of the learning community also transforms accordingly. Thus, appropriation can completely solve internalization problems 1 and 2. The appropriation process analyzed so far reveals a certain degree of the process. However, it is difficult to say that problem 3 internalization, that is, how to know the thoughts and intentions of others, has been clarified. In the next section, we consider the process of appropriation based on intersubjectivity and try to solve internalization problem 3.

10.5 Relationship Between Appropriation Process and Intersubjectivity

How can people know the thoughts and intentions of others? This is an issue at the starting point of appropriation also. We consider this from the perspective of intersubjectivity. Lerman (1996) argues that intersubjectivity has three aspects: aspects that become a subject through social practice, aspects of cognition contextualized in practice, and aspects of mathematics as cultural knowledge (Lerman, 1996, pp.142–147). Understanding others through intersubjectivity in practice leads to self-construction. Historical and cultural restrictions affect the connection and these restrictions are applied in mathematics learning. In other words, intersubjectivity refers to the ability to understand the thoughts of others in practice. Intersubjectivity in mathematics learning is the starting point of mathematics practice for the learning community. In the learning community, self and others are not separate but connected in practice (Roth & Radford, 2011).

How is intersubjectivity possible while understanding the thoughts of others? Steffe and Thompson (2000) argue that intersubjectivity can be built through dialogue and interaction with others. However, Lerman (2000, 2001) shows that intersubjectivity occurs in an individual's mind before the actual interaction takes place with others. The timing for intersubjectivity to occur could differ.

Let us examine this difference from the perspective of developmental psychology. In developmental psychology, intersubjectivity is defined generally as a "process in which mental activity - including motives and emotions - is transferred between minds" (Legerstee, 2009, p. 3). This concept of intersubjectivity is an important factor in the development of the theory of mind and is divided into two types, primary intersubjectivity and secondary intersubjectivity, for discussion.

Primary intersubjectivity is the "innate or early-developing sensory-motor capacities that bring us into relation with others and allow us to interact with them" (Gallagher, 2013, p.60). When a two- or three-month-old baby tries to convey his/her subjective mood through certain expression, the caregiver, for example, the mother, generates expressions that follow the baby's expressions. Infants have then been reported to pay attention to their caregiver's expressions and imitate them (Trevarthen, 1979). The caregiver's expression is also a specialized form tailored to the baby and complements the baby's expressions (Trevarthen, 1979). This period of primary intersubjectivity is the binary relational period when the interaction between the baby-other and baby-objects becomes conspicuous (Legerstee, 2005).

Secondary intersubjectivity develops around 9 to 12 months after birth, connecting infants- objects-others and helping them gain new awareness (Trevarthen & Hubley, 1978). It is based on primary intersubjectivity. For example, assume that an infant is playing with blocks in front of his mother. If he happens to pile up the blocks well, his mother will exaggerate the act, and smile for him. His attention will then shift from the building blocks to his mother, intuitively noticing his mother's joy and praise, and again shifting his attention to the building blocks in an uplifting mood. This shift of infant-objects-mother's attention and the accompanying intuitive understanding of intention can help the infant gradually become aware of the triad relationship with self, objects, and others. "In secondary intersubjectivity, interaction is shaped by joint attention and the surrounding environment" (Gallagher, 2013, p.64).

In developmental psychology discussions of intersubjectivity, human beings are considered to have the following two abilities: the ability to intuitively see the intentions of others within the binary relationship of baby-caregiver and baby-object through innate or early-developing primary intersubjectivity, and the ability to intuitively know the intentions of others and be aware of the objects in the interaction of the triad relationship with others regarding the objects around the first year of life. This discussion of intersubjectivity in developmental psychology clarifies that intersubjectivity should not be regarded as consent through children's dialogue. As intersubjectivity occurs in infants without language speech, it is the ability to intuitively notice the intentions of others even when interacting with others without language. Thus, intersubjectivity should not be viewed as taken-as-shared (Cobb, 1999) through interaction in the learning community, including language and reasoning (Lerman, 2000). We will next consider the relationship between appropriation and intersubjectivity.

According to the extended model in Fig. 10.1, sign use is conventionalized in the public/collective domain and can be appropriated in the private/collective domain. Appropriation will be further advanced in the private/individual domain. The conventionalized sign is used in the private/collective domain as a mere imitation without reflection. In other words, it is the stage where the learner intuitively grasps the meaning of signs as conventionalized by the learning community and begins to use them in the same way. This is the intersubjective understanding of the sign use conventionalized in the learning community. In developmental psychology, intuitive awareness without language use is called intersubjectivity, but in normal mathematical learning, learning progresses through language. Therefore, the intersubjectivity of the learner at school can be reconsidered as intuitively becoming aware of the meaning of the learning object in the interaction using language. In the private/individual domain, the meaning captured intersubjectively is reconstructed through reflective thinking with connecting learners' own knowledge and experiences, and the method of sign use is reconsidered. Historical and cultural constraints can influence reflective thinking. The human-specific ability of intersubjectivity allows us to understand the intentions of others and deepen the meaning of the individual based on them. This is in line with the sociocultural approach principle of prioritizing the social aspects of the development of meaning (Vygotsky, 1978).

So far, we discussed appropriation as a concept to overcome internalization problem 3. Thus, appropriation was shown to involve two processes: intersubjectivity and reflective thinking. Correspondingly in Figure 10.1, it can be said that the first appropriation from the public/collective domain to the private/collective domain is made mainly by intersubjectivity, and the second appropriation to the private/individual domain is made mainly by reflective thinking. As the appropriation process has been clarified to some extent, the problem of internalization can be overcome. Then, by clarifying the relationship between appropriation and intersubjectivity, a connection can be created between individual learners and the learning community, as in participatory appropriation (Rogoff, 1995). This connection will allow us to continue to create new meanings dynamically and mutually.

10.6 Two Meanings of Deviation in Appropriation

In this section, we focus on the deviation in appropriation and discuss it from the perspective of the weaknesses and strengths of appropriation. First, we discuss deviation as a weakness. From the discussion so far, it is clear that the concept of speaker and community may easily deviate from the concept of listener in the process of appropriation. This can be observed from Tables 10.1 and 10.2 too. These deviations may become even greater as the dialogue progresses and could be a major problem.

Here, we refer to the extended model in Fig. 10.1. A learner who does not speak but only listens to a dialogue does not pass through the public/individual domain in Fig. 10.1. However, appropriation begins after passing through the public/collective domain. Thus, learners can appropriate and enrich their own concepts to some extent by just listening to the utterances of others. After appropriation begins, the focus changes to whether to pass through the private/individual domain before listening to the next utterance of another person. One can go through the private/individual domain to the public/collective domain or to go to the public/collective domain without passing through that domain. In Fig. 10.2, the former learner follows the path of the "dashed line a." After appropriation begins, this process is as follows: reflect on the existing conventions and experiences, advance the appropriation, transform one's own concept, and listen to the next utterance. Clearly, this listener is thinking reflectively while listening to the utterance of others. A listener who self- regulates his/her own concept while listening to others' stories using reflective thinking is called an active listener (Kosko, 2014). An active listener fulfills the feature of dynamic composition. However, we cannot find a process that leads to the learning community from individual thoughts because this learner does not disclose his/her thoughts to others. In Fig. 10.2, no arrow connects the private/individual domain to the public/individual domain. Thus, the learning community and learner are connected not in both directions, but in only one direction, that is, from the public/collective domain. In this situation, mutual composition does not work well. In addition, because learners do not disclose their thoughts, the gap in appropriation remains. As the dialogue progresses, the gap is likely to increase. Active listeners who only listen to dialogues can deepen their learning; on the other hand, if they have differences with speakers, they are unlikely to be able to reduce differences. Further, they cannot contribute to concept formation of the learning community. This is a problem for learners who are active listeners.

Next, we consider the latter learner. This learner listens to the thoughts of others, begins to appropriate them without reflective thinking, listens to the next utterance of others, and re- appropriates them without reflective thinking. This is a cycle through the "dashed line b" in Fig. 10.2. Such learners often focus solely on problem solving and the superficial methods of others. For example, they may obtain the answer to a problem from another person and feel relieved if the answer is the same, or may ask another person how to solve the problem and follow the solution without inquiring further. If the answer is different or they have a question about how to solve the problem, reflective thinking intervenes to find a reason. In this case, the learner becomes a learner as active listener and hence becomes the former learner. The learner who does not think reflectively has little effect on the dynamic composition of the concept. Moreover, because the appropriation here is not bidirectionally connected to the learning community, it does not have a mutual composition. In other words, appropriation of learners who do not work reflective thinking just by listening to the thoughts of others becomes very superficial. This is the problem of the latter learner.

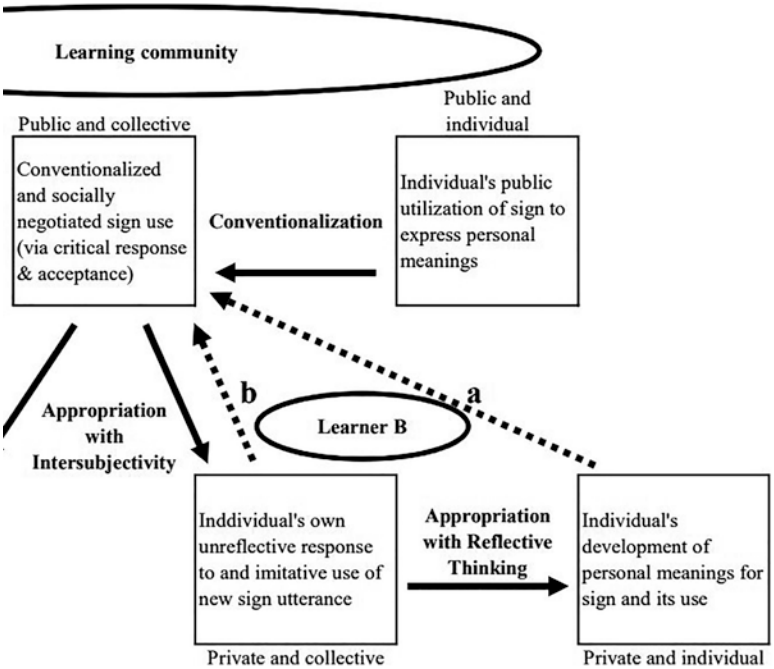

Fig. 10.2 An extended model for the learner as a non-speaking listener

Thus, we highlight the features of appropriation once again. Appropriation is the inner working where the thoughts of individuals who are connected from the public/collective domain to the private/individual domain via the private/collective domain work together. Sufficient appropriation requires two functions: intersubjectivity and reflective thinking.

So far, we considered two problems with appropriation deviation. However, the appropriation should be in the appropriate direction of the aim of learning. How can we ensure that our appropriation is proper? The answers to this question are presented in Tables 10.1 and 10.2. From a comparison of the number of concepts in Tables 10.1 and 10.2, Table 10.1 is extremely small. This indicates that if the appropriation is repeated for the speaker of the dialogue, the probability of approaching the proper use of sign is high. A more proper sign use can be achieved by repeating the appropriation only by listening, but the probability of achievement would be low because there are many types of concepts that can occur after appropriation. Repeating the appropriation as speaker of the dialogue may enhance its appropriateness. If we are asked, "Why does dialogue deepen mathematics learning?," we can answer this from the standpoint of a sociocultural approach: "If you repeat your appropriation as a speaker of dialogue, you can use the concept of mathematics appropriately because the appropriateness of appropriation increases." However, if we are asked, "Will mathematics learning deepen without dialogue?," our answer could be, “It’s not that it does not deepen, but it’s less likely that the direction of deepening is appropriate than if you were the speaker of the dialogue.” The appropriation can be made more proper by becoming the speaker of the dialogue.

Second, we consider the deviation of appropriation as a feature. The appropriation process is referred to as a “quite general process that can account for the emergent creativity of social interactions and the growth of flexible expertise in learners” (Newman et al., 1989, p.143). The process of becoming an adaptive proficient can be explained by the chain of appropriation and the reduction in its gap by becoming the speaker of the dialogue. In this study, we focus on its creativity. Where is creativity related to the appropriation process? This is its own deviation. Table 10.2 lists some of the possibilities of a wide variety of deviations. This includes the gap between the concepts of the learners interacting with each other and between the concepts of the learning community as seen by each learner. The cause of these deviations lies in the learners’ historical and cultural constraints. By interacting according to these deviations, each learner may misunderstand what he/she is talking about. Simultaneously, the learner may create a new concept not included in the speaker’s concept. In other words, the deviation of appropriation can be the source of creativity that leads to new ideas that the speaker did not intend. Fig. 10.2 shows the possibility of creating a new concept based on the deviation from the original concept. Therefore, deviation in appropriation can be a good feature.

10.7 Reconsideration of Appropriation Features

As mentioned earlier, appropriation is a “process that has as its end result the individual’s reproduction of historically formed human properties, capacities, and modes of behavior” (Leontyev, 1981, p.422).” In this definition, the word “historical” is supposed to express the historical and cultural constraints and mutual composition. However, the phrase “as its end result” obscures the dynamic composition. In addition, in this study, we reconsidered the process of appropriation based on two features, dynamic composition and mutual composition, and the extended model in Fig. 10.1, which roughly depicts the process of appropriation. The following characteristics have been pointed out in this research so far: the possibility of conceptual deviation between the speaker, listener, and learning community; intersubjectivity and reflective thinking in appropriation; appropriateness of appropriation; and creativity of appropriation. Therefore, the features of appropriation are summarized in the following six features.

Feature 1 (dynamic composition):Gradually forming one’s concept by speaking, while borrowing the concept of others.

Feature 2 (mutual composition):The concept of learning community is formed in the process of forming the concept of self.

Feature 3 (constraints and deviations):

Deviations occur in individual concept generation owing to historical and cultural restrictions of the learning community.

Feature 4 (intersubjectivity and reflective thinking):
Appropriation is begun with the awareness of others' intentions from intersubjectivity and is transformed one's concept through own reflective thinking.
Feature 5 (appropriateness):
Becoming a speaker in dialogue enhances the appropriateness of appropriation.
Feature 6 (creativity):Deviation in appropriation creates new ideas.

10.8 Future Research

In this study, we reconsidered the concept of appropriation from a sociocultural perspective. We thus clarified the process of appropriation and presented its six features. These features will be useful when analyzing the process of appropriation. In addition, because many studies in the literature are related to the formation of individual concepts, a future research topic could be to compare and consider the process of appropriation of the learning community and individual learners. This comparative study will allow the formulation of a lesson design that would be easy for all children to understand. Objectification and subjectification are concepts of similar to appropriation that can be considered. Research on appropriation is an ongoing process.

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Chapter 11
Towards a Philosophy of Algorithms as an Element of Mathematics Education

Regina D. Möller and Peter Collignon

11.1 Different Roles of Algorithms Throughout History

Before we concentrate on the meaning of algorithms in math education and in math classes, we outline the history of algorithms within mathematics to give a basis for the discussion. Nowadays the concept of algorithms is used in a rather broad range, since many subject matters refer to this notion, and they shape it along the respective desired usefulness or requirements. In the following, we describe the changes within history.

The history of algorithms as part of mathematics started in the very early days, as the systematic solving of mathematical problems is part of doing mathematics. The various algorithms were of different nature because they had different roles. For example, in Ancient Greece, Euclid (300 BC) gave a rule to calculate the greatest common divisor of two given natural numbers. Another well-known algorithm is the sieve of Eratosthenes, a step-by-step procedure selecting prime numbers from a given finite subset of the natural numbers. Both are widely accepted as subject matters on the level of elementary math classes throughout Europe and beyond. These examples show already the role of algorithms as simplifying tools for arithmetic procedures.

Several hundred years later, Al-Khwarizmi (approx. 780–850 AD) presented many mathematical applications for traders. Moreover, his oeuvre shows his insight in the significance of the digit "zero" (Vogel, 1963), which had an important impact on performing algorithms. Some of the mathematical rules were later on translated

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into Latin with the title “Algorithmi,” containing the artificial word “arithmos” that is derived from “number” and the name of the mathematician (cf. Chabert, 1999).

All these algorithms were published for a circle of interested contemporary mathematicians, scientists, and philosophers and for future reference. At that time, these publications of various algorithms stood for a common approach regarding mathematical thinking and communicating.

Later, the published elementary calculations of Adam Ries (1492–1559 AD), who wrote a booklet (1574) on basic arithmetic operations (Deschauer, 1992), are an outstanding example for the use of algorithms because this publishing had great influence on his German-speaking contemporaries. He provided the reader with algorithmic procedures with respect to the four basic arithmetic operations (Möller, 2002), which entails addition, subtraction, multiplication, and division. His publication had caused an “educational” enlightenment for the contemporary citizenship because the role that algorithms played in his books had a strong informative and educational impact at the same time. Knowing algorithms at that time helped the citizens to compute independently, and they did not need to hire a “Rechenmeister” any more in order to get arithmetic problems solved by paying for the solutions (Deschauer, 1992).

Further efforts followed a hundred years later from Leibniz (1646–1716) who used a binary code to design a predecessor of what we now call a computer. Ada Lovelace (1815–1852) followed him two centuries later as the first woman to write a complex version of what we name today a computer code. In the middle of the twentieth century, John von Neuman (1903–1957) and Alan Turing (1912–1954) built the ground for theoretical computer science. Eventually the development during the last 70 years shows a huge field of algorithmic applications.

In the second half of the twentieth century, the technical development of computers entailed an accelerated development of algorithms. Their distribution occurred rather hidden by new technical devices such as mainframe computers, home- and personal computers, and, later on, tablets and mobile phones. The use of the algorithms immanent in various tools was often taken for granted.

The role that algorithms have played since the middle of the last century is a fundamentally different one compared to the one they played only several decades before. Then, the knowledge of algorithmic procedures helped the citizens to become independent because they could solve the problems independently. Similarly, concepts such as the greatest common divisor and the least common multiple were taught and the related algorithms applied. However, the contemporary user of applications employs algorithms of which he is often not even aware.

As observed, algorithms play an important role in the development of mathematics. Naturally, they therefore are an essential subject in math classes. Taking into account the significance of fundamental ideas (Führer, 1997) as well as the postulate that math classes should develop and foster general education (Heymann, 2013), the concept of algorithms should be therefore in the center of didactical considerations.

11.2 The Traditional Role of Algorithms in Math Classes

As clarified in the first section, the algorithms themselves and their significance in everyday life as well as in mathematics classes have changed over the centuries. During the period of the ancient Greek mathematics, algorithms consisted of arithmetic procedures. In the twentieth century, we face a new phenomenon, as there is a strong tendency hiding algorithms behind technical devices. In how far do these changes influence the impacts of algorithms, especially in mathematics classes?

In the last 30 years, the role of algorithms, nowadays understood as a fundamental idea and their importance for mathematics education, has undergone substantial changes. This gives reason for defining an appropriate postulate for actual math classes as response to everyday life experiences, influenced or even determined by algorithms. From a philosophical point of view, new questions arise that can be considered within the framework of (post-) modernism and within a constructivist approach since the nature of algorithms suggests this access of reflection.

By analogy with the spiral principle, often applied to teach numbers, it seems plausible to structure the school curricula with algorithmic endeavors likewise in a spiral way, following Bruner (1976). Modern mathematics lessons are characterized by the use of algorithms and their careful reflection, accompanied by various devices. This relates to primary and secondary level as well.

Already in elementary school, even the four basic arithmetic operations, performed on paper, are of algorithmic nature. Furthermore, prime numbers are found within a finite set that is called the sieve of Eratosthenes. This includes electronic devices such as pocket calculators and modern hard- and software in general as well as non-electronic instruments such as the abacus and calculi. Additionally, algorithmizing mathematical teaching on the primary level is increasingly supported by a variety of software. Contemporary examples for the primary level are the ActivInspire, ClassFlow (Promethean), and the Learning Suite software (SMART). Textbook publishers (Klett, Cornelsen, etc.) provide interactive whiteboard images on their websites, offer digital material to accompany the textbooks they publish, and in some cases also have learning programs in their range. The Calliope Mini is available on the manufacturer's website with a collection of various applications in the classroom. With its Education series, LEGO also has various offers for the use of programmable, digital artifacts. Additionally worth mentioning are the products Dash, Ozobot, Thymio, Bee-Bot, Makey-Makey, Little-Bits, Cubelets, Matatalab Coding Set, mTiny, Scratch, and Snap, all of which were developed with a view to possible use in educational contexts and some of which already have suggestions for use in mathematics lessons.

In secondary school, linear equations can be solved by a step-by-step-procedure. Additionally, students on this level need to sketch curves, and they perform this task often in a way that is more or less algorithmic. With the Gaussian algorithm, systems of equations are solved. In recent years, students have started using geometric and algebraic software (CAS, DGS, Mathematica), often not being aware of the hidden algorithms driving these tools. Essentially, they use the tools like a black box.

The use of devices supporting the teaching and learning of mathematics came a long way. Long before the tools were electrified, various instruments suited for the realization of certain algorithms mattered. Moreover, one can observe a methodical algorithmization of mathematics itself, independent of machines (Krämer, 1988). In the following, we aim at a common view on more or less modern electronic devices, traditional appliances based on mechanical principles and eventually the formalization of algorithmic approaches within mathematics.

11.3 Changes in the Phenomena of Algorithms Resulting in Challenges for Math Classes

The role that algorithms play within technical devices is a fundamentally different one compared to the one they played before. Nowadays, the user of applications uses algorithms of which he is often not even aware.

The content of the curriculum for algorithms in math classes neither gives an overview nor does it exemplify typical algorithms used in todays' IT tools. Looking at the entire school time, we therefore question the role the curricula play in regard to the fundamental idea of algorithm. Our critique aims at the non-fulfilled task in this field since it does not reflect the actual common use in rapidly changing times (Ernest, 2016, p. 12).

One could divide the devices into those with and those without electricity. In earlier days, the focus of teaching within mathematics classes was influenced by the use of devices such as the abacus, slide rules, and logarithmic tables. Generally speaking, algorithmic thinking starts in school with the four basic calculations, after all in written form. However, these calculations are usually done without naming them as algorithms.

The naming of algorithms in school (or math classes) started just some 30 years ago in Germany which goes along with the discussion of the fundamental ideas. Especially in the last 10 years, tools such as bots and others were introduced in math classes with the idea that schoolchildren need to learn algorithmic thinking. With the help of these tools, the children are expected to get a first access to algorithms and even to simple programming. One assumes that this fosters algorithmic thinking, which should start on an elementary level.

Against the background of the temporary German separation, we present in the following a very short overview that depends on the different pedagogical and philosophical background of the Federal and Democratic German Republic, respectively. It is an example for two independent developments and the partly insufficient reflection.

Weigand (2003) provides an overview of the role of the pocket calculator in the federal states of the former Federal Republic of Germany in comparison to the former German Democratic Republic. On the basis of these considerations, lessons for the present and future use of the computer in mathematics education can be derived.

In Germany, the slide rule was admitted as an aid on an equal footing with logarithmic tables in 1925. In the Federal Republic of Germany, these tools were replaced by a standardized slide rule in 1958 by resolution of the Conference of Ministers of Education and Cultural Affairs. They were subsequently also used in lower secondary education. The first pocket calculator came onto the market in the Federal Republic of Germany in 1972; in 1974, sales figures were at a temporary peak. It was approved for use in mathematics classes starting in 1976, usually from grade 7. In the German Democratic Republic, the pocket calculator ("SR1") was introduced into the school curriculum in 1985, also usually from grade 7.

As goals, the German Society for Didactics of Mathematics (GDM) demanded in a statement in 1978:

- The facilitation of experimental student activities within the framework of learning by discovering ("Entdeckendes Lernen") and problem solving
- A concrete numerical starting point for conceptual formations
- The realistic handling of application tasks by means of numbers appropriate to reality
- The relief of activities which are not of central importance for the solution of the task at hand
- Access to algorithmic thinking
- Problem-adequate practical phases

In retrospect, it can be said that these goals were only partially achieved. A long-term careful preparation and a timely and elaborated didactical theory were and are missing. Essentially, previous exercise and task formats were now also processed with the help of the calculator. Weigand (2003) merely notes a certain increase in trigonometric tasks to be treated numerically, accompanied by an increasing emphasis on stochastic topics.

In the German Democratic Republic, a report of the Academy of Pedagogical Sciences was published in 1979 calling for empirical monitoring of the introduction of the pocket calculator. In Fanghänel and Flade (1979), the notion of "arithmetic culture" was emphasized in this context, which was directed against a surrender of arithmetic skills. It is noticeable that early on the discussion was also devoted to the "design" of pocket calculators, for example, regarding the question of whether it should abbreviate the arithmetic processes of calculating percentages. What remained controversial was the role of calculating aids for weaker students, who on the one hand (superficially) benefited from the pocket calculator, but on the other hand – partly – had their strengths in relation to "simple" arithmetic.

Both in the Federal Republic and in the German Democratic Republic, the introduction of the pocket calculator was viewed critically by parts of the teaching community. Motives ranged from concerns about students' arithmetic skills to a feared distortion of the concerns of mathematics education. It has not been sufficiently investigated to what extent subjective motives such as individual insecurity already played a role. Roughly speaking, the ideological differences in these two societies are reflected by the educational politics. It may be assumed that many teachers were aware that a didactically consistent introduction of the calculator would have to result in far-reaching changes with regard to the teaching procedures.

In the Federal Republic, such changes have been addressed by Kirsch (1985), among others, and he stated that now the interest even turned to computers "even before the existence of simple pocket calculators and their – actual or desirable – effects on mathematics teaching had been satisfactorily worked out or realized in practice."

11.4 Towards a Philosophical Approach of the Meanings of Algorithms

In the following, we start observing several aspects of algorithms that enlighten the relationships to neighboring sciences.

11.4.1 Preliminary Philosophical Considerations

On the basis of the current situation, characterized by the wide availability of computing speed and storage space, the algorithms, though hidden, generate an enormous force. This fact leads to dramatic changes in everyday life and modern economy (O'Neill, 2016). Furthermore, this process will continue, provided the computing speed and the storage space is growing at the current rate.

All these developments lead to social and educational effects and phenomena of which we have no previous experiences. These observations call for an investigation from a philosophical point of view. With his contribution on philosophy of mathematics education, Paul Ernest (2016) provides us with many and rather broad questions for further detailed reflections. One of his questions refers to the relationship between mathematics and society. This holds, because all devices of information technology rely on mathematics in digital processes. Another of his essential questions centers on the role of teaching and learning in fostering or hindering critical citizenship. He also questions the functions that mathematics performs in today's society, and which of these are intended and visible. In the case of modern use of algorithms, transparency is a delicate issue for very different reasons; it is due to complexity culminating in hidden data collections or data mining.

Looking at educational aspects, Ernest (2016) poses the question: Does mathematics education have an adequate and suitable philosophy of technology in order to accommodate the essential issues raised by information and communication technology? This is of high relevance because the idea of the algorithm is a fundamental one for math classes (Führer, 1997). For more than three decades, the notion has been extended in particular regarding its meaning in daily routines. Meanwhile, there is a great discrepancy between algorithms belonging to the curricular mathematical topics versus the algorithms that build the foundation of modern information technology, carrying us as far as the simulation of a human brain (e.g., artificial intelligence).

Against this background, the question arises in how far mathematics instruction should consider these challenges and find new ways to address them, that is, to what extent and with which mathematical tools this might be achievable. The answers to these questions are of crucial importance because society expects math topics and their teaching to contribute to general education throughout the school years (Heymann, 2013).

According to Heinrich Winter (1996), who stressed three basic experiences ("Grundvorstellungen"), mathematics education should be designed in such a way that it enables us to perceive and understand phenomena of the world around us, which concern or should concern all of us, from nature, society, and culture, in a significant way (first basic experience). There is no doubt that increasing digitization produces phenomena that shape our contemporary everyday experiences, and they do concern and influence us. These observations lead to the following questions: How can mathematics education adequately meet the requirements arising therefrom? How can it contribute to encountering the phenomena of digitization from an enlightened perspective and with sound judgments?

For decades, the aim of many mathematics educators, especially Heinrich Winter and Fischer and Malle (1985), consists in fostering a deep and integrated mathematical understanding, in particular on the field of applications. The intentions were focused on the interaction of mathematics, the issues of its applications and everyday situations.

Engel (1977) pointed out early the necessity of a theoretical reappraisal of the concept of algorithm. Shortly after the introduction of pocket calculators into German school teaching and at the beginning of an increasing algorithmization of mathematics teaching in the context of the emphasis on applications, he put the algorithms as a fundamental idea on the same level as the concept of function: At the turn of the century (meant here: the turn from the 19th to the 20th century – authors' note), Felix Klein initiated a reform of mathematics education. The reform movement adapted the keyword "functional thinking." The concept of functions was to permeate the entire subject matter as a guiding concept. Due to the widespread use of computers and calculators, the time has come for the next reform represented by the keyword "algorithmic thinking". The concept of algorithm should serve as a guiding concept for school mathematics. We have to rethink the whole school material from the algorithmic point of view (Engel, 1977, p. 5. Translation by the authors).

Engel wrote this in the 1970s. It should be remembered that the Meran Reform, especially stressing functional thinking (Vollrath, 1989), was not systematically incorporated into school teaching in Germany until after the Second World War. From some mathematics educators' perspective, it can even be considered a failure (Krüger, 2000).

11.4.2 Further Philosophical Observations

We need to pay attention to the conditions that caused today's "explosion of algorithms," which is what Lyotard (1984) calls the "postmodern perspective." Accordingly, postmodernism calls for a skeptical view. For this purpose, we need to take notice of the underlying meta-narratives, on which many different fields have an impact. Therefore, analyzing (that means deconstructing) meta-narratives requires retracing historical developments and the decisions made on the way. This endeavor reveals choices that were taken along the historical genesis within each relevant field. Information technology, the technical realization of algorithms, is largely driven by the belief that the age of technology would bring nothing but progress. This conviction about the ongoing technical progress is closely related to the meta-narrative of perpetual economic growth.

What is needed today is a discussion about the role of contemporary mathematics instruction, especially if mathematics education still holds the claim of being part of a broad general education, preparing also for future demands. This is achievable through a social contract facing the new challenges arisen. It has to be clarified to what extent school education should be responsible with regard to its pedagogical task in general (Skovsmose, 2011).

At the same time, there is a need of including the curricula of other subjects involved, such as information technology, social studies, and even, given an appropriate opportunity, philosophy and theory of science. The main issue in question is not of technical nature, but it is about the impact on math classes and phenomena that were completely unknown until recently and that involve fascinating opportunities and enormous threats at the same time. In any case, the perception of (mathematics) education must reflect the corresponding processes.

The modern teaching of algorithms can be split into two aspects: one, which is taught on the elementary level with a close relation to school mathematics and in terms of a fundamental idea (Führer, 1997), and the other, where sophisticated algorithms built in technical devices create new and complex phenomena. This observation needs a reflection, especially a philosophical one, in how far math classes should build a fundamental knowledge for these new phenomena under a mathematical view.

11.4.3 Algorithms and Their Connections Within Sciences

Considering the two aspects mentioned above in detail, one can identify the relationships between the notion of algorithms and the field of mathematics itself, the curricula for math education and, of course, the modern field of computer science. All these relationships can be found in the broader framework of real-life situations, whereby the use and the impact of algorithms as well as the related consequences have undergone tremendous changes.

Nowadays, the algorithms behind apps and programs are mostly hidden. Everyday examples are cash machines, ticket automats, and recent apps collecting data relating to pandemic processes, not to mention social media. Many more algorithms can be part of business plans and therefore be looked upon trade secrets. Examples are search machines as well as scoring procedures of any kind (O'Neill, 2016). As mentioned above, there are algorithmic procedures behind modern machines and digitized tools that the average person is often not aware of in everyday life. Meanwhile, the number of hidden algorithms radically outnumbers those known and keeps growing daily (O'Neill, 2016).

With the ongoing digitization, algorithms must not only be looked upon as a part of mathematics but also and self-evidently as an important part of computer science. They constitute a major role for programming and are therefore a basis of contemporary information technology. Figure 11.1 shows these relationships and also some further correlations.

From the perspective of general and mathematics education, the concept of algorithms as part of mathematics is consequently a part of mathematics curricula. This means it is of importance to clarify what kind of algorithmic knowledge is necessary for general education. In the light of contemporary discussions around digitization, the concept of algorithms turns out to be an important factor in modern mathematics education. Looking at the curriculum, it is also apparent that algorithmic thinking is by far not stressed enough.

Within teacher education, especially on the elementary level, it becomes often apparent that the understanding of scientific theory as a particular formation of how the world can be looked upon and can be interpreted with a set of particular mathematics concepts and can be extended. Students often regard mathematics only as a set of rules to be followed. Therefore, the subject of algorithms should keep the freedom of search, rather than following a given scheme. According to Freudenthal (1983), elementary algorithms could be (re-)invented by students. Here, like in other fields of mathematics classes, it is a question of how much, or how early within the learning process, such aspects of algorithmic thinking should be a subject. Recently, bots and similar devices are programmed by pupils in math classes, even on the primary level (Möller et al., 2022).

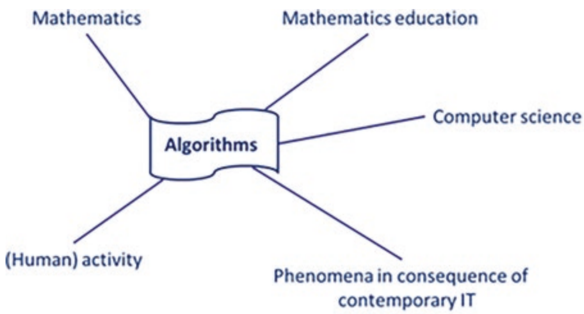

Fig. 11.1 Algorithms and their connections to related disciplines, illustrated in the form of a simple mind map

11.5 Conclusion and Outlook

Moreover, in teacher education, it is of special didactical interest to explore to what extent teacher students recognize and understand algorithms while doing mathematics. It is especially important for future teacher to evaluate when algorithms can be looked upon as "black boxes" and when they should be explored in detail. The development of this kind of educational judgment is an essential part for providing general education within math classes. This idea could be put into the center of further developments of algorithmic thinking activities. Along with Freudenthal's "Mathematics as a pedagogical task" (Freudenthal, 1973), one should also consider the teaching and fostering of algorithmic thinking as a pedagogical task.

The philosophy of mathematics education – and in particular the constructivist view – should establish a basis for adjusting mathematics instructions appropriately (Roberge & Seyfert, 2017). With their aspiration about the computational construction of reality, we observe that a constructivist explanation of these new phenomena is in the very nature of things.

This may enable advanced students to adopt an adequate epistemological perspective. In this context, the modeling aspect is included as a consequence of the constructivist view on the subjects under (mathematical) investigation (cf. Ernest, 2016). Among other things, this approach facilitates an unconventional access to Freudenthal's method of reinvention (Freudenthal, 1983).

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Chapter 12
The Times of Transitions in the Modern Education

Małgorzata Marciniak

12.1 Introduction

As the confusing times roll through, we reflect more and more about what happened during the years when the restrictions motivated by pandemic flipped around the reality for most of us. As an educator and a journal editor, I am tempted to reflect on transformations in mathematics education based on my own observations and on professional publications. During the past 2 years, a substantial fraction of article submissions centered around challenges of successful conversion from in person to remote education. They were frequently recalling an overwhelming drama of the first months of the sudden and urgent transitions from in person to remote modality of instruction. The energy behind the changes was rushed thus resembling revolutions and remaining in opposition to slow-paced transformations motivated by professional development.

Revolutionary changes are not exclusive to political or socio-economical human history but are frequently observed in nature as giant stellar collisions, mass extinctions of species on Earth, or in the growth of natural sciences as presented in 1962 by Kuhn (2012). Multiple attempts have been made to dispute, modify, or generalize the original idea of scientific revolutions. In the light of that concept, Ellis and Berry III (2005) discussed the paradigm shifts in mathematics education in the USA related to what is meaningful in mathematics and how it should be tested. This work sees the entire education, and in particular, mathematics education as a science with certain paradigms that can experience slow modifications motivated by professional development or revolutionary changes motivated by other factors, for example, the pandemic. To give a suitable background to the idea of revolutionary

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moments in education, in particular mathematics education, I will analyze three pivoting situations from history: introduction of compulsory education, secular education, and public education available to everybody. The shift of the paradigms which took place in the past, currently shape our presence and future. Then I will present the current moment and its short-term implications making some mental simulations of future possibilities.

12.2 Pandemic Articles and Observations

To give a brief overview of sample pandemic-themed articles, I will use examples from the Mathematics Teaching-Research Journal (MTRJ), since I am an editor and have insight into the publications. The first pandemic-themed submission arrived as early as in spring 2020 and analyzed instructor's creative processes which bloomed due to the change of circumstances. Marciniak (2020) saw the pandemic as another opportunity to further develop instructor's creative thought in the mathematics classroom. Another creativity paper was published in the fall 2020 issue of MTRJ and discussed from the perspective of the theoretical framework the moments of students' creative insights observed by a mathematics professor Baker et al. (2020) during remote classes. Another submission from the same issue of the journal discusses advantages and disadvantages of the remote teaching emphasizing aspects relevant for students' success in remote classes (Fuchs & Tsaganea, 2020). Wang (2021) analyzes benefits and hindrances of the remote mathematics instruction and goes few steps further in her investigations by collecting data throughout few semesters. Here the author connects learning outcomes with various aspects of remote learning environment, in particular students' class activities.

Assessment and self-reflections are highlighted by Caspari-Sadeghi et al. (2022) as a key to students' success in online learning of mathematics. Ariyanti and Santoso (2020) claim that remote learning of mathematics created lower scores and more frustration among high school students in Indonesia. But on the other hand, teachers' digital skills were discovered by Khanal et al. (2021) as the main factor of learning outcomes and positive experience in mathematics classes in Nepal. Only one submission discussed truly challenging situations experienced by Indonesian students and teachers from rural regions. After performing few hundred interviews by WhatsApp with math teachers, Tanujaya et al. (2021) suggested blended learning instead of purely remote learning due to insufficient digital infrastructure: poor Wi-Fi signal and unreliable electronic skills and devices.

The education during pandemic was equally challenging from the perspective of parents, pupils, and educators with each group facing their own difficulties and finding their own solutions. In academic publications, the educators express their progress while the others, parents and pupils remain silent. While speaking with parents and grandparents about challenges, I heard of difficulties managing their work time with teenagers and kids being crammed in small spaces with limited access to technology. During one of committee meetings performed via zoom, I heard a daughter of my colleague complaining that the parent took the computer for too long preventing her from playing video games. Apparently, the time for work and play of various family members was confused due to overlapping schedules.

The educators had to manage their classes using a different modality, provide instructions, assign homework, create quizzes and tests remotely, grade students work, and then provide feedback. While some converted their courses with minor difficulties, others had really a hard time managing technology entirely new for them. The population of teachers split into those who before pandemic knew and used technology daily and those who had to learn it within a week.

The students reported fatigue due to a large amount of time spent at the desk and looking at computer screen, insufficient learning skills while tempting video games and social media being right at their reach. With experience, all three groups learned to manage their time, attention, and resources while admittedly looking forward for the end of this enforced new reality.

12.3 Kuhn Theory

Looking through the history, one can see that pivoting moments and sudden changes, motivated by circumstances or revolutionary mandates, created by the governments shaped the education more intensely than the slow growth. This idea stays in an alignment with the theory of Thomas Kuhn of the revolutions in sciences, where the times of peaceful growth are alternated with times of stormy reorganizations.

In brief, Kuhn's idea of revolutionary growth of sciences combines slow growth within certain range of the paradigms together with sudden, pivoting growth that influences and eventually changes the paradigms. Kuhn calls the times of slow, regular growth "normal science" and the pivoting moments are for him the revolutionary times of the "paradigm shifts."

This alternating nature of the shape of the growth of science reminds of a self-sharpening mind, which gets ready to sharpen after accumulating new experiences and then improves itself by reorganizing certain aspects but keeping others. Similarly, having limited tools of discovery, early science was limited to questions it could ask but once the horizon expanded the toolbox enlarged, and renewed science could ask more insightful questions. This renewal is a repetitive event. It appears that education, and in particular mathematics education, goes through similar processes of alternating transformations that resemble slow growth and revolutionary leaps. Slow growth in education is based on discussions and conversations leading to reforms. The leaps are based on sudden changes which may result is spectacular effects.

The phases of growth of education can be expressed as follows. Pre-paradigmatic phase of education took place at the start when no general paradigms are well established. During those times, education was very much individual and depending on the capacity of the teachers and the students. This phase was experienced before education became compulsory. Once education became compulsory, certain standards were set by the educators (government or religious institutions) and education reached the stage of stabilization. When applied to science, this phase would be called normal science. But when applied to education we could call it normalized education. Well-functioning paradigms can remain for some time in science and education, but once a form of crisis appears the paradigms must be revised. In sciences, crisis usually is a result of deep insight of some scientists who announce their revolutionary results. But in case of education crisis may be a result of political or socio-economical transformations which appear to be independent of the work of teachers and students. However, in fact, they are a result of accumulative mindset growth of the entire population, accomplished thank to education of the previous generations. Once the new paradigms are set, the new normalized period begins. But just like in sciences, the accomplishments in the light of new paradigms may not be comparable with the previous ones due to incommensurability of the paradigms. Thinking in terms of science one would not be able to properly compare the value of work of astronomers who worked in the heliocentric or in the geocentric theories. Similarly, in case of the times of the pandemic, it is difficult to properly compare students' accomplishments before and during the pandemic since the styles of teaching and testing were so distinctive.

For the purpose of further discussion, a brief description of certain paradigms which function in (mathematics) education is provided below. Some, such as the details of the curriculum and the ways of testing, are always timely and generate heated discussions.

1. Structure and funding of education. Shall it be public or private? Who should fund schools and furnish teachers' salaries?
2. Evaluation and accreditation of schools. Which schools are "good" and how it is measured? How can schools improve?
3. The detailed content of the curriculum: what is taught and how. This paradigm is particularly important for mathematics education due to challenging nature of successful teaching and learning math which requires effort from both, the teachers and the students.
4. Recruitment and evaluation of students. Who is accepted at the schools and remains there. What knowledge and skills are tested and how. The problem of accurately testing math skills remains highly disputable due to sensitivity of the human mind to external disturbances from the environment and internal disturbances created by anxiety.
5. Selection, training, and evaluation of teachers. How the traits of "good" teachers can be encouraged by professional development.
6. Assessment of the efficiency and established ways of modifying the points above. Since every paradigm goes through certain slow transformations, it would be valuable to verify whether such paradigm is somehow improving. Mathematics teachers frequently receive negative feedback from students and parents who express their dissatisfaction related to the difficulties with course material and low grades. Up to what extent, if any, should the content of math lessons be dictated by students' low achievements?

While discussing pivoting moments in the history of education, I will point toward that paradigms that were in the center of the revolution.

12.4 Pivoting Moments in History

Touching upon few pivoting moments from the history of education help with understanding the importance of challenging times in the process of shaping modern education.

12.4.1 Compulsory Education

In a common language, we speak about education being mandatory for all children within a certain age range. Professional language calls it compulsory education. It has been a relatively recent accomplishment (since 1739) to introduce compulsory education for children within certain age range. Three European countries, namely, Denmark, Prussia, and Austria implemented it already in the eighteenth century and a long list of others joined in the nineteenth century. Still, a number of countries worldwide joined in the twentieth century with just few exceptions existing in the modern times. One of them is the Vatican City, which is the residence of the pope and apparently does not have to worry about educating young citizens. The introduction of compulsory education was imposed by the governments partially to avoid child labor and partially to prepare future citizens to contribute to the developing industries. In many countries, this change overlapped with the early growth of industrialism and factories but gaining momentum of its own, motivated the growth of thought and technology.

The origins and motivations of the early compulsory education in the first country that introduced it (Denmark) took place in the medieval ages. Early twelfth- and thirteenth century catholic schools already existed in Denmark, but their motivation was to prepare the students for theological studies teaching mainly reading and writing in Latin and Greek. New needs appeared with the spread of Pietistic Lutheranism which required common people to study the Bible and apply biblical doctrines for individual piety. Thus, even before introducing compulsory education, Denmark already developed and tested in practice the concept of educating common people in its 240 Calvary schools. The growth of philanthropy induced by the philosophical movement of Enlightenment brought additional funding and allowed education for all children. After Denmark, other countries began expanding the concept of educating common people to the concept of educating all people.

At that time in history, mathematics in a form of basic arithmetic was often taught together with natural sciences. Introducing compulsory education did not influence the subject but significantly changed the form and quality of mathematics exposition. Learning got converted from being tuned to the needs of an individual to normalized lecturing for nonhomogeneous groups. This destroyed the fine connection between the teacher and the pupil. Moreover, enlarging the distance between the teacher and the pupils reduced teacher's insight into attention and the process of learning.

It was and still is a duty of a local community to organize a suitable space for the school and pay for the teacher(s) who provide teaching instructions. In all countries, before compulsory education was introduced, some education already existed for the rich or the select few, who received non-standardized schooling from tutors and governesses. It was the poor children for whom the mandatory education really made a significant difference. Since compulsory schooling in its early years was introduced in countries with rural profiles and most poor people lived in villages, the first school calendars were tightly coordinated with the farmers' almanac of planting and harvesting. This way, the summer months were left for harvesting while the winter months were left for schooling.

Introduction of compulsory education was certainly a revolutionary change for all three groups, the parents, the pupils, and the teachers. The parents suddenly had to give their children away to the schooling institutions, exposing educational insufficiencies of their own family. Lack of intellectual and material resources together with signs of child abuse and neglect became evident to the entire community. Imagine overwhelmed and living-in-poverty adults, who did not give much attention to their children and did not care for their growth, mainly focusing on making daily living were suddenly called to give their children away, hold responsible for their behavior, and provide books, clothing, and respect the teachers.

I can only imagine the reaction of free-running children raised mainly outdoors and used to heavy farm work who suddenly had to obey strict rules of desk-and-chair schooling days, namely, not used to sitting still for an hour, not used to obey a stranger, not trained in focusing attention on abstract ideas. And most importantly, not seeing how these abstract concepts could possibly be useful in their daily life at a farm, while performing the same tasks as their parents. Even such basic skills as reading and writing may have been seen as unnecessary and fruitless.

At the same time, the children of the rich had to obey the same rules and participate in mandatory schooling abandoning the cozy feeling of home schooling. For most of them, it may have been quite a shock to realize the depth of the poverty of other children living in the near proximity. While mandatory schooling opened up the world for the poor, it as well opened up the eyes of the rich children to the extreme unevenness of the resources bringing confusing feelings or exaltation mixed with empathy.

The tutors of the rich went through quite a challenging moment shifting from individual, non-standardized education to group education based on school programs and uniform curricula. Lack of freedom was probably not the biggest issue for new instructors but lack of experience managing a group of pupils not used to rigid discipline. That was probably the motivation of strict discipline rules in the early classrooms.

Some criticized compulsory education claiming that this style of group learning does not support individual strengths and violates children's freedom. Others claiming that mandatory education is a form of a political game and control over entire populations. But not everybody knows that the concept of compulsory education was already presented by Plato in *The Republic* (c. 424–c. 348 BCE) as explained by Allen (1989). In Plato's mind, the perfected society required their perfected citizens go through a process of perfection, which had to be accomplished by popular and mandatory education. This concept was remodeled by Marsilio Ficino, a Catholic priest and scholar during Renaissance. Similarly, Enlightenment philosopher, Jean-Jacques Rousseau was advocating for mandatory education for all.

At those times, it took over 2000 years for an abstract concept of compulsory education to crystallize and become reality.

One of the arguments of the times of the introduction of the compulsory education was that students should be able to read the Bible on their own, which, ironically, motivated as a counteraction, another pivoting moment in the history of education, which is a conversion from religious to secular education.

12.4.2 From Religious to Secular Education

This process was much more complicated and very much dependent on the local region and culture. Moreover, religious education still exists and is doing very well within many groups and cultures, for example in Jewish communities as in Rosenak (2011). So, the transformation is not based on disappearance of the religious education but on rebalancing the time and efforts from learning religious subjects to learning in terms separated from religious authorities.

Originally religious education was created by the religious authorities for a variety of purposes with the main intention of being somehow useful within the religion-ruled society. Not limited to western world neither enclosed within certain historical brackets, religious education may have taught rituals, beliefs, or doctrines. Considering the Medieval Age education as an example, students then studied reading, writing, and arithmetic together with theology and the art of conversation. Due to established nature of arithmetic, the conversion to secular education did not affect teaching of mathematics as much as if affected humanities or natural sciences.

In the times, when sciences were limited to few books and writing skills were scarce within society, religious education took a significant part in the growth of individuals. Actually, when the educated people were gathering around religious centers such as orders and churches, it was assumed that the core of their education was religious.

Secular education has its roots in broadly understood secularism. To explain the motivations of the secular education, it is necessary to have a close look at certain aspects of secularism in politics and the so-called separation of church and state. Philosophical view of secularism is based on interpreting life as a result of the material world and not in terms of spiritual or religious aspects. It emphasizes equality before law and neutrality toward all religions but is not necessarily related to atheistic or anticlerical views.

Looking at this philosophical phenomenon chronologically, one needs to mention medieval periods of secularism of Islamic countries dated as early as the tenth century. The Western world discovered secularism during Enlightenment period and was very much influenced by the thoughts of John Locke (1689) contained in “A Letter Concerning Toleration” where he argues that the state should treat all citizens equally and should not discriminate based on religious beliefs. The idea calls for politicians to make decisions based on natural reasons withdrawing the religious motivations. Modern societies have been growing increasingly secular due to economic development, social progress, and increasingly secular education. Following political transformations, state-managed schools had to align with the philosophical trends of secularization creating secular societies.

While secular education is likely to focus on evidence-based science and non-religious literature, it may still contain lessons about religion or religions. But the main difference between religious and scientific education is the position of authority. While religion heavily relies on dogmas, science is based on evidence. Certainly, authorities play a key role in interpreting the evidence, in validating theories, and in defining new paradigms of science. But at the very moment when new evidence arrives, new theories are formulated, the new authorities override the previous. This allows science to transform and adapt to new circumstances. The growth of science and the scientific revolutions have been accelerating over last few centuries, while religions have apparently been not. Hence, separating education and religion gave the first one acceleration in the direction of the unknown, while the second one retrieved into the cozy space of the well known.

12.4.3 Public Education Available for Everybody

It is the democracy that determined the broad availability of schooling, and it is the availability of schooling that motivated the democracy. Thus, the progress of availability of education was growing together with the civil rights of women and minorities, as elaborated by Rousseau (1992). At the early stages of development of education, it was provided mainly to rich white males discriminating everybody else. Low income, race or ethnicity, gender, or remote geographic location may be reasons for children not receiving education.

The most popular and well-known theme in availability of public education is the education of females. Even if education for girls from rich and influential families was not questioned, somehow the same favors were not extended to girls from other social groups. It was the growth of emancipation during the Enlightenment era which brought expanded availability for schooling of girls. At first, these schools carried a different curriculum than those for boys, but once females began graduating from established institutions and becoming role models for younger generations, there was no return to the previous barriers. The changes were slow but consistent making education more and more inclusive.

Change of this paradigm brought even more students to the mathematics classroom making it even more nonhomogeneous, thus even more challenging for the teachers. But at the same time, it opened space for new ideas of contextualization. Nowadays, in a form of ethnomathematics, we treasure connections of mathematics with folk art, dance, and music.

But making schooling available to everyone did not mean that everybody could learn anything whenever they needed or wanted. This became possible just recently when internet resources became overloaded with broadly available content. With affordable prices of ample data plans and convenient digital devices, the access to these vast contents is in hands of everybody.

Now within a reach of a broad scope of users are vast resources for learning and studying. Any day and anytime one can find how to perform multiplication of matrices, use chopsticks, or make a silver mirror. These resources are free, easily searchable, and provide a variety of teaching styles on multiple learning levels for students of arbitrary background. Numerous people have their channels to share instructions and explanations of a range of academic topics. One can choose a teacher of their favorite gender, ethnicity, language, point of view, and other aspects, which aspects may not appear relevant to the learning process but somehow make learning more appealing.

The quality of these instructions may sometimes be questionable, but the caliber of the internet libraries appears quite spectacular. The children of the world have never had so many English teachers. As a researcher, who clearly does not know everything, I was trained by my students to "google everything" regardless of whether I know or do not know much about the topic and always found something new in the vast internet resources. The growth of internet resources and their availability has been stably expanding over the last 20 years. The interested parties, that is, teachers and students seem to be consistently involved in the exchange of information via internet resources. The teachers gaining millions of viewers of their education channels seem to earn their pride for having a wide public. And the students seem to treasure experienced instructors who can get the message across.

Vast availability of digital math lectures of various levels and styles allows students to repeatedly listen and read lessons from different teachers. Thus, students can find favorite styles of exposition suitable for their type of attention.

The truly dangerous aspect of vast internet resources is broad availability of certain information and instructions dangerous if fell into irresponsible hands. However, what educators find the most concerning is a non-intellectual style of the content of social media. At the same time, others claim that relevant information found on Facebook or Tik-Tok is otherwise unavailable to them.

12.5 The Current Pivoting Moment

For the sake of completeness of the presentation, I am tempted to briefly describe the current pivoting or revolutionary moment in the development of modern education. In March 2020, teachers and students worldwide were directed to perform schooling remotely. Preparation for such event was highly uneven among teachers and students with some of them having all digital skills, the software, and the hardware at hand. While others may lack digital skills on the required level, appropriate software, sufficiently fast computers, or Wi-Fi connection of a large capacity. Fortunately to the interested parties, certain aspects of the digital education were already introduced to the curriculum. Online platforms for homework, quizzes, tests, and course materials were already popular in certain locations, including web attendance records and grade rosters.

Based on my close proximity, I would claim that mental preparation for such a sudden change was entirely lacking among all individuals, maybe excluding those who went through intense meditation or military training and were able to adopt to the quickly changing reality without internal resistance. Since at same time, the entire life of everybody changed, equally students and teachers were struggling with multiple aspects of their personal and professional lives. This made the entire experience spectacularly uniform across the globe.

The execution of the traditional curriculum proved to be rather challenging in multiple aspects. Teachers reported that they simply could not cover the entire material, could not monitor students' progress during classes due to lack of cameras on students' devices. And the greatest concern was the inability to verify academic integrity of students' assignments. Exams and tests that used to be high-stakes concern became a matter of internet search.

As a result of remote learning, students could access class video lessons from home and watch them multiple times while remaining in a comfortable environment of their homes. From the perspective of challenging mathematics topics, this was a great advantage. But strict testing environment could not be executed anymore, which caused doubts about the validity of the exams. In mathematics, variability of possible solutions to basic textbook question is quite limited, making students solutions quite alike, if not indistinguishable. To rescue the rationale of math testing in general, test questions required revisions to expose individual differences among solutions. This could be done by asking for step-by-step explanations with complete English sentences or by adding a question for discussion.

Since the world was waiting for the pandemic to end quickly with a vaccine and medications, the teachers and the governments did not assume that the state of remote instructions will be permanent. Thus, the actions carried out were considered by all interested parties to be provisional and temporary.

What is spectacular about this moment in comparison to other pivoting moments in education is that the solutions of the challenges of the pandemic did not follow previously established paths of progress. There was no extensive preparation of the teachers, no adjustments of the curriculum, no preparation of the hardware, software, and no established assessment of the newly developed teaching techniques. During the first semester of the remote instructions, teachers were not evaluated since there were no suitable directions from the institutions to perform such a task.

Returning to campus after the pandemic felt even more surrealistic than converting to remote work. Things were supposed to be back to normal, but they were only up to certain extent. We, the teachers and the students, were again together in a classroom but with masks. Some students fell ill and with positive COVID test results and they were denied access to the college. Accommodating their needs required truly hybrid modality of the instruction that could accommodate both, students physically in the room and students remotely on zoom. I began accommodating students who must (often due to health or quarantine requirements) attend the class remotely even if the class is scheduled as “in person.” I began giving lectures with a laptop and a projector instead of a classic “markers on a whiteboard” lectures. This way students from the classroom could see the writing, hear the lecture, have access to my One Note writing. Even if I shared with my student class videos from previous semesters, they appreciated live recordings from our class. I began a new habit of giving unsupervised daily quizzes since they were available on Blackboard and all students could access them, including those students who attended remotely. My classes began a truly hybrid modality, when students were allowed to attend either in person or remotely. This was due to relaxing the discipline during the tests, since we spent three semesters adjusting the test questions, so they become somehow difficult to copy.

After speaking with students about their low-test results, I realized that they simply forgot how to prepare for proctored exams without access to their notes and books. We agreed that during the times of transition, it is wise to permit cheat-sheets or notes at least during some exams to allow more time for adjustments to new testing styles. This supported my intuition that learning itself is very much influenced by the style and capacity of testing.

My appreciation of having an immediate insight into the appearance of students' attention significantly increased due to my awareness that remote teaching strips entirely that information. During a discussion with students about benefits of in-person learning, I learned that they do not take class interactions for granted and sincerely participate in class activities.

At the same time, I see the culture of the entire university changing. In the past, research meetings were conducted remotely only if the interested parties could not meet in person due to distance or illness. Now most seminar meetings with colleagues from the same town take place on zoom only because this modality takes off the burden of commuting, reserving a room, and we are now all used to it. College and departmental meetings mostly have both attendance options, in person and remote. Meeting rooms are now equipped with rotating cameras and wide range microphones which allow group meetings. Now I wonder whether this trend is internationally popular.

12.6 Summary

Compared to previous pivoting moments from the history of education, the recent transition to remote learning had one important feature: it happened to all countries approximately at the same time and lasted for a similar length, which was not a matter of days or weeks but months and years. In 2021, remote education happened rather suddenly and without sufficient preceding discussions of its features. It is true that certain aspects of the remote instruction already existed and were on advanced experimental stages but for most instructors the "in-person" teaching was the default modality. All previous pivoting moments were analyzed and discussed, carefully planned, and prepared from the perspective of the students, the teachers, and the parents. In all cases, the described pivoting moments in education were somehow imposed on the society by the governments. In some cases, they were in one way or another, following and motivating the changes of the society.

What significantly distinguishes the remote teaching during the pandemic from other pivoting moments is the temporality of the modality of learning but with the underlying thought that up to certain extent some aspects can be used when the reality returns to normal. The other pivoting moments had a permanent set up with the underlying thought that the idea will expand. Thus, while introducing compulsory education the ministry assumed that it will eventually apply not only to white boys but white girls and later boys and girls of color. Similarly, secularization of education in terms of the content and the style of thinking had to progress. With the first stage being simply an introduction of secular subjects to the curriculum, the second stage being a reduction of the religious subjects, and the third stage removing all religious mentions from the books and lecture contents. While speaking with my colleagues, we realized that the growth of popular technology and the speed of Wi-Fi networks were insufficient just a year or two before the pandemic to accommodate remote teaching and learning in such capacity. Thus, the pandemic took place just at the right time for the remote teaching to bloom. Similarly, other pivoting moments of the history of education happened just at the right time when the society was ready to accept and accommodate them. This is what in my understanding characterizes the revolutionary leaps in the process of transformation of education. All the tools were already existent, and it was the traditional nature of the teachers resisting the new ideas of remote instruction. At the very moment, when the threats of the virus are declining, the world seems to embrace the new reality of digital learning combined with traditional modality of instructions. With numerous names (blended learning, hybrid courses, synchronous meetings, asynchronous courses) and varieties, one of a kind the education seems to eventually become individualized to fit the needs of the individual students.

In my view, education of mathematics will be strongly impacted by the evident needs for new and improved way of thinking about valid testing methods. On a level of college instruction, mathematics testing could shift away from being entirely based on computational skills and convert to project-based evaluation. Since in a long term, new testing methods influence the way of teaching, this should as well impact the way mathematics is taught.

At this very moment, while returning to the in-person teaching, I am finding the digital tools very useful and encouraging. This makes me think that the changes imposed during the pandemic on students, teachers, and the entire structure of education will find their places in new designs. Are we heading to the era of truly hybrid modalities of combined in-person and remote attendances?

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Chapter 13
Some Examples of Mathematical Paradoxes with Implications for the Professional Development of Teachers

Yenealem Ayalew

13.1 Introduction

The writing of this chapter was initiated to bring an alternative framework for a collaborative professional development of teachers and possible implications for optimal learning. I wanted to see the concept *mathematics teacher education* differently. The phrase may be re-considered as *education of teacher of mathematics [for students]*. Thus, there are at least three underlying terms "mathematics," "teacher," and education." Of course, the idea is that it would be possible by educator or trainer and ultimately for students, the end users. An interplay of the three thoughts is this: mathematics as the subject, teacher as the agent, and education as the program. Implicitly, the thought "mathematics teacher education" encloses two more notions: educator and student.

Then, the preparation, professional development, teaching, or empowerment of teacher educators would have impact. Which order gives more sound – mathematics-teacher-education or education-teacher-mathematics? It can be the basis of the discussion and shortly illustrated as follows (Fig. 13.1).

We have seen *mathematics teacher education* as the *education [by a trainer] of teacher of mathematics [for students]*. The overall mapping is attributed to the philosophy of education that assumes "creative imagination" as a vital component (Degu, 2020). Again, if we regard "creative imagination" of an educator as the key concept, then, the introduction would turn to sensitizing the concept; and thus, the approach would be the other way round. In this regard, the following picture might illustrate it better (Fig. 13.2).

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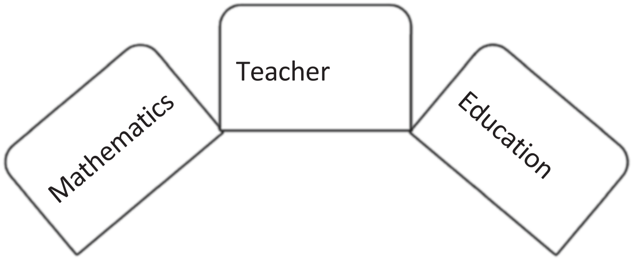

Fig. 13.1 Emphasizing on the mathematics-teacher-education so that the ultimate goal of professional development would be envisioned

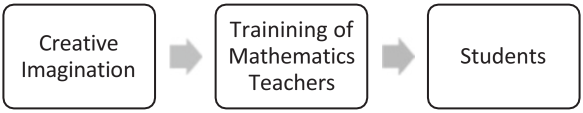

Fig. 13.2 The process of using "Creative Imagination" in teacher's professional development

Since, the end users of a teacher education program are students in schools, in order to benefit them with the best of service, seeking a creative imagination would be relevant (Degu, 2020). In this chapter, I advocate a sociocultural approach to creative imagination and mathematical paradoxes. Yet, the former would be included in the latter concept.

13.2 The Essence of Mathematics

The term *mathematics* is very common; yet, it may be hard to forward full elucidation. I consider some definitions from known dictionaries. The Oxford English Dictionary offline application defines mathematics as "a science (group of related sciences) dealing with the logic of quantity, shape, and arrangement." The Collins English Dictionary and Thesaurus defines it as "a group of related sciences, including algebra, geometry, and calculus, concerned with the study of number, quantity, shape, and space and their interrelationships by using a specialized notation." The Merriam-Webster 6.5 dictionary considers the subject as "the science of numbers and their operations, interrelations, combinations, generalizations, and abstractions and of space configurations and their structure, measurement, transformations, and generalizations." So, the term *number* and *quantification* are being referred commonly to represent the concept of Mathematics. In a sense, it may be regarded as a science of computation and operation.

On the other hand, mathematics may be taught as provision of skills to learn and create shared meanings by way of socializing the field. It is because of the fact that it is an integral part of our society in one or another way (König, 2016) as it possesses a central importance in our society. It shapes and influences many areas of our daily life. The field has been shaping the world in which we live; the world in turn shapes the discipline of mathematics (Greenwald & Thomley, 2012). Therefore, Mathematics is a human activity, a social phenomenon, part of human culture, and intelligible in a social context (Ernest, 1991). This is because as the societal problems evolve, new mathematical solutions would be created. Thus, a mathematical knowledge is socially and culturally situated (Clarke et al., 2015). In short, it is a cultural phenomenon; it is a set of ideas, connections, and relationships that we can use to make sense of the world (Boaler, 2016). Then, mathematical expressions can be and often are interpreted when applying mathematics in the real world (Baber, 2011). In such an interpretive paradigm, the basic principle is meaning making; interpretations are embedded in and dependent on language.

In this early twenty-first century, applied mathematicians seem to be far from exhausting the potential of mathematics to change and advance society (Greenwald & Thomley, 2012). So far, many developments in mathematics that raised philosophic questions are at times discussed in public (König, 2016) and found relevant for the proper schooling. Yet, there could be more mathematical discoveries or inventions that would serve for societal developments.

In this chapter, I look at mathematics through the lens of larger societal structures such as nations, cultures, and educational systems. In other ways, I tempt to explore the societal structures within mathematics, such as notions of proof, certainty, and success (Greenwald & Thomley, 2012). In turn, this goes to the issue of subjectivity in mathematics education which had been initiated by a number of scholars (Brown, 2001, 2011; Lerman, 2018; Williams, 1993). The inspiring point is that Brown and Lerman mentioned the use of multitude of filters in the didactics of mathematics. In this regard, teachers, students, researchers, and the subject mathematics are main driving forces in the didactics of mathematics and discourse. It is clear that meanings are derived from social interaction and modified through interpretation (Corbin & Strauss, 2008). Accordingly, multiple meanings could be created for the same object. For instance, scholars' conceptions of Euclid's fifth postulate have created disagreement among geometers and, as a result, Euclidean, Absolute, Elliptical, and Hyperbolic geometries were discovered.

Mathematical meaning is produced in discourse (Brown, 2001). Since, thinking mathematically is about interpreting situations (Lesh & Doerr, 2003), a mathematics education researcher is likely advocating the interpretive paradigm. However, mathematical representations are considered to be mathematically conventional, or standard, when they are based on assumptions and conventions shared by wider mathematical community. In a sense, it would be difficult to testify the truth-values of such assertions. How then do we relate to [the conventional] mathematics (Brown, 2011)? Of course, it is possible to do so. For instance, in Calculus teaching and learning, interpreting and characterizing the behavior and structure of solution functions are important goals. The rigorous study of initial-value problems would imply the effect of varying parameters on the solution space. The aforementioned three mathematical expressions are attempts of mathematization as social process. Similarly, mathematization provides another challenge for mathematics education as it becomes important to develop a critical position to mathematical rationality as well as new approaches to the construction of meaning.

Therefore, mathematics is about explorations, conjectures, and interpretations (Boaler, 2016); hence *meaning making* (Prediger, 2007) has great potential in bringing a perspective. Mathematical expressions can be and often are interpreted when applying mathematics in the real world (Baber, 2011). Social studies develop hermeneutically; its activity is inherently interpretive (Lerman, 2018). In the interpretive paradigm, the basic principle is meaning making; interpretations are embedded in and dependent on language.

13.3 Mathematical Paradoxes

Currently, mathematics education research is mostly concerned with two questions (Ernest, 2020):

- What is mathematical truth and how do we justify and explain it, and above all, how do we come to know it?
- How can we best and most effectively teach and facilitate the learning of mathematics?

These questions could be treated based on our understanding of the essence of mathematics. There is an argument that assumes mathematical knowledge as a humanly constructed (Ernest, 1991). That may enable us to see mathematics as multiple realities, relative truths, complexities, and ambiguities (Degu, 2020). However, such a freedom may lead to the existence of paradoxes. The paradoxes are deductive arguments that end in contradictions (Clark, 2012; Weber, 2021). We may think of sorts of a paradox. The most fascinating one is that it reveals a genuine problem in our understanding of its subject matter (Clark, 2012; Rayo, 2019), on the other hand, a boring paradox that leads nowhere due to a superficial mistake and is no more than a nuisance (Rayo, 2019). In this regard, Clark (2012) regarded any puzzle which has been called a "paradox," even if on examination it turns out not to be genuinely paradoxical. The most interesting paradoxes of all are those that reveal a problem interesting enough to lead to the development of an improved theory.

Euclid's fifth postulate was originally stated as:

> That if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles. (Merzbach & Boyer, 2011)

The idea could be illustrated as follows (Fig. 13.3).

The postulate states that if the lines l_1 and l_2 are extended without limit, they would meet at point. The postulate would result in a triangle. In other words, without loose of generality, three interior angles (of a triangle) would be created (Fig. 13.4).

Yet, the most familiar one is a re-statement and an interpretation of Euclid's fifth postulate by William Playfair's (1759–1823): "through a given point outside a given line, exactly one line may be constructed parallel to the given line" (Katz, 2009). It can be illustrated as follows (Fig. 13.5).

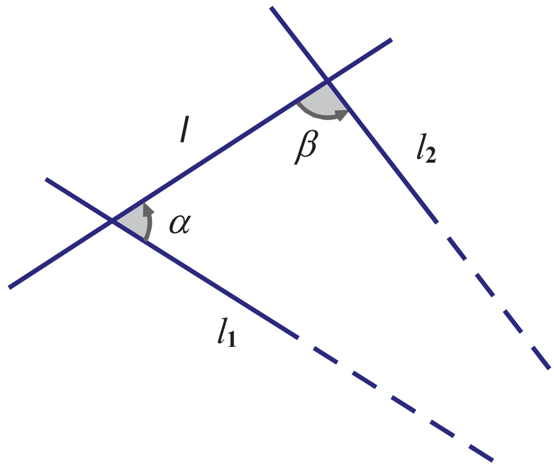

Fig. 13.3 Understanding Euclid's fifth postulate

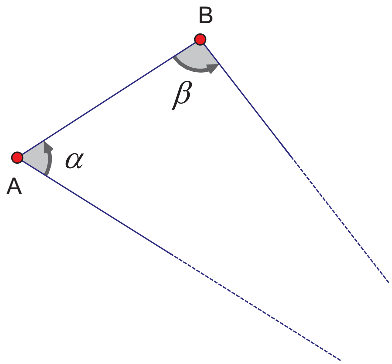

Fig. 13.4 Euclid's fifth postulate: a triangle

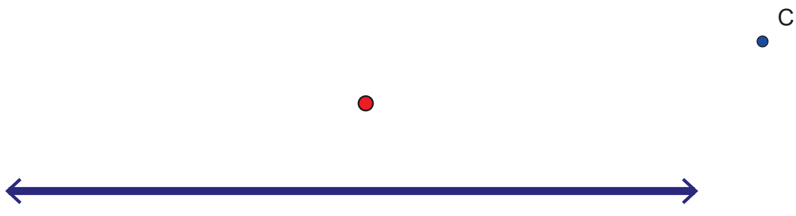

Fig. 13.5 An interpretation of Euclid's fifth postulate

The leading question is then: how many lines do really pass through *a given point outside a given line* and parallel to the given line be constructed? One? Two? Many? Infinite? What does "infinite" itself mean? As I noted earlier, the conceptions of the fifth postulate continued to create disagreement among geometers. That is why Euclidean Geometry, Elliptical Geometry, and Hyperbolic geometry were discovered. On the other hand, Absolute (Neutral) Geometry takes no position or stand on parallelism. Based on such disparities, for instance, the sum of the interior angles of a triangle is either 180°, less than 180° or greater than 180°. Since angles are quantified in numbers, the sum of numbers is a big deal. Thus, the main essences of mathematics, quantification and computation, are being questioned. In this regard, more examples are presented in the following sections.

13.4 Taking One Plus One as an Example

It is time for a renewal of the study of the politics of numbers (Mennicken & Salais, 2022). My intention here is to extend the meaning making for 1 + 1 by digging in to views of mathematics.

One Plus One in Group Theory
In the set of integers Z with a binary operation the usual addition, 1 + 1 is in Z. In this regard, 1 + 1 = 2.

Yet, 0 is the identity element and every element is invertible, for instance, 1 + (−1) = 0. It follows that, 1 + 1 = (1 + 1) + 0 = 1 + 1 + (−1 + 1).

Again, (1 + 1) = (1 + 1) + 0 + 0 + 0 + 0 + ⋯.

Hence, I have 1 + 1 = 1 + 1 + (−1 + 1) + (−1 + 1) + (−1 + 1) + ⋯.

By the associative property of the group,

$$\begin{aligned}1+1&=1+1+(-1+1)+(-1+1)+(-1+1)+\cdots\\&=1+(1+-1)+(1+-1)+(1+-1)+\cdots\\&=1+0+0+0+\cdots\\&=1\end{aligned}$$

Thus, 1 + 1 is not necessarily 2; it could be 1. We have 1 + 1 = 1.

However, if the group is (Z_2, +) whereby Z_2 = {0,1}; we get 1 + 1 = 0 (*modulo* 2). In short, we may write 1 + 1 = 0.

One Plus One in Number Theory
Most of the time, we use base 10 numeral system. However, a binary (base 2) representation using the digits 1 and 0 is essential in computer science. In a sense, information is expressed in terms of 00, 01, 10, and 11. Here, I capture 1 + 1 as $(10)_2$. In short, 1 + 1 = 10. That is a different perspective. Yet, *modulo* 2 and *base* 2 convey the same message; in both cases, 1 + 1 = 0. Thus, 1 + 1 becomes 10.

One Plus One in Logic
Paradoxes in (mathematical) logic have forced logicians to modify existing theories to rid themselves of troubling inconsistencies (Farlow, 2014). The trouble may

happen when the judgment is subjective in its nature. For instance, the knower [paradox] can mention (Clark, 2012):

> *If K is true it is false, because I know it; so, it is false. But, since I know that, I know it is false, which means it is true. So, it is both true and false.*

In the example, there are more than two sentences. Since, the conclusion given is "true and false," it is difficult to consider it as a mathematical statement (proposition) where by a sentence is either true or false, but not both.

Instead of "true," we may choose "T" in short form. Similarly, instead of "false," we may take "F" as a short form. Besides, we may associate values 1 and 0, respectively, for "true" and "false." Then, the dichotomy of true/false can be expressed as follows in a *characteristic function*.

$$\chi : A \to \{0,1\} \text{ such that } \chi(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

The truth-value table of combination of statements denoted by p and q can be put as follows.

The information in Fig. 13.6 is given by either FF, FT, TF, TT or 00, 01, 10, 11. The operation "+" could be assumed by the *logical conjunction* "and". In this case,

False and False is False. That is: 0 + 0 = 0

False and True is False; 0 + 1 = 0.

True and False is False;1 + 0 = 0.

True and True is True; 1 + 1 = 1.

Since "true and true" is logically true, we have: 1 + 1 = 1.

One Plus One in Geometry

One of the ancient geometers, Archimedes, had approximate evaluation of the ratio of the circumference to diameter for a circle (Merzbach & Boyer, 2011). If we think of a circle with minimized radius so that its area is 1 units-square and a hexagon each with one units-square, the combined area would be then represented by 1 + 1. On the other hand, the union of the areas would be the circular region. Hence, 1 + 1 = 1.

p	*q*
F	F
F	T
T	F
T	T

p	*q*
0	0
0	1
1	0
1	1

Fig. 13.6 Different scenarios of "true" and "false" with their corresponding representations using 0 and 1

One Plus One in Sequence and Series

I have considered the geometric series:

$$1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4+\ldots$$

Since $1 + a + a^2 + a^3 + \ldots$ for $|a| < 1$ converges to $\frac{1}{1-a}$, the geometric series becomes 2.

The limit value of $\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4+\ldots$ as n goes on indefinitely is 1 (Rayo, 2019).

By taking out the common factor, I get:

$$\frac{1}{2}\left[1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4+\ldots\right]$$

The expression in the bracket is equivalent to: $\dfrac{1}{1-\frac{1}{2.}}$

This leads to the equation: $\frac{1}{2}\left[\dfrac{1}{1-\frac{1}{2}}\right]=1$

Therefore $\underbrace{1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4+\ldots}_{1}=1+1=2$. That would result in the sum 2.

Once again,

$$1+1=1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4+\ldots$$

I take the common factor in

$$1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4+\ldots$$

$$=1+\frac{1}{2}\left[1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4+\ldots\right]$$

$$=1+\frac{1}{2}\left[1+\frac{1}{2}\left(1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4+\ldots\right)\right]$$

$$=1+\frac{1}{2}\left[1+\frac{1}{2}\left(1+\frac{1}{2}\left(1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4+\ldots\right)\right)\right]$$

$$=1+\frac{1}{2}\left[1+\frac{1}{2}\left(1+\frac{1}{2}\left(1+\frac{1}{2}\left[1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4+\ldots\right]\right)\right)\right]$$

$$=\left(1+\frac{1}{2}\right)\times\left(1+\frac{1}{2}\right)\times\left(1+\frac{1}{2}\right)\times\ldots=\left(1+\frac{1}{2}\right)^n=\left(\frac{3}{2}\right)^n$$

Therefore, computing the limit value of the geometric series, $\lim_{n\to\infty}\left(\frac{3}{2}\right)^n$, we have:

$$1+1=1+\lim_{n\to\infty}\left(\frac{3}{2}\right)^n=1+\infty=\infty$$

We get two conclusions: 1 + 1 = 2; 1 + 1 = ∞. What an illusion! Many of the most fascinating paradoxes involve infinity (Clark, 2012).

One Plus One in Applied Mathematics

In molecular biology, there is a term "Endosymbiosis" as a form of symbiosis in which two cells live together in nature, one inside the other (Archibald, 2014). Then, Endosymbiosis acts to bring evolutionarily distinct lineages together in a manner that can lead to the generation of entirely new organisms. That is, one plus one equals one.

So far, we saw how "One Plus One" varies in different scenarios including Applied Mathematics, Group Theory, Number Theory, Logic, Geometry, Sequence and Series. The results were 0, 1, 2, 3, 10, or ∞. The summary is given below.

$$1+1=\begin{cases}0\,(modulo\ 2)\\ 1\text{ as it happens in logic and Endosymbiosis}\\ 2\text{ as it happens in the usual addition}\\ 3\text{ in the case of synergy or collaborative work}\\ 10\,(\text{in base }2)\text{ or simply }(10)_2\\ \infty\text{ when infinite geometric series is considered}\end{cases}$$

Yet, there might be more alternatives too. The more varied the input, the more unexpected the combinations, the more creative the ideas. If new ideas come with new combinations of existing ideas, the more connections we can create, the more ideas we can generate (Trott, 2015). However, the claim needs to have some underpinning reason (Ernest, 2016). So, what makes us to believe in the truth of 1 + 1 = 2? The above scenarios have shown us the need to acknowledging plurality or subjective truths.

13.5 Associated Meanings in Mathematics Education

From the examples of paradoxes reported in this chapter, we can look for some implications in mathematics education. How is a meaning given to a mathematical conception? Or, what are the criterions to consider it as mathematical idea? The answer may be as follows: mathematical relationships are reasoned with subjective meanings and arguments. Then, the art would be the selection of a level of representation which is appropriate for a given task, both in terms of the information available and in terms of the reasoning required (Forbus, 2008). The formulae can be generated while lots of opinions are gathered. Generally, such qualitative values could be inculcated while mathematics education is associated with the daily life students.

The *question of meaning* with respect to mathematics education, the issue becomes more complex (Kilpatrick et al., 2005). On the one hand, we may claim that an activity has meaning as part of the curriculum, while students might feel that the same activity is totally devoid of meaning. For instance, what is meant by infinity? Mathematics education treats the term *infinite* not as a real number, but as the quality of being unlimited in size (Wapner, 2005). That suggests seeking abundance in our activities. It also gives place for qualitative value than quantification. Then, a change in teaching takes time, imagination, courage, and honest reflection on what works and what does not. So, a mathematics teacher or educator is expected to see the essence of the subject differently. In turn, that would determine the how of teaching and learning processes.

Above all, the key question to be posed is as follows: is it possible to bring qualitative mathematical relations in to classroom context? The science of learning and teaching mathematics can propose value judgments and normative prescriptions that can inform teachers of mathematics on the provision of study processes, and teachers will take up, or not, those judgments and prescriptions and interpret and apply them as they see fit (Lerman, 2018).

In this twenty-first century, applied mathematicians are far from exhausting the potential of mathematics to change and advance society (Greenwald & Thomley, 2012) Some developments in mathematics raised philosophic questions. There is a controversy concerning the certainty of mathematical knowledge and what it means (Ernest, 2016). Education is becoming more available, to members of society, in more places, and more ways than ever in human history (Naidoo, 2021) Yet, higher education is currently designed to meet the needs of old time (Gleason, 2018).

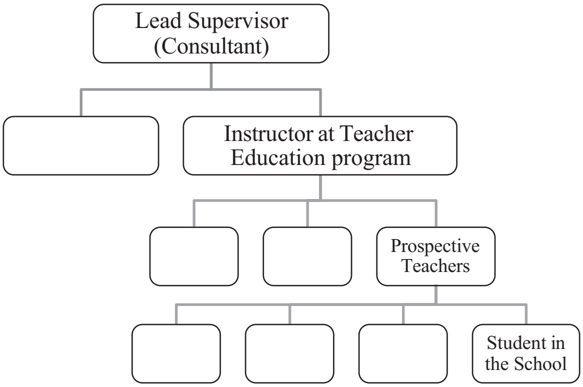

Fig. 13.7 Multi-level collaboration in the professional development of mathematics teachers

On the other hand, we are expected to plan and work for tomorrow. Yet, the future is uncertain; we do not know what will happen to the practice of education. Both teachers and students are required to possess critical skills to achieve success within the educational environment (Naidoo, 2021). Then, if we have a sense of the drivers that will influence society, schooling, and teacher education, we can begin to imagine possibilities for teacher education futures (Schuck et al., 2018). I think, the main movers and shakers in the education system are teachers and teacher educators. So, their creative imaginations would have greater impacts on the end users.

13.6 Professional Competencies of Mathematics Teachers

Since the teacher is a key stakeholder in the student's learning process, his/her capability shall be of a concern. Then, the teacher training program and competencies of teacher educators would be relevant to address. The following areas of standards (Ayalew, 2017) were framed for a mathematics teacher educator in Ethiopian context.

- *Personality and Professional Ethics*
- *Language and Communication*
- *School Mathematics Education*
- *Guidance and Support*
- *Assessment and Feedback*
- *Partnership and Collaboration*
- *Professional Development*
- *Teacher Education*

It was assumed by then that the standards could be used for determining courses to educators of mathematics teachers and evaluation of performance. Since the mathematics teacher educators are not from the same field of specialization, it is difficult to write down the detailed certification criterions. Yet, it is possible to point out the standards with regard to teaching and toward mathematics teacher education. In short, what is expected of mathematics teacher educators in this era? How could they work and learn together? The underling analysis goes to consider an investigation and a formulation of the creative imagination of educators. For instance, groups of mathematics teachers may be engaged in lesson studies.

So, managing such a collaboration may demand addressing interdisciplinary perspectives some of which are mathematics educator's point of view, philosophy, the science of learning, teacher education, and classroom observation. Accordingly, it demands the collaboration and involvement of faculty members. To enable prospective mathematics teachers to make stronger connections with the profession, an educator worked collaboratively with a practicing teacher by co-teaching one cohort of pre-service teachers studying primary mathematics education (Downton et al., 2018). Hence, an ideal collaborative work model could enable us to justify such assumptions (Fig. 13.7).

This hierarchy is based on an inspiration from the Ethiopian "traditional" (Orthodox Church) education peer-teaching and multi-stage teaching approaches. By this hierarchy, I assume that the supervisor (senior researcher in mathematics teacher education), research fellows (post-doctoral, doctoral or Master's degree candidates), practicing teachers (in-service mathematics teachers, action researchers, or teachers engaged in continuous professional development activities), and students are community of practice.

Here, in this era, an imagination is needed to construct activities, build a system, and anticipate conversations and actions that will bring learners' inquiry to fulfilment, enabling growth toward desirable skills and understandings. However, the selection of participating educators and teachers would be determined by the kind recommendation to the supervisor. In order to run such a scheme properly, communication and progress reports might to be performed regularly. Students would need to be exposed to and be stimulated in learning through technology-enabled pedagogy and technology-based tools to enhance the development of technology within educational environments. Teachers would also need to be proficient in using technology-enabled pedagogy and technology-based tools (Naidoo, 2021). They need to be professionally developed in acquiring skills of critical thinking, creativity, collaboration, communication, information literacy, media literacy, technology literacy, and flexibility. Thus, a multi-level collaborative framework would shape personal understandings of those who aspire to become mathematics teacher educators.

In this regard, concepts including imaginative leadership, facilitation, construction, and co-construction of knowledge might be worth to consider. This is because mathematics teacher educator's roles, competencies, and challenges are composite functions of practicing teachers' experience and students' expectation (Ayalew, 2017). Thus, senior researchers are expected not study on teachers but to work with them (Superfine, 2019). I argue that seeking creative imaginers would be vital to push the teaching profession in to a higher level. This view might be linked to "Futures Research" which seeks to provide insights that might help to change the present and direct the future (Schuck et al., 2018). In other words, the purpose is to systematically explore, create, and test both possible and desirable futures to improve decisions. There are numerous futures research methods used to gain understandings of possibilities in teacher education (Schuck et al., 2018); some of which are horizon scanning, driver analysis, Delphi panels, scenario production, and back casting.

13.7 Conclusions

The lines of thoughts communicated in this chapter may be labeled into two: mathematical paradox of infinity and social meaning of mathematics. "It is time for a renewal of the study of the politics of numbers" (Mennicken & Salais, 2022). In a sense, such an insight of multi-varied nature of heading to numbers in different situations is the present-day matter. This chapter has elaborated the sum of one and one. The discussion relies on an eclectic position of the different philosophies of mathematics. For instance, the mathematical statements "1 + 1 = 2" is wrong for Fictionalism because there are no such objects. On the other hand, the conclusion "1 + 1 > 2" is wrong in Absolutist view. Since a paradox is an argument that appears to be valid and impossible for valid reasoning to take us from true premises to a false conclusion (Rayo, 2019), it is a sure sign for at least one of the following mistakes: our premises are not really true; our conclusion is not really false; or our reasoning is not really valid. Therefore, a mathematical truth may not be taken for granted in the classroom instruction. Hence, there are multiple truths in mathematics. On the other hand, a habit of working together and self-study would afresh the existing practice in the teaching and learning process. Besides, there is a need of "transformation" of a community of practice into a community of inquiry (Goodchild, 2014); it can be a great concern.

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Part III
Philosophy of Critical Mathematics Education, Modelling and Education for Sustainable

Chapter 14
A Performative Interpretation of Mathematics

Ole Skovsmose

14.1 Introduction

In the way it was traditionally developed, the philosophy of mathematics appears two dimensional by concentrating on *ontological* and *epistemological* issues. I certainly consider such issues important, but I want to highlight the need of developing a four-dimensional philosophy of mathematics by adding a *sociological* and an *ethical* dimension. Such a philosophy has been presented in *Connecting Humans to Equations: A Reinterpretation of the Philosophy of Mathematics* (Ravn & Skovsmose, 2019). However, Ole Ravn and I do not claim that a philosophy of mathematics only contains four dimensions. One could, for instance, consider the relevance of an aesthetic dimension.

The ontological dimension addresses the nature of mathematical objects by focussing on questions like the following: What is a number? What is a point? What is a function? The epistemological dimension addresses the nature of mathematical knowledge by concentrating on questions like the following: How do we come to know about abstract mathematical objects? What is the nature of mathematical deduction? Is mathematical knowledge absolute? The sociological dimension addresses the social formation of mathematical knowledge by raising questions like the following: Are mathematical truths time-dependent? Do metaphysical or religious convictions have an impact on mathematical theorising? Does technological

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development form mathematical research approaches?[1] The ethical dimension considers the mathematical formation of the social by asking questions like the following: What is the impact of bringing mathematics into action? Could such an impact be problematic, if not disastrous? How might mathematics, brought into action, form our life-worlds?

The notion of life-world was introduced in 1936 by Edmund Husserl (1970) when he published the original German version of *The Crisis of European Sciences and Transcendental Phenomenology*. The notion has been further explored by phenomenology and by social theorising inspired by the phenomenological outlook. Husserl presents the life-world as an original experienced world, not structured by scientific or any other kinds of systematic knowledge. In *The Structures of the Life-World*, Alfred Schutz and Thomas Luckmann (1973) point out that a life-world is to be understood as "that province of reality which the wide-awake and normal adult simply takes for granted in the attitude of common sense" (p. 3). However, my use of the notion of life-world is different from the one inspired by phenomenology. I see a life-world as a complexity of socially structured living conditions. It can be structured by economic, political, religious, ideological, cultural, and discursive factors. For a clarification of my interpretation of life-world, see my book *Foregrounds: Opaque Stories About Learning* (Skovsmose, 2014b).

In this chapter, I suggest a *performative interpretation of mathematics*, which will establish the ethical dimension as being crucial to a philosophy of mathematics. In different ways – and by means of different terminologies – I have indicated a performative interpretation of mathematics. In Skovsmose (1994), I talk about the "formatting power of mathematics". Later, I talk about "mathematics in action" and explore different features of such actions. In Ravn and Skovsmose (2019), we analysed forms of "mathematics-based fabrications".[2] A performative interpretation contrasts a *descriptive interpretation of mathematics*, which highlights that mathematics is a unique tool for describing natural phenomena: by means of mathematics, one can capture the laws of nature. With respect to technological or architectural constructions, mathematics has also been considered a crucial descriptive tool. For instance, descriptive geometry addresses how to make two-dimensional presentations of three-dimensional objects and, in this way, how to provide blueprints of any kind of construction. According to a descriptive interpretation, mathematics can be considered as the language of both science and technology. In contrast, a performative interpretation of mathematics highlights that, by means of mathematics, one performs interventions on reality. Mathematics is not only a means for description, but also a means for action. Mathematics brought into action might generate a range of diverse implications, which require ethical reflections.

[1] An important initial step in beginning to explore the sociological dimension of a philosophy of mathematics was taken by Wittgenstein (1978, 1989). More steps were taken by Restivo (1992); Restivo et al. (1993); and Ernest (1998).

[2] For a careful presentation of metamathematics performativities, see also Yasukawa et al. (2012, 2016) and Part 4 "Mathematics and Power" in Skovsmose (2014a, pp. 199–280).

In the following, I start outlining three well-established positions in the philosophy of mathematics, namely, logicism, formalism, and intuitionism. All of these positions are fully concentrated on ensuring mathematics is imbued with certainty by focussing on ontological and epistemological questions. In order to move beyond this two-dimensional thinking, I present a performative interpretation of language, which for me serves as an inspiration for suggesting a performative interpretation of mathematics. This interpretation I outline with respect to both advanced mathematics and school mathematics. This brings me to recognise that mathematics brings about a pervasive formation of our life-worlds. As a result, the ethical dimension of a philosophy of mathematics becomes crucial.

14.2 The Three Magi: Merry Mathematics!

The three classic positions in the philosophy of mathematics – logicism, formalism, and intuitionism – are paradigmatic examples of two-dimensional thinking. "Philosophy of mathematical practice" is a recent trend that critically reacts to the three classic positions, although it does not move far beyond two-dimensional analyses. Neither logicism, formalism, intuitionism, nor the philosophy of mathematical practice has engaged in any performative interpretation of mathematics, and consequently these positions were all oblivious to ethical issues. Against this backdrop, I find it relevant to present a performative interpretation of mathematics that will bring the philosophy of mathematics out of its self-created ethical vacuum.

The *logicist programme* was launched in three steps by Gottlob Frege. First, he wanted to make sure that mathematical deductions had the genuine logical format, and in the *Begriffsschrift*, first published in German in 1879, Frege (1967) indicated how logic itself could be organised as an axiomatic system. The German word *Begriff* means "concept" and *Schrift* means" writing", so literally speaking *Begriffsschrift* means "concept writing". Frege claimed that all valid forms of deduction would appear as theorems in this system. Next, in *Die Grundlagen der Arithmetics*, published in German in 1884, Frege (1978) showed how the notion of number could be defined by means of logical notions. Finally, in *Grundgesetze der Arithmetik I-II*, Frege (1893, 1903) showed in detail how mathematics could be presented as a deductive system based upon a foundation of logic.

By organising logic as an axiomatic system, showing how basic mathematical notions could be defined by logical notions, and how mathematical theorems could be derived from logical theorems, Frege outlined the whole logicist programme. Logicism confronted psychologism, which interprets mathematical knowledge as being derived from empirical observations. In *A System of Logic,* published in 1843, John Stuart Mill (1970) formulated the psychologist position, and in *The Nature of Mathematical Knowledge*, Philip Kitcher (1983) revived this position. In *Principia Mathematica,* published in three volumes, Alfred Whitehead and Bertrand Russell (1910-1913) reworked all the technical details of the three-step programme presented by Frege. A difficulty for finalising the logicist programme in a satisfactory way emerged in terms of set-theoretical paradoxes. In *Principia Mathematica*, such paradoxes were prevented by means of a theory of types. This was, however, a deviation from the genuine logicist programme, as the theory of types is not a pure logical theory, but rather a heuristic device. *Principia Mathematica* is the most ambitious attempt to present mathematics as an ahistorical logical structure, assumed in order to ensure mathematics was imbued with eternal certainty.

Like logicism, formalism also tried to infuse mathematical knowledge with certainty. This concern was provoked by the appearance of paradoxes within what were thought to be solid mathematical structures. A point of departure for the *formalist programme* was to recognise problems with respect to the axiomatisation of geometry as presented in Euclid's *Elements*. It appeared that Euclid had used more than the five explicitly stated axioms for proving the theorems. In addition, he had used some intuitive conceptions of the properties of lines, planes, and space. In *Grundlagen der Geometrie*, published in 1899, David Hilbert (1968) presented a revised axiomatic system, containing in total twenty axioms. By doing so, he made the implicit axioms used by Euclid explicit. In this way, Hilbert wanted to eliminate the leftovers of intuition accompanying mathematical deductions. Intuition was under suspicion, allowing paradoxes to emerge in mathematical theory building.

In order to eliminate intuition from mathematical theory building in general, and not only from geometry, Hilbert launched the metamathematical approach.[3] According to this approach, mathematical theories should be properly axiomatised, as Hilbert himself had done with respect to Euclidian geometry. Then, the axiomatised theories should be represented as formal systems written in a symbolic language of the same type as the one presented in *Principia Mathematica*. The formal representations of mathematical theories should be analysed, in particular with respect to consistency and completeness. With respect to consistency, the ambition was to demonstrate that it was not possible to prove a theorem T as well as $\neg T$. With respect to completeness, the ambition was to demonstrate that for any possible formula T, it was possible to prove either T or $\neg T$. By constructing such metamathematical arguments, it was Hilbert's ambition to vaccinate mathematical theories against possible paradoxes.

However, in 1936 Kurt Gödel (1962) showed that Hilbert's ambition was illusory by demonstrating that, in the case where a mathematical theory has a certain degree of complexity – so complex that it contains the theory of numbers – then if it was consistent, it would be incomplete. In other words, a consistent theory would always contain a formula T, where neither T nor $\neg T$ could be proven. Nevertheless, embedded in Hilbert's metamathematical programme, formalism emerged as a general philosophy of mathematics by claiming that the formal representations of mathematical theories are the real mathematical theories. Genuine mathematics is captured by formal languages, while other forms of expressing mathematics are preliminary and only serve as approximations to mathematics.[4]

[3] For a careful presentation for metamathematics, see Kleene (1971).

[4] For a concise presentation of a formalist philosophy of mathematics, see Curry (1970)

By publishing the article "Intuitionism and Formalism" in 1913, L.E J. Brouwer launched the *intuitionist programme*. He was also concerned about the emergence of paradoxes within mathematics but suggested quite a different route out of the foundational crisis than Hilbert. Brouwer found that the paradoxes had emerged because invalid forms of logical deduction had been applied in mathematics. Brouwer, however, was not embracing any logicist programmes, as he found that the logical system as presented in both the *Begriffsschrift* and *Principia Mathematica* had incorporated invalid logical principles.

Brouwer did not see the need to eliminate intuition from mathematics but argued that a proper format of mathematical intuition had to be established. For instance, Brouwer did not consider indirect proof to have general validity. The approach of making an indirect proof of a theorem T is to assume non-T and to demonstrate that this assumption leads to a contradiction. On this basis, one claims to have proven T. However, according to Brouwer this is not a valid mathematical argument, as mathematical insight must be based on constructive processes. This means that one could be in the situation where neither T nor non-T has been proven. As a consequence, Brouwer claimed that the logical principle $p \vee \neg p$ is not valid in general with respect to mathematics. When any such invalid arguments are eliminated from mathematics, the paradoxes will evaporate. In "Intuitionism and Formalism", Brouwer argued that this would be the case by showing how different recognised paradoxes would disappear when assuming an intuitionist approach to mathematics.

Brouwer positioned intuition in the centre of mathematical theory building by claiming that mathematics emerges through mental constructions. According to intuitionism, such constructions constitute the real mathematics. Quite contrary to formalism, Brouwer considered formalised systems as being imprecise and at times as mischievous representations of genuine mathematics. To think of formalism as being *the* mathematics would be similar to thinking of a sheet music as being *the* music. Brouwer did not associate intuition with any degree of uncertainty. To him, a proper use of intuition was the best way of securing certainty in mathematics. Already in 1905 in *Life, Art, and Mysticism*, he emphasised this conviction to the claim that "truth is always the same to those who understand" (Brouwer, 1996, p. 404). In his *Cambridge Lectures on Intuitionism*, Brouwer (1996) gave a revised presentation of intuitionism. In *Intuitionism: An Introduction*, Arend Heyting (1971) gives a captivating introduction to the whole programme.

Different as they are, logicism, formalism, and intuitionism have all struggled to establish mathematics with certainty. The three of them have provided paradigmatic illustrations of what a two-dimensional philosophy of mathematics could look like.[5] These Three Kings, the Magi, worshipped mathematics as being a sublime science, whose eternal qualities could be grasped by concentrating on the intrinsic properties of mathematics.

[5] A two-dimension philosophy of mathematics has been elaborated further on, both in terms of detailed analyses and in terms of comprehensive presentations. See, for instance, Bernacerraf and Putnam (1964), Brown (2008), George and Velleman (2002), Jacquette (2002), Körner (1968), Mehlberg (1960), and Shapiro (2000).

The idolisation of mathematics resonates not only with the idolisation of the newborn Jesus, but also the idolisation of science in general that was part of the Modern outlook. Only a few philosophers during that period of time, Friedrich Nietzsche being one of them, confronted this position. In *Die Fröhliche Wissenschaft*, first published in 1882, he ironised over the modern idolisation of science. The expression *Fröhliche Weihnachten* is the German version of "Merry Christmas", and by talking about "Merry Science", Nietzsche portrays the worshipping of science as ridiculous. Missing Nietzsche's irony, *Die Fröhliche Wissenschaft* is translated into English as *The Gay Science* (Nietzsche, 1974). Nietzsche cannot associate any supernatural qualities with science. It is a human – all too human – affair, to use an expression that was the title of another of his books.[6]

The philosophy of mathematical practice challenged the Three Magi. This trend was initiated by Ruben Hersh (1979) when he published the article "Some Proposals for Reviving the Philosophy of Mathematics". He found that the philosophical investigations developed by the Three Magi all suffered from philosophical inbreeding. Instead, the philosophy of mathematics should be rooted in what was taking place in mathematical research practices. Hersh wanted to establish a philosophy of mathematics relevant to mathematicians, rather than to philosophers. The philosophy of mathematical practice developed rapidly after Hersh's initiation. Important steps were presented in *New Directions in the Philosophy of Mathematics*, edited by Thomas Tymoczko (1986), and detailed elaborations of a range of topics are presented in *The Philosophy of Mathematical Practice* edited by Paolo Mancosu (2008).[7]

In my summary of philosophical positions, I have not referred to Husserl's philosophy of mathematics as expressed in *Logische Untersuchungen*, published in two volumes in 1900 and 1901 (see Husserl, 2001). The reason is that this philosophy, although having had a huge impact on the formulation of phenomenology, has not been part of traditional debate in the philosophy of mathematics. The exclusion of Husserl might partly be due to Frege's accusation that Husserl was assuming a psychologist's position – an accusation that troubled Husserl and which he tried to show was unjustified.[8]

It might be that the philosophy of mathematical practice paved the way for the sociological dimension by, for instance, considering the impact of computers on mathematical research. Still, the philosophy of mathematical practice did not address ethical issues. Hersh (1990) published a paper with the title "Mathematics and Ethics". The questions he addressed, however, concern what can briefly be referred to as "the professional conducts of mathematicians" and not the broader issues concerning the social impact of mathematics. Like the Three Magi, the philosophy of mathematical practice operated in an ethical vacuum. However, in the

[6] See Nietzsche (1986).

[7] See also Ferreirós (2016) for discussing the very concept of mathematical practices, and Carter (2019) for providing an overview.

[8] Expositions of Husserl's philosophy of mathematics can be found in Centrone (2010), Haddock (2006), and Hartimo (2021).

Handbook of the History and Philosophy of Mathematical Practice edited by Bharath Sriraman (2020), one finds one important exception from this claim, namely Paul Ernest's (2020) chapter "The Ethics of Mathematical Practice".

From now on, we will concentrate on challenging the agenda of the philosophy for mathematics as set by the Three Magi. I am going to move beyond a two-dimensional philosophy by exploring an ethical dimension.[9] An important step in doing so is to suggest a performative interpretation of mathematics, and my inspiration for doing so emerged from a performative interpretation of language.

14.3 A Performative Interpretation of Language

A performative interpretation of language contradicts a descriptive interpretation that sees language as providing "pictures" of reality. In the *Tractatus Logico-Philosophicus* first published in 1922 in a German-English parallel edition, Ludwig Wittgenstein (1992) presented such a picture theory of language. Here, Wittgenstein did not talk about language in the plural, but about *the* language. He had in mind the formal language of logic and mathematics as presented by Frege in the *Begriffsschrift* and by Whitehead and Russell in *Principia Mathematica*. According to Wittgenstein, such a language has the capacity to depict reality.

In the *Tractatus* we find a descriptive interpretation of mathematics, which resonates with both logicism and formalism. We are going to confront this descriptive interpretation with a performative interpretation of mathematics. The *Tractatus* is composed around seven principal statements, where Statement 6 states that what can be expressed can be expressed in a formal language, while Statement 7 claims that one should be silent about the rest. As the *Tractatus* itself is not written in any formal language, the book concludes with a demolition of its own validity.

Wittgenstein himself came to doubt his descriptive interpretation of language, and in *Philosophical Investigations*, published posthumously in 1953, he questioned both the supremacy of formal language and the idea that language provides pictures of reality. The notion that Wittgenstein (1953) is using for abandoning the descriptive interpretation of language is *language game*. Through this notion, Wittgenstein highlights that one is *doing* something when using language. One is "gaming". One is performing. In *Philosophical Investigations*, Wittgenstein does not propose any particular priority with respect to the formal language of mathematics. He does not talk about *the* game, but rather about language games in the plural. The formal

[9] For other attempts to move beyond a two-dimensionality of philosophy of mathematics see, for instance, Bueno and Linnebo (2009), Hacking (2014), Lakatos (1976), and Linnebo (2017). It is important to observe that the philosophy of mathematics education does not operate within any two-dimensionality but establishes itself with many dimensions. This is apparent when considering the publication in the *Philosophy of Mathematics Education Journal* edited by Paul Ernest. See also Ernest (2018a), and Ernest et al. (2016).

language of mathematics constitutes just some possible language game amongst many other possibilities.

Apparently independent of Wittgenstein's formulations, John Austin also suggested a performative interpretation of language. Certainly, Austin was fully aware of what he was up against; thus he translated Frege's *Die Grundlagen der Arithmetik* into English as *The Foundations of Arithmetic*. Austin's whole approach is condensed in the title of a collection of some of his articles: *How to Do Things With Words* (Austin, 1962). His point is that the descriptive interpretation of language is inadequate. We are *doing* things by means of words. Austin finds that a statement can have a locutionary content, an illocutionary power, and a perlocutionary effect. For instance, when making a promise, one is informing another about the content of a promise (its locutionary content), but one is doing more than that. One is *promising* something. A promise has an illocutionary power by putting the person who makes the promise under an obligation. Finally, a promise has a perlocutionary effect, which refers to the effect the promise might have. For instance, a person who listens to the promise being made might doubt whether the promise will be kept. Austin reaches the conclusion that any statement – and not only statements like making a promise, making an accusation, and cracking a joke – has locutionary contents, an illocutionary power, and a perlocutionary effect. Performativity is not something we choose our linguistic expressions to have or not have. Performativity is embedded as an integral part of any use of language.

The performative interpretation was elaborated further by John Searle (1969), who talked about *speech acts*. This notion grasps explicitly the idea that one *acts* by speaking. By means of speech act theory, Searle provides a rounding off of the performative interpretation of language within the outlook of analytical philosophy. However, this rounding off also opened the way towards much broader performative interpretations of language.

Different versions of discourse theory present a much more radical perspective on what can be done by means of language.[10] This perspective was anticipated by Friedrich Nietzsche, when he characterised truth as a "mobile army of metaphors" (Nietzsche, 2010, §1). A discourse can also be considered a mobile army of metaphors, and it can support all kinds of ideologies. Discourses can be mischievous; they can be the carrier of questionable preconceptions; and simultaneously they can be powerful. Pierre Bourdieu (1991) talks about *symbolic power*, which highlights that discourses constitute real-life interventions and format what we see and how we act.

Slavoy Žižek (2008) talks about *symbolic violence*, as he finds that language can be aggressive. He states the following:

> Language simplifies the designating thing, reducing it to a single feature. It dismembers the thing, destroying its organic unity, treating its parts and properties as autonomous. (p. 61)

[10] For presentations of discourse analysis, see, for instance, Jørgensen and Phillips (2002); and Torfing (1999). I have interpreted mathematics as being a discourse (see Chap. 14 in Skovsmose, 2014a, pp. 199–214), and also ethnomathematics as being discourse (Skovsmose, 2015).

In this condensed formulation, Žižek brings together several metaphoric expressions of discursive acts.[11] Language might be used for designating things. According to the classic understanding of language, to "designate" is the principal function of language. Žižek's point, however, is that "designating" is not a simple and transparent process. "Designating" can transform what is then redesignated, as *language simplifies the designated thing.* In other words, one should not expect language to provide an accurate picturing of reality: language might distort and misrepresent.

Language might simplify reality by reducing what it is supposed to describe into a *single feature*. As an illustration, one can think of the many labels by means of which groups of people are singled out as being "immigrants", or "blacks", or "criminals". Such labels might include not only references to different groups of people, but also brutal acts of classification, deprivation, and accusation. Žižek also highlights that language *dismembers the thing, destroying its organic unity*. Language is a means for making dissections and for cutting things into pieces. As an illustration, one can consider how meticulous descriptions of work processes can turn into scrupulous suggestions for further automatisation and, consequently, for firing people. Such descriptions may strip workers of their human qualities and conceptualise them as more or less efficient components of a production machinery.

According to Žižek, *language treats parts and properties as autonomous*. In order to illustrate this metaphor, one can think of language as a means for paying attention to something, implying that other things may be ignored. Discourses are selective, formal languages as well. The many databases that provide resources for big companies' promotional strategies operate with particular information about, say, people's searches on the Internet, quite apart from whatever else might be relevant to consider with respect to the person. The management of the so-called "big data" has turned into a paradigmatic example of how bits and pieces of information are processed in non-transparent ways and turned into a basis for decision-making with huge impacts.

Precisely what Žižek had in mind when stating that *language treats parts and properties as autonomous*, we cannot know. It might be a pointed remark towards the conception of language adopted by analytic philosophy. According to this conception, the world is composed of logical independent facts, and the basic role of language is to depict these facts. The idea was advocated by Russell (1905) in the paper "On Denoting", where he tries to show that a formal language has an analytical power in depicting facts that no natural language can demonstrate. This idea was also advocated by Wittgenstein in the *Tractatus* where the statement §1.1 claims "The world is the totality of facts, not of things", and §2.061 states "Atomic facts are independent of one another". Furthermore, it is Wittgenstein's conviction that only a formal language is able to depict the totality of facts. This whole outlook was elaborated in detail by Rudolf Carnap (2003) in *Der Logische Aufbau der Welt*, first published in 1928. Carnap made the draft of the book during the period 1922-1925

[11] In Chap. 15 "Symbolic Power, Robotting, and Surveilling" in Skovsmose (2014a, pp. 215–230), I have commented about the same quotation by Žižek. Here I draw on these comments.

after he, as a member of the Vienna Circle, had studied the *Tractatus*. Using updated terminology, one can claim that the *Tractatus* presents the world as a collection of "big data", which can be depicted by formal systems. This is a way of treating parts and properties as autonomous.

Language might be powerful and violent by remaining silent about something. As a consequence, I find it relevant to talk about *symbolic silencing*, and also to consider this as being a powerful symbolic act. As an illustration of what this could mean, I can refer to an observation made by Thomas Eisensee and David Strömberg (2007). They point out that for every person killed by a volcano, nearly 40,000 people have to die of hunger to get the same probability of coverage in the USA television news. The global hunger problem can be addressed in different ways; being silent about it is one possible and cynical way.

By addressing something and by naming it, one might take an important step in entering a transformative process. For addressing forms of oppression and atrocities, linguistic articulation is important. Such acts I refer to as *symbolic articulations*. In the *Pedagogy of the Oppressed*, the original Portuguese version of which was published in 1968, Paulo Freire (2005) states:

> Dialogue is the encounter between men, mediated by the world, in order to name the world. (p. 88)

> [I]t is in speaking their word that people, by naming the world, transform it, dialogue imposes itself as the way by which they achieve significance as human beings. (p. 88)

Freire talks about speaking the *word* and naming the *world*, which appears to be a rather a passive activity. But it is not. To name the world is a political act by means of which one may call attention to cases of oppression and social injustice that otherwise might remain hidden behind a curtain of silence. Patterns of oppression can be normalised by being ignored. Freire's concept of naming the world is a paradigmatic expression of what it could mean to break a symbolic silence by means of a symbolic articulation. Such an articulation might have a profound socio-political impact, as an initial step in transforming the world. Freire also talks about "reading the word" and "reading the world", which is also a way of expressing a performative interpretation of language.[12]

Symbolic power can be acted out as symbolic violence, symbolic silence, symbolic articulation, and certainly in many other ways as well. In general, I will talk about *symbolic acts*, whatever the kind of language and acts we are dealing with. Symbolic acts form world views, decision-making, and actions. *I think of a performative interpretation of language as a suggestion of what to consider when looking at language through philosophical lenses. It is a suggestion to not only pay attention to syntactical patterns, grammatical rules, and semantical structures, but also to think about what is acted out by means of language. It is a suggestion to consider the full impact of symbolic acts and how they might form our life-worlds.*

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[12] See Freire and Macedo (1987). I am not aware of Freire referring explicitly to the performative interpretation of language as suggested by Wittgenstein, Austin, and Searle. Freire's performative interpretation of language appears to be inspired by other resources.

14.4 Performances Through Advanced Mathematics

The performative interpretation of language invites for a performative interpretation of mathematics. Such an interpretation I have investigated in terms of bringing mathematics into action (see, for instance, Skovsmose, 2014a, 2015, 2020a, b). In his discussion of semiotics, Paul Ernest (2021) has indicated the possibility of formulating a performative interpretation of signs. I find this suggestion extremely interesting as this interpretation might bring about a general point of departure for both a performative interpretation of language and a performative interpretation of mathematics. It might be possible to see mathematics brought into action as a symbolic act, and what has been said about symbolic acts in general – and about symbolic violence, symbolic silencing, and symbolic articulation in particular – might make sense with respect to mathematics. I talk about the "performativity of mathematics" as one talk about the "performativity of language". Naturally, neither mathematics nor language do anything by themselves: it is people doing mathematics and using the language that drives the performances. But it is convenient to use these more compressed formulations.

Mathematics forms an integral part of daily-life economic transactions. The apparently simple act of paying with a credit card takes place on top of a huge amount of mathematical algorithms put into operation. The code of the card is registered when we insert the card in the slot machine and press a few numbers. The amount to be paid is subtracted from the account associated to the card, and an equivalent amount enters the account of the shop. Everything is apparently due to a few movements of hands and fingers. However, behind this surface of simplicity a complex system of mathematical algorithms is brought into action.[13] Such algorithms do not just describe what is taking place; they are not merely picturing anything; the algorithms are in fact performing what is taking place.

In *Weapons of Math Destruction*, Cathy O'Neil (2016) points out that by speeding out of human control, mathematics-based automatics might be one of the causes of the economic collapse that took place in 2008. Mathematical algorithms form an integral part of decision-making and risk-taking at the stock market. While previously decisions about selling and buying were based on human interventions, they are now, to a large extent, automatised by means of mathematical algorithms. *Algorithmic trading has become a common phenomenon, and* decision-making and risk-taking have turned into advanced mathematical disciplines.[14] In this way, the whole stock market has turn into a gigantic economic experimental laboratory, where nobody can maintain an overview of what is taking place. The mathematisation of the stock market is not just a simple improvement of already existing

[13] The duality between the apparent simple actions and the mathematical complexity of the underlying technology has been expressed in terms of notions of mathematisation and demathematisation. See Jablonka and Gellert (2007).

[14] For a detailed presentation of algorithmic trading, see Johnson (2010). See also Miller (2014) for a discussion of mathematics-based risk management.

procedures; it is a complete mathematics-based reconfiguration of that which is taking place. Economic crises might be one of the consequences.[15]

Mathematics formats production processes. In the film *Modern Times*, Charlie Chaplin operates as an integral part of a production machinery, and quite literally in one scene in the film, he gets swallowed up by the machinery. This “swallowing up” can be taken metaphorically as being an ironic portrayal of modern forms of production. However, it can also be interpreted quite literally. Today, in many almost-automatised production processes, mathematics-based algorithms keep the processes running. This applies to the production of cars, mobile phones, plastic boxes, whatever. As part of overall automatisation, workers are assigned particular tasks. They are swallowed up by the production machinery by being put into the gaps of otherwise automatised processes. Workers get dismembered and reduced to components of a production process. The organic unity of the workers’ life-worlds gets destroyed through the treatment of its parts and properties as autonomous. As also presented in *Modern Times*, we are dealing with violent processes.[16]

Cryptography has been applied in all historical periods. The recurring question has been: How can we send important information in a form that nobody other than the intended receiver can decode? In periods of war, this question becomes urgent. Many techniques have been tried out, invisible ink being just one example. During the Second World War, a most advanced cryptographic approach was developed by the Germans. They had constructed a coding and decoding device, the Enigma Machine, that was efficient to handle. The Enigma system was implemented by means of a mechanical device with so high a degree of complexity that it appeared impossible to break the code. Nevertheless, the code was broken, with direct implications for the course of the war. Later, mechanical cryptographic procedures were substituted by mathematics-based algorithmic procedures. This made it possible to encode and decode huge amounts of information and, at the same time, ensure a new degree of security. Computer-based cryptography, often referred to as modern cryptography, is based on profound number-theoretical insight, for instance, concerning the distribution of prime numbers and the computational complexity of factorising a natural number that appears as a multiplication of two unknown prime numbers, say of around 100 digits each. Modern cryptography is crucial to modern warfare, but it has many other applications as well. It is crucial with respect to the transfer of any kind of information: of money, of personal data, of business information, and of research results. The whole approach of modern cryptography is based on mathematical algorithms brought into operation.[17]

The examples to which I have referred up to now concern mathematics, first of all, as an integral part of some algorithmic procedures. However, mathematics might also exercise a performative power by shaping the ways we perceive and interpret

[15] For a discussion of mathematics and crises, see Skovsmose (2021).

[16] For a discussion of high-tech workplace surveillance, see Parenti (2001). For a discussion of the move from the assembly line to just-in-time procedures, see Lanigan (2007).

[17] For a discussion of modern cryptography, see Skovsmose and Yasukawa (2009).

our environment and how we come to act accordingly. Mathematics might, in fact, form our world-views. In the past, mathematics was crucial for particularising a geocentric view of the universe. It was, however, also a tremendous mathematical achievement to present a heliocentric world view. Nicolaus Copernicus demonstrated how such a view could be formulated mathematically. As a result of this view, the orthodoxy of the Catholic Church was challenged. Albert Einstein's theory of relativity is still an example of a mathematical formation of a world-view, a world-view with a tremendous impact on war technology. A different kind of mathematics-based impact on our conception of nature has been pointed out by Richard Barwell (2013), who highlights that with mathematical climate models, we provide interpretations of climate change, which might lead to strategies – and also to inadequate strategies – for trying to cope with such changes.

Today, mathematics has a direct impact on the formation of our knowledge. It need not only be with respect to the construction of broader world-views, but also with respect to the composition of bits and pieces of information. Mathematics is an integral part of computer-based information processing, and the Google search engine can serve an example. We might assume that, when searching on the net, we are first of all guided by our own decisions. But we are not. We are simultaneously guided by a page-ranking device, which determines what pages we will be presented with and in what order, when typing a search word. The page ranking is operated by means of a conglomerate of mathematical algorithms, which also can be modified in order to serve particular business and economic and ideological interests and priorities. As with respect to the workers' life-worlds, the possible organic unity of knowledge gets destroyed by the treatment of its parts and properties as autonomous bits and pieces of information. The operation of the Google search engine is just one example of mathematics-based information processing.[18] What news, what sports events, and what commercials that we get exposed to are an algorithmic fabrication. In this way, mathematics becomes part of the formation of our day-to-day information and consequently of our preoccupations, our interests, our attentions, and also of what we remain unaware. Symbolic silence, as well as symbolic articulation, has turned into mathematics-based fabrications.

Investigations of mathematical algorithms are crucial for a performative interpretation of mathematics.[19] A profound characteristic of a mathematical algorithm was formulated by Alan Turing (1937), who provided a step-by-step presentation of what a calculating machine could do. The presentation was purely theoretical, and not based on any empirical investigations. The Turing machine, as the calculating machine soon became named, provides an abstract definition, not only of a mathematical algorithm, but simultaneously of the electronic computer. The Turing machine makes it possible to interpret the computer as a complexity of

[18] For a discussion of the Google page ranking, see Langville and Meyer (2012); and Ravn and Skovsmose (2019).

[19] For an investigation of mathematical algorithms, see Möller and Collignon (2019, 2021).

mathematical algorithms, and I see any form of computing as a direct expression of bringing mathematics into action.

Could a mathematical model create a description of reality without interfering with this reality? Could a mathematical model "picture" reality in accordance with the picture theory as presented by Wittgenstein? A mathematical model has been described as a triple (R, M, f), consisting of a set of empirical objects, R, a set of mathematical entities, M, and a function f: $R \rightarrow M$, which relates reality and mathematics.[20] Thus, the function f is ascribed the role of doing the picturing. Such a characteristic of a mathematical model does not indicate much performativity beyond the very act of "picturing". However, when a mathematical model is used, say, for describing a production process with the purpose of identifying further possibilities for automatisation, it does many more things than "picturing" an actual process. It captures some features of the situation as being important, say the measured productivity of the individual worker; it relegates some features as being irrelevant, say the family situation of the workers; and it stipulates connections between different variables, say between the level of payment and the level of productivity. The performativity of such a model cannot be expressed only in terms of "picturing" but needs to be expressed also in terms of "capturing", "relegating", and "stipulating". Conceptions like "simplifying", "dismembering", and "destroying" also come to mind.

One could assume that the situation would be different if a mathematical model was meant to "picture" not a social but a natural phenomenon. However, if we consider a mathematical model by means of which possible climate changes are discussed, one will also find that the model does more than "picturing". It captures some connections as being important, relegates other features to irrelevance, and stipulates the nature of certain connections.[21] So, even if we start from a picture theory of mathematical modelling, we come to acknowledge that the model does much more than picturing. A mathematical model operates as a symbolic language game among many other possible language games, and each game might provide different forms of symbolic acts. A symbolic language game might be extremely powerful. It may be useful to some and violent to others.

By means of mathematics, we format the way we deal with economic transactions, production processes, cryptography, world-views, information processing, computing, and modelling. We are dealing with powerful performatives of mathematics. Mathematics may operate as a mobile army of metaphors that covers what we are asserting and doing with a deceitful glitter of objectivity and neutrality. By means of mathematics, symbolic power can contribute to a deep-structuring of our life-worlds.

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[20] See Niss (1989); and Blum and Niss (1991) for such a characteristic of a mathematical model. For an overview of the present discussion of mathematical modelling, see Cevikbas et al. (2022).

[21] For a brief presentation of different components of a climate model, see, for instance, MacKenzie (2007).

14.5 Performances Through School Mathematics

Any kind of mathematics can be brought into action, including school mathematics. When talking about school mathematics, I have in mind elementary mathematics, and not only the mathematics that is presented in a school context, but also, for instance, in a work context. However, the very expression "elementary mathematics" might be accompanied by negative connotations, so I prefer instead to talk about "school mathematics". With reference to Freire, Eric Gutstein (2006) talks about *reading and writing the world with mathematics*. This is an expression of a performative interpretation of mathematics directly inspired by Freire's performative interpretation of language. By means of mathematics, one can articulate a range of socio-economic facts and thereby open them up to scrutiny. By means of mathematics, one can take the initial steps of writing the world. With this expression, Gutstein refers to changing the world towards a more just society. Gutstein presents a range of examples of classroom practices where he engages students in reading and writing the world with mathematics. In this way, he elaborates in great detail upon a performative interpretation of school mathematics, which he combines with an activist approach to education.[22]

In the article "Bringing Critical Mathematics to Work: But Can Numbers Mobilise?", Keiko Yasukawa and Tony Brown (2012) raise a crucial question with respect to a performative interpretation of school mathematics: Can numbers mobilise? They answer this question positively by illustrating how mathematics can help to provide a critical reading of work conditions. Yasukawa and Brown investigate a situation where a group of workers was dissatisfied with their working situation and, in particular, with the way in which "the work was constructed by a particular mathematical model" (p. 255). The model determined the workers' salary as based on certain measures of their productivity. Yasukawa and Brown analyse how, by articulating this dissatisfaction, it becomes possible to reach an understanding of how the model maintained an "inequitable an exploitative situation at the workplace" (p. 261).

Yasukawa and Brown distinguish between four different kinds of mathematical knowledge related to a work situation. *First,* they consider the mathematical knowledge that makes it possible for the workers to become qualified for the job. This is the mathematics which makes up part of their formal qualification. For a bank assistant, it could be accounting, for a woodworker it could be trigonometry, and for an engineer it could be calculus. (It might be questionable to consider calculus as school mathematics. However, I see initial courses in calculus as being transitions between school mathematics and advanced mathematics.) *Second*, they consider the mathematics that is actually brought into action in the workplace. For a bank assistant, it might be accounting, although it might well be in a quite different manner from the accounting that is part of the bank assistant's formal education. *Third*, they consider the mathematics that is relevant for the worker in order to operate in the

[22] See also Gutstein (2009, 2012); and Gutstein and Peterson (2006).

labour market. This includes the mathematics relevant for controlling salaries and for verifying tax payments. It might also include the mathematics relevant for dealing with everyday issues, such as drawing up a budget, paying in instalments, and understanding medical prescriptions. *Fourth*, they consider the mathematics that is relevant for "reading" the politics of the workplace. This is the mathematics that is relevant for critically addressing the way the company determines salaries, specifies working conditions, and measures levels of productivity. The existence of this fourth kind of mathematical knowledge brings Yasukawa and Brown to claim that numbers can mobilise. It is this kind of mathematical knowledge that Gutstein has in mind when talking about reading and writing the world with mathematics.

One can ask: Can equations mobilise? As part of one of his projects for social justice, Gutstein engaged his students in a project concerning mortgages and foreclosures. The starting point for the project was the difficulties that haunted the neighbourhood in Chicago where the project took place. Gutstein (2018) condensed the focus of the project in the following way:

> Our class was trying to determine whether a family earning the median income in the neighbourhood (around 32,000 US Dollars a year) could afford a home mortgage of 150,000 US Dollars, with an interest rate of 6 % a year on a 30 years loan. According to the US Department of Housing and Urban Development, paying more that 30% on one's income for housing is considered a "hardship" (p. 131).

Based on this information, the students, around 17-18 years old, calculated that a family could pay 808 dollars a month in mortgages without hardship. As the text indicates, the family is taking a loan of 150,000 dollars. However, if we assume that the family pays what it is able to pay without hardship, it would not be possible for them to pay off the loan. The students found that after paying 808 dollars a month for 30 years, the total payment would add up to 291,000 dollars. However, the family would still owe the bank about 92,000 dollars. This whole observation was summarised in the following equation:

$$150.000 - 291.000 = 92.000$$

What are we to think of this equation? It seems that the family has to pay and pay and pay, without seeing any end to the payment. Such observations could lead to questions like: Who sets the agenda for such economic transactions? Who has installed such equations? Are we dealing with an example of economic exploitation? This equation came to mobilise the students.[23]

The positive answers to the two questions: "Can numbers mobilise?" and "Can equations mobilise?" indicate that the performativity of school mathematics might include an articulation of socio-political issues.[24] By means of mathematics, students might come to see new aspects of their life conditions. They might recognise that taken-for-granted conditions can be questioned and maybe changed. The

[23] See also Gutstein (2016). For a short presentation of the project, see also Skovsmose (2020a).

[24] For other examples of how critical mathematics education might be brought into action, see, for instance Andersson and Barwell (2021).

students' world-views might be changed, and they might come to act in different ways. In this sense, school mathematics might form the students' life-worlds.

One can, however, imagine quite different performatives from school mathematics, for instance, the formation of a socio-political silence. Such a silence can be related to the school mathematics tradition dominated by the exercise paradigm. According to this paradigm, solving exercises establishes the main route for learning mathematics. The typical mathematical exercise includes information provided in numbers, and this information is always considered correct and exact. If an exercise informs the price of apples, it is irrelevant for the students to look up other prices at the supermarket. The information given in numbers is always sufficient for solving the exercise. There is no need for the students to search for additional information for solving an exercise. All the information provided in numbers is also necessary for solving the exercise. Finally, an exercise has one – and only one – correct answer.

Within the school mathematics tradition, exercises can refer to mathematical properties (of a triangle, of a function, of an equation), but they can also be contextualised. However, we are dealing with a contextualisation referring to invented prices, invented distances, invented travels, and invented salaries. The inventions are made by the author of the textbook for the only purpose of formulating an exercise. It is never relevant for the students to critically reconsider the contextualisation. The school mathematics tradition is accompanied by a deep symbolic silence with respect to socio-political issues.

The school mathematics tradition stimulates an "ideology of certainty". By this expression, I refer to the conviction that information presented in numbers can be trusted and that calculations based on numbers can be considered objective and neutral. During the school years, a student might well be exposed to around 10,000 exercises. The output of this need not be any deeper understanding of mathematics but could rather be a faith in the ideology of certainty. This ideology might be part of the performativity of the school mathematics tradition.

Symbolic violence might also be a feature of the school mathematics tradition. Such violence is exercised when mathematics education assumes a role as gate keeper, determining who is going to get access to further education and to the better jobs in society. Alexandre Pais highlights that mathematics education takes place in a society where capitalist structures and accompanying ideologies permeate whatever is taking place. As an illustration of what this might imply, he considers the broadly celebrated slogan "mathematics for all". According to Pais (2012), this slogan covers a cynical irony by concealing the "obscenity of a school system that year after year throws thousands of people into the garbage bin of society under the official discourse of an inclusionary and democratic school" (p. 58). This obscenity is a case of symbolic violence.

How could mathematics education adjust the students so that they could fit into the existing patterns of capitalist production and consumption? One element of such an adjustment is to establish and maintain a silence about socio-political issues. "I have referred to an explicit adjustment to the capital order of things in terms of a "prescription readiness". The notion of prescription readiness has been elaborated upon in Skovsmose (2008), where I investigate the functioning of mathematics education with respect to the knowledge market. This readiness includes a disposition to adjust to guidelines and information provided in numbers. By working through the 10,000 exercises with all information to be taken as given, students get habituated into a prescription readiness. When entering the labour market, such a readiness makes it easier to fit the workers into any kind of production process, the content of which is pre-defined. I find that the formation of the coming workforce is still a case of symbolic violence rooted in the school mathematics tradition.

I think of a performative interpretation of mathematics as a suggestion for what to consider when looking at mathematics through philosophical lenses. It is a suggestion for not only paying attention to deductive structures, possibilities for reaching certainties, and mathematical correctness, but also to what is acted out by means of mathematics. It is a suggestion for considering the full impact of mathematics brought into action. It is a suggestion for considering how symbolic acts routed in mathematics might form our life-worlds.

14.6 Ethical Reflections

Performatives can be associated with any kind of mathematics, just as they can be associated with any kind of language. The impacts of performatives can be of very different natures, and therefore ethics becomes an integral part of a philosophy of mathematics. In Skovsmose (2020a), I have pointed out that mathematics research and mathematics education at universities and faculties are often conducted in an ethical vacuum, as they do not engage in ethical reflections with respect to mathematics.[25] In the following, I will pay particular attention to ethical reflections related to what I have referred to as "advanced mathematises".

Reflections with respect to the impact of mathematics. Mathematics-based acts may have any kind of impact. They might be insignificant, but they might also be powerful. Such symbolic acts can be cynical, expensive, benevolent, interesting, risky, disastrous, aggressive, generous, and creative. While the performativity of mathematics can be expected, the very nature of this performativity is contested. Furthermore, it is relevant to talk about both an actual and a potential impact of mathematics in order to highlight that the consequences of bringing mathematics into action might not be observed immediately. Advanced number theory was developed as a purely theoretical discipline long before it became identified as a powerful basis for cryptography. Both the potential and the actual impact of mathematics need to be critically addressed.

Reflections with respect to different groups of people. Symbolic acts based on mathematics can have different qualities, but it is important to be aware that the

[25] For a discussion of mathematics and ethics, see also Ernest (2016, 2018b); and Skovsmose (2020b).

same symbolic act can have very different impacts for different groups of people. When brought into action for restructuring working processes, mathematics might add efficiency to the processes and ensure a higher profit for the investors. The restructuring might create possibilities for reducing the workforce, implying that some workers might get fired. It might change the content of the work for those that remain at the workplace. It might establish working tasks that can be completed by unskilled workers and, in this way, reduce the value and power of the workers.

Reflections with respect to the acting subject. In many cases, the acting subject appears to be straightforward to identify. It can be a person, a group of people, an institution, a company, a government, a president, a dictator. However, when mathematics is brought into action, the situation might be different. In many cases, it seems that mathematics tends to hide an action subject. As an example, one can consider the collapse in 2008 of the financial market: Who was the acting subject in this situation? If we assume O'Neil's interpretation, this economic crisis was triggered by certain mathematical algorithms accelerating out of control. However, was the acting subject the mathematicians creating the algorithms? Was it rather the computer specialists that implemented the algorithms? Was it the financial institutions – the banks and hedge fund companies – that directed the whole process? Or should one rather think of the acting subject in terms of some overall neo-liberal priorities? When reflecting on mathematics-based actions, I find it important to address issues concerning the possible, but maybe hidden, acting subjects. This is not only relevant when addressing economic issues, but relevant with respect to any kind of symbolic acts. Reflections with respect to the possible acting subject makes up a crucial part of ethical discussions.

Reflections with respect to possible intentions behind the action. Such intentions might be implicit, but they might also be stated explicitly. Still, the explicitly stated intentions need not be the real intentions behind an action, which might be less noble than those explicitly presented. As an illustration, one can consider a possible research project in number theory. Mathematicians might well express themselves in mathematical terms referring to the importance of making further theoretical discoveries of mathematical numbers. However, if a group of mathematicians should make an application for funding for a number-based theoretical project, they might express themselves differently. They might refer to some possible applications of number theory and highlight the usefulness with respect to cryptography. The research funding agency might pay particular attention to some of the possible applications; this could be with respect to possible military applications, maybe expressed in terms of the project's relevance for national security. The multiplicity of possible intentions that can be associated with the same mathematical activity is a general phenomenon. Whatever kind of mathematical activity one has in mind, one should not expect to be able to associate it to a singular set of intentions. It is important for ethical reflections to address the multiplicity of possible intentions for engaging in a mathematical activity.

Reflections with respect ethical reflections themselves. When discussing whether an action is good or bad, it seems consequential to pay attention to the implication of the action. This idea can be generalised to the ethical position referred to as utilitarianism, which claims that the ethical value of an action can been judged by considering the full impact of the action. This position has also been referred to as being a *teleological* one. This position confronts a *deontological* one, which claims that a priori to any discussion of implications, there exist some principles that need to be considered when judging an action. The deontological perspective has deep philosophical and also religious roots. The *Bible*, as well as the *Quran*, has been claimed to contain guidelines, according to which any kind of action needs to be judged. The deontological principle has been advocated by many philosophers, Immanuel Kant being one of them. Kant, however, did not look into the *Bible* in order to identify ethical principles, but did so through analytical investigations. Originally, the teleological perspective in the form of utilitarianism was presented by Jeremy Bentham. Being an atheist, Bentham argued that it was not necessary to look into the *Bible* or to consider any other forms of holy demands. To judge whether an act was good or bad was a completely human affair that could be adequately based on empirical observations.

The reflections that I have presented so far with respect to bringing mathematics into action have primarily been of a teleological format. However, this perspective might be insufficient if one wants to address not only the actual, but also the potential, impact of mathematics. Thus, it seems an irrational affair to try to judge a mathematics-based action on its impact when we have still not witnessed any such impact. But what kind of a priori principle should one consider? By making these comments, I want to indicate that there are deep uncertainties associated with ethical reflections regarding bringing mathematics into action. Still, I consider such reflections be necessary.

Ethical reflections concern the potential and actual impact of mathematics. This impact might be contested, and it might be ambiguous, by affecting different groups of people in radical different ways. Ethical reflections concern how symbolic acts rooted in mathematics might influence our ways of interpreting and acting. They concern how mathematics forms our life-worlds.

The performative interpretation of mathematics establishes ethics as a crucial element of a philosophy of mathematics. Mathematics can be powerful, fallible, and mischievous. Contrary to what the Three Magi assumed, mathematics is a human – all too human – affair.

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Chapter 15
Reflective Knowing in the Mathematics Classroom: The Potential of Philosophical Inquiry for Critical Mathematics Education

Nadia Stoyanova Kennedy

15.1 Introduction

"Mathemacy" is an essential competence in critical mathematics education that goes beyond mathematical knowledge and skills, or even the application of mathematics for solving real-world problems and interpreting outcomes. It requires "reflective knowledge," which demands a capacity to make critical judgments about the social consequences of the use of mathematical tools (Skovsmose, 1994). In Skovsmose's words "… mathemacy, as a radical construct, has to be rooted in the spirit of critique and the project of possibility that enables people to participate in the understanding and the transformation of their society, and therefore, mathemacy becomes a precondition for social and cultural emancipation" (1994, p. 27).

Many researchers have expounded on the importance of reflective awareness of mathematics as tool and as a sort of grammar for describing and predicting reality (e.g., Davis & Hersh, 1986; Geller & Jablonka, 2007). Given its increasing influence on our highly technological society, the dangers of neglecting to develop such an awareness are clear (e.g., Davis & Hersh, 1986; Skovsmose, 1994; D'Ambrosio, 1999; Ernest, 2018, 2019). However, Skovsmose (1994) points out that reflective knowing is not an ingredient of mathematical or technological knowledge, as its focus lies outside them. An understanding of the particular way of seeing and understanding the world through the lens of mathematics and of the role it plays in society is not to be found *in* mathematics. We need to look "beyond" mathematics for discursive environments that cultivate reflective knowing and for instruments that can be helpful in grasping the role of mathematics in reading and understanding the world. In this chapter, I examine more closely the concept of reflective knowing and argue that philosophical inquiry in collaborative group settings may be an

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appropriate vehicle to facilitate critical reflection in the classroom—reflection that is focused on understanding mathematics as a tool, on its role in society, and on the implications of using mathematical and technological knowledge in addressing social problems. I then offer a brief description of what philosophical inquiry is, the context in which it is conducted, and a framework for conducting philosophical inquiry in the mathematics classroom by engaging school students in encounters with contestable questions related to mathematics and its role in society.

15.2 On Reflective Knowing

Skovsmose's (1994) notion of *mathemacy* is akin to Paolo Freire's (2005) notion of critical literacy. The latter has a socio-political dimension, in that it presupposes an awareness of the complexity of broad ideological forces, of the functions of power, and of the universal problem of injustice. Critical literacy relies on dialogue, reflection, and pointed analysis of controversial and important issues in students' lives. Such dialogues function to expose students to multiple perspectives and engage them in a critique of oppressive structures, in the interest of gaining a deeper understanding of the social and political forces that drive them. In this sense, critical consciousness or "*conscientização*" (Freire, 2005) is as much about raising awareness as about empowerment, here understood as the ability to act in the world in the interest of changing it.

Similarly, Skovsmose's (2007) vision of *mathemacy* as a competency is not solely functional—that is, about learning mathematics in order to use it in one's personal and professional life. It is also a critical competency, one that engages people in critique and allows them to participate in "reading the world" through an emancipatory lens, using mathematics to question deeply seated assumptions and beliefs, examine implicitly held values, "talk back" to authorities, and imagine alternatives. Mathemacy is understood as empowering—a competency that opens the door to active participation in a democratic society.

As formulated by Skovsmose (1994), mathemacy includes mathematical, technological, and reflective knowing. Mathematical knowing "refers to the competencies we normally describe as mathematical skills," and technological knowing "refers to the ability to apply mathematics and formal methods in pursuing technological aims" (p. 100- 101). The latter relies on mathematization and the interpretive act of modeling real-world phenomena in purely mathematical terms. In his view, mathematics as a field of knowledge has a "formatting power" that structures all aspects of social life in non- transparent ways. He argues that mathematical and technological knowledge—the competency to solve mathematical problems and model real world phenomena with mathematics—is limited in addressing this power apart from a critical approach.

Mathematics does not offer an impartial reading of the world; rather it offers a particular worldview and epistemology, which may serve various specific political, economic, or other interests. In our postmodern world, economics, politics, culture, subjectivity, and identity are deeply informed by meanings filtered through the conceptual lenses of quantification and mathematics as its most formidable facilitator. Mathematical models regularly determine whether people get a job or a loan, whether they get into a school, how much they pay for insurance, or how long they spend in prison. These models often reflect biases and racial prejudices and can perpetrate profound unfairness and bigotry (O'Neil, 2016; Noble, 2018). Other concerns about mathematical models are related to issues of "social availability" of mathematical knowledge (Geller & Jablonka, 2007, p. 1). Technology often renders mathematical processes and their human and material dimensions invisible, codifying them in "packages" that execute mathematical procedures, which makes the processes and results difficult to question. Technology renders mathematical packages into "black boxes" of implicit underlying mathematical processes that are "frozen" in objects and technological structures and apparatuses (Gellert & Jablonka, 2007), and not accessible to the general public—a practice that shields them from accountability and, as such, susceptible to concealing corrupt practices. Not only do the "black boxes" render the human dimension of their use opaque, but they also make the embedded interests and values that the models reflect invisible. We are increasingly faced with what Keithel et al. (1993) refer to as "implicit mathematics," the biases of which are not visible. Such processes help increase gaps between "experts" and "non-experts," between the designers who understand these structures and the rest of us who do not, thus reinforcing antidemocratic divisions between an "expert" elite and consumers (Skovsmose, 2007), thus precluding democratic participation.

Since mathematics and technology are key tools in developing the infrastructure of a technological society, their impact is implicated in our approach to every kind of social problem—whether environmental crises; gender, racial, and income (in) equality; public schooling; and social welfare programming—and play a ubiquitous role in understanding these phenomena and influencing them (e.g., O'Neil, 2016; Noble, 2018; Skovsmose, 2019; Andersson & Barwell, 2021). Moreover, mathematics plays a dual role in that it has both descriptive and prescriptive powers; while claiming objective neutrality, it in fact directly stipulates actions and behavior. Not only does mathematics have descriptive and prescriptive powers, but it also becomes embodied in our thinking and action (Fisher, 2007). As such, it may lead to blind acceptance of the products of mathematization and a lack of awareness of how mathematics operates and its effects on society and subjectivity. Unless there is a recognition of these aspects of the operation of mathematics and its implicit epistemology, only narrow views of the discipline will result, and mathematics will be forever associated with the myth of objectivity, truth, and a value-free form of judgment. Excavation of the invisible dimensions associated with the uses of mathematics is necessary, as Skovsmose aptly puts it, because "... mathematical and technological knowledge are born 'shortsighted'" (1994, p. 99). Reflective knowing acts to counter the covert effects of mathematics and its hegemonic epistemological power. "It has to do," he argues, "with the evaluation and general discussion of what is identified as a technological aim and the social and ethical consequences of pursuing that aim with selected tools" (p. 101). It represents the competence required for the ability to "take a justified stand" in discussions that concerns technological questions (p. 101).

Indeed, some types of reflective knowing associated with mathematical problem-solving and modeling are grounded in mathematical and technological knowledge and are referred to by Skovsmose as the "technical evaluation of a model" (Skovsmose, 2019, p. 10). These types of reflection concern questions related to calculation correctness, appropriate choice of algorithms, and the reliability of the solutions and are representative of the meta-reflection typically considered part and parcel of good mathematical practice (Skovsmose, 1994, p. 118). However, other types of reflective knowing extend beyond the boundaries of mathematics. Skovsmose describes these as focused on questions as to whether mathematics needs to be used in a particular situation, as well as the broader consequences of its uses. These concern a critique of the social consequences of enacted mathematics and require a close examination of the assumptions, values, norms, and ethics that are implicit in the mathematization process. Inquiry into such questions and concerns qualify as philosophical rather than mathematical and require philosophical rather than mathematical judgments. As such, it would seem that promoting reflective knowing in the mathematics classroom would necessitate, not only that students learn to use mathematical methods to explore social, environmental, or economic questions but also that they reflect as well on mathematics as a tool and its role and uses in society.

Skovsmose argues that a kind of mathematical archeology needs to take place: first in uncovering the mathematics, which may be opaque, used in a given situation, and second, in drawing attention to and discussing the manifestation of its formatting power in that situation. Archeological excavation, as Anthony Giddens (1994) points out, involves digging deep and taking all pieces out, then cleaning and establishing connections between them, which is a critical interpretive task. This process would require a different set of lenses than the mathematical. As Stephen Toulmin (1961) aptly observes: "We see the world through them [the lenses] to such an extent that we forget what it would look like without them: our very commitment to them tends to blind us to other possibilities" (p. 101). The only way to gain a different perspective and to "unthink" the ideas filtered through these mathematical lenses is to remove them for purposes of analysis. As Toulmin observes, "It is impossible to focus both on them [the lenses] and through them at the same time" (p. 101). On the other hand, the anthropologist and semiotician Gregory Bateson (2002) advocates for bringing different perspectives together. He uses the metaphor of stereoscopic vision to promote the idea of combining perspectives in order to achieve an "extra depth," particularly when one of these perspectives offers an outsider's breadth of vision on the other.

In this chapter, I advocate for combining mathematical with philosophical perspectives. I consider this approach to be compatible with the goals of critical mathematics education in regard to the development of mathemacy and of the habits and dispositions that promote and sustain democratic citizenship. Other scholars have argued for including philosophy in the teaching of mathematics. Prediger (2005), for example, has argued for including reflection in the mathematical classroom in the area of mathematical content and practice, epistemology, and the philosophical base of mathematics. Skovsmose (2020) considers the overwhelming emphasis in mathematics courses on doing mathematics without reflecting on its possible social impacts as a manifestation of "inserting mathematics into an ethical vacuum" and as extremely problematic. Ernest (2018) proposes that aspects of philosophy of mathematics and especially the ethics of mathematics be included in school and university mathematics curricula.

I argue here for the inclusion of the practice of philosophical inquiry in the context of collective and deliberative dialogue in the mathematics classroom and propose that such an approach offers a pathway whereby students are encouraged to become more aware of the complex relationship between mathematics and society, more critical in their understanding of mathematics and its products, and better equipped for participation in democratic forms of life. I consider this approach to be compatible with the goals of critical mathematics education in regard to the development of mathemacy and of the habits and dispositions that promote and sustain democratic citizenship. Thus, I suggest that philosophical inquiry can act as a vehicle that facilitates reflective knowing in the mathematical domain. In what follows, I will explore one practical form of philosophical inquiry in educational settings that lends itself particularly well to this project.

15.3 Philosophical Inquiry, Dialogue, and Judgment

For a model for conducting philosophical inquiry, we turn to a program founded in the 1970s and internationally known as *Philosophy for Children* (P4C).[1] The program uses a pedagogical format known as "community of philosophical inquiry" (CPI) (Lipman, 2003), conceptualized as a communal dialogical process that bases its practice on philosophical questions and concepts generated by students themselves in response to a stimulus in the form of a text or other media. The program founders— Matthew Lipman and Ann Sharp (1980)—developed a CPI curriculum consisting of philosophically provocative stories written for children and thematized teacher manuals that focus on central, common, and contestable concepts found there such as friendship, freedom, justice, beauty, persons, mind, body, authority, conflict, truth, and "real" (Lipman, 1981; Lipman et al., 1980). Although Lipman and Sharp did not envision philosophical discussions focused on questions concerning mathematics or its impact on society, I suggest that the methodology (not the curriculum per se) can be used to engage students in critical examination of the role of mathematics in society, as well as mathematics in general—in other words, to conduct meta-mathematical reflections through dialogical inquiry.

[1] For more information about the Philosophy for Children (P4C) program, see the Institute for the Advancement of Philosophy for Children (IAPC, https://www.montclair.edu/iapc/); the International Council of Philosophical Inquiry with Children (ICPIC, https://www.icpic.org/); Philosophy Learning and Teaching Organization (PLATO, https://www.plato-philosophy.org/).

In a classroom CPI, the philosophical discourse in question is reconstructed as more practical and more accessible to students than is traditionally the case. The inquiry begins with some kind of problem which gives rise to a more general philosophical question—what is the most reasonable thing to believe or to value or to do in this case?—and ends in some kind of satisfactory resolution or fulfillment in the nature of a judgment. The latter tends to be a proposition about what should be believed or valued or done with regard to the original problem. In P4C methodology, inquiry begins with students generating questions based on a reading of an excerpt of a curricular novel, followed by determining the group's interests and creating an agenda for discussion.

Philosophical inquiry focuses on "big questions" (Wiggins & McTighe, 2005) or *common*, *central*, and *contestable* (CCC) questions (Splitter & Sharp, 1995): they are common to all humans in some form, central to our understanding of ourselves and of the world, and contestable, as they do not have one simple answer. They are also questions that lend themselves best to communal, collaborative deliberation in a setting in which participants are engaged in putting forward and evaluating arguments and arriving at reasonable judgments as to those arguments through dialogue. Philosophical inquiry so understood relies on the Deweyan proposition that ethical, aesthetic, political, and other philosophical dimensions describe facets of most people's ordinary experience rather than some esoteric experiences separated from the ordinary (Dewey, 1934; Lipman, 2003). It also takes on Dewey's notion that inquiry should connect to student's own questions and respond to genuine perplexity and doubt (Dewey, 1910/1997).

In the context of the mathematics classroom, philosophical inquiry can serve as a vehicle for an inquiry focused on contestable questions that are not discipline-specific but act to question the outcomes of the uses of mathematics in society. Some examples of questions might be "What part does mathematics play in the way we organize our lives?" or questions related to social justice such as the following: "What is fair wealth distribution? How should wealth be distributed?" These can be qualified as philosophical questions to the extent that they explore political and ethical dimensions of our experience with mathematics. Such a project draws heavily on Dewey's idea that inquiry should begin with a particular experience—in this case an experience that involves a mathematical activity. The mathematical activity makes for a shared group event, and the problematization that follows in the discussion can motivate students to inquire further in a search for answers to related, relevant, contestable questions. In other words, the mathematical activity acts to structure the inquiry that follows it, or to use Dewey's term, to "occasion" it.

In my own experience with conducting philosophical inquiry in the context of a mathematics classroom, I have used written texts about mathematics or posed mathematical tasks the experience of which occasions the generation of students' philosophical questions (e.g., Kennedy, 2018, 2020). What is seen and felt as problematic and perplexing must reflect the experiences of the group of students—not only those experiences related to the mathematical activity, but previous personal school and out-of- school experiences as well. Above all, the initial goal of inquiry with contested questions related to mathematics is to help students reflect on and challenge deep-seated assumptions, critically explore values and social practices, and discuss social alternatives.

The process is teleological—it aims at a product. As Lipman observes, "Inquiry is a self-corrective practice in which a subject matter is investigated with the aim of discovering or inventing ways of dealing with what is problematic. The products of inquiry are judgments." (Lipman, 2003, p. 184). The ideal immediate goal is for the participants to arrive at one or more reasonable philosophical judgments regarding the questions or issues that occasioned the dialogue. For example, with regard to ethical inquiry, Lipman et al. insist that, "Students must not only be encouraged to express their beliefs …. but to discuss and analyze them, considering the reasons for and against holding them, until they can arrive at reflective value judgments that are more firmly founded and defensible than their original preferences may have been" (Lipman et al., 1980, p. 47). The product of the inquiry should be a collective goal, guided by the Socratic dictum that we "follow the argument where it leads" rather than being solely invested in a one's own personal argument.

"Inquiry dialogue," as Douglas Walton (1998) calls it, is truth-directed and aims at arriving collectively at a conclusion or judgment on a common, central, and contestable issue that is deemed the most reasonable and acceptable by the community (Gregory, 2007). For a philosophical judgment to be reasonable, it must be well reasoned, well informed, and personally meaningful (Lipman, 2003). Judgments are justified in part by their reliance on sound arguments and good evidence. For a judgment to be reasonable, it must be informed by multiple and diverse perspectives and able to withstand the evaluation and critique of the dialogical community. The discourse of CPI assumes the Pragmatist view that good thinking is social and that the ability to think well is acquired through participation in a thinking community where one is both challenged and assisted in the effort to be clearer, more consistent, and more creative (e.g., Dewey, 1910/1997; Pierce, 1958). In that individual thinking is limited and susceptible to error, it is likely to be strengthened by being made accountable to a community of peers (Lipman et al., 1980). Because there is no authority external to the community of inquiry that can correct limitations or shortsightedness, the evolution of participants' insights and skills is the only means of producing and improving the arguments and the judgements in play. The reasonableness of the community's judgment depends on the ongoing reconstruction of the participants' individual and shared understandings, through individual and group self- correction.

The teacher has the important role of organizing and facilitating the inquiry. He or she introduces a stimulus—a text, a mathematical activity, or a list of questions related to a previous activity. She may invite students to pose their own questions related to previously collectively experienced mathematical activity or suggest a question for discussion and encourage students to offer arguments. In a CPI, teachers invite students to propose, evaluate, and build on each other's arguments and to agree and disagree in a spirit of ongoing, collaborative search for truth. They invite students to make certain logical and dialogical moves, which they model in the course of the discussion: to ask questions, agree or disagree, give reasons, offer a hypothesis or explanation, or make a statement, offer an example or a counterexample, make comparisons, classify/categorize, identify an assumption, offer a definition, make a distinction, self-correct, among others (Kennedy, 2013). They also facilitate the sequencing of student moves by inviting students to make interventions, by encouraging them to connect and respond to what has been said, by making or asking for clarifications or restatements, by offering or asking for summarization, and by managing turn taking.

The teacher acts to orchestrate the inquiry with the goal of focusing and moving the inquiry further through scaffolding the inquiry process with questions, counter-examples, restatements, and summarizations. In cases in which the community might align with only one side of an argument, the facilitator may offer a different perspective, another possible argument, or counterexample. For example, she might say "And what would you say if someone said……", or "What about thinking about this issue in a different way, for example…." Teachers act as facilitators of the dialogue and, as Lipman puts it, as " . . . 'pedagogically strong but philosophically self-effacing' so that they can strengthen the reasoning and judgement of their students, thereby getting them to think for themselves, while at the same time the teachers try to avoid indoctrinating their pupils with their own personal opinions." (1993, p. 296) Frequently the facilitator may need, in order to keep the discussion philosophical and productive, to negotiate between her own inclination and the desire of students to discuss questions that range across disciplinary boundaries and are not fruitfully addressed by dialogue, for example, questions that already have definitive answers, questions that can only be answered by calculation, observation, or experiment, and questions that are psychological and concern feelings and moods. As participants in a genuine pursuit of meaning regarding questions that they have taken up by their own volition, students come to develop personally meaningful judgments. Such judgments are of another order from "accepted truth" typically passed on to them by the teacher. Instead, they are developed and pursued through engaged participation in collective, deliberative, reflective thinking about mathematics and its impact on current social structures, and a result of careful evaluation of the ideas and assumptions aired by the group in response to an important question.

15.4 A Framework for Philosophical Inquiry in Mathematics

In this section, I offer a *Framework for Philosophical Inquiry in Mathematics* as shown in Table 15.1. The Framework is intended for use in engaging students with philosophical questions about mathematics, its uses in society, and its relations to other disciplines. It adopts and extends Skovsmose's (2001) notion of "landscapes of mathematical investigation" which may refer respectively (1) to mathematical concepts per se; (2) to invented (textbook) mathematics situations such as word problems, and (3) to real-world situations described mathematically, as in mathematical modeling. In his view, mathematical activities can be guided by any of these three references and can be narrowly constructed as exercises or as discussions. I

Table 15.1 Framework for philosophical inquiry in mathematics

Philosophical inquiry landscape	Sample questions	Sample goals
1. With reference to mathematical concepts	What is number? Is infinity a number? What can or cannot be expressed in numbers? Is there a connection between symmetry and beauty? Is symmetry important for plants, animals or life in general? If so, how? Where does mathematics come from? Is mathematics the best way to understand the world? Is it a way to understand ourselves?	Explore questions related to the nature of mathematics; the power of mathematical knowledge in describing, understanding, and predicting phenomena; its limitations; its formatting power on the society and ourselves
2. With reference to invented (textbook) situations	What are we assuming? Might different assumptions change our interpretations of the situation? What are the most reasonable assumptions?	Problematize the "simple" and "narrow" frame of the situation, investigate assumptions, and consider how different assumptions might change outcomes
3. With reference to real situations described mathematically	What is a mathematical description (model)? What is omitted during mathematical modeling? Is there anything that a model might miss or misrepresent? What are the assumptions under which the model operates? What criteria have been used to judge that your mathematical model is successful? What values underlie this model?	Problematize mathematical descriptions (models) as "accurate" representation of a situation and as products of mathematical abstraction Facilitate understanding that different criteria may be at play in judging the success of such models and that value judgments may prioritize one model over another

use Skovsmose's concept of "landscapes of investigation" to conceptualize landscapes of *philosophical* investigation.

In the framework offered below, each of the three landscapes offers a discursive space for engaging in communal philosophical inquiry and for encountering the sorts of questions mentioned below (Table 15.1).

15.4.1 Inquiry with Reference to Mathematical Concepts

Inquiries related to specific mathematical concepts might encourage a search for meanings that have not traditionally been the focus of school mathematics. For example, aesthetic inquiry into mathematical notions, like symmetry or fractals, offers the possibility of exploring potential connections between symmetry and beauty, patterning and intrinsic order. Other, ontological inquiries may focus on interdisciplinary connections—biology for example, in which we explore the mathematically expressible relations inherent in the world of plants, animals, or life in general.

One could go further and discuss the following questions with ethical implications: Does mathematical knowledge of the world and the use of a mathematical approach to studying the world actually inhibit or distract us from other, equally or, in some cases, more powerful forms of knowledge? Is mathematics always a good lens to look through when we try to understand the world? Can mathematics be helpful in trying to understand ourselves? Is there anything that it might miss?

Mathematics students commonly question its usefulness, which is related, in turn, to its evident relation to the lived world that we all inhabit in more or less the same spatio- temporal way. Our goal is to encourage questions that go deeper than the utilitarian uses of mathematics—questions that stimulate critical inquiry into our culturally constructed and transmitted beliefs and assumptions about mathematics and to heighten awareness of both its power and limitations (Kennedy, 2018). Such inquiry can open dialogue about the implications of the uses of mathematics in society and about the ways subjectivity and lifeworld are informed by meanings filtered through the conceptual lenses of number, measurement, probability, and mathematical modeling.

15.4.2 Inquiry with Reference to Invented (Textbook) Situations

Textbook word problems typically refer to situations that have been invented for the purpose of a mathematical exercise. Usually, such exercises do not invite students to bring their own personal experiences and practical understanding to the classroom. However, there is much to gain from mathematics problems that are treated as ambiguous and open to interpretation. Kennedy (2012) offers examples of tasks that were designated for use as straightforward mathematical exercises but were treated as "texts" to be interpreted. One task used in a group discussion with middle school students reads as follows: *A frog finds itself at the bottom of a 30-foot well. Each hour, it climbs 3 feet and slips back 2 feet. How many hours would it take the frog to get out?* Students were asked to work collaboratively on interpreting this problem through offering definitions and identifying implicit assumptions. A major benefit of this form of inquiry is the by-product of the experience of collective engagement, which could be anything from an understanding that a mathematical task could be often "read" and interpreted in different ways, to the realization that assumptions buried deep in the statement of a problem can make a distinct difference in terms of its final solution.

15.4.3 Inquiry with Reference to Real Situations Described Mathematically

Typically, the inquiry landscape that relates to real-life situations is framed by mathematical tasks involving real data. Kennedy (2022) describes an activity with middle school students who were asked to describe their room mathematically. The descriptions volunteered by students varied: some made drawings portraying the colors and the patterns of their wallpaper, flowers on their desks, etc. Others drew rectangles with smaller rectangles inside them, showing the placements of their bed and desk; yet others offered a list of measurements of the length, width, and square footage of the room. These representations were then used to initiate a comparison between them and to discuss the question: What is a mathematical description? How is it different from another, say an artist's or a poet's descriptions?

Often enough one discussion opens the door to more questions from the group. For example, collaborative inquiry into the question, "What is a mathematical description?," has led to new questions such as the following: "Why can't math descriptions have all the information in them? Are math descriptions useful if they don't have all the information in them? How do we know which is the best math description? Why do people use math descriptions? Are mathematical descriptions always helpful? What can we gain by using them? What can be lost in using them? Can a math description be harmful?"

Similarly, when discussing mathematics modeling tasks with high school students, we might widen the inquiry to a deliberation on the extent to which such models describe the "real world," which in turn suggests the obvious metaphysical question. Such discussions might help students develop a better understanding of models as products of mathematical abstraction that necessarily omit or limit part of the reality they describe. Often some of the judgments that the model developers make are value judgments; therefore, it is crucial to examine how different values may prioritize one model over another. Ethical inquiries could be extended to consider the fairness of a model. For example, a purely mathematical model of a given set of benefits distributions could prompt inquiry as to whether it is fair to all possible beneficiaries. What are the assumptions under which the model operates? Are there some possible beneficiaries who might have been left out? Kennedy (2020) offers an example of an activity using US wealth distribution data, which was introduced to initiate an ethical inquiry based on the question: What constitutes equitable (fair) distribution of human goods? The inquiry invoked several contestable concepts such as fairness in matters of wealth, responsibility for others, basic needs, living in an ethical manner, and the impact of unequal wealth distribution on society—questions that emerged in the course of the inquiry and which ended with a number of further questions that could be pursued in the future.

To summarize, the *Framework for Philosophical Inquiry* offers a format that encourages students to engage with contestable questions about the field of mathematics, both in its internal relations and its relation to the world.

15.5 The Potential Role of Philosophical Inquiry in the Classroom

As has been suggested, philosophical inquiry in the classroom can be orchestrated as a form of collective, intellectually rigorous and engaging dialogue focused on common, central and contestable questions, challenging and critiquing implicit assumptions, and dedicated to the reconstruction of concepts (Lipman, 2003), thereby facilitating the acquisition of an enriched overarching view of mathematics and its connections to other school disciplines. It may aid in opening a "wider horizon of interpretation" that includes a critical dimension (Kennedy, 2018). Philosophical dialogue requires an "outsider" perspective and thus promises to furnish a more global view of mathematics, its nature, and its instrumentarium. Such a perspective allows for the examination of the epistemological assumptions that influence the role of mathematics in social reproduction—most importantly in normalizing instrumental and calculative ways of thinking (Ernest, 2018)—and thus in organizing everyday experience. A widened perspective includes the examination of mathematics as a cultural product, and philosophical inquiry promises to facilitate that examination, leading to a widened, deepened, more nuanced understanding of the discipline, thereby offering a space for the critical examination of its role in society, and the political implications of its uses.

Mathematics as a system and method includes not only the mode of access to the products of mathematization but also the mode of studying, using, interpreting, and evaluating them; thus, as has been pointed out by many researchers, mathematics also acquires prescriptive and symbolic power (Davis & Hersh, 1986; Skovsmose, 1994). Unless these aspects of mathematics are brought into the open and discussed, there is an obvious danger of our students turning into uncritical consumers of mathematics with little or no understanding of the worldview it reinforces and with no critical competence to judge mathematical productions and prescriptions. If participation in a democratic society is not restricted to following formal procedures of elections and government, but is understood as participating in direct democracy or, in Deweyan terms, as "a mode of associated living," then citizens must be able to critically appraise and scrutinize not only the math instrumentarium and its uses but also the implications of those uses. The urgency of the need for the inclusion of critical inquiry into the ethical, social, and political aspects of the uses of mathematics is particularly salient now, as we examine how mathematical and technological forms of decision-making may be implicated in societal poverty, income inequality, racial injustice, social inequities, climate change, world hunger, educational gaps, and to what extent uncritical and unethical exploitation of mathematical knowledge may exacerbate these deep-seated problems. Inquiry into the ethical questions that pervade an increasingly mathematized world not only promises to raise awareness of the dangers of deceptively "value-free" mathematics but also promises to open dialogue about the moral responsibility of the creators and users of mathematical and technological knowledge.

Yet another inquiry that may have far-reaching implications for the development of the critical mathematical subject, and the prospects for the emergence of a truly autonomous citizenry, is epistemological. "Personal epistemology," a term now commonly used to denote personal beliefs about knowledge and knowing (Kuhn et al., 2000), is increasingly recognized as a powerful hidden shaper of student expectations and learning practices, and thus as crucial for individual learning and development. Epistemological inquiry could also potentially function as "under-laborer"—to use Ernest's term (2018)—that is, to help clear obstacles to learning mathematics that stem from false beliefs and misconceptions about its practice that make a crucial difference for success or failure in the classroom, in particular, received notions about who can do math and is good at it. Ernest's use of the term "under-laborer" is particularly fitting in identifying philosophical dialogue as a potential force in the reconstruction of students' beliefs, attitudes, and images of mathematics, which could work as a powerful mechanism in breaking the "failure cycles" (negative attitudes — > reduced learning —> mathematical failure) that he describes. Since philosophical inquiry is premised on students' own questions and on their active engagement in the rethinking and the reconceptualizing of received views, philosophical dialogue typically involves a process of taking apart and putting together, weighing differences, reformulating, and reconceptualizing. Such a collaborative engagement in developing students' personal views promises to produce deeper engagement with the discipline and world view of mathematics and thus to influence future mathematical experiences.

Finally, philosophical inquiry promises to play a key role in the reconstruction of beliefs about oneself as a mathematics learner. The discipline has become a forbidding gatekeeper for many economic, educational, and political opportunities for students, many of whom have developed self-narratives that act to prevent them from identifying themselves as capable math learners. As such, disrupting such self-narratives and working proactively to reconstruct negative mathematical identities represent an important educational task. Collaborative philosophical inquiry acts to challenge these narratives and to facilitate reflection and ongoing reconstruction, and thus represents a potent mechanism for nurturing students' mathematical identities.

In short, philosophical inquiry could potentially play a role in developing an expanded and more critical view of mathematics—one that offers more meaningful connections and interactions with students' personal experiences and which opens a view of the ethical and political implications of the use of mathematics for society which is essential to a healthy democracy. As such, dialogical philosophical inquiry represents a potentially transformative classroom practice in engaging, challenging, and reconstructing students' views of mathematics, as well as their beliefs, attitudes, identities, and level of engagement with the discipline.

15.6 Conclusion

In his masterwork *Democracy and Education,* Dewey (1916) highlighted the role of education in fostering a democratic society, which depends on the development of individuals as responsible citizens able to make informed and intelligent decisions leading to the public good. Although he recognized the importance of education in the development of citizens who are able to think for themselves, his educational model is based on scientific as opposed to philosophical inquiry and thought. However, contemporary critical mathematics education researchers point out that mathematical literacy, if solely understood as acquiring mathematical and technological (i.e., "scientific") knowledge, is limited in promoting democratic competencies (e.g., Skovsmose, 1994, 2007; D'Ambrosio, 1999; Ernest, 2018). In respect to mathematics education, such literacy should include an analytical dimension that facilitates a critique of mathematics in its philosophical and social implications. In this chapter, I have examined Skovsmose's notion of reflective knowing as associated with critique. I have suggested that philosophical inquiry offers a vehicle for conducting such a critique and briefly outlined a methodology for communal and collective deliberations in a classroom setting designed to facilitate such an inquiry. Finally, I have offered a framework for philosophical inquiry in the mathematics classroom, extending Skovsmose's (2001) notion of "landscapes of mathematical investigation" to include the practice of engaging students in philosophical inquiry into common, central, and contestable questions related to the field of mathematics, both in its internal relations and its relation to the world. In short, conducting philosophical inquiry as a complement to existing mathematical practice in the classroom promises to provide another, crucial dimension to mathematics education, in that it represents a vehicle for questioning and critique and offers a discursive and pedagogical space dedicated to developing an enriched and expanded view of mathematics, as well as a deeper understanding of its connections to other school disciplines, to society, and to self.

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Chapter 16
Mathematical Modelling: A Philosophy of Science Perspective

Uwe Schürmann

16.1 Introduction

The (analytical) separation between mathematics and reality can be found in numerous publications on mathematical modelling. For instance, PISA, the Programme for International Student Assessment (OECD, 2009), uses the following diagram in its mathematical framework (Fig. 16.1), where mathematics and the real world are considered to be separate domains.

Also, the introduction of the 14th study of the International Commission on Mathematical Instruction (ICMI) on modelling and applications (Blum et al., 2007) shows a modelling cycle distinguishing between mathematics and an extra-mathematical world. Additionally, this separation is also postulated in various contributions to the volumes of the International Community of Teachers of Mathematical Modelling and Application (ICTMA).

Figure 16.2 presents a modelling cycle by Blum and Leiß, which is frequently cited in German-language literature on mathematical modelling and is used (sometimes modified or extended) in various works (cf. Greefrath, 2011; Ludwig & Reit, 2013). Borromeo Ferri (2006) offers a carefully elaborated overview of many of these modelling cycles. It is clear from this overview that the (analytical) separation between mathematics and reality is omnipresent in the reconstruction of modelling processes.

In contrast, only a few publications are questioning this separation. For instance, Biehler et al. (2015) analyse modelling processes in engineering classes and conclude from their analysis that it is rather inadequate to separate mathematics and the "rest of the world" as well as to divide modelling processes into certain distinct phases. From

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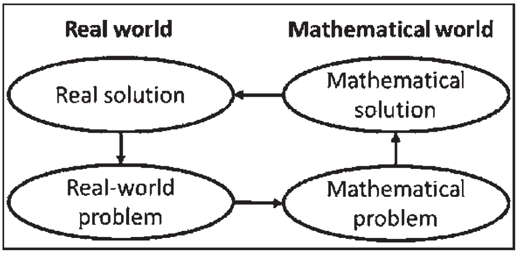


Fig. 16.1 Modelling cycle in PISA's theoretical framework (OECD, 2009, p. 105)

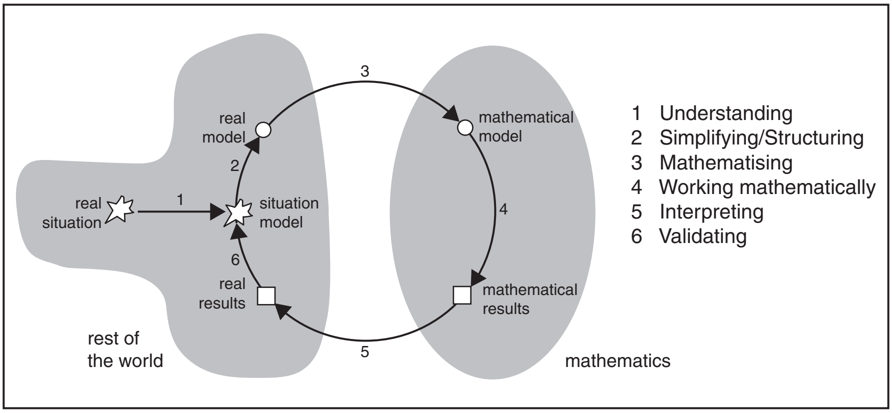


Fig. 16.2 Modelling cycle by Blum and Leiß (2006)

their point of view, mathematical aspects must be considered during the step of simplification (part of the "rest of the world" in most of the modelling cycles), already. This theoretical insight is supported by a subsequent empirical investigation by the authors. Furthermore, Voigt (2011, p. 868) identifies the analytical separation between mathematics and reality as a problem that can only be solved if we take a close look at the area between the "rest of the world" and "mathematics". Consequently, he considers this "intermediate realm" as substantial. Voigt strongly advocates examining not the separation but the connection between these two spheres.

This builds the starting point: In the following, the relationship between mathematics and reality will be explored in more detail. In this way, the question is posed whether and to what extent the analytical separation of mathematics and reality can be justified or whether it should be supplemented or even replaced by an alternative interpretation of the relationship between mathematics and reality.

16.1.1 Orientation

The separation between mathematics and reality, as found in many modelling cycles, can be interpreted in at least three different ways.

1. As an ontological separation according to which mathematics by its very nature would have to be distinguished from reality, the real world, or the rest of the world
2. As an analytical separation primarily serving to describe modelling activities adequately, i.e. to be able to empirically research them
3. As a separation that makes sense from a constructive point of view and serves to support learners while working on modelling tasks

The three interpretations mentioned are neither mutually exclusive nor mutually dependent. Nevertheless, the author hypothesises that when mathematics education research considers mathematics and reality as two distinct realms promoting a constructive point of view becomes more likely. Each of the three interpretations mentioned is problematised in the literature against the background of different perspectives. For instance, Voigt (2011, p. 869) asks whether in placing the "real situation" at the beginning of the modelling process—far from mathematics—the ideal of an everyday life orientation is expressed, under which one imagines that mathematics develops out of an everyday life untainted by any mathematics. Such notions are undermined in various contributions to mathematics education research. Niss (1994, p. 371), for example, mentions that mathematics is confronted with a "relevance paradox". On the one hand, mathematics is becoming more and more relevant and, at the same time, more and more irrelevant, since mathematics plays a pivotal role in the development of technical devices, yet the operation of these technical devices no longer requires mathematical literacy. Keitel's (1989) pair of terms de-/mathematisation points in the same direction. However, these terms emphasise the social significance of mathematics more strongly and problematise the use of supposedly realistic mathematics tasks in the classroom. Keitel introduces the pair of terms de-/mathematisation to describe those processes leading to mathematics—in terms of mechanisation and automation— increasingly determining our living environment (mathematisation). At the same time, mathematics increasingly disappears from everyday life (demathematisation) since the skills that were previously required are henceforth taken from humans by a technical device. Skovsmose and Borba (2004) critically examine the ideological effect of mathematics and its teaching within social contexts. They argue that if mathematics is considered a perfect system and an infallible tool for solving real problems, political control is in favour.

So, the separation between mathematics and reality cannot be understood as a fixed boundary, at least not within social contexts. A domain that is part of the "rest of the world" can be mathematised very soon. Since students gain experience in their mathematised environment way before mathematical concept formation processes take place in the classroom, the everyday life orientation of mathematics education, as outlined by Voigt with critical intent, should rather be rejected.

Another problematising perspective on the relationship between mathematics and reality is offered by those historical-philosophical approaches that are usually assigned to postmodernism. These approaches explicate the historical contexts from which a specific, prevailing image of mathematics has emerged. Deleuze (Deleuze, 1994; Deleuze & Guattari, 1987), for instance, sees a problematising side of mathematics alongside the prevailing axiomatising formalisation of mathematics.

By using historical examples— first and foremost the development of calculus by Leibniz—he elaborates on the possibility of dynamic mathematics emerging from concrete problems (cf. de Freitas, 2013; Smith, 2006). Châtelet (2000) highlights the representational side of mathematics by using historical examples to illustrate several ways in which mathematics' innovations and concepts are strongly dependent on the mathematical tools and forms of graphic representation used at a given time. In doing so, he interprets diagrams as a section of a sequence of physical gestures and thus relates the formal side of mathematics to its material and, above all, physical basis.

De Freitas and Sinclair (2014) take up this idea when they map out their didactics of mathematical concepts. They emphasise the material and ontological side of mathematics in addition to the logical and epistemological. Schürmann (2018a) points in the same direction as he attempts to show that mathematical models, in particular, do not merely serve knowledge, but should also be understood as entities, i.e. in addition to their epistemological function, their ontological side needs consideration, too. Furthermore, Schürmann (2018b) deals with the origin of historical knowledge formations that may have contributed to the separation of mathematics and reality. Using Frege's logicism and Hilbert's formalist programs as paradigms (Frege, 1884, 1892; Hilbert, 1903) against the background of what Foucault calls the episteme of modernity (Foucault, 1996) this separation is understood as a reaction to the relativisation of mathematical truth claims within the nineteenth century.

The literature cited here clarifies that the boundary between mathematics and reality is historically conditioned. A further problematisation of the separation of mathematics and reality emerges from those empirical studies focusing on individual modelling processes. Regarding these studies, students already consider relationships between the mathematical content and parts of the real world long before setting up a mathematical model. Biehler et al. (2015) and Meyer and Voigt (2010) give a critique of the analytical separation of mathematics and reality based on this empirical finding.

16.1.2 Focus

Since mathematics education research on modelling is largely detached from the philosophical discussion on models, which goes on for more than 100 years,[1] this chapter elucidates the separation of mathematics and reality against the background of the philosophy of science on models.

Here, the philosophy of science is understood as a subdomain of philosophy in which the validity claims of empirical sciences and mathematics are scrutinised, for

[1] In order to prove this thesis, the author has reviewed the bibliographies of all contributions in the ICTMA volumes published so far. It turns out that none of these contributions refer to relevant works from the philosophy of science.

instance, by reconstructing scientific theories. However, the philosophy of science concerning the humanities (e.g. hermeneutics) is excluded, even though such an approach may be of interest under certain conditions.[2]

Additionally, the following is mainly about the relationship between mathematics and reality in the context of theories and models. An epistemological question of the perception of reality is not raised here, although it is not intended to deny the importance of fundamental epistemological questions for the understanding of mathematical modelling.

The approach is to take up considerations from the philosophy of science on the relationship between theories, models, and reality and apply them to mathematics education research. For this purpose, two central views within analytical philosophy, the syntactic and the semantic view, are juxtaposed and related to mathematical modelling in the classroom. This selection is not intended to question divergent approaches, such as the pragmatic view on models (cf. Gelfert, 2017; Winther, 2016). The restriction to the two views mentioned above is merely for pragmatic reasons. Even these two views can only be outlined here. However, their discussion provides valuable information for answering the following questions:

1. *Epistemological question:* Is the analytical separation between mathematics and reality, often found in mathematics education research on modelling, tenable as such against the background of analytical philosophy, or does it need to be revised or at least relativised?
2. *Methodological question:* Does the discussion on the syntactic and semantic view on models and theories offer new insights into the description of mathematical modelling in the classroom? In particular, can methodological tools be derived that describe modelling processes more appropriately and accurately?

A third, rather constructive question, arising from an assumed separation between mathematics and reality, is excluded here. It is not asked whether the separation between mathematics and reality supports the learners in the processing of modelling tasks.

16.2 Analysis

Large parts of the philosophy of science's discussion on models have their origins in model theory, a subdomain of mathematical logic. To also grasp scientific models and theories, mathematical logic's angle, formerly focused on formal languages, was widened. From now on, natural and scientific languages are considered as well, i.e. formal languages are understood as subsets of natural languages.

[2] Frigg and Salis (2019), for example, compare models with (literary) fiction.

The syntax of a language *L* consists of its vocabulary and the rules for forming well-defined expressions in *L*. The semantics of *L* allows the interpretation of well-defined expressions by mapping them to another relational structure *R*. Thus, on the one hand, well-defined expressions from *L* are made comprehensible, and, on the other hand, these expressions can be examined within *L* for their validity. Then, the distinction between syntax and semantics initially leads to two opposing (but related) views on models and theories, the syntactic and the semantic view. The syntactic view on scientific theories was developed primarily by representatives of the Vienna Circle. Due to this, this view on theories and models is closely connected to logical positivism or logical empiricism,[3] which had a huge impact on the philosophy of science in the twentieth century until the 1960s (cf. Gelfert, 2017). Very likely, the achievements of the natural sciences in conjunction with the rapidly developing axiomatic-formal mathematics at the beginning of the twentieth century were decisive for the increasing influence of logical positivism.

The semantic view on theories and models has emerged largely in response to the syntactic view and its associated obstacles (some of them will be discussed below). The main difference between the two may be that the syntactic view attempts to describe theory building in an idealised form, while the semantic approach tries to outline theory building in terms of scientific practice. Due to the large amount of literature, it is necessary to select among the authors referred to in this chapter. From the syntactic view, the oeuvre of Rudolf Carnap is considered paradigmatic (Carnap, 1939, 1956, 1958, 1969). The analysis of the semantic view is based on the works of Patrick Suppes (1957, 1960, 1962, 1967).

16.2.1 Carnap's Syntactic View on Models

From Carnap's (1969, pp. 255 ff., 1958; see also Suppe, 1971) syntactic point of view, theories can be reconstructed based on propositions. A theory is formulated in a language *L* that consists of two sub-languages, the theoretical language L_T and the observational language L_O. The descriptive constants of L_T are named theoretical terms or *t*-terms. Those of L_O are called "observable" (Carnap, 1969, p. 225), observational terms or *o*-terms (Carnap, 1969, p. 255). *O*-terms denote observable objects or processes and the relations between them, e.g. "Zurich", "cold" and "heavy". *T*-terms are those that cannot be explicitly defined by *o*-terms, i.e. they cannot be derived from perception. Carnap's given examples are fundamental terms of theoretical physics such as "mass" or "temperature" (Carnap, 1958, p. 237). This distinction leads to three different types of propositions:

[3] Even though the Vienna Circle's members did not use the term "logical positivism" for themselves, this chapter does not distinguish between logical positivism and logical empiricism. Creath (2017) points out that a distinction between the two terms along theoretical assumptions and sociological viewpoints cannot be made meaningfully anyway.

1. Observational propositions containing *o*-terms but no *t*-terms
2. Mixed propositions containing *t*-terms and *o*-terms
3. Theoretical propositions containing *t*-terms but no *o*-terms

According to this approach, a theory in language *L* is based on two types of postulates: the theoretical or *t*-postulates and the correspondence or *c*-postulates, also called correspondence rules (Carnap, 1969) or protocol theorems (Carnap, 1932). *T*- postulates are pure *t*-propositions, i.e. they belong to type (3) of the three types of propositions listed above. *T*-postulates comprise all fundamental laws of a theory. For instance, these can be the fundamental laws of classical mechanics or the main laws of thermodynamics. *T*-postulates are therefore the axioms of a theory. They are taken for granted. All statements *s* that can be derived purely syntactically from the *t*-postulates also belong to L_T. The derivation of such statements is based on syntactic rules, which can contain further rules of formation in addition to mathematical rules. L_T in itself has no (empirical) meaning. The meaning of *t*-terms is only given indirectly using L_O. Carnap assumes that *o*-terms refer to directly observable or at least almost directly observable physical objects or processes and relations between them (Carnap, 1969, pp. 225 ff.). In the following, this direct interpretation will be called *d*-interpretation. Thus, the semantics of *o*-terms is directly given. It is not possible to derive empirical statements from theoretical statements, i.e. from propositions of type (3), it is not possible to conclude propositions of type (1) without further ado. Rules are needed, the so-called *c*-postulates, to connect *t*-terms with *o*-terms. For instance, Carnap (1969, p. 233) mentions the measurement of electromagnetic oscillations of a certain frequency, which is made visible by the display of a certain colour. *C*-postulates thus connect something visible with something invisible. Nevertheless, they do not thereby make the invisible itself visible.

The *t*-term to be interpreted remains theoretical. This kind of interpretation has therefore to be distinguished from the *d*-interpretation of the *o*-term. Moreover, the interpretation remains incomplete since it is always possible to establish further rules to connect *t*-terms with *o*-terms. Since the interpretation of *t*-terms using *c*-postulates is partial, it is called *p*-interpretation in the following.

To Carnap, it is important not to confuse *c*-postulates with definitions (Carnap, 1956, p. 48). The definition of *t*-terms itself is theoretical and can only be given adequately within L_T. A *t*-term is interpreted logically within L_T, which is why this kind of interpretation is called *l*-interpretation in the following. It is not possible to define a *t*- term completely by relating it to *o*-terms via *c*-postulates. Carnap gives us the following explanation: The terms of geometry as defined by Hilbert are entirely theoretical. However, if they are used within an empirical theory, their empirical use would have to be introduced with the help of *c*-postulates. However, no geometric *o*-term, such as "ray of light" or "taut string", corresponds to the theoretical properties of the *t*-term straight line (Carnap, 1969, p. 236).

Equipped with this repertoire of concepts, Carnap's understanding of empirical theories can be defined.

> A theory is a proposition. This proposition is the conjunction of the two propositions *T* and *C*, where *T* is the conjunction of all *t*-postulates and *C* is the conjunction of all *c*-postulates. (Carnap, 1969, p. 266, translation by the author)

To emphasise this connection, Carnap uses the abbreviation *TC* for theories. Now that we have a clear and distinctive definition of what Carnap calls a theory, we go on to explicate Carnap's view on models. Carnap distinguishes descriptive models of physics, which are built from real objects like a model ship, from scientific models in a contemporary sense. As in mathematics and logic, a model in the natural sciences in the twentieth century was understood to be an "abstract, conceptual structure". In this sense, a model is a simplified description of a (physical, economic, sociological, or other) structure in which abstract concepts are mathematically connected (Carnap, 1969, pp. 174–175).

By highlighting the importance of non-Euclidean geometry for physics, especially for the development of the theory of relativity, Carnap infers that it is not disadvantageous for theories if they cannot be visualised without difficulty. In this way, he opposes the idea that models are a sort of visualisation. For Carnap, the visualising character of models is only a makeshift or a didactic aid that merely brings the benefit of being able to think about theories in vivid pictures (Carnap, 1939, p. 210). According to Carnap, models only play a significant role in the development of empirical theories if they establish a connection between L_T and L_O. These "constructing models" (Carnap, 1959, p. 204) serve the *p*- interpretation of *t*-terms and, in this sense, are nothing else than *c*-postulates.

16.2.2 Suppes' Semantic View on Models

The objections to the syntactic view are numerous (cf. Achinstein, 1963, 1965; Suppe, 1971, 1989, 2000; Suppes, 1967; van Fraassen, 1980; see also Liu, 1997; Winther, 2016). Some of these objections are:

1. The formalisation of theories as linguistic entities is inadequate and obscures the underlying structures of theories.
2. Theory testing is oversimplified in the syntactic view since it is assumed that propositions from L_O can be directly linked to phenomena.
3. The pure distinction between *o*- and *t*-terms is not tenable if the characterisation of *o*-terms or *t*-terms is insufficient.
4. *P*-interpretation remains undefined and all possible ways to define *p*-interpretation lead to inconsistencies in the syntactic view.

The semantic view on theories and models can essentially be understood as a reaction to the shortcomings of the syntactic view outlined here (Gelfert, 2017; Portides, 2017). Thus, the meta-mathematical description of theories through formal languages is (largely) rejected in the semantic view. While the syntactic view tries to describe scientific theories in logical languages, the semantic approach asks what kind of mathematical models are used in the sciences (Winther, 2016). Mathematical tools are available for the direct analysis of such structures. In contrast, a reformulation of a theory in a specific formal language tends to be impractical, especially for those theories with rather complicated structures (Suppes, 1957, pp. 248–249).

Moreover, a direct description of mathematical structures may be independent of a particular language. From Suppes' semantic point of view, a theory is composed of a set of set-theoretic structures satisfying the different linguistic formulations of a theory. Worth mentioning that besides this conception of the semantic view, at least one differing semantic approach—the so-called state-space approach—exists (e.g. van Fraassen, 1980), which describes physical systems by vectors. In the semantic view, a model of a theory is a structure and should not be confused with the linguistic description of that structure. Propositions of a theory, expressed in a particular linguistic formulation, are merely interpreted within that structure.

> [A] model of a theory may be defined as a possible realization in which all valid sentences of the theory are satisfied, and a possible realization of the theory is an entity of the appropriate set-theoretical structure. (Suppes, 1962; see also Suppes, 1957, 1960, p. 253)

This emphasises the importance of models for theory building, and along with it the importance of nonlinguistic structures overall. Furthermore, Suppes points out that theories cannot be related directly to experimental data. Accordingly, the *d*-interpretation of *o*-terms in experimental settings is dismissed. Rather, this connection is only established indirectly via what Suppes calls models of data (Suppes, 1962). While models of a theory are possible realisations of a theory, models of data are possible realisations of experimental data. By this conception, Suppes circumvents objection (2), as listed above. In addition, even if a hierarchy between these different types of models is assumed, they are nevertheless connected by an isomorphism between the two types of models (for a critique of this connection by isomorphism, see Suárez, 2003). Objections (3) and (4) are discussed in more detail in the following sections "Theoretical and Empirical Concepts" and "Correspondence Rules and Partial Interpretation".

16.2.3 Theoretical and Empirical Concepts

The separation between *o*- and *t*-terms is challenged from different perspectives. Putnam (1962), for instance, indicates the possibility of formulating theories that do not contain any *t*-terms. He quotes Darwin's theory of evolution as an example. He thus questions whether the separation of L_O and L_T is at all necessary. Consequently, theories that manage without *t*-terms could also not be reconstructed as the proposition *TC* in Carnap's sense.

Putnam then goes on to say that the mere distinction between *o*- and *t*-terms is not sufficient at all. He points out that terms that do not belong to L_O cannot be considered *t*- terms automatically.

Moreover, it is unclear which criterion separates L_T from L_O. Carnap assumes that from a pragmatic point of view, a clear distinction can usually be made between the two (Carnap, 1969, p. 255). It is only decisive whether a term designates a directly or at least indirectly observable entity. Otherwise, it is a *t*-term. According to Achinstein (1965), this criterion is not exhaustive. For instance, an electron, usually a non-observable term, can be considered observable in certain contexts and under certain conditions. He concludes that the term electron cannot be unambiguously assigned to either L_T or L_O. Rather, the conditions for *o*-terms must be made explicit in more detail.

Therefore, Achinstein discusses another criterion that could justify the separation into *o*- and *t*-terms. *T*-terms could be distinguished from *o*-terms based on their theoretical character (cf. Hanson, 1958). According to this distinction, a term would be theory-laden and thus a *t*-term if it cannot be understood without its theoretical background. To Achinstein, even this distinction is not sufficient to divide *o*- and *t*-terms more clearly. A term can be essential in the context of a certain theory, while in another corresponding theory, it is rather independent. Thus, for each term, it must be made clear which theory in particular forms the background. Putnam (1962) also argues that there are no terms that belong exclusively to L_O. For instance, the colour red, which is considered an *o*-term in everyday language, is a *t*-term (red corpuscles) in Newton's corpuscular theory of light. So, the question is posed how to define *t*-terms more precisely.

Another criticism of the syntactic view deals with the possibility to make a theory-free perception at all. This focuses upon the syntactic view's assumption that *o*-terms can be interpreted by direct or at least indirect observation of real phenomena. Seen from the syntactic perspective, *o*-terms must be interpreted with direct reference to real phenomena, since indirect observation by instruments already implies *l*-interpretation.

To perceive objects without recourse to a theoretical background is questioned by other authors. Can there be such a thing as mere observation or does observation always require interpretation of sensory impressions? Hanson (1958, pp. 5–13) gives us various examples here: two biologists looking at an amoeba may see different things because of their different theoretical backgrounds, Tycho Brahe who would not recognise the telescope in a cylinder, as Kepler presumably would, etc. Hanson goes on by describing optical perceptions. He explains that seeing as a mere perception on the retina is always already an interpretation as soon as it enters consciousness. This also illustrates that observational concepts cannot be related to objects directly.

16.2.4 Correspondence Rules and Partial Interpretation

According to the syntactic view, *o*-terms are connected to *t*-terms by correspondence rules (Carnap's *c*-postulates). The assumption is that correspondence rules are the *p*-interpretation of a *t*-term. However, not all *t*-terms of a theory have to be partially interpretable. While an *o*-term must always be directly interpretable, *t*-terms may exist without *c*-postulate partially interpreting them. Such *t*-terms are only indirectly connected with L_O by being connected in L_T with other *t*-terms that can be partially interpreted. For instance, the square root of 2 is unobservable. Nevertheless, it can be indirectly connected with *o*-terms via an *l*-interpretation if it is interpreted as the side length of the square with the area 2. For the *c*-postulates of a theory, Carnap (1956) formulates the following rules.

1. The set of *c*-postulates of a theory must be finite.
2. All *c*-postulates must be logically compatible with the *t*-postulates.
3. The *c*-postulates do not contain terms neither belonging to L_T nor to L_O.
4. Each *c*-postulate must contain at least one *t*-term and *o*-term.

However, apart from the explanation by examples and these rules for *c*-postulates, Carnap does not define more clearly what is meant by *p*-interpretation. This lack of clarification is criticised by various authors (cf. Achinstein, 1963, 1965; Putnam, 1962). Hence, Putnam discusses three ways to define *p*-interpretation:

1. [T]o 'partially interpret' a theory is to specify a non-empty class of intended models. If the specified class has one member, the interpretation is complete; if more than one, properly partial.
2. To partially interpret a term P could mean […] to specify a verification-refutation procedure.
3. Most simply, one might say that to partially interpret a formal language is to interpret part of the language (e.g. to provide translations into common language for some terms and leave the others mere dummy symbols). (Putnam, 1962)

Definition 1 Putnam objects to the first definition. To define a class of models similar in structure to the theory in parts, (a) mathematical concepts, theoretical by definition, are required, and the argument would become circular. Furthermore, he points out (b) that models require certain broad-spectrum terms (e.g. physical object or physical quantity). Such terms cannot be defined a priori, as Quine (1957) illustrates by the meta-concept "science". Accordingly, it is possible that such terms do not acquire their meaning through *p*-interpretation in a particular model, but within a theoretical framework based on the conventions of a research community. Consequently, logical positivists like Carnap must reject such concepts as metaphysical. Ultimately, it refers (c) to the problem that a theory with an empty class of models can no longer be called false, but merely meaningless.

Definition 2 According to Putnam, the second understanding of *p*-interpretation also proves to be unsustainable. If for every concept or proposition a procedure for its confirmation or its refutation is specified, this would lead to curious statements against the background of the philosophical position of verificationism as advocated by Carnap. According to verificationism, only those (synthetic) statements may be true that can be empirically verified. Using the example of the sun and the helium it contains, Putnam draws attention to the following problem. Although it is possible to prove that the sun contains helium, no procedure can be used to prove that helium exists in every part of the sun. If this confirming or refuting procedure is missing, the truth value is indeterminate. In consequence, one would have to claim that the sun contains helium, whereas it cannot be said for parts of the sun, whether there is helium or not.

Definition 3 The third and last possible definition of *p*-interpretation, that L_T is only interpreted in parts, is rejected by Putnam in just one sentence. Such a view would lead to certain theoretical terms ultimately having no meaning at all. A part of L_T would be interpreted into everyday language, for example, and the remaining part of the *t*-terms would merely consist of dummy terms.

16.3 Modelling in Mathematics Classroom from a Syntactic Point of View

In the following, mathematical modelling in the classroom is interpreted against the background of Carnap's syntactic view, while bearing in mind criticism from a semantic point of view. For that, the posed epistemological and methodological questions are focused. Since Carnap's syntactic view first and foremost describes an ideal of empirical sciences, modifications must be made to transfer this to modelling in mathematics education. Axiomatised mathematics cannot be assumed for mathematics teaching, but mathematics in L_T that students master. Furthermore, it is not assumed that an understanding of mathematics in Carnap's formal sense prevails among the students. To describe a modelling process, it is sufficient to reformulate students' usage of terms in Carnap's sense. In this context, mathematical terms used by students in theoretical regards are classified as *t*-terms. Those that refer to observable objects are classified as *o*- terms.

The problem of theoretical terms is serious. Nevertheless, when it comes to mathematical modelling, most of the terms used are mathematical terms and therefore of theoretical nature. Thus, mathematical concepts in school also have a certain theoretical character if students can *l*-interpretate them to a certain extent. Likewise, students can understand that mathematical concepts are in principle unobservable, even if they can be illustrated. However, Achinstein's (1963, 1965) and Putnam's (1962) objections to the separation of theoretical and empirical terms remain considered insofar that the *t*-terms used in the context of mathematical modelling are always *t*-theoretical. In means of students' modelling processes, this implies that *t*-terms are dependent on the mathematics available to students.

Even when transferring Carnap's syntactic view to the description of learners' mathematical modelling, Putnam's objections (1962, p. 245) to different definitions of *p*- interpretation are still considered. If *p*-interpretation of mathematical terms is considered as building a set of intended models, theoretical terms are required indeed. However, this science-theoretical problem concerns the consistency of the syntactic view of theory building. This problem may be less important when it comes to *p*-interpretation within modelling processes taking place in the mathematics classroom. In fact, trying to provide an appropriate procedure for confirming or refuting each *t*-term can lead to some odd statements. For modelling problems in the classroom, however, this can also be a rather subordinate problem. Mathematics lessons usually consider those parts of reality for which such confirmation or refutation procedures exist. Furthermore, the third definition of *p*-interpretation, interpreting parts of L_T and leaving the remaining terms aside, is rather a duty for mathematics teaching than a real objection. Every *t*-term of mathematics should be made semantically accessible to students. Here, the psychological argument is that interpretation of mathematical content through its application leads to an improved and deeper understanding of such content (cf. Blum, 1996, p. 21–22).

16.3.1 Epistemological Question

The purpose of this chapter is to prove if the separation between mathematics and reality, often found in mathematics education research on modelling, is tenable as such against the background of analytical philosophy. By discussing Carnap's syntactic view of theories and models, it becomes clear that this separation needs to be revised.

Carnap's syntactic view captures more precisely the connection between mathematics and reality. In contrast to the dichotomous separation between mathematics and reality in many modelling cycles, there is at least a twofold gradation from mathematics in L_T, via empirical-mathematical concepts in L_O, to real-world phenomena. In a modelling process, (school) mathematics is to be understood as the theoretical (part of a) language with which students can proceed syntactically. Reality, or the "rest of the world", is henceforth divided into an observational language, which itself is not yet a reality, and a part that is identified with real-world phenomena (Fig. 16.3).

Bearing this picture in mind, the criticism of many so-called modelling tasks in mathematics textbooks can be justified by the fact that no real problem is actually solved by the students in the context of a modelling process. Most likely, those tasks take place only in the sphere of theoretical and observational language. While the translation process between these two parts of the language is crucial for making sense of pure mathematical concepts, it does not involve any connection to real-world phenomena. This insight is probably obscured by an overly simplistic juxtaposition of reality and mathematics in many modelling cycles.

Moreover, Carnap's interpretation of models in science as *c*-postulates, marking the area between theoretical and observational language, and Suppes' objection that models of data, marking the area between observational language and real-world phenomena, have to be considered as well. While three models appear in the modelling cycle proposed by Blum and Leiß ("situation model" and "real model" in the realm of reality, and "mathematical model" in the realm of mathematics, Fig. 16.2), we can now capture more accurately the nature of models in mathematical modelling processes. Models are translation rules both for the translation between

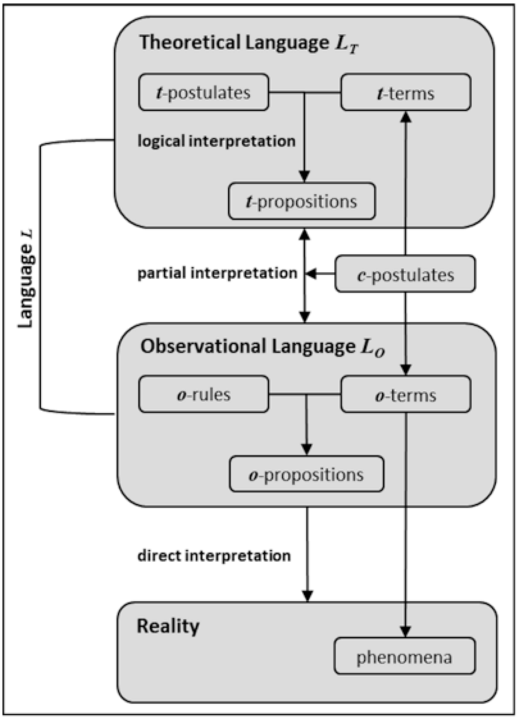


Fig. 16.3 Carnap's syntactic view on theories visualised

theoretical and observational language and for the translation between real-world phenomena and observational language.

Here, the crucial point is that the connection between the two domains of the language at issue is given by assumed rules, not by nature. This finding circumvents lots of epistemological obstacles (e.g. questions about the nature of mathematical terms and their possible empirical origin do not need to be answered for the syntactic view to work). Finally, the competencies described in the modelling cycle can now be interpreted against the background of the previous discussion of theories and models. Working mathematically (step 4 in the modelling cycle according to Blum & Leiß, 2006) can be identified with the *l*-interpretation, mathematising as a transition from the "rest of the world" to "mathematics" (step 3, ibid.) and interpreting as a transition into the opposite direction (step 5, ibid.) is associated with Carnap's *p*-interpretation. The decisive difference is that *p*-interpretation does not indicate the transition from mathematics to reality and vice versa but only a transition between two parts of a language. The *d*- interpretation is pivotal in making the transition from L_O to real-world phenomena (step 1, ibid.). Here, models of data are crucial.

It becomes obvious why, as Meyer and Voigt (2010) note, connections from mathematics need to be considered already in the step of simplification. Learners work with *o*-terms in the step of simplification. However, these must be connected, even implicitly, with *t*-terms. They form what Voigt (2011) calls the "intermediate area".

16.3.2 Methodological Question

The methodological question of whether the discussion on the syntactic and semantic views on models and theories offers new insights for the description of mathematical modelling in the classroom can now be answered against the background of the previous discussion. The goal is to derive methodological tools that describe modelling processes more appropriately and accurately compared to standard modelling cycles. To this end, Carnap's syntactic view and Suppes' criticism of it are considered.

One of the main objections to the syntactic view is that the formalisation of theories as linguistic entities tends to be inadequate because it obscures the underlying structures of theories. While this objection may be crucial for discussion in the philosophy of science, the attempt to focus on the underlying (mathematical) structures tends to be a hindrance when it comes to empirical research in mathematics education. Students' utterances (written, spoken, or expressed by gestures) can be directly observed, whereas the underlying mathematical structures can only be conjectured. With its distinction between theoretical and observational terms, the syntactic view provides a tool for a more detailed analysis of students' utterances. For instance, when a student uses the word "triangle", it is decisive whether the word is used in a theoretical way, for example, in a mathematical theorem, or whether it is used in a sentence to describe real-world phenomena. At this point, Suppes' objection to the theoretical-observational distinction must be considered. The discussion in the section on "Theoretical and Empirical Concepts" shows very briefly that it cannot be said that a concept, by its nature, belongs to either L_T or L_O. Nevertheless, the distinction holds when the theoretical or observational character of a term is considered against the background of the theory T in question. Stegmüller's (1970) solution to this problem is that a term can be called T-theoretical (or T-observable) in the case that T is the theory under consideration. The theoretical character of a term depends on the theory we are talking about. Carnap's definition of a theory (TC is the conjunction of T and C, while T is the conjunction of all t-postulates and C is the conjunction of all c-postulates) reminds us that a clear description of the theoretical background taught to students is necessary when mathematical modelling processes are captured empirically. The question is what theoretical tools (i.e. mathematical theorems, procedures, etc.) are on the theoretical side and what correspondence rules for translation between L_T and L_O are accessible to students.

In order to take a closer look at students' modelling processes, it is necessary to reconsider Carnaps' notion of the connection between L_T and L_O given by c-postulates. Although for Carnap, an ideal theory only includes c-postulates, i.e. axioms that translate between L_T and L_O, he points out that it is not essential for this connection that correspondence rules have the character of an axiom.

> The particular form chosen for the rules C is not essential. They might be formulated as rules of inference or as postulates. (Carnap, 1956, p. 47)

We can thus distinguish between the individual mental models expressed in students' utterances and the more normative models thought in class to make sense of pure mathematical concepts and to give students the ability to solve real-world problems.

From a normative point of view, it is necessary to describe rather abstract models that fit a wide range of situations. This involves the four rules for the formation of *c*- postulates (see section "Correspondence Rules and Partial Interpretation"). Under these conditions, the goal is to formulate as many (but still independent) c-postulates as possible, so that as many situations as possible that fit a notion of pure mathematics are covered by a certain set of *c*-postulates. Putnam's main objections to this understanding of partial interpretation (specifying a class of intended models) are that (a) pure mathematical terms (e.g. set) are needed and the procedure would become circular, and that (b) broad-spectrum terms are needed (e.g. physical object, physical quantity, etc.) that cannot be defined a priori and whose meaning cannot be given by partial interpretation via correspondence rules. While these objections can seriously affect the syntactic view when the focus is on the normative description of an ideal theory, they tend not to negatively affect the goal of describing students' modelling processes. Rather, these objections remind us that every description of individual modelling processes and even the establishment of normative models are limited by the framing through inherently broad-spectrum terms and (meta-) mathematical terms in use.

Bearing in mind, that correspondence rules not necessarily need to be formulated in a set of axioms, this offers an opportunity to analyse students' (implicit) use of correspondence rules in modelling processes. As we will see, this provides a methodological tool that leads to different results than the analyses that depend on standard modelling cycles and their inherent epistemological assumptions. Let us take a look at the mathematics task from a textbook for fifth and sixth graders:

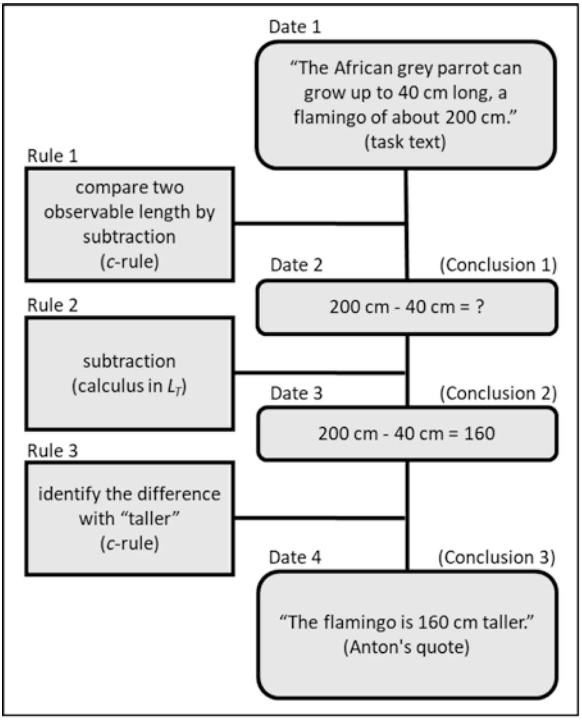
> The African grey parrot can grow up to 40 cm long; a flamingo of about 200 cm. How many times bigger is the flamingo compared to the grey parrot? (Prediger, 2009, p. 6, translation by the author)

From a normative point of view, the area of mathematics addressed in this task can be narrowed down to the structure of natural numbers in connection with multiplication $(N, \cdot)$. On the observational side, questions can be formulated such as "how often does one length fit into another?" or "how many times larger is this length compared to another?". The connection between L_T and L_O is then given by a *p*-interpretation containing at least two *c*-rules. Thus, *c*-rule c_1 connects—for instance—the number 1 with the observable length of 1 cm, while *c*-rule c_2 connects multiplication with a temporal- successive action (e.g. "an empirical length is juxtaposed until the length used for comparison is reached"). The model at issue here is the description of the structure $(N, \cdot)$ using the linguistic means from L_O, given by c_1 and c_2. If T is the conjunction of all true propositions in L_T about the structure $(N, \cdot)$ and C is the conjunction of c_1 and c_2, the theoretical background of the task is given by the conjunction TC. This interpretation of $(N, \cdot)$ by c_1 and c_2 remains partial. In contrast, an *l*-interpretation of $(N, \cdot)$ within L_T, (e.g. as the addition of equal addends) is complete. Again, it should be mentioned that c_1 and c_2 do not connect mathematics with reality, but theoretical terms of mathematics (*t*-terms) with empirical terms (*o*-terms). Hence, it is a purely linguistic connection. With this revision of the task in mind, we can now analyse individual utterances of students confronted with this task. To do this, individual modelling is analysed by reconstructing the reasoning within the modelling according to the Toulmin scheme (Toulmin, 1996). Thereby, the use of *c*-rules—whether implicit or explicit—has to be taken into account.

Due to the limitation of a book chapter, the focus is on a single case study, the student Anton. Anton is interviewed while solving the task (cf. Prediger, 2009). At the beginning of the interview, Anton soon says, "The flamingo is 160 cm taller". The genesis of this statement can be reconstructed with the help of the Toulmin scheme as follows (Fig. 16.4).

Against the background of common modelling cycles, Anton's statement must be interpreted as an individual construction of whether a situation model, a real model, or a mathematical model. However, a rational reconstruction shows that none of Anton's possible considerations can be the mere result of modelling processes taking place exclusively in the "rest of the world". Anton's statement cannot be interpreted without (implicit) translations between L_T and L_O.

Fig. 16.4 Anton's statement rationally reconstructed



16.4 Conclusion and Outlook

Against the background of the syntactic view on theories and models and its critique by the adherents of the semantic view of theory building, mainstream modelling cycles and their inherent epistemological assumptions about the relation between mathematics and reality have been problematised. The goal of the chapter is to show that the description of students' modelling processes cannot rely on a simple separation between mathematics and reality. The syntactic view, as offered by Carnap, indicates that distinguishing between the theoretical and observational side of a language can be helpful in capturing the translation processes of students that take place when mental models are used to interpret pure mathematical terms, and vice versa, to interpret the empirical part of a language through the means of mathematics.

Based on a single case study, it was shown that the twofold separation between L_T and L_O and the connection via *c*-rules—in combination with Toulmin's scheme—provides a methodological tool to investigate students' translations between mathematics understood as a theoretical language and everyday language and the empirical use of mathematical terms contained therein. In detail, this attempt allows us to reconstruct also those more implicit translation steps that are necessary to explain subsequent explicit utterances and that would remain hidden against the background of mainstream modelling cycles.

To give an outlook: While this brief chapter has paid attention to the multiple translations between the theoretical and empirical sides of a language used in modelling processes, the connection of *o*-terms with real-world phenomena was omitted to a large extent. In order to get a comprehensive picture of all the translations taking place in modelling processes, this connection needs to be described and problematised in more detail. Follow-up questions arise when not only the epistemological and ontological aspects but also constructive aspects of mathematical modelling are considered. Here, questions may arise concerning the design of textbook tasks to promote students' modelling skills, the teaching of adequate models for proper connection between pure mathematical terms and everyday language, and whether and what meta-knowledge about mathematical modelling should be taught in the class.

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Chapter 17
Education for Sustainable Development (ESD) in Mathematics Education: Reconfiguring and Rethinking the Philosophy of Mathematics for the Twenty-First Century

Hui-Chuan Li

17.1 Background

The aspect that emphasises a focus on sustainable development to transform education is not new, as formal education, for at least the last 50 years, has been challenged to engage with a range of economic, social, and environmental concerns. The United Nations (UN) calls for the inclusion of sustainable development into all areas of teaching and learning can date back to *Agenda 21* (UN, 1992). In *Agenda 21*, it identified four major imperatives to begin the work of Education for Sustainable Development (ESD): (1) improve basic education, (2) reorient existing education to address sustainable development, (3) develop public understanding and awareness, and (4) training (*ibid*, 1992).

The term ESD has been used internationally and by UNESCO to refer to the incorporation of information on sustainable development into the curriculum, information on issues such as climate change, disaster risk reduction, biodiversity, poverty reduction, and sustainable consumption. Over the past two decades, formal education systems have begun to take ESD into account as part of their responsibility (Li & Tsai, 2022). For instance, the international ESD agenda has informed curriculum developments in Australia and Scotland. Sustainability as a cross-curricular priority was introduced into Australia's curriculum in 2010. In Scotland, "Learning for Sustainability (LfS) [is] cross-cutting themes in Scotland's CfE [Curriculum for Excellence] which provides an overarching philosophical, pedagogical and practical framework for embedding ESD in the school curriculum" (Bamber et al., 2016, p. 5). LfS as a core component of teachers' professional standards has been

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embedded at all levels of Scottish Education since 2012 in response to UNESCO's call for action.

The Global Action Programme (GAP) on ESD, which ran from 2015 to 2019, deployed a two-fold approach to multiply and to scale up the ESD action:

1. To reorient education and learning so that everyone has the opportunity to acquire the knowledge, skills, values and attitudes that empower them to contribute to sustainable development.
2. To strengthen education and learning in all agendas, programmes and activities that promote sustainable development (UNESCO, 2018).

In 2015, all the UN member states have approved the 2030 Agenda for Sustainable Development. This Agenda is an ambitious plan that sets out for countries, the UN system, and all other actors to stimulate action over the period 2016 to 2030. It consists of 17 interconnected Sustainable Development Goals (SDGs), further broken down into 169 targets, to be met by 2030 with the intention of achieving inclusive, people-centred, and sustainable development with no one left behind. The concept of ESD has also brought a new focus to education policy and practice, often referred to as adjectival education, for example, development education, global citizenship education, peace education, environmental education, and climate change education (Evans, 2019).

The overarching goal of ESD is to integrate an awareness of sustainable development issues into all aspects of education so that students are empowered to make informed decisions in their daily lives. In the 2030 Agenda for sustainable development (UN, 2015), it stipulates that by 2030 all learners must:

> Acquire knowledge and skills needed to promote sustainable development, including among others through education for sustainable development and sustainable lifestyles, human rights, gender equality, promotion of a culture of peace and non-violence, global citizenship, and appreciation of cultural diversity and of culture's contribution to sustainable development. (p.19)

How can mathematics education help learners of all ages to respond to sustainable development challenges, to lead healthy lives, to nurture sustainable livelihoods, and to achieve human fulfilment for all? The answer is not straightforward because moving toward a mathematics education that incorporates ESD requires a paradigm shift in the philosophy of mathematics in general (Ernest, 2018; Skovsmose, 2019) and in the objective of mathematics education in particular (Gellert et al., 2018; Lyons et al., 2003). Without a philosophy for incorporating ESD into mathematics education, it is not reasonable to expect teachers to appreciate that they—in addition to teaching the subject—also have a responsibility to provide students with opportunities to apply newly gained subject matter expertise to the wider societal, ecological, equity, and economic issues that they encounter outside the classroom.

Therefore, this chapter first will review the current state of ESD in mathematics education. Second, it will look at philosophical theories that have been developed to explain the meaning of mathematics with respect to what mathematics is and what it means to understand mathematics. Third, it will discuss issues about equity and social justice in mathematics education. Fourth, it will talk about interdisciplinary learning and STEM education and draw attention to the question of whether the existing philosophical views of mathematics can be applied to understand the role of mathematics in interdisciplinary learning and/or STEM education. Finally, it will call for reconfiguring and rethinking the philosophy of mathematics for the twenty-first century, followed by a concluding remark.

17.2 Current State of Education for Sustainable Development in Mathematics Education

We live in the dawning of the information age, and we must ask what set of skills will be most appropriate for the twenty-first century and beyond. As the complexity of daily life increases, the balance shifts in favour of skills such as critical and creative thinking. As our social interactions become more diverse, globalised, and virtual, the balance shifts increasingly in favour of collaboration and communication. Moreover, we live during an emerging climate crisis, one which has led to a growing demand for us to attend to the problems of ecological sustainability. ESD, as defined by UNESCO, emphasises students' engagement in discussion, analysis, and the application of learning and knowledge through interdisciplinary activities (Laurie et al., 2016). It mandates the use of participatory, interdisciplinary teaching and learning methods. The objective is to promote competencies such as critical thinking, imagining future scenarios and making decisions in a collaborative way, with the aim of empowering learners to take informed decisions and responsible actions for environmental integrity, economic viability, and a just society, for present and future generations, while respecting cultural diversity (UNESCO, 2012).

While there is general agreement on the benefits of a sustainable mathematic education (e.g. Renert, 2011), there is a lack of clarity on what it is or what it looks like in the twenty-first century (Gellert, 2011; Li & Tsai, 2022; Petocz & Reid, 2003). Research has reported that incorporating ESD into mathematics education is proving difficult because lessons that involve sustainable development discussion and interdisciplinary activities are time-consuming and they can be challenging to teach, even for experienced teachers (Li & Tsai, 2022). In addition, the widespread interest in ESD has led to different terms and concepts being used to express the idea of sustainable development in students' learning. Researchers across a wide range of subjects (including mathematics) understand the concept of ESD in the curriculum in a variety of different ways (Petocz & Reid, 2003). The integration of ESD into the teaching and learning of mathematics is, however, more controversial—for example, ESD integration is especially problematic given that the UNESCO definition includes objectives whose implementation is the subject of contentious political debate, objectives such as the reduction of poverty and the establishment of equity and social justice.

ESD, as defined by UNESCO, is an interdisciplinary approach to learning and teaching that develops students' knowledge in collaborative and novel ways with the aim of empowering them to see sustainable development as a way of thinking about the world, as a way of guiding their actions and decision-making processes—as far as ESD is concerned, the development of proficiency in mathematics is not the objective. However, in school mathematics, national curriculum assessments play a powerful role in providing criterion measures of attainment for both students and schools, and results are often used by policymakers for school accountability purposes (e.g. Department of Education, 2020). Moreover, enhancing academic outcomes by "teaching to the test" in mathematics education has become an increasingly common phenomenon in many education systems across the world (Tsai & Li, 2017).

Yaro et al. (2020) state that the teachers in their project would discount or downplay the mathematical tasks for peace and sustainability, as they viewed those tasks "as outside the realm of formal school sanctioned activities" (p.227). Indeed, teachers may be concerned by the social justice component of ESD, by the requirement to pursue an interdisciplinary approach rather than a teacher-centred approach, by the need to gain subject matter expertise in subjects other than mathematics, and by the necessity of omitting some existing material from the current mathematics curriculum if time constraints are to be satisfied. The disparity between the formal school curriculum and ESD approaches invites me to examine whether a sustainable mathematics education is acceptable, or is indeed necessary, and, perhaps, to rethink what a vision for mathematics education should be in the twenty-first century.

How might mathematics contribute to our understanding of, and our responses to, sustainable development challenges? Earlier, Gellert (2011) calls for "new mathematics to improve our perception, control and regulation of the problematic situation" (p.20). More recently, Barwell and Hauge (2021) point out that critical mathematics education research has paid little attention to questions of environmental sustainability and contend:

> [F]or mathematics education to adequately address issues like climate change, ideas from critical mathematics education need to be supplemented with a theorisation of the nature of science and its role in society in the context of complex environmental problems such as the threat of climate change. (p.169)

Therefore, as a starting point for thinking about the connection between mathematics and climate change, Barwell and Hauge propose a set of principles for teaching mathematics in the context of climate change based on critical mathematics education and on the theory of post-normal science, as shown in Table 17.1.

Li and Tsai (2022) point out that, at present, the integration of ESD into mathematics education is the exception rather than the rule and suggest that one reason for this hesitancy is that there are no existing philosophic theories for doing so, as ESD integration would require a redefinition of the scope of mathematics. It is thus worth revisiting the questions of what mathematics is and what it means to understand mathematics.

Table 17.1 Principles for teaching mathematics in the context of climate change

Forms of authenticity	Students should have opportunities to use problems about climate change that students find relevant in their lives Students should have opportunities to work with real data as much as possible Students' own ideas and values should have a central role Students should have opportunities to engage in meaningful debate relating to climate change
Forms of participation	Students should participate in mathematics Students should actively participate in their classrooms Students should actively participate in their communities Students should actively engage with and participate in public debate
Reflecting on and with mathematics	Students should have opportunities to reflect on how mathematics is useful Students should have opportunities to reflect on the limits of mathematics Students should consider the role of values in mathematics

Adapted from Barwell and Hauge (2021), p.177

17.3 What Mathematics Is and What It Means to Understand Mathematics

What is mathematics? It is hard to answer this question since there are a variety of philosophical theories that have been developed to explain the meaning of mathematics with respect to what mathematics is—for example, logicism, intuitionism, formalism, Platonism, constructivism, and Husserlian phenomenology. To logicists, mathematics, or part of it, is essentially logic (Russell, 1902). However, intuitionists contend that, rather than logic, mathematics should be defined as a mental activity (Snapper, 1979). For formalists, the subject matter of mathematics is its symbols, which are neither of logic nor of intuition (Giaquinto, 2002). Platonism is based on the idea that mathematics is "conceived of as a pure body of knowledge, independent of its environment, and value- free" (Renert, 2011, p.20). From the Platonist perspective, mathematics holds the key to certain, indubitable knowledge. However, the certainty of Platonism has been questioned by a number of mathematicians and philosophers (Ernest, 1991, 2021; Zakaria & Iksan, 2007). For example, social constructivists have argued that mathematics is the theory of form and structure that arises within language and mathematical applications play an integral role in human social life.

Husserlian phenomenologists consider mathematical experience a constitutional explanation that analyses the ideal structures involved in our meaningful experience of the world. Hartimo (2006) highlights that "to Husserl, investigating the origin of the concepts of number and multiplicity means giving a detailed description of the related concrete phenomena and the process of abstraction from the concrete phenomena" (p.329). Phenomenologists in mathematics education do not attempt to start from individual components and posit any further entities but analyse the perception for conscious representation of a spatiotemporal world. In Husserl's *The*

Crisis of the European Sciences and Transcendental Phenomenology published by the Cambridge University Press (Moran, 2012), Husserl is concerned over how the scientific application of formal mathematics had changed the very conception of modernity and argues that the world for us all should be one that allows for mutual understanding, for action, and for the development of communicative rationality.

Ernest (2020) asserts that the philosophy of mathematics has largely concerned itself with pure mathematics, although, in recent years, it has started to focus on the issues associated with applied mathematics. Indeed, some researchers, such as Hardy (1967), focus on pure mathematics, an abstract neutral subject that has value independent of possible applications. However, others, such as Skovsmose (2019), have focused on applied mathematics and the impact that mathematics has on human life. Pure mathematics consists of abstract mathematical objects and is essentially an art, a creative process, one in which its learners have to envisage what putative sequence of steps might lead from a set of premises to a conclusion—deductive reasoning plays a role equivalent to the artist's paintbrush. Nevertheless, this perspective, influenced by Platonism, considers neither the methods used to teach mathematics nor the role of mathematics in modelling physical, environmental, and ecological processes (Li & Tsai, 2022). For example, almost everything taught in school mathematics—from arithmetic to algebra, to differential calculus—involves to some extent the use of deductive reasoning, a skill that is useful for decision-making in general. However, too often the focus is on memorising the steps required to solve a particular class of problems—without an appreciation of the modelling of real-world scenarios.

Skovsmose (2019) offers a performative perspective on the philosophy of mathematics. Ravn and Skovsmose (2019) have developed the position that mathematics is performative in the sense that its adoption and application by human beings can inform decision-making, leading to altered political priorities and transformed social realities. The mathematical modelling of real-world scenarios encourages a careful consideration of what factors are relevant and which ones are more or less important—it discourages decision-making driven by emotion and impulsive, instinctive patterns of thought. The use of mathematical modelling and measurement is performative, and through its application, mathematics plays an integral role in human social life (Ernest, 2020; Jankvist & Niss, 2020; Hagena, 2015; Skovsmose, 2019). Nevertheless, such an integral role of mathematics in the society has not yet been widely accepted. For example, in a TV show, a mathematician was asked: "What is the point of your practice?" and "What are your social responsibilities?". The mathematician answered:

> The point of mathematics is to pursue mathematical truth. This is truth about the perfect world of mathematical objects, which are timeless, perfect and exist in their own realm, untouched by material changes and decay. The truth we pursue can be exquisitely beautiful, but we are merciless in our rigour and only the most robust proofs can be accepted as demonstrating the truth of our results. Our responsibility is to pursue mathematical knowledge and beauty beyond all earthly bounds, knowing that we enrich human culture inestimably. Although our responsibilities end with the pursuit of mathematics for its own sake, it often turns out that the most abstract theories have surprising and very fruitful applications in science, technology and in modelling the physical world. But this is a happy

> accident, discovered only after we have revealed the truth. We pursue truth and beauty, but the only good we have to worry about is being good *mathematicians*. (cited from Ernest, 2021, p.3)

Ernest (2021) prompted by the mathematician's interview above and raised questions as to "Does mathematics only have to concern itself with truth and beauty and not with the good of humanity in a broader sense? Has mathematics no social responsibilities, no debt to the society that nurtures and funds it? Is ethics irrelevant to pure mathematics, as many mathematicians think?" (p.3). Ernest's questions draw my attention to the important question of what social responsibilities mathematics should have in the society, which I will discuss in more detail below.

17.4 Equity, Social Justice, and Mathematics Education

Equity and social justice are core components of ESD. However, as Naresh and Kasmer (2018) argue, "mathematics in school settings is mostly perceived and presented as an elite body of knowledge stripped of its rich social, cultural, and historical connections" (p.309). Berry (2018) used an analysis-critical race theory as a lens to critically examine policies and reforms in mathematics education in the US over the past few decades and concluded:

> Economic, technological, and security interests were, and continue to be, drivers of many policies and reforms. These policies and reforms situated mathematics education in a nationalistic position of being color-blind, in a context where race, racism, conditions, and contexts do not matter […]. Despite the evidence that racism and marginalization exist in schools and communities, many still adhere to the belief that color-blind policies and pedagogical practices will best serve all students. (p.16)

A recent report published in 2018 by the Social Mobility Commission in the UK revealed that "the attainment gap [in mathematics] widens as disadvantaged children fall further behind" (Social Mobility Commission, 2018, p.6). Research on the links between post-16 students' earnings progression and social and economic development has shown consistently that a disproportionately high number of disadvantaged students experience poor earnings progression (e.g. Department for Education, 2018). This inequality is concerning, especially as it can increase the risk of violent conflict (Montacute, 2020; Poverty Analysis Discussion Group, 2012). The Smith review published in 2017 by the Department of Education in England has confirmed that mathematics has a uniquely privileged status in society which has a big impact on future earnings:

> There is strong demand for mathematical and quantitative skills in the labour market at all levels […] Adults with basic numeracy skills earn higher wages and are more likely to be employed than those who fail to master basic quantitative skills. Higher levels of achievement in mathematics are associated with higher earnings for individuals and higher productivity. (Smith, 2017, p.6)

Questions arise as to "whether mathematics education contributes to social injustices and whether equity in mathematics education is an economic necessity or a moral obligation" (Berry, 2018, p.5). Gutiérrez (2013) argues that mathematics and mathematics teaching is political, including the curriculum we choose, the activities we assign, and the education systems we organise. In line with Gutiérrez (2013), I also argue that school mathematics is political. It places people in socially valued mathematical rationalities and forms of knowing and consequently decides the idea of what mathematics is and how people might relate to it (or not). Mathematics becomes an important element of larger processes of selection of people that schooling operates in society (Valero, 2018). In addition, policies and reforms in mathematics education have been largely motivated by the desire to compete in a global economy, which also reinforces the privileged status of mathematics in society and promotes a labour market that would only benefit a select few.

Tsai and Li (2017) note that "curriculum reforms have been one of the most common approaches adopted by policy makers when trying to promote change and improvement in school mathematics programs" (p.1264). This may thus lead to the situation in that the mathematics content that was taught, and the methods used to teach in schools were closely connected to standardised tests. Partly because of this, there would be little/no scope for teachers to introduce concepts/topics that are not examined in the standardised tests in school—for example, Schoenfeld and Kilpatrick (2013) argue that the likelihood of implementing inquiry-based curriculum in the US was small due to the considerations of preparing students for particular tests in the different states.

In mathematics education, the concept of increasing equity within mathematics education is not new. As called by Gutstein et al. (2005), "each of us has a responsibility to both think about and act on issues of equity" (p.98). Ernest (2021) also contends that ethics is widely perceived as irrelevant in mathematics. He examines the role and need for ethics in mathematical practice from some ethically sensitive areas and problematic categories with respect to mathematics and its applications, and then concludes that "we mathematicians have a vital role to play in keeping governments and corporations ethical and honest" (*ibid*, p.34). In recent years, the discourses in combating injustices and creating a just society have been working to move mathematics education research "from equity as choice to equity as an intentional collective professional responsibility" (Aguirre et al., 2017, p.128). Aguirre et al. (2017) identify four political acts that illustrate the essential role of equity as an explicit responsibility of mathematics education researchers. They are as follows: (1) *enhance mathematics education research with an equity lens*, (2) *acquire the knowledge necessary to do genuine equity work*, (3) *challenge the false dichotomy between mathematics and equity*, and (4) *expand the view of what counts as "mathematics"*.

In ESD, it aims to draw attention to the purpose of the world's education and moves the aim of education from simply a matter of training people for the global economy to engage citizens in and for society. As Khan (2014) suggests, "it [ESD] supports the acquisition of knowledge to understand our complex world; the development of interdisciplinary understanding, critical thinking and action skills to address these challenges with sustainable solutions" (p.11). Thus, ESD calls for an interdisciplinary approach and a reorienting of the existing disciplines and pedagogies to motivate people to "become proactive contributors to a more just, peaceful, tolerant, inclusive, secure and sustainable world" (UNESCO, 2014, p.15). Therefore, in the next section, I will look at research on interdisciplinary learning and STEM education in mathematics education, and then draw attention to the question of whether the existing philosophical views of mathematics can be applied to understand the role of mathematics in interdisciplinary learning.

17.5 Interdisciplinary Learning, STEM Education, and Mathematics Education

In mathematics education, as Williams et al. (2016) note, interdisciplinary learning is "a relatively new field of research in mathematics education, but one that is becoming increasingly prominent internationally because of the political agenda around Science Technology, Engineering and Mathematics (STEM)" (p.1). Over the past few decades, studies have highlighted the benefits of interdisciplinary STEM learning for transferring what students learn, for improving problem-solving abilities, for developing STEM knowledge in more flexible and novel ways, and for promoting a better understanding of STEM ideas in real-life situations (e.g. Hobbs et al., 2019; Nakakoji & Wilson, 2018; Saavedra & Opfer, 2012; Williams et al., 2016). Saavedra and Opfer (2012) state:

> Students must apply the skills and knowledge they gain in one discipline to another and what they learn in school to other areas of their lives. A common theme is that ordinary instruction doesn't prepare learners well to transfer what they learn, but explicit attention to the challenges of transfer can cultivate it. (p.10)

It is also worth mentioning that there is an increasing awareness of out-of-school, university-led programme value in enhancing student interest and understanding of STEM and its applications (Baran et al. 2019). Jensen and Sjaastad (2013) argue that secondary school students who had opportunities to attend out-of-school STEM programmes at local universities or further education colleges helped to increase their awareness of STEM careers and could picture themselves as scientists. To further illustrate the role that universities can play in promoting interdisciplinary STEM learning for secondary school students, in the following, I will describe some preliminary findings from a pilot study of my own STEM and Sustainability (STEMS) project for students (ages 15–18) in Scotland.

In this STEMS project, the contents of all of the workshops were designed by me and colleagues at my university, in discussion with the workshop instructors before implementing each of the workshops. All the workshops were conducted on Saturdays at a university. Students volunteered to attend this project. There were no selection criteria in recruiting workshop participants, but priority was given to students from disadvantaged backgrounds. The results of the student pre- and post-project questionnaires indicated that there were statistically significant differences in students' understanding of sustainability and in their attitudes towards STEM subjects after they attended the project, with a higher mean score in the post-project questionnaire. Table 17.2 shows the main themes and activities of the workshops. Examples of student feedback on each activity are also presented.

Due to the scope of this chapter, I will not describe each of the workshop activities here, but it is worth discussing some students' feedback on this workshop activity: "capture-mark-recapture discussion and card game". The activity consisted of (1) discussing biodiversity and the future of species issues, (2) understanding the proportion concept for the capture-mark-recapture method, (3) considering statistical assumptions for the capture-mark-recapture method (e.g. why is the result

Table 17.2 The main themes and activities of the workshops and student feedback

Theme	Main activity	Examples of student feedback
Theme 1 The volume of life on Earth – Biodiversity	Global goal string activity Capture-mark-recapture discussion and card game Carbon cycle game	"All goals are connected and everything impacts one another" "It was good to learn about sampling techniques and its practical use and its pros and cons depending on its use in different situations" "It was a fun and smart way to teach the carbon cycle"
Theme 2 A renewable, biodegradable fuel – Biodiesel	Bioenergy crossword game Controversial discussion: Jatropha – a solution or a false hope? Using graphs to represent and interpret biodiesel production	"I enjoyed crossword most as I was good as it" "A good debate. [it's] interesting and encouraged me to think about the topic in detail "Recapped my knowledge of drawing graphs – I further developed my knowledge of Jatropha and biodiesel"
Theme 3 A clean source of renewable energy – wind power	SDG 7 affordable and clean energy: why it matters? Benefits and drawbacks of offshore wind farms Wind turbines: how many blades experiment	"Clean energy is important as we need to think about the effects and impact on the world because of it" "Clean source of power, but it can 'ruin' landscapes and affect biodiversity" "It allowed me to problem solving and learn about the physics behind the wind power generation methods"
Theme 4 Mathematics and climate changes – what the mathematics is telling us	Climate changes art gallery How high school mathematics gives insights into climate changes Relay game: what students have learned from the workshops	"Recalled climate changes knowledge, but don't like drawing" "I learnt more about mathematical concepts and know climate changing is happening at a faster rate than predicted" "Revisited past workshops to revise knowledge – team building games are fun"

unlikely to be exactly true?), and (4) using the card game to help students know how to do the capture-mark-recapture method. Two examples of student feedback on the activity that were collected from their post-workshop interviews are presented as follows:

> I was surprised to know that it [the answer] is unlikely to be exactly true. I thought the maths would always give you the correct answer.

> The only real…real understanding I had of sustainability before was from a biology lesson years ago. This [Capture-Mark-Recapture discussion and card game] really helped me to understand it and also engage with it with some practical exercises and ideas which I found very interesting and enjoyable.

From the student feedback above, it shows that the practical exercises (e.g. card games) and interdisciplinary approaches had really helped them to understand sustainability, although they had been taught about it in their biology lesson before. The student feedback above also made a reference to "I thought that maths would always give you the correct answer". While it is not surprising that students were not aware of the uncertainty of mathematics, the results of the student interviews suggested that, partly because little statistics was taught to them or statistical knowledge was something that was crammed into their heads in school mathematics, they did not understand the concept that underlies the operation when performing the capture-mark-recapture method, even those who were able successfully to perform the operation. It is also worth mentioning here that, while statistical data is presented in the news on a daily basis, error bars around the data points are rarely included on the grounds that most people would not understand their significance. Similarly, when a change in some quantity—be it a car accident or the COVID case rate—is reported, there is no accompanying statement as to whether the change is statistically significant on the grounds that most people would not understand the concept. It may therefore be difficult for people to come to appreciate that measurement without an associated uncertainty is often of little value.

As I have mentioned earlier, in line with other researchers (e.g. Gutiérrez, 2013), school mathematics is political. Thus, in typical school mathematics, certain branches of mathematics are overrepresented (e.g. algebra) and others underrepresented (e.g. approximation, statistics, probability, discrete mathematics, and fractal mathematics). Take approximation and discrete mathematics as examples; when mathematics is applied to solve real-world problems, it invariably involves the use of approximation. In addition, discrete mathematics gives approximations for the size of some measurements. Students should come to understand that approximations produce inexact results and sometimes unreliable results. Arguably, discrete mathematics and approximation should be taught as a part of mathematics. Regarding statistical teaching in schools, Batanero et al. (2011) argue that "many teachers unconsciously share a variety of difficulties and misconceptions with their students with respect to fundamental statistical ideas" (p.409) because they do not actually teach much statistics and rarely use statistics to analyse data.

This is cause for concern as, if students in their education were routinely taught some basic statistics, then news feeds would be enhanced and students would have the capacity to make informed, data-driven decisions—and, more importantly, they would develop a readiness to question the validity of data-based assertions unaccompanied by a statistical justification. There is little doubt that most topics in schools help to teach deductive and inductive reasoning. However, it is arguable that the typical school-based mathematics curriculum tends to view students' learning from the narrow sense of the scores achieved, rather than from the standpoint of educational development such as transfer of learning. It is therefore not surprising that mathematics teaching has often been criticised for being teacher-centred, content-oriented, and academic-driven (e.g. Petocz & Reid, 2003) and students nowadays are highly dependent and passive and have difficulty in constructing meaning and understanding from their learning.

Like school mathematics, I also argue that STEM education is political too. As Williams et al. (2016) describe, "in almost all countries now politicians see education in terms of preparation of a workforce for a competitive industrial sector, and STEM is seen as the route to more value-adding industries, especially in knowledge economies" (p.1).

There is general agreement among the majority of the general public that the development of mathematical skills plays an essential role in helping the younger generations function effectively in today's technological society and, hence, that the acquisition of mathematical skills contributes to economic growth and prosperity—for example, the deductive reasoning skills developed by solving mathematical equations are essential when it comes to writing and debugging computer programs. However, several mathematics researchers have raised concerns that this focus on using mathematics for problem-solving is too narrow (e.g. Berry, 2018; Ernest, 2020; Gellert, 2011, Gutstein, 2012, Martin, 2015, Skovsmose, 2019).

Gellert (2011) contends that "thinking of mathematics only as a powerful tool for solving economic problems is a truncated conception of mathematics-in-society" (p.20). Swanson (2019) points out that "a common theme which emerges from the case studies is that of a potential for mathematics to disappear, or to become a mere tool, within such [STEM] activities" (p.157). The National Council of Supervisors of Mathematics (NCSM) and National Council of Teachers of Mathematics (NCTM) in the US have also highlighted that although it is recognised that all the subjects within the STEM education are important, it is necessary to affirm "the essential role of a strong foundation in mathematics as the centre of any STEM education program" (NCSM & NCTM, 2018, p.1).

A question arises as to whether the existing philosophical views of mathematics can be applied to understand the role of mathematics in interdisciplinary learning. This question remains unanswered. It is thus important for us to think about how mathematics interrelates with the other disciplines and contexts involved in the context of interdisciplinary learning. In particular, if ESD is to be integrated into mathematics education, we need a new philosophy to address questions as to what mathematics can help people gain insights into the role that mathematics plays in shaping human society and how mathematics can help people take up sustainable development challenges.

17.6 A Call for Reconfiguring and Rethinking the Philosophy of Mathematics for the Twenty-First Century

How sustainable is mathematics? It is still unclear—when it comes to classroom practice—what ESD in mathematics education should look like in the twenty-first century. As argued by Barwell (2018), "there has been little sustained mathematics education research" (p.156). My review of ESD-related studies has confirmed these findings: I found that the number of studies in ESD relating to mathematics in general, and to mathematics education in particular, is few and disproportionate to the number of studies that document problems with its implementation—problems for which solutions are needed urgently. Over the past few decades, while there has been substantial preparatory work when it comes to adapting ESD for disparate subjects and education systems (Laurie et al., 2016), there has been limited progress when it comes to embedding such work within school curricula (Hunt et al., 2011; Summers, 2013). Furthermore, research shows that the quantity and quality of ESD provision in teacher education have been "patchy" (Bamber et al., 2016).

In the field of mathematics and mathematics education, it is widely accepted that pure mathematics is neutral and value-free; however, the situation becomes more nuanced once mathematics is applied to real-world scenarios. Ernest (2020) notes that "one of the traditional problems of the philosophy of mathematics is the question of how wholly abstract mathematics can have any effect on the world" (p.79). As discussed in previous sections, traditionally, mathematics was taught as a static body of knowledge and unquestionable truth. However, it has in the last century been questioned that there can be no complete provable body of mathematical knowledge, no unification of what is provable and what is true (e.g. Gödel's incompleteness theorem). It is difficult to understand how the uncertainty of mathematics has surfaced in the philosophy of mathematics education.

In Husserl's later work, he claims that his idea of life-world can be used as a fundamental and novel phenomenon previously invisible to the sciences. Husserl insists:

> There is a specific and entirely new science of the life-world itself [..] that would, among other things, offer a new basis for grounding the natural and human sciences. There never has been such an investigation of the lifeworld as subsoil (Untergrund) for all forms of theoretical truth […] The life-world demands a different type of investigation that goes beyond the usual scientific treatment of the natural or human world. It must be descriptive of the life-world in its own terms, bracketing conceptions intruding from the natural and cultural sciences, and identifying the 'types' (Type) and 'levels' (Stufe) that belong to it (cited from Moran, 2012, p.223)

In line with Husserl's idea of the life-world, I also argue that there is clearly a need to demand a different type of investigation that goes beyond the existing philosophies of mathematics and mathematics education so that it can align more closely with twenty-first--century learning priorities.

In the previous section, I have shown that the interdisciplinary ESD approaches can assist the sustainable development movement by cultivating in students the skills needed in mathematics as well as across different subjects/disciplines to make informed judgments about sustainable development when faced with contradictory information, data, and opinions. I believe that the integration of ESD into mathematics education has the potential to make a significant contribution to the sustainable development movement. To this end, I argue that the philosophy of mathematics and mathematics education needs to be reconfigured and re-envisioned so as to accommodate social, economic, and environmental dimensions. Topics that would help students to acquire relevant knowledge, to practice critical thinking, to manage uncertainty, and to act in a measured and responsible manner when faced with a pending crisis should also be included in school mathematics, with the aim of better preparing students to address the global issues that they will face during their lifetimes.

17.7 Concluding Marks

This chapter by no means claims to be a fully comprehensive study of how the philosophy of mathematics and mathematics education should be reconfigured or re-envisioned in order to appropriately integrate the relevant aspects of ESD into mathematics and mathematics education. However, its discussions may offer relevant information on rethinking the existing philosophies to respond to a growing demand to integrate ESD into mathematics teaching and learning.

As I have discussed in previous sections—the “teaching to the test” culture in mathematics education, the dominant philosophy of pure mathematics and the issues in mathematics education regarding equity, social justice and interdisciplinary learning—all of which have shown that the incorporation of ESD, as defined by UNESCO, into mathematics education is far from simple. Arguably, if we are to meet the current sustainable development challenges, we must broaden the focus and find a new philosophy of mathematics to incorporate ESD into mathematics education.

Over the past two decades, research has only started to address the fundamental questions facing the incorporation of ESD into mathematics teaching and learning. The work is still in progress, and there is certainly still much work to be done. This chapter may only be one amongst several attempts to achieve this goal. More research will, no doubt, contribute further to the understanding of this highly complex and demanding aspect of mathematics education and to the routine teaching of ESD within mathematics classrooms. Of course, more research is also needed to focus on the critical thinking and problem-solving skills involved in mathematical modelling and measurement, on how mathematics is applied within other disciplines, and on how mathematics provides a lingua franca that supports effective, ethical communication between disparate communities across the world.

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Part IV
Philosophy of Mathematics Education in Diverse Perspectives, Cultures, and Environments

Chapter 18
Asé O'u Toryba 'Ara Îabi'õnduara!

Antonio Miguel, Elizabeth Gomes Souza, and Carolina Tamayo Osorio

18.1 Introduction

In this chapter, we carry out a therapeutic-grammatical investigation of the philosophical problem related to the belief in the supposed uniqueness and universality of Western logical-formal mathematics and the main philosophical arguments that support it, with the purpose of deconstructing it based on a decolonial counter-argument. This counter-argument does not claim to constitute a new philosophy of mathematics that can support a new perspective of the philosophy of mathematics education, it does give visibility, in general terms, to a therapeutic-decolonial way of educating and of educating oneself mathematically in school, through the non-disciplinary problematization of algorithmic cultural practices historically invented as adequate responses to normative social problems emerging in different *forms of life*. In this sense, such algorithmic-normative practices can be seen as mathematical *language* games,[1] in Wittgenstein's sense.

[1] Wittgenstein (2017, § 7) coined the expression "language games" to refer to the totality of language and the activities intertwined with it, proposing to redirect the words from their metaphysical use to their everyday use. The expression *language-game* seeks to highlight, with the word "game," the importance of language praxis, that is, it seeks to highlight language as a theatrical and performative practice, and, with this, the multiplicity of activities in which the language is inserted. Wittgenstein's deconstructionist, nondogmatic, and non-essentialist therapeutic philosophizing

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This decolonial counter-argument highlights the claim that such practices and the ways in which they affect different *forms of life* in the contemporary world should be the focus of therapeutic problematizations of a school mathematics education that intends to be decolonial. In line with Miguel and Tamayo (2020, p. 5), we are seeing and wanting to explore here a decolonial aspect in the therapeutic attitude,

> not only because this attitude is being practiced here by us to deconstruct discourses and practices that contemporary decolonial discourse sees as colonizers, but also because we are seeing its decolonizing aspect as the purpose that guides our way of practicing it.

Methodologically referenced by the works of the Austrian philosopher Ludwig Wittgenstein, the therapeutic-grammatical investigation that we present here is characterized through a *neither this/nor that*, because it cannot be seen either as a philosophical investigation guided by a general and prescriptive scientific method – empirical-verificationist or theoretical-logical-fundamentalist – nor as proceeding in an irrational and anarchic way, since it does not allow itself to be guided either by a previously defined method, nor by a universal method that could be later described and detached to be applied in other problems. "There is not a single philosophical method, though there are indeed methods, different therapies, as it were" (Wittgenstein, 2009, § 133).

Each therapeutic investigation – even when it uses sources or data directly collected in field research, which is not the case here – invents its own ethical-aesthetic style of philosophizing and does not use such sources or data to attest or support a supposed final thesis on the investigated problem to be defended by the researcher-therapist. In fact, the ethical-aesthetic style of Wittgenstein's philosophizing makes the reader co-responsible for the developments generated by reading his work. It is Wittgenstein himself who expresses this desire, in the following way, in the preface to *Philosophical Investigations*: "I should not like my writing to spare other people the trouble of thinking. But if possible, to stimulate someone to thoughts of his own" (Wittgenstein, 2009, Preface, p. 4[e]).

Another important point to be highlighted here is that a therapeutic-grammatical investigation of a philosophical problem has nothing to do with an investigation of a psychological nature, since it operates on the "grammars" – that is, on the set of

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does not allow itself to be categorically defined even one of the key terms of his philosophizing, namely, the expression "language games." However, instead of a categorical definition, Wittgenstein provides an ostensive definition, that is, numerous examples of language games, as in passage 23 of the Philosophical Investigations: "[…] The word "language-game" is used here to emphasize the fact that the speaking of language is part of an activity, or of a form of life. Consider the variety of language-games in the following examples, and in others: Giving orders, and acting on them; Describing an object by its appearance, or by its measurements; Constructing an object from a description (a drawing); Reporting an event; Speculating about the event; Forming and testing a hypothesis; Presenting the results of an experiment in tables and diagrams; Making up a story; and reading one; Acting in a play; Singing rounds; Guessing riddles; Cracking a joke; telling one; Solving a problem in applied arithmetic; Translating from one language into another; Requesting; thanking; cursing; greeting; praying […]."

guiding rules – of the different language games made public (the different "public philosophies," in Paul Ernest's sense) in the field of scientific-academic activities.

It is therefore necessary to warn readers about the analogies that could be wrongly established between therapeutic-grammatical investigations and psychotherapies, whatever their theoretical affiliations, as well as between them and the maieutic Socratic method in the way it was didactically and cognitively practiced in Plato's dialogues, guided by the purpose of "convincing" Socrates' interlocutors – or rather, to impose them – about the essential and irrefutable truth of a certain point of view (thesis) related to the problem in focus.

Even though the written presentation of our therapeutic investigation in the form of a dialogical-argumentative text can maintain "family resemblances" (Wittgenstein, 2009, § 67) with the philosophical-empirical-fallibilist investigation practiced in the dialogical-dialectical text of *Proofs and Refutations* by Imre Lakatos, there is a subtle and radical difference between both that needs to be highlighted here. The repeated application of the confident rational method of "prove and refute" triggered by Lakatos, guided by his paradoxical purpose of "proving" the skeptical thesis that a mathematical proof never proves what it purports to prove, is in manifest inverse contrast with our "confident conviction."

In a "methodologically skeptical" therapeutic investigation – by refusing to define a method, whether rational or irrational, that can guide it – there is not exactly a thesis to be proved, defended, verified, or refuted, but only a problem to be better understood, clarified or, as the case may be, to be dissolved as a problem, through the decolonial counter-argument, case by case.

The Wittgensteinian therapeutic-grammatical perspective begins from the conception that "philosophy must not interfere in any way with the actual use of language, so it can in the end only describe it. For it cannot justify it either. It leaves everything as it is" (Wittgenstein, 2017, § 124):

> Wittgenstein's writing styles are aphoristic, non-conceptual, polyphonic, and questioning. This writing profile leads his different readers to different assertions, in a self-responsible process of understanding his writings. [...] A style of philosophizing that deconstructs images, concepts, conceptions elaborated from a fixed, dogmatic, representational, and external spectrum to the practice or phenomenon studied. The aphoristic and dialogical way of conducting this deconstruction calls to look how the practices occur. "Don't think, but look". (Wittgenstein, 2017, § 66), see how the language works? (Souza et al., 2022, p. 278)

By extension, a therapeutic investigation is not driven by the dogmatic purpose of convincing readers of the reasonableness or truth of a point of view over the others that are manifested in the debate, nor is there a privileged interlocutor who had this kind of power.

Finally, therapeutic research, such as the one we carry out here, does not claim for itself a status of empirical-verificationist or theoretical-logical-fundamentalist scientificity. This is because, on the one hand, we do not believe that the theoretical discourses produced in the field of philosophical investigation – for example, philosophies of mathematics, philosophies of education, and philosophies of mathematics education – are endowed with the pretentious imperialist power of grounding, to validate or rectify the effective ways of practicing mathematics and mathematics education in different fields of human activity, including the field of school education.

On the other hand, we think that the greater or lesser social and political relevance, credibility, acceptability, effects, and affects of philosophical discourses reside in their performative powers to modify habits, customs, beliefs, ideologies, and behaviors, both on a personal and social-institutional level.

Hence, the choice we made to present our philosophical investigation through a therapeutic-dialogical-problematizing writing endowed, in our view, with performative power in relation to standard conventional presentations, seen as "scientifically correct or acceptable" by the scientific-editorial field of the contemporary academic world. "Hence our choice for what is in front of our vision, without worrying about what is supposedly hidden. These voices always tell us – "Look! – Look!" (De Jesus, 2015, p. 52).

We choose six persons will participate in this debate: Oiepé, Mokoi, Mosapyr, Irundyk, Mbó, and Opá kó mbó. Their names correspond, respectively, to the numerals one, two, three, four, five, and ten in the ancient Tupi language, today considered one of the two most important linguistic branches of the hundreds of different languages currently spoken by indigenous communities in Brazil. Other remote interlocutors are invited to participate in the debate, or they enter the room uninvited. They are identified by acronyms formed by the first letters of their names and surnames.

Although several public and published points of view related to the problem we investigated have been presented, defended, or refuted by the different face-to-face or remote interlocutors participating in the debate, none of these points of view – among others, those of Hogben, Aleksandrov, Lakatos, Perminov, and Raju or even those of Wittgenstein himself – could be attributed to us as researcher-therapists or authors of this collective text, which does not mean that each of us, as individual researchers, did not make different decolonial choices to conduct our research, teaching and teacher formation activities.

The purpose of a therapeutic-grammatical investigation is the rejection of every form of dogmatism. Therefore, this is not a text that dogmatically defends a formalist, fallibilist, ethnomathematics, decolonial or any other point of view in relation to the investigated problem, even though the indigenous interlocutors who discuss this problem are undoubtedly committed to an anthropophagic-decolonial critique regarding the belief in the uniqueness and universality of western logical-formal mathematics and the main philosophical arguments that support it.

It does not, however, prevent them from disagreeing with each other, whether in relation to various aspects of this criticism or in relation to aspects of a possible therapeutic-decolonial proposal to educate and to educate oneself mathematically, at school, through the nondisciplinary problematization of normative cultural practices. In fact, in our therapeutic-dialogical-problematizing text, we try to take care as much as possible not to mischaracterize the principles that characterize a therapeutic-grammatical investigation.

18.2 Therapeutic Debate

Irundyk – I think there is a common belief that has guided Western school mathematics education, namely, to assume as true and, therefore, as potentially applicable only those propositions that have been logically-deductively proved. It is, therefore, the value and power attached to logical proof that imparts to the proposition the status of truth, so that such a belief produces its own picture of the truth of a proposition as always being relative to the deductive system to which it belongs, without coming into conflict with the others.

Mbó – It's true! It is this belief that has been the engine of the entire history of colonizing Western mathematics education…

Mokoi – If the East is, in fact, an invention of the West, as argued by Said (2003), I think that such a belief could also be extended to the global school mathematics education, which is nothing but the axiomatized set of lies logico-deductively proved propositions that European colonizers told us about. This history, therefore, does not represent us, it only represents them.

Mosapyr – For my part, I think that such a belief is neither Eastern nor Western, but only colonial. It should no longer be imposed on us by school mathematics education. After all, as Lakatosian fallibilism has said, a mathematical proof never proves.

IL (Lakatos, 2015, p. 152) – It has not yet been sufficiently realised that present mathematical and scientific education is a hotbed of authoritarianism and is the worst enemy of independent and critical thought. While in mathematics this authoritarianism follows the deductivist pattern, in science it operates through the inductivist pattern.

Oiepé – I agree! For Lakatos, a proof can always be refutable through the presentation of local counterexamples – that is, those that refute one or more of its passages – or global ones, that is, those that refute the proposition itself. And even if a proof can be rectified, it will be, ad infinitum, open to logical criticism and refutation.

Opá kó mbó – Lakatos' fallibilism is nothing but an extension of Popperian fallibilism as a philosophy of science into the domain of the philosophy of mathematics. It is curious to note, however, that if Popperian fallibilism is widespread in scientific communities, Lakatosian fallibilism has never even shaken the mathematical community's confidence in the infallible performative power of a deductive proof. It seems, then, that no form of philosophical skepticism, even in the West, has merely unsettled the solidity of mathematics, usually considered universal, unique, and true.

IL (Lakatos, 2015, p. 4–5) – For more than two thousand years there has been an argument between dogmatists and sceptics. The dogmatists hold that – by the power of our human intellect and/or senses – we can attain truth and know that we have attained it. The sceptics on the other hand either hold that we cannot attain the truth at all (unless with the help of mystical experience), or that we cannot know if we can attain it or that we have attained it. In this great debate, in which arguments are time and again brought up to date, mathematics has been the proud fortress of dogmatism.

Whenever the mathematical dogmatism of the day got into a 'crisis', a new version once again provided genuine rigour and ultimate foundations, thereby restoring the image of authoritative, infallible, irrefutable mathematics, 'the only Science that it has pleased God hitherto to bestow on mankind'.

Mokoi – I think this happens because Lakatosian fallibilism is really a fragile skeptical philosophy, actually incapable to destabilize logic formal mathematics.

VP (Perminov, 1988, pp. 500–508) – Reliability of mathematical proofs can be called in question on the basis of different arguments. […] The principal argument by Lakatos, which is at the same time a proper empiricist argument, consists in stating that a mathematical proof is never liberated from the meaningful context and, consequently, from implicit assumptions. But the presence of implicit assumptions within the proof may result in its refutation by counterexamples of local or global nature. […] This argument is by all means true. But the principal problem is whether we can completely get rid of such implicit assumptions giving rise to counterexamples without total formalization of theory. Lakatos' response is definitely negative. We feel that he answers the question this way because he identifies all types of intuition with empirical intuition, and for this reason any meaningfulness is to him dangerous and undermines the validity of reasoning. In this case, any proof is not fully justified and any attempt to make it such leads to regression to infinity. Actually, the meaningfulness in mathematics differs essentially from that in natural sciences. At a certain level of evolution, mathematical proof is purified of all assumptions except those apodictically reliable ones. But this kind of meaningfulness cannot give rise to counterexamples. […] Another argument by Lakatos against the rigor of proof, which may be called methodological, procceds from the distinction between rigor of the proof and that of the proof analysis. Lakatos was convinced that by increasing the rigor of the proof analysis we always call in question what had been previously accepted as indubitable, we narrow down the ultimate justification layer and therefore reduce to the level of the unrigorous what had earlier seemed rigorous and final. […] But the possibility of conceptualizing the intuitive does not mean that we can correct or reject its content. Carrying out the logical analysis of arithmetic we do not reject the praxiologically accepted elementary truths. […] In mathematics there exists the area of ultimately valid, the area of apodictic truths that cannot be limited or corrected by logical analysis. […] Logical formalization of the theory is adequate only when it does not distort its present content. […] Lakatos' idea of the relative character of the justification layer […] proceeds from the erroneous concept of the intuitive as unavoidably invalid and subject to logical corrections. The ultimate justification layer in mathematics is only related to categorical and praxeological intuition and in the long run to fundamental categorical distinctions; it cannot be changed by means of external (logical or epistemological) criticism and in a sense remains unchanged through the entire history of mathematics. […] In his criticism of the infallibility of mathematics Lakatos resorts to another argument, namely, to the fact of historical changeability of the criteria of rigor. If the latter evolve, then the assertion of the ultimate rigor of a proof seems to lose sense. This argument is […] based on the false premise that takes for granted that new criteria of rigor are able to eliminate mathematical results obtained before their acceptance. But this is not supported by the history of mathematics. This reasoning does not take into account in a due way the logical criteria are secondary as compared to the contents of mathematics and that they are introduced only under the condition of preserving the content achieved that is ultimately based on the praxeological and categorical concepts.

Mbó – I am inclined to disagree with Perminov's critique. It is based on the postulation of the so-called "apodictic" or "irrefutable" truths – such as "logical intuitions" and "praxeological intuitions". These would compose an alleged "last instance of justification" of the proof.

Opá kó mbó – I think it would be necessary to clarify what Perminov would be understanding by "logical" and "praxeological" intuitions, claimed to be unrectifiable, as opposed to "empirical" and "conceptual intuitions", open to criticism and logical analysis and correction.

Mbó – I'll try to explain. For Perminov, "intuition" is any plausible reasoning or inference that can be true or false. They are based on implicit assumptions, sometimes legitimate and sometimes illegitimate, that are always amenable to rectification. In such cases, we should make them explicit, so that illegitimate implicit assumptions on which they are eventually based may come to light and be corrected.

Oiepé – Could you give us examples of intuitive inference amenable to rectifications?

Mbó – Empirical intuitive inferences, which are based on implicit assumptions associated with empirical inductive generalizations, are rectifiable. The Hungarian mathematician Farkas Bolyai proved, in 1832, that every polygon can be decomposed into polygonal parts that, arranged in a certain way, reproduce another polygon that occupies the same area as the first. Based on empirical-inductive reasoning, we could think that we could extend this theorem from two-dimensional space to three-dimensional space, assuming that it would also remain valid for polyhedra. But this turns out to be false, as the German mathematician Max Dehn has proved. The so-called "conceptual" intuitions are also rectifiable, that is, those that resort to methods of direct visualization of non-obvious propositions already proven within a logical-axiomatic theory, to assess their plausibility. An example would be the construction of Euclidean models by Henri Poincaré and Félix Klein to visualize non-obvious propositions of Lobachevsky's geometry. Such models are only accepted if they are compatible with the assumptions and propositions of the other system they represent.

Oiepé – And what would be the apodictic or unrectifiable intuitive inferences?

Mbó – These are the so-called "logical", "categorical" and "praxeological" intuitive inferences. The "logics" are those that are based on assumptions that make it possible to logically reconcile two or more facts that are not necessarily compatible. They decree, so to speak, compatibility. This is the case, for example, of propositions such as: "the product of two negative numbers is a positive number"; "the square root of 2 is equal to two raised to the fractional exponent ½". These propositions are inferences "decreed true" so that they do not contradict other praxiologically accepted arithmetical propositions, such as the commutative and associative properties of addition and multiplication of natural numbers and the distributive property of multiplication in relation to the addition and subtraction of natural numbers. On the other hand, the "categorical" intuitions, which are also unrectifiable, are those that express the desire shared by the community of mathematicians to preserve, in their proofs, the "empirical- perceptual obviousness" of certain evidence referring to spatial relationships between figures accessible to vision, such as, for example: "two distinct straight lines intersect at most at one point".

Mokoi – But in Riemannian geometry, two straight lines always intersect at two points.

Mbó – That's right… But for categorical intuitions to be preserved, mathematicians invent Euclidean geometric models or resort to existing models – a spherical surface, for example – in which non-obvious propositions of non-Euclidean geometries can be visualized in a Euclidean way.

Oiepé – Another example of a categorical intuitive proposition is that present in the proof of Proposition 1, from Book 1 of *The Elements*. To construct the equilateral triangle, Euclid implicitly assumes, based on an "empirical-perceptual obviousness", that the two circles with centers at each end of the side of the triangle should intersect. When Hilbert set out to rectify the logical flaws in the proofs provided by Euclid, he had to make this legitimate categorical intuitive assumption explicit, elevating it to the status of an axiom of his formalized Euclidean geometry.

Opá kó mbó – The conclusion I am reaching is that "ultimately", all these intuitions referred to by Perminov could be reduced to praxeological intuitions, the only ones about which we have not yet been clarified. And if this "intuition" of mine is correct, then why, for him, would only *some of them* be unrectifiable and not all of them? And, if they are all, where would the need to logically prove what has already passed through the sieve of the praxeologically irrefutable come from?

Irundyk – I think your questioning touches on the "Achilles' heel" of formalist mathematicians, logicists, intuitionists, conventionalists, etc. Or rather, it touches on "Hilbert's heel", or Cauchy's, or Dedekind's, or Whitehead's, or Russell's, … And let's stop right here, because this list is going forever, it tends *to infinity*. It is going to Cantor's paradise, from where all of them are supposed to have left and to where all of them would like to return to dwell for all eternity.

Mbó – That's right… What Perminov calls "intuitive praxeological propositions" are precisely the guiding rules of the ways of informally practicing mathematics by different peoples and civilizations throughout history. What we would call, for example, cultural practices and which he, referring to Kant, prefers to call "intuitions". But if, as Opá kó mbó said, they are all reducible to praxeological intuitions, either they would all be open to logical and epistemological analysis and rectification, or none would be.

IL (Lakatos, 2015, p. 5, our italics) – Our modest aim is to elaborate the point that informal, quasi-empirical mathematics does not grow through a monotonous increase of the number of indubitably established theorems but through the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations. Since, however, *metamathematics is a paradigm of informal,* *quasi-empirical mathematics* just now in rapid growth, the essay, by implication, will also challenge modern mathematical dogmatism.

YP (Perminov, 1988, pp. 500–508, our italics) – Nobody seems to defend the possibility of such a revision [of the praxiologically accepted mathematical contents or propositions that formalism formalizes] and nevertheless, in general philosophical speculations concerning mathematics we are quite willing to agree that "nothing is absolute" and, therefore, assume the principal possibility of this revision. Such a divergence between the practical attitude and the general philosophical outlook in contemporary mathematics is accounted for by a number of circumstances and in the first place by the non-critical transfer of the generally scientific methodological propositions into the sphere of mathematics. Mathematical theory, in contrast to empirical science, represents a specific artificial world with strictly defined elements and a finite number of their properties. Within this artificial world we can establish final relations subject to verifying by means of finite procedures. [...] Thus, we have all grounds to assert that the overwhelming part of mathematical results at work are actually ultimately justified in the sense that their proofs are fully guaranteed against refutation in the future, and this is true not only of specially verified formalized mathematics but also of common meaningful mathematics whose proofs are acknowledged to be sufficiently convincing.

Oiepé – I think that Perminov just echoes the formalist belief in the relevance of theory and the conception of truth as consistency within a deductive system. In such a world, a proposition can only be true if it is logically proved, after its due conceptualization and insertion into a formal deductive system. Therefore, in the so- called "scientific" mathematics, there can be no absolute truth of any isolated proposition, because the truth of a proposition is always relative to the truths of other propositions postulationally accepted as true, as well as relative to the rules of inference prevailing in formal classical logic, the only ones accepted as true. Perminov and the formal mathematicians think that they can escape the criticisms of the Lakatosian fallibilists through the procedure of total formalization of mathematical theories called "naive" or "intuitive". Thus, according to them, only adequate formalization of geometry (as, for example, Hilbert's), or Arithmetic (as, for example, Peano's or Russell-Whitehead's) could demonstrate unequivocally and infallibly the truths of geometric or arithmetic propositions.

Irundyk – Let me conclude that the criticism that Perminov refers to the Lakatosian fallibilism's desire for logical revision of mathematical propositions also applies to the logical desire expressed by Perminov, that it would be the existence and rigor of logical proof that would ensure the truth of such propositions and the trust we place in them.

Mosapyr – Totally agree! Even if Perminov defends that philosophical discourse does not have the performative power to modify the effective ways of practicing mathematics in the different fields of human activity, even if he does not explicitly claim any philosophical perspective guiding his criticism. I think that all philosophical perspectives that see practices as the last instance of justification of mathematical knowledge – and, among them, above all, the historical-dialectical materialist perspectives – paradoxically tend to consider 'mathematics' only the justificationist narratives or meta-narratives – rhetorical or formal-symbolic – about these practices. Everything happens as if cultural practices could only acquire a status of scientificity after having been enclosed by such narratives or metanarratives.

LH (Hogben, 1971, p. 3) – If we mean by science the written record of man's understanding of nature, its story begins five thousand years ago. Western science is thus a fabric to which threads of many colours have contributed before Britain, North America and Northern Europe were literate. Egypt and Mesopotamia, the Phoenician colonies and the Greek-speaking world of Mediterranean antiquity, the civilizations of China, India and the Moslem world supplied warp and woof in turn before Christendom began to make its own contribution.

LH (Hogben, 1973, Forward) – The first volume in this series, *Beginnings and Blunders*, traced the story of the skills our ancestors acquired, and the skills they left in their wake, *before science began*. This book starts with the need for a calendar to regulate the seasonal order of seed-sowing and flock-tending. For the construction of such a calendar men of the New Stone Age doubtless drew on the knowledge gained throughout many millennia from the scanning of the night skies by men who were still nomads and from the observation of the sun's seasonal changes when village life began. The art of keeping a written record of the lapse of time took shape about 5000 years ago and was the achievement of the priestly guardians of the calendar in the temple sites of Egypt and Iraq. An incidental, but not itself useful, by-product of this phase of infant science was the art offorecasting correctly the occurrence of eclipses.

Mosapyr – Note that, although Hogben recognizes the performance of astronomical practices by still nomadic human beings, he does not see such practices as scientific, given that he would only tend to attribute such a status to them when they acquire a linguistic or symbolic character.

Irundyk – I think that this way of legitimizing a practice as scientific is typically colonizing, because it ends up, by extension, exclusively empowering the different linguistic communities specialized in producing disciplinary narratives and metanarratives about practices that are effectively carried out in different fields of human activity.

Mosapyr – Furthermore, the way in which Hogben sees and characterizes science as a discourse aimed at understanding "nature", complements and explains the misunderstandings and ideological strategies triggered by the colonizing discourses of science and mathematics to empower themselves and differentiate themselves from other types of cultural practices.

Irundyk – Your clarification suggests that this way of characterizing science, at the same time that it institutes and produces an abstract, unitary and universal image of "nature", also comes to see it as an intelligible systemic totality, internally structured and teleologically preorganized phenomena that could be explained by different linguistic communities that turned to investigate and decipher this supposed universal *modus operandis* of nature. If, in the domain of the natural sciences, since Newton's *Principia*, the different scientific theories – these great and pretentious metanarratives of the *modus operandis* of nature – have taken up and continued the cosmogonies of the pre-Socratic Greek philosophers, in the domain of mathematics, this means that since Russell and Whitehead's *Principia*, the supposedly universal mathematical metanarrative of pure mathematicians or contemporary theorists does nothing more than to continue the Pythagorean-Platonic "cosmotheogony".[2]

Mosapyr – I tend to conclude that if Hogben, Lakatos and Perminov, in different ways, try to remove a supposed universal mathematics from its logical-formal cage to see it imbricated in human practices and activities, this is only to re-imprison it in rhetorical or symbolic-formal (meta)narratives cages.

Irundyk – I think, however, that none of these narratives, even when they wish to, are capable of radically breaking with the alleged universality and uniqueness of the axiomatic-formal metanarrative of Western mathematics.

Mosapyr – In addition to these colonizing critiques of the equally colonizing philosophy of mathematical fallibilism that we are discussing here, I think that Lakatos – and even less so Perminov – do not recognize the existence of mathematics other than Western logico-formal mathematics, but only defending the invariant method of proofs and disproofs to explain the way mathematics is invented and developed. Thus, even if Lakatos was indeed intent on challenging mathematical formalism, he paradoxically does nothing more than to recognize it as the unique and exclusive logic of the discovery and development of "informal and quasi-empirical mathematics", since, for him, unlike the absolute power of mere internalist criticism, other external factors – economic, technological, political, ideological, religious, legal, environmental, warlike, ethical, aesthetic, class struggles, etc. – would not play any relevant role in this development.

RCK (Raju, 2011a, p. 274–279, our italics) – Does formalism, then, provide a universal metaphysics? Now, it is an elementary matter of commonsense that metaphysics can never be universal. However, the case of 2 + 2 = 4 is often naively cited as "proof" of the universality of mathematics. This is naïve because the practical notion of 2 which derives as an abstraction from the empirical observation of 2 dogs, 2 stones etc. has nothing whatsoever to do with formal mathematics. […] The circuits on a computer chip routinely implement an arithmetic in which 1 + 1 = 0 (exclusive disjunction), or 1 + 1 = 1 (inclusive disjunction). Thus, formally, it is necessary to specify that the symbols 2, +, and 4 relate to Peano's postulates. Trying to specify this brings in the metaphysics of infinity—a real computer (with finite memory, not a Turing machine) can never implement Peano arithmetic, because the notion of a natural number cannot be finitely specified. Thus, formalism does not provide a universal metaphysics. However, the philosophy of mathematics as metaphysics, combined with the myth of mathematics as universal truth, helped to promote a particular brand of metaphysics as universal. This is problematic

[2] In this passage, Irundyk is using the word "cosmotheogony" to refer to any metaphysical narrative that, analogously to the Pythagorean and Platonic cosmogonies, explains and justifies the emergence, organization and functioning of nature – or, in this case, the functioning and justification of linguistic-symbolic systems of a logical-deductive character – based on diffuse forces and supernatural divine entities, or else, on linguistic-argumentative concepts that are supposed to be non-factual, immaterial and timeless.

because while formal mathematics is no longer explicitly religious, like mathesis,[3] its metaphysics remains religiously biased. On post-Crusade Christian rational theology, it was thought that God is bound by logic (cannot create an illogical world) but is free to create empirical facts of his choice. Hence, Western theologians came to believe that logic (which binds God) is "stronger" than empirical facts (which do not bind God). […] *The metaphysics of formal math is aligned to post-Crusade Western theology which regarded metaphysics as more reliable than physics. In sharp contrast, all Indian systems of philosophy, without any exception, accept the empirical (pratyaksa) as the first means of proof (pramana) while the "Lokayata" reject inference/deduction as unreliable. So, Indian philosophy considered empirical proof as more reliable than logical inference*. Thus, the contrary idea of metaphysical proof as "stronger" than empirical proof would lead at one stroke to the rejection of all Indian systems of philosophy. This illustrates how the metaphysics of formal math is not universal but is biased against other systems of philosophy. Now, deductive inference is based on logic, but which logic? *Deductive proof lacks certainty unless we can answer this question with certainty*. Russell thought, like Kant, that logic is unique and comes from Aristotle. However, *one could take instead Buddhist or Jain logic, or quantum logic, or the logic of natural language, none of which is 2-valued. The theorems that can be inferred from a given set of postulates will naturally vary with the logic used*: for example, all proofs by contradiction would fail with Buddhist logic. One would no longer be able to prove the existence of a Lebesgue non-measurable set, for example. *This conclusively establishes that the metaphysics of formal math is religiously biased, for the theorems of formal mathematics vary with religious beliefs*. Furthermore, *the metaphysics of formal math has no other basis apart from Western culture: it can hardly be supported on the empirical grounds it rejects as inferior!* The religious bias also applies to the postulates. In principle, a formal theory could begin with any postulates. However, in practice, those postulates are decided by authoritative mathematicians in the West, as in Hilbert's synthetic geometry. The calculus, as taught to millions of school students today, is based on the notion of limits and the continuum. As noted by Naquib al-Attas, the idea of an infinitely divisible continuum is contrary to the beliefs of Islamic thinkers like al-Ghazali and al-Ashari who believed in atomism. (In fact, the calculus originated in India with similar atomistic beliefs: that the subdivisions of a circle must stop when they reach atomic proportions.) This does not affect any practical application: all practical application

[3] According to Raju, "mathematics" is an originally Greek word that was invented by the early Pythagoreans to mean a "techne of mathesis," that is, a "technique of learning." It was later used by the Platonic Socrates, in the famous dialogue "Menon," to name and characterize his maieutic method of learning, which consisted in making the student recall knowledge that he had supposedly learned in previous lives, through a skillful rhetorical-verbal argumentation by his teacher-interlocutor. As we can see, the maieutic method was based on the Platonic theory of reminiscence, which believed in the idea of the reincarnation of the soul from the movement of renewal of the cosmos.

of calculus today can be done using computers which use floating point numbers which are "atomistic", being finite.

Neglecting small numbers is not necessarily erroneous, since it is not very different from neglecting infinitesimals in a non-Archimedean field, and can be similarly formalised, since the (formal) notion of infinitesimal is not God-given but is a matter of definition; but students are never told this. They are told that any real calculation done on a computer is forever erroneous, and the only right way to do arithmetic is by using the metaphysics of infinity built into Peano's postulates or the postulates of set theory. Likewise, they are taught that the only right way to do calculus is to use limits. Thus, *school students get indoctrinated with the Western theological biases about infinity built into the notions of formal real numbers and limits, which notions are of nil practical value for science and engineering which require real calculation.* In contrast to this *close linkage of mathematics to theology in the West*, most school math (arithmetic, algebra, trigonometry, calculus) actually originated in the non-West for practical purposes. From "Arabic numerals" (arithmetic algorithms) to trigonometry and calculus, this math was imported by the West for the practical advantages it offered (to commerce, astronomy, and navigation). This practically-oriented non-Western mathematics (actually ganita, or hisab), had nothing to do with religious beliefs such as mathesis. However, because of its different epistemology it posed difficulties for the theologically-laced notion of mathematics in the West. For example, from the sulba sutra to Aryabhata to the Yuktibhasa, *Indian mathematics freely used empirical means of proof*. Obviously, *an empirical proof will not in any way diminish the practical value of mathematics. However, trying to force-fit this practical, non-Western math into Western religiously-biased ideas about math as metaphysical made the simplest math enormously complicated*.

JF (Ferreirós, 2009, p. 380–381) – Raju thus proposes to deviate from classical logic, taking into account the empirical, in search of 'the logic of the empirical world'. Although we cannot enter into the question in any detail, let me sketch an argument that Raju does not seem to consider. As usually understood, logic is not concerned with the world, but with assertions about the world — or more generally, with representations of phenomena. Logic is not a reflection on ontological matters, but on language and representations. And when it comes to representing, it seems most natural to consider just two options: a representation can be either adequate (to some degree of accuracy) or inadequate, tertium exclusum. This way of grounding bivalent logic, by the way, has little to do with culturally charged conceptions of God or religion or the mind. (That, however, is not to say that there have been no historical connections between Western mathematics and religion; but in my view the topic should be pursued along a line different from Raju's insistence on the alleged theological basis of Western notions of logic and proof). The author asserts that abolishing the separation between mathematics and empirical science 'is fatal to the present-day (Western) notion of mathematics'. This suggests that he is not well acquainted with relevant philosophical literature, such as Quine or Putnam, or relevant historical figures like Riemann, Poincare, Weyl, or even Hilbert. The assertion is linked with his insistence throughout on considering formalism and a formal notion of proof as the quintessence of modern mathematics, but this view, perhaps natural in a computer scientist, is highly dubious and ignores too many other aspects of the discipline. […] On the basis of his twofold criticism of the ideal of proof (based on its theological underpinnings and its reliance on bivalent logic), Raju concludes that mathematics is best conceived as calculation, not proof. […] Since in Raju's view the choice of logic must depend on empirical considerations, and logic in turn determines inference and proof, he finds reason to believe that the Western separation of proof from the empirical is fundamentally wrong. Hence his preference for the traditional Indian notion of pramana, and also his insistence throughout this work that 'deduction will forever remain more fallible than induction'. In this reviewer's opinion, the argument remains far from convincing.

Opá kó mbó – I think that José Ferreirós' criticism of C. K. Raju does not hold up. When Raju proposes to deviate from classical binary logic, taking into account the empirical, in search of the "logic of the empirical world", so as not to separate mathematics from the empirical sciences, and JF opposes him with the argument that bivalent classical logic – and therefore the logic of the essentialist principles of identity, tertium exclusum and non-contradiction – is not concerned with the world, but with statements about the world – that is, with language, representations and propositions – he does nothing more than commit himself to a representationalist picture of language and a linguistic picture of logic, pictures that Wittgenstein himself, already in the *Tractatus*, was willing to abandon and which, in fact, he abandoned in his later works.

Mosapyr – I tend to agree with you, but I think that in speaking of "logic of the empirical world", Raju seems to approach Lakatos' thesis of the existence of a guiding logic for the informal practice of mathematics. As far as Lakatos is concerned, it is neither a deductivist nor an inductivist logic, but rather a fallibilist logic of proofs and refutations which is triggered to deconstruct confidence in the deductive proof. Raju also discredits both deduction and induction, although deduction, especially one based on binary logic, seems to him more fallible than induction. He even presents a potent argument that shows us where and why classical binary logic fails: "It is a well known principle of two-valued logic (which is used in the current method of mathematical proof) that any desired conclusion, whatsoever, may be derived from contradictory assumptions, which is why theologians use them so often" (Raju, 2021, p. 18).

Mokoi – To me, Raju is saying that if the first proposition of the truth table of the conditional operator of classical logic is a contradiction, that is, $p \wedge (\neg p)$, whatever the truth value (V or F) of the second proposition is, the conditional will always be true. This means that if you start from a contradiction in your premises, you will be able to conclude anything. See, then, what he means by this is that you can logically prove any fake news!!! And that you can use classical logic not only as a (theo)logic, but also as a (ideo)logic!

Mosapyr – With this, I understand that Raju's decolonial critique of Western mathematics goes beyond a properly logical or socio-anthropological-cultural critique. And even though this political-decolonial critique is made for defending mathematical multiculturalism, Raju, unlike ethnomathematicians, does not see the practice of Western mathematics as another ethnomathematics, among others. I tend to agree with him on this point, because the axiomatic-formal metanarrative of Western mathematics can never support the different ways in which mathematics is actually invented and practiced in different fields of human activity.

Mokoi – This means that it would no longer be necessary to strive for any other logic to support these different effective ways of inventing and practicing mathematics. In my view, there is not even "a logic of the empirical world" and, even less, different logics that support different mathematical practices, because a mathematical practice does not need and cannot be founded. Thus, if on the one hand, Perminov and Ferreirós believe that a logical-deductive proof in fact proves a proposition that one wishes to prove, Lakatos and Raju believe that the proof proves nothing. For Lakatos logical proof does not prove because it is always open to criticism or refutation by counterexamples, whereas for Raju logical proof does not prove because any proposition one wishes to prove can be inferred from contradictory assumptions.

LW (Wittgenstein apud Waismann, 1979, p. 33) – In mathematics there are not, first, propositions that have sense by themselves and, second, a method to determine the truth or falsity of propositions; there is only a method, and what is called a proposition is only an abbreviated name for the method.

Mosapyr – In fact, Pythagoras' proposition, for example, is nothing more than an algorithm, a technique, an "abbreviated method" to calculate the area of a square built on the hypotenuse of a right triangle as a function of the areas of the squares built on the legs. And if, as Wittgenstein says, the proposition is just an abbreviation of a method, that is, of a way of doing something, then there can be no difference between the method and an alleged "proof" of the method's effectiveness. Therefore, the alleged "proof" does not prove, because a mathematical proposition is simply an algorithm, a way of doing something that does not need to be proved or justified.

Opá kó mbó – I agree with you that there is no "logic of the empirical world" nor different logics that underlie different mathematical practices. But I disagree when you say that a mathematical proposition does not need to be grounded because it is a cultural practice, a method, a way of doing it. I think that such a practice does not need to be logically grounded; however, it needs to be somehow validated to have been seen and elected as a "good practice", that is, in Raju's terms, as a "practice that works". Raju talks about "empirical proofs" of validating a mathematical practice, and I wonder what he means by that.

AA (Aleksandrov et al., 1999, p. 61–62) – The concepts of arithmetic correspond to the quantitative relations of collections of objects. These concepts arose by way of abstraction, as a result of the analysis and generalization of an immense amount of practical experience. […] The conclusions of arithmetic are so convincing and unalterable, because they reflect experience accumulated in the course of unimaginably many generations and have in this way fixed themselves firmly in the mind of man and in language.

Mosapyr – If, for Raju, "empirical proof" means the same as "empirical-inductive proof" – since he seems to believe that ordinary induction based on generalized empirical abstractions of facts validated by reiterated experience is less fallible than non-factual logical deduction – I think that he would not only be dissolving differences between mathematics and natural sciences, but also committing himself to an untenable and repeatedly questioned empirical-inductivist philosophy of science or, at best, to a Popperian empirical-fallibilist philosophy. This would expose him, therefore, to all the criticisms that these philosophies have received throughout history, notably the one to which Frege (1960) submitted the inductivist empiricism with which John Stuart Mill (2009) tried to "found" arithmetic.

LW (Wittgenstein apud Waismann, 1979, p. 33) – Most people think that complete induction is merely a way of reaching a certain proposition; that the method of induction is supplemented by a particular inference saying, therefore, that this proposition applies to all numbers. Here I ask the question, What about this 'therefore'? There is no 'therefore' here! *Complete induction is the proposition to be proved*, it is the whole thing, not just the path taken by the proof. *This method is not a vehicle for getting anywhere*.

Mosapyr – On the other hand, it may be that by "empirical proof" Raju means "visual proofs" or static "perceptual proofs" or, more broadly, dynamic "mechanical proofs", analogous to those that peoples of antiquity produced to record graphically (in clay tablets; on the walls of temples and tombs; in leather, fabric, or papyrus; etc.), certain practices, techniques, or algorithms that have been repeatedly successful in different fields of human activity (agriculture, astronomy, navigation, construction, commerce, etc.). Such types of proofs, which I prefer to call "praxeological proofs", came, over time, to be verbally described through oral or written alphabetic languages and to receive additional support from local logical-rhetorical-verbal arguments, that is, without that they were embedded in any axiomatic-deductive system (proofs of Heron of Alexandria, Pappus of Alexandria, etc.) that logically connected them with each other.

Opá kó mbó – If this is the case, then I would tend to give credit to Raju, but on condition that a methodological-statutory difference is made between "empirical-inductive proof" and "praxeological proof". And when I speak of "praxeological proofs" it is to distinguish them from either "empirical-inductive proofs" or "probabilistic proofs" defended by historical-dialectical materialist philosophies committed to classical or asymptotic conceptions of truth – seen, respectively, as a reliable reflection or as an asymptotic approximation of empirical-natural or social facts through verbal language – either of "conventionalist proofs" committed to conceptions of truth as a consensus of expert communities, or even of "pragmatic proofs" committed to the conception of truth as accommodation of empirical-natural or social facts to human or socio-community purposes. The problem I see, for example, with Richard Rorty's postmodern pragmatism is that it continues to speak with the voice of the European imperialist colonizer who, arrogantly distinguishing and distancing himself from the natural beings and forms of life that we, human beings, constitute with them, presupposes being able to continue wishing to arbitrarily and unilaterally impose on them as many "true" discourses as there are human purposes. Thus, if Rorty no longer sees language as a "mirror of nature" (Rorty, 1981), he continues to see nature as an unproblematic multiplicity of mirrors of language: *RR* – (Rorty, 1998, our italics) – For us pragmatists, there can and must be *thousands of ways to describe things and people – as many purposes as we have relating to things and people*. But *this plurality is not problematic*, it does not raise philosophical problems, nor does it fragment knowledge. […] Reality is one, but descriptions of it are countless […], because human beings have and must have different goals.

Mosapyr – So, I speak of "praxeological proofs" to refer to cultural practices that reveal themselves as capable of providing satisfactory or adequate technologic solutions to problems that arise in different fields of human activity. I think that such proofs do not submit cultural practices to any regimes of truth, not because we could impose – supposedly à la Rorty – our purposes and desires on the natural and technological beings that participate in them, allowing or preventing the contemplation of our desires and purposes, but because I see such proofs as the last instances of justification of themselves and, therefore, not needing and not being able to be grounded. If a cultural practice has been invented and repeatedly practiced, it is because it is already an adequate response to a given difficulty and, for that reason, it functions as an *unequivocal know-how*, as a *technique* or *algorithm*, as a *norm* that, if followed strictly speaking, it must lead to the intended purpose. In this sense, any attempt to ground – logically, philosophically – a practice, or to suppose it fallible and rectifiable, shown itself as meaningless.

LW (Wittgenstein, 2017, PI-217, our italics) – "How am I able to follow a rule?" If this is not a question about causes, then it is about the justification for my acting in this way in complying with the rule. *Once I have exhausted the justifications, I have reached bedrock*, *and my spade is turned*. Then I am inclined to say: "*This is simply what I do*". (Remember that we sometimes demand explanations for the sake not of their content, but of their form. Our requirement is an architectural one; the explanation a kind of sham corbel that supports nothing).

Mosapyr – There is no *single* "logic of the empirical world", no *single* "dialectic of nature". Nor there is a "logic of praxis", as if human praxis were one and there was a single logic that guides the relationships between human beings and other natural and technological beings. We could, if we wished, speak of *logics of practices*, to clarify that each cultural practice is guided by rules of an *idiosyncratic praxeological logic*. But where would such rules come from, if not from *agreements among forms of life* manifested in the *dialectical interactions* between humans and other natural beings involved in the language games that constitute such practices, so that the purposes of each game are achieved? A *praxeological logic* is not a linguistic-propositional logic, but a logic of "this is how I act", of "this is how I *should* act", if I want to achieve *such* purpose. *LW* (Wittgenstein, 2017, PI-130, our italics) – Our clear and simple language-games *are not preliminary studies for a future regimentation of language as it were, first approximations, ignoring friction and air resistance*. Rather, the language games stand there as objects of comparison which, through similarities and dissimilarities, are meant to throw light on features of our language.

LW (Wittgenstein, 2017, PI-131, our italics) – For we can avoid unfairness or vacuity in our assertions only by presenting the model as what it is, as an object of comparison as a sort of yardstick; *not as a preconception to which reality must correspond*. (*The dogmatism into which we fall so easily in doing philosophy*).

LW (Wittgenstein, 2017, PI-107, our italics) – We have got on to slippery ice where there is no friction, and so, in a certain sense, the conditions are ideal; but also, just because of that, *we are unable to walk. We want to walk: so we need friction. Back to the rough ground*!

LW (Wittgenstein, 2017, PI-126, our italics) – *Philosophy just puts everything before us, and neither explains nor deduces anything*. – Since everything lies open to view, there is nothing to explain. *For whatever may be hidden is of no interest to us*. *The name "philosophy" might also be given to what is possible before all new discoveries and inventions*.

Irundyk – Are you trying to say that 'praxiological proof' is 'technological proof'? And that Raju, instead of trying to dissolve mathematics in the empirical sciences, empiricizing it, should, on the contrary, dissolve the empirical sciences in mathematics, normalizing them?

Mosapyr – What I mean is that the distinctions between *knowing* and *knowing how*, between *pure* and *applied sciences*, between *empirical* and *normative* sciences, between *science* and *technology*, are false. Knowledge is only knowledge if it is know- how in a normative language game that is played in a form of life, be it 'real' or 'virtual', and not in a *possible fictional* form of life. And there is no know-how that is not guided by a *technique*, *algorithm* or *invented norm*.

Oiepé – Following your line of reasoning, Mosapyr, Raju's claim that practicing a decolonizing mathematics education would consist in orienting it towards a new philosophy of mathematics – empirical-inductivist, empirical-fallibilist, pragmatic or dialectical historical-materialist – would be wrong, since you argue that a *praxeological image* of mathematics should not strictly be seen as a new philosophy. Wouldn't it, however, be a new philosophy of technology?

Mosapyr – At least it would not be a dogmatic philosophical orientation of mathematics or mathematics education. We could say that it is a non-dogmatic therapeutic philosophical orientation, since dogmatism does not combine with decolonialism.

Oiepé – Here, I wonder if what LW calls *dogmatism* is not the same as what Raju calls *theologization*. And if it seems to me legitimate to make such an identification, I could also say that what LW points out as the legitimate limit of philosophizing, so that we do not fall into dogmatism, would be exactly what Raju denounces as the removal or placement in parentheses of the "empirical world" on the part of *scholastic*, *theological* or *logical-formal* mathematics practiced by Western mathematicians. For, as LW says, when our ordinary language decides to go on holidays, it stops working on the hard and rocky soil of the praxeological "civil world", and our discourse tends to commit the greatest metaphysical blunders and follies, it loses its common sense, turning itself into a biased, partial, ideological, and theological nonsense.

Mbó – Our discourse tends to become fake news, as I prefer to say.

Oiepé – If it is indeed true that Western mathematics is a huge fake news, then, for Wittgenstein, fake news would be just a *false step in philosophizing*.

Mokoi – The conclusion I reach is the same as that of Raju: that the theological narrative of formal mathematics is not just a *false step in philosophizing*, but a false step in ideologizing… a *colonizing ideology*.

LW (Wittgenstein, 1976, p. 13–14) – I am proposing to talk about the foundations of mathematics. An important problem arises from the subject itself: How can I – or anyone who is not a mathematician – talk about this? What right has a philosopher to talk about mathematics? One might say: From what I have learned at school – my knowledge of elementary mathematics – I know something about what can be done in the higher branches of the subject. […] People who have talked about the foundations of mathematics have constantly been tempted to make prophecies-going ahead of what has already been done. As if they had a telescope with which they can't possibly reach the moon, but can see what is ahead of the mathematician who is flying there. That is not what I am going to do at all. In fact, I am going to avoid it at all costs; it will be most important not to interfere with the mathematicians. I must not make a calculation and say, "That's the result; not what Turing says it is". […] One might think that I am going to give you, not new calculations but a new interpretation of these calculations. But I am not going to do that either. I am going to talk about the interpretation of mathematical symbols, but I will not give a new interpretation. Mathematicians tend to think that interpretations of mathematical symbols are a lot of jaw-some kind of gas which surrounds the real process, the essential mathematical kernel. A philosopher provides gas, or decoration-like squiggles on the wall of a room. I may occasionally produce new interpretations, not in order to suggest they are right, but in order to show that the old interpretation and the new are equally arbitrary. I will only invent a new interpretation to put side by side with an old one and say, "Here, choose, take your pick". I will only make gas to expel old gas.[4] *Opá kó mbó* – Well… based on this speech by LW, I will then make my arbitrary *decolonial choice*: I will produce no more chatter about the foundations of mathematics.

And even less about mathematics. Because I came to the conclusion that neither mathematics nor mathematics education do not really need a philosophy…

Mokoi – I think differently! It is precisely *in order not to take the false step* that we need not a new philosophy… and, even less, an old dogmatic, theological, ideological, logical philosophy that proves to be yet *another new way of chatting*, but rather *a new way of philosophizing*… a *therapeutic philosophizing*, similar to that practiced by LW himself!

Mbó – But isn't philosophizing doing philosophy? In philosophizing, LW makes and practices a philosophy…

Mosapyr – For me, what is at stake is whether we need foundations, after all. This necessity can be seen as a false problem in philosophy. But we can, yes, practice a non- dogmatic, non-theological, non-ideological, non-logical philosophy. A philosophy without foundations, a new way of philosophizing…

[4] The way in which LW sought to challenge the fundamentalist view of mathematics in his introductory speech to the set of classes on the foundations of mathematics taught by him in Cambridge, from 1929 to 1944.

LW (Wittgenstein, 2017, PI-109) – Philosophy is a struggle against the bewitchment of our understanding by the resources of our language.

LW (Wittgenstein, 2017, PI-119) – The results of philosophy are the discovery of some piece of plain nonsense and the bumps that the understanding has got by running up against the limits of language. They a these bumps a make us see the value of that discovery.

LW (Wittgenstein, PI-124) – Philosophy must not interfere in any way with the actual use of language, so it can in the end only describe it. For it cannot justify it either. It leaves everything as it is. It also leaves mathematics as it is, and no mathematical discovery can advance it. A "leading problem of mathematical logic" is for us a problem of mathematics like any other.

Mbó – I think, then, that what is at stake here is to think about other ways of seeing and practicing philosophy, different from those that seek foundations and essences… Ways that do not tempt us to move away from the rough ground of life, from forms of life.

LW (Wittgenstein, 2017, PI-309) – What is your aim in philosophy? To show the fly the way out of the fly-bottle.

Mokoi – But LW does not show the fly how to get out of the bottle by teaching him one way or another, an old or new philosophy, but by pointing out the *ways to avoid*, signaling him how to philosophize deconstructively, in a non-dogmatic way, repeatedly diving into the river of doubt… and not being easily carried away by the current of the river…

LW (Wittgenstein, 2011, p. 8) – I must plunge into the water of doubt again and again.

Oiepé – But isn't it Raju himself who insists that we should educate and educate ourselves mathematically – in a critical and decolonizing way – by making mathematics oriented towards a new philosophy? When he says that mathematics education should not be separated from science education, what new philosophy would that be? A new scientism? A new computational pragmatism?

RCK (Raju, 2011b, p. 282–283, our italics) – This theological Western view of math was globalised by the political force of colonialism. It was stabilised by Macaulay's well known intervention with the education system, and the continued support for it is readily understood on Huntington's doctrine of soft-power. And this way of teaching math continues to be uncritically followed to this day even after independence. This is *the first attempt to try to re-examine and critically re-evaluate the Western philosophy of math and suggest an alternative to European ethnomathematics*. The new philosophy proposed by this author has now been renamed "zeroism", to emphasize that it is being used for its practical value, and does not depend upon (the interpretation of) any Buddhist texts about sunyavada. *A key idea is that of mathematics as an adjunct physical theory*. *Another key idea is that, like infinitesimals, small numbers may be neglected, as in a computer calculation, but on the new grounds that ideal representations are erroneous, for they can never be achieved in reality* (which is continuously changing). (Exactly what constitutes a discardable "small" number, or a "practical infinitesimal", is decided by the context, as with formal infinitesimals or order-counting.) *This is the* *antithesis of the Western view that mathematics being "ideal" must be "perfect"*, and that only metaphysical postulates for manipulating infinity (as in set theory), laid down by authoritative Western mathematicians, are reliable, and all else is erroneous.

LW (Wittgenstein, 2017, PI-218) – Whence the idea that the beginning of a series is a visible section of rails invisibly laid to infinity? Well, we might imagine rails instead of a rule. And infinitely long rails correspond to the unlimited application of a rule.

CKR (Raju, 2021, p. 44–45) – Though colonial education supposedly came for the sake of science most people entirely overlook the actual consequences. The fact is that after nearly 2 centuries of colonial education the net result is (a) widespread mathematical illiteracy and (b) belief in all sorts of superstitions and myths about mathematics. A typical such belief is that mathematics is universal and cannot be decolonized [...]. This total mathematical illiteracy among the colonially educated is combined with two deep-seated superstitions: (a) the superstition (most manifest in Wikipedia) that the West and only the West is trustworthy, and (b) the belief that any change from blindly imitating the West can only be for the worse. ("Doomsday awaits the unbelievers"). The colonized hence resist change. For example, a stock argument of the ignorant against change, and in favour of current math, is that "it works". But what exactly works? The ignorant don't understand how rocket trajectories, for example, are calculated. They conflate normal and formal math, the way rationalists conflate normal reason (reason plus facts) with formal or church reason (reason minus facts, faith-based reason). What works (and works better) is NOT the formal mathematics of proof but the normal mathematics of calculation (much of it imported by Europe from India for its practical value, starting from elementary arithmetic algorithms). [...] A simple rule of the thumb is that anything which can be done on a computer (such as calculation of rocket trajectories) is normal mathematics, and most practical applications of math today involve computers. The bigger problem is this: from this position of the darkest ignorance wrapped in the deepest superstition, even discussing an alternative to Western ethnomathematics is taboo for the colonized.

LW[5] – David Hilbert said that "no one will expel us from the paradise created by Cantor". I must tell you that I would not dream of expelling someone from this paradise. I would do something quite different. I would try to show him that it is not

[5] The dialogue between the specters of LW and Alan Turing (AT) that followed is a partial reconstitution of excerpts from the lectures on philosophy of mathematics given by LW in 1939, at the University of Cambridge. Such reconstitution is based on Monk (1991, p. 401–428) and on (Wittgenstein, 1998). According to Monk, the purpose of these classes was the deconstruction of the idolatry of science, since LW considered it the most salient symptom of the decadence of Western culture. Turing had also given, at the same institution, a course entitled "Fundamentals of Mathematics," which was nothing more than an introduction to mathematical logic and the technique of proving theorems of a rigorously axiomatic system. The more general context of the conversation between AT and LW that we selected here was the possibility of reinterpreting mathematics in the light of Cantor's work, which was seen by LW as a quagmire of philosophical confusion. The problem around which the conversation revolves is the role of contradiction in the calculations of logical-formal mathematics and in human activities.

a paradise – so that he would then leave on his own initiative. I would say, “You are welcome; but look around.”

LW – The mathematical problems of what have been called the foundations of mathematics have, for us, as much foundation as a painted stone supports a painted tower. *LW* – The mathematician discovers nothing. A mathematical proof does not establish the truth of a conclusion; it only fixes the meaning of certain signs. The “inexorability” of mathematics, therefore, does not consist in the certain knowledge of mathematical truths, but in the fact that mathematical propositions are grammatical. For example, to deny that “two plus two equals four” is not to disagree with a widely held opinion on a matter of fact; is to reveal ignorance of the meaning of the terms involved. Do you understand me, Turing?

AT – I understand, but I don’t agree that it’s just a matter of giving new meanings to words.

LW – You don’t object to anything I say. You agree with every word. But it disagrees with the idea you believe underlies it.

AT – You cannot be confident about applying your calculus until you know that there is no hidden contradiction in it.

LW – There seems to me to be an enormous mistake there. For your calculus gives certain results, and you want the bridge not to break down. I’d say things can go wrong in only two ways: either the bridge breaks down or you have made a mistake in your calculation – for example you multiplied wrongly. But you seem to think there may be a third thing wrong: the calculus is wrong.

AT – No. What I object to is the bridge falling down.

LW – But how do you know that it will fall down? Isn’t that a question of physics? It may be that if one throws dice in order to calculate the bridge it will never fall down.

AT – If one takes Frege’s symbolism and gives someone the technique of multiplying in it, then by using a Russell paradox he could get a wrong multiplication.

LW – This would come to doing something which we would not call multiplying. You give him a rule for multiplying and when he gets to a certain point he can go in either of two ways, one of which leads him all wrong.

AT – You seem to be saying, that if one uses a little common sense, one will not get into trouble.

LW – That is NOT what I mean at all. My point is rather that a contradiction cannot lead one astray because it leads nowhere at all. One cannot calculate wrongly with a contradiction, because one simply cannot use it to calculate. One can do nothing with contradictions, except waste time puzzling over them.

Mbó – I came to the conclusion that there is no fundamental divergence between Raju and LW, regarding the way of seeing and evaluating Western logical-formalized mathematics, said to be unique and universal: an *infinite* chatter… an *infinite* ‘meta-chatter’, empty, unproductive, useless, contradictory, ideological and colonizing coated with a surface layer of so-called “pure”, “rigorous”, “scientific”, “impartial” and “neutral” varnish. The difference between the two is consists in that Raju tends to approximate, or even to see as indistinct, scientific and mathematical practices, whereas Wittgenstein seems to make a subtle and not radical distinction between them.

Mosapyr – And in your view, what would that distinction be?

Mbó – For LW, scientific and mathematical practices seem to interact with natural beings, making them participate in language games aimed at fulfilling different purposes. In scientific language games, human participants, to achieve different reproducible social purposes, seek to interact dialectically with natural beings involved in the game, searching for patterns and regularities in the ways they behave in these interactions. In turn, in mathematical language games, human beings invent and impose patterns of regularity on other beings involved in the game, so that normative social purposes are unequivocally achieved. For both Raju and LW, mathematics actually invented and practiced in different fields of human activity cannot be seen as a transposition or application of fundamentalist logico-formal mathematics. Even because, when we participate in language games guided by normative purposes, in any context of human activity, the rules that guide our actions are not rules of formal logic, based on the principle of identity, the principle of the excluded-third and the principle of non-contradiction. They are rules or empirical statements that we invent or follow, so that the intended purposes, in each situation, are unequivocally achieved. We cannot, therefore, confuse the different normative grammars of each mathematical language game with the normative grammar of classical logic that guides the language games of western logico-formal mathematics. Hence, normativity is not synonymous with logico-formal coherence. A bridge may or may not fall, not because the algorithms or calculations used in its construction respected or transgressed the principles of classical logic, but because they were or were not followed correctly or for other unforeseen reasons. Between the well-founded or ill-founded foundations of bridges and architectural buildings and the supposed logical-formal foundations of Western mathematics there is an impassable abyss, but one that would not even need to be bridged. You can be an excellent demonstrator of theorems, but not even know how to build a toy bridge… Conversely, you can be an exceptional architect who designs and builds bridges and aqueducts – as Eupalinos did, in the sixth century BC – and not knowing how to demonstrate a theorem of the most elementary or verbally and rigorously enunciate the definition of angle. I think, then, that the central misunderstanding that runs through our entire discussion concerns the role that classical binary logic, and therefore contradiction, plays in the effective ways of inventing and practicing mathematics in different fields of human activity.

LW (Wittgenstein, PI-125, our italics) – *It is not the business of philosophy to resolve a contradiction by means of a mathematical or logico-mathematical discovery*, but to render surveyable the state of mathematics that troubles us a the state of affairs before the contradiction is resolved. (And in doing this one is not sidestepping a difficulty). Here the fundamental fact is that we lay down rules, a technique, for playing a game, and that then, when we follow the rules, things don't turn out as we had assumed. So that we are, as it were, entangled in our own rules. This entanglement in our rules is what we want to understand: that is, to survey. *It throws light on our concept of meaning something. For in those cases, things turn out otherwise than we had meant, foreseen*. That is just what we say when, for example, a contradiction appears: "That's not the way I meant it". *The civil status of a contradiction, or its status in civic life that is the philosophical problem.*

Mosapyr – I think what LW is *trying to say* is the same as Raju: that we should throw away the pure, ideal, intelligible, metaphysical and 'teleotheological' ladder that gives us access to the logical world of 'frictionless ice' and face, without fear, the problems and challenges posed by the "civil world" of forms of life, the "civil world" of human activities, the praxeological "civil world", which Raju, unfortunately, does not seem to distinguish from the "empirical world".

Oiepé – What would LW mean by the expression "civil world"? He also uses it in another passage of his work, this time in connection with mathematics:

LW (RFM-IV-2, 1998, our italics) – I want to say: it is essential to mathematics that be made *civil uses* of its signs as well. It is the use outside mathematics and, so, the meanings of signs, that makes the sign-games into mathematics.

Mosapyr – I think what LW meant by this is that a mathematical language game is not characterized by the types of objects or beings that participate in it, but by what we can do algorithmically – that is, mechanically – with them in a normative game of language. And, in this sense, knitting a blouse can be seen as a mathematical language game, even though the objects that participate in it, that is, threads, needles, etc., have nothing to do with objects or concepts of Western logical-formal mathematics. And what can legitimately characterize it as a mathematician is the existence of a knitting algorithm that, if followed to the letter, allows us to make a blouse, according to the previously planned model.

Oiepé – I see that this new image of mathematics allows for an unlimited and unusual expansion of what both Lakatos and Perminov understood by "informal mathematics", as well as what Raju calls "normal mathematics" ("mathematics plus facts"), as opposed to "formal mathematics", that is, to mathematics abstracted or independent of the facts of the empirical world. I understand that LW extends mathematical language practices or games to the entire "civil world", which I identify with the world of fields of human activity, that is, with *forms of life*.

Irundyk – I conclude from our therapeutic conversation about the possibilities of deconstructing the supposed uniqueness and universality of Western logical-formal mathematics and the philosophies that support it that an indiscipline therapeutic- decolonial way of educating and educating oneself mathematically, of researching in mathematics education and of forming educators need neither a philosophy of mathematics nor a philosophy of mathematics education.

Mbó – I disagree! We need both an *anti-colonizing philosophy* of mathematics and an *anti-colonizing philosophy* of education!

Mokoi – Don't you think, Mbó, that we would need to discuss the implications that an *anti-colonial way of philosophizing* would bring to the classroom?

TM (Macaulay, 1835) – I have travelled across the length and breadth of India and I have not seen one person who is a beggar, who is a thief, such wealth I have seen in this country, such high moral values, people of such high caliber, that I do not think we would ever conquer this country, unless we break the very back bone of this nation, which is her spiritual and cultural heritage, and therefore, I propose that we replace her old and ancient education system, her culture, for if the Indians think that all that is foreign and English is good and greater than their own, they will lose their self esteem, their native culture and they will become what we want them, a truly dominated nation. […] Higher studies … [need a] language not vernacular… What then shall that language be? One-half of the committee maintain that it should be the English. The other half strongly recommend the Arabic and Sanscrit… I have no knowledge of either Sanscrit or Arabic; but I have done what I could to form a correct estimate of their value. I have read translations of the most celebrated Arabic and Sanscrit works… I am quite ready to take the oriental learning at the valuation of the orientalists themselves. I have never found one among them who could deny that a single shelf of a good European library was worth the whole native literature of India and Arabia.

CKR (Raju, 2011b, p. 21–22) – In India, Western soft power and the colonial education project began with Macaulay in 1835. The BJP[6] election manifesto for the previous election stated that Macaulay admired Indian civilization, but wanted to "break [its] very backbone", by introducing English education. The BJP manifesto stated "India's prosperity, its talents and the state of its high moral society can be best understood by what Thomas Babington Macaulay stated in his speech of February 02, 1835, in the British Parliament. Such falsehoods do not help fight academic imperialism: a true understanding of the causes is needed to cure the malaise. Macaulay, a racist to the core, and an admirer of other racists like Locke and Hume (both of whom he cites in his infamous *Minute of 1835*), had nothing nice to say about Indian civilization or the then- prevailing system of Sanskrit and Arabic education in India.

Mbó – Help me Lord![7]: Christian-theological mathematics that *exponentially raised* to Christian-theological mathematics education resulted in the only textbook of our Western mathematics education: the Bible, that is, "Euclid"! Down with Bishop Sardinha![8]

OA[9] – Down with all the importers of canned counsciousness. The palpable existence of life. And the pre-logical mentality for Mr. Lévy-Bruhl to study. We

[6] BJP (Bharatiya Janata Party) is one of the two main political parties that have ruled the Republic of India since 2014. It is a right-wing nationalist party that from 2020 has become the largest political party in the country in terms of representation in national parliament and state legislatures

[7] The interjection originally used in Portuguese is "Cruz credo" ("Cross Creed"). Originating in Catholic- Christian culture, it literally means "I believe in the cross." It is used to express fear, disgust, or repugnance at something. (https://www.significados.com.br/cruz-credo/).

[8] This is Pero Fernandes Sardinha (1496–1556), a Portuguese theologian who was provider and vicar general in India and the first bishop of Brazil. The ship in which he was returning to Brazil on July 16, 1556 sank in the Coruripe River, in the State of Alagoas. The crew were captured by the Caeté Indians who *literally killed and ate them*.

[9] Excerpts from the *Cannibalist Manifesto* (ANDRADE, 1928) by the Brazilian writer Oswald de Andrade. Ironically decolonial, the manifesto was a reaction of Brazilian modernist artists to cultural colonization, at the same time that European modernist artists began to criticize the cultural values and standards of Western civilization. It was published in 1928, after the *Dada Manifestos* by the German writer Hugo Ball and the Romanian poet Tristan Tzara and the *Surrealist Manifesto* by the French writer André Breton.

want the Carib[10] Revolution. Greater than the French Revolution. The unification of all productive revolts for the progress of humanity. Without us, Europe wouldn't even have its meager declaration of thr rights of man. […] We never permitted the birth of logic among us. […] Down with the truth of missionary peoples, defined by the sagacity of a cannibal, the Viscount of Cairu[11]: – It's a lie told again and again. But those who cames here weren't crusaders.[12] They were fugitives from a civilization we are eating, because we are strong and vindictive like the Jabuti.[13]

Irundyk – These important clarifications helped to reinforce my point of view that a decolonizing school mathematics education does not need an anti-colonizing philosophy neither of mathematics nor of mathematics education, but only a *therapeutic- cannibalist philosophizing* that clarifies and problematizes the effects and affections – environmental, political, legal, technological, ideological, ethnic, ethical, aesthetic, etc. – of carrying out mathematical practices – that is, of language games aimed at fulfilling normative purposes – on the different forms of life in the contemporary world.

Mosapyr – This speech reverberates Oswald de Andrade's *cannibalist cry* – contained in his way of deconstructing the false binary opposition "tupi or not tupi, that is the question"[14] imposed by Shakespearean Hamlet on the colonized peoples of *Abya Yala lands*[15] – and the *therapeutic cry*[16] that LW launches against Frazer's

[10] *Caraíba* designates both one of the first indigenous communities with which the Portuguese came into contact in Brazil, and a linguistic family to which several Brazilian tribes belonged.

[11] José da Silva Lisboa, nineteenth-century Brazilian liberal economist who supported the expulsion of the Jesuits from Brazil by the Marquis of Pombal.

[12] Portuguese coin made of gold or silver.

[13] Reptile that inhabits Brazilian forests; in some indigenous cultures, represents perseverance and strength.

[14] Third sentence of the *Cannibalist Manifesto* (ANDRADE, 1928). The next sentence is: "Against all catechesis!".

[15] Term used by the Guna indigenous people – who inhabit the territories of Panama and Colombia – to refer to the continent where they lived since before the arrival of Columbus and other Europeans. "Abya Yala" comes from the words "Abe" (blood) and "Yala" (space, territory), etymologically meaning "land in full maturity" or "land of vital blood." The term is also used as an appeal and support for the autonomy and epistemic decolonization of indigenous populations, following the example of the Guna Revolution of 1925. Even after the clarification of the historiographical controversy surrounding the attribution of the name *America* to the lands of Abya Yala, it was the female version of the name "Américo" that ended up naming the new continent, as Martin Waldseemüller attests, in whose hands a letter from Bartolomeu Marchionni had fallen, telling the following *fake news* that ended up deciding the name of the new continent: "Currently, the parts of the Earth called Europe, Asia and Africa have already been fully explored and another part was discovered by Amerigo Vespuccio, as can be seen in the accompanying maps. And as Europe and Asia are named after women, I see no reason why we cannot call this part *Amerige*, that is, *the land of Amerigo*, or *America*, in honor of the sage who discovered it." (https://pt.wikipedia.org/wiki/Abya_Yala); (https://pt.wikipedia.org/wiki/Americo_Vespucci); (https://pt.wikipedia.org/wiki/Bartolomeu_Marchionni)

[16] Irundyk speaks with the conviction that the whole of LW's work – which has as its emblematic decolonial mark the therapeutic-grammatical critique he directs to the monumental work entitled

colonizing scientistic dogmatism. This *therapeutic-cannibalist philosophizing* is not a form of problematization that could be seen as ideologically or doctrinally oriented, nor supposedly neutral, pluralist or multicultural, but a problematization committed to a vital political-praxeological pan-ethics oriented towards the extinction of inequalities and discrimination and for the promotion of a dignified, democratic and sustainable life for all lives and forms of life on the planet.

Mokoi – I agree and add that from the algorithmic-praxeological perspective of our decolonial critique of Western logical-formal mathematics, mathematics in the plural are no longer seen as any kind of linguistic or logical-symbolic-formal narratives or meta- narratives and are seen as a multiplicity of algorithmic, autonomous, complete, independent and non-competing cultural practices that prove capable of solving a set of normative social problems that require such practices to function as a standard of rectification, not of themselves, but of the *iteration of themselves*. That is why such practices are not open to rectification.

Opá kó mbó – Following this line of reasoning, I could say, for example, that counting practices vary not only depending on the objects to be counted, but also on the guiding purposes of counting, on the available mediating elements, etc.: practices of counting fish are obviously distinct from practices of counting the amount of lightning that falls in a given geographic region, in a given time interval. Counting practices for the same object may also vary depending on the forms of life in which they are carried out: fish counting practices vary among different indigenous communities and may differ from those practiced by fishing industries, environmental groups, etc. All these practices also differ from those invented by the community of professional mathematicians to count finite or infinite abstract number sets, random events, etc. I remembered a speech by Olo Wintiyape, an indigenous Guna from Colombia:

OW – *(*Olo Wintiyape apud Tamayo, 2017, p. 93) – We don't have a generic standard for establishing correspondence relationships, as in Western mathematics, but we establish correspondence relationships just like you do. However, these relationships cannot be independent of the quality of the object because, in the Guna

The Golden Bough, from the Scottish anthropologist James George Frazer (1966) – can be seen not exactly as a philosophy or a new philosophy, but as a (self)therapeutic philosophizing about a set of philosophical problems, among them, the basic problem of language (Miguel & Tamayo, 2020). The core of LW's (2011) critique falls on the scientistic assumption that guides Frazer's *Whig* historiographical-anthropological narrative, which leads him to construct a false image of Nemi's practice of succession of the priesthood, which is produced through an illegitimate mechanism of 'scientification' that transforms a dogmatic- religious practice of a symbolic-ritualist nature into a scientific hypothesis open to empirical verification or refutation […] LW's objection to the Frazerian desire to scientifically explain a dogmatic-religious practice is from the the same nature as his objection to Frazer's inverse desire to dogmatically explain a scientific hypothesis for which it is possible to accumulate a set of empirical evidence that reinforces or refutes it (Miguel et al., 2020, p. 518). The colonizing character of Frazer's work can be equated with that of the British historian Thomas Macaulay, from which Frazer borrows the epigraph and literary style for his work.

cosmovision, classification is vital to know the world. This means that, even with this similarity, there are, above all, differences that prevent us from saying that our counting practices can be seen as numerical systems, just as the West understands this concept that even I would not be able to explain what it really means.

CT (Tamayo, 2017, p. 243) – Such usage follows the rules of the grammar of their culture and not the grammar of the counting practice of school mathematics. Words that describe counting practices are related to the qualities of the objects involved in counting. According to Dule grammar, a description of the quality of countable objects is a way of knowing the world. The act of telling is seen as cosmogonic, that is, as a base of historical, botanical, theological, agricultural and artistic knowledge. On the other hand, the practice of *waga* counting, as it is used in school, refers to the action of counting the number of elements in a set of objects. In other words, in several academic mathematics texts we will find that counting is carried out by successively *corresponding* to an object in a collection, a number of the natural succession.

Mokoi – What Tamayo describes also points us to the implications for human and non-human lives generated by this way of conceiving mathematics. We are experiencing the extinction of all forms of life around us, especially indigenous, riverside, quilombola, black and Amazonian life forms with all their cosmovisions. And these ways of philosophizing that I think are echoed in the transgressive legacy of LW come to *postpone the end of the world*, as Ailton Krenak (2020), a Brazilian indigenous of the Krenak people says.

Opá kó mbó – An *identity* that is neither absolute nor relative, neither local nor universal, neither true nor false, neither rational nor irrational, neither logical nor ideological, of a mathematics – that is, what characterizes, singularizes, defines, and differs from all other mathematics – it is not determining, decisively or unconditionally from the order of the historical, the spatiotemporal, the territorial, the geopolitical, the contextual, the communitarian, or from the ethno-community identity, but actually from the order of the algorithmic-praxeological, from the iterable in different temporalities, spatialities and contextualities, that is, from the order of technical reproducibility and the desire for unequivocal and unambiguous control of the actions and interactions of the participants of an algorithmic-normative language game. In fact, it is always good to remember that a mathematical practice is unequivocal, but not univocal, because it is always possible to achieve the same normative purpose through different algorithms.

Oiepé – By the way, it's always good to remember that *Joy is the casting out nines*!, as Oswald de Andrade told us in his 1928 *Cannibal Manifest* (Andrade, 1928).

Mbó – It is always good to remember that, since a calculation endowed with the power to verify the correctness of the application of another calculation is assumed, the "casting out nines rule" is nothing but a "fake news" that nothing proofs… Joy is the proof of itself!

Irundyk – It is true! But it is also true that the joy of some is almost always the sadness of the majority. Therefore, it can always be *cannibalically* evaluated and rectified on a case-by-case basis. That's why we are cannibals! *ASÉ O'U TORYBA 'ARA ÎABI'ÕNDUARA*![17]

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[17] Aphorism "We eat every joy of every day" written in the ancient Tupi language (Barbosa, 1956, p. 9). "Currently, more than 160 languages and dialects are spoken by indigenous peoples in Brazil. […] Before the arrival of the Portuguese, however, in Brazil alone this number must have been close to a thousand. In the colonization process, the Tupinambá language, as it is the most spoken language along the Atlantic coast, it was incorporated by a large part of the settlers and missionaries, being taught to the indigenous people in the missions and recognized as the General Language or Nheengatu. Until today, many words of Tupi origin are part of the vocabulary of Brazilians." Contemporary Brazilian linguists often refer to it as a large "linguistic trunk", alongside Macro-Jê and 19 other language families that do not present sufficient degrees of similarities to be able to be grouped into trunks (https://pib.socioambiental.org/pt/L%C3%ADnguas).

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Chapter 19
Mathematics Education and Ubuntu Philosophy: The Analysis of Antiracist Mathematical Activity with Digital Technologies

Maurício Rosa

19.1 Introduction

We begin by presenting what we understand by "mathematics education," because this act of educating is concerned with "[…] the developing and the nurturing for development, including in this process the ways of intervening, so that nutrition is satisfied and strengthens a direction, which is that of development" (Bicudo, 2003a, p.33, my translation[1]). In other words, we must develop ourselves to evolve as human beings, and, therefore, we understand that, initially, we need to recognize everyone as different/diverse and assume this difference/diversity without categorizing people into what arbitrarily may be considered as "normal" or even not. According to Davis (2013, p.1):

> We live in a world of norms. Each of us endeavors to be normal or else deliberately tries to avoid that state. We consider what the average person does, thinks, earns, or consumes. We rank our intelligence, our cholesterol level, our weight, height, sex drive, bodily dimensions along some conceptual line from subnormal to above average. We consume a minimum daily balance of vitamins and nutrients based on what an average human should consume. Our children are ranked in school and tested to determine where they fit into a normal curve of learning, of intelligence. Doctors measure and weigh them to see if they are above or below average on the height and weight curves. There is probably no area of contemporary life in which some idea of a norm, mean, or average has not been calculated. To understand the disabled body, one must return to the concept of the norm, the normal body. So much of writing about disability has focused on the disabled person as the object of study, just as the study of race has focused on the person of color. But as with recent scholarship on race,

[1] "[…] o desenvolver e o nutrir para o desenvolvimento, incluindo nesse processo os modos de intervir para que a nutrição se dê a contento e fortaleça uma direção, que é a do desenvolvimento."

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> which has turned its attention to whiteness and intersectionality, I would like to focus not so much on the construction of disability as on the construction of normalcy. I do this because the "problem" is not the person with disabilities; the problem is the way that normalcy is constructed to create the "problem" of the disabled person.

Moreover, this very "problem" is not with people with disabilities only, nor with black people, neither whomsoever. The problem actually relies on how we create or "calculate" the sense of normality. That is, as Frankenstein (1983) asks us, Which perspective do we build or "calculate" normality from? What interests with? Who does possibly care about this normality? Who is interested in this "calculation"? So, it is through mathematical calculation that we establish normality, but the reference to affirm what is and what is not normal is under a white, Eurocentric, male-heterosexual, and "without disabilities" standards. For example, the idea of establishing a mask of beauty through the golden ratio seems to forget the different, ignoring the massive amount of people, who would not fit in, as well as what it may mean to each of them not to fit in. We need to think about how to bring mathematics to help thinking about these issues of values, ethics, and meanings of standards.

Our focus is on educating through mathematics, on progressing as a person/people, and on understanding this mental, habitual, ideological, unconscious, or extremely conscious calculation. By "progressing," we assume the dynamic process as a human being/as human beings, not materially as within the capitalistic perspective. Therefore, progressing needs to be understood as a common good, a social good, promoting freedom for all. Freedom to think, act, and learn is a political act: such act can be intentionally practiced in mathematics classes, once we all need to progress mathematically, that is, learn to measure and, above all, progress through mathematics. That is, we should learn to measure the common good, the social good, promoting a variety of materializations of freedom(s) for all without making anyone feel out of place.

In other words, as professionals acting in the mathematics education field, we need to educate mathematically aiming at what really matters to educate through mathematics (Rosa, 2008, 2018). Thus, "mathematics," capitalized, disciplinary, powerful, unique, and finished (Rosa & Bicudo, 2018) needs to be transgressed and transformed into a "strange mathematics." We need to update our perception in order to highlight a new perception of their concept of mathematics, the one which makes sense, that is, discussing the mathematics (with lowercase letters (Rosa & Bicudo, 2018), which allow us to understand differences among people, understanding the value of the other that happens to emerge in a resistance space to prejudice, discrimination, homophobia, transphobia, and racism. According to Louro (2021, p.91, my translation[2]):

[2] "A resistência não será mais procurada apenas naqueles espaços explicitamente articulados como políticos. Por certo não se negará a importância de espaços ou movimentos que, declaradamente, se colocam no contraponto da imposição de normas heterossexuais, [brancas e androcêntricas] mas se passará a observar também outras práticas e gestos (ensaiados de outros tantos pontos) como capazes de se constituir em políticas de resistência."

> Resistance [the attempt to prevent racism, for instance] will no longer be sought only in those spaces explicitly articulated as political. Certainly, the importance of spaces or movements that, avowedly, are placed in the counterpoint of the imposition of heterosexual, [white and androcentric] norms will not be denied, but other practices and gestures (rehearsed from as many points) will also be observed as capable of constitute resistance policies.

The rehearsing can take place in education spaces with teachers who teach mathematics or who will teach mathematics in the future (initial education), in order to start philosophically "queering" the curriculum, the pedagogies, the very taught "mathematics," and the so called "one that must be taught" from nowadays. Such a rehearsing for resistance can reach and alter educational spaces through the insubordination of practices based, for example, on epistemological reflections about what mathematics is being taught, how it has been taught, and through what approach.

The point hereby is constituting a disposition for the non-conformation with certain given terms and for the refusal to adjust oneself to social impositions commonly taken as "natural." It is important to ask, for example, how can education with mathematics teachers promote reflections that support the fight against compulsory heterosexuality, misogyny, ageism, prejudice against people with disabilities, and, as the focus of this chapter, racism from the teachers? How can these reflections reach educational spaces, in order to produce an antiracist mathematics education? How to create activities, environments, and resources that come to disrupt, transgress, and provoke political and social reflections like this one? Reflections that will "ubuntu" mathematics? How to think of and foster a mathematics, that is for everyone, of everyone, with everyone, that is not done individually, that needs the other, and that needs an "us"? A mathematics that provides the understanding of the difference of skin color, race, and ethnicity, as just one more difference and that values it as one of the ways of being, in order to understand that there is no human group falling out of this these differences among their integrants.

What are those differences for? Do they really matter?

Therefore, our initial "estrangement" (or questioning confrontation) takes place as we perceive the need for a more fruitful dialogue about politics and society in the initial and continuing education with teachers of mathematics, because it seems to us that there is a *habitus* (Bourdieu, 1991a, b) of mathematics classes regarding about what should be taught, what in fact should be mathematics or what should be named mathematics and what it means in society. This *habitus* (Bourdieu, 1991a, b) presents to us a way of understanding mathematics fundamentally as a mechanical structure of calculations and exercises, of closed problems resolution, and of a method of applying formulas and having nothing to do with any social-political matters regarding sex, gender, and sexuality, as well as with disability and age, and, for matters of this chapter, with race issues. There is, in my view, a focus centered on mathematical contents historically constituted and evidenced as essential in a mathematics class, leaving aside the understanding of "why" were these "contents" picked? Where do they conduce us? Why were they constituted in the present way? Who did bring them to the classroom? And finally, and most relevant, How do I transform my mathematics class? How can I make mathematics itself strange? These considerations are what assign intentionality to this research, as they are investigated, theorized, destabilized, and possibly transgressed.

The experience with digital technologies (TD) was one of the possible paths we chose to take. It reveals different crossroads in providing huge diversity and possibilities for thinking, reflection, and criticism. From this perspective, in Rosa (2008, p.52, my translation[3]) it is already possible to glimpse the perspective that:

> […] the cybernetic world enhances the vision of modern physics of unconnected time/space, represented by Castells (2005) as a space of flows, in addition to highlighting the idea of multiple identities translated into online identities. This makes me think about the changes in conceptions of time, space, identity that have been taking place and that are now more evident with the information age, in which the Internet is a prominent actor.

In other words, Rosa (2008) and Rosa and Lerman (2011) highlight the role of DT as potentiators of identity performance, because they investigated how the construction of online identities is shown, through the RPG (role-playing game) online to the teaching and learning of definite integral (mathematical concept of differential and integral calculus), for example. In their research, they consider the performing practice with the Online RPG as "the playful process in this online mathematics education situation [, which] calls for understanding mathematical knowledge in interaction with the setting, as a social construction" (Rosa & Lerman, 2011, p.83). In this way, the construction of identities shows itself in transformation, immersion, and agency (Rosa, 2008) since these actions emerged from the identity performances unveiled in the digital environment.

In Rosa (2008), it is possible to posit that the performance, the creation, and the construction of identities both in the RPG and in the world also take place through the construction of bodies. The construction of bodies in the world is reinforced by Dumas (2019, p. 2, my translation[4]) when referring to the conceptual construction of the "black body," because, according to the author:

> The colonizers' resolution of this issue was based, in a way, on a definition of the body inventing a race, not everyone's, but the people to be enslaved. For this, the list of criteria already applied in Greece, for example, was not used. But the criterion based exclusively on the particularity of the African people: their territorial origin and the body defined by skin color, to the black phenotype.

[3] "[…] mundo cibernético potencializa a visão da física moderna de tempo/espaço não desvinculados, representado por Castells (2005) como espaço de fluxos, além de evidenciar a ideia de identidades múltiplas traduzidas em identidades *online*. Isso me faz pensar nas mudanças de concepções de tempo, espaço, identidade que vêm acontecendo e que agora são mais evidenciadas com a era da informática, na qual a Internet é ator proeminente."

[4] "A resolução dessa questão por parte dos colonizadores foi pautada, de certa forma, numa definição de corpo inventando uma raça, não a de todos, mas a do povo a ser escravizado. Para isso não foi usado o elenco de critérios já aplicados na Grécia, por exemplo. Mas, o critério baseado exclusivamente na particularidade do povo africano: a sua origem territorial e o corpo definido pela cor da pele, ao fenótipo negro."

In this way, the fabrication of bodies does not go beyond the playfulness of a game but permeates the intention of a group, its desires, and interests. According to Louro (2021, p. 80, my translation[5]):

The bodies considered “normal” and “common” are also produced through a series of artifacts, accessories, gestures, and attitudes that society arbitrarily established as adequate and legitimate. We all use artifices and signs to present ourselves, to say who we are and who others are.

However, who is interested in defining what is normal? How to define this normality? What are bodies that matter? This last question was inspired by Butler (2019) whose book has the title “Bodies that Matter: on the discursive limits of sex.”

Identities are also shaped by bodies, which are conditioned by artifacts, accessories, gestures, and attitudes, which can remarkably transgress spaces and strongly evidence freedom, that is, the meaning of politics itself (Arendt, 2002). In this bias, in terms of education and mathematics education linked to re-signifying bodies, we believe that DT can also become artifices and signs of the constitution of these bodies, present ourselves, and carry out identity performances, in order to learn from these performances, respecting differences, understanding them. Increasingly, this learning happens with DT, which are already mobile and ubiquitous (Rosa & Caldeira, 2018). Also, DT are considered to enhance the constitution of knowledge (Rosa, 2018), mainly because they present a multimodal language, which favors the feasibility and possibility of thinking of and perceiving themselves as transgressors, as enabling different imaginary, constructed, and invented realities.

In addition, we understand that mathematics education can highlight necessary dimensions for humanitarian development, both the political and social dimensions. This can be evidenced in this act of education, that is, the act of development which mathematics is taken as a reflective resource, language, and/or field of study articulated with digital technologies (DT).

Thus, we show in this chapter how mathematics education can encourage/provoke the understanding/constitution of the social responsibility of students in the face of social issues, such as structural racism, which consistently permeates the majority of our realities, including the educational reality. In this way, we analyze an antiracist mathematical activity with digital technologies that discusses the diversity of skin colors as something that belongs to each one and everybody at the same time, as a structure that connects us. Regarding this, we use the African philosophy Ubuntu, which does not conceive the existence of a being independent of the other, but of a “being” that thinks, acts, and lives with others, *be-ing-becoming*, that is, a *be-ing-becoming* that promotes a transformation in reality from its agency with others, with nature, with life. For some people, the central idea of this philosophy may seem to be ignoring human individuality, focusing efforts on the social, and disregarding subjectivity. But that’s not what happens, the Ubuntu philosophy

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[5] “Os corpos considerados “normais” e “comuns” são, também, produzidos através de uma série de artefatos, acessórios, gestos e atitudes que uma sociedade arbitrariamente estabeleceu como adequados e legítimos. Todos nós nos valemos de artifícios e de signos para nos apresentarmos, para dizer quem somos e dizer quem são os outros.”

makes it clear that subjectivity is important and that each one assumes their responsibility and performs their actions according to their desires and conceptions, a self-centered conception. However, it is different from the individualism we know. It goes beyond the individual/collective duality because the Ubuntu philosophy signals that there is an interconnection among human existences, assuming the integrity of these existences as a premise.

Thus, in the first section, we highlight the hegemonic historicity of white bodies in society, in mathematics, and in the ways of doing mathematics, discussing this white hegemony and situating our issue in terms of structural racism. Then, we moved on to digital technologies, focusing mainly on possibilities of educational and mathematical educational discussion about racism and its interconnections. Moreover, we present and apply the Ubuntu philosophy as a theoretical contribution to support our analysis. We first highlight the intersections of this philosophy with educational possibilities, and we show how mathematics education can help to understand the conceptions of this philosophy. In view of this, we present hereby one definition for the so-called antiracist mathematical activities with DT, which consists of:

> Mathematical-Activities-with-Digital-Technologies [that] can be developed considering cultural aspects of a given context. These activities consider Digital Technologies (DT) participants in the cognitive process, that is, DT are not mere auxiliaries, they are not considered tools that expedite or motivating source of the educational process, exclusively. They condition the production of mathematical knowledge. (Rosa & Mussato, 2015, p.23, my translation[6])

That is, according to Rosa (2020) more than tools, DTs are taken as resources, processes, and environments of a destining of revealing, as a revelation of what can be created, imagined, and discovered. In addition, these mathematical activities with DT assume an adjective which is the word "antiracist," precisely because we consider the political field to which they are linked, because this is understood as the social space where the struggle takes place through speech and action, that is:

> Knowledge of the social world and, more precisely, the categories which make it possible, are the stake par excellence of the political struggle, a struggle which is inseparably theoretical and practical, over the power of preserving or transforming the social world by preserving or transforming the categories of perception of that world. (Bourdieu, 1991a, b, p.236)

Thus, we theoretically analyze the proposal of an antiracist mathematical activity with DT, to answer our research question "how to discuss racism in a mathematics class with Digital Technologies in a way that mathematical concepts support the discussion?" Therefore, we thought that an antiracist mathematical activity with DT provokes discussions about skin color and think through mathematics about socially

[6] "Atividades-Matemáticas-com-Tecnologias-Digitais [que] podem ser desenvolvidas considerando aspectos culturais de um determinado contexto. Essas atividades consideram as Tecnologias Digitais (TD) partícipes do processo cognitivo, ou seja, as TD não são meras auxiliares, não são consideradas ferramentas que agilizam ou fonte motivadora do processo educacional, exclusivamente. Elas condicionam a produção do conhecimento matemático."

constructed ideas like racism. We understand that the analysis of this mathematical activity based on Ubuntu philosophy can provide important reflections for mathematics teachers and researchers in mathematics education regarding the role of mathematics in the world and, mainly, the role of mathematics in favor of an antiracist movement. It is worth taking into consideration that mathematics is not neutral, just as our positioning is not neutral. According to Shapiro (2021), Desmond Tutu, Archbishop Emeritus, who received the Nobel Peace Prize in 1984 stated "If you are neutral in situations of injustice, you have chosen the side of the oppressor." Although you think you have been neutral, you have definitely picked a side for your on and that was the oppressor's side. This may also happen, if we take neutrality for not taking position between two parts of a contradiction or taking both sides, in order to set a balance and try to avoid the contradiction by promoting a triangle between us and the two parts, because this would be more comfortable to us than making a decision. From this perspective, we understand that when neutrality per se seeks balance and equality of conditions and is part of a field of equality of conditions, the act of not choosing, not taking a position, or taking both sides can be considered neutral. But when this "neutrality" is in a disproportionate field, where there is a supremacy of power, the act of calling oneself neutral is only a way of hiding the position already taken, that is, on the side of the oppressor. We must then reflect on our role as mathematics educators and on our social responsibility and political *hexis* and make ourselves aware of historical power struggles (Bourdieu, 1991a, b), specifically related to skin color-bound issues. Recovering the speech of those who be oppressed in these disputes is necessary and using mathematics to understand this.

19.2 The Temporality/Spatiality Marked by a White Mathematics

When we talk about temporality/spatiality, we are closely linked to the idea of historicity, which according to Bicudo (2003b, p.75, my translation[7]) is:

> […] the feature of being historical – it is founded on the way of being of the pre-sence (dasein), understood as the human being who always is in the world temporally. […] The sense of pre-sence (dasein) attributed to this being, who we humanly are, is articulated by the understanding of what happens, of what happens in the over there, which is spatiality and temporality constituted in the ways in which one lives the space and the time.

In other words, encompassed by the presence of the human being in the world, we turn to the ways of being in the world, and among these, we highlight one

[7] "[…] caráter de ser histórico – está fundada no modo de ser da pre-sença, entendida como o ser humano que sempre é no mundo temporalmente. […] O sentido de pre-sença atribuído a este ser que humanamente somos é articulado pela compreensão do que ocorre, do que se dá no aí que é espacialidade e temporalidade constituídas nos modos de ele viver o espaço e o tempo."

specific way: the way called whiteness, which has been said to carry with it, among other things, the mathematical "truth," the right, the correct, what is "normal" to understand, and the "correct" lens through which we read and write the world. Sometimes, the historically constituted whiteness is not perceived, or assumed, many times it is even denied. Nonetheless:

> This lack of attention to whiteness leaves it invisible and neutral in documenting mathematics as a racialized space. Racial ideologies, however, shape the expectations, interactions, and kinds of mathematics that students experience. (Battey & Leyva, 2016, p. 49)

This means that the "mathematics" worked in the school, which is "white"(i.e., a "white mathematics," because it is presented as coming from the theoretical formulation of white peoples, of European origin), reinforces the symbolic power (Bourdieu, 1991a, b) attributed to white men, when it reveals that the origin of theorems, the construction of mathematics as a field and as a human achievement is fundamentally a consequence of the work of these white men.

There is no emphasis lying on any African peoples or African mathematics at all, despite very imposing works, such as the architecture of pyramids, for instance. This non-emphasis silently attributes greater power and valorization to European achievements and covers the real intention of the subjugating act, as if the subjugation was a spontaneous process, which "naturally" can be considered as unimportant, not interesting, something that took place unnoticedly, and that has no value or awareness, that is, a "neutral" act.

Confirming this, Powell (2002, p. 4) reveals:

> […] the mathematics presented in extant mathematical papyri from ancient Egypt most probably has preserved the mathematical ideas of an African elite. Nevertheless, mainstream, Eurocentric historians of mathematics have largely discounted these ideas. […] the importance of Africa's contribution to mathematics and the central role of that contribution to the mathematics studied in schools have not received the attention and understanding that befit them. As an example, documentary evidence of insightful and critical algebraic ideas developed in ancient Egypt exists, but little of this information has been made available to students studying mathematics, at any level.

What is recognized is that the enormous contributions of non-European development of mathematics have not been credited as deserved. The Arabic numeral system, historically written as the Hindu-Arabic numeral system, was carried into Europe by Muslims/Arabs. It is notable and it is recognized that algebra and trigonometry (which were studied by Arabs) were fundamentally developed by Hindu mathematicians and astronomers. But the empowerment of the skin color of these mathematicians is not valued as it should be in math classes. The fact that the mathematics curriculum neglects black and brown peoples' achievements leads us to infer that mathematics is perceived by all students (blacks, whites, etc.), subjectively, as a legitimate act and cognitive product exclusively done by white.

Furthermore, "mathematics" is, in many cases, empirically identified as a "difficult subject," "made for a few ones" and "intended to be understood only by the intelligent ones." This increases the symbolic power and the discrepancies it provokes even more (Bourdieu, 1991a, b), fostering ideological believes about mathematics itself and about the intrinsic logic of subtracting any black culture from deserved evaluative positions.

Corroborating this, Kivel (2011, p.282) states:

> Our curricula also omit the history of white colonialism as colonialism, and they don't address racism and other forms of exploitation. People of color are marginally represented as token individuals who achieved great things despite adversity rather than as members of communities of resistance. The enormous contributions people of color have made to our society are simply not mentioned. For example, Arab contributions to mathematics, astronomy, geology, mineralogy, botany, and natural history are seldom attributed to them. The Arabic numbering system, which replaced the cumbersome and limited Roman numeral system— along with trigonometry and algebra, which serve as cornerstones of modern mathematics— were all contributions from Muslim societies. As a result, young people of color do not see themselves at the center of history and culture. They do not see themselves as active participants in creating this society.

The fact that young people of color very currently do not consider themselves as an agent of the historical and cultural process, due to the non-presentation of information about black culture in school and other places like media, mainly due to the absence of this information from the common vocabulary, allows us to affirm that not feeling part is intrinsic to the idea of structural racism. Structural racism is considered by Almeida (2021) as a sort of racism transcending the scope of individual action, which does not require the intention of manifestation and allows legal responsibility not to materialize, although ethical and political responsibility is not excluded. Thus, racialized subjects are conceived as members of the social system surrounded by structural racism, so that the dimension of power stands out in terms of identifying one group over the other. Notwithstanding, we consider that structural racism manifests itself through a habitus that, according to Bourdieu (1991b, p.54):

> […] produces individual and collective practices - more history - in accordance with the schemes generated by history. It ensures the active presence of past experiences, which, deposited in each organism in the form of schemes of perception, thought and action, tend to guarantee the 'correctness' of practices and their constancy over time, more reliably than all formal rules and explicit norms.

Rephrasing this idea, once we assume habitus exists and is responsible for producing and/or reproducing the practices of an individual and their group or groups, everyone in these groups would be sharing the same premises and values on which they produce and/or reproduce these practices and their ways of being. In this sense, the intention or non-intention to practice or not to practice a racist action is directly linked not only to the agent's subjective world but also to the racial structure of his temporality/spatiality, and what results from the practice of an agent will be the product of the complex operation that considers in advance the values of the groups in which the agents belong (Lima, 2019).

Nevertheless, racism is hereby understood as a systematic social-ideological apparatus that discriminates, having race as a parameter, and its manifestation takes place through practices (conscious or unconscious) that result in disadvantages or privileges to each of the antagonistic groups, likewise to the individuals within these groups, who often happen to be stigmatized or flattered on the cause of the race, in which they are classified (Almeida, 2021). Thus, according to Kivel (2011, p. 19):

> Racism is based on the concept of whiteness - a powerful fiction enforced by power and violence. Whiteness is a constantly shifting boundary separating those who are entitled to have certain privileges from those whose exploitation and vulnerability to violence is justified by their not being white.

The race, a modern phenomenon in the mid-sixteenth century, gains meaning in the mercantile expansion, which later transforms the European into the "universal man" and which, under the contribution of the Enlightenment, materializes the conditions of comparison and classification of the most different human groups, under criteria mostly conditioned to physical and cultural characteristics. Whiteness becomes one of the criteria for distinguishing what is civilized from savage, which would later be called civilized or primitive (Almeida, 2021).

In addition, the ideological perspective of some philosophers reinforces somehow this approach, for example, the considerations made by the philosopher Hegel about Africans, who are described as "without history, bestial, and wrapped in ferocity and superstition." References to "bestiality" and "ferocity" demonstrate the trend to depose humanity from black people quite common at the colonial time, by associating them (including their physical characteristics) and their cultures with animals or even insects. The science made in the universities then assigned a very reliable tonic to racism and, therefore, to its process of dehumanization that preceded discriminatory practices or genocides then and until this day (Almeida, 2021, p.28–29).

The embargo on dehumanization starts from the religious domain, and there was a definition of the body based on the idea that the soul would be essential in the legitimation and qualification of being a human. Thus, religion, in connivance with the political sphere of the time, invents the criteria that define which group would be the holder of superiority. The invention of the black body or the animalized, objectified body contributes to legitimizing an economic project that assumed the human workforce. The religious doctrine only corroborated this idea, supporting it with the production of legitimation arguments (Dumas, 2019). Nevertheless, the sciences also served to support the explanation of dehumanization, according to the classification process by race, that is, white skin and tropical climate, according to biology and geography, would favor the emergence of so-called immoral behaviors, of violent and lascivious nature, as well as the identification of low intelligence. So, the closer to nature, the more primitive and the less civilized (Almeida, 2021).

When examining the racist internal structure of mathematics education, it is necessary to discover how we can educate through mathematics, using mathematics to understand what racism is, how it was disseminated, for what reasons, and with what interests and to be socially responsible as a proposal and as a way of being, thinking, and existing that does not exclude, oppress, and dehumanize anyone. Perhaps, the prevailing force of what we currently live with digital technologies (DT) can be a fruitful path. If we consider the role of DT in cases such as George Floyd (BBC NEWS, 2020), for example, we assume that there are possibilities for change and articulation through communication. That is, although the world is in a process of continuous technological advancement, it still struggles with issues of citizenship, a subjective condition for those who, as a member of a State, should be assured of constant enjoyment of a defined right set, able to allow him to participate in political life. However, there is still evidence of the absence of social and political conditions for this, particularly in cases arising from discrimination/prejudice, as was the case of the American George Floyd, who was murdered in a typically racist act. In this sense, Floyd's case mobilized the world in terms of protests against racism, precisely potentiated by the existence of the internet and all the technological apparatus that sustains it. For those reasons, we consider that highlighting the possibilities that are supported by the DT, in relation to social responsibility in terms of mathematics education, is one of the paths we wish to follow.

19.3 Digital Technologies and Racism: What Does Mathematics Education Have to Do with It?

Digital technologies have long been studied in the field of education and, in particular, mathematics education, bringing the potential of the educational experience with these resources. According to Rosa (2020, p. 3):

> The evolution of the digital domain, in the form of the computer network, has been appropriated by educators, and from this, the research about the possibilities of digital technologies (DT) brings to Education in Brazil and worldwide (de Oliveira, 2002; Kenski, 2003; Laurillard, 2008; Mansur, 2001; Underwood, 2009, among others) and has been conducted for decades. Specifically, in mathematics education, many studies point to the prominent potentialities of DT, inserting cyberspace in this context (Bairral, 2002, 2004; Bicudo & Rosa, 2015; Borba, 2004; Borba & Villarreal, 2005; Burton, 2009; Chronaki & Christiansen, 2005; Simmons, Jones Jr, & Silver, 2004; Zullato, 2007). Cyberspace can enhance the construction of online worlds and identities (Rosa, 2008; Rosa & Lerman, 2011; Rosa & Maltempi, 2006), as well as enabling the creation of a differentiated time/space for communication, interaction (Bicudo, 2018; Castells, 2003, 2005; Lévy, 2000) and, consequently, education (Hoyos, 2012; Tallent-Runnels et al., 2006).

Thus, these studies represent a wide range of research, which explore, in the field of mathematics education, issues concerning mathematics, the teaching, and learning of mathematics, as well as the training of mathematics teachers, in face-to-face and online modality. These studies in educational terms can open up possibilities for discussing racism, but they do not effectively do so. In this sense, we want to draw attention to the large gap that exists regarding research on the potential of DT as resources that can corroborate the understanding of social responsibility in the face of racism and also in mathematics classes. For example, society worldwide has faced a pandemic (Covid-19), and, in the meantime, we believe the pandemic brought to light the need for education as a foundation for raising awareness of the social responsibility of each one about the "whole" and the indispensability of a political stance, which is consistent with the common good. Assuming such responsibility as a math educator may allow one to understand in advance that "Trying to solve math problems in an over-dwelled hovel in a slum is very different than doing so in a spacious, luxurious apartment with a veranda" (Borba, 2021, our translation). From this perspective, we assume the term "responsibility" as described in the Abbagnano dictionary of philosophy (2007, p.855, my translation[8]), that is, "Possibility of predicting the effects of one's own behavior and of correcting it based on such prediction" and, respectively, its adjective, that is, the term "social" as "That which belongs to society or has in view its structures or conditions" (Abbagnano, 2007, p.912, my translation[9]). Thus, the possibility of predicting the effects of their behavior in relation to society, in this case, in relation to racism, in view of their structures or conditions, and correcting them is what we seek in educational terms, specifically, mathematical educational ones.

However, the need for social responsibility does not arise with the pandemic but is only emphasized by it. Part of society has been questioning itself for some time in relation to this understanding/constitution of social responsibility, even before the pandemic. However, now some questions stand out: what is the responsibility of mathematics education regarding the education of a group of students, belonging to a postcolonial society based on the process of (im)position of some bodies on having access or success in important spaces (depending on whether they are white or black bodies)? What responsibility do we have/assume in mathematics education in relation to structural racism (also evidenced during the pandemic)?

From this perspective, as already stated, we assume mathematics education as the act of educating (oneself) mathematically or educating (oneself) through mathematics (Rosa, 2008), which neither suppresses nor displaces the subjects involved in this act/process. Educators, teachers, students, and others involved in this act/process make mathematics education, act in relation to mathematics education, and become mathematics education. Thus, they need to question themselves about the social responsibility of mathematics education, that is, their own social responsibility.

In this sense, everyone (me, you, the teacher, the researcher in mathematics education, whoever reads this chapter, etc) is part of this. We, as mathematics teachers, need to perceive this responsibility as the primacy of knowledge and articulate educational possibilities that contribute to this understanding/constitution of social responsibility in the mathematics classroom, taking mathematics as a contribution and digital technologies as a means of enhancing this understanding/constitution.

The DT participation in this process is not limited to the use of them like auxiliary objects merely (which would be the appliance of a resource only for fulfilling instructions, a request or habits, with few or noncritical reflection on what is being done), but is understood like the experience with DT that is translated as perception, feeling, reflection, thinking, etc. On the contrary, the participation of DT becomes

[8] "Possibilidade de prever os efeitos do próprio comportamento e de corrigi-lo com base em tal previsão."

[9] "Que pertence à sociedade ou tem em vista suas estruturas ou condições. Neste sentido, fala-se em 'ação S.', 'movimento S.', 'questão S.'. etc.".

an articulating act under an intentionality which conceives the technological resource as a participant in the constitution of knowledge. This means that we constitute knowledge with the world, with the digital technologies that are in the world, and not about the world, alone, so that these technologies simply help us to think about something (Rosa, 2008, 2018). In this case, for example, we moved from the evaluation of the Internet as a mere apparatus, as a communication technology, to the understanding of digitality as a technique, technology, and process of modern life; in other words, we understand the internet as a participant in a digitally mediated society. That is, we understand the internet as a capital, within a capitalist regime that seeks profit, but, more than that, it also becomes a symbolic capital and needs to be critically understood as able to potentialize stigmas or privileges, as well as interests and objectives of those who rule any hegemony in question, depending on its conduction (e.g., skin color). This focus places the internet at the heart of society's digital transformations and also links it directly to the domain of the sociology of race, ethnicity, and racism. We say this because there is network capital that shapes a global racial hierarchy varying according to spatial geographies and the privatization of public and economic life. Internet technologies are central to the political economy of race and racism, as these technologies nowadays are at the base of politics and the concept of the capital (as we currently experience capitalism). Although we realize that digital transformation marks race and racism, transformation movements often leveraged colorblind racism, be whited racial projects, white racial frames, and implicit prejudice, that is, among other factors, the implicit aesthetic promoted many kinds of "white business" (business conducted under the will, desire, and perspective of white men). On the contrary, it would suffice to say that each one is important and none is perfect, once there are many prejudicing acts that happen with the internet, although there are many "important people" that use it for good things too. Indeed, the study of race and racism in the digital society must theorize network scale, obfuscation logics, and predatory inclusion mechanisms (McMillan Cottom, 2020). Or even, it should focus its efforts on raising awareness of diversity and valuing it.

For us, then, it is important focusing on diversity. So, in this chapter, our movement of understanding/constitution of social responsibility in mathematics education with digital technologies is in line with what Rosa (2022) proposes in his research. The author investigated how the process of understanding/constituting the social responsibility of mathematics teachers in cybereducation has shown itself through the analysis of cinematographic products. The perspective is linked to the structural racism that inhabits our reality, including the educational aspects. In this sense, based on the concept of cybereducation with mathematics teachers (defined as education seen under different dimensions and which assumes the work with DT from the perspective of being with, thinking with, and knowing how to do with DT), Rosa (2022) analyzed a participant of the subject/extension course "Macro/Micro Exclusions/Inclusions in Mathematics Education with Digital Technologies" held in 2021 in ERE mode (emergency remote education) at the Federal University of Rio Grande do Sul, Brazil. Specifically, analyzing one teacher from a group, Rosa (2022) understands that DT, in this case, cinematographic products, enhanced the understanding/constitution of the social responsibility of a mathematics teacher. The teacher was being-with-the-movie, consequently, with the people represented on the movie when watched and analyzed a specific film about racism. The teacher lived the experiences of the characters' lives, and, in this way, the understanding of the "ubuntu" philosophy happened, without dichotomizing, in the sense of not conceiving the existence of a being independent of the other but of a "being" that thinks, feels, and experiences with others and with the world. This bridge projected among mathematics education, digital technologies, and structural racism places us in the understanding/constitution of social responsibility, which appropriates the experience with DT in order to sustain itself in the ways of being, thinking, and knowing how to do it. It is also situated in opposition to racism through a mathematical foundation and through the support of technological resources. So that allows us to go further, seeking mathematics that can favor this very understanding/constitution. Before it, we need to discuss one of the theories that are linked to the way in which we can overcome racism is the African philosophy of Ubuntu.

19.4 Ubuntu Philosophy and Mathematics Education: Possible Interconnections

We envision a conception of the world, that is, a philosophy that, in our view, breaks with the Eurocentric and colonial idea of individuality as a primer thing and necessary to win in life and meritocracy as well. We bring up the Ubuntu philosophy, which becomes an ethical and pedagogical stance that evokes the idea of "being," in order to launch itself into existence even before materializing it, however, already launching itself into this materiality: there is a movement directed to people (each one with their individuality in direction to others) and the relationships between them. Each one becomes a be-ing-becoming (Ramose, 2002) marked by uncertainty once they are anchored in the search for understanding the cosmos in a constant struggle for harmony. This cosmic harmony encompasses politics, religion, and law, and those spheres are, by their turn, based on experience and the concept of this very harmony.

According to Noguera (2012, p.147, my translation[10]):

> Undoubtedly, the idea of ubuntu became widely known through free software for computers, characterized mainly by the proposal of offering an operating system that could easily be used by anyone. This essay is not about that; but, of |Ubuntu as a way of life: a possibility to exist together with other people in a non-egoistic way, an antiracist and polycentric community existence.

[10] "Sem dúvida, a ideia de ubuntu ficou amplamente conhecida através do software livre para computadores, caracterizado principalmente pela proposta de oferecer um sistema operacional que possa ser utilizado facilmente por qualquer pessoa. Este ensaio não trata disso; mas, de ubuntu como uma maneira de viver, uma possibilidade de existir junto com outras pessoas de forma não egoísta, uma existência comunitária antirracista e policêntrica."

In the same way, our research, even encompassing digital technologies, does not deal specifically with free software with that name, but the philosophy that portrays a way of living, a possibility of existing, be-ing-becoming without racism, seeking to get rid of any forms of discrimination, prejudice and/or selfishness, as well as perspectives based on a single way of being, of showing oneself, in unique and absolute terms. This is the perception of mathematics education that we believe teachers need to assume, discuss, and enact in their classes and perhaps make their students understand and apprehend it. In our view, it is also linked to the abdication of an exclusive "mathematics," a single "mathematics," finished, disciplinary, as well as a single way of seeing the world and others (a Eurocentric view, which takes as an adequate subject: "[…] a being of civilization, [male-]heterosexual, Christian, a being of mind and reason" (Lugones, 2014, p. 936)). So, the classroom should be filled with many kinds of mathematics.

Racism, for example, treats its origins through a Eurocentric vision that has historically signified any indigenous inhabitant out of Europe as nonhuman or less than a human. This construct contained one of the first epistemic acts of violence, for those peoples were and often still are perceived as beings absent from culture, living as beings closer to animals (Moraes & Biteti, 2019). Currently, it is notable that:

> […] racism is always structural, that is, it is an element that integrates the economic and political organization of society […] it provides meaning, logic and technology for the reproduction of forms of inequality and violence that shape the contemporary social life. (Almeida, 2021, p.20–21, author's emphasis, my translation[11])

On the other hand, reflecting on the Ubuntu philosophy becomes the action of assimilating the place of the decentralized "being" in the global context and seeking to abandon the legacies of a dominant discourse, understanding/constituting a knowledge that understands that people are not alone on the planet, much less that there are privileged societies in cognitive terms, due to coloniality. What constitutes Ubuntu philosophy is otherness, and thus is what:

> […] it constitutes my relationship with the other, in which the place of man is decentralized, removing him from the central place, demarcating his relationships with other beings. Thus, Ubuntu would not be a humanist ethics focused on man, but a way of being/with the other, with nature, with life. (Moraes & Biteti, 2019, p.138. my translation[12])

Also, it is necessary to understand that:

> Ubuntu is a be-ing-becoming, a come-to-be-ing-becoming, which promotes a transformation in reality from its agency with others. In its structure, ubuntu is made in time, promoting

[11] "[…] o racismo é sempre estrutural, ou seja, de que ele é um elemento que integra a organização econômica e política da sociedade […] fornece o sentido, a lógica e a tecnologia para a reprodução das formas de desigualdade e violência que moldam a vida social contemporânea."

[12] […] constitui minha relação com o outro, na qual se descentraliza o lugar do homem, o retirando do lugar central, demarcando suas relações com outros seres. Assim, o ubuntu não seria uma ética humanista concentrada no homem, mas um modo de ser/com o outro, com a natureza, com a vida (Moraes & Biteti, 2019, p.138).

> maintenance and transformations as measure of the do-doing, acting in constant continuity in its being in the world. (Moraes & Biteti, 2019, p.138, my translation[13])

We hope that there will be a chance for a new mathematics class developed by teachers who recognize the other as themselves and who encourage this same recognition on the part of their students, in order to experience mathematics as a way of showing it, educating mathematically oneself and, mainly, educating oneself through mathematics (Rosa, 2008) in the face of situations of racism, for example, and in favor of recognizing and respecting diversity.

Mathematically, it is important to have as a premise what Ngomane (2019, p.66). says:

> As human beings, we share our planet with 8 million different species, but we ourselves are pretty unique. With around 200 countries in the world – official number vary – and roughly 6,500 different spoken languages (and with an infinite number of cultural differences), what we all have in common is this: diversity.

Furthermore, according to Jojo (2018, p.256) "Ubuntu as a philosophy, is a way of thinking about what it means to be human, and how humans, are connected to each other or should behave toward others." Even though we are different people, we have a responsibility to each other, with well-being, and if you are not well, I am not well either. I am not without you, I am not without us. The idea of empathy materializes, not only as putting oneself in someone else's shoes but considering the other intrinsically linked to one's own being. According to Gade (2012, p. 257):

> Reality in Ubuntu is informed by the power derived from embracing people's cultural philosophy (Bopape, 1990). It is an aspect of the African people's culture. Batswana, Bapedi, Basotho refers to this culture as 'botho', while the Nguni's (Xhosa's, Zulu's, Ndebele's; and Swazi groups refer to it as 'Ubuntu'. For my convenience in this paper, I'll use Ubuntu. It embraces concepts like: "Umntu ngumtu ngabantu" which is literally translated as: 'a person is a person through other people'. This implies that it is through the support from other people that a person is able to achieve his/ her goals. This reality is therefore based on collaboration, togetherness, and working collectively, through which the best results can be achieved. When this philosophy is well explored and understood by a mathematics teacher, it culminates in the latter putting maximized effort to ensure that the classroom environment welcomes students' errors and questions while it also promotes engagement with problems posed and boosts their reputation under general.

The adoption of the Ubuntu philosophy can permeate the mathematics class, if this class is understood/constituted through premises, such as social responsibility, respect for diversity, and the belonging of being in the world with others. So, Jojo (2018), for example, studied fifteen 8th grade math teachers from South Africa in order to understand how their classroom environments and practices were transformed through Ubuntu, in relation to teaching geometry. The study reveals that teachers have transformed their approach to different math topics to explore

[13] "O ubuntu é um ser-sendo, um vir-a-ser sendo, que promove uma transformação na realidade a partir de seu agenciamento com outrem. Em sua estrutura, o ubuntu se faz no tempo, promovendo manutenções e transformações na medida em que faz-fazendo, agindo em constante continuidade no seu estar no mundo."

diversity in their classrooms and bring each student inclusively to understand the meaning of interpreting geometric terms with their application through sharing and modified valuing by each student's contribution. It is important to highlight here that the mathematical concepts to be developed, in this case, geometric, depart from the "universal mathematics," European, white, and historically (only) Greek. This is not bad, on the contrary, if I am not without us, Africans are not without Europeans, and the Europeans shouldn't be without Africans. In this way, they do not abdicate the knowledge produced historically, even if the official history does not actually portray the movements of temporality/spatiality that took place. The annihilation of knowledge or non-appropriation of it is not defended for us; it is defended by the recognition of the constitution of black, yellow, red, Latin, and indigenous knowledge, in order to conceive equity and nondiscrimination, non-exclusion, non-subordination, and non-coloniality (or better, decoloniality). Jojo (2018) argues about the different types of representations of the human activity of doing mathematics and states that these exist in our surroundings and should be explored for a meaningful understanding of geometry.

In turn, Osibodu (2020) investigated whether and how young people in sub-Saharan Africa use mathematics in understanding social issues related to the African continent. With five young people from sub-Saharan Africa, over a semester, the author of this research developed her study. She took as a theoretical reference the decolonial theory from an African perspective, which, according to the author, encompasses decolonial perspective structures, such as the Ubuntu philosophy that decenters power. Osibodu's study (2020) did not focus on learning new mathematics; rather, it sought to investigate what knowledge young people draw on in their exploration of social issues. As a result, the research highlights the focus on sub-Saharan youth's need to reread and rewrite their African world with and without mathematics. To encompass these results, young people were invested in rewriting narratives about the African continent, raising African indigenous knowledge. The act of rewriting has led to epistemic freedom and cognitive justice – an essential component of social justice – that corrects the loss of indigenous African knowledge. Despite this, there was still tension in recognizing and accepting indigenous African forms of knowledge, along with the belief that mathematics taught at their school was quite neutral.

The traces of coloniality were manifested in Osibodu's survey (2020), and one of the phrases expressed by one of the participants was "wherever you see a right angle, it means a white man has been there" (p. 56), or, that is, this participant's mathematical perception of his space made him affirm this and understand the mathematics produced by his own people, because all houses, for example, were built for his people in circular forms. In this case, for us, it is a clear example of what we call ethnomathematics. According to D'Ambrosio (2001) apud Powell (2002, p. 3-4):

> Ethnomathematics encompasses in this reflection on decolonialization and in the search for real possibilities of access for the subaltern, for the marginalized e for the excluded. The most promising strategy for education, in societies that are in transition from subordination to autonomy, is to reestablish the dignity of its individuals, recognizing and respecting their

> roots. To recognize and respect the roots of an individual does not mean to ignore and reject the roots of the other, but, in a process of synthesis, to reinforce one's own roots. (p. 42, author's translation)

Thus, our reflection recognizes different mathematical production ways. We do not want to substitute the Eurocentric mathematics for others. Notwithstanding, we aim to produce reflections about ethnomathematics and to relate the way of understanding the correlations between possible mathematics and the concept of Ubuntu, because, as Jojo (2018) declares:

> Ubuntu is a unifying concept within South African people culture and thus deserves prominence in the curriculum in all respects. When teachers demonstrate the understanding of the historical development of mathematics in various social and cultural contexts both in urban and rural settings, students will not feel threatened and will exercise their both their minds and confidence in the classroom. Through Ubuntu, the teachers worked as collective humans with their students, restored their self-worth, and their ways of thinking from hegemonic structures, and facilitated their ability to articulate what they do and think about in order to provide a foundation for their productive individual participation in the classroom. I will also borrow from the Bapedi' expression that talks to cooperation and solidarity attributes of Ubuntu when they say, "Tau tsa hloka seboka di fenywa ke nare e hlotsa". Literally explained this means that even a limping buffalo can beat lions without unity. Figuratively, this implies that unity is strength or simple tasks may remain impossible unless there is cooperation. In mathematics classrooms an environment of social organization, cooperation, communication; sharing of ideas; and solidarity where students are free to express themselves without fear of belittling is necessary to improve their performance and bring them into better understanding of the different concepts. In that environment their critical thinking skills are enabled, and they learn to ask the 'why' and 'how come' questions. (p.259–260)

With that, we started to discuss an antiracist mathematical activity with DT, which under our interpretation brings significant aspects of the Ubuntu philosophy weaving the ways of be-ing-becoming with the other, with the we, and with the mathematical reasoning. This weaving leads us to consider that the student may perceive diversity and assimilate the humanization of this diversity.

19.5 An Antiracist Mathematical Activity with DT

The antiracist mathematical activity with DT that we present in this chapter was inspired by the photographic exhibition Humanae carried out by Angélica Dass. According to the exhibition website, Dass (2022) reveals that:

> Humanæ is a photographic work in progress by artist Angélica Dass, an unusually direct reflection on the color of the skin, attempting to document humanity's true colors rather than the untrue labels "white", "red", "black" and "yellow" associated with race. It's a project in constant evolution seeking to demonstrate that what defines the human being is its inescapably uniqueness and, therefore, its diversity. The background for each portrait is tinted with a color tone identical to a sample of 11 x 11 pixels taken from the nose of the subject and matched with the industrial pallet Pantone®, which, in its neutrality, calls into question the contradictions and stereotypes related to the race issue. More than just faces

> and colors in the project there are almost 4,000 volunteers, with portraits made in 20 different countries and 36 different cities around the world, thanks to the support of cultural institutions, political subjects, governmental organizations and non-governmental organizations. The direct and personal dialogue with the public and the absolute spontaneity of participation are fundamental values of the project and connote it with a strong vein of activism. The project does not select participants and there is no date set for its completion. From someone included in the Forbes list, to refugees who crossed the Mediterranean Sea by boat, or students both in Switzerland and the favelas in Rio de Janeiro. At the UNESCO Headquarters, or at a shelter. All kinds of beliefs, gender identities or physical impairments, a newborn or terminally ill, all together build Humanae. All of us, without labels.

Through the Humanae project, we observed the possibilities that the diversity discussed in the project itself could list and favor the understanding/constitution of social responsibility in the face of racism, precisely, understanding the constitution of the skin color of each individual without labels and, at the same time, mathematically common to all.

In this perspective, we investigated a color recognition application (from now on "app"), and among those found in the gallery of applications for Android and IOS systems, we chose the app called "Identificação de Cor" (color identification) once it proved to be user-available with an easy handling interface. In Fig. 19.1, we can see the app's interface:

Initially, we have the key "Bloquear Anúncios" (block ads), then "Abrir Ficheiro" (open file), "Identificação em tempo real" (real-time identification), "Lista de cores" (color list), and "Criador de cor" (color creator). Our objective with this app (or other ones that work in the same way) is to be able to proceed like Angélica Dass in

Fig. 19.1 "Identicação de cor" app interface

her photographic work Humanae and, through that, bring mathematics as a basis for understanding colors. In other words, with this app, we want to be able to photograph skin tones, using photos of different people's nose tips (in our case, students' noses), in order to obtain the digital color recognized by the technological resource. The app operates by targeting a single pixel right in the middle of the photo and revealing instantaneously values between 0 and 255 of the basic hues (red, green, and blue) composed of the color that was recognized.

Our antiracist mathematical activity with DT:

First moment, exposition debate the activity can begin with the presentation of Dass' work, generating a debate about the exhibition and mentioning that there is a clear question of percentages (mathematics) present in the color of each one's skin.

Second moment, invitation the teacher performs an invitation to do the same that Dass did but in the classroom. This is "would the class agree to divide into a group and carry out this color identification?" Attention: regarding the students who are not willing to participate, the teacher can asking them the reason of participation refuse. Then, whether the teacher's awareness-raising argument is not convincing, or the student's justification is plausible, the teacher must accommodate these students in the groups as a technical reporter, that is, these students will not stop investigating and thinking together, although they will only not participate in the photography phase.

Third moment, app's download regarding the students participating in the photography phase, it is necessary to form groups, check if at least one student has a smartphone, and request them to download the "Identificação de Cor" app (or another app that works in the same way).

Fourth moment, formation of the groups with the app installed, the activity begins with the formation of groups. A random formation of the groups is recommended, so that racially mixed groups emerge. The group size and selection method depend on the teacher's perception and the size of the class, and it would be relevant to note how many girls, boys, blacks, whites, and different stereotypes existing in the classroom are compounding each group.

Fifth moment, confirmation of participation and ethical clarifications the teacher explains the activity and asks if anyone feels uncomfortable taking a photo of the tip of their noses and the teacher precisely clarifies that not the image of the person's face will ever be collected. The focus of the activity will be exclusively on the color of the nose, of the nose tip.

Sixth moment, request for the activity report and explanation of how to do this report it is necessary to explain to the group that there will be a report on the research to be delivered and to reveal procedures to do that ((1) creation of pseudonyms; (2) inserting the photos taken by cellphones; (3) using the app to measure the quantitative of red, green, and blue from each identified color; (4) showing the calculation of percentages of red, green, and blue from each identified color; (5) discussing the questions presented by the teacher (a, b, c…l)). It is important that the teacher reads all activities (until the 13th moment) and uses the app before taking it to the classroom.

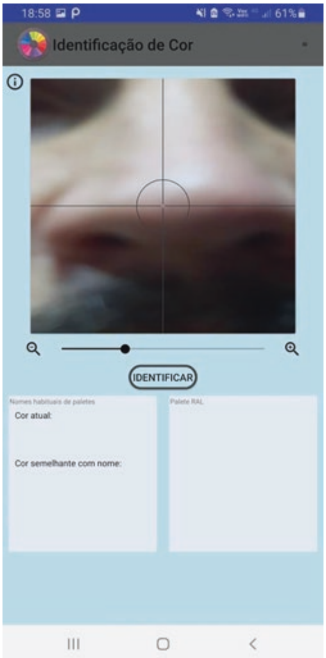
Seventh moment, creation of pseudonyms all participants/students will create pseudonyms to appear in the report, that is, each color will be identified by a pseudonym and not by the name of the subject involved.

Eighth moment, explanation of the reason for the choice of pseudonym also, it would be important to discuss why each pseudonym was chosen. As a plus, it would be interesting if the pseudonyms they chose were currently names of popular people, who took a stand against racism at some point in history.

Ninth moment, production of pictures the students take a photo of the tip of the nose of their classmates and capture the cell phone screen, so that the image composes the report. This photo must be taken using the "Identificação de Cor" app in the "Identificação Em tempo real" mode, as shown in Fig. 19.2.

Before capturing the smartphone screen, each student needs to click on "Identificar" (identify), so that the screen capture already brings the identified colors, namely: "Cor atual" (current color), "Cor semelhante com nome" (similar color with name) and Palette RAL. Figure 19.3 provides an example of personal color identification.

Fig. 19.2 "Identificação em tempo real" (real-time identification)



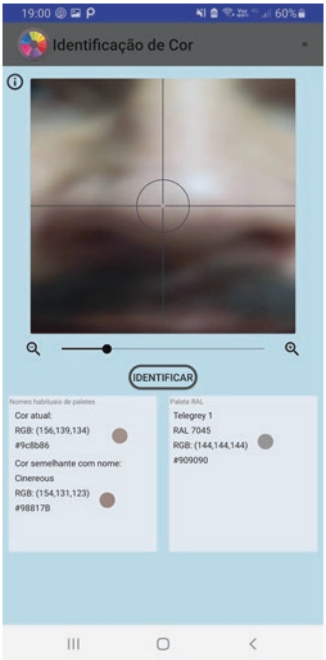


Fig. 19.3 "Cor atual" (actual color), "Cor semelhante com nome" (similar color with name) and Palette RAL

Tenth moment, internet research about the RGB color system the teacher asks the group to search about the RGB color system on the internet.

Eleventh moment, reflection on different color systems in order to guide the search, confront the group with these questions: what does each letter (R, G, and B) mean? What are the other systems? What are the differences between the systems? What is the number of levels in the RGB system? The answers are shown in the report.

Twelfth moment, reproduce the color identified the teacher asks the students that they reproduce the color identified by the app (identification resource), in the color creator resource, in order to learn to numerically configure a color using the RGB system. So, the teacher conducts the group report through the individual analysis of the participants. Initially, get the "Cor atual" (current color) of each participant and reproduce the color in the mode "Criador de cor" (color creator) of the app. As it is shown in Fig. 19.4, we can reproduce generated color in the app:

Thirteenth moment, reflection on the process through questions from the generation of photos and color recognition of all the participants of the groups, questions need to be launched. We remind the teacher that the questions presented here are just suggestions, each teacher is free to change them and add others, in a way that satisfies their reality. However, we call attention to the fact that mathematical discussion is a way of understanding diversity and equity, from the Ubuntu perspective. Tutu (2004) notes:

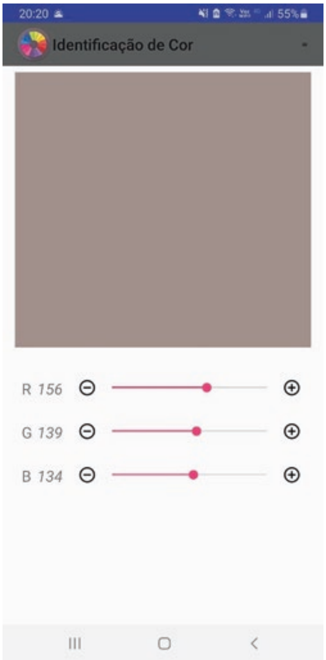


Fig. 19.4 "Criador de cor" (color creator)

> I am human because I belong. I participate, I share". A person with Ubuntu is open and available to others, affirming of others, does not feel threatened that others are able and good, for he or she belongs in a greater whole and is diminished when others are humiliated or diminished, when others are tortured or oppressed, or treated as if they were less than who they are. (p. 31)

By allowing the students to understand that there is a hint of red and/or green and/or blue in everyone's skin color, it will be possible to realize that the chromatic spectrum found in a rainbow represents much more than the pettiness of our imposed binarity of black and white. Thus, it is important to question and discuss mathematically:

(a) How much red does each photo have in its pigmentation? What is the percentage of red in relation to the total (100%) of the three colors?
(b) How much green does each photo have in its pigmentation? What is the percentage of green in relation to the total (100%) of the three colors?
(c) How much blue does each photo have in its pigmentation? What is the percentage of blue in relation to the total (100%) of the three colors?
(d) What do these percentages represent? What can we conclude individually?
(e) If we compare the participants of the group, what can we say about the percentage of red? What can we say about the percentage of green? What about the percentage of blue?
(f) What can we say in terms common to all participants in the group?

(g) Could any identified color exist without the three elements (RGB)? What does this explain to us?
(h) Did any participant in the group have their RGB value identified as (0,0,0)? Likewise, did any participant in the group have their RGB value identified as (255,255,255)? What percentage of participants met these criteria? What do you have to say about that percentage?
(i) Also, have you noticed that the colors displayed on the screen are not exact, and they depend on the brightness and contrast settings and may vary from one screen to another in relation to the camera and device? What does this reveal in relation to any color study?
(j) Can we say that someone is currently white or black?
(k) Does having more or less red in skin color pigmentation change what a person is? Likewise green or blue?
(l) Why are there people who suffer and are still killed today because of their skin color? What do you think about this? What mathematical explanations would you give about skin color that would help to understand what people may be like in terms of skin color? What actions would you like to take regarding racism?

This would be an activity in which the discussion of percentage, proportion, and interval would be intertwined with the discussion of diversity and understanding/constitution of social responsibility in relation to racism. Then, we move on to our considerations, which already carry out the analysis of the activity in the face of the Ubuntu philosophy and the perspective of social responsibility.

19.6 Antiracist Mathematical Educational Movements: Be-ing-Becoming Antiracist

"How to discuss racism in a mathematics class with digital technologies in a way that mathematical concepts support the discussion?" was the research question announced in this chapter. To seek to answer it, we drew a line of discussion about the historicity of a "white mathematics," which is taken as "neutral," sometimes under neutrality attributed to it as if it were an "entity" without any relation to humanity. Also, on the other hand, the "white mathematics" is evidenced under a single bias, that is, that of univocal production and undervalue judgment, with a great symbolic power, on the part of an exclusively white group. In the meantime, we discussed the structural racism that encompasses this way of thinking about mathematics itself. These "mathematics" were mostly consolidated by white men and by the intelligent people in the core of Enlightenment Europe, as if African mathematical production did not ever exist and as if black people were not intelligent, dehumanizing them, although it was already known back then that some mathematics contents had been hijacked from African ancient people (Powell, 2002).

In contrast to this structural racist movement, internal to the scientific field and which extends to the field of common sense, through a habitus concerning the dominant group, we are seeking to understand/constitute social responsibility in the face of racism, discussing it in an educational space of mathematics with digital technologies. We seek not only to value mathematics but also to value the philosophy that emerges from other places, Ubuntu, which manifests itself in a convergent way with the issues raised by decoloniality and ethnomathematics. Thus, we also apprehend "white mathematics" because it has its value and because we want dialogue. We understand the role of "we" as a key piece, as a tree trunk that sustains the fruitful possibility of philosophy itself, and we advance with the potential for the articulation of digital technologies for the debate of racism, for example.

In this way, we created an antiracist mathematical activity with DT as a way of discussing racism in a math class with digital technologies. In other words, the activity lists the exploration of percentages as a means of thinking about skin color when in the 13th moment, which aims at the reflection, the questions "a," "b," "c," and "d" request the calculation of the percentage of red, green, and blue that each individual has in the color identified in the tip of your nose, as a means of concluding that they all have red, green, and blue in their own color (questions "e" and "f"). This conclusion permeates the digital transformation with the "Identificação de Cor" app and marks race and racism with different issues in terms of reflective intent. There is a movement of transformation of colorblind racism, because it is desired to highlight the difference as a focus in this activity, but not with white racial frames and implicit prejudice, that is, among other factors, the implicit aesthetics that would promote white power. But as a source of understanding that everybody's skin color is composed of red, green, and blue, however, each complexion has its individual nuances, and it has nothing of value over one another. Thus, there can be no perfection or supremacy, for all of us matter equally.

> However, the percentages of the same colors in the activity cause race and racism to be questioned in the digital society, which allows one to understand the logics of obfuscation and to be aware of the possible mechanisms of predatory inclusion. (McMillan Cottom, 2020)

In terms of the Ubuntu philosophy, the antiracist mathematical activity with DT allows the understanding of the colors of the group, class, community, population, and world as a way for each one to live, a possibility of existing together with other people in a non-egoistic way, as we all have red, green and blue, and in an antiracist and polycentric community existence (Noguera, 2012), because red or green or blue are not just present on one person. They are on everyone's skin: we are altogether red and/or green and/or blue! This understanding is dealt with in the activity (questions "f" and "g") and it goes further, since it highlights that racism is always structural, that is, it already brings the possibility of who will try to value possible "whites," that is, the white color that would have the maximum levels of red, green, and blue or devalue the black color that would not have red, green, and blue pigment, raising questions about this (questions "h" and "j"). Questions that predict structural racism can emerge in the classroom itself, because it is an element that integrates the economic and political organization of society and it provides the meaning, logic, and technology for the reproduction of forms of inequality and violence that shape contemporary social life (Almeida, 2021, p.20–21). However, it is up to the teacher to be able to raise other questions if the idea of valuing white, as formed by maximum values of red, green, and blue (255, 255, 255) on the scale, and the devaluation of black, as formed by minimum values of the scale (0,0,0), happens. Questions must be linked to the validity of arguments. That is, if you value a white complexion, where can it currently be found? Who would be this truly white person? Would red 255, green 255, and blue 255 finally be matched? Do these white people really exist? (question "j"). Questions like these would serve to show each one, once again, that even if one wants to attribute "power," "value" to the white color because it assumes the maximum values in the RGB scale, in fact the real write and black doesn't appear in human beings. Which photo presented or presents these colors? In other words, the power relationship (attributed to one color and not to another) does not come from mathematics. This, on the contrary, shows that numerical differences do not express value or power, they simply express that the colors are different assuming percentages of colors (red, green, and blue) similar to all, once all have red, green, and blue in their pigmentation.

In addition, the participation of DT in this process is not limited to the use for the use, but it also becomes an articulating act under an intention that conceives the technological resource, that is, the app "color identification" as a participant in the constitution of knowledge. This means that we acquire knowledge about the diversity of colors with the app; only with it we determine how much red, green, and blue there is in a color, to later calculate the respective percentages and proportionality of each one in the final composition. We constitute knowledge with the digital technologies that are in the world and not about the world alone, so that these technologies help us to think about something else (Rosa, 2008, 2018). In this case, the antiracist mathematical activity with DT even supplied questioning and varied use of the technology itself (question "i"), that is, critical thinking about the digital aspect also needs to be highlighted.

However, issues involving the idea of equity make up the very awareness of social responsibility for those who suffer without having a real and just reason (questions "k" and"l"). In this perspective, Rosa (2022) sustains that DT enhance the understanding/constitution of the social responsibility of mathematics students who are involved when they being-with-the-App presenting his understood about the "Ubuntu" philosophy, without dichotomizing it. In this way, the author doesn't conceive the existence of a being independent of the other, but of a "being" that thinks, acts and lives with others.

Finally, we understand that there is a way to discuss racism in a mathematics class with digital technologies, for example, with the "Identificação de Cor" app or another similar, so that mathematical concepts such as percentage and proportionality of base colors that make up people's colors support the discussion. That is, is there a difference? Is there a reason for the other to be discriminated against? Why are there people who suffer and are still killed these days because of their skin color? What do you think about this? What mathematical explanations would you give about skin color in order to help elucidate what people are like in terms of skin color? What actions would you like to take regarding racism (question “l”)? With this, we understand that it is high time to ask ourselves about it and take these questions to mathematics classes.

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Chapter 20
Philosophy, Rigor, and Axiomatics in Mathematics: Imposed or Intimately Related?

Min Bahadur Shrestha

20.1 Introduction

Ever since mathematics began being developed, mathematicians have seemed to be relatively unconcerned with philosophy, as reflected in a Socratic dialogue (Rényi, 2006) in which ancient Greek philosopher Socrates mentions that the leading mathematicians of Athens do not understand what their subject is about. Plato, being devoted to philosophy in general and to the philosophy of mathematics in particular, was motivated by Socrates.

Plato's contribution of Platonic thought about mathematics, or Platonism, has descended through the centuries as the basis of the philosophy of mathematics. The Greek concept of the deductive–axiomatic model that culminated in Euclid's *Elements* was a paradigm of mathematical certainty until only recently. In *Elements*, Euclid developed a magnificent axiomatic and logical system that served as the sole model for establishing mathematical certainty until the end of the nineteenth century (Ernest, 1991: 1). Perhaps the most evident modern feature of *Elements* is the axiomatic method, which stands at the core of modern mathematics (Mueller, 1969). Along with axiomatics, rigor has been a major requirement in formal mathematics. Although the term *rigor* is usually associated with advanced mathematics, even in that domain it seems to be largely accepted without much discussion. Moreover, it seems to me that the concept of rigor is applied less prominently than the concept of axiomatics.

Against that background, in this chapter I examine the terms *rigor* and *axiomatic(s)* in detail. The extent to which rigor and axiomatization should be achieved seems to depend mostly on how mathematics is viewed. By extension, how mathematics is viewed is a relatively philosophical question. In that light, this chapter seeks to examine the philosophical reflection on rigor and axiomatics in

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mathematics. To that end, I first review the concepts of rigor and the axiomatic method in mathematics. Although both concepts have largely been associated with Western mathematical traditions as their integral components, Hindu mathematical traditions, despite lacking formal axiomatic proof, have also contributed significantly to the development of mathematics. Thus, rigor and reasoning in the Hindu development of mathematics also warrant examination to understand the status of the axiomatic method in mathematics and its relationship with philosophy.

To that purpose, the chapter is organized into four subsections—"Rigor in Mathematical Argumentation," "The Axiomatic Method," "Rigor and Reasoning in Traditional Hindu Mathematics," and "Philosophical Reflection," in that order—followed by a summary and conclusion. When appropriate, pedagogical concerns are also addressed to clarify the discussion from an educational point of view.

20.2 Rigor in Mathematical Argumentation

While searching for the precise meaning of *rigor* as used in mathematical argumentation, I came across an article written by Philip Kitcher (1981) that addresses the concept of rigor in a rather conventional way. Kitcher (1981, 1) states that "central to the idea of rigorous reasoning is that it should contain no gaps, that it should proceed by means of elementary steps." He argues that the argument in reasoning is rigorous when and only when the sequence of statements leads to the conclusion and every statement is either a premise or a statement obtainable from previous statements by means of elementary logical inference. In mathematical proofs, as a kind of argument required to convince readers of the truth of mathematics, the rigor in reasoning is aimed at establishing correct, consistent results. More recently, a wider basis of proof has been taken into consideration that includes not only a cognitive basis but also cultural and psychological bases (Joseph, 1994: 194). Nevertheless, writers of standard textbooks on mathematics seem to be primarily interested in the conventional mode, which has an exclusively cognitive basis.

To illustrate how formal deductive proof employs rigor in mathematical derivation, the following example demonstrates how to prove that 1 + 1 = 2. In doing so, the example reveals how rigorous proof is based on assumptions of axioms and/or postulates, along with definitions, as well as logical rules of inference (Ernest, 1991: 4–6). What is immediately noticeable is that proof is needed to establish a set of rules in advance of stating definitions, axioms, and rules of logical inference. After that, the proof can be developed in 10 steps (Ernest, 1991: 5). The example raises the question of what rigor indicates and what advantages tediously providing such proof for the fundamental facts of arithmetic affords. From a pedagogical perspective, my experiences of teaching pre-service students seeking master's of education degrees in mathematics have revealed the problem of convincing many students to recognize the value of and commit to providing such seemingly ridiculous rigor. As I see it, such rigor implies the development of consecutive steps without any lapses of reasoning in the formation of the argument. Alternatively, the sequence of the steps in the argument should be capable of connecting statements to form the integrity of the proof. However, the concept of rigor, like many other concepts, is a relative one and may depend on many factors. Although Paul Ernest (1991: 5) has developed 10 steps to show that 1 + 1 = 2, Reuben Hersh (1999: 254) has used only three steps to prove that 2 + 2 = 4, and that discrepancy indicates that rigor in proof also depends on the situation under consideration and other factors. Even so, the traditional view on ideal mathematical knowledge can explain why such rigor is needed; it maintains that by constructing rigorous proofs of known truths, mathematicians at all levels can improve their knowledge of those truths either by coming to know them a priori as certain or at least by making their knowledge more certain than it was before (Kitcher, 1981).

Kitcher refers to that type of mathematical thinking as "deductivism," which consists of two claims. On the one hand, it holds that all mathematical knowledge can be obtained by deduction from first principles and that such an approach is the optimal route to gaining mathematical knowledge because it is less vulnerable to empirical knowledge. Kitcher adds that though mathematicians can use deductive and/or axiomatic proof in mathematics, such proof lacks an epistemological basis, which deductivist theses attribute to first principles. A lack of an epistemological basis means that neither first principles nor their bases can be discerned as being true, and it is clear that so many such constraints are imposed in mathematics in developing deductive proofs. Interestingly enough, mathematical proofs have been useful in computer programming and formatting the power of mathematics in packages (Skovsmose, 2010). Although deductivists' way of producing mathematics is generally not how mathematicians produce mathematics, the deductivist approach is nevertheless a natural way to unravel the network of interconnections between various facts and to exhibit the essential logical skeleton of the structure of mathematics (Courant & Robbins, 1941/1996: 216).

The discovery of non-Euclidean geometry based on the logical consequences of Euclid's fifth postulate laid a strong foundation for the development of deductive–axiomatic rigor in mathematics. Although the development of non-Euclidean geometry has been an ingenious reconstruction in mathematics, one that has broadened horizons in the field as well as in the philosophy of mathematics, it has been limited to the field of geometry as the twin brother of Euclidean geometry. The most important aspect of rigor in that context implies the strict logical consequences of Euclid's fifth postulate and its two alternative postulates as being the postulates of non-Euclidean geometries. Moreover, whereas Euclid's *Elements* defines basic geometric objects, including points and lines, David Hilbert's work does not. Hilbert does not define *line*; on the contrary, he axiomatizes it by writing "Two distinct points determine a line." Hilbert's efficiency, as a modern geometer, lies in recognizing that a line, as such a simple geometric figure, cannot be defined satisfactorily, but it can be axiomatized.

Rigor in mathematics seems to have become understood differently depending on the course of the development of different areas of mathematics. In the past 200 years, the development of analysis, as a constructive method in mathematics independent from deductivism, has been made rigorous by identifying and defining many concepts and thereby allowing more refined development. By contrast, in the name of generalization and formalization, constructively developed analysis has been situated within deductive structures as a means to formalize it and lend it rigor. Such a representation of mathematics has indeed become more robust and rigorous with the work of Nicolas Bourbaki, the collective pseudonym of some young formalist mathematicians in France most active in the mid-twentieth century; however, it also seems to have become detached from the position from which it was developed and from the purpose for which it was first used. For example, on the topic of rigor in mathematics, E. T. Bell (1934) made the following remark in his article:

> No mathematical purist can dispute that the place of rigor in mathematics is in mathematics, for this assertion is tautological, and therefore, according to Wittgenstein, it must be of the same stuff that pure mathematical truths are made (p. 600).

The statement clearly reveals that, for mathematical purists, the rigor of mathematics is inherent in mathematics and is made of the same core elements as mathematical truths. By extension, Bell seems unsatisfied with the development of rigor in mathematics textbooks for graduate students:

The present plight of mathematical learning—instruction and research—in regard to the whole question of rigor is strangely reminiscent of Robert Browning's beautiful but somewhat dumb little heroine Pippa in the dramatic poem *Pippa Passes* (p. 600).

Thus, though as beautiful as a heroine in the field of poetry, rigor is not as beautiful in mathematics education. Similar to Bell, many scholars and mathematics educators, especially in the twentieth century, have shown concern with the difficulties of teaching and learning mathematics due to the excessive emphasis on formalism, which draws upon the axiomatic method as well as rigor. In response, Hans Freudenthal, in his contributions to make mathematics more educational, particularly in *Mathematics as an Educational Task* (1973), critically examined the use of mathematics for educational purposes. Meanwhile, intuitionists Richard Courant and Herbert Robbins, in their epoch-defining book *What Is Mathematics?* (1941/1996), sought to emphasize the intuitive and constructive nature of mathematics. Nevertheless, Jerome Bruner, a US learning theorist and well-known scholar, expressed anxiety over the frustrating situation in mathematics education bought about by an insistence on formalism. In his preface to Zoltan P. Dienes's book *An Experimental Study of Mathematical Learning* (1964), Bruner remarks:

> I comment on the difference between Dr. Dienes and some of the others of us who have tried our hand at revising mathematical teaching. Perhaps it can be summed up by saying that Dr. Dienes is much more distrustful of "formalism" than some of the rest of us.

Such situation reveals that the more formal, rigorous development of mathematics has long been regarded as a hindrance to the development of mathematics education. Many students seem to have difficulties understanding the essence of rigor in mathematical arguments involved in proofs. It seems that rigor becomes especially confusing in proofs of common truths—for example, showing that $1 + 1 = 2$. Nearly all students in my classes have been confused, if not perplexed, while trying to develop an understanding of that proof, and, even after being aided with prompts and hints, few have shown an understanding of its justification (Shrestha, 2019). In fact, Kitcher (1981) has even suggested testing the hypothesis that demanding rigorous proofs of known truths is just simply confusing.

It has recently been suggested that proofs in mathematics, to be convincing arguments, need not only a cognitive basis but also cultural and psychological bases (Joseph, 1994: 194). Social constructivism, as the philosophy of mathematics (Ernest, 1991: 98), has particularly supported such novel interpretations of the genesis of mathematical knowledge. Beyond that, Imre Lakatos's *Proof and Refutation* (1976), a common source for explaining the genesis of mathematical knowledge, shows how mathematics has developed into well-arranged forms marked by formalism and rigor. Through such lenses, answers to the questions "What is rigor?" and "How much rigor is needed in mathematics?" are not already fixed or given but the products of a dynamic discourse determined by the society in which mathematicians seek to show the purpose of the validity of mathematical knowledge. Because an axiomatic basis has played a starring role in the development of formal, rigorous mathematics, my next consideration is the axiomatic foundation of mathematics.

20.3 The Axiomatic Method

Robert and Glenn James's *Mathematics Dictionary* (1988) defines *axioms* in the context of mathematical systems as the basic propositions from which all other propositions can be derived. Axioms are independent, primitive statements in the sense that it is impossible to deduce one axiom from another. Euclid's fifth postulate is a well-known example of an axiom, one that took mathematicians and geometers roughly two millennia to determine whether it was indeed a postulate or could be proved as a theorem. Only in the nineteenth century did they conclude that the fifth postulate was in fact a postulate independent of the other four and could not be proven using the others. Added to that, they discovered that, unlike the fifth postulate, the other postulates could be stated in ways leading to other systems of geometries (e.g., hyperbolic and elliptical). Thus, the fifth postulate is an outstanding demonstration of the independent role of axioms and postulates in mathematical structure.

Because the conclusion of a proof of a theorem is a logical implication of the truth of the axioms, such models of deriving truth are termed *axiomatic model*s, and both Euclid's *Elements* and representative modern works such as Hilbert's *Grundlagen der Geometrie* showcase statements postulated as starting points and everything else derived from them (Mueller, 1969). Ever since the development of mathematical theory, much has been written on the axiomatic method (Wilder, 1967). Perhaps most prominently, Bourbaki's version of 1953, as quoted by Weintraub (1998) in his article, reflects that the axiomatic point of view in mathematics appears as a storehouse of abstract forms of the mathematical structures which happens without our knowledge that certain aspects of empirical reality fit themselves into these forms as if through a kind of preadaptation.

The views of Bourbaki clearly maintain that the axiomatic basis of mathematical structures represents certain aspects of empirical reality. Such an interpretation essentially derives from a Platonic view of mathematics: that mathematical truths are the propagation of preexisting reality. Here is the version of Hilbert, a master of axiomatic mathematics in relation to the role of the axiomatic method in scientific thought and mathematics:

> I believe: anything at all that can be the object of scientific thought becomes dependent on the axiomatic method, and thereby indirectly on mathematics, as soon as it is ripe for the formation of a theory. By pushing ahead to ever deeper layers of axioms ... we also win ever-deeper insights into the essence of scientific thought itself, and we become ever more conscious of the unity of our knowledge. In the sign of the axiomatic method, mathematics is summoned to a leading role in science (Weintraub, 1998: 1844).

Thus, according to Hilbert, the axiomatic method is not merely a means to establish scientific reasoning but also to gain profound insights into scientific thought in the form of axiomatic thinking.

Axiomatic method has also been used in social science, especially, in economics, in the assumption that the relationship between rigor and truth require an association of rigor with axiomatic development of economic theories since axiomatization was seen as the path to new scientific discovery (Weintraub, 1998:1845). The general form of the axiomatic method as used in sociology applies to a set of propositions summarizing current knowledge in a given field and for generating additional knowledge deductively (Coster & Leik, 1964). In that light, the axiomatic method can be viewed as a means to deduce deeper knowledge in Hilbert's sense of the word. Wilder (1967) stresses the role of the axiomatic method in introducing increased abstraction. Wilder points out that Greek mathematics performed the role of providing foundation as well as consistency. But, Ian Mueller (1969) argues that the axiomatic method used in Euclid's *Elements* differs from Hilbert's modern axiomatics even though both begin by postulating statements and deriving everything else from those postulates. Whereas modern mathematics, including Hilbert geometry, has the formal–hypothetical character of modern axiomatics, ancient Greek mathematics was not a hypothetical science in the same sense. For the Greeks, mathematical assertions were true and of interest only because they were true, which explains why Euclidean axioms and common notions are taken as self-evident truths. Wilder's position on the nature of ancient Greek mathematical assertions somewhat differs. He argues what Euclid and later mathematicians described was not a physical reality but a mental model of what their perception of physical reality had suggested. Thus, it did confront logical difficulties because the mathematical model and the logic used were derived from reality (Wilder, 1967: 124).

Wilder divides the axiomatic method into three types according to the degree of formalization. In the first, called the "Euclidean" type, the primitive terms are not treated as undefined, and the model is described only in the sense of a mental model: a model of the perception of the physical model. The second type of axiomatic method, the "naïve axiomatic" type, is the one in which we are careful to list primitive terms (such as, definition, axiom/postulate) needed to develop arguments, but we list neither logical nor set- theoretic rules by which we shall abide (say, for example, when introducing Group in modern algebra text, definitions and postulates are stated, and then theorems are proved without listing logical rules).

The second type of method is taken as the principal tool for modern research in all branches of mathematics, especially algebra, topology, and analysis. Wilder (1967: 126) continues: "The third type of axiomatics in which the logical apparatus not only enters the discussion, but is explicitly formalized, has provided one of the major tools in the foundational research." Summarizing his views on the types of axiomatic methods, Wilder concludes that modern mathematics, despite its abstraction, increasingly resembles applied science and thus interprets the three types of methods as being at work in science in one form or another.

However, Abraham Seidenberg (1975), a historian of mathematics, takes another view. He argues that though we have all been told since childhood that Euclid developed the axiomatic method and that conclusion seems amiss when *Elements* is viewed with modern hindsight (p. 263). Referring to Seidenberg's (1975) article, Yehuda Rav (2008: 138), in his own article, mentions that Euclid's Book I is not an axiomatic basis for the theorems but a theory of geometric construction for they serve to control the straightedge and compass construction. Unlike Wilder, Rav claims that it is fairly common for mathematicians to derive theorems from axioms by using valid rules of logic. However, he adds, problems arise from taking the view that ….. lacks actual evidence from the day-to-day proof of mathematicians, most of whom are not logicians and could hardly name any rule or axiom of logic, much less relate them to their proofing practices. Nevertheless, despite contrasting views that whether or not mathematics was already organized on the basis of explicit axioms, there is no question that deductive proof from some accepted principles was required at least from Plato's time (Rav, 2008: 136).

In the past two centuries, the increasing emphasis seems to have been placed on the axiomatic and rigorous development of mathematics. Against that excessive emphasis on rigor, axiomatics, and formalism in mathematics, new views have emerged that take into account the historical and cultural development of mathematics. In that line of thinking, the development of non-European mathematical traditions, including Hindu mathematics, has been found to be important because they, especially the Hindu tradition, address rigor in the absence of the axiomatic method. To address that trend, the next section examines rigor and reasoning in that tradition before the chapter commits to any philosophical reflections, namely, as a means to examine the relationships between philosophy, rigor, and axiomatics in non-European mathematical traditions.

20.4 Rigor and Reasoning in Traditional Hindu Mathematics

Recent literature has yielded more comprehensive interpretations of mathematics and philosophy. Among the most important interpretations is that mathematics is an intellectual cultural product whose various routes of development can be critically examined from social and cultural perspectives. Such sociocultural interpretations have shed new light on mathematics, particularly on the contributions of non-European traditions to its overall development. As a result, mathematics in non-European civilizations, including Hindu mathematics, also known as "Indian mathematics," has especially received sustained attention. In the history of mathematics, the lack of axiomatic–deductive proofs has been a common charge against Hindu mathematics, which has consequently been downgraded in the ranks of mathematical traditions. Carl Boyer (1968: 238), in *A History of Mathematics*, furnishes evidence of that trend:

> Although in Hindu trigonometry there is evidence of Greek influence, the Indians seem to have had no occasion to borrow Greek geometry, concerned as they were with simple mensurational in some form as well as axiomatic– deductive method of proof rules.

To show the treatment of Hindu mathematics by numerous Western writers of the history of mathematics, Joseph (1994) has pointed out that many commonly available books on the field's history either declare or imply that whatever the achievements of Hindu mathematics, they have never had any notion of proof.

Indian historians of Hindu mathematics have remarked that the greatest charge against Indian geometry in particular and mathematics in general is indeed the lack of deductive–axiomatic proof that was so beloved to the ancient Greeks (Amma, 1999: 3). However, Amma adds that some commentaries and independent work have preserved proofs and derivations of complicated mathematical series showing that early Indian mathematicians were also not satisfied unless they could prove and derive results. On that topic, Amma mentions an important difference between Greek proof and its Hindu counterpart, that whereas the former is built upon a few self-evident axioms, the latter aims at convincing students about the validity of theorems by way of visual demonstration. A more recent critical interpretation of Indian mathematical traditions, one especially examining the *Gaṇita-Yukti-Bhāṣā* (Ramasubramanium et al., 2008: 267), holds the following:

> Many of the results and algorithms discovered by the Indian mathematics have been discovered in some detail. But little attention has been paid to the methodology and foundations of Indian mathematics. There is hardly any discussion of the processes by which Indian mathematicians arrive and justify their results and procedures. And, almost no attention is paid to the philosophical foundations of Indian mathematics, and the Indian understanding of the nature of mathematical objects, and validation of mathematical results and procedures.

Regarding the history of mathematics on the Indian subcontinent, much attention has also been given to extremely large numbers and significant developments in algebra and trigonometry. Ernest (2009: 200) speculates that attention to large numbers with decimal fractions, possibly made possible by virtue of an advanced decimal place value system, might have aided in conceptualizing a large number of series expansions in Kerala between the fourteenth and sixteenth centuries, as well as contributed much of the basis for calculus, which is traditionally attributed to mathematicians in seventeenth- and eighteenth-century Europe. Nevertheless, the lack of deductive–axiomatic proof in Hindu mathematical traditions has been viewed as a great lapse. The disparity in interpretations raises the question of whether the development of Indian mathematics indeed addressed valid, rigorous mathematical proofs, as well as the question of whether mathematics can be rigorous without an axiomatic foundation.

To begin to answer those questions, I first review the reasoning involved in *upapatti*, in the Hindu mathematical tradition. Although the tradition lacks formal proof in the sense of Greek-based Western mathematics, it has *upapattis* as a means of convincing argumentation to show students the validity of theorems through visual demonstration as an acceptable form of proof in geometry (Amma, 1999: 3). Joseph (1994: 197) mentions that roughly 2000 years ago, a great deal of attention in Indian mathematics was paid to providing *upapattis*, some of which have recently been noted by European scholars of Indian mathematics. In the following subsections, I examine some of the notable features of the status of proof, rigor, and reasoning in Indian mathematics in relation to upapattis.

Nature of upapattis A upapatti is a means of establishing the validity of mathematical truths and removing doubts about such validity. Indian mathematicians agree that results in mathematics cannot be accepted to be valid unless they are supported by a upapatti or *yukti*, a Sanskrit term used to denote some scheme of demonstration (Ramasubramanium et al., 2008: 288) that, even today, refers to some method or technique to show how results hold true and/or how a problem can be solved. In geometry, *Upapatti* seems to be a construction able to reveal how results come to be true or a series of steps involved in algorithms with justification (e.g., the Euclidean algorithm for finding the greatest common divisor, called *kuttaka* in Hindu mathematics).

As mentioned by Amma (1999: 3), proof in Indian mathematics aims at convincing students of the validity of theorems with a visual demonstration of geometry, mostly viewed as a construction to show how results hold true. In fact, most students seem to rely on the scheme of construction to understand how results come to be true and/or how facts hold true. Even mathematicians are said to do so to convince themselves first, while systematic proof or demonstration is required only at a later stage. That is, many basic properties (e.g., axioms, definitions, and rules of inference) are taken for granted while working and only explicitly formulated for publication. If those properties are omitted, however, then the core structure of mathematical derivation contains the scheme to show how facts come to be true. In a sense, it is hardly different from what medieval Hindu mathematicians and astronomers did.

Upapattis in geometry are not a Euclidean type of proof or demonstration but can be somewhat compared with constructions in Euclidean geometry (e.g., construction of a triangle with a straight edge and compass) with justification. The problem, albeit not in the form of a theorem, is given with a brief scheme of how to make the construction. For example, “To draw a square equal to the difference of two squares” (Amma, 1999: 45), it states:

> Wishing to deduct a square from a square one should cut off a segment by the side of the square to be removed. One of the lateral side of the segment is drawn diagonally across to touch the other lateral side. The portion of the side beyond this point should be cut off. (English translation based on the *Apastamba Sulvasutra*)

In demonstrations of geometry (e.g., to convert a rectangle into a square), the construction of schemes involves terms such as “to cut off a part,” “juxtaposed,” and “fill in” in ways similar to making paper cuttings and fillings. As such, it does not show any abstract concept of demonstration such as that in Greek geometry. Because geometry seems to have a relatively calculative nature concentrating on metric notions of length and area and using algebraic notion, geometric algebra could be a major contribution to early Indian mathematics, as mentioned by Seidenberg (1975: 289):

> Becker accepts, though hesitatingly, an early date for the Indian geometry (the 8th century B.C.). Since the sulvasutra have the theorem of Pythagoras, Becker looks to old- Babylonia for the source of Indian geometry. … The Sulvasutras convert a rectangle (say, a by b) in a typical geometro–algebraic way; the old Babylonian would simply multiply *a* by *b* and take the square root. The Indian priest constructs the side of the required square; the old Babylonian computes it. So, the geometric algebra of the Indians could not very well have come from the old Babylonians. And it could not have very well come from the Greeks, at least not in 8th century B.C. And if the Greeks got its geometric algebra from the Indians, then the history of Greek mathematics has to be rewritten.

That excerpt captures the nature of Hindu mathematics and its style of reasoning. Upapattis, as the means of the demonstration and justification of mathematical truth, use both geometrical and algebraic approaches in deriving, for example, the Pythagorean theorem, as done by the great twelfth-century Hindu mathematician Vaskaracharya (Joseph, 1994). The combined geometrical–arithmetical approach can also be observed in the development of infinite series expansion, including the series for the expansion of π (π = 4–4/3 + 4/5–4/7 + …) by mathematicians in southern India in the fifteenth century (Amma, 1999: 166), regarded as one of the more sophisticated developments of mathematical reasoning based on similar triangles and the summation of series. It is also a prime example of the extension of mathematics to infinite processes.

Nrsimha Daivajńa (1507) explains that the *phala* (“objective”) of a upapatti is *pánditya* (“scholarship”) and the removal of doubt that can lead one to reject misinterpretations made by others due to *bhranti* (“confusion”), among other causes (as cited by Ramasubramanium et al., 2008). Such an interpretation of upapattis seems to have two major functions: to develop advanced intellectual integrity and to be free from any error or lack of clarity. In a sense, it relates to the process of ensuring rigor, for it needs to be checked against errors in mathematical practice. The purpose of that mathematical development is to make mathematics valid. On that note, establishing the validity of mathematical knowledge by consensus among mathematicians and astronomers seems to have long been unique in Hindu mathematical traditions. Indeed, large gatherings of scholars, especially in the Jain community, were often held to (dis)confirm the validity of whatever subject was being discussed.

Increased attention to numbers and algorithms More attention to numbers, algorithms, and calculation has been a common characteristic in the development of Hindu mathematics, presumably made possible by the development of a place value numeration system as a fundamental basis for the development of mathematics in general. Ernest (2009) highlights the importance of Indian emphasis on numbers and calculation for the possible development of infinite series expansions by Kerala mathematicians between the fourteenth and sixth centuries. However, as mentioned, Hindu mathematics has commonly been criticized for lacking proof for valid mathematical results despite the field's having developed numerous algorithms for facilitating calculation. Indeed, Hindu mathematics places greater emphasis on numbers and algorithms than methodology and foundations for reasoning. To understand the nature of the Hindu mathematical tradition regarding proof, rigor, and reasoning, it may be helpful to consider the underlying perspective.

Clarifying the nature and validation of mathematical knowledge, *Gaṇita-Yukti-Bhāṣā* (Ramasubramanium et al., 2008) mentions that the classical Indian understanding of the nature and validation of such knowledge seems to be rooted in the larger epistemological perspective developed by the Nyaya school of Indian logic. The distinguishing features of Nyaya logic important to the present discussion include the logic of cognitions (*jnana*), not propositions as in the Greek system, and the lack of any concept of pure formal validity distinguished from material truth. The text adds that Nyaya logic does not distinguish necessary from contingent truths (i.e., analytic and synthetic truths) or accord the logic (*tarka*) of proof by contradiction, as used in the development of Greek mathematics. Although the method of proof by contradiction is used occasionally, no upapattis support the existence of any mathematical object merely on the basis of logic alone (Ramasubramanium et al., 2008: 289). Instead, such objects seem to be partly based on the nature of mathematical knowledge, as considered in the next subsection.

The nature of mathematical knowledge Because the development of Hindu mathematics did not subscribe to any concept of absolute certainty but to an elevated intellectual level of mathematics free from confusion achieved by sharing in assemblies of mathematicians, it seems that rigor in mathematical proofs or upapattis developed in a way somewhat similar to Lakatos's understanding of the genesis and justification of mathematical knowledge. In modern terms, such knowledge may thus be regarded as a quasi-empiricist type of knowledge. In any case, it differs entirely from the Greeks' notion of indubitable and infallible mathematical knowledge. In the Hindu mathematical tradition, the term for *mathematics* (*Ganita)* literally means "the science of calculation," and the field ranks supreme among all of the secular sciences (Datta and Singh, 1935: 7) in reference to *Vedanga Jyotisa* (1200 B.C.E.): "As the crests on the heads of the peacocks, as the gems on the hoods of snakes, so is the Ganita as the top of the sciences known as the Vedangas."

Even though mathematical knowledge is given a supreme position among all secular sciences in the Hindu tradition, it has no concept of absoluteness, which may be one reason why the Hindu mathematical tradition did not subscribe to the concept of absolute truth as in Greek tradition. In turn, the tradition had no need to subscribe to the notion of perfect rigor in mathematical derivation based on self-evident truths and the rules of logical inference as done by the Greeks. That difference in the nature of mathematical knowledge seems to be a primary reason for the difference in rigor and reasoning between Hindu and Greek mathematics.

20.5 Philosophical Reflection

The philosophy of mathematics generally does not treat specific mathematical questions but instead attempts to present thoughts, produced through reflection, on what mathematics is and what mathematicians do and to contemplate the present state of affairs in mathematics. It is not mathematics; on the contrary, it is about mathematics, as mentioned by Rényi (2006) in "A Socratic Dialogue on Mathematics." It might be one of the reasons why Hersh, in the preface of his book *What Is Mathematics Really?* (1999), states that Richard Courant and Herbert Robbins in their book, *What Is Mathematics?* Courant and Robbins (1941/1996), do not answer the question raised by their book's title, a book nevertheless praised by Albert Einstein and Herman Weyl as a work approaching perfection and an astonishing examination of the extent of what mathematics is. Although *What Is Mathematics?* has been exceptionally useful for understanding many fundamental concepts of mathematics, it does not explicitly deal with the question of what mathematics is, because that question is basically one of philosophy. The task of the philosophy of mathematics is to "reflect on" and "account for" the nature of mathematics (Ernest, 1991: 3), as Ernest (1998) examines in detail. In this section, my focus is to examine philosophical reflections on axiomatic models and the concept of rigor that form the basis of formal mathematics.

The twentieth-century development of the three schools of logicism, formalism, and constructivism (incorporating intuitionism) was primarily guided by the purpose of establishing a firm foundation for mathematics with absolute certainty. By extension, Ernest (1998: 53) characterizes philosophers of mathematics as philosophically inclined professional mathematicians, who, with their foundationalist programs, focus on the philosophy of mathematical concerns and problems. However, the philosophers of mathematics that developed those schools could not establish a foundation of absolute truth in mathematics despite numerous attempts, because Kurt Gödel's incompleteness theorem checkmated the foundationalists' programs, especially the Hilbert program (Hersh, 1999: 138). If philosophers other than mathematicians were involved in the development of the philosophy of mathematics, then the case could have been different. Arguably, the interest and intention of well-known mathematicians involved in the establishment of the foundation of the philosophy of mathematics created a new perspective on the relationship between mathematics and philosophy. The maverick philosopher Hersh (1999: 151) quotes Hilbert's motive for engaging in the philosophy of mathematics:

> I wanted certainty in the kind of ways in which people want religious faith. I thought that certainty is more likely to be found in mathematics than elsewhere.
>
> … Having constructed an elephant upon which the mathematical world would rest, I found the elephant tottering, and proceeded to construct a tortoise to keep the elephant from falling.

That quotation reveals that the motive behind Hilbert's formalistic philosophy was to discover the means of achieving certainty in mathematics. However, that motivation indicates an intention of imposition, not one of internal necessity.

In a sense, the axiomatic method seems to be intimately related to mathematics as hereditary stresses as demanded for its own sake. In another sense, the method is not genuinely inherent in mathematics. In studying the evolution of mathematical concepts, Wilder (1967) distinguishes two types of influences: cultural influences, which he referred to as "hereditary," and environmental influences, meaning that the development of most early mathematics, including arithmetic and geometry in their primitive forms, was due to environmental aspects. By contrast, the axiomatic development of mathematics, such as captured in Euclid's *Elements*, was chiefly due to hereditary stresses. In that case, the hereditary aspect indicates that the arrangement of mathematical concepts in the form of structures (e.g., in Euclid's *Elements*) was mostly forced by an internal need to cope with paradoxes (e.g., Zeno's paradoxes) and problems, including the problem of the incommensurability of the sides and diagonals of rectangles and squares. Wilder (1967: 115) writes, "Most historians seem to agree that crises, attendant upon the attempts to cope with paradoxes such as those of Zeno, compelled the formulation of a basic set of principles upon which to erect the geometrical edifice." The axiomatic method is, without a doubt, the single most important contribution of ancient Greece to mathematics, which tends to deal with abstractions and which recognizes that proof by deductive reasoning offers a foundation for mathematical reasoning (Kleiner, 1991).

Both Wilder and Kleiner attribute the development of the axiomatic method to the necessity of mathematics. Mueller, by contrast, critically counters Zoltan Szabo's position that the shift from empirical to pure mathematics was closely connected with the idealistic, anti-empirical character of Eleatic and Platonic philosophy. He insists that a Euclidean derivation is a thought experiment of a certain kind, an experiment intended to show either that a certain operation can be performed or that a certain kind of object has a certain property, and hence Euclidean derivations are quite different from Hilbertian ones, which are usually said to involve no use of spatial intuition. He mentions that the evolution of the axiomatic method is explicable solely in terms of the desire for clarity and order in geometry while the philosophical conceptions of mathematics, such as those of Plato and Aristotle, were more probably the result of philosophically colored reflection on mathematical practice than causes of that practice. Referring to Kline (2008), Yehuda Rav (2008) mentions that it is generally accepted that the organization of mathematical knowledge on a deductive basis has roots in the teaching of Plato. He further refers to Kline (2008) to mention that Euclid who organizes the *Elements* in the third century B.C.E lived in Alexandria and it is quite certain that he was trained as a student in Plato's academy.

Another historian of mathematics, Seidenberg, seems to be one of the harshest critics of the axiomatic development of Euclidean geometry. In his 1975 article "Did Euclid's *Elements*, Book I, Develop Geometry Axiomatically?", Seidenberg states that we have all been told in childhood that Euclid developed the axiomatic method, but that conclusion seems amiss after viewing *Elements* with modern hindsight. According to Seidenberg, the first three postulates—"To draw a straight line from any one point to any point," "To produce a finite straight line continuously in a straight line," and "To describe a circle with any center and distance"—are bona fide axioms in the sense that they serve to control the straight edge and compass; however, they are not axioms for the development of geometry and indeed reveal nothing about space. Referring to Seidenberg's article, Yehuda Rav (2008: 137) quotes the same paragraph and summarizes it by asking, "Could it be that, by insisting on the axiomatic method, we are viewing *The Elements* from a false perspective." On the contrary, another historian of mathematics, T. L. Health, admired the genius of Euclid and concluded that the fifth postulate was a postulate, not a theorem to be proven, which took 20 centuries of attempts to finally realize (as cited by Seidenberg, 1975: 271).

The above examples are representative of not only contrasting views on the development of the axiomatic method in the pioneering work of Euclid's *Elements* but also of the necessity of the axiomatic method as the working basis of mathematics. Yehuda Rav (2008:30) argues as to where the axioms do come and where the axioms do not come in, and, where it seems to be indispensable and where subsidiary. He says unlike the indispensable place of axioms in foundational studies and mathematical logic in general, when looked at other branches of mathematics, it is striking what subsidiary role, if any, is played by axioms other than in geometry or in the introduction of structural axioms that define the subject matter of the theory (such as in group theory). In analyzing the situation, he writes, from its starts analysis stands out as an example of non-axiomatic edifice, but Dedekind's essay on continuity and irrational numbers was intended as an axiomatic basis of analysis. He claims that analysis has never been axiomatized as a deductive theory as it developed. The rigorization and axiomatization of calculus were made mostly in 19th and 20th century with arithmetization and giving precise definitions of the key concepts (such as, function, limit, continuity, derivative, and integral) and developing rigorous proofs.

What seems to me is that the axioms and definitions are also needed in mathematical development for they characterize/specify the abstract mathematical entities (for the mathematical system) which is a kind of creation/construction in mathematics. Otherwise, how could one characterize mathematical notion like groups, real numbers, and complex numbers? It is important to note that intuitively appealing set of counting numbers were needed to be axiomatized as Peano's postulates only in the nineteenth century to lay foundation for arithmetic. Such situations might also indicate that the axiomatic basis of mathematics became necessary at some point in its development sooner or later depending on the nature and development of the subject considered. Peano's postulates characterized the notion of counting numbers, provided the basis for systematically going forward to any counting numbers, and gave a basis for mathematical induction. The other thing to be noted is that the perceptive philosophers, like Plato and Aristotle, might have perceived the necessity of deductive reasoning to go forward from some basic assumptions which have been found most dependable in mathematics. Not only the modern foundationist philosophers (like Frege, Russell, Hilbert, and Godel) but also logicians and mathematicians have also felt the necessity of an axiomatic model in mathematics. Although it is said that working mathematicians rarely work on the basis of an axiomatic model in most areas of mathematical construction, most mathematicians seem to have strong faith in it. In many cases, a working mathematician may create mathematics constructively, and then in order to justify and communicate it to the circle of mathematicians, he/she may need to seek a convincing basis which in turn draws on an axiomatic basis as the valid reference. This is why both the mathematician and the philosopher may have some common concerns on the rigor and axiomatic basis of mathematics.

Traditionally, rigor in reasoning is based on axiomatics, as demonstrated in Kitcher's (1981) definition, because central to the idea of rigorous reasoning is that it should contain no gaps and that it should proceed by means of elementary steps. In the development of the mentioned three schools, especially formalism and logicism, the rigorous development of mathematical thinking has been the primary function of the philosophy of mathematics and has been intimately related to the fulfillment of the philosophical purpose of achieving the absolute certainty of mathematical knowledge. But, rigor is not only dependent on axiomatics; however, as illustrated by Kitcher (1981), Isaac Newton and Gottfried Wilhelm Leibniz (the founders of calculus) felt problems in his interpretation of a derivative. Kitcher shows how Newton faced problems in interpreting the derivative in a rigorous way. In that case, rigor might have been considered differently in geometry and calculus. The arithmetical interpretation of the notion of integration independent from geometrical intuition might be regarded as rigorous in analysis insofar as it further clarifies and generalizes the concept in the domain of real numbers. The development of calculus in the nineteenth century made many concepts clear by formulating concepts through definitions, and intuitive concepts explicitly formulated that helped to conceptualize them precisely resulted in the further conceptualization of the subject. Such cases might represent rigor in non-axiomatic settings. If we consider rigor in light of its precise meaning as in calculus and analysis, then it might seem to be more internal to mathematics and independent from philosophy.

Much of the literature indicates that the development of the Greek axiomatic method was closely connected with the dialectical method in Greek philosophy. Referring to Aristotle's *Posterior Analytics*, Mueller quotes the Greek philosopher's view insisting that the assumptions of science be not merely true but also primary, immediate, and more known causes of the conclusion drawn from them. Such views indicate the influence of philosophy on the axiomatic method and on the development of rigor in modern mathematics. Such is especially the case of the Eurocentric mathematical tradition being the custodian to ancient Greek thinking. Since the Eurocentric mathematics curriculum has been globalized in the name of valid mathematics, thus in practice, it has been the only mathematics that we think about and perform. That circumstance explains the necessity of examining the nature of the mathematical development of non-European culture, which is found to differ in nature from Western mathematics. In that context, the development of Hindu mathematics seems to be an important one for shedding light on the nature of mathematical knowledge.

Hindu mathematics, despite being rich and contributing significantly to the development of mathematics in general, lacks formal axiomatic proof in mathematics, as mentioned. As such, various questions arise: If so, does not the development of Hindu mathematics contain valid and rigorous proof? And what then is the philosophical basis behind them? The Hindu mathematical proof of the upapatti, a means of convincing argumentation for students to grasp the validity of theorems via visual demonstration as an acceptable form of proof in geometry (Amma, 1999: 3), is compatible with recent views on thinking and teaching mathematics, in which proofs are recognized as convincing arguments in constructive ways. With upapattis as a means of establishing the validity of mathematical truths and removing doubt, Indian mathematicians agree that results in mathematics cannot be accepted to be valid unless they are supported by upapattis as mentioned in Ganitayuktibhasa (Ramasubramanium et al., 2008: 288). On that topic, Ramasubramanium et al., refers to Nrsimha Daivajńa's (1507) assertion that the *phala* ("objective") of a upapatti is *pánditya* ("scholarship") and the removal of doubt that can lead one to reject misinterpretations made by others due to *bhranti* ("confusion"), among other causes. Thus, rigor is achieved through an elevation of intellect accompanied by the removal of any confusion or error, while validity is achieved by consensus among mathematicians.

Such a view on mathematical thinking seems to be somewhat similar to the quasi-empiricist view on mathematics stating that mathematics is a dialogue between people tackling mathematical problems (Lakatos, 1976). The quasi-empirical nature of Indian mathematics, at least to some extent, makes it analogous to the natural sciences. In the chapter "The Genre of Indian Mathematics," Khim Plofker (2009) mentions, via an Indian source, that Indian mathematics mostly served as the handmaid of astronomy, while credit for divorcing mathematics from astronomy is particularly due to Bakhshali manuscript and mathematicians Mahāvīra, from the ninth century, and Sridhara, from the ninth and tenth centuries. As a result, Hindu mathematics developed in the service of religion. The religio-astronomical orientation of Hindu mathematics differed from that of the development of Western mathematics, which seems to have been motivated by a combination of mathematics and theology beginning with Pythagoras and, from there, characterized religious philosophy in ancient Greece, in Europe in the Middle Ages, and the West in modern times through Immanuel Kant (Russell, 1957). According to Russell (1957:37), there has been an intimate blending of religion and reasoning, of moral aspiration and logical admiration for what is timeless, all of which comes from

Pythagoras and distinguishes the intellectualized theology of Europe from the more straightforward mysticism of Asia. Russell characterizes the nature of Western intellectual thinking as that which gave rise to the timeless truth of mathematics. That view on the development of Western mathematics indicates both a philosophical and mathematical basis for the development of the axiomatic method and rigor based on it, as well as differs from the relatively naive basis of the Hindu mathematical tradition.

20.6 Summary and Conclusion

Rigor in reasoning and the axiomatic basis of proof seem to be central to a mathematical proof in general. Whereas rigor is primarily guided by an intention to achieve the flawless derivation of mathematical truths, the axiomatic model of proof seems to have resulted from ancient Greeks' intellectual and cultural tradition motivated particularly by the philosophical thinking of Plato and Aristotle. The dynamic between them can be examined along at least two lines of thinking. On the one hand is the thinking of Mueller (1969), which insists that the evolution of the axiomatic method is explicable solely in terms of the desire for clarity and order in geometry and that the philosophical conceptions of mathematics, including those of Plato and Aristotle, were more probably the result of philosophically colored reflection on mathematical practice than on the causes of that practice. On the other hand is Wilder's (1967) thinking that the development of the Greek axiomatic method was closely connected with the development of the dialectical method in Greek philosophy, which is also Szabo's (1969) view. Wilder (1967: 115) writes "Most historians seem to agree that crises, attendant upon the attempts to cope with paradoxes such as those of Zeno, compelled the formulation of a basic set of principles upon which to erect the geometrical edifice." He pointed out that the role of the axiomatic method in Greek mathematics seems to have been a twofold objective, the provision of foundation which at the same time met the current charge of inconsistency, where the later one may have been the motivating factor (p.117). However, formal axiomatic thinking in mathematics from the twentieth century was guided by the purpose of establishing consistent mathematical truths, which, in effect, demanded rigor based on axiomatic and formal logic. In turn, the twentieth-century foundationalists such as Hilbert and Russell (with Whitehead) put forth great effort in different ways to lay a firm foundation for the absolute certainty of mathematical knowledge and protect it from contradictions and antinomies that arose at the turn of the century (Ernest, 1991: 8). But the humanist/maverick philosopher Reuben Hersh mentions that such attempts have been checkmated by Godel incompleteness theorems. Maverick and social constructivist thinkers have explained the mathematical rigorous development as a cultural function.

Social constructivists view the precise development of formal reasoning in mathematics as indicating a higher level of cultural growth. To explain its cause, Sal Restive (1994: 216) writes that the greater the level of cultural growth, the greater the distance between the material ground and its symbolic representation, and the more that the boundaries separating mathematics worlds from each other and from social worlds thicken and become increasingly impenetrable. Such an interpretation explains why pure mathematics seems to be isolated from the social world. A similar reason might apply to its relationship with philosophy, because the philosophy of mathematics is also an outcome of cultural growth that addresses mathematics from a different perspective. Because philosophy is not mathematics but about mathematics as seen from a distant position (Rényi, 2006), philosophy takes a distant view of mathematics, one that seems to be remote but is in fact powerful and provides the grounds for the existence and justification of mathematical truths by characterizing the nature of mathematical knowledge. Nevertheless, in the case of twentieth-century foundationalist philosophers such as Frege, Hilbert, and Russell, philosophy plays a different role. Indeed, well-known mathematicians such as Hilbert sought ground on which to establish mathematics as being absolutely true. To that purpose, he purposively imposed his theory of meta-mathematics to make that foundation rigorous. Being a mathematician–philosopher, he also attempted to rescue philosophy by providing that firm foundation and gave rigorous treatment to Euclidean geometry, a prime example of a modern axiomatic model. Thus, his meta-mathematics can be viewed as an imposition for the purpose of creating absolute rigor in proof. In that and other ways, philosophy has been related to rigor and the axiomatic method.

By contrast, the remarkable development of Hindu mathematics (Almeida & Joseph, 2009), one without axiomatic rigor or any well-formed philosophical presumptions, tells a different story of the development of mathematics. Even though Hindu mathematics lacks proofs based on axiomatics, it developed reasoning for the clarification and validation of mathematical truths in upapattis, a form of convincing argumentation. Hindu mathematics was also developed in the service of religion and bears a religio-astronomical orientation (Amma, 1999: 4). Even so, the religio-astronomical orientation of Hindu mathematics differs from the orientation of the development of Western mathematics, which seems to have been motivated by a combination of mathematics and theology beginning with Pythagoras. According to Russell (1957: 37), the intimate combination of religion and reasoning, of moral aspiration and logical admiration for what is timeless, comes from Pythagoras and distinguishes the intellectualized theology of Europe from the more straightforward mysticism of Asia. Thus, the development of timeless truth in mathematics has been based on axiomatics and logical rigor, largely motivated by the desire for absolute truth in mathematics. The axiomatic method is taken to be the single greatest contribution of ancient Greek thinking and thus remains dominant in mathematics. By the same token, not having been motivated by such thinking, even with the active contact of India and Greeks in centuries past, is viewed as being one of the great lapses of Indian scholarship (Amma, 1999: 4).

Due to the lack of any deductive or axiomatic structure in mathematical results, Hindu mathematics may have missed opportunities to face logical problems such as mathematics in Greece faced during its development. After all, the method of proof by contradiction is used rarely and only to show the nonexistence of certain entities.

From a sociocultural perspective, that development should not be viewed as a lapse, but as a different route toward mathematics that did not require using the Greek style of deductive–axiomatic thinking. Because Hindu methods were pedagogically oriented to convince students of the validity of mathematics, many well-known scholars composed commentaries in addition to their original contributions, which also became of pedagogical use. The oral transmission of knowledge in *gurukul* education also preserved those methods. Although those methods may seem to represent naive mathematical thinking, one not bothered with the problem of absolute certainty and geared toward the solution of problems without any presumed conception of ideal mathematics, they might also direct us toward thinking in alternative ways. At the very least, they convey that an emphasis on rigor and axiomatics is not the basic universal character of mathematical thinking or teaching.

In light of the above discussion, it seems that the axiomatic method and the concept of rigor based on it are not inherently contingent in mathematics but instead motivated by a particular intellectual–cultural development. Even so, rigor as the flawless, clear, precise, and organized elevation of intellectual thinking in mathematics seems to be more innate to mathematics in the sense that both proof and upapatti share the common purpose of justification and the elevation of intellect (*budhi-vridhi*). Despite differences in the nature of their development, they seem to share the common basis of mathematical objectivity, as also seen in the development of Egyptian geometry, which was guided by the purpose to measure land, and the development of Hindu geometry, guided by the religious purpose of making altars and fireplaces. Such objectivity in mathematics might be a common motivation among mathematicians, and the logical axiomatic method used in mathematics is the rule for organizing and preserving the certainty that mathematicians value.

Indeed, most mathematicians seem to believe in some kind of certainty in their mathematical discoveries: that certain sudden "A-ha!" or "Eureka!" moment during their mathematical thinking (Byers, 2007: 329). However, maverick philosophers Hersh and Steiner (2011: 54) interpret that feeling of certainty as an aesthetic pleasure—a satisfaction with deep, clear thinking—and as simply the emotional roller coaster of discovery. Nevertheless, belief in the certainty of mathematical knowledge seems to be common among users of mathematics, including teachers and students, most likely, I think, due to the usefulness and dependability of mathematics (Shrestha, 2019). Certainty also seems to be the common motivation of most mathematicians and philosophers, though they differ in their vocations. Ancient Hindu thinkers ranked *Ganita* as being supreme knowledge above all other knowledge but did not subscribe to the notion of absoluteness. The axiomatic method, as a special attribute of the ancient Greeks' intellectual development and its pursuit of absolute truth, seems to have a unique philosophical orientation in addition to a mathematical basis. However, exceeding emphasis on axiomatics and rigor based on it, including that endorsed by Hilbert and Russell (with Whitehead), seems to be exceedingly intentional and can be viewed as an intended imposition (motivated by the desire to meet the crisis in the foundation caused by the set-theoretic foundation), even if all mathematical constructions in a sense are intentional to some extent.

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Chapter 21
Idealism and Materialism in Mathematics Teaching: An Analysis from the Socio-epistemological Theory

Karla Sepúlveda Obreque and Javier Lezama Andalon

21.1 Introduction

Teaching of mathematics is a part of social sciences. However, mathematical knowledge itself does not have a clear connotation or at least a single acceptance by the entire population. Philosophy of mathematics has tried to find ontological and epistemological answers to the problem of mathematics and its nature. The discernment of its origin and nature have generated different philosophical positions throughout history. Sometimes these positions have not only been different but also contrary to each other. For Zalamea (2021), the efforts of the philosophy of mathematics have focused on answering issues related to its being and nature, but it is still pending to deal with historical and phenomenological issues related to this knowledge.

These ways of thinking and understanding knowledge, reality, and mathematical knowledge, in particular, are present and expressed in the classroom during the activity of teaching mathematics. Its manifestations are not always perceptible and can become unconscious in teachers and students. Sometimes they are deliberate choices made by teachers according to their ways of thinking or understanding reality.

Two philosophical currents that understand mathematical knowledge in a different and contrary way are idealism and materialism. Both are present in the teaching of mathematics in the classroom. The aim of this chapter is to reflect on their expressions in the classroom and their implications. The reflection we present is based on an approach of the socio-epistemological theory.

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Idealism as a philosophical current is commonly related to the philosophy of Plato, who proposes that the authentic reality is in the world of ideas, and not in the reality sensible to our senses. This idea corresponds to *a priori* conceptions of knowledge. For Plato, ideas are the true reality, because unlike the sensible and changing world captured by our senses, ideas are eternal and immutable. From this perspective, the senses give us knowledge of the particular, but the universal is only attainable through reason. For its part, German idealism, with representatives such as Fichte, Schelling, or Hegel, rejects the notion of noumenon, nothing exists beyond known reality. For Echegoyen (1997) German idealism enhances the active role of the cognizing subject and affirms that all aspects of known reality are a consequence of its activity. For his part, Zalamea (2021) explains that epistemological idealism does not need to rely on real correlates, because its truth values are ideas.

In opposition to idealism, materialism as a philosophical current grants reality a material character, which is governed by the laws of motion of matter. For Reyes (2020), in materialism, consciousness has a secondary character. For philosophical materialism, matter is not created by the materialization of an "absolute idea" or of a "universal spirit," it exists eternally and develops by the laws that govern movement. Philosophical thought, matter, nature, and being are an objective reality that exists outside our consciousness and is independent of it. Thus, thought is a product of matter and is elaborated by our brain. In other words, knowledge is a posteriori.

Social epistemology as a theory of educational mathematics deals with the study of didactic phenomena linked to mathematical knowledge, assuming the legitimacy of all forms of knowledge, whether popular, technical, or cultured, since it considers that they, as a whole, constitute human wisdom (Cantoral et al., 2014). The socio-epistemological research program differs from classical programs because it is concerned with explaining the social construction of mathematical knowledge and institutional diffusion. Considering that mathematical knowledge is the result of a process of social construction in situated contexts gives social epistemology a character of a materialistic theory.

Socio-epistemological theory is concerned with the study of man doing mathematics in specific contexts. With this, it makes up for the lack of attention that the philosophy of mathematics has paid to the historical contexts and situations that give rise to mathematical knowledge.

Socio-epistemological studies are characterized by problematizing knowledge, historicizing it, and dialectizing it. This theory takes elements from mathematics and social sciences. From mathematics, it considers its cultural dimension, and from the social sciences, it considers the acceptance of the construction of shared meanings. A preliminary statement of the socio-epistemological research program is that mathematics is a human creation situated in particular socio-historical contexts.

For some, reality and knowledge have been understood as matters in *general*. For example, for idealism concepts such as immutability and universality are endorsements of absolute general truths. Sometimes it has been thought that these *general matters* are governed by general laws, such as the invariable laws of nature.

Newton and Linnaeus thought that nature and organic beings were general and immutable matters, but today we know that this is not the case. Nature and all organisms not only mutate, but their changes are the product of their own stories. That is, the current state of all things and reality are a *particular* historical result.

In Smith (1982) and Ricardo (2001), reality and knowledge are understood as the product of the individual action of man on his environment to obtain subsistence conditions. This way of thinking was called Robinsonades by the nineteenth-century materialists.

Marx (2007) explain that in the Robinsonades of the eighteenth century, presented a subject isolated from society who manages to subsist alone in nature, becoming the starting point of history. This idea that conceives the possibility of an individual without a social bond is qualified by materialists as something unrealizable. To refute the idea of an individual devoid of a social bond, Marx (2007) compares it to the possibility of the emergence of language without individuals.

According to materialist positions, the existence of a human being without social fabric isn't possible. The current social density makes impossible the existence of the *individual* outside of society. The social reality and the set of historical elements of a particular moment are those who imprint the determining characteristics on the production processes and the products involved in these processes. The products as products and at the same time inputs for new productions are influenced by their historical context of production. The same happens with the production of mathematical knowledge, as a social, historical, and territorially situated production.

In this way, knowledge understood as a human product arises from a production process where the man who produces it is also transformed, going from a producer of knowledge to a product of his own production process. In sum, man is no longer the starting point of history, but a product of history. In short, man and knowledge exist and are the product of their social-historical reality *in particular*. This way of understanding knowledge happens for the production of mathematical knowledge and situates it as a historical product.

The socio-epistemological theory does not consider mathematical knowledge "in general," because from the historical understanding of the production processes, mathematical knowledge is understood as a "particular situated" product. Socio-epistemology conceives man as a gregarious being, part of a historically determined social group, which is why it conceives of knowledge as a particular historical product. Socio- epistemological studies understand and address mathematical knowledge in four dimensions: epistemological, didactic, social, and cognitive.

A principle of socio-epistemology that accounts for this way of thinking is the principle of epistemic relativity. This principle validates various ways of knowing and meaning in mathematics. Other socio-epistemological principles that show the materialistic nature of the theory are contextualized rationality, progressive resignification, and the normative principle of social practice. Regarding the value of mathematical knowledge, for Cantoral (2013) it consists in the use that people can make of it. Production and use value form an indivisible unit in the very production of mathematical knowledge.

The socio-epistemological theory seeks to explain the social construction of mathematical knowledge, considering it as an epistemological alternative that incorporates *the social* component. Understanding *the social* not only as the idea of interaction between people of a social group, but from socio-epistemology it is described as the set of regulations that regulate the behavior of groups. In the interview with OEI, Cantoral (2019) explains that these groups can be the inhabitants of a geographic region or the students of a course in a primary school. This chapter will analyze the presence of idealistic and materialistic positions in the teaching of school mathematics from a socio- epistemological perspective.

21.2 Mathematical Knowledge: A Vision from Idealism and Materialism

Idealism and materialism as philosophical currents, like the rest of philosophy, seek the ultimate meaning of the consciousness of the world and of all forms of existence. In the case of mathematical knowledge, these currents of thought problematize the duality "universality or particularity." Zalamea (2021) explains that there is a tension between the uniqueness or multiplicity of the objects and methods of mathematics, as well as of mathematical thought in general. He adds that resolving this binary situation is not simple, since it requires a complex analysis of reality in an integrated manner.

Reflecting on the dualities in mathematical knowledge, on the existence of the plural or the singular, the objective or the subjective of this knowledge, the universal or the particular of mathematics, is not something new. For centuries philosophers have been dealing with these questions and have disagreed among themselves. For example, Engels (1960) mentions that for Dühring philosophy is about the development of the highest form of consciousness of the world and of life and includes in a broad sense the principles of all knowledge and all will. If ideas are accepted as they are, it is necessary to be able to answer what those principles are and what their origin is. Those could come from the mind and thought or from the physical world. Solving issues like this occupy philosophy and at the same time divide it.

The different ways of understanding reality are related to the way in which we understand the origin and nature of knowledge. For his part, Zalamea (2021) states that it is not convenient to unilaterally choose one current of thought or another. According to him, in mathematics there is a mixture between realism and idealism that does not allow us to have ontologies or a priori epistemologies without first observing all mathematical knowledge in detail. Regarding whether mathematics comes from the world of *a priori* ideas or from the physical world, Celluci (2013) states that there is evidence from cognitive science that as a result of biological evolution we have "core knowledge systems" that are phylogenetically ancient, innate, and universal. These systems capture the primary information of the positive integer system and the Euclidean plane geometry system. He calls these “natural systems.” He adds that there is also an artificial mathematics, which is constructed by humans and is not the result of their biological evolution. It is rather the result of the need to develop models, maps, or symbol systems. From the socio-epistemological theory, the concept of “social practice” refers to the activity of the human being in the environment in which it develops. Cantoral (2013) explains that social practices generate mathematical knowledge. Some social practices that are identified in different human groups are count, measure, compare, approximate, predict, equate, infer, visualize, and anticipate.

Mathematical knowledge, however, it’s conceived, is a philosophical matter par excellence. When studying mathematical knowledge in philosophical study, we find that the main difference between the different principles of philosophy lies in the origin attributed to them. For idealists, the construction of the world and reality come from thought, mental constructions, and invariant categories that precede man and his history. For materialists this is not so, for them knowledge and mathematical knowledge in particular, does not come from thought detached from the external world. They come from human action on the external world, so the principles that govern it are not the starting point of knowledge, but the historical result of man’s productive work. In this way, mathematical knowledge is not a way of thinking that is applied to nature and history but is a product that is obtained from them.

In a critique of idealism from a materialist, Marx’s position uses the concept of *ideology* or *false consciousness of reality* to refer to what he describes as an inverted understanding of reality. He criticizes those who try to explain from the development of ideas, issues of nature, or the human person. This understanding of things would be *inverted* because it is based on unjustified basic ideas or assumptions on which its validity depends. In the formal constructions of mathematical knowledge, there are also basic discursive determinations called *axioms* which are established as a starting point.

The axioms correspond to truths that are accepted without proving them, and they are not provable from mathematics itself. These postulates are the product of our language and are established as truths that obey our intention to validate them. The axioms provide the discursive base with which mathematicians can advance in the establishment of theorems and discover the logical consequences of the conventions initiated from the terms of a theory. Arboleda (2002) discusses the difficulty of using the axiomatic- deductive method in mathematics research and teaching. He explains that it is possible to use the axiomatic method to base mathematics on a reduced number of simple principles, but emphasizes the importance of verifying in teaching and researching the agreement between the logical definition of an object and its experimental representation; by this he refers to the function called “deaxiomatization” proposed by Frechet. For his part, Celluci (2013) analyzes the inconsistency of the axiomatic method which he describes as the deduction of a group of basic axioms, which must be assumed to be consistent in order to justify a statement. In this way, Celluci validates the analytical method, because for this method the hypotheses used in the solution of a mathematical problem do not have to belong to the same field of the problem, which broadens its resolution possibilities by integrating mathematics to other areas of reality.

In these types of issues lie the differences between idealists and materialists about mathematical knowledge. For some, this arises from logical inferences based on elementary a priori assumptions, and for others, mathematical knowledge is the result of a historical process of material intervention in nature that establishes transitory truths a posteriori.

In the philosophy of mathematics, platonic ideas dominate where mathematical objects exist governed by their own relationships with each other, those that occur independently of us and the physical world. For Platonism, mathematical meanings are explained in relation to their truth conditions that are justified or denied in mathematics itself. The opposite of this is to justify the meaning of a mathematical object considering the conditions that allow obtaining proofs of truth. Idealism as a philosophical current from its beginnings with Plato to the German idealism of the eighteenth century shares the idea that the knowledge of phenomena should tend to the ideal, should be, considering that the state of perfection of things is found in Metaphysical space. Idealism as a philosophical current from its beginnings with Plato to the German idealism of the eighteenth century shares the idea that the knowledge of phenomena should tend to the ideal, to the "must be," considering that the state of perfection of things is found in metaphysical space. Engels (1961) in his *Dialectics of Nature* criticizes metaphysical mathematicians and mentions them as a mixture of remnants of old philosophies that boast of unshakable results using imaginary magnitudes. For example, holding that a=a as a principle of identity is only true in ideal terms, because no organism or object is equal to itself in nature, except at the same instant. An indisputable truth of material reality is that everything is constantly changing. Therefore, a correct mathematical philosophy should deal with the dialectical relationship of identity and difference in order to explain the state of things.

The general laws that are possible to establish from the verification of ideal objects are not really general in all time or all space. Its validity is limited to the space of ideas. For example, the concept of infinity corresponds to an ideal object. The set of Natural numbers (N) contains infinite elements. In reality there is no infinity, the universe is not infinite, for some it is expanding, and for others it is oscillating; therefore, there can be no straight lines or parallel lines or any infinite length in nature.

If these different ways of understanding mathematical knowledge are considered as opposing philosophical positions, it is worth asking which of them is correct. Answering that question requires a deep and complex analysis that cannot be taken lightly. To elucidate which current is true, it is necessary to establish what is meant by truth and to carefully review the historical development of each of them and the multiple elements that affect them, at least.

In relation to whether mathematical knowledge corresponds to ideal notions or to historically situated constructions from material reality, in Aboites and Aboites (2008), we find some questions that may help us to answer this dilemma:

> If Bohr's theory of the atom talks about nuclei and electrons, are those nuclei and electrons real and do they exist as the theory predicts? Or, is Bohr's theory of the atom just a useful tool for calculating the optical spectra of light atoms like hydrogen and helium, regardless of whether what the theory says about electrons spinning around nuclei exists in reality or not? Does the number Pi exist independently of whether there are human minds to conceive it, what is the relationship between logic and mathematics, are they the same thing, is mathematical knowledge just a game of chance, and are they the same thing? Are they the same? Is mathematical knowledge just a game based on symbols and rules? Do Gödel's incompleteness theorems affect what we can or cannot say about the world? Is the logic of our thinking unique? Is mathematics essential to science, or can science be done without mathematics? Is mathematics part of a web of knowledge or is it independent of the world? (Aboites & Aboites, 2008, p.11)

In this chapter it is not intended to establish the character of truth of each one of them; however, from socio-epistemology we are interested in investigating if they are present in the teaching of school mathematics.

21.3 Philosophical Expressions of the School Curriculum

The school curriculum is a construct of a philosophical nature. This expression of the school itself contains ontological, epistemological, teleological, and axiological implications. With all this, the school determines the type of human being that it intends to produce from its anthropogenic function. With the curriculum, the school validates a type of knowledge as official, establishes the goals of the social subject, and attempts to institute official social values. In this way, students are trained from the school duty expressed through ethical assumptions of the curriculum that determine the socially standardized moral subject.

Our interest is to analyze epistemologically the expression of idealism and materialism in the teaching of mathematical knowledge. In consideration of the socio-epistemological character of our analysis, it is necessary to consider the didactic, social, and cognitive dimensions. For this, we will analyze some elements of the Chilean school curriculum.

The subject that deals with teaching mathematical knowledge in the Chilean school is called mathematics. The singular of the name is an epistemic evidence that shows that in the Chilean school only one mathematics is accepted as official and is taught. The first learning objectives of this subject indicate that children must count, read, compare, and order numbers in a field less than 100. The implicit character given to the idea of number accounts for abstract, independent, or metaphysical objects. In the consideration of the number as a mathematical object, a monocultural position is appreciated that does not include other forms of counting other than the use of the natural numbers N.

The natural numbers N included in the curriculum are a set that is established logically from the Peano Axioms. This axiomatic does not deal with defining the number, the same thing happens in the curriculum in the first courses. Students read, order, and compare numbers without knowing the definition of numbers. They incorporate the concept of number with pictorial and concrete supports, sensing its possible meaning, without receiving a formal definition. The complexity of this lies in the difficulty of knowing and understanding abstract ideas if you do not have a clear definition of them.

The axiomatic of the natural numbers *N* contains the notion of infinity in the third axiom. By defining that every natural number has a successor that is also a natural number, it is established that it is a set with infinite elements. The notion of infinity is an abstraction that does not exist in nature, just as numbers do not exist. Its understanding requires the creation of mental structures associated with abstract ideas about ideal objects not possible to verify in the physical world. The students of the first school levels do not have the necessary evolutionary development to understand this type of ideas. Trying to make them understand them demands a great didactic effort from teachers who must use concrete representations to ensure that children have an understanding of the natural numbers *N*. In this sense, knowing and understanding mathematical objects proposed from idealism present great difficulties of understanding for school children of the initial courses.

The idea of infinity present in school mathematics is unaware of the natural world and physical laws as we know them. The laws that govern the phenomena of nature are mainly geocentric, and the development of science has made it possible to establish general laws that cover places outside the earth, but not the entire universe, so the idea of eternal or infinite laws is not possible. The infinite from mathematics is an abstraction, a creation of thought that exists in the world of ideas. This abstraction can be understood as an eternal repetition, an enormous magnitude, or the permanent development of something and brings with it the idea of movement. Perhaps the only way to explain the notion of infinity in relation to material reality is with the eternal state of movement and change. Didactic work with children will be more successful if they manage to relate what they need to learn with their context and their material reality.

The idea of numbers has been typical of various cultures and human groups throughout history. There are Roman, Egyptian, Mayan, and Mapuche numbers, to name a few. Each culture has created them with the intention of counting or quantifying, their differences lie in formal matters such as their writing, their base system, or their relationship with axiological or religious matters of the human groups that created them. For example, zero has been the object of study for having different meanings in the different numbering systems. For Aczel (2016) zero is "the greatest intellectual achievement of the human mind" (p. 201).

In Villamil and Riscanevo (2020), we find that in Egypt zero was used as a reference value in construction plans to refer to the base level. They add that in Chinese civilization zero was interpreted as the absence of elements. In Mesoamerica, the Mayans also used the zero and represented it with a snail shell. For Duque (2013) this symbol was associated with the cycle of the mollusk that was coming to an end. The Mayans also symbolized zero with a corn seed that represented the beginning and end of the cycle of a seed before changing levels and becoming a flower, which was understood as a spiritual level. Zero as we know it in the current decimal system is related to *nothing*. Villamil and Riscanevo (2020) quote Betti (2017) saying that the symbol for zero was created by Ptolemy who used the first letter omicron with which the word ουδεν begins, which refers to nothing. These diverse meanings about zero account for socially and historically situated constructions, related to the material environments and cultural meanings of different human groups. In these constructions we observe the presence of axiological elements that are the result of a material way of life.

For students at the school stage, understanding the idea of the absence of elements in a set and its symbolism or the absence of value in some position of a number in the decimal system, are complex abstract ideas. These ideas may attempt to be represented concretely, but they do not exist in the physical world as they are described as mathematical objects. If, in addition, the mathematical notions presented to students are axiologically neutral, a new difficulty is added, because the material world in which they live and the things that surround them are related to values, which are a historical result of life in society.

The school curriculum contains a large number of idealistic elements and *a priori* conceptions of mathematical objects. Understanding mathematical constructions as abstract and independent objects is a mono-epistemic expression that does not consider their social value and can cause cognitive difficulties in students. In the review of the school mathematics curriculum, we find that there is no evidence of a materialist position in it. On the contrary, the learning objectives of the different school levels contain various mathematical objects, all of which do not exist in the material or social reality of the student body. Despite the didactic guidelines that suggest contextualizing the teaching, the learning objectives propose the understanding of the mathematical objects themselves. To facilitate knowing and understanding the mathematical concepts of the school curriculum, a didactic work is required that relates them to the social and material reality of the students so that they make sense to them. The socio-epistemological theory as an epistemic alternative incorporates the social component, dealing with the mathematical knowledge put into play and promoting the significance of mathematical objects based on the use made of them. This materialist epistemological current recognizes different rationalities, validates different ways of knowing, and gives meaning to the teaching of mathematics, accepting the progressive redefinition of knowledge.

21.4 A Look at Classroom Work

The importance of analyzing the presence of philosophical expressions in the work of teachers lies in the anthropogenic implications of school education. Teachers with their speech legitimize another way of understanding knowledge and reality, and that influences students. Cantoral et al. (2015) point out that the school mathematical discourse shared by teachers validates the introduction of mathematical knowledge in the educational system and legitimizes a new system of reason.

In Sepúlveda's doctoral research (2021), classes of a group of teachers from southern Chile were observed. Then they reflected with them on mathematical knowledge. From this experience we can say that teachers, in general, are not aware of the epistemology they have about mathematical knowledge. Nor aware of the philosophical content of the statements he or she makes to students about mathematical knowledge.

In order to try to understand whether there is a presence of one or another philosophical current in the teaching of mathematics, we observed how teachers develop the school curriculum with students. We placed ourselves in the first school years and in the teaching of numbers and basic operations. For example, we looked at the way teachers taught counting. They ask children to read the numbers printed on sheets stuck on the walls of the classroom. In this activity the children start counting from zero. This work carried out by the students is related to the memory process. The absence of concrete material to count prevents children from intuiting that it is not possible to use zero to count something. The ability to count is the individual assignment of labels in sequence to the elements of a set, where the last label represents the cardinal (Caballero, 2006, p. 27). According to this, it cannot be counted when there are no elements to label. Counting using zero would be counting *nothing* and that is not possible in the physical world. The implications of zero and the concept of *nothing*, in students who are still in the evolutionary stage of concrete operations, are theoretically impossible.

In this, a lack of reflection by teachers between mathematical objects and their relationship with the physical world is observed. For example, Dummett (1986) mentions that when we know that there are 5 men and 7 women in a room, we say that there are 12 people. To know the total number of people in the room, counting was not used, addition was used. Something similar is found in Quidel and Sepúlveda (2016) who comment that when a Mapuche person was asked how he counted the sheep he had, he indicated that because of their color. If there were the amounts corresponding to each color, then there were 30 of his sheep. In this case, what the Mapuche person does to count is to add. This gives us evidence of the relative value of knowledge and allows us to understand that formal knowledge and knowledge put to use are not the same.

In the teaching of addition and subtraction, zero is also a number that can be difficult to understand. When working with concrete elements, you cannot add zero objects, however, in addition as a mathematical object, there is adding zero ($a + 0 = a$). Zero is the neutral element of addition. In the case of subtraction, mathematically zero can be subtracted, however, in a material context if I do not take anything away, I am not subtracting. In this a difference is observed between the understanding of numbers as ideal objects *a priori* and numbers as objects in use *a posteriori*. In this type of situation, philosophical assumptions present in teaching and the constant idealism-materialism tension are revealed.

In relation to how teachers teach basic operations, something similar happens in relation to the epistemic tensions mentioned above. To teach addition, teachers tell children that adding is *increase* and subtracting is *taking away*. This is true when dealing with a collection of concrete elements. Teachers explain multiplication as repeated addition and division as condensed subtraction. Indeed, this is fulfilled in the set of Natural numbers (N). But, for multiplication or division cases such as

$0.6 \times 0.\bar{3}$ ó $^{0.8}7$, the above does not hold. The teaching of addition or basic operations as a priori ideal mathematical objects have certain properties that are not possible to find or demonstrate in a material reality. For example, in school rationality, mathematical objects, adding is equal to subtracting, $a + \ -b = a - b$, in other words, in the addition/subtraction relationship, it is always an addition. Similarly, the division of $\frac{a}{b} = a \times \frac{1}{b}$ or the power $x^2 = \sqrt[2]{x^4}$. Mathematically, every number can be represented as the power of another: $y = a^x$, or it also happens that a power with a negative exponent is equal to the inverse of the number raised to a positive exponent, $a^{-n} = \frac{1}{a^n}$. In this way you can continue establishing relationships of opposites between numbers.

The logical relationships between mathematical objects occur at the level of ideas, they are not always reproducible or possible to represent in relation to the physical world or the material conditions of existence. However, mathematics does deal with the abstraction of numerical characteristics from physical properties. Engels (1961) points out that the concept of variable magnitude introduced by Descartes causes a turning point that introduces movement and dialectics to mathematics.

Another issue observed in math classes is that teachers present the number as something eminently quantitative. Indeed, the number is a quantitative entity, but numbers are distinguished by their qualitative characteristics. In this way 4 is equal to 4, but at the same time, it can be 4 times 1, it can also be 2 times 2, and the square root of 16. That is, a 4 can be the cardinality of a set, the result of an addition, the power of a number, or the root of another number. In addition, 1 is the basic number of any numbering system, and for 1 it is true that $1^2, \sqrt{1}, 1^{-1}$ is always equal to 1. The same thing happens with any power raised to 0, $a^0 = 1$. With this, the numerical value that can be something objective becomes subjective, and its ontology begins to depend on the context, even if this context is ideal.

In order to perform operations with fractions, 11-year-old students know the prime numbers. These numbers correspond to a nominal qualitative category. They differ by their characteristics from the rest of the numbers, just like even numbers or multiples of 3 do, to cite an example. To refer to the use of the qualitative in mathematics, Engels (1961) mentions how the terms *infinitely small* or *infinitely large* are used, introducing qualitative differences even as qualitative antitheses of an insurmountable type. That is, the terms refer to immeasurably different quantities, where the number is not enough to determine the difference and mathematics must use qualitative arguments to establish the truth of the magnitude.

Understanding mathematical knowledge and school mathematics in particular can be a work of deep reflection. In the doctoral research of Sepúlveda (2021), it is mentioned that in order to know the epistemology that teachers have about mathematical knowledge, they were asked how they understand mathematical knowledge, its origin and its nature.

All the teachers stated that they had never thought about these issues; however, they tried to respond from their intuition. Here are some of the teachers' statements:

P1 I think that mathematics is one, independent of the context in which it works, because its properties and representations are universal.
P2 It is an exact science; its nature is the scientific logic that explains the universe.
P3 Their origin is independent, they exist by themselves, they should only be discovered and understood by humans.
P4 It is in everything that surrounds us, man has discovered it when there is a need.
P5 They exist in everything, and man has been discovering them.
P6 Its origin must be discovered and deciphered by man.
P7 There is one mathematics, and it is universal.
P8 There is a mathematics that is universal, its language is unique.
P9 As a science there is one, which studies different topics, but has a universal language.

Teachers in general do not reflect on the mathematical knowledge they teach, this is due to a lack of time to do so, to management teams that do not favor reflection in schools, or to professional training processes focused on doing and not on reflecting on what it does. When these teachers respond spontaneously, they declare mathematical knowledge as one, that is, they do not recognize other mathematics outside of official mathematics. This shows a mono-epistemic position of knowledge. They also declare that mathematics is a universal knowledge, thereby ignoring the situated character of mathematics as a contextualized human production. They add that they exist by themselves and must be discovered. This is an acceptance of apriorism as a philosophical option. The result of observing a group of teachers during their teaching work showed an important presence of idealistic assumptions in their work and statements. Being unaware of their epistemic ideas, it could be thought that they are a product of the teaching tradition and the general acceptance of mathematical knowledge as ideal objects independent of the physical world.

Despite the historical tension between idealism and materialism, the observed discourses tend to privilege a vision of mathematics as a body of universal and immutable objects. To advance in the recognition of human activity in the construction of mathematical knowledge and in the acceptance of its social value, it is necessary to move towards relative epistemological positions. Not doing so and continuing to depreciate the mathematical forms of many is also a form of symbolic violence towards the other that must end (Sepúlveda & Lezama, 2021, p. 18).

21.5 Final Considerations

Idealism and materialism are two philosophical currents that understand reality in the opposite way. In the teaching of school mathematics, a classical position of understanding knowledge in ideal terms prevails. The absence of a reflection of this fact by the teachers, both in their initial training and in the institutional and personal processes of professionalization, makes an idealistic presentation of mathematics persist in the school mathematical discourse, a fact that makes it difficult to construct meanings. of both mathematical objects and their operational processes. There are efforts by teachers to give context to mathematical objects in order to make them meaningful for students; however, the didactic work needs to continue advancing in considering the mathematical knowledge put into use in real material contexts.

Mathematics has played a leading role in the historical processes of humanity and despite the fact that part of it corresponds to ideal or abstract assumptions, it clearly responds to the needs of the development of natural sciences and the technological advance necessary for human subsistence. We consider that the understanding of mathematics as an ideal knowledge may respond to a hereditary factor of culture that has been passed down through generations and that it is necessary to review so that mathematical knowledge reaches a wide level of democratization and thus achieves in society its status of popular and technical use as well as wise.

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Chapter 22
Cognitive and Neurological Evidence of Nonhuman Animal Mathematics and Implications for Mathematics Education

Thomas E. Ricks

22.1 Introduction

The belief that "only humans do mathematics" permeates the field of mathematics education (Ricks, 2021; Scheiner & Pinto, 2019). This human-centric perspective is manifest throughout the literature by such phrases as mathematics is a human activity (Freudenthal, 1973, 2002), a human construct (Abbott, 2013), a human construction (Longo, 2003), a human discipline (Fey, 1994), a human endeavor (Dehaene, 2011), a human enterprise (Noddings, 1985), a human invention (Bridgman, 1927), a human potential (Simon, 2007), and a human social activity (Tymoczko, 1980). Many authors directly state their own personal beliefs about the issue; for example, Dörfler (2007) asserted it is a "trivial fact mathematics is a human activity. Under all circumstances mathematics is done and produced by human beings" (p 105). Other authors imply similar beliefs by summarizing the human-centric positions of others, especially Freudenthal (Boaler, 2008; Cobb et al., 2008; Freudenthal, 1973) Thinking of mathematics as human activity has improved mathematical pedagogy because emphasizing the *humanness* of the mathematical process refocuses the pedagogy of the subject on the way students make sense of mathematics (Steffe, 1990). But is the mathematics education maxim that mathematics is a uniquely human creation and activity scientifically accurate? If not, how might animal mathematics matter for mathematics education?

Much recent scientific research suggests that many nonhuman animals (henceforth, just animals) mathematize as part of their natural behavior (Nieder, 2021). I organize this chapter—a meta-analysis of literature on the subject—around two types of recent scientific evidence for animal mathematics emerging from the field of animal and/or comparative research: *cognitive* and *neurological* studies

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evidencing legitimate animal mathematizing. Cognitive animal research investigates the psychological underpinnings of animal cognition by developing hypotheses about the way animal minds think, by observing animal behavior, and making inferences about the unobservable animal thought processes necessary to produce such behavior. Neurological animal research investigates the biophysical underpinnings of animal cognition to describe the way animal brains function, by developing schematics of neurological structure based on observable brain signatures. Together, these two lines of scientific evidence reinforce each other and demonstrate at least some forms of animal mathematics exist. I conclude the chapter by considering some implications animal mathematics research may have for the field of mathematics education.

22.2 Cognitive Research on Animal Mathematics

Recent cognitive research on a wide range of animals provides substantial evidence of animal mathematical capacity. These non-neurological animal cognition studies investigate animal mathematics through careful scientific experiments that observe mathematical behavior as animals complete research tasks. These studies are peer-reviewed, appear in reputable scholarly publications on animal and/or comparative cognition, and control for conflating experimental variables to avoid the possibility of anthropomorphizing animal behavior. I describe two principal areas of observable mathematical behavior mentioned consistently in animal cognitive research: *numerical discrimination* and *basic mathematical competence*.

22.2.1 Numerical Discrimination

Many animals demonstrate the cognitive capacity to numerically discriminate during experimental trials that control for potentially confounding non-numerical variables like size, density, or luminance. The great apes (Hanus & Call, 2007; Tomonaga & Matsuzawa, 2002) and other nonhuman primates (hereafter, just *primates*) like monkeys (Gazes et al., 2018) all demonstrate the capacity to numerically discriminate, as well as other mammals (Chacha et al., 2020), birds (Tornick et al., 2015), fish (Seguin & Gerlai, 2017), cuttlefish (Yang & Chiao, 2016), frogs (Stancher et al., 2015), salamanders (Uller et al., 2003), lizards (Miletto Petrazzini et al., 2018), and insects (Howard, 2018). Two distinct cognitive systems are believed to play a role in animal numerical discrimination: *subitizing* and the *approximate magnitude system*.

Subitizing Many animals demonstrate subitizing (also *subitising*, *exact number system* (ENS), *object file system* (OFS), or *object tracking system* (OTS) (Nieder, 2020a))—the rapid, highly accurate, cognitive capacity to numerate small groups of objects—a special form of numerical discrimination also present in humans from birth (Brannon & Terrace, 1998). The typical (including human) subitizing range is four or less elements (Dehaene, 2011), but some species naturally subitize past the normal subitizing range, including pigeons (up to five), budgerigars and jackdaws (up to six), and ravens, Amazon and African grey parrots, magpies, and squirrels (up to seven) (Al Aïn et al., 2009; Davis & Pérusse, 1988; Hassenstein, 1974; Hassmann, 1952; Howard, 2018; Nieder, 2005). This uncanny natural ability for numerosity was first investigated (in humans) by Jevons (1871) and later coined *subitizing* (Latin for *sudden*) by Kaufman et al. (1949). Dehaene (2011) popularized the term for mathematics educators with his *The Number Sense* book (de Freitas & Sinclair, 2016) that detailed subitizing capacities in human infants and a few common animal species, like rats and pigeons. Howard et al. (2018) calls subitizing a "counting mechanism with low quantities" (p 11). Subitizing appears to be widespread in the animal kingdom; De Cruz (2006) claims that every animal species tested so far for this capacity has demonstrated it.

Subitizing is legitimate mathematics because it allows animals to discriminate specific numerosities that convey "the cardinality of [a] set" of perceived items (Nieder & Dehaene, 2009, p 186). Subitizing offers a concrete description of animal numerical capacity and is supported by a hundred years of intensive scientific research (Clements et al., 2019; Jevons, 1871). And lest we dismiss subitizing as non-mathematical, attempts by humans to mimic this simple counting capacity in computer vision have required sophisticated human mathematical algorithms—including convoluted neural networks (Zhang et al., 2015)—and is an active area of ongoing mathematical research (Pezzelle, 2018).

Approximate Magnitude System The ability of animals to discriminate numerosity is not limited to the subitizing range. Much research confirms (Dehaene, 1992; Gallistel & Gelman, 1992; Nieder, 2020a) that many animals possess an approximate magnitude system (AMS), a capacity to represent any quantity (with no upper limit!) as a rough estimation—also referred to as the *analog magnitude system*, *analog format*, or the *approximate number system* (ANS) (Cantlon, 2012). Whatever its name, the nature of this capacity differs from the exact number system (ENS) of subitizing; the AMS always entails fuzzy estimation (akin to probability distributions) instead of recognizing discrete quantity. Like subitizing, the AMS is another example of an underlying, phylogenetically-embedded, biologically-determined, numerical capacity shared by both animals and humans (Nieder, 2013).

If metaphorically represented on a number line, the AMS perceives an observed (*objective*) countable quantity as a logarithmically symmetric continuous possibility (*subjective*) centering on the objective quantity and tapering off to overlap and conflate with nearby surrounding numerical possibilities. The larger the objective quantity observed, the greater the potential spread of the probable possibility, and the more likely the observer is to confuse the objective quantity with similar quantities, something called the *numerical size effect* (Nieder, 2020a). If an animal compares two quantities, the accuracy of differentiating their difference (or recognizing their similarity) decreases as the quantities converge numerically. For example, Stancher et al. (2015) describe how frogs struggle to differentiate between areas of food having four versus three items but regularly choose six food-item areas over three food-item areas. This capacity to increasingly differentiate between numerosities as the difference between the numerosities increases is known as the *numerical distance effect* (Nieder, 2020a). Numerous studies document this similarity of animals and humans to represent discrete, potentially countable quantities as nebulous, indistinct judgments of numerosity (Cantlon, 2012; Dehaene, 1992; Gallistel & Gelman, 1992).

Surprisingly, the capacity to discriminate numerosity is also documented in some rather tinier-than-amphibians-and-reptiles creatures. For example, much research on the lowly honeybee demonstrates their ability to discriminate numerosities within the normal subitizing range as they navigate and track the number of landmarks while foraging (Chittka & Geiger, 1995; see also Bortot et al., 2019; Howard et al., 2018; Howard et al., 2020). Howard et al. (2018) documents that with training, honeybees outperformed capacities previously reported in the literature, including "differentiating a correct choice consistent with rule learning compared to an incorrect choice consistent with associative mechanisms" (pp 212–213), "discriminat[ing] challenging ratios of number[s]" (p 213), "learn[ing] numerical rules" (p 213), applying numerical rules "to value zero numerosity" (p 213), "perform[ing] simple arithmetic" (p 213), and "associat[ing] a symbol [with] a specific quantity" (p 213).

Although research publications documenting animal numerosity capacity are obviously more prolific for species that are easier to study in laboratory settings, like rats, mice, small fish, domesticated chicks, and honeybees (Howard, 2018), these studies help demonstrate that numerical discrimination is widespread in the animal kingdom, and at least some forms of mathematical cognition do not require large, human-like brains.

22.2.2 Basic Mathematical Competence

Once cognitively aware of numerosities through subitizing or the AMS, many animals then utilize those numerosities to demonstrate basic mathematical competence (Nieder, 2013). In this way, animals continue to mathematize in the original Freudenthal (2002) usage that mathematics is applying common sense to quantitatively act upon perceived realities. For example, Nieder (2013) reports:

> Beyond [AMS's] discrete quantities, nonhuman primates can also grasp continuous-spatial quantities, such as length [and] relations between quantities resulting in proportions… [M]onkeys perform primitive arithmetic operations such as processing numerosities according to quantitative rules… it is well accepted that numerical competence is… found in animals. (p 2–3)

Additionally, studies on two hundred wild, semi-free rhesus monkeys document the ability of these primates to discriminate normal subitizing-range (four or less) numerosities without training as evidenced by their reliable ability—after watching researchers slowly deposit food items one by one within two containers—to choose the container with the greater number of food items (Hauser et al., 2000). Chimpanzees, however, can discriminate up to 10 items when choosing the larger of two food containers that have also been filled one by one (Beran, 2001; Beran & Beran, 2004). The numerical and problem-solving capacity of the chimpanzee has been known for many decades (Boysen, 1988, 1992; Boysen & Berntson, 1989; Boysen et al., 1993; Dooley & Gill, 1977; Ferster, 1964; Matsuzawa, 1985; Muncer, 1983; Rumbaugh et al., 1987; Woodruff & Premack, 1981).

Clearly, humans do much mathematics that animals cannot do, primarily because humans have a refined ability for symbolization (Nieder, 2020b). But animals not doing all of human mathematics should not prevent the realization of the mathematics that they can do. Describing what mathematical capacities different animals manifest and how they relate to developing human mathematical capacities is in its infancy. Keeping track of *which* animals do *what* mathematics presents challenges, especially as no descriptive theory of mathematical sophistication is universally accepted. Many different ways of organizing animal mathematical capacity and linking it to what humans can do—at which age-level or developmental level—have been proposed. Howard (2018, p 12–13, 195, 197, modified and adapted for prose) elegantly summarized animal cognition research on numerical mathematical competence by creating an intermeshed matrix of *eight numerical concepts*: zero numerosity, quantical cognition, subitizing, approximate (or analogue) magnitude system, arithmetic, numerical cognition, numerical competence, and true counting) with 23 unique *numerical tasks* (the italicized text that follows will be explained later in the chapter): (1) sensory representation of zero; (2) *categorical understanding of "nothing"*; (3) *quantitative understanding of zero numerosity*; (4) symbolic and mathematical use of zero; (5) *use of nonnumerical cues correlated with number*; (6) *quantity discrimination of numerosities below five elements*; (7) *subitizing*; (8) quantity discrimination which obeys Weber's Law; (9) *discriminate numbers above four*; (10) [serially] counting above four; (11) spontaneous arithmetic-like reasoning; (12) *symbolic representation of numbers in arithmetic*; (13) *symbol and number matching*; (14) *exact number use*; (15) *arithmetic problems*; (16) nominal number use; (17) *ordinality [ranking of sets]*; (18) *cardinality [valuation of sets]*; (19) *novel representation of number*; (20) *transfer to novel numbers*; (21) procedural translation of numbers; (22) modality transfer of numbers; and (23) symbolic representation and quantitative valuing of symbols. Animal research studies have documented that various animals have accomplished at one time or another 21 of the 23 unique numerical tasks. No evidence exists yet for animal *symbolic and mathematical use of zero*, or *nominal number use*; only humans have demonstrated these capacities. Considering that it took thousands of years for humans to develop modern mathematical competencies of zero, it should not be surprising that human use of zero is not yet documented in animal behavior. Further, nominal number use—an integral part of the symbol-laden, modern human culture—has not been seen in animal behavior either.

Intriguingly, honeybees demonstrate mathematical capacity in all eight *numerical concept* categories, and of the 23 unique *numerical tasks*, research confirms honeybees can manifest at least 14 of them (italicized in the list above, Howard, 2018, p 197; see p 173 for preliminary evidence of *ordinality of sets*, which might make 15 total). If we add one more task completion—the lowly *sensory representation of zero*—which I assume bees fulfill, considering their competence with zero numerosity (no one has yet studied sensory representation of zero in honeybees)—honeybees manifest mathematical capacity in over two-thirds of the categories. Imagine! All that, done by the little, humble honeybee, with a tiny brain over 80,000 times smaller than the human brain (roughly one million neurons compared with humans' 86 billion neurons) (Azevedo et al., 2009; Menzel & Giurfa, 2001). For such a tiny brain, the bee brain manifests significant mathematical power; humans have yet to develop comparable mathematics to match the honeybee's zero mathematical competence through machine learning. Researchers (Schmicker & Schmicker, 2018) have built a three-layered neural network to mimic the findings about "the quantitative value of an empty set" (Howard, 2018, p 212) that honeybees manifest through standard animal cognition training; the neural network was "trained using the same stimuli and protocol" (Howard, 2018, p 212) that the bee experiments used. Howard et al. (2018) admits: "We still have a lot to learn from biologically evolved processing systems, such as the honeybee brain, as while bees… took less than 100 trials to learn the task, *the simple neural network took about 4 million trials to learn the same task*" (p 212, emphasis added). And experts in the field of honeybee cognition believe with proper experimental setups and further research, honeybees may accomplish several more of the remaining unique numerical tasks, although their short lifetime limits training opportunities (Howard, 2018). This high ratio (roughly two-thirds) of numerical mathematical task-completion found in honeybees but not other species may be as much a function of the ease of using honeybees for animal cognition research as a measure of their mathematical competence; other animals may demonstrate similar or greater mathematical capacity once research is conducted on those more difficult-to-study species.

As the field of animal cognition continues to mature, more and more mathematical capacities are being discovered in more and more species. The abilities to subitize and approximate numerosity have obvious evolutionary (e.g., natural selection) advantages: "Numerical competence... is of adaptive value. It enhances an animal's ability to survive by exploiting food sources, hunting prey, avoiding predation, navigating, and persisting in social interactions" (Nieder, 2020a, p 605). For the animal cognition researcher, the question is no longer *if* animals mathematize, but *how* (Nieder, 2013).

22.3 Neurological Research on Animal Mathematics

Neurological research on animal brains (conducted while animals manifest mathematical-like behavior) strengthens the argument that animals are indeed doing legitimate mathematics by documenting "the neuronal mechanisms of numerical competence" (Nieder & Dehaene, 2009, p 186). Two branches of neurological evidence support the legitimacy of animal mathematics: "number neurons" and the similarity between human and animal brain activation while mathematizing.

22.3.1 Number Neurons

Many researchers have documented so-called number neurons (Dehaene, 2002, 2011; Nieder, 2013; Piazza & Dehaene, 2004) in various animal species' brains while performing mathematical tasks. Number neurons are specific neurons in special regions of animals' brains that spike electrically when the animal receives numerical stimuli in visual or auditory form (e.g., dots on a screen, sequential tones), or mentally numerates body motions, like limb motions (Nieder et al., 2002; Sawamura et al., 2002). First identified in domestic cat brains (Thompson et al., 1970), number neurons have also been documented in crow (Ditz & Nieder, 2015), rhesus monkey (Nieder, 2013), and Japanese macaque monkey brains (Sawamura et al., 2002). To find the number neurons, researchers surgically insert super-thin wires into specific math-related brain regions and slowly adjust the depth of the wire until a number neuron (tuned to the desired numerosity researchers wish to study) is identified. These specialized nerve cells are "tuned to the number of [sensory] items [experienced] show[ing] maximum [signal] activity to one of the presented quantities—a neuron's preferred numerosity—and a progressive drop off as the quantity [becomes] more remote from the preferred number" (Nieder & Dehaene, 2009, pp 188–189).

Researchers found, for example, in a monkey's prefrontal cortex (Nieder & Merten, 2007), a neuron tuned to the quantity of 20 (as well as one tuned to the quantity of 6, another to 4, and one for 2); the fact that specific neurons in a monkey's brain differentiate a visual spread of 20 dots from other presentations of dots is admittedly quite impressive. Some number neurons encode for perceived numerosity (*input modality*, like viewing dots on a screen or hearing a sequence of tones) and others for psychophysical movement (*output modality*, such as keeping track of body movements) (Nieder & Miller, 2004; Sawamura et al., 2002). In certain regions of the brains of monkeys, number neurons cluster in high percentages; Viswanathan and Nieder (2013) found in the *ventral intraparietal area* (VIP) of the *intraparietal sulcus* (IPS) that 10% of sampled neurons evidenced exclusively numerosity-selective spiking. In one study, (Nieder et al. 2006), three separate types of number neurons were found: (1) a type only encoding numerical information perceived simultaneously (*spatial*: as in a spread of dots on a screen), (2) a type only encoding numerical information perceived sequentially (*temporal*: as in food items placed in a bowl one by one, a series of auditory tones, or a sequence of flashing lights), and (3) a third type of number neuron that *integrated* the numerosities encoded by the first two types of neurons, for storage in memory.

The colloquial term "number neurons" can be a bit misleading, because these neurons encode more than just numerosity. Tudusciuc and Nieder (2007) discovered that roughly one-fifth of neurons in the monkey IPS spiked for numerosity and/or continuous quantity. Findings by other researchers demonstrate number neurons can encode for nonnumerical parameters such as size, luminance, angle, position, and density (Cohen Kadosh & Henik, 2006; Pinel et al., 2004; Zago et al., 2008). Bongard and Nieder (2010) found that individual monkey prefrontal cortex number neurons "can flexibly represent highly abstract mathematical rules" (p 2279), such as *greater than/less than* concepts, helping monkeys at the macroscale to consciously "understand relations between numerosities and how to apply them successfully in a goal-directed manner" (p 2279). Some researchers have found interlinked number neurons (Diester & Nieder, 2008) that mutually inhibit or mutually reinforce each other's numerate spike potential. Brains are vast interconnected structures of neurons, so no real surprise to find interconnected neurons; what is surprising is that the interactions form a type of intricate back-and-forth neuronal dialogue instead of just signals cascading from one nerve cell to the next. Both neurons "talk" to each other simultaneously to influence each other during the entire numeration-spiking process. Diester and Nieder (2008) posit this process enables more refined tuning by number neurons to their preferred numerosity. These interlinked number neurons evidence mathematical communication at the smallest inter-cellular level.

In 2013, Viswanathan and Nieder controlled for the possibility that neurons were manifesting trained animal conditioning (instead of numerosity) by designing a clever experiment testing monkeys' sense of color with colored dots; while running these color experiments (that varied the numerosity of the same-colored dots), the number neurons of each monkey were also being recorded electronically. Even though these monkeys had not been trained to discriminate numerosity, and could not differentiate numerosity behaviorally (the monkeys were tested later for their numerical capacity with the same dot patterns—this time in black—and showed scores no better than random chance), *the number neurons in monkeys' brains encoded the number of dots appearing on the colored screens*. This series of experiments confirm that monkey number neurons intuitively recognize numerosity, even though the animal has not received numerical training by researchers.

Recent studies have investigated "number neurons" in humans. In 2004, noninvasive functional magnetic resonance imaging (fMRI) studies on human brain blood flow patterns by Piazza et al. (2004) suggested the existence of similar tuning curves for numerosity in human brains to those operating in animals. In 2009, Jacob's and Nieder's work hinted human number neurons exist from evidence gathered through electrocorticography readings of masses of spiking neurons; groups of neurons were behaving similarly to the way number neurons would. In 2013, Shum et al. described the surgical implantation of 157 electrodes on the underside of the brains of seven epilepsy patients, a rare example of human intracranial electroencephalography. They reported "identifying the precise anatomical location of neurons with a preferential response to visual numerals… embedded within a larger pool of neurons that respond nonpreferentially to visual symbols that have lines, angles, and curves" (Shum et al., 2013, p 6712). Then in 2018, Kutter et al. reported finding individual number neurons in human neurosurgical patients, specifically "585 single neurons in... nine human subjects performing... calculation tasks" (p 754). They summarize their findings thus: “Using single-cell recordings in subjects performing a calculation task, we have shown that single neurons in... humans are tuned to numerical values in nonsymbolic dot displays. The data about nonsymbolic number coding from humans can now be compared to those of nonhuman primates” (Kutter et al., 2018, p 758).

22.3.2 Similar Brain Signatures

In addition to the existence of number neurons in humans, animal brain-imaging signatures are similar to those of humans while performing similar mathematical tasks. Numerous studies document that the very same regions activate in both primates’ and humans’ brains when research subjects (primate or human) perform a similar mathematical activity (Arsalidou & Taylor, 2011; Nieder, 2021; Tudusciuc & Nieder, 2007). Such comparisons are possible because human and primate brains share similar brain structure (Azevedo et al., 2009). Neurological research suggests two similar areas in primate and human brains activate when performing mathematical activity: the intraparietal sulcus (IPS) and the prefrontal cortex (PFC) (Amalric & Dehaene, 2019; Arsalidou & Taylor, 2011; Dehaene, 2011; Eger, 2016; Hyde et al., 2010; Nieder, 2012, 2013, 2016; Nieder & Dehaene, 2009; Nieder et al. 2006; Piazza et al., 2007; Shum et al., 2013). In humans, these regions are known for a variety of mathematical competencies, such as algebraic thinking (Maruyama et al., 2012; Monti et al., 2012).

Because these two regions play different roles in processing numerosity, the IPS and PFC manifest unique neuroimaging signatures when both primate and human subjects perform mathematical tasks. For example, the IPS processes more nonsymbolic numerosity while the PFC is connected with more symbolic number processing; additionally, the IPS processes the nonsymbolic numerosity *before* the PFC, suggesting that the PFC gets its numerosity information directly from the IPS (Nieder & Dehaene, 2009). This neuroimaging evidence suggests that humans, even with our powerful and unique symbolic mathematical capabilities not manifest by any animal, still utilize portions of our brains prior to symbolization similar to the way animals mathematize; humans are thus subconsciously doing mathematics like animals—all the time—despite the seemingly non-animal-ness of our conscious, symbol-heavy mathematics. Kutter et al. (2018) posit: “Our human-specific symbolic number skills… spring from nonsymbolic set size representations…. suggest[ing] number neurons as neuronal basis of human number representations that ultimately give rise to number theory and mathematics” (p 753).

In summary, animals not only produce mathematical behavior like humans, implying similar cognitive mathematizing (Cantlon, 2012; Nieder, 2020a), but the way animals’ brains function while producing such behavior is also similar to how human brains do the very same mathematics (Autio et al., 2021), strengthening the argument that animals indeed do at least some forms of mathematizing.

22.4 Discussion: Animal Mathematics Matters for Mathematics Education

Clearly, animals cannot match human mathematical capacity in many areas, especially the sophisticated, symbol-heavy, precision-based mathematics so prevalent in modern technological societies. Why would animal mathematics, therefore, matter for the field of mathematics education? Most obviously, cognitive and neurological animal mathematics studies legitimize the existence of animal mathematics and challenge the common mathematics education belief that only humans mathematize. Thus, the first reason animal mathematics matters for mathematics education is that continued antagonism by mathematics education against the acceptance of animal mathematics is increasingly anachronistic and scientifically inaccurate. It is time for mathematics education to familiarize itself with the burgeoning literature supporting animal mathematics. Secondly, animal mathematics studies deepen and enrich our understanding about the philosophy of human mathematics and related human mathematics education. Animal mathematics studies raise intriguing questions about the nature of mathematics, where it comes from, and how it is so unreasonably effective (Wigner, 1960). Such research on animal mathematics supports previous researchers' beliefs—e.g., Piaget—on the potential biological roots of mathematical development (Brannon, 2014; Duda, 2017; Piaget, 1971).

But perhaps the most important reason studies about animal mathematics matter for mathematics education is these studies reveal the similarity between human and animal mathematics. Together, the comparative cognition and neurological research evidence suggest that humans share with animals basic, underlying mathematical capacities continuously operating in the cognitive/neurological background, even when humans do more sophisticated, human-unique mathematics. Neonates, nonverbal infants, children, adults—even professional mathematicians—manifest animal-like mathematics while performing human mathematics (Nieder & Dehaene, 2009). Thus, animal mathematics matters for mathematics education not so much because animals *sometimes* mathematize like humans, but because humans *always* mathematize like animals.

For example, the presence of the numerical distance effect by "human number neurons… supports the hypothesis that high-level human numerical abilities are rooted in biologically determined mechanisms" (Kutter et al., 2018, p 759). This evidence suggests that human symbolic mathematics is an outgrowth of evolutionarily ancient mathematics that we share with animals, being "deeply rooted in our neuronal heritage as primates and vertebrates" (Nieder, 2020a, p 28).

All humans—regardless of the type, sophistication, or level of mathematizing, and regardless of the age, developmental status, or training (including the professional mathematicians!)—appear to mathematize with foundational, animal-like numerosity processing at the autonomous neuronal level, because human "symbolic number cognition [is] grounded in neuronal circuits devoted to deriving precise numerical values from approximate numerosity representations" (Kutter et al., 2018, p 759). The fact that humans share with animals a distinct "math-responsive network" (Amalric & Dehaene, 2019) activated during both sophisticated, human-only mathematical thinking as well as the more mundane—and that neuroimaging differentiates these human brain regions from human semantic networks (Nieder & Dehaene, 2009)—suggest that human mathematics consists of more than just the symbolic processing that only humans possess.

Human mathematical thinking is fundamentally rooted in biological core systems shared with animals that always activate during any type of human mathematizing (Dehaene et al., 2006; Xu & Spelke, 2000). These shared forms of mathematical thinking—so different from the symbol-heavy, precision-based modern mathematics so emphasized in contemporary mathematics classrooms—are always percolating in the encephalic background. They form consistent, powerful ways of mathematizing about our world that need more attention in mathematics education pedagogy and scholarship. Our curricular over-emphasis on symbols, terms, procedures, and precision already disenfranchises students, especially those manifesting neurodiversity (de Freitas & Sinclair, 2016); we should make room in our curriculum and research for more understanding of the animal-like mathematics that *all* humans manifest *whenever* they mathematize. Animal mathematics acceptance by mathematics educators has the potential to influence many theories of and in mathematics education (Bikner-Ahsbahs & Vohns, 2019). In particular, animal mathematics research illuminates similar underlying neuronal circuitry responsible for human mathematical thinking. Further, animal mathematics research is already improving understanding of various mathematical disabilities such as *dyscalculia* (Anobile et al., 2018; Ansari, 2008; Butterworth, 2005; Butterworth et al., 2011; Castaldi et al., 2018; Kucian et al., 2011; Mazzocco et al., 2011; Piazza et al., 2010; Rubinsten & Sury, 2011) and opens avenues for productive translational research (Bisazza & Santacà, 2022) to improve student learning outcomes (Iuculano & Cohen Kadosh, 2014; Piazza et al., 2013).

22.5 Conclusion

This chapter has challenged the common, contemporary mathematics education belief that only humans create and do mathematics by highlighting emerging cognitive and neurological evidence that animal mathematics exists and is similar to at least some forms of human mathematics. Further, comparative cognitive and neurological research suggest that all humans manifest animal-like mathematics even when doing human-only mathematics. Accepting animal mathematics by the mathematics education community will augment the work of mathematics education by illuminating better the foundational cognitive and neurological manner in which humans create and do mathematics.

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Part V
Concluding

Chapter 23
Living in the Ongoing Moment

Bronisław Czarnocha and Małgorzata Marciniak

The time of preparation of this manuscript was marked by the appearance of an advanced artificial intelligence (AI) product, in the form of ChatGPT. This new, unusual, yet somehow expected feature, has confronted all of us, humans and in particular the educators with its power and possible impact upon mathematics classrooms. From the point of view of the philosophy of mathematics education, AI inspires more questions and directions for research investigations, possibly making a topic for another book. Thus, this summary chapter is written in a very particular way creating more questions within the themes provided by the authors.

Looking back at the process of the creation of the book, we see that it was very fortunate that AI became vastly available after the main chapters have been written because they represent the state of our philosophizing before AI has appeared. So now when the manuscript has been completed and all chapters are assembled, we, the editors, can look back at our work and reflect on the concept of the current book taking into account the new AI component. This brings a host of new questions, all of them centered on how the philosophy of mathematics education can guide us to the doorstep of the new scientific and cultural revolution that AI is clearly announcing. Marciniak (Chap. 12) discusses the paradigm shifts in the history of mathematics education, and it seems to be clear that fundamental paradigm shifts are upon us. At present, the parameters and scopes of those changes are very difficult to assess as we do not know yet either the possibilities or limits of AI's impact on humans. In the further parts of the summary chapter, we will approach the question of the

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philosophy of mathematics education and AI by articulating explicit and implicit theme concerns of the chapters' authors.

But wait. Who are the authors? Who are the mathematics education philosophers? What inspires them and what are the methods of their works? Are they young or old in age or philosophical experience, do they work in academia, do they write about class practice or analyze statistical data? What are they trying to accomplish with their writing? Bicudo (Chap. 3) emphasizes the realization of philosophical thinking. Thus, we ask, what even is this philosophy of mathematics education? What topics should be discussed? Watson (Chap. 6) skillfully locates mathematics education in a contemporary system theory, while Czarnocha (Chap. 8) suggested that philosophy of the domain starts from examination of its fundamental problems. Thus, a new fundamental problem has appeared during the time of writing the book: what issues does mathematics education need to consider in the light of the skills of ChatGPT and other advanced chatbots, which are with no doubt coming in the near future (as in the new Bing searching engine).

The book displays a plethora of themes in philosophy of mathematics education and a cornucopia of ideas bringing together authors from multiple cultures, nationalities, and countries across five continents. As expected, these authors think in many directions in their own, unique ways, about a variety of topics, carrying their ideas across through as many pages as they need. Some authors zoom out spreading their work across many topics, while others zoom in and focus on one, well-defined theme. Others reflect on the past, present, and future, while some simply find one time frame and one place to address the needs of mathematics education that are important to them. Thus, we hear. We listen to their voices when they strive to answer the question that has been nurturing teachers for centuries. Why is mathematics, this queen of sciences, somehow not so likable to the students? Why does she cause such a stress and anxiety among students? What can the teachers do to ease the presence of the queen and make her more accessible and more approachable to future generations?

One of the most frequent complaints of students about mathematics is that it is disconnected from reality. However, it is physics, the source of many mathematical ideas and ultimately the king of sciences, that brings these connections forward. Will AI be able to bridge the gap between theory and reality and thus make mathematics, the queen of sciences, not only likable but a fully fledged member of the human community? Ernest (Chap. 1) is asking a relevant question of the ontology of our subject that is philosophy of mathematics education: "The ontological problem of mathematics education concerns persons. What is the nature and being of persons, including both children and adults, or precisely the mathematical identity of mathematicians and the developing mathematical identities of students? What are these mathematical identities and how are they constituted?" What is the mathematical identity of the ChatGPT or its more advanced versions? How can the mathematical identity of a chatbot contribute to the development and formation of the mathematical identity of a human student? That question becomes a bit more acute taking into account John Mason's (Chap. 5) focus on the process of abstraction as a component of mathematical identity. Abstracting is here seen as a change of a relationship between the person and the set of words, and hence a change of the relationship with self-constructed "virtual objects." Can AI creatures undertake such a change of mathematical identity through abstracting? More precisely, can AI creatures abstract? If we look at abstraction categorically as a forgetting functor, AI should have no problem with it. We wonder what could be the AI creature's relationship or its change with the AI-constructed "virtual" mathematical object. Would there be a relationship?

Goldin (Chap. 7) raises the ante asking for the self-integrity of philosophy of mathematics education; in particular he examines the question of mathematical validity and objectivity on one hand and its sociocultural origins, on the other. While the relationship between the two has been the place of strong disagreements, Goldin searches for the integrated approach where both principles, seemingly contradictory, are shown the possibility to coexist. How would AI creatures relate to such a situation, generating a compromising approach to the two seemingly incompatible principles, or objectivity and cultural subjectivity? Creation of such a compromise requires bisociative thinking that arises simultaneously out of two unconnected matrices of thought, a concept explored by Czarnocha in (Chap. 8). This is a creative process, and if AI creatures can do such constructions, it would mean they are indeed creative. Are they? Would they? Creativity is held as the uniquely human capacity, on the very border between humans and automation. Can AI creatures have experience – if such a term makes sense here? Or if creativity is the feature differentiating humans from AI, should education withdraw from its traditional testing approaches and focus on facilitating creativity? Similar questions are brought forward by the concept of internalization of mathematics via the process of learning as studied by Baker (Chap. 9). Maybe instead of teaching more content, we may focus on teaching more intensely even if it leads to writing poems about the Pythagorean Theorem. This remains within the curriculum inquiry of what to teach and how to measure progress. The sociocultural grounds of mathematics and mathematics education raised by Goldin and pursued by authors such as Miguel et al. (Chap. 18), Maurício Rosa (Chap. 19), and Obreque and Andalon (Chap. 21) bring new questions and concerns related to the AI industry. Given that AI creatures are made by us humans, we face the challenge of how to avoid imparting to it our own sociocultural biases. Who and how verifies whether AI may be biased against people of color or indigenous students? Similarly, we can ask whether it may be biased against females or LGBTQ. Who trains AI's non-bias toward all possible groups of people? Is the ethics of AI, even within a small scope of education, a subject of training? Or a subject of firm rules imprinted into the system as unchangeable? Who and based on what principles makes decisions about such setup? More than that, if asked, would AI display ethical judgments and what would be the value of such display? Taking into account the Ubuntu philosophy of life as essentially communal, and rejecting individualistic features, how to embrace AI's development globally? How could AI accommodate a vast sociocultural milieu remaining ethical towards all subjects? Finally, pursuing the ideas of Obreque and Andalon, the question arises whether AI creatures operate on an idealistic basis having their ideas implanted a priori by humans or it is a materialistic creature that derives its knowledge from the motion of matter within its circuitry.

Implications of futuristically advanced and broadly available AI are quite unpredictable, although the Commander Data of the Second-Generation Star Trek series, the human android explores the horizon of these implications. This is a worthy question since interactions with AI are inevitably going to change future humans. While thinking today about future education, we would like to take that change into consideration, even if this task seems quite challenging, if even possible. We already are aware of the influence of technology on young people and are aware of some who spend their childhood playing video games. However, they still realize the value of human interactions and miss them during remote instruction. Are such human-AI interactions working for the progress of the human mind? Shall we limit such interactions for the sake of the mental and social health of students? According to Matsushima (Chap. 10) dialogues among humans improve mathematics learning. Would a similar feature take place for interactions between humans and AI? Fearing that interactions with AI may disturb someone's view of reality, one can ask: what even is reality? Can mathematical modeling be used to model the reality that is coming? In his work (Chap. 16) Schürmann discusses classroom-level mathematical modeling and its philosophical aspects mainly in the light of the relationship between the model and the actual observations. But now, the multifold of realities that contain the real world and the mathematical world are being expanded by the reality of IA. Can AI model itself if asked about it, or in other words, what are possible states of self-awareness of the AI creatures? What would be the validity of such AI-made models in the view of syntactics? Commander Data of Star Trek was fully aware of his being as the complex interplay of many algorithms connected with the complete absence of feelings, of human feelings, and that fact was of some concern to him as he could not bridge it. On the other hand, as we know from the works of Chamberlin, Liljedahl, and Savic (2022), affect plays a fundamental role in the process of learning mathematics, in the development of mathematical intimacy, and in bonding with mathematics. What would be the role of AI here?

Nevertheless, if reality can be influenced by education, then rethinking education is mandatory for the sake of future generations. And educational paradigms should be defined anew. In this new education, what would be the place for Education for Sustainable Development (ESD) as discussed by Hui Chuan Li (Chap. 17)? Considering the fast pace of the ongoing changes, one could even doubt whether ESD can be set up before its value expires. May the goal of matching education with the needs of society or an individual be an ever-moving target? With the most disputable topics related to the values of performative skills, Ole Skovsmose (Chap. 14) introduces the concept of performative mathematics opening a variety of discussions. What new features of performative analysis will AI bring? Will AI be able to perform mathematics the same way we do? In all aspects? Or maybe studying the struggle of AI will allow a better understanding of the difficulties of students' performance. Or having handy and always available AI to perform for us, shall we withdraw from seeing value in performative skills? But can AI perform the entire spectrum of (mathematical) thinking available to humans? For example, can AI perform inquiry? Produce mathematical knowledge in the way presented by Stoyanova-Kennedy (Chap. 15)? Can AI pose meaningful problems and reason about solution methods? We asked ChatGPT to design a few problems related to graphing certain functions, and the results were unimpressive; thus, we began to wonder whether AI will ever be able to draw meaningful conclusions. Understand (mathematical) jokes? Could AI understand mathematical paradoxes valued so much by Yenealem Ayalev (Chap. 13)? Will AI ever understand infinity? What strategy should be applied in education then? Should teaching mathematics include more paradoxes as an attempt to differentiate human learning from artificial learning? Or this would be a missed attempt and possibly, philosophizing will remain the only human activity not mastered by AI? Discussion of the limitations of AI brings a fundamental question of whether AI can ever become sentient and develop any form of self-awareness. Can AI ever be aware of their states of mind? Can AI be aware of its own subtle leaps of awareness the same way humans are, as presented by Hausberger and Patras (Chap. 4)?

Maybe when discussing teaching mathematics, one should follow the idea of separating mathematical formalism and intuition. Min Bahadur Shresta (Chap. 20) draws our attention to the fact that formal mathematics is a relatively recent (nineteenth and twentieth centuries) discovery emphasizing the fact that for centuries, humans have been performing mathematics without formalism and axiomatization. It is quite clear that while AI can master logical aspects of mathematics, it is likely that humans can master intuitive mathematics much better than AI. Should mathematics education then reduce logical math and emphasize intuitive math? Being not burdened anymore by the necessity of carrying all algebraic manipulations flawlessly, shall we expect that students will joyfully engage in abstract mathematical thinking? Somehow, based on our current experience as college teachers, we do not see it coming anytime soon since the current attitude skews more toward avoidance than engagement into abstracting. But maybe it is just an intermediate stage between old and new education. However, intuitive performance carries a serious challenge for education as it is difficult to measure its progress. And measuring progress is a valid factor of education for society. How should the progress of students' learning be evaluated then? Can AI help with the matter of valid assessment? Interestingly, humans are not the only species performing mathematics intuitively. Thomas E. Ricks (Chap. 22) describes animals performing mathematics, but they do not worry about the quality of their performance and certainly not the grades. Potentially, animal brains developing mathematical thinking can give hints on how little children can develop mathematical thinking in an organic, natural way. If animals can mathematize, then is it possible that AI could mathematize on its own? On the more practical side, could AI in the future act as a skillful interpreter between animal languages and human languages so we can, in particular, understand the development of mathematization in animals?

In this big excitement about AI, we still need to remember that it is just an algorithm, a finite sequence of rigorous instructions performed by a machine. The nature of human thinking is not exactly of this type, since we do not always carry the same rigor and the finiteness of the instructions is uncertain. Following suggestions by

Regina D. Möller and Peter Collignon (Chap. 11), should future education significantly increase the teaching of algorithms? To what extent? Shall students learn the peaceful history of programming instead of the violent history of human wars and uprisings? And learn programming languages instead of human languages? Programming languages may be quite useful but are applications at the center of the education we want to design?

One of the central issues in our initial discussion of the relationship between mathematics education and artificial intelligence is creativity. The previous discussion of feelings, intuition, and algorithms lead to the role of AI creatures in mathematical creativity, within the emerging domain of philosophy of creativity. Will interaction with AI enhance human mathematical creativity or will limit our creative endeavors? That depends a bit on the degree to which the AI creatures themselves are creative and how they can facilitate human creativity. At present, Margaret Boden (2004) assures us that AI can go as far as "to seem being creative" without yet any indication as to whether "it is creative." How does this matter for mathematics education? There certainly is a difference in the quality of interactions between interacting with the entity that seems to be creative and the one that is creative. It seems that imitation of creativity only permits receiving the results of the process and prevents co-creation, which is the most joyful aspect of true creativity.

All these topics touched by the authors may be disputed in the light of theories of scientific revolutions as performed just like by Otte and Radu (Chap. 2), from Popper to Heisenberg. Hopefully AI will have something to suggest in relation to future transformations of sciences and education.

While feeling very present in this ongoing moment and philosophizing about the future, using previously acquired knowledge, we find space to observe one more leap of mind. Following it, we emerge in an imaginary world of a Polish futurist Stanislaw Lem. His book *Fables for Robots* (1964) contains stories written by and for AI. We find this futuristic literature very soothing as it fits exactly between clairvoyance and philosophy of the future influences of AI on human reality.

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