Appendix: Basic Terminology

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In this appendix, we present definitions of basic terminology used in the book for the reader's convenience. For a given set W, $x \in W$ means that x is an *element* of W.

5.1 Convergence

(1) Let M be a set. A real-valued function d defined on $M \times M$ is said to be a *metric* if
 (i) $d(x, y) = 0$ if and only if $x = y$ for $x, y \in M$;
 (ii) (symmetry) $d(x, y) = d(y, x)$ for all $x, y \in M$;
 (iii) (triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in M$.

Here, $X_1 \times X_2$ denotes the *Cartesian product* of two sets X_1 and X_2 defined by

$$X_1 \times X_2 := \left\{ (x_1, x_2) \,\middle|\, x_i \in X_i \text{ for } i = 1, 2 \right\}.$$

The set M equipped with a metric d is called a *metric space* and denoted by (M, d) if one needs to clarify the metric. Let W be a product of metric spaces of (M_i, d_i) $(i = 1, \ldots, m)$, i.e.,

$$\begin{aligned} W &= \prod_{i=1}^{m} M_i = M_1 \times \cdots \times M_m \\ &:= \left\{ (x_1, \ldots, x_m) \,\middle|\, x_i \in M_i \text{ for } i = 1, \ldots, m \right\}. \end{aligned}$$

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This W is metrizable, for example, with a metric

$$d(x, y) = \left(\sum_{i=1}^{m} d_i(x_i, y_i)^2 \right)^{1/2}$$

for $x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_m) \in W$. If M_i is independent of i, i.e., $M_i = M$, then we simply write W as M^m.
A subset A of M is said to be *open* if for any $x \in A$ there is $\varepsilon > 0$ such that the *ball* $B_\varepsilon(x) = \{ y \in M \mid d(y, x) \leq \varepsilon \}$ is included in A. If the *complement* A^c is open, then A is said to be *closed*. The complement A^c is defined by

$$A^c = M \backslash A := \{ x \in M \mid x \notin A \}.$$

For a set A, the smallest closed set including A is called the *closure* of A and denoted by $\overline{A}$. Similarly, the largest open set included in A is called the *interior* of A and denoted by int A or simply by $\mathring{A}$. By definition, $A = \overline{A}$ if and only if A is closed, and $A = \mathring{A}$ if and only if A is open. The set $\overline{A} \backslash \mathring{A}$ is called the *boundary* of A and denoted by ∂A. For a subset B of a set A, we say that B is *dense* in A if $\overline{B} = A$. A set A in M is *bounded* if there is $x_0 \in M$ and $R > 0$ such that A is included in $B_R(x_0)$. For a *mapping* f from a set S to M (i.e., an M-valued function defined on S), f is said to be *bounded* if its *image* $f(S)$ is bounded in M, where

$$f(S) = \{ f(x) \mid x \in S \}.$$

(2) Let V be a real vector space (a vector space over the field $\mathbf{R}$). A nonnegative function $\| \cdot \|$ on V is said to be a *norm* if
 (i) $\|x\| = 0$ if and only if $x = 0$ for $x \in V$;
 (ii) $\|cx\| = |c| \|x\|$ for all $x \in V$ and all $c \in \mathbf{R}$;
 (iii) (triangle inequality) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.
The vector space V equipped with a norm $\| \cdot \|$ is called a *normed vector space* and denoted by $(V, \| \cdot \|)$ if one needs to clarify the norm. By definition,

$$d(x, y) = \|x - y\|$$

is a metric. A normed vector space is regarded as a metric space with the foregoing metric.

(3) Let $\{z_j\}_{j=1}^\infty$ be a sequence in a metric space (M, d). We say that $\{z_j\}_{j=1}^\infty$ *converges* to $z \in M$ if for any $\varepsilon > 0$ there exists a natural number $n = n(\varepsilon)$ such that $j \geq n(\varepsilon)$ implies $d(z, z_j) < \varepsilon$. In other words,

$$\lim_{j \to 0} d(z, z_j) = 0.$$

We simply write $z_j \to z$ as $j \to \infty$, or $\lim_{j\to\infty} z_j = z$. If $\{z_j\}_{j=1}^{\infty}$ converges to some element, we say that $\{z_j\}_{j=1}^{\infty}$ is a *convergent sequence*.

(4) Let f be a mapping from a metric space (M_1, d_1) to another metric space (M_2, d_2). We say that $f(y)$ *converges* to $a \in M_2$ as y tends to x if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$d_2\,(f(y), a) < \varepsilon \quad \text{if} \quad d_1(y, x) < \delta.$$

We simply write $f(y) \to a$ as $y \to x$ or $\lim_{y\to x} f(y) = a$. If

$$\lim_{y\to x} f(y) = f(x),$$

then f is said to be *continuous* at $x \in M_1$. If f is continuous at all $x \in M_1$, then f is said to be *continuous* on M_1 (with values in M_2). The space of all continuous functions on M_1 with values in M_2 is denoted by $C(M_1, M_2)$.

(5) Let $\{z_j\}_{j=1}^{\infty}$ be a sequence in a metric space (M, d). We say that $\{z_j\}_{j=1}^{\infty}$ is a *Cauchy sequence* if for any $\varepsilon > 0$ there exists a natural number $n = n(\varepsilon)$ such that $j, k \geq n(\varepsilon)$ implies $d(z_j, z_k) < \varepsilon$. It is easy to see that a convergent sequence is always a Cauchy sequence, but the converse may not hold. We say that the metric space (M, d) is *complete* if any Cauchy sequence is a convergent sequence.

(6) Let $(V, \|\cdot\|)$ be a normed vector space. We say that V is a *Banach space* if it is complete as a metric space. The norm $\|\cdot\|$ is often written as $\|\cdot\|_V$ to distinguish it from other norms if we use several norms. We simply write $z_j \to z$ in V (as $j \to \infty$) if $\lim_{j\to\infty} \|z_j - z\|_V = 0$ and $z \in V$ for a sequence $\{z_j\}_{j=1}^{\infty}$. We often say that z_j converges to z *strongly* in V (as $j \to \infty$) to distinguish this convergence from other weaker convergences discussed later.

(7) Let V be a real vector space. A real-valued function $\langle\cdot,\cdot\rangle$ defined on $V \times V$ is said to be an *inner product* if
   (i) $\langle x, x\rangle \geq 0$ for all $x \in V$;
   (ii) $\langle x, x\rangle = 0$ if and only if $x = 0$;
   (iii) (symmetry) $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in V$;
   (iv) (linearity) $\langle c_1x_1 + c_2x_2, y\rangle = c_1\langle x_1, y\rangle + c_2\langle x_2, y\rangle$ for all $x_1, x_2, y \in V$, $c_1, c_2 \in \mathbf{R}$.

By definition, it is easy to see that

$$\|z\| = \langle z, z\rangle^{1/2}$$

is a norm. The space with an inner product is regarded as a normed vector space with the foregoing norm. If this space is complete as a metric space, we say that V is a *Hilbert space*. The Euclidean space $\mathbf{R}^N$ is a finite-dimensional Hilbert space equipped with a standard inner product. It turns out that any finite-dimensional Hilbert space is "isomorphic" to $\mathbf{R}^N$. Of course, a Hilbert space is an example of a Banach space.

(8) Let V be a Banach space equipped with norm $\|\cdot\|$. Let V^* denote the totality of all continuous linear function(al)s on V with values in $\mathbf{R}$. (By the Hahn–Banach theorem, the vector space V^* has at least one dimension. Incidentally, Mazur's theorem in the proof of Lemma 1.19 in Sect. 1.2.3 is another application of the Hahn–Banach theorem.)
The space V^* is called the *dual space* of V. Let $\{z_j\}_{j=1}^{\infty}$ be a sequence in V^*. We say that $\{z_j\}_{j=1}^{\infty}$ converges to $z \in V^*$ $*$-*weakly* if

$$\lim_{j\to\infty} z_j(x) = z(x)$$

for any $x \in V$. We often write $z_j \overset{*}{\rightharpoonup} z$ in V^* as $j \to \infty$. Such a sequence $\{z_j\}_{j=1}^{\infty}$ is called a $*$-*weak convergent sequence*. The dual space V^* is equipped with the norm

$$\|z\|_{V^*} := \sup\left\{z(x) \;\middle|\; \|x\| = 1,\ x \in V\right\} = \sup_{\|x\|=1} z(x).$$

The space V^* is also a Banach space with this norm. Here, for a subset A in $\mathbf{R}$, by $a = \sup A$ we mean that a is the smallest real member that satisfies $a \geq x$ for any $a \in A$. In other words, it is the least upper bound of A. The notation sup is the abbreviation of the *supremum*. Similarly, $\inf A$ denotes the greatest lower bound of A, and it is the abbreviation of the *infimum*. If $\sup A = a$ with $a \in A$, we write $\max A$ instead of $\sup A$. The same convention applies to inf and min.
Since V^* is a Banach space, there is a notion of convergence in the metric defined by the norm. To distinguish this convergence from $*$-weak convergence, we say that $\{z_j\}_{j=1}^{\infty}$ converges to z *strongly* in V^* if

$$\lim_{j\to\infty} \|z_j - z\|_{V^*} = 0,$$

and it is simply written $z_j \to z$ in V^* as $j \to \infty$. By definition, $z_j \to z$ implies $z_j \overset{*}{\rightharpoonup} z$, but the converse may not hold.

(9) Let A be a subset of a metric space M. The set A is said to be (sequentially) *relatively compact* if any sequence $\{z_j\}_{j=1}^{\infty}$ in A has a convergent subsequence in M. If, moreover, A is closed, we simply say that A is *compact*. When A is compact, it is always bounded. When A is a subset of $\mathbf{R}^N$, it is well known as the Bolzano–Weierstrass theorem that A is compact if and only if A is bounded and closed. However, if A is a subset of a Banach space V, such an equivalence holds if and only if V is of finite dimension. In other words, a bounded sequence of an infinite-dimensional Banach space may not have a (strongly) convergent subsequence.

There is a compactness theorem (Banach–Alaoglu theorem) that says if $\{z_j\}_{j=1}^\infty$ in a dual Banach space V^* is bounded, i.e.,

$$\sup_{j\geq 1} \|z_j\|_{V^*} < \infty,$$

then it has a $*$-weak convergent subsequence (Exercise 1.9).

(10) Let V be a Banach space and V^* denote its dual space. Let $\{x_k\}_{k=1}^\infty$ be a sequence in V. We say that $\{x_k\}_{k=1}^\infty$ converges to $x \in V$ *weakly* if

$$\lim_{k\to\infty} z(x_k) = z(x)$$

for all $z \in V^*$. We often write $x_k \rightharpoonup x$ in V as $k \to \infty$. Such a sequence is called a *weak convergent sequence*.

If a Banach space W is a dual space of some Banach space V, say, $W = V^*$, there are two notions, weak convergence and $*$-weak convergence. Let $\{z_j\}_{j=1}^\infty$ be a sequence in W. By definition, $z_j \overset{*}{\rightharpoonup} z$ (in W as $j \to \infty$) means that $\lim_{j\to\infty} z_j(x) = z(x)$ for all $x \in V$ while $z_j \rightharpoonup z$ (in W as $j \to \infty$) means that $\lim_{j\to\infty} y(z_j) = y(z)$ for all $y \in W^* = (V^*)^*$.

The space V can be continuously embedded in $V^{**} = (V^*)^*$. However, V may not be equal to V^{**}. Thus, weak convergence is stronger than $*$-weak convergence. If $V = V^{**}$, then both notions are the same. The space V is called *reflexive* if $V = V^{**}$.

(11) If V is a Hilbert space, it is reflexive. More precisely, the mapping $x \in V$ to $z \in V^*$ defined by

$$z(y) = \langle x, y\rangle, \quad y \in V$$

is a linear isomorphism from V to V^*, which is also norm preserving, i.e., $\|z\|_{V^*} = \|x\|$. This result is known as the Riesz–Fréchet theorem. Thus, the notions of weak convergence and $*$-weak convergence are the same.

(12) Let f be a real-valued function in a metric space M. We say that f is *lower semicontinuous* at $x \in M$ if

$$f(x) \leq \liminf_{y\to x} f(y) := \liminf_{\delta\downarrow 0} \left\{ f(y) \;\middle|\; d(y,x) < \delta \right\},$$

where $\lim_{\delta\downarrow 0}$ denotes the limit as $\delta \to 0$ but restricted to $\delta > 0$. Even if f is allowed to take $+\infty$, the definition of the lower semicontinuity will still be valid. If f is lower semicontinuous for all $x \in M$, we simply say that f is lower semicontinuous on M. If $-f$ is lower semicontinuous, we say that f is *upper semicontinuous*.

(13) Let $f = f(t)$ be a function of one variable in an interval I in $\mathbf{R}$ with values in a Banach space V. We say that f is *right differentiable* at $t_0 \in I$ if there is $v \in V$ such that

$$\lim_{h \downarrow 0} \| f(t_0 + h) - f(t_0) - vh \| \Big/ h = 0$$

provided that $t_0 + h \in I$ for sufficiently small $h > 0$. Such v is uniquely determined if it exists and is denoted by

$$v = \frac{\mathrm{d}^+ f}{\mathrm{d}t}(t_0).$$

This quantity is called the *right differential* of f at t_0. The function $t \mapsto \frac{\mathrm{d}^+ f}{\mathrm{d}t}(t)$ is called the *right derivative* of f. The left differentiability is defined in a symmetric way by replacing $h \downarrow 0$ with $h \uparrow 0$. Even if both right and left differentials exist, they may be different. For example, consider $f(t) = |t|$ at $t_0 = 0$. The right differential at zero is 1, while the left differential at zero is -1. If the right and left differentials agree with each other at $t = t_0$, we say that f is *differentiable* at $t = t_0$, and its value is denoted by $\frac{\mathrm{d}f}{\mathrm{d}t}(t_0)$. The function $t \mapsto \frac{\mathrm{d}f}{\mathrm{d}t}(t)$ is called the *derivative* of f. If f depends on other variables, we write $\partial f / \partial t$ instead of $\mathrm{d}f/\mathrm{d}t$ and call the *partial derivative* of f with respect to t.

5.2 Measures and Integrals

(1) For a set M, let 2^M denote the family of all subsets of H. We say that a function μ defined on 2^M with values in $[0, \infty]$ is an (outer) *measure* if
   (i) $\mu(\emptyset) = 0$;
   (ii) (countable subadditivity) $\mu(A) \le \sum_{j=1}^{\infty} \mu(A_j)$ if a countable family $\{A_j\}_{j=1}^{\infty}$ *covers* A, where $A_j, A \in 2^M$. In other words, A is included in a union of $\{A_j\}_{j=1}^{\infty}$, i.e., a point of A must be an element of some A_j.

   Here, $\emptyset$ denotes the empty set.

(2) A set $A \in 2^M$ is said to be μ-*measurable* if

$$\mu(S \cap A^c) + \mu(S \cap A) = \mu(S)$$

for any $S \in 2^M$. Let M_0 be a metric space. A mapping f from M to M_0 is said to be μ-*measurable* if the *preimage* $f^{-1}(U)$ of an open set U of M_0 is μ-measurable. Here,

$$f^{-1}(U) := \left\{ x \in M \;\middle|\; f(x) \in U \right\}.$$

A set A with $\mu(A) = 0$ is called a *μ-measure zero set*. If a statement $P(x)$ for $x \in M$ holds for $x \in M \backslash A$ with $\mu(A) = 0$, we say that $P(x)$ holds for *μ-almost every* $x \in M$ or shortly a.e. $x \in M$. In other words, P holds in M outside a μ-measure zero set. In this case, we simply say that P holds *almost everywhere* in M.

Let $\mathcal{M}$ be the set of all μ-measurable sets. If we restrict μ just to $\mathcal{M}$, i.e., $\overline{\mu} = \mu|_{\mathcal{M}}$, then $\overline{\mu}$ becomes a measure on $\mathcal{M}$. Since in this book we consider $\mu(A)$ for a μ-measurable set A, we often say simply a measure instead of an outer measure.

(3) Let A be a subset of $\mathbf{R}^N$. Let $\mathcal{C}$ be a family of closed cubes in $\mathbf{R}^N$ whose faces are orthogonal to the x_i-axis for some $i = 1, \ldots, N$. In other words, $C \in \mathcal{C}$ means

$$C = \left\{ (x_1, \ldots, x_N) \in \mathbf{R}^N \;\middle|\; a_i \le x_i \le a_i + \ell \; (i = 1, \ldots, n) \right\}$$

for some $a_i, \ell \in \mathbf{R}$. Let $|C|$ denote its volume, i.e., $|C| = \ell^N$. We set

$$\mathcal{L}^N(A) = \inf \left\{ \sum_{j=1}^{\infty} |C_j| \;\middle|\; \{C_j\}_{j=1}^{\infty} \text{ covers } A \text{ with } C_j \in \mathcal{C} \right\}.$$

It turns out that $\mathcal{L}^N(C) = |C|$; it is nontrivial to prove $\mathcal{L}^N(C) \ge |C|$. It is easy to see that $\mathcal{L}^N$ is an (outer) measure in $\mathbf{R}^N$. This measure is called the *Lebesgue measure* in $\mathbf{R}^N$. It can be regarded as a measure in the flat torus $\mathbf{T}^N = \prod_{i=1}^N (\mathbf{R}/\omega_i \mathbf{Z})$. For a subset A of $\mathbf{T}^N$, we regard this set as a subset A_0 of the *fundamental domain* (i.e., the *periodic cell* $[0, \omega_1) \times \cdots \times [0, \omega_N)$). The Lebesgue measure of A is defined by $\mathcal{L}^N(A) = \mathcal{L}^N(A_0)$. Evidently, $\mathcal{L}^N(\mathbf{T}^N) = \omega_1 \cdots \omega_N$, which is denoted by $|\mathbf{T}^N|$ in the proof of Lemma 1.21 in Sect. 1.2.5.

(4) In this book, we only use the Lebesgue measure. We simply say measurable when a mapping or a set is $\mathcal{L}^N$-measurable. Instead of writing $\mathcal{L}^N$-a.e., we simply write a.e. Let Ω be a measurable set in $\mathbf{T}^N$ or $\mathbf{R}^N$, for example, $\Omega = \mathbf{T}^N$. Let f be a measurable function on Ω with values in a Banach space V. Then one is able to define its integral over Ω. When $V = \mathbf{R}^m$, this integral is called the Lebesgue integral. In general, it is called the Bochner integral of f over Ω. Its value is denoted by $\int_\Omega f \, d\mathcal{L}^N$ or, simply, $\int_\Omega f \, dx$; See, for example, [90, Chapter V, Section 5]. If $\Omega = \mathbf{T}^N$ and f is continuous, this agrees with the more conventional Riemann integral. For $p \in [1, \infty)$ and a general Banach space V, let $\tilde{L}^p(\Omega, V)$ denote the space of all measurable functions f with values in V such that

$$\|f\|_p = \left(\int_\Omega \|f(x)\|^p \, dx \right)^{1/p}$$

is finite. If $\|f\|_p$ is finite, we say that f is pth *integrable*. If $p = 1$, we simply say f is integrable. If $f \in \tilde{L}^1(\Omega, V)$, we say that f is *integrable* in Ω. We identify two functions $f, g \in \tilde{L}^p(\Omega, V)$ if $f = g$ a.e. and define $L^p(\Omega, V)$ from $\tilde{L}^p(\Omega, V)$ by this identification. It is a fundamental result that $L^p(\Omega, V)$ is a Banach space equipped with the norm $\|\cdot\|_p$. When $V = \mathbf{R}$, we simply write $L^p(\Omega)$ instead of $L^p(\Omega, V)$. The case $p = \infty$ should be handled separately. For a general Banach space V, let $\tilde{L}^\infty(\Omega, V)$ denote the space of all measurable functions f with values in V such that

$$\|f\|_\infty = \inf\left\{\alpha \,\middle|\, \mathcal{L}^N\left(\left\{x \in \Omega \,\middle|\, \|f(x)\|_V > \alpha\right\}\right) = 0\right\}$$

is finite. By the same identification, the space $L^\infty(\Omega, V)$ can be defined. This space $L^\infty(\Omega, V)$ is again a Banach space. Key theorems in the theory of Lebesgue integrals used in this book include the Lebesgue dominated convergence theorem and Fubini's theorem. Here, we give a version of the *dominated convergence theorem*.

Theorem 5.1

Let V be a Banach space. Let $\{f_m\}_{m=1}^\infty$ be a sequence in $L^1(\Omega, V)$. Assume that there is a nonnegative function $\varphi \in L^1(\Omega)$ independent of m such that $\|f_m(x)\|_V \le \varphi(x)$ for a.e. $x \in \Omega$. If $\lim_{m\to\infty} f_m(x) = f(x)$ for a.e. $x \in \Omega$, then

$$\lim_{m\to\infty} \int_\Omega f_m(x)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x.$$

In other words, $\lim_{m\to\infty} \left\| \int_\Omega f_m\,\mathrm{d}x - \int_\Omega f(x)\,\mathrm{d}x \right\|_V = 0.$

Usually, V is taken as $\mathbf{R}$ or $\mathbf{R}^N$, but it is easy to extend to this setting. For basic properties of the Lebesgue measure and integrals, see for example a classical book of Folland [42]. We take this opportunity to clarify $*$-weak convergence in L^p space. A basic fact is that $(L^p(\Omega))^* = L^{p'}(\Omega)$ for $1 \le p < \infty$, where $1/p + 1/p' = 1$. Note that $p = \infty$ is excluded, but $(L^1)^* = L^\infty$. Since L^p is reflexive for $1 < p < \infty$, weak convergence and $*$-weak convergence agree with each other. Let us write a $*$-weak convergence in L^∞ explicitly. A sequence $\{f_m\}$ in $L^\infty(\Omega)$ $*$-weakly converges to $f \in L^\infty(\Omega)$ as $m \to \infty$ if and only if

$$\lim_{m\to\infty} \int_\Omega f_m\varphi\,\mathrm{d}x = \int_\Omega f\varphi\,\mathrm{d}x$$

for all $\varphi \in L^1(\Omega)$. For detailed properties of L^p spaces, see, for example, [19, Chapter 4].

For a Banach space V-valued L^p function, we also consider its dual space. That is, we have

$$\left(L^p(\Omega, V)\right)^* = L^{p'}(\Omega, V^*) \quad \text{for} \quad 1 \le p < \infty$$

with $1/p + 1/p' = 1$. (This duality—at least for reflexive V—can be proved along the same line as in [19, Chapter 4], where V is assumed to be $\mathbf{R}$. For a general Banach space V, see, for example, [35, Chapter IV].) We consider $*$-weak convergence in $L^\infty(\Omega, V)$ with $V = L^q(U)$, $1 < q \le \infty$, where Ω is an open interval $(0, T)$ and U is an open set in $\mathbf{T}^N$ or $\mathbf{R}^N$ since this case is explicitly used in Chap. 2. A sequence $\{f_m\}$ in $L^\infty(\Omega, L^q(U))$ $*$-weakly converges to $f \in L^\infty(\Omega, L^q(U))$ as $m \to \infty$ if and only if

$$\lim_{m\to\infty} \int_0^T \int_U f_m(x,t)\varphi(x,t)\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_U f(x,t)\varphi(x,t)\mathrm{d}x\,\mathrm{d}t$$

for $\varphi \in L^1\left(\Omega, L^{q'}(U)\right)$. (Note that the space $L^p(\Omega, L^q(U))$ is identified with the space of all measurable functions φ on $\Omega \times U$ such that $\int_0^T \|\varphi\|_{L^q(U)}^p(t)\,\mathrm{d}t < \infty$ or $\int_0^T \left(\int_U |\varphi(x,t)|^q\,\mathrm{d}x\right)^{p/q}\,\mathrm{d}t < \infty$ for $1 \le p, q < \infty$.)

(5) Besides the basic properties of the Lebesgue integrals, we frequently use a few estimates involving L^p-norms. These properties are by now standard and found in many books, including [19]. For example, we frequently use the *Hölder inequality*

$$\|fg\|_p \le \|f\|_r \|g\|_q$$

with $1/p = 1/r + 1/q$ for $f \in L^r(\Omega)$, $g \in L^q(\Omega)$, where $p, q, r \in [1, \infty]$. Here, we interpret $1/\infty = 0$. In the case $p = 1$, $r = q = 2$, this inequality is called the *Schwarz inequality*. As an application, we have *Young's inequality* for a convolution

$$\|f * g\|_p \le \|f\|_q \|g\|_r$$

for $f \in L^q(\mathbf{R}^N)$, $g \in L^r(\mathbf{R}^N)$ with $1/p = 1/q + 1/r - 1$ and $p, q, r \in [1, \infty]$; see, for example, [45, Chapter 4]. In this book, we use this inequality when $\mathbf{R}^N$ is replaced by $\mathbf{T}^N$.

(6) In analysis, we often need an approximation of a function by smooth functions. We only recall an elementary fact. The space $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ for $p \in [1, \infty)$; see, for example, [19, Corollary 4.23]. However, it is not dense in $L^\infty(\Omega)$.