Hamilton–Jacobi Equations

4

In the last chapter, we discussed uniqueness in a special class of weak solutions called entropy solutions for scalar conservation laws, which are quasilinear first-order equations. The notion of a weak solution and an entropy solution is based on integration by parts or a variational principle.

In this chapter, we consider another class of nonlinear first-order equations whose nonlinearity is very strong and not quasilinear. Such an equation is often called the Hamilton–Jacobi equation. It is in general impossible to introduce the notion of a weak solution by integration by parts. Instead, we introduce a notion of a weak solution based on the maximum principle. Such a notion was first introduced by Crandall and Lions [29] in the early 1980s as a viscosity solution and has been extensively studied.

In this chapter, we study uniqueness problems of viscosity solutions for several types of equations. We first observe that one-dimensional evolutionary Hamilton–Jacobi equations are formally an integration of a one-dimensional scalar conservation law. We then discuss the uniqueness issue for its stationary form, the eikonal equation, as well as evolutionary Hamilton–Jacobi equations. We also discuss a scalar conservation law and its generalization from the viewpoint of viscosity solutions to handle jump discontinuities.

4.1 Hamilton–Jacobi Equations from Conservation Laws

In this section, we derive a fully nonlinear equation of first order called a Hamilton–Jacobi equation from a conservation law. We shall give another interpretation of the entropy condition. We also derive a kind of stationary problem, the eikonal equation.

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4.1.1 Interpretation of Entropy Solutions

We consider a conservation law for a real-valued function $u = u(x, t)$, $x \in \mathbf{R}$, $t > 0$ of the form

$$u_t + (f(u))_x = 0, \tag{4.1}$$

where f is a given real-valued continuous function on $\mathbf{R}$. We integrate from 0 to x to get

$$\tilde{U}_t + f(\tilde{U}_x) = f\,(u(0, t)) \tag{4.2}$$

if we set $\tilde{U}(x, t) = \int_0^x u(y, t)\,\mathrm{d}y$. We set

$$U(x, t) = \tilde{U}(x, t) - \int_0^t f\,(u(0, s))\,\mathrm{d}s$$

and obtain

$$U_t + f(U_x) = 0. \tag{4.3}$$

This equation is fully nonlinear and called an (evolutionary) Hamilton–Jacobi equation. This is simply a formal procedure since u may jump at $x = 0$, so the value $f\,(u(0, t))$ is not well defined.

We consider a Riemann problem for (4.1) with initial condition

$$u(x, 0) = u_0(x), \tag{4.4}$$

with

$$u_0(x) = \begin{cases} -\alpha, & x < 0, \\ \alpha, & x > 0, \end{cases}$$

where $\alpha \in \mathbf{R}$, $\alpha \neq 0$. To simplify the presentation, we set $f(u) = u^2/2$, which corresponds to the case of the Burgers equation. As we already observed in Chap. 3,

$$u(x, t) = u_0(x)$$

is an entropy solution to (4.1) with (4.4) if $\alpha < 0$. It is not an entropy solution when $\alpha > 0$. For $\alpha > 0$ the entropy solution is a rarefaction wave u_R of the form

$$u_R(x, t) = \begin{cases} -\alpha, & x < x_\ell(t), \\ \dfrac{x}{t}, & x_\ell(t) < x < x_r(t), \\ \alpha, & x_r(t) < x, \end{cases}$$

where $x_\ell(t) = -\alpha t$ and $x_r(t) = \alpha t$. It is continuous for $t > 0$. Thus, if $\alpha > 0$, then $U = U_R$ with

$$U_R = \int_0^x u_R(y,t)\,\mathrm{d}y$$

solves (4.3) since $u_R(0,t) = 0$. However, if $\alpha < 0$, then the term $f(u(0,t))$ should be interpreted as $f(u(+0,t))$ $(= f(u(-0,t)) = f(\alpha)$. Thus,

$$\tilde{U}(x,t) = \int_0^x u(y,t)\,\mathrm{d}y = \int_0^x u_0(y)\,\mathrm{d}y$$

solves (4.2) with the right-hand side $f(u(0,t)) = f(\alpha)$. We thus conclude that

$$V(x,t) = \int_0^x u(y,t)\mathrm{d}y - f(\alpha)t = v_0(x) - f(\alpha)t,$$

with

$$v_0(x) = \begin{cases} -\alpha x, & x < 0, \\ \alpha x, & x > 0, \end{cases}$$

solves (4.3) with initial datum $v_0(x)$. Although V even "solves" (4.3) for $\alpha > 0$, its derivative u is not an entropy solution of (4.1). We have two solutions U_R and V with the same initial datum v_0 if $\alpha > 0$. We would like to choose a solution whose spatial derivative is an entropy solution.

We recall that an entropy solution is obtained as a vanishing viscosity method. In other words, it is as a limit of the ε-approximated equation

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon.$$

As previously, we set

$$U^\varepsilon(x,t) = \int_0^x u^\varepsilon(y,t)\,\mathrm{d}y - \int_0^x f\left(u^\varepsilon(0,s)\right)\mathrm{d}s$$

and obtain

$$U_t^\varepsilon + f(U_x^\varepsilon) = \varepsilon U_{xx}^\varepsilon.$$

By the construction of an entropy solution, it is clear that our solutions U_R for $\alpha > 0$ and V for $\alpha < 0$ are obtained as a limit $\lim_{\varepsilon\downarrow 0} U^\varepsilon$, at least formally.

4.1.2 A Stationary Problem

We continue to assume that $f(u) = u^2/2$. The solution V in Sect. 4.1.1 we found does not change its profile. It is a translative solution of (4.3) or a soliton-like solution. If we consider $W = V + f(\alpha)t$, then W solves

$$-f(\alpha) + f(W_x) = 0. \tag{4.5}$$

This is a stationary Hamilton–Jacobi equation. For $\alpha < 0$, this solution is obtained as a limit of aforementioned vanishing viscosity approach, while for $\alpha > 0$, it is not obtained as such a limit.

Although so far we assume for simplicity that $f(u) = u^2/2$, all arguments in Sects. 4.1.1 and 4.1.2 work for a general convex function f with $f(\sigma) = f(-\sigma)$ for all $\sigma \in \mathbf{R}$ and $f(0) = 0$ with modification of the explicit formula of the rarefaction wave u_R.

The equation $f(U_x) = g(x)$ is often called *the eikonal equation*. If $f(u) = u^2/2$, then this is of the form $|U_x| = \sqrt{2g}$. In multidimensional cases, it is of the form

$$|\nabla U| = G \quad \text{in} \quad \Omega,$$

where G is a given function defined in a domain Ω in $\mathbf{R}^N$.

4.2 Eikonal Equation

In this section, we begin with a one-dimensional eikonal equation and then introduce a notion of viscosity solution to distinguish jumps of derivatives. We conclude this section by proving uniqueness (comparison principle) based on a kind of doubling-variables argument, unlike in Chap. 3.

4.2.1 Nonuniqueness of Solutions

We consider a very simple example of the eikonal equation

$$\left|\frac{\mathrm{d}u}{\mathrm{d}x}\right| - 1 = 0 \quad \text{in} \quad (-1, 1) \tag{4.6}$$

with the Dirichlet boundary condition

$$u(\pm 1) = 0. \tag{4.7}$$

Here, $u = u(x)$ is a real-valued function defined for $x \in (-1, 1)$. It is clear that there is no C^1 solution. If one allows continuous functions satisfying the equation except at finitely many points, there are infinitely many solutions (even if nonnegative solutions are considered). For example,

$$\begin{aligned} u_0(x) &= 1 - |x|, & & |x| \le 1, \\ u_k(x) &= \frac{1}{2^k} a(2^k x), \quad & k = 1, 2, \ldots, & \; |x| \le 1, \end{aligned}$$

with

$$a(y) = \max \left\{ 1 - |y - (2m+1)| \;\middle|\; m \in \mathbf{Z} \right\},$$

are such solutions (Fig. 4.1). One would like to choose a typical solution of (4.6) and (4.7). One natural solution is a distance function from the boundary ± 1, which corresponds to u_0. See Exercise 4.8.

To conclude that a solution is unique, we must impose extra conditions like an entropy condition, which is obtained using a vanishing viscosity method. We consider for $\varepsilon > 0$

$$\left| \frac{\mathrm{d}u}{\mathrm{d}x} \right| - 1 = \varepsilon \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \quad \text{in} \quad (-1, 1). \tag{4.8}$$

Then it is easy to see that (4.8) under (4.7) admits a unique C^2 solution u_ε. Indeed, it can be written as

$$u_\varepsilon(x) = \begin{cases} 1 - x + \varepsilon(e^{-1/\varepsilon} - e^{-x/\varepsilon}), & 0 \le x \le 1, \\ 1 + x + \varepsilon(e^{-1/\varepsilon} - e^{x/\varepsilon}), & -1 \le x < 0. \end{cases}$$

The uniqueness can be proved using the uniqueness of the initial value problem of ordinary differential equations in Sect. 1.1. (We do not give details here. The uniqueness can also be proved by the maximum principle for second-order ordinary differential equations; see, for example, [84].) If we take its limit as $\varepsilon \to 0$, then

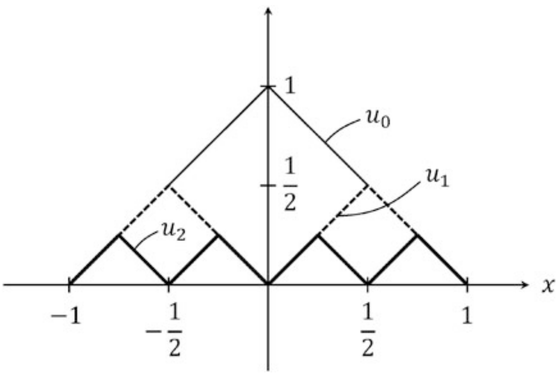

Fig. 4.1 Graphs of u_k

evidently $u_\varepsilon(x) \to u_0(x)$. We would like to choose u_0 as a "reasonable" solution of (4.6) with (4.7). It is desirable to check whether or not it is a reasonable solution without approximation. In other words, we must find a suitable notion like entropy solution to choose a reasonable solution.

4.2.2 Viscosity Solution

We consider a general Hamilton–Jacobi equation in a *domain* (i.e., connected open set) Ω in $\mathbf{R}^N$ of the form

$$H(x, \nabla u) = 0. \tag{4.9}$$

Here H is a (real-valued) continuous function in $\overline{\Omega} \times \mathbf{R}^N$, and $\nabla u = (\partial_1 u, \dots, \partial_N u)$ is the gradient of a scalar function $u = u(x)$, $x \in \Omega$. To motivate the definition of a viscosity solution, we consider a C^2 solution u and consider $\varphi \in C^2(\Omega)$ such that $\max_{\overline{\Omega}}(u - \varphi) = (u - \varphi)(\hat{x})$ for some $\hat{x} \in \Omega$. We know that at the maximum point $\hat{x}$

$$\begin{aligned} \nabla(u - \varphi)(\hat{x}) = 0, &\quad \nabla^2(u - \varphi)(\hat{x}) \le O \\ \text{or} \quad \nabla u(\hat{x}) = \nabla\varphi(\hat{x}), &\quad \nabla^2 u(\hat{x}) \le \nabla^2\varphi(\hat{x}). \end{aligned}$$

Here, $\nabla^2 u = (\partial_{x_i} \partial_{x_j} u)$ denotes the $N \times N$ Hessian matrix of u and O denotes the $N \times N$ zero matrix. For two symmetric matrices A and B, we say that $A \le B$ if the corresponding quadratic form for $B - A$ is nonnegative, i.e.,

$$\langle \eta, (B - A)\eta \rangle \ge 0$$

for all $\eta \in \mathbf{R}^N$. Let Δ denote the Laplace operator, i.e., $\Delta u = \sum_{i=1}^N \partial_i^2 u$. Assume that a solution u of (4.9) is obtained as a vanishing viscosity approach, more precisely u is given a limit of u^ε as $\varepsilon \downarrow 0$, and u^ε solves

$$H(x, \nabla u^\varepsilon) = \varepsilon \Delta u^\varepsilon.$$

Let $x_\varepsilon \in \Omega$ be a maximum point of $u^\varepsilon - \varphi$ in $\overline{\Omega}$. Assume that $x_\varepsilon \to \hat{x}$ as $\varepsilon \downarrow 0$. Then,

$$H\left(x_\varepsilon, \nabla\varphi(x_\varepsilon)\right) \le \varepsilon \Delta\varphi(x_\varepsilon)$$

since $\nabla^2 u \le \nabla^2\varphi$ at $x = x_\varepsilon$ implies $\Delta u \le \Delta\varphi$ at $x = x_\varepsilon$; we here note that $\Delta u = \operatorname{tr}(\nabla^2 u)$ and $\Delta\varphi = \operatorname{tr}(\nabla^2\varphi)$. Since u is a limit of u^ε and $x_\varepsilon \to \hat{x}$, we only obtain

$$H\left(\hat{x}, \nabla\varphi(\hat{x})\right) \le 0$$

instead of $H\left(\hat{x}, \nabla\varphi(\hat{x})\right) = 0$. Based on this observation, we arrive at the following definition of a viscosity solution.

Definition 4.1

A function $u \in C(\Omega)$ is said to be a *viscosity subsolution* of (4.9) in Ω if

$$H\left(\hat{x}, \nabla\varphi(\hat{x})\right) \leq 0$$

whenever $(\varphi, \hat{x}) \in C^1(\Omega) \times \Omega$ fulfills $\max_\Omega(u - \varphi) = (u - \varphi)(\hat{x})$. A function $u \in C(\Omega)$ is said to be a *viscosity supersolution* of (4.9) in Ω if

$$H\left(\hat{x}, \nabla\varphi(\hat{x})\right) \geq 0$$

whenever $(\varphi, \hat{x}) \in C^1(\Omega) \times \Omega$ fulfills $\min_\Omega(u - \varphi) = (u - \varphi)(\hat{x})$. If u is a viscosity sub- and supersolution, then u is said to be a *viscosity solution*.

It is easy to see that the C^1 function u is a viscosity subsolution if and only if u is a subsolution, i.e.,

$$H\left(x, \nabla u(x)\right) \leq 0 \quad \text{in} \quad \Omega.$$

We now check the example in the last subsection, where

$$H(x, \nabla u) = \left|\frac{\mathrm{d}u}{\mathrm{d}x}\right| - 1.$$

It is easy to see that u_k is a viscosity subsolution in $(-1, 1)$, but it is not a viscosity supersolution in $(-1, 1)$, except $k = 0$. Thus, among $\{u_k\}$, u_0 is the only viscosity solution.

Note that the notion of viscosity solution for $\left|\frac{\mathrm{d}u}{\mathrm{d}x}\right| - 1 = 0$ and $1 - \left|\frac{\mathrm{d}u}{\mathrm{d}x}\right| = 0$ is different. In fact, $-u_0$ is a viscosity solution of $1 - \left|\frac{\mathrm{d}u}{\mathrm{d}x}\right| = 0$, but it is not a viscosity solution of $\left|\frac{\mathrm{d}u}{\mathrm{d}x}\right| - 1 = 0$ (Exercise 4.1).

4.2.3 Uniqueness

We now consider the uniqueness problem for the eikonal equation

$$|\nabla u| - f(x) = 0 \quad \text{in} \quad \Omega. \tag{4.10}$$

Let $\partial\Omega$ denote the *boundary* of Ω.

Theorem 4.2 (Comparison principle)
Let Ω be a bounded domain in $\mathbf{R}^N$. Assume that $f \in C(\overline{\Omega})$ is positive in Ω. Let $u \in C(\overline{\Omega})$ and $v \in C(\overline{\Omega})$ be a viscosity sub- and supersolution of (4.10), respectively. If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in $\overline{\Omega}$. In particular, for a given continuous boundary value g on $\partial\Omega$, a viscosity solution u of (4.10) in $C(\overline{\Omega})$ with $u = g$ on $\partial\Omega$ is unique.

Proof. We shall prove that $u \leq v$ in $\overline{\Omega}$. Since $\overline{\Omega}$ is compact, by continuity, u and v are bounded (by Weierstrass' theorem). By adding a suitable constant, we may assume that u and v are nonnegative, i.e., $u, v \geq 0$ in $\overline{\Omega}$.

It suffices to prove that $\lambda u \leq v$ in $\overline{\Omega}$ for all $\lambda \in (0, 1)$ since $\lim_{\lambda\uparrow 1} \lambda u = u$ in $\overline{\Omega}$. Note that $u_\lambda = \lambda u$ is a viscosity solution of

$$|\nabla u| - \lambda f(x) = 0 \quad \text{in} \quad \Omega. \tag{4.11}$$

We shall fix λ in the sequel.

Although it is logically unnecessary, we first prove that $u_\lambda \leq v$ in $\overline{\Omega}$ when $v \in C^1(\Omega)$ because it reveals the merit of using u_λ instead of u. If $u_\lambda \leq v$ were false, then the function $u_\lambda - v$ would take a positive maximum at some $x_* \in \overline{\Omega}$. (The existence of a maximum follows from Weierstrass' theorem since $\overline{\Omega}$ is compact.) On the boundary $\partial\Omega$, we know $u_\lambda \leq v$, so $x_* \in \Omega$. Since u_λ is a viscosity subsolution of (4.11), by definition,

$$|\nabla v(x_*)| - \lambda f(x_*) \leq 0.$$

Since v is a classical subsolution of (4.10), we see that

$$|\nabla v(x_*)| - f(x_*) \geq 0.$$

Subtracting the second inequality from the first, we end up with $-\lambda f(x_*) + f(x_*) \leq 0$, which yields a contradiction since $\lambda < 1$ and $f > 0$ on Ω. Unfortunately, this argument does not work if v is not C^1.

To overcome this difficulty, we introduce a doubling-variables method (which is, of course, different from Kružkov's for conservation law). We note that if α is large, then $-\Phi_\alpha$ is sufficiently large, i.e., $\Phi_\alpha \ll 0$ away from the diagonal set $\left\{(x, x) \mid x \in \overline{\Omega}\right\}$. We consider

$$\Phi_\alpha(x, y) = u_\lambda(x) - v(y) - \alpha|x - y|^2$$

for a large positive number $\alpha > 0$. Assume that $u_\lambda \le v$ in $\overline{\Omega}$ would be false. Since we assume $u \ge 0$, we see that $u_\lambda \le v$ on $\partial\Omega$. Thus, there would exist $x_0 \in \Omega$ such that $m = \Phi_\alpha(x_0, x_0) > 0$. This would imply

$$\max_{\overline{\Omega}\times\overline{\Omega}} \Phi_\alpha \ge m > 0.$$

Let $(x_\alpha, y_\alpha) \in \overline{\Omega}\times\overline{\Omega}$ be a maximizer of Φ_α over $\overline{\Omega}\times\overline{\Omega}$, i.e., $\max \Phi_\alpha = \Phi_\alpha(x_\alpha, y_\alpha)$. Such (x_α, y_α) exists because of Weierstrass' theorem. Since $m > 0$ and both u and v are bounded as $\alpha \to \infty$, it is easy to see that $\alpha|x_\alpha - y_\alpha|^2$ is bounded. In particular, $x_\alpha - y_\alpha \to 0$ as $\alpha \to \infty$. Since Ω is bounded so that $\{x_\alpha\}$ is bounded, by compactness (Bolzano–Weierstrass theorem), there is a subsequence $\{x_{\alpha'}\}$ of $\{x_\alpha\}$ converging to some $\hat{x} \in \overline{\Omega}$. Similarly, $\{y_{\alpha'}\}$ has a subsequence $\{y_{\alpha''}\}$ converging to some $\hat{y} \in \overline{\Omega}$. Since $x_\alpha - y_\alpha \to 0$, we see that $\hat{x} = \hat{y}$. We shall denote $\{x_{\alpha''}\}$, $\{y_{\alpha''}\}$ by $\{x_{\alpha'}\}$, $\{y_{\alpha'}\}$ for simplicity.

Since we have assumed that $u_\lambda \le v$ on $\partial\Omega$, we see that $\hat{x} \notin \partial\Omega$. In fact, if $x_{\alpha'}, y_{\alpha'} \to \hat{x} \in \partial\Omega$, then, by the continuity of u and v, we see

$$m \le \limsup_{\alpha'\to\infty} \Phi_{\alpha'}(x_{\alpha'}, y_{\alpha'}) \le \limsup_{\alpha'\to\infty} (u_\lambda(x_{\alpha'}) - v(y_{\alpha'})) = u_\lambda(\hat{x}) - v(\hat{x}) \le 0,$$

which is a contradiction (Fig. 4.2).

We take α sufficiently large so that $x_\alpha, y_\alpha \in \Omega$. Since Φ is maximized at x_α, y_α, we see that the function

$$x \mapsto u_\lambda(x) - \varphi_\alpha(x), \quad \varphi_\alpha(x) = v(y_\alpha) + \alpha|x - y_\alpha|^2$$

takes its maximum at x_α and the function

$$y \mapsto v(y) - \psi_\alpha(y), \quad \psi_\alpha(y) = u_\lambda(x_\alpha) - \alpha|x_\alpha - y|^2$$

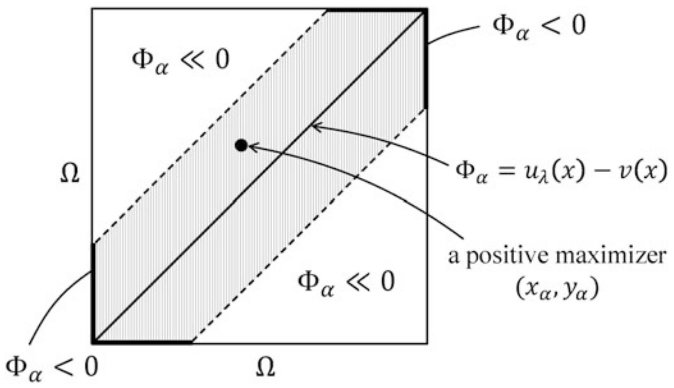

Fig. 4.2 Values of Φ_α

takes its minimum at y_α. By the definition of viscosity sub- and supersolutions, we conclude that

$$|\nabla \varphi_\alpha(x_\alpha)| - \lambda f(x_\alpha) \le 0,$$
$$|\nabla \psi_\alpha(y_\alpha)| - f(y_\alpha) \ge 0.$$

Subtracting the second inequality from the first and observing that $\nabla_x \varphi_\alpha(x_\alpha) = \nabla_y \psi_\alpha(y_\alpha)$, we now obtain

$$-\lambda f(x_\alpha) + f(y_\alpha) \le 0.$$

Since $x_{\alpha'} \to \hat{x}$ and $y_{\alpha'} \to \hat{x}$, sending $\alpha' \to \infty$ yields

$$-\lambda f(\hat{x}) + f(\hat{x}) \le 0.$$

If $f > 0$ on Ω, this leads to a contradiction since $\lambda < 1$. We thus conclude that $\lambda u \le v$ for all $\lambda \in (0, 1)$, which implies $u \le v$ in $\overline{\Omega}$.

Suppose that there are two solutions, u_1 and $u_2 \in C(\overline{\Omega})$, of (4.10) with $u_1 = u_2 = g$ on $\partial\Omega$. By the comparison just proved, we observe that $u_1 \le u_2$ and $u_2 \le u_1$ in $\overline{\Omega}$. This implies $u_1 = u_2$. The proof is now complete. □

The assumption $f(x) > 0$ for all $x \in \Omega$ is essential. If f takes a zero at some point of Ω, the uniqueness actually fails. In fact, if one considers

$$\begin{cases} \left|\dfrac{\mathrm{d}u}{\mathrm{d}x}\right| - |x| = 0, & |x| < 1 \\ u(\pm 1) = 0, \end{cases}$$

then

$$v_a(x) = \min\left\{(1 - x^2)/2, a + x^2/2\right\}$$

is a viscosity solution for all $a \in [-1/2, 1/2]$ (Fig. 4.3). It turns out that there is at most one solution if all its values on the set $\{x \mid f(x) = 0\}$ are prescribed; see the last paragraph of Sect. 4.5.1.

Note also that there may be no solution for given boundary data. Indeed, if we consider (4.6) in $(-1, 1)$ with $u(-1) = 0$, $u(1) = 3$, then there is no viscosity solution $u \in C[-1, 1]$ satisfying this boundary value. One must interpret the boundary condition in some weak sense.

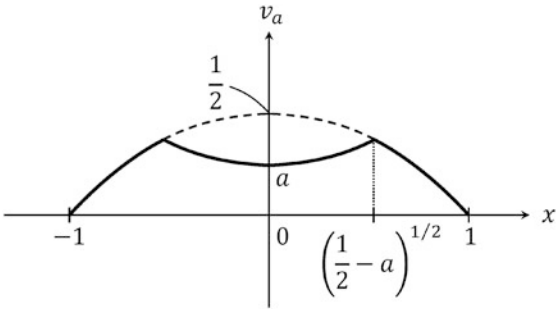

Fig. 4.3 Graphs of v_a

4.3 Viscosity Solutions of Evolutionary Hamilton–Jacobi Equations

In this section, we consider an evolutionary Hamilton–Jacobi equation and discuss the uniqueness of viscosity solutions under the periodic boundary condition. The proof is similar to that in the last section.

4.3.1 Definition of Viscosity Solutions

We consider an evolutionary Hamilton–Jacobi equation of the form

$$u_t + H(x, \nabla u) = 0 \quad \text{in} \quad \Omega \times (0, T), \tag{4.12}$$

where Ω is a domain in $\mathbf{R}^N$ or $\mathbf{T}^N$, which imposes a periodic boundary condition. Here we continue to assume that H is a (real-valued) continuous function in $\overline{\Omega} \times \mathbf{R}^N$; ∇u denotes the (spatial) gradient of a scalar function $u = u(x, t)$ defined on $\Omega \times (0, T)$, i.e., $\nabla u = (\partial_1 u, \ldots, \partial_N u)$.

Definition 4.3

A function $u \in C(\overline{Q})$ with $Q = \Omega \times (0, T)$ is said to be a *viscosity subsolution* of (4.12) in Q if

$$\varphi_t(\hat{x}, \hat{t}) + H\left(\hat{x}, \nabla\varphi(\hat{x}, \hat{t})\right) \le 0$$

whenever $\left(\varphi, (\hat{x}, \hat{t})\right) \in C^1(Q) \times Q$ fulfills $\max_Q(u - \varphi) = (u - \varphi)(\hat{x}, \hat{t})$. A function $u \in C(\overline{Q})$ is said to be a *viscosity supersolution* of (4.12) in Q if

$$\varphi_t(\hat{x}, \hat{t}) + H\left(\hat{x}, \nabla\varphi(\hat{x}, t)\right) \ge 0$$

whenever $\left(\varphi, (\hat{x}, \hat{t})\right) \in C^1(Q) \times Q$ fulfills $\min_Q(u - \varphi) = (u - \varphi)(\hat{x}, \hat{t})$. If u is a viscosity sub- and supersolution, we say that u is a *viscosity solution*.

4.3.2 Uniqueness

We now present a comparison principle for (4.12) under the periodic boundary condition to simplify the situation. We set $Q_0 = \Omega \times [0, T)$ for later convenience.

Theorem 4.4 (Comparison principle)
Let $\Omega = \mathbf{T}^N$. Assume that $H(x, p)$ is continuous in $\mathbf{T}^N \times \mathbf{R}^N$. Assume that

$$|H(x, p) - H(y, p)| \le \eta\left((1 + |p|)\,|x - y|\right) \ \textit{for all}\ (x, p) \in \mathbf{T}^N \times \mathbf{R}^N,$$

where η is a modulus, i.e., $\eta(s) > 0$ for $s > 0$ and $\eta(s) \downarrow 0$ as $s \to 0$. Let $u \in C(Q_0)$ and $v \in C(Q_0)$ be viscosity sub- and supersolutions of (4.12), respectively. If $u \le v$ at $t = 0$, then $u \le v$ in Q_0. In particular, a solution to (4.12) with given initial datum $g \in C(\mathbf{T}^N)$ is unique.

Proof. As in the proof of Theorem 4.2, since the uniqueness (the second statement) easily follows from the comparison principle (the first statement), we just give a proof for the comparison principle. We may assume that $u, v \in C(\overline{Q})$ by taking T smaller. We consider

$$\Phi(x, t, y, s) = u(x, t) - v(y, s) - \alpha|x - y|^2 - \beta|t - s|^2 - \gamma/(T - t) - \gamma/(T - s)$$

for sufficiently large $\alpha, \beta > 0$ and sufficiently small $\gamma > 0$.

Assume that $u \le v$ in Q were false. Then for sufficiently small γ, there exists $(x_0, t_0) \in Q$ such that $\Phi(x_0, t_0, x_0, t_0) > 0$. We shall fix such γ. Then this would imply

$$\max_{\overline{Q}\times\overline{Q}} \Phi = m_{\alpha\beta} > 0.$$

Let $(x_{\alpha\beta}, t_{\alpha\beta}, y_{\alpha\beta}, s_{\alpha\beta}) \in \overline{Q} \times \overline{Q}$ be a maximizer of Φ over $\overline{Q} \times \overline{Q}$. As in the proof of Theorem 4.2, we see that $\alpha|x_{\alpha\beta} - y_{\alpha\beta}|^2 + \beta|t_{\alpha\beta} - s_{\alpha\beta}|^2$ is bounded as $\alpha \to \infty$, $\beta \to \infty$. In particular, $x_{\alpha\beta} - y_{\alpha\beta} \to 0$, $t_{\alpha\beta} - s_{\alpha\beta} \to 0$ as $\alpha \to \infty, \beta \to \infty$.

As in the proof of Theorem 4.2, $t_{\alpha\beta}, s_{\alpha\beta} > 0$ for sufficiently large α, β because of the initial condition.

We next observe that

$$\alpha|x_{\alpha\beta} - y_{\alpha\beta}|^2 + \beta|t_{\alpha\beta} - s_{\alpha\beta}|^2 \to 0 \tag{4.13}$$

as $\alpha \to \infty$, $\beta \to \infty$. In fact, since $m_{\alpha\beta} \geq \Phi(x, t, x, t)$, we see that

$$\limsup_{\substack{x-y\to 0\\ t-s\to 0}} (u(x,t) - v(y,s) - \gamma/(T-t) - \gamma/(T-s)) - m_{\alpha\beta} \leq 0.$$

Setting $(x, t, y, s) = (x_{\alpha\beta}, t_{\alpha\beta}, y_{\alpha\beta}, s_{\alpha\beta})$, we obtain

$$\limsup_{\alpha,\beta\to\infty} \left\{ \Phi(x_{\alpha\beta}, t_{\alpha\beta}, y_{\alpha\beta}, s_{\alpha\beta}) + \alpha|x_{\alpha\beta} - y_{\alpha\beta}|^2 + \beta|t_{\alpha\beta} - s_{\alpha\beta}|^2 - m_{\alpha\beta} \right\} \leq 0,$$

which yields (4.13) since $\Phi(x_{\alpha\beta}, t_{\alpha\beta}, y_{\alpha\beta}, s_{\alpha\beta}) = m_{\alpha\beta}$.

We take α, β sufficiently large so that $t_{\alpha\beta}, s_{\alpha\beta} > 0$. Since Φ is maximized at $(x_{\alpha\beta}, t_{\alpha\beta}), (y_{\alpha\beta}, s_{\alpha\beta})$, we see that

$$(x, t) \mapsto u(x,t) - \varphi_{\alpha\beta}(x,t),$$
$$\varphi_{\alpha\beta}(x,t) = v(y_{\alpha\beta}, s_{\alpha\beta}) + \alpha|x - y_{\alpha\beta}|^2 + \beta|t - s_{\alpha\beta}|^2 + \gamma/(T-t)$$

takes its maximum at $(x_{\alpha\beta}, t_{\alpha\beta})$. Similarly,

$$(y, s) \mapsto v(y,s) - \psi_{\alpha\beta}(y,s),$$
$$\psi_{\alpha\beta}(y,s) = u(x_{\alpha\beta}, t_{\alpha\beta}) - \alpha|x_{\alpha\beta} - y|^2 - \beta|t_{\alpha\beta} - s|^2 - \gamma/(T-s)$$

takes its minimum at $(y_{\alpha\beta}, s_{\alpha\beta})$. By the definition of viscosity sub- and supersolutions, we conclude that

$$2\beta(t_{\alpha\beta} - s_{\alpha\beta}) + \gamma/(T - t_{\alpha\beta})^2 + H\left(x_{\alpha\beta}, 2\alpha(x_{\alpha\beta} - y_{\alpha\beta})\right) \leq 0,$$
$$2\beta(t_{\alpha\beta} - s_{\alpha\beta}) - \gamma/(T - s_{\alpha\beta})^2 + H\left(y_{\alpha\beta}, 2\alpha(x_{\alpha\beta} - y_{\alpha\beta})\right) \geq 0.$$

Subtracting the second inequality from the first, we conclude that

$$\gamma/(T - t_{\alpha\beta})^2 + \gamma/(T - s_{\alpha\beta})^2 \leq \eta\left(\left(1 + 2\alpha|x_{\alpha\beta} - y_{\alpha\beta}|\right)|x_{\alpha\beta} - y_{\alpha\beta}|\right)$$

by the assumption of continuity of H with respect to x. Since $\alpha|x_{\alpha\beta} - y_{\alpha\beta}|^2 \to 0$, $|x_{\alpha\beta} - y_{\alpha\beta}| \to 0$, and $T - t_{\alpha\beta} \leq T$, we conclude that

$$\gamma/T^2 + \gamma/T^2 \leq 0,$$

which yields a contradiction. We thus conclude that $u \leq v$ in Q_0. □

In the proofs of both comparison principles (Theorems 4.2 and 4.4), a key property is that

$$\nabla_x \varphi_\alpha(x_\alpha) = \nabla_y \psi_\alpha(y_\alpha)$$

for Theorem 4.2 and

$$\nabla_x \varphi_{\alpha\beta}(x_{\alpha\beta}, t_{\alpha\beta}) = \nabla_y \psi_{\alpha\beta}(y_{\alpha\beta}, s_{\alpha\beta}), \quad \partial_t \varphi_{\alpha\beta}(x_{\alpha\beta}, t_{\alpha\beta}) = \partial_s \psi_{\alpha\beta}(y_{\alpha\beta}, s_{\alpha\beta})$$

for Theorem 4.4, which follow from

$$\nabla_x |x-y|^2 = -\nabla_y |x-y|^2, \quad \partial_t (t-s)^2 = -\partial_s (t-s)^2.$$

For second derivatives, we have

$$\nabla_x^2 |x-y|^2 = \nabla_y^2 |x-y|^2 \neq -\nabla_y^2 |x-y|^2.$$

This prevents us from extending the foregoing proofs directly to the second-order problems.

4.4 Viscosity Solutions with Shock

In this section, we continue to study the uniqueness of a solution for an evolutionary Hamilton–Jacobi equation whose expected solution may develop jump discontinuities called shocks like conservation laws. We first recall the notion of viscosity solutions for semicontinuous functions.

4.4.1 Definition of Semicontinuous Functions

We consider an evolutionary Hamilton–Jacobi equation of the form

$$u_t + H(x, t, u, \nabla u) = 0 \quad \text{in} \quad Q = \Omega \times (0, T), \tag{4.14}$$

where Ω is a domain in $\mathbf{R}^N$ or $\mathbf{T}^N$ and H is a continuous function that may also depend on t and u. For a function $u : Q \to \mathbf{R} \cup \{\pm\infty\}$ (i.e., with values in $\mathbf{R} \cup \{\pm\infty\}$), let u^* denote the *upper semicontinuous envelope*, i.e.,

$$u^*(x, t) = \limsup_{\varepsilon \downarrow 0} \left\{ u(y, s) \;\middle|\; |y - s| < \varepsilon,\ |t - s| < \varepsilon,\ (y, s) \in Q \right\}$$

for $(x, t) \in \overline{Q}$. Similarly, $u_*(x, t)$ denotes the *lower semicontinuous envelope*, i.e., $u_*(x, t) = -(-u)^*(x, t)$ (Exercise 4.3).

Definition 4.5

A function $u : Q \to \mathbf{R} \cup \{\pm\infty\}$ is said to be a viscosity subsolution of (4.14) in Q if $u^* < \infty$ on $\overline{Q}$ and

$$\varphi_t(\hat{x}, \hat{t}) + H\left(\hat{x}, \hat{t}, u^*(\hat{x}, \hat{t}), \nabla\varphi(\hat{x}, \hat{t})\right) \le 0 \tag{4.15}$$

whenever $\left(\varphi, (\hat{x}, \hat{t})\right) \in C^1(Q) \times Q$ fulfills $\max_Q(u^* - \varphi) = (u^* - \varphi)(\hat{x}, \hat{t})$. A viscosity supersolution is defined by replacing u^*, ∞, $u^*(\hat{x}, \hat{t})$, $\le$, max by u_*, $-\infty$, $u_*(\hat{x}, \hat{t})$, $\ge$, min, respectively. If u is a viscosity sub- and supersolution, we say that u is a viscosity solution.

It is easy to extend Theorem 4.4 to such a discontinuous solution. Moreover, if $r \mapsto H(x, t, r, p)$ is nondecreasing, then u dependence is also allowed.

Theorem 4.6 (Comparison principle)
Assume that $H = H(x, r, p)$ is continuous in $\mathbf{T}^N \times \mathbf{R} \times \mathbf{R}^N$. Assume that $r \mapsto H(x, r, p)$ is nondecreasing and satisfies

$$\left|H(x, r, p) - H(y, r, p)\right| \le \eta\big((1 + |p|)\,|x - y|\big),\ \ p \in \mathbf{R}^N,\ x, y \in \mathbf{T}^N,\ r \in \mathbf{R}$$

for some modulus η. Let $u : Q \to \mathbf{R} \cup \{-\infty\}$ and $v : Q \to \mathbf{R} \cup \{+\infty\}$ be viscosity sub- and supersolutions of (4.14), respectively. If $u^* \le v_*$ at $t = 0$, then $u^* \le v_*$ in $Q_0 = \Omega \times [0, T)$. In particular, a solution to (4.14) with $u^*|_{t=0} = u_*|_{t=0} = g \in C(\mathbf{T}^N)$ is unique and continuous in Q_0.

The proof of $u^* \le v_*$ in Q_0 is the same as that of Theorem 4.4, replacing u and v with u^* and v_*, respectively, before comparing the inequalities

$$2\beta(t_{\alpha\beta} - s_{\alpha\beta}) + \gamma/(T - t_{\alpha\beta})^2 + H\left(x_{\alpha\beta}, u^*(x_{\alpha\beta}, t_{\alpha\beta}), 2\alpha(x_{\alpha\beta} - y_{\alpha\beta})\right) \le 0,$$
$$2\beta(t_{\alpha\beta} - s_{\alpha\beta}) - \gamma/(T - s_{\alpha\beta})^2 + H\left(y_{\alpha\beta}, v_*(y_{\alpha\beta}, s_{\alpha\beta}), 2\alpha(x_{\alpha\beta} - y_{\alpha\beta})\right) \ge 0.$$

By the choice of $x_{\alpha\beta}, t_{\alpha\beta}, y_{\alpha\beta}, s_{\alpha\beta}$, we know $u^*(x_{\alpha\beta}, t_{\alpha\beta}) > v_*(y_{\alpha\beta}, s_{\alpha\beta})$. If $r \mapsto H(x, r, p)$ is nondecreasing, we may replace $v_*(y_{\alpha\beta}, s_{\alpha\beta})$ with $u^*(x_{\alpha\beta}, t_{\alpha\beta})$ so that both inequalities are comparable. The remaining part is the same.

If u_1 and u_2 are solutions with initial datum g, the comparison principle implies $u_1^* \le u_{2*}$ and $u_2^* \le u_{1*}$. Thus, $u_1 = u_2 \in C(Q_0)$.

We may weaken the monotonicity assumption that $r \mapsto H(x, t, r, p)$ is nondecreasing by a weaker assumption such that $r \mapsto H(x, r, p) + \lambda r$ is nondecreasing for some $\lambda \in \mathbf{R}$ by modifying the structure assumption for H. The main idea to extend the proof is the change of dependent variables u, v by $e^{-\lambda t}u, e^{-\lambda t}v$. However, if H does not satisfy such monotonicity assumptions, the uniqueness may not hold in general.

4.4.2 Example for Nonuniqueness

We consider a scalar conservation law (3.2). Here we assume that f is a given strict convex C^1 function in the sense that $f' \in C(\mathbf{R})$ is (strictly) increasing. The equation can be written in the form of (4.14), with

$$H(x,t,r,p) = f'(r)p.$$

For $u_r < u_\ell$, we consider

$$u_s(x,t) = \begin{cases} u_\ell, \ x < st, \\ u_r, \ x \geq st; \end{cases}$$

see Fig. 4.4. If the speed s satisfies $f'(u_r) \leq s \leq f'(u_\ell)$, then u_s is a viscosity solution in $\mathbf{R} \times (0,\infty)$. (However, it is not a weak solution unless s satisfies the Rankine–Hugoniot condition, i.e., $s = s_*$, with

$$s_* = \frac{f(u_\ell) - f(u_r)}{u_\ell - u_r};$$

see Lemma 3.8.) This shows the nonuniqueness of viscosity solutions. Of course, this is not a direct counterexample of the comparison principle discussed previously since these functions are neither periodic nor continuous up to initial data, but it is easy to construct such an example under the periodic conditions with continuous initial data. In Chap. 3, we introduced the notion of an entropy solution and proved that it was unique. In this example, u_{s_*} is an entropy solution, while u_s with $s \neq s_*$ is not even a weak solution. We shall introduce a notion of a proper solution so that the speed of the jump satisfies the Rankine–Hugoniot condition and entropy condition.

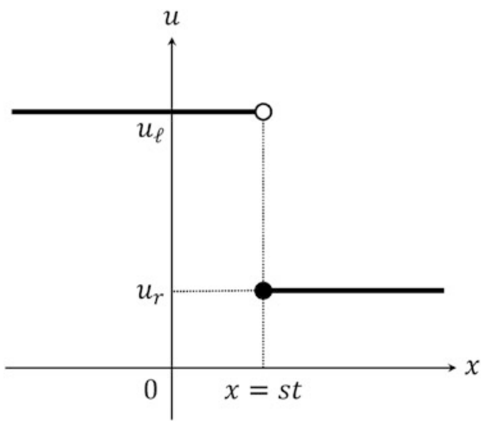

Fig. 4.4 Graph of u_s at time t

4.4.3 Test Surfaces for Shocks

In the definitions of viscosity solutions, we test a possibly irregular function u by a smoother function φ (called a test function) from both above and below; see, for example, Definition 4.5. If u is allowed to be discontinuous, as we saw in Sect. 4.4.2, such tests are not enough. To overcome this difficulty, we also test shocks. For simplicity, we consider a one-dimensional setting. In the case where u is discontinuous at Γ, as in the paragraph after Definition 3.2, but Γ may not be smooth, we test the shock Γ from both the right and left (or inside or outside with respect to the orientation ν^ℓ) by a smoother curve called a test curve (Fig. 3.4). The speed of test curves (surfaces) will be given by the Rankine–Hugoniot condition or entropy condition.

For a given point $(x_0, t_0) \in Q$ and $\rho > 0$, $\delta > 0$, let $\{S_t\}_{t\in\Lambda}$ be a smooth family of smooth hypersurfaces in $\mathring{B}_\rho(x_0) \subset \Omega$ with $x_0 \in S_{t_0}$, where $\Lambda = \Lambda_\delta(t_0) = (t_0 - \delta, t_0 + \delta)$, and $B_\rho(x_0)$ denotes a closed ball of radius ρ in $\mathbf{R}^N$ centered at $x_0 \in \mathbf{R}^N$. Let $\mathbf{n} = \mathbf{n}(\cdot, t)$ denote the unit normal vector field of S_t that gives the orientation of S_t; we assume that $\mathbf{n}(\cdot, t)$ depends on t at least continuously. Assume that $\mathring{B}_\rho(x_0)\backslash S_t$ consists of two domains. Let D_t denote one of these domains such that $\partial D_t = S_t$ in $\mathring{B}_\rho(x_0)$ and its inward normal agrees with $\mathbf{n} = \mathbf{n}(\cdot, t)$ for $t \in \Lambda_\delta(t_0)$. We call D_t a *region* associated with $(S_t, \mathbf{n}(\cdot, t))$. It is uniquely determined for given ρ and δ.

We simply say that $\{(S_t, \mathbf{n}(\cdot, t))\}$ is an *evolving hypersurface* through (x_0, t_0).

Definition 4.7

(i) Let $u : Q \to \mathbf{R} \cup \{-\infty\}$ be upper semicontinuous and $(x_0, t_0) \in Q$. For $\mu < u(x_0, t_0)$, we say that an evolving hypersurface $\{(S_t, \mathbf{n}(\cdot, t))\}$ through (x_0, t_0) is an *upper test surface* of u at (x_0, t_0) with level μ if

$$u(x, t) \le \mu \quad \text{in} \quad D_t \times \{t\}$$

for some $\rho > 0$ and $\delta > 0$, where D_t denotes the region associated with $(S_t, \mathbf{n}(\cdot, t))$.

(ii) Let $v : Q \to \mathbf{R} \cup \{+\infty\}$ be lower semicontinuous and $(x_0, t_0) \in Q$. For $\mu > v(x_0, t_0)$, we say that an evolving hypersurface $\{(S_t, \mathbf{n}(\cdot, t))\}$ at (x_0, t_0) is a *lower test surface* of v at (x_0, t_0) with level μ if

$$v(x, t) \ge \mu \quad \text{in} \quad D_t \times \{t\}$$

for some $\rho > 0$, and $\delta > 0$, where D_t denotes the region associated with $(S_t, \mathbf{n}(\cdot, t))$. See Fig. 4.5.

If $u(\cdot, t)$ jumps across a hypersurface Σ_t, such a surface Σ_t is often called a shock surface. In this case, one may take Σ_t as a test surface if Σ_t is regular enough.

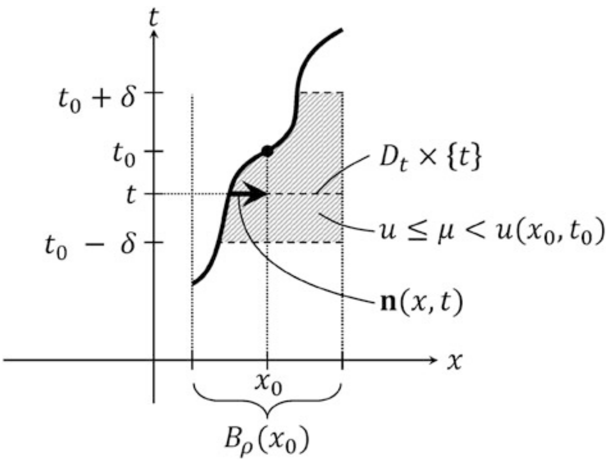

Fig. 4.5 Upper test surface

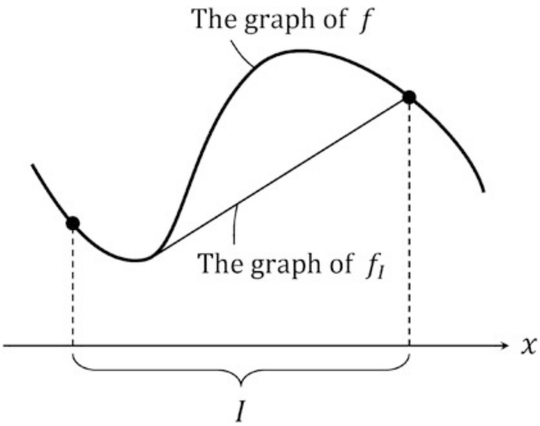

Fig. 4.6 Convexification

4.4.4 Convexification

To give a rigorous definition of solutions, we recall a few properties of convexification. Let f be a function defined on $\mathbf{R}$. Let I be a bounded closed interval. Let $f_I : I \to \mathbf{R}$ denote the *convex hull* (*convexification*) of f in I, i.e., f_I is the greatest convex function on I less than or equal to I (Fig. 4.6). By definition, $f_I = f$ in I if I is a singleton.

Lemma 4.8

(i) *If f is continuous in I, then $f_I = f$ on ∂I and f_I is continuous in I.*
(ii) *If f is C^1, then f_I is C^1 in I.*
(iii) *For $f \in C^1[a,d]$ $(-\infty < a < d < \infty)$,*

$$f_I'(x) \geq f_J'(x) \quad \text{for} \quad x \in I \cap J,$$

with $I = [a,b]$, $J = [c,d]$, $a \leq c \leq b \leq d$, where $'$ denotes the derivative. (At the boundary, the derivative is interpreted as the right or left derivative.)
(iv) *For $f \in C^1(\mathbf{R})$, the function $F(a,b,x) = f_{[a,b]}'(x)$ is continuous in*

$$\left\{(p,q,x) \in \mathbf{R}^3 \;\middle|\; p \leq q,\ p \leq x \leq q\right\}.$$

The proofs are elementary. They are safely left to the reader; see [46, Lemma 2.1].

4.4.5 Proper Solutions

To define a proper solution, we recall a *recession function* of $p = \nabla u$ variable for the Hamiltonian $H : Q \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$, i.e.,

$$H_\infty(x,t,r,p) = \lim_{\lambda \downarrow 0} \lambda H(x,t,r,p/\lambda).$$

See Exercise 4.2. We always assume that H_∞ exists and is continuous in its variables. By definition, $H_\infty(x,t,r,\sigma p) = \sigma H_\infty(x,t,r,p)$ for $\sigma > 0$, i.e., positively homogeneous of degree one in p. Indeed,

$$H_\infty(x,t,r,\sigma p) = \lim_{\lambda \downarrow 0} \lambda H\left(x,t,r,\sigma p/\lambda\right) = \lim_{\lambda' \downarrow 0} \sigma \lambda' H\left(x,t,r,p/\lambda'\right).$$

Let $f(r) = f(r;x,t,p)$ be a primitive of $H_\infty(x,t,r,p)$ as a function of r. For a closed interval I, let f_I denote the convexification of f in I. Since f_I is C^1 in I by Lemma 4.8 (ii), so that f_I' is continuous on I, we set

$$H^I(x,t,r,p) := f_I'(r;x,t,p), \quad r \in I, \quad (x,t) \in Q, \quad p \in \mathbf{R}^N$$

and call H^I a *relaxed Hamiltonian* in I. This is independent of the choice of a primitive f, so it is well defined. Since H_∞ is positively homogeneous of degree one, so is H^I, i.e.,

$$H^I(x, t, r, \sigma p) = \sigma H^I(x, t, r, p)$$

for all $\sigma > 0$, $(x, t) \in Q$, $r \in I$, $p \in \mathbf{R}^N$. If $r \mapsto H(x, t, r, p)$ is nondecreasing so that $f(r)$ is convex, then the relaxed H^I agrees with H_∞ for any choice of I.

Definition 4.9

(i) Let $u : Q \to \mathbf{R} \cup \{-\infty\}$ be a viscosity subsolution of (4.14) in Q. We say that u is *proper subsolution* of (4.14) if the inequality

$$V(x_0, t_0) + H^I \left(x_0, t_0, u^*(x_0, t_0), -\mathbf{n}(x_0, t_0)\right) \leq 0 \qquad (4.16)$$

holds whenever $(x_0, t_0) \in Q$ admits an upper test surface $\{(S_t, \mathbf{n}(\cdot, t))\}$ of u^* at (x_0, t_0) with the level $\mu\ (< u^*(x_0, t_0))$, where $I = \left[\mu, u^*(x_0, t_0)\right]$. Here, $V = V(x_0, t_0)$ denotes the normal velocity of $\{S_t\}$ at (x_0, t_0) in the direction of $\mathbf{n}(x_0, t_0)$, and H^I denotes the relaxed Hamiltonian.

(ii) For a viscosity supersolution $v : Q \to \mathbf{R} \cup \{+\infty\}$ of (4.14) in Q, we say that v is a *proper supersolution* of (4.14) if the inequality

$$-V(x_0, t_0) + H^I\ (x_0, t_0, v_*(x_0, t_0), \mathbf{n}(x_0, t_0)) \geq 0 \qquad (4.17)$$

holds whenever $(x_0, t_0) \in Q$ admits a lower test surface $\{(S_t, \mathbf{n}(\cdot, t))\}$ of v_* at (x_0, t_0) with level $\mu\ (> v_*(x_0, t_0))$, where $I = [v_*(x_0, t_0), u]$.

(iii) If u is a proper sub- and supersolution, we say that u is a *proper solution*. The notion of proper sub- and supersolution is reduced to classical viscosity sub- and supersolutions respectively if the function is continuous.

Remark 4.10

(i) If (4.16) is fulfilled with $I = \left[\mu, u^*(x_0, t_0)\right]$, then (4.16) holds for all $I' = [\mu, \sigma]$ provided that $\sigma \geq u^*(x_0, t_0)$ by Lemma 4.8 (iii).

(ii) If $\{(S_t, \mathbf{n}(\cdot, t))\}$ is an upper test surface of u^* at (x_0, t_0) with level μ, then it is also an upper test surface with level μ' for any $\mu' \in \left[\mu, u^*(x_0, t_0)\right]$. Thus, for a proper subsolution, the inequality

$$V(x_0, t_0) + H^J\left(x_0, t_0, u^*(x_0, t_0), -\mathbf{n}(x_0, t_0)\right) \leq 0$$

with $J = \left[\mu', u^*(x_0, t_0)\right]$ is valid. By Lemma 4.8 (iv), letting $\mu' \uparrow u^*(x_0, t_0)$ yields

$$V(x_0, t_0) + H_\infty\left(x_0, t_0, u^*(x_0, t_0), -\mathbf{n}(x_0, t_0)\right) \leq 0, \qquad (4.18)$$

since $H^J = H_\infty$ if J is a singleton. This inequality holds for any upper test surfaces $\{(S_t, \mathbf{n}(\cdot, t))\}$ at (x_0, t_0).

(iii) Suppose that $r \mapsto H(x, t, r, p)$ is nondecreasing so that $H^I = H_\infty$ for any I. If (4.18) holds for any upper test surface $\{(S_t, \mathbf{n}(\cdot, t))\}$ at (x_0, t_0) with level $\mu < u^*(x_0, t_0)$, then (4.16) holds for μ by the monotonicity of H in r. Thus, u is a proper subsolution if u is a viscosity subsolution and (4.18) holds for any upper test surface $\{(S_t, \mathbf{n}(\cdot, t))\}$ at (x_0, t_0) with level $\mu < u^*(x_0, t_0)$. In fact, if $r \mapsto H(x, t, r, p)$ is nondecreasing, then every subsolution is a proper subsolution, as stated subsequently in Theorem 4.11.

(iv) For a semiclosed interval $(0, T]$, it is possible to define a proper solution in $Q' = \Omega \times (0, T]$. For $u : Q' \to \mathbf{R} \cup \{\pm\infty\}$, we say that u is a proper subsolution of (4.14) in Q' if it is a viscosity subsolution of (4.14) in Q' (i.e., (4.15) holds for $\left(\varphi, (\hat{x}, \hat{t})\right) \in C^1(Q') \times Q'$ satisfying $\max_Q(u^* - \varphi) = (u^* - \varphi)(\hat{x}, \hat{t})$ with Q replaced by Q') and (4.16) holds for upper test surface $\{(S_t, \mathbf{n}(\cdot, t))\}$ at $(x_0, t_0) \in Q'$ with level μ $(< u^*(x_0, t_0))$. If $t_0 = T$, the family $\{(S_t, \mathbf{n}(\cdot, t))\}$ should be interpreted as being smooth in $(T - \delta, T]$.

If $r \mapsto H(x, t, r, p)$ is nondecreasing, then a proper subsolution is a conventional viscosity subsolution under an asymptotic homogeneity assumption on H as $|p| \to \infty$.

Theorem 4.11 (Consistency)
For $H \in C(Q \times \mathbf{R} \times \mathbf{R}^N)$, assume that $r \mapsto H(x, t, r, p)$ is nondecreasing in $\mathbf{R}$ for all $(x, t) \in Q$, $p \in \mathbf{R}^N$. Assume that $\lambda H(x, t, r, p/\lambda)$ converges to H_∞ locally uniformly in $Q \times \mathbf{R} \times \mathbf{R}^N$ as $\lambda \downarrow 0$. In other words,

$$\lim_{\lambda \downarrow 0} \sup_{(x,t,r,p) \in K} \left| \lambda H\left(x, t, r, \frac{p}{\lambda}\right) - H_\infty(x, t, r, p) \right| = 0 \tag{4.19}$$

for every compact set K in $Q \times \mathbf{R} \times \mathbf{R}^N$. If u and v are viscosity sub- and supersolutions of (4.14) in Q, then u and v are respectively proper sub- and supersolutions of (4.14) in Q.

▶ **Remark 4.12** By (4.19), the function H_∞ is continuous in its variables. In particular, by the homogeneity of H_∞,

$$H_\infty(x, t, r, 0) = \lim_{\sigma \downarrow 0} H_\infty(x, t, r, \sigma) = \lim_{\sigma \downarrow 0} H_\infty(x, t, r, 1) = 0.$$

By definition,

$$H^I(x, t, r, 0) = 0$$

for any closed interval I.

Proof. The proof of a viscosity supersolution is similar to that of a viscosity subsolution, so we only present the proof of a viscosity subsolution. By Remark 4.10 (iii), it suffices to prove (4.18). Let $\{(S_t, \mathbf{n}(\cdot, t))\}$ be an upper test surface at $(x_0, t_0) \in Q$ of u^* with level μ $(< u^*(x_0, t_0))$. Let D_t be a region associated with $(S_t, \mathbf{n}(\cdot, t))$. We set

$$D = \bigcup_{t \in \Lambda} D_t \times \{t\} \subset \mathring{B}_\rho(x_0) \times \Lambda, \quad \Lambda = (t_0 - \delta, t_0 + \delta).$$

We take another upper test function $\{(S'_t, \mathbf{n}'(\cdot, t))\}$ with level μ at (x_0, t_0) and $\mathbf{n}(x_0, t_0) = \mathbf{n}'(x_0, t_0)$ such that

$$(x_0, t_0) \in S' \quad \text{and} \quad S' \setminus \{(x_0, t_0)\} \subset D \quad \text{with} \quad S' = \bigcup_{t \in \Lambda} S'_t \times \{t\}.$$

(By construction S'_t touches S_t only at time t_0 at point x_0.) Let D'_t denote a region associated with $(S'_t, \mathbf{n}'(\cdot, t))$. To construct an appropriate test function[1] for u^*, we use a signed distance function of $D' = \bigcup_{t \in \Lambda} D'_t \times \{t\} \subset \mathring{B}_\rho(x_0) \times \Lambda$ defined by

$$d(x, t) = \begin{cases} \operatorname{dist}\left((x, t), \partial D'\right), & x \in D', \\ -\operatorname{dist}\left((x, t), \partial D'\right), & x \notin D'. \end{cases}$$

From this point forward, by $\partial D'$ we mean the boundary of D' in $\mathring{B}_\rho(x_0) \times \Lambda$. Since $\partial D'$ is smooth, so is d in $\mathring{B}_\rho(x_0) \times \Lambda$ for sufficiently small $\delta > 0$ and $\rho > 0$; see, for example, [67]. We fix $\mu' \in (\mu, u^*(x_0, t_0))$ and define

$$\varphi_L(x, t) = \max\left(-Ld(x, t) + u^*(x_0, t_0), \mu'\right)$$

for $L > 0$ (Fig. 4.7). The function $\varphi_L(x, t)$ is smooth outside D' in a small neighborhood of (x_0, t_0). Since u^* is upper semicontinuous, there is a maximizer (x_L, t_L) of $u^* - \varphi_L$ in $B_\rho(x_0) \times \overline{\Lambda}$, where $\overline{\Lambda} = [t_0 - \delta, t_0 + \delta]$. Sending $L \to \infty$ we see that $0 \leq \max(u^* - \varphi_L) \to 0$ and $\operatorname{dist}\left((x_L, t_L), \partial D'\right) \to 0$. Since $(x_L, t_L) \notin D'$, this implies $(x_L, t_L) \to (x_0, t_0)$. Moreover, $u^*(x_L, t_L) \to u^*(x_0, t_0)$ since u^* is upper semicontinuous and $u^*(x_L, t_L) \geq u^*(x_0, t_0)$. Thus, for sufficiently large L the function $u^* - \varphi_L$ takes its local maximum at $(x_L, t_L) \in \mathring{B}_\rho(x_0) \times \Lambda$, and at (x_L, t_L) the function φ_L is smooth.

---

[1] For a subset A in a metric space M equipped with distance d, the distance function $\operatorname{dist}(x, A)$ from A is defined by

$$\operatorname{dist}(x, A) := \inf\left\{d(x, y) \mid y \in A\right\}.$$

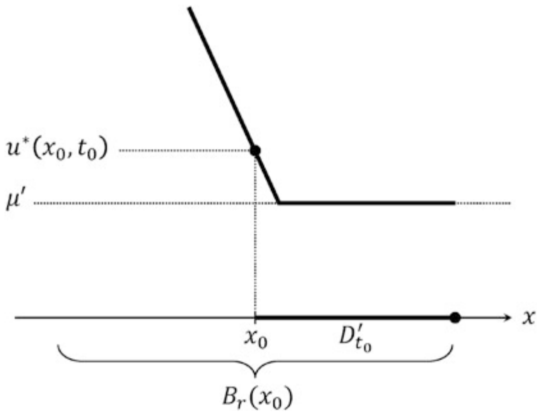

Fig. 4.7 Graph of φ_L at $t = t_0$

If u is a viscosity subsolution, then

$$\partial_t \varphi_L(x_L, t_L) + H\big(x_L, t_L, u^*(x_L, t_L), \nabla\varphi_L(x_L, t_L)\big) \le 0.$$

Dividing by $|\nabla\varphi(x_L, t_L)| = L$ and sending $L \to \infty$ yields

$$V + H_\infty\big(x_0, t_0, u^*(x_0, t_0), -\mathbf{n}(x_0, t_0)\big) \le 0$$

since the convergence $\lambda H(x, t, r, p/\lambda) \to H_\infty(x, t, r, p)$ is locally uniform in (x, t, r, p) as $\lambda \downarrow 0$ and

$$\frac{\nabla\varphi_L(x_L, t_L)}{|\nabla\varphi_L(x_L, t_L)|} \to -\mathbf{n}(x_0, t_0), \qquad (x_L, t_L) \to (x_0, t_0),$$
$$u^*(x_L, t_L) \to u^*(x_0, t_0), \quad \frac{\partial_t\varphi(x_L, t_L)}{|\nabla\varphi(x_L, t_L)|} \to V(x_0, t_0)$$
$$\text{as} \quad \lambda \downarrow 0.$$

We have thus proved (4.18). □

4.4.6 Examples of Viscosity Solutions with Shocks

We consider a scalar conservation law (3.2), where f is a given strict convex C^1 function. For $a < b$ we set

$$u_N(x, t) = \begin{cases} a, & x < st, \\ b, & x \ge st, \end{cases}$$

$$u_S(x, t) = \begin{cases} b, & x \le st, \\ a, & x > st. \end{cases}$$

If the speed s satisfies the Rankine–Hugoniot condition, i.e.,

$$s = \frac{f(b) - f(a)}{b - a},$$

then u_S is a viscosity solution, while u_N is not a viscosity solution even if s satisfies the Rankine–Hugoniot condition. If s satisfies the Rankine–Hugoniot condition, then u_N is still a weak solution (defined in Definition 3.2) of (3.2).

Proposition 4.13
Assume that $f' \in C(\mathbf{R})$ is (strictly) increasing and $a < b$. If $s = (f(b) - f(a))/(b - a)$, then u_S is a proper solution of (3.2). If $s < f'(b)$, then u_N is not even a viscosity supersolution of (3.2). If $s > f'(a)$, then u_N is not even a viscosity subsolution of (3.2).

Proof. It is easy to see that u_S is a viscosity solution. Thus, it suffices to check the speed of a test surface for shocks. Let (x_0, t_0) be a point on a shock, i.e., $x_0 = st_0, t_0 > 0$. The line $S_t = \{x = st\}$ itself is a test surface of u_S and u_N at (x_0, t_0) with level a. All other test surfaces at (x_0, t_0) are tangent to $\{S_t\}$, so by Remark 4.10 (ii) it suffices to estimate the normal velocity of $\{S_t\}$. Equation (3.2) can be written

$$u_t + H(u, \nabla u) = 0$$

if we set $H(r, p) = f'(r)p$. If we consider u_E, then $\mathbf{n} = 1$, so that

$$H(r, \mathbf{n}) = -f'(r).$$

Since $-f$ is concave,

$$\frac{d}{dr}(-f)_I(r) = -\frac{f(b) - f(a)}{b - a}, \quad r \in I = [a, b],$$

which yields $H^I(r, -1) = -s$ by the definition of s. Since $V(x_0, t_0) = c$, we now observe that

$$V(x_0, t_0) + H^I(b, -1) = 0.$$

Thus, u_S is a proper subsolution. A symmetric argument shows that $(u_S)_*$ is a proper supersolution.

It is easy to see that u_N is not a viscosity subsolution or a viscosity supersolution for the range indicated in the statement. □

It is well known (Exercise 3.6) that the entropy solution u with initial datum $u|_{t=0} = u_N|_{t=0}$ is a rarefaction wave solution

$$u_R(x,t) = \begin{cases} a, & x < f'(a)t, \\ (f')^{-1}(x/t), & f'(a)t \le x < f'(b)t, \\ b, & x \ge f'(b)t, \end{cases}$$

where f'^{-1} denotes the inverse function of f'. This function u is a continuous viscosity solution, so there are no jumps. Consequently, there are no test surfaces for shocks. Thus, u_R is automatically a proper solution. For u_R and u_S, the notions of proper and entropy solutions agree with each other. More generally, it turns out that notions of proper and entropy solutions essentially agree for initial-value problems [46]. We do not touch on this problem in this book.

4.4.7 Properties of Graphs

To derive some comparison principle, it is convenient to consider graphs of proper solutions. For a function $u : Q \to \mathbf{R} \cup \{\pm\infty\}$, we associate a function on $\Omega \times \mathbf{R} \times (0, T)$ of the form

$$i_u(x,z,t) = \begin{cases} 0, & z \le u(x,t), \\ -\infty, & z > u(x,t). \end{cases}$$

The set

$$\{i_u = 0\} := \left\{(x,z,t) \in \Omega \times \mathbf{R} \times (0,T) \;\middle|\; i_u(x,z,t) = 0\right\}$$

is called the *subgraph* of u and denoted by $\operatorname{sg} u$. Similarly, we set

$$I_u(x,z,t) = \begin{cases} 0, & z \ge u(x,t), \\ \infty & z < u(x,t). \end{cases}$$

The set

$$\{I_u = 0\} := \left\{(x,z,t) \in \Omega \times \mathbf{R} \times (0,T) \;\middle|\; I_u(x,z,t) = 0\right\}$$

is called the *supergraph* of u and denoted by $\operatorname{Sg} u$. This set $\{I_u = 0\}$ is usually called the epigraph of u, and I_u is called the *indicator function* of $\operatorname{Sg} u$ in convex analysis. By definition, $\operatorname{sg} u$ is closed if and only if u is upper semicontinuous. The closure of $\operatorname{sg} u$ equals the subgraph of u^*, i.e., $\overline{\operatorname{sg} u} = \operatorname{sg} u^*$. Similarly, $\overline{\operatorname{Sg} u} = \operatorname{Sg} u_*$. By definition, for a function $u : Q \to \mathbf{R} \cup \{\pm\infty\}$, we see that $i_{u^*} = (i_u)^*$, so i_{u^*} is always upper semicontinuous. Similarly, I_{u_*} is always lower semicontinuous.

For later convenience, we first recall the (left) accessibility of a viscosity solution of (4.14).

Proposition 4.14
Assume that H in (4.14) is continuous. Let u be a viscosity subsolution of (4.14) in $Q = \Omega \times (0, T)$. Then u^* is left accessible *at each $(x_0, t_0) \in Q$, i.e., there is a sequence $\{(x_j, t_j)\}_{j=1}^{\infty} \subset Q$ such that $x_j \to x_0$, $t_j \uparrow t_0$, $u^*(x_j, t_j) \to u^*(x_0, t_0)$ as $j \to \infty$.*

This follows from the fact that u is a viscosity subsolution of (4.14) in $\Omega \times (0, T']$ for any $T' < T$ and that such a u is left accessible at $t = T'$. We do not give the proof here. For the complete proof, see [22]; see also [47, §3.2.2].

As an application, we obtain some information of functions testing i_{u^*}.

Lemma 4.15
Assume the same hypothesis as that of Proposition 4.14. Then i_{u^*} is left accessible in $\hat{Q} = \Omega \times \mathbf{R} \times (0, T)$.

Proof. Assume that $i_{u^*}(x_0, z_0, t_0) = 0$ at $(x_0, z_0, t_0) \in \hat{Q}$, so that $u^*(x_0, t_0) > -\infty$. Since u^* is left accessible at (x_0, t_0) by Proposition 4.14, there is a sequence $\{(x_j, t_j)\}_{j=1}^{\infty} \subset Q$ such that $x_j \to x_0$, $t_j \uparrow t_0$, $u^*(x_j, t_j) \to u^*(x_0, t_0)$ as $j \to \infty$. Since $i_{u^*}(x_0, z_0, t_0) = 0$, we see $z_0 \le u^*(x_0, t_0)$. If $u^*(x_0, t_0) \in \mathbf{R}$, then we take

$$z_j = u^*(x_j, t_j) - \big(u^*(x_0, t_0) - z_0\big) \le u^*(x_j, t_j)$$

and observe that $i_{u^*}(x_j, z_j, t_j) = 0$ and $z_j \to z_0$. If $u^*(x_0, t_0) = \infty$, then $z_0 \le u^*(x_j, t_j)$ for sufficiently large j. In this case, we set $z_j = z_0$. We thus conclude that

$$i_{u^*}(x_j, z_j, t_j) = 0, \quad x_j \to x_0, \quad z_j \to x_0, \quad t_j \uparrow t_0$$

as $j \to \infty$. If $i_{u^*}(x_0, z_0, t_0) = -\infty$ so that $z_0 > u^*(x_0, t_0)$, then $z_0 > u^*(x_j, t_j)$ for sufficiently large j. Thus, taking $z_j = z_0$, we see that $i_{u^*}(x_j, z_j, t_j) = -\infty$. We now conclude that i_{u^*} is left accessible in $\hat{Q}$. □

We next check what kind of equations a test function of i_{u^*} satisfies.

Lemma 4.16

Assume that H is continuous and H_∞ exists. Let u be a proper subsolution of (4.14) in $Q = \Omega \times (0, T)$. For $\Phi \in C^1(Q)$ assume that $i_{u^*} - \Phi$ takes its maximum over $\hat{Q}$ at $(\hat{x}, \hat{z}, \hat{t})$, i.e.,

$$\max_{\hat{Q}}(i_{u^*} - \Phi) = (i_{u^*} - \Phi)(\hat{x}, \hat{z}, \hat{t}).$$

Then $\partial_z \Phi(\hat{x}, \hat{z}, \hat{t}) \le 0$ holds.

(A) Assume that $\hat{\nabla}\Phi(\hat{x}, \hat{z}, \hat{t}) \neq 0$, where $\hat{\nabla}\Phi = (\nabla_x \Phi, \partial_z \Phi)$. Then $\hat{z} \le u^*(\hat{x}, \hat{t})$. Moreover,

(i) If $\partial_z \Phi(\hat{x}, \hat{z}, \hat{t}) \neq 0$, then $\hat{z} = u^*(\hat{x}, \hat{t})$ and $\partial_z \Phi(\hat{x}, \hat{z}, \hat{t}) < 0$. Moreover,

$$\tau + H\left(\hat{x}, \hat{t}, u^*(\hat{x}, \hat{t}), p\right) \le 0, \tag{4.20}$$

with $\tau = -(\partial_t \Phi / \partial_z \Phi)(\hat{x}, \hat{z}, \hat{t}) \in \mathbf{R}$, $p = -(\nabla_x \Phi / \partial_z \Phi)(\hat{x}, \hat{z}, \hat{t}) \in \mathbf{R}^N$.

(ii) If $\partial_z \Phi(\hat{x}, \hat{z}, \hat{t}) = 0$ and $\hat{z} < u^*(\hat{x}, \hat{t})$, then

$$\partial_t \Phi(\hat{x}, \hat{z}, \hat{t}) + H^I\left(\hat{x}, \hat{t}, u^*(\hat{x}, \hat{t}), \nabla_x \Phi(\hat{x}, \hat{z}, \hat{t})\right) \le 0, \tag{4.21}$$

with $I = \left[\hat{z}, u^*(\hat{x}, \hat{t})\right]$.

(iii) Assume (4.19). Assume that $\partial_z \Phi(\hat{x}, \hat{z}, \hat{t}) = 0$ and $\hat{z} = u^*(\hat{x}, \hat{t})$. Then inequality (4.21) holds.

(B) Assume (4.19). If $\hat{\nabla}\Phi(\hat{x}, \hat{z}, \hat{t}) = 0$, then $\partial_t \Phi(\hat{x}, \hat{z}, \hat{t}) \le 0$.

Symmetric statements hold for proper supersolutions.

Proof. Since $i_{u^*}(x, z, t)$ is nonincreasing in z, $(i_{u^*} - \Phi)(\hat{x}, z, \hat{t})$ cannot take its maximum at $\hat{z}$ if $\partial_z \Phi(\hat{x}, \hat{z}, \hat{t}) > 0$. Thus, $\partial_z \Phi(\hat{x}, \hat{z}, \hat{t}) \le 0$.

(A) The point $(\hat{x}, \hat{z}, \hat{t})$ belongs to the boundary of the subgraph $\operatorname{sg} u^*$ since $\hat{\nabla}\Phi(\hat{x}, \hat{z}, \hat{t}) \neq 0$. For $\ell = \Phi(\hat{x}, \hat{z}, \hat{t})$ the ℓ-level set of Φ touches $\operatorname{sg} u^*$ at $(\hat{x}, \hat{z}, \hat{t})$, and the sublevel set $\{\Phi < \ell\}$ does not intersect $\operatorname{sg} u^*$. Thus, $\hat{z} \le u^*(\hat{x}, \hat{t})$. From this point forward, for a function F defined in $\hat{Q}$, by $\{F < \ell\}$ (resp. $\{F \le \ell\}$) we mean the set

$$\left\{w \in \hat{Q} \;\middle|\; F(w) < \ell\right\} \quad \text{(resp. } \left\{w \in \hat{Q} \;\middle|\; F(w) \le \ell\right\}\text{)}.$$

(i) By the definition of $\operatorname{sg} u^*$, the first statement is clear. Since

$$\partial_z \Phi(\hat{x}, \hat{z}, \hat{t}) < 0,$$

the ℓ-level set of Φ can be written as the graph of an implicit function $Z = Z(x, t)$ near $(\hat{x}, \hat{z}, \hat{t})$. By the geometry of the ℓ-level set of Φ and $\operatorname{sg} u^*$, $u^* - Z$ takes its local maximum at $(\hat{x}, \hat{t})$. Since Z is an implicit function satisfying $\Phi(x, Z(x, t), t) = \ell$, we see $\partial_t Z(\hat{x}, \hat{t}) = \tau$ and $\nabla Z(\hat{x}, \hat{t}) = p$. Since u^* is a subsolution, we get (4.20).

(ii) This is a crucial part of this lemma. We may assume that $\Phi(\hat{x}, \hat{z}, \hat{t}) = 0$ and $\hat{x} = 0$ without loss of generality. Since $\nabla_x \Phi(0, \hat{z}, \hat{t}) \neq 0$, by rotation we may assume that

$$\nabla_x \Phi / |\nabla_x \Phi| = (-1, 0, \ldots, 0) \quad \text{at} \quad (0, \hat{z}, \hat{t}).$$

We set

$$\Psi(x, z, t) := (x_1 - R)^2 + x_2^2 + \cdots + x_N^2 + (z - \hat{z})^2 + (t - \hat{t})^2 + A(t - \hat{t}) - R^2.$$

For a suitable choice of $R > 0$ and $A \in \mathbf{R}$, a ball $B = \{\Psi \leq 0\}$ touches $\operatorname{sg} u^*$ only at $(0, \hat{z}, \hat{t})$ i.e., $\operatorname{sg} u^* \cap B = \{(0, \hat{z}, \hat{t})\}$ and $B \subset \{\Phi \leq 0\}$; see Fig. 4.8. Thus,

$$\partial_t \Psi / |\nabla_x \Psi| = \partial_t \Phi / |\nabla_x \Phi| \quad \text{at} \quad (0, \hat{z}, \hat{t}). \tag{4.22}$$

By the choice of B we observe that

$$u(x, t) \leq \hat{z}$$

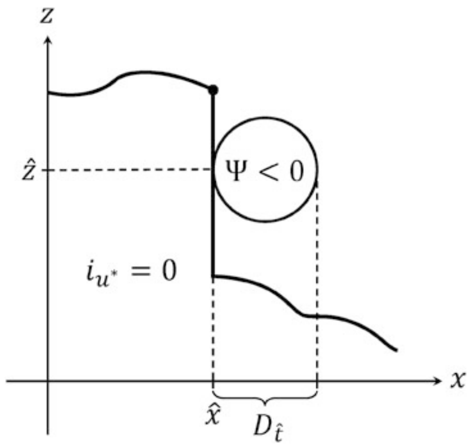

Fig. 4.8 Ball $\{(x, z) \in \Omega \times \mathbf{R} \mid \Psi(x, z, \hat{t}) < 0\}$

for $x \in D_t = \{x \in \Omega \mid \Psi(x, \hat{z}, t) < 0\}$. We take S_t as the boundary of D_t and $\mathbf{n}$ is the inward normal of D_t. By definition, $\{(S_t, \mathbf{n}(\cdot, t))\}$ is an evolving hypersurface through $(0, \hat{t})$, and D_t is a region associated with $\{(S_t, \mathbf{n}(\cdot, t))\}$. Then $\{(S_t, \mathbf{n}(\cdot, t))\}$ is an upper test surface with level $\hat{z}$ of u^* at $(0, \hat{t})$ and $\mathbf{n}(\cdot, \hat{t})$ at 0 equals $(1, 0, \ldots, 0)$. By (4.22), the normal velocity (in the direction of $\mathbf{n}(\cdot, \hat{t})$) V of $S_{\hat{t}}$ at $\hat{x} = 0$ equals

$$V = \partial_t \Psi / |\nabla_x \Psi| = \partial_t \Phi / |\nabla_x \Phi| \quad \text{at} \quad (0, \hat{z}, \hat{t}).$$

By the definition of a proper subsolution, we see that

$$V + H^I\left(0, \hat{t}, u^*(0, \hat{t}), -\mathbf{n}\right) \le 0, \tag{4.23}$$

with $I = \left[\hat{z}, u^*(0, \hat{t})\right]$. Since

$$V = \frac{\partial_t \Phi}{|\nabla_x \Phi|}(0, \hat{z}, \hat{t}), \quad \mathbf{n} = -\frac{\nabla_x \Phi}{|\nabla_x \Phi|}(0, \hat{z}, \hat{t}),$$

we conclude that (4.23) yields (4.21); here we invoke the homogeneity of $H^I(x, t, r, p)$ in p, i.e., $H^I(x, t, r, \sigma p) = \sigma H^I(x, t, r, p)$ for $\sigma > 0$.

(iii) We modify Ψ. Let $\tilde{\Psi}$ be a C^1 function defined by

$$\tilde{\Psi}(x, z, t) := \begin{cases} \Psi(x, z, t), & \text{if} \quad z \le \hat{z} \\ \Psi(x, z, t) - (z - \hat{z})^2, & \text{if} \quad z > \hat{z}, \end{cases}$$

so that $\partial_z \tilde{\Psi} \le 0$. Since $\operatorname{sg} u^*$ is a subgraph, the set $\{\tilde{\Psi} \le 0\}$ still touches $\operatorname{sg} u^*$ only at $(0, \hat{z}, \hat{t})$. For $\varepsilon > 0$ we set

$$\Psi_\varepsilon(x, z, t) = \tilde{\Psi}(x, z, t) - \varepsilon(z - \hat{z}).$$

Let $(x_\varepsilon, z_\varepsilon, t_\varepsilon)$ be a maximizer of $i_{u^*} - \Psi_\varepsilon$. Since $i_{u^*} - \tilde{\Psi}$ takes a strict maximum at $(0, \hat{z}, \hat{t})$, by a convergence of maximum points (e.g., [47, Lemma 2.2.5] and Exercise 4.4) $(x_\varepsilon, z_\varepsilon, t_\varepsilon) \to (0, \hat{z}, \hat{t})$ as $\varepsilon \to 0$. Since

$$\partial_z \Psi_\varepsilon(x, z, t) = 2 \min(z - \hat{z}, 0) - \varepsilon < 0,$$

we apply (i) to get $z_\varepsilon = u^*(x_\varepsilon, t_\varepsilon)$ and

$$\partial_t \Psi_\varepsilon(x_\varepsilon, z_\varepsilon, t_\varepsilon) + \lambda_\varepsilon H\left(x_\varepsilon, z_\varepsilon, t_\varepsilon, \frac{\nabla_x \Psi_\varepsilon(x_\varepsilon, z_\varepsilon, t_\varepsilon)}{\lambda_\varepsilon}\right) \le 0,$$

with $\lambda_\varepsilon = -\partial_z \Psi_\varepsilon(x_\varepsilon, z_\varepsilon, t_\varepsilon)$ for small $\varepsilon > 0$. By (4.19), letting $\varepsilon \to 0$ yields

$$\partial_t \Psi(0, \hat{z}, \hat{t}) + H_\infty \left(0, \hat{t}, \hat{z}, \nabla_x \Psi(0, \hat{z}, \hat{t})\right) \le 0$$

since $\lambda_\varepsilon \downarrow 0$. The desired inequality follows from (4.22) and $\nabla_x \Psi / |\nabla_x \Psi| = \nabla_x \Phi / |\nabla_x \Phi|$ at $(0, \hat{z}, \hat{t})$ since H_∞ is positively homogeneous in the variable $\nabla_x \Psi$.

(B) We may assume that Φ is a separable type of the form

$$\Phi(x, z, t) = \psi(x, z) + a(t), \quad (x, z, t) \in \hat{Q}.$$

We may assume that $i_{u^*} - \Phi$ takes its strict maximum at $(\hat{x}, \hat{z}, \hat{t})$ by replacing Φ by

$$\Phi + |x - \hat{x}|^2 + (z - \hat{z})^2 + (t - \hat{t})^2.$$

We consider a shift Φ_ζ of Φ by defining

$$\Phi_\zeta(x, z, t) = \Phi(x - \xi, z - \eta, t),$$

with $\zeta = (\xi, \eta) \in \mathbf{R}^N \times \mathbf{R}$. By the convergence of maximum points, there is a sequence $(x_\zeta, z_\zeta, t_\zeta)$ converging to $(\hat{x}, \hat{z}, \hat{t})$ as $\zeta \to 0$ such that

$$\max_{\hat{Q}} (i_{u^*} - \Phi_\zeta) = (i_{u^*} - \Phi_\zeta)(x_\zeta, z_\zeta, t_\zeta).$$

Suppose that there is a sequence $\zeta_j \to 0$ such that

$$\hat{\nabla} \Phi_{\zeta_j}(x_{\zeta_j}, z_{\zeta_j}, t_{\zeta_j}) \neq 0.$$

Since $(x_{\zeta_j}, z_{\zeta_j}, t_{\zeta_j}) \to (\hat{x}, \hat{z}, \hat{t})$, we see

$$\lim_{j \to \infty} \hat{\nabla} \Phi_{\zeta_j}(x_{\zeta_j}, z_{\zeta_j}, t_{\zeta_j}) = \hat{\nabla} \Phi(\hat{x}, \hat{z}, \hat{t}) = 0.$$

If there is a subsequence ζ_{j_k} such that

$$\partial_z \Phi_{\zeta_{j_k}}(x_{\zeta_{j_k}}, z_{\zeta_{j_k}}, t_{\zeta_{j_k}}) < 0,$$

we apply (A) (i) with $\Phi_{\zeta_{j_k}}$ at $(x_{\zeta_{j_k}}, z_{\zeta_{j_k}}, t_{\zeta_{j_k}})$ to get $z_{\zeta_j} = u^*(x_{\zeta_j}, t_{\zeta_j})$ and

$$\tau_k + H(x_{\zeta_{j_k}}, t_{\zeta_{j_k}}, z_{\zeta_{j_k}}, p_k) \le 0,$$

with

$$\tau_k = \partial_t \Phi_{\zeta_{j_k}}(x_{\zeta_{j_k}}, z_{\zeta_{j_k}}, t_{\zeta_{j_k}}) \,\big/\, \lambda_k,$$
$$p_k = \nabla_x \Phi_{\zeta_{j_k}}(x_{\zeta_{j_k}}, z_{\zeta_{j_k}}, t_{\zeta_{j_k}}) \,\big/\, \lambda_k,$$
$$\lambda_k = -\partial_z \Phi_{\zeta_{j_k}}(x_{\zeta_{j_k}}, z_{\zeta_{j_k}}, t_{\zeta_{j_k}}) \ (> 0).$$

In other words,

$$\partial_t \Phi_{\zeta_{j_k}}(x_{\zeta_{j_k}}, z_{\zeta_{j_k}}, t_{\zeta_{j_k}}) + \lambda_k H\left(x_{\zeta_{j_k}}, t_{\zeta_{j_k}}, z_{\zeta_{j_k}}, \nabla_x \Phi_{\zeta_{j_k}}(x_{\zeta_{j_k}}, z_{\zeta_{j_k}}, t_{\zeta_{j_k}}) \,\big/\, \lambda_k\right) \le 0.$$

By (4.19), sending $k \to \infty$, we obtain

$$\partial_t \Phi(\hat{x}, \hat{z}, \hat{t}) + H_\infty\left(\hat{x}, \hat{t}, \hat{z}, \nabla_x \Phi(\hat{x}, \hat{z}, \hat{t})\right) \le 0$$

since

$$\partial_t \Phi(\hat{x}, \hat{z}, \hat{t}) = \lim_{j\to\infty} \partial_t \Phi_{\zeta_j}(x_{\zeta_j}, z_{\zeta_j}, t_{\zeta_j}),$$
$$\nabla_x \Phi(\hat{x}, \hat{z}, \hat{t}) = \lim_{j\to\infty} \nabla_x \Phi_{\zeta_j}(x_{\zeta_j}, z_{\zeta_j}, t_{\zeta_j}) = 0,$$
$$\partial_z \Phi(\hat{x}, \hat{z}, \hat{t}) = \lim_{j\to\infty} \partial_z \Phi_{\zeta_j}(x_{\zeta_j}, z_{\zeta_j}, t_{\zeta_j}) = 0.$$

Since $H_\infty(\hat{x}, \hat{t}, \hat{z}, 0) = 0$ by Remark 4.12, we now conclude that $\partial_t \Phi(\hat{x}, \hat{z}, \hat{t}) \le 0$.

If $\partial_z \Phi_{\zeta_j}(x_{\zeta_j}, z_{\zeta_j}, t_{\zeta_j}) = 0$ for sufficiently large j, we apply (A) (ii) and (iii) to conclude that $\partial_t \Phi(\hat{x}, \hat{z}, \hat{t}) \le 0$ since $H^I(\hat{x}, \hat{t}, \hat{z}, 0) = 0$ by Remark 4.12.

It remains to discuss the case where

$$\hat{\nabla} \Phi_\zeta(x_\zeta, z_\zeta, t_\zeta) = 0$$

for sufficiently small ζ. We shall prove that Φ is independent of x and z near $(\hat{x}, \hat{z}, \hat{t})$. In other words, ψ is constant near $(\hat{x}, \hat{z})$. If so, we are able to conclude that

$$\partial_t \Phi(\hat{x}, \hat{z}, \hat{t}) = \partial_t a(\hat{t}) \le 0$$

since otherwise it would contradict the left accessibility of i_{u^*} in Lemma 4.15.

To show that Φ is spatially constant near $(\hat{x}, \hat{z}, \hat{t})$, we invoke the following constancy lemma; see [43, Lemma 7.5], where C^2 regularity of ϕ is assumed. This lemma is implicitly used in [21].

Lemma 4.17
Let K be a compact set in $\mathbf{R}^m$ ($m \geq 2$), and let h be a real-valued upper semicontinuous function on K. Let ϕ be a C^1 function on $\mathbf{R}^d$ with $1 \leq d < m$. Let G be a bounded domain in $\mathbf{R}^d$. For each $\zeta \in G$ assume that there is a maximizer $(r_\zeta, \rho_\zeta) \in K$ of

$$H_\zeta(r, \rho) = h(r, \rho) - \phi(r - \zeta)$$

over K such that $\nabla\phi(r_\zeta - \zeta) = 0$. Then

$$h_\phi(\zeta) = \sup\left\{H_\zeta(r, \rho) \;\middle|\; (r, \rho) \in K\right\}$$

is constant in G.

We set $m = N + 2$, $d = N + 1$, and

$$K = \left\{(x, z, t) \in \mathbf{R}^m \;\middle|\; |x - \hat{x}| + |z - \hat{z}| + |t - \hat{t}| \leq \delta,\ i_{u^*}(x, z, t) = 0\right\} \subset \hat{Q}$$

for some $\delta > 0$. We take $\varepsilon > 0$ small so that $|\zeta| < \varepsilon$ implies

$$\hat{\nabla}\Phi_\zeta(x_\zeta, z_\zeta, t_\zeta) = 0.$$

We then set

$$G = \left\{\zeta \in \mathbf{R}^d \;\middle|\; |\zeta| < \varepsilon\right\}$$

and take

$$h(r, \rho) = i_{u^*}(r, \rho) - a(\rho) = -a(\rho) \quad \text{on} \quad K$$
$$\phi(r) = \psi(r) \quad \text{on} \quad \mathbf{R}^{N+1},$$

with $r = (x, z)$, $\rho = t$. Here, we extend ψ outside $\Omega \times (0, T)$ so that the extended function is C^1 in $\mathbf{R}^{N+1}$. Since $(r_\zeta, \rho_\zeta) = (x_\zeta, z_\zeta, t_\zeta)$ is a minimizer of

$$H_\zeta(r, \rho) = h(r, \rho) - \phi(r - \zeta)$$

over K, with $\nabla\phi(r_\zeta - \zeta) = 0$, applying Lemma 4.17 implies that

$$h_\phi(\zeta) = \sup\left\{H_\zeta(r, \rho) \;\middle|\; (r, \rho) \in K\right\}$$

is constant in G. This implies that ψ is constant for r such that $|r - \hat{r}| < \varepsilon$, i.e., $|x - \hat{x}|^2 + |z - \hat{z}|^2 < \varepsilon$. The proof of Lemma 4.16 is now complete. □

Proof of Lemma 4.17 By definition,

$$H_\zeta(r_\eta, \rho_\eta) \le h_\phi(\zeta) = h(r_\zeta, \rho_\zeta) - \phi(r_\zeta - \zeta) \quad \text{for} \quad \zeta, \eta \in G.$$

Since

$$h_\phi(\eta) = H_\eta(r_\eta, \rho_\eta) = h(r_\eta, \rho_\eta) - \phi(r_\eta - \eta) = H_\zeta(r_\eta, \rho_\eta) + \phi(r_\eta - \zeta) - \phi(r_\eta - \eta),$$

we observe that

$$h_\phi(\eta) \le h_\phi(\zeta) + \phi(r_\eta - \zeta) - \phi(r_\eta - \eta).$$

Since $\nabla\phi(r_\eta - \eta) = 0$ and ϕ is C^1,

$$\left|\phi(r_\eta - \eta) - \phi(r_\eta - \zeta)\right| \le \omega\,(|\eta - \zeta|)\,|\eta - \zeta|$$

with some modulus ω, i.e., $\omega(0) = 0$, $\omega \ge 0$, $\omega(\sigma) \to 0$ as $\sigma \to 0$. Here, ω can be taken to be independent of η since $\nabla\phi$ is uniformly continuous on any bounded set. We thus observe that

$$h_\phi(\eta) - h_\phi(\zeta) \le \omega\,(|\eta - \zeta|)\,|\eta - \zeta|.$$

Changing the role of η and ζ, we end up with

$$\left|h_\phi(\eta) - h_\phi(\zeta)\right| \le \omega\,(|\eta - \zeta|)\,|\eta - \zeta|$$

for all $\eta, \zeta \in G$. We now conclude that h_ϕ is constant on G. □

4.4.8 Weak Comparison Principle

As an application of Lemmas 4.15 and 4.16, we present here a version of a comparison principle for periodic functions. Unlike the earlier comparison principle (Theorem 4.4), the following comparison principle does not imply the uniqueness of a solution. We consider (4.14) with $\Omega = \mathbf{T}^N$ and H independently of x, t, i.e.,

$$\partial_t u + H(u, \nabla u) = 0 \quad \text{in} \quad Q = \mathbf{T}^N \times (0, T). \tag{4.24}$$

Theorem 4.18 (Weak comparison principle)
Assume that $H = H(r, p)$ *is continuous and for* $M > 0$ *there exists a constant* C_M

$$\left|H(r, p) - H(r', p)\right| \le C_M |r - r'| \, (|p| + 1)$$

for $r, r' \in \mathbf{R}$*, with* $|r|, |r'| \le M$*,* $p \in \mathbf{R}^N$*. Assume that* (4.19) *holds for every compact set* K *in* $Q \times \mathbf{R} \times \mathbf{R}^N$*. Let* u *and* v *be bounded proper sub- and supersolutions of* (4.24)*, respectively. If* $u^*(x, 0) < v_*(x, 0)$ *for all* $x \in \mathbf{T}^N$*, then* $u^* < v_*$ *in* Q*.*

The proof is rather involved compared with that of Theorem 4.4. We present here only the idea of the proof.
The Idea of the Proof. We may assume that $u = u^*$, $v = v_*$. Instead of considering u and v, we consider i_u and I_v defined in Sect. 4.4.7. We consider

$$\Psi(x, z, t, y, w, s) := i_u(x, z, t) - I_v(y, w, s) - \Xi(x, z, t, y, w, s),$$

$$\Xi(x, z, t, y, w, s) := \alpha|x - y|^2 + \alpha|z - w|^2 + \alpha(t - s)^2 + \sigma/(T - t),$$

where $(x, z), (y, w) \in \mathbf{T}^N \times \mathbf{R}$ and $t, s \in (0, T)$; here, α and σ are positive parameters. We argue by contradiction. We fix $\sigma > 0$. We argue in the same way as in the proof of Theorem 4.4 and conclude that a maximizer of Ψ is away from $t = 0$, $s = 0$ for sufficiently large α since initially $i_u(x, z, 0) \le I_v(x, z, 0)$, $(x, z) \in \mathbf{T}^N \times \mathbf{R}$. We divide the situation into two cases.

Case 1. There is a sequence $\alpha_j \to \infty$ such that at a maximum of Ψ in $\left(\mathbf{T}^N \times \mathbf{R} \times (0, T)\right)^2$ the gradient $(\nabla_x \Xi, \partial_z \Xi) = 0$.
Case 2. For sufficiently large α, $(\nabla_x \Xi, \partial_z \Xi) \ne 0$ at a maximizer of Ψ.

In the first case, one gets a contradiction by Lemma 4.16 (B). The second case is itself further subdivided into two cases.

Case 2A. For sufficiently large α, $\partial_z \Xi \ne 0$ at a maximizer of Ψ.
Case 2B. There is a sequence $\alpha_j \to \infty$ such that $\partial_z \Xi = 0$ at a maximizer of Ψ.

To derive a contradiction in Case 2A, we use Lemma 4.16 (A) (i).
In Case 2B, we invoke the property of proper solutions. We provide a detailed proof in this case. Let $(\hat{x}, \hat{z}, \hat{t}, \hat{y}, \hat{w}, \hat{s})$ be a maximizer of Ψ, with $\hat{t}, \hat{s} > 0$. We have $\partial_z \Xi(\hat{x}, \hat{z}, \hat{t}, \hat{y}, \hat{w}, \hat{s}) = 0$, so that $\hat{z}$ must agree with $\hat{w}$. By Lemma 4.16, $\hat{z} \le u(\hat{x}, \hat{t})$, $\hat{w} \ge v(\hat{y}, \hat{s})$, so that $v(\hat{y}, \hat{s}) \le u(\hat{x}, \hat{t})$.
We shall fix α so that $\hat{t}, \hat{s} > 0$. We first note that

$$a_0 = (\hat{x}, \hat{v}, \hat{t}, \hat{y}, \hat{v}, \hat{s}) \quad \text{and} \quad a_1 = (\hat{x}, \hat{u}, \hat{t}, \hat{y}, \hat{u}, \hat{s}),$$

with

$$\hat{u} = u(\hat{x}, \hat{t}), \quad \hat{v} = v(\hat{y}, \hat{s}),$$

are also maximizers of Ψ. Indeed, since $i_u(\hat{x}, z, \hat{t}) = 0$ for all $z \le \hat{u}$ and $I_v(\hat{y}, w, \hat{s}) = 0$ for all $w \ge \hat{v}$, Ψ must take the same value for $z, w \in \mathbf{R}$ satisfying $z \le \hat{u}$, $w \ge \hat{v}$, and $z = w$. In particular, a_0 and a_1 are maximizers of Ψ since $\hat{v} \le \hat{u}$.

Since Ψ is maximized at a_0, we apply Lemma 4.16 (A) (ii) and (iii) to a function

$$(x, z, t) \mapsto i_u(x, z, t) - \Xi(x, z, t, \hat{y}, \hat{w}, \hat{s}) - I_v(\hat{y}, \hat{w}, \hat{s})$$

to conclude

$$\partial_t \Xi + H^I(\hat{u}, \nabla_x \Xi) \le 0 \quad \text{at} \quad a_0, \tag{4.25}$$

with $I = [\hat{v}, \hat{u}]$. Similarly, we have

$$-\partial_s \Xi + H^I(\hat{v}, -\nabla_y \Xi) \ge 0 \quad \text{at} \quad a_1. \tag{4.26}$$

Note that $\nabla_x \Xi(a_0) = -\nabla_y \Xi(a_1)$, $\partial_t \Xi(a_0) + \partial_s \Xi(a_1) = \sigma/(T - \hat{t})^2$. Thus, subtracting (4.26) from (4.25) yields

$$\sigma/(T - \hat{t})^2 \le 0$$

since $H^I(r, p)$ is nondecreasing in r and $\hat{v} \le \hat{u}$. This yields a contradiction to $\sigma > 0$. This is the end of the idea of the proof.

4.4.9 Comparison Principle and Uniqueness

In general, the uniqueness of a solution does not hold even if H is independent of u for discontinuous solutions. As shown in [10], a solution of

$$u_t + (x - t)|u_x| = 0,$$

starting with a characteristic function 1_I of some closed interval I, is not unique, where $1_I(x) = 1$ for $x \in I$ and $1_I(x) = 0$ for $x \notin I$. This is related to fattening phenomena for a level-set flow of a curvature flow equation; see, for example, [47]. Some additional condition is necessary to guarantee the uniqueness of the initial value problem for (4.24).

Theorem 4.19 (Strong comparison principle)
Assume the same hypothesis as that of Theorem 4.18 concerning u, v, and H. Assume furthermore that

$$-H(r,p) \geq c\sqrt{1+p^2} \quad \textit{with some} \quad c>0 \quad \textit{for all} \quad p \in \mathbf{R}^N \ r \in \mathbf{R}.$$

If $u^*(x,0) \leq (v_*)^*(x,0)$ for all $x \in \mathbf{T}^N$, then $u^* \leq (v_*)^*$ in $Q_0 = \mathbf{T}^N \times [0,T)$. If $(u^*)_*(x,0) \leq v_*(x,0)$ for all $x \in \mathbf{T}^N$, then $(u^*)_* \leq v_*$ in Q_0.

The proof requires several fundamental properties of viscosity solutions, so we provide only a sketch of the proof.
Sketch of the Proof. We provide the proof only where $u^* \leq (v_*)^*$ at $t = 0$ since the proof for the remaining case is symmetric. Again we may assume that $u = u^*$ and $v = v_*$. Since v is a viscosity supersolution of (4.24), it is a viscosity supersolution of

$$w_t - c\sqrt{1+|\nabla w|^2} = 0$$

by our assumption. This equation has a strong comparison principle (e.g., [10]), so the solution is unique even among semicontinuous functions; see, for example, [51]. The unique upper semicontinuous solution of the w equation with initial datum w_0 is given by

$$w(x,t) = \sup\left\{x \in \mathbf{R} \mid d\big((x,z), \overline{\mathrm{sg}\, w_0}\big) \leq ct\right\},$$

where $\overline{\mathrm{sg}\, w_0}$ denotes the closure of the subgraph $\mathrm{sg}\, w_0$ of w_0 defined in Sect. 4.4.7. Heuristically, this is easy to observe since our w equation requires that the graph of w moves with upward normal velocity $V = c$. If one interprets this equation as a surface evolution equation or front propagation of a set E_0, then the set E_t at time t is the set of all points whose distance from E_0 is less than or equal to ct. For more details, see [46]. Since v is a viscosity supersolution of the w equation, the comparison principle for the w equation with initial datum $w_0(x) = v^*(x,0)$ implies that $v \geq w$ in $\mathbf{T}^N \times (0,T)$. This implies that for $\delta \in (0,T)$ there is $\rho > 0$ that satisfies

$$v(x,t) \geq v^*(x,0) + \rho \quad \text{for all} \quad x \in \mathbf{T}^N, \quad t \geq \delta. \tag{4.27}$$

We shift v in time and set

$$v_\delta(x,t) = v(x,t+\delta), \quad t > 0.$$

Evidently, v_δ is a proper supersolution of (4.24) in $\mathbf{T}^N \times (0, T - \delta)$. Assume that $u \le v^*$ at $t = 0$. By (4.27), we see that $u \le v^* \le v_\delta - \rho$ at $t = 0$. Since v_δ is lower semicontinuous up to $t = 0$, applying weak comparison Theorem 4.18, we obtain $u < v_\delta$ in $\mathbf{T}^N \times [0, T - \delta)$. Sending δ to zero, we conclude that $u \le v^*$ in Q_0 since $\liminf_{\delta \downarrow 0} v_\delta \le v^*$ by the definition of upper semicontinuous function. This is the end of the sketch of the proof.

For a given function $u_0 : \mathbf{T}^N \to \mathbf{R} \cup \{\pm\infty\}$ we say that $u : \mathbf{T}^N \times (0, T) \to \mathbf{R} \cup \{\pm\infty\}$ is a *solution* of (4.24) with initial datum u_0 if u is a proper solution of (4.24) and

$$u^*(x, 0) = (u_*)^*(x, 0) = (u_0)^*(x),$$

$$u_*(x, 0) = (u^*)_*(x, 0) = (u_0)_*(x).$$

Our comparison principle implies the uniqueness of a solution.

Theorem 4.20
Assume the same hypothesis as that of Theorem 4.19 concerning H. Let u be a bounded solution of (4.24) with initial datum u_0. Then u^* and u_* are unique. Moreover, $(u_*)^* = u^*$, and $(u^*)_* = u_*$.

Proof. Let v be another solution. Since $u^* \le (v_*)^*$ at $t = 0$, the strong comparison principle (Theorem 4.19) implies that $u^* \le (v_*)^* (\le v^*)$ in $\mathbf{T}^N \times (0, T)$. Replacing the roles of v and u yields $v^* \le u^*$. We thus conclude that $v^* = u^*$. Moreover, $u^* \le (u_*)^* \le u^*$ implies $(u_*)^* = u^*$. A symmetric argument implies the uniqueness of u_* and $(u^*)_* = u_*$. □

There are several other situations in which the conclusion of the comparison principle holds. For example, it applies to a conservation law starting with a special class of initial data. The reader is referred to [46] for further examples.

4.5 Notes and Comments

4.5.1 A Few References on Viscosity Solutions

The theory of viscosity solutions is by now a standard tool to study nonlinear (degenerate) elliptic parabolic partial differential equations of second order as well as first-order equations like Hamilton–Jacobi equations, where expected solutions are not smooth. The notion of a viscosity solution was first introduced by [29] (see also [28]) in a different way for first-order Hamilton–Jacobi equations with a nonconvex Hamiltonian. See the book by Lions [71] for the early stage and the one by Barles [9] for the development of the theory. One of the original applications of the theory is to characterize the value function of control theory and differential games for ordinary differential equations as a unique nondifferentiable solution of Hamilton–Jacobi equations. The reader is referred to the book by Bardi and Capuzzo-Dolcetta [7] as well as [36, Chapter 10].

The extension to second-order equations is not straightforward. It takes several years to overcome the substantial difficulty of obtaining the key comparison principle. The reader is referred to the well-written review article of Crandall, Ishii, and Lions [26] and a shorter review by Ishii [59] for the development of the theory. There are accessible textbooks by Koike [63, 64]. The second-order problem relates to stochastic controls. For this type of application see the books by Fleming and Soner [41] and Morimoto [76]. The theory of viscosity solution also gives a mathematical foundation [21, 37] for a level-set flow of the mean curvature flow equations, which was introduced numerically by [81]. For this topic the reader is referred to the book [47], which includes a necessary survey of viscosity solutions. See also the lecture notes of Bardi et al. [8], where various applications, including a level-set method, are presented.

In Sect. 4.2.3, we provide an example of where uniqueness fails for the eikonal equation. For the eikonal equation (4.10), uniqueness with given boundary data is valid provided that the value of a solution on the set $\{x \mid f(x) = 0\}$ is prescribed. This has its origins in the book [71, Section 5.5]. This type of uniqueness and comparison principle is generalized by [38] and [60] for various Hamilton–Jacobi equations with convex Hamiltonians; see also the recent book [87]. This type of comparison principle is roughly stated as follows. If a subsolution u and a supersolution v have an order $u \leq v$ on the (projected) Aubry set $\mathcal{A}$ other than on boundaries, then $u \leq v$ in a whole domain. The Aubry set is a notion related to the Hamilton system corresponding to the Hamilton–Jacobi equation. It consists of an equilibrium set and a point having a sequence of closed curves converging at this point whose Euclidean length is bounded from below, but the corresponding action integral converges to zero. For the eikonal equation (4.10), the Aubry set is simply the equilibrium set $\{x \mid f(x) = 0\}$.

4.5.2 Discontinuous Viscosity Solutions

The contents up to Sect. 4.3 are basic materials on viscosity solutions. An elegant proof of the uniqueness of the eikonal equation (Sect. 4.2.3) is due to [58].

Definition 4.5 of the viscosity solution for a semicontinuous function was introduced by [56, 57]. Although this notion is very convenient when it comes to constructing a continuous solution by Perron's method [57], it is not enough to establish the uniqueness of a solution for discontinuous initial data even if the Hamiltonian $H = H(x, t, r, p)$ is independent of r, i.e., independent of the value of the unknown function. There are several approaches to recovering uniqueness among semicontinuous functions. When $H = H(x, t, p)$ is convex or concave with respect to p, a notion of solutions is introduced by [11] and

[12], so that the solution is unique among semicontinuous functions. For a general $H = H(x, t, p)$, uniqueness was established in [51] based on a level-set method; the main assumption in [51] is that the recession function H_∞ exists.

A proper viscosity solution to handle solution with shock is introduced by [46] to describe a kind of bunching phenomenon of growing crystals on a surface. The contents of Sect. 4.4 are essentially taken from [46]. Since there are many errors in [46], we take this opportunity to correct them. For example, in [46, Proposition 2.5], it was claimed that u_N is a viscosity solution when the speed of a shock comes from the Rankine–Hugoniot condition. However, this statement is wrong. As in Proposition 4.13, this u_N is not even a conditional viscosity solution. Also, "left accessibility" in Proposition 4.14 was written as "right accessibility" in [46]. We also give a detailed proof of Lemma 4.15 (ii) in this book.

There is an interesting way to interpret a proper viscosity solution as an evolution of its graph. If we rewrite the equation for the evolution of the graph, then the graph may not stay as the graph of a function, and the function becomes multivalued. It is natural to think that there is a very singular vertical diffusion that prevents such a phenomenon and causes shocks. This idea is useful for the formulation of a proper viscosity solution [88]. A discussion of the theoretical background of the topic can be found in [44]. There is another approach to interpreting a solution with a shock by introducing an obstacle to prevent overturn [15]. An extension of proper solutions to second-order problems is not yet available.

4.6 Exercises

4.1 Find the unique viscosity solution of

$$\begin{cases} \left|\dfrac{\mathrm{d}u}{\mathrm{d}x}\right| - 1 = 0 \quad \text{in} \quad (-1, 1), \\ u(\pm 1) = 0 \end{cases}$$

and

$$\begin{cases} 1 - \left|\dfrac{\mathrm{d}u}{\mathrm{d}x}\right| = 0 \quad \text{in} \quad (-1, 1), \\ u(\pm 1) = 0. \end{cases}$$

4.2 For a function $f(p) = \sqrt{1 + |p|^2}$ ($p \in \mathbf{R}^N$) calculate the recession function $f_\infty(p)$.

4.3 Prove that the upper semicontinuous envelope f^* of a real-valued function f in an open set $\Omega \subset \mathbf{R}^N$ is actually upper semicontinuous and that it is smallest among all upper semicontinuous functions greater than or equal to f.

4.4 Let $\{f_m\}$ be a sequence of real-valued continuous functions in $\overline{\Omega}$, where Ω is an open set in $\mathbf{R}^N$. Assume that f_m converges to a continuous function f

uniformly in $\overline{\Omega}$ as $m \to \infty$, i.e.,

$$\lim_{m\to\infty} \sup_{x\in\overline{\Omega}} |f_m(x) - f(x)| = 0.$$

Assume that there is $\hat{x}$ such that $f(x) \le f(\hat{x})$ for all $x \in \Omega$ and that $f(x) = f(\hat{x})$ if and only if $x = \hat{x}$. In other words, f takes its strict maximum at $\hat{x}$. Then there is a point $x_m \in \Omega$ that converges to $\hat{x}$ such that $\max_{\overline{\Omega}} f_m = f_m(x_m)$ and $\lim_{m\to\infty} f_m(x_m) = f(\hat{x})$.

4.5 Let Ω be a domain in $\mathbf{R}^N$. Assume that $u_m \in C(\Omega)$ converges to $u \in C(\Omega)$ locally uniformly in Ω as $m \to \infty$. Let u_m be a viscosity subsolution of (4.9). Show that u is a viscosity subsolution of (4.9).

4.6 Assume that $f_m \in C([0, 1])$ converges to f uniformly in $[0, 1]$ as $m \to \infty$, i.e.,

$$\lim_{m\to\infty} \sup_{0\le x\le 1} |f_m(x) - f(x)| = 0.$$

Let $\{x_j\}_{j=1}^{\infty}$ be a sequence in $[0, 1]$ converging to $\hat{x}$ as $j \to \infty$. Show that

$$\lim_{\substack{m\to\infty \\ j\to\infty}} f_m(x_j) = f(\hat{x}).$$

In other words, show that for any $\varepsilon > 0$, there are numbers m_0 and j_0 such that

$$\left| f_m(x_j) - f(\hat{x}) \right| < \varepsilon$$

for all $m \ge m_0$, $j \ge j_0$.

4.7 Let P_1 be the space of all affine functions on $\mathbf{R}^N$, i.e.,

$$P_1 = \left\{ a \cdot x + b \mid a \in \mathbf{R}^N,\ b \in \mathbf{R} \right\}.$$

Let M be a nonempty subset of P_1. Set

$$f(x) = \sup \left\{ p(x) \;\middle|\; p \in M \right\}, \quad x \in \mathbf{R}^N, \tag{4.28}$$

and assume that $f(x)$ is finite. Show that f is a convex function. Show that any real-valued convex function on $\mathbf{R}^N$ is of the form (4.28), with a suitable choice of M.

4.8 Let Ω be a bounded domain in $\mathbf{R}^N$. Let d be the distance function from the boundary $\partial\Omega$, i.e.,

$$d(x) = \inf \left\{ |x - y| \;\middle|\; y \in \partial\Omega \right\}.$$

Show that $d \in C(\overline{\Omega})$ is the unique viscosity solution of $|\nabla u| = 1$ in Ω with $u = 0$ on $\partial\Omega$.

4.9 Let $g \in C(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ be a given function. Show that

$$u(x,t) = \inf\left\{ g(y) + \frac{|x-y|^2}{2t} \;\middle|\; y \in \mathbf{R}^N \right\}$$

is a viscosity solution of

$$v_t + \frac{1}{2}|\nabla v|^2 = 0$$

in $\mathbf{R}^N \times (0, \infty)$. The function u is often called an inf-convolution of g.

4.10 Show that

$$u(x,t) = t - |x|, \quad x \in \mathbf{R}, \quad t > 0,$$

is not a viscosity solution of

$$v_t - |\partial_x v| = 0 \tag{4.29}$$

in $\mathbf{R} \times (0, \infty)$, although u satisfies (4.29) outside $x = 0$.