3 Uniqueness of Solutions to Initial Value Problems for a Scalar Conversation Law

In Chap. 2, we discussed the uniqueness of a weak solution to a transport equation, which is linear and of the first order. In this chapter, we consider scalar conservation laws, which are quasilinear but still of the first order. The major difference between the linear transport equations with a divergence-free (solenoidal) coefficient and a conservation law lies in the uniqueness problem of a weak solution. For the transport equation, it is unique under a very weak regularity assumption. However, for a conversation law, it may not be unique under a reasonable regularity assumption allowing discontinuities. To recover uniqueness, one must introduce an extra condition, called an entropy condition, that is not a regularity condition. Another difference is that the solution may develop singularity even if the initial datum are smooth for a conservation law but the solution is smooth for the transport equation if all data and coefficients are smooth.

In this chapter, we introduce a scalar conservation law and observe that a discontinuity –called a shock– may develop in finite time. To track the whole evolution, we need to introduce a weak solution. However, unfortunately, weak solutions may not be unique. To recover uniqueness, we introduce the "entropy condition" and the notion of an "entropy solution." After discussing the entropy condition, we prove the uniqueness of an entropy solution. To avoid technical complications, we discuss uniqueness in a periodic setting. A key idea in proving uniqueness is a method of doubling variables that is due to Kružkov [68]. The contents of this chapter are essentially taken from a book [53] by Holden and Risebro, with the modification that the uniqueness is discussed in a periodic setting. This topic is also discussed in [36, Chapter 11], with an emphasis on systems of conservation laws.

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3.1 Entropy Condition

In this section, we introduce a scalar conservation law and discuss the discontinuity of a solution. If initial datum are smooth, we are able to solve the equation locally in time, but it may develop discontinuity. To track evolution globally in time, we introduce the notion of a weak solution by integration by parts. We notice that uniqueness may be violated. There are several types of discontinuity. We only allow a particular type of discontinuity that satisfies the entropy condition. This eventually leads to the notion of an entropy solution.

3.1.1 Examples

We consider a flow map $x(t, X)$ generated by a vector field u on $\mathbf{R}^N$, i.e.,

$$\dot{x}(t, X) = u\,(x(t, X), t) \quad \text{for} \quad t > 0, \quad x(0, X) = X,$$

where $\dot{x}(t, X) = \frac{\partial}{\partial t}x(t, X)$. The coordinate by X is often called the Lagrangian coordinate, while the coordinate by x is called the Euler coordinate.

Assume that there is no acceleration. Physically speaking, there is no force by Newton's law. Then

$$\ddot{x}(t, X) = 0 \quad \text{or} \quad \frac{\partial^2}{\partial t^2}x(t, X) = 0,$$

where the partial derivative is taken in the Lagrangian coordinate. We shall write this law for $u(x, t)$ for the Euler coordinate. Since

$$\ddot{x} = \nabla_x u \cdot \dot{x} + u_t \quad \text{with} \quad \dot{x} = u(x, t) \quad \text{or}$$

$$\ddot{x}^i = \sum_{j=1}^{N} \partial_{x_j} u^i \dot{x}^i + u_t^i \quad \text{with} \quad \dot{x}^i = u^i(x, t),$$

where the partial derivative in the direction of x, t of u is in the Euler coordinate, we see that $\ddot{x} = 0$ is equivalent to saying that

$$u_t + u \cdot \nabla_x u = 0 \quad \text{or} \quad u_t^i + \sum_{j=1}^{N} u^j \partial_{x_j} u^i = 0, \quad 1 \le i \le N.$$

If $N = 1$, this is simply

$$u_t + u\,u_x = 0 \quad \text{or} \quad u_t + \left(\frac{u^2}{2}\right)_x = 0, \tag{3.1}$$

which is called the Burgers equation. Here $u_x = \partial u/\partial x$. This equation is a typical example of a (scalar) conservation law

$$u_t + f(u)_x = 0, \tag{3.2}$$

where f is a function of u and $f(u)_x = \frac{\partial}{\partial x}(f(u)) = \frac{\partial}{\partial x}(f \circ u)(x)$. In (3.1), $f(u) = u^2/2$.

We give another derivation of a conservation law modeling a traffic flow. We consider the simplest situation: a road having only one lane parameterized by a single coordinate x. All cars are assumed to move in only one direction, that of increasing x. Let $\rho(x,t)$ be the (number) density of cars at location x and time t. The number of cars in the interval $[a,b]$ at time t corresponds to $\int_a^b \rho(x,t)\mathrm{d}x$. Let $v(x,t)$ be the velocity of the car at x. The rate of cars passing a point x at some time t is given by $v(x,t)\rho(x,t)$. Thus, the change ratio of the number of cars in $[a,b]$ should be

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \rho(x,t)\mathrm{d}x = -\left(v(b,t)\rho(b,t) - v(a,t)\rho(a,t)\right).$$

Since the right-hand side equals $-\int_a^b (v\rho)_x \mathrm{d}x$ and since (a,b) is arbitrary, we get

$$\rho_t + (\rho v)_x = 0, \tag{3.3}$$

which is a typical mass conservation law, for example, in fluid mechanics. (In a multidimensional setting, it must be that

$$\rho_t + \operatorname{div}(\rho v) = 0,$$

which is the fundamental mass conservation law in science. Here v is a vector field.) In the simplest model, the velocity v is assumed to be a given function of the (number) density ρ only. This one-dimensional model may approximate the situation where the road is uniform with no obstacles like signals, crossings, or curves forcing cars to slow down. We postulate that there is a uniform maximal speed $v_{\max}$ for any car. If traffic is light, a car will approach this maximal speed, but the car will have to slow down if the number of cars increases. If ρ reaches some value $\rho_{\max}$, all cars must stop. Thus, it is reasonable to assume that v is a monotone decreasing function of ρ such that $v(0) = v_{\max}(> 0)$, $v(\rho_{\max}) = 0$. The simplest function is a linear function, i.e.,

$$v(\rho) = v_{\max}(1 - \rho/\rho_{\max}) \quad \text{for} \quad \rho \in [0, \rho_{\max}] \tag{3.4}$$

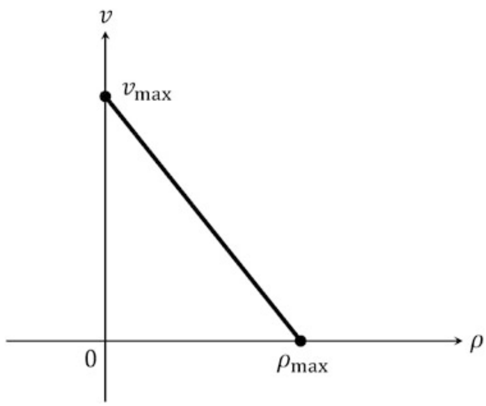


Fig. 3.1 Profile of V

(Figure 3.1). If $\tilde{u} = \rho/\rho_{max}$, $\tilde{x} = v_{max}x$ is normalized, the resulting normalized equation of (3.3) with (3.4) for $\tilde{u} = \tilde{u}(\tilde{x}, t)$ is of the form

$$\tilde{u}_t + (\tilde{u}(1 - \tilde{u}))_{\tilde{x}} = 0 \quad \text{for} \quad \tilde{u} \in [0, 1].$$

For further reference, we rewrite this equation as

$$u_t + (u(1 - u))_x = 0 \tag{3.5}$$

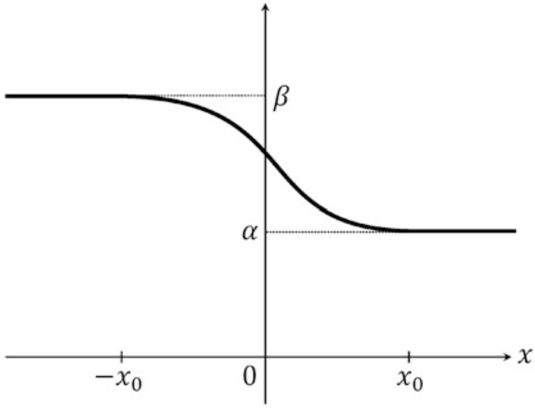
by writing $u = \tilde{u}$, $x = \tilde{x}$. The Burgers equation (3.1) is obtained by setting $\tilde{u} = \frac{1}{2}(1 - u)$, $\tilde{x} = x$.

3.1.2 Formation of Singularities and a Weak Solution

An important feature of conservation law (3.1) is that the solution may become singular in finite time.

Proposition 3.1
Assume that f is smooth in $\mathbf{R}$ and that its second derivative f'' is positive in an interval $[\alpha, \beta]$, which is nontrivial, i.e., $\alpha < \beta$. Let $u_0 \in C^\infty(\mathbf{R})$ be nonincreasing and $u_0(x) = \beta$ for $x < -x_0$ and $u_0(x) = \alpha$ for $x > x_0$ with some $x_0 > 0$. Then there exists a unique smooth solution u of (3.2), with $u(0, x) = u_0(x)$, for $x \in \mathbf{R}$ satisfying $\alpha \le u \le \beta$ in $\mathbf{R} \times (-T_0, T_1)$, with some $T_0, T_1 > 0$, but the maximal (forward) existence time T_1 must be finite.

Fig. 3.2 Graph of u_0



Proof. We consider the equation for $v \in \mathbf{R}$ of the form

$$v = u_0\left(x - f'(v)t\right) \tag{3.6}$$

for a given $x, t \in \mathbf{R}$. Here, f' denotes the derivative of f when f depends on just one variable. See Fig. 3.2 for the profile of u_0. This equation has a unique solution $\bar{v} \in [\alpha, \beta]$ for all $x \in \mathbf{R}$ provided that t is sufficiently small, say, $|t| < t_0$, with some $t_0 > 0$ by the implicit function theorem [67]. Indeed, differentiating

$$F(v, x, t) = v - u_0\left(x - f'(v)t\right)$$

with respect to v we get

$$\frac{\partial F}{\partial v}(v, x, t) = 1 + u_0{}'\left(x - f'(v)t\right) f''(v)t.$$

This is bounded away from zero uniformly in x and small t, allowing negative t, say, $|t| < t_0$ since f'' is bounded in $[\alpha, \beta]$ and $u_0{}'$ is bounded. Then, by the implicit function theorem, we get a unique $v = \bar{v}$, solving (3.6).

We shall write $\bar{v} = u(x, t)$ since $\bar{v}$ depends on (x, t). Since $\bar{v}$ solves (3.6), we see that $F\left(u(x, t), x, t\right) = 0$ for $x \in \mathbf{R}$, t, with $|t| < t_0$. Since F depends on v, x and t smoothly, we conclude that u is smooth in $\mathbf{R} \times (-t_0, t_0)$ by the smooth dependence of parameters in the implicit function theorem. (The curve $z = x - f'\left(u_0(z)\right)t$ in the xt-plane with a parameter $z \in \mathbf{R}$ is often called a *characteristic curve* (Fig. 3.3). The value of u on each characteristic curve $z = x - f'\left(u_0(z)\right)t$ equals the constant $u_0(z)$ by (3.6). Unlike the linear equation (2.6), the characteristic curve may depend on the initial datum u_0.)

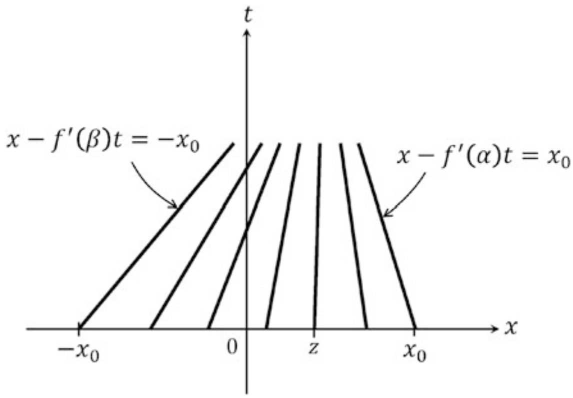


Fig. 3.3 Characteristic curves

Differentiating both sides of (3.6) by setting $v = u(x, t)$, we get

$$u_t = u_0{}' \left(x - f'(u)t\right) \left(-f''(u)u_t t - f'(u)\right),$$
$$f'(u)u_x = u_0{}' \left(x - f'(u)t\right) \left(-f''(u)u_x t + 1\right) f'(u).$$

Adding both sides we get

$$u_t + f'(u)u_x = u_0{}' \left(x - f'(u)t\right) \left(-f''(u)t \left(u_t + f'(u)u_x\right)\right).$$

From this identity we see that u solves (3.2) in $\mathbf{R} \times (-t_0, t_1)$, with $u(x, 0) = u_0(x)$, $x \in \mathbf{R}$, if we choose a sufficiently small $t_1 \in (0, t_0)$. Indeed, this identity implies $u_t + f'(u)u_x = 0$ unless $u_0' \left(u - f'(u)t\right) \left(-f''(u)t\right) = 1$. However, the last identity does not hold for $t < 0$ since $u_0' \leq 0$ and $f''(u) > 0$, and also for small $t > 0$ independent of x since u_0' and $f''\left(u(x, t)\right)$ are bounded. Thus, we get (3.2).

The uniqueness can be proved easily since the difference $w := u_1 - u_2$ of two solutions u_1 and u_2 solves

$$w_t + (pw)_x = 0, \quad w|_{t=0} = 0,$$

with

$$p(x, t) = \int_0^1 f'\left(u_2 + \theta(u_1 - u_2)\right) \mathrm{d}\theta,$$

which is smooth and bounded with its derivatives. Indeed,

$$f(u_1) - f(u_2) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta} \left(f\left(u_2 + \theta(u_1 - u_2)\right)\right) \mathrm{d}\theta = pw$$

so we get the preceding w equation by subtracting equation (3.2) for u_2 from that for u_1. We next apply an idea of the method of characteristics (see Chap. 2, especially the paragraph including (2.6)) to this w equation

$$w_t + pw_x + p_x w = 0.$$

In general, it is more involved since p depends on time t. Here we simply use it as a change of variables to remove the w_x term. Let $x = x(t, X)$ be the unique solution of

$$\dot{x} = p(x, t) \text{ for small } |t|, \quad x(0) = X.$$

We set

$$W(X, t) := w\,(x(t, X), t)$$

and observe that

$$\frac{\partial W}{\partial t} = w_t + pw_x.$$

The w equation is transformed to

$$W_t + qW = 0, \quad W|_{t=0} = 0$$

for small $|t|$, where $q = p_x\,(x(t, X), t)$. This is a linear ordinary differential equation, so the uniqueness (Proposition 1.1) yields $W \equiv 0$. Thus, $w \equiv 0$ on $\mathbf{R} \times (-\delta, \delta)$ for small $\delta > 0$. A similar argument implies that the time interval $[t_-, t_+]$ where uniqueness $w = 0$ holds is open. Thus, $w \equiv 0$ on $(-t_0, t_1)$, i.e., $u_1 \equiv u_2$ on $\mathbf{R} \times (-t_0, t_1)$.

By (3.6) we see that

$$\begin{aligned} u(x, t) = \beta \quad &\text{for} \quad x - f'(\beta)t < -x_0, \\ u(x, t) = \alpha \quad &\text{for} \quad x - f'(\alpha)t > x_0. \end{aligned}$$

Since $\alpha < \beta$, for sufficiently large t the two characteristic curves $x_0 = x - f'(\alpha)t$ and $-x_0 = x - f'(\beta)t$ merge (Fig. 3.3). Let $t = t_*$ be a number such that $f'(\alpha)t_* + x_0 < f'(\beta)t_* - x_0$. Then $u(\cdot, t_*)$ has two values, α and β, on $\big(f'(\alpha)t_* + x_0, f'(\beta)t_* - x_0\big)$. Thus, $t_1 < t_*$. This implies that the (forward) maximal existence time for a smooth solution is finite. □

We shall consider the initial value problem to (3.2) for $t > 0$. By Proposition 3.1, we must introduce a notion of a weak solution as in Definition 2.3 to track the whole evolution of a solution.

Definition 3.2

Assume that $f \in C(\mathbf{R})$. For $u_0 \in L^\infty(\mathbf{R})$, we say that $u \in L^\infty(\mathbf{R} \times (0, T))$ is a *weak solution* of (3.2) with initial datum u_0 if

$$\int_{\mathbf{R}\times(0,T)} \{\varphi_t u + \varphi_x f(u)\}\, \mathrm{d}x\mathrm{d}t + \int_{\mathbf{R}} \varphi|_{t=0}\, u_0 \mathrm{d}x = 0 \tag{3.7}$$

for all $\varphi \in C_c^\infty(\mathbf{R} \times [0, T))$. If u_0 and u is periodic in x, i.e., a function on $\mathbf{T} = \mathbf{R}/\omega_1\mathbf{Z}$ with some $\omega_1 > 0$, then φ should be taken from $C_c^\infty(\mathbf{T} \times [0, T))$.

We shall discuss the speed of jump discontinuity. Its speed is represented by the magnitude of the jump, and such a representation is called the *Rankine–Hugoniot condition*. Let $x(t)$ be a C^1 function defined on an interval $[t_0, t_1]$, with $t_0 < t_1$, $t_0, t_1 \in \mathbf{R}$. Let $D = J \times (t_0, t_1)$ be an open set containing the graph of $x(t)$ in (t_0, t_1), where J is an open interval in $\mathbf{R}$. We set

$$D_r = \{(x, t) \in D \mid x > x(t)\},$$
$$D_\ell = \{(x, t) \in D \mid x < x(t)\},$$
$$\Gamma = \overline{D_r} \cap \overline{D_\ell}.$$

Here, Γ is simply the graph of the curve $x = x(t)$. See Fig. 3.4.

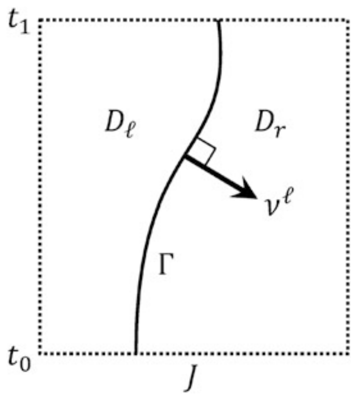


Fig. 3.4 Sets D_ℓ, D_r and Γ

Lemma 3.3
Let $f \in C(\mathbf{R})$ be given. Let u be C^1 in $\overline{D_r}$ and $\overline{D_\ell}$, and let u satisfy (3.7) for all $\varphi \in C_c^\infty(D \times (t_0, t_1))$. Then

$$\dot{x}(t)(u_\ell - u_r) = f_\ell - f_r \tag{3.8}$$

for $t \in (t_0, t_1)$, with

$$u_\ell = \lim \{u(y,s) \mid (y,s) \to (x(t), t), (y,s) \in D_\ell\} \text{ (left limit)},$$
$$u_r = \lim \{u(y,s) \mid (y,s) \to (x(t), t), (y,s) \in D_r\} \text{ (right limit)},$$

and $f_\ell = f(u_\ell)$, $f_r = f(u_r)$. (The speed $s = \dot{x}(t)$ is called the speed of the shock.) Conversely, if u satisfies (3.2) in D_r and D_ℓ and satisfies (3.8), then u satisfies (3.7) for all $\varphi \in C_c^\infty(D \times (t_0, t_1))$.

Proof. Since u is a classical solution of (3.2) in each D_i $(i = r, \ell)$, integration by parts yields

$$\int_{D_i} \{\varphi_t u + \varphi_x f(u)\} \, \mathrm{d}x\mathrm{d}t = \int_{\partial D_i} (\nu_t u + \nu_x f(u)) \, \varphi \mathrm{d}\mathcal{H}^1$$
$$= \int_\Gamma \left(\nu_t^i u_i + \nu_x^i f_i\right) \varphi \mathrm{d}\mathcal{H}^1,$$

where (ν_x^i, ν_t^i) is an external unit normal of ∂D_i. Here, $\mathrm{d}\mathcal{H}^1$ denotes the line element of the curve $x = x(t)$. Since u is a "weak solution" of (3.2) in D (i.e., u satisfies (3.7) for all $\varphi \in C_c^\infty(D \times (t_0, t_1))$), we see that

$$\int_\Gamma \left\{\left(\nu_t^r u_r + \nu_t^\ell u_\ell\right) + \left(\nu_x^r f_r + \nu_x^\ell f_\ell\right)\right\} \varphi \mathrm{d}\mathcal{H}^1 = 0.$$

Since $\nu^r = -\nu^\ell$ and φ is arbitrary, we now conclude (cf. Exercise 2.3) that

$$\nu_t^\ell (u_\ell - u_r) + \nu_x^\ell (f_\ell - f_r) = 0.$$

Since

$$\left(\nu_x^\ell, \nu_t^\ell\right) = (1, -\dot{x}(t)) / \left(1 + (\dot{x}(t))^2\right)^{1/2},$$

the desired relation (3.8) follows. Checking this argument carefully, we see the converse is easily obtained. The relation (3.8) is called the *Rankine–Hugoniot condition*. □

3.1.3 Riemann Problem

We consider the following special initial value problem for (3.2), which is called the *Riemann problem*. The initial datum we consider are

$$u_0(x) = \begin{cases} u_\ell, \ x < 0, \\ u_r, \ x > 0, \end{cases} \tag{3.9}$$

where u_ℓ and u_r are constants, i.e., $u_\ell, u_r \in \mathbf{R}$.

For simplicity, we assume that $u_\ell > u_r$ in this subsection. It is easy to see that

$$u_S(x,t) = \begin{cases} u_\ell, \ x < x(t), \\ u_r, \ x > x(t) \end{cases} \tag{3.10}$$

is a weak solution of (3.2) with (3.9) provided that $x(t) = t(f_\ell - f_r)/(u_\ell - u_r)$ by (3.8). If $u_r < u_\ell$ and f is convex, it turns out that this is the only weak solution. However, in the case where $u_r < u_\ell$ and f is concave, there is another weak solution called a *rarefaction wave*. Instead of writing a general form of a solution, we just restrict ourselves to the traffic flow equation (3.5) where $f(u) = u(1-u)$. In this case, the function

$$u_R(x,t) = \begin{cases} u_\ell, & x < x_\ell(t), \\ \dfrac{1}{2} - \dfrac{x}{2t}, & x_\ell(t) \le x \le x_r(t), \\ u_r, & x > x_r(t), \end{cases} \tag{3.11}$$

with $x_\ell(t) = \left(\frac{1}{2} - u_\ell\right)2t$, $x_r(t) = \left(\frac{1}{2} - u_r\right)2t$, is a weak solution of (3.2) with (3.9) provided that $u_r < u_\ell$ (Figs. 3.5 and 3.6). This is easy to check since there is no jump and $1/2 - x/(2t)$ solves equation (3.2) in the region $x_\ell < x < x_r$. The question is which is reasonable as a "solution." Of course, it depends on the physics we consider. For the traffic flow problem, consider the case where $u_r = 0$ and $u_\ell = 1$. The solution u_S in this case is time-independent since $f_\ell(0) = f_\ell(1) = 0$,

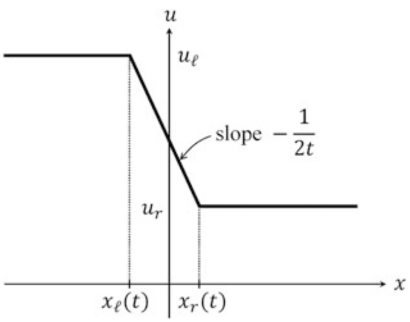


Fig. 3.5 A rarefaction wave u_R at time t

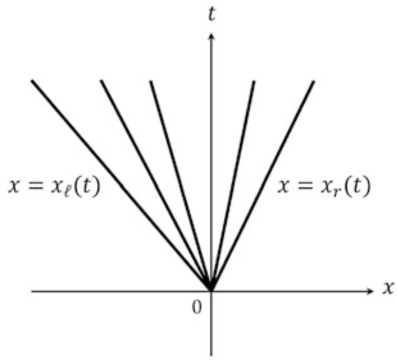


Fig. 3.6 Characteristic curves

so that $x(t) = 0$. Is it natural to stop even if there are no cars in front of us? There is no signal. From our intuition, u_R looks like a more reasonable solution. The question is how we determine this.

3.1.4 Entropy Condition on Shocks

We consider the viscous regularization of (3.2) of the form

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon \tag{3.12}$$

with initial datum u_0 of the form of (3.9). We are interested in the case where the limit tends to u_S as $\varepsilon \to 0$. We seek the solution u^ε of the form

$$u^\varepsilon(x,t) = U\left(\frac{x - st}{\varepsilon}\right), \tag{3.13}$$

where s is the shock wave speed $s = (f_\ell - f_r)/(u_\ell - u_r)$ determined by the Rankine–Hugoniot condition. The function $U = U(\xi)$ in (3.13) must satisfy

$$- sU_\xi + (f(U))_\xi = U_{\xi\xi}$$

if u^ε solves (3.12), where $U_\xi = (\mathrm{d}/\mathrm{d}\xi)U(\xi)$. Integrate both sides to get

$$U_\xi = -sU + f(U) + C_0, \tag{3.14}$$

where C_0 is a constant of integration. (We consider this equation assuming that f is C^1, so that its initial value problem admits only one C^1 solution (Proposition 1.1). If U is C^1, then the right-hand side of (3.14) is C^1, so that U is C^2.) We postulate that u_S is the limit of u^ε as $\varepsilon \downarrow 0$; then we should have

$$u_S(x,t) = \lim_{\varepsilon\to 0} U\left((x-st)/\varepsilon\right) = \begin{cases} u_\ell \text{ for } & x < st, \\ u_r \text{ for } & x > st, \end{cases}$$

so

$$\lim_{\xi\to-\infty} U(\xi) = u_\ell, \quad \lim_{\xi\to\infty} U(\xi) = u_r.$$

If we postulate U is monotone, we have

$$\lim_{\xi\to\pm\infty} U_\xi(\xi) = 0$$

since $\lim_{\xi\to\pm\infty} U_\xi$ always exists by (3.14). (The monotonicity follows from the maximum principle for the derivative of u^ε.) Letting $\xi \to \pm\infty$ in (3.14), we obtain

$$C_0 = su_\ell - f_\ell = su_r - f_r.$$

The last equality also gives the Rankine–Hugoniot condition. Thus, we obtain an ordinary differential equation for U with boundary condition at $\pm\infty$ of the form

$$\frac{\mathrm{d}}{\mathrm{d}\xi} U(\xi) = -s\left(U(\xi) - u_\ell\right), + \left(f\left(U(\xi)\right) - f_\ell\right), \tag{3.15}$$

$$U(\infty) = u_r, \quad U(-\infty) = u_\ell.$$

Definition 3.4

If there exists a solution U of (3.15) with $U(\infty) = u_r$, $U(-\infty) = u_\ell$ ($u_r \neq u_\ell$), we say that u_S in (3.10) with $x(t) = st$, $s = (f_\ell - f_r)/(u_\ell - u_r)$ satisfies a *traveling wave entropy condition.*

We shall derive an equivalent condition for u_r and u_ℓ, so that u_S satisfies a traveling wave entropy condition.

Proposition 3.5

Let $f \in C^1(\mathbf{R})$. Assume $u_\ell < u_r$ (resp. $u_r < u_\ell$). Let u_S be of the form of (3.10), with $x = st$, $s = (f_\ell - f_r)/(u_\ell - u_r)$, where $f_\ell = f(u_\ell)$ and $f_r = f(u_r)$. Then u_S fulfills the traveling wave entropy condition if and only

(continued)

Proposition 3.5 (continued)
if the graph of $f(u)$ lies above (resp. below) the straight line segment joining (u_ℓ, f_ℓ) and (u_r, f_r), i.e.,

$$f(u) > f_\ell + s(u - u_\ell) = f_r + s(u - u_r)$$

$$(\text{resp. } f(u) < f_\ell + s(u - u_\ell) = f_r + s(u - u_r))$$

for all $u \in (u_\ell, u_r)$ (resp. $u \in (u_r, u_\ell)$).

Proof. Assume that $u_\ell < u_r$. We first observe that U_ξ does not vanish. Indeed, if there were ξ_0 such that $U_\xi(\xi_0) = 0$, then $a = U(\xi_0)$ should satisfy

$$-s(a - u_\ell) + (f(a) - f_\ell) = 0.$$

Thus, $U \equiv a$ is a solution to (3.15), which is unique by Proposition 1.1. Thus, U must be a constant that cannot achieve at least one of the boundary conditions $U(\infty) = u_r$, $U(-\infty) = u_\ell$. Thus, $U_\xi(\xi) > 0$ for all ξ. This implies

$$f_\ell + s(u - u_\ell) < f(u)$$

for $u \in (u_\ell, u_r)$. Recalling the Rankine–Hugoniot condition, $s = (f_\ell - f_r)/(u_\ell - u_r)$, we observe the desired condition (Fig. 3.7). The converse is easy. The case $u_r < u_\ell$ is parallel. □

If f is convex, this condition is equivalent to saying that $f'(u_r) < s < f'(u_\ell)$ for $u_r < u_\ell$. This is a classical entropy condition for convex f. In the case of concave f

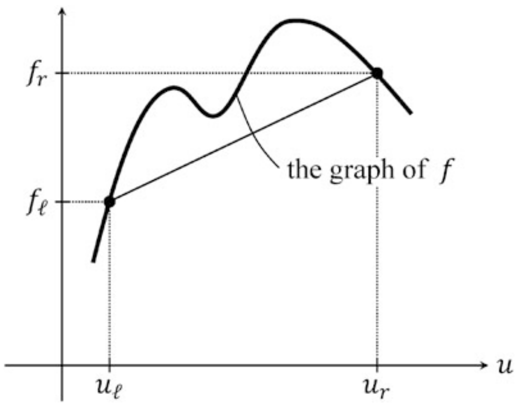


Fig. 3.7 Profile of f on (u_ℓ, u_r)

like the traffic flow problem (3.5), if $u_r < u_\ell$, then u_S does NOT fulfill the traveling wave entropy condition.

In the next section, we discuss the Kružkov entropy solution, which combines such an entropy condition and the notion of a weak solution, so that one can check the entropy condition for a general function whose jump (shock) curves are not regular.

In Proposition 3.5, we discuss an equivalent condition when u_S satisfies the traveling wave entropy solution. One can write this equivalent condition in a synthetic way as

$$s|k - u_\ell| < \operatorname{sgn}(k - u_\ell)\,(f(k) - f(u_\ell))$$

for all k between u_ℓ and u_r. Here, sgn denotes the *sign function* defined by

$$\operatorname{sgn} x = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

(Of course, one may replace u_ℓ with u_r in the above inequality.) One may write this another way to express the condition similarly. Let $a = a(x,t)$ be a function defined in $D = J \times (t_0, t_1)$, where J is an open interval in $\mathbf{R}$. Let $[\![a]\!]$ denote the difference between the limit from D_r and D_ℓ, i.e.,

$$\begin{aligned} [\![a]\!](x,t) &:= a_r(x,t) - a_\ell(x,t), \quad (x,t) \in \Gamma = \overline{D_r} \cap \overline{D_\ell}, \\ a_r(x,t) &= \lim \left\{ a(y,s) \;\middle|\; (y,s) \to (x,t),\ (y,s) \in D_r \right\}, \\ a_\ell(x,t) &= \lim \left\{ a(y,s) \;\middle|\; (y,s) \to (x,t),\ (y,s) \in D_\ell \right\}. \end{aligned}$$

Proposition 3.6
Let $f \in C^1(\mathbf{R})$. Consider the Riemann problem. The function u_S in (3.10) satisfies the traveling wave entropy condition if and only if

$$s[\![|u-k|]\!] \ge [\![\operatorname{sgn}(u-k)\,(f(u) - f(k))]\!] \text{ for all } k \in (u_\ell, u_r) \text{ if } u_\ell < u_r$$
$$(\text{resp. } k \in (u_r, u_\ell) \text{ if } u_r < u_\ell) \text{ and}$$
$$s[\![|u-k|]\!] = [\![\operatorname{sgn}(u-k)\,(f(u) - f(k))]\!] \text{ for all } k \notin (u_\ell, u_r) \text{ if } u_\ell < u_r$$
$$(\text{resp. } k \notin (u_r, u_\ell) \text{ if } u_r < u_\ell)$$

for $u = u_S$, where $x(t) = st$ with $s = (f_\ell - f_r)/(u_\ell - u_r)$, $f_\ell = f(u_\ell)$, $f_r = f(u_r)$.

Proof. We only give a proof when $u_\ell < u_r$ since the proof for $u_\ell > u_r$ is symmetric.

"If" part. Choosing k between u_ℓ and u_r, we obtain

$$s\left(-(u_\ell - k) - (u_r - k)\right) < -(f_r - f(k)) - (f_\ell - f(k))$$

or

$$\overline{f} + s(k - \overline{u}) < f(k).$$

Here, $\overline{f} = (f_r + f_\ell)/2, \overline{u} = (u_r + u_\ell)/2$. This implies that the graph of $f(u)$ must lie above the straight line segment between (u_ℓ, f_ℓ) and (u_r, f_r). Proposition 3.5 now implies that u_S satisfies the traveling wave entropy condition.

"Only if" part. Since the Rankine–Hugoniot condition holds,

$$s[\![|u-k|]\!] = [\![\operatorname{sgn}(u-k)\,(f(u) - f(k))]\!]$$

for any constant k not between u_ℓ and u_r. For constants k between u_ℓ and u_r, if the traveling wave entropy condition holds, then, by Proposition 3.5, we have

$$\begin{aligned} f(k) &> s(k - u_\ell) + f(u_\ell) \quad \text{and} \\ f(k) &> s(k - u_\ell) + f(u_r), \end{aligned}$$

so that

$$f(k) - sk > \overline{f} - s\overline{u}.$$

Then we obtain

$$s[\![|u-k|]\!] > [\![\operatorname{sgn}(u-k)\,(f(u) - f(k))]\!].$$

□

▶ **Remark 3.7** This proposition says that for the Riemann problem, the solution satisfying the traveling entropy condition is exactly the Kružkov entropy solution defined later.

3.2 Uniqueness of Entropy Solutions

We first derive two equivalent definitions of an entropy solution. One is based on what we call an entropy pair, and the other is its modification due to Kružkov. The first condition is easily motivated by a vanishing viscosity approximation. We derive this condition by a formal argument. Then we introduce Kružkov's entropy condition and discuss the equivalence of both definitions. We conclude this section by proving the uniqueness of an entropy solution. The key idea is a kind of doubling variable argument.

3.2.1 Vanishing Viscosity Approximations and Entropy Pairs

We consider the initial value problem of a scalar conservation law of the form

$$u_t + f(u)_x = 0 \qquad \text{in} \quad Q = \mathbf{T} \times (0, T), \tag{3.16}$$

$$u|_{t=0} = u_0 \qquad \text{on} \quad \mathbf{T}. \tag{3.17}$$

Here $u = u(x, t)$ is a real-valued function on Q. In other words, to simplify the presentation, u is periodic in x. The flux function f is always assumed to be at least a locally Lipschitz (real-valued) function.

To obtain a solution, we consider a parabolic approximation

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon, \tag{3.18}$$

$$u^\varepsilon\big|_{t=0} = u_0 \tag{3.19}$$

for $\varepsilon > 0$. We expect that a reasonable solution will be obtained as a limit of the solution of (3.18), (3.19) as $\varepsilon \to 0$. Since ε looks like a viscosity coefficient in fluid dynamics, this approximation is often called a vanishing viscosity approximation.

It is well known that (3.18) and (3.19) admit a global solution u^ε that is smooth for $t > 0$ for any given $u_0 \in L^\infty(\mathbf{T})$ provided that f is smooth. For a moment we assume that f is smooth, so that u^ε is smooth for $t > 0$ (the initial condition should be understood in a weak sense); see, for example, standard monographs on parabolic equations [70, 72]. We take a real-valued smooth function η defined on $\mathbf{R}$, and consider a composite function $\eta(u^\varepsilon) = \eta \circ u^\varepsilon$. Since u^ε satisfies (3.18), $\eta(u^\varepsilon)$ must solve

$$\eta(u^\varepsilon)_t + \eta'(u^\varepsilon) f'(u^\varepsilon) u_x^\varepsilon = \varepsilon \eta'(u^\varepsilon) u_{xx}^\varepsilon. \tag{3.20}$$

Since $\eta(u^\varepsilon)_{xx} = \eta'(u^\varepsilon)u_{xx}^\varepsilon + \eta''(u^\varepsilon)(u_x^\varepsilon)^2$, we observe that $\eta(u^\varepsilon)_{xx} \geq \eta'(u^\varepsilon)u_{xx}^\varepsilon$ provided that η is *convex*.

Assume that η is now convex, and take a function q such that $q' = \eta' f'$. Then (3.20) yields

$$\eta(u^\varepsilon)_t + q(u^\varepsilon)_x = \varepsilon \eta'(u^\varepsilon) u_{xx}^\varepsilon \leq \varepsilon \eta(u^\varepsilon)_{xx}. \tag{3.21}$$

We multiply (3.21) by a nonnegative function $\varphi \in C_c^\infty(Q_0)$ on $Q_0 = \mathbf{T} \times [0, T)$ and integrate by parts to get

$$\int_Q \left\{\varphi_t \eta(u^\varepsilon) + \varphi_x q(u^\varepsilon)\right\} \mathrm{d}x\mathrm{d}t + \int_{\mathbf{T}} \varphi|_{t=0}\, \eta(u_0)\mathrm{d}x \geq -\varepsilon \int_Q \varphi_{xx} \eta(u^\varepsilon)\mathrm{d}x\mathrm{d}t.$$

Here, we present a formal argument. The following argument can be justified if, for example, $\sup_Q |u^\varepsilon|$ is bounded in ε and if u^ε tends to u almost everywhere (a.e.) in Q as $\varepsilon \downarrow 0$. Sending ε to zero we get

$$\int_Q \{\varphi_t \eta(u) + \varphi_x q(u)\}\, \mathrm{d}x\mathrm{d}t + \int_{\mathbf{T}} \varphi|_{t=0}\, \eta(u_0)\mathrm{d}x \geq 0 \tag{3.22}$$

for any $\varphi \in C_c^\infty(Q_0)$ with $\varphi \geq 0$. In Q this condition implies

$$\eta(u)_t + q(u)_x \leq 0 \tag{3.23}$$

in a distribution sense, which means $-\eta(u)_t - q(u)_x$ is a nonnegative Radon measure in Q.

This argument can be extended when η is merely convex by an approximation for (3.22). (Incidentally, the inequality for taking $\eta(n)_{xx} \geq \eta'(u)u_{xx}$ for $\eta(u) = |u|$ is known as the Kato inequality $\Delta|w| \geq (\operatorname{sgn} w)\Delta w$ in a distribution sense. This inequality is also obtained by approximately $|u|$, by, for example, $\sqrt{|u|^2 + \delta}$, $\delta > 0$. See Exercise 3.8.)

Inequality (3.23) is trivially fulfilled if u solves (3.16) and u is *smooth*. However, it will turn out that this inequality distinguishes admissible jumps and nonadmissible jumps when u is discontinuous. We thus reach the following definition.

Definition 3.8

Let f be a locally Lipschitz function on $\mathbf{R}$.

(1) A pair of functions (η, q) is an *entropy pair* for (3.16) if η is convex and q is a primitive (antiderivative) of $\eta' f'$, i.e., $q' = \eta' f'$.

(2) Let $u \in L^\infty(Q)$ be a weak solution of (3.16), (3.17) with initial datum $u_0 \in L^\infty(\mathbf{T})$. Let (η, q) be an entropy pair for (3.16). We say that u is an *entropy solution* of (3.16), (3.17) if u satisfies (3.22) for all $\varphi \in C_c^\infty(Q_0)$, with $\varphi \geq 0$, where $Q_0 = \mathbf{T} \times [0, T)$.

3.2.2 Equivalent Definition of Entropy Solution

For a convex function η on $\mathbf{R}$, (η, q) is an entropy pair (for (3.16)) if we set

$$q(w) = \int_k^w \eta'(\tau) f'(\tau)\mathrm{d}\tau.$$

The function q is uniquely determined by η up to an additive constant. If we take $\eta(w) = |w - k|$ for $k \in \mathbf{R}$, then we have

$$q(w) = \operatorname{sgn}(w - k)\,(f(w) - f(k))\,.$$

It is clear that if u is an entropy solution, then it must satisfy

$$\int_Q \{\varphi_t |u-k| + \varphi_x (\mathrm{sgn}(u-k)(f(u)-f(k)))\} \,\mathrm{d}x\mathrm{d}t + \int_{\mathbf{T}} \varphi|_{t=0} |u_0 - k| \mathrm{d}x \geq 0 \tag{3.24}$$

for all $k \in \mathbf{R}$ and all $\varphi \in C_c^\infty(Q_0)$, with $\varphi \geq 0$. This condition is often called the *Kružkov entropy condition*. It is equivalent to the definition of an entropy solution.

Proposition 3.9
Let f be a locally Lipschitz function. Let $u \in L^\infty(Q)$ be a weak solution of (3.16), (3.17) with initial datum $u_0 \in L^\infty(\mathbf{T})$. Then u is an entropy solution if and only if u satisfies the Kružkov entropy condition, i.e., (3.24) for all $k \in \mathbf{R}$ and for all $\varphi \in C_c^\infty(Q_0)$, with $\varphi \geq 0$.

Proof. Since the "only if" part is trivial, we shall prove the "if" part. For η we set a linear functional

$$\Lambda(\eta) = \int_Q \{\varphi_t \eta(u) + \varphi_x q(u)\} \,\mathrm{d}x\mathrm{d}t + \int_{\mathbf{T}} \varphi|_{t=0}\, \eta(u_0)\mathrm{d}x$$

for a fixed $\varphi \in C_c^\infty(Q_0)$ and u_0. This quantity $\Lambda(\eta)$ is determined by η and is independent of the choice of q provided that (η, q) is an entropy pair. The Kružkov entropy condition (3.24) implies

$$\Lambda(\eta_i) \geq 0$$

for all $\eta_i(w) = \alpha_i |w - k_i|$, $k_i \in \mathbf{R}$, $\alpha_i \geq 0$, $i = 1, \cdots, m$. Thus,

$$\Lambda\left(\sum_{i=1}^m \eta_i\right) = \sum_{i=1}^m \Lambda(\eta_i) \geq 0$$

since $(\sum_{i=1}^m \eta_i, \sum_{i=1}^m q_i)$ is an entropy pair if (η_i, q_i) is an entropy pair. Since u is a weak solution, we see that $\Lambda(\eta) = 0$ if $\eta(w) = \alpha w + \beta$, $\alpha, \beta \in \mathbf{R}$. Thus, the convex piecewise linear function η of the form

$$\eta(w) = \alpha w + \beta + \sum_{i=1}^m \eta_i(w) \tag{3.25}$$

satisfies $\Lambda(\eta) \geq 0$.

As stated at the end of this subsection (Lemma 3.10), we notice that any piecewise linear convex function is of the form (3.25) provided that there is only a finite number of nondifferentiable points. We thus conclude that $\Lambda(\eta) \geq 0$ for any piecewise linear convex function η. Since a convex function η is approximable (Exercise 3.5) by a piecewise linear convex function $\{\zeta_j\}_{j=1}^{\infty}$ (having finitely many nondifferentiable points) locally uniformly in $\mathbf{R}$, we conclude that

$$\Lambda(\eta) = \lim_{j\to\infty} \Lambda(\zeta_j) \geq 0$$

since u is bounded. □

Lemma 3.10
Let η be a piecewise linear convex function in $\mathbf{R}$ with m nondifferentiable points. Then there are $\alpha_i \geq 0$, α_i, β_i, $k_i \in \mathbf{R}$ for $1 \leq i \leq m$ such that

$$\eta(w) = \alpha w + \beta + \sum_{i=1}^{m} \eta_i(w), \qquad \eta_i(w) = \alpha_i |w - k_i| + \beta_i.$$

Proof. This can be easily proved by induction of numbers m of nondifferentiable points of a piecewise linear convex function ξ_m. If $m = 0$, it is trivial. Let $\{k_i\}_{i=1}^{m}$ be the set of all nondifferentiable points of ξ_m. We may assume that $k_1 < k_2 < \cdots < k_m$. Assume that $m \geq 1$. Taking α, β, and α_1 in a suitable way, we see that

$$\xi_m(w) = \alpha w + \beta + \eta_1(w) \quad \text{for} \quad -\infty < w < k_2,$$

where k_2 is the second smallest nondifferentiable point of ξ_m; $k_2 = \infty$ if there is no such point (Fig. 3.8).

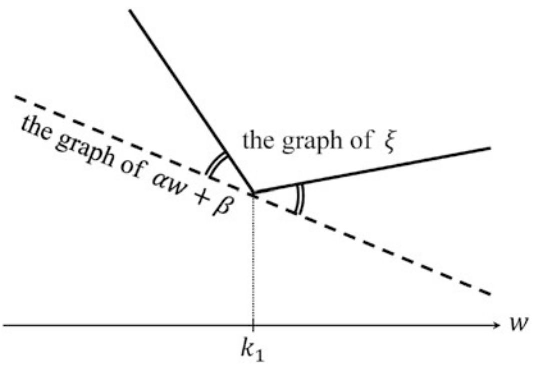


Fig. 3.8 Profile of graphs

We set

$$\xi(w) = \alpha w + \beta + \eta_1(w) \quad \text{for} \quad w \in \mathbf{R}.$$

Since ξ_m is convex and ξ is linear for $s > k_1$, $\xi_m - \xi$ is still convex and nonnegative and $\xi_m - \xi = 0$ on $(-\infty, k_2)$. Moreover, the number of nondifferentiable points of $\xi_m - \xi$ is $m - 1$, so by induction we conclude that ξ_m is of the form of (3.25). □

3.2.3 Uniqueness

We are now in a position to state our main uniqueness result as an application of the L^1-contraction property.

Theorem 3.11
Assume that f is locally Lipschitz. Let u and $v(\in L^\infty(Q))$ be an entropy solution of (3.16), (3.17) *with initial datum u_0 and v_0, respectively. Assume that $u(\cdot, t) \to u_0$ and $v(\ , t) \to v_0$ as $t \to 0$ in the sense of L^1-convergence. Then*

$$\|u - v\|_{L^1(\mathbf{T})}(t) \leq \|u_0 - v_0\|_{L^1(\mathbf{T})}. \tag{3.26}$$

In particular, a bounded entropy solution of (3.16), (3.17) *is unique. (The assumption of L^1-continuity as $t \downarrow 0$ is unnecessary but we assume it to simplify the proof.)*

Proof. We double the variables of a test function φ. Let $\phi = \phi(x, t, y, s)$ be a nonnegative function such that $\phi \in C_c^\infty(Q_0 \times Q_0)$. Since u is an entropy solution of (3.16), (3.17), the Kružkov entropy condition implies

$$\int_Q \{|u - k|\phi_t + q(u, k)\phi_x\} \, \mathrm{d}x\mathrm{d}t + \int_{\mathbf{T}} \phi(x, 0, y, s)|u_0 - k| \mathrm{d}x \geq 0$$

when $q(u, k) = \operatorname{sgn}(u - k)\,(f(u) - f(k))$. Plugging in $k = v(y, s)$ and integrating in (y, s), we get

$$\int_Q \int_Q \{|u(x, t) - v(y, s)|\, \phi_t + q\,(u(x, t), v(y, s))\, \phi_x\} \, \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s$$
$$+ \int_Q \int_{\mathbf{T}} \phi(x, 0, y, s)\, |u_0(x) - v(y, s)| \, \mathrm{d}x\mathrm{d}y\mathrm{d}s \geq 0.$$

The same inequality holds for v; in other words, we have

$$\int_Q\int_Q \left\{|u(x,t)-v(y,s)|\,\phi_s + q\,(u(x,t),v(y,s))\,\phi_y\right\} \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s + \int_{\mathbf{T}}\int_Q \phi(x,0,y,0)\,|v_0(y)-u(x,t)|\,\mathrm{d}x\mathrm{d}t\mathrm{d}y \geq 0.$$

Adding these two inequalities yields

$$\int_Q\int_Q \left\{|u(x,t)-v(y,s)|\,(\phi_t+\phi_s) + q(u,v)(\phi_x+\phi_y)\right\} \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s + \int_Q\int_{\mathbf{T}} |u_0(x)-v(y,s)|\,\phi(x,0,y,s)\,\mathrm{d}x\mathrm{d}y\mathrm{d}s + \int_{\mathbf{T}}\int_Q |u(x,t)-v_0(y)|\,\phi(x,t,y,0)\,\mathrm{d}x\mathrm{d}t\mathrm{d}y \geq 0. \tag{3.27}$$

Our strategy is as follows. We set

$$J_1 := \int_Q\int_Q \left\{|u(x,t)-v(y,s)|\,(\phi_t+\phi_s) + q(u,v)(\phi_x+\phi_y)\right\} \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s$$
$$J_2 := \int_Q\int_{\mathbf{T}} |u_0(x)-v(y,s)|\,\phi(x,0,y,s)\,\mathrm{d}x\mathrm{d}y\mathrm{d}s$$
$$J_3 := \int_{\mathbf{T}}\int_Q |u(x,t)-v_0(y)|\,\phi(x,t,y,0)\,\mathrm{d}x\mathrm{d}t\mathrm{d}y.$$

For a given $t_0 \in (0,T)$, we would like to take a suitable ϕ so that J_i equals I_i $(i=1,2,3)$, with

$$I_1 := -\int_{\mathbf{T}} |u(x,t_0)-v(x,t_0)|\,dx$$
$$I_2 := \frac{1}{2}\int_{\mathbf{T}} |u_0(x)-v_0(x)|\,dx, \quad I_3 := I_2.$$

Since (3.27) says $J_1+J_2+J_3 \geq 0$, we have

$$I_1+I_2+I_3 \geq 0, \quad \text{i.e.,}$$

$$-\int_{\mathbf{T}} |u(x,t_0)-v(x,t_0)|\,\mathrm{d}x + \frac{1}{2}\int_{\mathbf{T}} |u_0(x)-v_0(x)|\,\mathrm{d}x + \frac{1}{2}\int_{\mathbf{T}} |u_0(x)-v_0(x)|\,\mathrm{d}x \geq 0. \tag{3.28}$$

This is simply the desired contraction property (3.26). Unfortunately, there is no good function ϕ. We need a sequence $\phi = \phi_{\varepsilon,\varepsilon',\varepsilon''}$ depending on three parameters $\varepsilon, \varepsilon', \varepsilon'' > 0$. Let ρ_ε be a Friedrichs' mollifier ρ_ε defined in §2.2 (see also Lemma 3.12 in what follows). We further assume symmetry, i.e., $\rho_\varepsilon(-\sigma) = \rho_\varepsilon(\sigma)$ for all $\sigma \in \mathbf{R}$. We set

$$\phi(x,t,y,s) = \rho_\varepsilon(x-y)\rho_{\varepsilon'}(t-s)\chi_{\varepsilon''}\left(\frac{t+s}{2} - t_0\right),$$

with $\chi_{\varepsilon''}(\tau) = \int_\tau^\infty \rho_{\varepsilon''}(\sigma)\mathrm{d}\sigma$. (We shall give a heuristic explanation as to why this choice is good after the proof.) It suffices to prove

$$\lim_{\varepsilon''\downarrow 0}\left(\lim_{\varepsilon'\downarrow 0}\left(\lim_{\varepsilon\downarrow 0} J_i(\varepsilon,\varepsilon',\varepsilon'')\right)\right) = I_i \quad \text{for} \quad i = 1,2,3,$$

to get (3.28) since $J_1 + J_2 + J_3 \geq 0$.

Since $\phi_x + \phi_y = 0$ and

$$\phi_t + \phi_s = \rho_\varepsilon(x-y)\rho_{\varepsilon'}(t-s)\chi'_{\varepsilon''}\left(\frac{t+s}{2} - t_0\right),$$

we observe that

$$\begin{aligned} J_1 &= J_1(\varepsilon,\varepsilon',\varepsilon'') \\ &= -\int_Q\int_Q |u(x,t) - v(y,s)|\,\rho_\varepsilon(x-y)\rho_{\varepsilon'}(t-s)\rho_{\varepsilon''}\left(\frac{t+s}{2} - t_0\right)\mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s. \end{aligned}$$

We apply the approximation lemma (Lemma 3.12 below) to conclude that

$$\lim_{\varepsilon''\downarrow 0}\left(\lim_{\varepsilon'\downarrow 0}\left(\lim_{\varepsilon\downarrow 0} J_1\right)\right) = I_1.$$

Similarly,

$$\lim_{\varepsilon\downarrow 0} J_2 = \int_0^T\int_{\mathbf{T}} |u_0(x) - v(x,s)|\,\rho_{\varepsilon'}(s)\chi_{\varepsilon''}\left(\frac{s}{2} - t_0\right)\mathrm{d}x\mathrm{d}s.$$

For a given $t_0 \in (0,T)$, we take $\varepsilon'' > 0$ small, say, $\varepsilon'' < \varepsilon''_0$, for some $\varepsilon''_0 > 0$, so that $\chi_{\varepsilon''}\left(\frac{s}{2} - t_0\right) = 1$ for all $s \in [0, t_0/2]$, $\varepsilon'' < \varepsilon''_0$. We take $\varepsilon' > 0$ sufficiently small so that supp $\rho_{\varepsilon'} \subset [0, t_0/2]$ to get

$$\lim_{\varepsilon\downarrow 0} J_2 = \int_0^T \left\{\int_{\mathbf{T}} |u_0(x) - v(x,s)|\,\mathrm{d}x\right\}\rho_{\varepsilon'}(s)\,\mathrm{d}s.$$

Since we have assumed that $v(\cdot, t) \to v_0$ in $L^1(\mathbf{T})$,

$$h(s) = \int_{\mathbf{T}} |u_0(x) - v(x, s)|\, \mathrm{d}x = \|u_0 - v(\cdot, s)\|_{L^1(\mathbf{T})}$$

is continuous at $s = 0$. We now apply Lemma 3.12 (ii) to conclude that

$$\lim_{\varepsilon' \downarrow 0} \left(\lim_{\varepsilon \downarrow 0} J_2 \right) = \frac{1}{2} h(0) = I_1$$

for $\varepsilon'' < \varepsilon_0''$. The proof for J_3 is the same. We now conclude that

$$\lim_{\varepsilon'' \downarrow 0} \left(\lim_{\varepsilon' \downarrow 0} \left(\lim_{\varepsilon \downarrow 0} J_i \right) \right) = I_i$$

so that $J_1 + J_2 + J_3 \geq 0$ implies (3.28). The proof is now complete. □

Let us say a few words about why we choose ϕ as earlier. It is convenient to use what is called a delta function δ. It is defined as a distributional derivative of a Heaviside function $1_{>0}$, i.e.,

$$\delta = D_x 1_{>0},$$

where $1_{>0}(x) = 1$ if $x > 0$ and $1_{>0}(x) = 0$ if $x \leq 0$. In other words,

$$\delta(\varphi) = - \int_{\mathbf{R}} \frac{\mathrm{d}\varphi}{\mathrm{d}x} 1_{>0}\, \mathrm{d}x \quad \text{for} \quad \varphi \in C_c^\infty(\mathbf{R}).$$

By definition, $\delta(\varphi) = -\int_0^\infty \frac{\mathrm{d}\varphi}{\mathrm{d}x} \mathrm{d}x = \varphi(0)$. We often write $\delta(\varphi)$ by $\int_{\mathbf{R}} \delta(x)\varphi(x)dx$, though δ cannot be identified with any integrable function. We would like to take

$$\phi(x, t, y, s) = \delta(x - y)\delta(t - s) 1_{>0} \left(t_0 - \frac{t + s}{2} \right).$$

Since $\phi_x + \phi_y = 0$, $\phi_t + \phi_s = -\delta(x - y)\delta(t - s)\delta(t - t_0)$, we see that

$$J_1 = - \int_{\mathbf{R}} |u(x, t_0) - v(x, t_0)|\, \mathrm{d}x = I_1.$$

Since u and v are not necessarily continuous, we must approximate δ by mollifiers. For $J_2 + J_3$, we have

$$J_1 + J_2 = \int_0^{t_0} \left\{ \int_{\mathbf{T}} |u_0(x) - v(x, s)|\, \mathrm{d}x \right\} \delta(-s) \mathrm{d}s$$

$$+ \int_0^{t_0} \left\{ \int_{\mathbf{T}} |v_0(x) - u(x,s)|\, \mathrm{d}x \right\} \delta(t)\, \mathrm{d}t$$

$$= \int_{-t_0}^{t_0} k(t)\delta(t)\, \mathrm{d}t = k(0),$$

with

$$k(t) = \begin{cases} \displaystyle\int_{\mathbf{T}} |v_0(x) - u(x,t)|\, \mathrm{d}x & \text{for } t > 0, \\ \displaystyle\int_T |u_0(x) - v(x,-t)|\, \mathrm{d}x & \text{for } t \le 0. \end{cases}$$

Since k is continuous at $t = 0$ and $k(0) = \|u_0 - v_0\|_{L^1(\mathbf{T})}$, we observe that

$$I_1 + \|u_0 - v_0\|_{L^1(\mathbf{T})} \ge 0.$$

In our proof, we discuss J_1 and J_2 separately, so we use symmetry to simplify the argument.

Lemma 3.12
Let ρ_ε be a Friedrichs' mollifier on $\mathbf{R}$ defined in Sect. 2.2. In other words, $\rho_\varepsilon(\sigma) = \varepsilon^{-1}\rho(\sigma/\varepsilon)$, where $\rho \in C_c^\infty(\mathbf{R})$ satisfies $\rho \ge 0$ and $\int_{\mathbf{R}} \rho dx = 1$.

(i) *Let $h \in L^\infty(\mathbf{T}^2)$ and $h(x, x - z) \to h(x,x)$ as $|z| \to 0$ for a.e. x. Then*

$$\lim_{\varepsilon \downarrow 0} \int_{\mathbf{T}} \int_{\mathbf{T}} h(x,y)\rho_\varepsilon(x-y)\, \mathrm{d}x\mathrm{d}y = \int_{\mathbf{T}} h(x,x)\, \mathrm{d}x.$$

Let $h \in L^\infty(\mathbf{R})$ be compactly supported. Assume that $h(x, x - z) \to h(x,x)$ as $|z| \to 0$ for a.e. x. Then

$$\lim_{\varepsilon \downarrow 0} \int_{\mathbf{R}} \int_{\mathbf{R}} h(x,y)\rho_\varepsilon(x-y)\, \mathrm{d}x\mathrm{d}y = \int_{\mathbf{R}} h(x,x)\, \mathrm{d}x.$$

(ii) *Assume further that $\rho(-\sigma) = \rho(\sigma)$ for $\sigma \in \mathbf{R}$. For $h \in L^\infty(0,T)$,*

$$\lim_{\varepsilon \downarrow 0} \int_0^T h(s)\rho_\varepsilon(s)\mathrm{d}s = \frac{1}{2} h(0)$$

provided that h is continuous at $s = 0$.

Proof of Lemma 3.12

(i) We may assume $\operatorname{supp} \rho \subset (-1, 1)$ by replacing σ with σ/ε' for small $\varepsilon' > 0$. We recall $\mathbf{T} = \mathbf{R}/\omega_1\mathbf{Z}$. We take $\varepsilon < \omega_1/4$, so that $\operatorname{supp} \rho \subset \left(-\frac{\omega_1}{4\varepsilon}, \frac{\omega_1}{4\varepsilon}\right)$. By this choice, the support of $\rho_\varepsilon(x-y)$ as a function of x, y is contained in a periodic cell $C = [-\omega_1/2, \omega_1/2) \times [-\omega_1/2, \omega_1/2)$ of $\mathbf{T}^2 = (\mathbf{R}/\omega_1\mathbf{Z})^2$. In particular,

$$\int_{-\omega_1/2}^{\omega_1/2} \rho_\varepsilon(x-y)\, dy = 1 \quad \text{for} \quad x \in (-\omega_1/2, \omega_1/2)\,.$$

We proceed with

$$\begin{aligned} I(\varepsilon) &:= \int_{\mathbf{T}}\int_{\mathbf{T}} h(x,y)\rho_\varepsilon(x-y)\,\mathrm{d}x\mathrm{d}y - \int_{\mathbf{T}} h(x,x)\,\mathrm{d}x \\ &= \iint_C (h(x,y) - h(x,x))\,\rho_\varepsilon(x-y)\,\mathrm{d}x\mathrm{d}y. \end{aligned}$$

Changing the variables of integration from (x, y) to (x, z) with $z = (x-y)/\varepsilon$, we obtain, by Fubini's theorem, that

$$\begin{aligned} |I(\varepsilon)| &\le \iint_C |h(x,y) - h(x,x)|\,\rho_\varepsilon(x-y)\,\mathrm{d}x\mathrm{d}y \\ &= \int_{|x|\le\omega_1/2} \left\{ \int_{|x-\varepsilon z|\le\omega_1/2} |h(x, x-\varepsilon z) - h(x,x)|\,\rho(z)\,\mathrm{d}x \right\} \mathrm{d}z \\ &\le \int_{|x|\le\omega_1/2} \left\{ \int_{|z|\le 1} |h(x, x-\varepsilon z) - h(x,x)|\,\rho(z)\,\mathrm{d}x \right\} \mathrm{d}z. \end{aligned}$$

Since the integrand is bounded by $2\|h\|_\infty$ (independent of $\varepsilon > 0$) and $h(x, x-\varepsilon z) \to h(x,x)$ for a.e. $x, z \in \mathbf{R}$ as $\varepsilon \downarrow 0$, we conclude that $I(\varepsilon) \to 0$ as $\varepsilon \to 0$ by the dominated convergence theorem. We thus obtained the first statement. The proof for the second statement is parallel.

(ii) By a change of the variable of integration, we see that

$$\int_0^T h(s)\rho_\varepsilon(s)\mathrm{d}s = \int_{0\le z\le 1} h(\varepsilon z)\rho(z)\mathrm{d}z$$

for sufficiently small $\varepsilon > 0$. Since $h(z) \to h(0)$ as $z \to 0$, we now obtain

$$\lim_{\varepsilon\downarrow 0} \int_0^T h(s)\rho_\varepsilon(s)\mathrm{d}s = \lim_{\varepsilon\downarrow 0} \int_{0\le z\le 1} h(\varepsilon z)\rho(z)\mathrm{d}z = h(0)\int_{0\le z\le 1} \rho(z)\mathrm{d}z$$

by the bounded convergence theorem. The result follows if we note that $\int_0^\infty \rho(z)\mathrm{d}z = 1/2$ by symmetry. □

Remark 3.13

(i) The definition of the entropy solution extends to a bounded function in $\mathbf{R}$ not necessarily periodic. Although the uniqueness result still holds provided that u_0 is in $L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$, the proof is more involved. For example, we must take ϕ to have compact support in the space direction; See, for example, [53] and [68].

(ii) All results here can be extended to multidimensional space. The conservation law for a real-valued function $u = u(x,t)$ is of the form

$$u_t + \operatorname{div} f(u) = 0 \quad \text{in} \quad \mathbf{T}^N \times (0,T) = \tilde{Q},$$

with $f(u) = \left(f^1(u), \ldots, f^N(u)\right)$. A pair of real-valued functions (η, q) defined on $\mathbf{R}$ is said to be an *entropy pair* for this equation if it satisfies $q' = \eta' f'$ $(i = 1, \ldots, N)$, $q = (q^1, \ldots, q^N)$ and η is a convex function. A function $u \in L^\infty(\tilde{Q})$ is said to be an entropy solution with initial datum $u_0 \in L^\infty(\mathbf{T}^N)$ if

$$\int_0^T\!\!\int_{\tilde{Q}} \left(\varphi_t \eta(u) + \sum_{i=1}^N q^i(u)\varphi_{x_i}\right) \mathrm{d}x\mathrm{d}t + \int_{\mathbf{T}^N} \varphi|_{t=0}\, \eta(u_0)\,\mathrm{d}x \geq 0$$

holds for all $\varphi \in C_c^\infty(\tilde{Q}_0)$, with $\varphi \geq 0$, and all entropy pairs. Here $\tilde{Q}_0 = \mathbf{T}^N \times [0,T)$ and $\varphi_{x_i} = \frac{\partial \varphi}{\partial x_i}$. The Kružkov entropy condition is of the form

$$\int_0^T\!\!\int_{\tilde{Q}} \left(\varphi_t |u-k| + \sum_{i=1}^N \operatorname{sgn}(u-k)\left(f^i(u) - f^i(k)\right)\varphi_{x_i}\right)\mathrm{d}x\mathrm{d}t + \int_{\mathbf{T}^N} \varphi|_{t=0}\, |u-k|\,\mathrm{d}x \geq 0$$

for all $\varphi \in C_c^\infty(\tilde{Q}_0)$ with $\varphi \geq 0$ and $k \in \mathbf{R}$; see, for example, [68].

3.3 Notes and Comments

Most of the contents in this chapter are taken from Holden and Risebro's book [53], where $\mathbf{T}$ is replaced by $\mathbf{R}$. The theory of conservation laws has a long history. A weak formulation for the Burgers equation traces back to Hopf [54], where a parabolic approximation was studied. The literature on the topic has grown considerably since then. The reader is referred to the book [53].

There are several ways to construct an entropy solution, for example, [53]. Of course, parabolic approximation is one way. Other methods are based on the finite difference method. A front tracking method was studied extensively by Holden and Risebro [53]; it approximates f by a piecewise function; this seems to be very effective even for systems of conservation laws. A completely different approach, called a kinetic construction (not contained in [53]), traces back to Brenier [14], as well as the second author and others [49], [50]. The idea involves introducing an extra variable, which may be interpreted as a microscopic variable. All the aforementioned methods work for scalar conservation laws in multidimensional spaces. Note that there is a very accessible introduction to conservation laws in the book [36, Chapter 11]. In [36], systems of conservation laws are discussed.

If one considers systems of conservation laws, the uniqueness of entropy solutions is difficult because there are interactions of waves. Neverthless, there are now several uniqueness results that go back to Bressan's seminal works [16], [17], where the main assumption is that the spatial total variation of a solution is small. The reader is referred to [17] or [53] for this topic.

3.4 Exercises

3.1 (Hopf–Cole transformation) Let u be a solution of the (viscous) Burgers equation $u_t + (u^2/2)_x = u_{xx}$. Let $w(x,t)$ be defined as

$$w(x,t) = \int_0^x u(y,t)\mathrm{d}y + \int_0^t \left(u_x(0,\tau) - u(0,t)^2/2\right) \mathrm{d}\tau.$$

Show that w satisfies

$$w_t + (w_x)^2/2 = w_{xx}$$

in $\mathbf{R} \times (0,\infty)$. Show that $v = \exp(-w/2)$ solves the heat equation $v_t = v_{xx}$.

3.2 Let u be a solution of $u_t + (u^2/2)_x = u_{xx}$ in $\mathbf{R} \times (0,\infty)$. Set $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$ for $\lambda > 0$. Show that u_λ solves the same equation as u. Set $v_\varepsilon(x,t) = v(\varepsilon x, \varepsilon t)$. Show that v_ε solves $v_t + (v^2/2)_x = \varepsilon^{-1} v_{xx}$ for $\varepsilon > 0$.

3.3 Consider (3.2), with $f(u) = u^2/\left(u^2 + (1-u)^2\right)$. Find the entropy solution to the Riemann problem with initial datum (3.9), where $u_\ell = 0$, $u_r = 1$. In this case, the equation is called the Buckley–Leverett equation. It is a simple model of two-phase fluid flow in a porous medium. The unknown u represents a ratio of saturation of one of the phases. It varies from zero to one. Note that f is neither convex nor concave. The expected solution has a rarefaction and shock simultaneously. Note that there is a numerical method based on the level-set approach [88] discussed in Sect. 4.5.2.

3.4 Consider an equation for $u = u(x,t)$ in $\mathbf{R} \times (0,\infty)$ of the form

$$u_t + (u^2/2)_x = -u$$

with initial datum u_0 in (3.9). Find the entropy solution when $u_\ell = 1$, $u_r = 0$. Consider the case where $u_\ell = 0$, $u_r = 1$. Find the entropy solution in this case.

3.5 Let ξ be a real-valued convex function on $\mathbf{R}$. Prove that there exists a sequence of piecewise linear convex functions $\{\eta_j\}_{j=1}^{\infty}$ such that
 (i) η_j converges to ξ locally uniformly in $\mathbf{R}$ as $j \to \infty$ and
 (ii) η_j has at most finitely many nondifferentiable points.

3.6 Let f be a strictly convex C^1 function in the sense that $f' \in C(\mathbf{R})$ is (strictly) increasing. We set

$$u_R(x,t) = \begin{cases} u_\ell, & x < f'(u_\ell)t \\ (f')^{-1}(x/t), & f'(u_\ell)t \le x < f'(u_r)t \\ u_r, & x \ge f'(u_r)t \end{cases}$$

for $u_\ell < u_r$. Show that this is a weak solution of the Riemann problem to (3.2) with initial datum u_0 defined in (3.9). This solution is called a *rarefaction wave* solution. Show that u_R is indeed an entropy solution by checking the Kružkov entropy condition.

3.7 Let ξ be a real-valued convex function on $\mathbf{R}$. Prove that ξ is Lipschitz continuous in any bounded interval (a, b).

3.8 Let u be a real-valued C^2 function on $\mathbf{R}^N$.
 (i) Let η be a real-valued C^2 convex function on $\mathbf{R}$. Show that

$$\Delta\eta(u) \ge \eta'(u)\Delta u \quad \text{in} \quad \mathbf{R}^N.$$

 (ii) Show that

$$\int_{\mathbf{R}^N} (\Delta\varphi)|u|\,\mathrm{d}x \ge \int_{\mathbf{R}^N} \varphi(\operatorname{sgn} u)\Delta u\,\mathrm{d}x$$

 for any $\varphi \in C_c^\infty(\mathbf{R}^N)$ and $\varphi \ge 0$.

3.9 Let ξ be a real-valued C^2 function on $\mathbf{R}^N$. Show that ξ is convex in $\mathbf{R}^N$ if and only if its Hessian matrix $\left(\frac{\partial^2 \xi}{\partial x_i \partial x_j}(x)\right)_{1\le i,j\le N}$ is nonnegative definite for all $x \in \mathbf{R}^N$, i.e.,

$$\sum_{1\le i,j\le N} \frac{\partial^2 \xi}{\partial x_i \partial x_j}(x) z_i z_j \ge 0$$

for all $z = (z_1, \ldots, z_N) \in \mathbf{R}^N$.

3.10 Give an example of a function $f \in C\left(\mathbf{R}^2 \backslash \{0\}\right)$ such that

$$a := \lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) \quad \text{and} \quad b := \lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right)$$

exists but $a \ne b$.