2 Ordinary Differential Equations and Transport Equations

We continue to consider a system of ordinary differential equations (1.1), but we are more interested in the map $X \mapsto x(t, X)$, which is often called a *flow map* generated by a vector field b. If the initial value problem for (1.1) admits a unique local-in-time solution in a time interval $I = (0, a)$ with some $a > 0$ independent of X, the flow map is well defined. In Sect. 1.1, we gave a few sufficient conditions so that the flow map is uniquely determined assuming the existence of solutions to (1.1) with a given initial datum. Roughly speaking, if the vector field b satisfies the Lipschitz condition or a weaker condition called the Osgood condition, then the flow map is well defined. Since the Lipschitz continuity of b in $\mathbf{R}^N$ is equivalent to saying that the first distributional derivative of b is in L^∞ (see [36, §5.8. b]), it can be written $b \in W^{1,\infty}(\mathbf{R}^N)$, where the $W^{m,p}(\Omega)$ denotes the Sobolev space of order $m = 0, 1, 2, \ldots$ in $L^p(\Omega)$.

In this chapter, we are interested in the question of whether (1.2) can be replaced by $\|b(\cdot, t)\|_{W^{1,p}} \leq M$, with finite $p \geq 1$. However, unfortunately, this does not guarantee uniqueness; this can be easily seen if one elaborates Example 1.5. This suggests that we need some extra conditions for b so that a flow map is well defined. It turns out that if $\operatorname{div} b = 0$ or at least $\operatorname{div} b$ is bounded, this is the case (under some growth assumptions on b at the space infinity) provided that we regard the flow map $X \mapsto x(t, X)$ for almost all X (almost everywhere (a.e.) X) in $\mathbf{R}^N$ not for all X. Such a theory was started by DiPerna and Lions [32] in the late 1980s.

In this section, we explain the uniqueness part of the theory of autonomous equations, i.e., b is independent of time. To simplify the problem, we further assume that b is periodic in space variables.

M.-H. Giga, Y. Giga, *A Basic Guide to Uniqueness Problems for Evolutionary Differential Equations*, Compact Textbooks in Mathematics,
https://doi.org/10.1007/978-3-031-34796-2_2

2.1 Uniqueness of Flow Map

We consider a vector field (or $\mathbf{R}^N$-valued function) b on $\mathbf{T}^N = \prod_{i=1}^N (\mathbf{R}/\omega_i \mathbf{Z})$, with $\omega_i > 0$ $(i = 1, \dots, N)$, i.e.,

$$b(x) = \left(b^1(x), \dots, b^N(x)\right) \quad \text{for} \quad x \in \mathbf{T}^N.$$

In other words, we assume b is periodic in the ith variable with the period ω_i. In this section, we always assume that

$$b^j \in W^{1,1}(\mathbf{T}^N) \quad \text{for} \quad 1 \le j \le N \quad \text{and} \quad \operatorname{div} b = 0 \quad \text{in} \quad \mathbf{T}^N. \tag{2.1}$$

We simply write the first condition by $b \in W^{1,1}(\mathbf{T}^N)$ instead of writing $b \in \left(W^{1,1}(\mathbf{T}^N)\right)^N$, though the latter is more precise notation. Here, $W^{1,p}(\mathbf{T}^N)$ is the L^p-Sobolev space introduced in Sect. 1.2.5. We are interested in discussing the uniqueness of a solution to (1.1) with b independent of t, i.e.,

$$\dot{x} = b(x)$$

or

$$\frac{\mathrm{d}x^i}{\mathrm{d}t}(t) = b^i\left(x^1(t), \dots, x^N(t)\right)$$

for $x(t) = \left(x^1(t), \dots, x^N(t)\right)$ under condition (2.1). However, under condition (2.1), a flow map $X \mapsto x(t, X)$ (generated by b) for a fixed time t may not be integrable on $\mathbf{T}^N$. In other words, each component of this map may not belong to $L^1(\mathbf{T}^N)$. To overcome this difficulty, we introduce a space

$$\mathcal{M} = \mathcal{M}(\mathbf{T}^N) := \big\{\phi : \mathbf{T}^N \to \mathbf{R} \mid \text{(Lebesgue) } \textit{measurable} \text{ and} \\ |\phi| < \infty \text{ a.e.}\big\}.$$

This space is metrizable. For example, if we define a *metric* d as

$$d(\phi, \psi) = \|\min\left(|\phi - \psi|, 1\right)\|_{L^1(\mathbf{T}^N)} \quad \text{for} \quad \phi, \psi \in \mathcal{M},$$

then $(\mathcal{M}, d)$ becomes a *metric space*. See Exercise 2.2 and 2.6. From this point forward, $\|\cdot\|_{L^p(\mathbf{T}^N)}$ (or $\|\cdot\|_{L^p}$) denotes the L^p-norm in $L^p(\mathbf{T}^N)$. The convergence in this metric corresponds to the *convergence in measure*, i.e., $d(\phi_j, \psi) \to 0$ as $j \to \infty$ implies for any $\delta > 0$

$$\mathcal{L}^N \left\{x \in \mathbf{T}^N \;\middle|\; |\phi_j - \psi|(x) > \delta\right\} \to 0 \quad \text{as} \quad j \to \infty,$$

where $\mathcal{L}^N$ denotes the N-dimensional Lebesgue measure; see Sect. 1.2.5 or Appendix 5.2 for a precise definition of a set in $\mathbf{T}^N$. For fixed t, we expect each component x^i of a solution $x = x(t, X)$ belongs to $\mathcal{M}$ as a function of X, i.e., the mapping

$$x[t] : X \mapsto x(t, X)$$

is expected to be in $\mathcal{M}^N$. We also expect the mapping $x : t \mapsto x[t]$ to be defined for all $t \in \mathbf{R}$, and it is continuous from $\mathbf{R}$ to $\mathcal{M}^N$, i.e., $x \in C(\mathbf{R}, \mathcal{M}^N)$. The reason we expect x to be defined for all t is that the value $x[t](X) = x(t, X)$ actually belongs to the compact space $\mathbf{T}^N$, which prevents what is called blow-up phenomena.

If b is divergence-free, i.e., solenoidal, then the flow map $x[t]$ must satisfy the volume-preserving property. In other words, for all $t \in \mathbf{R}$,

$$\mathcal{L}^N \left(\left\{ z \in \mathbf{T}^N \;\middle|\; x[t]z \in A \right\} \right) = \mathcal{L}^N(A) \tag{2.2}$$

for any (Lebesgue) measurable set A. More generally,

$$\int_{\mathbf{T}^N} \psi\,(x(t, X))\,\mathrm{d}X = \int_{\mathbf{T}^N} \psi(y)\,\mathrm{d}y$$

for any measurable function ψ on $\mathbf{T}^N$. See Exercise 2.8. (In general, for a Lebesgue measurable set A, $f^{-1}(A) = \left\{ z \in \mathbf{T}^N \;\middle|\; f(z) \in A \right\}$ may not be Lebesgue measurable for a Lebesgue measurable function f. The volume-preserving property implicitly guarantees that $x[t]^{-1}(A)$ will be Lebesgue measurable if A is Lebesgue measurable.) The property (2.2) is obtained by $\operatorname{div} b = 0$. Here is a formal argument assuming that x is C^1 in t and X. We set $F = (F_{ij}) = \left(\partial x^i / \partial X^j\right)$ for the flow map $x = x(t, X)$. (This is a *Jacobi matrix* of the flow map $X \mapsto x(t, X)$.) By the area formula (or change of variable of integration), to see (2.2), it suffices to prove that $\det F = 1$ for all t, where $\det F$ denotes the determinant of F. Let $\operatorname{tr} M$ denote the *trace* of $N \times N$ metrics M, i.e., it is the sum of the diagonal components of M. By elementary calculus, we see that

$$\frac{\mathrm{d}}{\mathrm{d}t} \det F = \det F \operatorname{tr} \left(\frac{\partial F}{\partial t} F^{-1} \right).$$

By Eq. (1.1), we see that

$$\frac{\partial F_{ij}}{\partial t} = \sum_{\ell=1}^{N} (\partial_\ell b_i)(x) F_{\ell j}.$$

Thus, $\dfrac{\mathrm{d}}{\mathrm{d}t} \det F = \operatorname{tr}(Db) \det F$. Here Db denotes the Jacobian matrix $(Db)_{ij} = \partial_j b_i$, $1 \leq i, j \leq N$. We note that $\operatorname{tr}(Db) = \operatorname{div} b$. If $\operatorname{div} b = 0$ so that $\operatorname{tr}(Db) = 0$, we now observe that det F is time independent. Since det $F = 1$ at $t = 0$, we now conclude that det $F = 1$ for all t. This formal argument is justified when the flow map $x[t] : X \mapsto x(t, X)$ is in $C^1(\mathbf{T}^N, \mathbf{T}^N)$. This is indeed true if b is C^1, and it is known as C^1 dependence with respect to the initial data; See, for example, [52, Chapter 5].

We must consider a solution $x = x(t, X)$ of the ordinary differential equation (1.1), which is only continuous but may not be in C^1 in the time variable t. If one weakens the notion of a solution, there is a chance we lose the uniqueness. To keep the uniqueness, we consider a special class of a solution that is often called a renormalized solution. We consider a mapping $t \mapsto x(t, \cdot)$ from $\mathbf{R}$ to $\left(\mathcal{M}(\mathbf{T}^N)\right)^N$. If this mapping is continuous, we simply write $x \in C\left(\mathbf{R}, \left(\mathcal{M}(\mathbf{T}^N)\right)^N\right)$. It is also possible to consider the mapping $X \mapsto x(\cdot, X)$ from $\mathbf{T}^N$ to $(C(\mathbf{R}))^N$. This mapping is often called a flow map.

Definition 2.1

Assume that $x \in C\left(I, \left(\mathcal{M}(\mathbf{T}^N)\right)^N\right)$. We say that x is a *(renormalized) solution* of (1.1) in $\mathbf{R}$ if

$$\frac{\partial}{\partial t}(\beta \circ x)(t, X) = D\beta\left(x(t, X)\right) b\left(x(t, X)\right) \quad \text{on} \quad \mathbf{R} \times \mathbf{T}^N, \tag{2.3}$$

$$(\beta \circ x)|_{t=0}(X) = \beta(X) \quad \text{on} \quad \mathbf{T}^N \tag{2.4}$$

for all $\beta \in C^1(\mathbf{T}^N, \mathbf{T}^N)$ such that $\beta \circ x \in L^\infty\left(\mathbf{R}, \left(\mathcal{M}(\mathbf{T}^N)\right)^N\right)$, where $\beta \circ x$ is a composite function defined by $(\beta \circ x)(t, X) = \beta\left(x(t, X)\right)$. Here $D\beta$ denotes the Jacobian matrix $(D\beta)_{ij} = \partial \beta^i / \partial x_j,\ 1 \le i, j \le N$.

The time variable in (2.3) should be interpreted in the sense of a distribution whose variables are t and X. In other words, (2.3) means that

$$\begin{aligned} &- \int_{\mathbf{T}^N}\int_{-\infty}^{\infty} \frac{\partial \varphi}{\partial t}(t, X)(\beta \circ x)(t, X)\, \mathrm{d}t\mathrm{d}X \\ &= \int_{\mathbf{T}^N}\int_{-\infty}^{\infty} \varphi(t, X) D\beta\left(x(t, X)\right) b\left(x(t, X)\right)\, \mathrm{d}t\mathrm{d}X \end{aligned}$$

for all $\varphi \in C_c^\infty(\mathbf{T} \times \mathbf{T}^N)$. Of course, if x is C^1 in t, then x must satisfy (2.3) for all β and X if and only if x is a solution to (1.1) with $x(0, X) = X$.

We need to explain the space $L^\infty\left(\mathbf{R}, \left(\mathcal{M}(\mathbf{T}^N)\right)^N\right)$. If V is a Banach space V, then let $L^p(\mathbf{R}, V)$ be the space of all pth integrable functions on $\mathbf{R}$ as defined in Appendix 5.2 (4) using a Bochner integral. Since $\mathcal{M}(\mathbf{T}^N)$ is not a normed space but just a metric space, we must extend the definition. The space $L^\infty\left(\mathbf{R}, \mathcal{M}(\mathbf{T}^N)\right)$ is the space of all measurable functions f on $\mathbf{R}$ with values in $\mathcal{M}(\mathbf{T}^N)$ such that

$d(f, 0)$ is in $L^\infty(\mathbf{R})$ as function of t. The space $L^\infty\left(\mathbf{R}, \mathcal{M}(\mathbf{T}^N)^N\right)$ is defined as $\left(L^\infty\left(\mathbf{R}, \mathcal{M}(\mathbf{T}^N)\right)\right)^N$.

Finally, we expect that the flow map will satisfy the group property, i.e., for any $t, s \in \mathbf{R}$,

$$x(t+s, X) = x\left(t, x(s, X)\right) \quad \text{for a.e.} \quad X. \tag{2.5}$$

Theorem 2.2
Assume that (2.1) holds.

(i) (Existence) Then there exists a unique $x = x(t, X)$, with

$$x \in C\left(\mathbf{R}, \left(\mathcal{M}(\mathbf{T}^N)\right)^N\right)$$

satisfying (2.2)–(2.5). In particular, there exists a renormalized solution to (1.1). Moreover, the mapping $X \mapsto (\beta \circ x)(\cdot, X)$ is in $L^1\left(\mathbf{T}^N, (C(I))^N\right)$ for β given in (2.3), (2.4), where I is an arbitrary closed bounded interval. Furthermore, for almost every $X \in \mathbf{T}^N$ the function $t \mapsto x(t, X)$ is in $\left(C^1(\mathbf{R})\right)^N$ and $\dfrac{\partial x}{\partial t} = b(x)$ on $\mathbf{R}$ as a function of t.

(ii) (Uniqueness) There is at most one (renormalized) solution x to (1.1) satisfying all properties in (i).

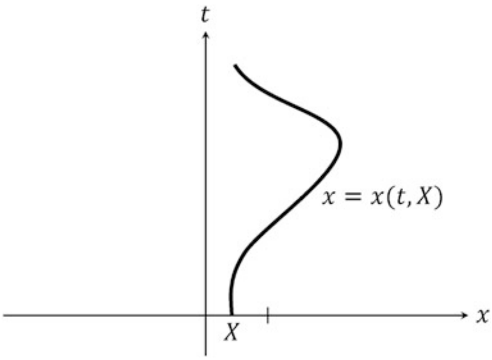
It is not difficult to see that the space $C(I)$ is regarded as a Banach space equipped with $\|\cdot\|_\infty$ norm since I is compact.

We shall focus on the uniqueness part of the proof. The main idea to prove the uniqueness is to show that the function $u_0\left(x(t, X)\right)$ depends only on $u_0 \in C^\infty(\mathbf{T}^N)$ for any choice of a real-valued function u_0. Since $u(X, t) = u_0\left(x(t, X)\right)$ solves a transport equation $u_t - b(X) \cdot \nabla_X u = 0$ with initial datum $u_0(X)$, the problem is reduced to the uniqueness of a (weak) solution to the transport equation with nonsmooth solenoidal coefficient b. Here, $u_t = \partial u / \partial t$, and ∇_X denotes the spatial gradient in X. We shall postpone the uniqueness proof of Theorem 2.2 to the end of Chap. 2.

For the reader's convenience, we show that $u(X, t)$ solves

$$u_t(X, t) - b(X) \cdot \nabla_X u(X, t) = 0 \tag{2.6}$$

Fig. 2.1 Characteristic curve



at least if $b \in C^1$ and x is C^1 in its variables $(t, X) \in \mathbf{R} \times \mathbf{T}^N$. We first prove that (2.6) at $t = 0$. By direct calculation,

$$u_t(X, t) = \sum_{i=1}^{N} \frac{\partial u_0}{\partial x^i}(x(t, X)) \frac{dx^i}{dt} = \sum_{i=1}^{N} \frac{\partial u_0}{\partial x^i}(x(t, X))\, b^i(x(t, X)),$$

$$\frac{\partial u}{\partial X^j}(X, t) = \sum_{i=1}^{N} \frac{\partial u_0}{\partial x^i}(x(t, X)) \frac{\partial x^i}{\partial X^j}(t, X).$$

At $t = 0$, $u_t(X, 0) = \sum_{j=1}^{N}(\partial_j u_0)(X) b^j(X)$, $\partial u/\partial X^j\big|_{t=0} = (\partial_j u_0)(X)$ since $\frac{\partial x^i}{\partial X^j}\Big|_{t=0} = \delta_{ij}$ ($\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$), so we have (2.6). We next set $u^s(X) = u(X, s)$ for $s \in \mathbf{R}$. Then, by the group property, we see that

$$u(X, t+s) = u_0(x(t+s, X)) = u_0(x(s, x(t, X))) = u^s(x(t, X)).$$

Applying the result for $t = 0$, with $u_0 = u^s$, we have

$$u_t(X, s) - b(X) \cdot \nabla_X u(X, s) = 0.$$

This yields (2.6). (The curve $x = x(t, X)$ is often called a *characteristic curve* of (2.6). It is easy to see that a solution u of (2.6) is constant along each characteristic curve, i.e., for a fixed X, the function $u(x(-t, X), t)$ is constant in t; see Fig. 2.1.)

2.2 Transport Equations

We are concerned with the uniqueness of a (weak) solution $u = u(x, t)$ to a transport equation

$$u_t - b(x) \cdot \nabla_x u = 0 \tag{2.7}$$

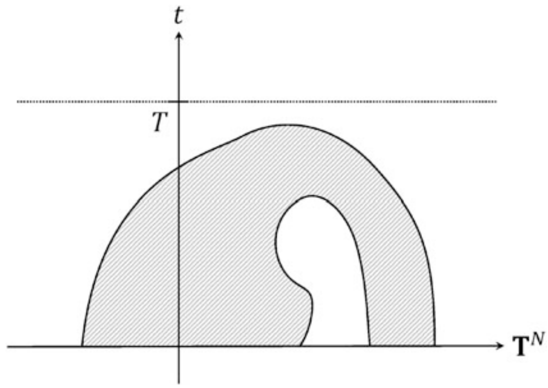


Fig. 2.2 Support of ϕ

or

$$u_t - \sum_{i=1}^{N} b^i(x) \frac{\partial u}{\partial x^i} = 0,$$

where $b = (b^1, \ldots, b^N)$; here we denote the independent variables by x instead of X. A notion of the weak solution u for (2.7) with initial datum $u_0 \in L^1(\mathbf{T}^N)$ is obtained by multiplying $\phi \in C_c^\infty\left(\mathbf{T}^N \times [0, T)\right)$ (i.e., $\operatorname{supp}\phi$ is compact in $\mathbf{T}^N \times [0, T)$) (cf. Fig. 2.2) and integrating over $\mathbf{T}^N \times [0, T]$. Indeed, we have

$$\int_0^T \int_{\mathbf{T}^N} \{\phi u_t - \phi\,(b(x) \cdot \nabla_x u)\}\, \mathrm{d}x\mathrm{d}t = 0.$$

Integrating by parts yields

$$-\int_0^T \int_{\mathbf{T}^N} \phi_t u \mathrm{d}x\mathrm{d}t - \int_{\mathbf{T}^N} \phi u_0 \mathrm{d}x + \int_0^T \int_{\mathbf{T}^N} (\operatorname{div}(b\phi))\, u\, \mathrm{d}x\mathrm{d}t = 0. \tag{2.8}$$

This formula leads to a definition of a weak solution to (2.7). If $\Omega = (0, T)$, then we simply write $L^p(0, T; V)$ instead of $L^p(\Omega, V)$, the space of all pth integrable functions on Ω with values in a Banach space V.

Definition 2.3

Let b be in $W^{1,p'}(\mathbf{T}^N)$, and let u_0 be in $L^p(\mathbf{T}^N)$ $(1 \leq p \leq \infty)$. If a function $u \in L^\infty\left(0, T; L^p(\mathbf{T}^N)\right)$ fulfills (2.8) for all $\phi \in C_c^\infty\left(\mathbf{T}^N \times [0, T)\right)$, then u is called a *weak solution* to (2.7) with initial datum u_0. The vector field b may not be divergence-free. Here p' is the conjugate exponent of p, i.e., $1/p + 1/p' = 1$. The integrability conditions guarantee that each term of (2.8) will be well defined as a usual Lebesgue integral. We interpret $1/\infty = 0$ so that $p = 1$ (resp. $p = \infty$) implies $p' = \infty$ (resp. $p' = 1$).

Of course, it is straightforward to extend the definition of a weak solution to an inhomogeneous problem of the form

$$u_t - b \cdot \nabla u = f$$

with initial datum u_0 and the inhomogeneous term $f \in L^1\left(0, T; L^1(\mathbf{T}^N)\right)$. We say that $u \in L^\infty\left(0, T; L^p(\mathbf{T}^N)\right)$ is a *weak solution* with initial datum u_0 if

$$-\int_0^T\int_{\mathbf{T}^N} \phi_t u \,\mathrm{d}x\mathrm{d}t - \int_{\mathbf{T}^N} \phi_t u_0 \,\mathrm{d}x + \int_0^T\int_{\mathbf{T}^N} (\operatorname{div}(b\phi))\, u \,\mathrm{d}x\mathrm{d}t = \int_0^T\int_{\mathbf{T}^N} f\varphi \,\mathrm{d}x\mathrm{d}t$$

for all $\varphi \in C_c^\infty\left(\mathbf{T}^N \times [0, T)\right)$.

In this section, we simply say that u is a solution to (2.7) if it is a weak solution of (2.8). We are now in a position to state our main uniqueness result.

Theorem 2.4
Let $1 \le p \le \infty$, and let $b \in W^{1,p'}(\mathbf{T}^N)$ be solenoidal, i.e., $\operatorname{div} b = 0$ in $\mathbf{T}^N$. Let $u \in L^\infty\left(0, T; L^p(\mathbf{T}^N)\right)$ be a solution to (2.7) with initial datum $u_0 \equiv 0$. Then $u \equiv 0$. (More precisely, $u(x,t) = 0$, a.e. $(x,t) \in \mathbf{T}^N \times (0,T)$.) In particular, if u_1 and u_2 are solutions to (2.7) with the same initial datum u_0, then $u_1 \equiv u_2$ since (2.7) is a linear equation.

A key observation is that $\theta \circ u$ solves (2.7) with initial datum $\theta \circ u_0$ provided that $\theta \in C^1(\mathbf{R})$, with $\theta' \in L^\infty(\mathbf{R})$. This is formally trivial since Eq. (2.7) is linear. However, in our setting, this property is nontrivial. Such a fact is often called a relabeling lemma. Here is a rigorous statement in this setting.

Lemma 2.5
Let $1 \le p \le \infty$, and let $b \in W^{1,p'}(\mathbf{T}^N)$. Assume that $u \in L^\infty\left(0, T; L^p(\mathbf{T}^N)\right)$ is a solution to (2.7) with initial datum $u_0 \in L^p(\mathbf{T}^N)$. Then $\theta \circ u \in L^\infty\left(0, T; L^p(\mathbf{T}^N)\right)$ is a solution to (2.7) with initial datum $\theta \circ u_0$ provided that $\theta \in C^1(\mathbf{R})$, with $\theta' \in L^\infty(\mathbf{R})$.

Admitting Lemma 2.5, we give a proof of Theorem 2.4 for $1 \le p < \infty$. The case $p = \infty$ is postponed to the next section.

Proof of Theorem 2.4 for $p < \infty$ A heuristic idea is to take $\theta(\sigma) = |\sigma|^p$ and observe that $|u|^p = \theta \circ u$ is a solution to (2.7) so that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbf{T}^N} |u|^p \,\mathrm{d}x = \int_{\mathbf{T}^N} \left(|u|^p\right)_t \,\mathrm{d}x = \int_{\mathbf{T}^N} \operatorname{div}\left(b|u|^p\right) \,\mathrm{d}x = 0$$

since $\operatorname{div} b = 0$ implies $\operatorname{div}(bw) = b \cdot \nabla w$ for a function w. However, since this θ is not C^1 or $\theta' \in L^\infty$, we must circumvent it.

We shall take a suitable C^1 function θ. By Lemma 2.5, the function $\theta \circ u$ is a solution of (2.7). Assume that $\theta(0) = 0$. By taking $\phi = \phi(t)$ (spatially constant function) in (2.8), we observe that

$$-\int_0^T \phi_t \left(\int_{\mathbf{T}^N} \theta(u)\mathrm{d}x\right) \mathrm{d}t - 0 + 0 = 0$$

since $\operatorname{div}(b\phi) = 0$. (This is the only place $\operatorname{div} b = 0$ is invoked.) This implies

$$\int_0^T \psi(t) \left(\int_{\mathbf{T}^N} \theta(u)\mathrm{d}x\right) \mathrm{d}t = 0$$

for any $\psi \in C_c^\infty((0,T))$ since we are able to take $\phi \in C^\infty([0,T])$ with $\operatorname{supp}\phi \subset [0,T)$ such that $\phi_t = \psi$. Indeed, it suffices to take $\phi = -\int_t^T \psi \mathrm{d}s$. Thus a fundamental lemma of the calculus of variations (cf. Exercise 2.3 or [19, Corollary 4.24]) implies that

$$\int_{\mathbf{T}^N} \theta(u)(x,t)\mathrm{d}x = 0 \quad \text{for a.e.} \quad t \in (0,T). \tag{2.9}$$

Since θ is required to be C^1 with bounded first derivative, for a given positive constant M we take

$$g(\sigma) := (|\sigma| \wedge M)^p\,,$$
$$\theta_\varepsilon(\sigma) := (\rho_\varepsilon * g)(\sigma) - (\rho_\varepsilon * g)(0) \quad \text{for} \quad \sigma \in \mathbf{R},$$

where $\rho_\varepsilon \in C_c^\infty(\mathbf{R})$ is a *Friedrichs' mollifier* in $\mathbf{R}$, i.e.,

$$\rho_\varepsilon(\sigma) = \frac{1}{\varepsilon}\rho(\sigma/\varepsilon), \quad \rho \geq 0, \quad \operatorname{supp}\rho \in (-1,1), \quad \int_{\mathbf{R}} \rho \mathrm{d}\sigma = 1$$

for $\varepsilon > 0$; see Fig. 2.3. Here $a_1 \wedge a_2 = \min(a_1, a_2)$ for $a_1, a_2 \in \mathbf{R}$. Since $\theta_\varepsilon \in C^\infty$ (Exercise 2.4), we plug such θ_ε into (2.9), and sending ε to zero (Exercise 2.5) yields

$$\int_{\mathbf{T}^N} (|u| \wedge M)^p\,(x,t)\mathrm{d}x = 0$$

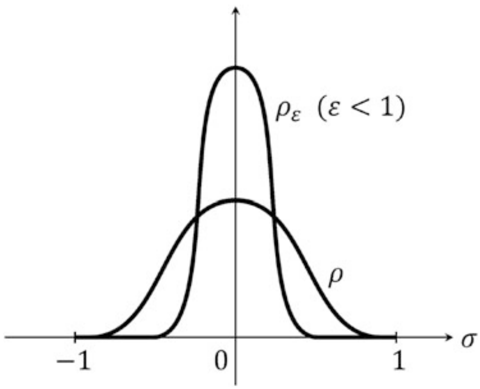


Fig. 2.3 A typical graph of ρ and ρ_ε

for a.e. t by the (Lebesgue) *dominated convergence theorem* (Theorem 5.1) because $\{|\theta_\varepsilon(u)|\}_{0<\varepsilon<1}$ is bounded in $L^\infty(\mathbf{T}^N)$. Since u is in $L^\infty\left(0, T; L^p(\mathbf{T}^N)\right)$, we send M to infinity and again use the dominated convergence theorem to conclude that

$$\int_{\mathbf{T}^N} |u|^p(x,t)\mathrm{d}x = 0, \quad \text{a.e.} \quad t \in (0,T).$$

Thus, we conclude that $u \equiv 0$ in $L^\infty\left(0, T; L^p(\mathbf{T}^N)\right)$. By Fubini's theorem, this is simply $u(x,t) = 0$ for a.e. $(x,t) \in \mathbf{T}^N \times (0,T)$. □

In the rest of this section, we shall prove the relabeling lemma (Lemma 2.5). A key idea is an approximation. From here, let ρ_ε be a Friedrichs' mollifier in $\mathbf{R}^N$, i.e., for $\varepsilon > 0$

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^N}\rho\left(\frac{x}{\varepsilon}\right), \quad \rho \in C_c^\infty(\mathbf{R}^N), \quad \int_{\mathbf{R}^N} \rho \,\mathrm{d}x = 1, \quad \rho \geq 0.$$

Lemma 2.6
Let $1 \leq p \leq \infty$, and let $b \in W^{1,\beta}(\mathbf{T}^N)$, with $\beta \geq p'$. Let $u \in L^\infty\left(0, T; L^p(\mathbf{T}^N)\right)$ be a solution of (2.7) with initial datum $u_0 \in L^p(\mathbf{T}^N)$. Let ρ_ε be a Friedrichs' mollifier in space variables. Then $u_\varepsilon = u * \rho_\varepsilon$ satisfies

$$\frac{\partial u_\varepsilon}{\partial t} - b \cdot \nabla u_\varepsilon = r_\varepsilon$$

with initial datum $u_{0\varepsilon} = u_0 * \rho_\varepsilon$, with some real-valued function r_ε converging to zero in $L^1\left(0, T; L^\alpha(\mathbf{T}^N)\right)$ as $\varepsilon \to 0$, where

$$1/\alpha = 1/\beta + 1/p \text{ if } \beta \text{ or } p \text{ is finite,}$$

$$1 \leq \alpha < \infty \text{ is arbitrary if } \beta = p = \infty.$$

Proof of Lemma 2.5 admitting Lemma 2.6 By Lemma 2.6, we observe that

$$\frac{\partial u_\varepsilon}{\partial t} - b \cdot \nabla u_\varepsilon = r_\varepsilon \to 0 \quad \text{in} \quad L^1\left(0, T; L^1(\mathbf{T}^N)\right);$$

in other words,

$$\lim_{\varepsilon \downarrow 0} \int_0^T \int_{\mathbf{R}^N} |r_\varepsilon| \, \mathrm{d}x \mathrm{d}t = 0.$$

We take $\theta \in C^1(\mathbf{R})$, with $\theta' \in L^\infty(\mathbf{R})$. Since u_ε is smooth in space, we see that

$$\frac{\partial}{\partial t}(\theta \circ u_\varepsilon) - b \cdot \nabla(\theta \circ u_\varepsilon) = r_\varepsilon \theta' \circ u_\varepsilon.$$

Since $\theta' \in L^\infty(\mathbf{R})$, the right-hand side $r_\varepsilon \theta' \circ u_\varepsilon \to 0$ in $L^1\left(0, T; L^1(\mathbf{R}^N)\right)$ as $\varepsilon \downarrow 0$, we formally conclude that $\theta \circ u$ is a solution to (2.7) with initial datum $\theta \circ u_0$. Of course, we must carry out these arguments in a weak form, (2.8). In other words, we send $\varepsilon \downarrow 0$ for

$$-\int_0^T \int_{\mathbf{T}^N} \phi_t (\theta \circ u_\varepsilon) \, \mathrm{d}x \mathrm{d}t - \int_{\mathbf{T}^N} \phi(\theta \circ u_{0\varepsilon}) \, \mathrm{d}x + \int_0^T \int_{\mathbf{T}^N} (\operatorname{div}(b\phi)) \, \theta \circ u^\varepsilon \, \mathrm{d}x \mathrm{d}t$$

$$= \int_0^T \int_{\mathbf{T}^N} \phi r_\varepsilon (\theta' \circ u_\varepsilon) \, \mathrm{d}x \mathrm{d}t \quad \text{with} \quad u_{0\varepsilon} = u_0 * \rho_\varepsilon$$

to get

$$-\int_0^T \int_{\mathbf{T}^N} \phi_t (\theta \circ u_\varepsilon) \, \mathrm{d}x \mathrm{d}t - \int_{\mathbf{T}^N} \phi(\theta \circ u_0) \, \mathrm{d}x + \int_0^T \int_{\mathbf{T}^N} (\operatorname{div}(b\phi)) \, \theta \circ u^\varepsilon \, \mathrm{d}x \mathrm{d}t = 0$$

for $\phi \in C_c^\infty\left(\mathbf{T}^N \times [0, T)\right)$. □

It remains to prove Lemma 2.6. For this purpose, it suffices to prove the convergence of commutators.

Lemma 2.7
Let ρ_ε denote a Friedrichs' mollifier in $\mathbf{R}^N$. Let $1 \le p \le \infty$, and let $b \in W^{1,\beta}(\mathbf{T}^N)$, with $\beta \ge p'$. If $w \in L^p(\mathbf{T}^N)$, then

$$R_\varepsilon(w, b) = (b \cdot \nabla w) * \rho_\varepsilon - b \cdot \nabla(w * \rho_\varepsilon) \to 0 \quad in \quad L^\alpha(\mathbf{T}^N)$$

as $\varepsilon \to 0$, where α is given as in Lemma 2.6; in particular, in the case $p = \infty$ so that $p' = 1$, $\alpha = \beta$ if $\beta < \infty$. Moreover, $\|R_\varepsilon(w, b)\|_{L^\alpha} \le C\|w\|_{L^p}\|Db\|_{L^\beta}$ with some $C > 0$ independent of sufficiently small ε.

Proof of Lemma 2.6 admitting Lemma 2.7 Direct calculation shows that

$$\frac{\partial}{\partial t}u_\varepsilon - b\cdot\nabla u_\varepsilon = R_\varepsilon(u,b);$$

in other words,

$$\begin{aligned}&-\int_0^T\int_{\mathbf{T}^N}\phi_t u_\varepsilon\,\mathrm{d}x\mathrm{d}t - \int_{\mathbf{T}^T}\phi u_{0\varepsilon}\,\mathrm{d}x + \int_0^T\int_{\mathbf{T}^N}(\operatorname{div}(b\phi))\,u_\varepsilon\,\mathrm{d}x\mathrm{d}t\\ &= \int_0^T\int_{\mathbf{T}^N}R_\varepsilon(u,b)\phi\,\mathrm{d}x\mathrm{d}t\end{aligned}$$

for all $\phi\in C_c^\infty\left(\mathbf{T}^N\times[0,T)\right)$. By the second statement of Lemma 2.7, we have

$$\|R_\varepsilon(u,b)\|_{L^\alpha}(t)\le C\|u\|_{L^p}(t)\|Db\|_{L^\beta}.$$

The right-hand side is bounded in t, so Lemma 2.7 yields

$$\lim_{\varepsilon\downarrow 0}\int_0^T\|R_\varepsilon(w,b)\|_{L^\alpha}(t)\,\mathrm{d}t = \int_0^T\lim_{\varepsilon\downarrow 0}\|R_\varepsilon(w,b)\|_{L^\alpha}(t)\,\mathrm{d}t = 0$$

by a dominated convergence theorem. (One immediately observes that b is allowed to depend on time t if $b\in L^1\left(0,T;W^{1,\beta}(\mathbf{T}^N)\right)$ and $u\in L^\infty\left(0,T;L^p(\mathbf{T}^N)\right)$. □

Proof of Lemma 2.7 We first observe that the term $(b\cdot\nabla w)*\rho_\varepsilon$ should be interpreted as

$$((b\cdot\nabla w)*\rho_\varepsilon)(x) = -\int_{\mathbf{T}^N}w(y)\operatorname{div}_y\{b(y)\rho_\varepsilon(x-y)\}\,\mathrm{d}y$$

since ∇w is not an integrable function. This identity is easily obtained if w is smooth by integration by parts. We proceed with

$$\begin{aligned}(b\cdot\nabla w)*\rho_\varepsilon - b\cdot\nabla(w*\rho_\varepsilon) &= \rho_\varepsilon*(b\cdot\nabla w) - \sum_{i=1}^N\left(\frac{\partial}{\partial x_i}\rho_\varepsilon * w\right)b^i\\ &= -\int_{\mathbf{T}^N}w(y)\left[\operatorname{div}_y\{b(y)\rho_\varepsilon(x-y)\} + b(x)\cdot(\nabla\rho_\varepsilon)(x-y)\right]\mathrm{d}y\\ &= \int_{\mathbf{T}^N}w(y)\left((b(y)-b(x))\cdot(\nabla\rho_\varepsilon)(x-y)\right)\mathrm{d}y - \int_{\mathbf{T}^N}(w\operatorname{div}b)*\rho_\varepsilon\,\mathrm{d}y\\ &= I + II.\end{aligned}$$

Here, we use the relation $\nabla_y(\rho_\varepsilon(x-y)) = -(\nabla\rho_\varepsilon)(x-y)$.

We next estimate I. Since ρ has compact support, $\rho_\varepsilon(x) = 0$ for $|x| \geq C\varepsilon$, with some C independent of ε. By changing ε by $c_0\varepsilon$ with some $c_0 > 0$, we may assume that $C < \min_i \omega_i/2$. This choice implies that the ball $B_C(z)$ centered at $z \in \mathbf{T}^N$ with radius C is contained in one periodic cell. We shall assume that $\varepsilon < 1$. By changing the variable of integration by $z = (y-x)/\varepsilon$, we obtain

$$\begin{aligned} I(x) &= \int_{|x-y|\leq C\varepsilon} \{(b(y) - b(x)) \cdot (\nabla\rho_\varepsilon)(x-y)w(y)\}\,\mathrm{d}y \\ &= \int_{|z|\leq C} \left\{ \frac{b(x+\varepsilon z) - b(x)}{\varepsilon} \cdot (\nabla\rho)(-z)w(x+\varepsilon z) \right\} \mathrm{d}z. \end{aligned}$$

Since $\nabla\rho$ is bounded, i.e., $C_0 := \|\nabla\rho\|_{L^\infty(\mathbf{R}^N)} < \infty$, we observe that

$$|I(x)| \leq C_0 \int_{|z|\leq C} k_\varepsilon(x,z)\,|w(x+\varepsilon z)|\,\mathrm{d}z$$

with $k_\varepsilon(x,z) = |b(x+\varepsilon z) - b(x)|\,/\varepsilon$. By the *Hölder inequality*

$$\|f \cdot 1\|^\alpha_{L^1(B_C)} \leq \|f\|^\alpha_{L^\alpha(B_C)} |B_C|^{\alpha-1},$$

we observe that

$$\begin{aligned} \frac{1}{C_0^\alpha}\|I\|^\alpha_{L^\alpha(\mathbf{T}^N)} &= \int_{\mathbf{T}^N} \left\{ \int_{|z|\leq C} k_\varepsilon(x,z)\,|w(x+\varepsilon z)|\,dz \right\}^\alpha \mathrm{d}x \\ &\leq \int_{\mathbf{T}^N} \int_{|z|\leq C} \{k_\varepsilon(x,z)\,|w(x+\varepsilon z)|\}^\alpha\,\mathrm{d}z\mathrm{d}x |B_C|^{\alpha-1} =: J^\alpha \end{aligned}$$

for $\alpha \in [1,\infty]$. Here, $|B_C| = \mathcal{L}^N(B_C)$ (i.e., the volume of a ball of radius C) and it equals $|S^{N-1}|C^N/N$. Applying the Hölder inequality for k_ε^α and $|w|^\alpha$ with $1/\alpha = 1/\beta + 1/p$, we see that

$$\begin{aligned} J &\leq |B_C|^{1-1/\alpha} \left\{ \int_{\mathbf{T}^N} \int_{|z|\leq C} k_\varepsilon(x,z)^\beta \mathrm{d}z\mathrm{d}x \right\}^{1/\beta} \left\{ \int_{\mathbf{T}^N} \int_{|z|\leq C} |w(x+\varepsilon z)|^p \mathrm{d}z\mathrm{d}x \right\}^{1/p} \\ &= C_1 \|w\|_{L^p(\mathbf{T}^N)} \left\{ \int_{\mathbf{T}^N} \int_{|z|\leq C} k_\varepsilon(x,z)^\beta \mathrm{d}z\mathrm{d}x \right\}^{1/\beta} \end{aligned}$$

with $C_1 = |B_C|^{1-1/\alpha}|B_C|^{1/p} = |B_C|^{1-1/\beta}$. Since

$$
\begin{aligned}
|b(x+\varepsilon z) - b(x)| &= \left\{ \sum_{i=1}^{N} \left| \int_0^1 \left\langle \nabla b^i(x+\varepsilon\sigma z), \varepsilon z \right\rangle \mathrm{d}\sigma \right|^2 \right\}^{1/2} \\
&\le \varepsilon |z| \left(\sum_{i=1}^{N} \left| \int_0^1 \left| \nabla b^i(x+\varepsilon\sigma z) \right| \mathrm{d}\sigma \right|^2 \right)^{1/2} \\
&\le \varepsilon |z| \int_0^1 |Db(x+\varepsilon\sigma z)|\, \mathrm{d}\sigma,
\end{aligned}
$$

we see that

$$
\begin{aligned}
\int_{\mathbf{T}^N} \int_{|z|\le C} k_\varepsilon(x,z)^\beta \mathrm{d}z\mathrm{d}x &\le \int_{\mathbf{T}^N} \int_{|z|\le C} \int_0^1 |Db(x+\varepsilon\sigma z)|^\beta \, d\sigma |z|^\beta \mathrm{d}z\mathrm{d}x \\
&= C_2^\beta \|Db\|_{L^\beta(\mathbf{T}^N)}^\beta \quad \text{with} \quad C_2 = \left(\int_{|z|\le C} |z|^\beta \mathrm{d}z \right)^{1/\beta}.
\end{aligned}
$$

Here, $|Db(x)|$ denotes the Euclidean norm of the $N \times N$ matrix $Db(x)$ in $\mathbf{R}^{N\times N}$. In other words, $|Db(x)|^2 = \operatorname{tr}\left(Db(x)Db(x)^T\right)$, where M^T denotes the transpose of a matrix M.

We now conclude that

$$
\begin{aligned}
\|I\|_{L^\alpha(\mathbf{T}^N)} \le C_0 J \le C_0 C_1 \|w\|_{L^p(\mathbf{T}^N)} \left\{ \int_{\mathbf{T}^N} \int_{|z|\le C} k_\varepsilon(x,z)^\beta \mathrm{d}z\mathrm{d}x \right\}^{1/\beta} \\
\le C_0 C_1 C_2 \|w\|_{L^p(\mathbf{T}^N)} \|Db\|_{L^\beta(\mathbf{T}^N)}.
\end{aligned}
$$

By *Young's inequality* for convolution, we have

$$
\|II\|_{L^\alpha} \le \|\rho_\varepsilon\|_{L^1} \|w \operatorname{div} b\|_{L^\alpha} \le 1 \|w \operatorname{div} b\|_{L^\alpha}.
$$

By the Hölder inequality,

$$
\|w \operatorname{div} b\|_{L^\alpha} \le \|w\|_{L^p} \|Db\|_{L^\beta}.
$$

Thus, the desired estimate $\|R_\varepsilon(w,b)\|_{L^\alpha} \le C\|w\|_{L^p}\|Db\|_{L^\beta}$ now follows.

It remains to prove that $\|R_\varepsilon(w,b)\|_{L^\alpha} \to 0$ as $\varepsilon \to 0$. This can be carried out by a density argument.

If $w \in W^{1,p}(\mathbf{T}^N)$, then both $(b\cdot\nabla w) * \rho_\varepsilon$ and $(b\cdot\nabla)(w * \rho_\varepsilon)$ converge to $(b\cdot\nabla)w$ in $L^\alpha(\mathbf{T}^N)$ as $\varepsilon \to 0$ since $f * \rho_\varepsilon \to f$ in $L^\alpha(\mathbf{T}^N)$ if $f \in L^\alpha(\mathbf{T}^N)$; see [19, Section 4.4]. Here, we invoke the property $\alpha < \infty$. Thus, $R_\varepsilon(w,b) \to 0$ in $L^\alpha(\mathbf{T}^N)$ provided that $w \in W^{1,p}(\mathbf{T}^N)$.

Suppose that $1 \leq p < \infty$. Since $W^{1,p}(\mathbf{T}^N)$ is *dense* in $L^p(\mathbf{T}^N)$ for $p < \infty$, for $w \in L^p(\mathbf{T}^N)$ there is a sequence $\{w_m\} \subset W^{1,p}(\mathbf{T}^N)$ converging to w in $L^p(\mathbf{T}^N)$. (This density follows from the fact that $f * \rho_{1/m} \in C^\infty(\mathbf{T}^N)$ and $f * \rho_{1/m} \to f$ in $L^p(\mathbf{T}^N)$ as $m \to \infty$ for $p \in [1, \infty)$. See [19, Section 4.4] and Appendix 5.2 (6).) Since

$$R_\varepsilon(w, b) = R_\varepsilon(w - w_m, b) + R_\varepsilon(w_m, b),$$

we observe that

$$\|R_\varepsilon(w, b)\|_{L^\alpha} \leq C_3 \|w - w_m\|_{L^p} \|Db\|_{L^\beta} + \|R_\varepsilon(w_m, b)\|_{L^\alpha}$$

by the estimate of Lemma 2.7 proved earlier with some $C_3 > 0$ independent of sufficiently small $\varepsilon > 0$. Sending $\varepsilon \downarrow 0$ yields

$$\limsup_{\varepsilon \downarrow 0} \|R_\varepsilon(w, b)\|_{L^\alpha} \leq C_3 \|w - w_m\|_{L^p} \|Db\|_{L^\beta} + 0.$$

Letting $m \to \infty$ yields a desired conclusion, i.e., $R_\varepsilon(w, b) \to 0$ as $\varepsilon \to 0$ in $L^\alpha(\mathbf{T}^N)$.

If $p = \beta = \infty$, then, by our assumption, $\alpha < \infty$. This case is reduced to the case $\alpha = p < \infty$, $\beta = \infty$.

It remains to prove the case $\beta < \infty$, but $p = \infty$. In this case, $\alpha = \beta$. Unfortunately, $w \in L^\infty(\mathbf{T}^N)$ cannot be approximated by an element of $W^{1,\infty}(\mathbf{T}^N)$ in the L^∞ sense. However, it is still possible to approximate in a weaker sense. That is, for any $w \in L^\infty(\mathbf{T}^N)$, there exists a sequence $w_n \in L^\infty(\mathbf{T}^N)$ such that $w_m \to w$ a.e. and $\|w_m\|_\infty \leq \|w\|_\infty$. (For example, it is enough to take $w_m = \rho_{1/m} * w$. See Exercise 2.10.) We are able to estimate

$$\limsup_{\varepsilon \to 0} \|R_\varepsilon(v, b)\|_{L^\alpha}^\alpha \leq C \int_{\mathbf{T}^N} |Db(x)|^\alpha \, |v(x)|^\alpha \, \mathrm{d}x \tag{2.10}$$

with some C independent of $v \in L^\infty(\mathbf{T}^N)$ and b. Indeed, from a similar argument as previously, the term corresponding to I is dominated by

$$\begin{aligned}
\frac{1}{C_0^\alpha} \|I\|_{L^\alpha(\mathbf{T}^N)}^\alpha &\leq \int_{\mathbf{T}^N} \int_{|z| \leq C} \{k_\varepsilon(x, z) \, |v(x + \varepsilon z)|\}^\alpha \, \mathrm{d}z \mathrm{d}x |B_C|^{\alpha - 1} \\
&\leq \int_{\mathbf{T}^N} \int_{|z| \leq C} |z|^\alpha \left(\int_0^1 |Db(x + \varepsilon \sigma z)|^\alpha \, |v(x + \varepsilon z)|^\alpha \, \mathrm{d}\sigma \right) \mathrm{d}z \mathrm{d}x |B_C|^{\alpha - 1},
\end{aligned}$$

where $\beta = \alpha$. We change the variable of integration by $\overline{x} = x + \varepsilon \sigma z$ to get

$$= \int_{\mathbf{T}^N} \int_{|z| \leq C} |z|^\alpha \int_0^1 |Db(\overline{x})|^\alpha \, |v\,(\overline{x} + (1 - \sigma)\varepsilon z)|^\alpha \, \mathrm{d}\sigma \mathrm{d}z \mathrm{d}\overline{x} |B_C|^{\alpha - 1}.$$

Since the shift operator is continuous in L^α norm,[1] we see that

$$v\left(\overline{x} + (1-\sigma)\varepsilon z\right) \to v(\overline{x})$$

for almost all $\overline{x} \in \mathbf{T}^N$ as $\varepsilon \to 0$ by taking a subsequence if necessary.[2] Since v is bounded and $|Db|^\alpha$ is integrable, by the dominated convergence theorem we conclude that the last term converges to

$$C_2^\alpha |B_C|^{\alpha-1} \int_{\mathbf{T}^N} |Db(x)|^\alpha \, |v(x)|^\alpha \, \mathrm{d}x \quad \text{as} \quad \varepsilon \downarrow 0.$$

A similar but much easier observation yields a similar estimate for II. We thus conclude (2.10).

By (2.10), we are able to estimate

$$\begin{aligned}
\limsup_{\varepsilon \downarrow 0} \|R_\varepsilon(w,b)\|_{L^\alpha} &\le \limsup_{\varepsilon \downarrow 0} \|R_\varepsilon(w - w_m, b)\|_{L^\alpha} \\
&\le C \int_{\mathbf{T}^N} |Db(x)|^\alpha \, |w(x) - w_m(x)|^\alpha \, \mathrm{d}x.
\end{aligned}$$

Since $\|w_m\|_\infty \le \|w\|_\infty$ and $w_m \to w$ a.e. as $m \to \infty$, by the dominated convergence theorem, we conclude that the right-hand side tends to zero. The proof for the convergence $\lim_{\varepsilon \downarrow 0} \|R_\varepsilon(w,b)\|_{L^\alpha} = 0$ is now complete. □

2.3 Duality Argument

In this section, we shall prove the uniqueness result (Theorem 2.4) of a solution to the transport equation (2.7) when it is bounded under the condition that the coefficient b of the transport term is merely in $W^{1,1}(\mathbf{T}^N)$. The argument presented so far does not work for $p = \infty$. We study the case where $p = \infty$ by a duality argument.

Let us explain a basic idea of the duality argument. This is a typical argument for uniqueness. Consider a linear operator S from $\mathbf{R}^n$ to $\mathbf{R}^m$. Suppose that we are asked to check whether this mapping is injective or one to one. Since S is linear, it is enough to show the kernel of S is just $\{0\}$. In other words, we are asked to prove that $Sx = 0$ implies $x = 0$. The main idea of the duality argument is to reduce the

[1] This is one of the fundamental properties of the Lebesque measure. It states that $\lim_{|y| \to 0} \|\tau_y f - f\|_{L^\alpha(\mathbf{T}^N)} = 0$ for $\alpha \in [1, \infty)$, where $(\tau_y f)(x) = f(x+y)$.

[2] If $f_\varepsilon \to f$ in $L^p(\mathbf{T}^N)$, there is a subsequence f_{ε_k} that converges to f a.e.

problem to the solvability of its dual problem $S^*z = y$ for all $y \in \mathbf{R}^n$. If there is a solution z, then

$$x \cdot y = x \cdot S^*z = Sx \cdot z = 0$$

for all $y \in \mathbf{R}^n$. This implies $x = 0$.

So to carry out this argument, we need some existence theorem for a dual problem.

Proposition 2.8
Let $1 \le p \le \infty$, and let $u_0 \in L^p(\mathbf{T}^N)$. Assume that $b \in W^{1,1}(\mathbf{T}^N) \cap L^{p'}(\mathbf{T}^N)$, with $\operatorname{div} b = 0$. Then there exists a solution $u \in L^\infty\big(0, T; L^p(\mathbf{T}^N)\big)$ of (2.7) with initial datum u_0. Here, $1/p + 1/p' = 1$.

Proof. A typical way to prove the existence of a solution under nonsmooth coefficients is as follows. We first approximate the problem by regularization, then take a limit of a solution to the approximate problem. We need a priori estimates to carry out the second step.

We begin with an a priori estimate assuming that b and u_0 are smooth. Let $x = x(t, X)$ be the flow map generated by b, i.e., $\dot{x} = b(x)$, with $x(0) = X$. If b is smooth, then by the smooth dependence of the initial data [52, Chapter 5], x is smooth in t and X. Moreover, by the uniqueness of the solution of (1.1) (Proposition 1.1), the group property (2.5) holds. We first recall that $u(X, t) = u_0\,(x(t, X))$ (uniquely) solves (2.7), i.e.,

$$u_t - b(X) \cdot \nabla_X u = 0$$

with smooth data

$$u|_{t=0}\,(X) = u_0(X)$$

if b is smooth. By this solution formula, it is clear that

$$\|u\|_{L^\infty}(t) \le \|u_0\|_{L^\infty}.$$

By the group property (2.5), we have $u_0(X) = u\,(x(-t, X), t)$. Thus, $\|u_0\|_{L^\infty} \le \|u\|_{L^\infty}(t)$, so that

$$\|u\|_{L^\infty}(t) = \|u_0\|_{L^\infty}.$$

Here, $\|u\|_{L^q}(t)$ denotes the norm of u in $L^q(\mathbf{T}^N)$ with a parameter t. By a formal argument, to obtain the volume-preserving property (2.2), we observed that the

Jacobian $\det F$ of the flow map equals one, where $F = (F_{ij}) = (\partial x^i / \partial X^j)$. Thus, by (2.2), we see that

$$\|u\|_{L^q}(t) = \|u_0\|_{L^q} \tag{2.11}$$

for $1 \leq q < \infty$. Combining the L^∞ estimate, we see (2.11) holds for all $q \in [1, \infty]$.

We next approximate our original b by $b_\varepsilon = b * \rho_\varepsilon$. Then there is a (unique) solution $u_\varepsilon(x, t) \in C^\infty\left(\mathbf{T}^N \times [0, T)\right)$ of

$$u_t - b_\varepsilon(x) \cdot \nabla_x u = 0 \tag{2.12}$$

with initial datum $u_{0\varepsilon} = u_0 * \rho_\varepsilon$ for $u_0 \in L^p(\mathbf{T}^N)$; the solution $u_\varepsilon(x, t)$ is given by $u_\varepsilon(x, t) = u_0\left(x_\varepsilon(t, x)\right)$. Here, x_ε is the flow map generated by b_ε, i.e., $\dot{x}_\varepsilon = b_\varepsilon(x_\varepsilon)$, with $x_\varepsilon(0) = x \in \mathbf{T}^N$. Since u_ε solves (2.12), it solves its weak form, i.e.,

$$-\int_0^T \int_{\mathbf{T}^N} \phi_t u_\varepsilon \mathrm{d}x\mathrm{d}t - \int_{\mathbf{T}^N} \phi u_{0\varepsilon} \mathrm{d}x + \int_0^T \int_{\mathbf{T}^N} \left(\mathrm{div}(b_\varepsilon \phi)\right) u_\varepsilon \mathrm{d}x\mathrm{d}t = 0 \tag{2.13}$$

for all $\phi \in C_c^\infty\left(\mathbf{T}^N \times [0, T)\right)$. We note that u_ε and $u_{0\varepsilon}$ satisfy the norm-preserving property (2.11).

Case 1 ($1 < p \leq \infty$). Since (2.11) for u_ε implies that $\{u_\varepsilon\}_{0<\varepsilon<1}$ is bounded in $L^\infty\left(0, T; L^p(\mathbf{T}^N)\right)$, by $*$-weak compactness, there is a subsequence $\{u_{\varepsilon'}\}$ converging to some u $*$-weakly in $L^\infty\left(0, T; L^p(\mathbf{T}^N)\right)$ for $p \in (1, \infty]$; see Appendix 5.2 (4) for $*$-weak convergence in $L^\infty\left(0, T; L^p(\mathbf{T}^N)\right)$. We now send ε' to zero in (2.13). It is easy to see that the first two terms of (2.13) converge to the first two terms of (2.8), respectively. The only difficulty lies in handling the last term. We proceed with

$$\int_0^T \int_{\mathbf{T}^N} \mathrm{div}(b_{\varepsilon'} \phi) u_{\varepsilon'} \mathrm{d}x\mathrm{d}t = \int_0^T \int_{\mathbf{T}^N} (b_{\varepsilon'} \cdot \nabla \phi) u_{\varepsilon'} \mathrm{d}x\mathrm{d}t \tag{2.14}$$

since $\mathrm{div}\, b_\varepsilon = \mathrm{div}(b * \rho_\varepsilon) = (\mathrm{div}\, b) * \rho_\varepsilon = 0$. By a standard property of the mollifier, we see that $b_\varepsilon \to b$ in $L^{p'}(\mathbf{T}^N)$ if $p \in (1, \infty]$ (e.g., [19, Section 4.4]). Since $u_{\varepsilon'} \rightharpoonup u$ $*$-weakly in $L^\infty\left(0, T; L^p(\mathbf{T}^N)\right)$, this implies that (2.14) converges to

$$\int_0^T \int_{\mathbf{T}^N} (b \cdot \nabla \phi) u \,\mathrm{d}x\mathrm{d}t = \int_0^T \int_{\mathbf{T}^N} \mathrm{div}(b\phi) u \,\mathrm{d}x\mathrm{d}t$$

as $\varepsilon' \downarrow 0$. Here, we invoked the property

$$\lim_{\varepsilon \downarrow 0} \int_0^T \int_{\mathbf{T}^N} f_\varepsilon g_\varepsilon \,\mathrm{d}x\mathrm{d}t = \int_0^T \int_{\mathbf{T}^N} f g \,\mathrm{d}x\mathrm{d}t$$

if $f_\varepsilon \to f$ in $L^1\left(0,T;L^p(\mathbf{T}^N)\right)$ and $g_\varepsilon \rightharpoonup g$ $*$-weakly in $L^\infty\left(0,T;L^p(\mathbf{T}^N)\right)$ as $\varepsilon \downarrow 0$. To see this property, we notice

$$f_\varepsilon g_\varepsilon - fg = (f_\varepsilon - f)g_\varepsilon + f(g_\varepsilon - g)$$

so that

$$\begin{aligned}&\left|\int_0^T\int_{\mathbf{T}^N} f_\varepsilon g_\varepsilon \,\mathrm{d}x\mathrm{d}t - \int_0^T\int_{\mathbf{T}^N} fg \,\mathrm{d}x\mathrm{d}t\right|\\ \le & \|f_\varepsilon - f\|_{L^1\left(0,T;L^{p'}(\mathbf{T}^N)\right)}\|g_\varepsilon\|_{L^\infty(0,T;L^p(\mathbf{T}^N))}\\ &+\left|\int_0^T\int_{\mathbf{T}^N} f(g_\varepsilon - g)\,\mathrm{d}x\mathrm{d}t\right|\end{aligned}$$

by the Hölder inequality (cf. Exercise 2.9). The first term tends to zero as $\varepsilon \downarrow 0$ since $\|g_\varepsilon\|_{L^\infty(0,T;L^p(\mathbf{T}^N))}$ is bounded and $f_\varepsilon \to f$ in $L^1\left(0,T;L^{p'}(\mathbf{T}^N)\right)$. The second term tends to zero as $\varepsilon \downarrow 0$ since $g_\varepsilon \rightharpoonup g$ $*$-weakly in $L^\infty\left(0,T;L^p(\mathbf{T}^N)\right)$. We thus obtain (2.8) when $p \in (1,\infty]$.

Case 2 ($p = 1$). In this case, boundedness in $L^1(\mathbf{T}^N)$ does not imply weak compactness. We approximate $u_0 \in L^1(\mathbf{T}^N)$ by $u_{0m} \in L^{\hat{p}}(\mathbf{T}^N)$, $\hat{p} > 1$ such that $\|u_0 - u_{0m}\|_{L^1} \to 0$ as $m \to \infty$. Let u_m^ε be an $L^\infty\left(0,T;L^{\hat{p}}(\mathbf{T}^N)\right)$ solution with initial datum u_{0m} for (2.12), which is given by

$$u_m^\varepsilon(x,t) = u_{0m}\left(x_\varepsilon(t,x)\right).$$

Since (2.12) is linear, we apply (2.11) to $u_m^\varepsilon - u_{m+1}^\varepsilon$ to get

$$\|u_m^\varepsilon - u_{m+1}^\varepsilon\|_{L^q}(t) = \|u_{0m} - u_{0m+1}\|_{L^q} \tag{2.15}$$

for all $1 \le q \le \hat{p}$. For $m = 1$, we take a subsequence as $\varepsilon \to 0$ to get U_1 satisfying (2.8) starting with u_{01} by Case 1. For $m = 2$, we take a further subsequence to get a U_2 satisfying (2.8) starting with u_{02}. We repeat the procedure and obtain $U_m \in L^\infty\left(0,T;L^{\hat{p}}(\mathbf{T}^N)\right)$ satisfying (2.8) with initial datum u_{0m} and moreover satisfying

$$\|U_m - U_{m+1}\|_{L^1}(t) = \|u_{0m} - u_{0m+1}\|_{L^1}$$

by (2.15). This implies that $\{U_m\}$ is a *Cauchy sequence* in $L^\infty\left(0,T;L^1(\mathbf{T}^N)\right)$. Since U_m solves (2.8), letting $m \to \infty$ yields the desired solution $U = \lim_{m\to\infty} U_m \in L^\infty\left(0,T;L^1(\mathbf{T}^N)\right)$ of (2.7) with initial datum $u_0 \in L^1(\mathbf{T}^N)$. □

Remark 2.9

(i) The assumption $b \in W^{1,1}(\mathbf{T}^N)$ is used only to define $\operatorname{div} b$ in $L^1(\mathbf{T}^N)$. It is enough to assume that $b \in L^{p'}(\mathbf{T}^N)$ with $\operatorname{div} b = 0$ in the distribution sense.

(ii) Proposition 2.8 is still valid for a solution of $u_t - b \cdot \nabla u = f$ (instead of (2.7)) provided that $f \in L^1\left(0, T; L^p(\mathbf{T}^N)\right)$. Its weak form is given right after Definition 2.3.

Proof of Theorem 2.4 for $p = \infty$ We shall prove that

$$\int_0^T \int_{\mathbf{T}^N} u\phi \mathrm{d}x\mathrm{d}t = 0$$

for all $\phi \in C_c^\infty\left(\mathbf{T}^N \times (0, T)\right)$. This implies $u = 0$ a.e. in $\mathbf{T}^N \times (0, T)$ by a fundamental lemma of the calculus of variations (Exercise 2.3 and [19, Corollary 4.24]).

We first consider a dual problem, which is a backward problem:

$$\frac{\partial \Phi}{\partial t} - b \cdot \nabla \Phi = \phi \ \text{ in } \ \mathbf{T}^N \times (0, T), \qquad \Phi|_{t=T} = 0 \ \text{ in } \ \mathbf{T}^N.$$

By Proposition 2.8 and Remark 2.9 (ii), there exists a solution

$$\tilde{\Phi} \in L^\infty\left(0, T; L^\infty(\mathbf{T}^N)\right)$$

for $\partial_t \tilde{\Phi} + b \cdot \nabla \tilde{\Phi} = \tilde{\phi}$ with $\tilde{\Phi}(x, 0) = 0$ for $\tilde{\phi}(x, t) = \phi(x, T - t)$. Setting $\Phi(x, t) = \tilde{\Phi}(x, T - t)$, we find a solution $\Phi \in L^\infty\left(0, T; L^\infty(\mathbf{T}^N)\right)$ to the preceding backward problem.

We regularized Φ and u by $\Phi_\varepsilon = \Phi * \rho_\varepsilon$ and $u_\varepsilon = u * \rho_\varepsilon$, respectively. The resulting equation for Φ_ε and u_ε is

$$\begin{aligned} &\frac{\partial \Phi_\varepsilon}{\partial t} - b \cdot \nabla \Phi_\varepsilon = \phi_\varepsilon + \psi_\varepsilon \text{ in } \mathbf{T}^N \times (0, T), && \Phi_\varepsilon|_{t=T} = 0 \ \text{ in } \ \mathbf{T}^N, \\ &\frac{\partial u_\varepsilon}{\partial t} - b \cdot \nabla u_\varepsilon = r_\varepsilon \text{ in } \ \mathbf{T}^N \times (0, T), && u_\varepsilon|_{t=0} = 0 \ \text{ in } \ \mathbf{T}^N \end{aligned}$$

with $\phi_\varepsilon = \phi * \rho_\varepsilon$, $\psi_\varepsilon = (b \cdot \nabla \Phi) * \rho_\varepsilon - b \cdot ((\nabla \Phi) * \rho_\varepsilon)$ and

$$r_\varepsilon = (b \cdot \nabla u) * \rho_\varepsilon - b \cdot ((\nabla u) * \rho_\varepsilon)\,.$$

We have $r_\varepsilon, \psi_\varepsilon \to 0$ (as $\varepsilon \downarrow 0$) in $L^1\left(0, T; L^1(\mathbf{T}^N)\right)$ by Lemma 2.6; the external term ϕ is also allowed in Lemma 2.6. Multiply the second equation by Φ_ε; integrating parts yields

$$-\int_0^T\int_{\mathbf{T}^N} \{u_\varepsilon(\phi_\varepsilon + \psi_\varepsilon) + r_\varepsilon \Phi_\varepsilon\}\, \mathrm{d}x\mathrm{d}t = 0;$$

here we invoke the property that $\operatorname{div} b = 0$. The term $\int_0^T\int_{\mathbf{T}^N} r_\varepsilon \Phi_\varepsilon\, \mathrm{d}x\mathrm{d}t$ tends to zero as $\varepsilon \downarrow 0$ since

$$\left|\int_0^T\int_{\mathbf{T}^N} r_\varepsilon \Phi_\varepsilon\, \mathrm{d}x\mathrm{d}t\right| \leq \|\Phi_\varepsilon\|_{L^\infty(0,T;L^\infty(\mathbf{T}^N))} \int_0^T \|r_\varepsilon\|_{L^1(\mathbf{T}^N)}(t)\, \mathrm{d}t$$

and $\|\Phi_\varepsilon\|_{L^\infty}(t) \leq \|\Phi\|_{L^\infty}(t)$ and $r_\varepsilon \to 0$ in $L^1\left(0, T; L^1(\mathbf{T}^N)\right)$. Similarly, the term $\int_0^T\int_{\mathbf{T}^N} u_\varepsilon \psi_\varepsilon\, \mathrm{d}x\mathrm{d}t$ tends to zero as $\varepsilon \downarrow 0$. Thus, sending ε to zero, we deduce

$$\int_0^T\int_{\mathbf{T}^N} u\phi \mathrm{d}x\mathrm{d}t = 0.$$

□

2.4 Flow Map and Transport Equation

In this section, we shall give the uniqueness of a flow map $X \mapsto x(t, X)$ stated in Theorem 2.2 (ii) using the transport equation. The following discussion admits the existence part (Theorem 2.2 (i)).

Proof of Theorem 2.2 (ii) If there are two different flow maps $x_1(t, X)$ and $x_2(t, X)$, then there is at least one $u_0 \in C^\infty(\mathbf{T}^N)$ such that $u_0(x_1(t, X)) \neq u_0(x_2(t, X))$ for some t and a set of X of positive measure.

For any $u_0 \in C^\infty(\mathbf{T}^N)$, we must prove the uniqueness of $u_0(x(t, X))$. Thanks to Theorem 2.4, it suffices to prove that $u(X, t) = u_0(x(t, X))$ is the unique (weak) solution of (2.7) in $L^\infty\left(0, T; L^\infty(\mathbf{T}^N)\right)$. For notational convenience in the rest of the proof, we will write the flow map $x(t, X)$ by $\varphi(t, x)$, so that the variable in (2.7) becomes x rather than X and $u(x, t) = u_0(\varphi(t, x))$.

For each $\psi \in C^\infty(\mathbf{T}^N)$, $h > 0$, $t \in \mathbf{R}$, we set

$$\begin{aligned}\Delta_h(t) &= \int_{\mathbf{T}^N} \frac{1}{h}\{u(x, t+h) - u(x, t)\}\, \psi(x)\mathrm{d}x \\ &= \int_{\mathbf{T}^N} \frac{1}{h}\{u_0(\varphi(t+h, x)) - u_0(\varphi(t, x))\}\, \psi(x)\mathrm{d}x.\end{aligned}$$

By the group property (2.5), we see that

$$\Delta_h(t) = \int_{\mathbf{T}^N} \frac{1}{h} \left\{ u_0\left(\varphi\left(t, \varphi(h, x)\right)\right) - u_0\left(\varphi(t, x)\right) \right\} \psi(x) \mathrm{d}x.$$

By the volume-preserving property (2.2), we see that

$$\int_{\mathbf{T}^N} u_0\left(\varphi\left(t, \varphi(h, x)\right)\right) \psi(x) \mathrm{d}x = \int_{\mathbf{T}^N} u_0\left(\varphi(t, z)\right) \psi\left(\varphi(-h, z)\right) \mathrm{d}z,$$

where we take $z = \varphi(h, x)$. Thus, we observe that

$$\Delta_h(t) = \int_{\mathbf{T}^N} \frac{1}{h} u(z, t) \left\{ \psi\left(\varphi(-h, z)\right) - \psi(z) \right\} \mathrm{d}z. \tag{2.16}$$

We next note that $(b \circ \varphi) \cdot (\nabla \psi \circ \varphi) \in L^\infty\left(\mathbf{R}, L^1(\mathbf{T}^N)\right)$ since φ has the volume-preserving property (2.2), which implies

$$\int_{\mathbf{T}^N} |b \circ \varphi(x)| \, \mathrm{d}x = \int_{\mathbf{T}^N} |b\left(\varphi(t, x)\right)| \, \mathrm{d}x = \int_{\mathbf{T}^N} |b(x)| \, \mathrm{d}x,$$

$$\int_{\mathbf{T}^N} |\nabla \psi \circ \varphi(x)| \, \mathrm{d}x = \int_{\mathbf{T}^N} |\nabla \psi\left(\varphi(t, x)\right)| \, \mathrm{d}x = \int_{\mathbf{T}^N} |\nabla \psi(x)| \, \mathrm{d}x.$$

Moreover,

$$\frac{\partial}{\partial t}(\psi \circ \varphi) = (b \circ \varphi) \cdot (\nabla \psi \circ \varphi)$$

since $\partial_t \varphi(t, x) = b\left(\varphi(t, x)\right)$ for a.e. $x \in \mathbf{T}^N$ and for all $t \in \mathbf{R}$ by (i). Thus,

$$\psi\left(\varphi(-h, z)\right) - \psi(z) = -\int_0^h b\left(\varphi(-\sigma, z)\right) \cdot (\nabla \psi)\left(\varphi(-\sigma, z)\right) \mathrm{d}\sigma.$$

We plug this formula into (2.16) to get

$$\Delta_h(t) = -\int_{\mathbf{T}^N} \frac{1}{h} u(z, t) \left[\int_0^h b\left(\varphi(-\sigma, z)\right) \cdot (\nabla \psi)\left(\varphi(-\sigma, z)\right) \mathrm{d}\sigma \right] \mathrm{d}z.$$

As previously, we invoke the volume-preserving property (2.2) and the group property (2.5) to get

$$\begin{aligned}
\Delta_h(t) &= -\int_{\mathbf{T}^N} \frac{1}{h} b(x) \cdot \nabla \psi(x) \int_0^h u\left(\varphi(\sigma, x), t\right) \mathrm{d}\sigma \, \mathrm{d}x \\
&= -\int_{\mathbf{T}^N} b(x) \cdot \nabla \psi(x) \frac{1}{h} \int_0^h u_0\left(\varphi\left(t, \varphi(\sigma, x)\right)\right) \mathrm{d}\sigma \, \mathrm{d}x \\
&= -\int_{\mathbf{T}^N} b(x) \cdot \nabla \psi(x) \frac{1}{h} \int_0^h u(x, t + \sigma) \mathrm{d}\sigma \, \mathrm{d}x.
\end{aligned}$$

Since $\varphi = \varphi(t, x)$ is continuous in $t \in \mathbf{R}$ for a.e. $x \in \mathbf{T}^N$,

$$\lim_{h \downarrow 0} \frac{1}{h} \int_0^h u(x, t + \sigma)\, \mathrm{d}\sigma = u(x, t) \quad \text{for all} \quad t$$

for a.e. $x \in \mathbf{T}^N$. Since $b \cdot \nabla\psi \in L^1\left(\mathbf{T}^N \times (0, T)\right)$ and u is bounded on $\mathbf{T}^N \times (0, T)$ independent of h, by the dominated convergence theorem we conclude that

$$\Delta_h(t) \to -\int_{\mathbf{T}^N} b(x) \cdot (\nabla\psi)(x) u(x, t)\, \mathrm{d}x \quad \text{as} \quad h \downarrow 0; \tag{2.17}$$

this convergence is *locally uniform* in $(0, T)$, i.e.,

$$\lim_{h \downarrow 0} \sup_{a \le t \le b} |\Delta_h(t) - \Psi(t)| = 0$$

for any $[a, b] \subset (0, T)$, where Ψ denotes the right-hand side of (2.17).

It is easy to see that

$$\Delta_h(t) \to \frac{\partial}{\partial t} \int_{\mathbf{T}^N} u(x, t)\psi(x)\mathrm{d}x \quad \text{as} \quad h \downarrow 0$$

in the sense of distribution as a function of t. We thus conclude that u satisfies (2.8) for $\phi(x, t) = \psi(x)\eta(t)$, with $\eta \in C_c^\infty([0, T))$. Thus, (2.8) is still valid for any $\phi \in C_c^\infty\left(\mathbf{T}^N \times [0, T)\right)$ since the linear span of the product type is dense in the class of test functions $C_c^\infty\left(\mathbf{T}^N \times [0, T)\right)$. □

▶ **Remark 2.10** In the case of $\mathbf{T}^N$, (2.3) for general β is not invoked for the uniqueness proof; we only use β = identity. However, if one considers the problem in $\mathbf{R}^N$ instead of $\mathbf{T}^N$, it is important to approximate the identity since, in general, only bounded β with bounded $|b(z)| / (1 + |z|)$ is allowed. This restriction is important to understand (2.3) in the distribution sense.

2.5 Notes and Comments

Remarks on Flow Maps and Transport Equations

The contents of Chap. 2 is an active area of current research. The construction of such a flow map $x = x(t, X)$ for non-Lipschitz vector field b is extended when b is just in BV spaces [2]. Although the flow map $x = x(t, X)$ is defined only for almost all $X \in \mathbf{T}^N$, it is known that x is Lipschitz in X with a small exceptional set [4]. The estimate is now quantified by [30]. It is of the following form. For given $T > 0$, $p > 1$, and small $\varepsilon > 0$ there is a compact set K such that $\mathcal{L}^N(\mathbf{T}^N \setminus K) < \varepsilon$ and

$$|x(t, X_1) - x(t, X_2)| \le \exp\left(C_N A_p(x)/\varepsilon^{1/p}\right) |X_1 - X_2|, \; X_1, X_2 \in K, \; t \in [0, T],$$

with C_N depending only on the dimension. Here,

$$A_p(x) = \left\{ \int_{\mathbf{T}^N} \left(\sup_{0 \le t \le T} \sup_{0 < r < 2} \frac{1}{\mathcal{L}^N\,(B_r(X))} \right.\right.$$
$$\left.\left. \int_{B_r(X)} \log\left(\frac{|x(t,X) - x(t,Y)|}{r} + 1 \right) dY \right)^p dX \right\}^{1/p}.$$

For simplicity, we assume that $B_2(X)$ covers the fundamental domain Ω of $\mathbf{T}^N$. The quantity $A_p(x)$ is uniformly controlled by $\|Db\|_{L^1}$ provided that $\operatorname{div} b = 0$. In [30], $\operatorname{div} b$ may not be zero, but some uniform compressibility for the flow map is assumed. Moreover, in [30], the flow map itself is studied directly without using the transport equations.

Our strategy for proving the uniqueness of the flow map in Chap. 2 is to reduce the uniqueness of the transport equation, as stated in Theorem 2.4. However, we warn the reader that the uniqueness of the transport equation fails if one considers a less regular vector field. In fact, if one relaxes the assumption

$$b \in W^{1,p'}(\mathbf{T}^N), \quad \operatorname{div} b = 0$$

by

$$b \in L^{p'}(\mathbf{T}^N), \quad Db \in L^{\tilde{p}}(\mathbf{T}^N), \quad \operatorname{div} b = 0,$$

with $1/p + 1/\tilde{p} > 1 - 1/(N-1)$, then the assertion of Theorem 2.4 fails. In other words, there is a nontrivial weak solution u to (2.7) with zero initial data. This is first proved by Modena and Székelyhidi, Jr. [75] using a convex integration method. A solution constructed there is not a renormalized solution, i.e., the assertion of Lemma 2.5 does not hold for their solution u. This can be understood as meaning there is a microscopic effect that cannot be captured by the macroscopic notion of a weak solution. Recently, a nonrenormalized weak solution was constructed by Drivas et al. [34] using a vanishing viscosity method with anomalous dissipation. As pointed out in [86], such a solution is produced by a microscopic effect. The notion of a weak solution is too weak to guarantee uniqueness even for linear transport equations. In a very recent preprint, Huysmans and Titi [55] proved that the uniqueness may fail even among renormalized solutions if one only assumes that $b = b(x,t)$ is bounded with $\operatorname{div} b = 0$. (Note that their b depends on time t.) They constructed two different solutions which are given as subsequential vanishing viscosity limits, of the same equation.

In the next two chapters, we will discuss scalar conservation laws and the Hamilton–Jacobi equations, where a naive "weak solution" may not be unique. For these equations one is able to recover uniqueness by considering a special class of weak solutions.

It is of current interest to show the nonuniqueness of weak solutions for various physically important nonlinear equations, even if the viscosity is included, for example, the Navier–Stokes equations [20]. However, it is not clear what kind of extra condition would guarantee uniqueness.

The contents of Chap. 2 are taken from the paper [32], where the problem is studied on $\mathbf{R}^N$. In this book, we consider the problem on $\mathbf{T}^N$ to simplify the situation. Lemma 2.7 is a crucial step of the argument and is often called DiPerna–Lion's lemma. A variant of this lemma is called Friedrichs' commutator lemma in [39, Section 11.19]. This type of lemma is useful for studying mass conservation laws for compressible flows.

2.6 Exercises

2.1 Give an example of the nonuniqueness of a solution to (1.1) with a given initial datum when $b \in \bigcap_{p\geq 1} W^{1,p}(\mathbf{R}^N)$.

2.2 Set

$$L = \left\{\phi : \mathbf{T}^N \to \mathbf{R} \;\middle|\; \phi \text{ is Lebesgue measurable}\right.$$
$$\left.\text{and } |\phi| < \infty \text{ a.e.}\right\}.$$

Set $d(\phi, \psi) = \|\min(|\phi - \psi|, 1)\|_{L^1(\mathbf{T}^N)}$. Show that (L, d) is a metric space.

2.3 Let f be a locally integrable function in $(0, T)$. Assume that

$$\int_0^T f(t)\psi(t)\,\mathrm{d}t = 0$$

for all $\psi \in C_c^\infty((0, T))$. Show that $f(t) = 0$ for almost all $t \in (0, T)$.

2.4 Let ρ_ε be a Friedrichs' mollifier. Let f be continuous on $\mathbf{R}$. Show that $\rho_\varepsilon * f$ is in $C^\infty(\mathbf{R})$.

2.5 In the context of Exercise 2.4, show that $\rho_\varepsilon * f$ converges to f locally uniformly in $\mathbf{R}$ as ε tends to zero.
See [45] for details of Exercises 2.3–2.5.

2.6 Let L be the space defined in Exercise 2.2. Set

$$\overline{d}(\phi, \psi) = \int_{\mathbf{T}^N} \frac{|\phi(x) - \psi(x)|}{1 + |\phi(x) - \psi(x)|}\,\mathrm{d}x.$$

Show that $(L, \overline{d})$ is a metric space.

2.7 Assume that $\{f_m\}_{m=1}^\infty$ is a sequence converging to f in $L^1(\mathbf{T}^N)$ as $m \to \infty$. In other words,

$$\lim_{m\to\infty} \int_{\mathbf{T}^N} |f_m(x) - f(x)|\,\mathrm{d}x = 0.$$

Show that $\{f_m\}$ converges to f in measure.

2.8 Assume that $\varphi : \mathbf{T}^N \to \mathbf{T}^N$ is a volume-preserving mapping. In other words, φ has the property that

$$\mathcal{L}^N\left(\left\{x \in \mathbf{T}^N \mid \varphi(x) \in A\right\}\right) = \mathcal{L}^N(A)$$

for any measurable set A. Show that

$$\int_{\mathbf{T}^N} \psi\left(\varphi(x)\right) \mathrm{d}x = \int_{\mathbf{T}^N} \psi(x) \, \mathrm{d}x$$

for any measurable function ψ on $\mathbf{T}^N$.

2.9 (i) For $p \in [1, \infty)$, let p' denote the conjugate exponent of p, i.e., $1/p + 1/p' = 1$. Assume that a sequence $\{f_m\}_{m=1}^\infty$ converges to f in $L^p(\mathbf{T}^N)$ as $m \to \infty$. In other words,

$$\lim_{m\to\infty} \int_{\mathbf{T}^N} |f_m(x) - f(x)|^p \, \mathrm{d}x = 0.$$

Assume that a sequence $\{g_m\}_{m=1}^\infty$ converges $*$-weakly to g in $L^{p'}(\mathbf{T}^N)$ as $m \to \infty$. In other words,

$$\lim_{m\to\infty} \int_{\mathbf{T}^N} g_m(x)\varphi(x) \, \mathrm{d}x = \int_{\mathbf{T}^N} g(x)\varphi(x) \, \mathrm{d}x$$

holds for all $\varphi \in L^p(\mathbf{T}^N)$. Show that

$$\lim_{m\to\infty} \int_{\mathbf{T}^N} f_m(x)g_m(x) \, \mathrm{d}x = \int_{\mathbf{T}^N} f(x)g(x) \, \mathrm{d}x.$$

(ii) Set $f_m(x) = \sin mx \in L^2(\mathbf{T})$, where $\mathbf{T} = \mathbf{R}/(2\pi\mathbf{Z})$. Show that $\{f_m\}_{m=1}^\infty$ converges weakly to 0 in $L^2(\mathbf{T})$ but

$$\lim_{m\to\infty} \int_{\mathbf{T}^N} f_m(x)^2 \, \mathrm{d}x \neq 0.$$

2.10 Let ρ_ε be a Friedrichs' mollifier. For $f \in L^\infty(\mathbf{T}^N)$, show that $\rho_\varepsilon * f$ converges to f a.e. as ε tends to zero. Moreover, show that

$$\|f\|_{L^\infty(\mathbf{T}^N)} = \lim_{\varepsilon\downarrow 0} \|f_\varepsilon\|_{L^\infty(\mathbf{T}^N)}, \quad \|f_\varepsilon\|_{L^\infty(\mathbf{T}^N)} \leq \|f\|_{L^\infty(\mathbf{T}^N)}.$$