The Supergeometric Algebra as the Language of Physics

Andrew J. S. Hamilton(✉)

JILA and Department of Astrophysical and Planetary Sciences, U. Colorado Boulder, Box 440, Boulder, CO 80309, USA
Andrew.Hamilton@colorado.edu
http://jila.colorado.edu/ ajsh/

Abstract. It is shown how the fermions and forces of Nature fit elegantly into the Supergeometric Algebra in 11+1 spacetime dimensions.

Keywords: Supergeometric Algebra · Geometric Algebra · Spinors

1 Introduction

The author's interest in spinors is sparked by the fact that spinors seem to be the fundamental objects from which physics is built. All known forms of matter (leptons and quarks) are made from spinors. And all known interactions, namely the three forces of the standard model, plus gravity, emerge from symmetries of spinors. The present paper, which is based on [1], shows how this works.

The present paper is a companion to [2], which presents a pedagogical introduction to the Supergeometric Algebra (SGA), the square root of the Geometric Algebra (GA). A central message of [2] is that a spinor, the fundamental representation of the group Spin(N) of rotations in N spacetime dimensions, is indexed by a bitcode with $[N/2]$ bits.

2 The Electron as a Dirac Spinor

A Dirac spinor is a spinor in 3+1 spacetime dimensions. It has $4/2 = 2$ bits, a boost bit ($\Uparrow$ or $\Downarrow$), and a spin bit ($\uparrow$ or $\downarrow$). A Dirac spinor is said to be right-handed if its boost and spin bits align, left-handed if they anti-align. Altogether, a Dirac spinor has $2^2 = 4$ complex components, or 8 real components. The 4 complex components of a Dirac electron, grouped into right- and left-handed (R and L) are:

$$e_R : \ e_{\Uparrow\uparrow} \, , \ e_{\Downarrow\downarrow} \ , \quad e_L : \ e_{\Downarrow\uparrow} \, , \ e_{\Uparrow\downarrow} \ . \tag{1}$$

The right- and left-handed components e_R and e_L are called the Weyl components of the electron, and they are massless. The massive electrons and positrons observed in Nature are linear combinations of right- and left-handed components.

D. W. Silva et al. (Eds.): ICACGA 2022, LNCS 13771, pp. 161–173, 2024.
https://doi.org/10.1007/978-3-031-34031-4_14

Electrons e and positrons $\bar{e}$ in their rest frames are complex conjugates of each other:

$$e_{\uparrow} = \tfrac{1}{\sqrt{2}}(e_{\Uparrow\uparrow} - i e_{\Downarrow\uparrow}) \ , \qquad e_{\downarrow} = \tfrac{1}{\sqrt{2}}(e_{\Downarrow\downarrow} - i e_{\Uparrow\downarrow}) \ , \tag{2a}$$

$$i\bar{e}_{\uparrow} = \tfrac{1}{\sqrt{2}}(e_{\Uparrow\uparrow} + i e_{\Downarrow\uparrow}) \ , \qquad i\bar{e}_{\downarrow} = \tfrac{1}{\sqrt{2}}(e_{\Downarrow\downarrow} + i e_{\Uparrow\downarrow}) \ . \tag{2b}$$

3 The Electron as a Spin(10) Spinor

That the chiral nature, right- or left-handed, of the electron should be taken seriously follows from the fact that only left-handed electrons feel the weak $\mathrm{SU}_L(2)$ force: right-handed electrons feel no weak force.

The standard model of physics is based on $\mathrm{U}_Y(1) \times \mathrm{SU}_L(2) \times \mathrm{SU}(3)$, the product of the hypercharge, left-handed weak, and color groups. At energies less the electroweak scale $\sim 100\,\mathrm{GeV}$, the symmetry of the hypercharge and weak groups breaks to the electromagnetic symmetry, $\mathrm{U}_Y(1) \times \mathrm{SU}_L(2) \rightarrow \mathrm{U}_{\mathrm{em}}(1)$. The method of electroweak symmetry breaking proposed by Weinberg (1967) [3], based on the so-called Higgs mechanism [4,5], has received spectacular experimental confirmation, culminating with the detection of the electroweak Higgs boson, with a mass 125 GeV, at the Large Hadron Collider in 2012 [6,7].

The success of the electroweak symmetry-breaking model prompted proposals in the mid-1970s that the three groups of the standard model would themselves become unified in a so-called Grand Unified Theory (GUT) group, at an energy that was estimated from the running of the three coupling parameters to be at $\sim 10^{14}$–10^{16} GeV. Three possible GUT groups fit the observed pattern of charges of fermions, of which the most unifying was Spin(10) (the covering group of SO(10)), first pointed out by [8,9]. The other two possible GUT groups, SU(5) proposed by [10], and the Pati-Salam group $\mathrm{Spin}(4) \times \mathrm{Spin}(6)$ proposed by [11], are subgroups of Spin(10).

As first pointed out by Wilczek in 1998 [12], and reviewed by Baez & Huerta [13], a spinor of Spin(10) is described by a bitcode with $10/2 = 5$ bits, consisting of 2 weak bits and 3 color bits. Wilczek and Baez & Huerta proposed different conventions for naming the bits. My own preference is to label the color bits r, g, b, following [13], and the weak bits y and z, inspired by the fact that y and z are infrared bands to be used by the Vera Rubin Observatory (the LSST) [14], for which first light is expected in 2025. The sequence $yzrgb$ is, in (inverse) order of wavelength,

$$y \sim 1000\,\mathrm{nm} \ , \quad z \sim 900\,\mathrm{nm} \ , \quad r \sim 600\,\mathrm{nm} \ , \quad g \sim 500\,\mathrm{nm} \ , \quad b \sim 400\,\mathrm{nm} \ . \tag{3}$$

This is an electron in Spin(10), labeled according to its $yzrgb$ bits (colored silver, bronze, red, green, blue):

$$\bar{e}_R : \ \downarrow\uparrow\downarrow\downarrow\downarrow \ , \quad \bar{e}_L : \ \uparrow\uparrow\downarrow\downarrow\downarrow \ , \quad e_R : \ \downarrow\downarrow\uparrow\uparrow\uparrow \ , \quad e_L : \ \uparrow\downarrow\uparrow\uparrow\uparrow \ . \tag{4}$$

Flipping all 5 $yzrgb$ bits flips between electron and positron. Flipping the y-bit flips between right- and left-handed. In the Spin(10) picture, each of the

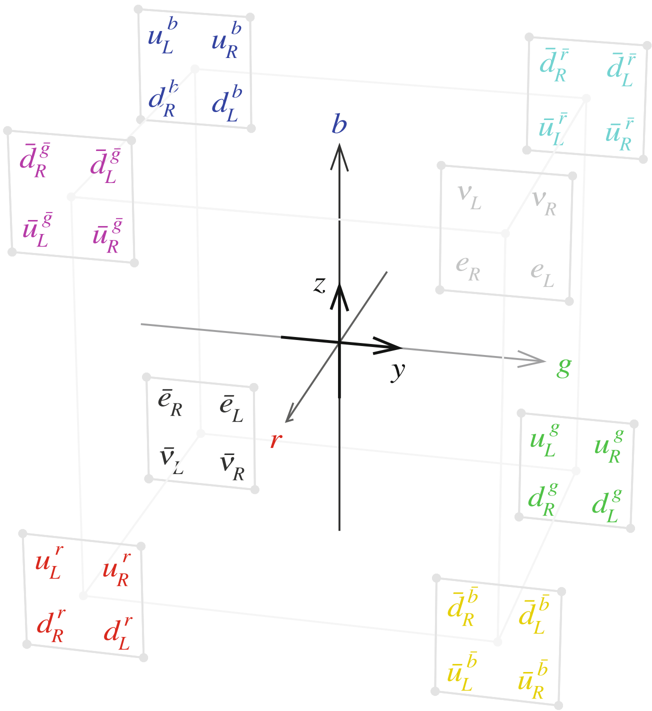

Fig. 1. The electron generation of 32 fermions arranged according to their Spin(10) $yzrgb$ charges. A Spin(11, 1) version of this figure is Fig. 2

right- and left-handed components is itself a Weyl spinor, with two complex components.

In the standard model, fundamental fermions come in 3 generations, the electron, muon, and tauon generations. The three generations of fermions differ only in their masses: the standard model charges of each generation replicate each other. Only fermions come in three generations. The gauge bosons that mediate the forces, the interactions between fermions, are the same for all generations: there is only one “boson generation.” This suggests that the 3 generations are not just another symmetry to be adjoined to the standard model. What causes the 3 generations remains a deep mystery of physics.

The lightest fermion generation is the electron generation. The fermions of the electron generation comprise 8 species, consisting of electrons and neutrinos, and 3 colors each of down and up quarks. Each of the 8 species comes in right- and left-handed varieties, and in particle and antiparticle versions, for a total of 32 fermion types. Each of those fermion types can be either spin-up or spin-down, for a total of 64 degrees of freedom. The pattern repeats for each of the 3 generations. Although no right-handed neutrino has been observed in Nature, the fact that the left-handed neutrino carries a non-zero mass strongly suggests that a right-handed neutrino should exist, since a purely left-handed neutrino would be massless.

Figure 1 shows the 32 fermions of the electron generation, arranged according to their $yzrgb$ charges. The same information illustrated in Fig. 1 is tabulated in the following Spin(10) chart, which arrays the fermions in columns according to the number of up-bits (compare Table 4 of [13]; see also [12]). The left element of each entry (before the colon) signifies which bits are up, from $-$ (no bits up, or $\downarrow\downarrow\downarrow\downarrow\downarrow$) in the leftmost (0) column, to $yzrgb$ (all bits up, or $\uparrow\uparrow\uparrow\uparrow\uparrow$) in the rightmost (5) column; the right element of each entry is the corresponding fermion, which comprise (electron) neutrinos ν, electrons e, and up and down quarks u and d, each in right- and left-handed Dirac chiralities R and L, and each in (unbarred) particle and (barred) antiparticle species, a total of $2^5 = 32$ fermions:

Fermions and their Spin(10) bitcodes, arranged by the number of up-bits

0	1	2	3	4	5	
$-:\ \bar{\nu}_L$	$y:\ \bar{\nu}_R$	$\bar{c}:\ \bar{u}^{\bar{c}}_L$	$y\bar{c}:\ \bar{u}^{\bar{c}}_R$	$zrgb:\ \nu_L$	$yzrgb:\ \nu_R$	(5)
	$z:\ \bar{e}_R$	$yz:\ \bar{e}_L$	$rgb:\ e_R$	$yrgb:\ e_L$		
	$c:\ d^c_R$	$yc:\ d^c_L$	$z\bar{c}:\ \bar{d}^{\bar{c}}_R$	$yz\bar{c}:\ \bar{d}^{\bar{c}}_L$		
		$zc:\ u^c_L$	$yzc:\ u^c_R$			

Here c denotes any of the three colors r, g, or b (one color bit up), while $\bar{c}$ denotes any of the three anticolors gb, br, or rg (two color bits up, the bit flip of a one-color-bit-up spinor).

The Spin(10) chart (5) of fundamental fermions is a Christmas puzzle of striking features. The most striking feature is that Dirac chirality (subscripted L or R in the chart) coincides with Spin(10) chirality. Spin(10) chirality counts whether the number of Spin(10) $yzrgb$ up-bits is even or odd: the even and odd columns of the chart (5) have respectively left- and right-handed Spin(10) chirality. In any GA, chirality is the eigenvalue, ± 1, of the pseudoscalar (normalized by a phase so the eigenvalues are real). The coincidence of Dirac and Spin(10) chiralities suggests that the pseudoscalars of the Dirac and Spin(10) geometric algebras are somehow the same, in contrast to the usual assumption that the Dirac and GUT algebras are distinct.

The second striking feature of the Spin(10) chart (5) is that standard-model transformations connect fermions vertically, while Lorentz transformations connect fermions (for the most part) horizontally. For example, electrons e and positrons $\bar{e}$ are arrayed along one row of the chart. Every Spin(N) group has a subgroup SU($[N/2]$) that preserves the number of up-bits [15]. The columns of the chart (5) are SU(5) multiplets within Spin(10), with dimensions respectively 1, 5, 10, 10, 5, 1. The standard-model group is a subgroup of SU(5). All standard-model interactions preserve the number of Spin(10) up-bits. With standard-model transformations arrayed vertically and spacetime transformations arrayed horizontally, the chart (5) seems to be signalling that the two are somehow connected.

The third striking feature of the Spin(10) chart (5) is that right- and left-handed versions of the same species (for example electrons e_R and e_L) differ by a flip of the y-bit. In the Spin(10) picture, electroweak symmetry breaking is a loss of y-symmetry. The electroweak Higgs field carries y-charge, and it gives mass to fermions by flipping their y-bit. This is prettier than the somewhat abstruse traditional description $\mathrm{U}_Y(1) \times \mathrm{SU}_L(2) \rightarrow \mathrm{U}_{\mathrm{em}}(1)$ of electroweak symmetry breaking.

4 The Electron as a Spin(11,1) Spinor

The two guises of each generation of fermions, on the one hand as spinors of the Spin$(3,1)$ Dirac algebra under Lorentz transformations, and on the other hand as spinors of the Spin(10) algebra under standard model transformations, cry out for unification in a common algebra. Each of the 2^5 entries in the Spin(10) chart (5) is a Weyl fermion with 2 components, so the unified algebra, if it exists, must have 6 bits and 12 dimensions. And since the Dirac algebra has a time dimension while Spin(10) has none, one of the extra dimensions must be a time dimension, and the extra bit must be a boost bit. The algebra must be that of Spin$(11,1)$ in 11+1 spacetime dimensions. The extra bit can be labeled the t-bit, or time bit.

The conclusion that the unified algebra should have 11+1 spacetime dimensions conflicts with the usual assumption that the Dirac and Spin(10) algebras combine as a direct product, in which case the 3+1 dimensions of the Dirac algebra and the 10 dimensions of the Spin(10) algebra would yield 13+1 spacetime dimensions.

The standard assumption that Dirac and GUT algebras combine as a direct product is motivated by the Coleman-Mandula no-go theorem [16,17], which says, roughly, that any gauge group that contains the Poincaré group of spacetime symmetries and admits non-trivial analytic elastic scattering is necessarily a direct product of the Poincaré group and a commuting group of internal symmetries. The Coleman-Mandula theorem generalizes to higher dimensions [18].

However, if the grand unified group is Spin$(11,1)$, then all grand symmetries are spacetime symmetries, and there are no additional internal symmetries, so the higher-dimensional Coleman-Mandula theorem [18] is satisfied trivially. After grand symmetry breaking, the Coleman-Mandula theorem requires only that spacetime and *unbroken* internal symmetries combine as a direct product. In the present context, the Coleman-Mandula theorem requires that the Dirac and standard-model algebras combine as commuting subalgebras of the Spin$(11,1)$ algebra.

Encouragement that 11+1 dimensions is the right number comes from the period-8 Cartan-Bott periodicity [19–21] of geometric algebras. which guarantees that the discrete symmetries of the Spin$(11,1)$ algebra are the same as those of the Dirac Spin$(3,1)$ algebra: the spinor metric is antisymmetric, while the conjugation operator is symmetric.

If indeed the unified algebra is that of Spin$(11,1)$, then the Spin(10) chart (5) cannot be quite right as it stands. Diagnosing the problem, and then solving it,

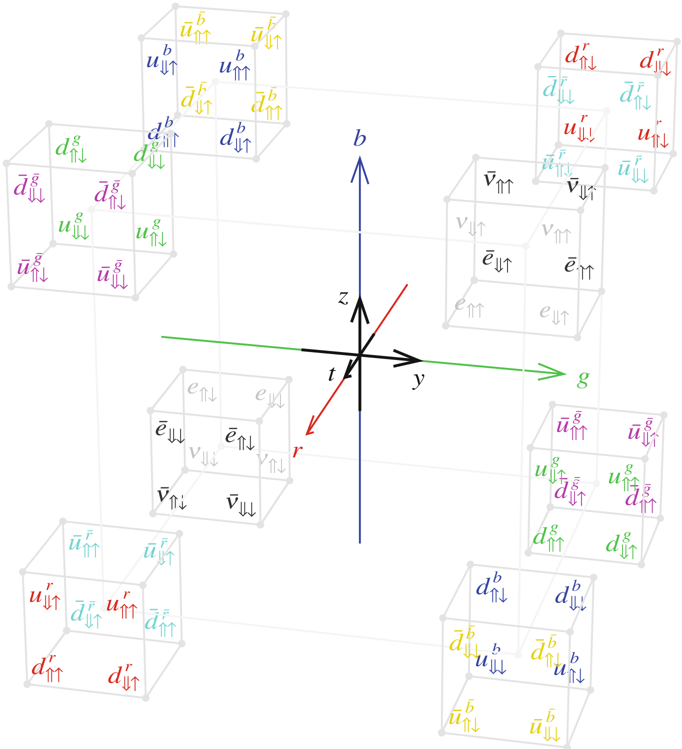

Fig. 2. The electron generation of $2^6 = 64$ fermions arranged according to their Spin$(11,1)$ $tyzrgb$ charges. This is similar to Fig. 1, but with the addition of the t-bit

is tricky. It's a Christmas puzzle. The loophole in the chart is that it assigns a definite charge to each fermion based on its Spin(10) charges, whereas the example of Eq. (2) shows that fermions and antifermions, which have opposite charges, are linear combinations of the same chiral components. Fermions and antifermions are distinguished by the fact that they are complex conjugates of each other; more precisely, the antiparticle of a spinor ψ is the anti-spinor $\bar{\psi} \equiv C\psi^*$, where C is the conjugation operator. The conjugation operator in Spin$(11,1)$ proves to be the same as the conjugation operator in Spin(10): the conjugation operator in Spin$(11,1)$ flips all bits except the time bit, so flips all 5 $yzrgb$ Spin(10) bits, as does the conjugation operator in Spin(10).

The solution to the unification problem is to replace each 2-component Weyl fermion in the Spin(10) chart (5) with a 2-component fermion with t-bit respectively up and down, with opposite Dirac boost but the same Dirac spin, a fermion and an antifermion. The Weyl companion of each fermion is identified as the fermion with all 6 $tyzrgb$ bits flipped. This is similar to the Dirac algebra, where the Weyl companion of for example the right-handed electron $e_{\Uparrow\uparrow}$ is its all-bit-flip partner $e_{\Downarrow\downarrow}$.

This is a Dirac electron in Spin$(11,1)$, labeled according to its $tyzrgb$ bits (colored gold, silver, bronze, red, green, blue):

$$
\begin{aligned}
&\bar{e}_{\Downarrow\downarrow} : \uparrow\downarrow\uparrow\downarrow\downarrow\downarrow \ , \quad \bar{e}_{\Uparrow\downarrow} : \uparrow\uparrow\uparrow\downarrow\downarrow\downarrow \ , \quad e_{\Uparrow\uparrow} : \uparrow\downarrow\downarrow\uparrow\uparrow\uparrow \ , \quad e_{\Downarrow\uparrow} : \uparrow\uparrow\downarrow\uparrow\uparrow\uparrow \ , \\
&e_{\Uparrow\downarrow} : \downarrow\downarrow\uparrow\downarrow\downarrow\downarrow \ , \quad e_{\Downarrow\downarrow} : \downarrow\uparrow\uparrow\downarrow\downarrow\downarrow \ , \quad \bar{e}_{\Downarrow\uparrow} : \downarrow\downarrow\downarrow\uparrow\uparrow\uparrow \ , \quad \bar{e}_{\Uparrow\uparrow} : \downarrow\uparrow\downarrow\uparrow\uparrow\uparrow \ .
\end{aligned}
\tag{6}
$$

Flipping the time bit t flips between electrons e and positrons $\bar{e}$. Flipping the y bit flips Dirac chirality. Flipping all 6 bits spatially rotates the spin of the electron (or positron) between up and down, preserving chirality.

Figure 2 illustrates one generation (the electron generation) of fermions of the standard model arranged according to their Spin$(11,1)$ $tyzrgb$ charges. The same information illustrated in Fig. 2 is tabulated in the following Spin$(11,1)$ chart of spinors, arranged in columns by the number of Spin(10) up-bits as in the earlier Spin(10) chart (5):

0	1	2	3	4	5
$- : \begin{matrix}\bar\nu_{\Uparrow\downarrow}\\ \nu_{\Downarrow\downarrow}\end{matrix}$	$y : \begin{matrix}\bar\nu_{\Downarrow\downarrow}\\ \nu_{\Uparrow\downarrow}\end{matrix}$	$\bar{c} : \begin{matrix}\bar u^{\bar c}_{\Uparrow\downarrow}\\ u^{c}_{\Downarrow\downarrow}\end{matrix}$	$y\bar{c} : \begin{matrix}\bar u^{\bar c}_{\Downarrow\downarrow}\\ u^{c}_{\Uparrow\downarrow}\end{matrix}$	$zrgb : \begin{matrix}\nu_{\Downarrow\uparrow}\\ \bar\nu_{\Uparrow\uparrow}\end{matrix}$	$yzrgb : \begin{matrix}\nu_{\Uparrow\uparrow}\\ \bar\nu_{\Downarrow\uparrow}\end{matrix}$
	$z : \begin{matrix}\bar e_{\Downarrow\downarrow}\\ e_{\Uparrow\downarrow}\end{matrix}$	$yz : \begin{matrix}\bar e_{\Uparrow\downarrow}\\ e_{\Downarrow\downarrow}\end{matrix}$	$rgb : \begin{matrix}e_{\Uparrow\uparrow}\\ \bar e_{\Downarrow\uparrow}\end{matrix}$	$yrgb : \begin{matrix}e_{\Downarrow\uparrow}\\ \bar e_{\Uparrow\uparrow}\end{matrix}$	
	$c : \begin{matrix}d^{c}_{\Uparrow\uparrow}\\ \bar d^{\bar c}_{\Downarrow\uparrow}\end{matrix}$	$yc : \begin{matrix}d^{c}_{\Downarrow\uparrow}\\ \bar d^{\bar c}_{\Uparrow\uparrow}\end{matrix}$	$z\bar{c} : \begin{matrix}\bar d^{\bar c}_{\Downarrow\downarrow}\\ d^{c}_{\Uparrow\downarrow}\end{matrix}$	$yz\bar{c} : \begin{matrix}\bar d^{\bar c}_{\Uparrow\downarrow}\\ d^{c}_{\Downarrow\downarrow}\end{matrix}$	
		$zc : \begin{matrix}u^{c}_{\Downarrow\uparrow}\\ \bar u^{\bar c}_{\Uparrow\uparrow}\end{matrix}$	$yzc : \begin{matrix}u^{c}_{\Uparrow\uparrow}\\ \bar u^{\bar c}_{\Downarrow\uparrow}\end{matrix}$		

(7)

whereas in the original Spin(10) chart (5) each entry was a 2-component Weyl spinor, in the Spin$(11,1)$ chart (7) the 2 components of each Weyl spinor appear in bit-flipped entries. For example, the right-handed electron e_R of the original chart is replaced by $e_{\Uparrow\uparrow}$, and its spatially rotated partner $e_{\Downarrow\downarrow}$ of the same chirality appears in the all-bit-flipped entry. Each entry still has two components, but in the Spin$(11,1)$ chart those two components differ by their t-bit; the upper component has t-bit up, the lower t-bit down. The net number of degrees of freedom remains the same, $2^6 = 64$.

In the unified Spin$(11,1)$ algebra, the Dirac boost and spin of a fermion are woven into the algebra, no longer dissociated from Spin(10). The Dirac boost $\Uparrow$ or $\Downarrow$ is the eigenvalue of the weak chiral operator $\varkappa_{tyz}$, which counts whether the number of tyz up-bits is odd or even. The Dirac spin $\uparrow$ or $\downarrow$ is the eigenvalue of the color chiral operator $\varkappa_{rgb}$, which counts whether the number of color rgb up-bits is odd or even. The weak and color chiral operators $\varkappa_{tyz}$ and $\varkappa_{rgb}$ are equal to weak and color pseudoscalars I_{tyz} and I_{rgb} modified by a phase factor to make their eigenvalues real:

$$
I_{tyz} \equiv -i\gamma_t^+\gamma_t^-\gamma_y^+\gamma_y^-\gamma_z^+\gamma_z^- \ = \ -\varkappa_{tyz} \equiv -\gamma_t \wedge \gamma_{\bar t} \wedge \gamma_y \wedge \gamma_{\bar y} \wedge \gamma_z \wedge \gamma_{\bar z} \ , \tag{8a}
$$

$$
I_{rgb} \equiv \gamma_r^+\gamma_r^-\gamma_g^+\gamma_g^-\gamma_b^+\gamma_b^- \ = \ -i\varkappa_{rgb} \equiv -i\gamma_r \wedge \gamma_{\bar r} \wedge \gamma_g \wedge \gamma_{\bar g} \wedge \gamma_b \wedge \gamma_{\bar b} \ . \tag{8b}
$$

The 12-dimensional pseudoscalar J is the product of the boost operator I_{tyz} and the spin operator I_{rgb},

$$\begin{aligned} J \equiv I_{tyz} I_{rgb} &= -i \gamma_t^+ \gamma_t^- \gamma_y^+ \gamma_y^- \gamma_z^+ \gamma_z^- \gamma_r^+ \gamma_r^- \gamma_g^+ \gamma_g^- \gamma_b^+ \gamma_b^- \\ &= i \varkappa_{12} \equiv i \gamma_t \wedge \gamma_{\bar t} \wedge \gamma_y \wedge \gamma_{\bar y} \wedge \gamma_z \wedge \gamma_{\bar z} \wedge \gamma_r \wedge \gamma_{\bar r} \wedge \gamma_g \wedge \gamma_{\bar g} \wedge \gamma_b \wedge \gamma_{\bar b} \ . \end{aligned} \tag{9}$$

In the Dirac algebra, the charge of a chiral fermion is ambiguous: a fermion and its antifermion partner, which have opposite charges, are linear combinations of the same chiral components, Eq. (2). The t-bit removes the ambiguity, specifying whether a fermion is going forwards or backwards in time. The charge of a fermion is determined unambiguously by its 6 $tyzrgb$ bits. In Spin(10), the standard-model charges of a fermion can be read off from its 5 $yzrgb$ bits. In Spin(11, 1), the standard-model charges are equal to Spin(10) charges multiplied by the color chiral operator $\varkappa_{rgb}$, as is evident from the fact that the spinors in the Spin(10) chart (5) are fermions (unbarred) or antifermions (barred) depending on whether their color chirality is odd or even.

In Spin(10), the 5 standard-model charges are eigenvalues of the 5 diagonal bivector generators of Spin(10),

$$\tfrac{1}{2} \gamma_i^+ \wedge \gamma_i^- = \tfrac{i}{2} \gamma_i \wedge \gamma_{\bar\imath} \ , \quad i = y, z, r, g, b \ . \tag{10}$$

In Spin(11, 1), standard-model charges are eigenvalues of the 5 diagonal bivectors (10) multiplied by the color chiral operator $\varkappa_{rgb}$. A consistent way to implement this modification, that leaves the bivector algebra of the standard model unchanged, is to multiply all imaginary bivectors $\gamma_i^+ \gamma_j^-$ in the Spin(10) geometric algebra by $\varkappa_{rgb}$, while leaving all real bivectors $\gamma_i^+ \gamma_j^+$ and $\gamma_i^- \gamma_j^-$ unchanged,

$$\gamma_i^+ \gamma_j^- \to \gamma_i^+ \gamma_j^- \varkappa_{rgb} \quad i, j = y, z, r, g, b \ . \tag{11}$$

Equivalently, replace the imaginary i in all Spin(10) bivectors by the color pseudoscalar $-I_{rgb} = i \varkappa_{rgb}$, Eq. (8b). A key point that allows this adjustment to be made consistently is that $\varkappa_{rgb}$ commutes with all standard-model bivectors. Note that $\varkappa_{rgb}$ does not commute with SU(5) bivectors that transform between leptons and quarks; but that is fine, because SU(5) is not an unbroken symmetry of the standard model.

The definitive proof that unification in Spin(11, 1) is consistent comes from expressing the 4 orthonormal vectors γ_m, $m = 0, 1, 2, 3$, of the Dirac algebra in terms of the 12 orthonormal vectors $\gamma_i^\pm$, $i = t, y, z, r, g, b$ of the Spin(11, 1) algebra:

$$\gamma_0 = i \gamma_t^- \ , \tag{12a}$$

$$\gamma_1 = \gamma_y^- \gamma_z^- \gamma_r^+ \gamma_g^+ \gamma_b^+ \ , \tag{12b}$$

$$\gamma_2 = \gamma_y^- \gamma_z^- \gamma_r^- \gamma_g^- \gamma_b^- \ , \tag{12c}$$

$$\gamma_3 = \gamma_t^+ \gamma_y^+ \gamma_y^- \gamma_z^+ \gamma_z^- \ . \tag{12d}$$

The Dirac vectors (12) all have grade 1 mod 4 in the Spin(11, 1) algebra. The multiplication rules for the Dirac vectors γ_m given by Eq. (12) agree with the usual multiplication rules for Dirac γ-matrices: the vectors $\boldsymbol{\gamma}_m$ anticommute, and their scalar products form the Minkowski metric. All the spacetime vectors $\boldsymbol{\gamma}_m$ commute with all standard-model generators modified per (11). The Dirac pseudoscalar I coincides with the Spin(11, 1) pseudoscalar J, Eq. (9),

$$I \equiv \boldsymbol{\gamma}_0 \boldsymbol{\gamma}_1 \boldsymbol{\gamma}_2 \boldsymbol{\gamma}_3 = J \ . \tag{13}$$

Thus the Dirac and standard-model algebras are subalgebras of the Spin(11, 1) geometric algebra, such that all Dirac generators commute with all standard-model generators modified per (11), consistent with the Coleman-Mandula theorem.

The time dimension (12a) is just a simple vector in the Spin(11, 1) algebra, but the 3 spatial dimensions (12b)–(12d) are all 5-dimensional. The spatial dimensions share a common 2-dimensional factor $\boldsymbol{\gamma}_y^- \boldsymbol{\gamma}_z^-$. Aside from that common factor, each of the 3 spatial dimensions is itself 3-dimensional: $\boldsymbol{\gamma}_r^+ \boldsymbol{\gamma}_g^+ \boldsymbol{\gamma}_b^+$, $\boldsymbol{\gamma}_r^- \boldsymbol{\gamma}_g^- \boldsymbol{\gamma}_b^-$, and $\boldsymbol{\gamma}_t^+ \boldsymbol{\gamma}_y^+ \boldsymbol{\gamma}_z^+$.

5 Predictions of the Spin(11,1) Theory

A first response to any new theory is, Does it make any predictions? Much of [1] is devoted to answering this question. The specific question is, what predictions can be made if the Grand Unified group is Spin(11, 1) and no additional ingredients are admitted? The condition of no additional ingredients is highly restrictive.

The end result is that the theory predicts the following sequence of symmetry breakings, at energies determined by the running of coupling parameters:

$$\begin{aligned} &\mathrm{Spin}(11,1) \xrightarrow[??]{} \mathrm{Spin}(10,1) \xrightarrow[10^{15}\,\mathrm{GeV}]{} \mathrm{Spin}(4) \times \mathrm{Spin}(6) \xrightarrow[10^{12}\,\mathrm{GeV}]{} \\ &\qquad \mathrm{U}_Y(1) \times \mathrm{SU}_L(2) \times \mathrm{SU}(3) \xrightarrow[100\,\mathrm{GeV}]{} \mathrm{U}_{\mathrm{em}}(1) \times \mathrm{SU}(3) \ . \end{aligned} \tag{14}$$

The top line of the sequence (14) is the prediction, while the bottom line is the standard model.

The addition of the 6th bit, the time bit t, to the 5 $yzrgb$ bits of Spin(10) adjoins to the bivectors of Spin(10) additional bivectors involving either or both of the two extra dimensions $\boldsymbol{\gamma}_t^\pm$. Of those bivectors, four commute with all the Dirac vectors $\boldsymbol{\gamma}_m$ defined by Eq. (12), and could therefore potentially play a role in the standard model. The four happen to have precisely the properties of the 4-component electroweak Higgs multiplet required by the Weinberg [3] model of electroweak symmetry breaking, motivating the identification of the electroweak Higgs field $\boldsymbol{H}$ as (with the bivectors being understood to be modified per (11) as usual)

$$\boldsymbol{H} \equiv H^{i^\pm} \boldsymbol{\gamma}_t^+ \boldsymbol{\gamma}_i^\pm \ , \quad i = y, z \ . \tag{15}$$

Electroweak symmetry breaking occurs when the Higgs field acquires a vacuum expectation value $\langle \boldsymbol{H} \rangle$ proportional to $\boldsymbol{\gamma}_t^+ \boldsymbol{\gamma}_y^-$,

$$\langle \boldsymbol{H} \rangle = \langle H \rangle \boldsymbol{\gamma}_t^+ \boldsymbol{\gamma}_y^- \varkappa_{rgb} \tag{16}$$

(the factor of $\varkappa_{rgb}$ from the modification (11), omitted from (15), is included here to avoid possible confusion). The electroweak Higgs field (16) carries y-charge, breaks y-symmetry, and generates masses for fermions by flipping their y-bit. The three remaining components of the Higgs multiplet are absorbed into the longitudinal components of the electroweak $W^{\pm}$ and Z bosons, giving them mass, while leaving the photon massless.

As long as spacetime is 4-dimensional, as in today's world, any intermediate gauge group on the path to grand unification must commute with all the Dirac vectors (12). The largest subgroup of Spin(11, 1) whose bivector generators, modified per (11), all commute with the Dirac vectors (12) is a product of weak and color groups Spin(5) × Spin(6) generated by, respectively, the ten bivectors formed from $\boldsymbol{\gamma}_t^+$ and $\boldsymbol{\gamma}_i^{\pm}$, $i = y, z$, and the fifteen bivectors formed from $\boldsymbol{\gamma}_i^{\pm}$, $i = r, g, b$. However, the subset of four Spin(5) bivectors $\boldsymbol{\gamma}_t^+\boldsymbol{\gamma}_i^{\pm}$ fail to commute with the field (18) that mediates grand symmetry breaking, so those bivectors are already eliminated as gauge fields (but not as scalar fields) at grand symmetry breaking. Thus the largest possible group on the path to grand unification is the product of extended weak and color groups, the Pati-Salam [11] group

$$\text{Spin}(4) \times \text{Spin}(6) \ . \tag{17}$$

The running of the three coupling parameters of the standard model indicates that unification to Spin(4) × Spin(6) should happen at 10^{12} GeV, so that unification does in fact happen. The energy 10^{12} GeV is comparable to that of the most energetic cosmic rays observed [22,23].

The general principles underlying symmetry breaking by the Higgs mechanism are: the Higgs field before symmetry breaking must be a scalar (spin 0) multiplet of the unbroken symmetry; one component of the Higgs multiplet must acquire a non-zero vacuum expectation value; components of the Higgs multiplet whose symmetry is broken are absorbed into longitudinal components of the broken gauge (spin 1) fields, giving those gauge fields mass; and unbroken components of the Higgs field persist as scalar fields, potentially available to mediate the next level of symmetry breaking.

In the sequence (14) of symmetry breakings, the primordial Higgs field is a scalar 66-component bivector multiplet of Spin(11, 1). The primordial Higgs field is the parent of all the other Higgs fields.

The field that breaks grand symmetry proves to be the Majorana-Higgs field $\langle \boldsymbol{T} \rangle$ proportional to the bivector $\boldsymbol{\gamma}_t^+\boldsymbol{\gamma}_t^-$,

$$\langle \boldsymbol{T} \rangle = -i \langle T \rangle \boldsymbol{\gamma}_t^+ \boldsymbol{\gamma}_t^- \varkappa_{rgb} \ , \tag{18}$$

the imaginary i coming from the time vector being timelike, $\boldsymbol{\gamma}_0 = i\boldsymbol{\gamma}_t^-$, and the factor $\varkappa_{rgb}$ from the modification (11). The Majorana-Higgs field (18) has the property that it commutes with all Spin(4) × Spin(6) fields, and fails to commute with all Spin(10) fields not in Spin(4) × Spin(6).

The Majorana-Higgs field $\langle \boldsymbol{T} \rangle$ carries t-charge, and is able to flip the t-bit of the right-handed neutrino, flipping the neutrino between itself and its left-handed antineutrino partner of opposite boost, giving the right-handed neutrino a so-called Majorana mass. Only the right-handed neutrino can acquire a Majorana mass, because only the right-handed neutrino possesses no conserved standard-model charge. A large Majorana mass for the right-handed neutrino can generate a small mass for the left-handed neutrino by the well-known see-saw mechanism proposed by [24].

The Majorana-Higgs field $\langle \boldsymbol{T} \rangle$ is available to drive cosmological inflation at the GUT scale. The running of weak and color coupling parameters implies that grand unification occurs at an energy of 3×10^{14} GeV. This unification energy is well within the upper limit on the energy scale $\mu_{\text{inflation}}$ of cosmological inflation inferred from the upper limit to B-mode polarization power in the cosmic microwave background measured by the Planck satellite [25, eq. (26)],

$$\mu_{\text{inflation}} \leq 2 \times 10^{16}\,\text{GeV} \ . \tag{19}$$

The first step in the symmetry-breaking sequence (14) is Spin(11, 1) $\rightarrow$ Spin(10, 1). The problem is that the Spin(11, 1) bivectors $\boldsymbol{\gamma}_t^+ \boldsymbol{\gamma}_i^\pm$ cannot be generators of a gauge (spin 1) field after grand symmetry breaking, because if they were, then their Higgs scalar (spin 0) counterparts would be absorbed into the gauge field after grand symmetry breaking, whereas the scalar counterparts apparently persist in the form of the electroweak Higgs multiplet (15).

The vector $\boldsymbol{\gamma}_t^+$, the spatial vector companion to the time vector $\boldsymbol{\gamma}_0 = i\boldsymbol{\gamma}_t^-$, stands out as the only spatial vector missing from the Spin(10) algebra. The solution to $\boldsymbol{\gamma}_t^+$ not generating any gauge symmetry is to assert that it behaves as a scalar dimension prior to grand symmetry breaking, so that the grand unified group is Spin(10, 1), not Spin(11, 1). Why this should be so is unclear. Possibly a non-trivial quantum field theory in higher dimensions requires 10+1 dimensions, as in M theory. Spin algebras live naturally in even dimensions, and one way to accommodate a spin algebra in 10+1 dimensions is to embed it in one extra dimension, 11+1 dimensions, and to treat the extra dimension, here $\boldsymbol{\gamma}_t^+$, as a scalar. The scalar dimension $\boldsymbol{\gamma}_t^+$, which anticommutes with the other 11 dimensions, plays the role of a time-reversal operator, essential to a consistent quantum field theory. It remains to be seen whether the Spin(11, 1) model can in fact be accommodated in M theory.

References

1. Hamilton, A.J.S., McMaken, T.: Unification of the four forces in the Spin(11,1) geometric algebra. Phys. Scr. **98**, 085306 (2023). https://doi.org/10.1088/1402-4896/acdaff
2. Hamilton, A.J.S.: The Supergeometric algebra: the square root of the geometric algebra. In: ENGAGE 2022 workshop at CGI 2022 (2022, accepted)
3. Weinberg, S.: A model of leptons. Phys. Rev. Lett. **19**, 1264–1266 (1967). https://doi.org/10.1103/PhysRevLett.19.126
4. Englert, F., Brout, R.: Broken symmetry and the mass of gauge vector mesons. Phys. Rev. Lett. **13**, 321–323 (1964). https://doi.org/10.1103/PhysRevLett.13.321

5. Higgs, P.W.: Broken symmetries and the masses of gauge bosons. Phys. Rev. Lett. **13**, 508–509 (1964) https://doi.org/10.1016/0550-3213(74)90038-8
6. Aad, G., et al.: (ATLAS Collaboration, 2934 authors): Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC. Phys. Lett. B **716**, 1–29 (2012). https://doi.org/10.1016/j.physletb.2012.08.020
7. Chatrchyan, S., et al. (CMS Collaboration, 2885 authors): Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC. Phys. Lett. B **716**, 30–61 (2012). arXiv:1207.7235, https://doi.org/10.3847/1538-4357/ab042c, https://doi.org/10.1016/j.physletb.2012.08.021
8. Georgi, H.: In Particles and Fields - 1974. In: Carlson, C.E. (ed.) Proceedings of Meeting of the APS Division of Particles and Fields, Williamsburg, Virginia (AIP, New York) (1975)
9. Fritzsch, H., Minkowski, P.: Unified interactions of leptons and hadrons. Ann. Phys. **93**, 193–266 (1975)
10. Georgi, H., Glashow, S.: Unity of all elementary-particle forces. Phys. Rev. Lett. **32**, 438–441 (1974)
11. Pati, J.C., Salam, A.: Lepton number as the fourth color. Phys. Rev. D **10**, 275–289 (1974)
12. Wilczek, F.: SO(10) marshals the particles. Nature **394**, 15 (1998). DOIurlhttps://doi.org/10.1038/27761
13. Baez, J.C., Huerta, J.: The algebra of grand unified theories. Bull. Am. Math. Soc. **47**, 483–552 (2010). arXiv:0904.1556
14. Ivezić, Ž., et al. (LSST Collaboration, 313 authors): LSST: from science drivers to reference design and anticipated data products. Astrophys. J. **873**, 11 (2019). arXiv:0805.2366, https://doi.org/10.3847/1538-4357/ab042c, arXiv:1806.00612 , https://doi.org/10.1140/epjc/s10052-018-5844-7
15. Atiyah, M.F., Bott, R., Shapiro, A.: Clifford modules. Topology **3**, 3–38 (1964)
16. Coleman, S., Mandula, J.: All possible symmetries of the S matrix. Phys. Rev. **159**, 1251–1256 (1967)
17. Mandula, J.E.: Coleman-Mandula theorem. Scholarpedia **10**(6), 7416 (1015). https://doi.org/10.4249/scholarpedia.7476
18. Pelc, O., Horwitz, L.P.: Generalization of the Coleman-Mandula theorem to higher dimension. J. Math. Phys. **38**, 139–172 (1997). https://doi.org/10.1063/1.531846
19. Eduard Study & Élie Cartan, Nombres complexes. Encyclopédie des sciences mathématiques; tome 1, volume 1, fascicule 3, pp. 353–411 (1908)
20. Bott, R.: The periodicity theorem for the classical groups and some of its applications. Adv. Math. **4**, 353–411 (1970). https://doi.org/10.1016/0001-8708(70)90030-7
21. Coquereaux, R.: Modulo 8 periodicity of real Clifford algebras and particle physics. Phys. Lett. B **115**, 389–395 (1982). https://doi.org/10.1016/0370-2693(82)90524-X
22. Telescope Array Collaboration: 126 authors, Indications of intermediate-scale anisotropy of cosmic rays with energy greater than 57 EeV in the Northern Sky measured with the surface detector of the telescope array experiment, Astrophys. J. **790**, L21 (2014). arXiv:1404.5890, https://doi.org/10.1088/2041-8205/790/2/l21
23. Batista, R.A., et al. (16 authors): Open questions in cosmic-ray research at ultrahigh energies. Front. Astron. Space Sci. **6**, 23 (2019). arXiv:1903.06714, https://doi.org/10.3389/fspas.2019.00023, https://doi.org/10.1103/physrevd.95.012004

24. Gell-Mann, M., Ramond, P., Slansky, R.: Complex spinors and unified theories. In: Freedman, D.Z., van Nieuwenhuizen, P. (eds.) Supergravity, pp. 315–321. North Holland, Amsterdam (1979). arXiv:1306.46694
25. Planck Collaboration (246 authors): Planck 2015 results. XX. Constraints on inflation, Astron. Astrophys. **594**, A20 (2016). arXiv:1502.02114, https://doi.org/10.1051/0004-6361/201525898