Equilibrium and Efficiency in Conflict Analysis Incorporating Permissibility

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Abstract. In this study, we aim to generalize the Nash equilibrium and efficiency established in conflict resolutions among decision-makers with permissibility in their preferences for possible outcomes on the framework of GMCR(Graph Model for Conflict Resolution). Obtaining sufficient information on preferences, especially in emergent crises, can often be daunting in real-world conflicts. However, identifying "unacceptable situations" is comparatively less challenging. Our proposed approach dichotomizes preferences into binary categories of "permissible" and "impermissible," exhibiting a particular aptness for decision-making in situations with limited or focused information that seek to prevent severe crises, particularly during the emergent phase or convergence point of conflicts. We provide propositions on the equilibrium and efficiency of permissibility analysis, introducing a novel approach using coarse decision theory. Overall, our study contributes significantly to improving the convenience and effectiveness of real-world conflict analysis.

Keywords: GMCR · permissible range · coarse decision theory

1 Introduction

Real-world decision-making often requires quick first-order decisions to prevent worst-case scenarios, even in the absence of sufficient information for a detailed analysis. We aim to develop a decision-making approach based on analyzing coarse information. To achieve this, we have introduced several new concepts. Firstly, we propose a method to describe states where unknown factors other than the primary decision maker (DM) impact the DM's state transitions [1, 2]. Secondly, we introduced a new state recognition concept that expands the DM's controllable choices beyond the binary values of true (T) or false (F) to include both (B) and none (N), thus accommodating contradictions [3]. Finally, we presented a concept for incorporating permissible ranges (PR) in the DM's preferences [4]. This paper explores a novel analytical approach that employs Graph Model for Conflict Resolution (GMCR) [5,6] in situations where there is insufficient information available regarding DMs' preferences. Building on the

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foundational concept of PR proposed in our previous study [4], this research presents several propositions to extend and generalize the approach.

To address preference uncertainty, GMCR has developed various approaches based on pairwise relationships of states, such as unknown [7–10], fuzzy [11–14], grey [15–18], and probabilistic [19,20] methods. The use of matrix representation facilitates more intricate categorization calculations and effectively tackles these uncertainties. Various other approaches have also been examined to manage uncertain preferences, including setting a permissible range for alternatives based on the committee framework in the context of simple game [21–25].

Nonetheless, due to their unique nature, there are inherent limitations in dealing with severe crises. These crises are either unprecedented or infrequent, resulting in restricted access to the information required for analysis. Additionally, the severity of the crisis renders empirical testing of the model implausible. For instance, while retrospective analysis of the simultaneous terrorist attacks in 2001 is feasible, evaluating experimentally the conditions under which the events occurred is impractical. Assuming complete knowledge of environmental dynamics, the optimal response to risk can be achieved through dynamic programming based on state transitions and the utility derived from those transitions in a given scenario. However, in cases where the information available is limited and the worst-case scenario is catastrophic, the selection of an appropriate model and the information partitioning utilized in the model must be carefully considered, given the constraints.

This study is grounded on the premise that adopting a coarse framework is rational and practical for decision-making in situations where information is scarce, and the aim is to avoid worst-case scenarios. Concerning the resolution of severe conflicts to prevent worst-case scenarios, the current approaches, for addressing uncertain preferences in GMCR [7–20], tend to augment the information categories required for managing uncertainty, which is antithetical to the objective of this study. Furthermore, the simple game-based approach [21–25] is a framework that is efficacious in situations originally intended for cooperative resolution and may not necessarily be applicable to analyzing non-cooperative, severe conflict scenarios. Within the framework of GMCR, this study presents a new and innovative approach to analysis by employing a smaller number of information categories and proposing several key propositions. These findings not only advance our understanding of conflict resolution study but also have the potential to establish valuable links with other theoretical perspectives, including coarse decision theory.

Section 2 begins with an exposition of the foundational principles that underpin the current research, focusing specifically on the core concepts of coarse information and decision-making systems. Subsequently, we review the framework for conflict analysis incorporating permissibility, which provides basic concepts. We scrutinize the relationship between the DMs' PRs, equilibrium, and efficiency, and then present generally valid propositions. In Sect. 3, the validity of the propositions is verified by applying them to the case of the Elmira environmental dispute: the case most frequently discussed in GMCR studies.

2 Underlying Concepts and Methods

2.1 Coarse Decision Theory

There is a great deal of insight to be gained from literature in the fields of economics, finance, and psychology about the models and information partitioning that DMs adopt and their validity [26–34]. Among them, *rational inattention* of DMs under limited information processing capacity, proposed by Sims [26] is remarkable. A priori, we know that it is impossible to solve the problem of temporal imprecision when considering the common knowledge that is the premise of the state of the world in decision-making. In this sense, it can be said to be reasonable to use a coarser granularity [32]. The coarser the criteria (fewer categories for each criterion), the lower the decision-making cost, even though the DM has to use more criteria [33]. The maximum number of alternative distinctions that can be generated considering the number of categories for each criterion is equal to the product of the number of categories for the criterion deployed. Theoretically, it can be said that there is a trade-off between categories and criteria when considering an efficient decision-making function with a limited amount of information. Obviously, in the analysis aimed at a solution that avoids the worst-case scenario, which is the subject of this study, a more reasonable solution can be obtained by reducing the number of criteria.

2.2 Graph Model for Conflict Resolution (GMCR)

GMCR is a framework consisting of four tuples: $(N, S, (A_i)_{i \in N}, (\succsim_i)_{i \in N})$ [5,6]. N is the set of all DMs, S denotes the set of all feasible states. (S, A_i) constitutes DM i', s graph G_i, where S is the set of all vertices and $A_i \subset S \times S$ is the set of all oriented arcs. (S, A_i) has no loops; $(s, s) \in A$ for each $s \in S$. The preferences of each DM are presented as $(\succsim_i)$, where the set of all DMs $N : |N| \geq 2$, set of all states $S : |S| \geq 2$, and preference of DM i satisfy reflectiveness, completeness, and transitivity. $s \succsim_i s'$: s is equally or more preferred to s' by DM i; $s \succ_i s'$: s is strictly preferred to s' by DM i; $s \sim_i s'$: s is equally preferred to s' by DM i. We assume that a rational DM desires the situation to change to a more favorable state and attempts to transition to the preferred state by repeating unilateral moves, which the DM exercises control over. For $i \in N$ and $s \in S$, we define DM i's reachable list from state s as the set $\{s' \in S \mid (s, s') \in A_i\}$, denoted by $R_i(s)$. $R_i(s)$ is the set of all the states in which DM i can move from s to s' in a single step. A unilateral improvement of DM i from state s is defined as an element of the reachable list of DM i from s (i.e., $s' \in R_i(s)$), where i strictly prefers state s' ($s' \succ_i s$). Therefore, the set of the unilateral improvement lists of DM i from state s is described as $\{s' \in R_i(s) | s' \succ_i s\}$ and denoted by $R_i^+(s)$. $\phi_i^+(s)$ denotes the set of all states that are more preferential for DM i to s described as $\{s' \in S \mid s' \succ_i s\}$, and $\phi_i^{\simeq}(s)$ denotes the set of all states that are at most equally preferential to state s, described as $\{s' \in S \mid s \succsim_i s'\}$. Moreover, $R_{N-\{i\}}(s)$ is defined as the set of all states that can be achieved by the sequences of unilateral moves of DMs other than DM i. Similarly, $R^+_{N-\{i\}}(s)$ is defined as the set of all states that can be achieved by the sequences of unilateral improvements of DMs other than DM i.

On the basis of the DMs' state transitions, we can obtain standard stability concepts : Nash stability (**Nash**) [35,36], general meta-rationality (**GMR**) [37], symmetric meta-rationality (**SMR**) [37], and sequential stability (**SEQ**) [38,39].

2.3 GMCR Incorporating Permissible Range (GMCR-PR)

Using the elements of GMCR presented in the previous subsection, we now define GMCR-PR.

Based on the properties of preferences in GMCR mentioned in Sect. 2.2 (reflexivity, completeness, and transitivity), defining DM's permissibility as a weak order on the set S of all possible states, then a non-empty subset of the set $L(S)$ of all weak orders on S can represent the permissible preference of a DM. Specifically, for any $i \in N$, a subset P_i that satisfies $\emptyset \neq P_i \subseteq L(S)$ can be considered as the permissible states of DM i. We refer to P_i as the permission of DM i, which is defined as a subset of the set of all linear orderings that includes the DM's actual preferences. This definition of DM's permission is motivated by the recognition that the true preferences of DMs are not always accurately known in real-world group decision-making situations. While new definitions relating to improvement are introduced, the general definitions of GMCR such as DM i's reachable list $R_i(s)$ and $\phi_i^{\simeq}(s)$ provided in Sect. 2.2 remain unchanged.

Definition 1 (Permissible States (PS)). *For any $i \in N$, Permissible States (PS) of DM i is a non-empty subset of L, denoted by P_i. A list $(P_i)_{i \in N}$ for each $i \in N$ represents DMs' PS, denoted by P.*

By imposing a permissible threshold, the state set S can be partitioned into two subsets: those that are permissible for the DM and those that are not. This partition can be interpreted that $|P_i| = 1$.

Definition 2 (Permissible Range (PR)). *We denote DM i's PR by P_i^k, that is, in a conflict, DM i allows up to the k^{th} most preferred state.*

GMCR-PR is represented by five tuples: DMs (N), a set of feasible states (S), a graph of DM i (A_i), the preferences of each DM i ($\succsim_i$), and a set of permissible preferences of DM i (P_i).

Definition 3 (GMCR-PR).

$$G = (N, S, (A_i)_{i \in N}, (\succsim_i)_{i \in N}, (P_i)_{i \in N}). \tag{1}$$

Example 1. Consider a conflict $G = (N, S, (A_i)_{i \in N}, (\succsim_i)_{i \in N}, (P_i)_{i \in N})$, where $N = \{a, b\}$, $S = \{1, 2, 3, 4\}$, $1 \succ_a 2 \succ_a 3 \succ_a 4$ and $4 \succ_b 3 \succ_b 2 \succ_b 1$. Suppose each DM's PR is P_a^2 and P_b^2 respectively, $P = \emptyset$.

Definition 4 (Reachable Lists in GMCR-PR). *DM i's* permissible reachable list *from $s \in S$ are subsets of S as follows:*

i. *DM i's reachable list from s to s' by unilateral moves in GMCR-PR is defined as in GMCR, $R_i(s) = \{s' \in S \mid (s, s') \in A_i\}$*

ii. *DM i's Unilateral Improvement in GMCR-PR (PUI) is a transition from a state $s \notin P_i$ to a state $s' \in P_i$ and is defined as follows:*

$$PR_i(s) = \{s' \in R_i(s) \mid (s, s') \in A_i, s \notin P_i, s' \in P_i\}. \tag{2}$$

A list of PUI by a DM other than DM i is represented as $PR_{N-i}(s)$.

iii. *DM i's list of states regarding s and s' being equally or less preferred is defined as in GMCR, $\phi_i^{\simeq}(s) = \{s' \in S \mid s \succsim_i s'\}$.*

When DM i has no PUI from state s, there are no further state transitions exist, thereby establishing stability.

Definition 5 (PNash). *For $i \in N$, state $s \in S$ is PNash stable for DM i, denoted by $s \in S_i^{PNash}$, if and only if*

$$PR_i(s) = \emptyset. \tag{3}$$

State s is PGMR stable for DM i when any PUI from state s of DM i may cause a state equal or less preferred state than s in the responses of the other DMs.

Definition 6 (PGMR). *For $i \in N$, state $s \in S$ is PGMR stable for DM i, denoted by $s \in S_i^{PGMR}$, if and only if*

$$\forall s' \in PR_i(s), R_{N-\{i\}}(s') \cap \phi_i^{\simeq}(s) \neq \emptyset. \tag{4}$$

When a state occurs where for any of DM i's PUI, another DM's countermove would result in a state equal or less favorable than s. Furthermore, regardless DM i's subsequent countermove, a state more favorable than s cannot occur, thereby establishing stability.

Definition 7 (PSMR). *For $i \in N$, state $s \in S$ is PSMR stable for DM i, denoted by $s \in S_i^{PSMR}$, if and only if*

$$\forall s' \in PR_i(s), \exists s'' \in R_{N-\{i\}}(s') \cap \phi_i^{\simeq}(s),\ R_i(s'') \subseteq \phi_i^{\simeq}(s). \tag{5}$$

When DM i has at least one PUI from state s, but states resulting from PUI of other DMs' responses from s' cause a state to be equal to or less preferable than s for DM i, then s is PSEQ stable for DM i.

Definition 8 (PSEQ). *For $i \in N$, state $s \in S$ is PSEQ stable for DM i, denoted by $s \in S_i^{PSEQ}$, if and only if*

$$\forall s' \in PR_i(s),\ PR_{N-\{i\}}(s') \cap \phi_i^{\simeq}(s) \neq \emptyset. \tag{6}$$

The chicken game in GMCR-PR can be represented as follows.

Example 2 (Chicken Game).
$(N, S, (A_i)_{i \in N}, (\succsim_i)_{i \in N}), N = \{1, 2\}, S = \{s_1, s_2, s_3, s_4\}$
$A_1 = \{(s_1, s_3), (s_3, s_1), (s_2, s_4), (s_4, s_2)\}$,
$A_2 = \{(s_1, s_2), (s_2, s_1), (s_3, s_4), (s_4, s_3)\}$,
DM_1's preference order $\succsim_1$: $s_3 \succ s_1 \succ s_2 \succ s_4$,
DM_2's preference order $\succsim_2$: $s_2 \succ s_1 \succ s_3 \succ s_4$.

Let us assume that both DMs have permissibility up to the states of the second preference order P^2, in stead of the linear order provided in the original game: $P_1 = \{s_1, s_3\}$, $P_2 = \{s_1, s_2\}$. Then, we have unilateral improvements for each DM as follows; $PR_1(s_1) = \emptyset, PR_1(s_2) = \emptyset, PR_1(s_3) = \emptyset, PR_1(s_4) = \emptyset$, $PR_2(s_1) = \emptyset, PR_2(s_2) = \emptyset, PR_2(s_3) = \emptyset, PR_2(s_4) = \emptyset$. Hence, Nash equilibrium is established in all states when P^2 is employed for each DM in the chicken game. The Table 1 summarizes the permissibility, reachability, PUI, and Nash equilibrium.

Table 1. Chicken Game in GMCR-PR - P_1^2, P_2^2

State		1	2	3	4
Permissibility	DM1	1	0	1	0
	DM2	1	1	0	0
$R_i(s)$	DM1	3	4	1	2
	DM2	2	1	4	3
$\phi_i^+(s)$	DM1		1,3		1,3
	DM2			1,2	1,2
PR_i	DM1				
	DM2				
Nash			E	E	
PNash		E	E	E	E

In our previous study [4], we examined the equilibrium and efficiency of 21 sets of 2×2 games classified as Class III in Rapoport and Guyer's taxonomy [40], in which neither DM has a dominant strategy for all combinations of the four permissible levels. Consequently, any conflict in the category was concluded to be resolved when both DMs set their threshold as P^3: "accept all states except the least favorable one." The following section develops the discussions on permissibility and equilibrium/efficiency based on these results and present propositions. Incorporating the concept of "permissibility" in the derivation of propositions concerning conflict resolution would enrich the spectrum of recommendations for resolving conflicts.

3 Nash Stability and Efficiency for Conflicts with Permissible Range

We present propositions for Nash stability and efficiency in conflict analysis incorporating PR. These propositions were prepared separately for cases with at least one state commonly permissible to all DMs in Subsect. 3.1, and cases without such a state in Subsect. 3.2.

Consider a conflict presented in GMCR-PR: $(N, S, (A_i)_{i\in N}, (\succsim_i)_{i\in N}, (P_i))$. Here, for $i \in N$, P_i denotes the set of all permissible states for DM i; Therefore, if $s \in S$ is permissible for DM i, then it is denoted by $s \in P_i$; otherwise, $s \notin P_i$.

3.1 Case with $\cap_{i\in N} P_i \neq \emptyset$

First, we consider the case with $\cap_{i\in N} P_i \neq \emptyset$; that is, there exists at least one state that is commonly permissible for all DMs. We have the following propositions:

Nash Stability

Proposition 1. State $s \in \cap_{i\in N} P_i$ is Nash equilibrium.

Proof. For $i \in N$, we have $R_i^+(s) = \emptyset$, because for all $s' \in S$, $s \succsim_i s'$. □

Proposition 2. Consider state $s' \notin \cap_{i\in N} P_i$. For $j \in N$, if $s' \in P_j$, then s' is Nash stable for DM j.

Proof. s' is Nash stable for DM j, because for all $s'' \in S$, $s' \succsim_j s''$. □

Proposition 3. Consider state $'s \notin \cap_{i\in N} P_i$. For $k \in N$, if $s' \notin P_j$, then s' is Nash stable for DM k if $R_k(s') \cap P_k = \emptyset$, and not if $R_k(s') \cap P_k \neq \emptyset$.

Proof. s' is Nash stable for DM k if $R_k(s') \cap P_k = \emptyset$, because we have $R_k^+(s') = \emptyset$ from $s' \notin P_k$ and for all $s'' \in R_k(s')$, $s'' \notin P_k$. s' is not Nash stable for DM k if $R_k(s') \cap P_k \neq \emptyset$, because we have $R_k^+(s') \neq \emptyset$ from $s' \notin P_k$ and there exists $s'' \in P_k(s')$ such that $s'' \in P_k$, which implies $s'' \succ_k s'$. □

For special cases in which each DM's PR includes all states except the least preferable one, we have Corollary 1 of Proposition 3.

Corollary 1 (Corollary of Proposition 3).
Consider cases that $P_i = S \backslash \{\min \succsim_i\}$ *for* $i \in N$*, where* $\min \succsim_i$ *denotes DM* i*'s least preferred state.* $s' = \min \succsim_i$ *is Nash stable for DM* i *if* $R_i(s') = \emptyset$*, and not if* $R_i(s') \neq \emptyset$*.*

Proof. If $R_i(s') = \emptyset$, then we always have $R_i^+(s') = \emptyset$, which means that s' is Nash stable for DM i. If $R_i(s') \neq \emptyset$, then we have that $R_i(s') \cap P_i \neq \emptyset$, because $R_i(s') \subseteq S \backslash \{s'\} = S \backslash \{\min \succsim_i\} = P_i$. Using the result of Proposition 3, we have that s' is not Nash stable for DM i. □

Efficiency. The following are propositions on the efficiency of states under the condition of $\cap_{i \in N} P_i \neq \emptyset$

Proposition 4. State $s \in \cap_{i \in N} P_i$ is weakly and strongly efficient.

Proof. In this case, for all $i \in N$ and all $s' \in S$, $s \succsim_i s'$. Therefore, $s' \succ_i s$ cannot be satisfied for any $i \in N$ and any $s' \in S$, which implies that s is weakly and strongly efficient. □

Proposition 5. Consider state $s' \notin \cap_{i \in N} P_i$. For $j \in N$, if $s' \in P_j$ (which implies that $s' \notin P_k$ for some $k \in N$), then s' is weakly efficient and not strongly efficient.

Proof. In this case, for all $i \in N$, $s \succsim_i s'$ and $s \succ_k s'$, because $s \in \cap_{i \in N} P_i$ and $s' \notin P_k$. This implies that s' is not strongly efficient. No $s'' \in S$ exists such that $s'' \succ s'$ for all $i \in N$, because $s' \in P_j$. This implies that s' is weakly efficient. □

Proposition 6. Consider state $s' \notin \cap_{i \in N} P_i$. If $s' \notin P_i$ for all $i \in N$, then s' is neither weakly nor strongly efficient.

Proof. In this case, for all $i \in N$, $s \succ_i s'$, because for all $i \in N$, and $s \in P_i$ and for all $i \in N$, $s' \notin P_i$. □

Worst Case Efficiency. For situations in which each DM's PR includes all cases except the least preferable one, we have Corollary 2 of Propositions 5 and 6.

Corollary 2 (Corollary of Propositions 5 and 6).
Consider the cases in which $P_i = S \backslash \{\min \succsim_i\}$ *for all* $i \in N$, *where* $\min \succsim_i$ *denotes DM* i*'s least preferred state.* $\min \succsim_i$ *is weakly efficient and not strongly efficient if* $\min \succsim_i \neq \min \succsim_j$ *for some* i *and* $j \in N$. $\min \succsim_i$ *is neither weakly nor strongly efficient if* $\min \succsim_i = \min \succsim_j$ *for all* i *and* $j \in N$.

Proof. In the case in which $\min \succsim_i \neq \min \succsim_j$ for some i and $j \in N$, $s' = \min \succsim_i \notin P_i$ and $s' \in P_j$. Then, by applying Proposition 5, we have that $\min \succsim_i$ is weakly efficient and not strongly efficient.

In the case in which $\min \succsim_i = \min \succsim_j$ for all i and $j \in N$, $s' = \min \succsim_i \notin P_i$ for all $i \in N$. Then, by applying Proposition 6, we have that $\min \succsim_i$ is neither weakly nor strongly efficient. □

3.2 Case with $\cap_{i \in N} \boldsymbol{P_i} = \emptyset$

Next, we consider the case with $\cap_{i \in N} P_i = \emptyset$, that is, there is no state that is commonly permissible for all DMs exists. We have the following propositions:

Nash Stability

Proposition 7. Consider state $s' \notin \cap_{i \in N} P_i$. For $j \in N$, if $s' \in P_j$, then s' is Nash stable for DM j.

Proof. s' is Nash stable for DM j because for all $s'' \in S$, $s' \succsim_j s''$. □

Proposition 8. Consider state $s' \notin \cap_{i \in N} P_i$. For $k \in N$, if $s' \notin P_j$, then s' is Nash stable for DM k if $R_k(s') \cap P_k = \emptyset$, and not if $R_k(s') \cap P_k \neq \emptyset$.

Proof. s' is Nash stable for DM k if $R_k(s') \cap P_k = \emptyset$, because we have $R_k^+(s') = \emptyset$ from $s' \notin P_k$ and for all $s'' \in R_k(s')$, $s'' \notin P_k$. s' is not Nash stable for DM k if $R_k(s') \cap P_k \neq \emptyset$, because we have $R_k^+(s') \neq \emptyset$ from $s' \notin P_k$ and there exists $s'' \in P_k(s')$ such that $s'' \in P_k$, which implies $s'' \succ_k s'$. □

For situations in which each DM's PR includes all states except the least preferable one, we have Corollary 3 of Proposition 8.

Corollary 3 (Corollary of Proposition 8).
Consider the cases in which $P_i = S \backslash \{\min \succsim_i\}$ *for all* $i \in N$*, where* $\min \succsim_i$ *denotes DM* i*'s least preferred state.* $s' = \min \succsim_i$ *is Nash stable for DM* i *if* $R_i(s') = \emptyset$*, and not Nash stable if* $R_i(s') \neq \emptyset$*.*

Proof. If $R_i(s') = \emptyset$, then we always have $R_i^+(s') = \emptyset$, which means that s' is Nash stable for DM i. If $R_i(s') \neq \emptyset$, then we have $R_i(s') \cap P_i \neq \emptyset$, because $R_i(s') \subseteq S \backslash \{\min \succsim_i\} = P_i$. Using the result of Proposition 8, we have that s' is not Nash stable for DM i. □

Efficiency. The following are propositions on the efficiency of states under the condition of $\cap_{i \in N} P_i = \emptyset$

Proposition 9. Consider state $s' \notin \cap_{i \in N} P_i$. For $j \in N$, if $s' \in P_j$ (which implies that $s' \notin P_k$ for some $k \in N$), then s' is weakly efficient.

Proof. No $s'' \in S$ exists such that $s'' \succ_i s'$ for all $i \in N$, because $s' \in P_j$. Thus, s' is weakly efficient. □

Proposition 10. Consider state $s' \notin \cap_{i \in N} P_i$. Assume that $N = \{j, k\}$, that is $|N| = 2$. Then, for $j \in N$, if $s' \in P_j$ (which implies that $s' \notin P_k$ for the other $k \in N$), then s' is strongly efficient.

Proof. Assume that there exists $s'' \in S$ such that $s'' \succsim_j s'$ and $s'' \succsim_k s'$, and that $s'' \succ_j s'$ or $s'' \succ_k s'$. Because $s' \in P_j$, it is impossible that $s'' \succ_j s'$. This implies that $s'' \succ_k s'$. Then, we must have that $s'' \in P_j$ and $s'' \in P_k$, which contradicts the condition that $\cap_{i \in N} P_i = \emptyset$. Therefore, s' is strongly efficient. □

With respect to the strong efficiency of state s' under the conditions of $\cap_{i \in N} P_i = \emptyset$, $s' \in P_j$ for some $j \in N$, $s' \notin P_k$ for some $k \in N$, and $|N| \geq 3$, see the following example. We see that s' may be strongly efficient depending on $(P_i)_{i \in N}$ in the following examples.

Example 3.

Case 1: Let $N = \{1, 2, 3\}$, $S = \{s_1, s_2, s_3\}$, and $P_1 = \{s_1, s_2\}$; $P_2 = \{s_2\}$; $P_3 = \{s_3, s_1\}$. In this case, $\cap_{i \in N} P_i = \emptyset$, and $s_1 \in P_1$; $s_1 \notin P_2$; $s_1 \notin P_3$. We see that s_1 is strongly efficient, because $s_1 \succ_3 s_2$ and $s_1 \succ_1 s_3$.
Case 2: Let $N = \{1, 2, 3\}$, $S = \{s_1, s_2, s_3\}$, and $P_1 = \{s_1, s_2\}$; $P_2 = \{s_2\}$; $P_3 = \{s_3\}$. In this case, $\cap_{i \in N} P_i = \emptyset$, and $s_1 \in P_1$; $s_1 \notin P_2$; $s_1 \notin P_3$. We see that s_1 is not strongly efficient because $s_2 \succsim_1 s_1$; $s_2 \succ_2 s_1$; $s_2 \succsim_3 s_1$.

Proposition 11. Consider state $s' \notin \cap_{i \in N} P_i$. If $s' \notin P_i$ for all $i \in N$, then s' is weakly efficient and not strongly efficient.

Proof. Assume that there exists $s'' \in S$ such that for all $i \in N$, $s'' \succ_i s'$. Then, we must have that for all $i \in N$, $s'' \in P_i$, which contradicts the condition that $\cap_{i \in N} P_i = \emptyset$. Thus, s' is weakly efficient. Because we assume that $P_j \neq \emptyset$ for all $j \in N$, we can take $s'' \in P_j$. Then, it is satisfied that $s'' \succ_j s'$ and $s'' \succsim_i s'$ for all $i \in N$, because $s' \notin P_i$ for all $i \in N$. Therefore, s' is not strongly efficient. □

Worst Case Efficiency

Corollary 4 (Corollary of Proposition 9 and Proposition 10). *Consider the cases that $P_i = S \backslash \{\min \succsim_i\}$ for all $i \in N$, in which $\min \succsim_i$ denotes DM i's least preferred state. Then, we have that $\min \succsim_i$ is weakly efficient. We also have that $\min \succsim_i$ is strongly efficient if $N = \{1, 2\}$.*

Proof. Under the conditions of $\cap_{i \in N} P_i = \emptyset$ and $P_i = S \backslash \{\min \succsim_i\}$ for all $i \in N$, we have that $S = \{\min \succsim_i \mid i \in N\}$, because otherwise, $x \in S \backslash \{\min \succsim_i \mid i \in N\}$ satisfies that $x \in \cap_{i \in N} P_i$, which contradicts the condition that $\cap_{i \in N} P_i = \emptyset$. Then, $S = \{\min \succsim_i \mid i \in N\}$ implies the results using Propositions 9 and 10. □

For strong efficiency in cases with $P_i = S \backslash \{\min \succsim_i\}$ for all $i \in N$ and $|N| \geq 3$, we have the following proposition:

Proposition 12. Consider cases that $P_i = S \backslash \{\min \succsim_i\}$ for all $i \in N$, where $\min \succsim_i$ denotes the DM i's least preferred state. Then, we have that $\min \succsim_i$ is strongly efficient, if $|N| \geq 3$.

Proof. Under the conditions of $\cap_{i \in N} P_i = \emptyset$ and $P_i = S \backslash \{\min \succsim_i\}$ for all $i \in N$, we have that $S = \{\min \succsim_i \mid i \in N\}$, because otherwise, $x \in S \backslash \{\min \succsim_i \mid i \in N\}$ satisfies $x \in \cap_{i \in N} P_i$, which contradicts the condition that $\cap_{i \in N} P_i = \emptyset$.

For all $s'' \in S = \{\min \succsim_i \mid i \in N\}$, there exists $i \in N$ such that $s'' = \min \succsim_i$, which implies that $s' \succ_i s''$. □

Table 2 summarizes the results for general cases in Subsects. 3.1 and 3.2, and Table 3 shows the results for the cases with $P_i = S \backslash \{\min \succsim_i\}$ for all $i \in N$ given by the corollaries in Subsects. 3.1 and 3.2.

Table 2. Interrelationships between Nash Stability and Efficiencies

	$s \in S$: $s \in \cap_{i \in N} P_i$	$s' \in S$: $s' \in P_j$ and $s' \notin P_k$	$s' \in S$: $\forall k \in N, s' \notin P_k$
If $\cap_{i \in N} P_i \neq \emptyset$:	Nash for all $i \in N$ (Proposition 1)	Nash for j (Proposition 2)	—
		Nash for k depending on $R_k(s')$ and P_k (Proposition 3)	
	w.eff. (Proposition 4)	w.eff. (Proposition 5)	NOT w.eff. (Proposition 6)
	s.eff. (Proposition 4)	NOT s.eff. (Proposition 5)	NOT s.eff. (Proposition 6)
If $\cap_{i \in N} P_i = \emptyset$:	—	Nash for j (Proposition 7)	—
		Nash for k depending on $R_k(s')$ and P_k (Proposition 8)	
	—	w.eff. (Proposition 9)	w.eff. (Proposition 11)
	—	s.eff. if $\lvert N \rvert = 2$ (Proposition 10); dep.on $(P_i)_{i \in N}$ if $\lvert N \rvert \geq 3$ (Ex. 3)	NOT s.eff. (Proposition 11)

Table 3. Nash stability and efficiencies of min $\succsim_i$ under the condition of $P_i = S \backslash \{\min \succsim_i\}$ for all $i \in N$

	$\exists i, j \in N$, $\min \succsim_i \neq \min \succsim_j$	$\forall i, j \in N$, $\min \succsim_i = \min \succsim_j$
If $\cap_{i \in N} P_i \neq \emptyset$:	Nash for i depending on $R_i(s')$ (Corollary 1)	
	w.eff. (Corollary 2)	NOT w.eff. (Corollary 2)
	NOT s.eff. (Corollary 2)	NOT s.eff. (Corollary 2)
If $\cap_{i \in N} P_i = \emptyset$:	Nash for i depending on $R_i(s')$ (Corollary 3)	
	w.eff. (Corollary 4)	—
	s.eff. (Corollary 4, Proposition 12)	—

4 Verification of Propositions in Application Cases

In this section, the propositions presented in Sect. 3 are verified by applying them to the Elmira conflict, a representative case of GMCR analysis [41,42].

Elmira Conflict. The Elmira conflict is an environmental contamination dispute in Ontario, Canada, upon which numerous studies have been conducted using GMCR. Three DMs are involved in the conflict: the Ministry of Environment (**M**), Uniroyal (**U**), and the local government (**L**). **M** discovered contamination and issued a control order to **U** that included a decontamination operation to be conducted by **U**. They desire to exercise their authority efficiently. **U** operates questionable chemical plants, and intends to exercise its right to object, aiming to lift or relax the control order. **L** represents diverse interest groups, and intends to protect the residents and the local industrial base. Table 4 summarizes all feasible states based on the DMs' options, while (Fig. 1) displays the corresponding conflict graph. In addition, the preference orders of the three DMs are given as follows: $\mathbf{M}: s_7 \succ s_3 \succ s_4 \succ s_8 \succ s_5 \succ s_1 \succ s_2 \succ s_6 \succ s_9$;

U : $s_1 \succ s_4 \succ s_8 \succ s_5 \succ s_9 \succ s_3 \succ s_7 \succ s_2 \succ s_6$; **L** : $s_7 \succ s_3 \succ s_5 \succ s_1 \succ s_8 \succ s_6 \succ s_4 \succ s_2 \succ s_9$.

Table 4. Elmira Conflict - Options and States

State		1	2	3	4	5	6	7	8	9
M	Modify	N	Y	N	Y	N	Y	N	Y	-
U	Delay	Y	Y	N	N	Y	Y	N	N	-
	Accept	N	N	Y	Y	N	N	Y	Y	-
	Abandon	N	N	N	N	N	N	N	N	Y
L	Insist	N	N	N	N	Y	Y	Y	Y	-

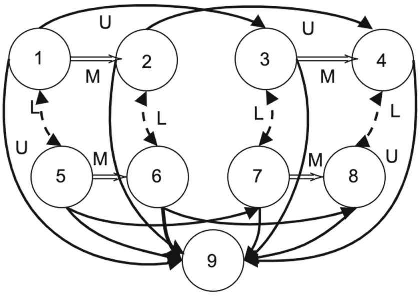

Fig. 1. Graph Model of Elmira conflict

Elmira Conflict Case-1: $\cap_{i \in N} \boldsymbol{P_i} \neq \emptyset$. We verified the propositions presented in Sect. 3.1 for the case with $\cap_{i \in N} P_i \neq \emptyset$ by examining the stability analysis of the Elmira conflict case-P_M^2, P_U^7, and P_L^5. Table 5 summarizes the permissibility, reachability, and PNash.

In conflicts where at least one state is permissible to all DMs, we determined the following propositions: 1) Proposition 1 concerns the permissible states for all DMs; thus, Nash equilibria are established at s_3 and s_7. 2) Propositions 2 and 3 concern states other than those verified in 1) that are permissible for each DM, and lead to Nash stability in s_1, s_5, s_8, and s_9. From 1) and 2), we can conclude that PNash equilibria hold for s_1, s_3, s_5, s_7, s_8, and s_9. This verification result is consistent with the GMCR-PR stability analysis presented in Table 5. In addition, we observe that the weak and strong Pareto efficiency proposed in Proposition 4 is consistent with the original results in Table 5. Each item in the table indicates the following.

- Permissibility: Boolean value denoting permissibility for DM i.

Table 5. Verification of Propositions: Elmira Conflict - P_M^2, P_U^7, P_L^5

State		1	2	3	4	5	6	7	8	9
Preference order	M	7	3	4	8	5	1	2	6	9
	U	1	4	8	5	9	3	7	2	6
	L	7	3	5	1	8	6	4	2	9
Permissibility	M	0	0	1	0	0	0	1	0	0
	U	1	0	1	1	1	0	1	1	1
	L	1	0	1	0	1	0	1	1	0
$R_i(s)$	M	2		4		6		8		
	U	3,9	$\overline{4}$, $\overline{9}$	9	9	7,9	$\overline{8}$, $\overline{9}$	9	9	
	L	5	6	7	$\overline{8}$	1	2	3	4	
PNash Equilibrium		E		E		E		E	E	E
Proposition 1(Nash)				E				E		
Proposition 2(Nash)		U,L			U	U,L			U,L	U
Proposition 3(Nash)		M			M	M			M	M,L
Proposition 4(eff.)				✓				✓		
Proposition 5(eff.)					✓					✓
Proposition 6(eff.)			✓				✓			

- $R_i(s)$: States where DM i can unilaterally transition (UM) from each state. The numbers indicate the number of states. The overbar signifies that the transition is PUI.
- PNash Equilibrium: Nash holds for all $i \in N$
- Prop. 1–3(Nash): E denotes equilibrium, M, U, and L indicate DMs who reached stability according to the proposition.
- Prop. 4–6(eff.): Weak and strong efficiency holds for the state with the check-mark.

Elmira Conflict Case-2: $\bigcap_{i \in N} \boldsymbol{P_i} = \boldsymbol{\emptyset}$. We verified the propositions by setting up a PR case P_M^2, P_U^2, and P_L^2 in the Elmira conflict.

Table 6 presents the stability analysis when the PR of all DMs is set to P^2. It is presented as a conflict without a single state that is commonly permissible for all DMs. Table 7 presents the correspondence between the stability and propositions in the P_M^2, P_U^2, P_L^2 case.

In conflicts where no state is permissible to all DMs, we determined the following regarding the propositions: 1) Proposition 7 is about permissible states for DM j; thus, Nash stability holds at s_1 for **U**, s_3 for **M** and **L**, s_4 for **U**, and s_7 for **M** and **L**. 2) Propositions 8 concerns states other than those verified in 1) that are permissible for DM j, and this proposition leads to Nash stability for **M**, **U**, and **L**. From 1) and 2), we can conclude that the PNash equilibra hold in s_1, s_3, s_4, s_5, s_6, s_7, s_8 and s_9. This verification result is consistent with the

Table 6. Elmira Conflict - Stability Analysis: P_M^2, P_U^2, P_L^2

State	1	2	3	4	5	6	7	8	9
M	0	0	1	0	0	0	1	0	0
U	1	0	0	1	0	0	0	0	0
L	0	0	1	0	0	0	1	0	0
PNash	✓		✓	✓	✓	✓	✓	✓	✓
PGMR	✓	✓	✓	✓	✓	✓	✓	✓	✓
PSMR	✓	✓	✓	✓	✓	✓	✓	✓	✓
PSEQ	✓		✓	✓	✓	✓	✓	✓	✓
Pareto	✓		✓	✓			✓		

Table 7. Verification of Propositions: Elmira Conflict - P_M^2, P_U^2, P_L^2

State		1	2	3	4	5	6	7	8	9
Preference order	M	7	3	4	8	5	1	2	6	9
	U	1	4	8	5	9	3	7	2	6
	L	7	3	5	1	8	6	4	2	9
Permissibility	M	0	0	1	0	0	0	1	0	0
	U	1	0	0	1	0	0	0	0	0
	L	0	0	1	0	0	0	1	0	0
$R_i(s)$	M	2		4		6		8		
	U	3,9	$\overline{4}$,9	9	9	7,9	8,9	9	9	
	L	5	6	7	8	1	2	3	4	
PNash Equilibrium		E		E	E	E	E	E	E	E
Proposition 7(Nash)		U		M,L	U			M,L		
Proposition 8(Nash)		M,L	M,L	U	M,L	M,U,L	M,U,L	U	M,U,L	M,U,L
Proposition 9(eff.)		✓		✓	✓			✓		
Proposition 11(eff.)			✓			✓	✓		✓	✓

GMCR-PR stability analysis shown in Table 6. In addition, we seek to confirm that the weak and strong Pareto efficiencies provided in Propositions 9 and 11 are consistent with the original results in Table 6.

This section examined the propositions presented in Sect. 3 and verified it to be consistent with the results of the GMCR-PR stability analyses of the Elmira conflict.

5 Conclusion

This study discussed the analysis capability with coarse information by introducing the concept of PR to GMCR. PR is set by placing a threshold on the preference, and the DM's preference is processed as binary information. Moreover, because the GMCR framework is retained, the resolution can be changed depending on the granularity of the information available. Introducing the concept of PR allows for the analysis to reflect implicit assumptions that are not part of the fundamental framework; describing a situation in which even a DM seeking reasonable resolution endeavors to avoid prolongation or escalation to converge the conflict by adjusting its permissible level is possible.

This paper focused on equilibrium and efficiency in the two cases of the presence or absence of commonly permissible states for all DMs. Future research topics include more complex issues, such as those in which permissibility differs from the initial judgment because of the availability of information after the determination from the first analysis.

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