An Analytical Approach for Calculating Crack Width of RC Members: Pure Shear Method

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Abstract. Nowadays Serviceability Limit State (SLS) of Reinforced Concrete (RC) structures is becoming a more and more important object of research among scientists and engineers. One of the most relevant topics of SLS is cracking due to its significant impact on the longevity and durability as well as aesthetic appearance of the structure. However, up until this day all theoretical models, dedicated to determining the most important characteristic of cracking behavior – crack width, are still highly debatable due to the high variation in the correspondence of the theoretical to experimental results. In this paper a new simple analytical approach for calculating average crack width of reinforced concrete members subjected to external short–term loading is presented. The new model is based on the concept of sole shear displacement occurring in the cover thickness of concrete due to the tangential stresses resulted by bonding. Theoretical crack width results obtained using the new approach are compared to the experimental results of tensile and bending members tested by different authors and the theoretical results of Eurocode 2 and Model Code 2010.

Keywords: Analytical model · Crack width · Cracking · Pure Shear Method · Short–term · Strain · Stress

1 Introduction

Nowadays, due to gradual improvement of materials resulting in lighter and more slender structures, the Serviceability Limit State (SLS) becomes more and more relevant topic among scientists and engineers. When designing reinforced concrete (RC) structures, the most important design aspect, which is closely related to SLS, is cracking control. Excessively large cracks cause problems related to aesthetics and even induces a feeling of unsafety. Moreover, moisture or other chemical materials penetrating through cracks provoke the corrosion of reinforcement which in time results in the reduction of the structural durability. To avoid the above – mentioned issues, crack widths must be limited. To achieve this goal, it is important to have a model which, knowing certain structural configuration as well as certain environmental and loading conditions, could predict the expected width of the cracks forming on the surface of RC member. From the past century up until this day a wide variety of cracking models have been developed. However, due to high scatter of the correspondence to the experimental results their accuracy is still highly debatable.

Considering the necessity to develop a more accurate and simple cracking model, which would be capable of predicting one of the most important parameters of the SLS – crack width, in this paper a new simplistic analytical approach, which is based on the concept of sole elastic shear stains of concrete, is presented.

2 Assumptions and Derivation of Pure Shear Method

To obtain a simple analytical expression for practically evaluating mean crack width of RC members subjected to short–term loads, the following model is based on the simplified assumptions listed below:

1. All cracks are considered to be fully formed and no new additional cracks can form as a result of a subjected external load.
2. All materials are elastic.
3. Longitudinal axial strains occurring in concrete due to external loads are neglected.
4. Only shear strains which are assumed to be constant across the cover and range along the longitudinal axis of the member occur in concrete.
5. Perfect bond (no slip) between reinforcement and concrete is considered, resulting in distortion of the concrete sections as a result of occurring shear deformation (Fig. 1).
6. Only surface crack widths of RC member are considered.
7. Shrinkage is neglected.
8. The mean crack width is predicted.
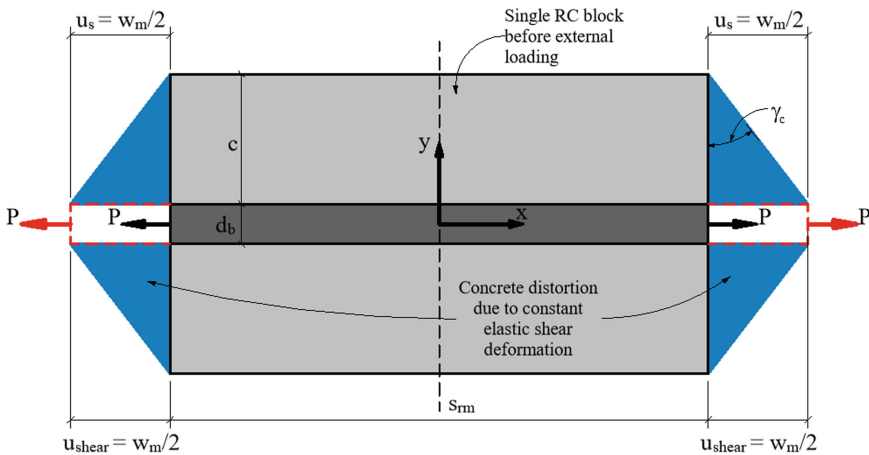
Fig. 1. A singly reinforced concrete block representing the mean region between two adjacent cracks spaced at s_{rm} (d_b– bar diameter, c – concrete cover, γ_c – shear angle, u_s – reinforcement displacement due to external loading, w_m – mean crack width, P – external short–term load)

Considering a tensile concrete block reinforced with a single reinforcing bar which is a representation of the average distance between two adjacent cracks (Fig. 1), and taking into account the assumptions 3–5, it can be stated that the axial displacement of

the reinforcing bar at any point x ($u_s(x)$) is equal to the shear displacement of concrete ($u_{shear}(x)$):

$$u_s(x) = u_{shear}(x). \tag{1}$$

Therefore, using the well–known classical expression (e.g., [1]) which describes a differential relationship between reinforcement strain and bonding stress:

$$\frac{d\varepsilon_s(x)}{dx} = \frac{4}{E_s d_b} \tau(x) \tag{2}$$

and considering that it is also mathematically true that the second derivative of reinforcement displacement ($u_s(x)$) is equal to the first derivative of reinforcement strain ($\varepsilon_s(x)$):

$$\frac{d^2 u_s(x)}{dx^2} = \frac{d\varepsilon_s(x)}{dx}, \tag{3}$$

a new expression can be written:

$$\frac{d^2 u_s(x)}{dx^2} = \frac{4}{E_s d_b} \tau(x), \tag{4}$$

where E_s is the elastic modulus of steel, d_b is bar diameter and $\tau(x)$ is bonding stress between reinforcement and concrete. According to the assumption 4 and Fig. 1, it can be stated that the shear angle ($\gamma_c(x)$) is equal to the ratio of shear displacement ($u_{shear}(x)$) to the cover thickness of concrete (c):

$$\gamma_c(x) = \frac{u_{shear}(x)}{c}. \tag{5}$$

Taking into consideration the assumption 2 as well as the Eqs. (1) and (5):

$$\tau(x) = \gamma_c(x) G_c = \frac{u_{shear}(x)}{c} G_c = \frac{u_s(x)}{c} G_c, \tag{6}$$

where G_c is the shear modulus of concrete. From the Elastic Theory it is well known that:

$$G_c = \frac{E_c}{2(1+v)}, \tag{7}$$

where E_c is the elastic modulus of concrete and v is Poisson's ratio (taken $= 0.2$ for uncracked concrete). Therefore, the Eq. (6) can be modified:

$$\tau(x) = \frac{u_s(x)}{c} \frac{E_c}{2(1+v)}. \tag{8}$$

Then, according to Eqs. (4) and (8), the following expression can be written:

$$\frac{d^2 u_s(x)}{dx^2} = \frac{2E_c}{E_s d_b c(1+v)} u_s(x). \tag{9}$$

To simplify the form of Eq. (9) a new variable is introduced:

$$K = \frac{2E_c}{E_s d_b c (1 + v)},$$ (10)

Rearranging the expression (9) results in a second order homogenous differential equation:

$$\frac{d^2 u_s(x)}{dx^2} - K u_s(x) = 0.$$ (11)

The solution to the Eq. (11) has a general form:

$$u_s(x) = C_1 e^{x\sqrt{K}} + C_2 e^{-x\sqrt{K}},$$ (12)

where C_1 and C_2 are the integration constants. To obtain the solution for a particular case shown in Fig. 1 the following boundary conditions, which take into consideration symmetry conditions and the reinforcement strain distribution, can be written:

$$u_s(0) = 0, \frac{du_s}{dx}\left(\frac{s_{rm}}{2}\right) = \varepsilon_s\left(\frac{s_{rm}}{2}\right) = \frac{P}{E_s A_s} = \varepsilon_{si},$$ (13)

where ε_{si} is reinforcement strain considering fully cracked section. Furthermore, taking into consideration the Eq. (12) and (13) the integration constants C_1 and C_2 can be determined. This results in the function of reinforcement displacement at any point x:

$$u_s(x) = \frac{\varepsilon_{si}}{\left(1 + e^{s_{rm}\sqrt{K}}\right)\sqrt{K}}\left(e^{(x+0.5s_{rm})\sqrt{K}} - e^{(0.5s_{rm}-x)\sqrt{K}}\right).$$ (14)

Taking into consideration Eq. (1) and Fig. 1, an expression, relating the average crack width and the displacement of reinforcement at the end of RC block, can be written:

$$w_m = 2u_s(0.5s_{rm}).$$ (15)

Lastly, taking into consideration the expressions (14) and (15), a formula for calculating average crack width of a tensile member is derived:

$$w_m = \frac{2\varepsilon_{si}}{\sqrt{K}}\frac{\left(e^{s_{rm}\sqrt{K}} - 1\right)}{\left(e^{s_{rm}\sqrt{K}} + 1\right)}.$$ (16)

The only unknown in the Eq. (16) is the average crack spacing (s_{rm}). However, mathematically, the maximum possible value of the term $\frac{\left(e^{s_{rm}\sqrt{K}}-1\right)}{\left(e^{s_{rm}\sqrt{K}}+1\right)}$ is equal to 1. In this case, seeking conservative results, the Eq. (16) can be simplified:

$$w_m = \frac{2\varepsilon_{si}}{\sqrt{K}} = \frac{2\sigma_s}{E_s\sqrt{K}}$$ (17)

where σ_s stress of reinforcement considering fully cracked section.

Equation (17) indicates that, similar to other theoretical models (e.g., [2–4]), the most important variable determining the crack width is reinforcement strain (ε_{si}). Other less important parameters are included in the variable K (Eq. (10)) which are modulus of elasticity of both steel (E_s) and concrete (E_c), bar diameter (d_b) and the cover thickness (c).

3 Verification of Model Using RC Ties and Beams

This section compares mean crack width (w_m) predictions by the proposed model against the test data of RC ties and beams subjected to short–term loading reported in the literature. The comparison also includes the crack width models from Eurocode 2 (EC2) [5] and MC 2010 (MC2010) [6].

3.1 Analysis of RC Ties

The analysis employs 125 RC ties from 2 experimental programs reported by Wu and Gilbert [7] (4 members) and Farra and Jaccoud [8] (121 members). All the members were singly reinforced and basically having the same nominal geometry: a square section of 100 × 100 mm and length of 1100 or 1150 mm. Tension on RC ties was exerted by pulling them by the ends of reinforcement. Main geometrical and material parameters of the ties such as cylinder strength (f_{cm}), modulus of elasticity of concrete (E_c) and reinforcement (E_s), bar diameter (d_b), concrete cover (c), reinforcement ratio (ρ), and free shrinkage strain (ε_{cs}) are presented in Table 1. It also should be noted that all the specimens were sorted by reinforcement ratio in ascending order.

Table 1. Main characteristics of test RC ties from the study of Wu and Gilbert [7] and Farra and Joccoud [8]

No	Authors	f_{cm}, MPa	E_c*, MPa	E_s, MPa	d_b, mm	c, mm	ρ, %	ε_{cs}, x10–6
1.–38	Farra and Joccoud	29.90–67.20	27000–41800	200000	10	45	0.79	–**
39.–40	Wu and Gilbert	21.56–24,73	21600–22400		12	44	1.13	28–249
41.–81	Farra and Joccoud	29.90–87.10	27000–41800		14	43	1.54	–
82.–83	Wu and Gilbert	21.56–24.73	21600–22400		16	42	2.01	28–249
84.–125	Farra and Joccoud	29.90–87.10	27000–41800		20	40	3.14	–

* Determined by a test
** No information

The accuracy of all 3 models – Eurocode 2 [5], Model Code 2010 [6] and Pure Shear Method (PSM) was judged by two parameters: while mean values represent consistency, the coefficient of variation quantifies the scatter. The predictions were expressed in terms of the normalized mean crack width. The statistical results are presented in Fig. 2 with crack widths taken at the reinforcement stress $\sigma_s = 250$ MPa representing the service load. For some of the members with the maximum reinforcement stress being below 250 MPa or stress at first cracking being above 250 MPa, the values closest to 250 MPa
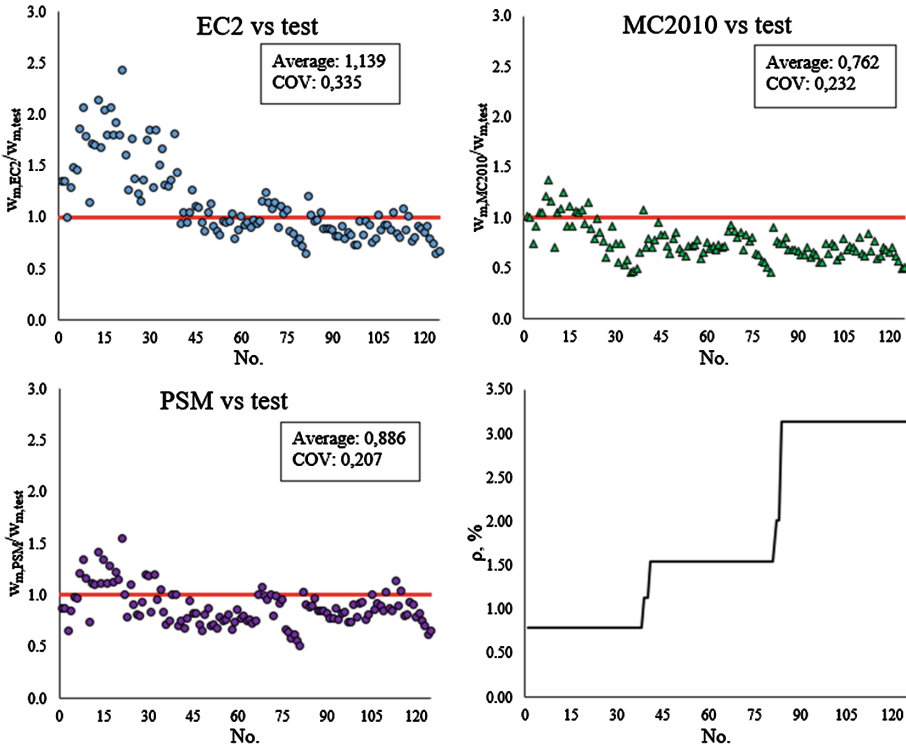
Fig. 2. Mean crack width (w_m) predictions for RC ties from the study of Wu and Gilbert [7] and Farra and Joccoud [8]

were taken. Two specimens were excluded from the original tests of Farra and Jaccoud [8] as σ_s exceeded 400 MPa at the formation of first crack.

The results presented in Fig. 2 indicate that the newly proposed model has demonstrated superior accuracy regarding the design code methods in terms of scatter. For the PSM, the coefficient of variation was 21% compared to 23 and 34% for Model Code 2010 [5] and Eurocode 2 [6], respectively. However, the mean value for the PSM was not safe with the prediction being ≈11% below the test. Model Code 2010 predictions were also unsafe (≈−24%), whereas only Eurocode 2 gave safe predictions (≈+14%). It is worth mentioning that the underestimation of the crack width using PSM could be the fact that the shrinkage of concrete was neglected (assumption 7).

3.2   Analysis of RC Beams

Although the model was originally derived using singly reinforced tensile member, due to practical reasons, it is worth checking its capability to predict mean crack width of bending members.

The analysis considers 4 RC beams from the well-known tests reported by Rüsch and Rehm [9]. Main characteristics of the beams are presented in the Table 2 and Fig. 3.

The bending members were reinforced with high–bond steel bars having similar bottom and lateral cover thicknesses. The members were tested under a four–point bending scheme with mean crack width (w_m) defined at the reinforcement level within the 2 m pure bending zone. The w_m–σ_s graphs for each of the beams are presented in Fig. 4. To quantify the prediction accuracy of different models, the numerical w_m values at steel stress $\sigma_s = 250$ MPa are given in the Table 3.

In general, the w_m predictions for three methods were quite accurate and close to each other. However, it should be noted that the Pure Shear Method gives a slightly lower variation compared to the other two theoretical models – Eurocode 2 [5] and Model Code 2010 [6].

Table 2. Bending members from the study of Rüsch and Rehm [9]

No	Name	h x b x L, mm	f_{cm}, MPa	$E_c{}^*$, MPa	E_s, MPa	d_b, mm	c, mm	ρ, %
1	R26	625x300x4500	14.35	24515	200000	16	35	0.46
2	R70	625x300x4500	14.18	24429	200000	26	25	0.60
3	R14	625x300x4500	13.84	24253	200000	16	30	0.46
4	R69	625x300x4500	13.50	24074	200000	26	25/26**	0.91

* Determined by Eurocode 2[5]
** Lateral cover thickness is equal to 25 mm, bottom – 26 mm
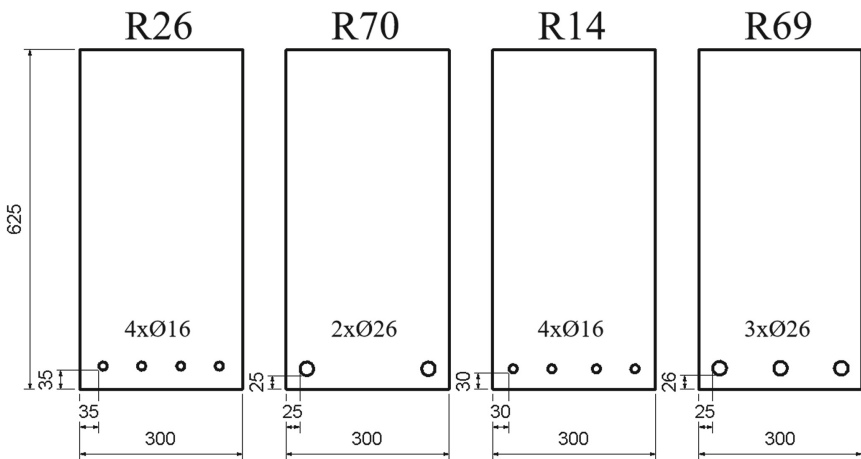
Fig. 3. Main cross section parameters of 4 beams reported by Rüsch and Rehm [9].
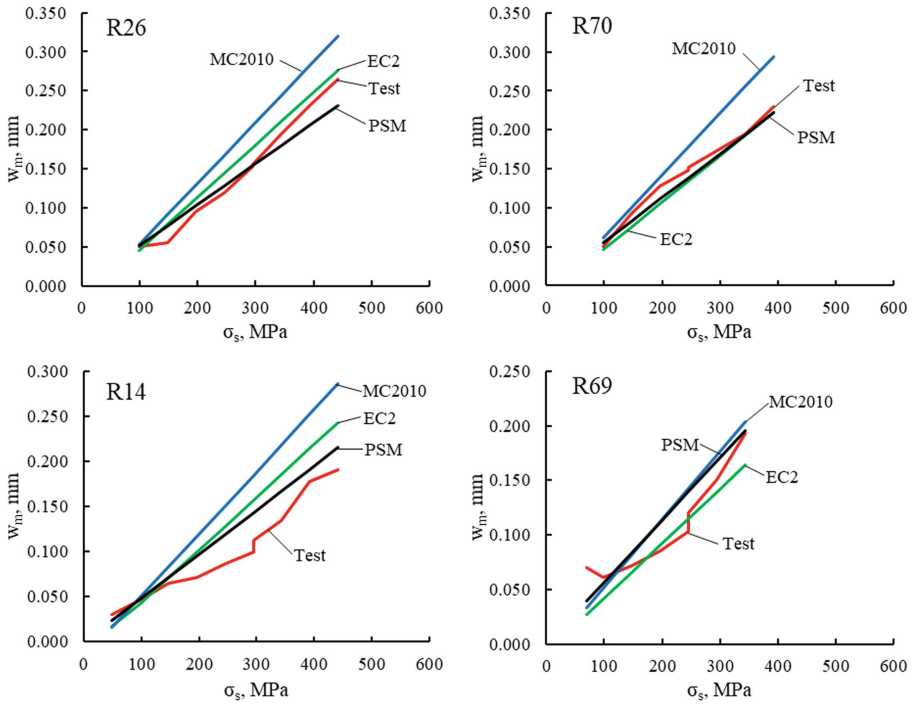
Fig. 4. Mean crack width (w_m) versus reinforcement stress (σ_s) for test beams from the study of Rüsch and Rehm [9]

Table 3. Statistical analysis of the average crack width of the bending members from the study of Rüsch and Rehm [9]

		Test	EC2	MC2010	PSM	EC2	MC2010	PSM
Name	σ_s, MPa	$w_{m,test}$, mm	$w_{m,EC2}$, mm	$w_{m,MC2010}$, mm	$w_{m,PSM}$, mm	$w_{m,EC2}/w_{m,test}$	$w_{m,MC2010}/w_{m,test}$	$w_{m,PSM}/w_{m,test}$
R26	250	0.123	0.148	0.171	0.131	1.205	1.395	1,067
R70		0.154	0.137	0.182	0.141	0.890	1.118	0.918
R14		0.088	0.131	0.154	0.122	1.484	1.747	1.380
R69		0.123	0.117	0.146	0.142	0.953	1.184	1.157
					Avg	1.133	1.377	1.131
					COV	0.239	0.194	0.171

4   Conclusions

- A new simplistic approach for modeling crack width of RC members subjected to short–term loading is proposed. The model assumes sole elastic shear stresses occurring in the cover thickness of concrete and, thus, allows to simply calculate one of the most essential parameters closely related to SLS – mean crack width.

- To check the capability of Pure Shear Method (PSM) to predict mean crack width of RC ties, a comparative analysis employing experimental results of 125 tensile members reported by Wu and Gilbert and Farra and Joccoud as well as theoretical results predicted by the new model, Eurocode 2 and Model Code 2010 was performed. It was indicated that, at the chosen reinforcement stress level of 250 MPa, representing service load, regarding the correspondence to the experimental results, PSM crack width predictions tends to give slightly lower variation ($COV = 21\%$) compared to other 2 models – Eurocode 2 ($COV = 34\%$) and Model Code 2010 ($COV = 23\%$). However, it was observed that the newly proposed model and Model Code 2010 on average tends to give slightly smaller mean crack width values compared to experiment.
- A possibility to predict mean crack width of bending members subjected to short–term loads using Pure Shear Method was also tested. For the mean crack width validation 4 bending members from the experimental program of Rüsch and Rehm were chosen. The predictions by the new model were also compared to the results obtained using Eurocode 2 and Model Code 2010. It was indicated that, at the chosen reinforcement stress level of 250 MPa, regarding the correspondence to the experimental results, the new model tends to give slightly lower variation ($COV = 17\%$) compared to other 2 models – EC2 ($COV = 24\%$) and MC2010 ($COV = 19\%$).

References

1. Fernández Ruiz, M., Muttoni, A., Gambarova, P.: Analytical modeling of the pre-and post-yield behavior of bond in reinforced concrete. ASCE J. Struct. Eng. **133**(Article), 1364–1372 (2007)
2. Gergely, P., Lutz, L.A.: Maximum crack width in reinforced concrete flexural members. Special Public. **20**, 87–117 (1968)
3. Clark, A.P.: Cracking in reinforced concrete flexural members. J. Proc. **52**(4), 851–862 (1956)
4. Frosch, R.J.: Another look at cracking and crack control in reinforced concrete. Struct. J. **96**(3), 437–442 (1999)
5. EN 1992–1–1. Eurocode 2: Design of concrete structures: Part 1–1: General rules and rules for buildings. CEN European Committee for Standardization. Brussels, Belgium (2004)
6. fib (International Federation for Structural Concrete). Model code for concrete structures 2010, Wiley, Berlin, 434 (2013)
7. Wu, H.Q., Gilbert, R.I.: An experimental study of tension stiffening in reinforced concrete members under short-term and long-term loads. UNICIV Report no, 28 (2008)
8. Farra, B., Jaccoud, J.P.: Influence du béton et de l'armature sur la fissuration des structures en béton: rapport des essais de tirants sous déformation imposée de courte durée. EPFL, Lausanne. Département de Génie Civil, École Polytechnique. Fédérale de Lausanne, Pub. No. 140 (1994)
9. Rüsch, H., Rehm, G.: Versuche mit Betonformstahlen. Deutscher Ausschuss fur Stahlbcton, Berlin (1964)