Unifying Reality and Virtuality: Constructing a Cohesive Metaverse Using Complex Numbers

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Abstract. In this position paper we present a novel mathematical framework for building metaverses, which is a potential way to unify reality and virtuality to create a cohesive whole universe. We argue that the nature of metaverses is inherently mathematical, and propose that the system of complex numbers could play a key role in constructing them. Specifically, we provide context for our argument and offer a supporting example, *the analytic signal*, to demonstrate how to construct its imaginary counterpart with the Hilbert transform to a given real signal and how to unify them to form a cohesive complex signal that facilitates the analysis of local dynamic behaviors of the signal. This framework has significant potential for building a metaverse. By leveraging the power of complex numbers, one can create a unified mathematical system that merges the physical and virtual worlds. We believe that this proposal will inspire further research and development of metaverses in this field and that our framework will contribute to the construction of a metaverse that offers unprecedented levels of interactivity and immersion.

Keywords: metaverse · complex numbers · analytic signal · Hilbert transform · phenomenology

1 Introduction

The concept of a metaverse has been popularized in science fiction and is often associated with the virtual world depicted in Neal Stephenson's novel Snow Crash, as well as the movie The Matrix. However, in recent years, the concept has gained momentum as advancements in VR and AR technologies have made it possible to create increasingly realistic and immersive virtual environments.

The potential applications of metaverses are vast and varied. They range from entertainment and gaming to education, healthcare, and even commerce. A metaverse could enable people from different parts of the world to interact in real-time, attend virtual events, and collaborate in the virtual space. It could

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also provide new opportunities for remote work and learning, and offer new ways for businesses to connect with customers.

As the concept of metaverses gains traction, it becomes increasingly clear that constructing one is a daunting technical challenge. To succeed in this endeavor, we must first answer several fundamental questions such as *What is a metaverse*? *What is the nature of a metaverse*? *How should a metaverse be constructed*?

With respect to the first question, there have been numerous discussions on the meaning of the term "metaverse." One way of discussing it is from the meaning of the word "metaverse" itself. The term is derived from two words: "-verse," meaning universe, and "meta-," meaning "after", "beside" [7]. While "meta-" has multiple meanings, the interpretation of "after" or "beside" is often adopted to describe a metaverse as a virtual world that exists *parallel* to the real world.

This is the most common understanding of a metaverse today and its realization accordingly is a virtual-reality (VR) type where a person's body is in the real world while his head is immersed in the virtual world. This separation of the body and mind can be seen an empirical evidence for the philosophical notion of Cartesian dualism, famously captured in Descartes' dictum, "cogito ergo sum" - I think, therefore I am [2]. This doctrine posits that mind and body are fundamentally distinct and that thinking and being are separate sets of phenomena. The mind is the seat of reason and meaning, and through observation, it assigns meaning to the world by relating it to abstract understandings of an idealized reality, which in turn informs a plan of action.

Heidegger rejected the doctrine and turned it around [2]. From his perspective, the meaningfulness of everyday experience lies not in the head, but in the world. It is a consequence of our mode of being, of the way in which we exist in the world. This is the new doctrine from phenomenology. To perceive and experience the world thinking and being must be fundamentally intertwined.

Hence, any virtual reality-type implementation may not be the most suitable approach to constructing a metaverse. Rather, the term "meta" carries a more profound connotation, denoting a system's capability to augment the comprehension of another system through explanatory enhancements. The objective of a metaverse, therefore, should be to unify the physical and virtual realms and further enhance the collective understanding of the world. By enabling people to coexist simultaneously in both worlds, we can unify them to create a harmonious whole where the virtual world complements and elevates the real world. While individuals can still opt to indulge in distinct experiences in each realm, the metaverse provides a pathway to merge the two worlds and generates a novel, integrated experience that transcends what either domain can offer in isolation. Such a realization is shown in Fig. 1. Thus, one can expect that such a metaverse opens up fresh avenues for unparalleled experiences.

Because we ask how to build a metaverse, we have to answer the question of what the nature of the metaverse is. As a concept that enhances our experience

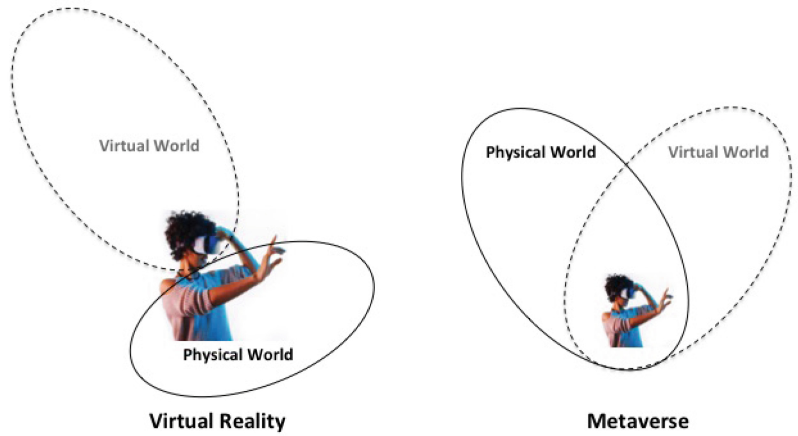

Fig. 1. Two different ways of implementing a metaverse. In the realization like virtual reality, the virtual world is isolated from the real world. The realization right illustrates how people coexist simultaneously in both worlds.

of the real world, a metaverse is not a new technology rather is a complex amalgam of various technologies, such as sensory technology, the internet, blockchain, big data, artificial intelligence, quantum technology, and more. Notably, these technologies rely heavily on mathematics, as seen in blockchain's use of digital numbers to establish trust (proof of trust). In the virtual world, physical laws can be ignored, as illustrated in the movie "Avatar," where floating mountains abound in Pandora. In this paper, we argue that the nature of a metaverse is inherently mathematical, that is, a metaverse is fundamentally rooted in mathematics. To build a metaverse we need to lay down a mathematical framework.

In this paper, we suggest using the system of complex numbers, more specifically, using the imaginary unit "i" to unify the real and world worlds into a cohesive whole:

$$\textit{Metaverse World}(\boldsymbol{M}) = \textit{Real World}(\boldsymbol{S}) + i\ \textit{Virtual World}(\boldsymbol{V})$$

or

$$M = S + i\,V$$

Here i is a mathematical operation of "unification". It will bring together or merge two worlds into a single cohesive whole and achieve harmony and consistency between the two worlds. In the following sections, we will discuss how to use i to unify two parts and particularly illustrate how to construct an imaginary part to complement a given real part.

2 Complex Numbers

What if -1 had a square root? We cannot find a real number whose square is negative, so we need to introduce a totally new type of number. The new quantity is called i, which is defined as the square root of -1. This gives rise to the expression of imaginary numbers in the form of $a + i\,b$, where a and b are real numbers. i is known as the imaginary unit, first introduced by Descartes [2].

A complex number of the form $a + i\,b$ consists of a real part a and an imaginary part b. A real number a can be regarded as a complex number $a + i0$, with an imaginary part of 0. Similarly, a purely imaginary number ib can be regarded as a complex number $0 + ib$, with a real part of 0. Here, we no longer treat a, b, as a pair of numbers, but rather as a single number, denoted by the symbol $z = a + ib$. It can be verified that complex numbers satisfy all algebraic rules [6].

The discovery of complex numbers in mathematics sparked skepticism and confusion initially due to their abstract nature. However, as scientific knowledge advanced, it became evident that complex numbers possess unique properties that make them essential in describing physical phenomena. In particular, in the latter half of the 20th century, the behavior of the subatomic world was found to be fundamentally governed by the laws of complex numbers. Furthermore, modern particle physics relies also on complex numbers as quantum numbers [6].

In the field of electrical engineering, understanding the dynamic relationship between current I and voltage V in AC circuits is of utmost importance. Although current and voltage can be represented by trigonometric functions such as $V = \sin \omega t$ and $I = \cos \omega t$, their relationship is not explored in depth. In fact, like two sides of a coin, these two quantities represent the same electrical phenomenon but just from different perspectives. Therefore, we can combine them with a complex number, $z = \cos \omega t + i \sin \omega t = e^{i\omega t}$, which provides a unified representation of the electrical phenomenon of AC circuits. Current and voltage are unified into a circle on the complex plane. At first glance, it appears that the time component has disappeared as only the circle is observed. However, what vanishes is not time but the time axis. The complex number $z = e^{i\omega t}$ represents the combination of current and voltage on the circle, creating a three-dimensional spiral signal varying over time.

As we observe the circle along the time axis, a point moving around the circle is seen, forming a spiral that advances with time. This spiral represents the combined current and voltage signals that vary over time, which cannot be observed without their unification into a three-dimensional spatial signal. The resulting three-dimensional spiral is an exciting and informative visualization that highlights the interplay between current and voltage in AC circuits.

By viewing current as analogous to the real world and voltage as the virtual world, the complex number z can be considered a realization of a metaverse. Furthermore, complex numbers could support the possibility of having multiple virtual environments, notwithstanding the existence of a singular real world. For example, a promising solution is to use generalized complex numbers, such as quaternions, for combining several virtual worlds. The form of Quaternions, a

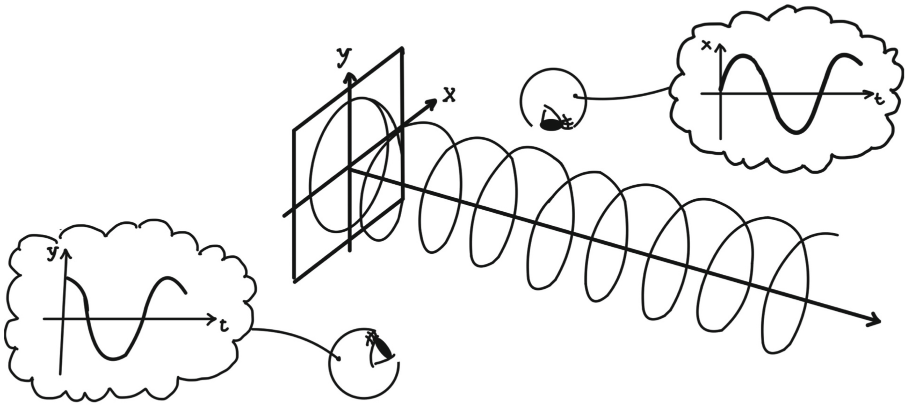

Fig. 2. AC is a three-dimensional spiral signal that changes over time. One can see its two components, current and voltage signals if one watches it from two different viewing angles (after [4]).

type of mathematical object, was invented by the Irish mathematician William Rowan Hamilton (1805–1865) [6], who discarded the commutativity of multiplication to form a four-dimensional vector space over the field of real numbers, expressed as $z = a + i\,b + j\,c + k\,d$. Quaternions have four independent "basis" elements, namely *1, i, j,* and *k*, that span the entire space. One can envision $a$representing the real space, while b, c, and d represent three virtual spaces. The algebraic structure of quaternions offers an elegant and powerful tool for performing calculations in the metaverse and multiple virtual spaces, especially for geometric construction problems.

Complex numbers offer a viable mathematical tool for constructing a metaverse. However, the crux is not about a simple combination of the real and imaginary rather lies in the synergy between the two parts to create a cohesive entity. An exquisite illustration of this synergy is the electromagnetic field F, which is described as $F = E + iB$ - where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, respectively, and together they form a perfectly unified entity. The interaction between these two fields is demonstrated clearly in the famous Maxwell's equations [6], which describe their behaviors in space and time:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \tfrac{\rho}{\epsilon_0} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\tfrac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \tfrac{\partial \mathbf{E}}{\partial t} \end{aligned} \tag{1}$$

where ρ is the charge density, $\mathbf{J}$ is the current density, ϵ_0 is the electric constant (also known as the vacuum permittivity), and μ_0 is the magnetic constant (also known as the vacuum permeability).

The interaction between the electric field $\mathbf{E}$ and the magnitude field $\mathbf{B}$ can be seen in Gauss's law, which states that the divergence of the electric field is proportional to the charge density at a point in space. This means that the electric field at a point in space is influenced by the distribution of charges around it. The magnitude field, on the other hand, is determined by the charge distribution itself. In this way, the electric and magnitude fields interact to produce the behavior of electric charges in space and time.

When it comes to metaverse applications, one can also expect E and B to be deeply intertwined. One of the most daunting challenges is constructing an imaginary counterpart to a given real part. While combining any two parts with the imaginary unit i can yield a complex number, generating a meaningful unification can be demanding. The question then arises as to whether there exists a *systematic* approach for creating an imaginary part to match a given real part. The answer to this question is affirmative, and in the following section, we provide an example from the field of signal processing to demonstrate how such a systematic approach is designed.

3 Construction of Analytic Functions

In this section, we present an illustrative example from the field of signal processing that showcases the potential of utilizing complex numbers to achieve meaningful unification. Our hope is that this example could inspire and inform efforts towards building a metaverse.

Real-world signals can often be complex and pose significant challenges for analysis. Fortunately, the development of Fourier Transform by the brilliant mathematician Josef Fourier revolutionized spectral analysis [1]. This powerful tool enables us to represent any infinite periodic function as a linear combination of sine and cosine functions, providing us with the ability to decompose continuous signals into a set of spectral components. By analyzing and processing individual sine or cosine functions, we can gain insights into the underlying patterns and structure of complex signals.

Suppose here we have a physical signal from the real world, denoted by $f(x)$ with a known frequency:

$$f(x) = Acos(\omega x), \quad A > 0. \tag{2}$$

The function $f(x)$ is a cosine function with amplitude A and angular frequency ω. Even though the function is very simple, there is no *direct* way to obtain information about the signal, for example, the amplitude A, from the signal [3]. This is simply because the instantaneous measurement of the function at any point, such as $x = x_0$, will result in any value between $-A$ and A, depending on x_0. The value of the function f at $x = x_0$ gives no information regarding whether the signal is at a local maximum or minimum, or whether f is increasing or decreasing in a neighborhood around x_0. This signal representation greatly restricts its usage in signal analysis.

Additionally, in the case of real physical signals, the negative frequencies they produce are physically meaningless. As a result, it is imperative to eliminate negative frequencies from real-world signals. In the field of communication and signal processing, a technique has been developed and implemented to suppress negative frequencies. This approach involves constructing an imaginary function, denoted as b, for a given real function a, and then combining them using the imaginary number i to form a complex number $a + ib$. The selection of b is not arbitrary and one practical way to determine it is to use the Hilbert transform of a.

The function created using this method is referred to as an **analytic function**. In the subsequent discussion, we shall introduce the Hilbert transform (HT) and demonstrate its application in constructing an analytic function. Our aim is to inspire the use of imaginary numbers in constructing a metaverse that comprises both real and imaginary worlds.

3.1 The Hilbert Transform

Different from the Fourier transform, the Hilbert transform is a mapping between two sets of functions.

The Hilbert transform H_i is defined as [3]

$$f_{H_i}(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(\tau)}{\tau - x}d\tau. \tag{3}$$

This is usually written as

$$f_{H_i}(x) = H_i\{f\} \tag{4}$$

Since the Hilbert transform simply maps one spatial function to another, we can calculate the Hilbert transform in the Fourier domain:

$$f_{H_i}(x) = f * \frac{-1}{\pi x} \tag{5}$$

It can be seen that the Hilbert transform of a function f can be obtained by convolving f with the function $\frac{-1}{\pi x}$. Therefore, the Hilbert transform is a just linear mapping.

Let F_{H_i} denote the Fourier transform of f_{H_i}. According to Eq. (4) we have:

$$F_{H_i}(u) = F(u) \cdot i\ sign(u) \tag{6}$$

Given that convolution in the spatial domain is equivalent to multiplication in the Fourier domain, the Fourier transform of f_{H_i} can be obtained by multiplying the Fourier transform F by the imaginary unit i, and then changing the sign of the resulting product for frequencies that are negative. Another way to explain this procedure is to rotate the argument of the frequency components by an angle of $\frac{\pi}{2}$ in the positive direction for positive frequencies and in the negative direction for negative frequencies [3]. It can be observed that applying the Hilbert transform twice simply changes the sign of a function.

Here are some typical functions and their corresponding Hilbert transforms [1,3]:

Function	**Hilbert transform**
$cos(x)$	$-sin(x)$
$sin(x)$	$cos(x)$
$\delta(x)$	$\frac{-1}{\pi x}$

(7)

With the Hilbert transform of a function f, we can define its analytic function f_A.

3.2 Analytic Signal

The analytic function f_A corresponding to the function f is defined as

$$f_A = f - \ if_{H_i} \tag{8}$$

From this definition, it is clear that the construction of an analytic function is a linear mapping. From the Hilbert transform table in the previous section, we can obtain the following correspondences: The analytic signal corresponding to the function $cosx$ is e^{ix}. The analytic signal corresponding to the function $sinx$ is $-ie^{ix}$. Since both $cosx$ and $i\ sinx$ correspond to the same analytic signal e^{ix}, the construction of an analytic function is obviously not a one-to-one mapping.

The definition of the analytic function f_A can be rewritten using convolution as

$$f_A = f * [\delta(x) + \frac{i}{\pi x}] \tag{9}$$

In the Fourier domain, this has the form

$$F_A = F \cdot [1 + sign(u)] = 2F \cdot step(u) \tag{10}$$

Hence, the analytic signal related to the function f can be obtained by eliminating all of its negative frequencies and multiplying by a factor of 2. It is important to note that this implies that an analytic function cannot be both real and non-zero. This is because the Fourier transform of a real function is always Hermitian, but F_A is not. Consequently, the Hilbert transform of a real function f is a one-to-one mapping, as the values of F for $u < 0$ can be derived from F for $u > 0$ [3].

In conclusion, we can observe that the analytic signal is only relevant for real signals. For such signals, the corresponding analytic signal is complex, with the real part being the original signal and the imaginary part being its Hilbert transform. An example of how to construct an analytic signal is presented in the following section.

3.3 Analytic Signals in Signal Analysis

Let us reconsider how to analyze a real-world signal $f(x)$ addressed in the beginning of Sect. 3.

$$f(x) = A\cos(\omega x), \quad A > 0. \tag{11}$$

To help signal analysis, we construct an analytic signal f_A for function f which could provide us with information on the local behavior of the signal.

We define its Hilbert transform

$$f_{H_i}(x) = -A\sin(\omega x), \tag{12}$$

and put them together

$$f_A(x) = A[\cos(\omega x) + i\sin(\omega x)] = Ae^{i\omega x}. \tag{13}$$

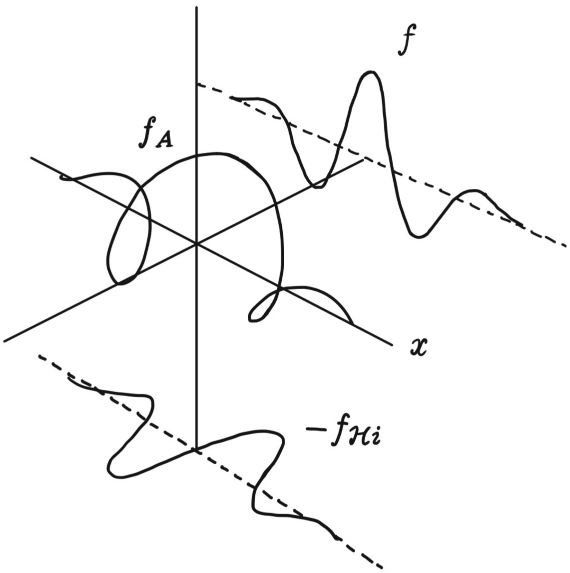

Fig. 3. The analytic signal f_A corresponding to a real function$f$. The Hilbert transform f_{H_i} of function f is shown in the imaginary domain with the reversed sign [3].

This means that the amplitude A can be directly obtained by

$$A = |f_A(x)| = \sqrt{[f(x)]^2 + [f_{H_i}(x)]^2}. \tag{14}$$

Note that this is a big advantage over function f whose amplitude A cannot be obtained directly.

Therefore, an analytic signal can provide a direct measure of the local behavior of the signal. For example, this information can be given by arg$[f_A]$ when the signal is a linear function of x. This is commonly referred to as the *phase* of f.

Here shows how to use the phase to infer the local behavior of the signal in some typical cases [3].

- If $arg[f_A] = 2\pi k$, the function f reaches its maximal value A at $x = x_0$
- If $arg[f_A] = \pi + 2\pi k$, the function f reaches its minimal value $-A$ at $x = x_0$
- If $arg[f_A] = \frac{\pi}{2} + 2\pi k$ then function f is just passing the zero going from negative to positive at $x = x_0$
- If $arg[f_A] = -\frac{\pi}{2} + 2\pi k$, the function f is just passing the zero going from positive to negative at $x = x_0$

For a real signal, its analytic signal is well-defined. It means for a given real signal, we can consider the absolute value, argument, and the arguments derivative of the corresponding analytic signal. Thus, we can make the following definitions:

Instantaneous amplitude of function $f(x)= |f_A(x)|$.

For a real signal in general, its instantaneous amplitude may not be a constant function. Similarly, the instantaneous phase of function f(x) is

Instantaneous phase of function $f(x)= arg[f_A(x)]$.

Finally, for any real function, such that its phase has a well-defined derivative with respect to x. We define

Instantaneous frequency of function $f(x)= \frac{d}{dx} arg[f_A(x)]$.

As with instantaneous amplitude, instantaneous frequency is usually not a constant function of x.

One may notice that for the example we discussed if we change x to t, the generated analytic signal is nothing but the AC signal. The Hilbert transform of the voltage signal is the current signal. The information of the local behavior and spatial frequency of a signal can be directly obtained from its counterpart analytic signal. Notice that local behavior and spatial frequency have always been considered important properties for signal processing and image processing.

4 Final Remarks

This position paper presents a mathematical framework for constructing a metaverse, which unifies reality and virtuality to create a cohesive whole universe. We argue that the nature of a metaverse is inherently mathematical, and propose that complex numbers could play a key role in constructing a metaverse. In the paper we offer a supporting example, the construction of analytic signals, to demonstrate how the power of complex numbers can be leveraged to create a unified mathematical system that merges the physical and virtual worlds. We believe that the proposed mathematical framework has significant potential for building a metaverse. We hope that this paper inspires further research and development in the field of metaverse construction and serves as a catalyst for the creation of new and innovative digital experiences that blur the line between reality and virtuality.

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