Chapter 9
Multiplication and Linear Integral Operators on L_p Spaces Representing Polynomial Covariant Type Commutation Relations

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Abstract Representations of polynomial covariant type commutation relations by pairs of linear integral operators and multiplication operators on Banach spaces L_p are constructed.

Keywords Multiplication operators · Integral operators · Covariance commutation relations

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9.1 Introduction

Commutation relations of the form

$$AB = BF(A) \tag{9.1}$$

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where A, B are elements of an associative algebra and F is a function of the elements of the algebra, are important in many areas of Mathematics and applications. Such commutation relations are usually called covariance relations, crossed product relations or semi-direct product relations. Elements of an algebra that satisfy (9.1) are called a representation of this relation in that algebra. Representations of covariance commutation relations (9.1) by linear operators are important for the study of actions and induced representations of groups and semigroups, crossed product operator algebras, dynamical systems, harmonic analysis, wavelets and fractals analysis and have applications in physics and engineering [4, 5, 20–22, 26–28, 34, 36, 45]. A description of the structure of representations for the relation (9.1) and more general families of self-adjoint operators satisfying such relations by bounded and unbounded self-adjoint linear operators on a Hilbert space use reordering formulas for functions of the algebra elements and operators satisfying covariance commutation relation, functional calculus and spectral representation of operators and interplay with dynamical systems generated by iteration of involved in the commutation relations [3, 6–8, 10, 11, 13–17, 29–34, 37–41, 45–58]. Algebraic properties of the commutation relation (9.1) are important in description of properties of its representations. For instance, there is a well-known link between linear operators satisfying the commutation relation (9.1) and spectral theory [44]. A description of the structure of representations for the relation (9.1) by bounded and unbounded self-adjoint linear operators on a Hilbert space, using spectral representation [2] of such operators, is given in [44] devoted to more general cases of families of commuting self-adjoint operators satisfying relations of the form (9.1).

In this paper we construct representations of (9.1) by pairs of linear integral and multiplication operators on Banach spaces L_p. Such representations can also be viewed as solutions for operator equations $AX = XF(A)$, when A is specified or $XB = BF(X)$ when B is specified. In contrast to [34, 45, 46, 58] devoted to involutive representations of covariance type relations by operators on Hilbert spaces using spectral theory of operators on Hilbert spaces, we aim at direct construction of various classes of representations of covariance type relations in specific important classes of operators on Banach spaces more general than Hilbert spaces without imposing any involution conditions and not using classical spectral theory of operators. This paper is organized in three sections. After the introduction, we present in Sect. 9.2 preliminaries, notations and basic definitions. In Sect. 9.3 we present the main results about construction of specific representations on Banach function spaces L_p.

9.2 Preliminaries and Notations

In this section we present some preliminaries, basic definitions and notations. For more details please read [1, 12, 18, 23, 24, 42, 43].

Let $S \subseteq \mathbb{R}$, ($\mathbb{R}$ is the set of real numbers), be a Lebesgue measurable set and let $(S, \Sigma, \tilde{m})$ be a σ-finite measure space, that is, S is a nonempty set, Σ is a σ-algebra with subsets of S, where S can be covered with at most countably many disjoint sets $E_1, E_2, E_3, \ldots$ such that $E_i \in \Sigma$, $\tilde{m}(E_i) < \infty$, $i = 1, 2, \ldots$ and $\tilde{m}$ is the Lebesgue measure. For $1 \leqslant p < \infty$, we denote by $L_p(S)$, the set of all classes of equivalent measurable functions $f : S \to \mathbb{R}$ such that $\int\limits_S |f(t)|^p dt < \infty$. This is a Banach space (Hilbert space when $p = 2$) with norm $\|f\|_p = \left(\int\limits_S |f(t)|^p dt\right)^{\frac{1}{p}}$. We denote by $L_\infty(S)$ the set of all classes of equivalent measurable functions $f : S \to \mathbb{R}$ such that there is a constant $\lambda > 0$, $|f(t)| \leq \lambda$ almost everywhere. This is a Banach space with norm $\|f\|_\infty = \text{ess sup}_{t\in S} |f(t)|$.

9.3 Operator Representations of Covariance Commutation Relations

Before we proceed with constructions of more complicated operator representations of commutation relations (9.1) on more complicated Banach spaces, we wish to mention the following two observations that, while being elementary, nevertheless explicitly indicate differences in how the different operator representations of commutation relations (9.1) interact with the function F.

Proposition 9.3.1 *Let $A : E \to E$ and $B : E \to E$, $B \neq 0$, be linear operators on a linear space E, such that any composition among them is well defined and consider $F : \mathbb{R} \to \mathbb{R}$ a polynomial. If $A = \alpha I$, then $AB = BF(A)$ if and only if $F(\alpha) = \alpha$.*

Proof If $A = \alpha I$, then

$$AB = \alpha I B = \alpha B,$$
$$BF(A) = BF(\alpha I) = BF(\alpha)I = F(\alpha)B.$$

We have then $AB = BF(A)$, $B \neq 0$ if and only if $F(\alpha) = \alpha$. □

Proposition 9.3.2 *Let $A : E \to E$ and $B : E \to E$ be linear operators such that any composition among them is well defined and consider a polynomial $F : \mathbb{R} \to \mathbb{R}$. If $B = \alpha I$, where $\alpha \neq 0$, then $AB = BF(A)$ if and only if F is a function such that $F(A) = A$.*

Proof If $B = \alpha I$ then

$$AB = A(\alpha I) = \alpha A,$$
$$BF(A) = \alpha I F(A) = \alpha F(A).$$

We have then $AB = BF(A)$ if and only if $F(A) = A$. □

9.3.1 Representations of Covariance Commutation Relations by Integral and Multiplication Operators on L_p Spaces

We consider first a useful lemma for integral operators.

Lemma 9.3.1 *Let $f : [\alpha_1, \beta_1] \to \mathbb{R}$, $g : [\alpha_2, \beta_2] \to \mathbb{R}$ be two measurable functions such that for all $x \in L_p(\mathbb{R})$, $1 \le p \le \infty$,*

$$\int_{\alpha_1}^{\beta_1} f(t)x(t)dt < \infty, \quad \int_{\alpha_2}^{\beta_2} g(t)x(t)dt < \infty,$$

where $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$, $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$. Set $G = [\alpha_1, \beta_1] \cap [\alpha_2, \beta_2]$. Then the following statements are equivalent:

(i) *For all $x \in L_p(\mathbb{R})$, where $1 \le p \le \infty$, the following holds*

$$\int_{\alpha_1}^{\beta_1} f(t)x(t)dt = \int_{\alpha_2}^{\beta_2} g(t)x(t)dt.$$

(ii) *The following conditions hold:*

a) *for almost every $t \in G$, $f(t) = g(t)$;*
b) *for almost every $t \in [\alpha_1, \beta_1] \setminus G$, $f(t) = 0$;*
c) *for almost every $t \in [\alpha_2, \beta_2] \setminus G$, $g(t) = 0$.*

Proof $(ii) \Rightarrow (i)$ follows from direct computation.

Suppose that (i) is true. Take $x(t) = I_{G_1}(t)$ the indicator function of the set $G_1 = [\alpha_1, \beta_1] \cup [\alpha_2, \beta_2]$. For this function we have,

$$\int_{\alpha_1}^{\beta_1} f(t)x(t)dt = \int_{\alpha_2}^{\beta_2} g(t)x(t)dt = \int_{\alpha_1}^{\beta_1} f(t)dt = \int_{\alpha_2}^{\beta_2} g(t)dt = \eta,$$

η is a constant. Now by taking $x(t) = I_{[\alpha_1,\beta_1]\setminus G}(t)$ we get

$$\int_{\alpha_1}^{\beta_1} f(t)x(t)dt = \int_{\alpha_2}^{\beta_2} g(t)x(t)dt = \int_{[\alpha_1,\beta_1]\setminus G} f(t)dt = \int_{\alpha_2}^{\beta_2} g(t)\cdot 0dt = 0.$$

Then $\int\limits_{[\alpha_1,\beta_1]\setminus G} f(t)dt = 0$. If instead $x(t) = I_{[\alpha_2,\beta_2]\setminus G}(t)$, then $\int\limits_{[\alpha_2,\beta_2]\setminus G} g(t)dt = 0$. We claim that $f(t) = 0$ for almost every $t \in [\alpha_1, \beta_1] \setminus G$ and $g(t) = 0$ for almost every $t \in [\alpha_2, \beta_2] \setminus G$. We take a partition $S_1, \ldots, S_n, \ldots$ of the set $[\alpha_1, \beta_1] \setminus G$

such that each set S_i, $i = 1, 2, 3, \ldots$ has positive measure. For each $x_i(t) = I_{S_i}(t)$, $i = 1, 2, 3, \ldots$ we have

$$\int_{\alpha_1}^{\beta_1} f(t)x(t)dt = \int_{\alpha_2}^{\beta_2} g(t)x(t)dt = \int_{S_i} f(t)dt = \int_{\alpha_2}^{\beta_2} g(t)\cdot 0dt = 0.$$

Thus, $\int_{S_i} f(t)dt = 0,\ i = 1, 2, 3, \ldots$. Since we can choose arbitrary partition with positive measure on each of its elements we have

$$f(t) = 0 \quad \text{for almost every } t \in [\alpha_1, \beta_1] \setminus G.$$

Analogously, $g(t) = 0$ for almost every $t \in [\alpha_2, \beta_2] \setminus G$. Then,

$$\eta = \int_{\alpha_1}^{\beta_1} f(t)dt = \int_{\alpha_2}^{\beta_2} g(t)dt = \int_G f(t)dt = \int_G g(t)dt.$$

Then, for all function $x \in L_p(\mathbb{R})$ we have

$$\int_G f(t)x(t)dt = \int_G g(t)x(t)dt \iff \int_G [f(t) - g(t)]x(t)dt = 0.$$

By taking $x(t) = \begin{cases} 1, & \text{if } f(t) - g(t) > 0, \\ -1, & \text{if } f(t) - g(t) < 0, \end{cases}$ for almost every $t \in G$ and $x(t) = 0$ for almost every $t \in \mathbb{R} \setminus G$, we get $\int_G |f(t) - g(t)|dt = 0$. This implies that $f(t) = g(t)$ for almost every $t \in G$. □

Remark 9.3.1 When operators are given in abstract form, we use the notation $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$ meaning that operator A is well defined from $L_p(\mathbb{R})$ to $L_p(\mathbb{R})$ without discussing sufficient conditions for it to be satisfied. For instance, for the following integral operator

$$(Ax)(t) = \int_{\mathbb{R}} k(t, s)x(s)ds$$

there are sufficient conditions on kernels $k(\cdot, \cdot)$ such that operator A is well defined from $L_p(\mathbb{R})$ to $L_p(\mathbb{R})$ and bounded [9, 18]. For instance, [18, Theorem 6.18] states the following: if $1 < p < \infty$ and $k : \mathbb{R} \times [\alpha, \beta] \to \mathbb{R}$ is a measurable function, $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, and there is a constant $\lambda > 0$ such that

$$\text{ess sup}_{s\in[\alpha,\beta]} \int\limits_{\mathbb{R}} |k(t,s)|dt \le \lambda, \quad \text{ess sup}_{t\in\mathbb{R}} \int\limits_{\alpha}^{\beta} |k(t,s)|ds \le \lambda,$$

then A is well defined from $L_p(\mathbb{R})$ to $L_p(\mathbb{R})$, $1 \le p \le \infty$ and bounded.

9.3.1.1 Representations When *A* is Integral Operator and *B* is Multiplication Operator

Proposition 9.3.3 *Let* $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$*,* $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$*,* $1 \le p \le \infty$*, be defined as follows, for almost all* $t \in \mathbb{R}$*,*

$$(Ax)(t) = \int\limits_{\alpha}^{\beta} k(t,s)x(s)ds, \quad (Bx)(t) = b(t)x(t), \quad \alpha,\beta \in \mathbb{R},\ \alpha < \beta,$$

where $k : \mathbb{R} \times [\alpha,\beta] \to \mathbb{R}$ *is a measurable function, and* $b : \mathbb{R} \to \mathbb{R}$ *is a measurable function. Consider a polynomial defined by* $F(z) = \delta_0 + \delta_1 z + \cdots + \delta_n z^n$*, where* $\delta_0, \delta_1, \ldots, \delta_n$ *are real numbers. We set*

$$k_0(t,s) = k(t,s), \qquad k_m(t,s) = \int\limits_{\alpha}^{\beta} k(t,\tau)k_{m-1}(\tau,s)d\tau, \quad m \in\{1,\ldots,n\}$$

$$F_n(k(t,s)) = \sum_{j=1}^{n} \delta_j k_{j-1}(t,s), \quad n \in \{1,2,3,\ldots\}. \tag{9.2}$$

Then $AB = BF(A)$ *if and only if*

$$\forall\, x \in L_p(\mathbb{R}) : \quad b(t)\delta_0 x(t) + b(t)\int\limits_{\alpha}^{\beta} F_n(k(t,s))x(s)ds = \int\limits_{\alpha}^{\beta} k(t,s)b(s)x(s)ds. \tag{9.3}$$

If $\delta_0 = 0$*, that is,* $F(z) = \delta_1 z + \cdots + \delta_n z^n$*, then the condition* (9.3) *reduces to the following: for almost every* (t,s) *in* $\mathbb{R} \times [\alpha,\beta]$*,*

$$b(t)F_n(k(t,s)) = k(t,s)b(s). \tag{9.4}$$

Proof By applying Fubini Theorem from [1] and iterative kernels from [25], We have

$$(A^2x)(t) = \int_\alpha^\beta k(t,s)(Ax)(s)ds = \int_\alpha^\beta k(t,s)\left(\int_\alpha^\beta k(s,\tau)x(\tau)d\tau\right)ds$$
$$= \int_\alpha^\beta \left(\int_\alpha^\beta k(t,s)k(s,\tau)ds\right)x(\tau)d\tau = \int_\alpha^\beta k_1(t,\tau)x(\tau)d\tau,$$

where $k_1(t,s) = \int_\alpha^\beta k(t,\tau)k(\tau,s)d\tau$. In the same way,

$$(A^3x)(t) = \int_\alpha^\beta k(t,s)(A^2x)(s)ds = \int_\alpha^\beta k(t,s)\left(\int_\alpha^\beta k_1(s,\tau)x(\tau)d\tau\right)ds$$
$$= \int_\alpha^\beta k_2(t,s)x(s)ds,$$

where $k_2(t,s) = \int_\alpha^\beta k(t,\tau)k_1(\tau,s)d\tau$. For every $n \geq 1$,

$$(A^nx)(t) = \int_\alpha^\beta k_{n-1}(t,s)x(s)ds,$$

where $k_m(t,s) = \int_\alpha^\beta k(t,\tau)k_{m-1}(\tau,s)d\tau, \quad m = 1,\ldots,n, \quad k_0(t,s) = k(t,s).$ Thus,

$$(F(A)x)(t) = \delta_0x(t) + \sum_{j=1}^n \delta_j(A^jx)(t) = \delta_0x(t) + \sum_{j=1}^n \delta_j \int_\alpha^\beta k_{j-1}(t,s)x(s)ds$$
$$= \delta_0x(t) + \int_\alpha^\beta F_n(k(t,s))x(s)ds,$$

where $F_n(k(t,s)) = \sum_{j=1}^n \delta_j k_{j-1}(t,s)$, for $n = 1,2,3,\ldots$. So, we can compute $BF(A)x$ and $(AB)x$ as follows:

$$(BF(A)x)(t) = b(t)(F(A)x)(t) = b(t)\delta_0 x(t) + b(t) \int_\alpha^\beta F_n(k(t,s))x(s)ds,$$

$$(ABx)(t) = A(Bx)(t) = \int_\alpha^\beta k(t,s)b(s)x(s)ds.$$

It follows that $ABx = BF(A)x$ if and only if condition (9.3) holds.

If $\delta_0 = 0$ then condition (9.3) reduces to the following:

$$\forall\, x \in L_p(\mathbb{R}) : \quad \int_\alpha^\beta b(t)F_n(k(t,s))x(s)ds = \int_\alpha^\beta k(t,s)b(s)x(s)ds.$$

Let $f(t,s) = b(t)F_n(k(t,s)) - k(t,s)b(s)$. By applying Lemma 9.3.1 we have for almost every $t \in \mathbb{R}$ that $f(t,\cdot) = 0$ almost everywhere. Since the set $N = \{(t,s) \in \mathbb{R} \times [\alpha,\beta] : \ f(t,s) \neq 0\} \subset \mathbb{R}^2$ is measurable and almost all sections $N_t = \{s \in [\alpha,\beta] : \ (t,s) \in N\}$ of the plane has Lebesgue measure zero, by the reciprocal Fubini Theorem [35], the set N has Lebesgue measure zero on the plane $\mathbb{R}^2$. □

Corollary 9.3.4 *For $M_1, M_2 \in \mathbb{R}$, $M_1 < M_2$ and $1 \le p \le \infty$, let $A : L_p([M_1, M_2]) \to L_p([M_1, M_2])$ and $B : L_p([M_1, M_2]) \to L_p([M_1, M_2])$ be nonzero operators defined, for almost all t, by*

$$(Ax)(t) = \int_\alpha^\beta k(t,s)x(s)ds, \quad (Bx)(t) = b(t)x(t), \quad \alpha, \beta \in \mathbb{R}, \ \alpha < \beta,$$

where $[M_1, M_2] \supseteq [\alpha, \beta]$, and $k(\cdot,\cdot) : [M_1, M_2] \times [\alpha,\beta] \to \mathbb{R}$, $b : [M_1, M_2] \to \mathbb{R}$ are given by

$$k(t,s) = a_0 + a_1 t + c_1 s, \qquad b(t) = \sum_{j=0}^{n} b_j t^j,$$

where n is non-negative integer, $a_0,\ a_1,\ c_1,\ b_j$ are real numbers for $j = 0, \ldots, n$. Consider a polynomial defined by $F(z) = \delta_0 + \delta_1 z + \delta_2 z^2$, where $\delta_0,\ \delta_1,\ \delta_2 \in \mathbb{R}$.

Then, $AB = BF(A)$ if and only if

$$\forall\, x \in L_p([M_1, M_2]) : \quad b(t)\delta_0 x(t) + b(t) \int_\alpha^\beta F_n(k(t,s))x(s)ds = \int_\alpha^\beta k(t,s)b(s)x(s)ds,$$

where $F_n(k(t,s))$ is given by (9.2).

If $\delta_0 = 0$, that is, $F(z) = \delta_1 z + \delta_2 z^2$ then the last condition reduces to the condition that for almost every (t, s) in $[M_1, M_2] \times [\alpha, \beta]$

$$b(t)F_2(k(t, s)) = k(t, s)b(s). \tag{9.5}$$

Condition (9.5) is equivalent to that $b(\cdot) \equiv b_0 \neq 0$ is a nonzero constant ($b_j = 0$, $j = 1, \ldots, n$) and one of the following cases holds:

(i) *if $\delta_2 = 0$, $\delta_1 = 1$, then $a_0, a_1, c_1 \in \mathbb{R}$ can be arbitrary;*
(ii) *if $\delta_2 \neq 0$, $\delta_1 = 1$, $a_1 \neq 0$, $c_1 = 0$, then*

$$a_0 = -\frac{\beta + \alpha}{2} a_1;$$

(iii) *if $\delta_2 \neq 0$, $\delta_1 = 1$, $a_1 = 0$, $c_1 \neq 0$, then*

$$a_0 = -\frac{\beta + \alpha}{2} c_1;$$

(iv) *if $\delta_2 \neq 0$, $\delta_1 \neq 1$, $a_1 \neq 0$, $c_1 = 0$, then*

$$a_0 = \frac{2 - 2\delta_1 - \delta_2(\beta^2 - \alpha^2)a_1}{2\delta_2(\beta - \alpha)};$$

(v) *if $\delta_2 \neq 0$, $\delta_1 \neq 1$, $c_1 \neq 0$, $a_1 = 0$, then*

$$a_0 = \frac{2 - 2\delta_1 - \delta_2(\beta^2 - \alpha^2)c_1}{2\delta_2(\beta - \alpha)};$$

(vi) *if $\delta_2 \neq 0$, $\delta_1 \neq 1$, $a_1 = 0$ and $c_1 = 0$, then*

$$a_0 = \frac{1 - \delta_1}{\delta_2(\beta - \alpha)}.$$

Proof Operator A is defined on $L_p[M_1, M_2]$, $1 \leq p \leq \infty$. Therefore, by applying [19, Theorem 3.4.10], we conclude that A is well defined. Moreover, kernel $k(\cdot, \cdot)$ is continuous on a closed and bounded set $[-M, M] \times [\alpha, \beta]$ and $b(\cdot)$ is continuous in $[M_1, M_2]$, so these functions are measurable. By applying Proposition 9.3.3 we just need to check when the condition (9.4) is satisfied for $k(\cdot, \cdot)$ and $b(\cdot)$. We compute

$$k_1(t, s) = \int_\alpha^\beta k(t, \tau)k(\tau, s)d\tau = \int_\alpha^\beta (a_0 + a_1 t + c_1\tau)(a_0 + a_1\tau + c_1 s)d\tau$$
$$= \int_\alpha^\beta [(a_0^2 + a_0 a_1 t + a_0 c_1 s + a_1 c_1 t s)$$

$$
\begin{aligned}
&+ (a_0a_1 + a_0c_1 + a_1^2 t + c_1^2 s)\tau + a_1c_1\tau^2]d\tau \\
=&(\beta-\alpha)(a_0^2 + a_0a_1t + a_0c_1s + a_1c_1ts) \\
&+ \frac{\beta^2-\alpha^2}{2}\cdot(a_0a_1 + a_0c_1 + a_1^2t + c_1^2s) \\
&+ \frac{\beta^3-\alpha^3}{3}a_1c_1 = \nu_0 + \nu_1t + \nu_2s + \nu_3ts, \qquad (9.6)
\end{aligned}
$$

where

$$
\begin{array}{ll}
\nu_0 = a_0^2(\beta-\alpha) + \frac{\beta^2-\alpha^2}{2}a_0(a_1+c_1) + a_1c_1\frac{\beta^3-\alpha^3}{3}, & \nu_2 = a_0c_1(\beta-\alpha) + c_1^2\frac{\beta^2-\alpha^2}{2}, \\
\nu_1 = a_1^2\frac{\beta^2-\alpha^2}{2} + a_1a_0(\beta-\alpha), & \nu_3 = a_1c_1(\beta-\alpha).
\end{array}
$$

Then, we have

$$
\begin{aligned}
&b(t)F_2(k(t,s)) = b(t)[\delta_1k(t,s) + \delta_2k_1(t,s)] = (a_0\delta_1 + \delta_2\nu_0)\sum_{j=0}^{n} b_jt^j \\
&+ (a_1\delta_1 + \delta_2\nu_1)\sum_{j=0}^{n} b_jt^{j+1} + (c_1\delta_1 + \delta_2\nu_2)\sum_{j=0}^{n} b_jt^js + \nu_3\delta_2\sum_{j=0}^{n} b_jt^{j+1}s \\
&= (\delta_1a_0 + \delta_2\nu_0)b_0 + (c_1\delta_1 + \nu_2\delta_2)b_0s + \sum_{j=1}^{n}[(\delta_1a_0 + \delta_2\nu_0)b_j + (\delta_1a_1 + \delta_2\nu_1)b_{j-1}]t^j \\
&+ \sum_{j=1}^{n}[(c_1\delta_1 + \nu_2\delta_2)b_j + \nu_3\delta_2b_{j-1}]t^js + (\delta_1a_1 + \delta_2\nu_1)b_nt^{n+1} + \nu_3\delta_2b_nt^{n+1}s \\
&k(t,s)b(s) = a_0\sum_{j=0}^{n} b_js^j + a_1\sum_{j=0}^{n} b_js^jt + c_1\sum_{j=0}^{n} b_js^{j+1} = a_0b_0 + a_1b_0t \\
&+ \sum_{j=1}^{n}(a_0b_j + c_1b_{j-1})s^j + \sum_{j=1}^{n} a_1b_js^jt + c_1b_ns^{n+1}.
\end{aligned}
$$

Thus we have $k(t,s)b(s) = b(t)F_2(k(t,s))$ for all $(t,s) \in [M_1, M_2] \times [\alpha, \beta]$ if and only if

$$
\begin{aligned}
a_0b_0 &= (a_0\delta_1 + \delta_2\nu_0)b_0 \\
a_1b_0 &= (a_0\delta_1 + \delta_2\nu_0)b_1 + (a_1\delta_1 + \delta_2\nu_1)b_0 \\
a_0b_1 + c_1b_0 &= (c_1\delta_1 + \delta_2\nu_2)b_0 \qquad (9.7) \\
a_1b_1 &= (c_1\delta_1 + \delta_2\nu_2)b_1 + \delta_2\nu_3b_0 \qquad (9.8) \\
0 &= a_0b_j + c_1b_{j-1}, \quad 2 \le j \le n \qquad (9.9) \\
0 &= (a_0\delta_1 + \delta_2\nu_0)b_j + (a_1\delta_1 + \delta_2\nu_1)b_{j-1}, \quad 2 \le j \le n \\
a_1b_j &= 0, \quad 2 \le j \le n \qquad (9.10)
\end{aligned}
$$

$$\begin{aligned} 0 &= c_1\delta_1 b_j + \delta_2\nu_3 b_{j-1} + \delta_2\nu_2 b_j \quad 2 \le j \le n \\ 0 &= a_1\delta_1 b_n + \delta_2\nu_1 b_n, \quad \text{if } n \ge 1 \\ c_1 b_n &= 0, \quad \text{if } n \ge 1 \\ 0 &= \delta_2\nu_3 b_n, \quad \text{if } n \ge 1. \end{aligned} \tag{9.11}$$

Suppose that $n \ge 1$. We proceed by induction to prove that $b_j = 0$, for all $j = 1, 2, \ldots, n$. For $i = 0$, we suppose that $b_n = b_{n-i} \ne 0$. Then from (9.10) we have $a_1 b_n = 0$ and thus $a_1 = 0$. From Eq. (9.11) we have $c_1 b_n = 0$ and thus $c_1 = 0$. From (9.9) we have $0 = a_0 b_n + c_1 b_{n-1} = a_0 b_n$ and thus $a_0 = 0$. This implies that $k(t, s) \equiv 0$, that is, $A = 0$. So for $i = 0$, $b_n = b_{n-i} \ne 0$ implies $A = 0$. Hence, $b_n = 0$. Let $1 < m \le n - 2$ and suppose that $b_{n-i} = 0$ for all $i = 1, 2, \ldots, m - 1$. Let us show that then $b_{n-m} = 0$. If $b_{n-m} \ne 0$, then from (9.10) we have $a_1 b_{n-m} = 0$ which implies $a_1 = 0$. From (9.9) and for $j = n - m + 1$ by induction assumption $a_0 b_{n-m+1} + c_1 b_{n-m} = c_1 b_{n-m} = 0$ which implies $c_1 = 0$. Therefore from (9.9) and for $j = n - m$ we have $a_0 b_{n-m} = 0$ which implies $a_0 = 0$. Then $k(t, s) \equiv 0$, that is $A = 0$. So we must have $b_{n-m} = 0$. If $m = n - 1$, then let us show that $b_{n-m} = b_1 = 0$. If $b_{n-m} \ne 0$ then (9.9) gives $c_1 b_{n-m} = c_1 b_1 = 0$ when $j = n - m + 1 = 2$. Then $c_1 = 0$ and by (9.8), since $\nu_2 = \nu_3 = 0$ we get $a_1 b_1 = 0$ which yields $a_1 = 0$. Therefore, (9.7) gives $a_0 b_1 = 0$ which yields $a_0 = 0$. Thus $A = 0$. Since $A \ne 0$, $b_1 = 0$ is proved. Thus $b(\cdot) = b_0$ is proved.

Since $B \ne 0$ and $B = b_0 I$ (multiple of identity operator), $b_0 \ne 0$ and the commutation relation is equivalent to $F(A) = A$. By (9.4) we have $F_2(k(t, s)) = k(t, s)$ which can be written as follows

$$\delta_1 k(t, s) + \delta_2 k_1(t, s) = k(t, s), \tag{9.12}$$

where $k(t, s) = a_0 + a_1 t + c_1 s$ and $k_1(t, s) = \nu_0 + \nu_1 t + \nu_2 s + \nu_3 ts$,

$$\begin{aligned} &\nu_0 = a_0^2(\beta - \alpha) + \tfrac{\beta^2-\alpha^2}{2} a_0(a_1 + c_1) + a_1 c_1 \tfrac{\beta^3-\alpha^3}{3}, && \nu_2 = a_0 c_1(\beta - \alpha) + c_1^2 \tfrac{\beta^2-\alpha^2}{2}, \\ &\nu_1 = a_1^2 \tfrac{\beta^2-\alpha^2}{2} + a_1 a_0(\beta - \alpha), && \nu_3 = a_1 c_1(\beta - \alpha). \end{aligned}$$

If $\delta_2 = 0$, then (9.12) becomes $(\delta_1 - 1)k(\cdot, \cdot) = 0$ and $A \ne 0$ yields $\delta_1 = 1$. Thus, if $\delta_2 = 0$ and $\delta_1 = 1$, then (9.12) is satisfied for any $a_0, a_1, c_1 \in \mathbb{R}$.

If $\delta_2 \ne 0$ and $\delta_1 = 1$ then (9.12) becomes $k_1(\cdot, \cdot) = 0$, that is, $\nu_0 = \nu_1 = \nu_2 = \nu_3 = 0$, where

$$\begin{aligned} &\nu_0 = a_0^2(\beta - \alpha) + \tfrac{\beta^2-\alpha^2}{2} a_0(a_1 + c_1) + a_1 c_1 \tfrac{\beta^3-\alpha^3}{3}, && \nu_2 = a_0 c_1(\beta - \alpha) + c_1^2 \tfrac{\beta^2-\alpha^2}{2}, \\ &\nu_1 = a_1^2 \tfrac{\beta^2-\alpha^2}{2} + a_1 a_0(\beta - \alpha), && \nu_3 = a_1 c_1(\beta - \alpha). \end{aligned}$$

Since $\alpha < \beta$, $a_1 c_1(\beta - \alpha) = 0$ is equivalent to either $a_1 = 0$ or $c_1 = 0$. If $a_1 \ne 0$, $c_1 = 0$, then

$$\begin{cases} \nu_0 = 0 \\ \nu_1 = 0 \\ \nu_2 = 0 \\ \nu_3 = 0 \end{cases} \Leftrightarrow \begin{cases} (\beta-\alpha)a_0^2 + \frac{\beta^2-\alpha^2}{2}a_0a_1 = 0 \\ (\beta-\alpha)a_1a_0 + \frac{\beta^2-\alpha^2}{2}a_1^2 = 0 \end{cases} \Leftrightarrow a_0 + \frac{\beta+\alpha}{2}a_1 = 0,$$

which is equivalent to $a_0 = -\frac{\beta+\alpha}{2}a_1$. If $a_1 = 0$, $c_1 \neq 0$, then

$$\begin{cases} \nu_0 = 0 \\ \nu_1 = 0 \\ \nu_2 = 0 \\ \nu_3 = 0 \end{cases} \Leftrightarrow \begin{cases} (\beta-\alpha)a_0^2 + \frac{\beta^2-\alpha^2}{2}a_0c_1 = 0 \\ (\beta-\alpha)c_1a_0 + \frac{\beta^2-\alpha^2}{2}c_1^2 = 0 \end{cases} \Leftrightarrow a_0 + \frac{\beta+\alpha}{2}c_1 = 0,$$

which is equivalent to $a_0 = -\frac{\beta+\alpha}{2}c_1$. If $a_1 = 0$, $c_1 = 0$, then $\nu_0 = \nu_1 = \nu_2 = \nu_3 = 0$ is equivalent to $a_0^2(\beta-\alpha) = 0$, that is, $a_0 = 0$. This implies $A = 0$. Therefore, $\delta_2 \neq 0$, $\delta_1 = 1$, $a_1 = c_1 = 0$ yields $A = 0$.

Consider $\delta_2 \neq 0$ and $\delta_1 \neq 1$, and note that (9.12) is equivalent to:

$$\begin{cases} a_0 = \delta_1 a_0 + \delta_2 a_0^2(\beta-\alpha) + \delta_2\frac{\beta^2-\alpha^2}{2}a_0(a_1+c_1) + \delta_2 a_1 c_1 \frac{\beta^3-\alpha^3}{3} \\ a_1 = \delta_1 a_1 + \delta_2 a_1^2 \frac{\beta^2-\alpha^2}{2} + \delta_2 a_1 a_0(\beta-\alpha) \\ c_1 = \delta_1 c_1 + \delta_2 a_0 c_1(\beta-\alpha) + \delta_2 c_1^2\frac{\beta^2-\alpha^2}{2} \\ 0 = \delta_2 a_1 c_1(\beta-\alpha). \end{cases} \tag{9.13}$$

Since $\alpha < \beta$ and $\delta_2 \neq 0$, equation $\delta_2 a_1 c_1(\beta-\alpha) = 0$ implies that either $a_1 = 0$ or $c_1 = 0$. If $\delta_2 \neq 0$, $\delta_1 \neq 1$, $a_1 \neq 0$ and $c_1 = 0$, then (9.13) becomes

$$\begin{aligned} a_0 &= \delta_1 a_0 + \delta_2 a_0^2(\beta-\alpha) + \delta_2\frac{\beta^2-\alpha^2}{2}a_0a_1 \\ a_1 &= \delta_1 a_1 + \delta_2 a_1^2\frac{\beta^2-\alpha^2}{2} + \delta_2 a_1 a_0(\beta-\alpha) \end{aligned}$$

which is equivalent to $1 = \delta_1 + \delta_2(\beta-\alpha)a_0 + \delta_2\frac{\beta^2-\alpha^2}{2}a_1$. Then,

$$a_0 = \frac{2 - 2\delta_1 - \delta_2(\beta^2-\alpha^2)a_1}{2\delta_2(\beta-\alpha)}.$$

If $\delta_2 \neq 0$, $\delta_1 \neq 1$, $a_1 = 0$ and $c_1 \neq 0$, then (9.13) becomes

$$\begin{aligned} a_0 &= \delta_1 a_0 + \delta_2 a_0^2(\beta-\alpha) + \delta_2\frac{\beta^2-\alpha^2}{2}a_0c_1 \\ c_1 &= \delta_1 c_1 + \delta_2 c_1^2\frac{\beta^2-\alpha^2}{2} + \delta_2 c_1 a_0(\beta-\alpha) \end{aligned}$$

which is equivalent to $1 = \delta_1 + \delta_2(\beta-\alpha)a_0 + \delta_2\frac{\beta^2-\alpha^2}{2}c_1$. Then,

$$a_0 = \frac{2 - 2\delta_1 - \delta_2(\beta^2 - \alpha^2)c_1}{2\delta_2(\beta - \alpha)}.$$

If $\delta_2 \neq 0$, $\delta_1 \neq 1$, $a_1 = 0$ and $c_1 = 0$, then $A \neq 0$ yields $a_0 \neq 0$ and (9.13) becomes

$$a_0 = \delta_1 a_0 + \delta_2 a_0^2(\beta - \alpha)$$

which is equivalent to $a_0 = \frac{1-\delta_1}{\delta_2(\beta-\alpha)}$. □

Remark 9.3.2 The integral operator given by $(Ax)(t) = \int\limits_{\alpha_1}^{\beta_1} k(t,s)x(s)ds$ for almost all t, where $k : [\alpha_1, \beta_1] \times [\alpha_1, \beta_1] \to \mathbb{R}$ is a measurable function that satisfies

$$\int\limits_{\alpha_1}^{\beta_1}\left(\int\limits_{\alpha_1}^{\beta_1} |k(t,s)|^q ds\right)^{\frac{p}{q}} dt < \infty,$$

by [19, Theorem 3.4.10] is well defined from $L_p[\alpha_1, \beta_1]$ to $L_p[\alpha_1, \beta_1]$, $1 < p < \infty$ and bounded.

Remark 9.3.3 If in the Corollary 9.3.4 when $0 \notin [M_1, M_2]$, one takes $b(t)$ to be a Laurent polynomial with only negative powers of t then there is no non-zero kernel $k(t,s) = a_0 + a_1 t + c_1 s$ (there is no $A \neq 0$ with such kernels) such that $AB = BF(A)$. In fact, let n be a positive integer and consider $b(t) = \sum\limits_{j=1}^{n} b_j t^{-j}$, where $t \in [M_1, M_2]$, $b_j \in \mathbb{R}$ for $j = 1, \ldots, n$ and $b_n \neq 0$. We set $k_1(t,s)$ as defined by (9.6). Then we have

$$b(t)F_2(k(t,s)) = b(t)[\delta_1 k(t,s) + \delta_2 k_1(t,s)] = (a_0\delta_1 + \delta_2\nu_0)\sum_{j=1}^{n} b_j t^{-j}$$

$$+ (a_1\delta_1 + \delta_2\nu_1)\sum_{j=1}^{n} b_j t^{-j+1} + (c_1\delta_1 + \delta_2\nu_2)\sum_{j=1}^{n} b_j t^{-j} s + \nu_3\delta_2 \sum_{j=1}^{n} b_j t^{-j+1} s$$

$$= (a_1\delta_1 + \delta_2\nu_1)b_1 + \nu_3\delta_2 b_1 s + \sum_{j=1}^{n-1}[(a_0\delta_1 + \delta_2\nu_0)b_j + (a_1\delta_1 + \delta_2\nu_1)b_{j+1}]t^{-j}$$

$$+(a_0\delta_1 + \delta_2\nu_0)b_n t^{-n} + \sum_{j=1}^{n-1}[(c_1\delta_1 + \delta_2\nu_2)b_j + \nu_3\delta_2 b_{j+1}]t^{-j}s + (c_1\delta_1 + \delta_2\nu_2)b_n t^{-n}s$$

$$k(t,s)b(s) = a_0\sum_{j=1}^{n} b_j s^{-j} + a_1\sum_{j=1}^{n} b_j s^{-j} t + c_1\sum_{j=1}^{n} b_j s^{-j+1}$$

$$= c_1 b_1 + \sum_{j=1}^{n-1} (a_0 b_j + c_1 b_{j+1}) s^{-j} + \sum_{j=1}^{n} a_1 b_j s^{-j} t + a_0 b_n s^{-n}.$$

Thus we have $k(t,s)b(s) = b(t)F_2(k(t,s))$ for almost every $(t,s) \in [M_1, M_2] \times [\alpha, \beta]$ if and only if

$$\begin{aligned}
c_1 b_1 &= a_1 \delta_1 b_1 + \delta_2 \nu_1 b_1, \\
0 &= \delta_2 \nu_3 b_1, \\
0 &= (a_0 \delta_1 + \delta_2 \nu_0) b_j + (\delta_1 a_1 + \delta_2 \nu_1) b_{j+1}, \quad 1 \le j \le n-1, \\
a_0 b_j + c_1 b_{j+1} &= 0, \quad 1 \le j \le n-1, \qquad (9.14) \\
0 &= c_1 \delta_1 b_j + \delta_2 \nu_2 b_j + \delta_2 \nu_3 b_{j+1}, \quad 1 \le j \le n-1, \\
a_1 b_j &= 0, \quad 1 \le j \le n, \qquad (9.15) \\
0 &= a_0 \delta_1 b_n + \delta_2 \nu_0 b_n, \\
0 &= a_0 b_n, \qquad (9.16) \\
0 &= c_1 \delta_1 b_n + \delta_2 \nu_3 b_n.
\end{aligned}$$

Since $b_n \neq 0$ then from (9.16) we have $a_0 b_n = 0$ and thus $a_0 = 0$. From (9.14) for $j = n-1$ we get $c_1 b_n = 0$ and thus $c_1 = 0$. Finally from (9.15) we have $0 = a_1 b_j$ for $j = n$ and thus $a_1 = 0$. This implies that $k(t,s) \equiv 0$, that is, $A = 0$. So $b_n \neq 0$ implies $A = 0$.

Corollary 9.3.5 *Let $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 < p < \infty$, be defined as follows, for almost all t,*

$$(Ax)(t) = \int_{\alpha}^{\beta} k(t,s)x(s)ds, \quad (Bx)(t) = b(t)x(t), \quad \alpha, \beta \in \mathbb{R}, \alpha < \beta,$$

where $k(t,s) : \mathbb{R} \times [\alpha, \beta] \to \mathbb{R}$ is a measurable function, and $b \in L_\infty(\mathbb{R})$ is a nonzero function such that the set $\operatorname{supp} b(t) \cap [\alpha, \beta]$ *has measure zero.*

Consider a polynomial defined by $F(z) = \delta_0 + \delta_1 z + \cdots + \delta_n z^n$, where $\delta_0, \ldots, \delta_n$ are real numbers. We set

$$k_0(t,s) = k(t,s), \qquad k_m(t,s) = \int_{\alpha}^{\beta} k(t,\tau) k_{m-1}(\tau, s) d\tau, \quad m = 1, \ldots, n,$$

$$F_n(k(t,s)) = \sum_{j=1}^{n} \delta_j k_{j-1}(t,s), \quad n = 1, 2, 3, \ldots$$

Then $AB = BF(A)$ *if and only if* $\delta_0 = 0$ *and the set*

$$\text{supp } b(t) \cap \text{supp } F_n(k(t,s))$$

has measure zero in $\mathbb{R} \times [\alpha, \beta]$.

Proof Suppose that the set supp $b \cap [\alpha, \beta]$ has measure zero. By Proposition 9.3.3 we have $AB = BF(A)$ if and only if condition (9.3) holds, that is,

$$\forall\, x \in L_p(\mathbb{R}): \quad b(t)\delta_0 x(t) + b(t)\int_\alpha^\beta F_n(k(t,s))x(s)ds = \int_\alpha^\beta k(t,s)b(s)x(s)ds,$$

almost everywhere. By taking $x(\cdot) = I_{[M_1,M_2]}(\cdot)b(\cdot)$, where $M_1, M_2 \in \mathbb{R}$, $M_1 < M_2$, $[M_1, M_2] \supset [\alpha, \beta]$, $\mu([M_1, M_2] \setminus [\alpha, \beta]) > 0$, $I_E(\cdot)$ is the indicator function of the set E, the condition (9.3) reduces to

$$I_{[M_1,M_2]}(\cdot)b^2(\cdot)\delta_0 = 0.$$

Since b has support with positive measure (otherwise $B \equiv 0$), then $\delta_0 = 0$. By using this, condition (9.3) reduces to the following

$$\forall\, x \in L_p(\mathbb{R}): \quad b(t)\int_\alpha^\beta F_n(k(t,s))x(s)ds = \int_\alpha^\beta k(t,s)b(s)x(s)ds.$$

By hypothesis the right hand side is equal zero. Then condition (9.3) reduces to

$$\forall\, x \in L_p(\mathbb{R}): \quad b(t)\int_\alpha^\beta F_n(k(t,s))x(s)ds = 0.$$

This is equivalent to

$$b(t)F_n(k(t,s)) = 0 \quad \text{for almost every } \ s \in [\alpha, \beta]. \tag{9.17}$$

By applying a similar argument as in the proof of Proposition 9.3.3 we conclude that condition (9.17) is equivalent to that the set

$$\text{supp } b(t) \cap \text{supp } F_n(k(t,s))$$

has measure zero in $\mathbb{R} \times [\alpha, \beta]$. □

Corollary 9.3.6 *Let* $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 \le p \le \infty$, *be defined as follows, for almost all* t,

$$(Ax)(t) = \int_{\alpha}^{\beta} a(t)c(s)x(s)ds, \quad (Bx)(t) = b(t)x(t), \quad \alpha, \beta \in \mathbb{R}, \alpha < \beta,$$

where $a : \mathbb{R} \to \mathbb{R}$, $c : [\alpha, \beta] \to \mathbb{R}$, $b : \mathbb{R} \to \mathbb{R}$ are measurable functions. Consider a polynomial defined by $F(z) = \delta_1 z + \delta_2 z^2 + \cdots + \delta_n z^n$, where $\delta_1, \ldots, \delta_n$ are real constants. We set $\mu = \int_{\alpha}^{\beta} a(s)c(s)ds$. Then, we have $AB = BF(A)$ if and only if the set

$$\operatorname{supp}[a(t)c(s)] \cap \operatorname{supp}\left[b(t)\sum_{j=1}^{n} \delta_j \mu^{j-1} - b(s)\right],$$

has measure zero in $\mathbb{R} \times [\alpha, \beta]$.

Proof We set $k(t, s) = a(t)c(s)$, so we have

$$k_0(t, s) = k(t, s) = a(t)c(s),$$

$$k_m(t, s) = \int_{\alpha}^{\beta} k(t, \tau)k_{m-1}(\tau, s)d\tau = a(t)c(s)\left(\int_{\alpha}^{\beta} a(s)c(s)ds\right)^{m}, \ m = 1, \ldots, n$$

$$F_n(k(t, s)) = \sum_{j=1}^{n} \delta_j k_{j-1}(t, s) = \sum_{j=1}^{n} \delta_j a(t)c(s)\left(\int_{\alpha}^{\beta} a(s)c(s)ds\right)^{j-1} \quad n = 1, 2, 3, \ldots$$

By applying Proposition 9.3.3 we have $AB = BF(A)$ if and only if

$$b(t)\sum_{j=1}^{n} \delta_j a(t)c(s)\left(\int_{\alpha}^{\beta} a(s)c(s)ds\right)^{j-1} = a(t)c(s)b(s) \iff$$

$$a(t)c(s)\left[b(t)\sum_{j=1}^{n} \delta_j \left(\int_{\alpha}^{\beta} a(s)c(s)ds\right)^{j-1} - b(s)\right] = 0$$

for almost every (t, s) in $\mathbb{R} \times [\alpha, \beta]$. The last condition is equivalent to the set

$$\operatorname{supp}[a(t)c(s)] \cap \operatorname{supp}\left[b(t)\sum_{j=1}^{n} \delta_j \left(\int_{\alpha}^{\beta} a(s)c(s)ds\right)^{j-1} - b(s)\right]$$

has measure zero in $\mathbb{R} \times [\alpha, \beta]$. We complete the proof by noticing that the corresponding set can be written as

$$\operatorname{supp}[a(t)c(s)] \cap \operatorname{supp}\left[b(t)\sum_{j=1}^{n}\delta_j\mu^{j-1} - b(s)\right],$$

where $\mu = \int\limits_{\alpha}^{\beta} a(s)c(s)ds.$ □

Example 9.3.7 Let $A : L_p(\mathbb{R}) \to L_p(\mathbb{R}),\ B : L_p(\mathbb{R}) \to L_p(\mathbb{R}),\ 1 < p < \infty$ be defined as follows, for almost all t,

$$(Ax)(t) = \int\limits_{0}^{2} a(t)c(s)x(s)ds, \quad (Bx)(t) = b(t)x(t),$$

where $a(t) = I_{[0,1]}(t)(1+t^2), c(s) = 1, b(t) = I_{[1,2]}(t)t^2$. Since kernel has compact support, we can apply [19, Theorem 3.4.10] and we conclude that operators A is well defined and bounded. Since function b has 4 as an upper bound then $\|B\|_{L_p} \le 4$. Hence operator B is well defined and bounded. Consider a polynomial defined by $F(z) = \delta_1 z + \cdots + \delta_n z^n$, where $\delta_1, \ldots, \delta_n$ are real constants. Then, the above operators does not satisfy the relation $AB = BF(A)$. In fact for $\lambda \neq 0$, by applying Corollary 9.3.6 and setting $\lambda = \sum\limits_{j=1}^{n} \delta_j(\beta-\alpha)^{j-1}$, we have

$$\operatorname{supp}\{b(t)\lambda - b(s)\} = (\mathbb{R}\times[1,2] \cup [1,2]\times[0,1]) \setminus W,$$

where $W = \{(t,s) \in [1,2]\times[1,2] :\ b(t)\lambda - b(s) = 0\}$ is a set of measure zero in the plane. Moreover, supp $a(t)c(s) = [0,1]\times[0,2]$. The set

$$\operatorname{supp}[a(t)c(s)] \cap \operatorname{supp}[b(t)\lambda - b(s)],$$

has positive measure in $\mathbb{R}\times[0,2]$.

Example 9.3.8 Let $A : L_p(\mathbb{R}) \to L_p(\mathbb{R}),\ B : L_p(\mathbb{R}) \to L_p(\mathbb{R}),\ 1 < p < \infty$ be defined as follows, for almost all t,

$$(Ax)(t) = \int\limits_{0}^{2} a(t)c(s)x(s)ds,\ (Bx)(t) = b(t)x(t),$$

where $a(t) = 2tI_{[0,2]}(t), c(s) = I_{[0,1]}(s), b(t) = I_{[1,2]}(t)t^2$. Since kernel has compact support, we can apply [19, Theorem 3.4.10] and, we conclude that operators A is well defined and bounded. Since function b has 4 as an upper bound then $\|B\|_{L_p} \le 4$. Hence operator B is well defined and bounded. Consider a polynomial defined by $F(z) = \delta_1 z + \cdots + \delta_n z^n$, where $\delta_1, \ldots, \delta_n$ are real constants. Then, the above operators satisfy the relation $AB = BF(A)$ if and only if $\sum\limits_{j=1}^{n} \delta_j = 0$. In fact, by applying Corollary 9.3.6 we have

$$\mu = \int\limits_0^2 a(s)c(s)ds = 1.$$

Hence, $\operatorname{supp}\{b(t) \cdot 0 - b(s)\} = \mathbb{R} \times [1, 2]$. Moreover, $\operatorname{supp} a(t)c(s) = [0, 2] \times [0, 1]$. The set $\operatorname{supp}[a(t)c(s)] \cap \operatorname{supp}[-b(s)]$, has measure zero in $\mathbb{R} \times [0, 2]$.

Example 9.3.9 Let $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 < p < \infty$ be defined as follows, for almost all t,

$$(Ax)(t) = \int\limits_0^2 a(t)c(s)x(s)ds, \quad (Bx)(t) = b(t)x(t),$$

where $a(t) = I_{[0,2]}(t)\sin(\pi t)$, $c(s) = I_{[0,1]}(s)$, $b(t) = I_{[1,2]}(t)t^2$. Since $a \in L_p(\mathbb{R})$ and $c \in L_q[0, 2]$, $1 < q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, by applying Hölder inequality we have that operator A is well defined and bounded. The function $b \in L_\infty$, so B is well defined and bounded because $\|B\|_{L_p} \le \|b\|_{L_\infty}$ we conclude that operator B is well defined and bounded. Consider a polynomial defined by $F(z) = \delta z^d$, where $\delta \neq 0$ is a real constant and d is a positive integer $d \ge 2$. Then, the above operators satisfy the relation $AB = \delta BA^d$. In fact, by applying Corollary 9.3.6 we have $\mu = \int\limits_0^2 a(s)c(s)ds = 0$. Hence, $\operatorname{supp}\{b(t) \cdot 0 - b(s)\} = \mathbb{R} \times [1, 2]$. Moreover, $\operatorname{supp} a(t)c(s) = [0, 2] \times [0, 1]$. The set $\operatorname{supp}[a(t)c(s)] \cap \operatorname{supp}[-b(s)]$, has measure zero in $\mathbb{R} \times [0, 2]$.

Example 9.3.10 Let $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 < p < \infty$, be defined as follows, for almost all t,

$$(Ax)(t) = \int\limits_\alpha^\beta I_{[\alpha,\beta]}(t)x(s)ds, \quad (Bx)(t) = I_{[\alpha,\beta]}(t)x(t), \quad \alpha, \beta \in \mathbb{R}, \alpha < \beta.$$

Since kernel has compact support, we can apply [19, Theorem 3.4.10] and, we conclude that operator A is well defined and bounded. Since $\|B\|_{L_p} \le 1$ then operator B is well defined and bounded. Consider a polynomial defined by $F(z) = \delta_1 z + \cdots + \delta_n z^n$, where $\delta_1, \ldots, \delta_n$ are constants. Then, the above operators satisfy the relation $AB = BF(A)$ if and only if $\sum\limits_{j=1}^{n} \delta_j (\beta - \alpha)^{j-1} = 1$. Indeed, if $a(t) = b(t) = I_{[\alpha,\beta]}(t)$, $c(s) = 1$ and

$$\lambda = \sum_{j=1}^{n} \delta_j \left(\int_{\alpha}^{\beta} a(s)c(s)ds \right)^{j-1} = \sum_{j=1}^{n} \delta_j (\beta - \alpha)^{j-1},$$

then from Corollary 9.3.6 we have the following:

- If $\lambda \neq 0, \lambda \neq 1$,

$$\text{supp}\,[b(t)\lambda - b(s))] = \left\{(t,s) \in \mathbb{R} \times [\alpha, \beta] : \ \lambda I_{[\alpha,\beta]}(t) \neq 1\right\} = \mathbb{R} \times [\alpha, \beta],$$

$$\text{supp}\ a(t)c(s) = \{(t,s) \in \mathbb{R} \times [\alpha, \beta] : \ I_{[\alpha,\beta]}(t) \neq 0\} = [\alpha, \beta] \times [\alpha, \beta].$$

  The set $\text{supp}\,[\lambda b(t) - b(s)] \cap \text{supp}\,[a(t)c(s)] = [\alpha, \beta] \times [\alpha, \beta]$ has positive measure.
- If $\lambda = 1$,

$$\begin{aligned}\text{supp}\,[b(t) - b(s)] &= \left\{(t,s) \in \mathbb{R} \times [\alpha, \beta] : \ I_{[\alpha,\beta]}(t) \neq 1\right\} \\ &= (\mathbb{R} \setminus [\alpha, \beta]) \times [\alpha, \beta].\end{aligned}$$

  The set $\text{supp}\,[b(t) - b(s)] \cap \text{supp}\,[a(t)c(s)]$ has measure zero in $\mathbb{R} \times [\alpha, \beta]$.
- If $\lambda = 0$,

$$\begin{aligned}\text{supp}\,[\lambda b(t) - b(s)] = \text{supp}\ b(s) &= \left\{(t,s) \in \mathbb{R}^2 : \ I_{[\alpha,\beta]}(s) \neq 0\right\} \\ &= \left\{(t,s) \in \mathbb{R}^2 : \ \alpha \leq s \leq \beta\right\}.\end{aligned}$$

  The set $\text{supp}\ b(s) \cap \text{supp}\,[a(t)c(s)] = [\alpha, \beta] \times [\alpha, \beta]$ has measure $(\beta - \alpha)^2$.

The conditions in the Corollary 9.3.6 are fulfilled only in the second case, that is, when $\lambda = 1$.

9.3.1.2 Representations When *A* is Multiplication Operator and *B* is Integral Operator

Proposition 9.3.11 *Let $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 < p < \infty$ be defined as follows, for almost all t,*

$$(Ax)(t) = a(t)x(t), \quad (Bx)(t) = \int_{\alpha}^{\beta} k(t,s)x(s)ds, \quad \alpha, \beta \in \mathbb{R}, \alpha < \beta,$$

where $a : \mathbb{R} \to \mathbb{R}$, $k : \mathbb{R} \times [\alpha, \beta] \to \mathbb{R}$ are measurable functions. Consider a polynomial defined by $F(z) = \delta_0 + \delta_1 z + \cdots + \delta_n z^n$, where $\delta_0, \delta_1, \ldots, \delta_n$ are constants. Then

$$AB = BF(A)$$

if and only if the set

$$\operatorname{supp}[a(t) - F(a(s))] \cap \operatorname{supp} k(t,s)$$

has measure zero in $\mathbb{R} \times [\alpha, \beta]$.

Proof We have for almost every $t \in \mathbb{R}$

$$(ABx)(t) = \int_{\alpha}^{\beta} a(t)k(t,s)x(s)ds$$

$$(A^n x)(t) = [a(t)]^n x(t)$$

$$(F(A)x)(t) = \sum_{i=0}^{n} \delta_i (A^i x)(t) = \left(\sum_{i=0}^{n} \delta_i [a(t)]^i \right) x(t) = F(a(t))x(t)$$

$$(BF(A)x)(t) = \int_{\alpha}^{\beta} k(t,s))F(a(s))x(s)ds.$$

Then we have $ABx = BF(A)x$ if and only if

$$\int_{\alpha}^{\beta} a(t)k(t,s)x(s)ds = \int_{\alpha}^{\beta} k(t,s)F(a(s))x(s)ds. \tag{9.18}$$

almost everywhere. By using Lemma 9.3.1 and by applying the same argument as in the final steps on the proof of Proposition 9.3.3, the condition (9.18) is equivalent to

$$a(t)k(t,s) = k(t,s)F[a(s)] \iff k(t,s)[a(t) - F(a(s))] = 0$$

for almost every (t, s) in $\mathbb{R} \times [\alpha, \beta]$.

Since the variables t and s are independent, this is true if and only if the set

$$\operatorname{supp}[a(t) - F(a(s))] \cap \operatorname{supp} k(t,s)$$

has measure zero in $\mathbb{R} \times [\alpha, \beta]$. □

Example 9.3.12 Let $A : L_p(\mathbb{R}) \to L_p(\mathbb{R}),\ B : L_p(\mathbb{R}) \to L_p(\mathbb{R}),\ 1 < p < \infty$ be defined as follows, for almost all t,

$$(Ax)(t) = I_{[\alpha,\beta]}(t)x(t), \quad (Bx)(t) = \int_{\alpha}^{\beta} I_{[\alpha,\beta]^2}(t,s)x(s)ds, \quad \alpha, \beta \in \mathbb{R}, \alpha < \beta$$

By using properties of norm and [19, Theorem 3.4.10], respectively, for operators A and B, we conclude that operators A and B are well defined and bounded. For a monomial defined by $F(z) = z^n$, $n = 1, 2, \ldots$, the above operators satisfy the relation $AB = BF(A)$. In fact, by setting $a(t) = I_{[\alpha,\beta]}(t)$, $k(t,s) = I_{[\alpha,\beta]^2}(t,s)$ we have

$$\text{supp}\,[a(t) - F(a(s))] = \big\{(t,s) \in \mathbb{R} \times [\alpha,\beta] : \; I_{[\alpha,\beta]}(t) \neq 1\big\} = (\mathbb{R} \setminus [\alpha,\beta]) \times [\alpha,\beta],$$

$$\text{supp}\; k(t,s) = [\alpha,\beta] \times [\alpha,\beta].$$

The set $\text{supp}\,[a(t) - F(a(s))] \cap \text{supp}\,[k(t,s)]$ has measure zero in $\mathbb{R} \times [\alpha,\beta]$. So the result follows from Proposition 9.3.11.

Example 9.3.13 Let $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 < p < \infty$ defined as follows, for almost all t,

$$(Ax)(t) = [\gamma_1 I_{[0,1/2)}(t) - \gamma_2 I_{[1/2,1]}(t)]x(t), \quad (Bx)(t) = \int_0^1 k(t,s)x(s)ds$$

$k : \mathbb{R} \times [0,1] \to \mathbb{R}$ is a Lebesgue measurable function such that B is well defined. The operator A is well defined and bounded. Consider a polynomial defined by $F(z) = \delta_0 + \delta_1 z$, where $\delta_0, \delta_1, \gamma_1, \gamma_2$ are constants such that

$$|\delta_0| + |\delta_1| + |\gamma_1| + |\gamma_2| \neq 0.$$

If $k(\cdot,\cdot)$ is a measurable function such that one of the following is fulfilled:

(i) $\delta_0 = -\delta_1\gamma_1$ and supp $k(t,s) \subseteq (\mathbb{R} \setminus [0,1]) \times [0,1/2]$;
(ii) $\delta_0 = \delta_1\gamma_2$ and supp $k(t,s) \subseteq (\mathbb{R} \setminus [0,1]) \times [1/2,1]$;
(iii) $\delta_0 + \delta_1\gamma_1 - \gamma_1 = 0$ and supp $k(t,s) \subseteq [0,1/2] \times [0,1/2]$;
(iv) $\delta_0 + \delta_1\gamma_1 + \gamma_2 = 0$ and supp $k(t,s) \subseteq [1/2,1] \times [0,1/2]$;
(v) $\delta_0 - \delta_1\gamma_2 - \gamma_1 = 0$ and supp $k(t,s) \subseteq [0,1/2] \times [1/2,1]$;
(vi) $\delta_0 - \delta_1\gamma_2 + \gamma_2 = 0$ and supp $k(t,s) \subseteq [1/2,1] \times [1/2,1]$,

then the above operators satisfy the relation $AB = BF(A)$.

In fact, putting $a(t) = \gamma_1 I_{[0,1/2)}(t) - \gamma_2 I_{[1/2,1]}(t)$ we have

$$[a(t) - F(a(s))] = \begin{cases} 0, & \text{if } \delta_0 = -\delta_1\gamma_1, \quad t \notin [0,1], \; s \in [0,1/2) \\ 0, & \text{if } \delta_0 = \delta_1\gamma_2, \quad t \notin [0,1], \; s \in [1/2,1] \\ 0, & \text{if } \delta_0 + \delta_1\gamma_1 - \gamma_1 = 0, \; t \in [0,1/2), \; s \in [0,1/2) \\ 0, & \text{if } \delta_0 + \delta_1\gamma_1 + \gamma_2 = 0, \; t \in [1/2,1), \; s \in [0,1/2] \\ 0, & \text{if } \delta_0 - \delta_1\gamma_2 - \gamma_1 = 0, \; t \in [0,1/2], \; s \in [1/2,1] \\ 0, & \text{if } \delta_0 - \delta_1\gamma_2 + \gamma_2 = 0, \; t \in [1/2,1], \; s \in [1/2,1] \\ \gamma_3, & \text{otherwise} \end{cases}$$

where γ_3 can be different from zero depending on the constants involved. Thus, in each condition we can choose $k(t,s) = I_S(t,s)$, where $S = \{(t,s) \in \mathbb{R} \times [0,1] : a(t) - F(a(s)) = 0\}$ and with a positive measure. Or for instance we can take:

(i) $k(t,s) = I_{[2,3]\times[0,1/2]}(t,s)$ if $\delta_0 = -\delta_1\gamma_1$;
(ii) $k(t,s) = I_{[2,3]\times[1/2,1]}(t,s)$ if $\delta_0 = \delta_1\gamma_2$;
(iii) $k(t,s) = I_{[0,1/3]\times[1/3,1/2]}(t,s)$ if $\delta_0 + \delta_1\gamma_1 - \gamma_1 = 0$;
(iv) $k(t,s) = I_{[2/3,1/2]\times[0,1/2]}(t,s)$ if $\delta_0 + \delta_1\gamma_1 + \gamma_2 = 0$;
(v) $k(t,s) = I_{[0,1/3]\times[2/3,1]}(t,s)$ if $\delta_0 - \delta_1\gamma_2 - \gamma_1 = 0$;
(vi) $k(t,s) = I_{[2/3,1]\times[2/3,1]}(t,s)$ if $\delta_0 - \delta_1\gamma_2 + \gamma_2$.

According to the definition, in all above cases the set

$$\text{supp}\,[a(t) - F(a(s))] \cap \text{supp}\ [k(t,s)]$$

has measure zero in $\mathbb{R} \times [0,1]$. So the result follows from Proposition 9.3.11.

Corollary 9.3.14 *Let* $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 < p < \infty$ *defined as follows, for almost all* t,

$$(Ax)(t) = a(t)x(t), \quad (Bx)(t) = \int_{\alpha}^{\beta} b(t)c(s)x(s)ds, \quad \alpha, \beta \in \mathbb{R}, \alpha < \beta,$$

where $a : \mathbb{R} \to \mathbb{R}$, $b : \mathbb{R} \to \mathbb{R}$, $c : [\alpha, \beta] \to \mathbb{R}$ *are measurable functions. For a polynomial defined by* $F(z) = \delta_0 + \delta_1 z + \cdots + \delta_n z^n$, *where* $\delta_0, \delta_1, \ldots, \delta_n$ *are real constants, we have*

$$AB = BF(A)$$

if and only if the set

$$\text{supp}\,[a(t) - F(a(s))] \cap \text{supp}\,[b(t)c(s)]$$

has measure zero in $\mathbb{R} \times [\alpha, \beta]$.

Proof This follows by Proposition 9.3.11. □

Example 9.3.15 Let $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 < p < \infty$ be defined as follows, for almost all t,

$$(Ax)(t) = a(t)x(t), \quad (Bx)(t) = \int_{\alpha}^{\beta} b(t)c(s)x(s)ds, \quad \alpha, \beta \in \mathbb{R}, \alpha < \beta,$$

where $a(t) = -1 + I_{[\alpha,\beta]}(t)$, $b(t) = I_{[\alpha-2,\alpha-1]}(t)$, $c(s) = 1$. We have that $a \in L_\infty(\mathbb{R})$ and so $\|A\|_{L_p} \le \|a\|_{L_\infty}$. Therefore, A is well defined and bounded. Since kernel has compact support in $\mathbb{R} \times [\alpha, \beta]$, we can apply [19, Theorem 3.4.10] and we conclude that operators B is well defined and bounded. Consider a polynomial defined by $F(z) = -1 + \delta_1 z$, where δ_1 is a real constant. Then the above operators satisfy the relation $AB = BF(A)$. In fact, for $(t, s) \in \mathbb{R} \times [\alpha, \beta]$ we have

$$F(a(s)) - a(t) = -\delta_1 + \delta_1 I_{[\alpha,\beta]}(s) - I_{[\alpha,\beta]}(t) = -I_{[\alpha,\beta]}(t).$$

Therefore, we have

$$\begin{aligned} \text{supp}\,[a(t) - F(a(s))] &= [\alpha, \beta] \times [\alpha, \beta], \\ \text{supp}\ b(t)c(s) = \text{supp}\ \ I_{[\alpha-2,\alpha-1]}(t) I_{[\alpha,\beta]}(s) &= [\alpha - 2, \alpha - 1] \times [\alpha, \beta]. \end{aligned}$$

The set $\text{supp}\,[a(t) - F(a(s))] \cap \text{supp}\,[I_{[\alpha-2,\alpha-1]}(t) I_{[\alpha,\beta]}(s)]$ has measure zero. So the result follows from Corollary 9.3.14.

Example 9.3.16 Let $A : L_p(\mathbb{R}) \to L_p(\mathbb{R}),\ B : L_p(\mathbb{R}) \to L_p(\mathbb{R}),\ 1 < p < \infty$ be defined as follows, for almost all t,

$$(Ax)(t) = a(t)x(t), \quad (Bx)(t) = \int_{\alpha}^{\beta} b(t)c(s)x(s)ds, \quad \alpha, \beta \in \mathbb{R}, \alpha < \beta,$$

where $a(t) = \gamma_0 + I_{\left[\alpha, \frac{\alpha+\beta}{2}\right]}(t)t^2$, γ_0 is a real number, $b(t) = (1 + t^2) I_{[\beta+1,\beta+2]}(t)$, $c(s) = I_{\left[\frac{\alpha+\beta}{2}, \beta\right]}(s)(1 + s^4)$. Consider a polynomial defined by $F(z) = \delta_0 + \delta_1 z$, where δ_0, δ_1 are real constants and $\delta_1 \neq 0$. If $\delta_0 = \gamma_0 - \delta_1\gamma_0$ then the above operators satisfy the relation

$$AB - \delta_1 BA = \delta_0 B.$$

In fact, A is well defined, bounded since $a \in L_\infty$ and this implies $\|A\|_{L_p} \leq \|a\|_{L_\infty}$. Operator B is well defined, bounded since $k(t, s) = b(t)c(s)$, $(t, s) \in \mathbb{R} \times [\alpha, \beta]$ has compact support and satisfies conditions of [19, Theorem 3.4.10]. If $\delta_0 = \gamma_0 - \delta_1\gamma_0$ then we have

$$\begin{aligned} F(a(s)) - a(t) &= \delta_0 + \gamma_0\delta_1 + \delta_1 I_{\left[\alpha, \frac{\alpha+\beta}{2}\right]}(s)s^2 - \gamma_0 - I_{\left[\alpha, \frac{\alpha+\beta}{2}\right]}(t)t^2 \\ &= \delta_1 I_{\left[\alpha, \frac{\alpha+\beta}{2}\right]}(s)s^2 - I_{\left[\alpha, \frac{\alpha+\beta}{2}\right]}(t)t^2. \end{aligned}$$

Then we have

$$\text{supp}\,[a(t) - F(a(s))] = \left(\mathbb{R} \times \left[\alpha, \frac{\alpha+\beta}{2}\right] \cup \left[\alpha, \frac{\alpha+\beta}{2}\right] \times \left[\frac{\alpha+\beta}{2}, \beta\right]\right) \setminus W,$$

where $W \subseteq \mathbb{R} \times [\alpha, \beta]$ is a set with Lebesgue measure zero, and

$$\text{supp}\ \ b(t)c(s) = \text{supp}\ \ (1+t^2)I_{[\beta+1,\beta+2]}(t)I_{\left[\frac{\alpha+\beta}{2},\beta\right]}(s)(1+s^4)$$
$$= [\beta+1,\beta+2] \times \left[\frac{\alpha+\beta}{2},\beta\right].$$

The set $\text{supp}\,[a(t) - F(a(s))] \cap \text{supp}\,[b(t)c(s)]$ has measure zero. So the result follows from Corollary 9.3.14.

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