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Sergei Silvestrov
Anatoliy Malyarenko *Editors*

Non-commutative and Non-associative Algebra and Analysis Structures

SPAS 2019, Västerås, Sweden, September 30–October 2

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Editors

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Preface

This volume is one of the long-term outcomes of the International Conference "Stochastic Processes and Algebraic Structures—From Theory Towards Applications" (SPAS 2019), which has been organized by the Division of Mathematics and Physics at the Mälardalen University in Västerås, Sweden on September 30–October 2, 2019, and of the follow-up research efforts, seminars, and activities on algebraic structures and applications developed following the ideas, research, and cooperation initiated at SPAS 2019. This top-quality focused international conference brought together a selected group of mathematicians and researchers from related subjects who actively contribute to the theory and applications of non-commutative and non-associative algebraic structures, methods, and models.

The scope of the volume is non-commutative and non-associative algebraic structures and their applications. The accompanying volume contains contributions to Stochastic Processes, Statistical Methods, and Engineering Mathematics.

The purpose of the book is to highlight the latest advances in non-commutative and non-associative algebra and non-commutative analysis structures with a focus on important mathematical notions, methods, structures, concepts, problems, and algorithms important in many other areas of mathematics, natural sciences, and engineering. The volume features mathematical methods and models from many important noncommutative and non-associative algebras and rings including various Hom-algebra structures such as Hom–Lie algebras, Hom–Lie superalgebras, color Hom–Lie algebras, Hom-bialgebra structures, and n-ary Hom-algebra structures, and related Hom-algebra structures, as well as other related non-commutative and non-associative algebra structures important in discrete and twisted generalizations of differential calculus, twisted derivations, quantum deformations of algebras, generalizations of Lie algebras, Lie superalgebras and color Lie algebras, semi-groups and group algebras, crossed product-type algebras, representations and applications of non-commutative and non-associative algebras, representations of commutation relations by special classes of operators on infinite-dimensional function spaces and

Banach spaces, computational algebra methods and their applications in the investigation of q-special functions and q-analysis, topology, dynamical systems, representation theory, operator theory and functional analysis, geometry, coding theory, information theory, and information analysis.

In Chap. 1, the index theory, important in representation theory and invariant theory, is extended to Hom–Lie algebras, for coadjoint and arbitrary representations, and index of multiplicative simple Hom–Lie algebras and semidirect products of Hom–Lie algebras is discussed.

Chapter 2 provides a construction procedure and examples of ternary Nambu–Poisson algebras and ternary Hom–Nambu–Poisson algebras from Poisson algebras and Hom–Poisson algebras equipped with a trace function satisfying some conditions.

In Chap. 3, several recent results concerning Hom–Leibniz algebra are reviewed, symmetric Hom–Leibniz superalgebra are introduced and some properties are obtained, classification of two-dimensional Hom–Leibniz algebras is provided, Centroids and derivations of multiplicative Hom–Leibniz algebras are considered including the detailed study of two-dimensional Hom–Leibniz algebras.

In Chap. 4, the representations of color Hom–Lie algebras are reviewed, the existence of a series of coboundary operators is demonstrated, the notion of a color omni-Hom–Lie algebra associated with a linear space and an even invertible linear map is introduced, characterization method for regular color Hom–Lie algebra structures on a linear space is examined and it is shown that the underlying algebraic structure of the color omni-Hom–Lie algebra is a color Hom–Leibniz algebra.

In Chap. 5, the decomposition theorem for the space of (σ, τ)-derivations of the group algebra $\mathbb{C}[G]$ of a discrete countable group G, generalizing the corresponding theorem on ordinary derivations on group algebras, is established in an algebraic context using groupoids and characters, several corollaries, and examples describing when all (σ, τ)-derivations are inner are obtained, and the cases of (σ, τ)-nilpotent groups and $(\sigma, \tau) - FC$ groups are considered in detail.

In Chap. 6, conditions for a color Hom–Lie algebra to be a complete color Hom–Lie algebra are obtained, the relationship between decomposition and completeness for a color Hom–Lie algebra is discussed, some conditions that the set of α^s-derivations of a color Hom–Lie algebra to be complete and simply complete are obtained, and conditions are derived for the decomposition into Hom-ideals of the complete multiplicative color Hom–Lie algebras to be unique up to order of Hom-ideals.

In Chap. 7, Hom-prealternative superalgebras and their bimodules are introduced, some constructions of Hom-prealternative superalgebras and Hom-alternative superalgebras and connections with Hom-alternative superalgebras are presented, bimodules over Hom-prealternative superalgebras are introduced, relations between bimodules over Hom-prealternative superalgebras and the bimodules of the corresponding Hom-alternative superalgebras are considered, and construction of bimodules over Hom-prealternative superalgebras by twisting is described.

Chapter 8 is devoted to the solution of certain equations on the set $\mathcal{H}$ of all quaternions. Using spectral analytic representations on $\mathcal{H}$, monomial equations, some quadratic equations, and linear equations on $\mathcal{H}$ are considered.

Chapter 9 is devoted to representations of polynomial covariant-type commutation relations by pairs of linear integral operators and multiplication operators on Banach spaces L_p.

Chapter 10 is concerned with representations of polynomial covariant-type commutation relations on Banach spaces L_p and $C[\alpha, \beta]$ $\alpha, \beta \in \mathbb{R}$ by operators of multiplication with piecewise functions, multiplication operators, and inner superposition operators.

In Chap. 11, nearly associative algebras are considered and are proved to be Lie-admissible algebras, two-dimensional nearly associative algebras are classified, and main classes are derived, The bimodules, matched pairs, and Manin triple of nearly associative algebras are studied and their equivalence with nearly associative bialgebras is proved. Furthermore, basic definitions and properties of nearly Hom-associative algebras are described.

Chapter 13 pertains to a study of the influence of Hom-associativity, involving a linear map twisting the associativity axiom, on (σ, τ)-derivations satisfying a (σ, τ)-twisted Leibniz product rule, factorization properties of elements in Hom-associative algebras, and zero divisors. Furthermore, new more general axioms of Hom-associativity, Hom-alternativity, and Hom-flexibility modulo kernel of a derivation are introduced leading to new classes of Hom-algebras motivated by (σ, τ)-Leibniz rule over multiplicative maps σ and τ and study of twisted derivations in arbitrary algebras and their connections to Hom-algebras structures.

Chapter 14 examines interactions between (σ, τ)-derivations via commutator and consider new n-ary structures based on twisted derivation operators. In particular, it is shown that the sums of linear spaces of $\left(\sigma^k, \tau^l\right)$-derivations and also of some of their subspaces, consisting of twisted derivations with some commutation relations with σ and τ, form Lie algebras, and with the semigroup- or group-graded commutator product, yield-graded Lie algebras when the sum of the subspaces is direct. These constructions of such Lie subalgebras spanned by twisted derivations of algebras are extended to twisted derivations of n-ary algebras. Furthermore, n-ary products defined by generalized Jacobian determinants based on (σ, τ)-derivations are defined, and n-Hom–Lie algebras associated with the generalized Jacobian determinants based on twisted derivations extending some results of Filippov to (σ, τ)-derivations are obtained. Moreover, commutation relations conditions are established for twisting maps and twisted derivations such that the generalized Jacobian determinant products yield (σ, τ, n)-Hom–Lie algebras, a new type of n-ary Hom-algebras different from n-Hom–Lie algebras in that the positions of twisting maps σ and τ are not fixed to positions of variables in n-ary products terms of the sum of defining identity as they were in Hom–Nambu–Filippov identity of n-Hom–Lie algebras.

Chapter 12 is devoted to q-analogues of some interesting formulas such as the $\mathbb{Z}_n$ components of the q-exponential function, factor-circulant matrices by the q-exponential of a permutation matrix, which has generalized q-hyperbolic functions as matrix elements, decomposition of functions with respect to the cyclic group of order n, q-analogues of inverse decomposition hypergeometric formulas by Osler and Srivastava, the q-Leibniz functional matrix, the q-difference operator, and q-analogues of hypergeometric series product formulas connected to the cyclic group decomposition.

Chapter 17, describes a computer program for doing network rewriting calculations, in its capacity as a tool used for scientific exploration—more precisely to systematically discover non-obvious consequences of the axioms for various algebraic structures. In particular, this program can cope with algebraic structures, such as bi- and Hopf algebras, that mix classical operations with co-operations.

Chapter 18 addresses a Hom-associative algebra built as a direct sum of a given Hom-associative algebra endowed with a non-degenerate symmetric bilinear form $\mathcal{B}$, double constructions, Hom–Frobenius algebras, and infinitesimal Hom-bialgebras, and double construction of Hom-dendriform algebras, also called double construction of Connes cocycle or symplectic Hom-associative algebra. The concept of biHom-dendriform algebras is introduced and discussed and their bimodules and matched pairs are also constructed, and related relevant properties are given.

Chapter 19 is devoted to properties of n-Hom–Lie algebras in dimension $n + 1$ allowing to explicitly find them and differentiate them, to classify them eventually. Some specific properties of $(n + 1)$-dimensional n-Hom–Lie algebra such as nilpotence, solvability, center, ideals, derived series, and central descending series are studied, the Hom–Nambu–Filippov identity for various classes of twisting maps in dimension $n + 1$ is considered, and systems of equations corresponding to each case are described. All four-dimensional 3-Hom–Lie algebras with some of the classes of twisting maps are computed in terms of structure constants as parameters and listed in a way that emphasizes the number of free parameters in each class, and also some detailed properties of the Hom-algebras are obtained.

In Chap. 20, the n-Hom–Lie algebras in dimension $n+1$ for $n = 4,5,6$ and nilpotent α with two-dimensional kernel are computed and some detailed properties of these algebras are obtained.

Chapter 22, introduces and gives some constructions of admissible Hom–Novikov–Poisson color Hom-algebras and Hom-Gelfand-Dorfman color Hom-algebras. Their bimodules and matched pairs are defined and the relevant properties and theorems are given. Also, the connections between the Hom–Novikov–Poisson color Hom-algebras and Hom–Gelfand–Dorfman color Hom-algebras are obtained. Furthermore, it is shown that the class of admissible Hom–Novikov–Poisson color Hom-algebras is closed under the tensor product.

Chapter 21 introduces a framework to study the deformation of algebras with anti-involution. Starting with the observation that twisting the multiplication of such an algebra by its anti-involution generates a Hom-associative algebra of type II, it formulates the adequate modules theory over these algebras and shows that there is a faithful functor from the category of finite-dimensional left modules of algebras with involution to finite-dimensional right modules of Hom-associative algebras of type II.

In Chap. 16, constructions of n-ary bialgebras and n-ary infinitesimal bialgebras of associative type and their Hom-analogs, generalizing the Hom-bialgebras and infinitesimal Hom-bialgebras are investigated. The main algebraic characteristics of n-ary totally, n-ary weak totally, n-ary partially, and n-ary alternate partially associative algebras and bialgebras, and their Hom-counterparts are described.

Chapter 23 provides the extension of the Wishart probability distributions in higher dimension based on the boundary points of the symmetric cones in Jordan algebras. The symmetric cones form a basis for the construction of the degenerate and non-degenerate Wishart distributions in the field of Herm $(m, \mathbb{C})$, Herm $(m, \mathbb{H})$, Herm $(3, \mathbb{O})$ denoting, respectively, the Jordan algebra of all Hermitian matrices of size $m \times m$ with complex entries, the skew field $\mathbb{H}$ of quaternions, and the algebra $\mathbb{O}$ of octonions. This density is characterized by the Vandermonde determinant structure and the exponential weight that is dependent on the trace of the given matrix.

Chapter 24 is concerned with induced ternary Hom-Nambu Lie algebras from Hom–Lie algebras and their classification. The induced algebras are constructed from a class of Hom–Lie algebra with a nilpotent linear map. The families of ternary Hom–Nambu–Lie arising in this way of construction are classified for a given class of nilpotent linear maps. In addition, some conditions on when morphisms of Hom–Lie algebras can remain as morphisms for the induced ternary Hom–Nambu–Lie algebras are given.

Chapter 25 is devoted to crossed product algebras of piecewise constant function algebras on the real line forming an increasing sequence of algebras of functions on the real line, which in case of invariance under bijection leading to an increasing sequence of crossed product algebras. A comparison of commutants (centralizers) in several cases is given.

Chapter 15 introduces a group key management protocol for secure group communications in a non-commutative setting using a group ring over the dihedral group with a twisted multiplication using a cocycle.

Chapter 26 studies the λ-constacyclic and skew λ-constacyclic codes of arbitrary length over a finite commutative non-chain ring.

Chapter 27 is devoted to a geometrical interpretation of the q-Wallis formula, and related estimates and conclusions about the number π_q.

Chapter 28 generalizes the results about generalized derivations of Lie algebras to the case of BiHom–Lie algebras. In particular, the classification of the generalized derivation of Heisenberg BiHom–Lie algebras is given. Classifications of two-dimensional BiHom–Lie algebra, centroids, and derivations of two-dimensional BiHom–Lie algebras are presented.

In Chap. 29, the construction of HNN-extensions of involutive Hom-associative algebras and involutive Hom–Lie algebras is described, and by using the validity of Poincaré-Birkhoff-Witt theorem for involutive Hom–Lie algebras, an embedding theorem is provided.

Chapter 30 is devoted to a path algebra of a quiver, left-sided (respective right-sided) ideals, and their Gröbner bases, two-sided ideals, a two-sided division algorithm and the two-sided Gröbner bases, and two-sided Buchberger's algorithm.

The volume is intended for researchers, graduate and Ph.D. students in the areas of Mathematics, Mathematical Physics, Information theory, Computer Science and Engineering, who are interested in a source of inspiration, cutting-edge research on algebraic structures, algebraic methods, and their applications. The book comprises selected refereed contributions from several research communities working in non-commutative and non-associative algebra and non-commutative analysis structures and their applications. The chapters cover both theory and applications and present a wealth of ideas, theorems, notions, proofs, examples, open problems, and findings on the interplay of algebraic structures with other parts of Mathematics and with applications. Presenting new methods and results, reviews of cutting-edge research, and open problems and directions for future research, the contributed chapters and the book as a whole will serve as a source of inspiration for a broad range of researchers and research students in algebra, noncommutative geometry, noncommutative analysis, non-commutative and non-associative algebraic structures and applied algebraic structures methods in computational and engineering mathematics, and relevant areas of mathematical and theoretical physics, information and computer science, natural science, and engineering.

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Chapter 1
Index of Hom-Lie Algebras

Hadjer Adimi and Abdenacer Makhlouf

Abstract The index is an important concept in representation theory and invariant theory. In this paper we extend the index theory to Hom-Lie algebras, we introduce the index theory in both cases, coadjoint and arbitrary representation. Moreover, we discuss Index of Multiplicative Simple Hom-Lie algebras and semidirect products of Hom-Lie algebras.

Keywords Hom-Lie algebra · Index · Representation

2020 Mathematics Subject Classification 17D30 · 17B61

1.1 Introduction

The notion of Hom-Lie algebra was introduced by Hartwig, Larsson, and Silvestrov in [13]. A Hom-Lie algebra is a triple $(\mathfrak{g}, [-,-], \alpha)$, where α is a linear self-map, in which the skewsymmetric bracket satisfies an $\alpha-$twisted variant of the Jacobi identity, called the Hom-Jacobi identity. When α is the identity map, the Hom-Jacobi identity reduces to the usual Jacobi identity, and $\mathfrak{g}$ is a Lie algebra. In [18] Makhlouf and Silvestrov introduced the notion of Hom-associative algebra defined by a triple $(\mathcal{A}, \mu, \alpha)$ in which α is a linear self-map of the vector space $\mathcal{A}$ and the bilinearity operation μ satisfies an α-twisted version of associativity. Associative algebras are a special case of Hom-associative algebras in which α is the identity map. The dual notion Hom-coalgebra was considered in [17].

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We start by considering representations of Hom-Lie algebras. The representation theory of an algebraic object reveals some of its profound structures hidden underneath. In [27] Y. Sheng defined representations of Hom-Lie algebras and corresponding Hom-cochain complexes. In particular, he obtains the adjoint representation and the trivial representation of Hom-Lie algebras. The definition was independently introduced in [4]. The index of a Lie algebra is an important concept in the representation theory and invariant theory. It was introduced by Dixmier in [8]. This theory has applications in invariant theory of invariants, deformations and quantum groups. A Lie algebra is called Frobenius if its index is 0, which is equivalent to say that there is functional in the dual such that the bilinear form B_F defined by $B_F(x, y) = F([x, y])$, is non-degenerate. Some results of the index are given in [1, 2]. The Frobenius algebras were studied by Ooms in [19]. Most index studies concern simple Lie algebras or their subalgebras. They were considered by many authors (see [7, 9–12, 19]). Note that a simple Lie algebra can never be Frobenius, but many subalgebras are. The index of a semisimple Lie algebra $\mathfrak{g}$ is equal to the rank of $\mathfrak{g}$.

The first main purpose of this paper is to introduce the index of Hom-Lie algebras. In the second Section we summarize the definitions and basics of Hom-Lie algebras from [13, 18, 32]. In the third Section, we study the index of Hom-Lie algebras. We introduce the notion of the index of Hom-Lie algebras in the case of coadjoint and an arbitrary representation. Moreover, we compare the index of Lie algebras with the index of Hom-Lie algebras obtained by twisting, and we discuss the index of Multiplicative Simple Hom-Lie algebras. In the last section we explore the coadjoint representations of semidirect products of Hom-Lie algebras and we give the index of semidirect products of Hom-Lie algebras.

1.2 Preliminary

We work in this chapter over an algebraically closed fields $\mathbb{K}$ of characteristic 0.

1.2.1 Hom-Lie Algebras

The notion of Hom-Lie algebra was introduced by Hartwig, Larsson and Silvestrov in [13] motivated initially by examples of deformed Lie algebras coming from twisted discretizations.

Definition 1.1 ([13]) A Hom-Lie algebra is a triple $(\mathfrak{g}, [-,-], \alpha)$ consisting of a vector space $\mathfrak{g}$, a skew-symmetric bilinear map $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ and a linear map $\alpha : \mathfrak{g} \to \mathfrak{g}$ satisfying the following Hom-Jacobi identity:

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0. \tag{1.1}$$

Let $(\mathfrak{g}_1, [-,-]_1, \alpha_1)$, $(\mathfrak{g}_2, [-,-]_2, \alpha_2)$ be two Hom-Lie algebras.
A linear map $\beta : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Hom-Lie algebra morphism if it satisfies

$$\begin{cases} \beta\,[x, y]_1 = [\beta(x), \beta(y)]_2\,, \ \forall x, y \in \mathfrak{g}_1 \\ \qquad\qquad \beta \circ \alpha_1 = \alpha_2 \circ \beta. \end{cases}$$

The map β is said to be a weak Hom-Lie algebras morphism if it satisfies only the first condition.

Remark 1.1 We recover classical Lie algebra when $\alpha = id_{\mathfrak{g}}$ and the identity (1.1) is the Jacobi identity

Definition 1.2 ([13]) A Hom-Lie algebra is called a multiplicative Hom-Lie algebra if is an algebra morphism, i.e. for any $x, y \in \mathfrak{g}$ we have $\alpha([x, y]) = [\alpha(x), \alpha(y)]$.

Definition 1.3 ([13]) A vector subspace $\mathfrak{H} \subset \mathfrak{g}$ is called a Hom-Lie sub-algebra of $(\mathfrak{g}, [-,-], \alpha)$ if $\alpha(\mathfrak{H}) \subset \mathfrak{H}$ and $\mathfrak{H}$ is closed under the bracket operation, i.e.,

$$[h, h'] \in \mathfrak{H}, \forall h, h' \in \mathfrak{H}.$$

Consider the direct sum of two Hom-Lie algebras.

Proposition 1.1 ([13]) *Given two Hom-Lie algebras* $(\mathfrak{g}, [-,-], \alpha)$ *and* $(\mathfrak{H}, [-,-], \beta)$*, there is a Hom-Lie algebra* $(\mathfrak{g} \oplus \mathfrak{H}, [-,-], \alpha + \beta)$*, where the skew-symmetric bilinear map* $[-,-] : \mathfrak{g} \oplus \mathfrak{H} \times \mathfrak{g} \oplus \mathfrak{H} \to \mathfrak{g} \oplus \mathfrak{H}$ *is given by*

$$[(x_1, y_1), (x_2, y_2)] = ([x_1, x_2], [y_1, y_2]), \ \forall x_1, x_2 \in \mathfrak{g}, y_1, y_2 \in \mathfrak{H},$$

and the linear map $(\alpha + \beta) : \mathfrak{g} \oplus \mathfrak{H} \to \mathfrak{g} \oplus \mathfrak{H}$ *is given by*

$$(\alpha + \beta)(x, y) = (\alpha(x), \beta(y)), \ \forall x \in \mathfrak{g}, y \in \mathfrak{H}.$$

A morphism of Hom-Lie algebras : $(\mathfrak{g}, [-,-], \alpha)$ *and* $(\mathfrak{H}, [-,-], \beta)$ *is a linear map* $\phi : \mathfrak{g} \to \mathfrak{H}$*, such that*

$$\phi[x, y] = [\phi(x), \phi(y)], \forall x, y \in \mathfrak{g} \tag{1.2}$$

$$\phi \circ \alpha = \beta \circ \phi \tag{1.3}$$

Denote by $\mathfrak{g}_\phi \subset \mathfrak{g} \oplus \mathfrak{H}$ *the graph of a linear map* $\phi : \mathfrak{g} \to \mathfrak{H}$.

Theorem 1.1 *Let* $\mathfrak{g} = (\mathfrak{g}, [-,-], \alpha)$ *be a Hom-Lie algebra and* $\beta : \mathfrak{g} \to \mathfrak{g}$ *be a weak Hom-Lie algebra morphism, then* $(\mathfrak{g}, \beta[-,-], \beta_\alpha)$ *is a Hom-Lie algebra.*

Corollary 1.1 ([32]) *Let* $(\mathfrak{g}, [-,-])$ *be a Lie algebra and* $\alpha : \mathfrak{g} \to \mathfrak{g}$ *be a Lie algebra endomorphism. Then* $\mathfrak{g}_\alpha = (\mathfrak{g}, [-,-]_\alpha, \alpha)$ *is a Hom-Lie algebra, where* $[-,-]_\alpha = \alpha \circ [-,-]$. *Moreover, suppose that* $\mathfrak{g}'$ *is another Lie algebra and that* $\alpha' : \mathfrak{g}' \to \mathfrak{g}'$

is a Lie algebra endomorphism. If $f : \mathfrak{g} \to \mathfrak{g}'$ *is a Lie algebras endomorphism that satisfies*

$$f \circ \alpha = \alpha' \circ f$$

then $f : (\mathfrak{g}, [-,-]_\alpha, \alpha) \to (\mathfrak{g}', [-,-]_{\alpha'}, \alpha')$ *is a morphism of multiplicative Hom-Lie algebras.*

The following example is obtained by using $\sigma-$derivation and it is not of the above type.

Example 1.1 (Jackson sl_2). The Jackson $q - sl_2$ is a $q-$deformation of the classical sl_2. This family of Hom-Lie algebras was constructed in [28] using a quasi-deformation scheme based on discretizing by means of Jackson $q-$derivations, a representation of $sl_2(\mathbb{K})$ by one-dimensional vector fields (first order ordinary differential operators) and using the twisted commutator bracket defined in [13]. It carries a Hom-Lie algebra structure but not a Lie algebra structure. It is defined with respect to a basis $\{x_1, x_2, x_3\}$ by the brackets and a linear map α such that

$[x_1, x_2] = -2qx_2$ $\qquad \alpha(x_1) = qx_1$
$[x_1, x_3] = 2x_3$ $\qquad \alpha(x_2) = q_2x_2$
$[x_2, x_3] = -\frac{1}{2}(1+q)x_1,$ $\qquad \alpha(x_3) = qx_3$

where q is a parameter in $\mathbb{K}$. If $q = 1$ we recover the classical Sl_2.

Proposition 1.2 *Let* $(\mathfrak{g}, [-,-], \alpha)$ *be a multiplicative Hom-Lie algebra with* α *bijective, then* $(\mathfrak{g}, \alpha^{-1}[-,-])$ *is a Lie algebra.*

Proof We set in Theorem 1.1, $\beta = \alpha^{-1}$. It shows that multiplicative Hom-Lie algebras with bijective twisting map correspond to a Lie algebras. The Lie algebra $(\mathfrak{g}, \alpha^{-1}[-,-])$ is called the induced Lie algebra.

1.2.2 Representations of Hom-Lie Algebras

Definition 1.4 ([4]) Let $(\mathfrak{g}, [-,-], \alpha)$ be a Hom-Lie algebra. A representation of $\mathfrak{g}$ is a triple $(\mathbb{V}, \rho, \beta)$, where $\mathbb{V}$ is a $\mathbb{K}-$vector space, $\beta \in End(\mathbb{V})$ and $\rho : \mathfrak{g} \to gl(\mathbb{V})$ is a linear map satisfying

$$\rho([x, y]) \circ \beta = \rho(\alpha(x)) \circ \rho(y) - \rho(\alpha(y)) \circ \rho(x), \forall x, y \in \mathfrak{g}.$$

In particular, a representation of a multiplicative Hom-Lie algebra $(\mathfrak{g}, [-,-]_\alpha, \alpha)$ on a vector space $\mathbb{V}$ is a representation of the Hom-Lie algebra satisfying in addition

$$\alpha\,(\rho(x)(v)) = \rho(\alpha(x))(\beta(v)), \quad \forall v \in \mathbb{V}, \quad \forall x \in \mathfrak{g}.$$

In the following, we explore the dual representations and coadjoint representations of Hom-Lie algebras.

Let $(\mathfrak{g}, [-,-], \alpha)$ be a Hom-Lie algebra and $(\mathbb{V}, \rho, \beta)$ be a representation of $\mathfrak{g}$. Let $\mathbb{V}^*$ be the dual vector space of $\mathbb{V}$. We define a linear map $\rho^* : \mathfrak{g} \to End(\mathbb{V}^*)$ by $\rho^*(x) = -^t\rho(x)$.

Proposition 1.3 ([4]) *Let* $(\mathfrak{g}, [-,-], \alpha)$ *be a Hom-Lie algebra and* $ad : \mathfrak{g} \to gl(\mathfrak{g})$ *be an operator defined for* $x \in \mathfrak{g}$ *by* $ad(x)(y) = [x, y]$. *Then* $(\mathfrak{g}, ad, \alpha)$ *is a representation of* $\mathfrak{g}$.

Indeed, the condition on the operator ad is equivalent to Hom-Jacobi condition. We call the representation defined in the previous proposition adjoint representation of the Hom-Lie algebra.

Proposition 1.4 ([4]) *Let* $(\mathfrak{g}, [-,-], \alpha)$ *be a Hom-Lie algebra and* $(\mathfrak{g}, ad, \alpha)$ *be the adjoint representation of* $\mathfrak{g}$, *where* $ad : \mathfrak{g} \to gl(\mathfrak{g})$. *We set* $ad^* : \mathfrak{g} \to gl(\mathfrak{g}^*)$ *and* $ad^*(x)(f) = -f \circ ad(x)$. *Then* $(\mathfrak{g}^*, ad^*, \alpha^*)$ *is a representation of* $\mathfrak{g}$ *if and only if*

$$\alpha([[x, y], z]) = [x, [\alpha(y), z]] - [y, [\alpha(x), z]] \quad \forall x, y, z \in \mathfrak{g} \tag{1.4}$$

Proposition 1.5 *Let* $(\mathfrak{g}, [-,-], \alpha)$ *be a Hom-Lie algebra and* $(\mathbb{V}, \rho, \beta)$ *be a representation of* $\mathfrak{g}$. *The triple* $(\mathbb{V}^*, \rho^*, \beta^*)$, *where* $\rho^* : \mathfrak{g} \to gl(\mathbb{V}^*)$ *is given by* $\rho^*(x) = -^t\rho^*(x)$, *defines a representation of the Hom-Lie algebra* $(\mathfrak{g}, [-,-], \alpha)$ *if and only if*

$$\beta \circ \rho([x, y]) = \rho(x)\rho(\alpha(y)) - \rho(y)\rho(\alpha(x)).$$

1.3 Index of Hom-Lie Algebras

1.3.1 For a Coadjoint Representation

Let $(\mathfrak{g}, [-,-], \alpha)$ be a Hom-Lie algebra. We assume that a coadjoint representation exists, that is Condition 1.4 in Proposition 1.4 is satisfied. Let f be a bilinear form on $\mathfrak{g}$ then set

$$\mathfrak{g}_f = \left\{x \in \mathfrak{g} : ad^*(x)(f) = 0\right\} = \{x \in \mathfrak{g} : f([x, y]) = 0, \forall y \in \mathfrak{g}\}.$$

Then we have the following definition.

Definition 1.5 The index of $\mathfrak{g}$, is defined by $\chi(\mathfrak{g}) = \min_{f \in \mathfrak{g}^*} \dim \mathfrak{g}_f$.

Example 1.2 (*Jackson* Sl_2) The index is given by $\chi(\mathfrak{g} = qSl_2) = \min_{f \in \mathfrak{g}^*} \dim \mathfrak{g}_f = 1$.

Indeed the associated matrix is of the form $\begin{pmatrix} 0 & -2qx_2 & 2x_3 \\ 2qx_2 & 0 & -\frac{1}{2}(1+q)x_1 \\ -2x_3 & \frac{1}{2}(1+q)x_1 & 0 \end{pmatrix}$.

It is of rank 2 hence the index equals 1.

1.3.2 For an Arbitrary Representation

Let $(\mathfrak{g}, [-,-], \alpha)$ be a Hom-Lie algebra, and $(\mathbb{V}, \rho, \beta)$ be a representation of $\mathfrak{g}$ where

$$\rho : \mathfrak{g} \to \mathfrak{g}l(\mathbb{V}), \qquad x \mapsto \rho(x) = \rho(x)(v) = x \cdot v.$$

We set $\mathfrak{g}_v = \{x \in \mathfrak{g} : x \cdot v = 0,\ v \in \mathbb{V}\}$ and $\mathfrak{g} \cdot v = \{x \cdot v : x \in \mathfrak{g},\ v \in \mathbb{V}\}$.

The set $\mathfrak{g}_v$ is the stabiliser of v. We say that $v \in \mathbb{V}$ is regular if $\mathfrak{g}_v$ has a minimum dimension, $\dim \mathfrak{g}_v = \min\{\dim \mathfrak{g}_w, w \in \mathbb{V}\}$. Since $\dim \mathfrak{g}_v + \dim \mathfrak{g} \cdot v = \dim \mathfrak{g}$, $v \in \mathbb{V}$ is regular if $\dim \mathfrak{g} \cdot v = \max_{w \in \mathbb{V}} \{\dim \mathfrak{g} \cdot w\}$.

So if we consider the dual representation of $\mathbb{V}$ on $\mathbb{V}^*$ when it exists, we have the following lemma.

Lemma 1.1 $\max_{f \in \mathbb{V}^*} \dim \mathfrak{g} \cdot f = \dim \mathfrak{g} - \min\left\{\dim \mathfrak{g}_f, f \in \mathbb{V}^*\right\}$.

Therefore, we define the index of a Hom-Lie algebra with respect to a given representation as:

Definition 1.6 The integer

$$\chi(\mathfrak{g}, \rho) = \dim \mathbb{V} - \max_{f \in \mathbb{V}^*} \{\dim \mathfrak{g}.f\} = \dim \mathbb{V} - \dim \mathfrak{g} + \min\left\{\dim \mathfrak{g}_f,\ f \in \mathbb{V}^*\right\}$$

is called the index of a representation $(\mathbb{V}, \rho, \beta)$ of the Hom-Lie algebra $(\mathfrak{g}, [-,-], \alpha)$.

Proposition 1.6 *Let* $(\mathfrak{g}, [-,-], \alpha)$ *be a Hom-Lie algebra, the index of* $\mathfrak{g}$, $\chi(\mathfrak{g}, \rho)$, *can be written*

$$\begin{aligned}\chi(\mathfrak{g}, \rho) &= \dim \mathbb{V} - \max_{f \in \mathbb{V}^*} \left\{\dim \mathfrak{g}_f\right\} = \min\left\{\dim \mathfrak{g}_f; f \in \mathfrak{g}^*\right\}, \\ &= \dim \mathbb{V} - rank_{\mathbb{K}(\mathbb{V})} \left(x_i \cdot v_j\right)_{ij}, \ \textit{(see Proposition 2.3, [19], p1)},\end{aligned}$$

where $\mathbb{K}(\mathbb{V})$ *is the quotient fields of the symmetric algebras* $S(\mathbb{V})$.

Proof Consider the bilinear form $\mathcal{B}$ with values in $\mathbb{V}$

$$\mathcal{B} = \mathcal{B}(\mathfrak{g}, \mathbb{V}) : \mathfrak{g} \times \mathbb{V} \to \mathbb{V}, \qquad (x, v) \mapsto x \cdot v.$$

Evaluating this form for an arbitrary element $f \in \mathbb{V}^*$ gives a form with values in $\mathbb{K}$.

It follows $\mathcal{B}_f : \mathfrak{g} \times \mathbb{V} \to \mathbb{V} \xrightarrow{f} \mathbb{K}$ and $\mathcal{B}_f(x, v) = f(x \cdot v)$. The kernel (resp. image) of $\mathcal{B}_f$ is $\mathfrak{g}_f$ (resp. $\mathfrak{g} \cdot f$). We have

$$\begin{aligned}\ker(\mathcal{B}_f) &= \mathfrak{g}_f = \{x \in \mathfrak{g}\ ;\ f(x \cdot v) = 0\} \text{ and} \\ Im(\mathcal{B}_f) &= \mathfrak{g} \cdot f = \{f(x \cdot v)\ ;\ x \in \mathfrak{g}, v \in \mathbb{V}\}.\end{aligned}$$

Hence, $\chi(\mathfrak{g},\rho) = \dim \mathbb{V} - \max_{f\in\mathbb{V}^*}(rank\mathcal{B}_f)$. Let $n = \dim \mathfrak{g}$ and $m = \dim \mathbb{V}$. Having chosen bases for $\mathfrak{g}$ and $\mathbb{V}$, we may regard $\mathcal{B}$ as $n \times m$-matrix with integer in $\mathbb{V}$, taking $\{x_1, \ldots, x_n\}$ a basis for $\mathfrak{g}$ and $\{v_1, \ldots, v_m\}$ a basis for $\mathbb{V}$,

$$\mathcal{B} = \left(x_i \cdot v_j\right)_{ij}, \qquad i = 1, \ldots, n,\ j = 1, \ldots, m \text{ and}$$
$$\mathcal{B}_f = \left(f\left(x_i \cdot v_j\right)\right)_{ij} \quad i = 1, \ldots, n,\ j = 1, \ldots, m$$

Therefore,

$$\begin{aligned}\chi(\mathfrak{g},\rho) &= \dim \mathbb{V} - \max_{f\in\mathbb{V}^*}\left(rank\left(f\left(x_i \cdot v_j\right)\right)_{ij}\right),\ i = 1, \ldots, n, j = 1, \ldots, m,\\ &= \dim \mathbb{V} - rank_{\mathbb{k}(\mathbb{V})}\left(x_i \cdot v_j\right)_{ij}, \qquad i = 1, \ldots, n, j = 1, \ldots, m.\end{aligned}$$

Hence, $\chi(\mathfrak{g},\rho) = \dim \mathbb{V} - rank\left(x_i \cdot v_j\right)_{ij},\ i = 1, \ldots, n, j = 1, \ldots, m.$

1.3.3 Index of Twisted Lie Algebras

Let $(\mathfrak{g}, [-,-])$ be a Lie algebra and $\alpha : \mathfrak{g} \to \mathfrak{g}$ be a Lie algebra morphism. According to Corollary 1.1, the twist $\mathfrak{g}_\alpha = (\mathfrak{g}, [-,-]_\alpha, \alpha)$, where $[x, y]_\alpha = \alpha[x, y]$, is a Hom-Lie algebra. We aim to compare the index of a Lie algebra with the index of the Hom-Lie algebra obtained by twisting.

Proposition 1.7 *Let $\mathfrak{g}_\alpha = (\mathfrak{g}, [-,-]_\alpha, \alpha)$ be a Hom-Lie algebra, and ad be the adjoint representation. Then*

$$\chi(\mathfrak{g}_\alpha) = n - rank_{\mathbb{k}(\mathbb{V})}\left(\alpha\left(\left[e_i, e_j\right]\right)\right)_{ij}.$$

Proof For all $x \in \mathfrak{g}$, ad_x is a $\mathbb{K}-$linear map and $\mathfrak{g}$ operate on $\mathfrak{g}^*$as

$$\mathfrak{g}\times\mathfrak{g}^* \to \mathfrak{g}^*, \qquad (x, f) \mapsto x \cdot f.$$

$$\forall y \in \mathfrak{g} : (x \cdot f)(y) = f\left([x, y]_\alpha\right),$$

$$\phi_f : \mathfrak{g}\times\mathfrak{g}\to\mathbb{K}, \qquad (x, y) \mapsto \phi_f(x, y) = f\left([x, y]_\alpha\right),$$

$$\mathfrak{g}_f = \left\{x \in \mathfrak{g}, f\left([x, y]_\alpha\right) = 0, \forall y \in \mathfrak{g}\right\},$$

or

$$Kerf = \left\{x \in \mathfrak{g}, f\left([x, y]_\alpha\right) = 0, \forall y \in \mathfrak{g}\right\}. \tag{1.5}$$

Particular cases: if the algebra is multiplicative then

$$f\left([x,y]_\alpha\right)=f\left(\alpha\left[x,y\right]\right)=\left(f\circ\alpha\right)\left([x,y]\right).$$

We denote the kernel of the map $(f\circ\alpha)$ by $\mathfrak{g}_{(f\circ\alpha)}$,

$$\mathfrak{g}_{(f\circ\alpha)}=Ker\left(f\circ\alpha\right)=\left\{x\in\mathfrak{g},\ \left(f\circ\alpha\right)\left([x,y]\right)=0,\forall y\in\mathfrak{g}\right\}, \tag{1.6}$$

and $Im\left(f\circ\alpha\right)=\left\{\left(f\circ\alpha\right)\left([x,y]\right),\forall x,y\in\mathfrak{g}\right\}$. Applying the rank theorem we obtain

$$\dim\mathfrak{g}=\dim\ker\left(f\circ\alpha\right)+\dim Im\left(f\circ\alpha\right),$$
$$\dim\ker\left(f\circ\alpha\right)=\dim\mathfrak{g}-\dim Im\left(f\circ\alpha\right)=n-\dim Im\left(f\circ\alpha\right).$$

Moreover, we have $\min\limits_{(f\circ\alpha)\in\mathfrak{g}^*}\left\{\dim\ker\left(f\circ\alpha\right)\right\}=n-\max\limits_{(f\circ\alpha)\in\mathfrak{g}^*}\left\{\dim Im\left(f\circ\alpha\right)\right\}$. We know that $\chi\left(\mathfrak{g}_\alpha\right)=\min\left\{\dim\ker\left(f\circ\alpha\right),\left(f\circ\alpha\right)\in\mathfrak{g}^*\right\}$. Then

$$\chi\left(\mathfrak{g}_\alpha\right)=n-\max\left\{\dim Im\left(f\circ\alpha\right),\quad\left(f\circ\alpha\right)\in\mathfrak{g}^*\right\}.$$

Let $B=\{e_1,\dots,e_n\}$ be a basis of $\mathfrak{g}$. For all $x,y\in\mathfrak{g}$ we have $x=\sum\limits_i x_ie_i,\ y=\sum\limits_j y_je_j$. Then

$$\begin{aligned}\left(f\circ\alpha\right)\left([x,y]\right)&=\left(f\circ\alpha\right)\left(\left[\sum_i x_ie_i,\sum_j y_je_j\right]\right)\\&=\left(x_1,\dots,x_n\right)\left(f\circ\alpha\right)\left(\left[e_i,e_j\right]\right)\begin{pmatrix}y_1\\\vdots\\y_n\end{pmatrix}\\&=X^tAY;\qquad\text{and } A=\left(\left(f\circ\alpha\right)\left(\left[e_i,e_j\right]\right)\right)_{ij}.\end{aligned}$$

Therefore

$$\begin{aligned}\chi\left(\mathfrak{g}_\alpha\right)&=n-\max rank\left(\left(f\circ\alpha\right)\left(\left[e_i,e_j\right]\right)\right)_{ij},\\&=n-rank\left(f\left(\alpha\left(\left[e_i,e_j\right]\right)\right)\right)_{ij},\\&=n-rank\left(f\left(\left[e_i,e_j\right]_\alpha\right)\right)_{ij},\\&=n-rank_{\mathbb{k}(\mathbb{V})}\left(\left[e_i,e_j\right]_\alpha\right)_{ij},\\&=n-rank_{\mathbb{k}(\mathbb{V})}\left(\alpha\left(\left[e_i,e_j\right]\right)\right)_{ij}.\end{aligned}$$

Then, $\chi\left(\mathfrak{g}_\alpha\right)=n-rank_{\mathbb{k}(\mathbb{V})}\left(\alpha\left(\left[e_i,e_j\right]\right)\right)_{ij}$.

Theorem 1.2 *Let* $(\mathfrak{g},[-,-])$ *be a Lie algebra,* $\alpha:\mathfrak{g}\to\mathfrak{g}$ *be a Lie algebra morphism and* $\mathfrak{g}_\alpha=(\mathfrak{g},[-,-]_\alpha,\alpha)$ *be the Hom-Lie algebra where* $[-,-]_\alpha=\alpha\circ[-,-]$. *Then* *we have* $\chi(\mathfrak{g}_\alpha) \geq \chi(\mathfrak{g})$. *Moreover if* f *is a regular vector of* $\mathfrak{g}$ *then it is a regular vector of* $\mathfrak{g}_\alpha$.

Proof Since $rank((f \circ \alpha)) \leq \min(rankf, rank\alpha)$,

$$rank((f \circ \alpha)) \leq rankf, \quad \chi(\mathfrak{g}_\alpha) \geq \chi(\mathfrak{g}).$$

Remark 1.2 (Case where α is bijective) Let $(\mathfrak{g}_\alpha, [-,-]_\alpha, \alpha)$ be a Hom-Lie algebra. If α is bijective then $\chi(\mathfrak{g}_\alpha) = \chi(\mathfrak{g})$. Indeed $rank((f \circ \alpha)) = rankf$. ($\alpha$ bijective, $Im(\alpha) = \mathfrak{g}$, so $Im(f \circ \alpha) = f(\mathfrak{g}) = Imf$, then $rank((f \circ \alpha)) = rankf$).

Example 1.3 (Morphism of Lie algebra and index). Let $(\mathfrak{g}, [-,-])$ be a Lie algebra, and $(\mathfrak{g}, [-,-]_\alpha, \alpha)$ be a Hom-Lie algebra and $\{x_1, x_2, ..., x_n\}$ be a fixed basis of $\mathfrak{g}$. We search morphisms corresponding to this algebra and calculate the index of $\mathfrak{g}_\alpha$ in this case. The twisting principle leads for the dimensional affine Lie algebra defined as $\mathfrak{g}_2^1 : [x_1, x_2] = x_2$ to two Hom-Lie algebras : the first one is the abelian Hom-Lie algebra $\mathfrak{g}_{2,\alpha,1}^1 : [x_1, x_2]_\alpha = 0$, and it is given by the homomorphism α defined, with respect to the previous basis by the following matrix $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$. The second are $\mathfrak{g}_{2,\alpha,2}^2$ defined as $\mathfrak{g}_{2,\alpha,2}^2 : [x_1, x_2]_\alpha = dx_2$, The homomorphism α is given by the following matrix $\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$. Hence, α is of the form $\alpha(x_1) = ax_1 + bx_2, \quad \alpha(x_2) = cx_1 + dx_2$.

The 3−dimensional Lie algebra defined by $\mathfrak{g}_3^1 : [x_1, x_2] = x_3$ leads to the Hom-Lie algebras: $\mathfrak{g}_{3,\alpha,3}^1 : [x_1, x_2]_\alpha = (a_1 b_2 - b_1 b_2)x_3$, and it is given by the homomorphism α defined, with respect to the previous basis by the following matrix $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 0 & 0 & a_1 b_2 - b_1 a_2 \end{pmatrix}$.

The Lie algebra $\mathfrak{g}_3^2 : [x_1, x_2] = x_2, \ [x_1, x_3] = \beta x_3, \ \beta \neq 0$ leads to four Hom-Lie algebras:

(i) $\mathfrak{g}_{3,\alpha,1}^2 : [x_1, x_2]_\alpha = 0, \qquad [x_1, x_3]_\alpha = 0$, this is an abelian Hom-Lie algebra, the homomorphism α is given by the following matrix $\begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

(ii) $\mathfrak{g}_{3,\alpha,2}^2 : [x_1, x_2]_\alpha = b_2 x_3, \qquad [x_1, x_3]_\alpha = 0$, the homomorphism α is given by the following matrix $\begin{pmatrix} \frac{1}{\beta} & b_1 & c_1 \\ 0 & 0 & c_2 \\ 0 & 0 & 0 \end{pmatrix}$.

(iii) $\mathfrak{g}_{3,\alpha,3}^2 : [x_1, x_2]_\alpha = 0, \qquad [x_1, x_3]_\alpha = \beta b_3 x_2$, the homomorphism α is given by the following matrix $\begin{pmatrix} \beta & b_1 & c_1 \\ 0 & 0 & 0 \\ a_3 & b_3 & 0 \end{pmatrix}$.

(iv) $\mathfrak{g}^2_{3,\alpha,4} : [x_1, x_2]_\alpha = b_2 x_2, \quad [x_1, x_3]_\alpha = \beta c_3 x_3$, the homomorphism α is given by the following matrix $\begin{pmatrix} 1 & b_1 & c_1 \\ 0 & b_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}$.

The Lie algebra $\mathfrak{g}^3_3 : [x_1, x_2] = x_2 + x_3, \quad [x_1, x_3] = x_3$ leads to two Hom-Lie algebras defined as

(i) $\mathfrak{g}^3_{3,\alpha,1} : [x_1, x_2]_\alpha = 0, \quad [x_1, x_3]_\alpha = 0$, this is an abelian Hom-Lie algebra, the homomorphism α is given by the following matrix $\begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

(ii) $\mathfrak{g}^3_{3,\alpha,2} : [x_1, x_2]_\alpha = b_2 x_2 + (b_2 + c_2) x_3, \quad [x_1, x_3]_\alpha = b_2 x_2$, the homomorphism α is given by the following matrix $\begin{pmatrix} 1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & b_2 \end{pmatrix}$.

The Lie algebra $\mathfrak{g}^3_4 : [x_1, x_2] = 2x_2, \quad [x_1, x_3] = -2x_3, \quad [x_1, x_3] = x_1$ leads to two Hom-Lie algebras defined as :

(i) $\mathfrak{g}^3_{4,\alpha,1} : [x_1, x_2]_\alpha = \frac{2}{b_3} x_2, \quad [x_1, x_3]_\alpha = -2 b_3 x_3, \quad [x_2, x_3]_\alpha = -x_1$, the homomorphism α is given by the following matrix $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & \frac{1}{b_3} \\ 0 & b_3 & 0 \end{pmatrix}$.

(ii) $\mathfrak{g}^3_{4,\alpha,2} : [x_1, x_2]_\alpha = \frac{2}{C_3} x_2, \quad [x_1, x_3]_\alpha = -2 c_3 x_3, \quad [x_2, x_3]_\alpha = x_1$, the homomorphism α is given by the following matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{c_3} & 0 \\ 0 & 0 & c_3 \end{pmatrix}$.

Hence α is of the form

$$\begin{aligned} \alpha(x_1) &= a_1 x_1 + b_1 x_2 + c_1 x_3, \\ \alpha(x_2) &= a_2 x_1 + b_2 x_2 + c_2 x_3, \\ \alpha(x_3) &= a_3 x_1 + b_3 x_2 + c_3 x_3. \end{aligned}$$

Evaluation of the index

In dimension 2, we have

$\chi\left(\mathfrak{g}^1_{2,\alpha,1}\right) = 2,$
$\chi\left(\mathfrak{g}^1_{2,\alpha,2}\right) = 0.$

In dimension 3, we have

(i) $\chi\left(\mathfrak{g}^1_{3,\alpha,1}\right) = \begin{cases} 1 \text{ if } a_1 b_2 - b_1 a_2 \neq 0, \\ 3 \text{ else.} \end{cases}$

(ii) $\chi\left(\mathfrak{g}^2_{3,\alpha,1}\right) = 0.$

$$\chi\left(\mathfrak{g}^2_{3,\alpha,2}\right) = \begin{cases} 1 \text{ if } c_2 \neq 0, \\ 3 \text{ else.} \end{cases}$$

$$\chi\left(\mathfrak{g}^2_{3,\alpha,3}\right) = \begin{cases} 1 \text{ if } b_3 \neq 0, \\ 3 \text{ else.} \end{cases}$$

$$\chi\left(\mathfrak{g}^2_{3,\alpha,4}\right) = \begin{cases} 1 \text{ if } b_2, c_3 \neq 0, \\ 3 \text{ else.} \end{cases}$$

(iii) $\chi\left(\mathfrak{g}^3_{3,\alpha,1}\right) = 0.$

$$\chi\left(\mathfrak{g}^3_{3,\alpha,2}\right) = \begin{cases} 1 \text{ if } b_2, c_2 \neq 0, \\ 3 \text{ else.} \end{cases}$$

(iv) $\chi\left(\mathfrak{g}^4_{3,\alpha,1}\right) = 1 \quad$ with $b_3 \neq 0,$

$\chi\left(\mathfrak{g}^4_{3,\alpha,2}\right) = 1 \quad$ with $c_3 \neq 0.$

1.3.4 Index of Multiplicative Simple Hom-Lie Algebras

Multiplicative simple Hom-Lie algebras were recently characterized in [6] and their representations studied in [3].

Definition 1.7 Let $(\mathfrak{g}, [-,-], \alpha)$ be a Hom-Lie algebra. A subspace $\mathfrak{h}$ of $\mathfrak{g}$ is called a Hom-Lie subalgebra of $(\mathfrak{g}, [-,-], \alpha)$ if $\alpha(\mathfrak{h}) \subseteq \mathfrak{h}$ and $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$. In particular, a Hom-Lie subalgebra $\mathfrak{h}$ is said to be an ideal of $(\mathfrak{g}, [-,-], \alpha)$ if $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$. A Hom-Lie algebra $\mathfrak{g}$ is called an abelian Hom-Lie algebra if $[x, y] = 0$ for any $x, y \in \mathfrak{g}$.

Definition 1.8 The set

$$C(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, y] = 0, [\alpha(x), y] = 0, \forall y \in \mathfrak{g}\}$$

is called the center of $(\mathfrak{g}, [-,-], \alpha)$.

Proposition 1.8 *Let* $(\mathfrak{g}, [-,-], \alpha)$ *be a multiplicative Hom-Lie algebra. Then* $(\ker(\alpha), [-,-], \alpha)$ *is an ideal.*

Proof Obviously $\alpha(x) = 0 \in \ker(\alpha)$ for any $x \in \ker(\alpha)$. Since $\alpha([x, y]) = [\alpha(x), \alpha(y)] = [0, y] = 0$ for any $x \in \ker(\alpha)$ and $y \in \mathfrak{g}$, we get $[x, y] \in \ker(\alpha)$. Therefore $(\ker(\alpha), [-,-], \alpha)$ is an ideal of $(\mathfrak{g}, [-,-], \alpha)$.

Definition 1.9 Let $(\mathfrak{g}, [-,-], \alpha)$ $(\alpha \neq 0)$ be a Hom-Lie algebra. A Hom-Lie algebra $(\mathfrak{g}, [-,-], \alpha)$ is called simple Hom-Lie algebra if $(\mathfrak{g}, [-,-], \alpha)$ has no proper ideals and is not abelian. A Hom-Lie algebra $(\mathfrak{g}, [-,-], \alpha)$ is called semisimple Hom-Lie algebra if $\mathfrak{g}$ is a direct sum of certain ideals.

Let $(\mathfrak{g}, [-,-], \alpha)$ be a multiplicative Hom-Lie algebra. By Proposition 1.8, α must be a monomorphism, thus α is an automorphism of $(\mathfrak{g}, [-,-], \alpha)$.

Definition 1.10 Let $(\mathfrak{g}, [-,-], \alpha)$ be a Hom-Lie algebra. The Lie algebra $(\mathfrak{g}, [-,-]')$ is called the induced Lie algebra of $(\mathfrak{g}, [-,-], \alpha)$ if

$$[x, y] = \alpha\left([x, y]'\right) = [\alpha(x), \alpha(y)]', \forall x, y \in \mathfrak{g}.$$

Proposition 1.9 *Let $(\mathfrak{g}, [-,-], \alpha)$ be a multiplicative simple Hom-Lie algebra. Define $[x, y]' = \alpha^{-1}([x, y])\, \forall x, y \in \mathfrak{g}$. Then $\left(\mathfrak{g}, [-,-]'\right)$is a Lie algebra and α is also a Lie algebra automorphism.*

Theorem 1.3 *The induced Lie algebra of a multiplicative simple Hom-Lie algebra $(\mathfrak{g}, [-,-], \alpha)$ is semisimple and can be decomposed into direct sum of isomorphic simple ideals. In addition α acts simply transitively on simple ideals of the induced Lie algebra.*

Theorem 1.4 *The index of a multiplicative simple Hom-Lie algebra $(\mathfrak{g}, [-,-], \alpha)$ is the same as the index of the induced Lie algebra of the multiplicative simple Hom-Lie algebra $\left(\mathfrak{g}, \alpha^{-1}[-,-]\right)$.*

Proof By Remark 1.2.

Hence, we have the following Proposition :

Proposition 1.10 *The index of a multiplicative simple Hom-Lie algebra $(\mathfrak{g}, [-,-], \alpha)$ is larger than 0.*

Proof Since a Simple Lie algebras is never Frobenius, then the index is larger than 0.

1.4 Index of Semidirect Products of Hom-Lie Algebras

In this section we introduce the adjoint and coadjoint representation of a semi-direct product of a Hom-Lie algebra $\mathfrak{g} \ltimes_\rho \mathbb{V}$.

Proposition 1.11 ([4]) *Let $(\mathfrak{g}, [-,-]_\mathfrak{g}, \alpha)$ be a Hom-Lie algebra and $(\mathbb{V}, \rho, \beta)$ be a representation of $\mathfrak{g}$. The direct sum $\mathfrak{g} \oplus \mathbb{V}$ with a bracket defined by*

$$[(x+u), (y+v)] = ([x, y]_\mathfrak{g}, \rho(x)(v) - \rho(y)(u)) \quad \forall x, y \in \mathfrak{g}, \quad \forall u, w \in \mathbb{V},$$

and the twisted map $\gamma : \mathfrak{g} \oplus \mathbb{V} \to \mathfrak{g} \oplus \mathbb{V}$ defined by

$$\gamma(x+w) = \alpha(x) + \beta(u), \quad \forall x \in \mathfrak{g}, \quad \forall u \in \mathbb{V}$$

is a Hom-Lie algebra.

We call the direct sum $\mathfrak{g} \oplus \mathbb{V}$ semi-direct product of $\mathfrak{g}$ and $\mathbb{V}$, it is denoted by $\mathfrak{g} \ltimes_\rho \mathbb{V}$. We identify the dual space $(\mathfrak{g} \ltimes_\rho \mathbb{V})^*$ with $\mathfrak{g}^* \oplus \mathbb{V}^*$.

Since $\mathfrak{g} \ltimes_\rho \mathbb{V}$ is a Hom-Lie algebra, the Hom-Jacobi condition on $x, y, z \in \mathfrak{g}$ and $u, v, w \in \mathbb{V}$ is

$$\circlearrowleft_{(x,u),(y,v),(z,w)} [\gamma(x+u), [y+v, z+w]],$$
$$= \circlearrowleft_{(x,u),(y,v),(z,w)} [\alpha(x) + \beta(u), [y,z]_{\mathfrak{g}} + \rho(y)(w) - \rho(z)(v)] = 0.$$

We can determine a representation of a semi-direct product of a Hom-Lie algebra $\mathfrak{g} \ltimes_\rho \mathbb{V}$. We call this representation the adjoint representation of semi-direct product of a Hom-Lie algebra $\mathfrak{g} \ltimes_\rho \mathbb{V}$, and it satisfies the condition

$$ad\left([x+u, y+v]\right) \circ \gamma = ad\left(\gamma\left(x+u\right)\right) \circ ad\left(y+v\right) - ad\left(\gamma\left(y+v\right)\right) \circ ad\left(x+u\right). \quad (1.7)$$

In the following, we explore coadjoint representations of the semi-direct product of a Hom-Lie algebra $\mathfrak{g} \ltimes_\rho \mathbb{V}$.

1.4.1 Coadjoint Representations

Let $\mathfrak{q} = \mathfrak{g} \ltimes_\rho \mathbb{V}$. We consider $\mathfrak{q}^* = \mathfrak{g}^* \oplus \mathbb{V}^*$, the dual space of q. An element of q^* is denoted by $\eta = (g, f)$; $\forall\, (x, v) \in \mathfrak{g} \ltimes_\rho \mathbb{V}$. We set $ad^* : \mathfrak{q} \to gl(\mathfrak{q}^*)$ defined by $ad^*\,(x+u)\,(\eta) = -\eta \circ ad(x+u)$ and $\gamma^* : \mathfrak{q}^* \to \mathfrak{q}^*$ an even homomorphism defined by $\gamma^*\,(\eta) = \eta \circ \gamma$. We compute the right hand side of (1.7):
$(ad^*(\gamma(x+u)) \circ ad^*(y+v) - ad^*(\gamma(y+v)) \circ ad^*(y+v))\,(\eta)\,(z+w)$
$= (ad^*(\gamma(x+u))\,(ad^*(y+v)\,(\eta)) - ad^*(\gamma(y+v))(ad^*(y+v)\,(\eta)))\,(z+w)$
$= -ad^*(y+v)\,(\eta)\,(ad(\gamma(x+u))\,(z+w)) + ad^*(x+u)ad\,(\eta)\,(\gamma(x+u))\,(z+w))$
$= \eta(ad(y+v)ad(\gamma(x+u))\,(z+w)) - \eta(ad(x+u)ad(\gamma(x+u))\,(z+w))$
$= \eta(ad(y+v)ad(\gamma(x+u)) - ad(x+u)ad(\gamma(x+u))\,(z+w))$.

On the other hand,

$$\left(\left(ad^*([x+u, y+v]\right)\gamma^*\right)(\eta))\,(z+w) = \left(ad^*([x+u, y+v]\right)(\eta \circ \gamma)\,(z+w)$$
$$= -\eta \circ \gamma\,(ad([x+u, y+v])\,(z+w)).$$

Thus condition (1.7) is satisfied. We call the representation ad^* the coadjoint representation.

We obtain the following corollary.

Corollary 1.2 *Let* $(\mathfrak{q}, [-,-], \gamma)$ *be a Hom-Lie algebra and* $(\mathfrak{q}, ad, \gamma)$ *be the adjoint representation of* $\mathfrak{q}$. *The triple* $(\mathfrak{q}, ad^*, \gamma^*)$ *defines a representation of* $(\mathfrak{q}, [-,-]\gamma)$ *if and only if*

$$\gamma \circ ad\left([x+u, y+v]\right) = ad\left(x+u\right) \circ ad\left(\gamma\left(y+v\right)\right) - ad\left(y+v\right) \circ ad\left(\gamma\left(x+u\right)\right).$$

We call the representation ad^* the coadjoint representation and it is given by

$$\begin{aligned}
&\left(ad^*_{\mathfrak{q}}(x,v)\right)(g,f) = \left(ad^*_{\mathfrak{g}}(x)(g) - v * f, x\cdot f\right),\\
&\mathfrak{g}\times\mathbb{V}^* \to \mathbb{V}^*,\quad (x,f)\mapsto x\cdot f,\\
&\mathbb{V}\times\mathbb{V}^* \to \mathfrak{g}^*,\quad (v,f)\mapsto v*f,\qquad \forall x\in\mathfrak{g} : (v*f)\,x = f\,(xv).
\end{aligned}$$

1.4.2 The Stabilizer of an Arbitrary Point of $\mathfrak{q}^*$

Let $\mathcal{K}_g$ denote the Kirillov form on $\mathfrak{g}$, i.e. $\forall\,(x_1,x_2)\in\mathfrak{g}$: $\mathcal{K}_g\,(x_1,x_2) = g\,[x_1,x_2]$. Then $\ker(\mathcal{K}_g) = \mathfrak{g}_g$, the stabiliser of g. If $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, then $\mathcal{K}_g|\mathfrak{h}$ can also be regarded as the Kirillov form associated with $g|\mathfrak{h}\in\mathfrak{h}^*$.

Proposition 1.12 *For any $\eta = (g,f)\in q$, we have*

$$\begin{aligned}
\mathfrak{q}_\eta &= \left\{(x,v)\in\mathfrak{g}\ltimes_\rho\mathbb{V}, ad^*_{\mathfrak{g}}(x,v)(g,f) = 0\right\}\\
&= \left\{(x,v)\in\mathfrak{g}\ltimes_\rho\mathbb{V}, \left(ad^*_{\mathfrak{g}}(x)(g) - v*f, x\cdot f\right) = 0\right\}\\
&= \left\{(x,v)\in\mathfrak{g}\ltimes_\rho\mathbb{V}, ad^*_{\mathfrak{g}}(x)(g) = v*f \text{ et } x\cdot f = 0\right\},
\end{aligned}$$

where $ad^*_{\mathfrak{g}}(x)(g) = g\,[x,y] = v*f,\ y\in\mathfrak{g}$ *and* $x\cdot f = 0 \Rightarrow x\in\ker\left(\mathcal{K}_g|\mathfrak{g}_f\right)$,

with $\mathcal{K}_g(x,f) = x\cdot f$. Then

$$\mathfrak{q}_\eta = \left\{(x,v)\in\mathfrak{g}\ltimes_\rho\mathbb{V}, ad^*_{\mathfrak{g}}(x)(g) = v*f, x\in\ker\left(\mathcal{K}_g|\mathfrak{g}_f\right)\right\}.$$

We denote by $\mathfrak{g}_f$ the kernel of $f\,([x,y])$, so $\left(\mathfrak{g}_f\right)^\perp = \mathfrak{g}\cdot f$ such that the space

$$\{v\in\mathbb{V}, v*f = 0\} = \{v\in\mathbb{V}, f\,(xv) = 0, \forall x\in\mathfrak{g}\} = (\mathfrak{g}\cdot f)^\perp = \ker\mathcal{B}_f.$$

It follows that $\mathfrak{q}_\eta$ is the direct sum of the space $(\mathfrak{g}\cdot f)^\perp$ and the space $\ker\left(\mathcal{K}_g|\mathfrak{g}_f\right)$, so $\mathfrak{q}_\eta = \ker\left(\mathcal{K}_g|\mathfrak{g}_f\right)\ltimes\ker\mathcal{B}_f$.

Proposition 1.13 *Let* $(\mathfrak{q},[-,-],\gamma)$ *be the Hom-Lie algebra defined above, then*

$$\chi_{\mathfrak{q}} = \chi_{\mathfrak{g}} + \chi_{(\mathfrak{g},\rho)}.$$

Remark 1.3 Let $(\mathfrak{g},[-,-],\alpha)$ be a Hom-Lie Algebra and $ad : \mathfrak{g}\to g\ell(\mathfrak{g})$ be the adjoint representation. We mean by $\mathfrak{g}\ltimes_{ad}\mathfrak{g}$ the Hom-Lie algebra of semi-direct product associated to the adjoint representation. then

$$\chi\,(\mathfrak{q}) = 2\chi\,(\mathfrak{g}).$$

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Chapter 2
On Ternary (Hom-)Nambu-Poisson Algebras

Hanene Amri and Abdenacer Makhlouf

Abstract In this paper we provide a procedure to construct ternary Nambu-Poisson algebras (resp. ternary Hom-Nambu-Poisson algebras) from Poisson algebras (resp. Hom-Poisson algebras) equipped with a trace function satisfying some conditions. Therefore, we give various examples of ternary Nambu-Poisson algebras (resp. ternary Hom-Nambu-Poisson algebras) using this construction.

Keywords Hom-Nambu-Poisson algebra

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2.1 Introduction

Poisson structures appear in various areas of different contexts, ranging from string theory, classical/quantum mechanics, differential geometry, abstract algebra, algebraic geometry and representation theory. In each one of these contexts, it turns out that the Poisson structure is not a theoretical artifact, but a key element which, unsolicited, comes along with the problem which is investigated and its delicate properties are in basically all cases decisive for the solution to the problem. Siméon Denis Poisson announced in 1809 that he had found an improvement in the theory of Lagrangian mechanics, which was being developed by Joseph-Louis Lagrange and Pierre-Simon Laplace. In that pioneering paper, Poisson introduced the notation

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$$(a, b) = \sum_{i=1}^{n} (\frac{\partial a}{\partial q_i} \frac{\partial b}{\partial p_i} - \frac{\partial a}{\partial p_i} \frac{\partial b}{\partial q_i}) \tag{2.1}$$

where a and b are two functions of the coordinates q_i and the conjugate quantities $p_i = \frac{\partial R}{\partial \dot{q}_i}$ for a mechanical system with Lagrangian function R. He proved that, if a and b are first integrals of the system, then (a, b) also is. This (a, b) is nowadays denoted by $\{a, b\}$ and called the Poisson bracket of a and b. Mathematicians of the 19th century already recognized the importance of this bracket. In particular, William Hamilton used it extensively to express his equations in an essay in 1835 on what we now call Hamiltonian dynamics. Carl Jacobi in his "Vorlesungen *ü*ber Dynamik" around 1842 showed that the Poisson bracket satisfies the famous Jacobi identity:

$$\{\{a, b\}, c\} + \{\{b, c\}, a\} + \{\{c, a\}, b\} = 0. \tag{2.2}$$

This important identity leads to the definition of a Poisson algebra as an algebra A equipped with a skew-symmetric binary bracket $\{\cdot, \cdot\} : A \times A \to A$, satisfying (2.2) for all a, b, c in A. In other words, a Poisson algebra is a Lie algebra $(A, \{\cdot, \cdot\})$, where $\{\cdot, \cdot\}$, with a bilinear associative commutative map $\mu : A \times A \to A$, satisfy the Leibniz rule $\{\mu(a, b), c\} = \mu(a, \{b, c\}) + \mu(\{a, c\}, b)$ for all a, b, c in A.

$n-$ary generalizations of Poisson structures go under the name of Nambu structures. Indeed the first instances appeared in the work of the physicists Nambu [23], which was considered from the algebraic point of view by Takhtajan in [27]. Nambu proposed a generalized Hamiltonian system based on a ternary product, the Nambu-Poisson bracket, which allows to use more that one hamiltonian. Quantization of Nambu-Poisson brackets were investigated in [16], it was presented in a novel approach of Zariski, this quantization is based on the factorization on $\mathbb{R}$ of polynomials of several variables.

A twisted generalization, called Hom-Nambu algebras, was introduced in [7]. This kind of algebras called Hom-algebras appeared as deformations of algebras of vector fields using σ-derivations. The first examples dealt with q-deformations of Witt and Virasoro algebras. Then Hartwig, Larsson and Silvestrov introduced a general framework and studied Hom-Lie algebras [19], in which Jacobi identity is twisted by a homomorphism. The corresponding associative algebras, called Hom-associative algebras were introduced in [21]. Non-commutative Hom-Poisson algebras were discussed in [30]. Ternary Hom-Nambu-Poisson algebras (non-commutative) were studied in [3]. Likewise, n-ary algebras of Hom-type were introduced in [7], see also [1, 2, 4, 28, 29]. The theory of induced ternary (Hom)-Nambu algebras by Hom-Lie algebras and more generally induced $(n + 1)$-ary algebra by n-ary Nambu algebras was developed in [4, 5, 20].

The aim of this paper is to construct ternary structures from binary structures, more precisely we will give a construction procedure of ternary Nambu-Poisson algebras from Poisson algebras. We explored examples in dimension 3, but it turns out there is no ternary Nambu-Poisson algebra which can be constructed from Poisson algebras. In dimension 4, we obtain several examples using Solvable Lie algebras classification [15]. The same procedure of construction is applied to construct ternary Hom-Nambu-Poisson algebras from Hom-Poisson algebras.

2.2 Ternary Nambu-Poisson Algebras Induced by Poisson Algebras

In this section we provide a way of constructing ternary Nambu-Poisson algebras from Poisson algebras. We recall some basic definitions.

Definition 2.1 A *Poisson algebra* is a triple $(A, \mu, \{\cdot, \cdot\})$ consisting of a $\mathbb{K}$-vector space A, a bilinear maps $\mu : A \times A \to A$ and a binary bracket $\{\cdot, \cdot\} : A \times A \to A$, such that

(i) (A, μ) is a commutative binary associative algebra,
(ii) $(A, \{\cdot, \cdot\})$ is a Lie algebra,
(iii) for all $x, y, z \in A$,

$$\{\mu(x, y), z\} = \mu(x, \{y, z\}) + \mu(\{x, z\}, y). \tag{2.3}$$

Condition (2.3) is called the Leibniz identity.

A non-commutative Poisson algebras is defined by the same axioms except the fact that the product is not assumed to be commutative.

Definition 2.2 A *ternary Nambu algebra* is a pair $(A, \{\cdot, \cdot, \cdot\})$ consisting of a $\mathbb{K}$-vector space A, and a trilinear map $\{\cdot, \cdot, \cdot\} : A \otimes A \otimes A \to A$ such that the following Nambu identity and skew-symmetry

$$\begin{aligned} \{x_1, x_2, \{x_3, x_4, x_5\}\} &= \{\{x_1, x_2, x_3\}, x_4, x_5\} + \{x_3, \{x_1, x_2, x_4\}, x_5\} \\ &\quad + \{x_3, x_4, \{x_1, x_2, x_5\}\}, \\ \{x_1, x_2, x_3\} &= sgn(\sigma)\{x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}\}, \quad \text{for all } \sigma \in \mathcal{S}_3 \end{aligned}$$

holds for all $x_1, x_2, x_3, x_4 \in A$.

Definition 2.3 A *ternary Nambu-Poisson algebra* is a triple $(A, \mu, \{\cdot, \cdot, \cdot\})$ consisting of a $\mathbb{K}$-vector space A, a bilinear map $\mu : A \times A \to A$ and a trilinear map $\{\cdot, \cdot, \cdot\} : A \otimes A \otimes A \to A$ such that

(i) (A, μ) is a commutative binary associative algebra,
(ii) $(A, \{\cdot, \cdot, \cdot\})$ is a ternary Nambu-Lie algebra,
(iii) the following Leibniz rule

$$\{x_1, x_2, \mu(x_3, x_4)\} = \mu(x_3, \{x_1, x_2, x_4\}) + \mu(\{x_1, x_2, x_3\}, x_4),$$

holds for all $x_1, x_2, x_3, x_4 \in A$.

A non-commutative ternary Nambu-Poisson algebras is defined by the same axioms except the fact that the product is not assumed to be commutative.

For simplicity, we choose the following notation for multiplication $x \cdot y$ instead of $\mu(x, y)$.

Definition 2.4 Let $\varphi : A^n \to A$ be an n-linear map and let $\tau : A \to \mathbb{K}$ be a linear map. We consider the map $\varphi_\tau : A^{n+1} \to A$ defined by

$$\varphi_\tau(x_1, ..., x_{n+1}) = \sum_{k=1}^{n+1}(-1)^k \tau(x_k)\varphi(x_1, ..., \widehat{x_k}, ..., x_{n+1}),$$

where $\widehat{x_k}$ means that x_k is excluded.

In particular

$$\{x, y, z\}_\tau = \tau(x)\{y, z\} + \tau(y)\{z, x\} + \tau(z)\{x, y\}.$$

We consider linear maps τ which have a generalized trace property.

Definition 2.5 We call a linear map $\tau : A \to \mathbb{K}$ a φ-trace (or trace function) if

$$\tau(\varphi(x_1, ..., x_n)) = 0,$$

for all $x_1, ..., x_n \in A$.

Lemma 2.1 *Let $\varphi : A^n \to A$ be a skew-symmetric n-linear map and $\tau : A \to \mathbb{K}$ a linear map. Then φ_τ is an $(n + 1)$-linear totally skew-symmetric map. Furthermore, if τ is a φ-trace map then τ is a φ_τ-trace map.*

We have the following construction procedure that allow to obtain ternary Nambu-Poisson algebras from binary brackets of Poisson algebras and trace functions.

Theorem 2.1 *Let $(A, \cdot, \{\cdot, \cdot\})$ be a unital Poisson algebra (resp. non-commutative unital Poisson algebra), assume that τ is a $\{\cdot, \cdot\}$-trace map on A, i.e. $\tau(\{x, y\}) = 0$ for all $x, y \in A$. Then $(A, \cdot, \{\cdot, \cdot, \cdot\}_\tau)$ is a ternary Nambu-Poisson algebra if and only if*

$$(\tau(x \cdot y)\mathbb{1} - \tau(y)x - \tau(x)y) \cdot (\{z, u\}) = 0. \tag{2.4}$$

We say that the ternary Nambu-Poisson algebra $(A, \cdot, \{\cdot, \cdot, \cdot\}_\tau)$ is induced by the Poisson algebra $(A, \cdot, \{\cdot, \cdot\})$.

Proof ([4, 20]) The ternary brackets $\{\cdot, \cdot, \cdot\}_\tau$ is skew-symmetric from Lemma 2.1, then we only have to prove that the Nambu identity and Leibniz identity are satisfied. First developing Nambu identity

$$\{x, y, \{z, u, v\}_\tau\}_\tau - \{\{x, y, z\}_\tau, u, v\}_\tau - \{z, \{x, y, u\}_\tau, v\}_\tau - \{z, u, \{x, y, v\}_\tau\}_\tau = 0,$$

gives 36 terms, 12 of these vanishes because τ is a trace function, $\tau(\{x, y\}) = 0$ for all $x, y \in A$. The remaining following 18 terms

$$\begin{aligned}
&\tau(x)\tau(z)(\{y, \{u, v\}\} - \{\{y, u\}, v\} - \{u, \{y, v\}\}) + \\
&\tau(y)\tau(z)(\{\{u, v\}, x\} - \{\{u, x\}, v\} - \{u, \{v, x\}\}) + \\
&\tau(x)\tau(u)(\{y, \{v, z\}\} - \{v, \{y, z\}\} - \{\{y, v\}, z\}) + \\
&\tau(y)\tau(u)(\{\{v, z\}, x\} - \{v, \{z, x\}\} - \{\{v, x\}, z\}) + \\
&\tau(x)\tau(v)(\{y, \{z, u\}\} - \{\{y, z\}, u\} - \{z, \{y, u\}\}) + \\
&\tau(y)\tau(v)(\{\{z, u\}, x\} - \{\{z, x\}, u\} - \{z, \{u, x\}\})
\end{aligned}$$

vanish by using Jacobi identity. The remaining 6 terms which can be written as

$$\begin{aligned}
&\tau(u)\tau(z)(\{v, \{x, y\}\} - \{\{x, y\}, v\}) + \\
&\tau(u)\tau(v)(\{\{x, y\}, z\} - \{z, \{x, y\}\}) + \\
&\tau(v)\tau(z)(\{\{x, y\}, u\} - \{u, \{x, y\}\})
\end{aligned}$$

vanish using the skew-symmetry of the brackets $\{\cdot, \cdot\}$. Hence the Nambu identity is satisfied.

Secondly we shall prove the Leibniz identity which is written as

$$\{x \cdot y, z, u\}_\tau = x \cdot \{y, z, u\}_\tau + \{x, z, u\}_\tau \cdot y.$$

By expanding, the left hand side is

$$\begin{aligned}
LHS &= \tau(x \cdot y)\{z, u\} + \tau(z)\{u, x \cdot y\} + \tau(u)\{x \cdot y, z\} \\
&= \tau(x \cdot y)\{z, u\} + \tau(z)(x \cdot \{u, y\} + \{u, x\} \cdot y) \\
&+ \tau(u)(x \cdot \{y, z\} + \{x, z\} \cdot y)) \\
&= \tau(x \cdot y)\{z, u\} + \tau(z)x \cdot \{u, y\} + \tau(z)\{u, x\} \cdot y \\
&+ \tau(u)x \cdot \{y, z\} + \tau(u)\{x, z\} \cdot y,
\end{aligned}$$

and the right hand side is

$$\begin{aligned}
RHS &= x \cdot \tau(y)\{z, u\} + x \cdot \tau(z)\{u, y\} + x \cdot \tau(u)\{y, z\} \\
&+ \tau(x)\{z, u\} \cdot y + \tau(z)\{u, x\} \cdot y + \tau(u)\{x, z\} \cdot y.
\end{aligned}$$

Hence, Leibniz identity with respect to ternary bracket is satisfied if and only if condition (2.4) holds.

Remark 2.1 If the algebra $(A, \cdot)$ is noncommutative, then Condition (2.4) should write

$$\tau(x \cdot y)\{z, u\} = \tau(y)x \cdot \{z, u\} + \tau(x)\{z, u\} \cdot y, \tag{2.5}$$

for all $x, y, z, u \in A$.

Corollary 2.1 *If $(A, \cdot)$ is unital and there exist $z \in \{A, A\}$ such that $x \cdot z \neq 0$ for all $x \in A$, then*

$$\tau(x \cdot y)\mathbb{1} - \tau(y)x - \tau(x)y = 0. \tag{2.6}$$

Proposition 2.1 *Let $(A, \cdot, \{\cdot, \cdot\})$ be a Poisson algebra and τ be a $\{\cdot, \cdot\}$-trace map. If there exists $z \in \{A, A\}$ such that $x \cdot z \neq 0$ for all $x \in A$, and if in addition $(A, \cdot)$ is unital, then $\tau(x) = 0$ for all $x \in A$.*

Proof Since $(A, \cdot)$ is a unital commutative algebra and there exists $z \in \{A, A\}$ such that $x \cdot z \neq 0$ for all $x \in A$, then

$$(2.4) \Leftrightarrow \tau(x \cdot y)\mathbb{1} - \tau(y)x - \tau(x)y = 0.$$

Let $x \neq 1$ and set $y = x$, then

$$\tau(x \cdot x)\mathbb{1} - 2\tau(x)x = 0.$$

Since x and $\mathbb{1}$ are linearly independent, it follows $\tau(x \cdot x) = 0$ and $\tau(x) = 0$ for all x in A.

Proposition 2.2 *Let $(A, \cdot, \{\cdot, \cdot\})$ be a Poisson algebra which is unital, τ be a $\{\cdot, \cdot\}$-trace map and there exists $z \in \{A, A\}$ such that $x \cdot z \neq 0$ for all $x \in A$. If e is an idempotent non proportional to $\mathbb{1}$, then $\tau(e) = 0$.*

Proof Let $(A, \cdot, \{\cdot, \cdot\})$ be a Poisson algebra, τ be a $\{\cdot, \cdot\}$-trace map and there exists $z \in \{A, A\}$ such that $x \cdot z \neq 0$ for all $x \in A$. Then from (2.1) and Corollary 2.1

$$\tau(x \cdot y)\mathbb{1} - \tau(y)x - \tau(x)y = 0. \tag{2.7}$$

If e is an idempotent i.e. $e^2 = e$, then (2.7) may be written as

$$\tau(e \cdot e)\mathbb{1} - \tau(e)e - \tau(e)e = 0,$$

Since

$$\tau(e)\mathbb{1} - 2\tau(e)e = 0,$$

we get

$$\tau(e)(\mathbb{1} - 2e) = 0,$$

which imply $\tau(e) = 0$.

Remark 2.2 In dimension 3, $\tau(x) = 0$ for all $x \in A$, that is all induced ternary Nambu-Poisson algebras from 3-dimensional Poisson algebras are trivial.

2.2.1 Examples

In this section we provide two examples of ternary Nambu-Poisson algebras induced by Poisson algebras using Theorem 2.1. These examples are 5-dimensional and 6-dimensional respectively. The construction of binary Poisson algebras were based on the classification of nilpotent Lie algebras in dimension five and six [14].

Example 2.1 Let $(A, \cdot, \{\cdot, \cdot\})$ be a Poisson algebra defined over a 5-dimensional vector space A spanned by $\{e_1, e_2, e_3, e_4, e_5\}$. The binary bracket which is skew-symmetric is defined by

$$\{e_1, e_2\} = e_3, \quad \{e_1, e_3\} = e_5, \quad \{e_2, e_4\} = e_5,$$

and the commutative multiplication is defined by

$$\begin{aligned} e_1 \cdot e_1 &= ae_5, \quad e_1 \cdot e_2 = be_5, \quad e_1 \cdot e_4 = ce_5, \\ e_2 \cdot e_2 &= de_5, \quad e_2 \cdot e_4 = fe_5, \quad e_4 \cdot e_4 = ge_5, \end{aligned}$$

where a, b, c, d, f, g are parameters in $\mathbb{K}$. The other products (resp. brackets) are obtained by commutativity (resp. skew-symmetry) or are equal to zero. Defining

$$\tau(e_1) = \gamma_1, \tau(e_2) = \gamma_2, \tau(e_2) = \gamma_3, \tau(e_3) = \tau(e_5) = 0,$$

for any $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{K}$, condition (2.4) is satisfied. Thus according to Theorem 2.1, we obtain a ternary Nambu-Poisson algebra defined by the following ternary brackets

$$\begin{aligned} \{e_2, e_1, e_3\}_\tau &= \gamma_2 e_5, \\ \{e_1, e_2, e_4\}_\tau &= \gamma_1 e_5 + \gamma_3 e_3, \\ \{e_1, e_3, e_4\}_\tau &= \gamma_3 e_5. \end{aligned}$$

Other brackets are obtained by skew-symmetry or are zero. Then $(A, \cdot, \{\cdot, \cdot, \cdot\}_\tau)$ is the induced ternary Nambu-Poisson algebra.

Example 2.2 Let $(A, \cdot, \{\cdot, \cdot\})$ be a Poisson algebra defined over a 6-dimensional vector space A spanned by $\{e_1, e_2, e_3, e_4, e_5, e_6\}$. The binary bracket which is skew-symmetric is defined by

$$\begin{aligned} \{e_1, e_2\} = e_3, &\quad \{e_1, e_3\} = e_4, \\ \{e_1, e_4\} = e_6, &\quad \{e_2, e_3\} = e_6, \\ \{e_2, e_5\} &= e_6, \end{aligned}$$

and the commutative multiplication is defined by

$$e_1 \cdot e_1 = ae_6, \quad e_1 \cdot e_2 = be_6, \quad e_1 \cdot e_5 = ce_6,$$
$$e_2 \cdot e_2 = de_6, \quad e_2 \cdot e_5 = fe_6, \quad e_5 \cdot e_5 = ge_6,$$

where a, b, c, d, f, g are parameters in $\mathbb{K}$. The other products (resp. brackets) are obtained by commutativity (resp. skew-symmetry) or are equal to zero.

Defining $\tau(e_1) = \gamma_1, \tau(e_2) = \gamma_2, \tau(e_5) = \gamma_3, \tau(e_3) = \tau(e_4) = \tau(e_6) = 0,$ for any $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{K}$, the condition (2.4) is satisfied. Thus, according to Theorem 2.1 we obtain a ternary Nambu-Poisson algebra defined by the following ternary brackets

$$\{e_1, e_2, e_3\}_\tau = \gamma_1 e_6 - \gamma_2 e_4,$$
$$\{e_1, e_2, e_4\}_\tau = -\gamma_2 e_6,$$
$$\{e_1, e_2, e_5\}_\tau = \gamma_1 e_6 + \gamma_3 e_3,$$
$$\{e_1, e_3, e_5\}_\tau = \gamma_3 e_4,$$
$$\{e_1, e_4, e_5\}_\tau = \gamma_3 e_6,$$
$$\{e_2, e_3, e_5\}_\tau = \gamma_3 e_6.$$

The other brackets are obtained by skew-symmetry or are equal to zero.

Then $(A, \cdot, \{\cdot, \cdot, \cdot\}_\tau)$ is the induced ternary Nambu-Poisson algebra.

We have also found an example of a non-commutative ternary Nambu-Poisson algebra induced by a non-commutative Poisson algebra .

Example 2.3 Let $(A, \cdot, \{\cdot, \cdot\})$ be a non-commutative Poisson algebra, defined over a 3-dimensional vector space A spanned by $\{e_1, e_2, e_3\}$. The binary bracket which is skew-symmetric is defined by

$$\{e_3, e_4\} = e_1, \quad \{e_1, e_3\} = e_2,$$

and the non-commutative multiplication is defined by

$$e_4 \cdot e_3 = ae_2, \quad e_4 \cdot e_4 = be_2,$$

where a, b are parameters in $\mathbb{K}$. The other brackets are obtained by skew-symmetry or are equal to zero. The other product are equal to zero. Defining

$$\tau(e_1) = \tau(e_2) = 0, \tau(e_3) = \gamma_1, \tau(e_4) = \gamma_2,$$

for all $\gamma_1, \gamma_2 \in \mathbb{K}$ the condition (2.4) is satisfied. Thus, according to Theorem 2.1, we obtain a ternary Nambu-Poisson algebra defined by the following ternary brackets

$$\{e_1, e_3, e_4\}_\tau = \gamma_2 e_2.$$

The other brackets are obtained by skew-symmetry or are equal to zero. Then $(A, \cdot, \{\cdot, \cdot, \cdot\}_\tau)$ is the induced ternary Nambu-Poisson algebra.

2.2.2 *Constructing Poisson and Ternary Nambu-Poisson Algebras from Solvable Lie Algebras*

We consider the 4-dimensional Lie algebra with basis $\{e_1, e_2, e_3, e_4\}$ brackets given by

$$\{e_1, e_4\} = ae_3, \quad \{e_3, e_4\} = e_1, \quad \{e_1, e_3\} = e_2,$$

where a is a parameter in $\mathbb{K}$, with all the other brackets obtained by skew-symmetry or are equal to zero. The products given a structure of Poisson algebra are of the form

with $a = 0$	with $a \neq 0$
$e_3 \cdot e_3 = \alpha e_2$	$e_4 \cdot e_4 = \alpha e_2$
$e_3 \cdot e_4 = \beta e_2$	
$e_4 \cdot e_4 = \gamma e_2$	

where α, β, γ are parameters in $\mathbb{K}$.

The other products are obtained by commutativity or are equal to zero. We set

$$\tau(e_1) = \tau(e_2) = \tau(e_3) = 0, \tau(e_4) = \gamma_4,$$

for all $\gamma_4 \in \mathbb{K}$. Using Theorem 2.1, we obtain the ternary Nambu-Poisson algebra defined by the following ternary bracket

$$\{e_1, e_3, e_4\}_\tau = \gamma_4 e_2.$$

The other brackets are obtained by skew-symmetry or are equal to zero.

2.3 Ternary Hom-Nambu-Poisson Algebras Induced by Hom-Poisson Algebras

In this section we consider the Hom-case of Poisson algebras. We provide a result which allows to construct ternary Hom-Nambu-Poisson algebras induced by Hom-Poisson algebras.

Definition 2.6 A Hom-associative algebra is a triple (A, μ, α), where A is a vector space, $\mu : A \times A \to A$ and $\alpha : A \to A$ are maps satisfying the Hom-associativity condition, that is

$$\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z)) \text{ for all } x, y, z \in A.$$

Definition 2.7 A Hom-Lie algebra is a triple $(A, [\cdot, \cdot], \alpha)$, where A is a vector space, $[\cdot, \cdot] : A \times A \to A$ and $\alpha : A \to A$ are maps such that the bracket is skew-symmetric and satisfying the Hom-Jacobi condition, that is

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0 \text{ for all } x, y, z \in A.$$

Remark 2.3 When α is the identity map, we recover the classical associativity and Jacobi conditions, then usual associative and Lie algebras.

Definition 2.8 A *Hom-Poisson algebra* is a quadruple $(A, \mu, \{\cdot, \cdot\}, \alpha)$ consisting of a vector space A, bilinear maps $\mu : A \times A \to A$ and $\{\cdot, \cdot\} : A \times A \to A$, and a linear map $\alpha : A \to A$ such that:

(i) (A, μ, α) is a commutative binary Hom-associative algebra,
(ii) $(A, \{\cdot, \cdot\}, \alpha)$ is a Hom-Lie algebra,
(iii) and for all $x, y, z \in A$,

$$\{\mu(x, y), \alpha(z)\} = \mu(\alpha(x)\{y, z\}) + \mu(\{x, z\}\alpha(y)). \tag{2.8}$$

The third condition is called Hom-Leibniz identity.

A non-commutative Hom-Poisson algebra is given by the same definition except the fact that the product is not assumed to be commutative.

A Hom-Poisson algebra $(A, \mu, \{\cdot, \cdot\}, \alpha)$ is said multiplicative if

$$\alpha(\{x, y\}) = \{\alpha(x), \alpha(y)\} \text{ and } \alpha \circ \mu = \mu \circ \alpha^{\otimes 2}.$$

Remark 2.4 When the Hom-algebra is multiplicative, we also say that α is multiplicative with respect to μ or an algebra morphism.

Remark 2.5 We recover the classical Poisson algebras when $\alpha = Id$.

The following construction, called Yau Twist, provides a way to construct a Hom-Poisson algebra using a Poisson algebra and a Poisson algebra morphism.

Theorem 2.2 *Let* $A = (A, \mu, \{\cdot, \cdot\})$ *be a Poisson algebra and* $\alpha : A \to A$ *be a linear map which is a Poisson algebra morphism. Then*

$$A_\alpha = (A, \mu_\alpha = \alpha \circ \mu, \{\cdot, \cdot\}_\alpha = \alpha \circ \{\cdot, \cdot\}, \alpha)$$

is a Hom-Poisson algebra.

Definition 2.9 ([7]) A *ternary Hom-Nambu algebra* is a triple $(A, \{\cdot, \cdot, \cdot\}, \widetilde{\alpha})$ consisting of a $\mathbb{K}$-vector space A, a ternary map $\{\cdot, \cdot, \cdot\} : A \times A \times A \to A$ and a pair of linear maps $\widetilde{\alpha} = (\alpha_1, \alpha_2)$ where $\alpha_1, \alpha_2 : A \to A$, satisfying

$$\{\alpha_1(x_1), \alpha_2(x_2), \{x_3, x_4, x_5\}\} = \{\{x_1, x_2, x_3\}, \alpha_1(x_4), \alpha_2(x_5)\}+ \\ \{\alpha_1(x_3), \{x_1, x_2, x_4\}, \alpha_2(x_5)\} + \{\alpha_1(x_3), \alpha_2(x_4), \{x_1, x_2, x_5\}\}. \tag{2.9}$$

We call the above condition the ternary Hom-Nambu identity.

Definition 2.10 A *ternary Hom-Nambu-Poisson algebra* $(A, \mu, \beta, \{\cdot, \cdot, \cdot\}, \widetilde{\alpha})$ is a tuple consisting of a vector space A, a ternary operation $\{\cdot, \cdot, \cdot\} : A \times A \times A \to A$, a binary operation $\mu : A \times A \to A$, a pair of linear maps $\widetilde{\alpha} = (\alpha_1, \alpha_2)$ where $\alpha_1, \alpha_2 : A \to A$, and a linear map $\beta : A \to A$ such that:

(i) (A, μ, β) is a binary commutative Hom-associative algebra,
(ii) $(A, \{\cdot, \cdot, \cdot\}, \widetilde{\alpha})$ is a ternary Hom-Nambu-Lie algebra,
(iii) $\{\mu(x_1, x_2), \alpha_1(x_3), \alpha_2(x_4)\} = \mu(\beta(x_1), \{x_2, x_3, x_4\}) + \mu(\{x_1, x_3, x_4\}, \beta(x_2))$.

A non-commutative Hom-Nambu-Poisson algebras is defined with same axioms except the fact that the product is non-commutative.

Remark 2.6 We recover the classical ternary Nambu-Poisson algebra when $\alpha_1 = \alpha_2 = \beta = Id$.

In the sequel we will mainly be interested in the class of ternary Hom-Nambu-Poisson algebras where $\alpha = \alpha_1 = \alpha_2 = \beta$, for which we refer by a quadruple $(A, \mu, \{\cdot, \cdot, \cdot\}, \alpha)$.

Definition 2.11 Let $(A, \mu, \{\cdot, \cdot, \cdot\}, \alpha)$ be a ternary Hom-Nambu-Poisson algebra. It is said to be *multiplicative* if

$$\alpha(\{x_1, x_2, x_3\}) = \{\alpha(x_1), \alpha(x_2), \alpha(x_3)\},$$
$$\alpha \circ \mu = \mu \circ \alpha^{\otimes 2}.$$

If in addition α is bijective then it is called *regular*.

Definition 2.12 Let $(A, \mu, \{\cdot, \cdot, \cdot\}, \alpha)$ and $(A', \mu', \{\cdot, \cdot, \cdot\}', \alpha')$ be two ternary Hom-Nambu-Poisson algebras. A linear map $f : A \to A'$ is a *morphism* of ternary Hom-Nambu-Poisson algebras if it satisfies for all x_1, x_2, x_3 in A :

$$f(\{x_1, x_2, x_3\}) = \{f(x_1), f(x_2), f(x_3)\}', \tag{2.10}$$
$$f \circ \mu = \mu' \circ f^{\otimes 2}, \tag{2.11}$$
$$f \circ \alpha = \alpha' \circ f. \tag{2.12}$$

It said to be a *weak morphism* if hold only the two first conditions.

We derive a construction procedure of ternary Hom-Nambu-Poisson algebra from binary brackets of a Hom-Poisson algebras and a trace function satisfying some compatibility conditions.

Theorem 2.3 *Let* $(A, \cdot, \{\cdot, \cdot\}, \alpha)$ *be a Hom-Poisson algebra. Assume that* τ *is a* $\{\cdot, \cdot\}$*-trace on A satisfying*

$$\tau(\alpha(x))\tau(y) = \tau(x)\tau(\alpha(y)), \tag{2.13}$$

$$\tau(x \cdot y)\alpha(\{z, u\}) = \alpha(x) \cdot \tau(y)\{z, u\} + \tau(x)\{z, u\} \cdot \alpha(y), \tag{2.14}$$

$$\tau(\alpha(z)) = \tau(z), \tag{2.15}$$

for all $x, y, z, u \in A$.. Then $(A, \cdot, \{\cdot, \cdot, \cdot\}_\tau, \alpha)$ is a ternary Hom-Nambu-Poisson algebra, and we say that it is induced by the Hom-Poisson algebra.

Proof From Lemma 2.1, $\{\cdot, \cdot, \cdot\}_\tau$ is skew-symmetric and the Hom-Nambu identity is proved using condition 2.13, see [4] for the details. One only has to prove that the Hom-Leibniz identity is satisfied. The Hom-Leibniz identity is:

$$\{x \cdot y, \alpha(z), \alpha(u)\}_\tau = \alpha(x) \cdot \{y, z, u\}_\tau + \{x, z, u\}_\tau \cdot \alpha(y).$$

By developing the left hand side, we obtain

$$\begin{aligned} LHS &= \tau(x \cdot y)\{\alpha(z), \alpha(u)\} + \tau(\alpha(z))\{\alpha(u), x \cdot y\} + \tau(\alpha(u))\{x \cdot y, \alpha(z)\} \\ &= \tau(x \cdot y)\{\alpha(z), \alpha(u)\} + \tau(\alpha(z))(\alpha(x) \cdot \{u, y\} + \{u, x\} \cdot \alpha(y)) \\ &+ \tau(\alpha(u))(\alpha(x) \cdot \{y, z\} + \{x, z\} \cdot \alpha(y)) \\ &= \tau(x \cdot y)\alpha(\{z, u\}) + \tau(\alpha(z))\alpha(x) \cdot \{u, y\} + \tau(\alpha(z))\{u, x\} \cdot \alpha(y) \\ &+ \tau(\alpha(u))\alpha(x) \cdot \{y, z\} + \tau(\alpha(u))\{x, z\} \cdot \alpha(y), \end{aligned}$$

and the right hand side is

$$\begin{aligned} RHS &= \alpha(x) \cdot \tau(y)\{z, u\} + \alpha(x) \cdot \tau(z)\{u, y\} + \alpha(x) \cdot \tau(u)\{y, z\} \\ &\quad + \tau(x)\{z, u\} \cdot \alpha(y) + \tau(z)\{u, x\} \cdot \alpha(y) + \tau(u)\{x, z\} \cdot \alpha(y). \end{aligned}$$

Using (2.14) and (2.15), the terms on the RHS cancel with those of the LHS. Hence, the Hom-Leibniz identity for the ternary bracket is satisfied, if and only if conditions (2.14) and (2.15) holds.

2.3.1 Examples

Using solvable 4-dimensional Lie algebras [5, 15] and the twisting principal, we were able to construct 4-dimensional Hom-Poisson algebras. Therefore, using the method described in Theorem 2.3, we provide examples of ternary Hom-Nambu-Poisson algebras induced by Hom-Poisson algebras.

Example 2.4 Let $(A, \cdot, \{\cdot, \cdot\}, \alpha)$ be a Hom-Poisson algebra defined over a 4-dimensional vector space A spanned by $\{e_1, e_2, e_3, e_4\}$. The binary bracket which is skew-symmetric is defined by

$$\{e_2, e_4\} = f_2^2 e_3,$$

and the commutative multiplication is defined by

$$e_2 \cdot e_2 = a f_2^2 e_3, \quad e_2 \cdot e_4 = b f_2^2 e_3, \quad e_4 \cdot e_4 = c f_2^2 e_3,$$

where $a, b, c, f_i \in \mathbb{K}$ are parameters. The other products (resp. brackets) are obtained by commutativity (resp. skew-symmetry) or are equal to zero. The linear map is defined by

$$\begin{aligned} \alpha(e_1) &= e_1 + f_1 e_3, & \alpha(e_2) &= f_2 e_2 + f_3 e_3, \\ \alpha(e_3) &= f_2^2 e_3, & \alpha(e_4) &= f_4 e_3 + f_2 e_4, \end{aligned}$$

where f_1, f_2, f_3, f_4 are parameters in $\mathbb{K}$, with $f_2 \neq 0$.

Defining

$$\tau(e_1) = \gamma, \tau(e_2) = \tau(e_3) = \tau(e_4) = 0,$$

for any $\gamma \in \mathbb{K}$, the conditions (2.13), (2.14) and (2.15) are satisfied. Thus according to Theorem 2.3, we obtain a ternary Hom-Nambu-Poisson algebra defined by the following ternary bracket

$$\{e_1, e_2, e_4\}_\tau = \gamma f_2^2 e_3.$$

The other brackets are obtained by skew-symmetric or are equal to zero.

We say that $(A, \cdot, \{\cdot, \cdot, \cdot\}_\tau, \alpha)$ is the ternary Hom-Nambu-Poisson algebra induced by the Hom-Poisson algebra.

Example 2.5 Let $(A, \cdot, \{\cdot, \cdot\}, \alpha)$ be a Hom-Poisson algebra defined over a 4-dimensional vector space A spanned by $\{e_1, e_2, e_3, e_4\}$. The binary bracket which is skew-symmetric is defined by

$$\{e_3, e_4\} = \lambda e_1 + \beta e_2, \quad \{e_1, e_3\} = e_2,$$

for all $\lambda, \beta \in \mathbb{K}$, and the commutative multiplication is defined by

$$e_3 \cdot e_3 = a e_2, \quad e_4 \cdot e_4 = b e_2,$$

for all $a, b \in \mathbb{K}$. The other products (resp. brackets) are obtained by commutativity (resp. skew-symmetric) or are equal to zero. We set also

$$\begin{aligned} \alpha(e_1) &= \lambda e_1 + \beta e_2, & \alpha(e_2) &= e_2, \\ \alpha(e_3) &= \alpha e_1 + \delta e_2 + \lambda e_3, & \alpha(e_4) &= -\rho\lambda e_1 + \omega e_2 + e_4, \end{aligned}$$

where $\lambda, \beta, \delta, \rho, \omega$ are parameters in $\mathbb{K}$. Defining

$$\tau(e_1) = \tau(e_2) = \tau(e_3) = 0, \tau(e_4) = \gamma,$$

for any $\gamma \in \mathbb{K}$, conditions (2.13), (2.14) and (2.15) are satisfied. Thus according to Theorem 2.3, we obtain a ternary Hom-Nambu-Poisson algebra defined by the following ternary brackets

$$\{e_1, e_3, e_4\}_\tau = \gamma e_2.$$

The other brackets are obtained by skew-symmetric or are equal to zero.

Now we consider the 3-dimensional Lie algebra twisting $\mathfrak{sl}(2)$ using Jackson derivation.

Example 2.6 (Jackson $\mathfrak{sl}(2)$) The Jackson $\mathfrak{sl}(2)$ is a q-deformation of the classical algebra $sl(2)$. It carries a Hom-Lie algebra structure but not a Lie algebra structure. It is defined with respect to a basis $\{x_1, x_2, x_3\}$ by the bracket and a linear map α such that

$$[x_1, x_2] = -2qx_2, \quad [x_1, x_3] = 2x_3, \quad [x_2, x_3] = -\frac{1+q}{2}x_1,$$

$$\alpha(x_1) = x_1, \quad \alpha(x_2) = \frac{q+1}{2}x_2, \quad \alpha(x_3) = \frac{q^{-1}+1}{2}x_3,$$

where q is a parameter in $\mathbb{K}$. If $q = 1$ we recover the classical $\mathfrak{sl}(2)$. This algebra is equipped with a non-commutative Hom-Poisson structure if $q = -1$ and it is given by the following non-commutative multiplication

$$x_1 \cdot x_2 = ax_1, \quad x_2 \cdot x_1 = -ax_2, \quad x_1 \cdot x_3 = ax_3,$$

where a is a parameter in $\mathbb{K}$. If $q \neq -1$ the multiplication is null. From this non-commutative Hom-Poisson algebra, we cannot construct a ternary bracket because the condition of Theorem 2.3 is not satisfied for $q = -1$.

Theorem 2.4 *Let $(A, \cdot, \{\cdot, \cdot, \cdot\}_\tau)$ be a ternary Nambu-Poisson algebra induced by a Poisson algebra $(A, \cdot, \{\cdot, \cdot\})$, and α a Poisson algebra endomorphism $\alpha : A \to A$ i.e. $\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)$ and $\alpha \circ \{x, y, z\}_\tau = \{\alpha(x), \alpha(y), \alpha(z)\}_\tau$. Then $(A, \cdot_\alpha, \{\cdot, \cdot, \cdot\}_{\tau,\alpha}, \alpha)$ is a Hom-Nambu-Poisson algebra.*

Proof See [3].

Example 2.7 Let $(A, \cdot, \{\cdot, \cdot\})$ be a Poisson algebra defined over a 4-dimensional vector space A spanned by $\{e_1, e_2, e_3, e_4\}$. The binary bracket which is skew-symmetric is defined by

$$\{e_3, e_4\} = e_1, \quad \{e_1, e_3\} = e_2,$$

and the commutative multiplication by

$$e_3 \cdot e_3 = ae_2, \quad e_3 \cdot e_4 = be_2, \quad e_4 \cdot e_4 = ce_2,$$

for all $a, b, c \in \mathbb{K}$. The other products (resp. brackets) are obtained by commutativity (resp. skew-symmetry) or are equal to zero. The endomorphism α is defined by

$$\begin{aligned} \alpha(e_1) &= e_1 + a_1 e_2, & \alpha(e_2) &= e_2, \\ \alpha(e_3) &= a_2 e_1 + a_3 e_2 + e_3, & \alpha(e_4) &= -a_1 e_1 + a_4 e_2 + e_4, \end{aligned}$$

where a_1, a_2, a_3, a_4 are parameters in $\mathbb{K}$. Hence, $(A, \cdot, \{\cdot, \cdot\}, \alpha)$ is a Hom-Poisson algebra. We set

$$\tau(e_1) = \tau(e_2) = \tau(e_3) = \gamma_1, \tau(e_4) = \gamma_2,$$

for any $\gamma_1, \gamma_2 \in \mathbb{K}$. Using Theorems 2.1 and 2.3, we obtain a ternary Nambu-Poisson algebra $(A, \cdot, \{\cdot, \cdot, \cdot\}_\tau)$ and a ternary Hom-Nambu-Poisson algebra $(A, \cdot, \{\cdot, \cdot, \cdot\}_\tau, \alpha)$ respectively defined by the following ternary brackets

$$\{e_1, e_3, e_4\}_\tau = \gamma_2 e_2.$$

The other brackets are obtained by skew-symmetry or are equal to zero.

The following diagram expresses the relationships between Twisting principle and the construction of induced ternary Hom-Nambu-Poisson algebras:

$$\begin{array}{ccc} (A, \cdot, \{\cdot, \cdot\}) & \xrightarrow{\alpha} & (A, \cdot, \{\cdot, \cdot\}, \alpha) \\ \tau \downarrow & & \downarrow \tau \\ (A, \cdot, \{\cdot, \cdot, \cdot\}_\tau) & \xrightarrow[\alpha]{} & (A, \cdot, \{\cdot, \cdot, \cdot\}_\tau, \alpha) \end{array}$$

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Chapter 3
Classification, Centroids and Derivations of Two-Dimensional Hom-Leibniz Algebras

Anja Arfa, Nejib Saadaoui, and Sergei Silvestrov

Abstract Several recent results concerning Hom-Leibniz algebra are reviewed, the notion of symmetric Hom-Leibniz superalgebra is introduced and some properties are obtained. Classification of 2-dimensional Hom-Leibniz algebras is provided. Centroids and derivations of multiplicative Hom-Leibniz algebras are considered including the detailed study of 2-dimensional Hom-Leibniz algebras.

Keywords Hom-Lie superalgebra · Hom-Leibniz superalgebra · Centroid · Derivation

MSC2020 Classification 17B61 · 17D30

3.1 Introduction

Hom-Lie algebras and quasi-Hom-Lie algebras were introduced first in 2003 in [45] where a general method for construction of deformations and discretizations of Lie algebras of vector fields based on twisted derivations obeying twisted Leibniz rule was developed motivated by the examples of q-deformed Jacobi identities in

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q-deformations of Witt and Visaroro and in related q-deformed algebras discovered in 1990'th in string theory, vertex models of conformal field theory, quantum field theory and quantum mechanics, and also in development of q-deformed differential calculi and q-deformed homological algebra [4, 38–42, 46, 48, 59–61]. The central extensions and cocycle conditions for general quasi-Hom-Lie algebras and Hom-Lie algebras, generalizing in particular q-deformed Witt and Virasoro algebras, have been first considered in [45, 55] and for graded color quasi-Hom-Lie algebras in [74]. General quasi-Lie and quasi-Leibniz algebras introduced in [56] and color quasi-Lie and color quasi-Leibniz algebras introduced in [57] in 2005, include the Hom-Lie algebras, the quasi-Hom-Lie algebras as well as the color Hom-Lie algebras, quasi-Hom-Lie color algebras, quasi-Hom-Lie superalgebras and Hom-Lie superalgebras, and color quasi-Leibniz algebras, quasi-Leibniz superalgebras, quasi-Hom-Leibniz superalgebras and Hom-Leibniz algebras. Hom-Lie algebras and Hom-Lie superalgebras and more general color quasi-Lie algebras interpolate on the fundamental level of defining identities between Lie algebras, Lie superalgebras, color Lie algebras and related non-associative structures and their deformations, quantum deformations and discretizations, and thus might be useful tool for unification of methods and models of classical and quantum physics, symmetry analysis and non-commutative geometry and computational methods and algorithms based on general non-uniform discretizations of differential and integral calculi. Binary Hom-algebra structures typically involve a bilinear binary operation and one or several linear unary operations twisting the defining identities of the structure in some special nontrivial ways, so that the original untwisted algebraic structures are recovered for the specific twisting linear maps. In quasi-Lie algebras and quasi-Hom-Lie algebras, the Jacobi identity contains in general six triple bracket terms twisted in special ways by families of linear maps, and the skew-symmetry is also in general twisted by a family of linear maps. Hom-Lie algebras is a subclass of quasi-Lie algebras with the bilinear product satisfying the Jacobi identity containing only three triple bracket terms twisted by a single linear map and the usual non-twisted skew-symmetry identity. Hom-Leibniz algebras arise when skew-symmetry is not required, while the Hom-Jacobi identity is written as Hom-Leibniz identity. Lie algebras and Leibniz algebras as a special case of Hom-Leibniz algebras are obtained for the trivial choice of the twisting linear map as the identity map on the underlying linear space. The Hom-Lie algebras, Hom-Lie superalgebras, Hom-Leibniz algebras and Hom-Leibniz superalgebras with twisting linear map different from the identity map, are rich and complicated algebraic structures with classifications, deformations, representations, morphisms, derivations and homological structures being fundamentally dependent on joint properties of the twisting maps as a unary operations and bilinear binary product intrinsically linked by Hom-Jacobi or Hom-Leibniz identities.

Hom-Lie admissible algebras have been considered first in [66], where the Hom-associative algebras and more general G-Hom-associative algebras including the Hom-Vinberg algebras (Hom-left symmetric algebras), Hom-pre-Lie algebras (Hom-right symmetric algebras), and some other new Hom-algebra structures have been introduced and shown to be Hom-Lie admissible, in the sense that the operation of commutator as new product in these Hom-algebras structures yields Hom-Lie algebras. Furthermore, in [66], flexible Hom-algebras and Hom-algebra generalizations of derivations and of adjoint derivations maps have been introduced, and the Hom-Leibniz algebras appeared for the first time, as an important special subclass of quasi-Leibniz algebras introduced in [56] in connection to general quasi-Lie algebras following the standard Loday's conventions for Leibniz algebras (i.e. right Loday algebras) [63]. In [66], moreover the investigation of classification of finite-dimensional Hom-Lie algebras have been initiated with construction of families of the low-dimensional Hom-Lie algebras. Since [45, 55–58, 66], Hom-algebra structures expanded into a popular area providing a new broad framework for establishing fundamental links between deformations and quantum deformations of associative algebras, various classes of non-associative algebras, super-algebras, color algebras, n-ary algebraic structures, non-commutative differential calculus and homological algebra constructions for associative and non-associative structures. Quadratic Hom-Lie algebras have been considered in [35] and representation theory, cohomology and homology theory of Hom-Lie algebras have been considered in [7, 8, 72, 78]. Investigation of color Hom-Lie algebras and Hom-Lie superalgebras and n-ary generalizations expanded [1–3, 5, 6, 9–27, 36, 44, 51–53, 62, 64, 65, 67, 68, 73–77, 80].

At the same time, in recent years, the theory and classification of Leibniz algebras, and also Leibniz superalgebras extending Leibniz algebras in a similar way as Lie superalgebras generalize Lie algebras, continued being actively investigated motivated in part by applications in Physics and by graded homological algebra structures of non-commutative differential calculi and non-commutative geometry [37, 47]. In [34], the description of centroid and derivations of low-dimensional Leibniz algebras using classification results is introduced. Note that a classification of 2-dimensional Leibniz algebras have been given by Loday in [63]. In dimension three there are fourteen isomorphism classes, and the list can be found in [31, 32, 71]. Furthermore, a classification of low-dimensional complex solvable Leibniz algebras can be found in [30, 31, 33, 49, 50, 50], and of two and three-dimensional complex Leibniz algebra is given in [29, 63].

The purpose of the present work is to study the classification of multiplicative 2-dimensional Hom-Leibniz algebras and to investigate centroids and derivations of Hom-Leibniz algebras and superalgebras and the concepts of left, right and symmetric (two-sided) Hom-Leibniz superalgebras. In Sect. 3.2, the left, right and symmetric (two-sided) Hom-Leibniz algebras and superalgebras generalizing the well known left, right and symmetric (two-sided) Leibniz algebras are defined and some of their properties are reviewed. In Section 3.5, we provide classification of multiplicative 2-dimensional Hom-Leibniz algebras. In Sect. 3.4, we review centroids and derivations of Hom-Leibniz superalgebras and some of their properties. In Sect. 3.6, we describe the algorithm to find centroids and derivations of Hom-Leibniz algebras, apply it to the classification of 2-dimensional Hom-Leibniz algebras from the previous section and use properties of the centroids of Hom-Leibniz algebras to categorize the algebra into having small and not small centroids.

3.2 Hom-Leibniz Algebras and Superalgebras

For exposition clarity, we assume throughout this article that all linear spaces are over a field $\mathbb{K}$ of characteristic different from 2, and just note that many of the results in this article hold as formulated or just with minor modifications for any field. Multilinear maps $f\colon V_1 \times \cdots \times V_n \to W$ on finite direct products and linear maps $F\colon V_1 \otimes \cdots \otimes V_n \to W$, on finite tensor products of linear spaces are identified via $F(v_1 \otimes \cdots \otimes v_n) = f(v_1, \ldots, v_n)$. A linear space V is said to be a G-graded by an abelian group G if $V = \bigoplus\limits_{g\in G} V_g$ for a family $\{V_g\}_{g\in G}$ of linear subspaces of V. For each $g \in G$, the elements of V_g are said to be homogeneous of degree $g \in G$, and the set of homogeneous elements is the union $\mathcal{H}(V) = \bigcup\limits_{g\in G} V_g$ all the spaces V_g of homogenous elements of degree g for all $g \in G$. For two G-graded linear spaces $V = \bigoplus\limits_{g\in G} V_g$ and $V' = \bigoplus\limits_{g\in G} V'_g$, a linear mapping $f : V \to V'$ is called homogeneous of degree d if $f(V_g) \subseteq V'_{g+d}$, for all $g \in G$. The homogeneous linear maps of degree zero (even maps) are those homogeneous linear maps satisfying $f(V_g) \subseteq V'_g$ for any $g \in G$. In the $\mathbb{Z}_2$-graded linear spaces $V = V_0 \oplus V_1$, also called superspaces, the elements of $V_{|j|}$, $|j| \in \mathbb{Z}_2$, are said to be homogenous of parity $|j|$. The space $End(V)$ is $\mathbb{Z}_2$-graded with a direct sum $End(V) = (End(V))_0 \oplus (End(V))_1$ where $(End(V))_{|j|} = \{f \in End(V) \mid f(V_i) \subset V_{i+j}\}$. The elements of $(End(V))$ are said to be homogenous of parity $|j|$.

An algebra $(A, \cdot)$ is called G-graded if its linear space is G-graded $A = \bigoplus\limits_{g\in G} A_g$, and $A_g \cdot A_h \subseteq A_{g+h}$ for all $g, h \in G$. Homomorphisms of G-graded algebras A and A' are homogeneous algebra morphisms of degree 0_G (even maps). Hom-superalgebras are triples (A, μ, α) consisting of a $\mathbb{Z}_2$-graded linear space (superspace) $A = A_0 \oplus A_1$, an even bilinear map $\mu : A \times A \to A$, and an even linear map $\alpha : A \to A$. An even linear map $f : A \to A'$ is said to be a weak morphism of hom-superalgebras if it is algebra structures homomorphism ($f \circ \mu = \mu' \circ (f \otimes f)$), and a morphism of hom-superalgebras if moreover $f \circ \alpha = \alpha' \circ f$. The important point for understanding of difference between algebras and Hom-algebras is that the properties preserved by the morphisms of specific Hom-algebras do not need to be preserved by all weak morphisms between these Hom-algebras, and the classifications of Hom-algebras up to weak isomorphisms (all algebra isomorphisms) and of Hom-algebras up to isomorphisms of Hom-algebras differ substantially in that the set of isomorphisms intertwining the twisting maps α and α' often is a proper subset of all isomorphisms, and thus classification of Hom-algebras up to isomorphism of Hom-algebras of different types typically contains much more isomorphism classes than weak isomorphism classes.

In any Hom-superalgebra $(A = A_0 \oplus A_1, \mu, \alpha)$,

$$\mu(A_0, A_0) \subseteq A_0, \quad \mu(A_1, A_0) \subseteq A_1, \quad \mu(A_0, A_1) \subseteq A_1, \quad \mu(A_1, A_1) \subseteq A_0.$$

Hom-subalgebras (graded Hom-subalgebras) of Hom-superalgebra (A, μ, α) are $\mathbb{Z}_2$-graded linear subspaces $I = (I \cap A_0) \oplus (I \cap A_1)$ of A obeying $\alpha(I) \subseteq I$ and $\mu(I, I) \subseteq I$. Hom-associator on Hom-superalgebra $(A = A_0 \oplus A_1, \mu, \alpha)$ is the even trilinear map $as_{\alpha,\mu} = \mu \circ (\mu \otimes \alpha - \alpha \otimes \mu) : A \times A \times A \to A$, acting on elements by $as_{\alpha,\mu}(x, y, z) = \mu(\mu(x, y), \alpha(z)) - \mu(\alpha(x), \mu(y, z))$, or $as_{\alpha,\mu}(x, y, z) = (xy)\alpha(z) - \alpha(x)(yz)$ in juxtaposition notation $xy = \mu(x, y)$. Since

$$|as_{\alpha,\mu}(x, y, z))| = |x| + |y| + |z|$$

for $x, y, z \in \mathcal{H}(A) = A_0 \cup A_1$, in any Hom-superalgebra $(A = A_0 \oplus A_1, \mu, \alpha)$ the following inclusions hold:

$$\begin{aligned} &as_{\alpha,\mu}(A_0, A_0, A_0) \subseteq A_0,\ as_{\alpha,\mu}(A_1, A_1, A_0) \subseteq A_0, \\ &as_{\alpha,\mu}(A_1, A_0, A_1) \subseteq A_0,\ as_{\alpha,\mu}(A_0, A_1, A_1) \subseteq A_0, \\ &as_{\alpha,\mu}(A_1, A_0, A_0) \subseteq A_1,\ as_{\alpha,\mu}(A_0, A_1, A_0) \subseteq A_1, \\ &as_{\alpha,\mu}(A_0, A_0, A_1) \subseteq A_1,\ as_{\alpha,\mu}(A_1, A_1, A_1) \subseteq A_1. \end{aligned}$$

Definition 3.1 ([45, 66]) Hom-Lie algebras are triples $(\mathcal{G}, [\cdot, \cdot], \alpha)$ consisting of a linear space $\mathcal{G}$ over a field $\mathbb{K}$, a bilinear map $[\cdot, \cdot] : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and a $\mathbb{K}$-linear map $\alpha : \mathcal{G} \to \mathcal{G}$ satisfying for all $x,\ y,\ z \in \mathcal{G}$,

$$[x, y] = -[y, x], \qquad \text{Skew-symmetry} \tag{3.1}$$

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0. \qquad \text{Hom-Jacobi identity} \tag{3.2}$$

Hom-Lie algebra is called a multiplicative Hom-Lie algebra if α is an algebra morphism, $\alpha([\cdot, \cdot]) = ([\alpha(.), \alpha(.)])$, meaning that $\alpha([x, y]) = [\alpha(x), \alpha(y)]$ for any $x, y \in \mathcal{G}$. Lie algebras are a very special subclass of multiplicative Hom-Lie algebras obtained for $\alpha = id$ in Definition 3.1.

Definition 3.2 ([5, 57]) Hom-Lie superalgebras are triples $(\mathcal{G}, [\cdot, \cdot], \alpha)$ which consist of $\mathbb{Z}_2$-graded linear space $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$, an even bilinear map $[\cdot, \cdot] : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and an even linear map $\alpha : \mathcal{G} \to \mathcal{G}$ satisfying the super skew-symmetry and Hom-Lie super Jacobi identities for homogeneous elements $x, y, z \in \mathcal{H}(\mathcal{G})$,

$$[x, y] = -(-1)^{|x||y|}[y, x], \qquad \text{Super skew-symmetry} \tag{3.3}$$

$$\begin{aligned} &(-1)^{|x||z|}[\alpha(x), [y, z]] + (-1)^{|y||x|}[\alpha(y), [z, x]] \\ &+ (-1)^{|z||y|}[\alpha(z), [x, y]] = 0. \ \text{Super Hom-Jacobi identity} \end{aligned} \tag{3.4}$$

Hom-Lie superalgebra is called multiplicative Hom-Lie superalgebra, if α is an algebra morphism, $\alpha([x, y]) = [\alpha(x), \alpha(y)]$ for any $x, y \in \mathcal{G}$.

Remark 3.1 In any Hom-Lie superalgebra, $(\mathcal{G}_0, [\cdot, \cdot], \alpha)$ is a Hom-Lie algebra since $[\mathcal{G}_0, \mathcal{G}_0] \in \mathcal{G}_0$ and $\alpha(\mathcal{G}_0) \in \mathcal{G}_0$ and $(-1)^{|a||b|} = (-1)^0 = 1$ for $a, b \in \mathcal{G}_0$. Thus, Hom-Lie algebras can be also seen as special class of Hom-Lie superalgebras when $\mathcal{G}_1 = \{0\}$.

Remark 3.2 From the point of view of Hom-superalgebras, Lie superalgebras is a special subclass of multiplicative Hom-Lie superalgebras obtained when $\alpha = id$ in Definition 3.2 which becomes the definition of Lie superalgebras as $\mathbb{Z}_2$-graded linear spaces $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$, with a graded Lie bracket $[\cdot,\cdot] : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ of degree zero, that is $[\cdot,\cdot]$ is a bilinear map obeying $[\mathcal{G}_i, \mathcal{G}_j] \subset \mathcal{G}_{i+j(mod2)}$, and for $x, y, z \in \mathcal{H}(\mathcal{G}) = \mathcal{G}_0 \cup \mathcal{G}_1$,

$$[x, y] = -(-1)^{|x||y|}[y, x], \qquad \text{Super skew-symmetry}$$
$$(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0. \qquad \text{Super Jacobi identity}$$

In super skew-symmetric superalgebras, the super Hom-Jacobi identity can be presented in the form of super Leibniz rule for the maps $ad_x = [x, \cdot] : \mathcal{G} \to \mathcal{G}$,

$$[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]].$$

Remark 3.3 Hom-Lie superalgebras are substantially different from Lie superalgebras, as all algebraic structure properties, morphisms, classifications and deformations become fundamentally dependent on the joint structure and properties of unary operation given by the linear map α and the bilinear product $[\cdot,\cdot]$ intricately linked via the α-twisted super-Jacobi identity (3.4).

In super skew-symmetric Hom-superalgebras, the super Hom-Jacobi identity can be presented in the form of super Hom-Leibniz rule for $ad_x = [x, \cdot] : \mathcal{G} \to \mathcal{G}$,

$$[\alpha(x), [y, z]] = [[x, y], \alpha(z)] + (-1)^{|x||y|}[\alpha(y), [x, z]] \qquad (3.5)$$

since, by super skew-symmetry (3.3), the following equalities are equivalent:

$$[\alpha(x), [y, z]] = [[x, y], \alpha(z)] + (-1)^{|x||y|}[\alpha(y), [x, z]],$$
$$[\alpha(x), [y, z]] - [[x, y], \alpha(z)] - (-1)^{|x||y|}[\alpha(y), [x, z]] = 0,$$
$$[\alpha(x), [y, z]] + (-1)^{|z|(|x|+|y|)}[\alpha(z), [x, y]] - (-1)^{|x||y|}[\alpha(y), [x, z]] = 0,$$
$$[\alpha(x), [y, z]] + (-1)^{|z|(|x|+|y|)}[\alpha(z), [x, y]] - (-1)^{|x||y|}[\alpha(y), [x, z]] = 0,$$
$$[\alpha(x), [y, z]] + (-1)^{|z|(|x|+|y|)}[\alpha(z), [x, y]] + (-1)^{|x||y|}(-1)^{|z||x|}[\alpha(y), [z, x]] = 0,$$
$$(-1)^{|z||x|}[\alpha(x), [y, z]] + (-1)^{|z||x|}(-1)^{|z|(|x|+|y|)}[\alpha(z), [x, y]] + (-1)^{|x||y|}[\alpha(y), [z, x]] = 0,$$
$$(-1)^{|x||z|}[\alpha(x), [y, z]] + (-1)^{|z||y|}[\alpha(z), [x, y]] + (-1)^{|y||x|}[\alpha(y), [z, x]] = 0,$$
$$(-1)^{|x||z|}[\alpha(x), [y, z]] + (-1)^{|z||y|}[\alpha(z), [x, y]] + (-1)^{|y||x|}[\alpha(y), [z, x]] = 0.$$

Remark 3.4 If skew-symmetry (3.1) does not hold, then (3.4) and (3.5) are not necessarily equivalent, defining different Hom-superalgebra structures. The Hom-superalgebras defined by just super algebras identity (3.5) without requiring super skew-symmetry on homogeneous elements are Leibniz Hom-superalgebras, a special class of general Γ-graded quasi-Leibniz algebras (color quasi-Leibniz algebras) first introduced in [56, 57].

3.3 Symmetric (Two-Sided) Hom-Leibniz Superalgebras

In this section, we define the left, right and symmetric (two-sided) Hom-Leibniz algebras Hom-Leibniz superalgebras and give examples of the symmetric Hom-Leibniz algebras and superalgebras. We also study some properties of centroids of Hom-Leibniz superalgebras.

With the skew-symmetry (3.1) satisfied, the Hom-Jacobi identity (3.2) can be presented in two equivalent ways, for all $x,\ y,\ z \in \mathcal{G}$,

$$[\alpha(x),[y,z]] = [[x,y],\alpha(z)] + [\alpha(y),[x,z]]\,, \qquad \text{(Left Hom-Leibniz)} \tag{3.6}$$
$$[\alpha(x),[y,z]] = [[x,y],\alpha(z)] - [[x,z],\alpha(y)]\,. \qquad \text{(Right Hom-Leibniz)} \tag{3.7}$$

Without skew-symmetry however, these identities lead to different types of Hom-algebra structures both containing Hom-Lie algebras as subclass obeying the skew-symmetry (3.1).

Definition 3.3 ([56, 66]) Left Hom-Leibniz algebras are triples $(L,[\cdot,\cdot],\alpha)$ consisting of a linear space L over a field $\mathbb{K}$, a bilinear map $[\cdot,\cdot] : L \times L \to L$ and a linear map $\alpha : L \to L$ satisfying (3.6) for all $x, y, z \in L$. Right Hom-Leibniz algebras are triples $(L,[\cdot,\cdot],\alpha)$ consisting of a linear space L and over a field $\mathbb{K}$, a bilinear map $[\cdot,\cdot] : L \times L \to L$ and a linear map $\alpha : L \to L$ satisfying (3.7) for all $x, y, z \in L$.

Remark 3.5 If $\alpha = id_L$, then a left (right) Hom-Leibniz algebra is just a left (right) Leibniz algebra [28, 69]. Any Hom-Lie algebra is both a left Hom-Leibniz algebras and a right Hom-Leibniz algebra. A left Hom-Leibniz algebra or a right Hom-Leibniz algebra is a Hom-Lie algebra if $[x,x]=0$ for all $x \in L$. For the field $\mathbb{K}$ of characteristic different from 2, left Hom-Leibniz algebras and right Hom-Leibniz algebras are Hom-Lie algebras are Hom-Lie algebras if and only if $[x,x]=0$ for all $x \in L$.

Definition 3.4 A triple $(L,[\cdot,\cdot],\alpha)$ is called a symmetric (or two-sided) Hom-Leibniz algebra if it is both a left Hom-Leibniz algebra and a right Hom-Leibniz algebra, that is if both (3.6) and (3.7) are satisfied for all $x,\ y,\ z \in L$.

Remark 3.6 A left Hom-Leibniz algebra $(L,[\cdot,\cdot],\alpha,\beta)$ is a symmetric Hom-Leibniz algebra if and only if, for all $x,\ y,\ z \in L$,

$$[\alpha(y),[x,z]] = -[[x,z],\alpha(y)].$$

Example 3.1 Let (x_1, x_2, x_3) be a basis of 3-dimensional space $\mathcal{G}$ over $\mathbb{K}$. Define a bilinear bracket operation on $\mathcal{G} \otimes \mathcal{G}$ by

$$[x_1 \otimes x_3, x_1 \otimes x_3] = x_1 \otimes x_1,$$
$$[x_2 \otimes x_3, x_1 \otimes x_3] = x_2 \otimes x_1,$$
$$[x_2 \otimes x_3, x_2 \otimes x_3] = x_2 \otimes x_2,$$

with the other necessary brackets being equal to 0. For any linear map α on $\mathcal{G}$, the triple $(\mathcal{G} \otimes \mathcal{G}, [\cdot, \cdot], \alpha \otimes \alpha)$ is not a Hom-Lie algebra but it is a symmetric Hom-Leibniz algebra.

In the following examples, we construct Hom-Leibniz algebras on a linear space $L \otimes L$ starting from a Lie or a Hom-Lie algebra L.

Proposition 3.1 ([54]) *For any Lie algebra* $(\mathcal{G}, [\cdot, \cdot])$*, the bracket*

$$[x \otimes y, a \otimes b] = [x, [a, b]] \otimes y + x \otimes [y, [a, b]]$$

defines a Leibniz algebra structure on the linear space $\mathcal{G} \otimes \mathcal{G}$.

Proposition 3.2 ([79]) *Let* $(L, [\cdot, \cdot])$ *be a Leibniz algebra and* $\alpha : L \to L$ *be a Leibniz algebra endomorphism. Then* $(L, [\cdot, \cdot]_\alpha, \alpha)$ *is a Hom-Leibniz algebra, where* $[x, y]_\alpha = [\alpha(x), \alpha(y)]$.

Using Propositions 3.1 and 3.2, we obtain the following result.

Proposition 3.3 *Let* $(\mathcal{G}, [\cdot, \cdot]')$ *be a Lie algebra and* $\alpha : \mathcal{G} \to \mathcal{G}$ *be a Lie algebra endomorphism. We define on* $\mathcal{G} \otimes \mathcal{G}$ *the following bracket on* $\mathcal{G} \otimes \mathcal{G}$,

$$[x \otimes y, a \otimes b] = \big[\alpha(x), [\alpha(a), \alpha(b)]'\big]' \otimes \alpha(y) + \alpha(x) \otimes \big[\alpha(y), [\alpha(a), \alpha(b)]'\big]'.$$

Then $(\mathcal{G} \otimes \mathcal{G}, [\cdot, \cdot], \alpha \otimes \alpha)$ *is a right Hom-Leibniz algebra.*

Example 3.2 Let us consider the Lie algebra $(A, [\cdot, \cdot])$ defined with respect to the basis $(e_i)_{1 \leq i \leq 3}$ by $[e_1, e_3] = e_1$, $[e_2, e_3] = e_1$. Let α be the Lie algebra morphism that is $\alpha([e_i, e_j] = [\alpha(e_i), \alpha(e_j)]$. The morphism is completely determined by a set of structure constants $\lambda_{i,j}$, that is $\alpha(e_j) = \sum_{i=1}^{n} \lambda_{i,j} e_i$. It turns out that it is defined by

$$\alpha(e_1) = \lambda_{11} e_1, \quad \alpha(e_2) = \lambda_{12} e_1 + (\lambda_{11} - \lambda_{12}) e_2, \quad \alpha(e_3) = \lambda_{13} e_1 + \lambda_{23} e_2 - e_3.$$

Take $\lambda_{11} = \lambda_{13} = \lambda_{23} = 1$, $\lambda_{12} = c$, where c is a parameter.

Now let us define the new multiplication of the Hom-Leibniz algebra as in the previous proposition. With the skew symmetry condition satisfied, it is defined as follows:

$$[e_3 \otimes e_3, e_1 \otimes e_3] = -2e_1 \otimes e_1 - e_1 \otimes e_2 + e_1 \otimes e_3 - e_2 \otimes e_1 + e_3 \otimes e_1$$

$$
\begin{aligned}
[e_3 \otimes e_3, e_2 \otimes e_3] &= -e_1 \otimes e_1 - e_1 \otimes e_2 + e_1 \otimes e_3 - e_1 \otimes e_1 \\
&\qquad - e_2 \otimes e_1 + e_3 \otimes e_1 \\
[e_1 \otimes e_3, e_1 \otimes e_3] &= [e_3 \otimes e_1, e_1 \otimes e_3] = -e_1 \otimes e_1 \\
[e_2 \otimes e_3, e_1 \otimes e_3] &= [e_3 \otimes e_2, e_1 \otimes e_3] = -ce_1 \otimes e_1 + (1-c)e_2 \otimes e_1 \\
[e_2 \otimes e_3, e_2 \otimes e_3] &= [e_3 \otimes e_2, e_2 \otimes e_3] = (-2c^2 + 1)e_1 \otimes e_1 \\
&\qquad - c(1-c)e_2 \otimes e_1 + (1-c)^2 e_2 \otimes e_1 \\
[e_3 \otimes e_3, e_1 \otimes e_3] &= -2e_1 \otimes e_1 - e_1 \otimes e_2 - e_2 \otimes e_1 + e_1 \otimes e_3 + e_3 \otimes e_1 \\
[e_3 \otimes e_3, e_2 \otimes e_3] &= -2e_1 \otimes e_1 - e_1 \otimes e_2 + (1-2c)e_1 \otimes e_3 \\
&\qquad - e_2 \otimes e_1 + e_3 \otimes e_1.
\end{aligned}
$$

The other brackets are null.

Proposition 3.4 *If $(L, [\cdot, \cdot])$ is a symmetric Leibniz algebra, $\alpha\colon L \to L$ a morphism of Leibniz algebra, and the map $\{\cdot, \cdot\}\colon L \times L \to L$ is defined by $\{x, y\} = [\alpha(x), \alpha(y)]$, for all $x,\ y \in L$, then $(L, \{\cdot, \cdot\}, \alpha)$ is a symmetric Hom-Leibniz algebra, called α-twist (or Yau twist) of L, and denoted by L_α.*

Proposition 3.5 *If $(A, [\cdot, \cdot], \alpha)$ be a multiplicative Hom-Leibniz algebra where α is invertible, then $(A, [\cdot, \cdot]' = \alpha^{-1}[\cdot, \cdot])$ is a Leibniz algebra and α is an automorphism with respect to $[\cdot, \cdot]'$. Hence, $(A, [\cdot, \cdot], \alpha)$ is of Leibniz type and $(A, [\cdot, \cdot]' = \alpha^{-1}[\cdot, \cdot])$ is its compatible Leibniz algebra.*

Proof The pair $(A, \alpha^{-1}[\cdot, \cdot])$ is a Leibniz algebra, since for all $x, y, z \in A$,

$$
\begin{aligned}
&[x, [y, z]']' = \alpha^{-1}[x, \alpha^{-1}([y, z])]] = \alpha^{-1}[\alpha^{-1} \circ x, \alpha^{-1} \circ [y, z]] \\
&= \alpha^{-2}[\alpha \circ x, [y, z]] = \alpha^{-2}([[x, y], \alpha(z)]] + [\alpha(y), [x, z]]) \\
&= \alpha^{-1}([\alpha^{-1}[x, y], z] + [y, \alpha^{-1}[x, z]]) = [[x, y]', z]' + [y, [x, z]']'.
\end{aligned}
$$

Moreover, α is an automorphism with respect to $[\cdot, \cdot]'$. Indeed

$$[\alpha(x), \alpha(y)]' = \alpha^{-1}[\alpha(x), \alpha(y)]) = \alpha^{-1} \circ \alpha[x, y] = \alpha(\alpha^{-1}[x, y]) = \alpha[x, y]'.$$

□

Remark 3.7 If α is multiplicative with respect to $[\cdot, \cdot]'$ but not invertible, and if $(A, [\cdot, \cdot])$ is left Hom-Leibniz algebra, then for $[\cdot, \cdot] = \alpha[\cdot, \cdot]'$, and $x, y, z \in A$,

$$[\alpha(x), [y, z]] = [[x, y], \alpha(z)] + [\alpha(y), [x, z]],$$

$$\alpha[\alpha(x), \alpha[y, z]']' = \alpha[\alpha[x, y]', \alpha(z)]' + \alpha[\alpha(y), \alpha[x, z]']', \tag{3.8}$$

$$\alpha^2([x, [y, z]']') = \alpha^2([[x, y]', z]' + [y, [x, z]']'), \tag{3.9}$$

$$\alpha^2([x, [y, z]']' - [[x, y]', z]' + [y, [x, z]']') = 0. \tag{3.10}$$

The identities (3.9) and (3.10) mean that the $[\cdot,\cdot]'$ does not necessarily define the left Leibniz algebra structure on A, but defines a slight generalisation of left Leibniz algebra structure on A, up to action of α^2, or equivalently, up to kernel of α^2. If α is a linear map that is not multiplicative with respect to $[\cdot,\cdot]'$, then (3.8) might not imply (3.9) and (3.10), which indicates in particular that multiplicativity is a highly restrictive extra condition leading to a very special subclass of Hom-algebras.

Proposition 3.6 *Let $(A,[\cdot,\cdot],\alpha)$ be a left Hom-Leibniz algebra and $\phi : A \to A$ be an invertible linear map. Then $(A,[\cdot,\cdot]' = \phi\circ[\phi^{-1}(\cdot),\phi^{-1}(\cdot)],\phi\alpha\phi^{-1})$ is a left Hom-Leibniz algebra isomorphic to the left Hom-Leibniz algebra $(A,[\cdot,\cdot],\alpha)$.*

Proof $(A,[\cdot,\cdot]' = \phi\circ[\phi^{-1}(\cdot),\phi^{-1}(\cdot)],\phi\alpha\phi^{-1})$ is a left Hom-Leibniz algebra, since for any $x,y,z\in A$,

$$\begin{aligned}
&[[x,y]',\phi\alpha\phi^{-1}(z)]' = \phi[\phi^{-1}[x,y]',\phi^{-1}\circ\phi\circ\alpha\circ\phi^{-1}(z)]\\
&= \phi[\phi^{-1}\circ\phi[\phi^{-1}(x),\phi^{-1}(y)],\phi^{-1}\circ\phi\circ\alpha\circ\phi^{-1}(z)]\\
&= \phi[[\phi^{-1}(x),\phi^{-1}(y)],\alpha\circ\phi^{-1}(z)]\\
&= \phi([\alpha\circ\phi^{-1}(x),[\phi^{-1}(y),\phi^{-1}(z)]] - [\alpha\circ\phi^{-1}(y),[\phi^{-1}(x),\phi^{-1}(z)]])\\
&= \phi([\phi^{-1}(\phi\alpha\phi^{-1}(x)),[\phi^{-1}(y),\phi^{-1}(z)]] - [\phi^{-1}(\phi\alpha\phi^{-1}(y)),[\phi^{-1}(x),\phi^{-1}(z)]])\\
&= \phi([\phi^{-1}(\phi\alpha\phi^{-1}(x)),\phi^{-1}\circ\phi[\phi^{-1}(y),\phi^{-1}(z)]]\\
&\qquad\qquad - [\phi^{-1}(\phi\alpha\phi^{-1}(y)),\phi^{-1}\circ\phi[\phi^{-1}(x),\phi^{-1}(z)]])\\
&= \phi([\phi^{-1}(\phi\alpha\phi^{-1}(x)),\phi^{-1}\circ\phi[\phi^{-1}(y),\phi^{-1}(z)]])\\
&\qquad\qquad - \phi([\phi^{-1}(\phi\alpha\phi^{-1}(y)),\phi^{-1}\circ\phi[\phi^{-1}(x),\phi^{-1}(z)]])\\
&= [\phi\alpha\phi^{-1}(x),[y,z]']' - [\phi\alpha\phi^{-1}(y),[x,z]']'.
\end{aligned}$$

The invertible linear map $\phi : A \to A$ is Hom-algebras morphism and thus isomorphism since

$$\begin{aligned}
&\phi[x,y] = \phi[\phi^{-1}\circ\phi(x),\phi^{-1}\circ\phi(y)] = [\phi(x),\phi(y)]',\\
&\phi\circ\alpha = (\phi\circ\alpha\circ\phi^{-1})\circ\phi.
\end{aligned}$$

Hence, $(A,[\cdot,\cdot],\alpha)$ and $(A,[\cdot,\cdot]' = \phi\circ[\phi^{-1}(\cdot),\phi^{-1}(\cdot)],\phi\alpha\phi^{-1})$ are isomorphic. □

Lemma 3.1 *For any bilinear map $[\cdot,\cdot] : A\times A \to A$ and linear map $\alpha : A \to A$ on a linear space A, if an linear map $\phi : A \to A$ is invertible, $\phi[\cdot,\cdot] = [\phi(\cdot),\phi(\cdot)]'$, and α is multiplicative with respect to $[\cdot,\cdot]$, that is $\alpha[\cdot,\cdot] = [\alpha(\cdot),\alpha(\cdot)]$ then $\phi\alpha\phi^{-1} : A \to A$ is multiplicative with respect to bilinear map $[\cdot,\cdot]' = \phi[\phi^{-1}(\cdot),\phi^{-1}(\cdot)] : A\times A \to A$.*

Proof For any $x,y\in A$,

$$\begin{aligned}\phi\alpha\phi^{-1}[x,y]' &= \phi\alpha\phi^{-1}\phi[\phi^{-1}(x),\phi^{-1}(y)] = \phi\alpha[\phi^{-1}(x),\phi^{-1}(y)]\\ &= [\phi\alpha\phi^{-1}(x),\phi\alpha\phi^{-1}(y)] = \phi^{-1}[\phi\phi\alpha\phi^{-1}(x),\phi\phi\alpha\phi^{-1}(y)]'\\ &= \phi^{-1}\phi[\phi\alpha\phi^{-1}(x),\phi\phi\alpha\phi^{-1}(y)]' = [\phi\alpha\phi^{-1}(x),\phi\alpha\phi^{-1}(y)]',\end{aligned}$$

and thus $\phi\alpha\phi^{-1} : A \to A$ is multiplicative with respect to bilinear map $[\cdot,\cdot]' = \phi[\phi^{-1}(\cdot),\phi^{-1}(\cdot)] : A \times A \to A$. □

Proposition 3.6 and Lemma 3.1 imply the following corollary about multiplicativity of Hom-Leibniz algebras.

Corollary 3.1 *If the left Hom-Leibniz algebra $(A,[\cdot,\cdot],\alpha)$ is multiplicative, $\phi : A \to A$ is invertible linear map, and ϕ is a week morphism of Hom-algebras (algebras morphism), that is $\phi[\cdot,\cdot] = [\phi(\cdot),\phi(\cdot)]'$, then the left Hom-Leibniz algebra $(A,[\cdot,\cdot]' = \phi\circ[\phi^{-1}(\cdot),\phi^{-1}(\cdot)],\phi\alpha\phi^{-1})$ is also multiplicative.*

Proposition 3.7 *Let $(A,[\cdot,\cdot],\alpha)$ be a left Hom-Leibniz algebra and let ψ be an automorphism of $(A,[\cdot,\cdot],\alpha)$. Then $\phi\psi\phi^{-1}$ is an automorphism of isomorphic left Hom-Leibniz algebra $(A,[\cdot,\cdot],\phi\alpha\phi^{-1})$ described in Proposition 3.6.*

Proof Let $\gamma = \phi\alpha\phi^{-1}$, ψ commute with α. We have,

$$\phi\psi\phi^{-1}\gamma = \phi\psi\phi^{-1}\phi\alpha\phi^{-1} = \phi\psi\alpha\phi^{-1} = \phi\alpha\psi\phi^{-1} = \phi\alpha\phi^{-1}\phi\psi\phi^{-1} = \gamma\phi\psi\phi^{-1}.$$

For any $x, y \in A$,

$$\begin{aligned}\phi\psi\phi^{-1}[\phi(x),\phi(y)]' &= \phi\psi\phi^{-1}\phi[x,y] = \phi\psi[x,y] = \phi[\psi(x),\psi(y)]\\ &= \phi([\phi^{-1}\phi\psi(x),\phi^{-1}\phi\psi(y)]) = [\phi\psi(x),\phi\psi(y)]'\\ &= [\phi\psi\phi^{-1}(\phi(x)),\phi\psi\phi^{-1}(\phi(y))]'.\end{aligned}$$

Hence $\gamma = \phi\psi\phi^{-1}$ is an automorphism of Hom-Leibniz algebras. □

Remark 3.8 Similar results to Propositions 3.1, 3.6 and 3.7 with similar proofs holds for the right Hom-Leibniz algebras and thus also for (two-sided) Hom-Leibniz algebras.

Next, we consider the notions of right, left symmetric and symmetric (two-sided) Hom-Leibniz superalgebras.

Definition 3.5 A triple $(L,[\cdot,\cdot],\alpha)$ consisting of a superspace L, an even bilinear map $[\cdot,\cdot] : L \times L \to L$ and an even superspace homomorphism $\alpha : L \to L$ (linear map of parity degree $0 \in \mathbb{Z}_2$) is called

(i) left Hom-Leibniz superalgebra if it satisfies for all $x, y, z \in L_0 \cup L_1$,

$$[\alpha(x),[y,z]] = [[x,y],\alpha(z)] + (-1)^{|x||y|}\,[\alpha(y),[x,z]]\,,$$

(ii) right Hom-Leibniz superalgebra if it satisfies

$$[\alpha(x), [y, z]] = [[x, y], \alpha(z)] - (-1)^{|y||z|} [[x, z], \alpha(y)] ,$$

(iii) symmetric Hom-Leibniz superalgebra if it is a left and a right Hom-Leibniz superalgebra.

Proposition 3.8 *A triple* $(L, [\cdot, \cdot], \alpha)$ *is a symmetric Hom-Leibniz superalgebra if and only if for all* $x, \ y, \ z \in L_0 \cup L_1$,

$$[\alpha(x), [y, z]] = [[x, y], \alpha(z)] + (-1)^{|x||y|} [\alpha(y), [x, z]] ,$$
$$[\alpha(y), [x, z]] = -(-1)^{(|x|+|z|)|y|} [[x, z], \alpha(y)] .$$

Example 3.3 Let $L = L_0 \oplus L_1$ be a 3-dimensional superspace, where L_0 is generated by e_1, e_2 and L_1 is generated by e_3. The product is given by

$$[e_1, e_1] = ae_1 + xe_2, \ [e_1, e_2] = [e_2, e_1] = -\frac{a}{x}[e_1, e_1], \ [e_2, e_2] = \left(\frac{a}{x}\right)^2 [e_1, e_1],$$
$$[e_3, e_3] = \frac{d}{x}[e_1, e_1], \quad [e_1, e_3] = [e_3, e_1] = [e_3, e_2] = [e_2, e_3] = 0.$$

Consider the homomorphism $\alpha \colon L \to L$ with the matrix $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{pmatrix}$ in the basis (e_1, e_2, e_3). Then $(L, [\cdot, \cdot], \alpha)$ is a symmetric Hom-Leibniz superalgebra.

3.4 Centroids and Derivations of Hom-Leibniz Superalgebras

In this section we consider centroids and twisted derivations of Hom-Leibniz superalgebras. The concept of centroids and derivation of Leibniz algebras is introduced in [34]. Left Leibniz superalgebras, originally were introduced in [43], can be seen as a direct generalization of Leibniz algebras. The left Hom-Leibniz superalgebras were recently considered in [70].

Recall that a linear superspace L over a field $\mathbb{K}$ is a $\mathbb{Z}_2$-graded linear space with a direct sum $L = L_0 \oplus L_1$. The elements of L_j, $j \in \mathbb{Z}_2$, are said to be homogenous of parity j. The parity of a homogeneous element x is denoted by $|x|$. For k homogeneous elements $x_1, \dots, x_k$ of L, $|(x_1, \dots, x_k)| = |x_1| + \dots + |x_k|$ is the parity of an element $(x_1, \dots, x_k)$ in L^k. The space $End(L)$ is $\mathbb{Z}_2$-graded with a direct sum $End(L) = (End(L))_0 \oplus (End(L))_1$, where $(End(L))_j = \{f \in End(L) \mid f(L_i) \subset L_{i+j}\}$. The elements of $(End(L))_j$ are said to be homogenous of parity j. Let L be a superspace and α an even linear map on L. Let $\Omega = \Omega_0 \oplus \Omega_1$ where $\Omega_0 = \{u \in (End(L))_0 \mid u \circ \alpha = \alpha \circ u\}$ and $\Omega_1 = \{u \in (End(L))_1 \mid u \circ$

$\alpha = \alpha \circ u\}$. As defined above, Ω is a graded linear subspace of $End(L)$. The homogenous elements of Ω are the elements of $\Omega_0 \cup \Omega_1$. If the map $\tilde{\alpha} \colon \Omega \to \Omega$ is defined by $\tilde{\alpha}(u) = \alpha \circ u$, then the map $\tilde{\alpha}$ is an even linear map (that is of degree zero), and hence belongs to $(End(L))_0$. Furthermore, $(\Omega, [\cdot, \cdot]')$ is a Lie superalgebra, and $(\Omega, [\cdot, \cdot]', \tilde{\alpha})$ is a Hom-Lie superalgebra with the bilinear super commutator bracket product defined by $[u, v]' = uv - (-1)^{|u||v|}vu$ for all homogeneous elements $u, v \in \Omega$. The super skew-symmetry holds as for homogeneous elements u, v in Ω,

$$[u, v]' = uv - (-1)^{|u||v|}vu = -(-1)^{|u||v|}(vu - (-1)^{|u||v|}uv) = -(-1)^{|u||v|}[v, u]'.$$

The super Jacobi identity for $(\Omega, [\cdot, \cdot]')$ holds, since for homogeneous elements u, v, w in Ω,

$$\begin{aligned}
[u, [v, w]']' &= [u, vw] - (-1)^{|v||w|}[u, wv] \\
&= \big(u(vw) - (-1)^{|u|(|v|+|w|)}(vw)u\big) \\
&\qquad - (-1)^{|v||w|}\big(u(wv) - (-1)^{|u|(|v|+|w|)}(wv)u\big) \\
&= u(vw) - (-1)^{|u|(|v|+|w|)}(vw)u - (-1)^{|v||w|}u(wv) \\
&\qquad + (-1)^{|v||w|+|u||v|+|u||w|}(wv)u,
\end{aligned}$$

and hence the first component of the graded Jacobi identity can be written as

$$\begin{aligned}
(-1)^{|u||w|}[u, [v, w]']' &= (-1)^{|u||w|}u(vw) - (-1)^{|u||v|}(vw)u \\
&\qquad - (-1)^{|v||w|+|u||w|}u(wv) + (-1)^{|v||w|+|u||v|}(wv)u,
\end{aligned}$$

and cyclic summation yields

$$\begin{aligned}
&\sum_{\circlearrowleft(u,v,w)} (-1)^{|u||w|}[u, [v, w]']' \\
&= (-1)^{|u||w|}[u, [v, w]']' + (-1)^{|w||v|}[w, [u, v]']' + (-1)^{|v||u|}[v, [w, u]']' \\
&= (-1)^{|u||w|}u(vw) - (-1)^{|u||v|}(vw)u - (-1)^{|v||w|+|u||w|}u(wv) \\
&\qquad + (-1)^{|v||w|+|u||v|}(wv)u \\
&+ (-1)^{|w||v|}w(uv) - (-1)^{|w||u|}(uv)w - (-1)^{|u||v|+|w||v|}w(vu) \\
&\qquad + (-1)^{|u||v|+|w||u|}(vu)w \\
&+ (-1)^{|v||u|}v(wu) - (-1)^{|v||w|}(wu)v - (-1)^{|w||u|+|v||u|}v(uw) \\
&\qquad + (-1)^{|w||u|+|v||w|}(uw)v = 0.
\end{aligned}$$

It remains to show that $(\Omega, [\cdot, \cdot]', \tilde{\alpha})$ satisfies the graded Hom-Jacobi identity. For all homogeneous elements $u, v, w \in \Omega$ and for $\tilde{\alpha} \in (End(L))_0$, similarly,

$$
\begin{aligned}
[\tilde{\alpha}(u), [v, w]']' &= [\alpha u, vw] - (-1)^{|v||w|}[\alpha u, wv] \\
&= \big(\alpha u(vw) - (-1)^{|u|(|v|+|w|)}(vw)\alpha u\big) \\
&\qquad - (-1)^{|v||w|}\big(\alpha u(wv) - (-1)^{|u|(|v|+|w|)}(wv)\alpha u\big) \\
&= \alpha u(vw) - (-1)^{|u|(|v|+|w|)}(vw)\alpha u \\
&\qquad - (-1)^{|v||w|}\alpha u(wv) + (-1)^{|v||w|+|u||v|+|u||w|}(wv)\alpha u.
\end{aligned}
$$

The first component of the super Hom Jacobi identity can be written as follows

$$
\begin{aligned}
(-1)^{|u||w|}[\tilde{\alpha}(u), [v, w]']' &= (-1)^{|u||w|}\alpha u(vw) - (-1)^{|u||v|}(vw)\alpha u \\
&\qquad - (-1)^{|w|(|u|+|v|)}\alpha u(wv) + (-1)^{|v||w|+|u||v|}(wv)\alpha u.
\end{aligned}
$$

Since u commutes with all elements of Ω, we get

$$
\begin{aligned}
&\sum_{\circlearrowleft(u,v,w)} (-1)^{|u||w|}[\tilde{\alpha}, [v, w]']' = \\
&= (-1)^{|u||w|}\alpha u(vw) - (-1)^{|u||v|}(vw)\alpha u - (-1)^{|v||w|+|u||w|}\alpha u(wv) \\
&\qquad + (-1)^{|v||w|+|u||v|}(wv)\alpha u \\
&+ (-1)^{|w||v|}\alpha w(uv) - (-1)^{|w||u|}(uv)\alpha w - (-1)^{|u||v|+|w||v|}\alpha w(vu) \\
&\qquad + (-1)^{|u||v|+|w||u|}(vu)\alpha w \\
&+ (-1)^{|v||u|}\alpha v(wu) - (-1)^{|v||w|}(wu)\alpha v - (-1)^{|w||u|+|v||u|}\alpha v(uw) \\
&\qquad + (-1)^{|w||u|+|v||w|}(uw)\alpha v = 0.
\end{aligned}
$$

Remark 3.9 Since, $\tilde{\alpha}([u, v]') = \alpha \circ (uv - (-1)^{|u||v|}(vu))$, and since

$$
[\tilde{\alpha}(u), \tilde{\alpha}(v)]' = \alpha u\alpha v - (-1)^{|\alpha u||\alpha v|}\alpha v\alpha u = \alpha^2 \circ (uv - (-1)^{|u||v|}(vu)) = \alpha^2 \circ [u, v]'
$$

holds because α is even and commutes with all elements of Ω, the multiplicativity of Hom-Lie superalgebras $(\Omega, [\cdot, \cdot]', \tilde{\alpha})$ is equivalent to $[u, v]' = uv - (-1)^{|u||v|}vu$ annihilating $(\alpha^2 - \alpha)$ in $End(L)$ for all homogeneous $u, v \in \Omega$. Thus, in particular, $(\Omega, [\cdot, \cdot]', \tilde{\alpha})$ is not necessarily multiplicative. However, if α is an idempotent (projection), $\alpha^2 = \alpha$, then $(\Omega, [\cdot, \cdot]', \tilde{\alpha})$ is multiplicative.

Definition 3.6 For any non-negative integer $k \geq 0$, an α^k-derivation of a Hom-Leibniz superalgebra $(L, [\cdot,\cdot], \alpha)$ is a homogeneous linear map $D \in \Omega$ satisfying, for all $x, y, z \in L_0 \cup L_1$,

$$D([x, y]) = [D(x), \alpha^k(y)] + (-1)^{|D||x|}[\alpha^k(x), D(y)]. \tag{3.11}$$

The set of all α^k-derivations of a Hom-Leibniz superalgebra L for all non-negative integers $k \geq 0$ is denoted by $Der(L) = \bigoplus_{k\geq 0} Der_{\alpha^k}(L)$.

We will refer sometimes to the elements of $Der(L)$ as derivations of Hom-Leibniz superalgebra L slightly abusing terminology for the convenience of the exposition.

Proposition 3.9 *If $(L, [\cdot,\cdot], \alpha)$ is a Hom-Leibniz superalgebra, then $Der(L)$ is Lie (resp. Hom-Lie) subsuperalgebra of $(\Omega, [\cdot,\cdot]')$ (resp. $(\Omega, [\cdot,\cdot]', \tilde{\alpha})$)*

Proof Let $d \in Der_{\alpha^k}(L)$ and $d' \in Der_{\alpha^l}(L)$. We have

$$\begin{aligned}
&d \circ d'([x, y]) = d\left([d'(x), \alpha^l(y)]\right) + (-1)^{|d'||x|} d\left([\alpha^l(x), d'(y)]\right)\\
&= [dd'(x), \alpha^{k+l}(y)] + (-1)^{|d||d'(x)|}[\alpha^k d'(x), d\alpha^l(y)]\\
&+ (-1)^{|d'||x|}[d\alpha^l(x), \alpha^k d'(y)] + (-1)^{|d'||x|}(-1)^{|d||x|}[\alpha^{k+l}(x), dd'(y)]\\
&= [dd'(x), \alpha^{k+l}(y)] + (-1)^{(|d|+|d'|)|x|}[\alpha^{k+l}(x), dd'(y)]\\
&+ (-1)^{|d|(|d'|+|x|)}[\alpha^k d'(x), d\alpha^l(y)] + (-1)^{|d'||x|}[d\alpha^l(x), \alpha^k d'(y)],\\
&d' \circ d([x, y]) = [d'd(x), \alpha^{k+l}(y)] + (-1)^{(|d'|+|d|)|x|}[\alpha^{k+l}(x), d'd(y)]\\
&+ (-1)^{|d'|(|d|+|x|)}[\alpha^l d(x), d'\alpha^k(y)] + (-1)^{|d||x|}[d'\alpha^k(x), \alpha^l d(y)].\\
&[d, d']([x, y]) = \left(d \circ d' - (-1)^{|d||d'|} d' \circ d\right)([x, y])\\
&= \left[[d, d'](x), \alpha^{k+l}(y)\right] + (-1)^{(|d'|+|d|)|x|}\left[\alpha^{k+l}(x), [d, d'](y)\right]\\
&+ (-1)^{|d|(|d'|+|x|)}[\alpha^k d'(x), d\alpha^l(y)] + (-1)^{|d'||x|}[d\alpha^l(x), \alpha^k d'(y)]\\
&- (-1)^{|d||d'|}(-1)^{|d'|(|d|+|x|)}[\alpha^l d(x), d'\alpha^k(y)]\\
&- (-1)^{|d||d'|}(-1)^{|d||x|}[d'\alpha^k(x), \alpha^l d(y)]\\
&= \left[[d, d'](x), \alpha^{k+l}(y)\right] + (-1)^{(|d'|+|d|)|x|}\left[\alpha^{k+l}(x), [d, d'](y)\right].
\end{aligned}$$

So, $[d, d']$ is an α^{k+l}-derivation of L. Clearly $[d, d'] \in \Omega$. Therefore, $Der(L)$ is Lie (resp. Hom-Lie) subsuperalgebras of $(\Omega, [\cdot,\cdot]')$ (resp. $(\Omega, [\cdot,\cdot]', \tilde{\alpha})$). □

Proposition 3.10 *Let $(L, [\cdot,\cdot], \alpha)$ be a left (resp. right) Hom-Leibniz superalgebra. For any $a \in L$ satisfying $\alpha(a) = a$, define $ad_k(a) \in End(L)$ and $Ad_k(a) \in End(L)$ for all $x \in L$ by*

$$ad_k(a)(x) = [a, \alpha^k(x)], \quad Ad_k(a)(x) = (-1)^{|a||x|}[\alpha^k(x), a].$$

Then $ad_k(a)$ (resp. $Ad_k(a)$) is an α^{k+1}-derivation.

Proof Let $(L, [\cdot,\cdot], \alpha)$ be a left Hom-Leibniz superalgebra and $a \in L_0 \cup L_1$ satisfying $\alpha(a) = a$. We have

$$ad_k(a)([x, y]) = [a, \alpha^k([x, y])] = [\alpha^{k+1}(a), [\alpha^k(x), \alpha^k(y)]].$$

If $(L, [\cdot,\cdot], \alpha)$ is a left Hom-Leibniz superalgebra, then

$$\begin{aligned}[\alpha^{k+1}(a), [\alpha^k(x), \alpha^k(y)]] &= [[\alpha^k(a), \alpha^k(x)], \alpha^{k+1}(y)] \\ &\quad +(-1)^{|a||x|}[\alpha^{k+1}(x), [\alpha^k(a), \alpha^k(y)]].\end{aligned}$$

Thus, $ad_k(a)([x, y]) = [ad_k(a)(x), \alpha^{k+1}(y)] + (-1)^{|a||x|}[\alpha^{k+1}(x), ad_k(a)(y)]$. Hence, $ad_k(a)$ is an α^{k+1}-derivation of the left Hom-Leibniz superalgebra L. If $(L, [\cdot,\cdot], \alpha)$ is a right Hom-Leibniz superalgebra, similarly, one can show that $Ad_k(a)$ is an α^{k+1}-derivation of the right Hom-Leibniz superalgebra L. □

Remark 3.10 If L is a symmetric Hom-Leibniz superalgebra, Then, $ad_k(a)$ and $Ad_k(a)$ are α^{k+1}-derivations.

Definition 3.7 The linear map $ad_k(a)$ (resp. $Ad_k(a)$) is called an inner left (resp. right) α^{k+1}-derivation of the left (resp. right) Hom-Leibniz superalgebra L, and $ad(L) = \bigoplus_{k\geq 0} ad_k(L)$ (resp. $Ad(L) = \bigoplus_{k\geq 0} Ad_k(L)$) denotes the subsuperalgebra of inner derivations of L.

Proposition 3.11 *If $(L, [\cdot,\cdot], \alpha)$ is a Hom-Leibniz superalgebra, then $ad(L)$ is a Lie (resp. Hom-Lie) subsuperalgebra of $(\Omega, [\cdot,\cdot]')$ (resp. $(\Omega, [\cdot,\cdot]', \tilde{\alpha})$)*

Definition 3.8 The α^k-centroid of a Hom-Leibniz superalgebra $(L, [\cdot,\cdot], \alpha)$ is

$$\Gamma_{\alpha^k}(L) = \left\{ \begin{array}{r} d \in \Omega \mid d([x, y]) = [d(x), \alpha^k(y)] = (-1)^{|d||x|}[\alpha^k(x), d(y)] \\ \forall x, y \in L_0 \cup L_1 \end{array} \right\}.$$

Denote by $\Gamma(L) = \bigoplus_{k\geq 0} \Gamma_{\alpha^k}(L)$ the centroid of $\mathcal{L}$.

Definition 3.9 Let $(L, [\cdot,\cdot], \alpha)$ be a Hom-Leibniz superalgebra. Then the α^k-centroid of L denoted by $C_{\alpha^k}(L)$ is defined by

$$C_{\alpha^k}(L) = \left\{ \begin{array}{r} d \in \Omega \mid d([x, y]) = [d(x), \alpha^k(y)] = (-1)^{|d||x|}[\alpha^k(x), d(y)], \\ \forall\, x, y \in L_0 \cup L_1 \end{array} \right\}. \tag{3.12}$$

Denote by $C(L) = \bigoplus_{k\geq 0} C_{\alpha^k}(L)$ the centroid of L.

Proposition 3.12 *If $(L, [\cdot,\cdot], \alpha)$ is a Hom-Leibniz superalgebra, then $C(L)$ is a Lie (resp. Hom-Lie) subsuperalgebras of $(\Omega, [\cdot,\cdot]')$ (resp. $(\Omega, [\cdot,\cdot]', \tilde{\alpha})$).*

Definition 3.10 Let $(L, [\cdot,\cdot], \alpha)$ be a Hom-Leibniz superalgebra and $d \in End(L)$. Then d is called a α^k-central derivation, if $d \in \Omega$ and

$$d([x, y]) = [d(x), \alpha^k(y)] = (-1)^{|d||x|}[\alpha^k(x), d(y)] = 0.$$

The set of all central derivations is denoted by $ZDer(L) = \bigoplus_{k\geq 0} ZDer_{\alpha^k}(L)$.

Lemma 3.2 *If L is a Hom-Leibniz superalgebra, $d \in Der_{\alpha^k}(L)$, $d \in Der_{\alpha^l}(L)$ and $\Phi \in C_{\alpha^l}(L)$, then*

(i) *$\Phi \circ d$ is an α^{k+l}-derivation of L.*
(ii) *$[\Phi, d]$ is an α^{k+l}-centroid of L.*
(iii) *$d \circ \Phi$ is an α^{k+l}-centroid if and only if $\Phi \circ d$ is a α^{k+l}-central derivation.*
(iv) *$d \circ \Phi$ is an α^{k+l}-derivation only if only $[d, \Phi]$ is a α^{k+l}-central derivation.*

Proof Let $d \in Der_{\alpha^k}(L)$ and $\Phi \in C_{\alpha^l}(L)$. Then

$$\Phi \circ d([x, y]) = \Phi\left([d(x), \alpha^k(y)]\right) + (-1)^{|d||x|}\Phi\left([\alpha^k(x), d(y)]\right).$$

Since $\Phi \in C_{\alpha^l}(L)$, we have

$$\begin{aligned}\Phi \circ d([x, y]) &= [\Phi(d(x)), \alpha^{k+l}(y)] + (-1)^{|d||x|}(-1)^{|\Phi||x|}[\alpha^{k+l}(x), \Phi(d(y))]\\ &= [\Phi \circ d(x)), \alpha^{k+l}(y)] + (-1)^{|\Phi\circ d||x|}[\alpha^{k+l}(x), \Phi \circ d(y))].\end{aligned}$$

Hence $\Phi \circ d$ is an α^{k+l}-derivation of L. The proof of the rest of the parts of lemma is similar to that of the first one. □

Theorem 3.1 *If L is a Hom-Leibniz superalgebra, then*

$$ZDer_{\alpha^k}(L) = Der_{\alpha^k}(L) \cap C_{\alpha^k}(L).$$

Proof Let $\Phi \in Der_{\alpha^k}(L) \cap C_{\alpha^k}(L)$. Then, for all $x, y \in L$,

$$\Phi([x, y]) = [\Phi(x), \alpha^k(y)] + (-1)^{|\Phi||x|}[\alpha^k(x), \Phi(y)], \quad \Phi([x, y]) = [\Phi(x), \alpha^k(y)].$$

Therefore, $[\alpha^k(x), \Phi(y)] = 0$. Hence, $\Phi([x, y]) = 0$ and $\Phi \in ZDer_{\alpha^k}(L)$. Conversely, let $\Phi \in ZDer_{\alpha^k}(L)$. Then,

$$\Phi([x, y]) = [\Phi(x), \alpha^k(y)] = (-1)^{|\Phi||x|}[\alpha^k(x), \Phi(y)] = 0.$$

Hence, Φ satisfies (3.11) and (3.12). So the result holds. □

3.5 Classification of Multiplicative 2-Dimensional Hom-Leibniz Algebras

First of all, note that the classifications of two and three-dimensional complex Leibniz algebras were studied in [29, 63]. In this section, the classification of the 2-dimensional left Hom-Leibniz algebras is obtained, and for each isomorphism class it is indicated whether the Hom-Leibniz algebras from this class are symmetric Hom-Leibniz algebras or not.

Proposition 3.13 *Every* 2*-dimensional left Hom-Leibniz algebra is isomorphic to one of the following nonisomorphic Hom-Leibniz algebras, with each algebra denoted by* L^i_j *where* i *is related to the linear map* α *and* j *being the number in the list.*

3.6 Centroids and Derivations of 2-Dimensional Multiplicative Hom-Leibniz Algebras

In this section we focus on the study of centroids and derivation of multiplicative Hom-Leibniz algebras of dimension 2 over $\mathbb{C}$. Note that if the odd component of Hom-Leibniz superalgebra is zero, it can be considered as Hom-Leibniz algebra.

Next, we introduce a parametric extension of the notion of α^k-derivations of Hom-Leibniz algebras and study it in more details with help of the computer computations.

Definition 3.11 Let $(L, [\cdot,\cdot], \alpha)$ be a Hom–Leibniz algebra and λ, μ, γ elements of $\mathbb{C}$. A linear map $d \in \Omega$ is a generalized α^k-derivation or a (λ, μ, γ)-α^k-derivation of L if for all $x, y \in L$ we have

$$\lambda\, d([x, y]) = \mu\, [d(x), \alpha^k(y)] + \gamma\, [\alpha^k(x), d(y)].$$

We denote the set of all (λ, μ, γ)-α^k-derivations by $Der^{(\lambda,\mu,\gamma)}_{\alpha^k}(L)$.

Remark 3.11 Clearly $Der^{(1,1,1)}_{\alpha^k}(L) = Der_{\alpha^k}(L)$ and $Der^{(1,1,0)}_{\alpha^k}(L) = \Gamma_{\alpha^k}(L)$.

Let $(L, [\cdot,\cdot], \alpha)$ be a n-dimensional multiplicative left Hom-Leibniz algebra. Let $\alpha^r(e_j) = \sum_{k=1}^{n} m_{kj} e_k$.

$L_1^1 : [e_1, e_1] = 0,$ $\alpha(e_1) = e_1,$ L_1^1 is not symmetric	$[e_1, e_2] = 0,$ $\alpha(e_2) = e_2.$	$[e_2, e_1] = e_1$,	$[e_2, e_2] = e_1,$
$^1_2 : [e_1, e_1] = 0,$ $\alpha(e_1) = e_1,$ L_2^1 is symmetric	$[e_1, e_2] = 0,$ $\alpha(e_2) = e_2.$	$[e_2, e_1] = 0$,	$[e_2, e_2] = e_1,$
$L_3^1 : [e_1, e_1] = 0,$ $\alpha(e_1) = e_1,$ L_3^1 is symmetric	$[e_1, e_2] = 0,$ $\alpha(e_2) = e_2.$	$[e_2, e_1] = e_1$,	$[e_2, e_2] = 0,$
$L_4^1 : [e_1, e_1] = 0,$ $\alpha(e_1) = e_1,$ L_4^1 is symmetric	$[e_1, e_2] = e_1,$ $\alpha(e_2) = e_2.$	$[e_2, e_1] = -e_1,$	$[e_2, e_2] = 0,$
$L_1^2 : [e_1, e_1] = 0,$ $\alpha(e_1) = 0,$ L_1^2 is not symmetric	$[e_1, e_2] = 0,$ $\alpha(e_2) = e_2.$	$[e_2, e_1] = e_1,$	$[e_2, e_2] = 0,$
$L_2^2 : [e_1, e_1] = e_1,$ $\alpha(e_1) = 0,$ L_2^2 is symmetric	$[e_1, e_2] = 0,$ $\alpha(e_2) = e_2.$	$[e_2, e_1] = 0,$	$[e_2, e_2] = 0,$
$L_3^2 : [e_1, e_1] = 0,$ $\alpha(e_1) = 0,$ L_3^2 is symmetric	$[e_1, e_2] = e_1,$ $\alpha(e_2) = e_1.$	$[e_2, e_1] = -e_1,$	$[e_2, e_2] = 0,$
$L_1^3 : [e_1, e_1] = e_1,$ $\alpha(e_1) = 0,$ L_1^3 is symmetric	$[e_1, e_2] = 0,$ $\alpha(e_2) = be_2.$	$[e_2, e_1] = 0,$	$[e_2, e_2] = 0,$
$L_2^3 : [e_1, e_1] = 0,$ $\alpha(e_1) = 0,$ L_2^3 is not symmetric	$[e_1, e_2] = 0,$ $\alpha(e_2) = be_2.$	$[e_2, e_1] = e_1,$	$[e_2, e_2] = 0,$
$L_3^3 : [e_1, e_1] = 0,$ $\alpha(e_1) = 0,$ L_3^3 is symmetric	$[e_1, e_2] = e_1,$ $\alpha(e_2) = be_2.$	$[e_2, e_1] = -e_1,$	$[e_2, e_2] = 0,$
$L_1^4 : [e_1, e_1] = 0,$ $\alpha(e_1) = ae_1 (a \notin \{0, 1\})$, L_1^4 is not symmetric	$[e_1, e_2] = 0,$ $\alpha(e_2) = e_2.$	$[e_2, e_1] = e_1,$	$[e_2, e_2] = 0,$
$L_2^4 : [e_1, e_1] = 0,$ $\alpha(e_1) = ae_1 (a \notin \{0, 1\})$, L_2^4 is not symmetric	$[e_1, e_2] = -e_1,$ $\alpha(e_2) = e_2.$	$[e_2, e_1] = e_1,$	$[e_2, e_2] = 0,$
$L_1^5 : [e_1, e_1] = 0,$ $\alpha(e_1) = a^2 e_1 (a \notin \{0, 1\})$, L_1^5 is symmetric	$[e_1, e_2] = 0,$ $\alpha(e_2) = ae_2.$	$[e_2, e_1] = 0,$	$[e_2, e_2] = e_1,$
$L_1^6 : [e_1, e_1] = 0,$ $\alpha(e_1) = 0$, L_1^6 is symmetric	$[e_1, e_2] = e_1,$ $\alpha(e_2) = e_1.$	$[e_2, e_1] = y_1 e_1,$	$[e_2, e_2] = z_1 e_1,$
$L_2^6 : [e_1, e_1] = 0,$ $\alpha(e_1) = 0$, L_2^6 is symmetric	$[e_1, e_2] = 0,$ $\alpha(e_2) = e_1.$	$[e_2, e_1] = e_1,$	$[e_2, e_2] = z_1 e_1,$
$L_3^6 : [e_1, e_1] = 0,$ $\alpha(e_1) = 0$, L_3^6 is symmetric	$[e_1, e_2] = 0,$ $\alpha(e_2) = e_1.$	$[e_2, e_1] = 0,$	$[e_2, e_2] = e_1,$
$L_1^7 : [e_1, e_1] = 0,$ $\alpha(e_1) = e_1$, L_1^7 is symmetric	$[e_1, e_2] = 0,$ $\alpha(e_2) = e_1 + e_2$.	$[e_2, e_1] = 0,$	$[e_2, e_2] = e_1,$

An element d of $Der^{(\delta,\mu,\gamma)}_{\alpha^r}(L)$, being a linear transformation of the linear space L, is represented in a matrix form $(d_{ij})_{1\leq i,j\leq n}$, that is $d(e_j)=\sum_{k=1}^{n} d_{kj}e_k$, for $j=1,\ldots,n$. According to the definition of the (δ,μ,γ)-α^r-derivation the entries d_{ij} of the matrix $(d_{ij})_{1\leq i,j\leq n}$ must satisfy the following systems of equations:

$$\sum_{k=1}^{n} d_{ik}a_{kj}=\sum_{k=1}^{n} a_{ik}d_{kj},$$

$$\delta\sum_{k=1}^{n} c_{ij}^{k}d_{sk}-\mu\sum_{k=1}^{n}\sum_{l=1}^{n} d_{ki}m_{lj}c_{kl}^{s}-\gamma\sum_{k=1}^{n}\sum_{l=1}^{n} d_{lj}m_{ki}c_{kl}^{s}=0,$$

where $(a_{ij})_{1\leq i,j\leq n}$ is the matrix of α and (c_{ij}^{k}) are the structure constants of L. First, let us give the following definitions.

Definition 3.12 A left Hom-Leibniz algebra is called characteristically nilpotent if the Lie algebra $Der_{\alpha^0}(L)$ is nilpotent.

Henceforth, the property of being characteristically nilpotent is abbreviated by CN.

Definition 3.13 Let L be an indecomposable left Hom-Leibniz algebra. We say that L is small if $\Gamma_{\alpha^0}(L)$ is generated by central derivation and the scalars. The centroid of a decomposable BiHom-Lie algebra is small if the centroids of each indecomposable factor is small.

Now we apply the algorithms mentioned in the previous paragraph to find centroid and derivations of 2-dimensional complex left Hom-Leibniz algebras. In this study we make use of the classification results from the previous section. The results are given in the following theorem. Moreover, we provide the type of $\Gamma_{\alpha^0}(L_i^j)$ and $Der_{\alpha^r}(L_i^j)$.

$$L_1^1: [e_1,e_1]=0,\quad [e_1,e_2]=0,\quad [e_2,e_1]=e_1,\quad [e_2,e_2]=e_1,$$
$$\alpha(e_1)=0,\quad \alpha(e_2)=e_2$$

α^r	$\Gamma_{\alpha^r}(L_1^1)$	Type of $\Gamma_{\alpha^0}(L_1^1)$	$Der_{\alpha^r}(L_1^1)$	–
$r\in\mathbb{N}$	$\begin{pmatrix}0 & c_1\\ 0 & c_2\end{pmatrix}$	–	$\begin{pmatrix}0 & 0\\ 0 & d_1\end{pmatrix}$	–

$$L_2^1: [e_1,e_1]=e_1,\quad [e_1,e_2]=0,\quad [e_2,e_1]=0,\quad [e_2,e_2]=0,$$
$$\alpha(e_1)=0,\quad \alpha(e_2)=e_2$$

α^r	$\Gamma_{\alpha^r}(L_1^1)$	Type of $\Gamma_{\alpha^0}(L_2^1)$	$Der_{\alpha^r}(L_1^1)$	–
$r \in \mathbb{N}$	$\begin{pmatrix} 0 & c_1 \\ 0 & c_2 \end{pmatrix}$	–	$\begin{pmatrix} 0 & d_1 \\ 0 & d_2 \end{pmatrix}$	–

$$L_3^1 : [e_1, e_1] = 0, \quad [e_1, e_2] = e_1, \quad [e_2, e_1] = -e_1, \quad [e_2, e_2] = 0,$$
$$\alpha(e_1) = 0, \quad \alpha(e_2) = e_2$$

α^r	$\Gamma_{\alpha^r}(L_3^1)$	Type of $\Gamma_{\alpha^0}(L_3^1)$	$Der_{\alpha^r}(L_3^1)$	–
$r \in \mathbb{N}$	$\begin{pmatrix} 0 & 0 \\ 0 & c_1 \end{pmatrix}$	–	$\begin{pmatrix} 0 & 0 \\ 0 & d_1 \end{pmatrix}$	–

$$L_1^2 : [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = e_1, \quad [e_2, e_2] = 0,$$
$$\alpha(e_1) = 0, \quad \alpha(e_2) = e_2$$

α^r	$\Gamma_{\alpha^r}(L_1^2)$	Type of $\Gamma_{\alpha^0}(L_1^2)$	$Der_{\alpha^r}(L_1^2)$	CN
$r \in \mathbb{N}$	$\begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}$	Not small	$\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$	Yes

$$L_2^2 : [e_1, e_1] = e_1, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = 0,$$
$$\alpha(e_1) = 0, \quad \alpha(e_2) = e_2$$

α^r	$\Gamma_{\alpha^r}(L_2^2)$	Type of $\Gamma_{\alpha^0}(L_2^2)$	$Der_{\alpha^r}(L_2^2)$	CN
$r \in \mathbb{N}$	$\begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}$	Small	$\begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix}$	Yes

$$L_3^2 : [e_1, e_1] = e_1, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = 0,$$
$$\alpha(e_1) = 0, \quad \alpha(e_2) = e_2$$

α^r	$\Gamma_{\alpha^r}(L_3^2)$	Type of $\Gamma_{\alpha^0}(L_3^2)$	$Der_{\alpha^r}(L_3^2)$	CN
$r \in \mathbb{N}$	$\begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}$	Not small	$\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$	Yes

$$L_1^3 : [e_1, e_1] = e_1, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = 0,$$
$$\alpha(e_1) = 0, \quad \alpha(e_2) = be_2$$

α^r	$\Gamma_{\alpha^r}(L_1^3)$	Type of $\Gamma_{\alpha^0}(L_1^3)$	$Der_{\alpha^r}(L_1^3)$	CN
$r \in \mathbb{N}$	$\begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}$	Small	$\begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix}$	Yes

$$L_2^3 : [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = e_1, \quad [e_2, e_2] = 0,$$
$$\alpha(e_1) = 0, \quad \alpha(e_2) = be_2 \ (b \neq 1)$$

α^r	$\Gamma_{\alpha^r}(L_2^3)$	Type of $\Gamma_{\alpha^0}(L_2^3)$	$Der_{\alpha^r}(L_2^3)$	CN
$r \in \mathbb{N}$	$\begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}$	Not small	$\begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix}$	Yes

$$L_3^3 : [e_1, e_1] = 0, \quad [e_1, e_2] = e_1, \quad [e_2, e_1] = -e_1, \quad [e_2, e_2] = 0,$$
$$\alpha(e_1) = 0, \quad \alpha(e_2) = be_2 \ (b \neq 1)$$

α^r	$\Gamma_{\alpha^r}(L_3^3)$	Type of $\Gamma_{\alpha^0}(L_3^3)$	$Der_{\alpha^r}(L_3^3)$	CN
$r \in \mathbb{N}$	$\begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}$	Not small	$\begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix}$	Yes

$$L_1^4 : [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = e_1, \quad [e_2, e_2] = 0,$$
$$\alpha(e_1) = ae_1, \quad \alpha(e_2) = e_2$$

α^r	$\Gamma_{\alpha^r}(L_1^4)$	Type of $\Gamma_{\alpha^0}(L_1^4)$	$Der_{\alpha^r}(L_1^4)$	CN
$r \in \mathbb{N}$	$\begin{pmatrix} c_1 & 0 \\ 0 & \frac{c_1}{a^r} \end{pmatrix}$	Small	$\begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix}$	Yes

$$L_2^4 : [e_1, e_1] = 0, \quad [e_1, e_2] = -e_1, \quad [e_2, e_1] = e_1, \quad [e_2, e_2] = 0,$$
$$\alpha(e_1) = ae_1, \quad \alpha(e_2) = e_2$$

α^r	$\Gamma_{\alpha^r}(L_2^4)$	Type of $\Gamma_{\alpha^0}(L_2^4)$	$Der_{\alpha^r}(L_2^4)$	CN
$r \in \mathbb{N}$	$\begin{pmatrix} c_1 & 0 \\ 0 & \frac{c_1}{a^r} \end{pmatrix}$	Small	$\begin{pmatrix} d_1 & 0 \\ 0 & 0 \end{pmatrix}$	Yes

$$L_1^5 : [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = 0 = e_1,$$
$$\alpha(e_1) = b^2 e_1, \quad \alpha(e_2) = be_2$$

α^r	$\Gamma_{\alpha^r}(L_1^5)$	Type of $\Gamma_{\alpha^0}(L_1^5)$	$Der_{\alpha^r}(L_1^5)$	CN
$r \in \mathbb{N}$	$\begin{pmatrix} c_1 & 0 \\ 0 & \frac{c_1}{b^r} \end{pmatrix}$	Small	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	Yes

$$L_1^6 : [e_1, e_1] = 0, \quad [e_1, e_2] = e_1, \quad [e_2, e_1] = z_1 e_1, \quad [e_2, e_2] = t_1 e_1,$$
$$\alpha(e_1) = 0, \quad \alpha(e_2) = e_2$$

α^r		$\Gamma_{\alpha^r}(L_1^6)$	Type of $\Gamma_{\alpha^0}(L_1^6)$	$Der_{\alpha^r}(L_1^6)$	CN
$r = 0$	$z_1 = -1$	$\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}$	Small	$\begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix}$	Yes
$r = 0$	$z_1 \neq -1$	$\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}$	Small	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	Yes
$r = 1$		$\begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix}$		$\begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix}$	
$r > 1$		$\begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix}$		$\begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix}$	

$$L_2^6 : [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = e_1, \quad [e_2, e_2] = t_1 e_1,$$
$$\alpha(e_1) = 0, \quad \alpha(e_2) = e_1$$

α^r	$\Gamma_{\alpha^r}(L_1^5)$	Type of $\Gamma_{\alpha^0}(L_1^5)$	$Der_{\alpha^r}(L_1^5)$	CN
$r = 0$	$\begin{pmatrix} c_1 & c_2 \\ 0 & c_1 \end{pmatrix}$	Not small	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	Yes
$r \geq 1$	$\begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix}$		$\begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix}$	

$$L_3^6 : [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = e_1, \quad [e_2, e_2] = e_1,$$
$$\alpha(e_1) = 0, \quad \alpha(e_2) = e_1$$

α^r	$\Gamma_{\alpha^r}(L_3^6)$	Type of $\Gamma_{\alpha^0}(L_3^6)$	$Der_{\alpha^r}(L_3^6)$	CN
$r=0$	$\begin{pmatrix} c_1 & c_2 \\ 0 & c_1 \end{pmatrix}$	Small	$\begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix}$	Yes
$r \geq 1$	$\begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix}$		$\begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix}$	

$$L_1^7 : [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = e_1,$$
$$\alpha(e_1) = e_1, \quad \alpha(e_2) = e_1 + e_2$$

α^r	$\Gamma_{\alpha^r}(L_1^7)$	Type of $\Gamma_{\alpha^0}(L_1^7)$	$Der_{\alpha^r}(L_1^7)$	CN
$r=0$	$\begin{pmatrix} c_1 & c_2 \\ 0 & c_1 \end{pmatrix}$	Small	$\begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix}$	Yes
$r \geq 1$	$\begin{pmatrix} c_1 & c_2 \\ 0 & c_1 \end{pmatrix}$		$\begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix}$	

Summarizing the obtained results, the dimensions of the spaces of α^r-derivations of Hom-Leibniz algebras in dimension 2 vary between 0, 1 and 2.

We have $\dim(Der_{\alpha^r}(L_i^j)) = 2$ in the following cases:

1) $Der_{\alpha^r}(L_1^2) = Der_{\alpha^r}(L_3^2) =< \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} >$, with $\alpha = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$,

2) $Der_{\alpha^r}(L_2^1) =< \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} >$, with $\alpha = id$,

3) $Der_{\alpha^r}(L_4^1) =< \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} >$, with $\alpha = id$.

For $\dim(Der_{\alpha^r}(L_i^j)) = 0$, we distinguish two cases:

1) $Der_{\alpha^r}(L_1^5)$, $r = 0$, $z_1 \neq -1$, with $\alpha = \begin{pmatrix} b^2 & 0 \\ 0 & b \end{pmatrix}$,

2) $Der_{\alpha^r}(L_2^6)$, $r = 0$ with $\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

For the other cases of $Der_{\alpha^r}(L_i^j)$, the dimension is equal to 1.

Moreover, $\dim(\Gamma_{\alpha^r}(L_i^j))$ vary between one and two.

We have $\dim(\Gamma_{\alpha^r}(L_i^j)) = 2$ for the following cases:

1) $\Gamma_{\alpha^r}(L_1^1) = \Gamma_{\alpha^r}(L_2^1) = \Gamma_{\alpha^r}(L_3^1) =< id, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} >, \alpha = id$,

2) $\Gamma_{\alpha^r}(L_2^6) = \Gamma_{\alpha^r}(L_3^6) =< id, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} >, \alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,

3) $\Gamma_{\alpha^r}(L_1^7) =< id, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} >, \alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

For the other cases of $\dim(\Gamma_{\alpha^r}(L_i^j))$ the dimension is equal to 1.

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Chapter 4
Color Hom-Lie Algebras, Color Hom-Leibniz Algebras and Color Omni-Hom-Lie Algebras

Abdoreza Armakan and Sergei Silvestrov

Abstract In this paper, the representations of color hom-Lie algebras have been reviewed and the existence of a series of coboundary operators is demonstrated. Moreover, the notion of a color omni-hom-Lie algebra associated to a linear space and an even invertible linear map have been introduced. In addition, characterization method for regular color hom-Lie algebra structures on a linear space is examined and it is shown that the underlying algebraic structure of the color omni-hom-Lie algebra is a color hom-Leibniz a algebra.

Keywords Color Hom-Lie algebras · Color Omni-Hom-Lie algebra · Color Hom-Leibniz algebra

2020 Mathematics Subject Classification 17B61 · 17D30 · 17B75 · 17A32

4.1 Introduction

The investigations of various quantum deformations or q-deformations of Lie algebras began a period of rapid expansion in 1980s stimulated by introduction of quantum groups motivated by applications to the quantum Yang-Baxter equation, quantum inverse scattering methods and constructions of the quantum deformations of universal enveloping algebras of semi-simple Lie algebras. Various q-deformed Lie algebras have appeared in physical contexts such as string theory, vertex models in conformal field theory, quantum mechanics and quantum field theory in the context

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of deformations of infinite-dimensional algebras, primarily the Heisenberg algebras, oscillator algebras and Witt and Virasoro algebras. In [3, 25–28, 31–33, 37, 38, 47–49], it was in particular discovered that in these q-deformations of Witt and Visaroro algebras and some related algebras, some interesting q-deformations of Jacobi identities, extending Jacobi identity for Lie algebras, are satisfied. This has been one of the initial motivations for the development of general quasi-deformations and discretizations of Lie algebras of vector fields using more general σ-derivations (twisted derivations) in [36].

Hom-Lie algebras and more general quasi-hom-Lie algebras were introduced first by Larsson, Hartwig and Silvestrov [36], where the general quasi-deformations and discretizations of Lie algebras of vector fields using more general σ-derivations (twisted derivations) and a general method for construction of deformations of Witt and Virasoro type algebras based on twisted derivations have been developed, initially motivated by the q-deformed Jacobi identities observed for the q-deformed algebras in physics, along with q-deformed versions of homological algebra and discrete modifications of differential calculi. Hom-Lie algebras, hom-Lie superalgebras, hom-Lie color algebras and more general quasi-Lie algebras and color quasi-Lie algebras where introduced first in [43, 44, 70]. Quasi-Lie algebras and color quasi-Lie algebras encompass within the same algebraic framework the quasi-deformations and discretizations of Lie algebras of vector fields by σ-derivations obeying twisted Leibniz rule, and the well-known generalizations of Lie algebras such as color Lie algebras, the natural generalizations of Lie algebras and Lie superalgebras. In quasi-Lie algebras, the skew-symmetry and the Jacobi identity are twisted by deforming twisting linear maps, with the Jacobi identity in quasi-Lie and quasi-hom-Lie algebras in general containing six twisted triple bracket terms. In hom-Lie algebras, the bilinear product satisfies the non-twisted skew-symmetry property as in Lie algebras, and the hom-Lie algebras Jacobi identity has three terms twisted by a single linear map, reducing to the Lie algebras Jacobi identity when the twisting linear map is the identity map. Hom-Lie admissible algebras have been considered first in [52], where in particular the hom-associative algebras have been introduced and shown to be hom-Lie admissible, that is leading to hom-Lie algebras using commutator map as new product, and in this sense constituting a natural generalization of associative algebras as Lie admissible algebras. Since the pioneering works [36, 42–45, 52], hom-algebra structures expanded into a popular area with increasing number of publications in various directions. Hom-algebra structures of a given type include their classical counterparts and open broad possibilities for deformations, hom-algebra extensions of cohomological structures and representations, formal deformations of hom-associative and hom-Lie algebras, hom-Lie admissible hom-coalgebras, hom-coalgebras, hom-Hopf algebras [4, 22, 34, 42, 46, 53–55, 67, 73, 75]. Hom-Lie algebras, hom-Lie superalgebras and color hom-Lie algebras and their n-ary generalizations have been further investigated in various aspects for example in [2, 4, 5, 7–15, 17–22, 24, 35, 39–41, 50–56, 58–60, 67–78].

In Sect. 4.2, we review basic concepts of hom-associative algebras, hom-modules and color hom-Lie algebras. In Sect. 4.3, (σ, τ)-differential graded commutative color algebras are defined and the classical result about the relation between Lie algebra structures and differential graded commutative color algebras structures is generalized to relation between color hom-Lie-algebras and (σ, τ)-differential graded commutative color algebras. In Sect. 4.4, representations of color hom-Lie algebras are considered, adjoint representation and its morphism interpretations are investigated, and hom-cochains, coboundary operators and cohomological complex are described, generalizing some results in [6, 67]. Moreover, the notion of a color omni-hom-Lie algebra associated to a linear space and an even invertible linear map is introduced, and it is shown that the color hom-Leibniz algebras appear as underlying algebraic structure of the color omni-hom-Lie algebras.

4.2 Hom-Associative Algebras, Hom-Modules and Color Hom-Lie Algebras

We start by recalling some basic concepts from [52, 55] where also various examples and properties of hom-associative algebras can be found. Throughout this paper, we use **k** to denote a commutative unital ring, for example a field.

Definition 4.1 The following notions will be used through the rest of the paper.

(i) A hom-module is a pair (M, α) consisting of an **k**-module M and a linear operator $\alpha : M \to M$.

(ii) A hom-associative algebra is a triple $(A, \cdot, \alpha)$ consisting of an **k**-module A, a bilinear map $\cdot : A \times A \to A$ called the multiplication and a linear operator $\alpha : A \to A$ which satisfies the hom-associativity condition

$$\alpha(x) \cdot (y \cdot z) = (x \cdot y) \cdot \alpha(z),$$

for $x, y, z \in A$.

(iii) A hom-associative algebra is called multiplicative if the linear map α is also an algebra morphism of $(A, \cdot)$, that is, $\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)$ for all $x, y \in A$.

(iv) A hom-associative algebra or a hom-module is called involutive if $\alpha^2 = \mathrm{id}$.

(v) Let (M, α) and (N, β) be two hom-modules. A **k**-linear map $f : M \to N$ is called a morphism of hom-modules if $f(\alpha(x)) = \beta(f(x))$ for all $x \in M$.

(vi) Let $(A, \cdot, \alpha)$ and $(B, \bullet, \beta)$ be two hom-associative algebras. A **k**-linear map $f : A \to B$ is called a morphism of hom-associative algebras if

1) $f(x \cdot y) = f(x) \bullet f(y)$,
2) $f(\alpha(x)) = \beta(f(x))$, for all $x, y \in A$.

(vii) If $(A, \cdot, \alpha)$ is a hom-associative algebra, then $B \subseteq A$ is called a hom-associative subalgebra of A if it is closed under the multiplication $\cdot$ and $\alpha(B) \subseteq B$. A submodule I is called a (two-sided) hom-ideal of A if $x \cdot y \in I$ and $y \cdot x \in I$ for all $x \in I$ and $y \in A$, and also $\alpha(I) \subseteq I$.

Definition 4.2 A hom-Lie algebra is a triple $(\mathfrak{g}, [\cdot,\cdot], \alpha)$, where $\mathfrak{g}$ is a vector space equipped with a skew-symmetric bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ and a linear map $\alpha : \mathfrak{g} \to \mathfrak{g}$, such that for all $x, y, z \in \mathfrak{g}$, the hom-Jacobi identity holds,

$$\sum_{\circlearrowleft\{x,y,z\}} [\alpha(x), [y, z]] = [\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0. \quad \text{hom-Jacobi identity}$$

A hom-Lie algebra is called a multiplicative hom-Lie algebra if α is an algebra morphism, that is, $\alpha([x, y]) = [\alpha(x), \alpha(y)]$ for any $x, y \in \mathfrak{g}$. We call a hom-Lie algebra regular if α is an automorphism. Moreover, it is called involutive if $\alpha^2 = id$. A linear subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is a hom-Lie sub-algebra of $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ if $\alpha(\mathfrak{h}) \subseteq \mathfrak{h}$ and $\mathfrak{h}$ is closed under the bracket operation, that is, $[x_1, x_2]_{\mathfrak{g}} \in \mathfrak{h}$, for all $x_1, x_2 \in \mathfrak{h}$. Let $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ be a multiplicative hom-Lie algebra. Let $\alpha^k = \underbrace{\alpha \circ \cdots \circ \alpha}_{k-times}$ denote the k-times composition of α by itself, for any nonnegative integer k, where $\alpha^0 = Id$ and $\alpha^1 = \alpha$. For a regular hom-Lie algebra $\mathfrak{g}$, let $\underbrace{\alpha^{-k} = \alpha^{-1} \circ \cdots \circ \alpha^{-1}}_{k-times}$.

Since color hom-Lie algebras generalize Lie color algebras, we recall first definition of Lie color algebras. For a Γ-graded linear space, $X = \bigoplus_{\gamma\in\Gamma} X_\gamma$, the elements of $H(X) = \bigcup_{\Gamma} X_\gamma$ are called homogenous of degree γ, for all $\gamma \in \Gamma$. Given a commutative group Γ (referred to as the grading group), a commutation factor on Γ with values in the multiplicative group $K \setminus \{0\}$ of a field K of characteristic 0 is a map $\varepsilon : \Gamma \times \Gamma \to K \setminus \{0\}$, satisfying for all $j, k, l \in \Gamma$,

1) $\varepsilon(j + k, l) = \varepsilon(j, l)\varepsilon(k, l)$,
2) $\varepsilon(j, l + k) = \varepsilon(j, l)\varepsilon(j, k)$,
3) $\varepsilon(j, k)\varepsilon(k, j) = 1$.

Definition 4.3 ([16, 29, 30, 57, 61–66]) A Γ-graded ε-Lie algebra (or a Lie color algebra) is a Γ-graded linear space $X = \bigoplus_{\gamma\in\Gamma} X_\gamma$, with a bilinear multiplication (bracket) $[\cdot,\cdot] : X \times X \to X$ satisfying for all homogeneous elements $x, y, z \in \bigcup_{\gamma\in\Gamma} X_\gamma$,

1) **Grading axiom:** $[X_j, X_k] \subseteq X_{j+k}, \ \forall j, k \in \Gamma$;
2) **Graded skew-symmetry:** $[x, y] = -\varepsilon(x, y)[y, x]$;
3) **Generalized Jacobi identity:**

$$\sum_{\circlearrowleft\{x,y,z\}} \varepsilon(z, x)[x, [y, z]] = \varepsilon(z, x)[x, [y, z]] + \varepsilon(y, z)[z, [x, y]] + \varepsilon(x, y)[y, [z, x]] = 0.$$

where $\varepsilon(x, y) = \varepsilon(j, k)$ for all $j, k \in \Gamma, x \in \Gamma_j, y \in \Gamma_k$.

Color hom-Lie algebras are a special class of general color quasi-Lie algebras (Γ-graded quasi-Lie algebras) defined first in [43, 44, 70].

Definition 4.4 ([1, 23, 24, 43, 44, 70, 77]) A color hom-Lie algebra is a quadruple $(\mathfrak{g}, [\cdot,\cdot], \varepsilon, \alpha)$ consisting of a Γ-graded linear space $\mathfrak{g}$, a bi-character ε, an even bilinear mapping

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

(that is, $[\mathfrak{g}_j, \mathfrak{g}_k] \subseteq \mathfrak{g}_{j+k}$, for all $j, k \in \Gamma$) and an even homomorphism $\alpha : \mathfrak{g} \to \mathfrak{g}$ such that for all homogeneous elements $x, y, z \in \bigcup_{\gamma\in\Gamma} \mathfrak{g}_\Gamma$,

1) ε-**skew symmetry:** $[x, y] = -\varepsilon(x, y)[y, x]$,
2) ε-**hom-Jacobi identity:** $\sum_{\circlearrowleft\{x,y,z\}} \varepsilon(z, x)[\alpha(x), [y, z]] = 0$.

where $\varepsilon(x, y) = \varepsilon(j, k)$ for all $j, k \in \Gamma$, $x \in \Gamma_j$, $y \in \Gamma_k$.

We call a color hom-Lie algebra *regular* if α is an automorphism. Let $\mathfrak{g} = \bigoplus_{\gamma\in\Gamma} \mathfrak{g}_\gamma$ and $\mathfrak{h} = \bigoplus_{\gamma\in\Gamma} \mathfrak{h}_\gamma$ be two Γ-graded color Lie algebras. A linear mapping $f : \mathfrak{g} \to \mathfrak{h}$ is said to be homogenous of the degree $\mu \in \Gamma$ if $f(\mathfrak{g}_\gamma) \subseteq \mathfrak{h}_{\gamma+\mu}$, for all $\gamma \in \Gamma$. If in addition, f is homogenous of degree zero, that is, $f(\mathfrak{g}_\gamma) \subseteq \mathfrak{h}_\gamma$ holds for any $\gamma \in \Gamma$, then f is said to be even. Let $(\mathfrak{g}, [\cdot,\cdot], \varepsilon, \alpha)$ and $(\mathfrak{g}', [\cdot,\cdot]', \varepsilon', \alpha')$ be two color hom-Lie algebras. A linear mapping of degree zero $f : \mathfrak{g} \to \mathfrak{g}'$ is called a morphism of color hom-Lie algebras if

1) $[f(x), f(y)]' = f([x, y])$, for all $x, y \in \mathfrak{g}$,
2) $f \circ \alpha = \alpha' \circ f$.

In particular, if α is a morphism of color Lie algebra to itself, then $(\mathfrak{g}, [\cdot,\cdot], \varepsilon, \alpha)$ is called a multiplicative color hom-Lie algebra. It will be useful to mention that if a hom-associative color algebra is defined as a triple (V, μ, α) consisting of a color space $V = \bigoplus_{\gamma\in\Gamma} V_\gamma$, an even bilinear map $\mu : V \times V \to V$ and an even homomorphism $\alpha : V \to V$ satisfying $\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z))$, for all homogeneous elements $x, y, z \in H(V) = \bigoplus_{\gamma\in\Gamma} V_\gamma$, then the hom-associativity holds actually for all $x, y, z \in V$, since Hom-associativity is multi-linear in its three arguments.

Example 4.1 ([1]) As in case of hom-associative and hom-Lie algebras, examples of multiplicative color hom-Lie algebras can be constructed by the standard method of composing multiplication with algebra morphism. Let $(\mathfrak{g}, [\cdot,\cdot], \varepsilon)$ be a color Lie algebra and α be a color Lie algebra morphism. Then $(\mathfrak{g}, [\cdot,\cdot]_\alpha := \alpha \circ [\cdot,\cdot], \varepsilon, \alpha)$ is a multiplicative color hom-Lie algebra.

As for an associative algebra and a Lie algebra, a hom-associative color algebra (V, μ, α) gives a color hom-Lie algebra by antisymmetrization. We denote this color hom-Lie algebra by $(A, [\cdot,\cdot]_A, \beta_A)$, where $\beta_A = \alpha$ and $[x, y]_A = xy - yx$, for all $x, y \in A$.

4.3 (σ, τ)-Differential Graded Commutative Color Algebra

Definition 4.5 Let A be an associative algebra, and let σ, τ denote two algebra endomorphisms on A. A (σ, τ)-derivation on A is a linear map $D : A \to A$ such that $D(ab) = D(a)\tau(b) + \sigma(a)D(b)$, for all $a, b \in A$. A σ-derivation on A is a (σ, id)-derivation.

In [36], hom-Lie algebra or more general quasi hom-Lie structures have been shown to arise in fundamental ways for σ-derivations on associative algebras. We define (σ, τ)-differential graded commutative color algebras as follows.

Definition 4.6 A (σ, τ)-differential graded commutative color algebra is a quadruple $(\mathcal{A}, \sigma, \tau, d_{\mathcal{A}})$ consisting of a Γ-graded commutative algebra $\mathcal{A}$, two algebra endomorphisms σ and τ of degree zero and an operator $d_{\mathcal{A}}$ of degree p such that

1) $d_{\mathcal{A}}^2 = 0$;
2) $d_{\mathcal{A}}$ commutes with σ and τ;
3) $d_{\mathcal{A}}(ab) = d_{\mathcal{A}}(a)\tau(b) + \varepsilon(a, b)\sigma(a)d_{\mathcal{A}}(b)$, for homogeneous $a, b \in \mathcal{A}$.

For a Γ-graded linear space $\mathfrak{g}$, denote by $\bigwedge \mathfrak{g}^* = \sum_k \bigwedge^k \mathfrak{g}^*$, $k \in \Gamma$ the exterior algebra of $\mathfrak{g}^*$. Now let $\mathfrak{g}$ be a color hom-Lie algebra, for an endomorphism β on $\mathfrak{g}$, its dual map $\beta^* : \mathfrak{g}^* \to \mathfrak{g}^*$ naturally extends to an algebra morphism, $(\beta^*\xi)(x_1, \dots, x_k) = \xi(\beta(x_1), \dots, \beta(x_k))$, for all $\xi \in \bigwedge \mathfrak{g}^*$ and all homogenous elements $x_1, \dots, x_k \in \mathfrak{g}$. Now, recall from [1, 12] that

$$d\xi(x_0, \dots, x_p) = \sum_{i<j} (-1)^{i+j} \theta_{ij}(x) \xi(\alpha(x_0), \dots, \alpha(x_{i-1}), \tag{4.1}$$

$$[x_i, x_j], \alpha(x_{i-1}), \dots, \widehat{x_j}, \dots, \alpha(x_p)), \tag{4.2}$$

for all $\xi \in \bigwedge \mathfrak{g}^*$ and all homogenous elements $x_1, \dots, x_k \in \mathfrak{g}$, where

$$\theta_{ij}(x) = \varepsilon(|x_{i+1}| + \cdots + |x_{j-1}|, |x_j|).$$

Proposition 4.1 *The following properties hold:*

(i) $d^2 = 0$,
(ii) $\alpha^* \circ d = d \circ \alpha^*$,
(iii) $d(\xi \wedge \eta) = d\xi \wedge \alpha^*\eta + \varepsilon(k, l)\alpha^*\xi \wedge d\eta$, *for all* $\xi \in \wedge^k \mathfrak{g}^*$ *and* $\eta \in \wedge^l \mathfrak{g}^*$.

Proof (i) The proof for (i) can be found in [1, 4].
(ii) Let $\xi \in \wedge^k \mathfrak{g}^*$. We have

$$\alpha^* \circ d\xi(x_1, \ldots, x_{k+1}) = d\xi(\alpha(x_1), \ldots, \alpha(x_{k+1}))$$
$$= \sum_{i<j} \varepsilon(i, j)\varepsilon(|x_1| + \cdots + |x_{i-1}|, |x_i|)\varepsilon(|x_1| + \cdots + |x_{j-1}|, |x_j|)\varepsilon(|x_i|, |x_j|)$$
$$\xi([\alpha(x_i), \alpha(x_j)], \alpha^2(x_1), \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, \alpha^2(x_{k+1}))$$
$$= \sum_{i<j} \varepsilon(i, j)\varepsilon(|x_1| + \cdots + |x_{i-1}|, |x_i|)\varepsilon(|x_1| + \cdots + |x_{j-1}|, |x_j|)\varepsilon(|x_i|, |x_j|)$$
$$\alpha^*\xi([x_i, x_j], \alpha(x_1), \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, \alpha(x_{k+1}))$$
$$= d(\alpha^*\xi)(x_1, \ldots, x_{k+1}).$$

(iii) We use induction on k. If $k = 1$, then $\xi \wedge \eta \in \wedge^{1+l}\mathfrak{g}^*$ and

$$d(\xi \wedge \eta)(x_1, \ldots, x_{l+2})$$
$$= \sum_{i<j} \varepsilon(i, j)\varepsilon(|x_1| + \cdots + |x_{i-1}|, |x_i|)\varepsilon(|x_1| + \cdots + |x_{j-1}|, |x_j|)\varepsilon(|x_i|, |x_j|)$$
$$\xi \wedge \eta([x_i, x_j], \alpha(x_1), \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, \alpha(x_{l+2}))$$
$$= \sum_{i<j} \varepsilon(i, j)\varepsilon(|x_1| + \cdots + |x_{i-1}|, |x_i|)\varepsilon(|x_1| + \cdots + |x_{j-1}|, |x_j|)\varepsilon(|x_i|, |x_j|)$$
$$\{\xi([x_i, x_j])\eta(\alpha(x_1), \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, \alpha(x_{l+2}))$$
$$+ \sum_{p<i} (-1)^p \varepsilon(|x_i| + |x_j| + |x_1| + \cdots + |x_{p-1}|, |x_p|)$$
$$\xi(\alpha(x_p))\eta([x_i, x_j], \alpha(x_1), \ldots, \widehat{x_p}, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, \alpha(x_{l+2}))$$
$$+ \sum_{i<p<j} (-1)^{p-1} \varepsilon(|x_j| + |x_1| + \cdots + |x_{p-1}|, |x_p|)$$
$$\xi(\alpha(x_p))\eta([x_i, x_j], \alpha(x_1), \ldots, \widehat{x_p}, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, \alpha(x_{l+2}))$$
$$+ \sum_{j<p} (-1)^{p-2} \varepsilon(|x_1| + \cdots + |x_{p-1}|, |x_p|)$$
$$\xi(\alpha(x_p))\eta([x_i, x_j], \alpha(x_1), \ldots, \widehat{x_p}, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, \alpha(x_{l+2}))\}$$
$$= d\xi \wedge \alpha^*\eta(x_1, \ldots, x_{l+2}) - \alpha^*\xi \wedge d\eta(x_1, \ldots, x_{l+2}).$$

Thus, $d(\xi \wedge \eta) = d\xi \wedge \alpha^*\eta + \varepsilon(1, l)\alpha^*\xi \wedge d\eta$ when $k = 1$. Now, suppose that for $k = n$,

$$d(\xi \wedge \eta) = d\xi \wedge \alpha^*\eta + \varepsilon(n, l)\alpha^*\xi \wedge d\eta.$$

Let $\omega \in \mathfrak{g}^*$. We have $\xi \wedge \omega \in \wedge^{n+l}\mathfrak{g}^*$ and

$$\begin{aligned} d(\xi \wedge \omega \wedge \eta) =& d\xi \wedge \alpha^*(\omega \wedge \eta) + \varepsilon(n,l)\alpha^*\xi \wedge d(\omega \wedge \eta) \\ =& d\xi \wedge (\alpha^*\omega \wedge \alpha^*\eta) + \varepsilon(n,l)\alpha^*\xi \wedge (d\omega \wedge \alpha^*\eta + \varepsilon(1,l)\alpha^*\omega \wedge d\eta) \\ =& (d\xi \wedge \alpha^*\omega + \varepsilon(n,l)\alpha^*\xi \wedge d\omega) \wedge \alpha^*\eta + \varepsilon(n+1,l)(\alpha^*\xi \wedge \alpha^*\omega)d\eta \\ =& d(\xi \wedge \omega) \wedge \alpha^*\eta + \varepsilon(n+1,l)\alpha^*\xi \wedge \omega) \wedge d\eta), \end{aligned}$$

which completes the proof. □

Part (iii) of the above proposition says that $(\wedge\mathfrak{g}^*, \alpha^*, \alpha^*, d)$ is an (α^*, α^*)-differential graded commutative algebra. The converse of the above conclusion is also true. Thus, we have the following theorem, which generalizes the classical result about the relation between Lie algebra structures and DGCA structures.

Theorem 4.1 *The triple* $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ *is a color hom-Lie algebra if and only if the quadruple* $(\wedge\mathfrak{g}^*, \alpha^*, \alpha^*, d)$ *is an* (α^*, α^*)*-differential graded commutative color algebra, where d is defined in* (4.1) *and the skewsupersymmetric bracket*

$$[\cdot,\cdot] : \wedge^2\mathfrak{g} \to \mathfrak{g}$$

is defined by

$$[[x_1, x_2]\eta] = -d\eta(x_1, x_2),$$

for all $\eta \in \mathfrak{g}^*$*,* $x_1, x_2 \in \mathfrak{g}$*.*

Proof According to Definition 4.6 and Proposition 4.1, we only need to prove the converse. Suppose that $(\wedge\mathfrak{g}^*, \alpha^*, \alpha^*, d)$ is an (α^*, α^*)-differential graded commutative color algebra. We have

$$\begin{aligned} d(\alpha^*\eta)(x_1, x_2) &= -[\alpha^*\eta, [x_1, x_2]] = -[\eta, \alpha([x_1, x_2])], \\ \alpha^* d\eta(x_1, x_2) &= -[\eta, [\alpha(x_1), \alpha(x_2)]]. \end{aligned}$$

Moreover, $\alpha([x_1, x_2]) = [\alpha(x_1), \alpha(x_2)]$, which implies that α is an algebra endomorphism. On the other hand, for homogenous elements $x_1, x_2, x_3 \in \mathfrak{g}$ and $\xi, \eta \in \mathfrak{g}^*$, we have

$$\begin{aligned} d(\xi \wedge \eta)(x_1, x_2, x_3) &= d\xi \wedge (\alpha^*\eta) - (\alpha^*\xi) \wedge d\eta(x_1, x_2, x_3) \\ =& \quad \varepsilon(1,2)d\xi(x_1, x_2)\eta(\alpha(x_3)) - \varepsilon(1,3)d\xi(x_1, x_3)\eta(\alpha(x_2)) \\ &+ \varepsilon(2,3)d\xi(x_2, x_3)\eta(\alpha(x_1)) - \varepsilon(1,2)\xi(\alpha(x_1))d\eta(x_2, x_3) \\ &+ \varepsilon(2,1)\xi(\alpha(x_2))d\eta(x_1, x_3) - \varepsilon(3,1)\xi(\alpha(x_3))d\eta(x_1, x_2) \\ =& - \varepsilon(1,2)\xi(x_1, x_2)\eta(\alpha(x_3)) + \varepsilon(1,3)\varepsilon(|x_1| + |x_2|, |x_3|)\xi(x_1, x_3)\eta(\alpha(x_2)) \\ &- \varepsilon(2,3)\varepsilon(|x_1| + |x_2|, |x_3|)\xi(x_2, x_3)\eta(\alpha(x_1)) \\ &+ \varepsilon(1,2)\xi(\alpha(x_1))d\eta([x_2, x_3]) \\ &- \varepsilon(2,1)\varepsilon(|x_1|, |x_3|)\xi(\alpha(x_2))d\eta([x_1, x_3]) \\ &+ \varepsilon(|x_1| + |x_2|, |x_3|)\varepsilon(|x_2|, |x_3|)\xi(\alpha(x_3))d\eta([x_1, x_2]) \end{aligned}$$

$$\begin{aligned} = \quad & \varepsilon(1,2)\xi \wedge \eta([x_1,x_2],\alpha(x_3)) \\ & + \varepsilon(1,3)\varepsilon(|x_1|+|x_2|,|x_3|)\varepsilon(|x_1|,|x_3|)\xi \wedge \eta([x_1,x_3],\alpha(x_2)) \\ & + \varepsilon(2,3)\varepsilon(|x_1|,|x_2|)\varepsilon(|x_1|+|x_2|,|x_3|)\varepsilon(|x_2|,|x_3|)\xi \wedge \eta([x_2,x_3],\alpha(x_1)). \end{aligned}$$

Therefore, we have

$$\begin{aligned} 0 = d(d\xi)(x_1,x_2,x_3) &= \varepsilon(1,2)d\xi([x_1,x_2],\alpha(x_3)) \\ & \quad + \varepsilon(1,3)\varepsilon(|x_1|+|x_2|,|x_3|)\varepsilon(|x_1|,|x_3|)d\xi([x_1,x_3],\alpha(x_2)) \\ & \quad + \varepsilon(2,3)\varepsilon(|x_1|,|x_2|)\varepsilon(|x_1|+|x_2|,|x_3|)\varepsilon(|x_2|,|x_3|)d\xi([x_2,x_3],\alpha(x_1)) \\ &= \varepsilon(1,2)d\xi([x_1,x_2],\alpha(x_3)) + \varepsilon(|x_2|,|x_3|)d\xi([x_1,x_3],\alpha(x_2)) \\ & \quad + \varepsilon(|x_2|+|x_3|,|x_1|)d\xi([x_2,x_3],\alpha(x_1)) \\ &= \xi(\varepsilon(1,2)[[x_1,x_2],\alpha(x_3)] + \varepsilon(1,3)[[x_1,x_3],\alpha(x_2)] + \varepsilon(2,3)[[x_2,x_3],\alpha(x_1)]). \end{aligned}$$

Thus,

$$\varepsilon(1,2)[[x_1,x_2],\alpha(x_3)] + \varepsilon(1,3)[[x_1,x_3],\alpha(x_2)] + \varepsilon(2,3)[[x_2,x_3],\alpha(x_1)] = 0,$$

which completes the proof. □

4.4 Representations of Color Hom-Lie Algebras

I this section, we are going to generalize some results from [6, 67]. We start by the definition of a representation of a color hom-Lie algebra.

Definition 4.7 Let $(\mathfrak{g}, [\cdot,\cdot], \varepsilon, \alpha)$ be a color hom-Lie algebra. A representation of $\mathfrak{g}$ is a triplet (M, ρ, β), where M is a Γ-graded linear space, $\beta \in End(M)_0$ and $\rho : \mathfrak{g} \to End(M)$ is an even linear map satisfying

$$\rho([x,y]) \circ \beta = \rho(\alpha(x)) \circ \rho(y) - \varepsilon(x,y)\rho(\alpha(y)) \circ \rho(x) \tag{4.3}$$

for all homogeneous elements $x, y \in \mathfrak{g}$.

Let $\mathfrak{g}$ be a Γ-graded linear space and let $\beta \in \mathfrak{gl}(\mathfrak{g})_{\bar{0}}$. For any homogenous elements $x, y \in \mathfrak{gl}(\mathfrak{g})$, define $[\cdot,\cdot]_\beta : \mathfrak{gl}(\mathfrak{g}) \times \mathfrak{gl}(\mathfrak{g}) \to \mathfrak{gl}(\mathfrak{g})$, by

$$[x,y]_\beta = \beta x \beta^{-1} y \beta^{-1} - \varepsilon(x,y)\beta y \beta^{-1} x \beta^{-1}$$

Recall from [1], the adjoint action on $\mathfrak{gl}(\mathfrak{g})$:

$$Ad_\beta(x) = \beta x \beta^{-1}$$

for an element x which satisfies $\alpha(x) = x$.

It is shown in [1, 4] that $(\mathfrak{g}, ad_k, \alpha)$ is a representation of $\mathfrak{g}$ which is called the adjoint representation of the color hom-Lie algebra $\mathfrak{g}$.

Proposition 4.2 *Let $\mathfrak{g}$ and $[\cdot,\cdot]_\beta$ be as described above. Then $(\mathfrak{gl}(\mathfrak{g}), [\cdot,\cdot]_\beta, Ad_\beta)$ is a regular color hom Lie algebra.*

Proof One can easily see that Ad_β is invertible, since $Ad_\beta \circ Ad_{\beta^{-1}} = \mathrm{id}$. Moreover,

$$\begin{aligned}
[Ad_\beta(x)Ad_\beta(y)]_\beta &= [\beta x\beta^{-1}, \beta y\beta^{-1}]_\beta \\
&= \beta^2 x\beta^{-1}y\beta^{-1}\beta^{-1} - \varepsilon(x,y)\beta^2 y\beta^{-1}x\beta^{-1}\beta^{-1} = Ad_\beta([x,y]_\beta)
\end{aligned}$$

for all $x, y \in \mathfrak{gl}(\mathfrak{g})$. Furthermore,

$$\begin{aligned}
&\sum_{\circlearrowleft\{x,y,z\}} \varepsilon(x,z)[[x,y]_\beta, Ad_\beta(z)]_\beta \\
&= \sum_{\circlearrowleft\{x,y,z\}} (\varepsilon(x,z)([\beta x\beta^{-1}, \beta y\beta^{-1}z\beta^{-1}]_\beta) - [\beta x\beta^{-1}, \varepsilon(z,y)\beta z\beta^{-1}y\beta^{-1}]_\beta) \\
&= \sum_{\circlearrowleft\{x,y,z\}} (\varepsilon(x,z)\beta^2 x\beta^{-1}y\beta^{-1}z\beta^{-1}\beta^{-1} - \varepsilon(z, y+x)\beta^2 x\beta^{-1}z\beta^{-1}y\beta^{-1}\beta^{-1} \\
&\quad - \varepsilon(x,y)\beta^2 y\beta^{-1}z\beta^{-1}x\beta^{-1}\beta^{-1} + \varepsilon(y, x+y)\beta^2 z\beta^{-1}y\beta^{-1}x\beta^{-1}\beta^{-1}) = 0.
\end{aligned}$$

This completes the proof. □

Theorem 4.2 *Let $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ be a color hom-Lie algebra, V a Γ-graded linear space and $\beta \in \mathfrak{gl}(V)_{\bar{0}}$. Then $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ on V with respect to β if and only if $\rho : (\mathfrak{g}, [\cdot,\cdot], \alpha) \to (\mathfrak{gl}(V), [\cdot,\cdot]_\beta, Ad_\beta)$ is a morphism of color hom-Lie algebras.*

Proof Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ on V with respect to β. One can see that

$$\rho(\alpha(x)) \circ \beta = \beta \circ \rho(x), \tag{4.4}$$

$$\rho([x,y]) \circ \beta = \rho(\alpha(x))\rho(y) - \varepsilon(x,y)\rho(\alpha(y))\rho(x). \tag{4.5}$$

Using (4.4), we get that $\rho \circ \alpha = Ad_\beta \circ \rho$. Moreover, due to (4.4) and (4.5),

$$\begin{aligned}
\rho([x,y]) &= \rho(\alpha(x))\beta \circ \beta^{-1}\rho(y)\beta^{-1} - \varepsilon(x,y)\rho(\alpha(y))\beta \circ \beta^{-1}\rho(x)\beta^{-1} \\
&= \beta\rho(x)\beta^{-1}\rho(y)\beta^{-1} - \varepsilon(x,y)\beta\rho(y)\beta^{-1}\rho(x)\beta^{-1} \\
&= [\rho(x), \rho(y)]_\beta.
\end{aligned}$$

Hence, ρ is a morphism of color hom-Lie algebras. The converse is shown easily in a similar way. □

Corollary 4.1 *Let $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ be a regular color hom-Lie algebra. Then the adjoint representation $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a morphism from $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ to $(\mathfrak{gl}(g), [\cdot,\cdot]_\alpha, Ad_\alpha)$.*

Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ on V with respect to $\beta \in \mathfrak{gl}(V)_{\bar{0}}$. Denote by $C^k(\mathfrak{g}; V)$, the set of all k-cochains on $\mathfrak{g}$, that is, all k-linear homogeneous maps $\varphi : \bigwedge^n(\mathfrak{g}) \to V$, satisfying

$$\varphi(x_1, \ldots, x_i, x_{i+1}, \ldots, x_k) = -\varepsilon(x_i, x_{i+1})\varphi(x_1, \ldots, x_{i+1}, x_i, \ldots, x_k).$$

Define for all $\theta \in \Gamma$,

$$C^n(\mathfrak{g}; V)_\theta = \{\varphi \in C^n(\mathfrak{g}; V) : |\varphi(x_1, \ldots, x_n)| = |x_1| + \ldots + |x_n| + \theta\}.$$

If $\beta \in \mathfrak{gl}(V)_{\bar{0}}$, we define $\bar{\beta}$ from $C^k(\mathfrak{g}; V)$ to itself using the k-cochains

$$\bar{\beta}(\varphi)(x_1, \ldots, x_k) = \beta \circ \varphi(x_1, \ldots, x_k)$$

for all $\varphi \in C^n(\mathfrak{g}; V)$. Moreover, using α, one can define

$$\bar{\alpha} : C^k(\mathfrak{g}; V) \to C^k(\mathfrak{g}; V)$$

$$\bar{\alpha}(\varphi)(x_1, \ldots, x_k) = \varphi(\alpha(x_1), \ldots, \alpha(x_k))$$

for all $\varphi \in C^n(\mathfrak{g}; V)$.

Definition 4.8 A k-hom-cochain on $\mathfrak{g}$ with values in V is a k-cochain $\varphi \in C^k(\mathfrak{g}; V)$ such that $\bar{\alpha}(\varphi) = \bar{\beta}(\varphi)$.

The set of all k-hom-cochains on $\mathfrak{g}$ with values in V is denoted by $C^k_{\alpha,\beta}(\mathfrak{g}; V)$. The action $\bullet : C^l_\alpha(\mathfrak{g}; V) \times C^k_{\alpha,\beta}(\mathfrak{g}; V) \to C^{k+l}_{\alpha,\beta}(\mathfrak{g}; V)$ is defined as follows. For $l = 1$,

$$\eta \bullet \varphi(x_1, \ldots, x_{k+1}) = \sum_i sgn(i)\eta(x_{i_1})\varphi(x_{i_{l+1}}, \ldots, x_{i_{k+1}}).$$

For $l \geqslant 2$,

$$\eta \bullet \varphi(x_1, \ldots, x_{k+1}) = \sum_i sgn(i)\kappa_j(x)\eta(x_{i_1}, \ldots, x_{i_l})\varphi(x_{i_{l+1}}, \ldots, x_{i_{k+1}})$$

for all $\eta \in C^l_\alpha(\mathfrak{g}; V)$, $\varphi \in C^k_{\alpha,\beta}(\mathfrak{g}; V)$, where

$$\lambda_j(x) = \varepsilon((\sum_{j=1}^{l} |x_1| + \ldots + |x_{k+1}|), |x_j|)(-1)^{\sum\limits_{1 \le p \ne q \le l} |x_p||x_q|}$$

and the summation is taken over (l, k)-shuffles.

The linear map $d_s : C^k_{\alpha,\beta}(\mathfrak{g}; V) \to C^{k+1}(\mathfrak{g}; V)$ is defined as follows

$$d_s\varphi(x_1,\dots,x_{k+1}) = \sum_{i=1}^{k+1} \kappa_j(x)\rho(\alpha^{s+k}(x_i))\varphi(x_1,\dots,\hat{x_i},\dots,x_{k+1})$$
$$+\sum_{i<j}(-1)^i\kappa_{ji}(x)\varphi([x_i,x_j],\alpha(x_1),\dots,\hat{x_i},\dots,\hat{x_j},\dots,\alpha(x_{k+1})),$$
$$\kappa_j(x)=\varepsilon(x_1+\dots+x_{j-1},x_j),\quad \kappa_{ji}(x)=\varepsilon(x_{j+1}+\dots+x_{i-1},x_i).$$

It is shown in [1, 4, 12] that d_s is a well-defined coboundry operator. Moreover, it can be easily checked that $\beta\circ d_s = d_{s+1}\circ\bar{\beta}$. It is also shown that d_s is an α^l derivation in the sense that $d_s(\eta\bullet\varphi) = d\eta\bullet\bar{\alpha}(\varphi)+\varepsilon(s,l)\eta\bullet d^{s+l}\varphi$.

Let $(\mathfrak{g},[\cdot,\cdot],\alpha)$ be a color hom-Lie algebra. Denote by $C^k_\alpha(\mathfrak{g})$ the set of all $\xi\in\bigwedge^k\mathfrak{g}^*$ for which $\alpha^*\xi=\xi$. Then the complex $(\bigoplus_k C^k_\alpha(\mathfrak{g}),d)$ is a subcomplex of $(\bigwedge\mathfrak{g}^*,d)$, where $(\bigwedge\mathfrak{g}^*,\wedge)$ is the exterior hom-algebra. This complex is considered to be the cohomological complex of $\mathfrak{g}$ in[1, 12].

Let $\rho:\mathfrak{g}\to\mathfrak{gl}(V)$ be a representation of the color hom-Lie algebra $(\mathfrak{g},[\cdot,\cdot],\alpha)$ on the Γ-graded linear space V with respect to $\beta\in\mathfrak{gl}(V)_{\bar{0}}$. Denote by $C^k(\mathfrak{g},V)$, The set of k-cochains on $\mathfrak{g}$ with values in V. Therefore, $C^k(\mathfrak{g},V)$ is spanned by all k-linear homogenous maps $\varphi:\Lambda^k\mathfrak{g}\to V$ for which one has

$$\varphi(x_1,\dots,x_i,x_{i+1},\dots,x_k)=\varepsilon(x_{i+1},x_i)\varphi(x_1,\dots,x_{i+1},x_i,\dots,x_k).$$

The following definition, will identify the notion of a color omni-hom-Lie algebra which will be used through the rest of the paper.

Definition 4.9 Let $\mathfrak{g}$ be a Γ-graded linear space and $\beta\in\mathfrak{gl}(\mathfrak{g})_{\bar{0}}$. A color omni-hom-Lie algebra is a quadruple $(\mathfrak{gl}(\mathfrak{g})\otimes\mathfrak{g},\delta_\beta,\{\cdot,\cdot\}_\beta,\langle\cdot,\cdot\rangle)$, where

$$\delta_\beta:\mathfrak{gl}(\mathfrak{g})\otimes\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g})\otimes\mathfrak{g}$$

is an even linear map satisfying

$$\delta_\beta(A+x)=Ad_\beta(A)+\beta(x)$$

for all $A+x\in\mathfrak{gl}(\mathfrak{g})\otimes\mathfrak{g}$,

$$\{\cdot,\cdot\}_\beta:\mathfrak{gl}(\mathfrak{g})\otimes\mathfrak{g}\times\mathfrak{gl}(\mathfrak{g})\otimes\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g})\otimes\mathfrak{g}$$

is a bilinear map satisfying

$$\{A+x,B+y\}_\beta=[x,y]_\beta+A(y)$$

for all $A+x, B+y\in\mathfrak{gl}(\mathfrak{g})\otimes\mathfrak{g}$, and

$$\langle\cdot,\cdot\rangle:\mathfrak{gl}(\mathfrak{g})\otimes\mathfrak{g}\times\mathfrak{gl}(\mathfrak{g})\otimes\mathfrak{g}\to\mathfrak{g}$$

is a supersymmetric bilinear V-valued pairing given by

$$\langle A+x, B+y\rangle = \frac{1}{2}(A(y) - \varepsilon(x, y)B(x)).$$

Note that the $\frac{1}{2}$ factor in the above definition, forces the bracket not to satisfy the ε-hom-Jacobi identity. Without it, we would obtain a color hom-Lie algebra.

Recall that a color hom-Leibniz algebra is a Γ-graded linear space V together with a morphism $\circ : L \otimes L \to L$ satisfying $L_\alpha \circ L_\beta \subseteq L_{\alpha+\beta}$, for all $\alpha, \beta \in \Gamma$, and the color Leibniz rule:

$$x \circ (y \circ z) = (x \circ y) \circ z + \varepsilon(x, y) y \circ (x \circ z)$$

for all homogeneous elements $x, y, z \in L$.

One can easily check that a color hom-Leibniz algebra is simply a hom-Lie color algebra when the map "$\circ$" is ε-skew-symmetric where the color Leibniz rule becomes the ε-hom-Jacobi identity.

Proposition 4.3 *Let V be a Γ-graded linear space. Then,*

(i) δ_β *is an algebra automorphism.*
(ii) $(\mathfrak{gl}(V) \oplus V, \{\cdot, \cdot\}_\beta, \delta_\beta)$ *is a color hom-Leibniz algebra. Moreover, we have*

$$\beta\langle A+u, B+v\rangle = \langle \delta_\beta(A+u), \delta_\beta(B+v)\langle.$$

Proof Since Ad_β is an algebra automorphism,

$$\begin{aligned}\delta_\beta(\{A+u, B+v\}_\beta) =& \delta_\beta([A, B]_\beta + A(v)) = Ad_\beta([A, B]_\beta) + \beta A(v)\\ =& [Ad_\beta(A),_\beta (B)]_\beta + \beta A(v).\end{aligned}$$

On the other hand,

$$\begin{aligned}\{\delta_\beta(A+u), \delta_\beta(B+v)\}_\beta &= \{Ad_\beta(A) + \beta(u), Ad_\beta(B) + \beta(v)\}_\beta\\ &= [Ad_\beta(A), Ad_\beta(B)]_\beta + Ad_\beta(A)\beta(v) = [Ad_\beta(A), Ad_\beta(B)]_\beta + \beta A(v).\end{aligned}$$

Therefore, δ_β is an algebra automorphism. Moreover,

$$\begin{aligned}\{\delta_\beta(A+u), \{B+v, C+w\}_\beta\}_\beta &= \{Ad_\beta(A) + \beta(u), [B, C]_\beta + B(w)\}_\beta\\ &= [Ad_\beta(A), [B, C]_\beta]_\beta + Ad_\beta(A)B(w) = [Ad_\beta(A), [B, C]_\beta]_\beta + \beta A\beta^{-1}B(w),\\ \{\{A+u, B+v\}_\beta, \delta_\beta(C+w)\}_\beta &= \{[A, B]_\beta + A(v), Ad_\beta(C) + \beta(w)\}_\beta\\ &= [[A, B]_\beta, Ad_\beta(C)]_\beta + [A, B]_\beta\beta(w)\\ &= [[A, B]_\beta, Ad_\beta(C)]_\beta + \beta A\beta^{-1}B(w) - \varepsilon(A, B)\beta B\beta^{-1}A(w),\end{aligned}$$

$$\begin{aligned}\{\delta_\beta(B+v),\{A+u,C+w\}_\beta\}_\beta &= \{Ad_\beta(B)+\beta(v),[A,C]_\beta+A(w)\}_\beta\\ &= [Ad_\beta(B),[A,C]_\beta]_\beta + Ad_\beta(B)A(w)\\ &= [Ad_\beta(B),[A,C]_\beta]_\beta + \beta B\beta^{-1}A(w),\end{aligned}$$

Since $(\mathfrak{gl}(V), [\cdot,\cdot]_\beta, Ad_\beta)$ is a color hom-Leibniz algebra, (ii) is proved. Furthermore,

$$\begin{aligned}\langle\delta_\beta(A+u),\delta_\beta(B+v)\rangle &= \langle Ad_\beta(A)+\beta(u), Ad_\beta(B)+\beta(v)\rangle\\ &= \frac{1}{2}(Ad_\beta(A)\beta(v)-\varepsilon(A,B)Ad_\beta(B)\beta(u)) = \frac{1}{2}\beta(A(v)-\varepsilon(A,B)B(u))\\ &= \beta\langle A+u,B+v\rangle.\end{aligned}$$

This completes the proof. □

Proposition 4.4 *Let* $e_1 = A + x$, $e_2 = B + y$, $e_3 = C + z$, *for* $A, B, C \in \mathfrak{gl}(V)$ *and* $x, y, z \in V$. *Define*

$$\begin{aligned}T(e_1,e_2,e_3) :=&\frac{1}{3}(\varepsilon(z,x)\langle\{e_1,e_2\}_\beta,e_3\rangle+\varepsilon(x,y)\langle\{e_2,e_3\}_\beta,e_1\rangle\\ &+\varepsilon(y,z)\langle\{e_3,e_1\}_\beta,e_2\rangle).\end{aligned}$$

Then T *coincides with the left hand side of the* ε*-hom-Jacobi identity.*

Proof First note that

$$\begin{aligned}T(e_1,e_2,e_3) =&\frac{1}{3}\varepsilon(z,x)\langle\{A+x,B+y\}_\beta,C+z\rangle+c.p.\\ =&\frac{1}{3}\varepsilon(z,x)\langle[A,B]_\beta+\frac{1}{2}(Ay-\varepsilon(x,y)Bx),C+z\rangle+c.p.\\ =&\frac{1}{6}\varepsilon(z,x)([A,B]_\beta z+\frac{1}{2}\varepsilon(x+y,z)C(Ay-\varepsilon(x,y)Bx))+c.p.\\ =&\frac{1}{6}\varepsilon(z,x)ABz-\frac{1}{6}\varepsilon(z,x)\varepsilon(x,y)BAz+\frac{1}{12}\varepsilon(y,z)CAy\\ &-\frac{1}{12}\varepsilon(y,z)\varepsilon(x,y)CBx+\frac{1}{6}\varepsilon(x,y)BCx-\frac{1}{6}\varepsilon(x,y)\varepsilon(y,z)CBx\\ &+\frac{1}{12}\varepsilon(z,x)ABz-\frac{1}{12}\varepsilon(z,x)\varepsilon(y,z)ACy+\frac{1}{6}\varepsilon(y,z)CAy\\ &-\frac{1}{6}\varepsilon(y,z)\varepsilon(z,x)ACy+\frac{1}{12}\varepsilon(x,y)BCx-\frac{1}{12}\varepsilon(x,y)\varepsilon(z,x)BAz\\ =&\frac{1}{4}\varepsilon(z,x)ABz-\frac{1}{4}\varepsilon(z,x)\varepsilon(x,y)BAz+\frac{1}{4}\varepsilon(y,z)CAy\\ &-\frac{1}{4}\varepsilon(y,z)\varepsilon(x,y)CBx+\frac{1}{4}\varepsilon(x,y)BCx-\frac{1}{4}\varepsilon(z,x)\varepsilon(y,z)ACy.\end{aligned}$$

On the other hand,

$$
\begin{aligned}
&\varepsilon(z,x)\{\{A+x, B+y\}_\beta, C+x\}_\beta + c.p.\\
=&\{\frac{1}{6}\varepsilon(z,x)[A,B]_\beta + \frac{1}{2}\varepsilon(z,x)(Ay-\varepsilon(x,y)Bx), C+z\}_\beta + c.p.\\
=&\varepsilon(z,x)[[A,B]_\beta, C]_\beta + c.p.\\
&+\frac{1}{2}(\varepsilon(z,x)[A,B]_\beta z - \frac{1}{2}\varepsilon(z,x)\varepsilon(x+y,z)C(Ay-\varepsilon(x,y)Bx))\\
&+\frac{1}{2}(\varepsilon(x,y)[B,C]_\beta x - \frac{1}{2}\varepsilon(x,y)\varepsilon(y+z,x)A(Bz-\varepsilon(y,z)Cy))\\
&+\frac{1}{2}(\varepsilon(y,z)[C,A]_\beta y - \frac{1}{2}\varepsilon(y,z)\varepsilon(x+z,y)B(Cx-\varepsilon(z,x)Az))\\
=&\frac{1}{4}\varepsilon(z,x)ABz - \frac{1}{4}\varepsilon(z,x)\varepsilon(x,y)BAz + \frac{1}{4}\varepsilon(y,z)CAy\\
&-\frac{1}{4}\varepsilon(y,z)\varepsilon(x,y)CBx + \frac{1}{4}\varepsilon(x,y)BCx - \frac{1}{4}\varepsilon(z,x)\varepsilon(y,z)ACy.
\end{aligned}
$$

This completes the proof. □

Finally, let V be an ε-graded linear space and recall that the graph of the adjoint operator is defined as the following

$$\mathcal{F}_\beta = \{ad_\beta(x) + x, \forall x \in \mathfrak{g}\} \subset \mathfrak{gl}(V) \otimes V,$$

which is an ε-graded subspace of $\mathfrak{gl}(V)\otimes V$. Let $\mathcal{F}_\beta^\perp$ denote the orthogonal complement of $\mathcal{F}_\beta$ with respect to $\langle .,.\rangle$.

Proposition 4.5 *The triple $(V, \{.,.\}_\beta, \beta)$ form a hom-Lie color algebra if and only if the graph of the adjoint representation on V is maximal isotopic, that is, $\mathcal{F}_\beta = \mathcal{F}_\beta^\perp$, and is closed with respect to the bracket $\{.,.\}_\beta$.*

Proof First, note that from the definition of the adjoint map we get

$$
\begin{aligned}
\langle ad_\beta(x)+x, ad_\beta(y)+y\rangle =&\frac{1}{2}(ad_\beta(x)y + \varepsilon(x,y)ad_\beta(y)x)\\
=&\frac{1}{2}([x,y]_\beta + \varepsilon(x,y)[y,x]_\beta).
\end{aligned}
$$

which indicated that $\mathcal{F}_\beta \subseteq \mathcal{F}_\beta^\perp$.

Now we will rewrite the ε-graded Jacobi identity on V.

$$
\begin{aligned}
\{ad_\beta(x)+x, ad_\beta(y)+y\} =&[ad_\beta(x), ad_\beta(y)]_\beta + \frac{1}{2}(ad_\beta(x)y - \varepsilon(x,y)ad_\beta(y)x)\\
=&[ad_\beta(x), ad_\beta(y)]_\beta + \frac{1}{2}([x,y]_\beta - \varepsilon(x,y)[y,x]_\beta)\\
=&[ad_\beta(x), ad_\beta(y)]_\beta + [x,y]_\beta.
\end{aligned}
$$

Therefore, this bracket is closed if and only if $[ad_\beta(x), ad_\beta(y)]_\beta = ad_\beta([x, y]_\beta)$ in which case for all homogenous elements $z \in V$ we have

$$\begin{aligned}
&[ad_\beta(x), ad_\beta(y)]_\beta(\beta(z)) - ad_\beta([x, y]_\beta)(\beta(z)) \\
&= ad_\beta(x)ad_\beta(y)(\beta(z)) - \varepsilon(x, y)ad_\beta(y)ad_\beta(x)(\beta(z)) - ad_\beta([x, y]_\beta)(\beta(z)) \\
&= ad_\beta(\beta(x))[y, z]_\beta - \varepsilon(x, y)ad_\beta(\beta(y))[x, z]_\beta - [[x, y]_\beta, \beta(z)]_\beta \\
&= [\beta(x), [y, z]_\beta]_\beta - \varepsilon(x, y)[\beta(y), [x, z]_\beta]_\beta - [[x, y]_\beta, \beta(z)]_\beta = 0.
\end{aligned}$$

This completes the proof. □

Note that the last proof can be rewritten by α instead of β which shows that the result is independent of the representation.

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Chapter 5
On (σ, τ)-Derivations of Group Algebra as Category Characters

Aleksandr Alekseev, Andronick Arutyunov, and Sergei Silvestrov

Abstract For the space of (σ, τ)-derivations of the group algebra $\mathbb{C}[G]$ of a discrete countable group G, the decomposition theorem for the space of (σ, τ)-derivations, generalising the corresponding theorem on ordinary derivations on group algebras, is established in an algebraic context using groupoids and characters. Several corollaries and examples describing when all (σ, τ)-derivations are inner are obtained. Considered in details are cases of (σ, τ)-nilpotent groups and (σ, τ)-FC groups.

Keywords Group algebra · Derivation · (σ, τ)-derivation · Groupoid · Character

MSC2020 Classification 16W25 · 13N15 · 16S34

5.1 Introduction

The general theory of derivations for C^*-algebras, W^*-algebras, Banach, normed and topological algebras, motivated by many parts of Mathematics and Mathematical Physics, has developed since 1950'th. Fundamental derivation theorems describing conditions for all or almost all derivations being inner, constructions of outer derivations and related cohomology methods and generalizations have been developed for

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C^*-algebras, W^*-algebras and some related classes of Banach, normed and topological algebras and their representations [1, 2, 8, 11, 15, 16, 22, 24–27, 30–34, 46, 47, 52–56].

For group algebras, the *derivation problem* is formulated as follows: "Under what conditions all derivations in a group algebra are inner?". There are many kinds of group algebras based on algebraic structure, topological structure, measure structures and choices of function spaces for the group algebra elements. For the group algebra $L_1(G)$, the derivation problem is important for investigations in measure theory and harmonic analysis, operator theory, operator algebras and cohomological constructions [15, Question 5.6.B, p. 746]. The derivation problem for $L_1(G)$ of a locally compact group G was considered in [40], where it was mentioned that all derivations of $L_1(G)$ are inner.

It is important to note that in the cited publications derivations are considered in topological context of the classes of algebras and modules equipped with normed or more general topological structures. If we consider the problem in algebraic way it is easy to find examples of non-inner derivations [4–6]. Algebraic view on the derivation problem is also presented in [3] together with more complete bibliography.

We consider in this article (σ, τ)-derivations, the linear operators on an associative algebra satisfying a generalized Leibniz rule $D(xy) = D(x)\tau(y) + \sigma(x)D(y)$ twisted by two linear maps σ and τ. The (σ, τ)-derivation operators include, for example, the ordinary derivations on commutative and non-commutative algebras, the q-difference and (p, q)-difference operators on algebras of functions, the superderivations, graded colored derivations and q-derivation on graded associative algebras. Since 1930s, (σ, τ)-derivations and subclass of σ-derivations have been shown to play a fundamental role in the theory of Ore extensions and iterated Ore extension rings and algebras, skew polynomial algebras and skew fields, Noetherian rings and algebras, differential and difference algebra, homological algebra, Lie algebras and Lie groups, Lie superalgebras and colored Lie algebras, operator algebras, noncommutative geometry, quantum groups and quantum algebras, differential geometry, symbolic algebra computations and algorithms q-analysis and q-special functions and numerical analysis [9, 10, 13, 14, 18, 28, 29, 35–37, 41–43, 45, 51]. The space of (σ, τ)-derivations and subspace of σ-derivations have been recently used in [12, 17, 21, 38, 39, 48, 49, 57, 58] in general constructions of quasi-Hom-Lie algebras and their central extensions, extending Witt and Virasoro Lie algebras to context of (σ, τ)-derivations, with special emphasize on (σ, τ)-derivations and σ-derivations on commutative algebras such as unique factorization domains, algebras of polynomials, Laurent polynomials and truncated algebras of polynomials in one or several variables.

In this paper, we consider (σ, τ)-derivations and σ-derivations of the group algebra $\mathbb{C}[G]$ of a discrete countable group G. We apply the approach proposed in [4–7, 44] for the description of derivations in group algebras to study the space of (σ, τ)-derivations of the group algebras.

Section 5.2 contains general definitions and preliminaries on (σ, τ)-derivations and group algebras considered in the article. In Sect. 5.3, we construct a groupoid Γ associated with the group algebra and the pair of maps (σ, τ) in the (σ, τ)-twisted

Leibniz rule for (σ, τ)-derivations, (σ, τ)-conjugacy classes and characters on this groupoid. We also construct an isomorphism between the space of (σ, τ)-derivations on a group algebra of countable group and the space of locally finite characters on the associated groupoid (Theorem 5.1). In Sect. 5.4, we define and describe quasi-inner (σ, τ)-derivations, as well as a class of not quasi-inner derivations. We prove Theorem 5.2 which describes the view of quasi-inner (σ, τ)-derivations. In Sect. 5.5 we consider the case of (σ, τ)-nilpotent groups and obtain a description of the (σ, τ)-derivation algebra (see Theorem 5.3). For the case of inner endomorphisms and a Heisenberg group, we calculate the (σ, τ)-derivations in the group algebra (see results in Sect. 5.5.3). The Sect. 5.6 is dedicated to the case of (σ, τ)-FC groups which are natural generalization of usual FC-groups on our "twisted" case of (σ, τ)-derivations. For this class of groups we show some simple properties (see Proposition 5.9) and construct a description of (σ, τ)-derivations (Theorem 5.4).

5.2 General Definitions and Preliminaries

In this section, we recall some basic definitions and properties of main objects studied in the rest of the article.

Definition 5.1 Let $\mathcal{A}$ be an associative algebra over a field $\mathcal{K}$ and (σ, τ) is a pair of $\mathcal{K}$-linear endomorphisms of $\mathcal{A}$. A (σ, τ)-*derivation* $D : \mathcal{A} \to \mathcal{A}$ is an $\mathcal{K}$-linear map such that the following twisted by (σ, τ) generalized Leibniz identity

$$D(ab) = D(a)\tau(b) + \sigma(a)D(b) \tag{5.1}$$

is satisfied for all $a, b \in \mathcal{A}$. If $\tau = id_{\mathcal{A}}$, the generalized Leibniz identity is twisted by one map σ as follows

$$D(ab) = D(a)b + \sigma(a)D(b), \tag{5.2}$$

and the $(\sigma, id_{\mathcal{A}})$-derivation D is called σ-*derivation*. If $\tau = \sigma = id_{\mathcal{A}}$, then the usual Leibniz identity

$$D(ab) = D(a)b + aD(b) \tag{5.3}$$

holds, and D is called a derivation on $\mathcal{A}$.

The set of all (σ, τ)-derivations on $\mathcal{A}$ is denoted by $\mathcal{D}_{(\sigma,\tau)}(\mathcal{A})$. The set of all σ-derivations is denoted by $\mathcal{D}_{\sigma}(\mathcal{A})$, and the set of all derivations is denoted by $\mathcal{D}(\mathcal{A})$

The set $\mathcal{D}_{(\sigma,\tau)}(\mathcal{A})$ of (σ, τ)-derivations on $\mathcal{A}$ is a linear subspace of the space of linear operators on $\mathcal{A}$ as a $\mathcal{K}$-linear space.

For any (σ, τ)-derivation on $\mathcal{A}$,

$$\begin{aligned}
D(a(bc) - (ab)c) &= D(a(bc)) - D((ab)c) = \\
&= D(a)\tau(bc) + \sigma(a)D(bc) - \big(D(ab)\tau(c) + \sigma(ab)D(c)\big) = \\
&= D(a)\tau(bc) + \sigma(a)\big(D(b)\tau(c) + \sigma(b)D(c)\big) \\
&\quad -\big(D(a)\tau(b) + \sigma(a)D(b)\big)\tau(c) - \sigma(ab)D(c) = \\
&= D(a)\big(\tau(bc) - \tau(b)\tau(c)\big) + \big(\sigma(a)\sigma(b) - \sigma(ab)\big)D(c).
\end{aligned}$$

Since $\mathcal{A}$ is associative, $a(bc) - (ab)c = 0$, and since D is linear, $D(a(bc) - (ab)c) = 0$, and hence

$$D(a)\big(\tau(bc) - \tau(b)\tau(c)\big) + \big(\sigma(a)\sigma(b) - \sigma(ab)\big)D(c) = 0. \tag{5.4}$$

Note that for the identity element e, in general, $D(e) = D(e \cdot e) = D(e)\tau(e) + \sigma(e)D(e)$, or equivalently, $D(e)(e - \tau(e)) = \sigma(e)D(e)$, implying that if $\tau(e) = \sigma(e) = e$ (as for example for group endomorphisms), then $D(e) = 0$.

In this article we will be interested only in (σ, τ)-derivations D with $\sigma, \tau \in$ **End**(G) and $D(e) = 0$.

If $\mathcal{A}$ is an associative algebra over a field $\mathcal{K}$, and σ and τ are algebra endomorphisms on $\mathcal{A}$, then for any $p \in \mathcal{A}$ the map $\delta_p : \mathcal{A} \to \mathcal{A}$ given by the (σ, τ)-twisted generalised commutator $\delta_p(x) = p\tau(x) - \sigma(x)p$, is a (σ, τ)-derivation on $\mathcal{A}$, since for each $x, y \in \mathcal{A}$, we have

$$\begin{aligned}
\delta_p(xy) = p\tau(xy) - \sigma(xy)p &= p\tau(x)\tau(y) - \sigma(x)\sigma(y)p = \\
&= p\tau(x)\tau(y) - \sigma(x)p\tau(y) + \sigma(x)p\tau(y) - \sigma(x)\sigma(y)p = \\
&= \delta_p(x)\tau(y) + \sigma(x)\delta_p(y).
\end{aligned}$$

Definition 5.2 (Inner (σ, τ)-derivation) If $\mathcal{A}$ is an associative algebra, and σ and τ are algebra endomorphisms on $\mathcal{A}$, then (σ, τ)-derivations δ_p for $p \in \mathcal{A}$ are called the inner (σ, τ)-derivations of $\mathcal{A}$.

Throughout this article, the group algebra $\mathbb{K}[G]$ of a group $(G, \cdot)$ over the field $\mathbb{K}$ means the linear space of mappings $f : G \to \mathbb{K}$ of finite support with the pointwise operations of multiplication by scalars and addition, and the algebra product defined as convolution

$$(f * g)(x) = \sum_{u \cdot v = x} f(u)g(v) = \sum_{u \in G} f(u)g(u^{-1}x). \tag{5.5}$$

where all sums are finite because f and g are of finite support. With these operations, the group algebra is an unital associative algebra with the algebra unity coinciding with the indicator function I_e of the group unity $e \in G$, that is $I(e) = 1_{\mathbb{K}}$ and $I(e) = 0_{\mathbb{K}}$ otherwise. The elements $f \in \mathbb{K}[G]$ often are conveniently presented as the formal linear combinations of elements of G with coefficients in $\mathbb{K}$ written as $\sum_{g \in G} f(g)g$ or $\sum_{g \in G} f_g g$ similar to usual way of writing polynomials and Laurent polynomials.

5.3 Groupoid and Characters

For any mapping $D : \mathbb{C}[G] \to \mathbb{C}[G]$ and any $g \in G$, the element $D(g) \in \mathbb{C}[G]$ can be written as

$$D(g) = \sum_{h \in G} \lambda_g^h h, \qquad \lambda_g^h \in \mathbb{C}.$$

When $D : \mathbb{C}[G] \to \mathbb{C}[G]$ is (σ, τ)-derivation, the (σ, τ)-twisted Leibniz rule (5.1) for $g_1, g_2 \in G \hookrightarrow \mathbb{C}[G]$ becomes

$$D(g_2 g_1) = D(g_2)\tau(g_1) + \sigma(g_2)D(g_1),$$
$$\lambda_{g_2 g_1}^h = \lambda_{g_2}^{h\tau(g_1^{-1})} + \lambda_{g_1}^{\sigma(g_2^{-1})h}. \tag{5.6}$$

Here we apply an approach developed in [4, 6, 7, 44] to describe derivation in terms of groupoid geometrical properties. We construct a groupoid Γ associated with the group algebra in the following way:

- $\mathrm{Obj}(\Gamma) = G$
- For all a and $b \in \mathrm{Obj}(\Gamma)$ a set of maps is $\mathbf{Hom}(a, b) = \{(u, v) \in G \times G \mid \sigma(v^{-1})u = a, u\tau(v^{-1}) = b\}$
- A composition of maps $\varphi = (u_1, v_1) \in \mathbf{Hom}(a, b)$, $\psi = (u_2, v_2) \in \mathbf{Hom}(b, c)$ is a map $\varphi \circ \psi \in \mathbf{Hom}(a, c)$, such that

$$\varphi \circ \psi = (u_2 \tau(v_1), v_2 v_1). \tag{5.7}$$

Remark 5.1 We do not have in formula (5.7) the map σ explicitly. But if the composition $\varphi \circ \psi$ exists, then $t(\psi) = s(\varphi)$. This means that

$$\sigma(v_1^{-1})u_1 = u_2 \tau(v_2^{-1}).$$

The last formula gives us a connection to map σ in (5.7).

We will be interested in some internal structure issues of the groupoid Γ. If morphism $(u, v) \in \mathbf{Hom}(a, b)$, then $u = \sigma(v)a$ and $u\tau(v^{-1}) = b$. Therefore we have

$$b = \sigma(v)a\tau(v^{-1}).$$

It gives the following description of subgroupoid $\Gamma_{[a]}$

$$\mathrm{Obj}(\Gamma_{[a]}) = \{\sigma(v)a\tau(v^{-1}) | v \in G\}. \tag{5.8}$$

Definition 5.3 A subset $[a]_{\sigma,\tau} = \{\sigma(g^{-1})a\tau(g) \mid g \in G\}$ is called a (σ, τ)-conjugation class of the u.

The set of (σ, τ)-conjugation classes is denoted $G^{(\sigma,\tau)}$. If two (σ, τ)-conjugacy classes intersect, then they coincide. So the set of group elements is represented as a disjoint union of (σ, τ)-conjugacy classes

$$\{G\} = \bigsqcup_{[u]_{\sigma,\tau} \in G^{(\sigma,\tau)}} [u]_{\sigma,\tau}.$$

The set $[a]_{(\sigma,\tau)} := \{\sigma(v)a\tau(v^{-1}) \mid v \in G\}$ can be understood as (σ, τ) analogue of conjugacy class. If the (σ, τ)-class $[a]$ contains just one element, then we will say that a is a (σ, τ)-central element of G.

It is easy to represent Γ as disjoint union of the groupoids $\Gamma_{[u]_{\sigma,\tau}}$ (see [4, 5]):

$$\Gamma = \bigsqcup_{[u]_{\sigma,\tau} \in G^{(\sigma,\tau)}} \Gamma_{[u]_{\sigma,\tau}} \tag{5.9}$$

Definition 5.4 A linear map $\chi : \mathbf{Hom}(\Gamma) \to \mathbb{C}$, such that

$$\chi(\varphi \circ \psi) = \chi(\varphi) + \chi(\psi), \tag{5.10}$$

is called a character on the groupoid Γ.

Characters with natural sum operations and multiplication by scalar form a vector space. It is natural to define the support of character χ in following way

$$supp\chi = \{\varphi \in \mathbf{Hom}(\Gamma) | \chi(\varphi) \neq 0\}. \tag{5.11}$$

Further we will be interested just in those characters which are connected with derivations.

Definition 5.5 The character χ such that, for fixed $v \in \mathrm{Obj}(C)$, $\chi(u, v) = 0$ for almost all $u \in \mathrm{Obj}(C)$ is called a locally finite character.

Let $X(\Gamma)$ be a space of the all locally finite characters on Γ. From formula (5.9) we get the decomposition of the space $X(\Gamma)$ in following way

$$X(\Gamma) = \bigoplus_{[u]_{\sigma,\tau} \in G^{(\sigma,\tau)}} X(\Gamma_{[u]_{\sigma,\tau}}), \tag{5.12}$$

where $X(\Gamma_{[u]_{\sigma,\tau}})$ denotes the locally finite characters supported in $\Gamma_{[u]_{\sigma,\tau}}$.

Consider a map $\Psi : \mathcal{D}_{(\sigma,\tau)}(\mathbb{C}[G]) \to X(\Gamma)$, such that if $D(g) = \sum_{h \in G} \lambda_g^h h$, then $\Psi(D)(h, g) = \lambda_g^h$. The map Ψ^{-1} is constructed in the same way,

$$\Psi^{-1}(\chi)(g) = \sum_{h \in G} \chi(h, g)h.$$

Theorem 5.1 *Consider the discrete countable group G with* $\sigma, \tau \in \mathbf{End}(G)$. *Then, the map* $\Psi : \mathcal{D}_{(\sigma,\tau)}(\mathbb{C}[G]) \to X(\varGamma)$ *is an isomorphism.*

Proof On the one hand, we show that $\Psi(D) \in X(\varGamma)$. Due to the definition of the groupoid $\varGamma$, there exists a following composition of maps:

$$(h\tau(g_1^{-1}), g_2) \circ (\sigma(g_2^{-1})h, g_1) = (h, g_2 g_1).$$

Using (5.6) one obtains

$$\begin{aligned} \Psi(D)(h, g_2 g_1) &= \lambda^h_{g_2 g_1} = \\ &= \lambda^{h\tau(g_1^{-1})}_{g_2} + \lambda^{\sigma(g_2^{-1})h}_{g_1} = \Psi(D)(h\tau(g_1^{-1}), g_2) + \Psi(D)(\sigma(g_2^{-1})h, g_1). \end{aligned}$$

The latter equation means that $\Psi(D)$ satisfies the property (5.10). Thus, $\Psi(D) \in X(\varGamma)$.

On the other hand, due to the property of locally finiteness,

$$\Psi^{-1}(\chi)(g) = \sum_{h \in G} \chi(h, g)h \in \mathbb{C}[G],$$

and $\Psi\Psi^{-1} = \mathrm{Id}_{X(\varGamma)}$, $\Psi^{-1}\Psi = \mathrm{Id}_{\mathcal{D}_{(\sigma,\tau)}(\mathbb{C}[G])}$. □

5.4 Quasi-inner (σ, τ)-Derivations

Recall that inner (σ, τ)-derivation δ_p is given by formula

$$\delta_p : x \mapsto p\tau(x) - \sigma(x)p.$$

By Theorem 5.1 the corresponding character is trivial on loops.

Proposition 5.1 *For the given inner* (σ, τ)*-derivation* δ_p *the corresponding character* $\Psi(\delta_p)$ *is trivial on loops, in the other words* $\forall a \in \mathrm{Obj}(C)$ *and* $\forall \varphi \in \mathbf{Hom}(a, a)$ *the value* $\Psi(\delta_p)(\varphi) = 0$.

Proof If $p\tau(g) = \sigma(g)p$, then $\Psi(\delta_p)(pg, g) = \lambda_g^{p\tau(g)} - \lambda_g^{\sigma(g)p} = 1 - 1 = 0$. Otherwise, if $\varphi \in \mathbf{Hom}(p, p)$ then $\varphi = (p\tau(g), p) = (\sigma(g)p, p)$. □

Not all locally finite characters which are trivial on loops are given by an inner derivation even for ordinary derivations (see [4, page 76 (example)]).

Definition 5.6 A (σ, τ)-derivation D is said to be a *quasi-inner* if the corresponding character $\Psi(D)$ is trivial on loops.

When mapping σ and τ are identity mappings (σ, τ)-derivations are equal to usual derivations on group algebra $\mathbb{C}[G]$. In this case quasi-inner derivations form an ideal which contains ordinary inner derivations ([7, Theorem 4.1]), and quasi-inner derivations can be easily calculated (see Theorem 5.2).

Definition 5.7 An element $a \in G$ is said to be (σ, τ)-central if $a\tau(v) = \sigma(v)a \ \forall v \in G$.

Proposition 5.2 *For the given group G, maps $\sigma, \tau \in \mathbf{End}(G)$, (σ, τ)-central element a and a homomorphism $\varphi : G \to \mathbb{C}$, the map $D(g) = \varphi(g)\sigma(g)a$ is a (σ, τ)-derivation.*

Proof Using first the definition of D, then (σ, τ)-centrality of the element a and then homomorphism property of φ and σ and finally again the definition of D one obtains:

$$\begin{aligned} D(g_1)\tau(g_2) + \sigma(g_1)D(g_2) &= \varphi(g_1)\sigma(g_1)a\tau(g_2) + \sigma(g_1)\varphi(g_2)\sigma(g_2)a = \\ &\varphi(g_1)\sigma(g_1)\sigma(g_2)a + \varphi(g_2)\sigma(g_1)\sigma(g_2)a = \\ &(\varphi(g_1) + \varphi(g_2))\sigma(g_1)\sigma(g_2)a = \varphi(g_1g_2)\sigma(g_1g_2)a = D(g_1g_2). \end{aligned}$$

which means that D is a (σ, τ)-derivation. □

Definition 5.8 The (σ, τ)-derivation D is called (σ, τ)-central.

The (σ, τ)-central derivations give us an example of noninner (σ, τ)-derivations. For the case of general group algebra they were studied in [5].

Proposition 5.3 *The nonzero (σ, τ)-central (σ, τ)-derivation D is not quasi-inner.*

Proof The character's value $\Psi(D)(\sigma(g)a, g) = \varphi(g)$, and since $(\sigma(g)a, g)$ is a loop, the (σ, τ)-derivation D is not quasi-inner. □

Quasi-inner (σ, τ)-derivations can be calculated in the following way. Consider a map $(u, v) \in \mathbf{Hom}(\Gamma)$. Let $t, s : \mathbf{Hom}(\Gamma) \to \mathrm{Obj}(\Gamma)$ be a target and source maps, such that $\phi = (u, v) : s(\phi) \to t(\phi)$. If the character χ is trivial on loops then exists function $P_\chi : \mathrm{Obj}(\Gamma) \to \mathbb{C}$ such that

$$\chi(\phi) = P_\chi(t(\phi)) - P_\chi(s(\phi)).$$

That means that if χ is locally finite character, then the following formula is valid for a quasi-inner (σ, τ)-derivation D_P:

$$D_P(g) = \sum_{h \in G} (P_\chi(t(h, g)) - P_\chi(s(h, g)))h = \sum_{h \in G} (P_\chi(h\tau(g^{-1})) - P_\chi(\sigma(g^{-1})h))h.$$

In other words we get the following statement.

Theorem 5.2 *If $D \in QInn(\Gamma)$, then there exists a finitely supported function $P : \mathrm{Obj}(\Gamma) \to \mathbb{C}$ such that for generators $g \in \mathbb{C}[G]$,*

$$D(g) = \sum_{h \in G} (P(h\tau(g^{-1})) - P(\sigma(g^{-1})h))h.$$

Remark 5.2 The function P from the theorem is not unique. It is determined up to the addition of a constant on each subgroupoid.

5.5 (σ, τ)-Nilpotent Groups

5.5.1 *General Case of (σ, τ)-Nilpotent Groups*

The following concepts are similar to the terms from classic group theory. Derivations in classic rank 2 nilpotent groups were studied in [5].

Definition 5.9 A subgroup $Z_{\sigma,\tau} = \{z \in G \mid \sigma(z)p = p\tau(z)\, \forall p \in G\}$ is called a (σ, τ)-center of the group G.

Of course $z \in Z_{\sigma,\tau}$ is equivalent to the fact that there is single object in subgroupoid $\Gamma_{[z]}$.

It is worth mentioning that the (σ, τ)-center is *not* a subgroup of (σ, τ)-central elements, which were introduced in Definition 5.7.

Definition 5.10 A subgroup $Z_{\sigma,\tau}(u) = \{z \in G \mid \sigma(z)u = u\tau(z)\}$, $u \in G$ is called a (σ, τ)-centraliser of the element u.

Proposition 5.4 *A subgroup $Z_{\sigma,\tau} \subseteq G$ is a normal subgroup.*

Proof Let us show that for $z \in Z_{\sigma,\tau}$ and $g \in G$, the element $gzg^{-1} \in Z_{\sigma,\tau}$ or in the other words $p\tau(gzg^{-1}) = \sigma(gzg^{-1})p\ \forall p \in G$. Indeed,

$$p\tau(gzg^{-1}) = p\sigma(z) = \sigma(\sigma^{-1}(p)z) =$$
$$\sigma(\tau^{-1}(\tau(\sigma^{-1}(p))\tau(z))) = \sigma(\tau^{-1}(\sigma(z)\tau(\sigma^{-1}(p)))) = \sigma(\tau^{-1}(\sigma(z)))p,$$
$$\sigma(gzg^{-1})p = \sigma(\tau^{-1}(\sigma(z)))\sigma(gg^{-1})p = \sigma(\tau^{-1}(\sigma(z)))p.$$

□

In accordance with the definition of Γ the source of the map $(\sigma(g)p, g)$ is the object p, the target is $\sigma(g)p\tau(g^{-1})$ and the inverse map $(\sigma(g)p, g)^{-1} = (p\tau(g^{-1}), g^{-1})$. That means that $g^{-1}Z_{\sigma,\tau}(p)g = Z_{\sigma,\tau}(\sigma(g)p\tau(g^{-1}))$.

Definition 5.11 A group G such that $G/Z_{\sigma,\tau}$ is abelian is called a (σ, τ)-nilpotent group with rank 2.

Proposition 5.5 *Consider the (σ,τ)-nilpotent group G with rank 2. Then all elements in $\mathrm{Obj}(\Gamma_{[u]_{\sigma,\tau}})$ have the same (σ,τ)-centralizer group or, in the other words, $Z_{\sigma,\tau}(\sigma(g)p\tau(g^{-1})) = gZ_{\sigma,\tau}(p)g^{-1} = Z_{\sigma,\tau}(p)$.*

Proof Consider an element $z_p \in Z_{\sigma,\tau}(p)$. Let $[z_p]$ be a class in quotient group $G/Z_{\sigma,\tau}$. Due to the fact, that $G/Z_{\sigma,\tau}$ is abelian, one obtain, that $[gz_pg^{-1}] = [z_p]$. Thus, there exists an element $z \in Z_{\sigma,\tau}$, such that $z_pz = gz_pg^{-1}$. That means, that $gz_pg^{-1} \in Z_{\sigma,\tau}(p)$. □

Corollary 5.1 *If there is a subgroupoid $\Gamma_{[u]_{\sigma,\tau}}$ with the infinite number of objects, then each character $\chi \in X(\Gamma_{[u]_{\sigma,\tau}})$ is trivial on loops.*

Proof The proof immediately follows from the statement that for the given (σ,τ)-derivation D the corresponding character $\Psi(D)$ has to be locally-finite. Consider an object $a \in \mathrm{Obj}(\Gamma_{[u]_{\sigma,\tau}})$. Then, since

$$Z_{\sigma,\tau}(\sigma(g)a\tau(g^{-1})) = Z_{\sigma,\tau}(a),$$

if there is a loop

$$(a\tau(g), g) \in \mathbf{Hom}(a, a),$$

then every set $\mathbf{Hom}(b, b)$ $\forall b \in \mathrm{Obj}(\Gamma_{[u]_{\sigma,\tau}})$ has a map

$$(b\tau(g), g) \in \mathbf{Hom}(b, b).$$

Due to the fact that

$$\chi(a\tau(g), g) = \chi(b\tau(g), g),$$

one obtains that if $\chi(a\tau(g), g) \neq 0$, then a character χ does not satisfy the property of locally-finiteness. □

We are going now to generalize the result of the corollary in a following theorem.

Theorem 5.3 *If G is rank 2 (σ,τ)-nilpotent group, then*

$$D_{(\sigma,\tau)} \cong \bigoplus_{|[a]_{(\sigma,\tau)}|<\infty} Z^*_{(\sigma,\tau)}(a) \bigoplus QInn(\Gamma), \tag{5.13}$$

where $Z^*_{(\sigma,\tau)}$ is the space of group characters of the centralizer $Z_{(\sigma,\tau)}(a)$, i.e. $Z^*_{(\sigma,\tau)} = \mathbf{Hom}(Z_{(\sigma,\tau)}, \mathbb{C})$, and $QInn(\Gamma)$ is a space of the all quasi-inner derivations on Γ.

Proof As we noted above

$$\Gamma = \bigsqcup \Gamma_{[u]_{\sigma,\tau}}.$$

This implies the following decomposition

$$X(\Gamma) = \bigoplus X(\Gamma_{[u]_{\sigma,\tau}}),$$

where $X(\Gamma_{[u]_{\sigma,\tau}})$ is the notation of locally finite characters supported in a subgroupoid $\Gamma_{[u]_{\sigma,\tau}}$. If the class $[a]_{(\sigma,\tau)}$ is infinite, then by Corollary 5.1 characters from the subspace $X(\Gamma_{[u]_{\sigma,\tau}})$ are quasi-inner.

Now consider finite class $[u]_{(\sigma,\tau)}$. In $\Gamma_{[u]_{\sigma,\tau}}$ each set of maps $\{(a\sigma(g), g) \mid a \in \mathrm{Obj}(\Gamma_{[u]_{\sigma,\tau}})\}$ is finite because the set of objects is finite. That means that each character on this subgroupoid is locally finite. Each equivalence class of the characters which have equals values on loops, i.e. $\chi_1 - \chi_2 \in QInn$, is defined by an element of the group $Z^*_{(\sigma,\tau)}(u)$.

Hence, we get that

$$D_{(\sigma,\tau)}(\Gamma_{[u]_{\sigma,\tau}}) \cong Z^*_{(\sigma,\tau)}(u) \bigoplus QInn(\Gamma_{[u]_{\sigma,\tau}}).$$

From the above the statement of the theorem follows. □

Remark 5.3 Remind that if $u \in Z_{\sigma,\tau}$ then the space $QInn(\Gamma_{[u]_{\sigma,\tau}})$ is trivial.

Group (σ, τ)-nilpotency is necessary for triviality of characters on loops on infinite subgroupoids. If our group G is not nilpotent then the right side of (5.13) is just a subspace in the space of all (σ, τ)-derivations.

Following to the papers cited in introduction of this article we will describe conditions of quasi-innerness of (σ, τ)-derivations.

Corollary 5.2 *For groups satisfying the conditions of the Theorem 5.3, all (σ, τ)-derivations are quasi-inner if and only if the following condition is satisfied: all (σ, τ)-centralizers are such that factor-group $Z_{(\sigma,\tau)}(a)/Z'_{(\sigma,\tau)}(a)$ is a periodic group.*

Here $Z'_{(\sigma,\tau)}(a)$ is a derived subgroup of $Z_{(\sigma,\tau)}(a)$. Recall that a periodic group is a group such that all elements have finite order.

Proof The group $Z_{(\sigma,\tau)}(a)/Z'_{(\sigma,\tau)}(a)$ is naturally abelian. The periodicity of a group is equivalent to triviality of the space of group characters $Z_{(\sigma,\tau)}(a)/Z'_{(\sigma,\tau)}(a)$. Triviality for each $a \in G$ of spaces $Z^*_{(\sigma,\tau)}(a)$ is equivalent to triviality of first term in (5.20). So from the Theorem 5.3 we get that all (σ, τ)-derivations are quasi-inner. □

5.5.2 The Case of Inner Endomorphisms

Let G be a discrete rank 2 nilpotent group and $\sigma, \tau \in \mathbf{Aut}(G)$ act for fixed elements $\tilde{\sigma}, \tilde{\tau} \in G$ as follows

$$\sigma(u) = \tilde{\sigma} u \tilde{\sigma}^{-1}, \tau(u) = \tilde{\tau} u \tilde{\tau}^{-1}.$$

Let $Z(G)$ be the usual center of G.

Remark 5.4 We remind, that the group G is said to be rank 2 nilpotent group, if and only if the quotient group $G/Z(G)$ is abelian. In our notation nilpotent rank 2 group is a rank 2 (id, id)-nilpotent group.

Proposition 5.6 *The usual center of G is equal to* $Z_{\sigma,\tau}$.

Proof Consider $z \in Z(G)$. Then

$$\sigma(z)a = \tilde{\sigma}z\tilde{\sigma}^{-1}a = za = az = a\tilde{\tau}z\tilde{\tau}^{-1} = a\tau(z).$$

Thus, $Z(G) \subseteq Z_{\sigma,\tau}$. Now consider $z \in Z_{\sigma,\tau}$. Then

$$\sigma(z)a = a\tau(z) \to z = \tilde{\sigma}^{-1}a\tilde{\tau}z\tilde{\tau}^{-1}a^{-1}\tilde{\sigma}$$

Since $\tilde{\sigma}G\tilde{\tau}^{-1} = G$ as sets, the latter equation holds for every $g \in G$. Thus, $Z(G) = Z_{\sigma,\tau}$. □

Corollary 5.3 *The discrete rank* 2 *nilpotent group G coupled with* (σ, τ) *pair is a rank* 2 (σ, τ)*-nilpotent group.*

Proof As we mentioned before, a quotient group $G/Z(G)$ is abelian and $Z(G) = Z_{\sigma,\tau}$. Thus, the given group is a rank 2 (σ, τ)-nilpotent group. □

5.5.3 Heisenberg Group

Our results and observations allow us to calculate all (σ, τ)-derivations in Heisenberg group. In the calculation we will use results in [5, Sect. 3.3]. Recall that the Heisenberg group H is a group of unitriangular integer matrices. We denote the group algebra as $\mathcal{H}$.

Classes $[u]_{\sigma,\tau}$ in Heisenberg group either consist of one element or infinite. That means that by Theorem 5.3

$$D_{\sigma,\tau}(\mathcal{H}) = ZDer_{\sigma,\tau} \bigoplus QInn,$$

where $ZDer_{\sigma,\tau}$ denotes (σ, τ)-central (σ, τ)-derivations from Definition 5.8.

Description of (σ, τ)-central derivations is quite simple.

The homomorphisms $\varphi_{\mu,\nu}$ to additive group of complex numbers look alike

$$\varphi_{\mu,\nu} : \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto (\mu a + \nu b). \tag{5.14}$$

The center of group H (both (σ, τ) and usual by Proposition 5.6) contains elements

$$z_r = \begin{pmatrix} 1 & 0 & r \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Consider

$$\tilde{\sigma} = \begin{pmatrix} 1 & \sigma_a & \sigma_c \\ 0 & 1 & \sigma_b \\ 0 & 0 & 1 \end{pmatrix}, \tilde{\tau} = \begin{pmatrix} 1 & \sigma_a & \tau_c \\ 0 & 1 & \sigma_b \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.15}$$

Proposition 5.7 *For each centralizer element $u = \sigma(g)u\tau(g^{-1})\ \forall g \in G$ there exists $r \in \mathbb{R}$, such that $u = z_r$. In the other words, (σ, τ)-centralzers and elements from $Z_{(\sigma,\tau)} = Z(G)$ become equal.*

Proof

$$u = \begin{pmatrix} 1 & u_a & u_c \\ 0 & 1 & u_b \\ 0 & 0 & 1 \end{pmatrix}, g = \begin{pmatrix} 1 & g_a & g_c \\ 0 & 1 & g_b \\ 0 & 0 & 1 \end{pmatrix},$$

$$\sigma(g)u\tau(g^{-1}) = \begin{pmatrix} 1 & u_a & u_c + g_a u_b - u_b g_a \\ 0 & 1 & u_b \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, from the equation $u = \sigma(g)u\tau(g^{-1})$, one gets $u_a = 0$, $u_b = 0$, $u_c = r$. □

The latter equation means that (σ, τ)-derivations given by (5.15) become equal to the usual derivations considered in [5].

So, we can calculate the (σ, τ)-central derivation $d^r_{\varphi_{\mu,\nu}}$ on the generator of algebra $\mathcal{H}$:

$$d^r_{\varphi_{\mu,\nu}} \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = (\mu a + \nu b) \begin{pmatrix} 1 & a & c + \sigma_a b - \sigma_b a + r \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.16}$$

5.6 (σ, τ)-FC Groups

The class of FC-groups is an interesting class of groups for which conditions are similar to condition of finite groups. More detailed study of FC-groups can be found in [19, 20, 50].

In classical group theory, FC-group is a group in which all conjugacy classes are finite. In this terms abelian group is a group where all conjugacy classes contain one element. These concepts are naturally carried to the case of (σ, τ)-groups.

Definition 5.12 1. Group G is a (σ, τ)-FC group if each (σ, τ)-conjugacy class $[u]_{\sigma,\tau} \in G^{(\sigma,\tau)}$ is finite.
2. Group G is a (σ, τ)-abelian (or (σ, τ)-commutative) if each class $[u]_{\sigma,\tau}$ contains single element.

For (σ, τ)-commutative group, the following identity holds

$$\sigma(v)u = u\tau(v), \ \forall\, u, v \in G. \tag{5.17}$$

Apparently the definition of (σ, τ)-FC group is introduced in this paper for the first time, and so we will show some properties of such groups. Let us give an example of a source of such groups.

Proposition 5.8 *Let endomorphisms σ, τ acting on a group G have a finite image. Then G is a (σ, τ)-FC group.*

Proof If images of endomorphisms σ, τ are finite, then each (σ, τ)-conjugacy class is finite by Definition 5.3. □

Standard FC-group may not be an (σ, τ)-FC group for arbitrary endomorphisms (σ, τ) even if σ and τ are inner. Let for fixed elements $x, y \in G, \sigma_x : g \to xgx^{-1}, \tau_y : g \to ygy^{-1}$. Then for $a \in G$ the corresponding (σ, τ)-conjugacy class looks like

$$[a]_{\sigma_x,\tau_y} = \{xvx^{-1}ayv^{-1}y^{-1} | v \in G\}. \tag{5.18}$$

So if G is an infinite FC-group, then we have infinite number of conjugacy classes, so typically there is an infinite number of elements in (σ, τ)-conjugacy class. However the following proposition holds.

Proposition 5.9 *A group G is a (σ_x, σ_x)-FC group if and only if G is a FC-group.*

Proof For the proof it is enough to see that for $x = y$ formula (5.18) can be rewritten in the following way

$$[a]_{\sigma_x,\sigma_x} = \{xvx^{-1}axv^{-1}x^{-1} | v \in G\}, \tag{5.19}$$

and on the right side we get

$$xvx^{-1}axv^{-1}x^{-1} = xvx^{-1}a(xvx^{-1})^{-1},$$

so elements of $[a]_{\sigma_x,\sigma_x}$ are contained in the usual conjugacy class of the element a. □

The following statements follow easily from Proposition 5.9.

Corollary 5.4 *For each $x \in G$,*

1. *For each group G holds that $[a]_{\sigma_x,\sigma_x} = [a]$, where $[a]$ is the usual conjugacy class.*
2. *If G is an abelian group then it is (σ_x, σ_x)-abelian.*

Remark 5.5 If group G is (σ, τ)-abelian it may not be abelian in the usual sense. An example of this is the case when $\sigma = \tau$ and an image of map σ subsets in the usual centre of the group G.

Now we will prove an analogue of Theorem 5.3 for (σ, τ)-FC groups.

Theorem 5.4 *If G is a finitely generated (σ, τ)-FC group, and σ, τ are endomorphisms of group G, then*

$$D_{(\sigma,\tau)} \cong \bigoplus_{[a]_{(\sigma,\tau)}} Z^*_{(\sigma,\tau)}(a) \bigoplus Inn(\Gamma). \tag{5.20}$$

Here $Z^*_{(\sigma,\tau)}(a)$ is the space of group characters of the (σ, τ)-centralizer $Z_{(\sigma,\tau)}(a)$ as in our Theorem 5.3.

Proof The proof of (5.20) in our theorem is similar to the proof of Theorem 5.3. Except for one moment: we have to proof that all quasi-inner (σ, τ)-derivations for the case of (σ, τ)-FC group are inner.

For quasi-inner (σ, τ)-derivations, the Theorem 5.2 is applicable. The set of objects in each subgroupoid $\Gamma_{[u]_{\sigma,\tau}}$ is finite. So the following formula holds:

$$d(g) = \sum_{h \in G} (P(h\tau(g^{-1})) - P(\sigma(g^{-1})h))h. \tag{5.21}$$

From formula (5.12) we have the following decomposition for derivation d:

$$d = \sum_{[u]_{\sigma,\tau} \in G^{(\sigma,\tau)}} d_{[u]_{\sigma,\tau}}, \tag{5.22}$$

where derivation $d_{[u]_{\sigma,\tau}}$ is supported in groupoid $\Gamma_{[u]_{\sigma,\tau}}$. First we will prove that each term is inner and then check that sum (5.22) is finite.

Consider the fixed term $d_{[u]_{\sigma,\tau}}$. The set of objects in $\Gamma_{[u]_{\sigma,\tau}}$ is finite so the right side in formula (5.21) is nonzero just for $h \in [u]_{\sigma,\tau}$. That implies that derivation $d_{[u]_{\sigma,\tau}}$ is inner and holds the formula

$$d_{[u]_{\sigma,\tau}}(g) = [\sum_{h \in [u]_{\sigma,\tau}} P(h)h, g]. \tag{5.23}$$

If G is a finite or abelian group proof of innerness of derivation d is trivial. So let G be an infinite group.

By assumption $G =< g_1, \dots, g_n >$ is a finitely generated group. If the support of character χ contains infinite number of subgroupoids, then for some $i \in 1, \dots, n$

character is not trivial on infinite number of morphisms of the form $(*, g_i)$, which contradicts with locally finiteness condition which is necessary for χ to yield the derivation. □

Corollary 5.5 *If G is a finite group then all (σ, τ)-derivations are inner.*

Proof If G is a finite group, then the set morphisms in our groupoid Γ is finite because the set $\mathbf{Hom}(\Gamma)$ is a Cartesian product $G \times G$. So the set of loops around each object is finite. But if $\zeta \in \mathbf{Hom}(a, a)$ and $\chi(\zeta) \neq 0$ then $\chi(\zeta^n) \neq 0$, so the set of loops $\{\zeta^n | n \in \mathbf{N}\}$ is infinite which is impossible.

That gives us triviality of the character χ on all loops and it remains to apply the Theorem 5.4. □

The general case when maps σ, τ are endomorphisms of group algebra was reviewed in [12]. In the cited paper, the following theorem (see Theorem 1.1) was proved.

Theorem 5.5 (Chaudhuri 2019) *Let G be a finite group and R be an integral domain with* 1 *with characteristic $p \geq 0$ such that p does not divide the order of G.*

1. *If R is a field and σ, τ are algebra endomorphisms of the group ring RG such that they fix the center $\mathcal{Z}(RG)$ elementwise, then every (σ, τ)-derivation of RG is (σ, τ)-inner.*
2. *If R is an integral domain that is not a field and σ, τ are R-linear extensions of group homomorphisms of G such that they fix $\mathcal{Z}(RG)$ elementwise, then every (σ, τ)-derivation of RG is (σ, τ)-inner.*

Note that if σ and τ are identical isomorphisms, then we get a well-known theorem that in group algebras for finite groups all derivations are inner.

Another natural application of Theorem 5.4 is a case of (σ, τ)-abelian group. It is easy to see that in (σ, τ)-abelian groups the (σ, τ)-commutator is trivial, so there are no inner (σ, τ)-derivations.

Corollary 5.6 *The σ, τ-derivation algebra $D_{(\sigma,\tau)}$ of (σ, τ)-abelian group coincides with (σ, τ)-central derivations.*

Proof It is easy to see that derivation is (σ, τ)-central if and only if its support contains single element. □

Considering Remark 5.5 we note that the case when G is abelian group is significantly different to the case of (σ, τ)-abelian group. The case of σ-derivations for abelian groups was studied in [57].

Now we can find criterion of innerness of (σ, τ)-derivations in (σ, τ)-FC group which is similar to Corollary 5.2.

Corollary 5.7 *For groups satisfying to conditions of the Theorem 5.4, all (σ, τ)-derivations are inner if and only if the following condition is satisfied: all (σ, τ)-centralizers are such that $\forall a \in G$, the abealianization $Z_{(\sigma,\tau)}(a)/Z'_{(\sigma,\tau)}(a)$ is a periodic group.*

Here $Z'_{(\sigma,\tau)}(a)$ is a derived subgroup of $Z_{(\sigma,\tau)}(a)$.

Proof The prove is similar to prove of Corollary 5.2. The group $Z_{(\sigma,\tau)}(a)/Z'_{(\sigma,\tau)}(a)$ is naturally abelian. So locally finiteness is equivalent to triviality for each $a \in G$ of the space of group characters $Z_{(\sigma,\tau)}(a)/Z'_{(\sigma,\tau)}(a)$. So from the Theorem 5.4 we get that all (σ, τ)-derivations are quasi-inner. □

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Chapter 6
Decomposition of Complete Color Hom-Lie Algebras

Ahmad Reza Attari Polsangi, Mohammad Reza Farhangdoost, and Sergei Silvestrov

Abstract In this paper, we study some equivalent conditions for a color hom-Lie algebra to be a complete color hom-Lie algebra. In particular, we discuss the relationship between decomposition and completness for a color hom-Lie algebra. Moreover, we check some conditions that the set of α^s-derivations of a color hom-Lie algebra to be complete and simply complete. Finally, we find some conditions in which the decomposition into hom-ideals of the complete multiplicative color hom-Lie algebras is unique up to order of hom-algebra.

Keywords Color hom-Lie algebra · Complete color hom-Lie algebra · Simple color hom-Lie algebra

MSC2020 Classification: 17B61 · 17D30 · 17B75 · 17B40 · 17B70

6.1 Introduction

Hom-Lie algebras and quasi-hom-Lie algebras were introduced first by Hartwig, Larsson, and Silvestrov in 2003 in [30] devoted to a general method for construction of deformations and discretizations of Lie algebras of vector fields and deformations of Witt and Virasoro type algebras based on general twisted derivations obeying twisted Leibniz rule, and motivated also by the examples of q-deformed Jacobi iden-

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tities in q-deformations of Witt and Visaroro algebras and in related q-deformed algebras discovered in 1990'th in string theory, vertex models of conformal field theory, quantum field theory and quantum mechanics, and q-deformed differential calculi and q-deformed homological algebra. Subsequently, in 2005, Larsson and Silvestrov introduced quasi-Lie and quasi-Leibniz algebras in [35] and color (graded) quasi-Lie and color (graded) quasi-Leibniz algebras in [36] which include as special subclasses the color hom-Lie algebras and hom-Lie superalgebras as well as quasi-hom-Lie color algebras, quasi-hom-Lie superalgebras allowing to treat within the same natural algebraic structure framework the hom-Lie algebras and quasi-hom-Lie algebras defined in [30] and Lie color and super algebras, color (graded) Leibniz algebras and superalgebras and their hom-algebra deformations. The central extensions and cocycle conditions have been first considered for quasi-hom-Lie algebras and hom-Lie algebras in [30, 34] and for graded color quasi-hom-Lie algebras in [49]. Hom-Lie admissible algebras have been considered first by Makhlouf and Silvestrov in [41], where the hom-associative algebras and more general G-hom-associative algebras including the Hom-Vinberg algebras (hom-left symmetric algebras), hom-pre-Lie algebras (hom-right symmetric algebras), and some other new Hom-algebra structures have been introduced and shown to be Hom-Lie admissible, in the sense that the operation of commutator as new product in these hom-algebras structures yields hom-Lie algebras. Furthermore, in [41], flexible hom-algebras have been introduced and connections to hom-algebra generalizations of derivations and of adjoint derivations maps have been considered, investigations of the classification problems for hom-Lie algebras have been initiated with constriction of families of the low-dimensional hom-Lie algebras. Following [30, 34–37, 41]. The extensions of representations theory, cohomology and homology theory of hom-Lie algebras have been considered in [6, 7, 42, 46, 53], and quadratic hom-Lie algebras have been considered in [27]. The area of hom-algebra structures expanded unifying and extending in non-trivial ways known and new classes of deformed associative and non-associative algebras, super-algebras and color (graded) algebras, n-ary algebraic structures and non-commutative and non-associative extensions and deformations of differential calculi and homological algebra constructions. Development of the theory of color hom-Lie algebras, hom-Lie superalgebras and their n-ary generalizations recently intensified in [1–5, 8–26, 28, 29, 31–33, 38–40, 43, 44, 48–52, 54]. In color quasi-Lie algebras, color quasi-hom-Lie algebras and color hom-Lie algebras in addition to graded (color) algebra structure with grading by an abelian group and bicharacter commutation factor modifying the skew-symmetry and Jacobi identities algebra, the skew-symmetry and the Jacobi identities are actually partial identities satisfied on homogeneous subspaces of the algebra grading and are furthermore twisted by the bicharacter commutation factor and by actions of more general families of deforming twisting linear maps, and with the twisted Jacobi identity in quasi-Lie and quasi-hom-Lie algebras containing six twisted triple bracket terms. Hom-Lie color algebras are special case of quasi-Lie color algebras where the graded bilinear product satisfies the colored (twisted) by bicharacter commutation factor skew-symmetry property as in Lie color algebras, but the hom-Lie color algebras Jacobi identity has only three terms twisted by a single linear map. Lie color algebras are a special case of hom-Lie color algebras when there is no linear twisting in Jacobi identity, meaning that the twisting linear map in the Jacobi identity is the identity map. For general twisting linear maps, the quasi hom-Lie color algebras are substantially different much richer families of more complicated varieties of in general non-associative algebraic structures with binary algebra operation and unary operations defined by twisting linear maps intricately interlinked via twisted skew-symmetry and twisted hom-Jacobi identities, and thus morphisms, classifications, deformations, representations, derivations and homological structures of quasi hom-Lie color algebras in the fundamental ways depend simultaneously on twisting maps unary operations and bilinear algebra structure operations. For instance, typically there are much more classes of non-isomorphic quasi hom-Lie color algebras, then non-isomorphic classes of Lie color algebras since in general only morphisms of the binary algebra structure which intertwine also twisting unary operations are morphisms of full quasi hom-Lie color algebra as hom-algebra structure with both binary and unary operations. The mathematical theory of hom-algebraic structures is interesting and important to develop in its own right as it provides unified approaches and new links between seemingly unrelated classes of associative and non-associative structures arising in different parts of mathematics. Moreover, Lie, super Lie and color Lie structures, their q-deformations and formal deformation theory and related generalizations of homological and geometric structures are important in development of fundamentals of quantum mechanics, quantum field theory, particle physics as well as symmetry analysis in classical and quantum physics models. Color hom-Lie algebras and more general color quasi-Lie algebras provide a unified framework for new families of non-associative structures, which interpolate on the fundamental level of defining identities between Lie algebras, Lie superalgebras, color Lie algebras and related non-associative structures and their deformations, quantum deformations and discritizations, and thus should be useful for development of unified approaches to algebraic models of classical and quantum physics, non-commutative geometry and symmetry analysis and computational methods and algorithms based on general non-uniform discretizations of differential and integral calculi.

In this article, we expand investigation of the interesting general classes of color hom-Lie algebras and their decompositions with respect to their hom-algebra substructures. Complete color hom-Lie algebras are considered and several equivalent conditions for a color hom-Lie algebra to be a complete color hom-Lie algebra are established. In particular, the relation between decomposition and completness for a color hom-Lie algebra is described. Moreover, some conditions for the set of α^s-derivations of a color hom-Lie algebra to be complete and simply complete are obtained. Furthermore, we find some conditions in which the decomposition into hom-ideals of the complete multiplicative color hom-Lie algebras is unique up to order of hom-ideals. Section 6.2 is devoted to relevant preliminaries on hom-associative algebras, hom-modules, color Hom-Lie algebras and their representations and derivations. In Sect. 6.3, the notion of a complete color hom-Lie algebra is presented, and the equivalent conditions for the completeness of $\mathfrak{g}_\gamma$ and $\mathfrak{g}$ are studied. Then conditions for a color hom-Lie algebra to be complete are considered by using the notion of holomorph color hom-Lie algebras and hom-ideals. After that a simply complete color hom-Lie algebras are defined and equivalence of a color hom-Lie algebra being simply complete or indecomposable is investigated. Then, we discuss the conditions for the $Der_{\alpha^{s+1}}(\mathfrak{g})$ to be complete and simply complete. Finally we find some conditions in which the decomposition into hom-ideals of the complete multiplicative color hom-Lie algebras is unique up to order of hom-ideals.

6.2 Preliminaries on Color Hom-Lie Algebras and Their Representation and Derivations

In the following we summarize some basic concept from [30, 34–36, 41] where also various examples and properties of color Hom-Lie and color Hom-associative algebraic structures can be found.

Throughout this article, all linear spaces are assumed to be over a field $\mathbb{K}$ of characteristic different from 2. A linear space V is said to be a Γ-graded by an abelian group Γ if there exists a family $\{V_j\}_{j\in\Gamma}$ of linear subspaces of V such that $V = \bigoplus_{j\in\Gamma} V_j$. The elements of V_j are said to be homogeneous of degree $j \in \Gamma$. The set of all homogeneous elements of V is denoted $\mathcal{H}(V) = \bigcup_{j\in\Gamma} V_j$. A linear mapping $f : V \to V'$ of two Γ-graded linear spaces $V = \bigoplus_{j\in\Gamma} V_j$ and $V' = \bigoplus_{j\in\Gamma} V'_j$ is called homogeneous of degree d if $f(V_j) \subseteq V'_{j+d}$, for all $j \in \Gamma$. Homogeneous linear maps of degree zero, $f(V_j) \subseteq V'_j$ for any $j \in \Gamma$, are also called even. An algebra $(A, \cdot)$ is said to be Γ-graded if its underlying linear space is Γ-graded, $A = \bigoplus_{j\in\Gamma} A_j$, and moreover $A_j \cdot A_k \subseteq A_{j+k}$, for all $j, k \in \Gamma$. A homomorphism $f : A \to A'$ of Γ-graded algebras A and A' is an algebra morphism homogeneous of degree 0_Γ (even) as a linear map.

Hom-modules are pairs (M, α) where M is an $\mathbb{K}$-module and $\alpha : M \to M$ is a linear operator. Hom-associative algebras are triples $(A, \cdot, \alpha)$ consisting of an $\mathbb{K}$-module A, a bilinear map $\cdot : A \times A \to A$ called multiplication and an even linear operator $\alpha : A \to A$ which satisfies the hom-associativity condition for all $x, y, z \in A$,

$$\alpha(x) \cdot (y \cdot z) = (x \cdot y) \cdot \alpha(z). \qquad \text{(hom-associativity)}$$

Hom-associative algebras or hom-modules with $\alpha^2 = id$ are called involutive. A $\mathbb{K}$-linear map $f : A \to B$ is called a morphism of hom-associative algebras $(A, \cdot, \alpha)$ and $(B, \times, \beta)$ if for all $x, y \in A$, $f(x \cdot y) = f(x) \times f(y)$ and $f(\alpha(x)) = \beta(f(x))$. If $(A, \cdot, \alpha)$ is a hom-associative algebra, then a linear subspace $B \subseteq A$ is called a hom-associative subalgebra of A if it is closed under both the multiplication $\cdot$ and the twisting map α, that is $B \cdot B \subseteq B$ and $\alpha(B) \subseteq B$. A hom-associative subalgebra I is called a hom-ideal of A if $x \cdot y \in I,\ y \cdot x \in I$ for all $x \in I,\ y \in A$, and $\alpha(I) \subseteq I$.

Definition 6.1 ([30, 34, 35, 41]) Hom-Lie algebras are triples $(\mathfrak{g}, [.,.], \alpha)$, where $\mathfrak{g}$ is a linear space, $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a bilinear map and $\alpha : \mathfrak{g} \to \mathfrak{g}$ is a linear map satisfying for all $x, y, z \in \mathfrak{g}$,

$$[x, y] = -[y, x] \qquad \text{(Skew-symmetry)}$$
$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0, \qquad \text{(Hom-Lie Jacobi identity)}$$

Hom-Lie algebra is called a multiplicative hom-Lie algebra if α is an algebra morphism, meaning that $\alpha([x, y]) = [\alpha(x), \alpha(y)]$ for any $x, y \in \mathfrak{g}$. Multiplicative hom-Lie algebra is called regular, if α is an automorphism.

Lie algebras form a special subclass of regular Hom-Lie algebras obtained when $\alpha = id$ in Definition 6.1 since $\alpha = id$ is clearly an algebra automorphism.

Now, we recall the notion of a color hom-Lie algebra as a generalization of Lie color algebras.

Definition 6.2 ([45, 47]) Given a commutative group Γ (grading group), a commutation factor (bi-character) on Γ with values in the multiplicative group $\mathbb{K} \setminus \{0\}$ of a field $\mathbb{K}$ of characteristic 0 is a map $\varepsilon : \Gamma \times \Gamma \to \mathbb{K} \setminus \{0\}$ satisfying for all $j, k, l \in \Gamma$ and $x \in X_j$, $y \in X_k$, $z \in X_l$ the following (bi-character) properties:

$$\varepsilon(j + k, l) = \varepsilon(j, l)\varepsilon(k, l), \quad \varepsilon(j, k + l) = \varepsilon(j, l)\varepsilon(j, k),$$
$$\varepsilon(j, k)\varepsilon(k, j) = 1$$

Γ-Graded ε-Lie algebra (Lie color algebra) is a Γ-graded linear space $X = \bigoplus_{j \in \Gamma} X_\gamma$, with a bilinear multiplication $[.,.] : X \times X \to X$ obeying for $j, k, l \in \Gamma$ and $x \in X_j$, $y \in X_k$, $z \in X_l$:

$$\text{Grading axiom:} \qquad [X_j, X_k] \subseteq X_{j+k}, \tag{6.1}$$
$$\text{Color skew-symmetry:} \qquad [x, y] = -\varepsilon(j, k)[y, x], \tag{6.2}$$

Color Jacobi identity:

$$\sum_{\text{cyclic}\{x,y,z\}} \varepsilon(z, x)[\alpha(x), [y, z]] = \\ \varepsilon(l, j)[x, [y, z]] + \varepsilon(k, l)[z, [x, y]] + \varepsilon(j, k)[y, [z, x]] = 0. \tag{6.3}$$

The elements of X_j are called homogenous of degree $j \in \Gamma$. Since the commutation factor ε yields a map from $(\bigcup_{j \in \Gamma} X_j) \times (\bigcup_{j \in \Gamma} X_j) \to \mathbb{K} \setminus \{0\}$ taking the value $\varepsilon(l, j)$ on all elements of $X_l \times X_j$, the notation convention $\varepsilon(z, x) = \varepsilon(l, j)$ is used for $z \in X_l$ and $x \in X_j$ where $(l, j) \in \Gamma \times \Gamma$.

Now, we recall the notion of a color hom-Lie algebra which is a special class of general color quasi-Lie algebras (Γ-graded quasi-Lie algebras) that where defined first in [36, 48, 49].

Definition 6.3 ([9, 14, 36, 48, 49]) Color hom-Lie algebras are defined as quadruples $(\mathfrak{g}, [.,.], \varepsilon, \alpha)$ consisting of a Γ-grade linear space $\mathfrak{g}$, an even (degree zero) bilinear mapping $[.,.] : \mathfrak{g} \times \mathfrak{g}$, meaning that $[\mathfrak{g}_a, \mathfrak{g}_b] \subseteq \mathfrak{g}_{a+b}$ for $a, b \in \Gamma$, a commutation factor (bi-character) ε and an even homomorphism $\alpha : \mathfrak{g} \to \mathfrak{g}$ such that for homogeneous elements $x, y, z \in \mathfrak{g}$,

$$[x, y] = -\varepsilon(x, y)[y, x], \qquad (\varepsilon - \text{skew symmetry}) \qquad (6.4)$$

$$\sum_{\text{cyclic}} \{x, y, z\}\varepsilon(z, x)[\alpha(x), [y, z]] = 0. \qquad (\varepsilon - \text{hom-Jacobi identity}) \qquad (6.5)$$

Definition 6.4 A color hom-Lie algebra $(\mathfrak{g}, [.,.], \varepsilon, \alpha)$ is called

(i) multiplicative color hom-Lie algebra, if α is a morphism of the color Lie algebra, that is $\alpha \circ [.,.] = [.,.] \circ \alpha^{\otimes 2}$ for any $x, y \in \mathfrak{g}$;
(ii) regular color Hom-Lie algebra if α is an Hom-algebra automorphism of the color Hom-Lie algebra.
(iii) involutive color Hom-Lie algebra if α is an involution, that is $\alpha^2 = Id$.

Example 6.1 A multiplicative color hom-Lie algebra can be constructed for example by the standard method of composing multiplication with algebra morphism as in case of hom-Lie algebras. Let $(\mathfrak{g}, [.,.], \varepsilon)$ be a color Lie algebra and α be a Lie color algebra morphism. Then $(\mathfrak{g}, [.,.]_\alpha := \alpha \circ [.,.], \varepsilon, \alpha)$ is a multiplicative hom-Lie color algebra.

An even linear map $f : \mathfrak{g} \to \mathfrak{g}'$, where $(\mathfrak{g}, [.,.], \varepsilon, \alpha)$ and $(\mathfrak{g}', [.,.]', \varepsilon', \alpha')$ are two color hom-Lie algebras is said to be a morphism of color hom-Lie algebras, if

(i) $f([x, y]) = [f(x), f(y)]'$, for all $x, y \in \mathfrak{g}$,
(ii) $f \circ \alpha = \alpha' \circ f$.

Hom-subalgebras of color hom-Lie algebra $(\mathfrak{g}, [.,.], \varepsilon, \alpha)$ are defined as Γ-graded linear subspaces closed under both α and $[.,.]$, that is $\alpha(I) \subseteq I$ and $[I, I] \subseteq I$. Hom-subalgebra I is called a hom-ideal of the color hom-Lie algebra $\mathfrak{g}$, if $[I, \mathfrak{g}] \subseteq I$, and notation $I \lhd \mathfrak{g}$ is used in this case. In color hom-Lie algebras, by ε-skew symmetry (6.4), $[I, \mathfrak{g}] \subseteq I$ is equivalent to $[\mathfrak{g}, I] \subseteq I$, since

$$\forall\, y=\sum_{k\in\Gamma} y_k \in I=\sum_{k\in\Gamma} I_k,\, x=\sum_{j\in\Gamma} x_j \in \mathfrak{g}=\oplus_{j\in\Gamma}\mathfrak{g}_j,\, y_k\in I_k,\, x_j\in\mathfrak{g}_j:$$

$$\begin{aligned}
[x,y] &= \sum_{j,k\in\Gamma}[x_j,y_k] \stackrel{(6.4)}{=} \sum_{j,k\in\Gamma}(-\varepsilon(j,k)[y_k,x_j] \\
&\in \sum_{k\in\Gamma}[I,\mathfrak{g}] \subseteq \sum_{k\in\Gamma} I = I, \quad \text{when} \quad [I,\mathfrak{g}]\subseteq I, \\
[x,y] &= \sum_{j,k\in\Gamma}[x_j,y_k] \stackrel{(6.4)}{=} \sum_{j,k\in\Gamma}(-\varepsilon(j,k)[y_k,x_j] \\
&\in \sum_{k\in\Gamma}[\mathfrak{g},I] \subseteq \sum_{k\in\Gamma} I = I, \quad \text{when} \quad [\mathfrak{g},I]\subseteq I.
\end{aligned}$$

Thus, in color hom-Lie algebras, all right or left hom-ideals are two-sided hom-ideals.

Color hom-Lie subalgebra I of a color hom-Lie algebra is called commutative if $[I,I]=0$. If I is not Abelian, then $[x,y]\neq 0$ for some non-zero elements $x,y\in I$.

Definition 6.5 ([13]) The center of a color hom-Lie algebra $\mathfrak{g}$ is defined as

$$C(\mathfrak{g})=\{x\in\mathfrak{g} : [x,\mathfrak{g}]=0\}.$$

The centralizer of a hom-ideal I in a color hom-Lie algebra $\mathfrak{g}$ is defined as

$$C_{\mathfrak{g}}(I)=\{x\in\mathfrak{g} : [x,I]=0\}.$$

In any color hom-Lie algebra $(\mathfrak{g}=\bigoplus_{\gamma\in\Gamma}\mathfrak{g}_\gamma,[.,.],\varepsilon,\alpha)$, the center is the centraliser of hom-ideal $\mathfrak{g}$ in $(\mathfrak{g},[.,.],\varepsilon,\alpha)$, that is $C(\mathfrak{g})=C_{\mathfrak{g}}(\mathfrak{g})$. For any hom-ideal I the centralizer $C_{\mathfrak{g}}(I)$ is a Γ-graded subspace $C_{\mathfrak{g}}(I)=\bigoplus_{\gamma\in\Gamma}(C_{\mathfrak{g}}(I)\cap\mathfrak{g}_\gamma)$, since

$$\begin{aligned}
&\forall\, y\in I,\ x=\sum_{j\in\Gamma} x_j\in\mathfrak{g}=\oplus_{j\in\Gamma}\mathfrak{g}_j,\ x_j\in\mathfrak{g}_j: \\
&\quad [x,y]=[\sum_{j\in\Gamma}x_j,y]=\sum_{j\in\Gamma}[x_j,y]=0 \Leftrightarrow \\
&\forall\,\gamma\in\Gamma, [x_\gamma,y]=-\sum_{j\in\Gamma/\{\gamma\}}[x_j,y]=\sum_{j\in\Gamma/\{\gamma\}}[-x_j,y]\in\mathfrak{g}_\gamma\cap\sum_{j\in\Gamma/\{\gamma\}}\mathfrak{g}_j\cap I=\{0\}\Leftrightarrow \\
&\quad [x_\gamma,y]=0 \quad \forall\,\gamma\in\Gamma \Leftrightarrow x_\gamma\in C_{\mathfrak{g}}(I)\cap\mathfrak{g}_\gamma,\ \gamma\in\Gamma.
\end{aligned}$$

In general, $[C_{\mathfrak{g}}(I),C_{\mathfrak{g}}(I)]\subseteq C_{\mathfrak{g}}(I)$ and $\alpha(C_{\mathfrak{g}}(I))\subseteq C_{\mathfrak{g}}(I)$ are not assured, since the equality $[[x_1,x_2],y]=0$ is not necessarily implied by $[x_1,y]=0$ and $[x_2,y]=0$, and $[x,y]=0$ does not necessarily imply $[\alpha(x),y]=0$ for $x_1,x_2,x\in\mathfrak{g}$ and $y\in I$.

Lemma 6.1 *Let* $(\mathfrak{g},[.,.],\varepsilon,\alpha)$ *be a color hom-Lie algebra. If* $(\mathfrak{g},[.,.],\varepsilon,\alpha)$ *is a multiplicative color hom-Lie algebra with surjective* α*, that is* $\alpha([.,.])=[\alpha(.),\alpha(.)]$ *and* $\alpha(\mathfrak{g})=\mathfrak{g}$*, then the center* $C(\mathfrak{g})$ *is a commutative hom-ideal in* $(\mathfrak{g},[.,.],\varepsilon,\alpha)$.

Proof The Γ-graded hom-subspace $C(\mathfrak{g}) = \bigoplus_{\gamma\in\Gamma}(C(\mathfrak{g}) \cap \mathfrak{g}_\gamma)$ of the color hom-Lie algebra $(\mathfrak{g}, [.,.], \varepsilon, \alpha)$ is closed under $[.,.]$ and α. Indeed, $\alpha(C(\mathfrak{g})) \subseteq C(\mathfrak{g})$, since the preimage set $\alpha^{-1}(y) \neq \emptyset$ of any $y \in \mathfrak{g}$ is non-empty by surjectivity of α, and

$$\begin{aligned}&\forall\, x \in C(\mathfrak{g}), y \in \mathfrak{g} :\\ &[\alpha(x), y] = [\alpha(x), \alpha(\alpha^{-1}(y))] = \alpha([x, \alpha^{-1}(y)]) = \alpha(\{0\}) = \{0\}.\end{aligned}$$

Moreover, $[C(\mathfrak{g}), C(\mathfrak{g})] = [C(\mathfrak{g}), \mathfrak{g}] = \{0\} \subseteq C(\mathfrak{g})$ by definition of the center. Hence, $C(\mathfrak{g})$ is commutative hom-ideal. □

Lemma 6.2 *If $(\mathfrak{g}, [.,.], \varepsilon, \alpha)$ is a multiplicative color hom-Lie algebra with α surjective on I, that is $\alpha([.,.]) = [\alpha(.), \alpha(.)]$ and $\alpha(I) = I$, then for any hom-ideal I in color hom-Lie algebra $(\mathfrak{g}, [.,.], \varepsilon, \alpha)$,*

(i) *$C_\mathfrak{g}(I)$ is a hom-ideal in color hom-Lie algebra $(\mathfrak{g}, [.,.], \varepsilon, \alpha)$.*
(ii) *$C(I) = C_I(I)$ is an commutative hom-ideal in the color hom-Lie algebra $(I, [.,.]_I, \varepsilon, \alpha_I)$, where $[.,.]_I$ and α_I are restrictions of $[.,.]$ and α to I.*
(iii) *If $(\mathfrak{g}, [.,.], \varepsilon, \alpha)$ is a multiplicative color hom-Lie algebra with surjective α, that is $\alpha([.,.]) = [\alpha(.), \alpha(.)]$ and $\alpha(\mathfrak{g}) = \mathfrak{g}$, then the center $C(\mathfrak{g})$ is an commutative hom-ideal in $(\mathfrak{g}, [.,.], \varepsilon, \alpha)$.*

Proof For a hom-ideal I, the Γ-graded hom-subspace $C_\mathfrak{g}(I) = \bigoplus_{\gamma\in\Gamma}(C_\mathfrak{g}(I)) \cap \mathfrak{g}_\gamma)$ of the color hom-Lie algebra $(\mathfrak{g}, [.,.], \varepsilon, \alpha)$ is closed under $[.,.]$ if $\alpha(I) = I$, since by super hom-Jacobi identity (6.5), definition of the centralizer, and the condition $I = \alpha(I)$ of surjectivity of the restriction of α on I,

$$\begin{aligned}&\forall\, x \in I \cap \mathcal{H}(\mathfrak{g}),\ \ y, z \in C_\mathfrak{g}(I) \cap \mathcal{H}(\mathfrak{g}) :\\ &[x, y] = 0,\ \ [\alpha(y), [x, z]] = [\alpha(y), 0] = 0, \Rightarrow\\ &[\alpha(x), [y, z]] = -\varepsilon(z, x)[[x, y], \alpha(z)] - \varepsilon(z, y)[\alpha(y), [x, z]] = 0, \Rightarrow\\ &[I, [C_\mathfrak{g}(I), C_\mathfrak{g}(I)]] \overset{\alpha(I)=I}{=} [\alpha(I), [C_\mathfrak{g}(I), C_\mathfrak{g}(I)]] = \{0\} \Rightarrow\\ &[C_\mathfrak{g}(I), C_\mathfrak{g}(I)] \subseteq C_\mathfrak{g}(I).\end{aligned}$$

The Γ-graded hom-subspace $C_\mathfrak{g}(I) = \bigoplus_{\gamma\in\Gamma}(C_\mathfrak{g}(I)) \cap \mathfrak{g}_\gamma)$ is closed under α, since definition of the centraliser, surjectivity $\alpha(I) = I$ of α on I and multiplicativity of α yield

$$\begin{aligned}&[\alpha(C_\mathfrak{g}(I)), I] = [\alpha(C_\mathfrak{g}(I)), \alpha(I)] = \alpha([C_\mathfrak{g}(I), I]) \in \alpha(\{0\}) = \{0\} \Rightarrow\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \alpha(C_\mathfrak{g}(I)) \subseteq C_\mathfrak{g}(I).\end{aligned}$$

So, $C_\mathfrak{g}(I)$ is a Γ-graded hom-subalgebra in the color hom-Lie algebra $(\mathfrak{g}, [.,.], \varepsilon, \alpha)$. Moreover,

$$\begin{aligned}
&\forall\, x \in I \cap \mathcal{H}(\mathfrak{g}),\ \ y \in \mathfrak{g} \cap \mathcal{H}(\mathfrak{g}), z \in C_{\mathfrak{g}}(I) \cap \mathcal{H}(\mathfrak{g}):\\
&[x, y] \in I,\ \ [\alpha(y), [x, z]] = [\alpha(y), 0] = 0, \Rightarrow\\
&[\alpha(x), [y, z]] = -\varepsilon(z, x)[[x, y], \alpha(z)] - \varepsilon(z, y)[\alpha(y), [x, z]] \in I, \Rightarrow\\
&[I, [\mathfrak{g}, C_{\mathfrak{g}}(I)]] \overset{\alpha(I)=I}{=} [\alpha(I), [\mathfrak{g}, C_{\mathfrak{g}}(I)]] \in I \Rightarrow [\mathfrak{g}, C_{\mathfrak{g}}(I)] \subseteq C_{\mathfrak{g}}(I).
\end{aligned}$$

Hence, $C_{\mathfrak{g}}(I)$ is a hom-ideal. □

We are going to need the following definition throughout the rest of the article.

Definition 6.6 ([13, 15]) A representation of the color hom-Lie algebra $(\mathfrak{g}, [.,.], \varepsilon, \alpha)$ on a Γ-graded linear space V with respect to $\beta : V \to V$, is an even linear map $\rho : \mathfrak{g} \to End(V)$ such that for all homogeneous $x, y \in \mathcal{H}(\mathfrak{g})$,

$$\begin{aligned}
\rho(\alpha(x)) \circ \beta &= \beta \circ \rho(x),\\
\rho([x, y]) \circ \beta &= \rho(\alpha(x)) \circ \rho(y) - \varepsilon(x, y)\rho(\alpha(y)) \circ \rho(x).
\end{aligned}$$

A representation V of $\mathfrak{g}$ is called irreducible or simple, if it has no nontrivial subrepresentations. Otherwise V is called reducible.

For any linear transformation $T : X \mapsto X$ of a set X, and any nonnegative integer s, the s-times composition is $T^s = T \circ \cdots \circ T\ (\mathrm{s}-times),\ T^0 = Id,\ T^1 = T$, and if T is invertible with inverse map $T^{-1}\mathfrak{g} \to \mathfrak{g}$, then $T^{-s} = T^{-1} \circ \cdots \circ T^{-1}\ (s - times)$.

Let $(\mathfrak{g}, [.,.]_{\mathfrak{g}}, \varepsilon, \alpha)$ be a color hom-Lie algebra and $\mathfrak{g} = \bigoplus\limits_{\gamma \in \Gamma} \mathfrak{g}_\gamma$. Then $End(\mathfrak{g})$ is equipped with the induced Γ-grading $End(\mathfrak{g}) = \bigoplus\limits_{\gamma \in \Gamma} (End(\mathfrak{g}))_\gamma$ where

$$(End(\mathfrak{g}))_\gamma = \{f \in End(\mathfrak{g}) | f(\mathfrak{g}_k) \subseteq \mathfrak{g}_{k+\gamma}\}.$$

Next, we recall the notion of α^s-derivations.

Definition 6.7 ([15]) Let $(\mathfrak{g}, [.,.]_{\mathfrak{g}}, \varepsilon, \alpha)$ be a color hom-Lie algebra. For any nonnegative integer s, a linear map $D : \mathfrak{g} \to \mathfrak{g}$, of degree d is called a homogeneous α^s-derivation of the multiplicative color hom-Lie algebra $(\mathfrak{g}, [.,.]_{\mathfrak{g}}, \varepsilon, \alpha)$, if for all homogeneous $x, y \in \mathcal{H}(\mathfrak{g})$,

$$\begin{aligned}
D(\mathfrak{g}_\gamma) &\subseteq \mathfrak{g}_{\gamma+d},\\
D \circ \alpha &= \alpha \circ D,\\
D([x, y]_{\mathfrak{g}}) &= [D(x), \alpha^s(y)]_{\mathfrak{g}} + \varepsilon(d, x)[\alpha^s(x), D(y)]_{\mathfrak{g}}.
\end{aligned}$$

Denote by $Der_{\alpha^s}(\mathfrak{g})$ the set of all α^s-derivations of the color hom-Lie algebra $(\mathfrak{g}, [.,.], \varepsilon, \alpha)$, and

$$Der(\mathfrak{g}) = \bigoplus_{s \geq -1} Der_{\alpha^s}(\mathfrak{g}).$$

For any $x \in \mathfrak{g}$ satisfying $\alpha(x) = x$, the map $ad_s(x) : \mathfrak{g} \to \mathfrak{g}$, for all $y \in \mathfrak{g}$ is defined by

$$ad_s(x)(y) = [x, \alpha^s(y)]_{\mathfrak{g}}.$$

Lemma 6.3 *Let* $(\mathfrak{g}, [., .]_{\mathfrak{g}}, \varepsilon, \alpha)$ *be a multiplicative color hom-Lie algebra. Then* $ad_s(a)$ *is an* α^{s+1}*-derivation, which we call inner* α^{s+1}*-derivation.*

Proof We have, using multiplicativity at the steps marked by $\stackrel{*}{=}$,

$$ad_s(a) \circ \alpha(x) = [a, \alpha^{s+1}(x)]_{\mathfrak{g}} = [\alpha(a), \alpha^s(x)]_{\mathfrak{g}} \stackrel{*}{=} \alpha([a, \alpha^s(x)]_{\mathfrak{g}}) = \alpha \circ ad_s(a)(x),$$

and

$$\begin{aligned}
&ad_s(a)([x, y]_{\mathfrak{g}}) = [a, \alpha^s[x, y]]_{\mathfrak{g}} \stackrel{*}{=} [a, [\alpha^s(x), \alpha^s(y)]]_{\mathfrak{g}} \\
&= -\varepsilon(a, y)(\varepsilon(x, a)[\alpha^{s+1}(x), [\alpha^s(y), a]_{\mathfrak{g}}]_{\mathfrak{g}} + \varepsilon(y, x)[\alpha^{s+1}(y), [a, \alpha^s(x)]_{\mathfrak{g}}]_{\mathfrak{g}} \\
&= -\varepsilon(a, y)(\varepsilon(x, a)\varepsilon(y, a)[\alpha^{s+1}(x), [a, \alpha^s(y)]_{\mathfrak{g}}]_{\mathfrak{g}} \\
&+ \varepsilon(y, x)\varepsilon(y, [a, y])[[a, \alpha^s(x)]_{\mathfrak{g}}, \alpha^{s+1}(y)]_{\mathfrak{g}} \\
&= [[a, \alpha^s(x)]_{\mathfrak{g}}, \alpha^{s+1}(y)]_{\mathfrak{g}} + \varepsilon(x, a)[\alpha^{s+1}(x), [a, \alpha^s(y)]_{\mathfrak{g}}]_{\mathfrak{g}} \\
&= [ad_s(a)(x), \alpha^{s+1}(y)]_{\mathfrak{g}} + \varepsilon(x, a)[\alpha^{s+1}(x), ad_s(a)(y)]_{\mathfrak{g}}.
\end{aligned}$$

Therefore, $ad_s(a)$ is an α^{s+1}-derivation. □

The set $Inn_{\alpha^{s+1}}(\mathfrak{g}) = \{[x, \alpha^s(.)]_{\mathfrak{g}} | \quad x \in \mathfrak{g}, \quad \alpha(x) = x\}$ is a linear space in $Der_{\alpha^{s+1}}(\mathfrak{g})$. For $T \in End(\mathfrak{g})$ and $T' \in End(\mathfrak{g})$, define the color commutator (ϵ-commutator) as

$$[T, T']_{End(\mathfrak{g})} = T \circ T' - \varepsilon(T, T')T' \circ T.$$

For $D \in Der(\mathfrak{g})$ and $D' \in Der(\mathfrak{g})$, denote the color commutator (ε-commutator) as

$$[D, D']_{\mathcal{D}} = D \circ D' - \varepsilon(D, D')D' \circ D. \tag{6.6}$$

With the above notation, $(Der(\mathfrak{g}), [., .]_{\mathcal{D}}, \varepsilon)$ is a color Lie algebra, in which the bracket is given by (6.6).

Proposition 6.1 ([49]) *Let* $(\mathfrak{g}, [., .]_{\mathfrak{g}}, \varepsilon, \alpha)$ *be a multiplicative color hom-Lie algebra and consider on* $Der(\mathfrak{g})$ *the endomorphism* $\tilde{\alpha}$ *defined by* $\tilde{\alpha}(D) = \alpha \circ D$, *then* $(Der(\mathfrak{g}), [., .]_{\mathcal{D}}, \varepsilon, \tilde{\alpha})$ *is a color hom-Lie algebra where* $[., .]_{\mathcal{D}}$ *is given by* (6.6).

Proof By above notation, we know $(Der(\mathfrak{g}), [., .]_{\mathcal{D}}, \varepsilon)$ is a color Lie algebra. Then by using the method of Example 6.1, $(Der(\mathfrak{g}), [., .]_{\mathcal{D}}, \varepsilon, \tilde{\alpha})$ will be a color hom-Lie algebra. □

6.3 Decomposition of Complete Hom-Lie Superalgebras

In this section, we recall the notion of a complete hom-Lie superalgebra and state some results about it.

Definition 6.8 Color hom-Lie algebra $(\mathfrak{g}, [.,.], \varepsilon, \alpha)$ is called a complete color hom-Lie algebra if $\mathfrak{g}$ satisfies the following two conditions:

$$C(\mathfrak{g}) = 0,$$
$$Der_{\alpha^{s+1}}(\mathfrak{g}) = ad_s(\mathfrak{g}).$$

Remark 6.1 If $\mathfrak{g}_\gamma$ is a complete Lie algebra, then it is not necessary that $(\mathfrak{g}, [.,.], \varepsilon, \alpha)$, where $\mathfrak{g} = \bigoplus_{\gamma\in\Gamma} \mathfrak{g}_\gamma$, be a complete color hom-Lie algebra.

Let $(\mathfrak{g}_\gamma, <.,.>, \alpha)$ be a semisimple hom-Lie algebra, $\mathfrak{h} = \sum_{j\in\Gamma/\{\gamma\}} \mathfrak{g}_j$ be a finite-dimensional linear space and in $\tilde{\alpha} : \mathfrak{g} = \mathfrak{g}_\gamma \oplus \mathfrak{h} \to \mathfrak{g} = \mathfrak{g}_\gamma \oplus \mathfrak{h}$ be an even endomorphism such that $\tilde{\alpha}|_{\mathfrak{g}_0} = \alpha$. Then by [4], $(\mathfrak{g}, [.,.], \varepsilon, \alpha)$ is a color hom-Lie algebra such that $[x, y] = 0$ for all $x \in \mathfrak{h}$, $y \in \mathfrak{g}$ and $[x, y] = <x, y>$ for all $x, y \in \mathfrak{g}_\gamma$ where $<.,.>$ is bracket operation of the hom-Lie algebra $\mathfrak{g}_\gamma$. Since $C(\mathfrak{g}) \neq 0$, $\mathfrak{g}$ is not complete color hom-Lie algebra but $\mathfrak{g}_\gamma$ is complete, i.e. $C(\mathfrak{g}_\gamma) = 0$ and $Der_{\alpha^{s+1}}(\mathfrak{g}_\gamma) = ad_s(\mathfrak{g}_\gamma)$.

Definition 6.9 A color hom-Lie algebra $(\mathfrak{g}, [.,.], \varepsilon, \alpha)$ is called solvable if $\mathfrak{g}^n = 0$ for some $n \in \mathbb{N}$, where $\mathfrak{g}^n$, the members of the derived series of $\mathfrak{g}$, are defined inductively: $\mathfrak{g}^1 = \mathfrak{g}$, and $\mathfrak{g}^n = [\mathfrak{g}^{n-1}, \mathfrak{g}^{n-1}]$ for $n > 1$.

Note that any commutative color hom-Lie algebra is solvable and for a multiplicative color hom-Lie algebra $\mathfrak{g}$, we have $\alpha(\mathfrak{g}^n) \subseteq \mathfrak{g}^n$ for any n.

The color hom-Lie algebra $(\mathfrak{g}, [.,.], \varepsilon, \alpha)$ is called semisimple if it does not contain any non-trivial solvable hom-ideal.

Let $\mathfrak{g}$ be a color hom-Lie algebra and let Φ be a bilinear form on $\mathfrak{g}$. Recall that Φ is called invariant if $\Phi([x, y], z) = \Phi(x, [y, z])$ for all $x, y, z \in \mathfrak{g}$. The invariant bilinear form associated to the adjoint representation of $\mathfrak{g}$ is called the Killing form on $\mathfrak{g}$.

Now, we check the condition in which the completeness of $\mathfrak{g}_\gamma$, where $\gamma \in \Gamma$ and $\mathfrak{g}$ are equivalent.

Theorem 6.1 *Let $\mathfrak{g} = \bigoplus_{\gamma\in\Gamma} \mathfrak{g}_\gamma$ be a multiplicative color hom-Lie algebra and $\mathfrak{g}_\beta$, $\beta \in \Gamma$, is a direct summand of $\mathfrak{g}$ with surjective α on $\mathfrak{g}$ and $\mathfrak{g}_\beta$. If $\mathfrak{g}$ has the non-degenerate Killing form, then $\mathfrak{g}_\beta$ is a complete hom-Lie algebra and $\mathfrak{g}$ is a complete color hom-Lie algebra.*

Proof We know $\mathfrak{g}$ has non-degenerate Killing form, thus $Der_{\alpha^{s+1}}(\mathfrak{g}) = ad_s(\mathfrak{g})$. Since α is surjective, $C(\mathfrak{g})$ is commutative hom-ideal, and so $C(\mathfrak{g})$ is solvable. Hence $C(\mathfrak{g}) = 0$. Thus $\mathfrak{g}$ is complete. Let Φ be a non-degenerate Killing form of $\mathfrak{g}$. Then the restriction of Φ to $\mathfrak{g}_\beta$ is the non-degenerate Killing form of $\mathfrak{g}_\beta$. Hence $\mathfrak{g}_\beta$ is semisimple hom-Lie algebra and $Der_{\alpha^{s+1}}(\mathfrak{g}_\beta) = ad_s(\mathfrak{g}_\beta)$. Since α is surjective, $C(\mathfrak{g}_\beta)$ is commutative and solvable, So $C(\mathfrak{g}_\beta) = 0$. Therefore $\mathfrak{g}_\beta$ is complete hom-Lie algebra. □

Proposition 6.2 *Let $\mathfrak{g}$ be a multiplicative color hom-Lie algebra and I be a complete hom-ideal of $\mathfrak{g}$ with surjective α on both $\mathfrak{g}$ and I. There exists a hom-ideal J such that $\mathfrak{g} = I \oplus J$.*

Proof Let $J = C_{\mathfrak{g}}(I)$. Then $C_{\mathfrak{g}}(I)$ is a hom-ideal of $\mathfrak{g}$ by Lemma 6.2. Since I is hom-ideal, $ad_s(x) \in Der_{\alpha^{s+1}}(I)$, for all $x \in \mathfrak{g},\ \gamma \in \Gamma$.

Since I is complete, $Der_{\alpha^{s+1}}(I) = ad_s(I)$, so there exists a α^{s+1}-derivation D in $Der_{\alpha^{s+1}}(I)$ such that $ad_s(x) = D$. Hence there exists $r \in I$ such that

$$D(t) = ad_s(x)(t) = [x, \alpha^s(t)] = [r, \alpha^s(t)],$$

for any $t \in I$. Then $[x - r, \alpha^s(t)] = 0$ and $x - r \in C_{\mathfrak{g}}(I) = J$. Thus $x = r + l$, for some $l \in J$. On the other hand, since I is complete $I \cap J = I \cap C_{\mathfrak{g}}(I) = C(I) = 0$. Therefore $\mathfrak{g} = I \oplus J$. □

Definition 6.10 Let $(\mathfrak{g}, [.,.], \varepsilon, \alpha)$ be a color hom-Lie algebra and $h(\mathfrak{g}) = \mathfrak{g} \oplus Der(\mathfrak{g})$. The even bilinear map (bracket) $[.,.]_h : h(\mathfrak{g}) \times h(\mathfrak{g}) \to h(\mathfrak{g})$ and a linear map $\alpha_h : h(\mathfrak{g}) \to h(\mathfrak{g})$ are defined in $h(\mathfrak{g})$ by

$$[x + D, y + E]_h = [x, y]_g + D(y) - \varepsilon(x, E)E(x) + [D, E]_{\mathcal{D}},$$

$$\alpha_h(x + D) = \alpha(x) + \alpha \circ D,$$

where $x, y \in \mathfrak{g}$, $D, E \in Der(\mathfrak{g})$ and $[.,.]_{\mathcal{D}}$ is bracket in $Der(\mathfrak{g})$ given by (6.6). With the above notation, $h(\mathfrak{g})$ is a color hom-Lie algebra. We call $h(\mathfrak{g})$ a holomorph color hom-Lie algebra.

We know that $(Der(\mathfrak{g}), [.,.]_{\mathcal{D}}, \varepsilon, \tilde{\alpha})$ is color hom-Lie algebra by Lemma 6.1. Therefore we have the following results.

Lemma 6.4 *Let $\mathfrak{g}$ be a multiplicative color hom-Lie algebra and $(h(\mathfrak{g}), [.,.]_h, \varepsilon, \alpha_h)$ be holomorph color hom-Lie algebra.*

(i) *If $C(\mathfrak{g}) = 0$, then $C(Der(\mathfrak{g})) = \{D \in Der(\mathfrak{g}) | [D, Der(\mathfrak{g})]_{\mathcal{D}} = 0\} = 0$.*
(ii) *$\mathfrak{g}$ is hom-ideal of $h(\mathfrak{g})$ and $h(\mathfrak{g})/\mathfrak{g} \simeq Der(\mathfrak{g})$.*
(iii) *$\mathfrak{g} \cap C_{h(\mathfrak{g})}(\mathfrak{g}) = C(\mathfrak{g})$.*

Proof Let $D \in (Der_{\alpha^s}(\mathfrak{g}))_i$, $i \in \Gamma$ and $D \in C(Der(\mathfrak{g}))$. Then $[D, Der(\mathfrak{g})]_{\mathcal{D}} = 0$. So $[D, ad_s(x)]_{\mathcal{D}} = 0$, hence $[D, ad_s(x)]_{\mathcal{D}}(y) = 0$, for all $x, y \in \mathfrak{g}$. Thus

$$
\begin{aligned}
&D(ad_s(x)(y)) - \varepsilon(D, x)ad_s(x)(D(y)) = 0, \Rightarrow\\
&D([x, \alpha^s(y)]) - \varepsilon(D, x)[x, \alpha^s(D(y))] = 0, \Rightarrow\\
&D([x, \alpha^s(y)]) = \varepsilon(D, x)[x, \alpha^s(D(y))], \Rightarrow\\
&[D(x), \alpha^{2s}(y)] + \varepsilon(D, x)[x, \alpha^s(D(y))] = \varepsilon(D, x)[x, \alpha^s(D(y))], \Rightarrow\\
&[D(x), \alpha^{2s}(y)] = 0 \overset{C(\mathfrak{g})=0}{\Rightarrow} D(x) = 0 \Rightarrow D = 0.
\end{aligned}
$$

Thus $C(Der(\mathfrak{g})) = 0$. Next, $\mathfrak{g} \lhd \mathfrak{g}$, so $\mathfrak{g}$ is a hom-ideal of $h(\mathfrak{g})$ and $h(\mathfrak{g})/\mathfrak{g} \simeq Der(\mathfrak{g})$. Now, let $x \in \mathfrak{g}$. Then

$$x \in C(\mathfrak{g}) \Longleftrightarrow [x, \mathfrak{g}]_{\mathfrak{g}} = 0 \Longleftrightarrow [x, \mathfrak{g}]_h = 0 \Longleftrightarrow x \in \mathfrak{g} \cap C_{h(\mathfrak{g})}(\mathfrak{g}).$$

Hence $\mathfrak{g} \cap C_{h(\mathfrak{g})}(\mathfrak{g}) = C(\mathfrak{g})$. □

Now, by using the notion of holomorph color hom-Lie algebras, we prove some equivalence conditions for a color hom-Lie algebra to be complete.

Definition 6.11 Let $\mathfrak{g}$, $\mathfrak{h}$ be two color hom-Lie algebras. We call $\mathfrak{e}$ an extension of the color hom-Lie algebra $\mathfrak{g}$ by $\mathfrak{h}$, if there exists a short exact sequence

$$0 \to \mathfrak{h} \to \mathfrak{e} \to \mathfrak{g} \to 0$$

of color hom-Lie algebras and their morphisms.

(i) An extension $0 \to \mathfrak{h} \overset{i}{\to} \mathfrak{e} \overset{p}{\to} \mathfrak{g} \to 0$ is called trivial extension if there exists an hom-ideal $I \subset \mathfrak{e}$ such that $\mathfrak{e} = Ker(p) \oplus I$.

(ii) An extension $0 \to \mathfrak{h} \overset{i}{\to} \mathfrak{e} \overset{p}{\to} \mathfrak{g} \to 0$ is called splitting extension if there exists an hom-supersubspace $S \subset \mathfrak{e}$ such that $\mathfrak{e} = Ker(p) \oplus S$.

Theorem 6.2 *For a multiplicative color hom-Lie algebra* $(\mathfrak{g}, [.,.], \varepsilon, \alpha)$ *with surjective* α*, the following conditions are equivalent:*

(i) $\mathfrak{g}$ *is a complete color hom-Lie algebra;*
(ii) *any splitting extension* $\mathfrak{e}$ *by* $\mathfrak{g}$ *is a trivial extension and* $\mathfrak{e} = \mathfrak{g} \oplus C_{\mathfrak{e}}(\mathfrak{g})$*;*
(iii) $h(\mathfrak{g}) = \mathfrak{g} \oplus C_{h(\mathfrak{g})}(\mathfrak{g})$.

Proof Let $\mathfrak{e}$ be a splitting extension by $\mathfrak{g}$ and assume (i) holds. Hence $\mathfrak{g} \lhd \mathfrak{e}$ and $C_{\mathfrak{e}}(\mathfrak{g}) \lhd \mathfrak{e}$. By (i), $C(\mathfrak{g}) = 0$, so $\mathfrak{g} \cap C_{\mathfrak{e}}(\mathfrak{g}) = 0$. Since $\mathfrak{g} \lhd \mathfrak{e}$, $ad_s(e)(\mathfrak{g}) \subset \mathfrak{g}$, for any $e \in \mathfrak{e}$. Then the restriction $ad_s(e)|_{\mathfrak{g}}$ is a derivation of $\mathfrak{g}$. Since $\mathfrak{g}$ is complete, thus $ad_s(e)|_{\mathfrak{g}}$ is a α^{s+1}-derivation of $\mathfrak{g}$. We set $\pi(e) = ad_s(e)|_{\mathfrak{g}}$, for all $e \in \mathfrak{e}$. Since $Der_{\alpha^{s+1}}(\mathfrak{g}) = ad_s(\mathfrak{g}) \simeq \mathfrak{g}$, the map π is a homomorphism from $\mathfrak{e}$ onto $Der_{\alpha^{s+1}}(\mathfrak{g})$ and $Ker(\pi) = C_{\mathfrak{e}}(\mathfrak{g})$. Thus $\mathfrak{e} = \mathfrak{g} \oplus Ker(\pi)$. Therefore $\mathfrak{e} = \mathfrak{g} \oplus C_{\mathfrak{e}}(\mathfrak{g})$. Suppose (ii) holds, then (iii) is obvious by setting $\mathfrak{e} = h(\mathfrak{g})$. Next, suppose (iii) holds. $C(\mathfrak{g}) = \mathfrak{g} \cap C_{h(\mathfrak{g})}(\mathfrak{g})$ by Lemma 6.4. From (iii), $C_{h(\mathfrak{g})}(\mathfrak{g}) \simeq h(\mathfrak{g})/\mathfrak{g}$. By Lemma 6.4 $h(\mathfrak{g})/\mathfrak{g} \simeq Der_{\alpha^{s+1}}(\mathfrak{g}) \simeq C_{h(\mathfrak{g})}(\mathfrak{g}) \simeq \mathfrak{g}$. Since $C(\mathfrak{g}) = 0$, then $\mathfrak{g} \simeq ad_s(\mathfrak{g})$. Thus $Der_{\alpha^{s+1}}(\mathfrak{g}) = ad_s(\mathfrak{g})$. Therefore $\mathfrak{g}$ is a complete color hom-Lie algebra. □

In the next theorem, we check the condition in which the completeness of $\mathfrak{g}$ and its ideals are equivalent.

Theorem 6.3 *Let* $(\mathfrak{g}, [.,.], \varepsilon, \alpha)$ *be a multiplicative color hom-Lie algebra and* $\mathfrak{g} = I \oplus J$, *where* I *and* J *are hom-ideals and* α *is surjective on* $\mathfrak{g}$, I *and* J. *Then*

(i) $C(\mathfrak{g}) = C(I) \oplus C(J)$;
(ii) *if* $C(\mathfrak{g}) = 0$, *then*

$$ad_s(\mathfrak{g}) = ad_s(I) \oplus ad_s(J),$$

$$Der_{\alpha^{s+1}}(\mathfrak{g}) = Der_{\alpha^{s+1}}(I) \oplus Der_{\alpha^{s+1}}(J);$$

(iii) $\mathfrak{g}$ *is complete if and only if* I *and* J *are complete.*

Proof (i) $C(I)$ and $C(J)$ are hom-ideals of $\mathfrak{g}$, by Lemma 6.2. $I \cap J = 0$, so $C(I) \cap C(J) = 0$. Let $a + b \in C(I) \oplus C(J)$, where $a \in C(I)$ and $b \in C(J)$. Thus $[a, I] = 0$ and $[b, J] = 0$. Let $m + n \in I \oplus J = \mathfrak{g}$, where $m \in I$ and $n \in J$. Then

$$[a + b, m + n] = [a + b, m] + [a + b, n] = [a, m] + [b, m] + [a, n] + [b, n] = 0,$$

since $a, m \in I$, $b, n \in J$ and $[b, m], [a, n] \in I \cap J = 0$. Therefore $a + b \in C(\mathfrak{g})$ and $C(I) \oplus C(J) \subseteq C(\mathfrak{g})$. Let $x = m + n \in C(\mathfrak{g})$, where $m \in I$ and $n \in J$. Then $[x, \mathfrak{g}] = [m + n, \mathfrak{g}] = [m + n, I + J] = 0$. Since $x \in C(\mathfrak{g})$ and $[n, I] \subseteq [J, I] = 0$, then

$$[m, I] = [x - n, I] = [x, I] - [n, I] = 0.$$

Hence $m \in C(I)$. By the same way, $n \in C(J)$. Thus $C(\mathfrak{g}) \subseteq C(I) \oplus C(J)$.
(ii) For $D \in Der_{\alpha^{s+1}}(I)$, we define an extended linear transformation on $\mathfrak{g}$ by setting $D(m + n) = D(m)$, for $m \in I$ and $n \in J$. So $D \in Der_{\alpha^{s+1}}(\mathfrak{g})$, $Der_{\alpha^{s+1}}(I) \subseteq Der_{\alpha^{s+1}}(\mathfrak{g})$ and $Der_{\alpha^{s+1}}(J) \subseteq Der_{\alpha^{s+1}}(\mathfrak{g})$. Let $m \in I_i, n \in J$ and $D \in (Der_{\alpha^{s+1}}(\mathfrak{g}))_j$, where $i, j \in \Gamma$. Since I, J are hom-ideals, then

$$[D(m), n] = D([m, n]) = [D(m), \alpha^{s+1}(n)] + \varepsilon(i, j)[\alpha^{s+1}(m), D(n)] \in I \cap J.$$

Since $I \cap J = 0$, then $[D(m), \alpha^{s+1}(n)] = [\alpha^{s+1}(m), D(n)] = 0$. Let $D(m) = m' + n'$, where $m' \in I$ and $n' \in C(J)$. Then

$$[D(m), \alpha^{s+1}(n)] = [m' + n', \alpha^{s+1}(n)] = [m', \alpha^{s+1}(n)] + [n', \alpha^{s+1}(n)] = 0.$$

By (i), $n' = 0$. Hence $D(m) = m' \in I$. Thus $D(I) \subseteq I$. By the same way, $D(J) \subseteq J$. Let $D \in Der_{\alpha^{s+1}}(\mathfrak{g})$ and $m + n \in I + J$, where $m \in I$ and $n \in J$. We define α^{s+1}-derivations E and F by setting

$$E(m + n) = D(m), \quad F(m + n) = D(n).$$

Obviously, $E \in Der_{\alpha^{s+1}}(I)$ and $F \in Der_{\alpha^{s+1}}(J)$. Then $D = E + F \in Der_{\alpha^{s+1}}(I) + Der_{\alpha^{s+1}}(J)$. Therefore $Der_{\alpha^{s+1}}(\mathfrak{g}) = Der_{\alpha^{s+1}}(I) \oplus Der_{\alpha^{s+1}}(J)$ as a linear space, since $Der_{\alpha^{s+1}}(I) \cap Der_{\alpha^{s+1}}(J) = 0$. Now we prove that $Der_{\alpha^{s+1}}(I)$ and $Der_{\alpha^{s+1}}(J)$ are hom-ideals of color hom-Lie algebra $Der_{\alpha^{s+1}}(\mathfrak{g})$. Let $E \in (Der_{\alpha^{s+1}}(I))_i$, $F \in (Der_{\alpha^{s+1}}(\mathfrak{g}))_j$ and $n \in J$. By using the commutator of α^{s+1}-derivations, we have

$$[F, E](n) = (F \circ E)(n) - \varepsilon(i, j)(E \circ F)(n) = 0.$$

Thus $Der_{\alpha^{s+1}}(I)$ is hom-ideal of $Der_{\alpha^{s+1}}(\mathfrak{g})$. Similarly $Der_{\alpha^{s+1}}(J)$ is hom-ideal of $Der_{\alpha^{s+1}}(\mathfrak{g})$.
(iii) Let $\mathfrak{g}$ be complete. Then $C(\mathfrak{g}) = 0$ and $C(I) = C(J) = 0$ by (i). By using $ad_s(\mathfrak{g}) = Der_{\alpha^{s+1}}(\mathfrak{g})$ and statements (i) and (ii), we have

$$ad_s(I) \oplus ad_s(J) = Der_{\alpha^{s+1}}(I) \oplus Der_{\alpha^{s+1}}(J).$$

Since $ad_s(I) \subseteq Der_{\alpha^{s+1}}(I)$ and $ad_s(J) \subseteq Der_{\alpha^{s+1}}(J)$, thus $ad_s(I) = Der_{\alpha^{s+1}}(I)$ and $ad_s(J) = Der_{\alpha^{s+1}}(J)$. Therefore I and J are complete color hom-Lie algebras. Conversely, let I and J are complete, then $C(\mathfrak{g}) = C(I) \oplus C(J) = 0$, by (i); and $Der_{\alpha^{s+1}}(\mathfrak{g}) = Der_{\alpha^{s+1}}(I) \oplus Der_{\alpha^{s+1}}(J) = ad_s(I) \oplus ad_s(J) = ad_s(\mathfrak{g})$, by (ii). □

Definition 6.12 Let $\mathfrak{g}$ be a complete color hom-Lie algebra. If any non-trivial hom-ideal of $\mathfrak{g}$ is not complete, then $\mathfrak{g}$ is called a simply complete color hom-Lie algebra.

A simple and complete color hom-Lie algebra is a simply complete color hom-Lie algebra. In the next theorem, we want to state the relation between simply complete color hom-Lie algebras and indecomposable complete color hom-Lie algebras.

Theorem 6.4 *Let* $(\mathfrak{g}, [., .], \varepsilon, \alpha)$ *be a complete multiplicative color hom-Lie algebra with surjective* α *on* $\mathfrak{g}$.

(i) $\mathfrak{g}$ *can be decomposed into the direct sum of simply complete hom-ideals.*
(ii) $\mathfrak{g}$ *is simply complete if and only if it is indecomposable.*

Proof (i) If $\mathfrak{g}$ is simply complete, then (i) holds. If $\mathfrak{g}$ is not simply complete, then by Proposition 6.2, there exists a nonzero minimal complete hom-ideal I of $\mathfrak{g}$ such that $\mathfrak{g} = I \oplus C_{\mathfrak{g}}(I)$. Since a hom-ideal of $C_{\mathfrak{g}}(I)$ is also a hom-ideal of $\mathfrak{g}$, by continuing this method for $C_{\mathfrak{g}}(I)$, we reach to the decomposition of $\mathfrak{g}$ into the simply complete hom-ideals.
(ii) If $\mathfrak{g}$ is simply complete, then it is indecomposable by (i). Conversely, if $\mathfrak{g}$ is indecomposable, then it has not any non-trivial hom-ideal. Hence $\mathfrak{g}$ is simply complete by Definition 6.12. □

Definition 6.13 A subspace I of a color hom-Lie algebra $\mathfrak{g}$ is called a characteristic hom-ideal of $\mathfrak{g}$, if $D(I) \subset I$ for all $D \subseteq Der(\mathfrak{g})$.

Lemma 6.5 *Let* $(\mathfrak{g}, [.,.], \varepsilon, \alpha)$ *be a multiplicative color hom-Lie algebra,* I *be a characteristic hom-ideal of* $\mathfrak{g}$ *and* α *is surjective on* $\mathfrak{g}$ *and* I*. Then* I *is hom-ideal of* $\mathfrak{g}$*.*

Proof Let $x, y \in I$, since α is surjective on I and $ad_s(\mathfrak{g})$ is a α^{s+1}-derivation, then

$$[x, y] \overset{\alpha(I)=I}{=} [x, \alpha^s(t)] = ad_s(x)(t) \in I,$$

where $t \in I$ and $\alpha^s(t) = y$. Thus $[I, I] \subseteq I$.
Next, $\alpha(I) \subseteq I$, since α is surjective on I. Let $y \in I$ and $a \in \mathfrak{g}$. Then

$$[a, y] \overset{\alpha(I)=I}{=} [a, \alpha^s(t)] = ad_s(a)(t) \in I,$$

where $t \in I$ and $\alpha^s(t) = y$. Thus $[\mathfrak{g}, I] \subseteq I$. Therefore I is a hom-ideal of $\mathfrak{g}$. □

Theorem 6.5 *Let* $(\mathfrak{g}, [.,.], \varepsilon, \alpha)$ *be a multiplicative color hom-Lie algebra with surjective* α*,* $C(\mathfrak{g}) = 0$ *and* $ad_s(\mathfrak{g})$ *be a characteristic hom-ideal of* $Der(\mathfrak{g})$*. Then* $Der(\mathfrak{g})$ *is complete. Furthermore, if* $\mathfrak{g}$ *is indecomposable and* $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$*, then* $Der(\mathfrak{g})$ *is simply complete.*

Proof $\mathfrak{g}$ has trivial center, so $\mathfrak{g} \simeq ad_s(\mathfrak{g})$. Let $\mathfrak{p} = Der(\mathfrak{g})$, then $\mathfrak{g} \lhd \mathfrak{p}$. Let $\mathfrak{q}$ be a splitting extension by $\mathfrak{p}$, i.e. $\mathfrak{p} \lhd \mathfrak{q}$. Hence for all $q \in \mathfrak{q}$, we have $ad_s(q) \in Der_{\alpha^{s+1}}(\mathfrak{p})$. $\mathfrak{g}$ is a characteristic hom-ideal of $\mathfrak{p}$, so there exists $p \in \mathfrak{p}$ such that $ad_s(\mathfrak{p})|_{\mathfrak{g}} = ad_s(\mathfrak{q})|_{\mathfrak{g}}$. Then $ad_s(p - q)|_{\mathfrak{g}} = 0$ and $p - q \in C_{\mathfrak{q}}(\mathfrak{g})$. Hence we have $\mathfrak{q} = \mathfrak{p} + C_{\mathfrak{q}}(\mathfrak{g})$. On the other hand, $\mathfrak{p} \cap C_{\mathfrak{q}}(\mathfrak{g}) = C_{\mathfrak{p}}(\mathfrak{g}) = 0$ and $\mathfrak{p} \lhd \mathfrak{q}$, thus $\mathfrak{q} = \mathfrak{p} \oplus C_{\mathfrak{q}}(\mathfrak{g})$. Hence $C_{\mathfrak{q}}(\mathfrak{g}) \subseteq C_{\mathfrak{q}}(\mathfrak{p})$ and we have $\mathfrak{q} = \mathfrak{p} \oplus C_{\mathfrak{q}}(\mathfrak{p})$. Therefore by Theorem 6.2, $\mathfrak{p} = Der(\mathfrak{g})$ is a complete color hom-Lie algebra. Now, assume that $Der(\mathfrak{g})$ is not simply complete. So there exists a simply complete hom-ideal I. By Proposition 6.2, there exists a hom-ideal J such that $\mathfrak{p} = I \oplus J$. For any $x, y \in \mathfrak{g}$, there exists $x_1, y_1 \in I$ and $x_2, y_2 \in J$ such that $x = x_1 + x_2$ and $y = y_1 + y_2$. Thus $[x, y] = [x_1 + x_2, y] = [x_1, y] + [x_2, y]$ such that $[x_1, y] \in I \cap \mathfrak{g}$ and $[x_2, y] \in J \cap \mathfrak{g}$. Hence $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] = (I \cap \mathfrak{g}) \oplus (J \cap \mathfrak{g})$. $\mathfrak{g}$ is indecomposable, then $I \cap \mathfrak{g} = 0$ or $J \cap \mathfrak{g} = 0$. Hence $\mathfrak{g} \subseteq J$ and $I \subseteq C_{\mathfrak{p}}(\mathfrak{g}) = 0$. Therefore by Theorem 6.4, $\mathfrak{p} = Der(\mathfrak{g})$ is simply complete. □

Definition 6.14 Let ψ be an endomorphism of a multiplicative color hom-Lie algebra $(\mathfrak{g}, [.,.], \varepsilon, \alpha)$. If for all $x \in \mathfrak{g}$ and any nonnegative integer t, $\psi ad_s(x) = ad_s(x)\psi$, then ψ is called an $\mathfrak{g}$-endomorphism of $\mathfrak{g}$.

With Definition 6.14, we get the following proposition.

Proposition 6.3 *Let* $(\mathfrak{g}, [.,.], \varepsilon, \alpha)$ *be a multiplicative color hom-Lie algebra and* I, J *be hom-ideals such that* $\mathfrak{g} = I \oplus J$*. If* π *is the projection into* I *with respect to this decomposition, then* π *is a* $\mathfrak{g}$*-endomorphism of* $\mathfrak{g}$*.*

Proof Let $g_1 = a_1 + b_1$ and $g_2 = a_2 + b_2$ such that, $a_1, a_2 \in I$ and $b_1, b_2 \in J$, then

$$\begin{aligned}\pi ad_s(l_1)(l_2) &= \pi[g_1, \alpha^s(g_2)] = \pi[a_1 + b_1, \alpha^s(a_2 + b_2)]\\ &= \pi[a_1 + b_1, \alpha^s(a_2) + \alpha^s(b_2)]\\ &= [\pi(a_1 + b_2), \pi(\alpha^s(a_2) + \alpha^s(b_2))]\\ &= [a_1, \alpha^s(a_2)] = ad_s(g_1)\pi(g_2).\end{aligned}$$

Therefore $\pi ad_s(g) = ad_s(g)\pi$, for all $g \in \mathfrak{g}$. □

Remark 6.2 Let ψ be an $\mathfrak{g}$-endomorphism of a finite-dimensional multiplicative color hom-Lie algebra $\mathfrak{g}$. Then there exists $k \in \mathbb{N}$, such that $\mathfrak{g} = Ker\psi^k \oplus Im\psi^k$.

Definition 6.15 A color hom-Lie algebra $(\mathfrak{g}, [.,.], \varepsilon, \alpha)$ is called indecomposable if it can not be written as direct sum of two nonzero hom-ideals.

Proposition 6.4 *Let $\mathfrak{g}$ be a finite-dimensional indecomposable multiplicative color hom-Lie algebra and $\psi_1, \ldots, \psi_n$ and $\sum_{i=1}^{j} \psi_i$ $(j = 1, \ldots, n)$ be $\mathfrak{g}$-endomorphisms of $\mathfrak{g}$ such that $\psi_1 + \cdots + \psi_n = id$. Then there exists an index k, such that $\psi_k \in Aut(\mathfrak{g})$.*

Proof The result is obtained by induction on n and using Remark 6.2. □

Theorem 6.6 *Let $(\mathfrak{g}, [.,.], \varepsilon, \alpha)$ be a finite-dimensional multiplicative color hom-Lie algebra with trivial center and assume that $\mathfrak{g}$ has decomposition of direct sum of hom-ideals, such that $\mathfrak{g} = I_1 \oplus I_2 \oplus \cdots \oplus I_m$ and $\mathfrak{L} = J_1 \oplus J_2 \oplus \cdots \oplus J_n$, where $I_1, I_2, \ldots, I_m$ and $J_1, J_2, \ldots, J_n$ are indecomposable. Then $m = n$ and if necessary, by permutation $I_i = J_i$ for $i = 1, \ldots, m$.*

Proof We prove by induction on n. If $n = 1$, then $\mathfrak{g}$ is indecomposable, so $m = n = 1$ and $I_1 = J_1 = \mathfrak{g}$. Suppose $n > 1$ and $m > 1$. Denote the projection of $\mathfrak{g}$ onto I_1 by π, the embedding of I_1 into $\mathfrak{g}$ by σ, the projection of $\mathfrak{g}$ onto J_i by ρ_i and the embedding of J_i into $\mathfrak{g}$ by τ_i. We know that $ad_s(a)(b) = [a, \alpha^s(b)]$. By using the Definition 6.14, $\pi, \rho_1, \rho_2, \ldots, \rho_n$ and $\sum_{i=1}^{j} \rho_i$ $(j = 1, 2, \ldots, n)$ are $\mathfrak{g}$-endomorphisms of color hom-Lie algebra $\mathfrak{g}$ and $\rho_1 + \rho_2 + \cdots + \rho_n = id_{\mathfrak{g}}$. Let $\bar{\pi}_i = \pi\tau_i = \pi|_{B_i}$ and $\bar{\rho}_i = \rho_i\sigma = \rho_i|_{I_1}$ for any $i = 1, 2, \ldots, n$. Then $\bar{\pi}_i\bar{\rho}_i$ and $\sum_{i=1}^{j} \bar{\pi}_i\bar{\rho}_i$ $(j = 1, 2, \ldots, n)$ are I_1-endomorphisms of I_1. For any $a \in I_1$, we have $\sum_{i=1}^{n} \bar{\pi}_i\bar{\rho}_i(a) = a$, so $\sum_{i=1}^{n} \bar{\pi}_i\bar{\rho}_i = id|_{I_1}$. Therefore, by Remark 6.2, there exists an index i such that $\bar{\pi}_i\bar{\rho}_i \in Aut(I_1)$. By permutation we can assume that $i = 1$. Hence $\bar{\rho}_1$ is injective. Let $I = I_2 \oplus I_3 \oplus \cdots \oplus I_m$ and $J = J_2 \oplus J_3 \oplus \cdots \oplus J_n$. Then $I_1 \cap J = 0$, since $\bar{\rho}_1$ is injective. So $I_1 \subseteq C_{\mathfrak{g}}(J)$. Therefore $I_1 = J_1$, since J_1 is indecomposable. Hence $I = J$. The result follows by inductive assumption. □

Corollary 6.1 *Let* $(\mathfrak{g}, [.,.], \varepsilon, \alpha)$ *be a multiplicative complete color hom-Lie algebra and* $\mathfrak{g} = I_1 \oplus \cdots \oplus I_m$*, where each* I_i *is a simple hom-ideal. Then this decomposition is unique up to the order of the hom-ideals.*

Proof Since $\mathfrak{g}$ is complete, $C(\mathfrak{g}) = 0$. Then the proof is straightforward from Theorem 6.6. □

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Chapter 7
Hom-Prealternative Superalgebras

Ibrahima Bakayoko and Sergei Silvestrov

Abstract The purpose of this paper is to introduce Hom-prealternative superalgebras and their bimodules. Some constructions of Hom-prealternative superalgebras and Hom-alternative superalgebras are given, and their connection with Hom-alternative superalgebras are studied. Bimodules over Hom-prealternative superalgebras are introduced, relations between bimodules over Hom-prealternative superalgebras and the bimodules of the corresponding Hom-alternative superalgebras are considered, and construction of bimodules over Hom-prealternative superalgebras by twisting is described.

Keywords Hom-prealternative superalgebra · Hom-alternative algebra · Bimodule

MSC 2020 Classification 17D30 · 17B61 · 17B60 · 17B62

7.1 Introduction

Hom-Lie algebras and more general quasi-Hom-Lie algebras were introduced first by Hartwig, Larsson and Silvestrov in [53] where a general approach to discretization of Lie algebras of vector fields using general twisted derivations (σ-derivations) and a general method for construction of deformations of Witt and Virasoro type algebras based on twisted derivations have been developed. The general quasi-Lie algebras, containing the quasi-Hom-Lie algebras and Hom-Lie algebras as subclasses, as well their graded color generalization, the color quasi-Lie algebras including color quasi-hom-Lie algebras, color hom-Lie algebras and their

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special subclasses the quasi-Hom-Lie superalgebras and hom-Lie superalgebras, have been first introduced in [53, 67–70, 98]. Subsequently, various classes of Hom-Lie admissible algebras have been considered in [78]. In particular, in [78], the Hom-associative algebras have been introduced and shown to be Hom-Lie admissible, that is leading to Hom-Lie algebras using commutator map as new product, and in this sense constituting a natural generalization of associative algebras as Lie admissible algebras leading to Lie algebras using commutator map. Furthermore, in [78], more general G-Hom-associative algebras including Hom-associative algebras, Hom-Vinberg algebras (Hom-left symmetric algebras), Hom-pre-Lie algebras (Hom-right symmetric algebras), and some other Hom-algebra structures, generalizing G-associative algebras, Vinberg and pre-Lie algebras respectively, have been introduced and shown to be Hom-Lie admissible, meaning that for these classes of Hom-algebras, the operation of taking commutator leads to Hom-Lie algebras as well. Also, flexible Hom-algebras have been introduced, connections to Hom-algebra generalizations of derivations and of adjoint maps have been noticed, and some low-dimensional Hom-Lie algebras have been described. Since the pioneering works [53, 67–70, 78], Hom-algebra structures have developed in a popular broad area with increasing number of publications in various directions. Hom-algebra structures are very useful since Hom-algebra structures of a given type include their classical counterparts and open broad possibilities for deformations, Hom-algebra extensions of homology and cohomology structures and representations, formal deformations of Hom-associative and Hom-Lie algebras, Hom-Lie admissible Hom-coalgebras, Hom-coalgebras, Hom-bialgebras and Hom-Hopf algebras, [6, 33, 45, 67, 72, 79–81, 94, 104, 106]. Hom-Lie algebras, Hom-Lie superalgebras, color Hom-Lie algebras, Hom-associative color algebras, Enveloping algebras of color Hom-Lie algebras, color Hom-Leibniz algebras, omni-Hom-Lie algebras, color omni-Hom-Lie algebras, biHom-Lie algebras, biHomassociative algebras, biHom-Frobenius algebras, Hom-Ore extensions Hom-algebras, Hom-alternative algebras, Hom-center-symmetric algebras, Hom-left-symmetric color dialgebras, Hom-dendriform algebras, Rota–Baxter Hom-algebras, Hom-tridendriform color algebras, Hom-Malcev algebras, Hom-Jordan algebras, Hom-Poisson algebras, Color Hom-Poisson algebras, Hom-Akivis algebras, Hom-Lie-Yamaguti algebras, nearly Hom-associative algebras, Hom-Gerstenhaber algebras and Hom-Lie algebroids, n-Lie algebras and Hom-Nambu-Lie algebras and other n-ary Hom-algebra structures have been further investigated in various aspects for example in [1–30, 32–37, 37–44, 46–52, 54–59, 61–67, 71, 73–75, 77–79, 79–84, 86–101, 103–113]. In particular, Color Hom-Poisson algebras [24] and modules over some color Hom-algebras [27], under the name of generalized Hom-algebras, have been considered. When the grading abelian group is $\mathbb{Z}_2$, the corresponding $\mathbb{Z}_2$-graded Hom-algebras are called Hom-superalgebras. Hom-Lie superalgebra structures such as Hom-Lie superalgebras and Hom-Lie admissible superalgebras [46], Rota-Baxter operator on pre-Lie superalgebras [2], Hom-Novikov superalgebras [102] have been considered in more details. Hom-alternative superalgebras have been considered in [1] as a $\mathbb{Z}_2$-graded version of Hom-alternative algebras [76] and their relationships with Hom-Malcev superalgebras and Hom-Jordan superalgebras are established [1].

The aim of this paper is to study the $\mathbb{Z}_2$-graded version of Hom-prealternative algebras and their bimodules. In Sect. 7.2, we recall some basic notions on Hom-alternative superalgebras and their bimodules. We prove that bimodules over Hom-alternative superalgebras are closed under twisting and direct product. We show that the tensor product of super-commutative Hom-associative superalgebras and Hom-alternative superalgebras is also a Hom-alternative superalgebra. Then we recall the definition of Hom-Jordan superalgebra. Section 7.3 is devoted to Hom-prealternative superalgebras and Hom-alternative superalgebras and their connections. We point out that to any Hom-prealternative superalgebra one may associate a Hom-alternative superalgebra, and conversely to any Hom-alternative superalgebra it corresponds a Hom-prealternative superalgebra via an $\mathcal{O}$-operator. Construction of Hom-prealternative superalgebras by composition is given. Bimodules over Hom-prealternative superalgebras are introduced, relations between bimodules over Hom-prealternative superalgebras and bimodules of the corresponding Hom-alternative superalgebras are considered, and a construction of bimodules over Hom-prealternative superalgebras by twisting is described.

7.2 Hom-Prealternative Algebras and Bimodules

In this section, we present important basic notions and provide some construction results for Hom-alternative superalgebras.

Firstly, let us recall necessary important basic notions and notations on graded spaces and algebras. Throughout this paper, all linear spaces are assumed to be over a field $\mathbb{K}$ of characteristic different from 2.

Definition 7.1 Let G be an abelian group. A linear space V is called G-graded if $V = \bigoplus_{a\in G} V_a$ for some family $(V_a)_{a\in G}$ of linear subspaces of V.

(i) An element $x \in V$ is said to be homogeneous of degree $a \in G$ if $x \in V_a$, and $\mathcal{H}(V) = \bigcup_{a\in G} V_a$ denotes the set of all homogeneous elements in V.

(ii) Let $V = \bigoplus_{a\in G} V_a$ and $V' = \bigoplus_{a\in G} V'_a$ be two G-graded linear spaces. A linear mapping $f : V \to V'$ is said to be homogeneous of degree b if $f(V_a) \subseteq V'_{a+b}$ for $a \in G$. If f is homogeneous of degree zero i.e. $f(V_a) \subseteq V'_a$ holds for any $a \in G$, then f is said to be even.

(iii) An algebra $(A, \cdot)$ is said to be G-graded if its underlying linear space is G-graded i.e. $A = \bigoplus_{a\in G} A_a$, and if furthermore $A_a \cdot A_b \subseteq A_{a+b}$ for $a, b \in G$.

(iv) A morphism $f : A \to A'$ of G-graded algebras A and A' is by definition an algebra morphism from A to A', which is moreover an even mapping.

Let A be a $\mathbb{Z}_2$-graded linear space with direct sum $A = A_0 \oplus A_1$. The elements of A_j, are said to be homogeneous of degree (parity) $j \in \mathbb{Z}_2$. The set of all homogeneous elements of A is $\mathcal{H}(A) = A_0 \cup A_1$. Usually $|x|$ denotes parity of a homogeneous element $x \in \mathcal{H}(A)$.

Definition 7.2 Hom-superalgebras are triples (A, μ, α) in which $A = A_0 \oplus A_1$ is a $\mathbb{Z}_2$-graded linear space ($\mathbb{K}$-superspace), $\mu : A \times A \to A$ is an even bilinear map, and $\alpha : A \to A$ is an even linear map.

(i) Let (A, μ, α) be a Hom-superalgebra. Hom-associator of A is the even trilinear map $as_{\alpha,\mu} : A \times A \times A \to A$ given by $as_{\alpha,\mu} = \mu \circ (\mu \otimes \alpha - \alpha \otimes \mu)$. In terms of elements, in usual juxtaposition notation $xy = \mu(x, y)$, the map $as_{\alpha,\mu}$ is given by

$$as_{\alpha,\mu}(x, y, z) = \mu(\mu(x, y), \alpha(z)) - \mu(\alpha(x), \mu(y, z)) = (xy)\alpha(z) - \alpha(x)(yz).$$

(ii) An even linear map $f : (A, \mu, \alpha) \to (A', \mu', \alpha')$ is said to be a weak morphism of Hom-superalgebras if $f \circ \mu = \mu \circ (f \otimes f)$, and a morphism of Hom-superalgebras if moreover $f \circ \alpha = \alpha' \circ f$.

(iii) Hom-superalgebra (A, μ, α) in which $\alpha : A \to A$ is moreover an endomorphism of the algebra structure μ is said to be multiplicative, and the algebra endomorphism condition

$$\alpha \circ \mu = \mu \circ (\alpha \otimes \alpha) \tag{7.1}$$

is called the multiplicativity of α with respect to μ.

Since the grading degree of Hom-associator $|as_{\alpha,\mu}(x, y, z))| = |x| + |y| + |z|$ for $x, y, z \in \mathcal{H}(A) = A_0 \cup A_1$ in any Hom-superalgebra $(A = A_0 \oplus A_1, \mu, \alpha)$,

$$as_{\alpha,\mu}(A_0, A_0, A_0) \subseteq A_0, \tag{7.2}$$
$$as_{\alpha,\mu}(A_1, A_0, A_0) \subseteq A_1, \tag{7.3}$$
$$as_{\alpha,\mu}(A_0, A_1, A_0) \subseteq A_1, \tag{7.4}$$
$$as_{\alpha,\mu}(A_0, A_0, A_1) \subseteq A_1, \tag{7.5}$$
$$as_{\alpha,\mu}(A_1, A_1, A_0) \subseteq A_0, \tag{7.6}$$
$$as_{\alpha,\mu}(A_1, A_0, A_1) \subseteq A_0, \tag{7.7}$$
$$as_{\alpha,\mu}(A_0, A_1, A_1) \subseteq A_0, \tag{7.8}$$
$$as_{\alpha,\mu}(A_1, A_1, A_1) \subseteq A_1. \tag{7.9}$$

Definition 7.3 Hom-associative superalgebras are those Hom-superalgebras $(A = A_0 \oplus A_1, \bullet, \alpha)$ obeying super ($\mathbb{Z}_2$-graded) Hom-associativity super identity for $x, y, z \in \mathcal{H}(A) = A_0 \cup A_1$:

$$as_{\alpha,\bullet}(x, y, z) = 0 \qquad \text{(super Hom-associativity)} \tag{7.10}$$

equivalent in juxtaposition notation $x \bullet y = \bullet(x, y)$ to

$$(x \bullet y) \bullet \alpha(z) = \alpha(x) \bullet (y \bullet z).$$

Hom-associativity super identity for Hom-superalgebras is equivalent to

$$as_{\alpha,\bullet}(A_i, A_j, A_k) = \{0_A\}, \quad i, j, k \in \mathbb{Z}_2. \tag{7.11}$$

Definition 7.4 Left Hom-alternative superalgebras are Hom-superalgebras $(A = A_0 \oplus A_1, \bullet, \alpha)$ obeying the left Hom-alternative super identity for $x, y, z \in \mathcal{H}(A) = A_0 \cup A_1$:

$$as_{\alpha,\bullet}(x, y, z) + (-1)^{|x||y|} as_{\alpha,\bullet}(y, x, z) = 0 \tag{7.12}$$

equivalent in juxtaposition notation $x \bullet y = \bullet(x, y)$ to

$$(x \bullet y) \bullet \alpha(z) - \alpha(x) \bullet (y \bullet z) = -(-1)^{|x||y|}((y \bullet x) \bullet \alpha(z) - \alpha(y) \bullet (x \bullet z)).$$

For $(x, y, z) \in A_{|x|} \times A_{|y|} \times A_{|z|}$, $|x|, |y|, |z| \in \mathbb{Z}_2$, the left super Hom-alternativity for $|x||y| = 0$ or $|x||y| = 1$ respectively is

$$|x||y| = 0 : (x, y, z) \in ((A_0 \times A_0) \cup (A_1 \times A_0) \cup (A_0 \times A_1)) \times A_k, \quad k \in \mathbb{Z}_2 :$$
$$(x \bullet y) \bullet \alpha(z) - \alpha(x) \bullet (y \bullet z) = -((y \bullet x) \bullet \alpha(z) - \alpha(y) \bullet (x \bullet z)), \tag{7.13}$$
$$|x||y| = 1 : \quad (x, y, z) \in A_1 \times A_1 \times A_k, \quad k \in \mathbb{Z}_2 :$$
$$(x \bullet y) \bullet \alpha(z) - \alpha(x) \bullet (y \bullet z) = (y \bullet x) \bullet \alpha(z) - \alpha(y) \bullet (x \bullet z). \tag{7.14}$$

Definition 7.5 Right Hom-alternative superalgebra is a Hom-superalgebra $(A = A_0 \oplus A_1, \bullet, \alpha)$ obeying the right Hom-alternative super identity for $x, y, z \in \mathcal{H}(A) = A_0 \cup A_1$:

$$as_{\alpha,\bullet}(x, y, z) + (-1)^{|y||z|} as_{\alpha,\bullet}(x, z, y) = 0, \tag{7.15}$$

which, in juxtaposition notation $x \bullet y = \bullet(x, y)$, is

$$(x \bullet y) \bullet \alpha(z) - \alpha(x) \bullet (y \bullet z) = -(-1)^{|y||z|}((x \bullet z) \bullet \alpha(y) - \alpha(x) \bullet (z \bullet y)).$$

For $(x, y, z) \in A_{|x|} \times A_{|y|} \times A_{|z|}$, $|x|, |y|, |z| \in \mathbb{Z}_2$, the left super Hom-alternativity for $|y||z| = 0$ or $|y||z| = 1$ respectively is

$$|y||z| = 0 : (x, y, z) \in A_k \times ((A_0 \times A_0) \cup (A_1 \times A_0) \cup (A_0 \times A_1)), \quad k \in \mathbb{Z}_2,$$
$$(x \bullet y) \bullet \alpha(z) - \alpha(x) \bullet (y \bullet z) = -((x \bullet z) \bullet \alpha(y) - \alpha(x) \bullet (z \bullet y)), \tag{7.16}$$
$$|y||z| = 1 : (x, y, z) \in A_k \times A_1 \times A_1, \quad k \in \mathbb{Z}_2,$$
$$(x \bullet y) \bullet \alpha(z) - \alpha(x) \bullet (y \bullet z) = ((x \bullet z) \bullet \alpha(y) - \alpha(x) \bullet (z \bullet y)). \tag{7.17}$$

Definition 7.6 Hom-alternative superalgebras are defined as both left and right Hom-alternative superalgebras.

Definition 7.7 Hom-flexible superalgebra is a Hom-superalgebra (A, μ, α) obeying the Hom-flexible super-identity for $x, y, z \in \mathcal{H}(A) = A_0 \cup A_1$:

$$as_{\alpha,\bullet}(x, y, z) + (-1)^{|x||y|+|x||z|+|y||z|} as_{\alpha,\bullet}(z, y, x) = 0, \tag{7.18}$$

which, in juxtaposition notation $x \bullet y = \bullet(x, y)$, is

$$(x \bullet y) \bullet \alpha(z) - \alpha(x) \bullet (y \bullet z) = -(-1)^{|x||y|+|x||z|+|y||z|}((z \bullet y) \bullet \alpha(x) - \alpha(z) \bullet (y \bullet x)).$$

For $(x, y, z) \in A_{|x|} \times A_{|y|} \times A_{|z|}$, $|x|, |y|, |z| \in \mathbb{Z}_2$, the left super Hom-alternativity for $|x||y| + |x||z| + |y||z| = 0$ or 1 respectively is

$$\begin{aligned}
&|x||y| + |x||z| + |y||z| = 0 : (x, y, z) \in (A_1 \times A_0 \times A_0) \cup (A_0 \times A_1 \times A_0) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cup (A_0 \times A_0 \times A_1) \cup (A_0 \times A_0 \times A_0), \\
&(x \bullet y) \bullet \alpha(z) - \alpha(x) \bullet (y \bullet z) = -((z \bullet y) \bullet \alpha(x) - \alpha(z) \bullet (y \bullet x)), \qquad (7.19) \\
&|x||y| + |x||z| + |y||z| = 1 : (x, y, z) \in (A_1 \times A_1 \times A_0) \cup (A_1 \times A_0 \times A_1) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cup (A_0 \times A_1 \times A_1) \cup (A_1 \times A_1 \times A_1), \\
&(x \bullet y) \bullet \alpha(z) - \alpha(x) \bullet (y \bullet z) = (z \bullet y) \bullet \alpha(x) - \alpha(z) \bullet (y \bullet x). \qquad (7.20)
\end{aligned}$$

Definition 7.8 A bimodule over a Hom-alternative superalgebra $(A, \bullet, \alpha)$ consists of a $\mathbb{Z}_2$-graded linear space V with an even linear map $\beta : V \to V$ and two even bilinear maps

$$\begin{aligned}
\succ : \quad & A \otimes V \to V & \qquad \prec : \quad & V \otimes A \to V \\
& x \otimes v \mapsto x \succ v & & v \otimes x \mapsto v \prec x
\end{aligned}$$

such that, for any homogeneous elements $x, y \in A$ and $v \in V$,

$$\begin{aligned}
&(v \prec x) \prec \alpha(y) + (-1)^{|x||v|}(x \succ v) \prec \alpha(y) - \\
&\quad (-1)^{|x||v|}\alpha(x) \succ (v \prec y) - \beta(v) \prec (x \bullet y) = 0,
\end{aligned} \tag{7.21}$$

$$\begin{aligned}
&\alpha(y) \succ (v \prec x) - (y \succ v) \prec \alpha(x) - \\
&\quad (-1)^{|x||v|}(y \bullet x) \succ \beta(v) + (-1)^{|x||v|}\alpha(y) \succ (x \succ v) = 0,
\end{aligned} \tag{7.22}$$

$$\begin{aligned}
&(x \bullet y) \succ \beta(v) + (-1)^{|x||y|}(y \bullet x) \succ \beta(v) - \\
&\quad \alpha(x) \succ (y \succ v) - (-1)^{|x||y|}\alpha(y) \succ (x \succ v) = 0,
\end{aligned} \tag{7.23}$$

$$\begin{aligned}
&\beta(v) \prec (x \bullet y) + (-1)^{|x||y|}\beta(v) \prec (y \bullet x) - \\
&\quad (v \prec x) \prec \alpha(y) - (-1)^{|x||y|}(v \prec y) \prec \alpha(x) = 0.
\end{aligned} \tag{7.24}$$

Remark 7.1 The notation $x \succ v$ means the left action of x on v and $v \prec x$ means the right action of x on v given by the linear operators on V defined by

$$L_{\succ}(x)v = x \succ v, \qquad R_{\prec}(x)v = v \prec x.$$

Bimodules over Hom-alternative superalgebras are closed under twisting in the sense of Theorem 7.1.

Theorem 7.1 *Let $(V, L_{\succ}, R_{\prec}, \beta)$ be a bimodule over the multiplicative Hom-alternative superalgebra $(A, \bullet, \alpha)$. Then, $(V, L^{\alpha}_{\succ}, R^{\alpha}_{\prec}, \beta)$ is a bimodule over A, where $L^{\alpha}_{\succ} = L_{\succ} \circ (\alpha^2 \otimes Id)$ and $R^{\alpha}_{\prec} = R_{\prec} \circ (\alpha^2 \otimes Id)$.*

Proof We only prove (7.21), as (7.22), (7.23), (7.24) are proved similarly. With

$$x \succeq v = L^{\alpha}_{\succ}(x)v = L_{\succ} \circ (\alpha^2 \otimes Id)(x \otimes v) = \alpha^2(x) \succ v,$$
$$v \preceq x = R^{\alpha}_{\prec}(x)v = R_{\prec} \circ (id \otimes \alpha^2)(v \otimes x) = v \prec \alpha^2(x),$$

for any $x, y \in A$ and any $v \in V$,

$$\begin{aligned}
&(v \preceq x) \preceq \alpha(y) + (-1)^{|x||v|}(x \succeq v) \preceq \alpha(y) \\
&\quad - (-1)^{|x||v|}\alpha(x) \succeq (v \preceq y) - \beta(v) \preceq (x \bullet y) = \\
&(v \prec \alpha^2(x)) \prec \alpha^3(y) + (-1)^{|x||v|}(\alpha^2(x) \succ v) \prec \alpha^3(y) \\
&\quad - (-1)^{|x||v|}\alpha^3(x) \succ (v \prec \alpha^2(y)) - \beta(v) \prec \alpha^2(x \bullet y) = \\
&(v \prec \alpha^2(x)) \prec \alpha^3(y) + (-1)^{|x||v|}(\alpha^2(x) \succ v) \prec \alpha^3(y) \\
&\quad - (-1)^{|x||v|}\alpha^3(x) \succ (v \prec \alpha^2(y)) - \beta(v) \prec (\alpha^2(x) \bullet \alpha^2(y)) = 0,
\end{aligned}$$

by using the multiplicativity of α in the last term, and then (7.21) for $\alpha^2(x)$ and $\alpha^2(y)$ in $(V, L_{\succ}, R_{\prec}, \beta)$.

For two $\mathbb{Z}_2$-graded linear spaces $V = \oplus_{a\in\mathbb{Z}_2} V_a$ and $V' = \oplus_{a\in\mathbb{Z}_2} V'_a$, the tensor product $V \otimes V'$ is also a $\mathbb{Z}_2$-graded linear space such that for $j \in \mathbb{Z}_2$,

$$(V \otimes V')_j = \sum_{j=a+a'} V_a \otimes V_{a'}.$$

Theorem 7.2 *Let $(A, \bullet, \alpha)$ be a super-commutative Hom-associative superalgebra and $(A', \bullet', \alpha')$ be a Hom-alternative superalgebra. Then the tensor product $(A \otimes A', *, \alpha \otimes \alpha')$, where for $x, y \in \mathcal{H}(A)$, $a, b \in \mathcal{H}(A')$,*

$$(\alpha \otimes \alpha')(x \otimes a) = \alpha(x) \otimes \alpha'(a),$$
$$(x \otimes a) * (y \otimes b) = (-1)^{|a||y|}(x \bullet y) \otimes (a \bullet' b),$$

is a Hom-alternative superalgebra.

Proof Let us set $X = x \otimes a,\ Y = y \otimes b,\ Z = z \otimes c\ \in \mathcal{H}(A) \otimes \mathcal{H}(A')$. Then,

$$\begin{aligned}
&as_{\alpha\otimes\alpha',*}(X, Y, Z) = as_{\alpha\otimes\alpha',*}(x \otimes a, y \otimes b, z \otimes c)\\
&= ((x \otimes a) * (y \otimes b)) * (\alpha \otimes \alpha')(z \otimes c) - (\alpha \otimes \alpha')(x \otimes a) * ((y \otimes b) * (z \otimes c))\\
&= \Big((x \otimes a) * (y \otimes b)) * (\alpha(z) \otimes \alpha'(c)) - (\alpha(x) \otimes \alpha'(a)) * ((y \otimes b) * (z \otimes c)\Big)\\
&= (-1)^{|a||y|}\Big((x \bullet y) \otimes (a \bullet' b)) * (\alpha(z) \otimes \alpha'(c))\\
&\qquad - (-1)^{|b||z|}(\alpha(x) \otimes \alpha'(a)) * ((y \bullet z) \otimes (b \bullet' c)\Big)\\
&= (-1)^{|a||y|+(|a|+|b|)|z|}((x \bullet y) \bullet \alpha(z)) \otimes ((a \bullet' b) \bullet' \alpha'(c))\\
&\qquad - (-1)^{|b||z|+|a|(|y|+|z|)}(\alpha(x) \bullet (y \bullet z)) \otimes (\alpha'(a) \bullet' (b \bullet' c)).\\
&as_{\alpha\otimes\alpha',*}(X, Y, Z) + (-1)^{|XY|}as_{\alpha\otimes\alpha',*}(Y, X, Z)\\
&= as_{\alpha\otimes\alpha',*}(x \otimes a, y \otimes b, z \otimes c) + (-1)^{|(x\otimes a)*(y\otimes b)|}as_{\alpha\otimes\alpha',*}(x \otimes a, y \otimes b, z \otimes c)\\
&= (-1)^{|a||y|+(|a|+|b|)|z|}((x \bullet y) \bullet \alpha(z)) \otimes ((a \bullet' b) \bullet' \alpha'(c))\\
&\quad - (-1)^{|b||z|+|a|(|y|+|z|)}(\alpha(x) \bullet (y \bullet z)) \otimes (\alpha'(a) \bullet' (b \bullet' c))\\
&\quad + (-1)^{(|x|+|a|)(|y|+|b|)+|b||x|+(|a|+|b|)|z|}((y \bullet x) \bullet \alpha(z)) \otimes ((b \bullet' a) \bullet' \alpha'(c))\\
&\quad - (-1)^{(|x|+|a|)(|y|+|b|)+|a||z|+|b|(|x|+|z|)}(\alpha(y) \bullet (x \bullet z)) \otimes (\alpha'(b) \bullet' (a \bullet' c)).
\end{aligned}$$

By super-commutativity, $x \bullet y = (-1)^{|x||y|} y \bullet x$, and Hom-associativity (7.10),

$$\begin{aligned}
&as_{\alpha\otimes\alpha',*}(X, Y, Z) + (-1)^{|XY|}as_{\alpha\otimes\alpha',*}(Y, X, Z) = (-1)^{|a||y|+|a||z|+|b||z|}\\
&\Big((x \bullet y) \bullet \alpha(z) - \alpha(x) \bullet (y \bullet z) + (-1)^{|x||y|}(y \bullet x) \bullet \alpha(z) - (-1)^{|x||y|}\alpha(y) \bullet (x \bullet z)\Big)\\
&\hfill \otimes (a \bullet' b) \bullet' \alpha'(c).
\end{aligned}$$

The left hand side vanishes by the left Hom-alternativity of A'. The right Hom-alternativity is proved similarly. □

Definition 7.9 ([31]) *Left averaging operator* over a Hom-alternative superalgebra $(A, \cdot, \alpha)$ is an even linear map $\beta : A \to A$ such that $\alpha \circ \beta = \beta \circ \alpha$ and for $x, y \in \mathcal{H}(A)$,

$$\beta(x) \cdot \beta(y) = \beta(\beta(x) \cdot y).$$

Right averaging operator over a Hom-alternative superalgebra $(A, \cdot, \alpha)$ is an even linear map $\beta : A \to A$ such that $\alpha \circ \beta = \beta \circ \alpha$ and for $x, y \in \mathcal{H}(A)$,

$$\beta(x) \cdot \beta(y) = \beta(x \cdot \beta(y)).$$

Averaging operator over a Hom-alternative superalgebra $(A, \cdot, \alpha)$ is both left averaging operator and right averaging operator, meaning an even linear map $\beta : A \to A$ such that $\alpha \circ \beta = \beta \circ \alpha$ and for $x, y \in \mathcal{H}(A)$,

$$\beta(\beta(x) \cdot y) = \beta(x) \cdot \beta(y) = \beta(x \cdot \beta(y)).$$

Proposition 7.1 *Let $(A, \cdot, \alpha)$ be a Hom-alternative algebra. Let $\beta : A \to A$ be an element of the centroid, an even linear map such that for $x, y \in \mathcal{H}(A)$,*

$$\beta \circ \alpha = \alpha \circ \beta, \tag{7.25}$$

$$\beta(x \cdot y) = \beta(x) \cdot y = x \cdot \beta(y). \tag{7.26}$$

Then $(A, \cdot_\beta = \beta \circ \cdot, \alpha)$ is a Hom-alternative superalgebra.

Proof Hom-alternativity means both left and right Hom-alternativity. The left and right Hom-alternativity of $(A, \cdot_\beta = \beta \circ \cdot, \alpha)$ are proved as follows. For $x, y, z \in \mathcal{H}(A)$,

$$\begin{aligned}
as_{\alpha,\cdot_\beta}(x, y, z) &= (x \cdot_\beta y) \cdot_\beta \alpha(z) - \alpha(x) \cdot_\beta (y \cdot_\beta z) \\
&= \beta(\beta(x \cdot y) \cdot \alpha(z)) - \beta(\alpha(x) \cdot \beta(y \cdot z)) \\
&\overset{(7.26)}{=} \beta((\beta(x) \cdot y)) \cdot \alpha(z) - \beta(\alpha(x)) \cdot (\beta(y) \cdot z) \\
&\overset{(7.26)}{=} (\beta(x) \cdot \beta(y)) \cdot \alpha(z) - \beta(\alpha(x)) \cdot (\beta(y) \cdot z) \\
&\overset{(7.25)}{=} (\beta(x) \cdot \beta(y)) \cdot \alpha(z) - \alpha(\beta(x)) \cdot (\beta(y) \cdot z) \\
&= as_{\alpha,\cdot}(\beta(x), \beta(y), z) \qquad (7.27) \\
&\text{(using left Hom-alternativity of } (A, \cdot, \alpha)) \\
&\overset{(7.12)}{=} -(-1)^{|x||y|} as_{\alpha,\cdot}(\beta(y), \beta(x), z) \\
&\text{(using proved in (7.27) for } (y, x, z)) \\
&= -(-1)^{|x||y|} as_{\alpha,\cdot_\beta}(y, x, z), \\
as_{\alpha,\cdot_\beta}(x, y, z) &= (x \cdot_\beta y) \cdot_\beta \alpha(z) - \alpha(x) \cdot_\beta (y \cdot_\beta z) \\
&= \beta(\beta(x \cdot y) \cdot \alpha(z)) - \beta(\alpha(x) \cdot \beta(y \cdot z)) \\
&\overset{(7.26)}{=} (\beta(x) \cdot y) \cdot \beta(\alpha(z)) - \beta(\alpha(x)) \cdot (y \cdot \beta(z)) \\
&\overset{(7.25)}{=} (\beta(x) \cdot y) \cdot \alpha(\beta(z)) - \alpha(\beta(x)) \cdot (y \cdot \beta(z)) \\
&= as_{\alpha,\cdot}(\beta(x), y, \beta(z)) \\
&\text{(using right Hom-alternativity of } (A, \cdot, \alpha)) \\
&\overset{(7.15)}{=} -(-1)^{|y||z|} as_{\alpha,\cdot}(\beta(x), \beta(z), y) \\
&\text{(using proved in (7.27) for } (x, z, y)) \\
&= -(-1)^{|y||z|} as_{\alpha,\cdot_\beta}(x, z, y).
\end{aligned}$$

□

Proposition 7.2 *Any Hom-alternative superalgebra* $(A, \cdot, \alpha)$ *with an averaging operator* ∂ *is a Hom-alternative superalgebra with respect to multiplication* $* : A \times A \to A$ *defined by* $x * y := x \cdot \partial(y)$ *and the same twisting map* α.

Proof For any $x, y, z \in \mathcal{H}(A)$,

$$\begin{aligned}(x * y) * \alpha(z) - \alpha(x) * (y * z) &= \alpha(x) \cdot (\partial(y) \cdot \partial(z)) - \alpha(x) \cdot \partial(y \cdot \partial(z))\\ &= \alpha(x) \cdot (\partial(y) \cdot \partial(z)) - \alpha(x) \cdot (\partial(y) \cdot \partial(z)) = 0.\end{aligned}$$

On the one hand, exchanging the role of x and y, yields

$$(x * y) * \alpha(z) - \alpha(x) * (y * z) + (-1)^{|x||y|}\Big((y * x) * \alpha(z) - \alpha(y) * (x * z)\Big) = 0.$$

On the other hand, exchanging the role of y and z, yields

$$(x * y) * \alpha(z) - \alpha(x) * (y * z) + (-1)^{|y||z|}\Big((x * z) * \alpha(y) - \alpha(x) * (z * y)\Big) = 0.$$

This completes the proof. □

Definition 7.10 ([1]) A Hom-Jordan superalgebra is a Hom-superalgebra $(A, \bullet, \alpha)$ satisfying super-commutativity and Hom-Jordan super identity for $x, y, z, t \in \mathcal{H}(A)$,

$$x \bullet y = (-1)^{|x||y|} y \bullet x, \quad \text{super-commutativity} \tag{7.28}$$

$$\sum_{\circlearrowleft(x,y,t)} (-1)^{|t|(|x|+|z|)} as_{\bullet,\alpha}(xy, \alpha(z), \alpha(t)) = 0, \quad \text{Hom-Jordan super identity} \tag{7.29}$$

where $\sum_{\circlearrowleft(a,b,c)}$ is the summation over cyclically permutated (a, b, c). Hom-Jordan super identity (7.29) in juxtaposition notation $x \bullet y = \bullet(x, y)$ is, for $x, y, z, t \in \mathcal{H}(A)$,

$$\sum_{\circlearrowleft(x,y,t)} (-1)^{|t|(|x|+|z|)}((x \bullet y) \bullet \alpha(z)) \bullet \alpha^2(t) = \sum_{\circlearrowleft(x,y,t)} (-1)^{|t|(|x|+|z|)}\alpha(x \bullet y) \bullet (\alpha(z) \bullet \alpha(t)).$$

Remark 7.2 If $(x, y, z, t) \in (A_0 \times A_0 \times A_0 \times A_0) \cup (A_1 \times A_1 \times A_1 \times A_1)$, then

$$(-1)^{|t|(|x|+|z|)} = (-1)^{|x|(|y|+|z|)} = (-1)^{|y|(|t|+|z|)} = 1,$$

and Hom-Jordan super identity is

$$\sum_{\circlearrowleft(x,y,t)} ((x \bullet y) \bullet \alpha(z)) \bullet \alpha^2(t) = \sum_{\circlearrowleft(x,y,t)} \alpha(x \bullet y) \bullet (\alpha(z) \bullet \alpha(t)). \tag{7.30}$$

Theorem 7.3 ([1]) *Any multiplicative Hom-alternative superalgebra is Hom-Jordan admissible, that is, for any multiplicative Hom-alternative superalgebra* $(A, \bullet, \alpha)$*, the Hom-superalgebra* $A^+ = (A, *, \alpha)$ *is a multiplicative Hom-Jordan superalgebra, where* $x * y = xy + (-1)^{|x||y|} yx$.

7.3 Hom-Prealternative and Hom-Alternative Superalgebras

In this section, we introduce Hom-prealternative superalgebras, give some construction theorems and study their connection with Hom-alternative superalgebras. The associated bimodules are also discussed.

7.3.1 Prealternative Superalgebras

Definition 7.11 A Hom-prealternative superalgebra is a quadruple $(A, \prec, \succ, \alpha)$ where A is a super vector space, $\prec, \succ: A \otimes A \to A$ are even bilinear maps and $\alpha : A \to A$ an even linear map such that, for any $x, y, z \in \mathcal{H}(A)$ and $x \bullet y = x \succ y + x \prec y$,

$$(x \bullet x) \succ \alpha(y) - \alpha(x) \succ (x \succ y) = 0, \tag{7.31}$$

$$(x \prec y) \prec \alpha(y) - \alpha(x) \prec (y \bullet y) = 0, \tag{7.32}$$

$$\begin{aligned}&(x \succ y) \prec \alpha(z) - \alpha(x) \succ (y \prec z) + \\ &\quad (-1)^{|x||y|}(y \prec x) \prec \alpha(z) - (-1)^{|x||y|}\alpha(y) \prec (x \bullet z) = 0,\end{aligned} \tag{7.33}$$

$$\begin{aligned}&(x \succ y) \prec \alpha(z) - \alpha(x) \succ (y \prec z) + \\ &\quad (-1)^{|y||z|}(x \bullet z) \succ \alpha(y) - (-1)^{|y||z|}\alpha(x) \succ (z \succ y) = 0.\end{aligned} \tag{7.34}$$

Definition 7.12 Let $(A, \prec, \succ, \alpha)$ and $(A', \prec', \succ', \alpha')$ be two Hom-prealternative superalgebras. An even linear map $f : A \to A'$ is said to be a morphism of Hom-prealternative superalgebras if, for $x, y \in \mathcal{H}(A)$,

$$\alpha' \circ f = f \circ \alpha, \quad f(x \prec y) = f(x) \prec' f(y) \text{ and } f(x \succ y) = f(x) \succ' f(y).$$

A Hom-prealternative superalgebra $(A, \prec, \succ, \alpha)$ in which $\alpha : A \to A$ is a morphism is called a multiplicative Hom-alternative superalgebra.

Remark 7.3 Axioms (7.31) and (7.32) can be rewritten respectively as

$$(x \bullet y) \succ \alpha(z) - \alpha(x) \succ (y \succ z) + (-1)^{|x||y|}(y \bullet x) \succ \alpha(z) - (-1)^{|x||y|}\alpha(y) \succ (x \succ z) = 0, \tag{7.35}$$

$$(x \prec y) \prec \alpha(z) - \alpha(x) \prec (y \bullet z) + (-1)^{|y||z|}(x \prec z) \prec \alpha(y) - (-1)^{|y||z|}\alpha(x) \prec (z \bullet y) = 0. \tag{7.36}$$

Remark 7.4 If $(A, \prec, \succ, \alpha)$ is a Hom-prealternative superalgebra, then $(A, \prec_\lambda = \lambda \cdot \prec, \succ_\lambda = \lambda \cdot \succ, \alpha)$ is also a Hom-prealternative superalgebra.

Using the following notations [60, 85]:

$$(x, y, z)_1 = (x \bullet y) \succ \alpha(z) - \alpha(x) \succ (y \succ z), \tag{7.37}$$
$$(x, y, z)_2 = (x \succ y) \prec \alpha(z) - \alpha(x) \succ (y \prec z), \tag{7.38}$$
$$(x, y, z)_3 = (x \prec y) \prec \alpha(z) - \alpha(x) \prec (y \bullet z), \tag{7.39}$$

the axioms in Definition 7.11 of Hom-prealternative superalgebras can be rewritten for $x, y, z \in \mathcal{H}(A)$ as

$$(x, x, z)_1 = (y, x, x)_3 = 0 \tag{7.40}$$
$$(x, y, z)_2 + (-1)^{|x||y|}(y, x, z)_3 = 0, \tag{7.41}$$
$$(x, y, z)_2 + (-1)^{|y||z|}(x, z, y)_1 = 0. \tag{7.42}$$

The following definition is motivated by [60, Definitions 17, 18].

Definition 7.13 A Hom-prealternative superalgebra $(A, \prec, \succ, \alpha)$ is said to be left Hom-alternative if

$$(x, y, z)_i + (-1)^{|x||y|}(y, x, z)_i = 0, \quad i = 1, 2, 3. \tag{7.43}$$

and right Hom-alternative if

$$(x, y, z)_i + (-1)^{|y||z|}(x, z, y)_i = 0, \quad i = 1, 2, 3. \tag{7.44}$$

Definition 7.14 A Hom-prealternative superalgebra algebra $(A, \prec, \succ, \alpha)$ is said to be flexible if

$$(x, y, x)_i = 0, \quad i = 1, 2, 3. \tag{7.45}$$

Theorem 7.4 *If $(A, \prec, \succ, \alpha)$ is a left Hom-prealternative superalgebra, then $Alt(A) = (A, \bullet, \alpha)$ is a left Hom-alternative superalgebra. If $(A, \prec, \succ, \alpha)$ is a right Hom-prealternative superalgebra, then $Alt(A) = (A, \bullet, \alpha)$ is a right Hom-alternative superalgebra.*

Proof For any $x, y, z \in \mathcal{H}(A)$,

$$\begin{aligned}
as_\bullet(z,x,y) &= (z \bullet x) \bullet \alpha(y) - \alpha(z) \bullet (x \bullet y)\\
&= (z \bullet x) \prec \alpha(y) + (z \bullet x) \succ \alpha(y) - \alpha(z) \prec (x \bullet y) - \alpha(z) \succ (x \bullet y)\\
&= (z \prec x + z \succ x) \prec \alpha(y) + (z \bullet x) \succ \alpha(y) -\\
&\qquad \alpha(z) \prec (x \bullet y) - \alpha(z) \succ (x \prec y + x \succ y)\\
&= ((z \prec x) \prec \alpha(y) - \alpha(z) \prec (x \bullet y)) + ((z \succ x) \prec \alpha(y) - \alpha(z) \succ (x \prec y)) +\\
&\qquad ((z \bullet x) \succ \alpha(y) - \alpha(z) \succ (x \succ y))\\
&= (z,x,y)_3 + (z,x,y)_2 + (z,x,y)_1 = -(-1)^{|x||y|}((z,y,x)_3 + (z,y,x)_2 + (z,y,x)_1)\\
&= -(-1)^{|x||y|} as_\bullet(z,y,x).
\end{aligned}$$

The left alternatively is proved analogously. □

Note that the left and right Hom-alternativity for dialgebras is not defined in the same way that the one of algebras with one operation; so the two terminologies must not be confused.

Proposition 7.3 *Let $(A, \prec, \succ, \alpha)$ be a flexible Hom-prealternative superalgebra. Then $(A, \bullet, \alpha)$ is a flexible Hom-alternative superalgebra.*

Theorem 7.5 *Let $(A, \prec, \succ, \alpha)$ be a Hom-prealternative superalgebra. Then $A' = (A, \prec', \succ', \alpha)$ is also a Hom-prealternative superalgebra with*

$$x \prec' y = (-1)^{|x||y|} y \succ x, \qquad x \succ' y = (-1)^{|x||y|} y \prec x.$$

Proof We prove only (7.33), as (7.31), (7.32) and (7.34) are proved similarly. For $x, y, z \in \mathcal{H}(A)$,

$$\begin{aligned}
&(x \succ' y) \prec' \alpha(z) - \alpha(x) \succ' (y \prec z) +\\
&\qquad (-1)^{|x||y|}(y \prec' x) \prec' \alpha(z) - (-1)^{|x||y|}\alpha(y) \prec' (x \bullet' z)\\
&= (-1)^{|x||y|}(y \prec x) \prec' \alpha(z) - (-1)^{|y||z|}\alpha(z) \succ' (z \succ y) +\\
&\qquad (x \succ y) \prec' \alpha(z) - (-1)^{|x||y|+|x||z|}\alpha(y) \prec' (z \bullet x)\\
&= (-1)^{|x||y|+(|x|+|y|)|z|}\alpha(z) \succ (y \prec x) - (-1)^{|y||z|+|x|(|y|+|z|)}(z \succ y) \prec \alpha(x) +\\
&\qquad (-1)^{(|x|+|y|)|z|}\alpha(z) \succ (x \succ y) - (-1)^{|x||y|+|x||z|+|y|(|x|+|z|)}(z \bullet x) \succ \alpha(y)\\
&= (-1)^{|x||y|+|x||z|+|y||z|}[\alpha(z) \succ (y \prec x) - (z \succ y) \prec \alpha(x) +\\
&\qquad (-1)^{|x||y|}\alpha(z) \succ (x \succ y) - (-1)^{|x||y|}(z \bullet x) \succ \alpha(y)] = 0
\end{aligned}$$

by axiom (7.34) for $(A, \prec, \succ, \alpha)$. □

Note that $Alt(A') = Alt(A)^{op}$, that is, $x \bullet' y = (-1)^{|x||y|} y \bullet x$, for $x, y \in \mathcal{H}(A)$.

Theorem 7.6 *Let $(A, \prec, \succ, \alpha)$ be a Hom-prealternative superalgebra. Let us define the operation $x \bullet y = x \prec y + x \succ y$ for any homogeneous elements x, y in A. Then $Alt(A) = (A, \bullet, \alpha)$ is a Hom-alternative superalgebra.*

Proof Let us prove the left alternativity. For any homogeneous $x, y, z \in A$,

$$\begin{aligned} as_\bullet(x, y, z) + (-1)^{|x||y|}as_\bullet(y, x, z) = \\ (x \prec y) \prec \alpha(z) + (x \succ y) \prec \alpha(z) + (x \bullet y) \succ \alpha(z) - \alpha(x) \prec (y \bullet z) \\ - \alpha(x) \succ (y \succ z) - \alpha(x) \succ (y \prec z) + (-1)^{|x||y|}(y \prec x) \prec \alpha(z) \\ + (-1)^{|x||y|}(y \succ x) \prec \alpha(z) + (-1)^{|x||y|}(y \bullet x) \succ \alpha(z) - (-1)^{|x||y|}\alpha(y) \prec (x \bullet z) \\ - (-1)^{|x||y|}\alpha(y) \succ (x \prec z) - (-1)^{|x||y|}\alpha(y) \succ (x \succ)z. \end{aligned}$$

The left hand side vanishes by using one axiom (7.33) and twice axiom (7.36). □

The Hom-alternative superalgebra $Alt(A) = (A, \bullet, \alpha)$ in Theorem 7.6 is called the associated Hom-alternative superalgebra of $(A, \prec, \succ, \alpha)$. We call $(A, \prec, \succ, \alpha)$ a compatible Hom-prealternative superalgebra structure on the Hom-alternative superalgebra $Alt(A)$.

Theorems 7.3, 7.5 and 7.6 yield the following corollary.

Corollary 7.1 *Let $(A, \prec, \succ, \alpha)$ be a multiplicative Hom-prealternative superalgebra. Then $(A, *, \alpha)$ is a multiplicative Hom-Jordan superalgebra with*

$$x * y = x \prec y + x \succ y + (-1)^{|x||y|}y \prec x + (-1)^{|x||y|}y \succ x.$$

Let us define the notion of $\mathcal{O}$-operator for Hom-alternative superalgebras.

Definition 7.15 Let (V, L, R, β) be a bimodule of the Hom-alternative superalgebra $(A, \bullet, \alpha)$. An even linear map $T : V \to A$ is called an $\mathcal{O}$-operator associated to (V, L, R, β) if for $u, v \in V$,

$$T(u) \bullet T(v) = T(L(T(u))v + R(T(v))u), \tag{7.46}$$
$$T \circ \beta = \alpha \circ T. \tag{7.47}$$

Theorem 7.7 *Let $T : V \to A$ be an $\mathcal{O}$-operator of the Hom-alternative superalgebra $(A, \bullet, \alpha)$ associated to the bimodule (V, L, R, β). Then $(V, \prec, \succ, \beta)$ is a Hom-prealternative superalgebra structure, where for $u, v \in V$,*

$$u \prec v = R(T(v))u \quad and \quad u \succ v = L(T(u))v.$$

Therefore, $(V, \bullet = \prec + \succ, \beta)$ is the associated Hom-alternative superalgebra of this Hom-prealternative superalgebra, and T is a homomorphism of Hom-alternative superalgebras. Furthermore, $T(V) = \{T(v), v \in V\} \subseteq A$ is a Hom-alternative subalgebra of $(A, \bullet, \alpha)$, and $(T(V), \prec, \succ, \alpha)$ is a Hom-prealternative superalgebra, where

$$T(u) \prec T(v) = T(u \prec v) \ \ \textit{and} \ \ T(u) \succ T(v) = T(u \succ v) \ \textit{for} \ u, v \in V$$

The associated Hom-alternative superalgebra $(T(V), \bullet = \prec + \succ, \alpha)$ is just the Hom-alternative subalgebra structure of $(A, \bullet, \alpha)$, and T is a homomorphism of Hom-prealternative superalgebras.

Proof For any homogeneous elements $u, w, w \in V$,

$$\begin{aligned}
&(u \succ v) \prec \beta(w) - \beta(u) \succ (v \prec w) + (-1)^{|u||v|}(v \prec u) \prec \beta(w) \\
&\qquad - (-1)^{|u||v|}\beta(v) \prec (u \bullet w) \\
&= (T(u)v)T\beta(w) - T\beta(u)(vT(w)) + (-1)^{|u||v|}(vT(u))T\beta(w) \\
&\qquad - (-1)^{|u||v|}\beta(v)T(uT(w) + T(u)w) \\
&= (T(u)v)\alpha(T(w)) - \alpha(T(u))(vT(w)) + (-1)^{|u||v|}(vT(u))\alpha(T(w)) \\
&\qquad - (-1)^{|u||v|}\beta(v)(T(u)T(w)) = 0.
\end{aligned}$$

The other identities are checked similarly, and the rest of the proof is easy. □

Definition 7.16 A Hom-alternative Rota-Baxter superalgebra of weight λ is a Hom-alternative superalgebra $(A, \cdot, \alpha)$ together with an even linear self-map $R : A \to A$ such that $R \circ \alpha = \alpha \circ R$ and

$$R(x) \cdot R(y) = R\Big(R(x) \cdot y + x \cdot R(y) + \lambda x \cdot y\Big).$$

Corollary 7.2 *Let $(A, \cdot, \alpha)$ be a Hom-alternative superalgebra and $R : A \to A$ a Rota-Baxter operator of weight 0 on A. Then*

(i) *$A_R = (A, \prec, \succ, \alpha)$ is a Hom-prealternative superalgebra, where*

$$x \prec y = x \cdot R(y), \quad x \succ y = R(x) \cdot y, \quad \textit{for} \ x, y \in \mathcal{H}(A).$$

(ii) *$(A, \bullet, \alpha)$ is a Hom-alternative superalgebra with $x \bullet y = R(x) \cdot y + x \cdot R(y)$.*

Proposition 7.4 *Let $(V, \prec, \succ, \beta)$ be a bimodule over the Hom-alternative superalgebra $(A, \cdot, \alpha)$ and $R : A \to A$ be a Rota-Baxter operator of weight 0 on A. Then, $(V, \lhd, \rhd, \beta)$, with $v \lhd x = v \prec R(x)$ and $x \rhd y = R(x) \succ v$, is a bimodule over $(A, \bullet, \alpha)$.*

Proof For any homogeneous elements $x, y \in A$ and $v \in V$,

$$\begin{aligned}
&(v \lhd x) \lhd \alpha(y) - \beta(v) \lhd (x \bullet y) \\
&= (v \prec R(x)) \prec R(\alpha(y)) - \beta(v) \prec R(R(x) \cdot y + x \cdot R(y)) \\
&= (v \prec R(x)) \prec \alpha(R(y)) - \beta(v) \prec (R(x) \cdot R(y)) \\
&= (-1)^{|x||y|}\Big((R(x) \succ v) \prec \alpha(y) - \alpha(x) \succ (v \prec y)\Big) \\
&= (-1)^{|x||y|}\Big((x \rhd v) \lhd \alpha(y) - \alpha(x) \rhd (v \lhd y)\Big).
\end{aligned}$$

The other identities are proved similarly. □

Theorem 7.8 *Suppose that $(A, \prec, \succ, \alpha)$ is a Hom-prealternative superalgebra, and let $\beta : A \to A$ be an even Hom-prealternative superalgebra endomorphism. Then $A_\beta = (A, \prec_\beta = \beta \circ \prec, \succ_\beta = \beta \circ \succ, \beta\alpha)$ is a Hom-prealternative superalgebra. Let $(A', \prec', \succ')$ be another prealternative superalgebra and $\alpha' : A \to A'$ be a prealternative superalgebra endomorphism. If $f : A \to A'$ is a prealternative superalgebra morphism satisfying $f \circ \beta = \alpha' \circ f$, then*

$$f : (A, \prec_\beta = \beta \circ \prec, \succ_\beta = \beta \circ \succ, \beta\alpha) \to (A', \prec'_{\alpha'} = \alpha' \circ \prec', \succ'_{\alpha'} = \alpha' \circ \succ', \alpha')$$

is a morphism of Hom-prealternative superalgebras.

Proof For $x, y, z \in \mathcal{H}(A)$,

$$\begin{aligned}
&(x \succ_\beta y) \prec_\beta \beta\alpha(z) - \beta\alpha(x) \succ_\beta (y \prec_\beta z)\\
&\quad = \beta((\beta(x) \succ \beta(y))) \prec \beta(\alpha(z)) - \beta(\alpha(x)) \succ \beta[(\beta(y) \prec \beta(z))\\
&\quad = \beta^2\Big((x \succ y) \prec \alpha(z) - \alpha(x) \succ (y \prec z)\Big)\\
&\quad = (-1)^{|x||y|}\beta^2\Big(\alpha(y) \prec (x \bullet z) - (-1)^{|x||y|}(y \prec x) \prec \alpha(z)\Big)\\
&\quad = (-1)^{|x||y|}\beta\Big(\beta\alpha(y) \prec \beta(x \bullet z) - (-1)^{|x||y|}\beta(y \prec x) \prec \beta\alpha(z)\Big)\\
&\quad = (-1)^{|x||y|}\beta\Big(\beta\alpha(y) \prec (x \bullet_\beta z) - (-1)^{|x||y|}(y \prec_\beta x) \prec \beta\alpha(z)\Big)\\
&\quad = (-1)^{|x||y|}\Big(\beta^2\alpha(y) \prec \beta(x \bullet_\beta z) - (-1)^{|x||y|}\beta(y \prec_\beta x) \prec \beta^2\alpha(z)\Big)\\
&\quad = (-1)^{|x||y|}\beta\alpha(y) \prec_\beta (x \bullet_\beta z) - (-1)^{|x||y|}(y \prec_\beta x) \prec_\beta \beta\alpha(z).
\end{aligned}$$

The other axioms are proved similarly. For the second part,

$$f \circ \prec_\alpha = f \circ \alpha \circ \prec = \alpha' \circ f \circ \prec = \alpha' \circ \prec' \circ (f \otimes f) = \prec'_{\alpha'} \circ (f \otimes f).$$

Analogues equalities hold for $\succ_\alpha$ and $\succ'_{\alpha'}$. □

Taking $\beta = \alpha^{2^n - 1}$ yields the following result.

Corollary 7.3 *Let $(A, \prec, \succ, \alpha)$ be a multiplicative Hom-prealternative superalgebra. Then,*

(i) *For $n \geq 0$, $A^n = (A, \prec^{(n)} = \alpha^{2^n-1} \circ \prec, \succ^{(n)} = \alpha^{2^n-1} \circ \succ, \alpha^{2^n})$ is a multiplicative Hom-prealternative superalgebra, called the n*th *derived multiplicative Hom-prealternative superalgebra.*
(ii) *For $n \geq 0$, $A^n = (A, \prec^{(n)} = \alpha^{2^n-1} \circ (\prec + \succ), \alpha^{2^n})$ is a multiplicative Hom-alternative superalgebra, called the n*th *derived multiplicative Hom-alternative superalgebra.*

7.3.2 Bimodules of Hom-Prealternative Superalgebras

Definition 7.17 Let $(A, \prec, \succ, \alpha)$ be a Hom-prealternative superalgebra. An A-bimodule is a super vector space V with an even linear map $\beta : V \to V$ and four even linear maps from A to the space $gl(V)$ of all even linear maps on V,

$$\begin{aligned}
&L_\succ : A \to gl(V) && L_\prec : A \to gl(V)\\
&\quad x \mapsto L_\succ(x)(v) = x \succ v, && \quad x \mapsto L_\prec(x)(v) = x \prec v,\\
&R_\succ : A \to gl(V) && R_\prec : A \to gl(V)\\
&\quad x \mapsto R_\succ(x)(v) = v \succ x, && \quad x \mapsto R_\prec(x)(v) = v \prec x,
\end{aligned}$$

satisfying the following relations:

$$L_\succ(x \bullet y + (-1)^{|x||y|} y \bullet x)\beta(v) = \\ L_\succ(\alpha(x))L_\succ(y) + (-1)^{|x||y|} L_\succ(\alpha(y))L_\succ(x), \tag{7.48}$$

$$R_\succ(\alpha(y))(L_\bullet(x) + (-1)^{|x||v|} R_\bullet(x))v = \\ L_\succ(\alpha(x))R_\succ(y)v + (-1)^{|x||v|} R_\succ(x \succ y)\beta(v), \tag{7.49}$$

$$R_\prec(\alpha(y))L_\succ(x) + (-1)^{|x||v|} R_\prec(\alpha(y))R_\prec(x) = \\ L_\succ(\alpha(x))R_\prec(y) + (-1)^{|x||v|} R_\prec(x \circ y)\beta(v), \tag{7.50}$$

$$R_\prec(\alpha(y))R_\succ(x)v + (-1)^{|x||v|} R_\succ(\alpha(y))L_\prec(x)v = \\ L_\prec(\alpha(x))R_\bullet(y)v + (-1)^{|x||v|} R_\succ(x \bullet y)\beta(v), \tag{7.51}$$

$$L_\prec(y \prec x)\beta(v) + (-1)^{|x||y|} L_\prec(x \succ y)\beta(v) = \\ L_\prec(\alpha(y))L_\bullet(x)v + (-1)^{|x||y|} L_\succ(\alpha(y))L_\succ(x)v, \tag{7.52}$$

$$R_\prec(\alpha(x))L_\succ(y) + (-1)^{|x||v|} L_\succ(y \succ x)\beta(v) = \\ L_\succ(y)R_\prec(x)v + (-1)^{|x||v|} L_\succ(\alpha(y))L_\succ(x)v, \tag{7.53}$$

$$R_\prec(\alpha(x))R_\succ(y)v + (-1)^{|x||y|} R_\succ(\alpha(y))R_\bullet(x)v = \\ R_\succ(y \prec x)\beta(v) + (-1)^{|x||y|} R_\succ(x \succ y)\beta(v), \tag{7.54}$$

$$L_\prec(y \succ x)\beta(v) + (-1)^{|x||v|} R_\succ(\alpha(x))L_\bullet(y)v = \\ L_\succ(\alpha(y))L_\prec(x)v + (-1)^{|x||v|} L_\succ(\alpha(y))R_\succ(y)v, \tag{7.55}$$

$$R_\prec(\alpha(x))R_\prec(y)v + (-1)^{|x||y|} R_\prec(\alpha(y))R_\prec(x)v = \\ R_\prec(x \bullet y + (-1)^{|x||y|} y \bullet x)\beta(v), \tag{7.56}$$

$$R_\prec(\alpha(y))L_\prec(x) + (-1)^{|y||v|} L_\prec(x \prec y)\beta(v) = \\ L_\prec(\alpha(x))(R_\bullet(y) + (-1)^{|y||v|} L_\bullet(y))v, \tag{7.57}$$

where $\bullet = \prec + \succ$ and, for homogeneous $x, y \in A$, $v \in V$,

$$x \bullet y = x \prec y + x \succ y, \quad L_\bullet = L_\succ + L_\prec, \quad R_\bullet = R_\succ + R_\prec.$$

Remark 7.5 Axioms (7.48)–(7.57) are respectively equivalent to

$$(x \bullet y + (-1)^{|x||y|} y \bullet x) \succ \beta(v) = \alpha(x) \succ (y \succ v) + (-1)^{|x||y|} \alpha(y) \succ (x \succ v), \tag{7.58}$$

$$(x \bullet v + (-1)^{|x||v|} v \bullet x) \succ \alpha(y) = \alpha(x) \succ (v \succ y) - (-1)^{|x||v|} \beta(v) \succ (x \succ y), \tag{7.59}$$

$$(v \prec x) \prec \alpha(y) + (-1)^{|x||v|} (x \succ v) \prec \alpha(y) = \beta(v) \prec (x \bullet y) + (-1)^{|x||v|} \alpha(x) \succ (v \prec y), \tag{7.60}$$

$$(x \prec v) \prec \alpha(y) + (-1)^{|x||v|} (v \succ x) \prec \alpha(y) = \alpha(x) \prec (v \bullet y) + (-1)^{|x||v|} \beta(v) \succ (x \bullet y), \tag{7.61}$$

$$(y \prec x) \prec \beta(v) + (-1)^{|x||y|} (x \succ y) \prec \beta(v) = \alpha(y) \prec (x \bullet v) + (-1)^{|x||y|} \alpha(x) \succ (y \prec v), \tag{7.62}$$

$$(y \succ v) \prec \alpha(x) + (-1)^{|x||v|} (y \bullet x) \succ \beta(v) = \alpha(y) \succ (v \prec x) + (-1)^{|x||v|} \alpha(y) \succ (x \succ v), \tag{7.63}$$

$$(v \succ) \prec \alpha(x) + (-1)^{|x||y|} (v \bullet x) \succ \alpha(y) = \beta(v) \succ (y \prec x) + (-1)^{|x||y|} \beta(v) \succ (x \succ y), \tag{7.64}$$

$$(y \succ x) \prec \beta(v) + (-1)^{|x||v|} (y \bullet v) \succ \alpha(x) = \alpha(y) \succ (x \prec v) + (-1)^{|x||v|} \alpha(y) \succ (v \succ x), \tag{7.65}$$

$$(v \prec x) \prec \alpha(y) + (-1)^{|x||y|} (v \prec y) \prec \alpha(x) = \beta(v) \prec (x \bullet y + (-1)^{|x||y|} y \bullet x), \tag{7.66}$$

$$(x \prec v) \prec \alpha(y) + (-1)^{|y||v|} (x \prec y) \prec \beta(v) = \alpha(x) \prec (v \bullet y + (-1)^{|y||v|} y \bullet v). \tag{7.67}$$

Proposition 7.5 *Suppose that $(A, \prec, \succ, \alpha)$ is a Hom-prealternative superalgebra. Then $(A, l_{\succ}, r_{\prec}, \alpha)$ is a bimodule of the associated Hom-alternative superalgebra $Alt(A) = (A, \bullet, \alpha)$.*

Proposition 7.6 *Let $(V, \prec, \succ, \beta)$ be a bimodule over the Hom-alternative superalgebra $(A, \bullet, \alpha)$ and $R : A \to A$ be a Rota-Baxter operator on A. Then $(V, 0, \rhd, 0, \lhd, \beta)$, with $x \rhd v = R(x) \succ v$ and $v \lhd x = v \prec R(x)$, is a bimodule over the Hom-prealternative superalgebra $A_R = (A, \prec, \succ, \alpha)$.*

Proof For any homogeneous elements $x, y \in A$ and $v \in V$,

$$\begin{aligned}
(v \lhd x) \lhd \alpha(y) + (x \rhd v) \lhd \alpha(y) \\
&= (v \prec R(x)) \prec R(\alpha(y)) + (R(x) \succ v) \prec R(\alpha(y)) \\
&= (v \prec R(x)) \prec \alpha(R(y)) + (R(x) \succ v) \prec \alpha(R(y)) \\
&= \beta(v) \prec (R(x) \cdot R(y)) + \alpha(R(x)) \succ (v \prec R(y)) \\
&= \beta(v) \prec R(R(x) \cdot y + x \cdot R(y)) + R(\alpha(x)) \succ (v \prec R(y)) \\
&= \beta(v) \lhd (R(x) \cdot y + x \cdot R(y)) + \alpha(x) \rhd (v \lhd y) \\
&= \beta(v) \lhd (x \succ y + x \prec y) + \alpha(x) \rhd (v \lhd y) \\
&= \beta(v) \lhd (x \cdot y) + \alpha(x) \rhd (v \lhd y).
\end{aligned}$$

The other axioms are proved in the same way. □

Theorem 7.9 *Suppose that* $(V, L_{\prec}, R_{\prec}, L_{\succ}, R_{\succ}, \beta)$ *be a bimodule over the Hom-prealternative superalgebra* $(A, \prec, \succ, \alpha)$*, and let* $Alt(A) = (A, \bullet, \alpha)$ *be the associated Hom-alternative superalgebra. Then*

(i) $(V, L_{\succ}, R_{\prec}, \beta)$ *is a bimodule over* $Alt(A)$.
(ii) $(V, L_{\bullet} = L_{\prec} + L_{\succ}, R_{\bullet} = R_{\prec} + R_{\succ}, \beta)$ *is a bimodule over* $Alt(A)$.
(iii) *If* (V, L, R, β) *is a bimodule of* $Alt(A)$*, then* $(V, 0, R, L, 0, \beta)$ *is a bimodule over* $(A, \prec, \succ, \alpha)$.

Proof For any homogeneous elements $x, y \in A$ and $v \in V$,
i The statement i follows from axioms (7.58), (7.60), (7.63) and (7.66). ii For ii, the axiom (7.23) is verified as follows,

$$\begin{aligned}
&(x \bullet y) \bullet \beta(v) + (-1)^{|x||y|}(y \bullet x) \bullet \beta(v) - \alpha(x) \bullet (y \bullet v) - (-1)^{|x||y|}\alpha(y) \bullet (x \bullet v) \\
&= (x \prec y) \prec \beta(v) + (x \succ y) \prec \beta(v) + (x \bullet y) \succ \beta(v) \\
&\quad + (-1)^{|x||y|}(y \prec x) \prec \beta(v) + (-1)^{|x||y|}(y \succ x) \prec \beta(v) + (-1)^{|x||y|}(x \bullet y) \succ \beta(v) \\
&\quad - \alpha(x) \prec (y \bullet v) - \alpha(x) \succ (y \prec v) - \alpha(x) \succ (y \succ v) \\
&\quad - (-1)^{|x||y|}\alpha(y) \prec (x \bullet v) - (-1)^{|x||y|}\alpha(y) \succ (x \prec v) - (-1)^{|x||y|}\alpha(y) \succ (x \succ v).
\end{aligned}$$

The left hand side vanishes by axioms (7.58) and (7.62). The other axioms are verified analogously: axiom (7.24) comes from axioms (7.64) and (7.66); axiom (7.22) comes from axioms (7.63), (7.65) and (7.67); axiom (7.21) comes from axioms (7.59), (7.60) and (7.61). iii It suffices to take $R_{\succ} = 0$ and $L_{\prec} = 0$. □

Theorem 7.10 *Let* $(V, L_{\prec}, R_{\prec}, L_{\succ}, R_{\succ}, \beta)$ *be a bimodule over the multiplicative Hom-prealternative superalgebra* $(A, \prec, \succ, \alpha)$*, and let* $Alt(A) = (A, \bullet, \alpha)$ *be the associated Hom-alternative superalgebra. For*

$$\begin{aligned}
L^{\alpha}_{\prec} &= L_{\prec} \circ (\alpha^2 \otimes Id), \quad L^{\alpha}_{\succ} = L_{\succ} \circ (\alpha^2 \otimes Id), \\
R^{\alpha}_{\prec} &= R_{\prec} \circ (\alpha^2 \otimes Id), \quad R^{\alpha}_{\succ} = R_{\succ} \circ (\alpha^2 \otimes Id),
\end{aligned}$$

$(V, L^{\alpha}_{\succ}, R^{\alpha}_{\prec}, \beta)$ *and* $(V, L^{\alpha}_{\bullet} = L^{\alpha}_{\prec} + L^{\alpha}_{\succ}, R^{\alpha}_{\bullet} = R^{\alpha}_{\prec} + R^{\alpha}_{\succ}, \beta)$ *are bimodules over* $Alt(A)$.

Proof We only prove (7.23) in detail, as the other axioms are verified similarly. Putting $\succ_{\alpha} = L^{\alpha}_{\succ}$, for any homogeneous elements $x, y \in A$ and $v \in V$,

$$\begin{aligned}
&(x \bullet y + (-1)^{|x||y|} y \bullet x) \succ_{\alpha} \beta(v) = \alpha^2(x \bullet y + (-1)^{|x||y|} y \bullet x) \succ \beta(v) \\
&\overset{(7.1)}{=} (\alpha^2(x) \bullet \alpha^2(y) + (-1)^{|x||y|}\alpha^2(y) \bullet \alpha^2(x)) \succ \beta(v) \\
&\overset{(7.58)}{=} \alpha^3(x) \succ (\alpha^2(y) \succ v) + (-1)^{|x||y|}\alpha^3(y) \succ (\alpha^2(x) \succ v) \\
&= \alpha(x) \succ_{\alpha} (y \succ_{\alpha} v) + (-1)^{|x||y|}\alpha(y) \succ_{\alpha} (x \succ_{\alpha} v).
\end{aligned}$$

□

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Chapter 8
Spectral Analysis of Equations over Quaternions

Ilwoo Cho and Palle E. T. Jorgensen

Abstract In this paper, we study how to solve certain equations on the set $\mathcal{H}$ of all quaternions. By using spectral analytic representations on $\mathcal{H}$, monomial equations, some quadratic equations, and linear equations on $\mathcal{H}$ are considered.

Keywords Quaternions · q-Spectral forms · Quaternionic equations

MSC 2020 Classification 47S05 · 46S05

8.1 Introduction

In this paper, we study how to solve certain equations on the set of quaternions. In particular, we are interested in monomial equations, quadratic equations with real coefficients, and linear equations. Let

$$\mathcal{C} = \left\{ x + yi : x, y \in \mathcal{R},\ i = \sqrt{-1} \right\}$$

be the set of all *complex numbers*, where $\mathcal{R}$ is the set of all *real numbers*.

The main purposes of this paper are (i) to study a representation of the set

$$\mathcal{H} = \left\{ x + yi + uj + vk \,\middle|\, \begin{array}{c} x, y, u, v \in \mathcal{R} \\ i^2 = j^2 = k^2 = -1, \\ ijk = -1 \end{array} \right\}$$

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of all *quaternions* (or *quaternion numbers*) realized on the 2-dimensional complex space $\mathbb{C} \times \mathbb{C}$ (ii) to consider *spectral properties* of realizations $[q] \in M_2(\mathbb{C})$ of quaternions $q \in \mathcal{C}$, (iii) to investigate how to solve some equations by using spectral properties (ii) on $\mathcal{C}$.

While quaternions are part of algebra, their applications are diverse, covering a number of neighboring areas, usually not considered as a part of algebra; e.g., analysis, spectral theory, geometry, and quantum physics. Our present focus has this in mind: (I) The use od quaternions in geometry and robotics (e.g., algorithms for iteration of systems of rotations, see Sect. 8.3); (II) their use in quantum systems of spin observables (see Sect. 8.4); and (III) how to solve some algebraic problems (see Sects. 8.5, 8.6 and 8.7). Our present purpose is to investigate a new spectral transform which will serve to link the algebra of quaternions to these applications. We shall also make use of certain compact Lie groups and their algebras, as tools in this endeavor.

8.1.1 Motivation

The study of the quaternions $\mathcal{H}$ is important not only in pure mathematics (e.g., [1, 2, 13, 14, 20]), but also in applied mathematics (e.g., [7, 23]). In particular, algebra on $\mathcal{H}$ is considered in e.g., [24]; analysis on $\mathcal{H}$ is studied in e.g., [15, 21]; and physics on $\mathcal{H}$ is observed in e.g., [5, 10]. Also, the matrices over the quaternions $\mathcal{H}$ have been studied (e.g., [8, 9, 18, 21]); and the eigenvalue problems on such matrices form an interesting branch of linear, or multilinear algebra (e.g., [1, 2, 16, 17, 19]).

Motivated by a representation of $\mathcal{H}$, introduced in [24], we studied eigenvalues of quaternions by regarding each quaternion as a (2×2)-matrix over $\mathbb{C}$ and classified $\mathcal{H}$ by such eigenvalues in [6]. And such classifications are characterized algebraically there. By applying the main results of [6], we here consider how to solve certain equations on $\mathcal{H}$.

8.1.2 Overview

In Sects. 8.2, 8.3 and 8.4, an algebraic representation $\left(\mathbb{C}^2, \pi\right)$ of the quaternions $\mathcal{H}$ is introduced, and the spectral analysis of the realizations of quaternions is reconsidered (Also, see [6]), for the self-containedness of the paper.

In Sect. 8.5, by applying the main results of previous sections, we study *monomial equations on* $\mathcal{H}$. The solutions of those equations are characterized.

The solutions of certain *quadratic equations on* $\mathcal{H}$ are characterized in Sect. 8.6. In particular, we are interested in the cases where given quadratic functions have $\mathcal{R}$-coefficients.

Finally, in Sect. 8.6, the solutions of linear equations on $\mathcal{H}$ are formulated in terms of the representation of $\mathcal{H}$.

8.2 Preliminaries

In this section, we study a representation of the quaternions $\mathcal{H}$. In particular, we view each quaternion $q \in \mathcal{H}$ as a (2×2)-matrix $[q] \in M_2(\mathbb{C})$ acting on the usual 2-dimensional space $\mathbb{C}^2$ (e.g., see [6, 24]).

8.2.1 The Quaternions $\mathcal{H}$

Let a and b be complex numbers,

$$a = x + yi \text{ and } b = u + vi \in \mathbb{C},$$

where $x, y, u, v \in \mathbb{R}$, and $i = \sqrt{-1}$ in $\mathbb{C}$.

From the complex numbers $a, b \in \mathbb{C}$, the corresponding quaternion $q \in \mathcal{H}$ is canonically constructed by

$$\begin{aligned} q &= a + bj = (x + yi) + (u + vi)j \\ &= x + yi + uj + vij \\ &= x + yi + uj + vk, \end{aligned}$$

in $\mathcal{H}$, where

$$i^2 = j^2 = k^2 = ijk = -1, \tag{8.1}$$

satisfying

$$ij = k \text{ in } \mathcal{H}.$$

This set $\mathcal{H}$ has a well-defined *addition* $(+)$, and *multiplication* $(\cdot)$; for any

$$q_l = a_l + b_l j \ \in \mathcal{H}, \text{with } a_l, b_l \in \mathbb{C},$$

(in the sense of (8.1)) for $l = 1, 2$, one has

$$q_1 + q_2 = (a_1 + a_2) + (b_1 + b_2)j,$$

and

$$q_1 q_2 = (a_1 a_2 - b_1 \overline{b_2}) + (a_1 b_2 + \overline{a_2} b_1)j \tag{8.2}$$

in $\mathcal{H}$, where $\overline{z}$ are the *conjugates of* $z \in \mathbb{C}$.

Remark that, different from $\mathbb{C}$, the multiplication $(\cdot)$ is noncommutative on $\mathcal{H}$, since

$$q_1 q_2 = (a_1 a_2 - b_1 \overline{b_2}) + (a_1 b_2 + \overline{a_2} b_1)j$$

and

$$q_2 q_1 = (a_2 a_1 - b_2 \overline{b_1}) + (a_2 b_1 + \overline{a_1} b_2) j,$$

by (8.2), and hence,

$$q_1 q_2 \neq q_2 q_1 \tag{8.3}$$

in $\mathcal{H}$, in general.

Remark 1 The quaternions $\mathcal{H}$ is a "noncommutative" field algebraically by (8.3). A noncommutative field $(F, +, \cdot)$ is an algebraic structure satisfying that: the algebraic pair $(F, +)$ forms an abelian group; and the pair $(F^{\times}, \cdot)$ forms a "non-abelian," or "noncommutative" group, where $F^{\times} = F \setminus \{0_F\}$, where 0_F is the $(+)$-identity of $(F, +)$; and $(+)$ and $(\cdot)$ are left-and-right distributed.

If $q \in \mathcal{H}$ is a quaternion (8.1), then one can define the *quaternion-conjugate* $\overline{q} \in \mathcal{H}$ by

$$\begin{aligned} \overline{q} &= \overline{a} - bj = (x - yi) - (u + vi) j \\ &= x - yi - ui - vi. \end{aligned} \tag{8.4}$$

So, one has that

$$\begin{aligned} q\overline{q} &= \left(|a|^2 + |b|^2\right) + \left(-ab + \overline{\overline{a}}b\right) j \\ &= |a|^2 + |b|^2, \\ &= |x|^2 + |y|^2 + \left|u^2\right| + |v|^2, \end{aligned}$$

and

$$\begin{aligned} \overline{q}q &= \left(|a|^2 + |b|^2\right) + \left(\overline{\overline{a}}b - ab\right) j \\ &= |a|^2 + |b|^2 \\ &= |x|^2 + |y|^2 + |u|^2 + |v|^2, \end{aligned} \tag{8.5}$$

by (8.2) and (8.4), where $|.|$ in the second equalities of (8.5) is the *modulus on* $\mathcal{C}$, and $|.|$ in the third equalities of (8.5) is the absolute value on $\mathcal{R}$. i.e.,

$$\overline{q}q = |a|^2 + |b|^2 = q\overline{q} \tag{8.6}$$

in $\mathcal{H}$, $\forall q \in \mathcal{H}$.

By (8.6), one can define the *quaternion-modulus* $\|.\|$ *on* $\mathcal{H}$ by

$$\|q\| = \sqrt{q\overline{q}}, \tag{8.7}$$

for all $q \in \mathcal{H}$.

The quaternion-modulus $\|.\|$ of (8.7) is a well-defined norm on$\mathcal{H}$ by (8.6).

If a quaternion q of (8.1) is nonzero, i.e., $q \neq 0$ in $\mathcal{H}$, then the quaternion-reciprocal q^{-1} of q

$$q^{-1} = \frac{1}{q} = \frac{\overline{q}}{q\overline{q}} = \frac{\overline{a}-bj}{|a|^2+|b|^2}$$
$$= \left(\frac{\overline{a}}{|a|^2+|b|^2}\right) + \left(\frac{-b}{|a|^2+|b|^2}\right) j \tag{8.8}$$

is well-defined in $\mathcal{H}$, by (8.6) and (8.7).

By (8.2) and (8.8), one can define the quaternion fractions

$$\frac{q_1}{q_2} = q_1 \left(\frac{1}{q_2}\right) = q_1 q_2^{-1} \text{in } \mathcal{H},$$

whenever $q_2 \neq 0$ in $\mathcal{H}$.

8.2.2 A Representation $(\mathbb{C}^2, \pi)$ of $\mathcal{H}$

In this section, we consider a representation of the quaternions $\mathcal{H}$, introduced in [24], realized on the 2-dimensional space $\mathbb{C}^2$ over $\mathbb{C}$. As in (8.1), let's view each quaternion $q \in \mathcal{H}$ as

$$q = a + bj \text{ in } \mathcal{H} \text{ with } a, b \in \mathbb{C}.$$

Define a morphism

$$\pi : \mathcal{H} \to M_2(\mathbb{C})$$

by

$$\pi(q) = \pi\ (a + bj =) \begin{pmatrix} a & -b \\ \overline{b} & \overline{a} \end{pmatrix}, \tag{8.9}$$

where $\overline{a} = x - yi$ and $\overline{b} = u - vi$ are the complex-conjugates of a and b in $\mathbb{C}$, respectively, and $M_2(\mathbb{C})$ is the *matricial ring* of all (2×2)-matrices over $\mathbb{C}$.

This morphism π of (8.9) satisfies that

$$\pi\ (q_1 + q_2) = \pi(q_1) + \pi(q_2),$$

and

$$\pi\ (q_1 q_2) = \pi(q_1)\pi(q_2), \tag{8.10}$$

for all $q_1, q_2 \in \mathcal{H}$, by (8.2). This relation (8.10) shows that the morphism π of (8.9) is a well-defined ring-homomorphism. i.e., a noncommutative field $\mathcal{H}$ is ring-homomorphic to the matricial ring $M_2(\mathbb{C})$.

It is easy to check that the quaternion-conjugate $\overline{q}$ of $q \in \mathcal{H}$ satisfies that

$$\pi\,(\overline{q}) = \pi\,(\overline{a} - bj) = \begin{pmatrix} \overline{a} & b \\ -\overline{b} & a \end{pmatrix}$$
$$= \begin{pmatrix} a & -b \\ \overline{b} & \overline{a} \end{pmatrix}^{*} = \pi(q)^{*}, \tag{8.11}$$

in $M_2(\mathbb{C})$, by (8.4), where A^* are the *adjoints* (or the conjugate-transposes) of $A \in M_2(\mathbb{C})$. It shows that the morphism π of (8.9) covers the quaternion-conjugacy (8.4) by (8.11). Furthermore, one can get that

$$\det\,(\pi(q)) = \det \begin{pmatrix} a & -b \\ \overline{b} & \overline{a} \end{pmatrix} = |a|^2 + |b|^2\,,$$

and hence, one can have

$$\|q\| = \sqrt{\det\,(\pi(q))}, \tag{8.12}$$

for all $q \in \mathcal{H}$, by (8.6) and (8.7). The relation (8.12) says that the quaternion-modulus $\|.\|$ of (8.7) is characterized by the determinant on $M_2(\mathbb{C})$.

Note that, by the very definition (8.9), π is injective. Indeed,

$$\pi(q_1) = \pi(q_2) \Longleftrightarrow \pi(q_1) - \pi(q_2) = O_2,$$
$$\Longleftrightarrow$$
$$\pi(q_1 - q_2) = O_2, \text{ by (8.10)},$$
$$\Longleftrightarrow$$
$$q_1 - q_2 = 0, \tag{8.13}$$

where O_2 is the zero matrix of $M_2(\mathbb{C})$.

Proposition 1 *The pair* $\left(\mathbb{C}^2, \pi\right)$ *is a representation of* $\mathcal{H}$.

Proof It is sufficient to prove that the morphism π of (8.9) is a well-defined operation-preserving homomorphism from $\mathcal{H}$ into $M_2(\mathbb{C})$. But it is shown by (8.10) and (8.11). Moreover, by (8.12), it preserves the norm property of $\mathcal{H}$ with respect to the determinant on $M_2(\mathbb{C})$, too. In other words, π is a bounded (or continuous) action of $\mathcal{H}$ acting on $\mathbb{C}^2$. Therefore, the morphism π is a well-defined noncommutative-field-action of $\mathcal{H}$ acting on $\mathbb{C}^2$, equivalently, the pair $(\mathbb{C}^2, \pi)$ is a representation of $\mathcal{H}$.

From below, for convenience, we denote the realization $\pi(q)$ of a quaternion $q \in \mathcal{H}$ by $[q]$. Let's define a subset $\mathcal{H}_2$ of $M_2(\mathbb{C})$ by the set of all realizations of $\mathcal{H}$. i.e.,

$$\mathcal{H}_2 \overset{def}{=} \{[q] \in M_2\,(\mathbb{C}) : q \in \mathcal{H}\} = \pi\,(\mathcal{H}) \tag{8.14}$$

By (8.9), (8.10) and (8.11), one can realize that π is a ring-homomorphism from $\mathcal{H}$ to $\mathcal{H}_2$. Moreover, they are isomorphic as noncommutative fields by (8.13).

Theorem 1 *The quaternions* $\mathcal{H}$ *and the subset* $\mathcal{H}_2$ *of (8.14) are isomorphic noncommutative fields. i.e.,*

$$\mathcal{H} \overset{NF}{=} \mathcal{H}_2. \tag{8.15}$$

where "$\overset{NF}{=}$*" means "being noncommutative-field-isomorphic."*

Proof Take the action π of (8.9) acting on $\mathbb{C}^2$. By the injectivity (8.13), two sets $\mathcal{H}$ and $\mathcal{H}_2$ are bijective (or equipotent), by (8.14). i.e., $\pi : \mathcal{H} \to \mathcal{H}_2$ is a bijection. Moreover, π is a well-defined ring-homomorphism from $\mathcal{H}$ onto $\mathcal{H}_2$, by (8.10). i.e., π is a noncommutative-field-isomorphism from $\mathcal{H}$ onto $\mathcal{H}_2$.

The above isomorphism theorem (8.15) illustrates that the quaternions $\mathcal{H}$ is regarded as a noncommutative field $\mathcal{H}_2$, embedded in $M_2(\mathbb{C})$. Remark that if $[q] \in \mathcal{H}_2$, for $q \in \mathcal{H}$, and if $q \neq 0$, then

$$\det([q]) = |a|^2 + |b|^2 \neq 0,$$

if and only if $[q]$ is invertible in $M_2(\mathbb{C})$, implying that all nonzero realizations of $\mathcal{H}_2$ are automatically invertible. It is easily checked that

$$[q]^{-1} = [q^{-1}], \tag{8.16}$$

in $\mathcal{H}_2$, by (8.8).

8.3 Quaternion-Spectral Forms

Let $\mathcal{H}_2$ be the noncommutative field (8.14), isomorphic to the quaternions $\mathcal{H}$, in the matricial ring $M_2(\mathbb{C})$. In this section, we regard each quaternion $q \in \mathcal{H}$ as a (2×2)-matrix $[q] \in \mathcal{H}_2$ by (8.15), and study spectral analysis on $\mathcal{H}_2$ (and hence, that on $\mathcal{H}$).

8.3.1 Quaternion-Spectral Forms of $\mathcal{H}$

In this section, by regarding the realizations $[q] \in \mathcal{H}_2$ of quaternions $q \in \mathcal{H}$ as (2×2)-matrices in $M_2(\mathbb{C})$, the spectra *spec* $([q])$ of $[q]$ are studied canonically. In the long run, the Jordan canonical forms of $[q]$ are formulated.

First, suppose that

$$a = x + 0i, \text{ and } b = 0 \text{ in } \mathbb{C},$$

i.e., the corresponding quaternion $q = a + bj \in \mathcal{H}$ is identified with the real number $q = x$ in $\mathcal{R}$. Then, by the noncommutative-field-action π of (8.9), one has that

$$\pi(q) = [q] = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = xI_2 \tag{8.17}$$

where I_2 is the identity matrix of $M_2(\mathbb{C})$.

Lemma 1 *Let $a = x + 0i$, and $b = 0$ in $\mathbb{C}$, and $q = a + bj = x$ in $\mathcal{H}$. Then the spectrum spec $([q])$ of the realization $[q] \in \mathcal{H}_2$ is the set*

$$spec\ ([q]) = \{x\} = \{q\} \tag{8.18}$$

Proof By (8.17), the realization $[q] \in \mathcal{H}_2$ of $q \in \mathcal{H}$ is the diagonal matrix xI_2 in $M_2(\mathbb{C})$. Therefore,

$$spec\ ([q]) = \{x\} \text{ in } \mathbb{C}.$$

So, the relation (8.18) holds.

Now, assume that

$$a = x + yi \text{ with } y \neq 0, \text{ and } b = 0 \text{ in } \mathbb{C},$$

and $q = a + bj = a + 0j \in \mathcal{H}$ is the corresponding quaternion. Then the realization $[q] \in \mathcal{H}_2$ is determined to be

$$[q] = \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} = \begin{pmatrix} x + yi & 0 \\ 0 & x - yi \end{pmatrix}, \tag{8.19}$$

in $\mathcal{H}_2$.

Lemma 2 *Let $a = x + yi$, with $y \neq 0$, and $b = 0$ in $\mathbb{C}$ and $q = a + bj \in \mathcal{H}$. If $[q] \in \mathcal{H}_2$ is the realization of q, then its spectrum satisfies*

$$spec\ ([q]) = \{a, \overline{a}\} = \{x + yi, x - yi\} \tag{8.20}$$

Proof By (8.19), the realization $[q]$ is a diagonal matrix in $M_2(\mathbb{C})$, and hence, a and $\overline{a}$ are the distinct eigenvalues of $[q]$ in $\mathbb{C}$. Thus, the spectrum (8.20) is obtained.

Now, suppose $q = a + bj \in \mathcal{H}$ is a quaternion (8.17), with

$$b \neq 0 \Longleftrightarrow b \in \mathbb{C}^{\times} = \mathbb{C} \setminus \{0\}, \tag{8.21}$$

and let

$$[q] = \begin{pmatrix} a & -b \\ b & \overline{a} \end{pmatrix} = \begin{pmatrix} x + yi & -u - vi \\ u - vi & x - yi \end{pmatrix} \in \mathcal{H}_2$$

be the realization of q. Observe that: for $z \in \mathbb{C}$,

$$\det([q] - zI_2) = \det\left(\begin{pmatrix} a & -b \\ \overline{b} & \overline{a} \end{pmatrix} - \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}\right)$$
$$= \det\left(\begin{pmatrix} a-z & -b \\ \overline{b} & \overline{a}-z \end{pmatrix}\right) = (a-z)(\overline{a}-z) - \left(-b\overline{b}\right)$$
$$= z^2 - (a+\overline{a})\, z + \left(|a|^2 + |b|^2\right), \text{ i.e.,}$$

$$\det([q] - zI_2) = z^2 - 2xz + \left(x^2 + y^2 + u^2 + v^2\right). \tag{8.22}$$

Consider the equation,

$$\det([q] - zI_2) = 0,$$

$\Longleftrightarrow$

$$z^2 - 2xz + \left(x^2 + y^2 + u^2 + v^2\right) = 0, \tag{8.23}$$

by (8.22). Then (8.23) has its solutions

$$z = \frac{2x \pm \sqrt{(-2x)^2 - 4\left(x^2 + y^2 + u^2 + v^2\right)}}{2},$$

$\Longleftrightarrow$

$$z = \frac{2x \pm \sqrt{4x^2 - 4x^2 - 4y^2 - 4u^2 - 4v^2}}{2},$$

$\Longleftrightarrow$

$$z = x \pm i\sqrt{y^2 + u^2 + v^2}. \tag{8.24}$$

Lemma 3 *Let $q = a + bj \in \mathcal{H}$ be a quaternion* (8.17) *with $b \in \mathbb{C}^{\times}$, realized to be $[q] \in \mathcal{H}_2$. Then the spectrum spec $([q])$ of $[q]$ is the subset,*

$$spec\,([q]) = \{\lambda, \overline{\lambda}\} \text{ of } \mathbb{C},$$

where

$$\lambda = x + i\sqrt{y^2 + u^2 + v^2}. \tag{8.25}$$

Proof Under hypothesis, the characteristic polynomial of the realization $[q]$ is identical to the quadratic function (8.22), providing (8.23). Thus one can get the eigenvalues

$$\lambda = x + i\sqrt{y^2 + u^2 + v^2},$$

and

$$\overline{\lambda} = x - i\sqrt{y^2 + u^2 + v^2},$$

in $\mathbb{C}$, by (8.24). So, the set-equality (8.25) holds.

By (8.25), we obtain the following result.

Corollary 1 *Let* $[q] \in \mathcal{H}_2$ *be the realization of a quaternion* $q \in \mathbb{H}$ *of (8.17) satisfying the condition (8.21). Then it has its Jordan canonical form,*

$$J([q]) = \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} \text{“in } \mathcal{H}_2\text{,”}$$

with

$$\lambda = x + i\sqrt{y^2 + u^2 + v^2}. \tag{8.26}$$

Proof By the condition (8.21) that $b \in \mathbb{C}^{\times}$ the quantity

$$\lambda = x + i\sqrt{y^2 + u^2 + v^2} \in \mathbb{C}$$

satisfies

$$y^2 + u^2 + v^2 \neq 0 \text{ in } \mathbb{R} \Longleftrightarrow \operatorname{Im}(\lambda) \neq 0 \text{ in } \mathbb{C},$$

where $Im(t)$ are the imaginary parts of $t \in \mathbb{C}$.

So, $\lambda \neq \overline{\lambda}$ in $\mathbb{C}$. Thus, the Jordan canonical form

$$J([q]) = \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} = \begin{pmatrix} x + i\sqrt{y^2 + u^2 + v^2} & 0 \\ 0 & x - i\sqrt{y^2 + u^2 + v^2} \end{pmatrix}$$

of $[q]$ is obtained “in $M_2(\mathbb{C})$,” by (8.25).

It implies that

$$J([q]) = \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} \text{“in } \mathcal{H}_2\text{,”}$$

by (8.14).

By the above results, the following theorem is obtained.

Theorem 2 *Let* $q = x + yi + uj + vk \in \mathbb{H}$ *be a quaternion with* $x, y, u, v \in \mathbb{R}$, *with its realization* $[q] \in \mathcal{H}_2$. *If either* $u \neq 0$, *or* $v \neq 0$ *in* $\mathbb{R}$, *then the Jordan canonical form* $J([q])$ *is*

$$J([q]) = \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} \text{“in } \mathcal{H}_2\text{,”}$$

with

$$\lambda = x + i\sqrt{y^2 + u^2 + v^2}. \tag{8.27}$$

Proof The formula (8.27) for $J([q])$ is obtained in $M_2(\mathbb{C})$ by (8.25) and (8.26). And this resulted matrix $J([q])$ is contained in $\mathcal{H}_2$ by (8.14).

Motivated by (8.27), one can define Jordan-canonical-form-like matrices of $\mathcal{H}_2$, by (8.18), (8.20) and (8.25),

Definition 1 Let $q = x + yi + uj + vk \in \mathbb{H}$ be a quaternion with $x, y, u, v \in \mathbb{R}$, realized to be $[q] \in \mathcal{H}_2$. Then the quaternion-spectral form (in short, the q-spectral form) of q is defined to be a matrix $\mathbf{q}$ of $\mathcal{H}_2$,

$$\mathbf{q} \overset{def}{=} \begin{cases} [q] & \text{if } u = 0 = v \text{ in } \mathbb{R} \\ \\ J([q]) & \text{if either } u \neq 0, \text{ or } v \neq 0 \text{ in } \mathbb{R}, \end{cases} \tag{8.28}$$

where $J([q])$ is the Jordan canonical form (8.27).

For example, if $-3,\ 1 + i \in \mathbb{H}$, then the q-spectral forms of -3 and $1 + i$ are

$$\mathbf{-3} = [-3] = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix},$$

respectively

$$\mathbf{1 + i} = [1 + i] = \begin{pmatrix} 1 + i & 0 \\ 0 & 1 - i \end{pmatrix},$$

in $\mathcal{H}_2$, by (8.20) and (8.28); while, if $1 - 2j \in \mathbb{H}$,

$$\mathbf{1 - 2j} = \begin{pmatrix} 1 + 2i & 0 \\ 0 & 1 - 2i \end{pmatrix},$$

in $\mathcal{H}_2$, by (8.27) and (8.28).

8.3.2 Similarity on q-Spectral Forms in $\mathcal{H}_2$

In this section, we consider the similarity on q-spectral forms "in $\mathcal{H}_2$." Throughout this section, let

$$a = x + yi, b = u + vi \in \mathbb{C}, \text{ with } x, y, u, v \in \mathbb{R},$$

and

$$q = a + bj = x + yi + uj + vk. \tag{8.29}$$

In Sect. 8.3.1, we showed that every quaternion $q \in \mathbb{H}$ of (8.29), realized to be $[q] \in \mathcal{H}_2$, has its q-spectral form,

$$\mathbf{q} = \begin{cases} [q] & \text{if } b = 0 \\ J([q]) & \text{if } b \neq 0, \end{cases} \tag{8.30}$$

of (8.28) in $\mathcal{H}_2$, where $J([q])$ is the Jordan canonical form (8.27).

Suppose $b \in \mathbb{C}^\times$ in (8.29). For $t \in \mathbb{C}^\times$, define a (2×2)-matrix $Q_t(q)$ by

$$Q_t(q) = \begin{pmatrix} t & -\overline{t\left(\frac{a-\lambda}{b}\right)} \\ t\left(\frac{a-\lambda}{b}\right) & \overline{t} \end{pmatrix}, \tag{8.31}$$

in $M_2(\mathbb{C})$.

By the assumption that $t, b \in \mathbb{C}^\times$, the nonzero matrix $Q_t(q)$ of (8.31) is well-defined in $M_2(\mathbb{C})$. Note that this matrix $Q_t(q)$ is invertible, since

$$\det(Q_t(q)) = |t|^2 \left(1 + \left|\frac{a-\lambda}{b}\right|^2\right) \neq 0 \tag{8.32}$$

by the condition that $t, b \in \mathbb{C}^\times$. Observe now that

$$\begin{aligned} &\begin{pmatrix} a & -b \\ \overline{b} & \overline{a} \end{pmatrix} \begin{pmatrix} t & -\overline{\left(\frac{a-\lambda}{b}\right)t} \\ \left(\frac{a-\lambda}{b}\right)t & \overline{t} \end{pmatrix} \\ &= \begin{pmatrix} at - (a-\lambda)t & -a\left(\frac{\overline{a}-\overline{\lambda}}{\overline{b}}\right)\overline{t} - b\overline{t} \\ \overline{a\left(\frac{\overline{a}-\overline{\lambda}}{\overline{b}}\right)\overline{t} + b\overline{t}} & \overline{at - (a-\lambda)t} \end{pmatrix} \\ &= \begin{pmatrix} \lambda t & \left(\frac{\overline{\lambda}a - |a|^2}{\overline{b}}\right)\overline{t} - b\overline{t} \\ \overline{\left(\frac{\overline{\lambda}a - |a|^2}{\overline{b}}\right)\overline{t} - b\overline{t}} & \overline{\lambda t} \end{pmatrix}, \end{aligned} \tag{8.33}$$

and

$$\begin{aligned} &\begin{pmatrix} t & -\overline{\left(\frac{a-\lambda}{b}\right)t} \\ \left(\frac{a-\lambda}{b}\right)t & \overline{t} \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} \\ &= \begin{pmatrix} \lambda t & -\overline{\left(\frac{a\lambda - \lambda^2}{b}\right)t} \\ \left(\frac{a\lambda-\lambda^2}{b}\right)t & \overline{\lambda t} \end{pmatrix}. \end{aligned} \tag{8.34}$$

In the computations (8.33) and (8.34), let's compare their (2, 1)-entries:

$$\left(\frac{\overline{\lambda}a-|a|^2}{\overline{b}}\right)\overline{t} - b\overline{t} = \overline{t}\left(\frac{(x+yi)\left(x-i\sqrt{y^2+u^2+v^2}\right)-(x^2+y^2)}{u-vi} - (u+vi)\right)$$
$$= \overline{t}\left(\frac{-ix\sqrt{y^2+u^2+v^2}+xyi+y\sqrt{y^2+u^2+v^2}-y^2}{u-vi} - \frac{u^2+v^2}{u-vi}\right)$$
$$= \overline{t}\left(\frac{\left(y\sqrt{y^2+u^2+v^2}-y^2-u^2-v^2\right)+i\left(xy-x\sqrt{y^2+u^2+v^2}+xy\right)}{u-vi}\right), \tag{8.35}$$

respectively

$$-\overline{\left(\frac{a\lambda-\lambda^2}{b}\right)t} = \overline{t}\,\overline{\left(\frac{-a\lambda+\lambda^2}{b}\right)}$$
$$= \overline{t}\left(\frac{\left(x-i\sqrt{y^2+u^2+v^2}\right)^2-(x-yi)\left(x-i\sqrt{y^2+u^2+v^2}\right)}{u-vi}\right)$$
$$= \overline{t}\left(\frac{x^2-2xi\sqrt{y^2+u^2+v^2}-(y^2+u^2+v^2)-\left(x^2-ix\sqrt{y^2+u^2+v^2}-xyi-y\sqrt{y^2+u^2+v^2}\right)}{u-vi}\right)$$
$$= \overline{t}\left(\frac{\left(y\sqrt{y^2+u^2+v^2}-y^2-u^2-v^2\right)+i\left(xy-x\sqrt{y^2+u^2+v^2}+xy\right)}{u-vi}\right). \tag{8.36}$$

By (8.35) and (8.36), the (2, 1)-entry of $[q]Q_t(q)$, and that of $Q_t(q)\mathbf{q}$ are identical, and hence,

$$[q]Q_t(q) = Q_t(q)\mathbf{q}, \tag{8.37}$$

in $M_2(\mathbb{C})$ by (8.33) and (8.34). Note that the (2×2)-matrix $Q_t(q)$ of (8.31) is contained in the noncommutative field $\mathcal{H}_2$ by (8.14) (which implies the invertibility (8.32) in $M_2(\mathbb{C})$ automatically), whenever $t, b \in \mathbb{C}^\times$.

Theorem 3 *Let $q = a + bj \in \mathbb{H}$ be a quaternion* (8.29) *with its realization $[q] \in \mathcal{H}_2$, and let $\mathbf{q} \in \mathcal{H}_2$ be the q-spectral form of q. If $b \neq 0$ in $\mathbb{C}$, then*

$$\mathbf{q} = Q_t(q)^{-1}[q]Q_t(q) \Longleftrightarrow [q] = Q_t(q)\mathbf{q}Q_t(q)^{-1}$$

in $\mathcal{H}_2$, where

$$Q_t(q) = \begin{pmatrix} t & -\overline{\left(\frac{a-\lambda}{b}\right)t} \\ \left(\frac{a-\lambda}{b}\right)t & \overline{t} \end{pmatrix} \in \mathcal{H}_2, \tag{8.38}$$

for all $t \in \mathbb{C}^\times$. Meanwhile, if $b = 0$ in $\mathbb{C}$, then

$$\mathbf{q} = [w]^{-1}[q][w], \tag{8.39}$$

where

$$w = w + 0j \in \mathcal{H}, \tag{8.40}$$

Proof First, suppose that $b = 0$ in $\mathbb{C}$, and hence, $q = a + 0j$ in $\mathcal{H}$. Then, by (8.30), the quaternion q has its q-spectral form,

$$\mathbf{q} = \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} = [q] \text{ in } \mathcal{H}_2.$$

Suppose $w \in \mathbb{C}^{\times}$ and $w = w + 0j \in \mathcal{H}$, realized to be $[w] \in \mathcal{H}_2$. Then

$$\begin{aligned} \mathbf{q} = [q] &= \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} = \begin{pmatrix} \frac{wa}{w} & 0 \\ 0 & \overline{\left(\frac{wa}{w}\right)} \end{pmatrix} \\ &= \begin{pmatrix} w & 0 \\ 0 & \overline{w} \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} \begin{pmatrix} w^{-1} & 0 \\ 0 & \overline{w^{-1}} \end{pmatrix} \\ &= [w][q][w]^{-1} = [w]\, \mathbf{q}\, [w]^{-1}, \end{aligned}$$

in $\mathcal{H}_2$. Therefore, the relation (8.39) holds true, whenever $w \in \mathcal{H}$ are in the sense of (8.39)′.

Assume now that $b \neq 0$ in $\mathbb{C}$. Then, for any $t \in \mathbb{C}^{\times}$, the corresponding matrices $Q_t(q)$ of (8.31) satisfy

$$Q_t(q)\mathbf{q} = [q]Q_t(q),$$

by (8.37). Thus, by the invertibility (8.32) of $Q_t(q)$,

$$Q_t(q)^{-1}\left(Q_t(q)\mathbf{q}\right) = Q_t(q)^{-1}[q]Q_t(q) \text{ in } \mathcal{H}_2,$$

if and only if

$$\mathbf{q} = Q_t(q)^{-1}[q]Q_t(q). \tag{8.41}$$

Therefore, the relation (8.38) holds by (8.41), whenever $b \neq 0$ in $\mathbb{C}$.

Remark 2 Let $z, a \in \mathbb{C}$ and $b \in \mathbb{C}^{\times}$, and let $q = a + bj \in \mathcal{H}$. Observe that if we regard $z \in \mathbb{C}$ as a quaternion $z + 0j \in \mathcal{H}$, then

$$[z][q] = \begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix} \begin{pmatrix} a & -b \\ \overline{b} & \overline{a} \end{pmatrix} = \begin{pmatrix} az & -bz \\ \overline{bz} & \overline{az} \end{pmatrix},$$

and

$$[q][z] = \begin{pmatrix} a & -b \\ \overline{b} & \overline{a} \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix} = \begin{pmatrix} az & -b\overline{z} \\ \overline{bz} & \overline{az} \end{pmatrix},$$

in $\mathcal{H}_2$. i.e.,

$$[z][q] \neq [q][z] \Longleftrightarrow [z] \neq [q][z][q]^{-1} \text{ in } \mathcal{H}_2,$$

in general. However, if $z \in \mathcal{R}^{\times} = \mathcal{R} \setminus \{0\}$ in $\mathcal{H}$, then

$$[z] = [q][z][q]^{-1} \text{ in } \mathcal{H}_2,$$

because all diagonal matrices with "real" entries are commuting with all matrices of $M_2(\mathcal{C})$. This consideration explains not only that the relations (8.38) and (8.39) are meaningful.

The above theorem shows that, for a quaternion $q \in \mathcal{H}$ with its q-spectral form $\mathbf{q} \in \mathcal{H}_2$, there exists at least one nonzero matrix $A \in \mathcal{H}_2$, such that

$$\mathbf{q} = A^{-1}[q]A, \, or, \, [q] = A\mathbf{q}A^{-1}, \tag{8.42}$$

in $\mathcal{H}_2$.

Corollary 2 *Let $q = a + bj \in \mathcal{H}$ be a quaternion* (8.29) *with $b \neq 0$ in $\mathcal{C}$, and let*

$$\lambda = x + i\sqrt{y^2 + u^2 + v^2} \in \mathcal{C} \text{ in } \mathcal{H}.$$

Then there exist

$$y_t = t + \left(-\overline{t\left(\frac{a-\lambda}{b}\right)}\right) j \in \mathcal{H},$$

for any $t \in \mathcal{C}^{\times}$, such that

$$q = y_t \lambda y_t^{-1}. \tag{8.43}$$

Proof Recall first that the noncommutative field $\mathcal{H}_2$ and the quaternions $\mathcal{H}$ are isomorphic by (8.15). So, a matrix $Q_t(q) \in \mathcal{H}_2$ of (8.32) is assigned to be a quaternion

$$y_t = t + \left(-\overline{t\left(\frac{a-\lambda}{b}\right)}\right) j \in \mathcal{H},$$

equivalently,

$$[y_t] = Q_t(q) \text{ in } \mathcal{H}_2, \text{ for all } t \in \mathcal{C}^{\times}.$$

In a similar manner, the q-spectral form $\mathbf{q}$ is assigned to be a quaternion

$$\lambda = \lambda + 0j \in \mathcal{H} \Longleftrightarrow [\lambda] = \mathbf{q} \text{ in } \mathcal{H}_2.$$

So, the formula (8.38) satisfies that

$$[q] = [y_t][\lambda][y_t]^{-1} \text{ in } \mathcal{H}_2, \text{ by (8.16)},$$

if and only if

$$[q] = [y_t \lambda y_t^{-1}] \text{ in } \mathcal{H}_2 \text{ by (8.10)},$$

if and only if

$$q = y_t \lambda y_t^{-1} \text{ in } \mathcal{H} \text{ by (8.15).}$$

Therefore, the relation (8.43) holds true.

The above corollary shows that the relation (8.38) on the noncommutative field $\mathcal{H}_2$ is equivalent to the relation (8.43) on the quaternions $\mathcal{H}$. It also shows the relation between a quaternion $q \in \mathcal{H}$ of (8.29) and an eigenvalue $\lambda \in \mathbb{C}$ of the realization $[q]$ of q. i.e., for any $q \in \mathcal{H}$ with its realization $[q] \in \mathcal{H}_2$, there exists at least one nonzero $q_0 \in \mathcal{H}$, such that

$$q = q_0 \, \lambda \, q_0^{-1} \text{ in } \mathcal{H},$$

where $spec\,([q]) = \{\lambda, \overline{\lambda}\}$ in $\mathbb{C}$, as in (8.42).

Definition 2 Let $q \in \mathcal{H}$ be a quaternion with its realization $[q] \in \mathcal{H}_2$, and let $\mathbf{q} = [\lambda] \in \mathcal{H}_2$ be the q-spectral form in $\mathcal{H}_2$. From below, the eigenvalue $\lambda \in \mathbb{C}$ of $[q]$ is called the quaternion-spectral value (in short, q-spectral value) of q.

The following example summarizes the above results.

Example 1 Let $q_1 = 2x + i - j + 3k \in \mathcal{H}$. Then the realization $[q_1] \in \mathcal{H}_2$ of q_1 has its spectrum

$$spec\,([q_1]) = \{\lambda, \overline{\lambda}\},$$

where

$$\lambda = 2 + i\sqrt{1^2 + (-1)^2 + 3^2} = 2 + \sqrt{11}i \in \mathbb{C}$$

is the q-spectral value of q_1, providing the q-spectral form,

$$\mathbf{q}_1 = \begin{pmatrix} 2 + \sqrt{11}i & 0 \\ 0 & 2 - \sqrt{11}i \end{pmatrix} \in \mathcal{H}_2.$$

So, if we take $1 \in \mathbb{C}^{\times}$ and the corresponding matrix $Q_1(q)$,

$$Q_1(q_1) = \begin{pmatrix} 1 & -\overline{\left(\frac{(2+i)-(2+\sqrt{11}\,i)}{-1+3i}\right)} \\ \left(\frac{(2+i)-(2+\sqrt{11}\,i)}{-1+3i}\right) & 1 \end{pmatrix} \in \mathcal{H}_2,$$

then

$$[q_1] = Q_1(q_1)\mathbf{q}_1 Q_1(q_1)^{-1} \text{ in } \mathcal{H}_2,$$

implying that

$$q_1 = \left(1 + \overline{\left(\frac{(1-\sqrt{11})i}{-1+3i}\right)} j\right) \left(2 + i\sqrt{11}\right) \left(1 + \overline{\left(\frac{(1-\sqrt{11})i}{-1+3i}\right)} j\right)^{-1},$$

in $\mathcal{H}$.

Meanwhile, if $q_2 = 1 - 3i + 0j + 0k \in \mathcal{H}$, then

$$spec\left([q_2]\right) = \{1 - 3i, 1 + 3i\},$$

satisfying

$$[q_2] = \begin{pmatrix} 1-3i & 0 \\ 0 & 1+3i \end{pmatrix} = \mathbf{q}_2,$$

where $\mathbf{q}_2$ is the q-spectral form of q_2, in $\mathcal{H}_2$.

So, for any nonzero diagonal matrix $D \in \mathcal{H}_2$

$$\begin{aligned} [q_2] &= [q_2]\left(DD^{-1}\right) = \left([q_2]D\right) D^{-1} \\ &= \left(D[q_2]\right) D^{-1} = D[q_2]D^{-1} = D\mathbf{q}_2 D^{-1}, \end{aligned}$$

implying that

$$q_2 = z(1 - 3i)z^{-1} \text{ in } \mathcal{H},$$

for all nonzero complex numbers $z \in \mathbb{C}^{\times} \subset \mathcal{H}$.

8.3.3 Quaternion-Spectral Equivalence

In this section, based on the main results of Sects. 8.3.1 and 8.3.2, we study q-spectral values from the quaternions $\mathcal{H}$. As before, we let an arbitrary fixed quaternion $q \in \mathcal{H}$ be in the sense of (8.29).

Let

$$q_1 = -2 + i - j + 3k \neq -2 - i + j - 3k = q_2,$$

in $\mathcal{H}$. Then these quaternions have their q-spectral values,

$$\lambda_1 = -2 + i\sqrt{1^2 + (-1)^2 + 3^2} = -2 + \sqrt{11}i,$$

and

$$\lambda_2 = -2 + i\sqrt{(-1)^2 + 1^2 + (-3)^2} = -2 + \sqrt{11}i,$$

in $\mathbb{C}$, respectively. So, these distinct quaternions q_1 and q_2 have the identical q-spectral values,

$$\lambda_1 = -2 + \sqrt{11}i = \lambda_2 \text{ in } \mathbb{C}.$$

Motivated by this, let's consider an equivalence relation on the quaternions $\mathcal{H}$. Define a relation $\mathcal{R}$ on $\mathcal{H}$ by

$$q_1 \mathcal{R} q_2 \overset{def}{\Longleftrightarrow} \lambda_1 = \lambda_2, \tag{8.44}$$

where λ_l are the q-spectral values of q_l, for $l = 1, 2$.

It is not difficult to check the relation $\mathcal{R}$ of (8.44) is indeed an equivalence relation on $\mathcal{H}$, because

$$q\mathcal{R}q, \text{ for all } q \in \mathcal{H};$$

and

$$q_1 \mathcal{R} q_2 \Longleftrightarrow \lambda_1 = \lambda_2 \Longleftrightarrow \lambda_2 = \lambda_1 \Longleftrightarrow q_2 \mathcal{R} q_1,$$

for all $q_1, q_2 \in \mathcal{H}$; and

$$q_1 \mathcal{R} q_2, \text{ and } q_2 \mathcal{R} q_3 \Longleftrightarrow \lambda_1 = \lambda_2 = \lambda_3$$

$\Longleftrightarrow$

$$\lambda_1 = \lambda_3 \Longleftrightarrow q_1 \mathcal{R} q_3,$$

for all $q_1, q_2, q_3 \in \mathcal{H}$, where λ_l are the q-spectral values of q_l, for all $l = 1, 2, 3$.

Definition 3 The equivalence relation $\mathcal{R}$ of (8.44) is called the quaternion-spectral equivalence relation (in short, the q-spectral relation) on $\mathcal{H}$. And two q-spectral equivalent quaternions q_1 and q_2 are said to be q-spectral related in $\mathcal{H}$.

Let $q_l = a_l + b_l j$ be q-spectral related quaternions in $\mathcal{H}$ with $b_l \neq 0$ in $\mathbb{C}$, and let $\lambda \in \mathbb{C}$ be the identical q-spectral value of q_l, for $l = 1, 2$. For any $t \in \mathbb{C}^{\times}$, there exists $y_{t,l} \in \mathcal{H}$ such that

$$q_l = y_{t,l} \lambda y_{t,l}^{-1}, \tag{8.45}$$

for all $l = 1, 2$, by (8.43). In particular,

$$y_{t,l} = t + \left(-\overline{t \left(\frac{a_l - \lambda}{b_l} \right)} \right) j \in \mathcal{H}, \tag{8.46}$$

for all $l = 1, 2$, by (8.38). So, one can have that

$$q_2 = y_{t,2} \lambda y_{t,2}^{-1} = y_{t,2} \left(y_{t,1}^{-1} y_{t,1} \right) \lambda \left(y_{t,1}^{-1} y_{t,1} \right) y_{t,2}^{-1}$$

by (8.45)

$$= \left(y_{t,2} y_{t,1}^{-1} \right) \left(y_{t,1} \lambda y_{t,1}^{-1} \right) \left(y_{t,1} y_{t,2}^{-1} \right)$$

by (8.15), i.e.,

$$q_2 = \left(y_{t,2} y_{t,1}^{-1} \right) q_1 \left(y_{t,2} y_{t,1}^{-1} \right)^{-1}, \tag{8.47}$$

in $\mathcal{H}$. Equivalent to (8.47), one obtains that

$$[q_2] = \left(Q_t(q_2)Q_t(q_1)^{-1}\right)[q_1]\left(Q_t(q_2)Q_t(q_1)^{-1}\right)^{-1},$$

in $\mathcal{H}_2$, where $Q_t(q_l)$ are in the sense of (8.31), for $l = 1, 2$.

Now, assume that either $b_1 = 0$, or $b_2 = 0$ in $\mathbb{C}$. Say $b_1 = 0$, and $b_2 \neq 0$ in $\mathbb{C}$. Then, since $q_1 \in \mathbb{C}$ in $\mathbb{H}$, the q-spectral value $\lambda = q_1$. So,

$$q_2 = y_{t,2}\lambda y_{t,2}^{-1} = y_{t,2}q_1 y_{t,2}.$$

If $b_1 = 0 = b_2$ in $\mathbb{C}$, then

$$q_1 = b_1 = \lambda = b_2 = q_2 \text{ in } \mathbb{C} \subset \mathbb{H},$$

under hypothesis, generalized to be

$$q_2 = yq_1y^{-1}, \text{ for all nonzero } y \in \mathbb{C}^{\times}.$$

Recall that two matrices A_1 and A_2 are *similar* in a matricial ring $M_n(\mathbb{C})$, for $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers, if there exists an invertible matrix $U \in M_n(\mathbb{C})$, such that

$$A_2 = UA_1U^{-1} \text{ in } M_n(\mathbb{C}).$$

Definition 4 Let $q_l \in \mathbb{H}$ be quaternions realized to be $[q_l] \in \mathcal{H}_2$, for $l = 1, 2$. The realizations $[q_1]$ and $[q_2]$ are said to be similar "in $\mathcal{H}_2$," if there exists a nonzero matrix $U \in \mathcal{H}_2$, such that

$$[q_2] = U[q_1]U^{-1} \tag{8.48}$$

"in $\mathcal{H}_2$." By abusing notation, two quaternions q_1 and q_2 are said to be similar in $\mathbb{H}$, if their realizations $[q_1]$ and $[q_2]$ are similar in the sense of (8.48).

Remark that, since $\mathcal{H}_2$ is a noncommutative field (in $M_2(\mathbb{C})$), if $U \in \mathcal{H}_2$ is a nonzero matrix, then it is automatically invertible by (8.32). So, the *similarity on* $\mathcal{H}_2$ (and hence, the similarity on $\mathbb{H}$) is determined by the similarity on $M_2(\mathbb{C})$ under restricted conditions. In this sense, since the similarity on $M_2(\mathbb{C})$ is an equivalence relation, the similarity on $\mathcal{H}_2$ (and hence, that on $\mathbb{H}$) is an equivalence relation.

Theorem 4 *Two quaternions q_1 and q_2 are q-spectral related, if and only if they are similar in the sense of (8.48) in $\mathbb{H}$. i.e., as equivalence relations,*

$$\mathcal{R} = similarity. \tag{8.49}$$

Proof ($\Rightarrow$) Suppose q_1 and q_2 are q-spectral related in $\mathbb{H}$. Then, by (8.47), they are similar in $\mathbb{H}$.

($\Leftarrow$) Suppose q_1 and q_2 are similar in $\mathbb{H}$, equivalently, assume that their realizations $[q_1]$ and $[q_2]$ are similar in $\mathcal{H}_2$. If λ_l are the q-spectral values of q_l, then $[q_l]$ and $[\lambda_l]$ are similar in the sense of (8.48) in $\mathcal{H}_2$, too, for all $l = 1, 2$. So, since the similarity on $\mathcal{H}_2$ is an equivalence relation, the q-spectral forms $[\lambda_1]$ and $[\lambda_2]$ are similar in $\mathcal{H}_2$. Since

$$[\lambda_l] = \begin{pmatrix} \lambda_l & 0 \\ 0 & \overline{\lambda_l} \end{pmatrix} \in \mathcal{H}_2, \text{ for } l = 1, 2,$$

the similarity of them guarantees that

$$[\lambda_1] = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \overline{\lambda_1} \end{pmatrix} = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \overline{\lambda_2} \end{pmatrix} = [\lambda_2],$$

by (8.48), and hence,

$$\lambda_1 = \lambda = \lambda_2 \text{ in } \mathbb{C}.$$

It means that $\lambda \in \mathbb{C}$ is the q-spectral value of both q_1 and q_2 in $\mathbb{H}$. Therefore, if q_1 and q_2 are similar in $\mathbb{H}$, then they are q-spectral related in $\mathbb{H}$.

By (8.49), we will use the q-spectral relation $\mathcal{R}$ of (8.44) on $\mathbb{H}$ and the similarity (8.48) on $\mathbb{H}$, alternatively.

8.3.4 Quaternion-Spectral Mapping Theorem

In this section, we consider q-spectral values more in detail. Throughout this section, we let

$$q = a + bj = x + yi + uj + vk \in \mathbb{H}$$

be a quaternion (8.29), with its q-spectral value,

$$\lambda = x + i\sqrt{y^2 + u^2 + v^2}, \text{ (if } b \neq 0),$$

or, $\lambda = a$ (if $b = 0$) in $\mathbb{C}$.

Now, let $\mathbb{C}[z]$ be the polynomial ring over $\mathbb{C}$, i.e.,

$$\mathbb{C}[z] = \{f(z) : f \text{ is a polynomial in } z \text{ over } \mathbb{C}\}.$$

It is well-known that if A is a matrix in $M_n(\mathbb{C})$, for $n \in \mathbb{N}$, and if $f \in \mathbb{C}[z]$, then

$$spec\,(f(A)) = f\,(spec(A))\,, \tag{8.50}$$

by the *spectral mapping theorem*, where the right-hand side of (8.50) means that

$$f\,(spec(A)) = \{f(t) : t \in spec(A)\}.$$

In the left-hand side of (8.50), a new matrix $f(A) \in M_n(\mathbb{C})$ is

$$a_k A^k + a_{k-1} A^{k-1} + ... + a_2 A^2 + a_1 A + a_0 I_n,$$

where I_n is the identity $(n \times n)$-matrix of $M_n(\mathbb{C})$, whenever

$$f(z) = a_k z^k + a_{k-1} z^{k-1} + ... + a_2 z^2 + a_1 z + a_0 \in \mathbb{C}[z].$$

More general to (8.50), if g is a continuous on $\mathbb{C}$, then

$$spec\,(g(A)) = g\,(spec(A)) \tag{8.51}$$

(e.g., [11, 12]). By (8.50) and (8.51), one can get that

$$spec\,(f\,([q])) = f\,(spec\,([q]))\,, \tag{8.52}$$

for all $f \in \mathbb{C}[z]$, for all $q \in \mathcal{H}$, realized to be $[q] \in \mathcal{H}_2$, "in $M_2(\mathbb{C})$."

Lemma 4 *Let g be a continuous function on $\mathbb{C}$, and let $q \in \mathcal{H}$. Then, by regarding the realization $[q] \in \mathcal{H}_2$ as a matrix of $M_2(\mathbb{C})$,*

$$spec\,(g\,([q])) = g\,(spec\,([q]))\,. \tag{8.53}$$

Proof The relation (8.53) is proven by (8.51) and (8.52).

Before proceeding, let's define the subset $\mathbb{C}_r[z]$ of $\mathbb{C}[z]$ by

$$\mathbb{C}_r[z] = \bigcup_{N=0}^{\infty} \{\sum_{n=0}^{N} a_n z^n \in \mathbb{C}[z] : a_0, a_1, ..., a_N \in \mathbb{R}\}. \tag{8.54}$$

Theorem 5 *Let $q \in \mathcal{H}$ be a quaternion (8.29) with its q-spectral value $\lambda \in \mathbb{C}$. For all $f \in \mathbb{C}_r[z]$, the quantity $f(\lambda)$ is the q-spectral value of $f(q)$, where $\mathbb{C}_r[z]$ is the subset (8.54) of $\mathbb{C}[z]$, and*

$$f(q) = \sum_{n=0}^{N} a_n q^n \in \mathcal{H}, \textit{whenever}\, f(z) = \sum_{n=0}^{N} a_n z^n \in \mathbb{C}[z].$$

Proof Let $b \neq 0$ in $\mathbb{C}$, and let $q = a + bj \in \mathcal{H}$ be a quaternion (8.29) with its q-spectral value $\lambda \in \mathbb{C}$ and let $h(z) \in \mathbb{C}[z]$. If $[q] \in \mathcal{H}_2$ is the realization of q, then

$$spec\,(h\,([q])) = \{h(\lambda), h\left(\overline{\lambda}\right)\} \text{ in } \mathbb{C},$$

by (8.53). Note however that, for $h(z) \in \mathbb{C}[z]$,

$$h\left(\overline{\lambda}\right) \neq \overline{h(\lambda)} \text{ in } \mathbb{C}, \text{ in general.}$$

(For instance, if $h(z) = iz$ in $\mathbb{C}[z]$, then $\overline{h(1+i)} = -1 - i \neq 1 + i = h\left(\overline{1+i}\right)$.)

However, if $f(z) = \sum_{n=0}^{N} a_n z^n \in \mathcal{C}_r[z]$ with $a_0, a_1, ..., a_N \in \mathcal{R}$, then

$$f\left(\overline{\lambda}\right) = \sum_{n=0}^{N} a_n \left(\overline{\lambda}\right)^n = \sum_{n=0}^{N} a_n \left(\overline{\lambda^n}\right)$$

$$= \sum_{n=0}^{N} \overline{(a_n \lambda^n)} = \overline{\sum_{n=0}^{N} a_n \lambda^n} = \overline{f(\lambda)},$$

in $\mathcal{C}$. It shows that, if $f(z) \in \mathcal{C}_r[z]$, then

$$spec\,(f\,([q])) = \{f(\lambda), f\left(\overline{\lambda}\right)\} = \{f(\lambda), \overline{f(\lambda)}\},$$

in $\mathcal{C}$, satisfying that

$$J\,(f\,([q])) = f\,(\mathbf{q}) \tag{8.55}$$

in $\mathcal{H}_2$, if and only if

$$\text{the } q\text{-spectral value of } f\,(q) = f\,(\lambda) \text{ in } \mathcal{C} \subset \mathcal{H},$$

where $\mathbf{q}$ is the q-spectral form of $[q]$ in $\mathcal{H}_2$, in general.

It is easy to verify that if $q = a + 0j = a \in \mathcal{C}$ in $\mathcal{H}$, with its q-spectral value $\lambda = a$ in $\mathcal{C}$, then

$$f(\overline{\lambda}) = f(\overline{a}) = \overline{f(a)} = \overline{f(\lambda)} \text{ in } \mathcal{C}, \forall f(z) \in \mathcal{C}_r[z].$$

i.e., $f(\lambda)$ is a q-spectral value of $f(q)$, too.

Therefore, the statement (8.55) holds.

Remark that the statement (8.55) holds for the polynomials of $\mathcal{C}_r[z]$, not those of $\mathcal{C}[z]$ (in general). Now, let $\mathcal{R}[x]$ be the polynomial ring over $\mathcal{R}$, i.e.,

$$\mathcal{R}[x] = \underset{N=0}{\overset{\infty}{\cup}} \{\sum_{n=0}^{N} a_n x^n : a_0, a_1, ..., a_N \in \mathcal{R}\}. \tag{8.56}$$

For any $f(x) = \sum_{n=0}^{N} a_n x^n \in \mathcal{R}[x]$, let's understand $f(z)$, or $f(q)$ as

$$f(z) = \sum_{n=0}^{N} a_n z^n \in \mathcal{C},$$

respectively,

$$f(q) = \sum_{n=0}^{N} a_n q^n \in \mathcal{H},$$

for all $z \in \mathcal{C}$, $q \in \mathcal{H}$. Then, the above theorem can be re-stated as follows.

Corollary 3 *Let $f(x) \in \mathcal{R}[x]$, where $\mathcal{R}[x]$ is the polynomial ring (8.56). If $q \in \mathcal{H}$ is a quaternion with its q-spectral value $\lambda \in \mathcal{C}$, realized to be $[q] \in \mathcal{H}_2$, then*

$$spec\,(f\,([q])) = \{f(\lambda), \overline{f(\lambda)}\}. \tag{8.57}$$

Proof Under hypothesis, the quantity $f(\lambda) \in \mathcal{C}$ is the q-spectral value of $f(q) \in \mathcal{H}$, by (8.55). Therefore, the set-equality (8.57) holds.

One may call the relation (8.57), the *quaternion-spectral mapping theorem.*

Theorem 6 *Let q_1 and q_2 be q-spectral related in $\mathcal{H}$ with their q-spectral value $\lambda \in \mathcal{C}$. If $f(x) \in \mathcal{R}[x]$, then $f(q_1)$ and $f(q_2)$ are q-spectral related in $\mathcal{H}$, with their identical q-spectral value $f(\lambda) \in \mathcal{C}$. Equivalently, if q_1 and q_2 are similar in $\mathcal{H}$, then $f(q_1)$ and $f(q_2)$ are similar in $\mathcal{H}$, for all $f(x) \in \mathcal{R}[x]$.*

Proof Let q_1 and q_2 be q-spectral related quaternions in $\mathcal{H}$. Assume that $\lambda \in \mathcal{C}$ is the q-spectral value of both q_1 and q_2. Then, for any $f(x) \in \mathcal{R}[x]$, the quantity $f(\lambda) \in \mathcal{C}$ is the q-spectral value of both $f(q_1)$ and $f(q_2)$ by (8.57). Therefore, two quaternions $f(q_1)$ and $f(q_2)$ are q-spectral related in $\mathcal{H}$. By (8.49), the q-spectral relation and the similarity are equivalent on $\mathcal{H}$. So, if q_1 and q_2 are similar, then $f(q_1)$ and $f(q_2)$ are similar in $\mathcal{H}$, for all $f(x) \in \mathcal{R}[x]$.

8.3.5 The Quaternion-Spectralization σ

Motivated by the main results of Sects. 8.3.3 and 8.3.4, a certain function from $\mathcal{H}$ to $\mathcal{C}$ is considered here. Define a function,

$$\sigma : \mathcal{H} \to \mathcal{C} \tag{8.58}$$

by the map, assigning each quaternion to its q-spectral value.

For example,

$$\begin{aligned}\sigma\,(1 + 0i + 2j - 3k) &= 1 + \sqrt{0^2 + 2^2 + (-3)^2}\; i \\ &= 1 + \sqrt{13}\, i,\end{aligned}$$

and

$$\sigma\,(-2 - i + 0j + 0k) = -2 - i,$$

etc..

Definition 5 We call the function σ of (8.58), the quaternion-spectralization (in short, the q-spectralization).

Let's consider the range of the q-spectralization σ.

Proposition 2 *The q-spectralization σ is surjective from $\mathcal{H}$ onto $\mathbb{C}$. i.e.,*

$$\sigma(\mathcal{H}) = \mathbb{C}. \tag{8.59}$$

Proof Let σ be the q-spectralization (8.58), and let $q = a + bj \in \mathcal{H}$ be a quaternion (8.29) with its q-spectral value $\lambda \in \mathbb{C}$. First, assume that $b = 0$ in $\mathbb{C}$, and hence, $q = a + 0j \in \mathbb{C}$ in $\mathcal{H}$. Then the realization $[q] \in \mathcal{H}_2$ satisfies

$$[q] = \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} = \mathbf{q} \text{ in } \mathcal{H}_2,$$

where $\mathbf{q}$ is the q-spectral form of q. So, by (8.58),

$$\sigma(q) = \sigma(a + 0j) = a = q. \tag{8.60}$$

Now, let $b \neq 0$ in $\mathbb{C}$ and $q = a + bj \in \mathcal{H}$, where

$$a = x + yi, b = u + vi \in \mathbb{C}.$$

Then

$$\sigma(q) = x + i\sqrt{y^2 + u^2 + v^2}, \tag{8.61}$$

in $\mathbf{H}_+$, by (8.27) and (8.58), where

$$\mathbf{H}_+ = \{z \in \mathbb{C} : Im(z) > 0\} \text{ in } \mathbb{C}.$$

Thus, by (8.60) and (8.61), we have

$$\sigma(\mathcal{H}) \subseteq (\mathbb{C} \cup \mathbf{H}_+) = \mathbb{C}.$$

It is easy to check that, for any $z \in \mathbb{C}$, there exists $z + 0j \in \mathcal{H}$, such that $\sigma(z) = z$, as in (8.60). Therefore,

$$\mathbb{C} \subseteq \sigma(\mathcal{H}).$$

Therefore, $\sigma(\mathcal{H}) = \mathbb{C}$, implying the surjectivity (8.59).

8.4 Some Algebraic Structures on $\mathcal{H}$

In this section, we consider structure theorem of a quotient structure of $\mathcal{H}$ under an equivalence relation $\mathcal{R}$ induced by our q-spectralization.

8.4.1 Classification of $\mathcal{H}$

Let $\mathcal{R}$ be the q-spectral relation (8.44) (which is the similarity (8.48)) on $\mathcal{H}$. Since $\mathcal{R}$ is an equivalence relation on $\mathcal{H}$, for a quaternion $q \in \mathcal{H}$, one can determine the corresponding equivalence class,

$$q^o \overset{denote}{=} q/\mathcal{R} \overset{def}{=} \{w \in \mathcal{H} : w\mathcal{R}q\}, \tag{8.62}$$

by collecting all q-spectral related quaternions of q. Then one can define the quotient set,

$$\mathcal{H}^o \overset{denote}{=} \mathcal{H}/\mathcal{R} \overset{def}{=} \{q^o : q \in \mathcal{H}\}, \tag{8.63}$$

where $q^o \in \mathcal{H}^o$ are the equivalence classes (8.62).

Remark that if $\sigma : \mathcal{H} \to \mathcal{C}$ is the q-spectralization (8.58), then the equivalence class $q^o \in \mathcal{H}^o$ of (8.62) satisfies that

$$q^o = \sigma(q)^o, \tag{8.64}$$

in $\mathcal{H}^o$, since $\sigma(q) = \sigma(q) + 0j + 0k$ in $\mathcal{H}$, and

$$\sigma\left(\sigma(q) + 0j + 0k\right) = \sigma(q) \tag{8.65}$$

in $\mathcal{C}$, for all $q \in \mathcal{H}$.

Theorem 7 *Let $\mathcal{H}^o$ be the quotient set (8.63). Then $\mathcal{H}^o = \mathcal{C}$, set-theoretically.*

Proof Let $\mathcal{H}^o$ be the quotient set (8.63). Then, by (8.59), (8.62), (8.64) and (8.65), this set can be re-expressed by

$$\mathcal{H}^o = \{\sigma(q)^o : q \in \mathcal{H}\} = \{\sigma(q)^o : \sigma(q) \in \mathcal{C}\},$$

i.e.,

$$\mathcal{H}^o = \{\lambda^o : \lambda \in \mathcal{C}\}, \tag{8.66}$$

because $q_1^o = q_2^o$, if and only if $\sigma(q_1) = \sigma(q_2)$, for $q_1, q_2 \in \mathcal{H}$.

Define now a function $F : \mathcal{H}^o \to \mathcal{C}$ by

$$F(\lambda^o) = \lambda, \text{ for all } \lambda^o \in \mathcal{H}^o,$$

where λ^o are in the sense of (8.66).

It is not hard to check that this function is a bijection by (8.59) and (8.66). i.e., two sets $\mathcal{H}^o$ and $\mathcal{C}$ are equipotent (or bijective). Thus, all equivalence classes $\lambda^o \in \mathcal{H}^o$ are understood to be $\lambda \in \mathcal{C}$, and vice versa. So, the set-equality,

$$\mathcal{H}^o = \mathbb{C}$$

holds.

The above theorem shows that each $\mathbb{C}$-number λ becomes a representative of all quaternions $q \in \mathcal{H}$ satisfying $\sigma(q) = \lambda$. i.e., the quaternions $\mathcal{H}$ is classified by the complex numbers $\mathbb{C}$.

8.4.2 The Quaternions $\mathcal{H}$ and the Lie Group $SU_2(\mathbb{C})$

Independent from the set-theoretical classification considered in Sect. 8.4.1, we here focus on the quaternion-multiplication $(\cdot)$ of (8.2), and an algebraic structure of $\mathcal{H}$ up to $(\cdot)$. In particular, we are interested in the connections between $\mathcal{H}$ and the Lie group $SU_2(\mathbb{C})$.

Define a subset $SU_2(\mathbb{C})$ of the matricial algebra $M_2(\mathbb{C})$ by

$$SU_2(\mathbb{C}) \overset{def}{=} \{g \in M_2(\mathbb{C}) : \det(g) = 1, g^* = g^{-1}\}, \tag{8.67}$$

where g^* are the matricial adjoints of g, where g^{-1} mean the matricial-inverses of g. i.e., this subset $SU_2(\mathbb{C})$ of (8.67) consists of all unitary matrices g satisfying

$$g^*g = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = gg^*, \text{ and } \det(g) = 1.$$

Now, let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU_2(\mathbb{C})$. Then, by definition, it satisfies that

$$\det(g) = ad - bc = 1, \tag{8.68}$$

and

$$\begin{aligned} g^*g &= \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} |a|^2 + |c|^2 & \overline{a}b + \overline{c}d \\ a\overline{b} + c\overline{d} & |b|^2 + |d|^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} |a|^2 + |b|^2 & a\overline{c} + b\overline{d} \\ \overline{a}c + \overline{b}d & |c|^2 + |d|^2 \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix} = gg^*. \end{aligned}$$

So, for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU_2(\mathbb{C})$,

$$ad - bc = 1,$$

and

$$|a|^2 + |c|^2 = 1 = |b|^2 + |d|^2\,,$$

and

$$a\overline{c} + b\overline{d} = 0 = \overline{a}c + \overline{b}d = \overline{a\overline{c} + b\overline{d}}, \tag{8.69}$$

in $\mathbb{C}$, by (8.68), if and only if

$$ad - bc = 1 = |a|^2 + |c|^2 = |b|^2 + |d|^2\,,$$

and

$$a\overline{c} + b\overline{d} = 0, \tag{8.70}$$

by (8.69).

Suppose $A = \begin{pmatrix} x & -y \\ \overline{y} & \overline{x} \end{pmatrix} \in M_2(\mathbb{C})$, with

$$\det(A) = |x|^2 + |y|^2 = 1. \tag{8.71}$$

Then it automatically satisfies the conditions in (8.70), by letting

$$a = x, b = -y, c = \overline{y}, \text{ and } d = \overline{x} \text{ in } \mathbb{C}.$$

i.e., such a matrix $A \in M_2(\mathbb{C})$ satisfying (8.71) is contained in $SU_2(\mathbb{C})$, satisfying the conditions (8.70).

Assume now that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C}) \setminus \{O_2\}$ satisfies

$$\det(A) = ad - bc = 1. \tag{8.72}$$

Under (8.72), suppose either

$$c \neq -\overline{b}, \text{ or } d \neq \overline{a} \text{ in } \mathbb{C}.$$

Then, one can get that

$$A^*A = \begin{pmatrix} |a|^2 + |c|^2 & \overline{a}b + \overline{c}d \\ a\overline{b} + c\overline{d} & |b|^2 + |d|^2 \end{pmatrix} \neq I_2,$$

since

$$|a|^2 + |c|^2 \neq |b|^2 + |d|^2 \text{ in } \mathbb{C},$$

respectively,

$$\overline{a}b + \overline{c}d \neq 0, \text{ or } a\overline{b} + c\overline{d} \neq 0.$$

Similarly, under (8.72),

$$AA^* \neq I_2 \text{ in } M_2(\mathbb{C}),$$

whenever either $c \neq -\overline{b}$, or $d \neq \overline{a}$ in $\mathbb{C}$.

Proposition 3 *The subset* $SU_2(\mathbb{C})$ *of* $M_2(\mathbb{C})$, *in the sense of* (8.67), *is identical to the subset,*

$$\left\{ \begin{pmatrix} a & -b \\ \overline{b} & \overline{a} \end{pmatrix} : a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}, \tag{8.73}$$

set-theoretically.

Proof By the discussions in the very above paragraphs, the subset $SU_2(\mathbb{C})$ of (8.67) is identical to the subset (8.73) of $M_2(\mathbb{C})$. Indeed, as we have seen, if we denote the set (8.73) by $\mathcal{S}$, then the following set-inclusion,

$$\mathcal{S} \subseteq SU_2(\mathbb{C}),$$

holds automatically.

Meanwhile, if $g \in SU_2(\mathbb{C})$ is not of the form as an element of $\mathcal{S}$, then it is not contained in $SU_2(\mathbb{C})$, which contradicts the assumption g is taken from $SU_2(\mathbb{C})$. i.e.,

$$SU_2(\mathbb{C}) \subseteq \mathcal{S}.$$

Therefore, set-theoretically, two subsets $SU_2(\mathbb{C})$ and $\mathcal{S}$ are identical in $M_2(\mathbb{C})$.

By the above proposition, without loss of generality, one can let

$$SU_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & -b \\ \overline{b} & \overline{a} \end{pmatrix} : a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\},$$

by (8.73).

It motivates our structure-characterization of the quaternions $\mathcal{H}$. It is not difficult to check that the subset $SU_2(\mathbb{C})$ forms a group under the matricial multiplication. Indeed, for all $g_1, g_2 \in SU_2(\mathbb{C})$,

$$g_1 g_2 \in SU_2(\mathbb{C}),$$

since

$$\det(g_1 g_2) = \det(g_1)\det(g_2) = 1,$$

and

$$(g_1g_2)^*(g_1g_2) = g_2^*g_1^*g_1g_2 = g_2^*I_2g_2 = I_2,$$

and similarly,

$$(g_1g_2)(g_1g_2)^* = I_2 \text{ in } SU_2(\mathbb{C});$$

and the inherited matricial multiplication on $SU_2(\mathbb{C})$ is associative; and it has its identity $I_2 \in SU_2(\mathbb{C})$; finally, since all elements of $SU_2(\mathbb{C})$ are unitary in the sense that:

$$g^*g = I_2 = gg^* \Longleftrightarrow g^* = g^{-1},$$

in $SU_2(\mathbb{C})$, the multiplication satisfies the inverse-property, implying that $SU_2(\mathbb{C})$, equipped with multiplication, forms a group. Actually, it is a typical example of *Lie groups* (e.g., [3, 4]).

Define now a new group $\mathcal{S}_2$ by

$$\mathcal{S}_2 \overset{def}{=} SU_2(\mathbb{C}) \cdot \mathbb{R}_+^{\times} = \left\{ \sqrt{\gamma} \cdot g : \gamma \in \mathbb{R}_+^{\times}, g \in SU_2(\mathbb{C}) \right\}, \tag{8.74}$$

where

$$\mathbb{R}_+^{\times} = \{ r \in \mathbb{R} : \gamma > 0 \}.$$

If $A \in \mathcal{S}_2$, then there exist $\gamma \in \mathbb{R}_+^{\times}$, and

$$g = \begin{pmatrix} a & -b \\ \overline{b} & \overline{a} \end{pmatrix} \in SU_2(\mathbb{C}), \text{ with } |a|^2 + |b|^2 = 1,$$

such that

$$A = \begin{pmatrix} \sqrt{\gamma}a & -\sqrt{\gamma}b \\ \overline{\sqrt{\gamma}b} & \overline{\sqrt{\gamma}a} \end{pmatrix} \overset{denote}{=} \begin{pmatrix} z & -w \\ \overline{w} & \overline{z} \end{pmatrix} \text{ in } \mathcal{S}_2,$$

satisfying

$$\det(A) = |z|^2 + |w|^2 = \gamma \cdot 1 > 0.$$

Now, let's understand the quaternions $\mathbb{H}$ as its isomorphic noncommutative field,

$$\mathcal{H}_2 = \left\{ [q] = \begin{pmatrix} a & -b \\ \overline{b} & \overline{a} \end{pmatrix} : q = a + bj \in \mathbb{H} \right\}$$

of (8.14), and consider the corresponding noncommutative multiplicative group,

$$\mathcal{H}_2^{\times} \overset{denote}{=} \left(\mathcal{H}_2^{\times}, \cdot \right),$$

where

$$\mathcal{H}_2^{\times} = \{[q] \in \mathcal{H}_2 : q \neq 0 \in \mathbb{H}\}, \tag{8.75}$$

and $(\cdot)$ is the multiplication (8.10) on $\mathcal{H}_2$.

Theorem 8 *Let $\mathcal{S}_2$ be the group* (8.74)*, and let $\mathcal{H}_2^{\times}$ be the group* (8.75) *induced by the quaternions $\mathbb{H}$. Then*

$$\mathcal{S}_2 \overset{Group}{=} \mathcal{H}_2^{\times}, \tag{8.76}$$

where "$\overset{Group}{=}$" means "being group-isomorphic to."

Proof Let $\mathcal{H}_2^{\times}$ be the group (8.75), and

$$[q] = \begin{pmatrix} a & -b \\ \overline{b} & \overline{a} \end{pmatrix} \in \mathcal{H}_2^{\times}$$

be a realization of an arbitrary quaternion,

$$q = a + bj \in \mathbb{H}.$$

Take a quantity,

$$\gamma_q = \det([q]) = |a|^2 + |b|^2 = \|q\| \text{ in } \mathbb{R}_+^{\times}.$$

Since $[q] \neq O_2$ in $\mathcal{H}_2$, indeed, the above quantity is contained in $\mathbb{R}_+^{\times}$. So, the element $[q] \in \mathcal{H}_2^{\times}$ is regarded as a matrix,

$$[q] = \begin{pmatrix} \gamma_q \left(\frac{a}{\gamma_q}\right) & \gamma_q \left(\frac{-b}{\gamma_q}\right) \\ \\ \gamma_q \left(\frac{\overline{b}}{\gamma_q}\right) & \gamma_q \left(\frac{\overline{a}}{\gamma_q}\right) \end{pmatrix}$$
$$= \gamma_q \begin{pmatrix} \left(\frac{a}{\gamma_q}\right) & \left(\frac{-b}{\gamma_q}\right) \\ \\ \left(\frac{\overline{b}}{\gamma_q}\right) & \left(\frac{\overline{a}}{\gamma_q}\right) \end{pmatrix} = \gamma_q \left[\left(\frac{a}{\gamma_q}\right) + \left(\frac{b}{\gamma_q}\right) j\right],$$

where

$$\frac{q}{\gamma_q} = \left(\frac{a}{\gamma_q}\right) + \left(\frac{b}{\gamma_q}\right) j = \frac{1}{\gamma_q}(a + bj) = \frac{1}{\gamma_q} \cdot q \in \mathbb{H}. \tag{8.77}$$

Observe that

$$\det\left(\left[\frac{q}{\gamma_q}\right]\right) = \frac{1}{\gamma_q} \det([q]) = \frac{\gamma_q}{\gamma_q} = 1,$$

and hence,

$$\left[\frac{q}{\gamma_q}\right] \in SU_2(\mathbb{C}). \tag{8.78}$$

By (8.77) and (8.78), one can conclude that, for any $[q] \in \mathcal{H}_2^\times$, there exists a unique

$$\left[\frac{q}{\det(q)}\right] \in SU_2(\mathbb{C}),$$

such the

$$[q] = \det(q)\left[\frac{q}{\det(q)}\right]. \tag{8.79}$$

By (8.79), we define a morphism,

$$\Phi : \mathcal{H}_2^\times \to \mathcal{S}_2 = SU_2(\mathbb{C}) \cdot \mathbb{R}_+^\times,$$

by

$$\Phi([q]) = (\det(q))\left[\frac{q}{\det(q)}\right] \tag{8.80}$$

in $\mathcal{S}_2$, for all $[q] \in \mathcal{H}_2^\times$ (or, for all $q \in \mathbb{H}^\times = \mathbb{H} \setminus \{0\}$).

Then, by (8.73) and (8.74), this morphism Φ of (8.80) is bijective. Moreover, it satisfies that

$$\begin{aligned}\Phi([q_1][q_2]) &= \Phi([q_1 q_2]) \\ &= (\det([q_1 q_2]))\left[\tfrac{q_1 q_2}{\det([q_1 q_2])}\right] \\ &= (\det([q_1][q_2]))\left[\tfrac{q_1 q_2}{\det([q_1][q_2])}\right] \\ &= (\det([q_1])\det([q_2]))\left[\tfrac{q_1 q_2}{\det([q_1])\det([q_2])}\right] \\ &= \left(\tfrac{\det([q_1])}{\det([q_1])}\right)\left(\tfrac{\det([q_2])}{\det([q_2])}\right)[q_1 q_2] \\ &= \left(\tfrac{\det([q_1])}{\det([q_1])}[q_1]\right)\left(\tfrac{\det([q_2])}{\det([q_2])}[q_2]\right),\end{aligned}$$

since π is a well-defined noncommutative-field-isomorphism from $\mathbb{H}$ onto $\mathcal{H}_2$, i.e.,

$$\Phi([q_1][q_2]) = (\Phi([q_1]))(\Phi([q_2])), \tag{8.81}$$

in $\mathcal{S}_2$, for all $[q_1], [q_2] \in \mathcal{H}_2^\times$.

Therefore, the bijection Φ of (8.80) is a group-homomorphism by (8.81), and hence, it is a group-isomorphism from $\mathcal{H}_2^\times$ onto $\mathcal{S}_2$. i.e., the relation (8.76) holds true.

By the above theorem, the following corollary is immediately obtained.

Corollary 4 *Let $\mathbb{H}^\times = (\mathbb{H}^\times, \cdot)$ be a noncommutative multiplicative group of non-zero quaternions. Then it is group-isomorphic to the group $\mathcal{S}_2$ of* (8.74).

Proof Since the multiplicative group $\mathbb{H}^\times$ is isomorphic to $\mathcal{H}_2^\times$, it is isomorphic to the group $\mathcal{S}_2$ by (8.76).

The above theorem and corollary provide a connection between the quaternionic group $\mathcal{H}^{\times}$ and the Lie group $SU_2(\mathbb{C})$ of (8.67).

Corollary 5 *Let $\mathcal{S}_2 = SU_2(\mathbb{C}) \cdot \mathcal{R}_+^{\times}$ be the group (8.74). Define a relation $\mathcal{C}$ on $\mathcal{S}_2$ by*

$$g_1 \mathcal{C} g_2 \Longleftrightarrow \exists\, g \in \mathcal{S}_2, \text{ such that } g_2 = g^{-1} g_1 g.$$

Then $\mathcal{C}$ is an equivalence relation on $\mathcal{S}_2$. And the corresponding quotient group,

$$\mathcal{S}_2^o \overset{def}{=} \mathcal{S}_2/\mathcal{C},$$

is group-isomorphic to the usual multiplicative "abelian" group $\left(\mathbb{C}^{\times}, \cdot\right)$, where $(\cdot)$ is the usual multiplication.

Proof Recall that the group $\mathcal{S}_2$ and the multiplicative group $\mathcal{H}^{\times} = \mathcal{H}_2^{\times}$ are isomorphic by (8.76). Note now that, under this isomorphic relation, the relation $\mathcal{C}$ on $\mathcal{S}_2$ is equivalent to the similarity on $\mathcal{H}_2$, which is equivalent to the q-spectral relation $\mathcal{R}$ of (8.44). Therefore, the quotient group $\mathcal{S}_2^o$ of $\mathcal{S}_2$ under $\mathcal{C}$ is isomorphic to the usual multiplicative abelian group $\left(\mathbb{C}^{\times}, \cdot\right)$.

By the above corollary, one obtains the following result.

Corollary 6 *The following diagram commutes;*

$$\begin{array}{ccc} \mathcal{S}_2 & & \\ \downarrow_{\pi} & \nwarrow^{\beta} & \\ \mathcal{S}_2^o & \underset{\Phi}{\simeq} & \mathbb{C}^{\times}, \end{array}$$

where π is the usual quotient map, and Φ is the group-isomorphism (8.80), and

$$\beta : \mathbb{C}^{\times} \to \mathcal{S}_2$$

is a group-homomorphism, defined by

$$\beta(\gamma) = \begin{pmatrix} \gamma & 0 \\ 0 & \overline{\gamma} \end{pmatrix} \in \mathcal{S}_2, \forall \gamma \in \mathbb{C}^{\times}.$$

Proof The proof is done by the very above corollary.

Now, define a set $\mathcal{S}_2^0$ by

$$\mathcal{S}_2^0 = SU_2(\mathbb{C}) \cdot \mathcal{R}_+, \tag{8.82}$$

where

$$\mathcal{R}_+ = \{r \in \mathcal{R} : r \geq 0\}.$$

Then, similar to the proof of (8.76), one can verify that

$$\mathcal{S}_2^0 \overset{\text{N.F}}{=} \mathcal{H}_2 \overset{\text{N.F}}{=} \mathbb{H},$$

by defining the extension Φ^0 of the group-isomorphism Φ of (8.80),

$$\Phi^0([q]) = \begin{cases} \det([q]) \left[\frac{q}{\det(q)}\right] & \text{if } q \neq 0 \text{ in } \mathbb{H} \\ O_2 & \text{if } q = 0 \text{ in } \mathbb{H}, \end{cases} \tag{8.83}$$

for all $q \in \mathbb{H}$, where $O_2 = [0]$ is the zero matrix of $\mathcal{H}_2$.

Theorem 9 *Let $\mathcal{S}_2^0 = SU_2(\mathbb{C}) \cdot \mathbb{R}_+$ be the set* (8.82). *Then*

$$\mathcal{S}_2^0 \overset{N.F}{=} \mathcal{H}_2 \overset{N.F}{=} \mathbb{H}. \tag{8.84}$$

Proof Like the proof of (8.76), it is not hard to show that the morphism Φ^0 of (8.83) is a noncommutative-field isomorphism from $\mathcal{H}_2$ onto $\mathcal{S}_2^0$, satisfying

$$\left(\mathcal{S}_2^0, +\right) \overset{\text{Group}}{=} (\mathcal{H}_2, +),$$

and

$$(\mathcal{S}_2, \cdot) \overset{\text{Group}}{=} \left(\mathcal{H}_2^{\times}, \cdot\right) \text{ (by (8.76))}.$$

Therefore, the first isomorphic relation of (8.84) holds. The second noncommutative-field isomorphic relation of (8.84) is shown by (8.15).

The above theorem also provides a connection between the quaternions $\mathbb{H}$ and the Lie group $SU_2(\mathbb{C})$. And it provides the following decomposition property on $\mathbb{H}$.

Theorem 10 *Let $q \in \mathbb{H}$ be a non-zero quaternion. Then there exists $g \in SU_2(\mathbb{C})$, such that*

$$q = \|q\| \, \pi^{-1}\left(\Phi^{-1}(g)\right) \tag{8.85}$$

in $\mathbb{H}$, where π is the q-spectral representation, and Φ is in the sense of (8.80)*, or* (8.83).

Proof By (8.84), the quaternions $\mathbb{H}$ and the noncommutative field $\mathcal{S}_2^0$ of (8.82) are isomorphic, and hence, all nonzero quaternions $\mathbb{H}^{\times}$ and nonzero elements $\mathcal{S}_2$ are group-isomorphic by (8.76). So, by (8.80), the relation (8.85) holds.

8.5 Monomial Equations on $\mathcal{H}$

In this section, by applying the main results of Sects. 8.3 and 8.4, we consider certain equations on the quaternions $\mathcal{H}$. In particular, we are interested in monomial equations,

$$h^n = q, \tag{8.86}$$

for all $n \in \mathcal{H}$, where h is a variable on $\mathcal{H}$, and

$$q = a + bj \in \mathcal{H},$$

with

$$a = t + si, b = u + vi \in \mathbb{C},$$

is an arbitrarily fixed quaternion.

If $n = 1$ in (8.86), then the corresponding monomial equation $h = q$ is trivial. So, we are not interested in the case where $n = 1$.

Assumption. From below, for any given monomial equations $h^n = q$ of (8.86), it is automatically assumed that $n \in \mathbb{N}_{>1}$ where

$$n \in \mathbb{N}_{>1} \overset{def}{=} \{k \in \mathbb{N} : k > 1\}.$$

□

A $\mathcal{H}$-variable h is understood to be its realization $[h]$ in $\mathcal{H}_2$,

$$[h] = \begin{pmatrix} z_1 & -z_2 \\ \overline{z_2} & \overline{z_1} \end{pmatrix} \in \mathcal{H}_2,$$

as an operator-variable on $\mathcal{H}_2$, where (z_1, z_2) is a 2-dimensional vector-variable on $\mathbb{C}^2$, equivalently,

$$z_1 = x_1 + y_1 i, \text{ and } z_2 = x_2 + y_2 i$$

are two distinct variables on $\mathbb{C}$, where (x_1, y_1, x_2, y_2) is a vector-variable on $\mathbb{R}^4$. So, (8.86) is equivalent to an equation,

$$[h^n] = [q] \Longleftrightarrow [h]^n = [q] \text{ on } \mathcal{H}_2,$$

if and only if

$$\begin{pmatrix} z_1 & -z_2 \\ \overline{z_2} & \overline{z_1} \end{pmatrix}^n = \begin{pmatrix} a & -b \\ \overline{b} & \overline{a} \end{pmatrix} \tag{8.87}$$

by (8.15), where $[q] \in \mathcal{H}_2$ is the realization of a fixed quaternion $q \in \mathcal{H}$ of (8.86).

Suppose a quaternion $h_0 \in \mathcal{H}$ is a solution of (8.86), i.e., assume

$$h_0^n = q, \Longleftrightarrow [h_0]^n = [q], \tag{8.88}$$

by (8.87). If the equality (8.88) holds for h_0, then

$$\sigma\left(h_0^n\right) = \sigma(q), \Longleftrightarrow \sigma(h_0)^n = \sigma(q), \tag{8.89}$$

in $\mathcal{H}$, where $\sigma : \mathcal{H} \to \mathcal{C}$ is the q-spectralization (8.58).

Now, let $h = x_1 + y_1 i + x_2 j + y_2 k$ be a $\mathcal{H}$-variable with its q-spectral value,

$$\sigma(h) = x_1 + i\sqrt{y_1^2 + x_2^2 + x_3^2},$$

as an unknown in $\mathcal{C}$, and let $\sigma(q) \in \mathcal{C}$ be the q-spectral value of a fixed quaternion $q \in \mathcal{H}$ of (8.86). Then (8.86) satisfies (as an equality)

$$[\sigma(h)]^n = [\sigma(q)] \tag{8.90}$$

in $\mathcal{H}_2$. And, by (8.89), (8.90) has two cases: (i) where $q \in \mathcal{C}$ in $\mathcal{H}$, and (ii) $q \notin \mathcal{C}$ in $\mathcal{H}$.

Assume first that $q = a + bi \in \mathcal{C}$ in $\mathcal{H}$. Then (8.90) is equivalent to

$$\begin{pmatrix} \left(x_1 + i\sqrt{y_1^2 + x_2^2 + y_2^2}\right)^n & 0 \\ 0 & \left(x_1 - i\sqrt{y_1^2 + x_2^2 + y_2^2}\right)^n \end{pmatrix}$$

$$= \begin{pmatrix} q & 0 \\ 0 & \overline{q} \end{pmatrix} = \begin{pmatrix} a + bi & 0 \\ 0 & a - bi \end{pmatrix}, \tag{8.91}$$

since $q = \sigma(q) \in \mathcal{C}$ in $\mathcal{H}$.

Assume now that $q = a + bi + uj + vk \in \mathcal{H}$, where either u or v is nonzero in $\mathcal{R}$. Then (8.90) goes to

$$\begin{pmatrix} \left(x_1 + i\sqrt{y_1^2 + x_2^2 + y_2^2}\right)^n & 0 \\ 0 & \left(x_1 - i\sqrt{y_1^2 + x_2^2 + y_2^2}\right)^n \end{pmatrix}$$

$$= \begin{pmatrix} a + i\sqrt{b^2 + u^2 + v^2} & 0 \\ 0 & a - i\sqrt{b^2 + u^2 + v^2} \end{pmatrix}, \tag{8.92}$$

since $\sigma(q) = a + i\sqrt{b^2 + u^2 + v^2}$ in $\mathcal{C}$

Lemma 5 *Let $h^n = q$ be a monomial equation (8.86) for $n \in \mathbb{N}_{>1}$. If $q = a + bi + 0j + 0k \in \mathcal{C}$ in $\mathcal{H}$, then solving (8.86) is to solve*

$$\left(x_1 + i\sqrt{y_1^2 + x_2^2 + y_2^2}\right)^n = a + bi. \tag{8.93}$$

Meanwhile, if $q = a + bi + uj + vk \in \mathcal{H}$*, where either* u *or* v *is nonzero in* $\mathcal{R}$*, then solving* (8.86) *is to solve*

$$\left(x_1 + i\sqrt{y_1^2 + x_2^2 + y_2^2}\right)^n = a + i\sqrt{b^2 + u^2 + v^2}. \tag{8.94}$$

Proof Solving an equation $h^n = q$ of (8.86) is to solve the equation

$$[\sigma(h)]^n = [\sigma(q)] \text{ on } \mathcal{H}_2 \text{of (8.90)},$$

by (8.88) and (8.89).

Suppose $q = a + bi + 0j + 0k \in \mathcal{H}$, equivalently, $\sigma(q) = q = a + bi \in \mathcal{C}$ in $\mathcal{H}$. So, (8.90) becomes

$$\begin{pmatrix} \sigma(h) & 0 \\ 0 & \overline{\sigma(h)} \end{pmatrix}^n = \begin{pmatrix} q & 0 \\ 0 & \overline{q} \end{pmatrix}$$

$\Longleftrightarrow$

$$\begin{pmatrix} \sigma(h)^n & 0 \\ 0 & \overline{\sigma(h)}^n \end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & \overline{q} \end{pmatrix}$$

$\Longleftrightarrow$

$$\begin{pmatrix} \sigma(h^n) & 0 \\ 0 & \overline{\sigma(h^n)} \end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & \overline{q} \end{pmatrix}$$

on $\mathcal{H}_2$, and hence,

$$\sigma(h^n) = q = \sigma(q)$$

$\Longleftrightarrow$

$$\left(x_1 + i\sqrt{y_1^2 + x_2^2 + y_2^2}\right)^n = a + bi.$$

Therefore, the statement (8.93) holds.

Meanwhile, if $q = a + bi + uj + vk \in \mathcal{H}$, where either u or v is nonzero in $\mathcal{R}$. Then

$$\sigma(q) = a + i\sqrt{b^2 + u^2 + v^2} \text{ in } \mathcal{C},$$

and hence, (8.90) is equivalent to

$$\sigma(h)^n = \sigma(q),$$

if and only if

$$\left(x_1 + i\sqrt{y_1^2 + x_2^2 + y^2}\right) = a + i\sqrt{b^2 + u^2 + v^2}.$$

Thus, the statement (8.94) holds.

Let $h = x_1 + y_1 i + x_2 j + y_2 k$ be a variable on $\mathcal{H}$, and $q = a + bi + uj + vk \in \mathcal{H}$, an arbitrarily fixed quaternion. By (8.93) and (8.94), solving an equation $h^n = q$ of (8.86) is to solve an equation

$$\sigma(h^n) = \sigma(q) \text{ on } \mathcal{C},$$

$$\Longleftrightarrow \quad \left(x_1 + i\sqrt{y_1^2 + x_2^2 + y_2^2}\right)^n = a + i\sqrt{b^2 + u^2 + v^2}. \tag{8.95}$$

i.e., (8.95) covers both (8.93) and (8.94).

Consider now the polar decomposition $re^{i\theta} \in \mathcal{C}$ of

$$\sigma(h) = x_1 + i\sqrt{y_1^2 + x_2^2 + y_2^2} \text{ in } \mathcal{C},$$

where $r \in \mathcal{R}_+$ and $\theta = \arg(\sigma(h))$, the argument of $\sigma(h)$, as a $\mathcal{R}$-variable acting on $\mathcal{R}_+$ and as a $\mathcal{R}$-variable acting on the closed interval $[0, 2\pi]$ of $\mathcal{R}$, respectively. One can have that

$$r = \sqrt{x_1^2 + y_1^2 + x_2^2 + y_2^2},$$

and

$$\theta = \cos^{-1}\left(\frac{x_1}{\sqrt{x_1^2 + y_1^2 + x_2^2 + y_2^2}}\right) = \sin^{-1}\left(\frac{\sqrt{y_1^2 + x_2^2 + y_2^2}}{\sqrt{x_1^2 + y_1^2 + x_2^2 + y_2^2}}\right). \tag{8.96}$$

Also, the $\mathcal{C}$-quantity $\sigma(q) = a + i\sqrt{b^2 + u^2 + v^2}$ has its polar decomposition $r_q e^{i\theta_q} \in \mathcal{C}$ with

$$r_q = \sqrt{a^2 + b^2 + u^2 + v^2},$$

and

$$\theta_q = \cos^{-1}\left(\frac{a}{\sqrt{a^2 + b^2 + u^2 + v^2}}\right) = \sin^{-1}\left(\frac{\sqrt{b^2 + u^2 + v^2}}{\sqrt{a^2 + b^2 + u^2 + v^2}}\right), \tag{8.97}$$

where $r_q \in \mathcal{R}_+$ and $\theta_q \in [0, 2\pi]$ are fixed quantities for $\sigma(q) \in \mathcal{C}$.

By (8.96) and (8.97), (8.95) is equivalent to

$$\left(re^{i\theta}\right)^n = r_q e^{i\theta_q} \Longleftrightarrow r^n e^{in\theta} = r_q e^{i\theta_q}$$

if and only if

$$r^n = \left(\sqrt{x_1^2 + y_1^2 + x_2^2 + y_2^2}\right)^n$$
$$= \sqrt{a^2 + b^2 + u^2 + v^2} = r_q,$$

and

$$\begin{aligned} n\theta &= n\cos^{-1}\left(\frac{x_1}{\sqrt{x_1^2+y_1^2+x_2^2+y_2^2}}\right) = n\sin^{-1}\left(\frac{\sqrt{y_1^2+x_2^2+y_2^2}}{\sqrt{x_1^2+y_1^2+x_2^2+y_2^2}}\right) \\ &= \cos^{-1}\left(\frac{a}{\sqrt{a^2+b^2+u^2+v^2}}\right) = \sin^{-1}\left(\frac{\sqrt{b^2+u^2+v^2}}{\sqrt{a^2+b^2+u^2+v^2}}\right) = \theta_q. \end{aligned} \tag{8.98}$$

By (8.93), (8.94) and (8.98), we obtain the following proposition.

Proposition 4 *Let $h^n = q$ be (8.86) on the quaternions $\mathcal{H}$, where*

$$h = x_1 + y_1 i + x_2 j + y_2 k$$

is a $\mathcal{H}$-variable, and

$$q = a + bi + uj + vk \in \mathcal{H}.$$

Then solving this equation is to solve the system,

$$\begin{cases} \left(x_1^2 + y_1^2 + x_2^2 + y_2^2\right)^{2n} = a^2 + b^2 + u^2 + v^2, \\ n\cos^{-1}\left(\frac{x_1}{\sqrt{x_1^2+y_1^2+x_2^2+y_2^2}}\right) = \cos^{-1}\left(\frac{a}{\sqrt{a^2+b^2+u^2+v^2}}\right) \end{cases} \tag{8.99}$$

Proof Solving (8.86) is to solve the system (8.99), by (8.95), (8.96), (8.97) and (8.98).

The above proposition shows that solving a monomial equation $h^n = q$ on $\mathcal{H}$, for $n \in \mathcal{N}$, is to solve the system,

$$\begin{cases} \|h\|^{2n} = \|q\|^2, \\ n\cos^{-1}\left(\frac{Re(h)}{\|h\|}\right) = \cos^{-1}\left(\frac{Re(q)}{\|q\|}\right), \end{cases} \tag{8.100}$$

by (8.99), where $\|.\|$ is the quaternion-modulus (8.7), and where

$$Re(w) \in \mathcal{R} \text{ and } Im(w) \in \mathcal{H} \setminus \mathcal{R}$$

mean the *real parts*, respectively, the *imaginary parts* of quaternions $w \in \mathcal{H}$ in the sense that:

$$Re(w) = t_1 \text{ and } Im(w) = t_2 i + t_3 j + t_4 k,$$

whenever

$$w = t_1 + t_2 i + t_3 j + t_4 k \in \mathcal{H},$$

for $t_1, t_2, t_3, t_4 \in \mathcal{R}$.

Corollary 7 *Solving a monomial equation $h^n = q$ of (8.86) is to solve a system,*

$$\begin{cases} \|h\|^{2n} = \|q\|^2, \\ n\cos^{-1}\left(\frac{Re(h)}{\|h\|}\right) = \cos^{-1}\left(\frac{Re(q)}{\|q\|}\right), \end{cases} \tag{8.101}$$

Proof Solving (8.86) is to solve the system (8.101), by (8.99) and (8.100).

Consider a system (8.101) under the same hypothesis. To satisfy $h^n = q$ in $\mathcal{H}$,

$$\|h\|^n = \|q\|,$$

and hence,

$$n\cos^{-1}\left(\frac{Re(h)}{\|h\|}\right) = \cos^{-1}\left(\frac{Re(q)}{\|q\|}\right),$$

$$\Longleftrightarrow$$

$$\cos^{-1}\left(\frac{Re(h)}{\|h\|}\right) = \frac{1}{n}\cos^{-1}\left(\frac{Re(q)}{\|q\|}\right),$$

$$\Longleftrightarrow$$

$$\frac{Re(h)}{\|h\|} = \cos\left(\frac{1}{n}\cos^{-1}\left(\frac{Re(q)}{\|q\|}\right)\right),$$

$$\Longleftrightarrow$$

$$\frac{Re(h)}{\|q\|^{\frac{1}{n}}} = \cos\left(\frac{1}{n}\cos^{-1}\left(\frac{Re(q)}{\|q\|}\right)\right),$$

$$\Longleftrightarrow$$

$$Re(h) = \|q\|^{\frac{1}{n}}\cos\left(\frac{1}{n}\cos^{-1}\left(\frac{Re(q)}{\|q\|}\right)\right). \tag{8.102}$$

So, if $h_0 = t_1 + t_2 i + t_3 j + t_4 k$ is a solution of $h^n = q$, then

$$t_1 = \|q\|^{\frac{1}{n}}\cos\left(\frac{1}{n}\cos^{-1}\left(\frac{Re(q)}{\|q\|}\right)\right), \tag{8.103}$$

in $\mathcal{R}$, by (8.102), since $t_1 = Re(h_0)$ in $\mathcal{R}$.

By (8.101) and (8.103),

$$t_1^2 + t_2^2 + t_3^2 + t_4^2 = \left(a^2 + b^2 + u^2 + v^2\right)^{\frac{n}{2}},$$

where t_1 is in the sense of (8.103), if and only if

$$t_2^2 + t_3^2 + t_4^2 = \left(a^2 + b^2 + u^2 + v^2\right)^{\frac{n}{2}} - t_1^2. \tag{8.104}$$

Thus, by (8.103) and (8.103), we obtain the following refined result of (8.99).

Theorem 11 *Let $h^n = q$ be a monomial equation (8.86) for $n \in \mathbb{N}_{>1}$. Then a solution*

$$h_0 = t_1 + t_2 i + t_3 j + t_4 k \in \mathcal{H} \text{ of } h^n = q$$

satisfy

$$t_1 = \|q\|^{\frac{1}{n}} \cos\left(\frac{1}{n}\cos^{-1}\left(\frac{Re(q)}{\|q\|}\right)\right),$$

and

$$t_2^2 + t_3^2 + t_4^2 = \|q\|^{\frac{n}{2}} - t_1^2. \tag{8.105}$$

Proof The implicit relation (8.105) of the solutions h_0 of $h^n = q$ is obtained by (8.103) and (8.104). Indeed, by applying (8.99) and (8.101), if $h_0 \in \mathcal{H}$ is a solution of the equation, then

$$h_0^n = q \text{ in } \mathcal{H},$$

if and only if

$$t_1 = \|q\|^{\frac{1}{n}} \cos\left(\frac{1}{n}\cos^{-1}\left(\frac{Re(q)}{\|q\|}\right)\right) \text{ in } \mathcal{R},$$

by (8.103), and

$$t_2^2 + t_3^2 + t_4^2 = \|q\|^{\frac{n}{2}} - t_1^2, \text{ in } \mathcal{R},$$

by (8.104).

The above theorem illustrates that a monomial equation $h^n = q$ can have infinitely many solutions in $\mathcal{H}$ satisfying (8.105), for $n \in \mathbb{N}_{>1}$.

Let $h^n = q$ be (8.86) on $\mathcal{H}$, and assume that

$$q = a + bi + uj + vk \in \mathcal{H}, \|q\| = 1. \tag{8.106}$$

Under the condition (8.106), if $h_0 = t_1 + t_2 i + t_3 j + t_4 k \in \mathcal{H}$ is a solution of $h^n = q$, then

$$t_1 = \cos\left(\frac{1}{n}\cos^{-1}(a)\right),$$

and

$$t_2^2 + t_3^2 + t_4^2 = 1 - \cos^2\left(\tfrac{1}{n}\cos^{-1}(a)\right) \tag{8.107}$$

in $\mathcal{R}$, by (8.105).

Corollary 8 *Let $h^n = q$ be a monomial equation (8.86). If $q \in \mathcal{H}$ is a quaternion satisfying the condition (8.106), and if*

$$h_0 = t_1 + t_2 i + t_3 j + t_4 k \in \mathcal{H}$$

is a solution of $h^n = q$, then

$$t_1 = \cos\left(\frac{1}{n}\cos^{-1}(Re(q))\right),$$

and

$$t_2^2 + t_3^2 + t_4^2 = \sin^2\left(\frac{1}{n}\cos^{-1}(Re(q))\right), \tag{8.108}$$

in $\mathcal{R}$.

Proof Under the condition (8.106), the proof of (8.108) is done by (8.105) and (8.107). Indeed, by the well-known trigonometric identity,

$$\cos^2\left(\frac{1}{n}\cos^{-1}(Re(q))\right) + \sin^2\left(\frac{1}{n}\cos^{-1}(Re(q))\right) = 1,$$

$\Longleftrightarrow$

$$\sin^2\left(\frac{1}{n}\cos^{-1}(Re(q))\right) = 1 - \cos^2\left(\frac{1}{n}\cos^{-1}(Re(q))\right).$$

Therefore, the relation (8.108) holds by (8.107).

Let $\mathcal{R}^3$ be the 3-dimensional vector space over $\mathcal{R}$, and let $(x, y, z) \in \mathcal{R}^3$ be a vector-variable. Consider the sphere formula,

$$x^2 + y^2 + z^2 = r^2,$$

with its center $(0, 0, 0)$, and its radius $r > 0$ in $\mathcal{R}$. Let

$$S_{r,0} \overset{def}{=} \{(x, y, z) \in \mathcal{R}^3 : x^2 + y^2 + z^2 = r^2\} \tag{8.109}$$

be such a sphere in $\mathcal{R}^3$.

The formula (8.108) means that, under (8.106), a solution $h_0 \in \mathcal{H}$ of the equation $h^n = q$ satisfies that

$$t_1 = \cos\left(\frac{1}{n}\cos^{-1}(Re(q))\right),$$

and

$$(t_2, t_3, t_4) \in S_{r,0}, r = \sqrt{1 - t_1^2}. \tag{8.110}$$

Corollary 9 *Let $h_0 = t_1 + t_2 i + t_3 j + t_4 k \in \mathcal{H}$ be a solution of a monomial equation $h^n = q$ of* (8.86)*. If $q \in \mathcal{H}$ satisfies the condition* (8.106)*, then*

$$t_1 = \cos\left(\frac{1}{n}\cos^{-1}(Re(q))\right) \text{ in } \mathcal{R},$$

and

$$(t_2, t_3, t_4) \in S_{r,0}, with r = \sqrt{1 - t_1^2}, \tag{8.111}$$

where $S_{r,0}$ is a sphere (8.109).

Proof The geometric characterization (8.111) of (8.108) is obtained by (8.110).

Example 2 (1) Let $q = \frac{1}{2} + 0i + 0j + \frac{\sqrt{3}}{2}k \in \mathcal{H}$. Consider a monomial equation

$$h^2 = q \text{ on } \mathcal{H}.$$

It is not hard to check that

$$\|q\| = \sqrt{\left(\frac{1}{2}\right)^2 + 0^2 + 0^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1.$$

Thus, by (8.105) and (8.108), if

$$h_0 = t_1 + t_2 i + t_3 j + t_4 k \in \mathcal{H}$$

is a solution of $h^2 = q$, then

$$t_1 = \cos\left(\tfrac{1}{2}\cos^{-1}\left(\tfrac{1}{2}\right)\right)$$
$$= \cos\left(\tfrac{1}{2}\cdot\tfrac{\pi}{3}\right) = \cos\left(\tfrac{\pi}{6}\right) = \tfrac{\sqrt{3}}{2},$$

and

$$t_2^2 + t_3^2 + t_4^2 = 1 - \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4}.$$

So, a solution h_0 is a quaternion

$$h_0 = \frac{\sqrt{3}}{2} + t_2 i + t_3 j + t_4 k \in \mathcal{H},$$

satisfying

$$t_2^2 + t_3^2 + t_4^2 = \frac{1}{4} \Longleftrightarrow (t_2, t_3, t_4) \in S_{\frac{1}{2},0}.$$

(2) Consider a monomial equation $h^3 = q$ on $\mathcal{H}$ where q is as in (1). If

$$h_0 = s_1 + s_2 i + s_3 j + s_4 k \in \mathcal{H}$$

is a solution of $h^3 = q$, then

$$s_1 = \cos\left(\frac{1}{3}\cos^{-1}\left(\frac{1}{2}\right)\right) = \cos\left(\frac{\pi}{9}\right),$$

and

$$s_2^2 + s_3^2 + s_4^2 = \sin^2\left(\frac{\pi}{9}\right).$$

So, a solution h_0 is a quaternion,

$$h_0 = \cos\left(\frac{\pi}{9}\right) + s_2 i + s_3 j + s_4 k \in \mathcal{H},$$

satisfying

$$s_2^2 + s_3^2 + s_4^2 = \sin^2\left(\frac{\pi}{9}\right).$$

(3) Consider now a monomial equation $h^2 = 1$ on $\mathcal{H}$. If

$$h_0 = t_1 + t_2 i + t_3 j + t_4 k \in \mathcal{H}$$

is a solution of $h^2 = 1$, then

$$t_1 = \cos\left(\frac{1}{2}\cos^{-1}(1)\right) = \begin{cases} \cos\left(\frac{0}{2}\right) = 1 & \text{or} \\ \cos\left(\frac{2\pi}{2}\right) = -1. \end{cases}$$

and

$$t_2^2 + t_3^2 + t_4^2 = \begin{cases} \sin^2(0) = 0 & \text{respectively} \\ \sin^2(\pi) = 0. \end{cases}$$

Therefore, the equation $h^2 = 1$ has its solutions

$$h = 1, \text{ or } -1.$$

(4) Consider a monomial equation $h^2 = -1$ on $\mathcal{H}$. If

$$h_0 = s_1 + s_2 i + s_3 j + s_4 k \in \mathcal{H}$$

is a solution of $h^2 = -1$, then

$$s_1 = \cos\left(\frac{1}{2}\cos^{-1}(-1)\right) = \cos\left(\frac{\pi}{2}\right) = 0,$$

and

$$s_2^2 + s_3^2 + s_4^2 = \sin^2\left(\frac{\pi}{2}\right) = 1.$$

Thus, all quaternions

$$0 + s_2 i + s_3 j + s_4 k \in \mathcal{H}, \text{ with } s_2^2 + s_3^2 + s_4^2 = 1$$

are the solutions of $h^2 = -1$.

Let's now consider the solvability of monomial equation.

Corollary 10 *A monomial equation $h^n = q$ is solvable on $\mathcal{H}$, if and only if*

$$\|q\|^{\frac{n}{2}} \geq \|q\|^{\frac{2}{n}} \cos\left(\frac{1}{n}\cos^{-1}\left(\frac{Re(q)}{\|q\|}\right)\right), \tag{8.112}$$

for $n \in \mathbb{N}_{>1}$.

Proof By (8.105), if $h_0 = t_1 + t_2 i + t_3 j + t_4 k \in \mathcal{H}$ is a solution of the monomial equation $h^n = q$, for $n \in \mathbb{N}_{>1}$, then

$$t_1 = \|q\|^{\frac{1}{n}} \cos\left(\frac{1}{n}\cos^{-1}\left(\frac{Re(q)}{\|q\|}\right)\right),$$

and

$$t_2^2 + t_3^2 + t_4^2 = \|q\|^{\frac{n}{2}} - t_1^2. \tag{8.113}$$

Thus, if

$$\|q\|^{\frac{n}{2}} - t_1^2 < 0 \Longleftrightarrow \|q\|^{\frac{n}{2}} < t_1^2 \text{ in } \mathcal{R},$$

then the second equality of (8.113) is undefined since the left-hand side is non-negative, but the right-hand side becomes negative in $\mathcal{R}$. i.e., the quaternion h_0 is undefined in $\mathcal{H}$. It implies that the equation $h^n = q$ has no solutions on $\mathcal{H}$, whenever $\|q\|^{\frac{n}{2}} < t_1^2$ in $\mathcal{R}$.

Conversely, assume that

$$\|q\|^{\frac{n}{2}} - t_1^2 \geq 0 \Longleftrightarrow \|q\|^{\frac{n}{2}} \geq t_1^2 \text{ in } \mathcal{R}.$$

Then the second equality of (8.113) is well-determined on $\mathcal{R}$. i.e., a quaternion h_0 satisfying $h_0^n = q$ does exist in $\mathcal{H}$.

Therefore, a equation $h^n = q$ has its solutions in $\mathcal{H}$, if and only if the condition (8.112) holds in $\mathcal{R}$.

Example 3 The above solvability condition (8.112) shows that every quadratic monomial equation $h^2 = q$ is solvable on $\mathcal{H}$. Indeed, by the boundedness of the cosine function on $\mathcal{R}$,

$$-1 \leq \cos\left(\frac{1}{n}\cos^{-1}\left(\frac{Re(q)}{\|q\|}\right)\right) \leq 1 \text{ in } \mathcal{R},$$

we have

$$0 \leq t_0 \overset{denote}{=} \cos^2\left(\frac{1}{n}\cos^{-1}\left(\frac{Re(q)}{\|q\|}\right)\right) \leq 1 \text{ in } \mathcal{R}.$$

It satisfies the condition (8.112) for $n = 2$, since

$$\|q\|^{\frac{2}{2}} \geq \|q\|^{\frac{2}{2}}\, t_0 \Longleftrightarrow \|q\| \geq \|q\|\, t_0 \text{ in } \mathcal{R}.$$

Since $q \in \mathcal{H}$ is arbitrary, every equation $h^2 = q$ is solvable on $\mathcal{H}$ by (8.112).

Also, one can get the following corollary of (8.112).

Corollary 11 *Let $h^n = q$ be a monomial equation on $\mathcal{H}$. If $\|q\| = 1$, then this equation is solvable on $\mathcal{H}$.*

Proof Let $h^n = q$ be a monomial equation on $\mathcal{H}$, where $\|q\| = 1$. If $h_0 = t_1 + t_2 i + t_3 j + t_4 k \in \mathcal{H}$ is a solution, then

$$t_1 = \cos\left(\frac{1}{n}\cos^{-1}\left(Re(q)\right)\right),$$

and

$$t_2^2 + t_3^2 + t_4^2 = 1 - t_1^2.$$

by (8.108) and (8.111). One can verify that

$$1 - t_1^2 \geq 0 \text{ in } \mathcal{R},$$

by the boundedness of the cosine function on $\mathcal{R}$. i.e., such an equation automatically satisfies the condition (8.112). Therefore, every equation $h^n = q$ with $\|q\| = 1$ is solvable on $\mathcal{H}$.

Under our q-spectral relation on $\mathcal{H}$, the following theorem is obtained.

Theorem 12 *Let $h^n = q$ be a monomial equation* (8.86) *for $n \in \mathbb{N}_{>1}$, and suppose $h_0 \in \mathcal{H}$ is a solution of $h^n = q$. If h_1 is similar to h_0 in the sense of* (8.48)*, then h_1 is a solution of $h^n = q$, i.e., it also satisfies $h_1^n = q$ in $\mathcal{H}$.*

Proof Recall that the similarity (8.48) on $\mathcal{H}$, and the q-spectral relation (8.44) are same as equivalence relations. So, if

$$h_1 = s_1 + s_2 i + s_3 j + s_4 k$$

is similar to

$$h_0 = t_1 + t_2 i + t_3 j + t_4 k$$

in $\mathcal{H}$, then they are q-spectral related, i.e.,

$$\sigma(h_0) = \sigma(h_1) \text{ in } \mathcal{C},$$

$\Longleftrightarrow$

$$t_1 + i\sqrt{t_2^2 + t_3^2 + t_4^2} = s_1 + i\sqrt{s_2^2 + s_3^2 + s_4^2}. \tag{8.114}$$

The equality (8.114) implies that

$$s_1 = t_1 \text{ in } \mathcal{R},$$

and

$$s_2^2 + s_3^2 + s_4^2 = t_2^2 + t_3^2 + t_4^2. \tag{8.115}$$

Also, recall that solving a given equation $h^n = q$ is to solve

$$\sigma(h)^n = \sigma(q). \tag{8.116}$$

By (8.115), one has that

$$\sigma(h_0)^n = \sigma(q) = \sigma(h_1)^n,$$

implying that h_1 is again a solution of $h^n = q$, by (8.116).

8.6 Certain Quadratic Equations on $\mathcal{H}$

In Sect. 8.5, we considered monomial equations $h^n = q$, for $n \in \mathbb{N}_{>1}$, where h is a $\mathcal{H}$-variable, and $q \in \mathcal{H}$ is fixed. By using the q-spectralization σ, we showed that if

$$h_0 = t_1 + t_2 i + t_3 j + t_4 k \in \mathcal{H}$$

is a solution of $h^n = q$, then

$$t_1 = \|q\|^{\frac{1}{n}} \cos\left(\frac{1}{n}\cos^{-1}\left(\frac{Re(q)}{\|q\|}\right)\right),$$

and

$$t_2^2 + t_3^2 + t_4^2 = \|q\|^{\frac{n}{2}} - t_1^2, \tag{8.117}$$

by (8.105), up to (8.112). In this section, we consider quadratic equations on $\mathcal{H}$,

$$h^2 + th + s = 0, \tag{8.118}$$

for fixed $t, s \in \mathcal{R}$.

Remark 3 (1) One may / can consider quadratic equation,

$$ah^2 + bh + c = 0 \tag{8.119}$$

on $\mathcal{H}$, for a $\mathcal{H}$-variable h, and $a \in \mathcal{R}^{\times}$ and $b, c \in \mathcal{R}$. But, (8.119) is equivalent to

$$h^2 + \frac{b}{a}h + \frac{c}{a} = 0 \text{ on } \mathcal{H}.$$

Thus, in the long run, studying the equations of (8.119) is to consider the quadratic equations (8.118) on $\mathcal{H}$.
(2) We now justify why we have restricted to the special cases where $t, s \in \mathcal{R}$ in (8.118). Here, our quaternion-spectral mapping theorem (8.57) would be applied. To do that, the coefficients of the left-hand side of (8.118) should be real numbers by (8.55).

As in Sect. 8.5, let's consider q-spectralizations. i.e., if

$$h_0 = t_1 + t_2 i + t_3 j + t_4 k \in \mathcal{H}, \tag{8.120}$$

is a solution of (8.118), then

$$h_0^2 + th_0 + s = 0 \text{ in } \mathcal{H},$$

$\Longleftrightarrow$

$$\left[h_0^2 + th_0 + s\right] = [0] \text{ in } \mathcal{H}_2,$$

$\Longleftrightarrow$

$$\left[\sigma(h_0^2 + th_0 + s)\right] = [0] = [\sigma(0)] \text{ in } \mathcal{H}_2,$$

$\Longleftrightarrow$

$$\left[\sigma(h_0)^2 + t\sigma(h_0) + s\right] = [0] \text{ in } \mathcal{H}_2,$$

$\Longleftrightarrow$

$$\sigma(h_0)^2 + t\sigma(h_0) + s = 0 \text{ in } \mathcal{C},$$

by (8.55) and (8.57), where

$$\sigma(h_0) = t_1 + i\sqrt{t_2^2 + t_3^2 + t_4^2} \in \mathbb{C} \tag{8.121}$$

is the q-spectral value of h_0.

Lemma 6 *Let $h = x_1 + y_1 i + x_2 j + y_2 k$ be a $\mathcal{H}$-variable, and let*

$$\sigma(h) = x_1 + i\sqrt{y_1^2 + x_2^2 + y_2^2}$$

be the q-spectral value of h, as an unknown in $\mathbb{C}$. Then solving a quadratic equation (8.118) is to solve

$$\sigma(h)^2 + t\sigma(h) + s = 0 \text{ on } \mathbb{C}.$$

Proof The proof is done by (8.57) and (8.121).

Now, let h_0 be a solution (8.122) of (8.118). Then

$$\sigma(h_0)^2 + t\sigma(h_0) + s = 0, \tag{8.122}$$

by the above lemma. Consider (8.122) in detail:

$$\sigma(h_0)^2 + t\sigma(h_0) + s = 0,$$

$$\Longleftrightarrow$$

$$\left(t_1 + i\sqrt{t_2^2 + t_3^2 + t_4^2}\right)^2 + t\left(t_1 + i\sqrt{t_2^2 + t_3^2 + t_4^2}\right) + s = 0,$$

$$\Longleftrightarrow$$

$$\left(t_1^2 - t_2^2 - t_3^2 - t_4^2 + tt_1 + s\right) + i(2t_1 + t)\sqrt{t_2^2 + t_3^2 + t_4^2} = 0. \tag{8.123}$$

Theorem 13 *Let $h_0 = t_1 + t_2 i + t_3 j + t_4 k \in \mathcal{H}$ be a solution (8.122) of a quadratic equation (8.118). Then either*

$$t_1 = -\frac{t}{2}, \text{ and } t_2^2 + t_3^2 + t_4^2 = s - \frac{t^2}{4},$$

or

$$t_1 = \frac{-t \pm \sqrt{t^2 - 4s}}{2}, t_2^2 + t_3^2 + t_4^2 = 0. \tag{8.124}$$

Proof By (8.121) and (8.122), if h_0 is a solution of $h^2 + th + s = 0$, then

$$\sigma(h_0)^2 + t\sigma(h_0) + s = 0 \text{ in } \mathbb{C},$$

where

$$\sigma(h_0) = t_1 + i\sqrt{t_2^2 + t_3^2 + t_4^2}$$

is the q-spectral value of h_0. And, this equality is equivalent to the equality (8.123), implying the system

$$\begin{cases} t_1^2 - t_2^2 - t_3^2 - t_4^2 + tt_1 + s = 0 \\ \\ (2t_1 + t)\sqrt{t_2^2 + t_3^2 + t_4^2} = 0. \end{cases}$$

The second equality of the above system shows that

$$t_1 = -\frac{t}{2}, \text{ or } t_2^2 + t_3^2 + t_4^2 = 0.$$

If $t_1 = -\frac{t}{2}$, then the first equality of the system can be re-written by

$$\frac{t^2}{4} - t_2^2 - t_3^2 - t_4^2 - \frac{t^2}{2} + s = 0,$$

$\Longleftrightarrow$

$$t_2^2 + t_3^2 + t_4^2 = s - \frac{t^2}{4} \text{ in } \mathcal{R}.$$

Meanwhile, if $t_2^2 + t_3^2 + t_4^2 = 0$, then the first equality becomes that

$$t_1^2 - (t_2^2 + t_3^2 + t_4^2) + tt_1 + s = 0,$$

$\Longleftrightarrow$

$$t_1^2 + tt_1 + s = 0 \text{ in } \mathcal{R},$$

$\Longleftrightarrow$

$$t_1 = \frac{-t \pm \sqrt{t^2 - 4s}}{2} \text{ in } \mathcal{R}.$$

Therefore, if $h_0 \in \mathcal{H}$ is a solution, then either

$$t_1 = -\frac{t}{2}, \text{ and } t_2^2 + t_3^2 + t_4^2 = s - \frac{t^2}{4},$$

or

$$t_1 = \frac{-t \pm \sqrt{t^2 - 4s}}{2}, \text{ and } t_2^2 + t_3^2 + t_4^2 = 0.$$

So, the relation (8.124) holds.

Example 4 (1) Consider a quadratic equation

$$h^2 + 3h - 2 = 0 \text{ on } \mathcal{H}.$$

By (8.124), if $h_0 = t_1 + t_2 i + t_3 j + t_4 k \in \mathcal{H}$ is a solution, then either

$$t_1 = -\frac{3}{2}, \text{ and } t_2^2 + t_3^2 + t_4^2 = -2 - \frac{9}{4} = -\frac{17}{4},$$

or

$$t_1 = \frac{-3 \pm \sqrt{17}}{2}, \text{ and } t_2^2 + t_3^2 + t_4^2 = 0.$$

Suppose $t_1 = -\frac{3}{2}$. Then, since $t_2, t_3, t_4 \in \mathcal{R}$, there does not exist $(t_2, t_3, t_4) \in \mathcal{R}^3$, such that

$$t_2^2 + t_3^2 + t_4^2 = -\frac{17}{4} \text{ in } \mathcal{R},$$

because the left-hand side is nonnegative in $\mathcal{R}$. It shows that t_1 cannot be $-\frac{3}{2}$. Therefore, the solutions of this equation are

$$\left(\frac{-3 \pm \sqrt{17}}{2}\right) + 0i + 0j + 0k \text{ in } \mathcal{H}.$$

(2) Consider a quadratic equation

$$h^2 + h + 1 = 0 \text{ on } \mathcal{H}.$$

By (8.124), if $h_0 = t_1 + t_2 i + t_3 j + t_4 k \in \mathcal{H}$ is a solution, then either

$$t_1 = -\frac{1}{2}, \text{ and } t_2^2 + t_3^2 + t_4^2 = 1 - \frac{1}{4} = \frac{3}{4}.$$

or

$$t_1 = \frac{-1 \pm \sqrt{-3}}{2}, \text{ and } t_2^2 + t_3^2 + t_4^2 = 0 \text{ in } \mathcal{R}.$$

Note that if $t_1 = 2^{-1}\left(-1 \pm \sqrt{-3}\right)$, then it is undefined in $\mathcal{R}$. So, it implies that all quaternions

$$\frac{1}{2} + \alpha_2 i + \alpha_3 j + \alpha_4 k, \text{ with } \alpha_2^2 + \alpha_3^2 + \alpha_4^2 = \frac{3}{4}$$

are the solutions.

Motivated by the above example, the following result is obtained.

Corollary 12 *Let* $h_0 = t_1 + t_2 i + t_3 j + t_4 k \in \mathcal{H}$ *be a solution of (8.118). If* $t^2 \geq 4s$, *then*

$$h_0 = \frac{-t \pm \sqrt{t^2 - 4s}}{2} + 0i + 0j + 0k. \tag{8.125}$$

Meanwhile, If $t^2 < 4s$, *then*

$$t_1 = -\frac{t}{2}, t_2^2 + t_3^2 + t_4^2 = s - \frac{t^2}{4}. \tag{8.126}$$

Proof If $h_0 \in \mathcal{H}$ is a solution of (8.118), then either

$$t_1 = -\frac{t}{2}, \text{ and } t_2^2 + t_3^2 + t_4^2 = s - \frac{t^2}{4},$$

or

$$t_1 = \frac{-t \pm \sqrt{t^2 - 4s}}{2}, \text{ and } t_2^2 + t_3^2 + t_4^2 = 0,$$

by (8.124).

Assume first that

$$t^2 \geq 4s \Longleftrightarrow s \leq \frac{t^2}{4} \Longleftrightarrow s - \frac{t^2}{4} \leq 0 \text{ in } \mathcal{R}.$$

If $s - \frac{t^2}{4} < 0$, then there does not exist $(t_2, t_3, t_4) \in \mathcal{R}^3$, such that

$$t_2^2 + t_3^2 + t_4^2 = s - \frac{t^2}{4}.$$

Thus, in such a case, a solution $h_0 \in \mathcal{H}$ needs to satisfy

$$t_1 = \frac{-t \pm \sqrt{t^2 - 4s}}{2}, \text{ and } t_2^2 + t_3^2 + t_4^2 = 0.$$

If $s - \frac{t^2}{4} = 0$, then

$$t_1 = \frac{-t \pm \sqrt{0}}{2} = -\frac{t}{2}, \text{ and } t_2^2 + t_3^2 + t_4^2 = 0.$$

i.e., if $t^2 \geq 4s$, then a solution h_0 satisfies that

$$t_1 = \frac{-t \pm \sqrt{t^2 - 4s}}{2}, \text{ and } t_2 = t_3 = t_4 = 0 \text{ in } \mathcal{R}.$$

Thus, the statement (8.125) holds.

Suppose now that

$$t^2 < 4s \iff s > \frac{t^2}{4} \iff s - \frac{t^2}{4} > 0 \text{ in } \mathcal{R}.$$

Then one can take infinitely many $(t_2, t_3, t_4) \in \mathcal{R}^3$ satisfying

$$t_2^2 + t_3^2 + t_4^2 = s - \frac{t^2}{4} \text{ in } \mathcal{R}.$$

Remark that since $t_2 - 4s < 0$, the real number t_1 cannot be identical to

$$\frac{-t \pm \sqrt{t^2 - 4s}}{2} \text{ in } \mathcal{R},$$

since it is undefined in $\mathcal{R}$. So, in this case, h_0 satisfies

$$t_1 = \frac{-t}{2}, \text{ and } t_2^2 + t_3^2 + t_4^2 = s - \frac{t^2}{4},$$

by (8.124). Therefore, the statement (8.126) holds.

The above corollary refines (8.124) by (8.125) and (8.126).

Theorem 14 *Every quadratic equation,*

$$ah^2 + bh + c = 0, \textit{ with } a \in \mathcal{R}^{\times} b, c \in \mathcal{R},$$

is solvable on $\mathcal{H}$.

Proof Let $ah^2 + bh + c = 0$ be a quadratic equation on $\mathcal{H}$, where h is a $\mathcal{H}$-variable, and $a \in \mathcal{R}^{\times}$ and $b, c \in \mathcal{R}$. Then it is equivalent to an equation,

$$h^2 + \left(\frac{b}{a}\right) h + \left(\frac{c}{a}\right) = 0.$$

So, the solvability of such a quadratic equation is that of an equation

$$h^2 + th + s = 0, \text{ with } t, s \in \mathcal{R}.$$

For a given two real numbers t and s, they satisfy either

$$t^2 \geq 4s, \text{ or } t^2 < 4s, \text{ in } \mathcal{R},$$

by the axiom of choice.

However, if $t^2 \geq 4s$ in $\mathcal{R}$, then the equation has its solution

$$\left(\frac{-t \pm \sqrt{t^2 - 4s}}{2}\right) + 0i + 0j + 0k \text{ in } \mathcal{H},$$

by (8.125); and if $t^2 < 4s$ in $\mathcal{R}$, then this equation has its solutions

$$-\frac{t}{2} + t_2 i + t_3 j + t_4 k \in \mathcal{H},$$

with

$$t_2^2 + t_3^2 + t_4^2 = s - \frac{t^2}{4},$$

by (8.126). Therefore, such an equation always has its solutions in $\mathcal{H}$, by (8.124).

The above theorem shows that all quadratic equations with real coefficients are solvable on the quaternions $\mathcal{H}$.

Consider following observations from (8.105) and (8.124).

Remark 4 Let $-s \in \mathcal{R}$. Consider a monomial equation

$$h^2 = -s \text{ on } \mathcal{H}.$$

By (8.105), if $h_0 = t_1 + t_2 i + t_3 j + t_4 k \in \mathcal{H}$ is a solution, then

$$t_1 = \|-s\|^{\frac{1}{2}} \cos\left(\frac{1}{2}\cos^{-1}\left(\frac{-s}{\|-s\|}\right)\right),$$

and

$$t_2^2 + t_3^2 + t_4^2 = \|-s\|^{\frac{2}{2}} - t_1^2.$$

Since $-s \in \mathcal{R}$, $\|-s\| = |-s|$ in $\mathcal{R}_+$, where $|.|$ is the absolute value on $\mathcal{R}$. So, either

$$t_1 = \sqrt{|-s|}\cos\left(\tfrac{1}{2}\cos^{-1}(-1)\right)$$
$$= \sqrt{|-s|}\cos\left(\tfrac{\pi}{2}\right) = 0,$$

or

$$t_1 = \sqrt{|-s|}\cos\left(\tfrac{1}{2}\cos^{-1}(1)\right)$$
$$= \begin{cases} \sqrt{|-s|}\cos(0) = \sqrt{|-s|}, & \text{or} \\ \sqrt{|-s|}\cos\left(\tfrac{\pi}{2}\right) = -\sqrt{|-s|}, & \end{cases}$$

and

$$t_2^2 + t_3^2 + t_4^2 = |-s| \text{ (if } t_1 = 0),$$

respectively

$$t_2^2 + t_3^2 + t_4^2 = 0 \text{ (if } t_1 = \text{ either } \sqrt{|-s|}, \text{ or } -\sqrt{|-s|}).$$

Therefore, the quaternions

$$\pm\sqrt{|-s|} + 0i + 0j + 0k \in \mathcal{H},$$

or

$$0 + t_2 i + t_3 j + t_4 k \in \mathcal{H}, t_2^2 + t_3^2 + t_4^2 = |-s| \tag{8.127}$$

are the solutions of the equation $h^2 = -s$.

Now, let's understand the above monomial equation $h^2 = -s$ as a quadratic equation,

$$h^2 + 0h + s = 0 \text{ on } \mathcal{H}.$$

If $h_0 = t_1 + t_2 i + t_3 j + t_4 k \in \mathcal{H}$ is a solution, then either

$$t_1 = -\frac{0}{2}, \text{ and } t_2^2 + t_3^2 + t_4^2 = s,$$

or

$$t_1 = \frac{-0 \pm \sqrt{0 - 4s}}{2} = \sqrt{-s}, t_2^2 + t_3^2 + t_4^2 = 0. \tag{8.128}$$

by (8.124).

If one compares (8.127) with (8.128), then he can realize that two relations are identical on $\mathcal{H}$.

8.7 Linear Equations on $\mathcal{H}$

In this section, we consider linear equations

$$q_1 h + q_2 = q_3 \text{ on } \mathcal{H},$$

where h is a $\mathcal{H}$-variable, and

$$q_1 \in \mathcal{H}^{\times} = \mathcal{H} \setminus \{0\}, q_2, q_3 \in \mathcal{H}. \tag{8.129}$$

Clearly, if $q_1 = 1$, and $q_2 = 0$ in $\mathcal{H}$, then (8.129) becomes the trivial equation $h = q_3$. So, we are not interested in such a case.

In a similar manner of Sects. 8.5 and 8.6, solving (8.129) is to solve

$$[q_1 h + q_2] = [q_3] \Longleftrightarrow [q_1][h] + [q_2] = [q_3]. \tag{8.130}$$

From below, let

$$h = z_1 + z_2 j, \text{ with two } \mathbb{C}\text{-variables } z_1, z_2,$$

and

$$q_l = a_l + b_l j \in \mathcal{H}, a_l, b_l \in \mathbb{C}, \tag{8.131}$$

for all $l = 1, 2, 3$.

By (8.130) and (8.131), solving (8.129) is to solve the matricial equation,

$$[q_1][h] + [q_2] = [q_3] \text{ in } \mathcal{H}_2,$$

$\Longleftrightarrow$

$$\begin{pmatrix} a_1 & -b_1 \\ \overline{b_1} & \overline{a_1} \end{pmatrix} \begin{pmatrix} z_1 & -z_2 \\ \overline{z_2} & \overline{z_1} \end{pmatrix} + \begin{pmatrix} a_2 & -b_2 \\ \overline{b_2} & \overline{a_2} \end{pmatrix} = \begin{pmatrix} a_3 & -b_3 \\ \overline{b_3} & \overline{a_3} \end{pmatrix},$$

$\Longleftrightarrow$

$$\begin{pmatrix} a_1 z_1 - b_1 \overline{z_2} + a_2 & -a_1 z_2 - b_1 \overline{z_1} - b_2 \\ \overline{b_1} z_1 + \overline{a_1 z_2} + \overline{b_2} & -\overline{b_1} z_2 + \overline{a_1 z_1} + \overline{a_2} \end{pmatrix} = \begin{pmatrix} a_3 & -b_3 \\ \overline{b_3} & \overline{a_3} \end{pmatrix}, \tag{8.132}$$

in $\mathcal{H}_2$.

Lemma 7 *Under* (8.131)*, solving a linear equation* $q_1 h + q_2 = q_3$ *of* (8.129) *is to solve a system,*

$$\begin{cases} a_1 z_1 - b_1 \overline{z_2} + a_2 = a_3 \\ a_1 z_2 + b_1 \overline{z_1} + b_2 = b_3. \end{cases} \tag{8.133}$$

Proof The system (8.133) is obtained by (8.132).

By the system (8.133), one can get the following equivalent systems

$$\begin{cases} a_1 z_1 - b_1 \overline{z_2} + a_2 = a_3 \\ \overline{b_1} z_1 + \overline{a_1 z_2} + \overline{b_2} = \overline{b_3}, \end{cases}$$

$\Longleftrightarrow$

$$\begin{cases} |a_1|^2 z_1 - \overline{a_1} b_1 \overline{z_2} + \overline{a_1} a_2 = \overline{a_1} a_3 \\ |b_1|^2 z_1 + \overline{a_1} b_1 \overline{z_2} + b_1 \overline{b_2} = b_1 \overline{b_3}, \end{cases} \tag{8.134}$$

where $|.|$ is the modulus on $\mathbb{C}$. Then

$$\left(|a_1|^2 + |b_1|^2\right) z_1 + \left(\overline{a_1}a_2 + b_1\overline{b_2}\right) = \overline{a_1}a_3 + b_1\overline{b_3},$$

$\Longleftrightarrow$

$$z_1 = \frac{\left(\overline{a_1}a_3 + b_1\overline{b_3}\right) - \left(\overline{a_1}a_2 + b_1\overline{b_2}\right)}{|a_1|^2 + |b_1|^2}, \tag{8.135}$$

in $\mathbb{C}$, by (8.134). So, one has that

$$a_1 z_1 - b_1\overline{z_2} + a_2 = a_3,$$

$\Longleftrightarrow$

$$a_1\left(\frac{\left(\overline{a_1}a_3 + b_1\overline{b_3}\right) - \left(\overline{a_1}a_2 + b_1\overline{b_2}\right)}{|a_1|^2 + |b_1|^2}\right) - b_1\overline{z_2} + a_2 = a_3,$$

$\Longleftrightarrow$

$$b_1\overline{z_2} = a_1\left(\frac{\left(\overline{a_1}a_3 + b_1\overline{b_3}\right) - \left(\overline{a_1}a_2 + b_1\overline{b_2}\right)}{|a_1|^2 + |b_1|^2}\right) + a_2 - a_3,$$

$\Longleftrightarrow$

$$\overline{z_2} = b_1^{-1}\left(a_1\left(\frac{\left(\overline{a_1}a_3 + b_1\overline{b_3}\right) - \left(\overline{a_1}a_2 + b_1\overline{b_2}\right)}{|a_1|^2 + |b_1|^2}\right) + a_2 - a_3\right), \tag{8.136}$$

in $\mathbb{C}$, by (8.135).

Theorem 15 *Let $q_1 h + q_2 = q_3$ be a linear equation (8.129), where q_1, q_2, q_3 are fixed quaternions (8.131). Then the solution h_0 of this equation is a quaternion*

$$h_0 = w_1 + w_2 j \in \mathcal{H}, \text{ with } w_1, w_2 \in \mathbb{C},$$

with

$$w_1 = \frac{\left(\overline{a_1}a_3 + b_1\overline{b_3}\right) - \left(\overline{a_1}a_2 + b_1\overline{b_2}\right)}{|a_1|^2 + |b_1|^2},$$

and

$$w_2 = \overline{b_1^{-1}\left(a_1\left(\frac{\left(\overline{a_1}a_3 + b_1\overline{b_3}\right) - \left(\overline{a_1}a_2 + b_1\overline{b_2}\right)}{|a_1|^2 + |b_1|^2}\right) + a_2 - a_3\right)}. \tag{8.137}$$

Proof The straightforward computations proves (8.137) by (8.133), (8.134), (8.135) and (8.136).

The above theorem is easy to be proven, but the solutions of arbitrary linear equations are formulated by (8.137).

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Chapter 9
Multiplication and Linear Integral Operators on L_p Spaces Representing Polynomial Covariant Type Commutation Relations

Domingos Djinja, Sergei Silvestrov, and Alex Behakanira Tumwesigye

Abstract Representations of polynomial covariant type commutation relations by pairs of linear integral operators and multiplication operators on Banach spaces L_p are constructed.

Keywords Multiplication operators · Integral operators · Covariance commutation relations

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9.1 Introduction

Commutation relations of the form

$$AB = BF(A) \tag{9.1}$$

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where A, B are elements of an associative algebra and F is a function of the elements of the algebra, are important in many areas of Mathematics and applications. Such commutation relations are usually called covariance relations, crossed product relations or semi-direct product relations. Elements of an algebra that satisfy (9.1) are called a representation of this relation in that algebra. Representations of covariance commutation relations (9.1) by linear operators are important for the study of actions and induced representations of groups and semigroups, crossed product operator algebras, dynamical systems, harmonic analysis, wavelets and fractals analysis and have applications in physics and engineering [4, 5, 20–22, 26–28, 34, 36, 45]. A description of the structure of representations for the relation (9.1) and more general families of self-adjoint operators satisfying such relations by bounded and unbounded self-adjoint linear operators on a Hilbert space use reordering formulas for functions of the algebra elements and operators satisfying covariance commutation relation, functional calculus and spectral representation of operators and interplay with dynamical systems generated by iteration of involved in the commutation relations [3, 6–8, 10, 11, 13–17, 29–34, 37–41, 45–58]. Algebraic properties of the commutation relation (9.1) are important in description of properties of its representations. For instance, there is a well-known link between linear operators satisfying the commutation relation (9.1) and spectral theory [44]. A description of the structure of representations for the relation (9.1) by bounded and unbounded self-adjoint linear operators on a Hilbert space, using spectral representation [2] of such operators, is given in [44] devoted to more general cases of families of commuting self-adjoint operators satisfying relations of the form (9.1).

In this paper we construct representations of (9.1) by pairs of linear integral and multiplication operators on Banach spaces L_p. Such representations can also be viewed as solutions for operator equations $AX = XF(A)$, when A is specified or $XB = BF(X)$ when B is specified. In contrast to [34, 45, 46, 58] devoted to involutive representations of covariance type relations by operators on Hilbert spaces using spectral theory of operators on Hilbert spaces, we aim at direct construction of various classes of representations of covariance type relations in specific important classes of operators on Banach spaces more general than Hilbert spaces without imposing any involution conditions and not using classical spectral theory of operators. This paper is organized in three sections. After the introduction, we present in Sect. 9.2 preliminaries, notations and basic definitions. In Sect. 9.3 we present the main results about construction of specific representations on Banach function spaces L_p.

9.2 Preliminaries and Notations

In this section we present some preliminaries, basic definitions and notations. For more details please read [1, 12, 18, 23, 24, 42, 43].

Let $S \subseteq \mathbb{R}$, ($\mathbb{R}$ is the set of real numbers), be a Lebesgue measurable set and let $(S, \Sigma, \tilde{m})$ be a σ-finite measure space, that is, S is a nonempty set, Σ is a σ-algebra with subsets of S, where S can be covered with at most countably many disjoint sets $E_1, E_2, E_3, \dots$ such that $E_i \in \Sigma$, $\tilde{m}(E_i) < \infty$, $i = 1, 2, \dots$ and $\tilde{m}$ is the Lebesgue measure. For $1 \leqslant p < \infty$, we denote by $L_p(S)$, the set of all classes of equivalent measurable functions $f : S \to \mathbb{R}$ such that $\int\limits_S |f(t)|^p dt < \infty$. This is a Banach space (Hilbert space when $p = 2$) with norm $\|f\|_p = \left(\int\limits_S |f(t)|^p dt\right)^{\frac{1}{p}}$. We denote by $L_\infty(S)$ the set of all classes of equivalent measurable functions $f : S \to \mathbb{R}$ such that there is a constant $\lambda > 0$, $|f(t)| \leq \lambda$ almost everywhere. This is a Banach space with norm $\|f\|_\infty = \text{ess sup}_{t\in S} |f(t)|$.

9.3 Operator Representations of Covariance Commutation Relations

Before we proceed with constructions of more complicated operator representations of commutation relations (9.1) on more complicated Banach spaces, we wish to mention the following two observations that, while being elementary, nevertheless explicitly indicate differences in how the different operator representations of commutation relations (9.1) interact with the function F.

Proposition 9.3.1 *Let $A : E \to E$ and $B : E \to E$, $B \neq 0$, be linear operators on a linear space E, such that any composition among them is well defined and consider $F : \mathbb{R} \to \mathbb{R}$ a polynomial. If $A = \alpha I$, then $AB = BF(A)$ if and only if $F(\alpha) = \alpha$.*

Proof If $A = \alpha I$, then

$$AB = \alpha IB = \alpha B,$$
$$BF(A) = BF(\alpha I) = BF(\alpha)I = F(\alpha)B.$$

We have then $AB = BF(A)$, $B \neq 0$ if and only if $F(\alpha) = \alpha$. □

Proposition 9.3.2 *Let $A : E \to E$ and $B : E \to E$ be linear operators such that any composition among them is well defined and consider a polynomial $F : \mathbb{R} \to \mathbb{R}$. If $B = \alpha I$, where $\alpha \neq 0$, then $AB = BF(A)$ if and only if F is a function such that $F(A) = A$.*

Proof If $B = \alpha I$ then

$$AB = A(\alpha I) = \alpha A,$$
$$BF(A) = \alpha IF(A) = \alpha F(A).$$

We have then $AB = BF(A)$ if and only if $F(A) = A$. □

9.3.1 Representations of Covariance Commutation Relations by Integral and Multiplication Operators on L_p Spaces

We consider first a useful lemma for integral operators.

Lemma 9.3.1 *Let $f : [\alpha_1, \beta_1] \to \mathbb{R}$, $g : [\alpha_2, \beta_2] \to \mathbb{R}$ be two measurable functions such that for all $x \in L_p(\mathbb{R})$, $1 \le p \le \infty$,*

$$\int_{\alpha_1}^{\beta_1} f(t)x(t)dt < \infty, \quad \int_{\alpha_2}^{\beta_2} g(t)x(t)dt < \infty,$$

where $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$, $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$. Set $G = [\alpha_1, \beta_1] \cap [\alpha_2, \beta_2]$. Then the following statements are equivalent:

(i) *For all $x \in L_p(\mathbb{R})$, where $1 \le p \le \infty$, the following holds*

$$\int_{\alpha_1}^{\beta_1} f(t)x(t)dt = \int_{\alpha_2}^{\beta_2} g(t)x(t)dt.$$

(ii) *The following conditions hold:*

a) *for almost every $t \in G$, $f(t) = g(t)$;*
b) *for almost every $t \in [\alpha_1, \beta_1] \setminus G$, $f(t) = 0$;*
c) *for almost every $t \in [\alpha_2, \beta_2] \setminus G$, $g(t) = 0$.*

Proof $(ii) \Rightarrow (i)$ follows from direct computation.

Suppose that (i) is true. Take $x(t) = I_{G_1}(t)$ the indicator function of the set $G_1 = [\alpha_1, \beta_1] \cup [\alpha_2, \beta_2]$. For this function we have,

$$\int_{\alpha_1}^{\beta_1} f(t)x(t)dt = \int_{\alpha_2}^{\beta_2} g(t)x(t)dt = \int_{\alpha_1}^{\beta_1} f(t)dt = \int_{\alpha_2}^{\beta_2} g(t)dt = \eta,$$

η is a constant. Now by taking $x(t) = I_{[\alpha_1,\beta_1]\setminus G}(t)$ we get

$$\int_{\alpha_1}^{\beta_1} f(t)x(t)dt = \int_{\alpha_2}^{\beta_2} g(t)x(t)dt = \int_{[\alpha_1,\beta_1]\setminus G} f(t)dt = \int_{\alpha_2}^{\beta_2} g(t) \cdot 0dt = 0.$$

Then $\int\limits_{[\alpha_1,\beta_1]\setminus G} f(t)dt = 0$. If instead $x(t) = I_{[\alpha_2,\beta_2]\setminus G}(t)$, then $\int\limits_{[\alpha_2,\beta_2]\setminus G} g(t)dt = 0$. We claim that $f(t) = 0$ for almost every $t \in [\alpha_1, \beta_1] \setminus G$ and $g(t) = 0$ for almost every $t \in [\alpha_2, \beta_2] \setminus G$. We take a partition $S_1, \ldots, S_n, \ldots$ of the set $[\alpha_1, \beta_1] \setminus G$

such that each set S_i, $i = 1, 2, 3, \dots$ has positive measure. For each $x_i(t) = I_{S_i}(t)$, $i = 1, 2, 3, \dots$ we have

$$\int_{\alpha_1}^{\beta_1} f(t)x(t)dt = \int_{\alpha_2}^{\beta_2} g(t)x(t)dt = \int_{S_i} f(t)dt = \int_{\alpha_2}^{\beta_2} g(t)\cdot 0dt = 0.$$

Thus, $\int_{S_i} f(t)dt = 0,\ i = 1, 2, 3, \dots$. Since we can choose arbitrary partition with positive measure on each of its elements we have

$$f(t) = 0 \quad \text{for almost every } t \in [\alpha_1, \beta_1] \setminus G.$$

Analogously, $g(t) = 0$ for almost every $t \in [\alpha_2, \beta_2] \setminus G$. Then,

$$\eta = \int_{\alpha_1}^{\beta_1} f(t)dt = \int_{\alpha_2}^{\beta_2} g(t)dt = \int_G f(t)dt = \int_G g(t)dt.$$

Then, for all function $x \in L_p(\mathbb{R})$ we have

$$\int_G f(t)x(t)dt = \int_G g(t)x(t)dt \Longleftrightarrow \int_G [f(t) - g(t)]x(t)dt = 0.$$

By taking $x(t) = \begin{cases} 1, & \text{if } f(t) - g(t) > 0, \\ -1, & \text{if } f(t) - g(t) < 0, \end{cases}$ for almost every $t \in G$ and $x(t) = 0$ for almost every $t \in \mathbb{R} \setminus G$, we get $\int_G |f(t) - g(t)|dt = 0$. This implies that $f(t) = g(t)$ for almost every $t \in G$. □

Remark 9.3.1 When operators are given in abstract form, we use the notation $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$ meaning that operator A is well defined from $L_p(\mathbb{R})$ to $L_p(\mathbb{R})$ without discussing sufficient conditions for it to be satisfied. For instance, for the following integral operator

$$(Ax)(t) = \int_{\mathbb{R}} k(t, s)x(s)ds$$

there are sufficient conditions on kernels $k(\cdot, \cdot)$ such that operator A is well defined from $L_p(\mathbb{R})$ to $L_p(\mathbb{R})$ and bounded [9, 18]. For instance, [18, Theorem 6.18] states the following: if $1 < p < \infty$ and $k : \mathbb{R} \times [\alpha, \beta] \to \mathbb{R}$ is a measurable function, $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, and there is a constant $\lambda > 0$ such that

$$\text{ess sup}_{s\in[\alpha,\beta]} \int\limits_{\mathbb{R}} |k(t,s)|dt \le \lambda, \quad \text{ess sup}_{t\in\mathbb{R}} \int\limits_{\alpha}^{\beta} |k(t,s)|ds \le \lambda,$$

then A is well defined from $L_p(\mathbb{R})$ to $L_p(\mathbb{R})$, $1 \le p \le \infty$ and bounded.

9.3.1.1 Representations When *A* is Integral Operator and *B* is Multiplication Operator

Proposition 9.3.3 *Let $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 \le p \le \infty$, be defined as follows, for almost all $t \in \mathbb{R}$,*

$$(Ax)(t) = \int\limits_{\alpha}^{\beta} k(t,s)x(s)ds, \quad (Bx)(t) = b(t)x(t), \quad \alpha,\beta \in \mathbb{R},\ \alpha < \beta,$$

where $k : \mathbb{R} \times [\alpha,\beta] \to \mathbb{R}$ is a measurable function, and $b : \mathbb{R} \to \mathbb{R}$ is a measurable function. Consider a polynomial defined by $F(z) = \delta_0 + \delta_1 z + \cdots + \delta_n z^n$, where $\delta_0, \delta_1, \ldots, \delta_n$ are real numbers. We set

$$k_0(t,s) = k(t,s), \qquad k_m(t,s) = \int\limits_{\alpha}^{\beta} k(t,\tau)k_{m-1}(\tau,s)d\tau, \quad m \in\{1,\ldots,n\}$$

$$F_n(k(t,s)) = \sum_{j=1}^{n} \delta_j k_{j-1}(t,s), \quad n \in \{1,2,3,\ldots\}. \tag{9.2}$$

Then $AB = BF(A)$ if and only if

$$\forall\, x \in L_p(\mathbb{R}) : \quad b(t)\delta_0 x(t) + b(t)\int\limits_{\alpha}^{\beta} F_n(k(t,s))x(s)ds = \int\limits_{\alpha}^{\beta} k(t,s)b(s)x(s)ds. \tag{9.3}$$

If $\delta_0 = 0$, that is, $F(z) = \delta_1 z + \cdots + \delta_n z^n$, then the condition (9.3) reduces to the following: for almost every (t,s) in $\mathbb{R} \times [\alpha,\beta]$,

$$b(t)F_n(k(t,s)) = k(t,s)b(s). \tag{9.4}$$

Proof By applying Fubini Theorem from [1] and iterative kernels from [25], We have

$$(A^2x)(t) = \int_\alpha^\beta k(t,s)(Ax)(s)ds = \int_\alpha^\beta k(t,s)\left(\int_\alpha^\beta k(s,\tau)x(\tau)d\tau\right)ds$$
$$= \int_\alpha^\beta \left(\int_\alpha^\beta k(t,s)k(s,\tau)ds\right)x(\tau)d\tau = \int_\alpha^\beta k_1(t,\tau)x(\tau)d\tau,$$

where $k_1(t,s) = \int_\alpha^\beta k(t,\tau)k(\tau,s)d\tau$. In the same way,

$$(A^3x)(t) = \int_\alpha^\beta k(t,s)(A^2x)(s)ds = \int_\alpha^\beta k(t,s)\left(\int_\alpha^\beta k_1(s,\tau)x(\tau)d\tau\right)ds$$
$$= \int_\alpha^\beta k_2(t,s)x(s)ds,$$

where $k_2(t,s) = \int_\alpha^\beta k(t,\tau)k_1(\tau,s)d\tau$. For every $n \geq 1$,

$$(A^nx)(t) = \int_\alpha^\beta k_{n-1}(t,s)x(s)ds,$$

where $k_m(t,s) = \int_\alpha^\beta k(t,\tau)k_{m-1}(\tau,s)d\tau, \quad m = 1,\dots,n, \quad k_0(t,s) = k(t,s).$ Thus,

$$(F(A)x)(t) = \delta_0 x(t) + \sum_{j=1}^n \delta_j (A^jx)(t) = \delta_0 x(t) + \sum_{j=1}^n \delta_j \int_\alpha^\beta k_{j-1}(t,s)x(s)ds$$
$$= \delta_0 x(t) + \int_\alpha^\beta F_n(k(t,s))x(s)ds,$$

where $F_n(k(t,s)) = \sum_{j=1}^n \delta_j k_{j-1}(t,s)$, for $n = 1,2,3,\dots$. So, we can compute $BF(A)x$ and $(AB)x$ as follows:

$$(BF(A)x)(t) = b(t)(F(A)x)(t) = b(t)\delta_0 x(t) + b(t)\int_{\alpha}^{\beta} F_n(k(t,s))x(s)ds,$$

$$(ABx)(t) = A(Bx)(t) = \int_{\alpha}^{\beta} k(t,s)b(s)x(s)ds.$$

It follows that $ABx = BF(A)x$ if and only if condition (9.3) holds.

If $\delta_0 = 0$ then condition (9.3) reduces to the following:

$$\forall\, x \in L_p(\mathbb{R}): \quad \int_{\alpha}^{\beta} b(t)F_n(k(t,s))x(s)ds = \int_{\alpha}^{\beta} k(t,s)b(s)x(s)ds.$$

Let $f(t,s) = b(t)F_n(k(t,s)) - k(t,s)b(s)$. By applying Lemma 9.3.1 we have for almost every $t \in \mathbb{R}$ that $f(t,\cdot) = 0$ almost everywhere. Since the set $N = \{(t,s) \in \mathbb{R}\times[\alpha,\beta]: \ f(t,s) \neq 0\} \subset \mathbb{R}^2$ is measurable and almost all sections $N_t = \{s \in [\alpha,\beta]: \ (t,s) \in N\}$ of the plane has Lebesgue measure zero, by the reciprocal Fubini Theorem [35], the set N has Lebesgue measure zero on the plane $\mathbb{R}^2$. □

Corollary 9.3.4 *For $M_1, M_2 \in \mathbb{R}$, $M_1 < M_2$ and $1 \le p \le \infty$, let $A : L_p([M_1, M_2]) \to L_p([M_1, M_2])$ and $B : L_p([M_1, M_2]) \to L_p([M_1, M_2])$ be nonzero operators defined, for almost all t, by*

$$(Ax)(t) = \int_{\alpha}^{\beta} k(t,s)x(s)ds, \quad (Bx)(t) = b(t)x(t), \quad \alpha,\beta \in \mathbb{R}, \ \alpha < \beta,$$

where $[M_1, M_2] \supseteq [\alpha,\beta]$, and $k(\cdot,\cdot) : [M_1, M_2]\times[\alpha,\beta] \to \mathbb{R}$, $b : [M_1, M_2] \to \mathbb{R}$ are given by

$$k(t,s) = a_0 + a_1 t + c_1 s, \qquad b(t) = \sum_{j=0}^{n} b_j t^j,$$

where n is non-negative integer, $a_0,\ a_1,\ c_1,\ b_j$ are real numbers for $j = 0,\ldots,n$. Consider a polynomial defined by $F(z) = \delta_0 + \delta_1 z + \delta_2 z^2$, where $\delta_0,\ \delta_1,\ \delta_2 \in \mathbb{R}$.

Then, $AB = BF(A)$ if and only if

$$\forall\, x \in L_p([M_1, M_2]): \quad b(t)\delta_0 x(t) + b(t)\int_{\alpha}^{\beta} F_n(k(t,s))x(s)ds = \int_{\alpha}^{\beta} k(t,s)b(s)x(s)ds,$$

where $F_n(k(t,s))$ is given by (9.2).

If $\delta_0 = 0$, that is, $F(z) = \delta_1 z + \delta_2 z^2$ then the last condition reduces to the condition that for almost every (t, s) in $[M_1, M_2] \times [\alpha, \beta]$

$$b(t)F_2(k(t, s)) = k(t, s)b(s). \tag{9.5}$$

Condition (9.5) is equivalent to that $b(\cdot) \equiv b_0 \neq 0$ is a nonzero constant ($b_j = 0$, $j = 1, \ldots, n$) and one of the following cases holds:

(i) *if $\delta_2 = 0$, $\delta_1 = 1$, then $a_0, a_1, c_1 \in \mathbb{R}$ can be arbitrary;*
(ii) *if $\delta_2 \neq 0$, $\delta_1 = 1$, $a_1 \neq 0$, $c_1 = 0$, then*

$$a_0 = -\frac{\beta + \alpha}{2} a_1;$$

(iii) *if $\delta_2 \neq 0$, $\delta_1 = 1$, $a_1 = 0$, $c_1 \neq 0$, then*

$$a_0 = -\frac{\beta + \alpha}{2} c_1;$$

(iv) *if $\delta_2 \neq 0$, $\delta_1 \neq 1$, $a_1 \neq 0$, $c_1 = 0$, then*

$$a_0 = \frac{2 - 2\delta_1 - \delta_2(\beta^2 - \alpha^2)a_1}{2\delta_2(\beta - \alpha)};$$

(v) *if $\delta_2 \neq 0$, $\delta_1 \neq 1$, $c_1 \neq 0$, $a_1 = 0$, then*

$$a_0 = \frac{2 - 2\delta_1 - \delta_2(\beta^2 - \alpha^2)c_1}{2\delta_2(\beta - \alpha)};$$

(vi) *if $\delta_2 \neq 0$, $\delta_1 \neq 1$, $a_1 = 0$ and $c_1 = 0$, then*

$$a_0 = \frac{1 - \delta_1}{\delta_2(\beta - \alpha)}.$$

Proof Operator A is defined on $L_p[M_1, M_2]$, $1 \leq p \leq \infty$. Therefore, by applying [19, Theorem 3.4.10], we conclude that A is well defined. Moreover, kernel $k(\cdot, \cdot)$ is continuous on a closed and bounded set $[-M, M] \times [\alpha, \beta]$ and $b(\cdot)$ is continuous in $[M_1, M_2]$, so these functions are measurable. By applying Proposition 9.3.3 we just need to check when the condition (9.4) is satisfied for $k(\cdot, \cdot)$ and $b(\cdot)$. We compute

$$k_1(t, s) = \int_\alpha^\beta k(t, \tau)k(\tau, s)d\tau = \int_\alpha^\beta (a_0 + a_1 t + c_1 \tau)(a_0 + a_1 \tau + c_1 s)d\tau$$
$$= \int_\alpha^\beta [(a_0^2 + a_0 a_1 t + a_0 c_1 s + a_1 c_1 t s)$$

$$
\begin{aligned}
&+ (a_0a_1 + a_0c_1 + a_1^2 t + c_1^2 s)\tau + a_1c_1\tau^2]d\tau \\
=&(\beta-\alpha)(a_0^2 + a_0a_1t + a_0c_1s + a_1c_1ts) \\
&+ \frac{\beta^2-\alpha^2}{2}\cdot(a_0a_1 + a_0c_1 + a_1^2t + c_1^2s) \\
&+ \frac{\beta^3-\alpha^3}{3}a_1c_1 = \nu_0 + \nu_1 t + \nu_2 s + \nu_3 ts, \qquad (9.6)
\end{aligned}
$$

where

$$
\begin{array}{ll}
\nu_0 = a_0^2(\beta-\alpha) + \frac{\beta^2-\alpha^2}{2}a_0(a_1+c_1) + a_1c_1\frac{\beta^3-\alpha^3}{3}, & \nu_2 = a_0c_1(\beta-\alpha) + c_1^2\frac{\beta^2-\alpha^2}{2}, \\
\nu_1 = a_1^2\frac{\beta^2-\alpha^2}{2} + a_1a_0(\beta-\alpha), & \nu_3 = a_1c_1(\beta-\alpha).
\end{array}
$$

Then, we have

$$
\begin{aligned}
&b(t)F_2(k(t,s)) = b(t)[\delta_1 k(t,s) + \delta_2 k_1(t,s)] = (a_0\delta_1 + \delta_2\nu_0)\sum_{j=0}^{n} b_j t^j \\
&+ (a_1\delta_1 + \delta_2\nu_1)\sum_{j=0}^{n} b_j t^{j+1} + (c_1\delta_1 + \delta_2\nu_2)\sum_{j=0}^{n} b_j t^j s + \nu_3\delta_2\sum_{j=0}^{n} b_j t^{j+1} s \\
&= (\delta_1 a_0 + \delta_2\nu_0)b_0 + (c_1\delta_1 + \nu_2\delta_2)b_0 s + \sum_{j=1}^{n}[(\delta_1a_0 + \delta_2\nu_0)b_j + (\delta_1a_1 + \delta_2\nu_1)b_{j-1}]t^j \\
&+ \sum_{j=1}^{n}[(c_1\delta_1 + \nu_2\delta_2)b_j + \nu_3\delta_2 b_{j-1}]t^j s + (\delta_1a_1 + \delta_2\nu_1)b_n t^{n+1} + \nu_3\delta_2 b_n t^{n+1} s \\
&k(t,s)b(s) = a_0\sum_{j=0}^{n} b_j s^j + a_1\sum_{j=0}^{n} b_j s^j t + c_1\sum_{j=0}^{n} b_j s^{j+1} = a_0b_0 + a_1b_0t \\
&+ \sum_{j=1}^{n}(a_0b_j + c_1b_{j-1})s^j + \sum_{j=1}^{n} a_1b_js^jt + c_1b_ns^{n+1}.
\end{aligned}
$$

Thus we have $k(t,s)b(s) = b(t)F_2(k(t,s))$ for all $(t,s) \in [M_1, M_2] \times [\alpha, \beta]$ if and only if

$$
\begin{aligned}
a_0b_0 &= (a_0\delta_1 + \delta_2\nu_0)b_0 \\
a_1b_0 &= (a_0\delta_1 + \delta_2\nu_0)b_1 + (a_1\delta_1 + \delta_2\nu_1)b_0 \\
a_0b_1 + c_1b_0 &= (c_1\delta_1 + \delta_2\nu_2)b_0 \qquad (9.7) \\
a_1b_1 &= (c_1\delta_1 + \delta_2\nu_2)b_1 + \delta_2\nu_3b_0 \qquad (9.8) \\
0 &= a_0b_j + c_1b_{j-1}, \quad 2 \le j \le n \qquad (9.9) \\
0 &= (a_0\delta_1 + \delta_2\nu_0)b_j + (a_1\delta_1 + \delta_2\nu_1)b_{j-1}, \quad 2 \le j \le n \\
a_1b_j &= 0, \quad 2 \le j \le n \qquad (9.10)
\end{aligned}
$$

$$
\begin{aligned}
0 &= c_1\delta_1 b_j + \delta_2\nu_3 b_{j-1} + \delta_2\nu_2 b_j \quad 2 \le j \le n \\
0 &= a_1\delta_1 b_n + \delta_2\nu_1 b_n, \quad \text{if } n \ge 1 \\
c_1 b_n &= 0, \quad \text{if } n \ge 1 \\
0 &= \delta_2\nu_3 b_n, \quad \text{if } n \ge 1.
\end{aligned} \tag{9.11}
$$

Suppose that $n \ge 1$. We proceed by induction to prove that $b_j = 0$, for all $j = 1, 2, \dots, n$. For $i = 0$, we suppose that $b_n = b_{n-i} \ne 0$. Then from (9.10) we have $a_1 b_n = 0$ and thus $a_1 = 0$. From Eq. (9.11) we have $c_1 b_n = 0$ and thus $c_1 = 0$. From (9.9) we have $0 = a_0 b_n + c_1 b_{n-1} = a_0 b_n$ and thus $a_0 = 0$. This implies that $k(t,s) \equiv 0$, that is, $A = 0$. So for $i = 0$, $b_n = b_{n-i} \ne 0$ implies $A = 0$. Hence, $b_n = 0$. Let $1 < m \le n-2$ and suppose that $b_{n-i} = 0$ for all $i = 1, 2, \dots, m-1$. Let us show that then $b_{n-m} = 0$. If $b_{n-m} \ne 0$, then from (9.10) we have $a_1 b_{n-m} = 0$ which implies $a_1 = 0$. From (9.9) and for $j = n-m+1$ by induction assumption $a_0 b_{n-m+1} + c_1 b_{n-m} = c_1 b_{n-m} = 0$ which implies $c_1 = 0$. Therefore from (9.9) and for $j = n - m$ we have $a_0 b_{n-m} = 0$ which implies $a_0 = 0$. Then $k(t,s) \equiv 0$, that is $A = 0$. So we must have $b_{n-m} = 0$. If $m = n-1$, then let us show that $b_{n-m} = b_1 = 0$. If $b_{n-m} \ne 0$ then (9.9) gives $c_1 b_{n-m} = c_1 b_1 = 0$ when $j = n-m+1 = 2$. Then $c_1 = 0$ and by (9.8), since $\nu_2 = \nu_3 = 0$ we get $a_1 b_1 = 0$ which yields $a_1 = 0$. Therefore, (9.7) gives $a_0 b_1 = 0$ which yields $a_0 = 0$. Thus $A = 0$. Since $A \ne 0$, $b_1 = 0$ is proved. Thus $b(\cdot) = b_0$ is proved.

Since $B \ne 0$ and $B = b_0 I$ (multiple of identity operator), $b_0 \ne 0$ and the commutation relation is equivalent to $F(A) = A$. By (9.4) we have $F_2(k(t,s)) = k(t,s)$ which can be written as follows

$$\delta_1 k(t,s) + \delta_2 k_1(t,s) = k(t,s), \tag{9.12}$$

where $k(t,s) = a_0 + a_1 t + c_1 s$ and $k_1(t,s) = \nu_0 + \nu_1 t + \nu_2 s + \nu_3 ts$,

$$
\begin{aligned}
\nu_0 &= a_0^2(\beta-\alpha) + \tfrac{\beta^2-\alpha^2}{2} a_0(a_1+c_1) + a_1 c_1 \tfrac{\beta^3-\alpha^3}{3}, & \nu_2 &= a_0 c_1(\beta-\alpha) + c_1^2 \tfrac{\beta^2-\alpha^2}{2}, \\
\nu_1 &= a_1^2 \tfrac{\beta^2-\alpha^2}{2} + a_1 a_0(\beta-\alpha), & \nu_3 &= a_1 c_1(\beta-\alpha).
\end{aligned}
$$

If $\delta_2 = 0$, then (9.12) becomes $(\delta_1 - 1)k(\cdot,\cdot) = 0$ and $A \ne 0$ yields $\delta_1 = 1$. Thus, if $\delta_2 = 0$ and $\delta_1 = 1$, then (9.12) is satisfied for any $a_0, a_1, c_1 \in \mathbb{R}$.

If $\delta_2 \ne 0$ and $\delta_1 = 1$ then (9.12) becomes $k_1(\cdot,\cdot) = 0$, that is, $\nu_0 = \nu_1 = \nu_2 = \nu_3 = 0$, where

$$
\begin{aligned}
\nu_0 &= a_0^2(\beta-\alpha) + \tfrac{\beta^2-\alpha^2}{2} a_0(a_1+c_1) + a_1 c_1 \tfrac{\beta^3-\alpha^3}{3}, & \nu_2 &= a_0 c_1(\beta-\alpha) + c_1^2 \tfrac{\beta^2-\alpha^2}{2}, \\
\nu_1 &= a_1^2 \tfrac{\beta^2-\alpha^2}{2} + a_1 a_0(\beta-\alpha), & \nu_3 &= a_1 c_1(\beta-\alpha).
\end{aligned}
$$

Since $\alpha < \beta$, $a_1 c_1(\beta-\alpha) = 0$ is equivalent to either $a_1 = 0$ or $c_1 = 0$. If $a_1 \ne 0$, $c_1 = 0$, then

$$\begin{cases} \nu_0 = 0 \\ \nu_1 = 0 \\ \nu_2 = 0 \\ \nu_3 = 0 \end{cases} \Leftrightarrow \begin{cases} (\beta-\alpha)a_0^2 + \frac{\beta^2-\alpha^2}{2}a_0a_1 = 0 \\ (\beta-\alpha)a_1a_0 + \frac{\beta^2-\alpha^2}{2}a_1^2 = 0 \end{cases} \Leftrightarrow a_0 + \frac{\beta+\alpha}{2}a_1 = 0,$$

which is equivalent to $a_0 = -\frac{\beta+\alpha}{2}a_1$. If $a_1 = 0$, $c_1 \neq 0$, then

$$\begin{cases} \nu_0 = 0 \\ \nu_1 = 0 \\ \nu_2 = 0 \\ \nu_3 = 0 \end{cases} \Leftrightarrow \begin{cases} (\beta-\alpha)a_0^2 + \frac{\beta^2-\alpha^2}{2}a_0c_1 = 0 \\ (\beta-\alpha)c_1a_0 + \frac{\beta^2-\alpha^2}{2}c_1^2 = 0 \end{cases} \Leftrightarrow a_0 + \frac{\beta+\alpha}{2}c_1 = 0,$$

which is equivalent to $a_0 = -\frac{\beta+\alpha}{2}c_1$. If $a_1 = 0$, $c_1 = 0$, then $\nu_0 = \nu_1 = \nu_2 = \nu_3 = 0$ is equivalent to $a_0^2(\beta-\alpha) = 0$, that is, $a_0 = 0$. This implies $A = 0$. Therefore, $\delta_2 \neq 0$, $\delta_1 = 1$, $a_1 = c_1 = 0$ yields $A = 0$.

Consider $\delta_2 \neq 0$ and $\delta_1 \neq 1$, and note that (9.12) is equivalent to:

$$\begin{cases} a_0 = \delta_1 a_0 + \delta_2 a_0^2(\beta-\alpha) + \delta_2\frac{\beta^2-\alpha^2}{2}a_0(a_1+c_1) + \delta_2 a_1c_1\frac{\beta^3-\alpha^3}{3} \\ a_1 = \delta_1 a_1 + \delta_2 a_1^2\frac{\beta^2-\alpha^2}{2} + \delta_2 a_1a_0(\beta-\alpha) \\ c_1 = \delta_1 c_1 + \delta_2 a_0c_1(\beta-\alpha) + \delta_2 c_1^2\frac{\beta^2-\alpha^2}{2} \\ 0 = \delta_2 a_1c_1(\beta-\alpha). \end{cases} \tag{9.13}$$

Since $\alpha < \beta$ and $\delta_2 \neq 0$, equation $\delta_2 a_1c_1(\beta-\alpha) = 0$ implies that either $a_1 = 0$ or $c_1 = 0$. If $\delta_2 \neq 0$, $\delta_1 \neq 1$, $a_1 \neq 0$ and $c_1 = 0$, then (9.13) becomes

$$a_0 = \delta_1 a_0 + \delta_2 a_0^2(\beta-\alpha) + \delta_2\frac{\beta^2-\alpha^2}{2}a_0a_1$$
$$a_1 = \delta_1 a_1 + \delta_2 a_1^2\frac{\beta^2-\alpha^2}{2} + \delta_2 a_1a_0(\beta-\alpha)$$

which is equivalent to $1 = \delta_1 + \delta_2(\beta-\alpha)a_0 + \delta_2\frac{\beta^2-\alpha^2}{2}a_1$. Then,

$$a_0 = \frac{2-2\delta_1-\delta_2(\beta^2-\alpha^2)a_1}{2\delta_2(\beta-\alpha)}.$$

If $\delta_2 \neq 0$, $\delta_1 \neq 1$, $a_1 = 0$ and $c_1 \neq 0$, then (9.13) becomes

$$a_0 = \delta_1 a_0 + \delta_2 a_0^2(\beta-\alpha) + \delta_2\frac{\beta^2-\alpha^2}{2}a_0c_1$$
$$c_1 = \delta_1 c_1 + \delta_2 c_1^2\frac{\beta^2-\alpha^2}{2} + \delta_2 c_1a_0(\beta-\alpha)$$

which is equivalent to $1 = \delta_1 + \delta_2(\beta-\alpha)a_0 + \delta_2\frac{\beta^2-\alpha^2}{2}c_1$. Then,

$$a_0 = \frac{2 - 2\delta_1 - \delta_2(\beta^2 - \alpha^2)c_1}{2\delta_2(\beta - \alpha)}.$$

If $\delta_2 \neq 0$, $\delta_1 \neq 1$, $a_1 = 0$ and $c_1 = 0$, then $A \neq 0$ yields $a_0 \neq 0$ and (9.13) becomes

$$a_0 = \delta_1 a_0 + \delta_2 a_0^2(\beta - \alpha)$$

which is equivalent to $a_0 = \frac{1-\delta_1}{\delta_2(\beta-\alpha)}$. □

Remark 9.3.2 The integral operator given by $(Ax)(t) = \int\limits_{\alpha_1}^{\beta_1} k(t,s)x(s)ds$ for almost all t, where $k : [\alpha_1, \beta_1] \times [\alpha_1, \beta_1] \to \mathbb{R}$ is a measurable function that satisfies

$$\int\limits_{\alpha_1}^{\beta_1} \left(\int\limits_{\alpha_1}^{\beta_1} |k(t,s)|^q ds \right)^{\frac{p}{q}} dt < \infty,$$

by [19, Theorem 3.4.10] is well defined from $L_p[\alpha_1, \beta_1]$ to $L_p[\alpha_1, \beta_1]$, $1 < p < \infty$ and bounded.

Remark 9.3.3 If in the Corollary 9.3.4 when $0 \notin [M_1, M_2]$, one takes $b(t)$ to be a Laurent polynomial with only negative powers of t then there is no non-zero kernel $k(t,s) = a_0 + a_1 t + c_1 s$ (there is no $A \neq 0$ with such kernels) such that $AB = BF(A)$. In fact, let n be a positive integer and consider $b(t) = \sum\limits_{j=1}^{n} b_j t^{-j}$, where $t \in [M_1, M_2]$, $b_j \in \mathbb{R}$ for $j = 1, \ldots, n$ and $b_n \neq 0$. We set $k_1(t,s)$ as defined by (9.6). Then we have

$$\begin{aligned}
b(t)F_2(k(t,s)) &= b(t)[\delta_1 k(t,s) + \delta_2 k_1(t,s)] = (a_0\delta_1 + \delta_2\nu_0)\sum_{j=1}^{n} b_j t^{-j} \\
&+ (a_1\delta_1 + \delta_2\nu_1)\sum_{j=1}^{n} b_j t^{-j+1} + (c_1\delta_1 + \delta_2\nu_2)\sum_{j=1}^{n} b_j t^{-j} s + \nu_3\delta_2 \sum_{j=1}^{n} b_j t^{-j+1} s \\
&= (a_1\delta_1 + \delta_2\nu_1)b_1 + \nu_3\delta_2 b_1 s + \sum_{j=1}^{n-1}[(a_0\delta_1 + \delta_2\nu_0)b_j + (a_1\delta_1 + \delta_2\nu_1)b_{j+1}]t^{-j} \\
&+ (a_0\delta_1 + \delta_2\nu_0)b_n t^{-n} + \sum_{j=1}^{n-1}[(c_1\delta_1 + \delta_2\nu_2)b_j + \nu_3\delta_2 b_{j+1}]t^{-j}s + (c_1\delta_1 + \delta_2\nu_2)b_n t^{-n} s
\end{aligned}$$

$$k(t,s)b(s) = a_0 \sum_{j=1}^{n} b_j s^{-j} + a_1 \sum_{j=1}^{n} b_j s^{-j} t + c_1 \sum_{j=1}^{n} b_j s^{-j+1}$$

$$= c_1 b_1 + \sum_{j=1}^{n-1} (a_0 b_j + c_1 b_{j+1}) s^{-j} + \sum_{j=1}^{n} a_1 b_j s^{-j} t + a_0 b_n s^{-n}.$$

Thus we have $k(t,s)b(s) = b(t)F_2(k(t,s))$ for almost every $(t,s) \in [M_1, M_2] \times [\alpha, \beta]$ if and only if

$$\begin{aligned}
c_1 b_1 &= a_1 \delta_1 b_1 + \delta_2 \nu_1 b_1, \\
0 &= \delta_2 \nu_3 b_1, \\
0 &= (a_0 \delta_1 + \delta_2 \nu_0) b_j + (\delta_1 a_1 + \delta_2 \nu_1) b_{j+1}, \quad 1 \le j \le n-1, \\
a_0 b_j + c_1 b_{j+1} &= 0, \quad 1 \le j \le n-1, && (9.14) \\
0 &= c_1 \delta_1 b_j + \delta_2 \nu_2 b_j + \delta_2 \nu_3 b_{j+1}, \quad 1 \le j \le n-1, \\
a_1 b_j &= 0, \quad 1 \le j \le n, && (9.15) \\
0 &= a_0 \delta_1 b_n + \delta_2 \nu_0 b_n, \\
0 &= a_0 b_n, && (9.16) \\
0 &= c_1 \delta_1 b_n + \delta_2 \nu_3 b_n.
\end{aligned}$$

Since $b_n \neq 0$ then from (9.16) we have $a_0 b_n = 0$ and thus $a_0 = 0$. From (9.14) for $j = n-1$ we get $c_1 b_n = 0$ and thus $c_1 = 0$. Finally from (9.15) we have $0 = a_1 b_j$ for $j = n$ and thus $a_1 = 0$. This implies that $k(t,s) \equiv 0$, that is, $A = 0$. So $b_n \neq 0$ implies $A = 0$.

Corollary 9.3.5 *Let $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 < p < \infty$, be defined as follows, for almost all t,*

$$(Ax)(t) = \int_{\alpha}^{\beta} k(t,s)x(s)ds, \quad (Bx)(t) = b(t)x(t), \quad \alpha, \beta \in \mathbb{R}, \alpha < \beta,$$

where $k(t,s) : \mathbb{R} \times [\alpha, \beta] \to \mathbb{R}$ is a measurable function, and $b \in L_\infty(\mathbb{R})$ is a nonzero function such that the set supp $b(t) \cap [\alpha, \beta]$ *has measure zero.*

Consider a polynomial defined by $F(z) = \delta_0 + \delta_1 z + \cdots + \delta_n z^n$, where $\delta_0, \ldots, \delta_n$ are real numbers. We set

$$k_0(t,s) = k(t,s), \qquad k_m(t,s) = \int_{\alpha}^{\beta} k(t,\tau) k_{m-1}(\tau, s) d\tau, \quad m = 1, \ldots, n,$$

$$F_n(k(t,s)) = \sum_{j=1}^{n} \delta_j k_{j-1}(t,s), \quad n = 1, 2, 3, \ldots$$

Then $AB = BF(A)$ *if and only if* $\delta_0 = 0$ *and the set*

$$\text{supp } b(t) \cap \text{supp } F_n(k(t,s))$$

has measure zero in $\mathbb{R} \times [\alpha, \beta]$.

Proof Suppose that the set supp $b \cap [\alpha, \beta]$ has measure zero. By Proposition 9.3.3 we have $AB = BF(A)$ if and only if condition (9.3) holds, that is,

$$\forall\, x \in L_p(\mathbb{R}): \quad b(t)\delta_0 x(t) + b(t)\int_\alpha^\beta F_n(k(t,s))x(s)ds = \int_\alpha^\beta k(t,s)b(s)x(s)ds,$$

almost everywhere. By taking $x(\cdot) = I_{[M_1,M_2]}(\cdot)b(\cdot)$, where $M_1, M_2 \in \mathbb{R}$, $M_1 < M_2$, $[M_1, M_2] \supset [\alpha, \beta]$, $\mu([M_1, M_2] \setminus [\alpha, \beta]) > 0$, $I_E(\cdot)$ is the indicator function of the set E, the condition (9.3) reduces to

$$I_{[M_1,M_2]}(\cdot)b^2(\cdot)\delta_0 = 0.$$

Since b has support with positive measure (otherwise $B \equiv 0$), then $\delta_0 = 0$. By using this, condition (9.3) reduces to the following

$$\forall\, x \in L_p(\mathbb{R}): \quad b(t)\int_\alpha^\beta F_n(k(t,s))x(s)ds = \int_\alpha^\beta k(t,s)b(s)x(s)ds.$$

By hypothesis the right hand side is equal zero. Then condition (9.3) reduces to

$$\forall\, x \in L_p(\mathbb{R}): \quad b(t)\int_\alpha^\beta F_n(k(t,s))x(s)ds = 0.$$

This is equivalent to

$$b(t)F_n(k(t,s)) = 0 \quad \text{for almost every } s \in [\alpha, \beta]. \tag{9.17}$$

By applying a similar argument as in the proof of Proposition 9.3.3 we conclude that condition (9.17) is equivalent to that the set

$$\text{supp } b(t) \cap \text{supp } F_n(k(t,s))$$

has measure zero in $\mathbb{R} \times [\alpha, \beta]$. □

Corollary 9.3.6 *Let* $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 \le p \le \infty$, *be defined as follows, for almost all* t,

$$(Ax)(t) = \int_{\alpha}^{\beta} a(t)c(s)x(s)ds, \quad (Bx)(t) = b(t)x(t), \quad \alpha, \beta \in \mathbb{R}, \alpha < \beta,$$

where $a : \mathbb{R} \to \mathbb{R}$, $c : [\alpha, \beta] \to \mathbb{R}$, $b : \mathbb{R} \to \mathbb{R}$ *are measurable functions. Consider a polynomial defined by* $F(z) = \delta_1 z + \delta_2 z^2 + \cdots + \delta_n z^n$, *where* $\delta_1, \ldots, \delta_n$ *are real constants. We set* $\mu = \int_{\alpha}^{\beta} a(s)c(s)ds$. *Then, we have* $AB = BF(A)$ *if and only if the set*

$$\text{supp}\,[a(t)c(s)] \cap \text{supp}\left[b(t)\sum_{j=1}^{n} \delta_j \mu^{j-1} - b(s)\right],$$

has measure zero in $\mathbb{R} \times [\alpha, \beta]$.

Proof We set $k(t, s) = a(t)c(s)$, so we have

$$k_0(t, s) = k(t, s) = a(t)c(s),$$

$$k_m(t, s) = \int_{\alpha}^{\beta} k(t, \tau)k_{m-1}(\tau, s)d\tau = a(t)c(s)\left(\int_{\alpha}^{\beta} a(s)c(s)ds\right)^{m}, \ m = 1, \ldots, n$$

$$F_n(k(t, s)) = \sum_{j=1}^{n} \delta_j k_{j-1}(t, s) = \sum_{j=1}^{n} \delta_j a(t)c(s)\left(\int_{\alpha}^{\beta} a(s)c(s)ds\right)^{j-1} \quad n = 1, 2, 3, \ldots$$

By applying Proposition 9.3.3 we have $AB = BF(A)$ if and only if

$$b(t)\sum_{j=1}^{n} \delta_j a(t)c(s)\left(\int_{\alpha}^{\beta} a(s)c(s)ds\right)^{j-1} = a(t)c(s)b(s) \iff$$

$$a(t)c(s)\left[b(t)\sum_{j=1}^{n} \delta_j \left(\int_{\alpha}^{\beta} a(s)c(s)ds\right)^{j-1} - b(s)\right] = 0$$

for almost every (t, s) in $\mathbb{R} \times [\alpha, \beta]$. The last condition is equivalent to the set

$$\text{supp}\,[a(t)c(s)] \cap \text{supp}\left[b(t)\sum_{j=1}^{n} \delta_j \left(\int_{\alpha}^{\beta} a(s)c(s)ds\right)^{j-1} - b(s)\right]$$

has measure zero in $\mathbb{R} \times [\alpha, \beta]$. We complete the proof by noticing that the corresponding set can be written as

$$\operatorname{supp}\,[a(t)c(s)] \cap \operatorname{supp}\left[b(t)\sum_{j=1}^{n}\delta_j\mu^{j-1} - b(s)\right],$$

where $\mu = \int\limits_{\alpha}^{\beta} a(s)c(s)ds.$ □

Example 9.3.7 Let $A : L_p(\mathbb{R}) \to L_p(\mathbb{R}),\ B : L_p(\mathbb{R}) \to L_p(\mathbb{R}),\ 1 < p < \infty$ be defined as follows, for almost all t,

$$(Ax)(t) = \int\limits_{0}^{2} a(t)c(s)x(s)ds, \quad (Bx)(t) = b(t)x(t),$$

where $a(t) = I_{[0,1]}(t)(1+t^2), c(s) = 1, b(t) = I_{[1,2]}(t)t^2$. Since kernel has compact support, we can apply [19, Theorem 3.4.10] and we conclude that operators A is well defined and bounded. Since function b has 4 as an upper bound then $\|B\|_{L_p} \le 4$. Hence operator B is well defined and bounded. Consider a polynomial defined by $F(z) = \delta_1 z + \cdots + \delta_n z^n$, where $\delta_1, \ldots, \delta_n$ are real constants. Then, the above operators does not satisfy the relation $AB = BF(A)$. In fact for $\lambda \neq 0$, by applying Corollary 9.3.6 and setting $\lambda = \sum\limits_{j=1}^{n} \delta_j(\beta-\alpha)^{j-1}$, we have

$$\operatorname{supp}\,\{b(t)\lambda - b(s)\} = (\mathbb{R}\times[1,2] \cup [1,2]\times[0,1]) \setminus W,$$

where $W = \{(t,s) \in [1,2]\times[1,2] :\ b(t)\lambda - b(s) = 0\}$ is a set of measure zero in the plane. Moreover, supp $a(t)c(s) = [0,1]\times[0,2]$. The set

$$\operatorname{supp}\,[a(t)c(s)] \cap \operatorname{supp}\,[b(t)\lambda - b(s)],$$

has positive measure in $\mathbb{R}\times[0,2]$.

Example 9.3.8 Let $A : L_p(\mathbb{R}) \to L_p(\mathbb{R}),\ B : L_p(\mathbb{R}) \to L_p(\mathbb{R}),\ 1 < p < \infty$ be defined as follows, for almost all t,

$$(Ax)(t) = \int\limits_{0}^{2} a(t)c(s)x(s)ds,\ (Bx)(t) = b(t)x(t),$$

where $a(t) = 2tI_{[0,2]}(t), c(s) = I_{[0,1]}(s), b(t) = I_{[1,2]}(t)t^2$. Since kernel has compact support, we can apply [19, Theorem 3.4.10] and, we conclude that operators A is well defined and bounded. Since function b has 4 as an upper bound then $\|B\|_{L_p} \le 4$. Hence operator B is well defined and bounded. Consider a polynomial defined by $F(z) = \delta_1 z + \cdots + \delta_n z^n$, where $\delta_1, \ldots, \delta_n$ are real constants. Then, the above operators satisfy the relation $AB = BF(A)$ if and only if $\sum_{j=1}^{n} \delta_j = 0$. In fact, by applying Corollary 9.3.6 we have

$$\mu = \int_0^2 a(s)c(s)ds = 1.$$

Hence, $\operatorname{supp}\{b(t)\cdot 0 - b(s)\} = \mathbb{R}\times[1,2]$. Moreover, $\operatorname{supp} a(t)c(s) = [0,2]\times[0,1]$. The set $\operatorname{supp}[a(t)c(s)] \cap \operatorname{supp}[-b(s)]$, has measure zero in $\mathbb{R}\times[0,2]$.

Example 9.3.9 Let $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 < p < \infty$ be defined as follows, for almost all t,

$$(Ax)(t) = \int_0^2 a(t)c(s)x(s)ds, \quad (Bx)(t) = b(t)x(t),$$

where $a(t) = I_{[0,2]}(t)\sin(\pi t)$, $c(s) = I_{[0,1]}(s)$, $b(t) = I_{[1,2]}(t)t^2$. Since $a \in L_p(\mathbb{R})$ and $c \in L_q[0,2]$, $1 < q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, by applying Hölder inequality we have that operator A is well defined and bounded. The function $b \in L_\infty$, so B is well defined and bounded because $\|B\|_{L_p} \le \|b\|_{L_\infty}$ we conclude that operator B is well defined and bounded. Consider a polynomial defined by $F(z) = \delta z^d$, where $\delta \neq 0$ is a real constant and d is a positive integer $d \ge 2$. Then, the above operators satisfy the relation $AB = \delta BA^d$. In fact, by applying Corollary 9.3.6 we have $\mu = \int_0^2 a(s)c(s)ds = 0$. Hence, $\operatorname{supp}\{b(t)\cdot 0 - b(s)\} = \mathbb{R}\times[1,2]$. Moreover, $\operatorname{supp} a(t)c(s) = [0,2]\times[0,1]$. The set $\operatorname{supp}[a(t)c(s)] \cap \operatorname{supp}[-b(s)]$, has measure zero in $\mathbb{R}\times[0,2]$.

Example 9.3.10 Let $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 < p < \infty$, be defined as follows, for almost all t,

$$(Ax)(t) = \int_\alpha^\beta I_{[\alpha,\beta]}(t)x(s)ds, \quad (Bx)(t) = I_{[\alpha,\beta]}(t)x(t), \quad \alpha,\beta \in \mathbb{R}, \alpha < \beta.$$

Since kernel has compact support, we can apply [19, Theorem 3.4.10] and, we conclude that operator A is well defined and bounded. Since $\|B\|_{L_p} \le 1$ then operator B is well defined and bounded. Consider a polynomial defined by $F(z) = \delta_1 z + \cdots + \delta_n z^n$, where $\delta_1, \ldots, \delta_n$ are constants. Then, the above operators satisfy the relation $AB = BF(A)$ if and only if $\sum_{j=1}^{n} \delta_j(\beta-\alpha)^{j-1} = 1$. Indeed, if $a(t) = b(t) = I_{[\alpha,\beta]}(t)$, $c(s) = 1$ and

$$\lambda = \sum_{j=1}^{n} \delta_j \left(\int_{\alpha}^{\beta} a(s)c(s)ds \right)^{j-1} = \sum_{j=1}^{n} \delta_j (\beta - \alpha)^{j-1},$$

then from Corollary 9.3.6 we have the following:

- If $\lambda \neq 0$, $\lambda \neq 1$,

$$\text{supp}\,[b(t)\lambda - b(s))] = \left\{(t,s) \in \mathbb{R} \times [\alpha,\beta] : \lambda I_{[\alpha,\beta]}(t) \neq 1\right\} = \mathbb{R} \times [\alpha,\beta],$$

$$\text{supp}\ a(t)c(s) = \{(t,s) \in \mathbb{R} \times [\alpha,\beta] : I_{[\alpha,\beta]}(t) \neq 0\} = [\alpha,\beta] \times [\alpha,\beta].$$

  The set supp $[\lambda b(t) - b(s)] \cap$ supp $[a(t)c(s)] = [\alpha,\beta] \times [\alpha,\beta]$ has positive measure.
- If $\lambda = 1$,

$$\begin{aligned}\text{supp}\,[b(t) - b(s)] &= \left\{(t,s) \in \mathbb{R} \times [\alpha,\beta] : I_{[\alpha,\beta]}(t) \neq 1\right\} \\ &= (\mathbb{R} \setminus [\alpha,\beta]) \times [\alpha,\beta].\end{aligned}$$

  The set supp $[b(t) - b(s)] \cap$ supp $[a(t)c(s)]$ has measure zero in $\mathbb{R} \times [\alpha,\beta]$.
- If $\lambda = 0$,

$$\begin{aligned}\text{supp}\,[\lambda b(t) - b(s)] = \text{supp}\ b(s) &= \left\{(t,s) \in \mathbb{R}^2 : I_{[\alpha,\beta]}(s) \neq 0\right\} \\ &= \left\{(t,s) \in \mathbb{R}^2 : \alpha \le s \le \beta\right\}.\end{aligned}$$

  The set supp $b(s) \cap$ supp $[a(t)c(s)] = [\alpha,\beta] \times [\alpha,\beta]$ has measure $(\beta - \alpha)^2$.

The conditions in the Corollary 9.3.6 are fulfilled only in the second case, that is, when $\lambda = 1$.

9.3.1.2 Representations When *A* is Multiplication Operator and *B* is Integral Operator

Proposition 9.3.11 *Let $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 < p < \infty$ be defined as follows, for almost all t,*

$$(Ax)(t) = a(t)x(t), \quad (Bx)(t) = \int_{\alpha}^{\beta} k(t,s)x(s)ds, \quad \alpha,\beta \in \mathbb{R}, \alpha < \beta,$$

where $a : \mathbb{R} \to \mathbb{R}$, $k : \mathbb{R} \times [\alpha,\beta] \to \mathbb{R}$ are measurable functions. Consider a polynomial defined by $F(z) = \delta_0 + \delta_1 z + \cdots + \delta_n z^n$, where $\delta_0, \delta_1, \ldots, \delta_n$ are constants. Then

$$AB = BF(A)$$

if and only if the set

$$\operatorname{supp}[a(t) - F(a(s))] \cap \operatorname{supp} k(t,s)$$

has measure zero in $\mathbb{R} \times [\alpha, \beta]$.

Proof We have for almost every $t \in \mathbb{R}$

$$(ABx)(t) = \int_{\alpha}^{\beta} a(t)k(t,s)x(s)ds$$

$$(A^n x)(t) = [a(t)]^n x(t)$$

$$(F(A)x)(t) = \sum_{i=0}^{n} \delta_i (A^i x)(t) = \left(\sum_{i=0}^{n} \delta_i [a(t)]^i\right) x(t) = F(a(t))x(t)$$

$$(BF(A)x)(t) = \int_{\alpha}^{\beta} k(t,s))F(a(s))x(s)ds.$$

Then we have $ABx = BF(A)x$ if and only if

$$\int_{\alpha}^{\beta} a(t)k(t,s)x(s)ds = \int_{\alpha}^{\beta} k(t,s)F(a(s))x(s)ds. \tag{9.18}$$

almost everywhere. By using Lemma 9.3.1 and by applying the same argument as in the final steps on the proof of Proposition 9.3.3, the condition (9.18) is equivalent to

$$a(t)k(t,s) = k(t,s)F[a(s)] \Longleftrightarrow k(t,s)[a(t) - F(a(s))] = 0$$

for almost every (t,s) in $\mathbb{R} \times [\alpha, \beta]$.

Since the variables t and s are independent, this is true if and only if the set

$$\operatorname{supp}[a(t) - F(a(s))] \cap \operatorname{supp} k(t,s)$$

has measure zero in $\mathbb{R} \times [\alpha, \beta]$. □

Example 9.3.12 Let $A : L_p(\mathbb{R}) \to L_p(\mathbb{R}),\ B : L_p(\mathbb{R}) \to L_p(\mathbb{R}),\ 1 < p < \infty$ be defined as follows, for almost all t,

$$(Ax)(t) = I_{[\alpha,\beta]}(t)x(t), \quad (Bx)(t) = \int_{\alpha}^{\beta} I_{[\alpha,\beta]^2}(t,s)x(s)ds, \quad \alpha, \beta \in \mathbb{R}, \alpha < \beta$$

By using properties of norm and [19, Theorem 3.4.10], respectively, for operators A and B, we conclude that operators A and B are well defined and bounded. For a monomial defined by $F(z) = z^n$, $n = 1, 2, \ldots$, the above operators satisfy the relation $AB = BF(A)$. In fact, by setting $a(t) = I_{[\alpha,\beta]}(t)$, $k(t, s) = I_{[\alpha,\beta]^2}(t, s)$ we have

$$\operatorname{supp}[a(t) - F(a(s))] = \{(t, s) \in \mathbb{R} \times [\alpha, \beta] : I_{[\alpha,\beta]}(t) \neq 1\} = (\mathbb{R} \setminus [\alpha, \beta]) \times [\alpha, \beta],$$

$$\operatorname{supp} k(t, s) = [\alpha, \beta] \times [\alpha, \beta].$$

The set $\operatorname{supp}[a(t) - F(a(s))] \cap \operatorname{supp}[k(t, s)]$ has measure zero in $\mathbb{R} \times [\alpha, \beta]$. So the result follows from Proposition 9.3.11.

Example 9.3.13 Let $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 < p < \infty$ defined as follows, for almost all t,

$$(Ax)(t) = [\gamma_1 I_{[0,1/2)}(t) - \gamma_2 I_{[1/2,1]}(t)]x(t), \quad (Bx)(t) = \int_0^1 k(t, s)x(s)ds$$

$k : \mathbb{R} \times [0, 1] \to \mathbb{R}$ is a Lebesgue measurable function such that B is well defined. The operator A is well defined and bounded. Consider a polynomial defined by $F(z) = \delta_0 + \delta_1 z$, where $\delta_0, \delta_1, \gamma_1, \gamma_2$ are constants such that

$$|\delta_0| + |\delta_1| + |\gamma_1| + |\gamma_2| \neq 0.$$

If $k(\cdot, \cdot)$ is a measurable function such that one of the following is fulfilled:

(i) $\delta_0 = -\delta_1\gamma_1$ and $\operatorname{supp} k(t, s) \subseteq (\mathbb{R} \setminus [0, 1]) \times [0, 1/2]$;
(ii) $\delta_0 = \delta_1\gamma_2$ and $\operatorname{supp} k(t, s) \subseteq (\mathbb{R} \setminus [0, 1]) \times [1/2, 1]$;
(iii) $\delta_0 + \delta_1\gamma_1 - \gamma_1 = 0$ and $\operatorname{supp} k(t, s) \subseteq [0, 1/2] \times [0, 1/2]$;
(iv) $\delta_0 + \delta_1\gamma_1 + \gamma_2 = 0$ and $\operatorname{supp} k(t, s) \subseteq [1/2, 1] \times [0, 1/2]$;
(v) $\delta_0 - \delta_1\gamma_2 - \gamma_1 = 0$ and $\operatorname{supp} k(t, s) \subseteq [0, 1/2] \times [1/2, 1]$;
(vi) $\delta_0 - \delta_1\gamma_2 + \gamma_2 = 0$ and $\operatorname{supp} k(t, s) \subseteq [1/2, 1] \times [1/2, 1]$,

then the above operators satisfy the relation $AB = BF(A)$.

In fact, putting $a(t) = \gamma_1 I_{[0,1/2)}(t) - \gamma_2 I_{[1/2,1]}(t)$ we have

$$[a(t) - F(a(s))] = \begin{cases} 0, & \text{if } \delta_0 = -\delta_1\gamma_1, \quad t \notin [0, 1], \quad s \in [0, 1/2) \\ 0, & \text{if } \delta_0 = \delta_1\gamma_2, \quad t \notin [0, 1], \quad s \in [1/2, 1] \\ 0, & \text{if } \delta_0 + \delta_1\gamma_1 - \gamma_1 = 0, \ t \in [0, 1/2), \ s \in [0, 1/2) \\ 0, & \text{if } \delta_0 + \delta_1\gamma_1 + \gamma_2 = 0, \ t \in [1/2, 1), \ s \in [0, 1/2] \\ 0, & \text{if } \delta_0 - \delta_1\gamma_2 - \gamma_1 = 0, \ t \in [0, 1/2], \ s \in [1/2, 1] \\ 0, & \text{if } \delta_0 - \delta_1\gamma_2 + \gamma_2 = 0, \ t \in [1/2, 1], \ s \in [1/2, 1] \\ \gamma_3, & \text{otherwise} \end{cases}$$

where γ_3 can be different from zero depending on the constants involved. Thus, in each condition we can choose $k(t,s) = I_S(t,s)$, where $S = \{(t,s) \in \mathbb{R} \times [0,1] : a(t) - F(a(s)) = 0\}$ and with a positive measure. Or for instance we can take:

(i) $k(t,s) = I_{[2,3]\times[0,1/2]}(t,s)$ if $\delta_0 = -\delta_1\gamma_1$;
(ii) $k(t,s) = I_{[2,3]\times[1/2,1]}(t,s)$ if $\delta_0 = \delta_1\gamma_2$;
(iii) $k(t,s) = I_{[0,1/3]\times[1/3,1/2]}(t,s)$ if $\delta_0 + \delta_1\gamma_1 - \gamma_1 = 0$;
(iv) $k(t,s) = I_{[2/3,1/2]\times[0,1/2]}(t,s)$ if $\delta_0 + \delta_1\gamma_1 + \gamma_2 = 0$;
(v) $k(t,s) = I_{[0,1/3]\times[2/3,1]}(t,s)$ if $\delta_0 - \delta_1\gamma_2 - \gamma_1 = 0$;
(vi) $k(t,s) = I_{[2/3,1]\times[2/3,1]}(t,s)$ if $\delta_0 - \delta_1\gamma_2 + \gamma_2$.

According to the definition, in all above cases the set

$$\text{supp}\,[a(t) - F(a(s))] \cap \text{supp}\ [k(t,s)]$$

has measure zero in $\mathbb{R} \times [0,1]$. So the result follows from Proposition 9.3.11.

Corollary 9.3.14 *Let* $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 < p < \infty$ *defined as follows, for almost all* t,

$$(Ax)(t) = a(t)x(t), \quad (Bx)(t) = \int_{\alpha}^{\beta} b(t)c(s)x(s)ds, \quad \alpha,\beta \in \mathbb{R}, \alpha < \beta,$$

where $a : \mathbb{R} \to \mathbb{R}$, $b : \mathbb{R} \to \mathbb{R}$, $c : [\alpha,\beta] \to \mathbb{R}$ *are measurable functions. For a polynomial defined by* $F(z) = \delta_0 + \delta_1 z + \cdots + \delta_n z^n$, *where* $\delta_0, \delta_1, \ldots, \delta_n$ *are real constants, we have*

$$AB = BF(A)$$

if and only if the set

$$\text{supp}\,[a(t) - F(a(s))] \cap \text{supp}\,[b(t)c(s)]$$

has measure zero in $\mathbb{R} \times [\alpha,\beta]$.

Proof This follows by Proposition 9.3.11. □

Example 9.3.15 Let $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 < p < \infty$ be defined as follows, for almost all t,

$$(Ax)(t) = a(t)x(t), \quad (Bx)(t) = \int_{\alpha}^{\beta} b(t)c(s)x(s)ds, \quad \alpha,\beta \in \mathbb{R}, \alpha < \beta,$$

where $a(t) = -1 + I_{[\alpha,\beta]}(t)$, $b(t) = I_{[\alpha-2,\alpha-1]}(t)$, $c(s) = 1$. We have that $a \in L_\infty(\mathbb{R})$ and so $\|A\|_{L_p} \le \|a\|_{L_\infty}$. Therefore, A is well defined and bounded. Since kernel has compact support in $\mathbb{R} \times [\alpha, \beta]$, we can apply [19, Theorem 3.4.10] and we conclude that operators B is well defined and bounded. Consider a polynomial defined by $F(z) = -1 + \delta_1 z$, where δ_1 is a real constant. Then the above operators satisfy the relation $AB = BF(A)$. In fact, for $(t, s) \in \mathbb{R} \times [\alpha, \beta]$ we have

$$F(a(s)) - a(t) = -\delta_1 + \delta_1 I_{[\alpha,\beta]}(s) - I_{[\alpha,\beta]}(t) = -I_{[\alpha,\beta]}(t).$$

Therefore, we have

$$\operatorname{supp}[a(t) - F(a(s))] = [\alpha, \beta] \times [\alpha, \beta],$$
$$\operatorname{supp}\ b(t)c(s) = \operatorname{supp}\ \ I_{[\alpha-2,\alpha-1]}(t)I_{[\alpha,\beta]}(s) = [\alpha - 2, \alpha - 1] \times [\alpha, \beta].$$

The set $\operatorname{supp}[a(t) - F(a(s))] \cap \operatorname{supp}[I_{[\alpha-2,\alpha-1]}(t)I_{[\alpha,\beta]}(s)]$ has measure zero. So the result follows from Corollary 9.3.14.

Example 9.3.16 Let $A : L_p(\mathbb{R}) \to L_p(\mathbb{R}),\ B : L_p(\mathbb{R}) \to L_p(\mathbb{R}),\ 1 < p < \infty$ be defined as follows, for almost all t,

$$(Ax)(t) = a(t)x(t), \quad (Bx)(t) = \int_{\alpha}^{\beta} b(t)c(s)x(s)ds, \quad \alpha, \beta \in \mathbb{R}, \alpha < \beta,$$

where $a(t) = \gamma_0 + I_{\left[\alpha, \frac{\alpha+\beta}{2}\right]}(t)t^2$, γ_0 is a real number, $b(t) = (1 + t^2)I_{[\beta+1,\beta+2]}(t)$, $c(s) = I_{\left[\frac{\alpha+\beta}{2},\beta\right]}(s)(1 + s^4)$. Consider a polynomial defined by $F(z) = \delta_0 + \delta_1 z$, where δ_0, δ_1 are real constants and $\delta_1 \neq 0$. If $\delta_0 = \gamma_0 - \delta_1\gamma_0$ then the above operators satisfy the relation

$$AB - \delta_1 BA = \delta_0 B.$$

In fact, A is well defined, bounded since $a \in L_\infty$ and this implies $\|A\|_{L_p} \leq \|a\|_{L_\infty}$. Operator B is well defined, bounded since $k(t, s) = b(t)c(s)$, $(t, s) \in \mathbb{R} \times [\alpha, \beta]$ has compact support and satisfies conditions of [19, Theorem 3.4.10]. If $\delta_0 = \gamma_0 - \delta_1\gamma_0$ then we have

$$\begin{aligned} F(a(s)) - a(t) &= \delta_0 + \gamma_0\delta_1 + \delta_1 I_{\left[\alpha, \frac{\alpha+\beta}{2}\right]}(s)s^2 - \gamma_0 - I_{\left[\alpha, \frac{\alpha+\beta}{2}\right]}(t)t^2 \\ &= \delta_1 I_{\left[\alpha, \frac{\alpha+\beta}{2}\right]}(s)s^2 - I_{\left[\alpha, \frac{\alpha+\beta}{2}\right]}(t)t^2. \end{aligned}$$

Then we have

$$\operatorname{supp}[a(t) - F(a(s))] = \left(\mathbb{R} \times \left[\alpha, \frac{\alpha+\beta}{2}\right] \cup \left[\alpha, \frac{\alpha+\beta}{2}\right] \times \left[\frac{\alpha+\beta}{2}, \beta\right]\right) \setminus W,$$

where $W \subseteq \mathbb{R} \times [\alpha, \beta]$ is a set with Lebesgue measure zero, and

$$\begin{aligned} \text{supp } \ b(t)c(s) &= \text{supp } \ (1+t^2) I_{[\beta+1,\beta+2]}(t) I_{\left[\frac{\alpha+\beta}{2},\beta\right]}(s)(1+s^4) \\ &= [\beta+1,\beta+2] \times \left[\frac{\alpha+\beta}{2}, \beta\right]. \end{aligned}$$

The set supp $[a(t) - F(a(s))] \cap$ supp $[b(t)c(s)]$ has measure zero. So the result follows from Corollary 9.3.14.

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Chapter 10
Representations of Polynomial Covariance Type Commutation Relations by Piecewise Function Multiplication and Composition Operators

Domingos Djinja, Sergei Silvestrov, and Alex Behakanira Tumwesigye

Abstract Representations of polynomial covariance type commutation relations are constructed on Banach spaces L_p and $C[\alpha, \beta]$ $\alpha, \beta \in \mathbb{R}$. Representations involve operators of multiplication with piecewise functions, multiplication operators and inner superposition operators.

Keywords Piecewise function multiplication operators · Covariance commutation relations

MSC2020 Classification: 47B33 · 47L80 · 81D05 · 47L65

10.1 Introduction

In many areas of applications there can be found relations of the form

$$ST = F(TS) \tag{10.1}$$

where S, T are elements of an associative algebra and $F : \mathbb{R} \to \mathbb{R}$ is a function satisfying certain conditions. For example, if $F(z) = z$, then S and T commute. If

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$F(z) = -z$, then S and T anti-commute. If $F(z) = \delta_0 + \delta_1 z$, then S and T satisfy the relation

$$ST - \delta_1 TS = \delta_0 I, \tag{10.2}$$

where δ_0, δ_1 are constants and I is the identity element. This relation is known as the deformed Heisenberg commutation relation [21]. If $\delta_0 = \delta_1 = 1$, then (10.2) reduces to the canonical Heisenberg commutation relation $ST - TS = I$ which is important in differential and integral calculus and quantum physics. If $\delta_0 = 0$ then (10.2) reduces to the quantum plane relation $ST = \delta_1 TS$. Elements of an algebra that satisfy (10.1) are called a representation of this relation in that algebra. Representations of covariance commutation relations (10.1) by linear operators are important for study of actions and induced representations of groups and semigroups, crossed product operator algebras, dynamical systems, harmonic analysis, wavelets and fractals analysis and have applications in physics and engineering [4, 5, 11, 22–24, 27–29, 35, 36, 43]. A description of the structure of representations for the relation (10.1) and more general families of self-adjoint operators satisfying such relations by bounded and unbounded self-adjoint linear operators on a Hilbert space use reordering formulas for functions of the algebra elements and operators satisfying covariance commutation relation, functional calculus and spectral representation of operators and interplay with dynamical systems generated by iteration of maps involved in the commutation relations [3, 7–9, 12, 13, 15–19, 30–35, 37–41, 43–56]. Algebraic properties of the commutation relation (10.1) are important in description of properties of its representations. For instance, there is a well-known link between linear operators satisfying the commutation relation (10.1) and spectral theory [43]. In case of $*$-algebras, when $S = X$ and $T = X^*$ where X is an element in the algebra, the relation (10.1) reduces to $XX^* = F(X^*X)$. This relation often can be transformed to relations of the form

$$AB = BF(A), \tag{10.3}$$

$$BA = F(A)B \tag{10.4}$$

for some other elements A, B of the $*$-algebra obtained from X and X^* using some transformations or factorizations in an appropriate functional calculus (see for example [35, 39, 43, 44, 55, 56] and references cited their). A description of the structure of representations of relation (10.3) by bounded and unbounded self-adjoint linear operators on Hilbert space by using spectral representation [2] of such operators is given in [43], where also more general families of commuting self-adjoint operators satisfying relation (10.3) with other operators on Hilbert spaces are considered using spectral theory and non-commutative analysis for bounded and unbounded operators on Hilbert spaces.

In this paper we construct representations of relation (10.3) and (10.4) by linear operators acting on Banach spaces L_p and $C[\alpha, \beta]$ for $\alpha, \beta \in \mathbb{R}$, and F is a polynomial. When $B = 0$, the relation (10.3) is trivially satisfied for any A. If $A = 0$ then the relation (10.3) reduces to $F(0)B = 0$. This implies either ($F(0) = 0$ and B can be any well defined operator) or $B = 0$. Thus, we focus on construction and properties of non-zero representations of (10.3). Such representations can also be viewed as solutions for operator equations $AX = XF(A)$, when A is specified or $XB = BF(X)$ when B is specified. We consider representations of (10.3) involving linear operators with piecewise function multiplication operators, multiplication operators and inner superposition operators. We derive conditions on the parameters or functional coefficients of operators so that they satisfy (10.3) for a polynomial F. In contrast to [35, 43, 44, 56] devoted to involutive representations of covariance type relations by operators on Hilbert spaces using spectral theory of operators on Hilbert spaces, we aim at direct construction of various classes of representations of covariance type relations in specific important classes of operators on Banach spaces more general than Hilbert spaces without imposing any involution conditions and not using classical spectral theory of operators.

This paper is organized in five sections, after the introduction, we present in Sect. 10.2 preliminaries, notations and basic definitions. In Sect. 10.3, we present representations involving piecewise function multiplication operators acting on L_p for $1 < p < \infty$. In Sect. 10.4, we construct representations involving inner superposition operators. These operators are important in wavelets analysis for instance. In Sect. 10.5 we construct representations by multiplication operators acting on L_p for $1 < p < \infty$ and the space of continuous functions.

10.2 Preliminaries and Notations

We use the following basic standard definitions and notations (see for example [1, 6, 10, 14, 20, 25, 26, 42]). Let $S \subseteq \mathbb{R}$, ($\mathbb{R}$ is the set of real numbers), be a Lebesgue measurable set and let $(S, \Sigma, \tilde{m})$ be a σ-finite measure space, that is, S is a nonempty set, Σ is a $\sigma-$algebra with subsets of S, where S can be covered with at most countably many disjoint sets $E_1, E_2, E_3, \ldots$ such that $E_i \in \Sigma$, $\tilde{m}(E_i) < \infty$, $i = 1, 2, \ldots$ and $\tilde{m}$ is the Lebesgue measure. For $1 \leqslant p < \infty$, we denote by $L_p(S)$, the set of all classes of equivalent measurable functions $f : S \to \mathbb{R}$ such that $\int_S |f(t)|^p dt < \infty$. This is a Banach space with norm $\|f\|_p = \left(\int_S |f(t)|^p dt\right)^{\frac{1}{p}}$. We denote by $L_\infty(S)$ the set of all classes of equivalent measurable functions $f : S \to \mathbb{R}$ such that there exists a constant $\lambda > 0$ for which $|f(t)| \leq \lambda$ for almost every t. This is a Banach space with norm $\|f\|_\infty = \text{ess sup}_{t\in S} |f(t)|$. We denote by $C[\alpha, \beta]$ the set of all continuous functions $f : [\alpha, \beta] \to \mathbb{R}$. This is a Banach space with norm $\|f\| = \max_{t\in[\alpha,\beta]} |f(t)|$.

10.3 Representations by Operators Involving Piecewise Functions

Remark 10.3.1 When operators are given in abstract form, we use the notation $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$ meaning that operator A is well defined from $L_p(\mathbb{R})$ to $L_p(\mathbb{R})$ without discussing sufficient conditions for that to be satisfied.

Proposition 10.3.1 *Let* $(\mathbb{R}, \Sigma, \tilde{m})$ *be the standard Lebesgue measure space on the real line. Let* $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 \le p \le \infty$ *be defined as follows*

$$(Ax)(t) = \sum_{i=1}^{l} \alpha_i a(t) I_{G_i} x(t), \qquad (Bx)(t) = \sum_{i=1}^{m} \beta_i b(t) I_{H_i} x(t), \tag{10.5}$$

where $G_i \in \Sigma$ *for each* $i = 1, \ldots, l$, $\tilde{m}(G_i \cap G_j) = 0$ *if* $i \ne j$, $H_i \in \Sigma$ *for each* $i = 1, \ldots, m$, $\tilde{m}(H_i \cap H_j) = 0$ *if* $i \ne j$, I_{G_k} *is the indicator function of the set* G_k, $a, b : \mathbb{R} \to \mathbb{R}$ *are measurable functions. Then, for a polynomial* $F_1(z) = \delta_0 + F(z)$ *with real coefficients* $\delta_i \in \mathbb{R}$, $i = 0, \ldots, n$, *where* $F(z) = \delta_1 z + \cdots + \delta_n z^n$, *we have* $AB = BF_1(A)$ *if and only if for almost every* t,

$$\sum_{i=1}^{l}\sum_{j=1}^{m} \alpha_i \beta_j a(t) b(t) I_{G_i \cap H_j}(t) = \sum_{j=1}^{m} \delta_0 \beta_j b(t) I_{H_j}(t) + \sum_{i=1}^{l}\sum_{j=1}^{m} \beta_j b(t) I_{G_i \cap H_j}(t) F(a(t)\alpha_i).$$

Proof Consider a monomial $M(z) = \delta z^n$, where n is a positive integer, and $\delta \in \mathbb{R}$. We compute AB, δA^n and $B\delta A^n$ as follows:

$$\begin{aligned}(ABx)(t) &= \sum_{i=1}^{l} \alpha_i a(t) I_{G_i}(t) \left(\sum_{j=1}^{m} \beta_j b(t) I_{H_j}(t) x(t) \right)\\ &= \sum_{i=1}^{l}\sum_{j=1}^{m} \alpha_i \beta_j a(t) b(t) I_{G_i \cap H_j}(t) x(t),\end{aligned}$$

$$(A^2 x)(t) = \sum_{i=1}^{l}\sum_{j=1}^{m} \alpha_i a(t) \alpha_j a(t) I_{G_i \cap G_j}(t) x(t) = \sum_{i=1}^{l} \alpha_i^2 a^2(t) I_{G_i}(t) x(t)$$

for almost all t. Therefore, for almost all $t \in \mathbb{R}$,

$$
\begin{aligned}
(\delta A^n x)(t) &= \sum_{i=1}^{l} \delta\alpha_i^n a^n(t) I_{G_i}(t) x(t),\\
(B\delta A^n x)(t) &= \sum_{j=1}^{m}\sum_{i=1}^{l} \beta_j b(t)\delta\alpha_i^n a^n(t) I_{G_i}^n(t) I_{H_j}(t) x(t)\\
&= \sum_{j=1}^{m}\sum_{i=1}^{l} \beta_j b(t)\delta\alpha_i^n a^n(t) I_{G_i\cap H_j}(t) x(t).
\end{aligned}
$$

Thus $AB = BM(A)$ if and only if for almost every $t \in \mathbb{R}$,

$$
\begin{aligned}
\sum_{i=1}^{l}\sum_{j=1}^{m} \alpha_i a(t)\beta_j b(t) I_{G_i\cap H_j}(t) &= \sum_{j=1}^{m}\sum_{i=1}^{l} \delta\alpha_i^n a^n(t)\beta_j b(t) I_{G_i}^n(t) I_{H_j}(t)\\
&= \sum_{i=1}^{l}\sum_{j=1}^{m} M(\alpha_i a(t))\beta_j b(t) I_{G_i\cap H_j}(t).
\end{aligned}
$$

Suppose now that $F_1(z) = \delta_0 + F(z)$, where $F(z) = \delta_1 z + \cdots + \delta_n z^n$, where δ_i, $i = 0, \ldots, n$ are constants. Then, for almost every $t \in \mathbb{R}$ we have

$$
\begin{aligned}
F_1(A)x(t) &= \delta_0 x(t) + F(A)x(t) = \delta_0 x(t) + \sum_{i=1}^{l} F(\alpha_i a(t) I_{G_i}(t))x(t)\\
BF_1(A)x(t) &= \delta_0 (Bx)(t) + BF(A)x(t)\\
&= \sum_{j=1}^{m} \delta_0 \beta_j b(t) I_{H_j}(t)x(t) + \sum_{i=1}^{l}\sum_{j=1}^{m} b(t)F(\alpha_i a(t))\beta_j I_{G_i\cap H_j}(t)x(t).
\end{aligned}
$$

Then $AB = BF_1(A)$ if and only if for almost every $t \in \mathbb{R}$,

$$
\begin{aligned}
&\sum_{i=1}^{l}\sum_{j=1}^{m} \alpha_i a(t) b(t)\beta_j I_{G_i\cap H_j}(t)\\
&= \sum_{j=1}^{m} \delta_0 b(t)\beta_j I_{H_j}(t) + \sum_{i=1}^{l}\sum_{j=1}^{m} F(\alpha_i a(t)) b(t)\beta_j I_{G_i\cap H_j}(t). \qquad \square
\end{aligned}
$$

Example 10.3.2 Let $A : L_p([1,3]) \to L_p([1,3])$, $B : L_p([1,3]) \to L_p([1,3])$, $1 \le p \le \infty$ be operators defined as follows, for almost every t,

$$
(Ax)(t) = \sum_{i=1}^{3} \alpha_i I_{G_i} x(t), \quad (Ax)(t) = \sum_{i=1}^{3} \beta_i I_{H_i} x(t),
$$

where

$$G_1 = [1, 3/2),\ G_2 = [3/2, 2],\ G_3 = (2, 3],\ H_1 = [1, 2],\\ H_2 = (2, 5/2],\ H_3 = (5/2, 3], \tag{10.6}$$

$\alpha_i,\ \beta_i \in \mathbb{R},\ i = 1, 2, 3$. Let $F(z) = \delta_0 + \delta_3 z^3$, where $\delta_0, \delta_3 \in \mathbb{R}$. By applying Proposition 10.5.1 we have $AB = B\delta_0 + \delta_3 BA^3$ if and only if

$$\sum_{i=1}^{3}\sum_{j=1}^{3} \alpha_i \beta_j I_{G_i\cap H_j}(t) = \delta_0 \sum_{j=1}^{3} \beta_j I_{H_j}(t) + \sum_{i=1}^{3}\sum_{j=1}^{3} \delta_3 \alpha_i^3 \beta_j I_{G_i\cap H_j}(t).$$

By simplifying this we have

$$\beta_1 \sum_{i=1}^{2} \alpha_i I_{G_i}(t) + \alpha_3 \sum_{i=2}^{3} \beta_i I_{H_i}(t)\\ = \sum_{i=1}^{2} \delta_0 \beta_j I_{H_j}(t) + \beta_1 \sum_{i=1}^{2} \delta_3 \alpha_i^3 I_{G_i}(t) + \alpha_3^3 \delta_3 \sum_{i=2}^{3} \beta_i I_{H_i}(t).$$

This is equivalent to

$$\begin{cases} \beta_1\alpha_1 = \delta_0\beta_1 + \beta_1\delta_3\alpha_1^3 \\ \beta_1\alpha_2 = \delta_0\beta_1 + \beta_1\delta_3\alpha_2^3 \\ \beta_2\alpha_3 = \delta_0\beta_2 + \beta_2\delta_3\alpha_3^3 \\ \beta_3\alpha_3 = \delta_0\beta_3 + \beta_3\delta_3\alpha_3^3 \end{cases} \Leftrightarrow \begin{cases} F(\alpha_i)\beta_1 = \alpha_i\beta_1, & i = 1, 2, \\ F(\alpha_3)\beta_i = \alpha_3\beta_i, & i = 2, 3. \end{cases}$$

This is equivalent to the following:

$$\beta_1 = 0, \quad \text{or} \quad F(\alpha_i) = \alpha_i,\ i = 1, 2,\\ F(\alpha_3) = \alpha_3, \quad \text{or} \quad \beta_j = 0,\ j = 2, 3.$$

In particular, this holds if $\beta_i \neq 0$, for $i = 1, 2, 3$ and $F(\alpha_i) = \alpha_i$ for $i = 1, 2, 3$. Therefore, the operators

$$(Ax)(t) = \sum_{i=1}^{3} \alpha_i I_{G_i} x(t), \quad (Bx)(t) = \sum_{i=1}^{3} \beta_i I_{H_i} x(t),$$

where $\beta_i \neq 0$, $i = 1, 2, 3$, each α_i obeys $F(\alpha_i) = \alpha_i$, $i = 1, 2, 3$, and the sets G_i, H_i, $i = 1, 2, 3$ are given by (10.6), satisfy $AB = \delta_0 B + \delta_3 BA^3$ for real constants δ_0 and δ_3.

In the following corollary we consider a case where both operators A and B are considered on the same partition.

Corollary 10.3.3 *Let* $(\mathbb{R}, \Sigma, \tilde{m})$ *be the standard Lebesgue measure space on the real line. Let* $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 \le p \le \infty$ *be defined as follows*

$$(Ax)(t) = \sum_{i=1}^{l} \alpha_i a(t) I_{G_i} x(t), \qquad (Bx)(t) = \sum_{i=1}^{l} \beta_i b(t) I_{G_i} x(t),$$

where $G_i \in \Sigma$ *for* $i = 1, 2, \ldots, l$, $\tilde{m}(G_i \cap G_j) = 0$ *for* $i \neq j$, I_{G_k} *is the indicator function of the set* G_k*, and* $a, b : \mathbb{R} \to \mathbb{R}$ *are measurable functions. Then, for a polynomial* $F_1(z) = \delta_0 + F(z)$, *where* $F(z) = \delta_1 z + \cdots + \delta_n z^n$, $\delta_i \in \mathbb{R}$, $i = 0, \ldots, n$, *the commutation relation* $AB = BF_1(A)$ *is satisfied if and only if*

$$\sum_{i=1}^{l} \alpha_i \beta_i a(t) b(t) I_{G_i}(t) = \sum_{i=1}^{l} b(t) F_1(a(t)\alpha_i) \beta_i I_{G_i}(t) \quad \text{for almost every } t.$$

Proof By applying Proposition 10.3.1 we have $AB = BF_1(A)$ if and only if

$$\sum_{i,j=1}^{l} \alpha_i a(t) b(t) \beta_j I_{G_i \cap G_j}(t) = \sum_{j=1}^{l} b(t) \delta_0 \beta_j I_{G_j}(t) + \sum_{j,i=1}^{l} b(t) F(\alpha_i a(t)) \beta_j I_{G_i \cap G_j}(t)$$

for almost every t. Since $G_i \cap G_j = \emptyset$ when $i \neq j$, the last condition becomes

$$\sum_{i=1}^{l} \alpha_i a(t) b(t) \beta_i I_{G_i}(t) = \sum_{i=1}^{l} b(t) \delta_0 \beta_i I_{G_i}(t) + \sum_{i=1}^{l} b(t) F(\alpha_i a(t)) \beta_i I_{G_i}(t)$$
$$= \sum_{i=1}^{l} b(t) \beta_i I_{G_i}(t) [\delta_0 + F(\alpha_i a(t))] = \sum_{i=1}^{l} b(t) \beta_i I_{G_i}(t) F_1(\alpha_i a(t)). \qquad \square$$

Corollary 10.3.4 *Let* $(\mathbb{R}, \Sigma, \tilde{m})$ *be the standard Lebesgue measure space on the real line. Consider the operators* A *and* B *defined in* (10.5) *where* $\beta_j \neq 0$ *for all* $j \in \{1, \ldots, m\}$. *Then, for a constant monomial* $F_1(z) = \delta_0$, *we have* $AB = BF_1(A)$ *if and only if for all* $i \in \{1, \ldots, l\}$, $j \in \{1, \ldots, m\}$, *the set*

$$\operatorname{supp}(\alpha_i a(t) - \delta_0) \cap \operatorname{supp} b(t) \cap (G_i \cap H_j)$$

has measure zero.

Proof We have $F_1(A)x(t) = \delta_0 x(t)$, for almost all t. Moreover, for almost every t

$$(ABx)(t) = \sum_{i=1}^{l} \alpha_i a(t) I_{G_i}(t) \left(\sum_{j=1}^{m} \beta_j b(t) I_{H_j}(t) x(t) \right) = \sum_{i=1}^{l} \sum_{j=1}^{m} \alpha_i \beta_j a(t) b(t) I_{G_i \cap H_j} x(t)$$

$$(BF_1(A)x)(t) = \sum_{j=1}^{m} \delta_0 \beta_j b(t) I_{H_j}(t) x(t),$$

Then, for almost every $t \in \mathbb{R}$ we have $AB = BF_1(A)$ if and only if

$$\sum_{i=1}^{l} \sum_{j=1}^{m} \alpha_i a(t) \beta_j b(t) I_{G_i \cap H_j}(t) = \sum_{j=1}^{m} \delta_0 \beta_j b(t) I_{H_j}(t),$$

which is equivalent to

$$\sum_{i=1}^{l} \alpha_i a(t) \beta_j b(t) I_{G_i \cap H_j}(t) = \delta_0 \beta_j b(t) I_{H_j}(t), \text{ for almost every } t \in \mathbb{R},\ j \in \{1, \dots, m\}.$$

Since $\beta_j \neq 0$ for all j, the last condition is equivalent to the following: for $t \in G_i \cap H_j$, $1 \leq i \leq l$, $1 \leq j \leq m$, $(\alpha_i a(t) - \delta_0) b(t) = 0$, which is equivalent to that the set $\operatorname{supp}(\alpha_i a(t) - \delta_0) \cap \operatorname{supp} b(t) \cap (G_i \cap H_j)$ has measure zero, for all $i \in \{1, \dots, l\}$, $j \in \{1, \dots, m\}$. □

Corollary 10.3.5 *Let* $(\mathbb{R}, \Sigma, \tilde{m})$ *the standard Lebesgue measure space on the real line. Let* $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$*,* $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$*,* $1 \leq p \leq \infty$ *be defined as follows, for almost every* t*,*

$$Ax(t) = \sum_{i=1}^{l} \alpha_i a(t) I_{G_i}(t) x(t), \qquad (Bx)(t) = I_{G_k}(t) b(t) x(t), \quad 1 \leq k \leq l$$

where $G_i \in \Sigma$ *for each* $i = 1, \dots, l$*,* $\tilde{m}(G_i \cap G_j) = 0$ *for* $i \neq j$*,* $I_{G_k}(t)$ *is the indicator function for the set* G_k*,* $a, b : \mathbb{R} \to \mathbb{R}$ *are measurable functions. For a polynomial* $F(z) = \delta_1 z + \dots + \delta_n z^n$*,* $\delta_j \in \mathbb{R}$*,* $j = 1, \dots, n$*, we have* $AB = BF(A)$ *if and only if*

$$\operatorname{supp} b \cap \operatorname{supp} [\alpha_k a - F(\alpha_k a)] \cap G_k$$

has measure zero.

Proof We write the operator B as follows

$$(Bx)(t) = I_{G_k}(t) b(t) x(t) = \sum_{j=1}^{l} \beta_j b(t) I_{G_j}(t) x(t),$$

almost everywhere, where $\beta_j = \begin{cases} 1, & j = k \\ 0, & \text{otherwise} \end{cases}$. By using Corollary 10.3.3 we have $AB = BF(A)$ if and only if for almost very t

$$\sum_{i=1}^{l} \alpha_i \beta_i a(t) b(t) I_{G_i}(t) = \sum_{i=1}^{l} b(t) F(a(t)\alpha_i) \beta_i I_{G_i}(t).$$

We can simplify the last relation as follows

$$\alpha_k a(t) b(t) I_{G_k}(t) = b(t) F(a(t)\alpha_k) I_{G_k}(t) \;\Leftrightarrow\; b(t)(a(t)\alpha_k - F(a(t)\alpha_k)) = 0,$$

for almost every $t \in G_k$, which is equivalent to the set

$$\text{supp } b \cap \text{supp } (\alpha_k a - F(\alpha_k a)) \cap G_k$$

has measure zero. □

Example 10.3.6 Let $A : L_p[0, 1] \to L_p[0, 1]$, $B : L_p[0, 1] \to L_p[0, 1]$, $1 \le p \le \infty$ be defined as follows

$$\begin{aligned}(Ax)(t) &= \alpha I_{[0,1/3]}(t)x(t) + \beta I_{(1/3,1/2]}(t)x(t) + \gamma I_{(1/2,1]}(t)x(t) \\ (Bx)(t) &= I_{(1/3,1/2)}(t)x(t),\end{aligned}$$

where α, β are constants and I_E is the indicator function of the set E. For a monomial $F(z) = z^n$, where n is a positive integer, we have

$$AB = BF(A) \quad (\text{that is } AB = BA^n)$$

if and only if $F(\beta) = \beta$. In fact, taking a partition $G_1 \cup G_2 \cup G_3$ where

$$G_1 = [0, 1/3], \quad G_2 = (1/3, 1/2), \quad G_3 = [1/2, 1],$$

we have $Ax(t) = (\alpha I_{G_1}(t) + \beta I_{G_2}(t) + \gamma I_{G_3}(t))x(t)$ and $Bx(t) = I_{G_2}(t)x(t)$. By applying Corollary 10.3.5 we have $AB = BF(A)$ if and only if $F(\beta) = \beta$.

In this case, we can also get the same result by the following direct computation. For almost every $t \in [0, 1]$,

$$\begin{aligned}
(ABx)(t) &= A(Bx)(t) = (\alpha I_{[0,1/3]}(t) + \beta I_{(1/3,1/2)}(t) + \gamma I_{(1/2,1]}(t))(Bx)(t) \\
&= (\alpha I_{[0,1/3]}(t) + \beta I_{1/3,1/2}(t) + \gamma I_{(1/2,1]}(t)) I_{(1/3,1/2)}(t)x(t) \\
&= \beta I_{(1/3,1/2)}(t)x(t) \\
(A^2x)(t) &= A(Ax)(t) = (\alpha I_{[0,1/3]}(t) + \beta I_{(1/3,1/2)}(t) + \gamma I_{(1/2,1]}(t))(Ax)(t) \\
&= (\alpha I_{[0,1/3]}(t) + \beta I_{(1/3,1/2)}(t) + \gamma I_{(1/2,1]}(t))((\alpha I_{[0,1/3]}(t) + \beta I_{(1/3,1/2)}(t) \\
&+ \gamma I_{(1/2,1]}(t))x(t)) \\
&= (\alpha^2 I_{[0,1/3]}(t) + \beta^2 I_{(1/3,1/2)}(t) + \gamma^2 I_{(1/2,1]}(t))x(t).
\end{aligned}$$

In general, for almost every t,

$$(A^n x)(t) = (\alpha^n I_{[0,1/3]}(t) + \beta^n I_{(1/3,1/2)}(t))x(t) + \gamma^n I_{[1/2,1]}(t)x(t).$$

Thus, for almost every t,

$$\begin{aligned}
(BF(A)x)(t) &= (BA^n x)(t) = B(A^n x)(t) = I_{(1/3,1/2)}(A^n x)(t) \\
&= \beta^n I_{(1/3,1/2)}(t)x(t) = F(\beta) I_{(1/3,1/2)}(t)x(t).
\end{aligned}$$

10.4 Representations Involving Inner Superposition Operators

In this section we will look at the relation $BA = F(A)B$ which has important applications in wavelets analysis for instance.

Proposition 10.4.1 *Let* $(\mathbb{R}, \Sigma, \tilde{m})$ *the standard Lebesgue measure space on the real line. Let* $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $p \geq 1$ *be defined as follows, for almost every* $t \in \mathbb{R}$,

$$(Ax)(t) = \sum_{i=-\infty}^{\infty} \alpha_i I_{G_i}(t)x(t-1), \qquad (Bx)(t) = \beta x(\gamma t),$$

where $G_i \in \Sigma$ *for each* $i \in \mathbb{Z}$, $\tilde{m}(G_i \cap G_j) = 0$ *for* $i \neq j$, $I_{G_k}(t)$ *is the indicator function for the set* G_k, $\beta, \gamma \in \mathbb{R} \setminus \{0\}$ *and* $\gamma > 0$, $\alpha_i \in \mathbb{R}$, *for all* $i \in \mathbb{Z}$. *For a monomial* $F(z) = \delta z^m$, *where* $\delta \in \mathbb{R} \setminus \{0\}$, m *is a positive integer, we have* $(BAx)(t) = (F(A)Bx)(t)$ *for almost every* t *if and only if*

$$\sum_{i=-\infty}^{\infty} \alpha_i I_{G_i}(\gamma t)x(\gamma t - 1) = \delta \sum_{i=-\infty}^{\infty} \alpha_i^m I_{G_i}(t)x(\gamma t - \gamma m). \qquad (10.7)$$

Proof For almost every $t \in \mathbb{R}$ we have

$$\begin{aligned}(A^2x)(t) &= A\left(\sum_{i=-\infty}^{\infty}\alpha_i I_{G_i}(t)x(t-1)\right) = \sum_{i=-\infty}^{\infty}\alpha_i I_{G_i}(t)(Ax)(t-1)\\ &= \sum_{i=-\infty}^{\infty}\alpha_i I_{G_i}(t)\sum_{j=-\infty}^{\infty}\alpha_j I_{G_j}(t)x(t-2) = \sum_{i=-\infty}^{\infty}\alpha_i^2 I_{G_i}(t)x(t-2).\end{aligned}$$

We suppose that for almost every $t \in \mathbb{R}$,

$$(A^m x)(t) = \sum_{i=-\infty}^{\infty}\alpha_i^m I_{G_i}(t)x(t-m).$$

for $m = 1, 2, \ldots$. Then we have, for almost every t,

$$\begin{aligned}(A^{m+1}x)(t) = A(A^m x)(t) &= A\left(\sum_{i=-\infty}^{\infty}\alpha_i^m I_{G_i}(t)x(t-m)\right)\\ &= \sum_{i=-\infty}^{\infty}\alpha_i I_{G_i}(t)\sum_{j=-\infty}^{\infty}\alpha_j^m I_{G_j}(t)x(t-m-1)\\ &= \sum_{i=-\infty}^{\infty}\alpha_i^{m+1} I_{G_i}(t)x(t-(m+1)).\end{aligned}$$

Moreover, we have for almost every $t \in \mathbb{R}$,

$$\begin{aligned}(BAx)(t) &= \beta\sum_{i=-\infty}^{\infty}\alpha_i I_{G_i}(\gamma t)x(\gamma t-1)\\ \delta(A^m Bx)(t) &= \delta\beta\sum_{i=-\infty}^{\infty}\alpha_i^m I_{G_i}(t)x(\gamma t-\gamma m).\end{aligned}$$

Therefore, $(BAx)(t) = (F(A)Bx)(t)$ for almost every t if and only if

$$\sum_{i=-\infty}^{\infty}\alpha_i I_{G_i}(\gamma t)x(\gamma t-1) = \delta\sum_{i=-\infty}^{\infty}\alpha_i^m I_{G_i}(t)x(\gamma t-\gamma m), \text{ for almost every } t. \tag{10.8}$$

□

Proposition 10.4.2 *Let* $A : L_p(\mathbb{R}) \to L_p(\mathbb{R}),\ \ B : L_p(\mathbb{R}) \to L_p(\mathbb{R}),\ \ 1 \le p \le \infty$ *defined as follows*

$$(Ax)(\cdot) = \alpha x(a(\cdot)), \quad (Bx)(\cdot) = \beta x(b(\cdot)),$$

where α, β *are non zero real numbers,* $a, b : \mathbb{R} \to \mathbb{R}$ *are continuous functions. For a monomial* $F(z) = \delta z^m$ *where* $\delta \in \mathbb{R} \setminus \{0\}$*,* $m \in \mathbb{Z}_{>0} = \{1, 2, \ldots\}$*, we have for* $x \in L_p(\mathbb{R})$ *and for almost every* t*,*

$$(BAx)(t) = (F(A)Bx)(t)$$

if and only if

$$x(a(b(t))) = \delta \cdot \alpha^{m-1} \cdot x(b(a^{\circ(m)}(t)).$$

Proof We have for almost every t

$$(BAx)(t) = \alpha\beta x(a(b(t))),$$
$$(A^2 x)(t) = \alpha^2 x(a(a(t))).$$

In the same way we have for almost every t

$$(A^m x)(t) = \alpha^m x(a^{\circ(m)}(t)),$$
$$\delta(A^m Bx)(t) = \delta\alpha^m \beta x(b(a^{\circ(m)}(t))).$$

Then, for all $L_p(\mathbb{R})$, $1 \le p \le \infty$, $ABx = BF(A)x$ if and only if for almost every t,

$$x(a(b(t))) = \delta \cdot \alpha^{m-1} x(b(a^{\circ(m)}(t))). \tag{10.9}$$

□

Example 10.4.3 Consider operators $A : L_2(\mathbb{R}) \to L_2(\mathbb{R})$, $B : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ be defined as follows, for almost every t,

$$(Ax)(t) = x(t-1), \quad (Bx)(t) = \gamma^{1/2} x(\gamma t), \quad \gamma > 0.$$

These operators are particular case of the corresponding ones of Proposition 10.4.2 when $a(t) = t - 1$, $b(t) = \gamma t$, $\alpha = 1$, $\beta = \gamma^{\frac{1}{2}}$. We get

$$b(a(t)) = \gamma t - 1, \quad a^{\circ(m)}(b(t)) = \gamma t - \gamma m, \quad m = 1, 2, 3, \ldots.$$

If $\gamma = \frac{1}{m}$ and $\delta = 1$, then we get on both sides $x(\frac{t}{m} - 1)$ and thus $BA = A^m B$ on $L_p(\mathbb{R})$. For example, when $\gamma = 1/2$ and $m = 2$, we have operators

$$(Ax)(t) = x(t-1), \quad (Bx)(t) = \frac{1}{2^{1/2}} x\left(\frac{1}{2}t\right),$$

which satisfy $BA = A^2B$.

10.5 Representations Involving Weighted Composition Operators

In this section we consider pairs of operators (A, B) which involve weighted composition operators, and the multiplication operator composed with the point evaluation functional $\upsilon_\gamma : x(t) \mapsto x(\gamma)$ ("boundary value" operator, "one-dimensional range" operator). These operators act on some spaces of real-valued functions $x(t)$ of one real variable by the formulas

$$(T_{w,\sigma}x)(t) = w(t)x(\sigma(t)), \quad T_{w,\upsilon_\gamma}x(t) = w(t)x(\gamma), \quad t, \gamma \in \mathbb{R}.$$

Note that the multiplication operator composed with the point evaluation functional is a special case of weighted composition operator in these notations since

$$T_{w,\upsilon_\gamma} = T_{w,\sigma}, \quad \text{for} \quad \sigma = \upsilon_\gamma.$$

Proposition 10.5.1 *Let* $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 \le p \le \infty$ *be defined for measurable functions* $a, b : \mathbb{R} \to \mathbb{R}$ *by*

$$(Ax)(\cdot) = a(\cdot)x(\cdot), \quad (Bx)(\cdot) = b(\cdot)x(\cdot).$$

For a polynomial $F(z) = \delta_0 + \delta_1 z + \cdots + \delta_n z^n$ *with coefficients* $\delta_0, \delta_1, \ldots, \delta_n \in \mathbb{R}$,

$$AB = BF(A),$$

if and only if, $\text{supp}\, b \cap \text{supp}\,[a - F(a)]$ *has measure zero.*

Proof For almost every $t \in \mathbb{R}$,

$$(ABx)(t) = A(Bx)(t) = a(t)b(t)x(t),$$
$$(A^2x)(t) = A(Ax)(t) = a(t)(Ax)(t) = [a(t)]^2x(t) = [a(t)]^2x(t),$$
$$(A^3x)(t) = A(A^2x)(t) = [a(t)]^3x(t).$$

Therefore, for $n \ge 1$ and for almost every t,

$$(A^nx)(t) = [a(t)]^nx(t), \ (BA^nx)(t) = B(A^nx)(t) = b(t)(A^nx)(t) = b(t)[a(t)]^nx(t).$$

Thus, we have for almost every t,

$$(BF(A)x)(t) = \sum_{k=0}^{n} \delta_k (B(A^k x))(t) = \sum_{k=0}^{n} \delta_k b(t)[a(t)]^k x(t) = b(t)F(a(t))x(t).$$

Then $AB = BF(A)$ if and only if for almost every t,

$$b(t)F(a(t)) = a(t)b(t).$$

This is equivalent to the set supp $b \cap$ supp $[a - F(a)]$ having measure zero. $\square$

10.5.1 Representations by Operators on $C[\alpha, \beta]$

In this subsection we consider pairs of operators (A, B) which involve weighted composition operators, and the multiplication operator composed with the point evaluation functional $\upsilon_\gamma : x(t) \mapsto x(\gamma)$ ("boundary value" operator, "one-dimensional range" operator). These operators act on some spaces of continuous real-valued functions of one real variable $x(t)$ by the formulas:

$$(T_{w,\sigma} x)(t) = w(t)x(\sigma(t)), \quad T_{w,\upsilon_\gamma} x(t) = w(t)x(\gamma), \quad t, \gamma \in \mathbb{R}.$$

Proposition 10.5.2 *Let $A : C[\alpha, \beta] \to C[\alpha, \beta]$, $B : C[\alpha, \beta] \to C[\alpha, \beta]$, $1 \le p \le \infty$ defined as follows*

$$(Ax)(t) = a(t)x(\upsilon(t)), \quad (Bx)(t) = b(t)x(\sigma(t)),$$

where α, β are real numbers, $\alpha < \beta$, $a, b, \upsilon, \sigma : [\alpha, \beta] \to [\alpha, \beta]$ are continuous functions. For a monomial $F(z) = \delta z^m$ where $\delta \in \mathbb{R} \setminus \{0\}$, $m = 1, 2, \ldots$, and for $x \in C[\alpha, \beta]$,

$$(ABx)(t) = (BF(A)x)(t)$$

for each $t \in [\alpha, \beta]$, if and only if

$$a(t)b(\upsilon(t))x(\sigma(\upsilon(t))) = \delta b(t)a(\sigma(t)) \cdot a(\upsilon(\sigma(t))) \cdot \ldots \cdot a(\upsilon^{\circ(m-1)}(\sigma(t)))x(\sigma(\upsilon^{\circ(m)}(t))).$$

Proof For each $t \in [\alpha, \beta]$

$$(ABx)(t) = a(t)b(\upsilon(t))x(\sigma(\upsilon(t))),$$
$$(A^2 x)(t) = a(t)a(\upsilon(t))x(\upsilon(\upsilon(t)))).$$

In the same way we have for $n \ge 1$ and for each $t \in [\alpha, \beta]$,

$$(A^n x)(t) = a(t)a(\upsilon(t))a(\upsilon^{\circ(2)}(t))\dots a(\upsilon^{\circ(n-1)}(t))x(\upsilon^{\circ(n)}(t)),$$
$$\delta(BA^m x)(t) = \delta b(t)a(\sigma(t))a(\upsilon(\sigma(t)))\cdot\ldots\cdot a(\upsilon^{\circ(m-1)}(\sigma(t)))x(\upsilon^{\circ(m)}(\sigma(t))).$$

Then $ABx = BF(A)x$ for all $x \in C[\alpha, \beta]$ if and only if for all $t \in [\alpha, \beta]$,

$$\begin{aligned} &a(t)b(\upsilon(t))x(\sigma(\upsilon(t))) \\ &\qquad = \delta b(t)a(\sigma(t))\cdot a(\upsilon(\sigma(t)))\cdot\ldots\cdot a(\upsilon^{\circ(m-1)}(\sigma(t)))x(\upsilon^{\circ(m)}(\sigma(t))). \end{aligned}$$

□

10.5.1.1 Representations when *B* is the Multiplication Operator

In this case $\sigma_B(t) = t$, $t \in \mathbb{R}$ is the identity map.

Lemma 10.5.1 *Let* $a, b \in C[\alpha, \beta]$, $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$. *If the set* $\Omega = \operatorname{supp} a \cap \operatorname{supp} b$ *has measure zero, then it is empty.*

Proof Without loss of generality we suppose that there exists $\alpha_0 \in]\alpha, \beta[$ such that $\alpha_0 \in \Omega$, that is, Ω is not empty. Then, $a(\alpha_0) \neq 0$ and $b(\alpha_0) \neq 0$. Since a, b are continuous functions, there is an open interval $V_{\alpha_0} =]\alpha_0 - \varepsilon, \alpha_0 + \varepsilon[\subseteq]\alpha, \beta[$, $\varepsilon > 0$ such that $a(t) \neq 0$ and $b(t) \neq 0$ for all $t \in V_{\alpha_0}$. Then the Lebesgue measure of Ω is positive. But this contradicts the hypothesis. Then the set Ω must be empty. □

Proposition 10.5.3 *Let* $A : C[\alpha, \beta] \to C[\alpha, \beta]$, $B : C[\alpha, \beta] \to C[\alpha, \beta]$ *defined by*

$$(Ax)(t) = a(t)x(\gamma), \quad (Bx)(t) = b(t)x(t),$$

where α, β *are real numbers,* $\alpha < \beta$, $\gamma \in [\alpha, \beta]$ *and* $a, b : [\alpha, \beta] \to \mathbb{R}$ *are continuous functions. Let* $F(z) = \delta_0 + \delta_1 z + \dots + \delta_n z^n$, $\delta_i \in \mathbb{R}$, $i = 0, 1, \dots, n$. *For* $x \in C[\alpha, \beta]$ *and* $t \in [\alpha, \beta]$,

$$(ABx)(t) = (BF(A)x)(t)$$

if and only if,

$$a(t)b(\gamma)x(\gamma) = \delta_0 b(t)x(t) + \delta_1 b(t)a(t)x(\gamma) + b(t)a(t)x(\gamma)\sum_{j=2}^{n}\delta_j [a(\gamma)]^{j-1}. \tag{10.10}$$

Proof We have

$$(ABx)(t) = A(Bx)(t) = a(t)(Bx)(\gamma) = a(t)b(\gamma)x(\gamma)$$
$$(A^m x)(t) = a(t)x(\gamma) + a(t)[a(\gamma)]^{m-1}x(\gamma), \quad m = 2, 3, \dots$$
$$((BA^m)x)(t) = (BA^m x)(t) = \begin{cases} b(t)a(t)x(\gamma), & m = 1 \\ b(t)a(t)[a(\gamma)]^{m-1}x(\gamma), & m = 2, 3, \dots \end{cases}$$
$$(BF(A)x)(t) = \delta_0 b(t)x(t) + \delta_1 b(t)a(t)x(\gamma) + \sum_{j=2}^{n} \delta_j b(t)a(t)[a(\gamma)]^{j-1}x(\gamma).$$

Then $(ABx)(t) = (BF(A)x)(t)$ if and only if

$$a(t)b(\gamma)x(\gamma) = \delta_0 b(t)x(t) + \delta_1 b(t)a(t)x(\gamma) + \sum_{j=2}^{n} \delta_j b(t)a(t)[a(\gamma)]^{j-1}x(\gamma). \quad \square$$

Corollary 10.5.4 *Let $A : C[\alpha, \beta] \to C[\alpha, \beta]$, $B : C[\alpha, \beta] \to C[\alpha, \beta]$ be defined by*

$$(Ax)(t) = a(t)x(\gamma), \quad (Bx)(t) = b(t)x(t),$$

where α, β are real numbers, $\alpha < \beta$, $\gamma \in [\alpha, \beta]$ and $a, b : [\alpha, \beta] \to \mathbb{R}$ are continuous functions such that $a(\gamma) \neq 0$, $b(\gamma) \neq 0$. Let $F(z) = \delta_1 z + \dots + \delta_n z^n$, where $\delta_i \in \mathbb{R}$, $i = 1, \dots, n$. The following statements hold.

(i) *If $b(\cdot)$ is not constant in any open interval included in $[\alpha, \beta]$, then*

$$AB \neq BF(A).$$

(ii) *If $b(\cdot)$ is constant in some open interval included in $[\alpha, \beta]$, then*

$$AB = BF(A)$$

if and only if $\text{supp } a \cap \text{supp } \left[b(\gamma) - b \cdot k_1\right] = \emptyset$, *where* $k_1 = \sum_{j=1}^{n} \delta_j [a(\gamma)]^{j-1}$.
In particular, if $b(\cdot)$ is identically non-zero constant in $[\alpha, \beta]$ then $AB = BF(A)$ if and only if $k_1 = 1$.

Proof Suppose that there exist two continuous functions $a, b : [\alpha, \beta] \to \mathbb{R}$, $a(\gamma) \neq 0$, $b(\gamma) \neq 0$ and a polynomial $F(z) = \delta_1 z + \dots + \delta_n z^n$ such that $ABx = BF(A)x$ for all $x \in C[\alpha, \beta]$. By applying Proposition 10.5.3, $(ABx)(t) = (BF(A)x)(t)$ if and only if for all $x \in C[\alpha, \beta]$,

$$a(t)b(\gamma)x(\gamma) = \sum_{j=1}^{n} \delta_j b(t)a(t)[a(\gamma)]^{j-1}x(\gamma).$$

This is equivalent to

$$a(t)b(\gamma) = \sum_{j=1}^{n} \delta_j b(t)a(t)[a(\gamma)]^{j-1} \Leftrightarrow a(t)\left[b(\gamma) - b(t)\sum_{j=1}^{n} \delta_j [a(\gamma)]^{j-1}\right] = 0,$$

which is equivalent to the set

$$\Omega = \text{supp } a \cap \text{supp}\left[b(\gamma) - b \cdot \sum_{j=1}^{n} \delta_j [a(\gamma)]^{j-1}\right]$$

has measure zero. Since a, b are continuous functions, by applying Lemma 10.5.1 we conclude that the set Ω is empty. We consider the following cases.

Case 1: If $b(\cdot)$ is not constant in any open interval included in $[\alpha, \beta]$ and

$$k_1 = \sum_{j=1}^{n} \delta_j [a(\gamma)]^{j-1} \neq 1,$$

then $\gamma \in \Omega$. This contradicts the fact that Ω is empty.
Case 2: Suppose $k_1 = 1$ and $b(\cdot)$ is not constant in any open interval included in $[\alpha, \beta]$ and without loss of generality suppose that $\gamma \in]\alpha, \beta[$. Since the functions a and $b(\gamma) - k_1 b(t)$ are continuous, then for some positive ε, we can find an open interval $]\gamma - \varepsilon,\ \gamma + \varepsilon[\subseteq]\alpha, \beta[$ such that the set

$$\text{supp } a \cap \text{supp}\left[b(\gamma) - b \cdot k_1\right] \cap]\gamma - \varepsilon,\ \gamma + \varepsilon[\neq \emptyset,$$

which is a contradiction.
Case 3: If $b(\cdot)$ is constant in an open interval contained in $[\alpha, \beta]$ then $AB = BF(A)$ if and only if

$$\Omega = \text{supp } a \cap \text{supp}\left[b(\gamma) - b \cdot \sum_{j=1}^{n} \delta_j [a(\gamma)]^{j-1}\right] = \emptyset.$$

Case 4: If $b(\cdot)$ is identically constant in $[\alpha, \beta]$ and $k_1 = 1$ then Ω is empty. If $k_1 \neq 1$ then $\Omega = \text{supp } a$ must be empty. This implies $a(\cdot) \equiv 0$, but, $a(\gamma) \neq 0$. That is a contradiction. Therefore, the condition of Proposition 10.5.3 is fulfilled.
□

Example 10.5.5 Let $A : C[0, 3] \to C[0, 3]$, $B : C[0, 3] \to C[0, 3]$ be defined as follows

$$(Ax)(t) = a(t)x(1), \quad (Bx)(t) = b(t)x(t),$$

where $a(t) = \begin{cases} t(2-t), & \text{if } 0 \le t \le 2 \\ 0, & \text{otherwise,} \end{cases}$ and $b(t) = \begin{cases} 1, & \text{if } 0 \le t \le 2 \\ (t-1)^3, & \text{if } 2 < t \le 3 \end{cases}$. Consider a polynomial $F(z) = \delta_1 z + \cdots + \delta_n z^n$, $\delta_i \in \mathbb{R}$, $i = 1, \ldots, n$. If $k_1 = \sum_{j=1}^{n} \delta_j = 1$, then these operators satisfy $AB = BF(A)$. Indeed, by Corollary 10.5.4 we have

$$\Omega = \text{supp } a \cap [b(1) - b] = \emptyset.$$

Corollary 10.5.6 *Let* $A : C[\alpha, \beta] \to C[\alpha, \beta]$, $B : C[\alpha, \beta] \to C[\alpha, \beta]$ *be defined by*

$$(Ax)(t) = a(t)x(\gamma), \quad (Bx)(t) = b(t)x(t),$$

where α, β *are real numbers,* $\alpha < \beta$, $\gamma \in [\alpha, \beta]$ *and* $a, b : [\alpha, \beta] \to \mathbb{R}$ *are continuous functions such that* $b(\gamma) \neq 0$. *Consider a polynomial* $F(z) = \delta_0 + \delta_1 z + \cdots + \delta_n z^n$, $\delta_i \in \mathbb{R}$, $i = 0, \ldots, n$. *If* $a(\gamma) = 0$, *then* $AB = BF(A)$ *if and only if* $\delta_0 = 0$ *and*

$$\Omega = \text{supp } a \cap \text{supp}\,[b(\gamma) - \delta_1 b] = \emptyset.$$

Proof By applying Proposition 10.5.3 we have $(ABx)(t) = (BF(A)x)(t)$ if and only if, for all $x \in C[\alpha, \beta]$,

$$a(t)b(\gamma)x(\gamma) = \delta_0 b(t)x(t) + \delta_1 b(t)a(t)x(\gamma) + \sum_{j=2}^{n} \delta_j b(t)a(t)[a(\gamma)]^{j-1}x(\gamma).$$

By taking $t = \gamma$ and if $a(\gamma) = 0$ we get $\delta_0 b(\gamma)x(\gamma) = 0$ for all $x \in C[\alpha, \beta]$. Since $b(\gamma) \neq 0$ then δ_0 must be zero. By using this, we remain with the equation

$$a(\cdot)b(\gamma)x(\gamma) = \delta_1 a(\cdot)b(\cdot)x(\gamma)$$

for all $x \in C[\alpha, \beta]$. This is equivalent to $a(\cdot)[b(\gamma) - \delta_1 b(\cdot)] = 0$, which itself is equivalent to the set $\Omega = \text{supp } a \cap \text{supp}\,[b(\gamma) - \delta_1 b]$ having measure zero. Since the functions involved are continuous then the set Ω is empty. □

Remark 1 According to Corollary 10.5.6, if $a(\gamma) = 0$ the role of the polynomial $F(\cdot)$ is only played by the coefficient δ_1. In particular, Corollary 10.5.6 establishes conditions for representations of the quantum plane relation $AB = \delta_1 BA$, for a real constant δ_1.

Corollary 10.5.7 *Let* $A : C[\alpha, \beta] \to C[\alpha, \beta]$, $B : C[\alpha, \beta] \to C[\alpha, \beta]$ *be defined by*

$$(Ax)(t) = a(t)x(\gamma), \quad (Bx)(t) = b(t)x(t),$$

where α, β *are real numbers,* $\alpha < \beta$, $\gamma \in [\alpha, \beta]$ *and* $a, b : [\alpha, \beta] \to \mathbb{R}$ *are continuous functions such that* $a(\gamma) \neq 0$. *Consider a polynomial* $F(z) = \delta_1 z + \cdots + \delta_n z^n$,

$\delta_i \in \mathbb{R}$, $i = 1, \ldots, n$. *Let* $k_1 = \sum_{j=1}^{n} \delta_j [a(\gamma)]^{j-1}$. *If* $b(\gamma) = 0$, *then* $AB = BF(A)$ *if and only if* $k_1 = 0$ *or* supp $a \cap$ supp $b = \emptyset$.

Proof By applying Proposition 10.5.3 we have $(ABx)(t) = (BF(A)x)(t)$ if and only if for all $x \in C[\alpha, \beta]$ and for all $t \in [\alpha, \beta]$,

$$a(t)b(\gamma)x(\gamma) = \sum_{j=1}^{n} \delta_j b(t)a(t)[a(\gamma)]^{j-1}x(\gamma).$$

By hypothesis this reduces to the following:

$$\forall x \in C[\alpha, \beta] : \ k_1 a(t)b(t)x(\gamma) = 0.$$

This is equivalent to $k_1 a(\cdot)b(\cdot) = 0$ which is equivalent to $k_1 = 0$ or to the set

$$\Omega_1 = \text{supp } a \cap \text{supp } b$$

having measure zero. Since a, b are continuous functions, the set Ω_1 is empty. □

Example 10.5.8 Let $A : C[0, 2] \to C[0, 2]$, $B : C[0, 2] \to C[0, 2]$ be defined as follows

$$(Ax)(t) = a(t)x(1/2), \quad (Bx)(t) = b(t)x(t),$$

where $a(t) = \begin{cases} t(1-t), & \text{if } 0 \le t \le 1 \\ 0, & \text{otherwise,} \end{cases}$ and $b(t) = \begin{cases} 0, & \text{if } 0 \le t \le 1 \\ (1-t)(t-2), & \text{if } 1 < t \le 2 \end{cases}$.
These operators satisfy $AB = BF(A)$ for any polynomial $F(z) = \delta_1 z + \cdots + \delta_n z^n$, $\delta_i \in \mathbb{R}$, $i = 1, \ldots, n$. Indeed this follows by Corollary 10.5.7.

Example 10.5.9 Let $A : C[-1, 1] \to C[-1, 1]$, $B : C[-1, 1] \to C[-1, 1]$ be defined by

$$(Ax)(t) = a(t)x(0), \quad (Bx)(t) = b(t)x(t),$$

where $a(t) = 1 + t^2$, $b(t) = t$. Consider a polynomial $F(z) = \delta_1 z + \cdots + \delta_n z^n$, $\delta_i \in \mathbb{R}$, $i = 1, \ldots, n$. If $k_1 = \delta_1 + \cdots + \delta_n = 0$, then these operators satisfy $AB = BF(A)$. Indeed this follows by Corollary 10.5.7.

Corollary 10.5.10 *Let* $A : C[\alpha, \beta] \to C[\alpha, \beta]$, $B : C[\alpha, \beta] \to C[\alpha, \beta]$ *be defined by*

$$(Ax)(t) = a(t)x(\gamma), \quad (Bx)(t) = b(t)x(t),$$

where α, β *are real numbers,* $\alpha < \beta$, $\gamma \in [\alpha, \beta]$ *and* $a, b : [\alpha, \beta] \to \mathbb{R}$ *are non-zero continuous functions. Consider a polynomial* $F(z) = \delta_0 + \delta_1 z + \cdots + \delta_n z^n$, $\delta_i \in \mathbb{R}$,

$i = 0, \dots, n$. *If* $a(\gamma) = b(\gamma) = 0$ *then* $AB = BF(A)$ *if and only if* $\delta_0 = 0$ *and either* $\delta_1 = 0$ *or the set* $\Omega = \text{supp } a \cap \text{supp } b = \emptyset$.

Proof By applying Proposition 10.5.3 we have $(ABx)(t) = (BF(A)x)(t)$ if and only if for all $x \in C[\alpha, \beta]$ and for all $t \in [\alpha, \beta]$,

$$a(t)b(\gamma)x(\gamma) = \delta_0 b(t)x(t) + \delta_1 b(t)a(t)x(\gamma) + \sum_{j=2}^{n} \delta_j b(t)a(t)[a(\gamma)]^{j-1}x(\gamma).$$

By hypothesis, this reduces to the condition

$$\forall x \in C[\alpha, \beta] : \ \delta_0 b(\cdot)x(\cdot) + \delta_1 a(\cdot)b(\cdot)x(\gamma) = 0. \tag{10.11}$$

If $\delta_0 \neq 0$ and $t_0 \in \text{supp } b$, then in an open interval $]t_0 - \varepsilon, t_0 + \varepsilon[\subset [\alpha, \beta]$ we have $x(t) = -\frac{\delta_1}{\delta_0} a(t)\zeta, \ t \in]t_0 - \varepsilon, t_0 + \varepsilon[$, where ζ is a constant. Since $x(\cdot)$ is continuous in $]t_0 - \varepsilon, t_0 + \varepsilon[$, thus $1 + x^2(\cdot)$ is also continuous in $]t_0 - \varepsilon, t_0 + \varepsilon[$, but the identity (10.11) is not valid for this function. This contradicts the condition (10.11). Since $b(\cdot)$ is not identically zero, this implies that $\delta_0 = 0$. By using this, the condition (10.11) reduces to the following condition: $\forall x \in C[\alpha, \beta] : \ \delta_1 a(t)b(t)x(\gamma) = 0$, which is equivalent to $\delta_1 = 0$ or to $\Omega = \text{supp } a \cap \text{supp } b = \emptyset$. □

Example 10.5.11 Let $A : C[0, 2] \to C[0, 2]$, $B : C[0, 2] \to C[0, 2]$ be defined by

$$(Ax)(t) = a(t)x(1), \quad (Bx)(t) = b(t)x(t),$$

where $a(t) = \sin(\pi t)$, $b(t) = t^2 - 1$. These operators satisfy $AB = BF(A)$ for any polynomial $F(z) = \delta_2 z^2 + \dots + \delta_n z^n$, $\delta_i \in \mathbb{R}$, $i = 2, \dots, n$ are real constants. Indeed this follows by Corollary 10.5.10.

10.5.1.2 Representations when *A* is Multiplication Operator

Proposition 10.5.12 *Let* $A : C[\alpha, \beta] \to C[\alpha, \beta]$, $B : C[\alpha, \beta] \to C[\alpha, \beta]$ *be defined by*

$$(Ax)(t) = a(t)x(t), \quad (Bx)(t) = b(t)x(\gamma),$$

where $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, $\gamma \in [\alpha, \beta]$ *and* $a, b : [\alpha, \beta] \to \mathbb{R}$ *are continuous functions. Consider a polynomial* $F(z) = \delta_0 + \delta_1 z + \dots + \delta_n z^n$, $\delta_i \in \mathbb{R}$, $i = 0, 1, \dots, n$. *Then,* $AB = BF(A)$ *if and only if* $\text{supp } [a - F(a(\gamma))] \cap \text{supp } b = \emptyset$.

Proof We have

$$(ABx)(t) = A(Bx)(t) = a(t)(Bx)(t) = a(t)b(t)x(\gamma)$$
$$(A^m x)(t) = [a(t)]^m x(t), \quad m = 0, \dots, n$$
$$(BA^m)x)(t) = (BA^m x)(t) = B(A^m x)(t) = \begin{cases} b(t)x(\gamma), & m = 0 \\ b(t)[a(\gamma)]^m x(\gamma), & m = 1, \dots, n \end{cases}$$
$$(BF(A)x)(t) = \delta_0 b(t)x(\gamma) + \sum_{j=1}^{n} \delta_j b(t)[a(\gamma)]^j x(\gamma) = b(t)F(a(\gamma))x(\gamma).$$

Then $(ABx)(t) = (BF(A)x)(t)$ if and only if for all $x \in C[\alpha, \beta]$ and for all $t \in [\alpha, \beta]$,

$$a(t)b(t)x(\gamma) = b(t)F(a(\gamma))x(\gamma).$$

This is equivalent to the equation $a(\cdot)b(\cdot) = b(\cdot)F(a(\gamma))$, which is equivalent to

$$\text{supp } [a - F(a(\gamma))] \cap \text{supp } b = \emptyset,$$

by Lemma 10.5.1. □

Example 10.5.13 Let $A : C[\alpha, \beta] \to C[\alpha, \beta]$, $B : C[\alpha, \beta] \to C[\alpha, \beta]$ be defined by

$$(Ax)(t) = a(t)x(t), \quad (Bx)(t) = b(t)x(\gamma),$$

where $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, $\gamma \in [\alpha, \beta]$ and $a, b : [\alpha, \beta] \to \mathbb{R}$ are continuous functions. Then, for a real constant ζ and a positive integer d,

$$AB = \zeta BA^d$$

if and only if $\text{supp } [a - \zeta a(\gamma)^d] \cap \text{supp } b = \emptyset$. This follows by Proposition 10.5.12.

Example 10.5.14 Let $A = A_\nu : C[0, 2] \to C[0, 2]$, $B : C[0, 2] \to C[0, 2]$ be defined by

$$(Ax)(t) = A_\nu x(t) = a(t)x(t), \quad (Bx)(t) = b(t)x(1/2),$$

where $a(t) = I_{[0,1]}(t)\sin(\pi t) + \nu$, $b(t) = I_{[1,2]}(t)\sin(\pi t)$, $\nu \in \mathbb{R}$, and $I_{[\alpha_1, \beta_1]}(t)$ is the indicator function of the interval $[\alpha_1, \beta_1]$. Consider a polynomial $F(z) = \delta_0 + \delta_1 z + \cdots + \delta_n z^n$, $\delta_i \in \mathbb{R}$, $i = 0, \dots, n$. If $\nu = F(1 + \nu)$, then, by Proposition 10.5.12, these operators satisfy $AB = BF(A)$. Note that the condition $\nu = F(1 + \nu)$ is equivalent to $\tilde{F}(1 + \nu) = 1 + \nu$, where $\tilde{F}(z) = F(z) + 1$, that is to $1 + \nu$ being a fixed point of $\tilde{F}(z)$, or equivalently to $1 + \nu$ being the root of $\tilde{F}(z) - z = 0$, which in terms of F is the same as $1 + \nu$ being a root of $F(z) - z + 1 = 0$. If this equation has roots in $\mathbb{R}$, then for such roots the corresponding operators A and B satisfy $AB = BF(A)$. If $F(z) = z - 1$, then $1 + \nu$ is a root of $F(z) - z + 1 = 0$

for any real number ν, since the equation becomes the equality $0 = 0$. Thus, for any $\nu \in \mathbb{R}$, the operators A and B satisfy $AB = B(A - I)$ which is equivalent to $AB - BA = -B$ and to $BA - AB = B$, where I is the identity operator. This can be also checked by direct computation. In fact, the operator A_ν corresponding to the parameter ν can be expressed as $A_\nu = A_0 + \nu I$. For $\nu = 0$, a and b have non-overlapping supports and hence $(A_0 Bx)(t) = x(\frac{1}{2})a(t)b(t) = 0$, and $(BA_0 x)(t) = b(t)(A_0 x)(\frac{1}{2}) = b(t)a(\frac{1}{2})x(\frac{1}{2}) = b(t)x(\frac{1}{2}) = B$, and thus $BA - AB = B$, and hence, for any $\nu \in \mathbb{R}$, $BA_\nu - A_\nu B = B(A_0 + \nu I) - (A_0 + \nu I)B = BA_0 + \nu B - A_0 B - \nu B = BA_0 - A_0 B = B$.

Corollary 10.5.15 *Let $A : C[\alpha, \beta] \to C[\alpha, \beta]$, $B : C[\alpha, \beta] \to C[\alpha, \beta]$ be defined by*

$$(Ax)(t) = a(t)x(t), \quad (Bx)(t) = b(t)x(\gamma),$$

where $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, $\gamma \in [\alpha, \beta]$ and $a, b : [\alpha, \beta] \to \mathbb{R}$ are continuous functions. Consider a polynomial $F(z) = \delta_0 + \delta_1 z + \cdots + \delta_n z^n$, $\delta_i \in \mathbb{R}$, $i = 0, \ldots, n$. If $a(\gamma) = 0$ then $AB = BF(A)$ if and only if $\operatorname{supp}[a - \delta_0] \cap \operatorname{supp} b = \emptyset$. *Furthermore, if $\delta_0 \neq 0$ then $AB = BF(A)$ yields $b(\gamma) = 0$.*

Proof This follows by Proposition 10.5.12. □

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Chapter 11
Nearly Associative and Nearly Hom-Associative Algebras and Bialgebras

Mafoya Landry Dassoundo and Sergei Silvestrov

Abstract Basic definitions and properties of nearly associative algebras are described. Nearly associative algebras are proved to be Lie-admissible algebras. Two-dimensional nearly associative algebras are classified, and main classes are derived. The bimodules, matched pairs and Manin triple of a nearly associative algebras are derived and their equivalence with nearly associative bialgebras is proved. Basic definitions and properties of nearly Hom-associative algebras are described. Related bimodules and matched pairs are given, and associated identities are established.

Keywords Nearly Hom-associative algebra · Nearly associative algebra · Bialgebra · Bimodule

MSC 2020 Classification: 17B61 · 17D30 · 17D25 · 17B62

11.1 Introduction

An algebra A with a bilinear product $\cdot : A \times A \to A$ is not necessarily associative (possibly non-associative) if possibly there are $x, y, z \in A$ obeying $(x \cdot y) \cdot z - x \cdot (y \cdot z) \neq 0$. If such $x, y, z \in A$ exist, then algebra is not associative. The term non-associative algebras is used often to mean all possibly non-associative algebras, including also the associative algebras. Associative algebras, Lie algebras, and Jordan algebras are well-known sub-classes of non-associative algebras in the sense of possibly not associative algebras [60].

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Hom-algebraic structures originated from quasi-deformations of Lie algebras of vector fields which gave rise to quasi-Lie algebras, defined as generalized Lie structures in which the skew-symmetry and Jacobi conditions are twisted. Hom-Lie algebras and more general quasi-Hom-Lie algebras where introduced first by Silvestrov and his students Hartwig and Larsson in [27], where the general quasi-deformations and discretizations of Lie algebras of vector fields using general twisted derivations, σ-derivations, and a general method for construction of deformations of Witt and Virasoro type algebras based on twisted derivations have been developed. The initial motivation came from examples of q-deformed Jacobi identities discovered in q-deformed versions and other discrete modifications of differential calculi and homological algebra, q-deformed Lie algebras and other algebras important in string theory, vertex models in conformal field theory, quantum mechanics and quantum field theory, such as the q-deformed Heisenberg algebras, q-deformed oscillator algebras, q-deformed Witt, q-deformed Virasoro algebras and related q-deformations of infinite-dimensional algebras [1, 16–22, 33, 34, 41–43].

Possibility of studying, within the same framework, q-deformations of Lie algebras and such well-known generalizations of Lie algebras as the color and super Lie algebras provided further general motivation for development of quasi-Lie algebras and subclasses of quasi-Hom-Lie algebras and Hom-Lie algebras. The general abstract quasi-Lie algebras and the subclasses of quasi-Hom-Lie algebras and Hom-Lie algebras, as well as their color (graded) counterparts, color (graded) quasi-Lie algebras, color (graded) quasi-Hom-Lie algebras and color (graded) Hom-Lie algebras, including in particular the super quasi-Lie algebras, super quasi-Hom-Lie algebras, and super Hom-Lie algebras, have been introduced in [27, 37–39, 63, 64]. In [48], Hom-associative algebras were introduced, generalizing associative algebras by twisting the associativity law by a linear map. Hom-associative algebra is a triple $(A, \cdot, \alpha)$ consisting of a linear space A, a bilinear product $\cdot : A \times A \to A$ and a linear map $\alpha : A \to A$, satisfying $a_{\alpha,\cdot}(x, y, z) = (x \cdot y) \cdot \alpha(z) - \alpha(x) \cdot (y \cdot z) = 0$, for any $x, y, z \in A$. In [48], alongside Hom-associative algebras, the Hom-Lie admissible algebras generalizing Lie-admissible algebras, were introduced as Hom-algebras such that the commutator product, defined using the multiplication in a Hom-algebra, yields a Hom-Lie algebra, and also Hom-associative algebras were shown to be Hom-Lie admissible. Moreover, in [48], more general G-Hom-associative algebras including Hom-associative algebras, Hom-Vinberg algebras (Hom-left symmetric algebras), Hom-pre-Lie algebras (Hom-right symmetric algebras), and some other Hom-algebra structures, generalizing G-associative algebras, Vinberg and pre-Lie algebras respectively, have been introduced and shown to be Hom-Lie admissible, meaning that for these classes of Hom-algebras, the operation of taking commutator leads to Hom-Lie algebras as well. Also, flexible Hom-algebras have been introduced, connections to Hom-algebra generalizations of derivations and of adjoint maps have been noticed, and some low-dimensional Hom-Lie algebras have been described. The enveloping algebras of Hom-Lie algebras were considered in [67] using combinatorial objects of weighted binary trees. In [29], for Hom-associative algebras and Hom-Lie algebras, the envelopment problem, operads, and the Diamond Lemma and Hilbert series for the Hom-associative operad and free algebra have been studied. Strong Hom-associativity yielding a confluent rewrite system and a basis for the free strongly hom-associative algebra has been considered in [28]. An explicit constructive way, based on free Hom-associative algebras with involutive twisting, was developed in [25] to obtain the universal enveloping algebras and Poincaré-Birkhoff-Witt type theorem for Hom-Lie algebras with involutive twisting map. Free Hom-associative color algebra on a Hom-module and enveloping algebra of color Hom-Lie algebras with involutive twisting and also with more general conditions on the powers of twisting map was constructed, and Poincaré-Birkhoff-Witt type theorem was obtained in [4, 5]. It is worth noticing here that, in the subclass of Hom-Lie algebras, the skew-symmetry is untwisted, whereas the Jacobi identity is twisted by a single linear map and contains three terms as in Lie algebras, reducing to ordinary Lie algebras when the twisting linear map is the identity map.

Hom-algebra structures include their classical counterparts and open new broad possibilities for deformations, extensions to Hom-algebra structures of representations, homology, cohomology and formal deformations, Hom-modules and hom-bimodules, Hom-Lie admissible Hom-coalgebras, Hom-coalgebras, Hom-bialgebras, Hom-Hopf algebras, L-modules, L-comodules and Hom-Lie quasi-bialgebras, n-ary generalizations of biHom-Lie algebras and biHom-associative algebras and generalized derivations, Rota–Baxter operators, Hom-dendriform color algebras, Rota–Baxter bisystems and covariant bialgebras, Rota–Baxter cosystems, coquasitriangular mixed bialgebras, coassociative Yang–Baxter pairs, coassociative Yang–Baxter equation and generalizations of Rota–Baxter systems and algebras, curved $\mathcal{O}$-operator systems and their connections with tridendriform systems and pre-Lie algebras, BiHom-algebras, BiHom-Frobenius algebras and double constructions, infinitesimal biHom-bialgebras and Hom-dendriform D-bialgebras, Hom-algebras has been considered from a category theory point of view [3, 8–15, 24, 26, 30, 31, 35–37, 40, 44–46, 49–52, 56, 57, 61, 62, 65–70].

This paper is organized as follows. In Sect. 11.2, basic definitions and fundamental identities and some elementary examples of nearly associative algebras are given. In Sect. 11.3, we derive the classification of the two-dimensional nearly associative algebras and main classes are provided. In Sect. 11.4, bimodules, duals bimodules and matched pair of nearly associative algebras are established and related identities are derived and proved. In Sect. 11.5, Manin triple of nearly associative algebras is given and its equivalence to the nearly associative bialgebras is derived. In Sect. 11.6, Hom-Lie-admissible, G-Hom-associative, flexible Hom-algebras, the result on Lie-admissibility of G-Hom-admissible algebras and subclasses of G-Hom-admissible algebras are reviewed. In Sect. 11.7, main definitions and fundamental identities of Hom-nearly associative algebras are given. Furthermore, the bimodules, and matched pair of the Hom-nearly associative algebras are derived and related properties are obtained.

11.2 Nearly Associative Algebras: Basic Definitions and Properties

Throughout this paper, for simplicity of exposition, all linear spaces are assumed to be over field $\mathbb{K}$ of characteristic is 0, even though many results hold in general for other fields as well unchanged or with minor modifications. An algebra is a couple (A, μ) consisting of a linear space A and a bilinear product $\mu : A \times A \to A$.

Definition 11.1 An algebra $(A, \cdot)$ is called nearly associative if, for $x, y, z \in A$,

$$x \cdot (y \cdot z) = (z \cdot x) \cdot y.$$

Example 11.1 Consider a two-dimensional linear space A with basis $\{e_1, e_2\}$.

1) Then, $(A, \cdot)$ is a nearly associative algebra, where $e_1 \cdot e_1 = e_1 + e_2$ and for all $(i, j) \neq (1, 1)$ with $i, j \in \{1, 2\}$, $e_i \cdot e_j = 0$.
2) The linear product defined on A by:

$$e_1 \cdot e_1 = e_2,\ e_1 \cdot e_2 = e_1 = e_2 \cdot e_1,\ e_2 \cdot e_2 = e_2,$$

is such that $(A, \cdot)$ is a nearly associative algebra.

Example 11.2 Let A be a three-dimensional linear space with basis $\{e_1, e_2, e_3\}$.

1) The linear space A equipped with the linear product defined on A by:

$$e_1 \cdot e_1 = e_2 + e_3,\ e_2 \cdot e_2 = e_1 + e_2 - e_3,\ e_3 \cdot e_3 = -e_1 + e_2$$

and for all $i \neq j$, $e_i \cdot e_j = 0$, where $i, j \in \{1, 2, 3\}$, is a nearly associative algebra.
2) The linear space A equipped with the linear product defined by

$$e_1 \cdot e_1 = e_2 - e_3,\ e_2 \cdot e_2 = e_2 + e_3,\ e_3 \cdot e_3 = e_1 - e_2 + e_3$$

and $e_i \cdot e_j = 0$, for all $i \neq j, i, j \in \{1, 2, 3\}$, is a nearly associative algebra.
3) The linear space A with the linear product defined by

$$e_1 \cdot e_1 = e_1 + e_2 + e_3,\ e_2 \cdot e_2 = e_1 + e_3,\ e_3 \cdot e_3 = e_1 + e_2$$

and $e_i \cdot e_j = 0$ for all $i \neq j,\ i, j \in \{1, 2, 3\}$, is a nearly associative algebra.

Definition 11.2 ([2, 23, 53–55, 58, 59]) An algebra $(A, \cdot)$ is called Lie admissible if $(A, [.,.])$ is a Lie algebra, where $[x, y] = x \cdot y - y \cdot x$ for all $x, y \in A$.

For a Lie admissible algebra $(A, \cdot)$, the Lie algebra $\mathcal{G}(A) = (A, [.,.])$ is called an underlying Lie algebra of $(A, \cdot)$.

It is known that associative algebras, left-symmetric algebras and anti-flexible algebras (center-symmetric algebras) are Lie-admissible [6, 7, 30].

Proposition 11.1 *Any nearly associative algebra is Lie-admissible.*

Proof The commutator $[.,.] : (v, w) \mapsto v \cdot w - w \cdot v$ is skew-symmetric on any algebra $(A, \cdot)$, and in a nearly associative algebra $(A, \cdot)$, for any $x, y, z \in A$,

$$
\begin{aligned}
&[x, [y, z]] + [y, [z, x]] + [z, [x, y]] \\
&= [x, y \cdot z - z \cdot y] + [y, z \cdot x - x \cdot z] + [z, x \cdot y - y \cdot x] \\
&= x \cdot (y \cdot z) - x \cdot (z \cdot y) - (y \cdot z) \cdot x + (z \cdot y) \cdot x \\
&\quad + y \cdot (z \cdot x) - y \cdot (x \cdot z) - (z \cdot x) \cdot y + (x \cdot z) \cdot y \\
&\quad + z \cdot (x \cdot y) - z \cdot (y \cdot x) - (x \cdot y) \cdot z + (y \cdot x) \cdot z \\
&= \{x \cdot (y \cdot z) - (z \cdot x) \cdot y\} + \{(y \cdot x) \cdot z - x \cdot (z \cdot y)\} \\
&\quad + \{y \cdot (z \cdot x) - (x \cdot y) \cdot z\} + \{z * (x \cdot y) - (y \cdot z) \cdot x\} \\
&\quad + \{(z \cdot y) \cdot x - y \cdot (x \cdot z)\} + \{(x \cdot z) \cdot y - z \cdot (y \cdot x)\} = 0.
\end{aligned}
$$

Therefore, $(A, [.,.])$ is a Lie algebra. □

Remark 11.1 In a nearly associative algebra $(A, \cdot)$, for $x, y \in A$,

$$
\begin{aligned}
L(x)L(y) &= R(y)R(x), \\
L(x)R(y) &= L(y \cdot x), \\
R(x)L(y) &= R(x \cdot y),
\end{aligned}
$$

where $L, R : A \to \text{End}(A)$ are the operators of left and right multiplications.

Definition 11.3 An anti-flexible algebra is a couple $(A, \cdot)$ where A is a linear space, and $\cdot : A \times A \to A$ is a bilinear product such that for all $x, y, z \in A$,

$$(x \cdot y) \cdot z - (z \cdot y) \cdot x = x \cdot (y \cdot z) - z \cdot (y \cdot x). \tag{11.1}$$

Using associator $a(x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z)$, (11.1) is equivalent to

$$a(x, y, z) = a(z, y, x). \tag{11.2}$$

In view of (11.2), anti-flexible algebras were called center-symmetric algebras in [30].

Proposition 11.2 *Any commutative nearly associative algebra is anti-flexible.*

Proof For all $x, y, z \in A$ in a commutative nearly associative algebra $(A, \cdot)$, by using nearly associativity, commutativity and again nearly associativity,

$$
\begin{aligned}
a(x, y, z) &= (x \cdot y) \cdot z - x \cdot (y \cdot z) = y \cdot (z \cdot x) - (z \cdot x) \cdot y = [y, z \cdot x] \\
&= [y, x \cdot z] = y \cdot (x \cdot z) - (x \cdot z) \cdot y = (z \cdot y) \cdot x - z \cdot (y \cdot x) = a(z, y, x)
\end{aligned}
$$

proves (11.2) meaning that $(A, \cdot)$ is anti-flexible. □

11.3 Classification of the Two-Dimensional Nearly Associative Algebras

In this section we compute some low-dimensional examples of near associative algebras contributing at the same time also towards investigation of classification of low-dimensional nearly associative algebras.

Theorem 11.1 *A two-dimensional algebra* $(A, \cdot)$ *with a basis* $\{e_1, e_2\} \in A$ *is nearly associative if and only if*

$$\begin{aligned} e_1 \cdot (e_1 \cdot e_1) &= (e_1 \cdot e_1) \cdot e_1, & e_1 \cdot (e_1 \cdot e_2) &= (e_2 \cdot e_1) \cdot e_1, \\ e_1 \cdot (e_2 \cdot e_1) &= (e_1 \cdot e_1) \cdot e_2, & e_2 \cdot (e_1 \cdot e_1) &= (e_1 \cdot e_2) \cdot e_1, \\ e_1 \cdot (e_2 \cdot e_2) &= (e_2 \cdot e_1) \cdot e_2, & e_2 \cdot (e_1 \cdot e_2) &= (e_2 \cdot e_2) \cdot e_1, \\ e_2 \cdot (e_2 \cdot e_1) &= (e_1 \cdot e_2) \cdot e_2, & e_2 \cdot (e_2 \cdot e_2) &= (e_2 \cdot e_2) \cdot e_2. \end{aligned}$$

Theorem 11.2 *Any two-dimensional nearly associative algebra over the field* $\mathbb{K} = \mathbb{C}$ *is isomorphic to one of the following nearly associative algebras*:

(i) *for all* $(\alpha, \beta) \in \mathbb{K}^2 \backslash \{(0, 0)\}$,

$$e_1 \cdot e_1 = \alpha e_2,\ e_1 \cdot e_2 = \beta e_1 = e_2 \cdot e_1,\ e_2 \cdot e_2 = \beta e_2,$$

(ii) *for all* $(\alpha, \beta) \in \mathbb{K}^2 \backslash \{(0, 0)\}$,

$$e_1 \cdot e_1 = \alpha e_1 + \beta e_2,\ e_1 \cdot e_2 = \beta e_1 + \alpha e_2 = e_2 \cdot e_1,\ e_2 \cdot e_2 = \alpha e_1 + \beta e_2,$$

(iii) *for all* $(\alpha, \beta, \gamma) \in \mathbb{K}^3$*, such that* $\gamma^2 + 4\alpha\beta \geq 0$,

$$\begin{aligned} e_1 \cdot e_1 &= \alpha e_1, \quad e_2 \cdot e_2 = \beta e_1 + \gamma e_2, \\ e_1 \cdot e_2 &= \frac{1}{2}\left(\gamma + \sqrt{\gamma^2 + 4\alpha\beta}\right) e_1 = e_2 \cdot e_1. \end{aligned}$$

Proof Equip the linear space A with the basis $\{e_1, e_2\}$, and for all $i, j \in \{1, 2\}$, set $e_i \cdot e_j = a_{ij} e_1 + b_{ij} e_2$, where $a_{ij} \in \mathbb{K}$ and $b_{ij} \in \mathbb{K}$. In addition, for all $i, j, k \in \{1, 2\}$, $a_{jk} a_{i1} + b_{jk} a_{i2} = a_{ki} a_{1j} + b_{ki} a_{2j}$, $a_{jk} b_{i1} + b_{jk} b_{i2} = a_{ki} b_{1j} + b_{ki} b_{2j}$. By Theorem 11.1,

$$\begin{cases} e_1 \cdot (e_1 \cdot e_1) = (e_1 \cdot e_1) \cdot e_1 \\ e_1 \cdot (e_1 \cdot e_2) = (e_2 \cdot e_1) \cdot e_1 \\ e_1 \cdot (e_2 \cdot e_1) = (e_1 \cdot e_1) \cdot e_2 \\ e_2 \cdot (e_1 \cdot e_1) = (e_1 \cdot e_2) \cdot e_1 \\ e_1 \cdot (e_2 \cdot e_2) = (e_2 \cdot e_1) \cdot e_2 \\ e_2 \cdot (e_1 \cdot e_2) = (e_2 \cdot e_2) \cdot e_1 \\ e_2 \cdot (e_2 \cdot e_1) = (e_1 \cdot e_2) \cdot e_2 \\ e_2 \cdot (e_2 \cdot e_2) = (e_2 \cdot e_2) \cdot e_2 \end{cases} \Longleftrightarrow \begin{cases} a_{11}a_{11} + b_{11}a_{12} = a_{11}a_{11} + b_{11}a_{21}, \\ a_{11}b_{11} + b_{11}b_{12} = a_{11}b_{11} + b_{11}b_{21}, \\ a_{12}a_{11} + b_{12}a_{12} = a_{21}a_{11} + b_{21}a_{21}, \\ a_{12}b_{11} + b_{12}b_{12} = a_{21}b_{11} + b_{21}b_{21}, \\ a_{21}a_{11} + b_{21}a_{12} = a_{11}a_{12} + b_{11}a_{22}, \\ a_{21}b_{11} + b_{21}b_{12} = a_{11}b_{12} + b_{11}b_{22}, \\ a_{11}a_{21} + b_{11}a_{22} = a_{12}a_{11} + b_{12}a_{21}, \\ a_{11}b_{21} + b_{11}b_{22} = a_{12}b_{11} + b_{12}b_{21}, \\ a_{22}a_{11} + b_{22}a_{12} = a_{21}a_{12} + b_{21}a_{22}, \\ a_{22}b_{11} + b_{22}b_{12} = a_{21}b_{12} + b_{21}b_{22}, \\ a_{12}a_{21} + b_{12}a_{22} = a_{22}a_{11} + b_{22}a_{21}, \\ a_{12}b_{21} + b_{12}b_{22} = a_{22}b_{11} + b_{22}b_{21}, \\ a_{21}a_{21} + b_{21}a_{22} = a_{12}a_{12} + b_{12}a_{22}, \\ a_{21}b_{21} + b_{21}b_{22} = a_{12}b_{12} + b_{12}b_{22}, \\ a_{22}a_{21} + b_{22}a_{22} = a_{22}a_{12} + b_{22}a_{22}, \\ a_{22}b_{21} + b_{22}b_{22} = a_{22}b_{12} + b_{22}b_{22} \end{cases}$$

$$\Longleftrightarrow \begin{cases} e(b-c) = 0, e(f-g) = 0, \\ h(b-c) = 0, d(b-c) = 0, \\ d(f-g) = 0, a(f-g) = 0 \\ (b-c)(b+c) = 0, \\ (f-g)(f+g) = 0 \\ e(b-c) + f(g-a) = 0 \\ d(a-g) + b(h-c) = 0 \\ a(b-c) = 0, h(f-g) = 0 \\ (bf-cg) = 0, bg = de = fc \end{cases} \Longleftrightarrow$$

$$\begin{cases} a = r_1, b = r_2, c = r_2, d = r_1, \\ e = r_2, f = r_1, g = r_1, h = r_2 \end{cases} \text{or} \begin{cases} a = r_1, b = r_2, c = r_2, d = r_2, \\ e = r_1, f = r_1, g = r_1, h = r_2 \end{cases}$$

or

$$\begin{cases} a = r_5, b = 0, c = 0, d = 0, \\ e = r_6, f = r_5, g = r_5, h = r_8 \end{cases} \text{or} \begin{cases} a = r_5, b = 0, c = 0, d = r_6, \\ e = 0, f = r_5, g = r_5, h = r_8 \end{cases}$$

or

$$\begin{cases} a = r_9, b = \dfrac{|r_{12}| + r_{12}}{2}, \\ c = \dfrac{|r_{12}| + r_{12}}{2}, d = 0, \\ e = r_{11}, f = 0, g = 0, h = r_{12} \end{cases} \text{or} \begin{cases} a = r_9, b = \dfrac{\sqrt{4r_{10}\,r_9 + {r_{12}}^2} + r_{12}}{2}, \\ c = \dfrac{\sqrt{4r_{10}\,r_9 + {r_{12}}^2} + r_{12}}{2}, d = r_{10}, \\ e = 0, f = 0, g = 0, h = r_{12} \end{cases}$$

or

$$\begin{cases} a = r_{13}, b = \dfrac{r_{16} - \sqrt{r_{16}{}^2}}{2}, \\ c = \dfrac{r_{16} - \sqrt{r_{16}{}^2}}{2}, d = 0, \\ e = r_{14}, f = 0, \\ g = 0, h = r_{16} \end{cases} \text{or} \begin{cases} a = r_{13}, b = \dfrac{r_{16} - \sqrt{r_{16}{}^2 + 4r_{13}\,r_{14}}}{2}, \\ c = \dfrac{r_{16} - \sqrt{r_{16}{}^2 + 4r_{13}\,r_{14}}}{2}, d = r_{14}, \\ e = 0, f = 0, g = 0, h = r_{16} \end{cases}$$

or

$$\begin{cases} a = r_{17}, b = 0, c = 0, d = 0, \\ e = r_{18}, f = 0, g = 0, h = 0 \end{cases} \text{or} \begin{cases} a = r_{17}, b = \sqrt{r_{17}\,r_{18}}, c = \sqrt{r_{17}\,r_{18}}, \\ d = r_{18}, e = 0, f = 0, g = 0, h = 0 \end{cases}$$

or

$$\begin{cases} a = r_{20}, b = 0, c = 0, d = 0, \\ e = r_{21}, f = 0, g = 0, h = 0 \end{cases} \text{or} \begin{cases} a = r_{20}, b = -\sqrt{r_{20}\,r_{21}}, \\ c = -\sqrt{r_{20}\,r_{21}}, d = r_{21}, \\ e = 0, f = 0, g = 0, h = 0 \end{cases}$$

or

$$\begin{cases} a = 0, b = 0, c = 0, d = 0, \\ e = r_{24}, f = 0, g = 0, h = r_{25} \end{cases} \text{or} \begin{cases} a = 0, b = 0, c = 0, d = r_{23}, \\ e = 0, f = 0, g = 0, h = r_{25} \end{cases}$$

or

$$\begin{cases} a = 0, b = r_{26}, c = r_{26}, d = 0, \\ e = r_{28}, f = 0, g = 0, h = r_{26} \end{cases} \text{or} \begin{cases} a = 0, b = r_{26}, c = r_{26}, d = r_{27}, \\ e = 0, f = 0, g = 0, h = r_{26} \end{cases}$$

or

$$\begin{cases} a = r_{29}, b = 0, c = 0, d = 0, \\ e = 0, f = 0, g = 0, h = 0 \end{cases}$$

with $a_{11} = a, a_{12} = b, a_{21} = c, a_{22} = d, b_{11} = e, b_{12} = f, b_{21} = g, b_{22} = h.$

This yields the statement of the theorem. □

11.4 Bimodules and Matched Pairs Nearly Associative Algebras

Definition 11.4 Let $(A, \cdot)$ be a nearly associative algebra. Consider the linear maps $l; r : A \to \mathrm{End}(V)$, where V is a linear space. A triple (l, r, V) is a bimodule of $(A, \cdot)$ if for all $x, y \in A$,

$$l(x)l(y) = r(y)r(x), \tag{11.3}$$
$$l(x)r(y) = l(y \cdot x), \tag{11.4}$$
$$r(x)l(y) = r(x \cdot y). \tag{11.5}$$

Example 11.3 Let $(A, \cdot)$ be a nearly associative algebra. The triple (L, R, A) is a bimodule of $(A, \cdot)$, where for any $x, y \in A$, $L(x)y = x \cdot y = R(y)x$.

Proposition 11.3 *Let (l, r, V) be a bimodule of a nearly associative algebra $(A, \cdot)$, where $l; r : A \to \text{End}(V)$ are two linear maps and V a linear space. There is a nearly associative algebra defined on $A \oplus V$ by, for any $x, y \in A$ and any $u, v \in V$,*

$$(x + u) * (y + v) = x \cdot y + l(x)v + r(y)u. \tag{11.6}$$

Proof Consider the bimodule (l, r, V) of the nearly associative algebra $(A, \cdot)$. For all $x, y, z \in A$ and $u, v, w \in V$,

$$(x + u) * ((y + v) * (z + w)) = x \cdot (y \cdot z) + l(x)l(y)w + l(x)r(z)v + r(y \cdot z)u \tag{11.7a}$$
$$((z + w) * (x + u)) * (y + v) = (z \cdot x) \cdot y + l(z \cdot x)v + r(y)l(z)u + r(y)r(x)w \tag{11.7b}$$

Using (11.3), (11.4) and (11.5) in (11.7a) and (11.7b) yields that $(A \oplus V, *)$ is a nearly associative algebra. □

Corollary 11.1 *Let (l, r, V) be a bimodule of a nearly associative algebra $(A, \cdot)$, where $l, r : A \to \text{End}(V)$ are two linear maps and V a linear space. Then there is a Lie algebra product on $A \oplus V$ given for $x, y \in A$ and $u, v \in V$ by*

$$[x + u, y + v] = [x, y]_{\cdot} + (l(x) - r(x))v - (l(y) - r(y))u. \tag{11.8}$$

Proof It is simple to remark that the commutator of the product defined in (11.6) is the product defined in (11.8). By taking into account Proposition 11.1, the Jacobi identity of the product given in (11.8) is satisfied. □

Definition 11.5 Let $(\mathcal{G}, [.,.]_{\mathcal{G}})$ be a Lie algebra. A representation of $(\mathcal{G}, [.,.]_{\mathcal{G}})$ over the linear space V is a linear map $\rho : \mathcal{G} \to \text{End}(V)$ satisfying for $x, y \in \mathcal{G}$,

$$\rho([x, y]_{\mathcal{G}}) = \rho(x) \circ \rho(y) - \rho(y) \circ \rho(x). \tag{11.9}$$

Proposition 11.4 *Let $(A, \cdot)$ be a nearly associative algebra and let V be a finite-dimensional linear space over the field $\mathbb{K}$ such that (l, r, V) is a bimodule of $(A, \cdot)$, where $l, r : A \to \text{End}(V)$ are two linear maps. Then, the linear map $l - r : A \to \text{End}(V)$, $x \mapsto l(x) - r(x)$ is a representation of the underlying Lie algebra $\mathcal{G}(A)$ of $(A, \cdot)$.*

Proof Let (l, r, V) be a bimodule of the nearly associative algebra $(A, \cdot)$. Then, for any $x, y \in A$,

$$\begin{aligned}(l(x) - r(x))(l(y) - r(y)) - (l(y) - r(y))(l(x) - r(x))\\ = l(x)l(y) - l(x)r(y) - r(x)l(y) + r(x)r(y)\\ - l(y)l(x) + l(y)r(x) + r(y)l(x) - r(y)r(x)\\ = -l(x)r(y) - r(x)l(y) + l(y)r(x) + r(y)l(x)\\ = -l(y \cdot x) - r(x \cdot y) + l(x \cdot x) + r(y \cdot x)\\ = (l - r)(x \cdot y - y \cdot x) = (l - r)([x, y]).\end{aligned}$$

Therefore, (11.9) is satisfied for $l - r = \rho$. □

Definition 11.6 If $(A, \cdot)$ is a nearly associative algebra and (l, r, V) its associated bimodule, with a finite-dimensional linear space V, then the dual maps $l^*, r^* : A \to \mathrm{End}(V^*)$ of linear maps l, r, are defined so that, for $x \in A, u^* \in V^*, v \in V$, $\langle l^*(x)u^*, v\rangle = \langle u^*, l(x)v\rangle$, $\langle r^*(x)u^*, v\rangle = \langle u^*, r(x)v\rangle$.

Proposition 11.5 *Let $(A, \cdot)$ be a nearly associative algebra and (l, r, V) be its bimodule. The following properties are equivalent.*

(i) *(r^*, l^*, V^*) is a bimodule of $(A, \cdot)$,*
(ii) *$l(x)r(y) = r(y)l(x)$, for all $x, y \in A$,*
(iii) *(l^*, r^*, V^*) is a bimodule of $(A, \cdot)$.*

Proof Let $(A, \cdot)$ be a nearly associative algebra and (l, r, V) be its associated bimodule consisting of a finite-dimensional linear space V and linear maps $l, r : A \to \mathrm{End}(V)$ obeying (11.3), (11.4) and (11.5).

If (r^*, l^*, V^*) is a bimodule of $(A, \cdot)$, with correspondences $l \to r^*$ and $r \to l^*$ obeying (11.3), (11.4) and (11.5), then for $x, y \in A, v \in V, u^* \in V^*$:

$$\begin{aligned}\langle l(x)r(y)v, u^*\rangle = \langle v, r^*(y)l^*(x)u^*\rangle = \langle v, r^*(y \cdot x)u^*\rangle\\ = \langle r(y \cdot x)v, u^*\rangle = \langle r(y)l(x)v, u^*\rangle.\end{aligned}$$

Therefore, the relation $l(x)r(y) = r(y)l(x)$ is satisfied.

Suppose $l(x)r(y) = r(y)l(x)$ for any $x, y \in A$. For $x, y \in A, v \in V, u^* \in V^*$:

$$\left\langle l^*(x)l^*(y)u^*, v\right\rangle = \left\langle u^*, l(y)l(x)v\right\rangle = \left\langle u^*, r(x)r(y)v\right\rangle = \left\langle r^*(y)r^*(x)u^*, v\right\rangle$$

yields $l^*(x)l^*(y) = r^*(y)r^*(x)$;

$$\begin{aligned}\left\langle l^*(x)r^*(y)u^*, v\right\rangle = \left\langle u^*, r(y)l(x)v\right\rangle = \left\langle u^*, l(x)r(y)v\right\rangle =\\ \left\langle u^*, l(y \cdot x)v\right\rangle = \left\langle l^*(y \cdot x)u^*, v\right\rangle\end{aligned}$$

yields $l^*(x)r^*(y) = l^*(y \cdot x)$;

$$\langle r^*(y)l^*(x)u^*, v\rangle = \langle u^*, l(x)r(y)v\rangle = \langle u^*, r(y)l(x)v\rangle = \langle u^*, r(y\cdot x)v\rangle = \langle r^*(y\cdot x)u^*, v\rangle$$

yields $r^*(y)l^*(x) = r^*(y\cdot x)$. Thus, with correspondences $r^* \to l$ and $l^* \to r$, (11.3), (11.4) and (11.5) are satisfied.

Similarly, one obtains the equivalence between $l(x)r(y) = r(y)l(x)$, for any $x, y \in A$, and (l^*, r^*, V^*) being a bimodule of $(A, \cdot)$. □

Remark 11.2 It is clear that $(L^*_\cdot, R^*_\cdot, A^*)$ and $(R^*_\cdot, L^*_\cdot, A^*)$ are bimodules of the nearly associative algebra $(A, \cdot)$ if and only if L and R commute.

Theorem 11.3 *Let $(A, \cdot)$ and $(B, \circ)$ be two nearly associative algebras. Suppose that (l_A, r_A, B) and (l_B, r_B, A) are bimodules of $(A, \cdot)$ and $(B, \circ)$, respectively, where $l_A, r_A : A \to \mathrm{End}(B)$, $l_B, r_B : B \to \mathrm{End}(A)$ are four linear maps satisfying for all $x, y \in A$, $a, b \in B$,*

$$r_B(l_A(x)a)y + y\cdot(r_B(a)x) - (l_B(a)y)\cdot x - l_B(r_A(y)a)x = 0, \quad (11.10a)$$
$$r_B(a)(x\cdot y) - y\cdot(l_B(a)x) - r_B(r_A(x)a)y = 0, \quad (11.10b)$$
$$l_B(a)(x\cdot y) - (r_B(a)y)\cdot x - l_B(l_A(y)a)x = 0, \quad (11.10c)$$
$$r_A(l_B(a)x)b + b\circ(r_A(x)a) - (l_A(x)b)\circ a - l_A(r_B(b)x)a = 0, \quad (11.10d)$$
$$r_A(x)(a\circ b) - b\circ(l_A(x)a) - r_A(r_B(a)x)b = 0, \quad (11.10e)$$
$$l_A(x)(a\circ b) - (r_A(x)b)\circ a - l_A(l_B(b)x)a = 0. \quad (11.10f)$$

Then, $(A\oplus B, *)$ is a nearly associative algebra, where for $x, y \in A$, $a, b \in B$,

$$(x+a)*(y+b) = (x\cdot y + l_B(a)y + r_B(b)x) + (a\circ b + l_A(x)b + r_A(y)a).$$

Proof For $x, y, z \in A$ and $a, b, c \in B$,

$$\begin{aligned}(x+a)*((y+b)*(z+c)) = {} & x\cdot(y\cdot z) + \{x\cdot(l_B(b)z) + r_B(r_A(z)b)x\} + l_B(a)(y\cdot z)\\ & +\{x\cdot(r_B(c)y) + r_B(l_A(y)c)x\} + l_B(a)(l_B(b)z) + r_B(b\circ c)x\\ & +l_B(a)(r_B(c)y) \quad + a\circ(b\circ c) + \{a\circ(l_A(y)c) + r_B(r_B(c)y)a\}\\ & +\{a\circ(r_A(z)b) + r_B(l_B(b)z)a\} + l_A(x)(l_A(y)c)\\ & +l_A(x)(b\circ c) + l_A(x)(r_A(z)b) + r_A(y\cdot z)a,\end{aligned}$$

$$\begin{aligned}((z+c)*(x+a))*(y+b) = {} & (z\cdot x)\cdot y + \{(l_B(c)x)\cdot y + l_B(r_A(x)c)y\} + r_B(x)(z\cdot x)\\ & +\{(r_A(a)z)\cdot y + l_B(l_A(z)a)y\} + l_B(c\circ a)y + r_B(b)(l_B(c)x)\\ & +r_B(b)(r_B(a)z)(c\circ a)\circ y + \{(l_A(z)a)\circ b + l_A(r_B(a)z)b\}\\ & +\{(r_A(x)c)\circ b + l_A(l_B(c)x)b\} + r_A(y)(r_A(x)c)\\ & +r_A(y)(c\circ a) + r_A(y)(l_A(z)a) + l_A(z\cdot x)b.\end{aligned}$$

With (11.10a)–(11.10f), and (l_A, r_A, B) and (l_B, r_B, A) being respectively bimodules of $(A, \cdot)$ and $(B, \circ)$, $(A\oplus B, *)$ is a nearly associative algebra. □

Definition 11.7 ([47]) Let $(\mathcal{G}, [.,.]_\mathcal{G})$ and $(\mathcal{H}, [.,.]_\mathcal{H})$ be two Lie algebras such that $\rho : \mathcal{G} \to \mathrm{End}(\mathcal{H})$ and $\mu : \mathcal{H} \to \mathrm{End}(\mathcal{G})$ are representations of $\mathcal{G}$ and $\mathcal{H}$, respectively.

A matched pair of Lie algebras $\mathcal{G}$ and $\mathcal{H}$ is $(\mathcal{G}, \mathcal{H}, \rho, \mu)$ such that ρ and μ obey for $x, y \in \mathcal{G}, a, b \in \mathcal{H}$,

$$\rho(x)[a,b]_{\mathcal{G}} - [\rho(x)a, b]_{\mathcal{H}} - [a, \rho(x)b]_{\mathcal{H}} \\ +\rho(\mu(a)x)b - \rho(\mu(b)x)a = 0, \tag{11.11a}$$

$$\mu(a)[x,y]_{\mathcal{G}} - [\mu(a)x, y]_{\mathcal{G}} - [x, \mu(a)y]_{\mathcal{G}} \\ +\mu(\rho(x)a)y - \mu(\rho(y)a)x = 0. \tag{11.11b}$$

Corollary 11.2 *Let $(A, B, l_A, r_A, l_B, r_B)$ be a matched pair of the nearly associative algebras $(A, \cdot)$ and $(B, \circ)$. Then $(\mathcal{G}(A), \mathcal{G}(B), l_A - r_A, l_B - r_B)$ is a matched pair of Lie algebras $\mathcal{G}(A)$ and $\mathcal{G}(B)$.*

Proof Let $(A, B, l_A, r_A, l_B, r_B)$ be a matched pair of the nearly associative algebras $(A, \cdot)$ and $(B, \circ)$. In view of Proposition 11.4, the linear maps $l_A - r_A : A \longrightarrow \mathrm{End}(B)$ and $l_B - r_B : B \longrightarrow \mathrm{End}(A)$ are representations of the underlying Lie algebras $\mathcal{G}(A)$ and $\mathcal{G}(B)$, respectively. Therefore, by direct calculation we have (11.11a) is equivalent to (11.10a)–(11.10c) and similarly, (11.11b) is equivalent to (11.10d)–(11.10f). □

Proposition 11.6 *For a nearly associative algebra $(A, \cdot)$, if there is a nearly associative algebra structure $\circ$ on its dual space A^*, and if the linear maps L and R commute, then $(A, A^*, R_\cdot^*, L_\cdot^*, R_\circ^*, L_\circ^*)$ is a matched pair of the nearly associative algebras $(A, \cdot)$ and $(A^*, \circ)$ if and only if, for $x, y \in A$, $a \in A^*$,*

$$L_\circ^*(R_\cdot^*(x)a)y - y \cdot (L_\circ^*(a)x) - (R_\circ^*(a)y) \cdot x - R_\circ^*(L_\cdot^*(y)a)x = 0, \tag{11.12a}$$

$$L_\circ^*(a)(x \cdot y) - y \cdot (R_\circ^*(a)x) - L_\circ^*(L_\cdot^*(x)a)y = 0, \tag{11.12b}$$

$$R_\circ^*(a)(x \cdot y) - (L_\circ^*(a)y) \cdot x - R_\circ^*(R_\cdot^*(y)a)x = 0. \tag{11.12c}$$

Proof Since L and R commute, according to Remark 11.2 and Proposition 11.5, both $(R_\cdot^*, L_\cdot^*, A^*)$ and $(L_\cdot^*, R_\cdot^*, A^*)$ are bimodules of $(A, \cdot)$. Setting $l_A = R_\cdot^*$, $r_A = L_\cdot^*$, $l_B = R_\circ^*$ and $r_B = L_\circ^*$ in Theorem 11.3 the equivalences among (11.10a) and (11.12a), (11.10b) and (11.12b), and finally (11.10c) and (11.12c) are straightforward. Besides, for any $x, y \in A$ and any $a, b \in A^*$, we have

$$\begin{aligned}
&\langle L_\circ^*(R_\cdot^*(x)a)y, b\rangle = \langle y, L_\circ(R_\cdot^*(x)a)b\rangle = \langle y, (R_\cdot^*(x)a) \circ b\rangle,\\
&\langle y \cdot (L_\circ^*(a)x), b\rangle = \langle R_\cdot(L_\circ^*(a)x)y, b\rangle = \langle y, R_\cdot^*(L_\circ^*(a)x)b\rangle,\\
&\langle (R_\circ^*(a)y) \cdot x, b\rangle = \langle R_\circ^*(a)y, R_\cdot^*(x)b\rangle = \langle y, (R_\cdot^*(x)b) \circ a\rangle,\\
&\langle R_\circ^*(L_\cdot^*(y)a)x, b\rangle = \langle L_\circ^*(b)x, L_\cdot^*(y)a\rangle = \langle y \cdot (L_\circ^*(b)x), a\rangle = \langle y, R_\cdot^*(L_\circ^*(b)x)a\rangle,\\
&\langle L_\circ^*(a)(x \cdot y), b\rangle = \langle R_\cdot(y)x, a \circ b\rangle = \langle x, R_\cdot^*(y)(a \circ b)\rangle,\\
&\langle y \cdot (R_\circ^*(a)x), b\rangle = \langle R_\circ^*(a)x, L_\cdot^*(y)b\rangle = \langle x, (L_\cdot^*(y)b) \circ a\rangle,\\
&\langle L_\circ^*(L_\cdot^*(x)a)y, b\rangle = \langle R_\circ^*(b)y, L_\circ^*(x)a\rangle = \langle x \cdot (R_\circ^*(b)y), a\rangle = \langle x, R_\cdot^*(R_\circ^*(b)y)a\rangle,\\
&\langle R_\circ^*(a)(x \cdot y), b\rangle = \langle L_\cdot(x)y, b \circ a\rangle = \langle y, L_\cdot(x)^*(b \circ a)\rangle,\\
&\langle (L_\circ^*(a)y) \cdot x, b\rangle = \langle L_\circ^*(a)y, R_\cdot^*(b)\rangle = \langle y, a \circ (R_\cdot^*(x)b)\rangle,\\
&\langle R_\circ^*(R_\cdot^*(y)a)x, b\rangle = \langle L_\circ^*(b)x, R_\cdot^*(y)a\rangle = \langle (L_\circ^*(b)x) \cdot y, a\rangle = \langle y, L_\cdot^*(L_\circ^*(b)x)a\rangle.
\end{aligned}$$

Then, (11.10a) holds if and only if (11.10d) holds, (11.10b) holds if and only if (11.10e) holds, and finally (11.10c) holds if and only if (11.10f) holds. □

11.5 Manin Triple and Bialgebra of Nearly Associative Algebras

Definition 11.8 A bilinear form $\mathfrak{B}$ on a nearly associative algebra $(A, \cdot)$ is called left-invariant if $\mathfrak{B}(x \cdot y, z) = \mathfrak{B}(x, y \cdot z)$, for all $x, y, z \in A$.

Proposition 11.7 *Let $(A, \cdot)$ be a nearly associative algebra. If there is a nondegenerate symmetric invariant bilinear form $\mathfrak{B}$ defined on A, then as bimodules of the nearly associative algebra $(A, \cdot)$, (L, R, A) and (R^*, L^*, A^*) are equivalent. Conversely, if (L, R, A) and (R^*, L^*, A^*) are equivalent bimodules of a nearly associative algebra $(A, \cdot)$, then there exists a nondegenerate invariant bilinear form $\mathfrak{B}$ on A.*

Definition 11.9 A Manin triple of nearly associative algebras is a triple of nearly associative algebras (A, A_1, A_2) together with a nondegenerate symmetric invariant bilinear form $\mathfrak{B}$ on A such that:

(i) A_1 and A_2 nearly associative subalgebras of A;
(ii) as linear spaces, $A = A_1 \oplus A_2$;
(iii) A_1 and A_2 are isotropic with respect to $\mathfrak{B}$, i.e. for any $x_1, y_1 \in A_1$ and any $x_2, y_2 \in A_2$, $\mathfrak{B}(x_1, y_1) = 0 = \mathfrak{B}(x_2, y_2) = 0$.

Definition 11.10 Let $(A, \cdot)$ be a nearly associative algebra. Suppose that $\circ$ is a nearly associative algebra structure on the dual space A^* of A and there is a nearly associative algebra structure on the direct sum $A \oplus A^*$ of the underlying linear spaces of A and A^* such that $(A, \cdot)$ and $(A^*, \circ)$ are subalgebras and the natural symmetric bilinear form on $A \oplus A^*$ given by $\forall x, y \in A; \forall a^*, b^* \in A^*$,

$$\mathfrak{B}_d(x + a^*, y + b^*) := \langle a^*, y\rangle + \langle x, b^*\rangle, \tag{11.13}$$

is left-invariant, then $(A \oplus A^*, A, A^*)$ is called a standard Manin triple of nearly associative algebras associated to $\mathfrak{B}_d$.

Obviously, a standard Manin triple of nearly associative algebras is a Manin triple of nearly associative algebras. By symmetric role of A and A^*, we have

Proposition 11.8 *Every Manin triple of nearly associative algebras is isomorphic to a standard one.*

Proposition 11.9 *Let $(A, \cdot)$ be a nearly associative algebra. Suppose that there is a nearly associative algebra structure $\circ$ on the dual space A^*. There exists a nearly associative algebra structure on the linear space $A \oplus A^*$ such that $(A \oplus A^*, A, A^*)$ is a standard Manin triple of nearly associative algebras associated to $\mathfrak{B}_d$ defined by (11.13) if and only if $(A, A^*, R^*_\cdot, L^*_\cdot, R^*_\circ, L^*_\circ)$ is a matched pair of nearly associative algebras.*

Theorem 11.4 *Let $(A, \cdot)$ be a nearly associative algebra such that the left and right multiplication operators commute. Suppose that there is a nearly associative* *algebra structure* $\circ$ *on its the dual space* A^* *given by* $\Delta^* : A^* \otimes A^* \to A^*$. *Then,* $(A, A^*, R^*_\cdot, L^*_\cdot, R^*_\circ, L^*_\circ)$ *is a matched pair of the nearly associative algebras* $(A, \cdot)$ *and* $(A^*, \circ)$ *if and only if* $\Delta : A \to A \otimes A$ *satisfies*

$$\begin{aligned}(R_\cdot(x) \otimes \mathrm{id} - \sigma(R_\cdot(x) \otimes \mathrm{id}))\Delta(y)\\ +(\mathrm{id} \otimes L_\cdot(y) - \sigma(\mathrm{id} \otimes L_\cdot(y)))\Delta(x) = 0,\end{aligned} \tag{11.14a}$$

$$\begin{aligned}(L_\cdot(x) \otimes \mathrm{id})\Delta(y) + \sigma(L_\cdot(y) \otimes \mathrm{id})\Delta(x) = \Delta(x \cdot y)\\ = \sigma(\mathrm{id} \otimes R_\cdot(x))\Delta(y) + (\mathrm{id} \otimes R_\cdot(y))\Delta(x).\end{aligned} \tag{11.14b}$$

Proof For any $a, b \in A^*$ and any $x, y \in A$,

$$\begin{aligned}
\langle (R_\cdot(x) \otimes \mathrm{id})\Delta(y), a \otimes b\rangle &= \langle y, (R^*_\cdot(x)a) \circ b\rangle = \langle L^*_\circ(R^*_\cdot(x)a)y, b\rangle,\\
\langle \sigma(R_\cdot(x) \otimes \mathrm{id})\Delta(y), a \otimes b\rangle &= \langle y, (R^*_\cdot(x)b) \circ a\rangle\\
&= \langle R^*_\circ(a)y, R^*_\cdot(x)b\rangle = \langle (R^*_\circ(a)y) \cdot x, b\rangle,\\
\langle (\mathrm{id} \otimes L_\cdot(y))\Delta(x), a \otimes b\rangle &= \langle x, a \circ (L^*_\cdot(y)b)\rangle = \langle y \cdot (L^*_\circ(a)x), b\rangle,\\
\langle \sigma(\mathrm{id} \otimes L_\cdot(y))\Delta(x), a \otimes b\rangle &= \langle x, b \circ (L^*_\cdot(y)a)\rangle = \langle R^*_\circ(L^*_\cdot(y)a)x, b\rangle.
\end{aligned}$$

Hence (11.12a) is equivalent to (11.14a).

Similarly, for any $x, y \in A$, $a, b \in A^*$,

$$\begin{aligned}
&\langle \Delta(x \cdot y), a \otimes b\rangle = \langle x \cdot y, a \circ b\rangle = \langle L^*_\circ(a)(x \cdot y), b\rangle = \langle R^*_\circ(b)(x \cdot y), a\rangle,\\
&\langle (L_\cdot(x) \otimes \mathrm{id})\Delta(y), a \otimes b\rangle = \langle y, (L^*_\cdot(x)a) \circ b\rangle = \langle L^*_\circ(L^*_\cdot(x)a)y, b\rangle,\\
&\langle \sigma(L_\cdot(y) \otimes \mathrm{id})\Delta(x), a \otimes b\rangle = \langle x, (L^*_\cdot(y)b) \circ a\rangle = \langle y \cdot (R^*_\circ(a)x), b\rangle,\\
&\langle \sigma(\mathrm{id} \otimes R_\cdot(x))\Delta(y), a \otimes b\rangle = \langle y, b \circ (R^*_\cdot(x)a)\rangle = \langle R^*_\circ(R^*_\cdot(x)a)y, b\rangle,\\
&\langle (\mathrm{id} \otimes R_\cdot(y))\Delta(x), a \otimes b\rangle = \langle x, a \circ (R^*_\cdot(y)b)\rangle = \langle (L^*_\circ(a)x) \cdot y, b\rangle.
\end{aligned}$$

Therefore, (11.12b) and (11.12c) and is equivalent to (11.14b). □

Remark 11.3 Obviously, if L and R commute, then L^* and R^* commute too and if in addition $\gamma : A^* \to A^* \otimes A^*$ is a linear maps such that its dual $\gamma^* : A \otimes A \to A$ defines a nearly associative algebra structure $\cdot$ on A, then Δ satisfies (11.14a) and (11.14b) if and only if γ satisfies for all $a, b \in A^*$,

$$\begin{aligned}
&(R_\circ(a) \otimes \mathrm{id} - \sigma(R_\circ(a) \otimes \mathrm{id}))\gamma(b) + (\mathrm{id} \otimes L_\circ(b) - \sigma(\mathrm{id} \otimes L_\circ(b)))\gamma(a) = 0,\\
&(L_\circ(x) \otimes \mathrm{id})\gamma(b) + \sigma(L_\circ(b) \otimes \mathrm{id})\gamma(a) =\\
&\qquad \gamma(a \circ b) = \sigma(\mathrm{id} \otimes R_\circ(a))\gamma(b) + (\mathrm{id} \otimes R_\circ(b))\gamma(a).
\end{aligned}$$

Definition 11.11 Let $(A, \cdot)$ be a nearly associative algebra in which the left and right multiplication operators L and R commute. A nearly anti-flexible bialgebra structure is a linear map $\Delta : A \to A \otimes A$ such that

1) $\Delta^* : A^* \otimes A^* \to A^*$ defines a nearly associative algebra structure on A,
2) Δ satisfies (11.14a) and (11.14b).

Theorem 11.5 *Let $(A, \cdot)$ be a nearly associative algebra in which the left and right multiplication operators commute. Suppose that there is a nearly associative algebra structure on A^* denoted by $\circ$ which defined a linear map $\Delta : A \to A \otimes A$. Then the following conditions are equivalent:*

(i) $(A \oplus A^*, A, A^*)$ *is a standard Manin triple of nearly associative algebras* $(A, \cdot)$ *and* $(A^*, \circ)$ *such that its associated symmetric bilinear form* $\mathfrak{B}_d$ *is defined by* (11.13).
(ii) $(A, A^*, R_\cdot^*, L_\cdot^*, R_\circ^*, L_\circ^*)$ *is a matched pair of nearly associative algebras* $(A, \cdot)$ *and* $(A^*, \circ)$.
(iii) (A, A^*) *is a nearly associative bialgebra.*

11.6 Hom-Lie Admissible, G-Hom-Associative, Flexible and Anti-flexible Hom-Algebras

Hom-Lie admissible algebras along with Hom-associative algebras and more general G-Hom-associative algebras were introduced, and Hom-associative algebras and G-Hom-associative algebras were shown to be Hom-Lie admissible in [48].

Hom-algebra is a triple (A, μ, α) consisting of a linear space A over a field $\mathbb{K}$, a bilinear product $\mu : A \times A \to A$ and a linear map $\alpha : A \to A$.

Definition 11.12 ([48]) Hom-Lie, Hom-Lie admissible, Hom-associative and G-Hom-associative Hom-algebras (over a field $\mathbb{K}$) are defined as follows:

1) Hom-Lie algebras are triples $(A, [.\,,.], \alpha)$, consisting of a linear space A over a field $\mathbb{K}$, bilinear map (bilinear product) $[.\,,.] : A \times A \to A$ and a linear map $\alpha : A \to A$ satisfying, for all $x, y, z \in A$,

$$[x, y] = -[y, x], \qquad \text{(Skew-symmetry)}$$
$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0. \qquad \text{(Hom-Jacobi identity)}$$

2) Hom-Lie admissible algebras are Hom-algebras (A, μ, α) consisting of possibly non-associative algebra (A, μ) and a linear map $\alpha : A \to A$, such that $(A, [.\,,.], \alpha)$ is a Hom-Lie algebra, where $[x, y] = \mu(x, y) - \mu(y, x)$ for all $x, y \in A$.
3) Hom-associative algebras are triples $(A, \cdot, \alpha)$ consisting of a linear space A over a field $\mathbb{K}$, a bilinear product $\mu : A \times A \to A$ and a linear map $\alpha : A \to A$, satisfying for all $x, y, z \in A$,

$$\mu(\mu(x, y), \alpha(z)) = \mu(\alpha(x), \mu(y, z)). \qquad \text{(Hom-associativity)} \qquad (11.15)$$

4) Let G be a subgroup of the permutations group $\mathcal{S}_3$. Hom-algebra (A, μ, α) is said to be G-Hom-associative if for $x_i \in A, i = 1, 2, 3$,

$$\sum_{\sigma\in G}(-1)^{\varepsilon(\sigma)}(\mu(\mu(x_{\sigma(1)},x_{\sigma(2)}),\alpha(x_{\sigma(3)}))-\mu(\alpha(x_{\sigma(1)}),\mu(x_{\sigma(2)},x_{\sigma(3)}))=0, \tag{11.16}$$

where $(-1)^{\varepsilon(\sigma)}$ is the signature of the permutation σ.

For any Hom-algebra (A,μ,α), the Hom-associator, called also α-associator of μ, is a trilinear map (ternary product) $a_{\alpha,\mu}: A\times A\times A\to A$ defined by

$$a_{\alpha,\mu}(x_1,x_2,x_3)=\mu(\mu(x_1,x_2),\alpha(x_3))-\mu(\alpha(x_1),\mu(x_2,x_3))$$

for all $x_1,x_2,x_3\in A$. The ordinary associator

$$a_\mu(x_1,x_2,x_3)=a_{\mathrm{id},\mu}(x_1,x_2,x_3)=\mu((x_1,x_2),(x_3))-\mu((x_1),\mu(x_2,x_3))$$

on an algebra (A,μ) is α-associator for the Hom-algebra $(A,\mu,\alpha)=(A,\mu,\mathrm{id})$ with $\alpha=\mathrm{id}: A\to A$, the identity map on A.

Using Hom-associator $a_{\alpha,\mu}$, Hom-associativity (11.15) is

$$a_{\alpha,\mu}(x,y,z)=\mu(\mu(x,y),\alpha(z))-\mu(\alpha(x),\mu(y,z))=0, \quad \text{(Hom-associativity)}$$

that is, $a_{\alpha,\mu}=0$, and G-Hom-associativity (11.16) is $\sum\limits_{\sigma\in G}(-1)^{\varepsilon(\sigma)}a_{\alpha,\mu}\circ\sigma=0$, where $\sigma(x_1,x_2,x_3)=(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)})$. If μ is the multiplication of a Hom-Lie admissible Lie algebra, then (11.16) is equivalent to $[x,y]=\mu(x,y)-\mu(y,x)$ satisfying the Hom-Jacobi identity, or equivalently,

$$\sum_{\sigma\in\mathcal{S}_3}(-1)^{\varepsilon(\sigma)}(\mu(\mu(x_{\sigma(1)},x_{\sigma(2)}),\alpha(x_{\sigma(3)}))-\mu(\alpha(x_{\sigma(1)}),\mu(x_{\sigma(2)},x_{\sigma(3)})))=0,$$

which may be written as $\sum\limits_{\sigma\in\mathcal{S}_3}(-1)^{\varepsilon(\sigma)}a_{\alpha,\mu}\circ\sigma=0$. Thus, Hom-Lie admissible Hom-algebras are $\mathcal{S}_3$-associative Hom-algebras. In general, for all subgroups G of the permutations group $\mathcal{S}_3$, all G-Hom-associative Hom-algebras are Hom-Lie admissible, or in other words, all Hom-algebras from the six classes of G-Hom-associative Hom-algebras, corresponding to the six subgroups of the symmetric group $\mathcal{S}_3$, are Hom-Lie admissible [48, Proposition 3.4]. All six subgroups of $\mathcal{S}_3$ are $G_1=\mathcal{S}_3(\mathrm{id})=\{\mathrm{id}\}$, $G_2=\mathcal{S}_3(\tau_{12})=\{\mathrm{id},\tau_{12}\}$, $G_3=\mathcal{S}_3(\tau_{23})=\{\mathrm{id},\tau_{23}\}$, $G_4=\mathcal{S}_3(\tau_{13})=\{\mathrm{id},\tau_{13}\}$, $G_5=\mathcal{A}_3$, $G_6=\mathcal{S}_3$ where $\mathcal{A}_3$ is the alternating group and τ_{ij} is the transposition of i and j. Table 11.1 summarises the defining identities and names for the classes G-Hom-associative algebras.

The skew-symmetric G_5-Hom-associative Hom-algebras and Hom-Lie algebras form the same class of Hom-algebras for linear spaces over fields of characteristic different from 2, since then the defining identity of G_5-Hom-associative algebras is equivalent to the Hom-Jacobi identity of Hom-Lie algebras when the product μ is skew-symmetric. A Hom-right symmetric (Hom-pre-Lie) algebra is the opposite algebra of a Hom-left-symmetric algebra. Hom-flexible algebras introduced in [48] is a generalization to Hom-algebra context of flexible algebras [2, 53, 55].

Table 11.1 *G*-Hom-associative algebras

Subgroup of $\mathcal{S}_3$	Hom-algebras class names	Defining Identity (notation: $\mu(a, b) = ab$)
$G_1 = \mathcal{S}_3(\mathrm{id})$	Hom-associative	$\alpha(x)(yz) = (xy)\alpha(z)$
$G_2 = \mathcal{S}_3(\tau_{12})$	Hom-left symmetric, Hom-Vinberg	$\alpha(x)(yz) - \alpha(y)(xz) = (xy)\alpha(z) - (yx)\alpha(z)$
$G_3 = \mathcal{S}_3(\tau_{23})$	$\mathcal{S}_3(\tau_{23})$-Hom-associative, Hom-right symmetric, Hom-pre-Lie	$\alpha(x)(yz) - \alpha(x)(zy) = (xy)\alpha(z) - (xz)\alpha(y)$
$G_4 = \mathcal{S}_3(\tau_{13})$	$\mathcal{S}_3(\tau_{13})$-Hom-associative, Hom-anti-flexible, Hom-center symmetric	$\alpha(x)(yz) - \alpha(z)(yx) = (xy)\alpha(z) - (zy)\alpha(x)$
$G_5 = \mathcal{A}_3$	$\mathcal{A}_3$-Hom-associative	$\alpha(x)(yz) + \alpha(y)(zx) + \alpha(z)(xy) = (xy)\alpha(z) + (yz)\alpha(x) + (zx)\alpha(y)$
$G_6 = \mathcal{S}_3$	Hom-Lie admissible	$\sum_{\sigma\in\mathcal{S}_3} (-1)^{\varepsilon(\sigma)} \big((x_{\sigma(1)}x_{\sigma(2)})\alpha(x_{\sigma(3)}) - \alpha(x_{\sigma(1)})(x_{\sigma(2)}x_{\sigma(3)})\big) = 0$

Definition 11.13 ([48]) Hom-algebra (A, μ, α) is called flexible if for $x, y \in A$,

$$\mu(\mu(x, y), \alpha(x)) = \mu(\alpha(x), \mu(y, x))). \tag{11.17}$$

Using the α-associator $a_{\alpha,\mu}(x, y, z) = \mu(\mu(x, y), \alpha(z)) - \mu(\alpha(x), \mu(y, z))$, the condition (11.17) may be written as

$$a_{\alpha,\mu}(x, y, x) = 0. \tag{11.18}$$

Since Hom-associator map $a_{\alpha,\mu}$ is a trilinear map,

$$a_{\alpha,\mu}(z - x, y, z - x) = a_{\alpha,\mu}(z, y, z) + a_{\alpha,\mu}(x, y, x) - a_{\alpha,\mu}(x, y, z) - a_{\alpha,\mu}(z, y, x),$$

and hence (11.18) yields

$$a_{\alpha,\mu}(x, y, z) = -a_{\alpha,\mu}(z, y, x) \tag{11.19}$$

in linear spaces over any field. Setting $x = z$ in (11.19) gives $2a_{\alpha,\mu}(x, y, x) = 0$, implying that (11.18) and (11.19) are equivalent in linear spaces over fields of characteristic different from 2. The equality (11.19) in terms of the Hom-algebra product μ is

$$\mu(\mu(x, y), \alpha(z)) - \mu(\alpha(x), \mu(y, z)) = \mu(\alpha(z), \mu(y, x)) - \mu(\mu(z, y), \alpha(x)).$$

Definition 11.14 Hom-algebra (A, μ, α) is called anti-flexible if for $x, y, z \in A$.

$$\begin{aligned} &\mu(\mu(x, y), \alpha(z)) - \mu(\mu(z, y), \alpha(x)) \\ &= \mu(\mu(\alpha(x), \mu(y, z)) - \mu(\mu(\alpha(z), \mu(y, x)). \end{aligned} \tag{11.20}$$

In terms of the Hom-associator $a_{\alpha,\mu}(x, y, z)$, (11.20) can be written as

$$a_{\alpha,\mu}(x, y, z) = a_{\alpha,\mu}(z, y, x). \tag{11.21}$$

Hom-anti-flexible algebras were first introduced in [48] as $\mathcal{S}_3(\tau_{13})$-Hom-associative algebras, the subclass of G-Hom-associative algebras corresponding to the subgroup $G = \mathcal{S}_3(\tau_{13}) \subset \mathcal{S}_3$ (see Table 11.1). In view of (11.21), anti-flexible algebras have been called Hom-center symmetric in [32].

Note that (11.21) differs from (11.19) by absence of the minus sign on the right hand side, meaning that for any y, the bilinear map $a_{\alpha,\mu}(., y, .)$ is symmetric on Hom-anti-flexible algebras and skew-symmetric on Hom-flexible algebras. Unlike (11.17) and (11.19) in Hom-flexible algebras, in Hom-anti-flexible algebras, (11.21) is generally not equivalent to the restriction of (11.21) to $z = x$ trivially identically satisfied for any x and y. In view of (11.21), Hom-anti-flexible algebras are called Hom-center-symmetric algebras in [32].

11.7 Nearly Hom-Associative Algebras, Bimodules and Matched Pairs

Definition 11.15 A nearly Hom-associative algebra is a triple $(A, *, \alpha)$, where A is a linear space endowed to the bilinear product $* : A \times A \to A$ and $\alpha : A \to A$ is a linear map such that for all $x, y, z \in A$,

$$\alpha(x) * (y * z) = (z * x) * \alpha(y).$$

Nearly Hom-associative algebras are Hom-Lie admissible.

Proposition 11.10 *Any nearly Hom-associative algebra* $(A, *, \alpha)$ *is Hom-Lie admissible, that is* $(A, [. , .], \alpha)$ *is a Hom-Lie algebra with* $[x, y] = x * y - y * x$ *for* $x, y \in A$.

Proof The commutator is skew-symmetric in any algebra, $[x, y] = x * y - y * x = -(y * x - x * y) = -[y, x]$. For $x, y, z \in A$ in a nearly Hom-associative algebra $(A, *, \alpha)$,

$$\begin{aligned}
&[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] \\
&= [\alpha(x), y * z - z * y] + [\alpha(y), z * x - x * z] + [\alpha(z), x * y - y * x] \\
&= \alpha(x) * (y * z) - \alpha(x) * (z * y) - (y * z) * \alpha(x) \\
&\quad + (z * y) * \alpha(x) + \alpha(y) * (z * x) - \alpha(y) * (x * z) \\
&\quad - (z * x) * \alpha(y) + (x * z) * \alpha(y) + \alpha(z) * (x * y) \\
&\quad - \alpha(z) * (y * x) - (x * y) * \alpha(z) + (y * x) * \alpha(z) \\
&= \{\alpha(x) * (y * z) - (z * x) * \alpha(y)\} + \{(y * x) * \alpha(z) - \alpha(x) * (z * y)\} \\
&\quad + \{\alpha(y) * (z * x) - (x * y) * \alpha(z)\} + \{\alpha(z) * (x * y) - (y * z) * \alpha(x)\} \\
&\quad + \{(z * y) * \alpha(x) - \alpha(y) * (x * z)\} + \{(x * z) * \alpha(y) - \alpha(z) * (y * x)\} = 0.
\end{aligned}$$

Therefore, $(A, [.,.], \alpha)$ is a Hom-Lie algebra. □

Commutative nearly Hom-associative algebras are Hom-anti-flexible.

Proposition 11.11 *If $(A, *, \alpha)$ is a commutative nearly Hom-associative algebra, then $(A, *, \alpha)$ is a Hom-anti-flexible algebra.*

Proof In a commutative nearly Hom-associative algebra $(A, *, \alpha)$.

$$\begin{aligned}
a_{\alpha,*}(x, y, z) &= (x * y) * \alpha(z) - \alpha(x) * (y * z) \\
&= \alpha(y) * (z * x) - (z * x) * \alpha(y) \quad \text{(nearly Hom-associativity)} \\
&= \alpha(y) * (x * z) - (x * z) * \alpha(y) \quad \text{(commutativity)} \\
&= (z * y) * \alpha(x) - \alpha(z) * (y * x) \quad \text{(nearly Hom-associativity)} \\
&= a_{\alpha,*}(z, y, x).
\end{aligned}$$

So any commutative nearly Hom-associative algebra is Hom-anti-flexible. □

Definition 11.16 A bimodule of a nearly Hom-associative algebra $(A, *, \alpha)$ is a quadruple (l, r, V, φ), where V is a linear space, $l, r : A \to \mathrm{End}(V)$ are two linear maps and $\varphi \in \mathrm{End}(V)$ satisfying the relations, for all $x, y \in A$,

$$\begin{aligned}
\varphi \circ l(x) = l(\alpha(x)) \circ \varphi, \quad \varphi \circ r(x) = r(\alpha(x)) \circ \varphi, \\
l(\alpha(x)) \circ l(y) = r(\alpha(y)) \circ r(x), \\
l(\alpha(x)) \circ r(y) = l(y * x) \circ \varphi, \\
r(\alpha(x)) \circ l(y) = r(x * y) \circ \varphi.
\end{aligned}$$

Proposition 11.12 *Consider a nearly Hom-associative $(A, *, \alpha)$. Let $l, r : A \to \mathrm{End}(V)$ be two linear maps such that V is a linear space and $\varphi \in \mathrm{End}(V)$. The* *quadruple* (l, r, V, φ) *is a bimodule of* $(A, *, \alpha)$ *if and only if there is a structure of a nearly Hom-associative algebra* $\star$ *on* $A \oplus V$ *given by, for all* $x, y \in A$ *and all* $u, v \in V$,

$$(\alpha \oplus \varphi)(x + u) = \alpha(x) + \varphi(u),$$
$$(x + u) \star (y + v) = (x * y) + (l(x)v + r(y)u).$$

Definition 11.17 A representation of a Hom-Lie algebra $(\mathcal{G}, [.,.]_{\mathcal{G}}, \alpha_{\mathcal{G}})$ on a linear space V with respect to $\psi \in \mathrm{End}(V)$ is a linear map $\rho_{\mathcal{G}} : \mathcal{G} \to \mathrm{End}(V)$ obeying for all $x, y \in \mathcal{G}$,

$$\rho_{\mathcal{G}}(\alpha_{\mathcal{G}}(x)) \circ \psi = \psi \circ \rho_{\mathcal{G}}(x), \tag{11.23}$$
$$\rho_{\mathcal{G}}([x, y]_{\mathcal{G}}) \circ \psi = \rho_{\mathcal{G}}(\alpha_{\mathcal{G}}(x)) \circ \rho_{\mathcal{G}}(y) - \rho_{\mathcal{G}}(\alpha_{\mathcal{G}}(y)) \circ \rho_{\mathcal{G}}(x). \tag{11.24}$$

Proposition 11.13 *Let* $(A, \cdot, \alpha)$ *be a nearly Hom-associative algebra and* V *be a finite-dimensional linear space over the field* $\mathbb{K}$ *such that* (l, r, φ, V) *is a bimodule of* $(A, \cdot, \alpha)$, *where* $l, r : A \to \mathrm{End}(V)$ *are two linear maps and* $\varphi \in \mathrm{End}(V)$. *Then the linear map* $l - r : A \to \mathrm{End}(V), x \mapsto l(x) - r(x)$ *is a representation of the underlying Hom-Lie algebra* $(\mathcal{G}(A), \alpha)$ *associated to the nearly Hom-associative algebra* $(A, \cdot, \alpha)$.

Proof Let $(A, \cdot, \alpha)$ be a nearly Hom-associative algebra and V a finite-dimensional linear space over the field $\mathbb{K}$ such that (l, r, φ, V) is a bimodule of $(A, \cdot, \alpha)$, where $l, r : A \to \mathrm{End}(V)$ are two linear maps and $\varphi \in \mathrm{End}(V)$. For all $x, y \in A$,

$$\begin{aligned}
&(l - r)(\alpha(x)) \circ \varphi = l(\alpha(x)) \circ \varphi - r(\alpha(x)) \circ \varphi \\
&= \varphi \circ l(x) - \varphi \circ r(x) = \varphi \circ (l - r)(x), \\
&(l - r)((\alpha(x))) \circ (l - r)(y) - (l - r)((\alpha(y))) \circ (l - r)(x) \\
&\quad = l(\alpha(x)) \circ l(y) - l(\alpha(x)) \circ r(y) - r(\alpha(x)) \circ l(y) + r(\alpha(x)) \circ r(y) \\
&\qquad - l(\alpha(y)) \circ l(x) + l(\alpha(y)) \circ r(x) + r(\alpha(y)) \circ l(x) - r(\alpha(y)) \circ r(x) \\
&\quad = \{l(\alpha(x)) \circ l(y) - r(\alpha(y)) \circ r(x)\} - l(\alpha(x)) \circ r(y) - r(\alpha(x)) \circ l(y) \\
&\qquad + \{r(\alpha(x)) \circ r(y) - l(\alpha(y)) \circ l(x)\} + r(\alpha(y)) \circ l(x) + l(\alpha(y)) \circ r(x) \\
&\quad = r(\alpha(y)) \circ l(x) - l(\alpha(x)) \circ r(y) + l(\alpha(y)) \circ r(x) - r(\alpha(x)) \circ l(y) \\
&\quad = r(y \cdot x) \circ \varphi - l(y \cdot x) \circ \varphi + l(x \cdot y) \circ \varphi - r(x \cdot y) \circ \varphi = (l - r)([x, y]) \circ \varphi.
\end{aligned}$$

Therefore, (11.23) and (11.24) are satisfied. □

Definition 11.18 Let $(\mathcal{G}, [.,.]_{\mathcal{G}}, \alpha_{\mathcal{G}})$ and $(\mathcal{H}, [.,.]_{\mathcal{H}}, \alpha_{\mathcal{H}})$ be two Hom-Lie algebras. Let $\rho_{\mathcal{H}} : \mathcal{H} \to \mathrm{End}(\mathcal{G})$ and $\mu_{\mathcal{G}} : \mathcal{G} \to \mathrm{End}(\mathcal{H})$ be Hom-Lie algebra representations, and $\alpha_{\mathcal{G}} : \mathcal{G} \to \mathcal{G}$ and $\alpha_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$ linear maps such that for all $x, y \in \mathcal{G}, a, b \in \mathcal{H}$,

$$\begin{aligned}\mu_{\mathcal{G}}(\alpha_{\mathcal{G}}(x))\,[a,b]_{\mathcal{H}} &= \left[\mu_{\mathcal{G}}(x)a, \alpha_{\mathcal{H}}(b)\right]_{\mathcal{H}} + \left[\alpha_{\mathcal{H}}(a), \mu_{\mathcal{G}}(x)b\right]_{\mathcal{H}} \\ &\quad -\mu_{\mathcal{G}}(\rho_{\mathcal{H}}(a)x)(\alpha_{\mathcal{H}}(b)) + \mu_{\mathcal{G}}(\rho_{\mathcal{H}}(b)x)(\alpha_{\mathcal{H}}(a)),\end{aligned} \tag{11.25a}$$

$$\begin{aligned}\rho_{\mathcal{H}}(\alpha_{\mathcal{H}}(a))\,[x,y]_{\mathcal{G}} &= \left[\rho_{\mathcal{H}}(a)x, \alpha_{\mathcal{G}}(y)\right]_{\mathcal{G}} + \left[\alpha_{\mathcal{G}}(x), \rho_{\mathcal{H}}(a)y\right]_{\mathcal{G}} \\ &\quad -\rho_{\mathcal{H}}(\mu_{\mathcal{G}}(x)a)(\alpha_{\mathcal{G}}(y)) + \rho_{\mathcal{H}}(\mu_{\mathcal{G}}(y)a)(\alpha_{\mathcal{G}}(x)).\end{aligned} \tag{11.25b}$$

Then, $(\mathcal{G}, \mathcal{H}, \mu, \rho, \alpha_{\mathcal{G}}, \alpha_{\mathcal{H}})$ is called a matched pair of the Hom-Lie algebras $\mathcal{G}$ and $\mathcal{H}$, and denoted by $\mathcal{H} \bowtie^{\rho_{\mathcal{H}}}_{\mu_{\mathcal{G}}} \mathcal{G}$. In this case, $(\mathcal{G} \oplus \mathcal{H}, [.\,,.]_{\mathcal{G}\oplus\mathcal{H}}, \alpha_{\mathcal{G}} \oplus \alpha_{\mathcal{H}})$ defines a Hom-Lie algebra, where

$$[(x+a),(y+b)]_{\mathcal{G}\oplus\mathcal{H}} = [x,y]_{\mathcal{G}} + \rho_{\mathcal{H}}(a)y - \rho_{\mathcal{H}}(b)x + [a,b]_{\mathcal{H}} + \mu_{\mathcal{G}}(x)b - \mu_{\mathcal{G}}(y)a.$$

Theorem 11.6 *Let $(A, \cdot, \alpha_A)$ and $(B, \circ, \alpha_B)$ be two nearly Hom-associative algebras. Suppose there are linear maps $l_A, r_A : A \to \mathrm{End}(B)$ and $l_B, r_B : B \to \mathrm{End}(A)$ such that (l_A, r_A, B, α_B) and (l_B, r_B, A, α_A) are bimodules of the nearly Hom-associative algebras $(A, \cdot, \alpha_A)$ and $(B, \circ, \alpha_B)$, respectively, satisfying the following conditions for $x, y \in A$, $a, b \in B$:*

$$\begin{aligned}&\alpha_A(x)\cdot(r_B(a)y) + (r_B(l_A(y)a)\alpha_A(x) \\ &-(l_B(a)x)\cdot\alpha_A(y) - l_B(r_A(x)a)\alpha_A(y) = 0,\end{aligned} \tag{11.26a}$$

$$\alpha_A(x)\cdot(l_B(a)y) + r_B(r_A(y)a)\alpha_A(x) - r_B(\alpha_B(a))(y\cdot x) = 0, \tag{11.26b}$$

$$l_B(\alpha_B(a))(x\cdot y) - (r_B(a)y)\cdot\alpha_A(x) - l_B(l_A(y)a)\alpha_A(x) = 0, \tag{11.26c}$$

$$\begin{aligned}&\alpha_B(a)\circ(r_A(x)b) + r_A(l_B(b)x)\alpha_B(a) \\ &-(l_A(x)a)\circ\alpha_B(b) - l_A(r_B(a)x)\alpha_B(b) = 0,\end{aligned} \tag{11.26d}$$

$$\alpha_B(a)\circ(l_A(x)b) + r_A(r_B(b)x)\alpha_B(a) - r_A(\alpha_A(x))(b\circ a) = 0, \tag{11.26e}$$

$$l_A(\alpha_A(x))(b\circ a) - (r_A(x)a)\circ\alpha_B(b) - l_A(l_B(a)x)\alpha_B(b) = 0. \tag{11.26f}$$

Then, there is a bilinear product defined on $A \oplus B$ for $x, y \in A$, $a, b \in B$, by

$$(x+a)*(y+b) = (x\cdot y + l_B(a)y + r_B(b)x) + (a\circ b + l_A(x)b + r_A(y)a)$$

such that $(A \oplus B, *, \alpha_A \oplus \alpha_B)$ is a nearly Hom-associative algebra.

Proof Let $(A, \cdot, \alpha_A)$, $(B, \circ, \alpha_B)$ be nearly Hom-associative algebras, (l_A, r_A, B, α_B) a bimodule of $(A, \cdot, \alpha_A)$ and (l_B, r_B, A, α_A) a bimodule of $(B, \circ, \alpha_B)$. For all $x, y \in A$ and all $a, b \in B$,

$$\begin{aligned}&(\alpha_A(x)+\alpha_B(a))*((y+b)*(z+c)) \\ &= \{(\alpha_A(x))\cdot(l_B(b)z) + r_B(r_A(z)b)\cdot(\alpha_A(x))\} \\ &\quad + \{(\alpha_A(x))\cdot(r_B(c)y) + r_B(l_A(y)c)\alpha_A(x)\} \\ &\quad + (\alpha_A(x))\cdot(y\cdot z) + l_B(\alpha_B(a))(y\cdot z) + l_B(\alpha_B(a))(l_B(b)z) \\ &\quad + l_B(\alpha_B(a))(r_B(c)y) + r_B(b\circ c)(\alpha_A(x)) \\ &\quad + \{(\alpha_B(a))\circ(l_A(y)c) + r_A(r_B(c)y)\alpha_B(a)\}\end{aligned}$$

$$
\begin{aligned}
&+\{(\alpha_B(a))\circ(r_A(z)b)+r_A(l_B(b)z)\alpha_B(a)\}\\
&+(\alpha_B(a))\circ(b\circ c)+r_A(y\cdot z)\alpha_B(a)+l_A(\alpha_A(x))(b\circ c)\\
&+l_A(\alpha_A(x))(l_A(y)c)+l_A(\alpha_A(x))(r_A(z)b),\\
&((z+c)*(x+a))*(\alpha_A(y)+\alpha_B(b))\\
&=\{(l_B(c)x)\cdot(\alpha_A(y))+l_B(r_A(x)c)\alpha_A(y)\}\\
&+\{l_B(l_A(z)a)\alpha_A(y)+(l_B(c)x)\cdot\alpha_A(y)\}\\
&+(z\cdot x)\cdot(\alpha_A(y))+l_B(c\circ a)(\alpha_A(y))+r_B(\alpha_B(b)(z\cdot x)\\
&+r_B(\alpha_B(b)(l_B(c)x)+r_B(\alpha_B(b)(r_B(a)z)\\
&+\{(l_A(z)a)\circ(\alpha_B(b))+l_A(r_B(a)z)\alpha_B\}\\
&+\{(r_A(x)c)\circ(\alpha_B(b))+l_A(l_B(c)x)\alpha_B(b)\}\\
&+(c\circ a)\circ(\alpha_B(b))+l_A(z\cdot x)(\alpha_B(b))+r_A(\alpha_A(y))(c\circ a)\\
&+r_A(\alpha_A(y))(l_A(z)a)+r_A(\alpha_A(y))(r_A(x)c).
\end{aligned}
$$

Using (11.26a)–(11.26f) and that (l_A, r_A, B, α_B) and (l_B, r_B, A, α_A) are bimodules of the nearly Hom-associative algebras $(A, \cdot, \alpha_A)$ and $(B, \circ, \alpha_B)$, respectively, we obtain that $(A \oplus B, *, \alpha_A \oplus \alpha_B)$ is a nearly associative algebra. □

Definition 11.19 A matched pair of nearly Hom-associative algebras $(A, \cdot, \alpha_A)$ and $(B, \circ, \alpha_B)$ is the octuple $(A, B, l_A, r_A, \alpha_B, l_B, r_B, \alpha_A)$, where $l_A, r_A : A \to \text{End}(B)$ and $l_B, r_B : B \to \text{End}(A)$ are linear maps such that (l_A, r_A, B, α_B) and (l_B, r_B, A, α_A) are bimodules of the nearly Hom-associative algebras $(A, \cdot, \alpha_A)$ and $(B, \circ, \alpha_B)$, respectively, and satisfying (11.26a)–(11.26f).

Corollary 11.3 *Let $(A, B, l_A, r_A, \alpha_B, l_B, r_B, \alpha_A)$ be a matched pair of nearly Hom-associative algebras $(A, \cdot, \alpha_A)$ and $(B, \circ, \alpha_B)$. Then, $(\mathcal{G}(A), \mathcal{G}(B), l_A - r_A, l_B - r_B, \alpha_A, \alpha_B)$ is a matched pair of the underlying Hom-Lie algebras $\mathcal{G}(A)$ and $\mathcal{G}(B)$ of the nearly Hom-associative algebras $(A, \cdot, \alpha_A)$ and $(B, \circ, \alpha_B)$.*

Proof Let $(A, B, l_A, r_A, \alpha_B, l_B, r_B, \alpha_A)$ be a matched pair of nearly Hom-associative algebras $(A, \cdot, \alpha_A)$ and $(B, \circ, \alpha_B)$. By Proposition 11.13, the linear maps $l_A - r_A : A \to \text{End}(B)$ and $l_B - r_B : B \to \text{End}(A)$ are representations of the underlying Hom-Lie algebras $(\mathcal{G}(A), \alpha_A)$ and $(\mathcal{G}(B), \alpha_B)$, respectively. Therefore, (11.25a) is equivalent to (11.26a), (11.26b) and (11.26c), and similarly, (11.25b) is equivalent to (11.26d), (11.26e) and (11.26f). □

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Chapter 12
On Generalized q-Hyperbolic Functions in the Spirit of Kapteyn, with Corresponding q-Lie Group

Thomas Ernst

Abstract There are many possible generalizations of hyperbolic functions in the literature, in recent years all these generalizations have been collected into a single definition. The purpose of this article is to find q-analogues of the most interesting formulas of this type, i.e. the $\mathbb{Z}_n$ components of the q-exponential function. There is a close connection to factor-circulant matrices by the q-exponential of a permutation matrix, which has generalized q-hyperbolic functions as matrix elements; this leads to the decomposition of functions with respect to the cyclic group of order n by Ben Cheikh and Kwasniewski. The latter formula is used to find q-analogues of inverse decomposition hypergeometric formulas by Osler and Srivastava. Furthermore, the q-Leibniz functional matrix from a previous paper of the author is equal to the q-exponential of the transpose of the permutation matrix times the q-difference operator. Finally, some q-analogues in general form of Bailey hypergeometric series product formulas connected to cyclic group decomposition are presented.

Keywords q-hyperbolic function · q-exponential · q-difference operator · q-Lie group

MSC2000 Classification Primary 33D15 · Secondary 15A15 · 15B99

12.1 Introduction

In the nineteenth century, a large number of generalizations of hyperbolic functions were considered, until Jacobus Cornelius Kapteyn (1851–1922) together with his fellow astronomer W. Kapteyn published a long paper [13] on the higher sine functions written in German. The formulas in [13] apply to the special case $\alpha = 1$ in our paper. We have written a q-analogue of [13] anyway, since the formulas are quite similar. More than hundred years later, Muldoon and Ungar [17] generalized

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Kapteyn's definitions and introduced the corresponding matrix approach. Several of Kapteyn's formulas are unique, and we have found similar q-analogues of some of them. Already Kapteyn was aware of the connection between cyclic decomposition of functions and generalized hyperbolic functions as well as the de Moivre theorem. Amazingly, these formulas look the same with the q-exponential function and the Ward numbers. Similar to the q-exponential function, there will be two q-hyperbolic functions as well as two q-Taylor formulas.

Remark 12.1 Kwasniewski [15, (23), (32), (46)] has found formulas equivalent to (12.14), (12.27), (12.44) in his own quantum plane notation.

This paper is organized as follows: In Sect. 12.1 we give a general introduction and the first definitions. In Sect. 12.2 we prove the most important formulas in the paper and introduce the decomposition with respect to the cyclic group.

In Sect. 12.3 we show that some of the previous formulas can also be expressed in matrix language. In Sect. 12.4 we relate to some inverse series relations by Osler and Srivastava, which imply some results of Bailey expressed in hypergeometric form. Finally, in Appendix 12.5 we give the historical background of the generalized hyperbolic functions.

We start with some of the definitions from our book.

Definition 12.1 ([3]) The q-exponential functions are defined by

$$\mathrm{E}_q(z) \equiv \sum_{k=0}^{\infty} \frac{1}{\{k\}_q!} z^k;\ \mathrm{E}_{\frac{1}{q}}(z) \equiv \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}}{\{k\}_q!} z^k. \tag{12.1}$$

The q-trigonometric functions are defined by

$$\mathrm{Cos}_q(x) \equiv \frac{1}{2}(\mathrm{E}_q(ix) + \mathrm{E}_q(-ix)). \tag{12.2}$$

$$\mathrm{Sin}_q(x) \equiv \frac{1}{2i}(\mathrm{E}_q(ix) - \mathrm{E}_q(-ix)). \tag{12.3}$$

The q-hyperbolic functions are defined by

$$\mathrm{Sinh}_q(x) \equiv \frac{1}{2}(\mathrm{E}_q(x) - \mathrm{E}_q(-x)) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{\{2n+1\}_q!}. \tag{12.4}$$

$$\mathrm{Cosh}_q(x) \equiv \frac{1}{2}(\mathrm{E}_q(x) + \mathrm{E}_q(-x)) = \sum_{n=0}^{\infty} \frac{x^{2n}}{\{2n\}_q!}. \tag{12.5}$$

Definition 12.2 ([3]) Let a and b belong to a commutative ring. The Nalli–Ward–AlSalam q-addition (NWA) is given by

$$(a \oplus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^k b^{n-k}, \ (a \ominus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^k (-b)^{n-k} \tag{12.6}$$

The Jackson–Hahn–Cigler q-addition (JHC) is given by

$$(a \boxplus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}} a^{n-k} b^k, \ (a \boxminus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}} a^{n-k} (-b)^k. \tag{12.7}$$

Definition 12.3 ([15, (43)], a q-analogue of [17, p. 4]) Let $\alpha \in \mathbb{C}^\star$. The α, q-hyperbolic function of order n and rth kind is defined by

$$\mathrm{F}^{\alpha}_{n,r,q}(x) \equiv \sum_{k=0}^{\infty} \frac{\alpha^k}{\{nk+r\}_q!} x^{nk+r}, \ r = 0, 1, \ldots, n-1, \ \mathrm{F}^{\alpha}_{n,0,q}(0) \equiv 1. \tag{12.8}$$

Definition 12.4 Another q-analogue of [17, p. 4]. Let $\alpha \in \mathbb{C}^\star$. The complementary α, q-hyperbolic function of order n and rth kind is defined by

$$\mathrm{F}^{\alpha}_{n,r,\frac{1}{q}}(x) \equiv \sum_{k=0}^{\infty} \frac{\alpha^k}{\{nk+r\}_q!} x^{nk+r} \mathrm{QE}\left(\binom{nk+r}{2}\right), \ r = 0, 1, \ldots, n-1,$$
$$\mathrm{F}^{\alpha}_{n,0,\frac{1}{q}}(0) \equiv 1. \tag{12.9}$$

We observe that $\mathrm{F}^{\alpha}_{n,r,q}(0) = \mathrm{F}^{\alpha}_{n,r,\frac{1}{q}}(0) = 0, \ r = 1, 2, \ldots, n-1$. The functions (12.8) and (12.9) are generalizations of $\mathrm{E}_q(x)$ and $\mathrm{E}_{\frac{1}{q}}(x)$, respectively.

Definition 12.5 The q-Mittag-Leffler function $E_{\gamma,q}(z)$ is defined by

$$E_{\gamma,q}(z) \equiv \sum_{k=0}^{\infty} \frac{z^k}{\Gamma_q(\gamma k + 1)}. \tag{12.10}$$

For γ real and positive, the series (12.10) is an entire function of the complex variable z.

The relation between the α, q-hyperbolic function of order n and first kind and the q-Mittag-Leffler function, a q-analogue of [17, p. 9], is $\mathrm{F}^{1}_{n,0,q}(x) = E_{n,q}(x^n)$.

Proof

$$E_{n,q}(x^n) = \sum_{k=0}^{\infty} \frac{x^{kn}}{\Gamma_q(nk+1)} = \sum_{k=0}^{\infty} \frac{x^{kn}}{\{nk\}_q!}. \tag{12.11}$$

This completes the proof.

12.2 q-Analogues of the Results by Muldoon, Ungar and Kapteyn

The following simple formula forms the basis of many results in this article. In the rest of the paper, we put $\omega_n \equiv \text{Exp}(\frac{2\pi i}{n})$. The reader should pay attention to the two inverse formulas (12.19) and (12.54).

Theorem 12.2.1 ([2, 1.4, 1.5]) *The decomposition with respect to the cyclic group of order n of an arbitrary complex function $f(z)$ admitting a Laurent expansion in an annulus I with center at the origin.*

$$f(z) = \sum_{k=0}^{n-1} f_{[n,k]}(z), \tag{12.12}$$

and

$$f_{[n,k]}(z) = \frac{1}{n} \sum_{m=0}^{n-1} \omega_n^{-km} f(z\omega_n^m), \;\; 0 < k < n. \tag{12.13}$$

Remark 12.2 An equivalent theorem was given in [20, p. 889].

Theorem 12.2.2 (A q-analogue of [17, (5) p. 5], [21, p. 302].) *The α, q-hyperbolic function of order n and rth kind form a set of n linearly independent solutions to the q-difference equation*

$$\text{D}_q^n f(x) = \alpha f(x), \tag{12.14}$$

with initial conditions

$$\text{D}_q^k f(0) = \begin{cases} 0, \;\; k \neq r, \;\; 0 \leq k \leq n-1, \\ 1, \;\; k = r. \end{cases} \tag{12.15}$$

We observe that the theory of differential (q-difference) equations gives the exponents $\alpha^{\frac{1}{n}}$ as solutions of (12.14); the initial conditions (12.15) are the best possible ones.

Theorem 12.2.3 (A q-analogue of [17, p. 5], [21, (6) p. 302]) *Furthermore, a cyclic permutation results from q-differentiation, apart from a factor α for $r = 0$:*

$$\mathrm{D}_q \mathrm{F}^{\alpha}_{n,r,q}(x) = \begin{cases} \mathrm{F}^{\alpha}_{n,r-1,q}(x), \ 0 < r \leq n-1, \\ \alpha \mathrm{F}^{\alpha}_{n,n-1,q}(x), \ r = 0. \end{cases} \tag{12.16}$$

We observe that formula (12.16) implies (12.14). The following formula relates some of the α, q-hyperbolic functions with $\alpha = \pm 1$.

Theorem 12.2.4 *A q-analogue of [17, (17), (18) p. 10]*

$$\mathrm{F}^{1}_{2m,r,q}(x) = \frac{1}{2}\left[\mathrm{F}^{1}_{m,r,q}(x) + \mathrm{F}^{-1}_{m,r,q}(x)\right], r = 0, 1, \ldots, m-1, \tag{12.17}$$

$$\mathrm{F}^{1}_{2m,r+m,q}(x) = \frac{1}{2}\left[\mathrm{F}^{1}_{m,r,q}(x) - \mathrm{F}^{-1}_{m,r,q}(x)\right], r = 0, 1, \ldots, m-1. \tag{12.18}$$

Theorem 12.2.5 (A q-analogue of [17, (6) p. 5], [13, (14) p. 809]. A q-generalization of Eulers' formula) *Let $\alpha^{\frac{1}{n}}$ denote the unique root given by the principal branch of the logarithm. Then*

$$\mathrm{E}_q\left(\alpha^{\frac{1}{n}} \omega_n^k x\right) = \sum_{r=0}^{n-1} \alpha^{\frac{r}{n}} \omega_n^{kr} \mathrm{F}^{\alpha}_{n,r,q}(x), \ k = 0, 1, \ldots, n-1. \tag{12.19}$$

Eulers' formula results from $n = 2, \alpha = -1, q = 1$. Formula (12.19) is n linear equations, because there are n nth roots of α.

Proof Use formula (12.12).

Our next aim is to q-deform some basic formulas by Kapteyn (and Nikodemo [18]).

Theorem 12.2.6 (Almost [15, (18)], a q-analogue of [13, (14) p. 809], [18, p.198].)

$$\mathrm{E}_q\left(\omega_n^k x\right) = \sum_{r=0}^{n-1} \omega_n^{rk} \mathrm{F}^{1}_{n,r,q}(x), \ k = 1, 2, \ldots, n. \tag{12.20}$$

Corollary 12.2.7 (A de Moivre type theorem. A q-analogue of [13, p. 814].)

$$\mathrm{E}_q\left(\omega_n^k x \overline{m}_q\right) = \left[\sum_{r=0}^{n-1} \omega_n^{rk} \mathrm{F}^{1}_{n,r,q}(x)\right]^m, \ k = 1, 2, \ldots, n. \tag{12.21}$$

Proof Use the formula (12.20) with $x \mapsto x\overline{m}_q$.

Corollary 12.2.8 (A q-analogue of [13, (32) p. 814].)

$$\sum_{r=0}^{n-1} \omega_n^{rk} \mathrm{F}^{1}_{n,r,q}(x\overline{m}_q) = \left[\sum_{r=0}^{n-1} \omega_n^{rk} \mathrm{F}^{1}_{n,r,q}(x)\right]^m, \ k = 1, 2, \ldots, n. \tag{12.22}$$

Proof Use the formula (12.20) with $x \mapsto x\overline{m}_q$, together with (12.21).

Theorem 12.2.9 (A q-analogue of [13, (7), (8) p. 808].)

$$\mathrm{F}^{\alpha}_{n,r,q}(\omega_n^m x) = \omega_n^{mr} \mathrm{F}^{\alpha}_{n,r,q}(x), \tag{12.23}$$

$$\mathrm{F}^{\alpha}_{n,r,q}\left(\omega_n^{\frac{2m-1}{2}} x\right) = \omega_n^{\frac{r(2m-1)}{2}} \mathrm{F}^{-\alpha}_{n,r,q}(x), \tag{12.24}$$

Proof We prove the first formula.

$$\text{LHS}=\sum_{k=0}^{\infty} \frac{x^{kn+r}}{\{kn+r\}_q!} \omega_n^{m(kn+r)} = \text{RHS}. \tag{12.25}$$

Theorem 12.2.10 (A q-analogue of [13, (9), (10) p. 808].)

$$\mathrm{F}^{\alpha}_{n,r,q}(-x) = (-1)^r \begin{cases} \mathrm{F}^{\alpha}_{n,r,q}(x), & n \text{ even} \\ \mathrm{F}^{-\alpha}_{n,r,q}(x), & n \text{ odd}. \end{cases} \tag{12.26}$$

The following simple formulas are all kind of convolutions in the second index r.

Theorem 12.2.11 (A q-Taylor expansion, a q-analogue of [13, (22) p. 811].)

$$\mathrm{F}^{\alpha}_{n,r,q}(x \oplus_q y) = \sum_{\lambda=0}^{n-1} \mathrm{F}^{\alpha}_{n,r-\lambda,q}(x) \mathrm{F}^{\alpha}_{n,\lambda,q}(y). \tag{12.27}$$

Proof

$$\begin{aligned} \text{LHS}=&\sum_{k=0}^{\infty} \frac{y^k}{\{k\}_q!} \mathrm{D}_q^k \mathrm{F}^{\alpha}_{n,r,q}(x) \\ =&\sum_{m=0}^{\infty} \sum_{\lambda=0}^{n-1} \frac{y^{mn+\lambda}}{\{mn+\lambda\}_q!} \alpha^m \mathrm{F}^{\alpha}_{n,r-\lambda,q}(x) = \text{RHS}. \end{aligned} \tag{12.28}$$

Corollary 12.2.12

$$\mathrm{F}^{\alpha}_{n,r,q}(x \oplus_q \omega_n^k y) = \sum_{\lambda=0}^{n-1} \omega_n^{\lambda k} \mathrm{F}^{\alpha}_{n,r-\lambda,q}(x) \mathrm{F}^{\alpha}_{n,\lambda,q}(y). \tag{12.29}$$

Proof Use the formula (12.23) together with (12.27).

Corollary 12.2.13 (A q-Taylor expansion, a q-analogue of [13, (23), (24) p. 811], [18, (6), p. 199].)

$$\mathrm{F}_{n,r,q}^{\alpha}(x \ominus_q y) = \sum_{\lambda=0}^{n-1}(-1)^{\lambda}\mathrm{F}_{n,r-\lambda,q}^{\alpha}(x)\begin{cases}\mathrm{F}_{n,\lambda,q}^{\alpha}(y), & n \text{ even.}\\ \mathrm{F}_{n,\lambda,q}^{-\alpha}(y), & n \text{ odd.}\end{cases} \tag{12.30}$$

Proof See the next proof.

Corollary 12.2.14 (Another q-analogue of [13, (23), (24) p. 811].)

$$\mathrm{F}_{n,r,q}^{\alpha}(x \boxminus_q y) = \sum_{\lambda=0}^{n-1}(-1)^{\lambda}\mathrm{F}_{n,r-\lambda,q}^{\alpha}(x)\begin{cases}\mathrm{F}_{n,\lambda,\frac{1}{q}}^{\alpha}(y), & n \text{ even.}\\ \mathrm{F}_{n,\lambda,\frac{1}{q}}^{-\alpha}(y), & n \text{ odd.}\end{cases} \tag{12.31}$$

Proof

$$\begin{aligned}\text{LHS}=&\sum_{k=0}^{\infty}\frac{(-y)^k}{\{k\}_q!}\mathrm{D}_q^k\mathrm{F}_{n,r,q}^{\alpha}(x)q^{\binom{k}{2}}\\ =&\sum_{m=0}^{\infty}\sum_{\lambda=0}^{n-1}\frac{(-y)^{mn+\lambda}}{\{mn+\lambda\}_q!}\alpha^m\mathrm{F}_{n,r-\lambda,q}^{\alpha}(x)\mathrm{QE}\left(\binom{mn+\lambda}{2}\right)=\text{RHS}.\end{aligned} \tag{12.32}$$

We are finished with the convolutions and find some corollaries.

Corollary 12.2.15 (A q-analogue of [13, (25) p. 812].)

$$\begin{aligned}&\mathrm{F}_{n,r,q}^{\alpha}(x \oplus_q y) \pm \mathrm{F}_{n,r,q}^{\alpha}(x \ominus_q y)\\ &=\sum_{\lambda=0}^{n-1}\mathrm{F}_{n,r-\lambda,q}^{\alpha}(x)\left[\mathrm{F}_{n,\lambda,q}^{\alpha}(y) \pm (-1)^{\lambda}\mathrm{F}_{n,\lambda,q}^{\alpha}(y)\right],\ n \text{ even.}\end{aligned} \tag{12.33}$$

A q-analogue of [13, (26) p. 812], [18, p. 205]

$$\begin{aligned}&\mathrm{F}_{n,r,q}^{\alpha}(x \oplus_q y) \pm \mathrm{F}_{n,r,q}^{\alpha}(x \ominus_q y)\\ &=\sum_{\lambda=0}^{n-1}\mathrm{F}_{n,r-\lambda,q}^{\alpha}(x)\left[\mathrm{F}_{n,\lambda,q}^{\alpha}(y) \pm (-1)^{\lambda}\mathrm{F}_{n,\lambda,q}^{-\alpha}(y)\right],\ n \text{ odd.}\end{aligned} \tag{12.34}$$

Remark 12.3 As Nicodemo points out, formulas (12.33) and (12.34) correspond to the multiplication formulas for trigonometric and hyperbolic functions.

Corollary 12.2.16 (A q-analogue of [13, (28) p. 812], [18, p. 203].)

$$\mathrm{F}_{n,r,q}^{\alpha}(x\overline{2}_q) = \sum_{\lambda=0}^{n-1}\mathrm{F}_{n,r-\lambda,q}^{\alpha}(x)\mathrm{F}_{n,\lambda,q}^{\alpha}(x). \tag{12.35}$$

Proof Put $x = y$ in formula (12.27).

Corollary 12.2.17 (A q-analogue of [13, (29), (30) p. 812].)

$$\mathrm{F}^{\alpha}_{n,r,q}(0) = \sum_{\lambda=0}^{n-1}(-1)^{\lambda}\mathrm{F}^{\alpha}_{n,r-\lambda,q}(x)\begin{cases}\mathrm{F}^{\alpha}_{n,\lambda,\frac{1}{q}}(x), & n \text{ even.}\\ \mathrm{F}^{-\alpha}_{n,\lambda,\frac{1}{q}}(x), & n \text{ odd.}\end{cases} \tag{12.36}$$

Proof Put $x = y$ in formula (12.31).

12.3 The Corresponding Matrix Formulas

In this paper we will use a few matrices, which are now defined.

Definition 12.6 ([17, p. 8]) The permutation matrix $\mathbf{A}_n(\alpha)$ is defined by

$$\mathbf{A}_n(\alpha) \equiv \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \alpha & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{12.37}$$

Definition 12.7 (A q-analogue of [17, p. 6].) The α, q hyperbolic matrix is defined by

$$\mathbf{H}^{\alpha}_{n,q}(x) \equiv \mathrm{E}_q(x\mathbf{A}_n(\alpha)). \tag{12.38}$$

We have the following two special cases.

Definition 12.8 (A q-analogue of [14, p. 253].) The $\mathcal{H}$ and $\mathcal{F}$ hyperbolic matrices are defined by

$$\mathcal{H}_{n,q}(x) \equiv \mathrm{E}_q(x\mathbf{A}_n(1)), \tag{12.39}$$

$$\mathcal{F}_{n,q}(x) \equiv \mathrm{E}_q(x\mathbf{A}_n(-1)). \tag{12.40}$$

The inverses of these hyperbolic matrices are found in formula (12.49).

Theorem 12.3.1 *An explicit formula for the factor-circulant matrix.*

$$\mathbf{H}^{\alpha}_{n,q}(x) = \begin{pmatrix} \mathrm{F}^{\alpha}_{n,0,q}(x) & \mathrm{F}^{\alpha}_{n,1,q}(x) & \mathrm{F}^{\alpha}_{n,2,q}(x) & \cdots & \mathrm{F}^{\alpha}_{n,n-1,q}(x) \\ \alpha\mathrm{F}^{\alpha}_{n,n-1,q}(x) & \mathrm{F}^{\alpha}_{n,0,q}(x) & \mathrm{F}^{\alpha}_{n,1,q}(x) & \cdots & \mathrm{F}^{\alpha}_{n,n-2,q}(x) \\ \alpha\mathrm{F}^{\alpha}_{n,n-2,q}(x) & \alpha\mathrm{F}^{\alpha}_{n,n-1,q}(x) & \mathrm{F}^{\alpha}_{n,0,q}(x) & \cdots & \mathrm{F}^{\alpha}_{n,n-3,q}(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha\mathrm{F}^{\alpha}_{n,1,q}(x) & \alpha\mathrm{F}^{\alpha}_{n,2,q}(x) & \alpha\mathrm{F}^{\alpha}_{n,3,q}(x) & \cdots & \mathrm{F}^{\alpha}_{n,0,q}(x) \end{pmatrix}. \tag{12.41}$$

Proof Let $0 < k < n$. Use the formula

$$\mathbf{A}_n(\alpha)^k = \begin{pmatrix} 0 & \vdots & \mathrm{I}_{n-k} \\ \cdots & + & \cdots \\ \alpha \mathrm{I}_k & \vdots & 0 \end{pmatrix}, \tag{12.42}$$

$$\text{where } \frac{1}{\alpha}\mathbf{A}_n(\alpha)^n = \mathrm{I}_n, \text{ the unit matrix.} \tag{12.43}$$

Theorem 12.3.2 (A q-analogue of [17, (11) p. 7], [14, p. 254].)

$$\mathbf{H}^{\alpha}_{n,q}(x)\mathbf{H}^{\alpha}_{n,q}(y) = \mathbf{H}^{\alpha}_{n,q}(x \oplus_q y), \; n \geq 1. \tag{12.44}$$

Proof Use the properties of the q-exponential function. Observe that the matrices $x\mathbf{A}_n(\alpha)$ and $y\mathbf{A}_n(\alpha)$ commute [9, p. 108].

Theorem 12.3.3 (A second q-analogue of [17, (11) p. 7], [14, p. 254].)

$$\mathbf{H}^{\alpha}_{n,q}(x)\mathbf{H}^{\alpha}_{n,\frac{1}{q}}(y) = \mathbf{H}^{\alpha}_{n,q}(x \boxplus_q y), \; n \geq 1. \tag{12.45}$$

Corollary 12.3.4

$$(\mathbf{H}^{\alpha}_{n,q}(x))^{-1} = \mathbf{H}^{\alpha}_{n,\frac{1}{q}}(-x). \tag{12.46}$$

Proof Use the formula $\mathbf{H}^{\alpha}_{n,q}(0) = \mathrm{I}_n$ together with (12.45).

Theorem 12.3.5 *The α, q hyperbolic matrices $\mathbf{H}^{\alpha}_{n,q}(x)$ form a q-Lie group [6] with multiplications ordinary matrix multiplication $\cdot$ given by (12.44) and twisted matrix multiplication $\cdot_q$ given by (12.45).*

Proof The associativity follows by the associativity of the two q-additions. The inverse matrix is given by (12.46). The unit is the unit matrix I_n by (12.46).

Theorem 12.3.6 *The α, q hyperbolic matrix $\mathbf{H}^{\alpha}_{n,q}(x)$ is the unique solution of the q-difference system*

$$\mathrm{D}^n_q M = \alpha M, \; \mathrm{D}^k_q M(0) = (\mathbf{A}_n(\alpha))^k, \; k = 0, 1, \ldots, n-1. \tag{12.47}$$

Remark 12.4 We call the equations (12.44) and (12.45) the matrix q-binomial theorem. We call matrices which satisfy (12.44) and (12.45) q-binomial matrices.

We can also express the relation between $\mathbf{H}^{\alpha}_{n,q}(x)$ and $\mathbf{A}_n(\alpha)$ in the form

$$\mathbf{H}^{\alpha}_{n,q}(x) = \sum_{k=0}^{n-1} \mathrm{F}^{\alpha}_{n,k,q}(x)\,(\mathbf{A}_n(\alpha))^k. \tag{12.48}$$

Corollary 12.3.7 (A q-analogue of [14, (14) p. 254].)

$$\mathcal{H}_{n,q}(x)^{-1} = \mathcal{H}_{n,\frac{1}{q}}(-x) \text{ and } \mathcal{F}_{n,q}(x)^{-1} = \mathcal{F}_{n,\frac{1}{q}}(-x). \tag{12.49}$$

In a previous paper [4], we introduced a special q-deformed ϵ matrix multiplication in connection with the q-Leibniz rule. Referring to the paper by Kalman and Ungar [12, p. 29], we can show that the q-Leibniz functional matrix is equal to the q-exponential of the transpose of the permutation matrix times the q-difference operator. First we repeat the definition:

Definition 12.9 ([4]) The q-deformed Leibniz functional matrix is given by

$$(\mathcal{L}_{n,q})[f(t,q)](i,j) \equiv \begin{cases} \frac{\mathrm{D}_{q,t}^{i-j} f(t,q)}{\{i-j\}_q!} & \text{if} i \geq j; \\ 0, & \text{otherwise} \end{cases} \quad i,j = 0, 1, \ldots, n-1. \tag{12.50}$$

Then we have by the q-Leibniz formula

$$(\mathcal{L}_{n,q})[f(t,q)g(t,q)] = (\mathcal{L}_{n,q})[f(t,q)] \cdot_\epsilon (\mathcal{L}_{n,q})[g(t,q)], \tag{12.51}$$

where in the matrix multiplication for every term which includes $\mathrm{D}_q^k f$, we operate with ϵ^k on g. We denote this by $\cdot_\epsilon$. We infer the lower triangular matrix equation

Theorem 12.3.8

$$\mathcal{L}_{n,q} = \mathrm{E}_q(\mathrm{D}_q \cdot (\mathbf{A}_n(0))^T), \tag{12.52}$$

where powers of the 'scalar' D_q *are interpreted as the operator* D_q^k *(which is always divided by the corresponding* q*-factorial).*

The analogues to the formulas in [12, pp. 29–30] exist, by formula (12.51).

We now introduce the Fourier matrix to be able to decompose functions with respect to the cyclic group of order n.

Definition 12.10 ([17, p. 9]) The Fourier matrix $\mathcal{F}_n$ is defined by

$$\mathcal{F}_n \equiv \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & \cdots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \cdots & \omega_n^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \cdots & \omega_n^{(n-1)^2} \end{pmatrix}. \tag{12.53}$$

Theorem 12.3.9 ([17, p. 10]) *The inverse of the Fourier matrix is* $\mathcal{F}_n^{-1} = n^{-1}\bar{\mathcal{F}}_n$.

Theorem 12.3.10 (A generalization of [15, (17) and p. 58], a q-analogue of [17, (7) p. 5], [13, (15) p. 809], [18, p. 198].)

$$\mathrm{F}^{\alpha}_{n,r,q}(x) = \frac{1}{n}\alpha^{-\frac{r}{n}} \sum_{k=0}^{n-1} \omega_n^{-rk} \mathrm{E}_q\left(\omega_n^k \alpha^{\frac{1}{n}} x\right). \tag{12.54}$$

Proof The system of n Eq. (12.19) can be written in matrix form

$$\begin{pmatrix} \mathrm{E}_q\left(\alpha^{\frac{1}{n}} x\right) \\ \mathrm{E}_q\left(\omega_n \alpha^{\frac{1}{n}} x\right) \\ \vdots \\ \mathrm{E}_q\left(\omega_n^{n-1} \alpha^{\frac{1}{n}} x\right) \end{pmatrix} = \mathfrak{F}_n \begin{pmatrix} \mathrm{F}^{\alpha}_{n,0,q}(x) \\ \alpha^{\frac{1}{n}} \mathrm{F}^{\alpha}_{n,1,q}(x) \\ \vdots \\ \alpha^{\frac{n-1}{n}} \mathrm{F}^{\alpha}_{n,n-1,q}(x) \end{pmatrix}. \tag{12.55}$$

The proof is concluded by using the formula for the inverse of $\mathfrak{F}_n$.

12.4 Connections with the Triangle Operator

We start with the Osler–Srivastava transformation formulas for q-hypergeometric functions. Osler [20, p. 890] proved the precursor of the following theorem in a slightly different notation.

Theorem 12.4.1 (Two inverse q-hypergeometric formulas with the $\triangle$ operator.) *Let* (a) *and* (b) *be two vectors of length* P. *Then*

$$\begin{aligned} &\frac{x^K \langle (a); q\rangle_K}{\langle (b); q\rangle_K} \sum_{m=0}^{\infty} \frac{\langle \triangle(q; n; (a+K); q\rangle_m}{\langle \triangle(q; n; (b+K); q\rangle_m} x^{nm} \\ &= \frac{1}{n}\sum_{j=0}^{n-1} \omega_n^{-Kj} {}_{P+1}\phi_P((a), 1; (b)|q; x\omega_n^j),\ 0 \le K < n. \end{aligned} \tag{12.56}$$

$$\begin{aligned} &{}_{P+1}\phi_P((a), 1; (b)|q; x) \\ &= \sum_{k=0}^{n-1} \frac{x^k \langle (a); q\rangle_k}{\langle (b); q\rangle_k} {}_{2nP+1}\phi_{2nP}\left[\begin{matrix} \triangle(q; n; (a+k)), 1 \\ \triangle(q; n; (b+k)) \end{matrix}\middle| q; x^n\right]. \end{aligned} \tag{12.57}$$

Proof To prove (12.56), use formula (12.13) together with [3, 6.80]. To prove (12.57), sum (12.56) for K from 0 to $N-1$ and use the orthogonality property.

Example 12.1 Put $K=0,\ n=2,\ a \mapsto 2a,\ b=1,\ P=1$ in (12.56) to get

$${}_4\phi_3\left[\begin{matrix} \triangle(q; 2; 2a) \\ \frac{1}{2}, \widetilde{\frac{1}{2}}, \widetilde{1} \end{matrix}\middle| q; x^2\right] = \frac{1}{2}\left(\frac{1}{(x; q)_{2a}} + \frac{1}{(-x; q)_{2a}}\right). \tag{12.58}$$

Put $K=1,\ n=2,\ a\mapsto 2a,\ b=1,\ P=1$ to get

$$\frac{x\{2a\}_q}{\{1\}_q}{}_4\phi_3\left[\begin{matrix}\triangle(q;2;2a+1)\\ \frac{3}{2},\widetilde{\frac{3}{2}},\widetilde{1}\end{matrix}\middle|q;x^2\right]=\frac{1}{2}\left(\frac{1}{(x;q)_{2a}}-\frac{1}{(-x;q)_{2a}}\right). \tag{12.59}$$

Put $K=0,\ n=2,\ a_1\mapsto 2a,\ a_2\mapsto 2b,\ b_1\mapsto 2c,\ b_2=1,\ P=2$ to get

$$\begin{aligned}&{}_8\phi_7\left[\begin{matrix}\triangle(q;2;2a,2b)\\ \triangle(q;2;2c),\frac{1}{2},\widetilde{\frac{1}{2}},\widetilde{1}\end{matrix}\middle|q;x^2\right]\\ &=\frac{1}{2}\left({}_2\phi_1(2a,2b;2c|q;x)+{}_2\phi_1\left(2a,2b;2c|q;-x\right)\right).\end{aligned} \tag{12.60}$$

Formula (12.57) can be expressed in another form.

Theorem 12.4.2 (A q-analogue of Srivastava [22, p. 194, (12)].) *Let* $\triangle^\star(q;N;j+1)$ *denote the factor with the parameter* $\frac{N}{N}$ *omitted. Then*

$$\begin{aligned}&{}_p\phi_{p-1}\left[\begin{matrix}\alpha_1;\alpha_2;\ldots;\alpha_p\\ \beta_1;\beta_2;\ldots;\beta_{p-1}\end{matrix}\middle|q;z\right]=\sum_{j=0}^{N-1}\frac{\langle(\alpha);q\rangle_j z^j}{\langle(\beta),1;q\rangle_j}\\ &{}_{2Np}\phi_{2N(p-1)+N-1}\left[\begin{matrix}\triangle(q;N;(\alpha+j))\\ \triangle(q;N;(\beta+j)),\triangle^\star(q;N;j+1)\end{matrix}\middle|q;z^N\right].\end{aligned} \tag{12.61}$$

Proof The left member is equal to

$$\begin{aligned}&\sum_{j=0}^{N-1}\sum_{n=0}^{\infty}\frac{\langle(\alpha);q\rangle_{nN+j}z^{nN+j}}{\langle(\beta),1;q\rangle_{nN+j}}\\ =&\sum_{j=0}^{N-1}\frac{\langle(\alpha);q\rangle_j z^j}{\langle(\beta),1;q\rangle_j}\sum_{n=0}^{\infty}\frac{\langle\triangle(q;N;\alpha+j);q\rangle_n z^{nN}}{\langle 1;q\rangle_n\langle\triangle(q;N;\beta+j);q\rangle_n\langle\triangle^\star(q;N;j+1)\rangle_n}.\end{aligned} \tag{12.62}$$

Theorem 12.4.3 (Almost a q-analogue of Tremblay, Fugère [23, pp. 846–847]. Decomposition of a product of two restricted q-hypergeometric functions into even an odd parts.) *The vectors* (a), (b), (c) *and* (d) *have lengths* p,t,r,s *respectively. Furthermore, assume that* λ *is a constant independent of* x *and*

$${}_p\phi_t\left[\begin{matrix}(a)\\ (b)\end{matrix}\middle|q;x\right]{}_r\phi_s\left[\begin{matrix}(c)\\ (d)\end{matrix}\middle|q;\lambda x\right]=\sum_{n=0}^{\infty}c_n x^n. \tag{12.63}$$

Then we obtain two new formulas (12.64) and (12.65) for each (12.63) identity.

$$
\begin{aligned}
&\sum_{n=0}^{\infty} c_{2n} x^n \\
&= \sum_{m=0}^{\infty} \frac{\langle \triangle(q; 2; (a)); q\rangle_m}{\langle \triangle(q; 2; 1, (b)); q\rangle_m} x^m \left(q^{\binom{2m}{2}}\right)^{t-p+1} \\
&\sum_{k=0}^{\infty} \frac{\langle \triangle(q; 2; (c)); q\rangle_k \lambda^{2k}}{\langle \triangle(q; 2; 1, (d)); q\rangle_k} x^k \left(q^{\binom{2k}{2}}\right)^{s-r+1} \\
&+x \frac{\langle (a); q\rangle_1}{\langle 1, (b); q\rangle_1} \sum_{m=0}^{\infty} x^m \frac{\langle \triangle(q; 2; (a+1)); q\rangle_m}{\langle \triangle(q; 2; 2, (b+1)); q\rangle_m} \left(-q^{\binom{2m+1}{2}}\right)^{t-p+1} \\
&\frac{\langle (c); q\rangle_1}{\langle 1, (d); q\rangle_1} \sum_{k=0}^{\infty} \frac{\langle \triangle(q; 2; (c+1)); q\rangle_k \lambda^{2k+1}}{\langle \triangle(q; 2; 2, (d+1)); q\rangle_k} x^k \left(-q^{\binom{2k+1}{2}}\right)^{s-r+1},
\end{aligned} \tag{12.64}
$$

$$
\begin{aligned}
&\sum_{n=0}^{\infty} c_{2n+1} x^n \\
&= \sum_{m=0}^{\infty} \frac{\langle \triangle(q; 2; (a)); q\rangle_m}{\langle \triangle(q; 2; 1, (b)); q\rangle_m} x^m \left(q^{\binom{2m}{2}}\right)^{t-p+1} \\
&\frac{\langle (c); q\rangle_1}{\langle 1, (d); q\rangle_1} \sum_{k=0}^{\infty} \frac{\langle \triangle(q; 2; (c+1)); q\rangle_k \lambda^{2k+1}}{\langle \triangle(q; 2; 2, (d+1)); q\rangle_k} x^k \left(-q^{\binom{2k+1}{2}}\right)^{s-r+1} \\
&+x \frac{\langle (a); q\rangle_1}{\langle 1, (b); q\rangle_1} \sum_{m=0}^{\infty} x^m \frac{\langle \triangle(q; 2; (a+1)); q\rangle_m}{\langle \triangle(q; 2; 2, (b+1)); q\rangle_m} \left(-q^{\binom{2m+1}{2}}\right)^{t-p+1} \\
&\sum_{k=0}^{\infty} \frac{\langle \triangle(q; 2; (c)); q\rangle_k}{\langle \triangle(q; 2; 1, (d)); q\rangle_k} \lambda^{2k} x^k \left(q^{\binom{2k}{2}}\right)^{s-r+1}.
\end{aligned} \tag{12.65}
$$

Example 12.2 (A q-analogue of [23, (11) p. 849].) We use formula [3, 10.120]

$$
\begin{aligned}
&{}_2\phi_1(\infty, \infty; \nu|q; x)\ {}_0\phi_1(-; \sigma|q; xq^{\sigma}) \\
&= {}_4\phi_3\left[\begin{matrix} \triangle(q; 2; \nu+\sigma-1) \\ \nu, \sigma, \nu+\sigma-1 \end{matrix}\middle| q; x\right].
\end{aligned} \tag{12.66}
$$

Put $(a) = 2\infty,\ (b) = \nu,\ (c) = \cdot,\ (d) = \sigma,\ \lambda = q^{\sigma}$ in (12.4.3) to get

$$
\begin{aligned}
&{}_{16}\phi_{15}\left[\begin{matrix}\triangle(q;4;\nu+\sigma-1),8\infty\\ \frac{1}{2},\widetilde{\frac{1}{2}},\widetilde{1},\triangle(q;2;\nu,\sigma,\nu+\sigma-1)\end{matrix}\middle|q;x\right]\\
&\quad={}_{8}\phi_{7}\left[\begin{matrix}8\infty\\ \frac{1}{2},\widetilde{\frac{1}{2}},\widetilde{1},\triangle(q;2;\nu)\end{matrix}\middle|q;x\right]\\
&\quad\sum_{k=0}^{\infty}\frac{x^k}{\langle\triangle(q;2;1,\sigma);q\rangle_k}\mathrm{QE}\left(2k\sigma+2\binom{2k}{2}\right)\\
&\quad+\frac{x}{(1-q)^2(1-q^\nu)(1-q^\sigma)}\,{}_{8}\phi_{7}\left[\begin{matrix}8\infty\\ \frac{3}{2},\widetilde{\frac{3}{2}},\widetilde{1},\triangle(q;2;\nu+1)\end{matrix}\middle|q;x\right]\\
&\quad\sum_{k=0}^{\infty}\frac{x^k}{\langle\triangle(q;2;2,\sigma+1);q\rangle_k}\mathrm{QE}\left((2k+1)\sigma+2\binom{2k+1}{2}\right).
\end{aligned}
\tag{12.67}
$$

A q-analogue of [23, (12) p. 849].

Put $(a)=2\infty,\ (b)=\nu,\ (c)=\cdot,\ (d)=\sigma,\ \lambda=q^\sigma$ in (12.65) to get

$$
\begin{aligned}
&{}_{16}\phi_{15}\left[\begin{matrix}\triangle(q;4;\nu+\sigma),8\infty\\ \frac{3}{2},\widetilde{\frac{3}{2}},\widetilde{1},\triangle(q;2;\nu+1,\sigma+1,\nu+\sigma)\end{matrix}\middle|q;x\right]=\frac{1}{\langle 1,\sigma,\nu;q\rangle_1}\\
&\times\ {}_{8}\phi_{7}\left[\begin{matrix}8\infty\\ \frac{1}{2},\widetilde{\frac{1}{2}},\widetilde{1},\triangle(q;2;\nu)\end{matrix}\middle|q;x\right]\sum_{k=0}^{\infty}\frac{x^k}{\langle\triangle(q;2;2,\sigma+1);q\rangle_k}\\
&\mathrm{QE}\left((2k+1)\sigma+2\binom{2k+1}{2}\right)\\
&+\frac{x}{(1-q)(1-q^\nu)}\,{}_{8}\phi_{7}\left[\begin{matrix}8\infty\\ \frac{3}{2},\widetilde{\frac{3}{2}},\widetilde{1},\triangle(q;2;\nu+1)\end{matrix}\middle|q;x\right]\\
&\sum_{k=0}^{\infty}\frac{x^k}{\langle\triangle(q;2;1,\sigma);q\rangle_k}\mathrm{QE}\left((2k)\sigma+2\binom{2k}{2}\right).
\end{aligned}
\tag{12.68}
$$

We now switch to Bailey. The motivation for formulas (12.77) and (12.78) is that they are q-analogues of Bailey's product expansions with generalized hyperbolic functions. First we q-deform some of Bailey's products of generalized hypergeometric series [1, p. 244].

Theorem 12.4.4 *In the product*

$$
{}_{p}\phi_{u}((\alpha);(\rho)|q;x)\ {}_{r}\phi_{s}((\beta);(\sigma)|q;cx),
\tag{12.69}
$$

the coefficient of x^n *is*

$$\frac{(-1)^{n(p+u+1)}\langle(\alpha);q\rangle_n}{\langle 1,(\rho);q\rangle_n}\sum_{m=0}^{n}\frac{(-1)^{m(1+r+s)}\langle -n,1-(\rho)-n,(\beta);q\rangle_m c^m}{\langle 1-(\alpha)-n,1,(\sigma);q\rangle_m}$$
$$\mathrm{QE}\left((s-r+p-u)\binom{m}{2}+m(p-u+n(u+1-p)-(\alpha)+(\rho))\right) \quad (12.70)$$
$$\mathrm{QE}\left((-p+u+1)\binom{n-m}{2}\right).$$

Proof The above expression is equal to

$$\sum_{m=0}^{n}\frac{\langle(\alpha);q\rangle_{n-m}\langle(\beta);q\rangle_m c^m\left[(-1)^{n-m}q^{\binom{n-m}{2}}\right]^{-p+u+1}}{\langle 1,(\rho);q\rangle_{n-m}\langle 1,(\sigma);q\rangle_m\left[(-1)^m q^{\binom{m}{2}}\right]^{r-s-1}}, \quad (12.71)$$

which can easily be shown to be equivalent to this.

In the case $p=u+1$, $r=s+1$ this specializes to the following. In the product

$$_p\phi_{p-1}((\alpha);(\rho)|q;x)\;_r\phi_{r-1}((\beta);(\sigma)|q;cx), \quad (12.72)$$

the coefficient of x^n is

$$\frac{\langle(\alpha);q\rangle_n}{\langle 1,(\rho);q\rangle_n}\,{}_{p+r}\phi_{p+r-1}\left[\begin{matrix}-n,1-(\rho)-n,(\beta)\\ 1-(\alpha)-n,(\sigma)\end{matrix}\middle|q;c\right]. \quad (12.73)$$

Theorem 12.4.5 (A q-analogue of [1, p. 246].) *In the product*

$$_p\phi_u((\alpha);(\rho)|q;x)\;_r\phi_s((\beta);(\sigma)|q;cx^j), \quad (12.74)$$

the coefficient of x^n *is*

$$\frac{(-1)^{n(p+u+1)}\langle(\alpha);q\rangle_n}{\langle 1,(\rho);q\rangle_n}$$
$$\sum_{m=0}^{[\frac{n}{j}]}\frac{(-1)^{m(1+r+s)}\langle -n,1-(\rho)-n);q\rangle_{jm}\langle(\beta);q\rangle_m c^m}{\langle 1-(\alpha)-n;q\rangle_{jm}\langle 1,(\sigma);q\rangle_m} \quad (12.75)$$
$$\mathrm{QE}\left((1+s-r)\binom{m}{2}+jm(1-(\alpha)+(\rho))\right)$$
$$\mathrm{QE}\left((-p+u+1)\left(\binom{n-jm}{2}-\binom{jm}{2}\right)\right).$$

This can also be written in a form without q-factors.

$$\frac{(-1)^{n(p+u+1)}\langle(\alpha);q\rangle_n}{\langle 1,(\rho);q\rangle_n}\phi\left[\begin{matrix}\triangle(q;j;-n,1-(\rho)-n),(\beta)\\ \triangle(q;j;1-(\alpha)-n),(\sigma)\end{matrix}\middle|q;\pm c\right]. \tag{12.76}$$

We now list five special implications of formula (12.75), which are q-analogues of Bailey's formulas [1, (3.2)–(3.6) p. 247]

$$\begin{aligned}&\mathrm{E}_q(x)_5\phi_4\left[\begin{matrix}1,4\infty\\ \triangle(q;2;k)\end{matrix}\middle|q;-x^2\right]\\ &=\sum_{n=0}^{\infty}\sum_{m=0}^{[\frac{n}{2}]}\frac{x^n(1-q)^{n-2m}\langle -n;q\rangle_{2m}(-1)^m}{\langle 1;q\rangle_n\langle k;q\rangle_{2m}}\mathrm{QE}\left(-\binom{2m}{2}+2mn\right),\end{aligned} \tag{12.77}$$

where $k\in\{1,2\}$.

$$\begin{aligned}&\mathrm{E}_q(x)_7\phi_6\left[\begin{matrix}1,6\infty\\ \triangle(q;3;k)\end{matrix}\middle|q;-x^3\right]\\ &=\sum_{n=0}^{\infty}\sum_{m=0}^{[\frac{n}{3}]}\frac{x^n(1-q)^{n-3m}\langle -n;q\rangle_{3m}(-1)^m}{\langle 1;q\rangle_n\langle k;q\rangle_{3m}}\mathrm{QE}\left(-\binom{3m}{2}+3mn\right),\end{aligned} \tag{12.78}$$

where $k\in\{1,2,3\}$.

12.5 Appendix: Discussion

This was the fourth example of a q-Lie group. Previous examples were q-Pascal matrices [5], q-Bernoulli and q-Euler matrices [7] and q-Appell polynomial matrices [8]. Twisted matrix Lie groups are very useful in physics. By the way, the Osler lemma is used by Masjed-Jamei and Koepf [16] to obtain explicit forms of two bivariate power-trigonometric series. The history of generalized hyperbolic functions started with Count Vincenzo Riccati in 1757 [11, p. 394], who in connection with the rectification of a hyperbola briefly considered these functions. Then various authors in Germany, independently of each other [11, p. 395], studied solutions of special cases of the generalized hypergeometric differential equation, often in connection with geometric applications. Olivier, [19] 1827, followed by Glaisher 1872 [10] found special cases of formulas (12.19) and (12.54) for $n = 3$. Olivier studied formula (12.27) and Glaisher studied formulas like (12.15). We summarize the conclusions of Appell, Glaisher 1879 and Nicodemi [11, p. 412]:

Theorem 12.5.1 *There exist extensions of the trigonometric functions, one category each of arbitrary many related functions, which satisfy an addition theorem.*

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Chapter 13
Divisibility in Hom-Algebras, Single-Element Properties in Non-associative Algebras and Twisted Derivations

Germán García Butenegro, Abdennour Kitouni, and Sergei Silvestrov

Abstract We compare and examine the influence of Hom-associativity, involving a linear map twisting the associativity axiom, on fundamental aspects important in study of Hom-algebras and (σ, τ)-derivations satisfying a (σ, τ)-twisted Leibniz product rule in connection to Hom-algebra structures. As divisibility may be not transitive in general not necessarily associative algebras, we explore factorization properties of elements in Hom-associative algebras, specially related to zero divisors, and develop an α-deformed divisibility sequence, formulated in terms of linear operators. We explore effects of the twisting maps σ and τ on the whole space of twisted derivations, unfold some partial results on the structure of (σ, τ)-derivations on arbitrary algebras based on a pivot element related to σ and τ and examine how general an algebra can be while preserving certain well-known relations between (σ, τ)-derivations. Furthermore, new more general axioms of Hom-associativity, Hom-alternativity and Hom-flexibility modulo kernel of a derivation are introduced leading to new classes of Hom-algebras motivated by (σ, τ)-Leibniz rule over multiplicative maps σ and τ and study of twisted derivations in arbitrary algebras and their connections to Hom-algebra structures.

Keywords Hom-algebra · Divisor · Twisted derivation

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13.1 Introduction, Definitions and Notations

In the past 20 years, the area of Hom-algebra structures initiated in 2003 in [4] become popular expanding research direction with Hom-algebra structures increasingly interlacing many areas of mathematics and mathematical physics. Hom-Lie algebras and more general quasi-Hom-Lie algebras were introduced first in 2003 in [4] in the course of construction of a general method for construction of deformations and discretizations of Lie algebras of vector fields based on twisted derivations satisfying twisted Leibniz rule, initially motivated by the examples of q-deformed Jacobi identities satisfied in q-deformations of Witt and Visaroro and in related q-deformed algebras discovered in 1990'th in string theory, vertex models of conformal field theory, quantum field theory and quantum mechanics, quantum calculus and q-analysis and q-deformed differential calculi and q-deformed homological algebra. In the course of this investigation, also the central extensions and cocycle conditions for general quasi-Hom-Lie algebras and Hom-Lie algebras, generalizing in particular q-deformed Witt and Virasoro algebras, have been first considered in [4, 6] and for graded color quasi-Hom-Lie algebras in [15]. At the same time, in 2004-2005, general quasi-Lie and quasi-Leibniz algebras where introduced in [7] and color quasi-Lie and color quasi-Leibniz algebras where introduced in [8] encompassing within the same algebraic structure along with the Hom-Lie algebras and the quasi-Hom-Lie algebras, also the color Hom-Lie algebras, quasi-Hom-Lie color algebras, quasi-Hom-Lie superalgebras and Hom-Lie superalgebras, and color quasi-Leibniz algebras, quasi-Leibniz superalgebras, quasi-Hom-Leibniz superalgebras and Hom-Leibniz algebras. Also, graded color quasi-Lie algebras of Witt type have been first considered in [14]. Hom-Lie admissible algebras, that is Hom-algebras consisting of an algebra and a linear map (homomorphism of linear space) such that the commutator bilinear product yields Hom-Lie algebra, have been considered first in 2006 in [12], where the Hom-associative algebras and more general G-Hom-associative algebras including the Hom-Vinberg algebras (Hom-left symmetric algebras), Hom-pre-Lie algebras (Hom-right symmetric algebras), and some other new Hom-algebra structures have been introduced and shown to be Hom-Lie admissible, in the sense that the operation of commutator as new product in these Hom-algebras structures yields Hom-Lie algebras. Furthermore, in [12], flexible Hom-algebras and Hom-algebra generalizations of derivations and of adjoint derivations maps have been introduced, and the Hom-Leibniz algebras appeared for the first time, as an important special subclass of quasi-Leibniz algebras introduced in [7] in connection to general quasi-Lie algebras following the standard Loday's conventions for Leibniz algebras (i.e. right Loday algebras) [11]. In [12], moreover the investigation of classification of finite-dimensional Hom-Lie algebras have been initiated with construction of families of the low-dimensional Hom-Lie algebras.

Binary Hom-algebra structures typically involve a bilinear binary operation and one or several linear unary operations twisting the defining identities of the structure in some special nontrivial ways, so that the original untwisted algebraic structures are recovered for the specific twisting linear maps. The Hom-Lie algebras, Hom-Lie superalgebras, Hom-Leibniz algebras and Hom-Leibniz superalgebras with twisting linear map different from the identity map, are rich and complicated algebraic structures with classifications, deformations, representations, morphisms, derivations and homological structures being fundamentally dependent on joint properties of the twisting maps as a unary operations and bilinear binary product intrinsically linked by Hom-Jacobi or Hom-Leibniz identities.

The fundamental basic properties, when extended to Hom-algebras, are modified in interesting ways as the defining identities are twisted by linear maps in Hom-algebras in special ways. The purpose of this work is to compare and examine the influence of associativity and more general Hom-associativity, involving a linear map twisting the associativity axiom, on different aspects of Hom-algebras and twisted derivations satisfying a twisted Leibniz product rule. Divisibility may be not transitive in general algebras lacking associativity. Thus, we explore factorization properties of elements in Hom-associative algebras, specially related to zero divisors, and develop an α-deformed divisibility sequence, formulated in terms of linear operators. Discretizations of derivatives often are (σ, τ)-derivations as the Leibniz product rule is typically twisted by two linear maps σ and τ. Invertibility of map τ in [2] was used as the cornerstone for construction of Hom-Lie algebras from (σ, τ)-derivations with two twisting maps. In this work, we expand on the effects of maps σ and τ on the space of twisted derivatives in general with or without invertibility assumptions about σ and τ. As several structural results on the space of (σ, τ)-derivations rely on associativity in the algebra, we unfold some partial results on the structure of (σ, τ)-derivations on arbitrary algebras based on a pivot element related to σ and τ. We take a reverse approach with respect to [2, 4], and examine how general an algebra $\mathcal{A}$ can be while preserving certain well-known relations between (σ, τ)-derivations. We generalize some results from the article by Larsson, Hartwig and Silvestrov [4] in search for general algebraic requirements.

For associative algebras, the associativity is deeply engrained in the algebraic structure implying in particular many important properties of divisibility relation. One of these properties is transitiveness of divisibility, which for any associative algebra $\mathcal{A}$ can be formulated as $p|q,\ q|s \Rightarrow p|s$ for all $p, q, s \in \mathcal{A}$. We will consider what happens with this and some other basic important properties of the divisibility relation for elements in Hom-algebras, and in particular in Hom-associative algebras where associativity is twisted to Hom-associativity by a linear map. We also study the relation between associativity and twisted derivation operators defined via (σ, τ)-twisted Leibniz product rule $D(fg) = D(f)\tau(g) + \sigma(f)D(g)$. Application of twisted derivations on algebras involves determining submodules of the space of (σ, τ)-derivations over the algebra and hopefully then use them to describe all (σ, τ)-derivations. This problem has intimate relations to existence and structure of divisors, greatest common divisors and related properties of the elements of the algebra. Assuming $(\tau - \sigma)(\mathcal{A}) \subseteq \mathcal{A}$ to have at least one invertible element or division by elements in that set to be possible, one can consider submodules of the form $k\frac{\tau - \sigma}{r}$. This idea can be developed into the systematic method for discovering quasi Hom-Lie algebras structure replacing Lie algebras of vector fields when derivations are discretized by twisted derivations satisfying twisted Liebniz rules. For example this leads to systematic construction of twisted generalizations of Witt and Virasoro algebras arising when discretising derivations of Laurent polynomials and considering quasi Hom-Lie algebras central extensions [2, 4, 9, 10, 13]. One of the interesting relations between D, σ and τ for commutative $\mathcal{A}$, that gives $\mathcal{D}_{\sigma\tau}(\mathcal{A})$ its module structure is $D(f)(\tau-\sigma)(g) = D(g)(\tau-\sigma)(f)$. We establish association and commutation conditions for elements in an algebra so that this relation is preserved around a pivot element $g_0 \in \mathcal{A}$. Authors in [2, 4] use a greatest common divisor (GCD) element of $(\tau-\sigma)(\mathcal{A})$ as pivot. This grants the existence of certain quotient that is in some sense invariant under change of arguments, facilitating the structure of single generated (1-dimensional) module on $\mathcal{A}\cdot D$ on unique factorization domains (UFD). Multiplicativity on σ and τ is intuitive (see, for example, the constructions made in [2]) as it minimizes interaction between σ and τ, but a similar, more general, quadratic relation $c(\sigma(fg)-\sigma(f)\sigma(g)) = c(\tau(fg)-\tau(f)\tau(g))$ for some element c in the center of $\mathcal{A}$ acting as a defect of homomorphism, will help expand this concept. This naive approach will later allow to define a wider, more finely threaded class of twisted derivations.

Throughout this work, $\mathbb{F}$ is a field of characteristic different from 2, and all linear spaces are over a field of characteristic other than 2. An algebra over a field $\mathbb{F}$ is a pair $(\mathcal{A}, *)$ consisting of a vector space $\mathcal{A}$ over $\mathbb{F}$ with a bilinear binary operation $\mu : \mathcal{A}\times\mathcal{A}\to\mathcal{A}, \mu(x, y)\mapsto x * y$. Juxtaposition xy is often used for multiplication for convenience of notations, when it is clear which multiplication it stands for. For any algebra, the left multiplication operator $L_x : \mathcal{A}\to\mathcal{A},\ L_x(y) = xy$ and the right multiplication operator $R_y : \mathcal{A}\to\mathcal{A},\ R_y(x) = xy$ are linear operators.

An algebra $\mathcal{A}$ is *left unital* if there is an element $1_L \in \mathcal{A}$ (left unity) such that $a = 1_L\cdot a$ for all $a\in\mathcal{A}$. An algebra is *right unital* if there is an element $1_R\in\mathcal{A}$ (right unity) such that $a = a\cdot 1_R$ for all $a\in\mathcal{A}$, and *unital* if it is both left and right unital. An algebra $\mathcal{A}$ is called *associative* if $x(yz) = (xy)z$ (associativity) holds for all $x, y, z\in\mathcal{A}$. An algebra is called *non-associative* if $x(yz)\neq(xy)z$ for some elements in the algebra. If $xy = yx$ (commutativity) for all $x, y\in\mathcal{A}$, the algebra is called *commutative*, and it is called *non-commutative* if for some elements $xy\neq yx$. If $xy = -yx$ (skew-symmetry or anti-commutativity) for all $x, y\in\mathcal{A}$, the algebra is called *skew-symmetric* (or *anti-commutative*).

Lie algebras are pairs $(\mathcal{A}, \langle\cdot,\cdot\rangle)$ consisting of a linear space $\mathcal{A}$ and a bilinear mapping $\langle\cdot,\cdot\rangle : \mathcal{A}\times\mathcal{A}\to\mathcal{A}$, satisfying for all $x, y, z\in\mathcal{A}$,

$$\langle x, y\rangle = -\langle y, x\rangle \qquad \textbf{Skew-symmetry}$$

$$\sum_{\circlearrowleft(x,y,z)} \langle x, \langle y, z\rangle\rangle = \langle x, \langle y, z\rangle\rangle + \langle y, \langle z, x\rangle\rangle + \langle z, \langle x, y\rangle\rangle = 0. \qquad \textbf{Jacobi identity}$$

where $\sum_{\circlearrowleft(x,y,z)}$ denotes the summation over cyclic permutations of (x, y, z).

In any algebra $(\mathcal{A}, *)$, the commutator defined by $[x, y] = [x, y]_- = xy - yx$ for any two elements $x, y \in \mathcal{A}$, is a bilinear map $[\cdot, \cdot] : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ defining a new algebra $(\mathcal{A}, [\cdot, \cdot])$ on the same vector space.

Elements of an algebra commute, $xy = yx$, if and only if their commutator is zero, $[x, y] = [x, y]_- = xy - yx = 0$. The center of an algebra $Z(\mathcal{A}) = Z(\mathcal{A}, *) = \{x \in \mathcal{A} \mid \forall y \in \mathcal{A} : xy = yx\}$, consisting of all those elements that commute with any element of an algebra $\mathcal{A}$, is a linear subspace of $\mathcal{A}$.

For any algebra, the commutator is skew-symmetric bilinear map, since $[x, y] = xy - yx = -(yx - xy) = -[y, x]$, and thus the new algebra $(\mathcal{A}, [\cdot, \cdot])$ is always a skew-symmetric algebra. If the algebra $(\mathcal{A}, *)$ is associative, then the new algebra $(\mathcal{A}, [\cdot, \cdot])$, with commutator bracket as multiplication, is a Lie algebra, that is the commutator on associative algebras satisfies not only skew-symmetry, but also the Jacobi identity of Lie algebras. *Lie admissible* algebras are those algebras for which the new algebra with commutator as product is a Lie algebra. So, in particular, all associative algebras are Lie admissible. There are many other classes of algebras which are Lie admissible.

If $[x, [y, z]] \neq [[x, y], z]$ for some elements in an algebra, then the commutator defines a non-associative product. For any elements,

$$\begin{aligned}
&[x, [y, z]] = x[y, z] - [y, z]x = x(yz - zy) - (yz - zy)x = x(yz) - x(zy) - (yz)x + (zy)x, \\
&[[x, y], z] = [x, y]z - z[x, y] = (xy - yx)z - z(xy - yx) = (xy)z - (yx)z - z(xy) + z(yx), \\
&[x, [y, z]] - [[x, y], z] = x(yz) - x(zy) - (yz)x + (zy)x - (xy)z + (yx)z + z(xy) - z(yx) \\
&\quad = x(yz) - (xy)z + (zy)x - z(yx) - x(zy) - (yz)x + (yx)z + z(xy) \\
&\text{(if the product is associative)} \\
&\quad = y(xz) - y(zx) + (zx)y - (xz)y = y[x, z] - [x, z]y = [y, [x, z]].
\end{aligned}$$

Thus, in associative algebras, the commutator is associative if and only if $[\mathcal{A}, \mathcal{A}] \subseteq Z(\mathcal{A})$ where $Z(\mathcal{A}) = Z(\mathcal{A}, *)$ is center of $(\mathcal{A}, *)$. This is the case in nilpotent algebras of degree 3, where $[\mathcal{A}, \mathcal{A}] \subseteq Z(\mathcal{A}) \Rightarrow [[\mathcal{A}, \mathcal{A}], \mathcal{A}] = 0, [\mathcal{A}, [\mathcal{A}, \mathcal{A}]] = 0$.

The associator $[\cdot, \cdot, \cdot]_{as} : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, defined by $[x, y, z]_{as} = x(yz) - (xy)z$, is a trilinear mapping, thus also defining a ternary algebra structure on $\mathcal{A}$. The associator can be expressed using the commutator of the left and right multiplication operators L_x and R_z,

$$[x, y, z]_{as} = x(yz) - (xy)z = L_x(yz) - R_z(xy) = L_x(R_z(y)) - R_z(L_x(y)) = [L_x, R_z](y).$$

Elements associate if their associator is 0. The associative algebras are those algebras in which associator is identically 0 on all elements, or equivalently in which all left and right multiplication operators commute. In an algebra, the *nucleus*, that is the set of elements that associate with all other elements,

$$\begin{aligned}
N(\mathcal{A}) &= \{a \in \mathcal{A} \mid [a, p, q]_{as} = 0, \forall\, p \in A, q \in A\} \\
&= \{a \in \mathcal{A} \mid [a, A, A]_{as} = [A, a, A]_{as} = [A, A, a]_{as} = 0\},
\end{aligned}$$

is an associative subalgebra, and moreover is a maximal associative subalgebra as outside it there are no elements not associating with some elements of the algebra $\mathcal{A}$. In non-associative algebras, the associator map is not identically 0, which makes association relations between elements of $\mathcal{A}$ troublesome to keep track of, specially those elements that do not necessarily associate with the whole $\mathcal{A}$. We introduce thus a weaker, partial nucleus notation to pinpoint those relations.

Definition 13.1 (*Relative nucleus*) For any subsets P, Q of an algebra $\mathcal{A}$, relative nucleus of $\mathcal{A}$ with respect to P and Q is defined as the following set,

$$N(\mathcal{A})|_{P,Q} := \{a \in \mathcal{A} | \, [a, p, q]_{as} = 0, \forall \; p \in P, q \in Q\}.$$

If $P = Q$, then we write $N(\mathcal{A})|_P = N(\mathcal{A})|_{P,Q}$, and if P or Q only have one element, then we identify the set with the element to ease the notation. Association between three elements $a, p, q \in \mathcal{A}$ can be expressed as $a \in N(\mathcal{A})|_{p,q}$. The relative nucleus contains the associative nucleus $N(\mathcal{A})$, since an element associating with all $\mathcal{A}$ naturally associates with any subset of it. The converse is not true in general.

Definition 13.2 ([4], *Definition 14*; [12], *Definition 1.3*) A Hom-Lie algebra is a triple $(\mathcal{A}, \mu, \alpha)$ consisting of a linear space $\mathcal{A}$ together with a bilinear mapping $\langle \cdot, \cdot \rangle_{\mathcal{A}} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, called product (and commonly referred as bracket) and a linear map $\alpha : \mathcal{A} \to \mathcal{A}$ such that for all $x, y, z \in \mathcal{A}$,

Skew-symmetry $\quad \langle x, y \rangle_{\mathcal{A}} = -\langle y, x \rangle_{\mathcal{A}},$

Hom-Jacobi identity $\quad \sum_{\circlearrowleft(x,y,z)} \langle \alpha(x), \langle y, z \rangle_{\mathcal{A}} \rangle_{\mathcal{A}}$

$$= \langle \alpha(x), \langle y, z \rangle \rangle + \langle \alpha(y), \langle z, x \rangle \rangle + \langle \alpha(z), \langle x, y \rangle \rangle = 0,$$

where $\sum_{\circlearrowleft(x,y,z)}$ denotes the summation over cyclic permutations of (x, y, z).

Definition 13.3 ([12], *Definition 1.1*) Hom-associative algebras are defined as triples $(\mathcal{A}, \mu, \alpha)$, where $\mathcal{A}$ is a linear space, $\mu : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is bilinear and $\alpha : \mathcal{A} \to \mathcal{A}$ is a linear map, such that for all $x, y, z \in \mathcal{A}$,

$$\alpha(x)(yz) = (xy)\alpha(z). \qquad \textbf{Hom-associativity}$$

If no confusion arises, we identify this triple with the associated vector space $\mathcal{A}$.

Hom-algebras with twisting map α are called *multiplicative* if α is an algebra homomorphism, that is, $\alpha(xy) = \alpha(x)\alpha(y)$ for all elements in the algebra. The action of linear map α twists fundamental identities of algebras into Hom-algebras. Similarly to the associative case, a twisted version of associator, the *Hom-associator* (α-Hom-associator, α-associator), is defined as $[x, y, z]^{\alpha}_{as}(x, y, z) = \alpha(x)(yz) - (xy)\alpha(z)$. In the special case, when $\alpha = \mathrm{Id}$, the Hom-associator becomes the ordinary associator $[x, y, z]^{\mathrm{Id}}_{as}(x, y, z) = [x, y, z]_{as} = x(yz) - (xy)z$, Hom-associative algebras are associative algebras, and Hom-Lie algebras are Lie algebras.

Leibniz product rule for derivative on an algebra of polynomials, or on the algebra of Laurent polynomials $\mathbb{C}[t, t^{-1}]$, or algebras of differentiable functions is used to define derivation operators on arbitrary algebras.

Definition 13.4 ([1]) A derivation on an algebra is an $\mathbb{F}$-linear operator $D : \mathcal{A} \to \mathcal{A}$ such that for every $f, g \in \mathcal{A}$, the Leibniz product rule holds

$$D(fg) = D(f)g + fD(g). \qquad \textbf{Leibniz product rule}$$

Discretization of derivative typically satisfy twisted Leibniz product rule instead of the Leibniz rule for ordinary derivations. For example, the Jackson q-derivative (Jackson q-difference operator) underlying the foundations of q-analysis,

$$D_q(f)(t) = \begin{cases} \dfrac{f(qt) - f(t)}{qt - t} \text{ and } M_t D_q(f)(t) = \dfrac{f(qt) - f(t)}{q - 1}, & q \neq 1 \\ Df(t) = f'(t) \quad q = 1 \end{cases},$$

acting on the algebra of polynomials $\mathbb{F}[t]$ or Laurent $\mathbb{F}[t, t^{-1}]$ or on some suitable function spaces, satisfies the twisted Leibniz rule,

$$D(fg) = D(f)g + \sigma_q(f)D(g), \text{ for } \sigma_q(f)(t) = f(qt).$$

which can be interpreted as a q-deformation of the Leibniz rule recovered for $q = 1$.

Definition 13.5 ([2], *Definition 1.1*) Let $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ be two linear maps. A (σ, τ)-derivation is a $\mathbb{F}$-linear operator $D : \mathcal{A} \to \mathcal{A}$ satisfying a (σ, τ)-twisted generalized Leibniz product rule

$$D(fg) = D(f)\tau(g) + \sigma(f)D(g). \qquad \sigma, \tau\textbf{-twisted Leibniz product rule}$$

If $\tau = \mathrm{Id}$, D is referred to as σ-derivation.

Jackson q-derivative operator is one of the important examples of the σ-derivations. For more examples of σ-derivations and (σ, τ)-derivations we refer to [2, 4, 9].

Definition 13.6 Let $(\mathcal{A}_1, *_1)$ and $(\mathcal{A}_2, *_2)$ be algebras over field $\mathbb{F}$ and $\sigma, \tau : \mathcal{A}_1 \to \mathcal{A}_2$ be linear maps. A (σ, τ)-derivation from $\mathcal{A}_1$ to $\mathcal{A}_2$ is a linear operator $D : \mathcal{A}_1 \to \mathcal{A}_2$ satisfying a (σ, τ)-twisted generalized Leibniz product rule

$$D(f *_1 g) = D(f) *_2 \tau(g) + \sigma(f) *_2 D(g).$$

Definition 13.7 Let $(\mathcal{A}, *)$ be an algebra over a field $\mathbb{F}$ and S be a linear subspace of $\mathcal{A}$, $\sigma, \tau : S \to \mathcal{A}$ be linear maps. A (σ, τ)-derivation from S to $\mathcal{A}$ is a linear operator $D : S \to \mathcal{A}$ satisfying a (σ, τ)-twisted generalized Leibniz product rule for all $f, g \in S$:

$$D(f * g) = D(f) * \tau(g) + \sigma(f) * D(g).$$

Definition 13.8 A linear map $\alpha : \mathcal{A} \to \mathcal{A}$ is left-invertible if there exists $\beta : \mathcal{A} \to \mathcal{A}$ such that $\beta \circ \alpha = id_{\mathcal{A}}$. Then β is called left inverse of α and denoted α_L^{-1}.

A linear map $\alpha : \mathcal{A} \to \mathcal{A}$ is right-invertible if there exists $\beta : \mathcal{A} \to \mathcal{A}$ such that $\alpha \circ \beta = id_{\mathcal{A}}$. Then β is called right inverse of α and denoted α_R^{-1}.

If $\alpha_L^{-1} = \alpha_R^{-1}$, α is said to be invertible and $\alpha^{-1} := \alpha_L^{-1} = \alpha_R^{-1}$ is called inverse of α.

Product relations within non-commutative algebras require some finesse dealing with. Up next we introduce notation for one-sided factors.

Definition 13.9 Let $p, q \in \mathcal{A}$. If there is a nonzero $k_r \in \mathcal{A}$ such that $p \cdot k_r = q$, then p is a *left divisor* of q, denoted $p \overset{L}{|} q$. If there is a nonzero $k_l \in \mathcal{A}$ such that $k_l \cdot p = q$, then p is a *right divisor* of q, denoted $p \overset{R}{|} q$. If p is both left and right divisor of q, it is a *two-sided divisor* of q, denoted $p|q$.

In commutative algebras, all divisors are two-sided.

Remark 13.1 In non-associative algebras, divisibility is not transitive in general. Considering left divisors, if $p \overset{L}{|} q$ and $q \overset{L}{|} s$, then $s = q \cdot k_s,\ q = p \cdot k_q \Rightarrow s = (p \cdot k_q) \cdot k_s$. This is not necessarily a multiple of p, so p is not necessarily a left divisor of s. If $[p, k_q, k_s]_{as} = 0$, then $s = p \cdot (k_q \cdot k_s)$, and thus $p \overset{L}{|} s$. If $s \neq 0$, then $k_q \cdot k_s \neq 0$. But, if $s = 0$, then $k_q \cdot k_s \neq 0$ cannot be guaranteed even in associative algebras, and hence it is not given that $p \overset{L}{|} 0$. Considering right divisors, if $p \overset{R}{|} q$ and $q \overset{R}{|} s$, then $s = k_s \cdot q,\ q = k_q \cdot p \Rightarrow s = k_s \cdot (k_q \cdot p)$, so p is not necessarily a right divisor of s. If $[k_s, k_q, p]_{as} = 0$, then $s = (k_s \cdot k_q) \cdot p$ and thus $p \overset{R}{|} s$. If $s \neq 0$, then $k_s \cdot k_q \neq 0$. But, if $s = 0$, then $k_s \cdot k_q \neq 0$ cannot be guaranteed even in associative algebras, and hence it is not given that $p \overset{R}{|} 0$. This behaviour is particularly interesting in Hom-associative algebras (see Lemmas 13.2 and 13.3).

An element $p \in \mathcal{A}$ is a *left zero divisor* in $\mathcal{A}$ if there is a nonzero element $k_0 \in \mathcal{A}$ such that $p \cdot k_0 = 0$, it is a *right zero divisor* in $\mathcal{A}$ if there is a nonzero element $k_0 \in \mathcal{A}$ such that $k_0 \cdot p = 0$ and it is a *zero divisor* in $\mathcal{A}$ if it is either a left or right zero divisor or both. A zero divisor in a (Hom-)algebra $\mathcal{A}$ is nullified by a subspace S of $\mathcal{A}$. We can see p as zero divisor of S, or as zero divisor of each element of S. Additionally, zero division is a relation between two elements: every left (resp.right) zero divisor p is nullified by an element q which is a right (resp.left) zero divisor nullified by p. The equality $p \cdot q = 0$ can be read as: *p is a left zero divisor of q and q is a right zero divisor of p.* In order to express these relations we group zero divisors into two different categories, attending to whether they divide single elements of $\mathcal{A}$ or an entire subspace.

Definition 13.10 For a subset S of $\mathcal{A}$, let

$Ann^L_{\mathcal{A}}(S) = \{p \in \mathcal{A} \mid \forall\, k_0 \in S : k_0 \neq 0,\ p \cdot k_0 = 0\}$	**Left annihilator**
$Ann^R_{\mathcal{A}}(S) = \{p \in \mathcal{A} \mid \forall\, k_0 \in S : k_0 \neq 0,\ k_0 \cdot p = 0\}$	**Right annihilator**
$\mathcal{L}_0(S) = \{p \in \mathcal{A} \mid \exists\, k_0 \in S : k_0 \neq 0,\ p \cdot k_0 = 0\}$	**Set of left zero divisors**
$\mathcal{R}_0(S) = \{p \in \mathcal{A} \mid \exists\, k_0 \in S : k_0 \neq 0,\ k_0 \cdot p = 0\}$	**Set of right zero divisors**
$\mathcal{L}_0(S) \cup \mathcal{R}_0(S)$	**Set of zero divisors**
$\mathcal{L}_0(S) \cap \mathcal{R}_0(S)$	**Set of two-sided zero divisors.**

If $Ann^L_{\mathcal{A}}(S) = Ann^R_{\mathcal{A}}(S)$, we denote it $Ann_{\mathcal{A}}(S)$.

Remark 13.2 The annihilators of subsets are linear spaces in any algebra. The sets of zero divisors, however, in general are not linear spaces. The sets of left and right zero divisors coincide for subsets in commutative algebras, while in non-commutative algebras they might differ.

Definition 13.11 An element $p \in \mathcal{A}$ of an algebra $\mathcal{A}$ is *left-regular* if it is not in $\mathcal{L}_0(\mathcal{A})$, it is *right-regular* if it is not in $\mathcal{R}_0(\mathcal{A})$ and it is *regular* if it is either left or right-regular or both.

Definition 13.12 (*GCD and one-sided GCD in algebras*) An element $r \in \mathcal{A}$ of an algebra $\mathcal{A}$ is a *left greatest common divisor* of a subset S, if

$$r \overset{L}{|} s \text{ for } s \in S,$$

$$q \overset{L}{|} s \text{ for } s \in S \Rightarrow q \overset{L}{|} r.$$

An element $r \in \mathcal{A}$ is a *right greatest common divisor* of a subset S, if

$$r \overset{R}{|} s \text{ for } s \in S,$$

$$q \overset{R}{|} s \text{ for } s \in S \Rightarrow q \overset{R}{|} r.$$

The set of all left greatest common divisors of a subset S is denoted by $gcd_L(S)$, and the set of all right greatest common divisors of a subset S is denoted by $gcd_R(S)$. Those elements that are both right and left greatest common divisors of a subset S are called *two-sided greatest common divisors*, and $gcd(S) = gcd_L(S) \bigcap gcd_R(S)$ is the set of all greatest common divisors of a subset S.

Definition 13.13 An algebra A over a field that is non-associative and has no zero divisors is known as non-associative domain.

Domains are usually defined as commutative associative rings, because that allows intuitive definitions of factorization and ideals. Non-commutative rings can be concretized into one-sided division rings (see for example [5]) using the proper construction of ideals.

Remark 13.3 Note that domain is used indistinctly for both rings and algebras. In both cases, domain property is a property of the ring of vectors.

Definition 13.14 A *GCD* domain is an associative ring without zero divisors (domain) such that every two elements admit a two-sided greatest common divisor.

13.2 Twisted Derivations. Review of Some Well-Known Results

Proposition 13.1 ([2], Lemma 1.4) *Let $\mathcal{A}$ be a UFD, c an element in the center of $\mathcal{A}$ and σ, τ two linear maps on $\mathcal{A}$. Then, $D : \mathcal{A} \longrightarrow \mathcal{A},\ f \mapsto D(f) = c(\tau(f) - \sigma(f))$ is a (σ, τ)-derivation.*

Proposition 13.2 *Let $\mathcal{A}$ be an algebra, $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ be linear maps, $D : \mathcal{A} \to \mathcal{A}$ be a (σ, τ)-derivation and $\alpha : \mathcal{A} \to \mathcal{A}$ be a homomorphism. Then,*

(i) *$\alpha \circ D$ is a $(\alpha \circ \sigma, \alpha \circ \tau)$-derivation.*
(ii) *$D \circ \alpha$ is a $(\sigma \circ \alpha, \tau \circ \alpha)$-derivation.*
(iii) *If σ is left-invertible, then $\sigma_L^{-1} \circ D$ is a $(id, \sigma_L^{-1} \circ \tau)$-derivation. If $\mathcal{A}$ is commutative, it is also a $(\sigma_L^{-1} \circ \tau)$-derivation.*
(iv) *If τ is left-invertible, then $\tau_L^{-1} \circ D$ is a $(\tau_L^{-1} \circ \sigma)$-derivation.*
(v) *If σ is a right-invertible homomorphism, then $D \circ \sigma_R^{-1}$ is a $(id, \tau \circ \sigma_R^{-1})$-derivation. If $\mathcal{A}$ is commutative, then $D \circ \sigma_R^{-1}$ is a $(\tau \circ \sigma_R^{-1})$-derivation.*
(vi) *If τ is a right-invertible homomorphism, then $D \circ \tau_R^{-1}$ is a $(\sigma \circ \tau_R^{-1})$-derivation.*

When $\sigma(x)a = a\sigma(x)$ for all $x, a \in \mathcal{A}$ (The same applies to τ, and $\mathcal{A}$ being commutative is a particular case), $\mathcal{D}_{\sigma,\tau}(\mathcal{A})$ carries a natural $\mathcal{A}$-module structure defined by $(\mathcal{A}, D) \mapsto a \cdot D,\ x \mapsto aD(x)$ This allows to give a very concrete structure to the space of derivations. In the more general case, $\mathcal{D}_{\sigma,\tau}(\mathcal{A})$ is considered as a vector space instead.

Lemma 13.1 *Let $\mathcal{A}$ be a commutative algebra. If linear maps $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ satisfy*

$$c(\sigma(fg) - \sigma(f)\sigma(g)) = c(\tau(fg) - \tau(f)\tau(g))$$

for some $c \in Z(\mathcal{A})$, and $D : \mathcal{A} \to \mathcal{A}$ is a (σ, τ)-derivation on $\mathcal{A}$, then, the equality $D(x)(\tau(y) - \sigma(y)) = 0$ holds for all $x \in ker(\tau - \sigma), y \in \mathcal{A}$. Moreover, if $\mathcal{A}$ has no zero divisors and $\sigma \neq \tau$, then $ker(\tau - \sigma) \subseteq ker(D)$.

The following proposition is a generalization of [2, Proposition 2.8].

Proposition 13.3 *Let $\mathcal{A}$ be a GCD domain. Let $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ be two different linear maps such that $\sigma \circ \tau = \tau \circ \sigma$. Also, let $r \in gcd((\tau - \sigma)(\mathcal{A}))$ such that, for all $f \in \mathcal{A}$, $\sigma(rf) = \sigma(r)\sigma(f)$, $\tau(rf) = \tau(r)\tau(f)$ and $D = \dfrac{\tau - \sigma}{r}$. Then $D(\sigma(f)) = S\sigma(D(f))$ and $D(\tau(f)) = T\tau(D(f))$ hold for $S = \dfrac{\sigma(r)}{r}$, $T = \dfrac{\tau(r)}{r}$ and all $f \in \mathcal{A}$.*

Proof Let $y \in \mathcal{A}$. Since τ and σ commute,

$$\sigma(r\frac{\tau(y)-\sigma(y)}{r}) = \sigma(\tau(y)-\sigma(y)) = \sigma(\tau(y)) - \sigma^2(y) = (\tau-\sigma)(\sigma(y)),$$
$$\tau(r\frac{\tau(y)-\sigma(y)}{r}) = \tau(\tau(y)-\sigma(y)) = \tau^2(y) - \tau(\sigma(y)) = (\tau-\sigma)(\tau(y))$$

Since $r|(\tau(r)-\sigma(r))$, if it divides $\tau(r)$ then it divides $\sigma(r)$ and viceversa. Assume that r divides $\tau(r)$ and divide both expressions above by r:

$$\frac{\sigma(r)}{r}\sigma(\frac{\tau-\sigma}{r}(y)) = \frac{\tau-\sigma}{r}(\sigma(y)) \Longleftrightarrow \frac{\sigma(r)}{r}\sigma(D(y)) = D(\sigma(y))$$
$$\frac{\tau(r)}{r}\tau(\frac{\tau-\sigma}{r}(y)) = \frac{\tau-\sigma}{r}(\tau(y)) \Longleftrightarrow \frac{\tau(r)}{r}\tau(D(y)) = D(\tau(y)).$$

□

Theorem 13.1 ([2], Theorem 2.6) *Let $\mathcal{A}$ be a commutative associative unital algebra and let D be a (σ,τ)-derivation of $\mathcal{A}$ with algebra morphisms $\sigma,\tau:\mathcal{A}\to\mathcal{A}$, satisfying, for all $f\in\mathcal{A}$:*

$$(\sigma\circ\tau)(f) = (\tau\circ\sigma)(f),$$
$$(D\circ\sigma)(f) = \delta(\sigma\circ D)(f),$$
$$(D\circ\tau)(f) = \delta(\tau\circ D)(f),$$

with $\delta\in\mathcal{A}$. The bracket $\langle\cdot,\cdot\rangle_{\sigma,\tau}:\mathcal{A}\cdot D\times\mathcal{A}\cdot D\to\mathcal{A}\cdot D$ defined by

$$\langle f\cdot D, g\cdot D\rangle_{\sigma,\tau} = (\sigma(f)D(g)-\sigma(g)D(f))\cdot D$$

endows the linear space $(\mathcal{A}\cdot D,\langle\cdot,\cdot\rangle_{\sigma,\tau},\overline{\sigma+\tau})$ with a structure of Hom-Lie algebra.

The following is a generalization of [4, Theorem 4].

Proposition 13.4 *Let $\mathcal{A}$ be a commutative associative UFD, $\sigma,\tau:\mathcal{A}\to\mathcal{A}$ different linear maps on $\mathcal{A}$ satisfying*

$$c(\sigma(fg)-\sigma(f)\sigma(g)) = c(\tau(fg)-\tau(f)\tau(g)) \tag{13.1}$$

for some c in the center of $\mathcal{A}$. Then $\mathcal{D}_{\sigma,\tau}(\mathcal{A})$ is free of rank 1 as an $\mathcal{A}$-module, and is generated by the following operator:

$$\Delta := c\frac{(\tau-\sigma)}{r} : x\longmapsto c\frac{(\tau-\sigma)(x)}{r},$$

where $r = gcd((\tau-\sigma)(\mathcal{A}))$.

Proof Firstly, we check that $cD = c\dfrac{\tau - \sigma}{r}$ is a (σ, τ)-derivation by property (13.1):

$$\begin{aligned} cD(fg) &= \frac{c(\tau-\sigma)(fg)}{r} = \frac{c(\tau(fg)-\sigma(fg))}{r} \overset{(13.1)}{=} \frac{c(\tau(f)\tau(g)-\sigma(f)\sigma(g))}{r} = \\ &= \frac{c((\tau(f)-\sigma(f))\tau(g)-\sigma(f)(\tau(g)-\sigma(g)))}{r} = c\frac{(\tau-\sigma)(f)}{r}\sigma(g)+ \\ &+ c\tau(f)\frac{(\tau-\sigma)(f)}{r} = (cD(f)\tau(g)+\sigma(f)cD(g)). \end{aligned}$$

If cD is a (σ, τ)-derivation over a commutative, associative and algebra $\mathcal{A}$ with unit then D is as well as long as c is invertible.

Secondly, assume $x \cdot c\dfrac{\tau - \sigma}{r} = 0$. If $\sigma \neq \tau$, there is $y \in \mathcal{A}$ such that

$$\sigma(y) \neq \tau(y) \to x \cdot c\frac{(\tau-\sigma)(y)}{r} = 0 \Rightarrow x = 0 \Rightarrow \mathcal{A} \cdot cD$$

is a free $\mathcal{A}$-module. Since cD is the only generator, it has rank 1.

Lastly, one must show that $D_{\sigma,\tau}(\mathcal{A}) \subseteq \mathcal{A} \cdot cD$. Indeed, let Δ be a (σ, τ)-derivation in $\mathcal{A}$, and let $a_\Delta \in \mathcal{A}$ such that $\Delta(f) = a_\Delta c\dfrac{(\tau-\sigma)(f)}{r}$, $f \in \mathcal{A}$. Two conditions must be satisfied:

(i) $(\tau - \sigma)(f)$ divides $cD(f)r$.
(ii) $\dfrac{cD(f)r}{c(\tau-\sigma)(f)} = \dfrac{cD(g)r}{c(\tau-\sigma)(g)}$ for $(\tau - \sigma)(f) \neq 0 \neq (\tau - \sigma)(g)$.

For the first part, let $f, g \in \mathcal{A}$ with $(\tau - \sigma)(f) \neq 0 \neq (\tau - \sigma)(g)$. We know the following:

$$\begin{aligned} 0 &= \Delta(fg - gf) = \Delta(f)\tau(g) + \sigma(f)\Delta(g) - \Delta(g)\tau(f) - \sigma(g)\Delta(f) = \\ &= \Delta(f)(\tau(g)-\sigma(g)) - \Delta(g)(\tau(f)-\sigma(f)) \Rightarrow \Delta(f)(\tau(g)-\sigma(g)) = \\ &= \Delta(g)(\tau(f)-\sigma(f)) \end{aligned}$$

Now we define the following function:

$$h : \mathcal{A} \times \mathcal{A} \to \mathcal{A}, \quad (z, w) \longmapsto gcd(\tau(z) - \sigma(z), \tau(w) - \sigma(w))$$

By election of f and g, $h(f, g) \neq 0$, so we can divide by $h(f, g)$ in the equation above:

$$\Delta(f)\frac{\tau(g)-\sigma(g)}{h(f,g)} = \Delta(g)\frac{\tau(f)-\sigma(f)}{h(f,g)} \Rightarrow$$

Now, it $gcd\left(\frac{\tau(g)-\sigma(g)}{h(f,g)}, \frac{\tau(f)-\sigma(f)}{h(f,g)}\right) = 1$, and so $\frac{\tau(f)-\sigma(f)}{h(f,g)}$ divides $\Delta(f) \Rightarrow (\tau-\sigma)(f)|cD(f)\cdot h(f,g)$.

Finally, let $S = \mathcal{A}\backslash ker(\tau-\sigma)$. Then $(\tau-\sigma)(f)|cD(f)\cdot h(f,S)$. But

$$\begin{aligned} gcd(h(f,S)) &= gcd(\{gcd((\tau-\sigma)(f),(\tau-\sigma)(s)|s\in S\}) = \\ &= gcd((\tau-\sigma)(S)\cup\{(\tau-\sigma)(f))\}) = \\ &= gcd((\tau-\sigma)(\mathcal{A})\cup\{(\tau-\sigma)(f)\}) = r \Rightarrow (\tau-\sigma)(f)|cD(f)r. \end{aligned}$$

In order to prove the second point we recall

$$0 = \Delta(fg-gf) = \Delta(f)(\tau(g)-\sigma(g)) - \Delta(g)(\tau(f)-\sigma(f)).$$

From which it is immediate that

$$\Delta(f)\frac{c(\tau(g)-\sigma(g))}{r} = \Delta(g)\frac{c(\tau(f)-\sigma(f))}{r} \Rightarrow \frac{cD(f)r}{c(\tau(f)-\sigma(f))} = \frac{cD(g)r}{c(\tau(g)-\sigma(g))},$$

as sought. Note that $\mathcal{A}$ being a UFD allows to drop the c if it is invertible, and therefore the result holds for D as well. □

13.3 Divisibility in Hom-Associative Algebras

Divisibility between three nonzero elements p, q, s of an arbitrary algebra $\mathcal{A}$ is, in general, not transitive. In particular, it only seems to be the case in general, when $p \in Z(\mathcal{A})$ - that is, divisibility in non-associative algebras requires considering to a certain extent associating elements in this regard.

The Hom-associative case is slightly more complex due to the action of the linear map α involved in the Hom-associativity axiom. For non-injective twisting maps α, some $p, p' \in \mathcal{A}$ satisfy $\alpha(p) = \alpha(p')$, in which case $\alpha(p)(qr) = (pq)\alpha(r)$, but also $\alpha(p)(qr) = (p'q)\alpha(r)$—this indicates a degree of non-uniqueness between triple products. Hom-associativity brings some interesting identities, the first of which is deeply linked to $\ker\alpha$.

Proposition 13.5 *Let $(\mathcal{A}, \mu, \alpha)$ be a Hom-associative algebra. Let $p, p' \in \mathcal{A}$ such that $\alpha(p) = \alpha(p')$. Then, for all $q, r \in \mathcal{A}$:*

$$((p-p')q)\alpha(r) = 0, \tag{13.2}$$

$$\alpha(r)(q(p-p')) = 0. \tag{13.3}$$

Proof From $\alpha(p) = \alpha(p')$, we apply Hom-associativity and obtain

$$0 = \alpha(p - p') \cdot (q \cdot r) = ((p - p') \cdot q) \cdot \alpha(r),$$
$$0 = (r \cdot q) \cdot \alpha(p - p') = \alpha(r) \cdot (q \cdot (p - p')).$$

□

If $\mathcal{A}$ is left-unital, replacing q with a left unity 1_L in (13.3) yields $\alpha(r)(p - p') = 0 \Rightarrow p - p' \in Ann^R_{\mathcal{A}}(\alpha(\mathcal{A}))$. If $\mathcal{A}$ is right-unital, replacing q with a right unity 1_R in (13.2) yields $(p - p')\alpha(r) = 0 \Rightarrow p - p' \in Ann^L_{\mathcal{A}}(\alpha(\mathcal{A}))$. If $\mathcal{A}$ is commutative, left and right annihilators coincide, which yields $p - p' \in Ann_{\mathcal{A}}(\alpha(\mathcal{A}))$.

Proposition 13.6 *Let $(\mathcal{A}, \mu, \alpha)$ be a Hom-associative algebra. Then for all $p \in \ker \alpha$ and $q, r \in \mathcal{A}$,*

$$(pq)\alpha(r) = 0, \quad \alpha(r)(qp) = 0.$$

Proof By Hom-associativity,

$$(pq)\alpha(r) = \alpha(p)(qr) = 0, \quad \alpha(r)(qp) = (rq)\alpha(p) = 0.$$

□

We present both left and right formulations for all results to account for non-commutative Hom-algebras properly. We examine Hom-association relations between elements of $\mathcal{A}$ to establish divisibility, and in particular, describe families of factors for certain elements or subspaces of $\mathcal{A}$.

Lemma 13.2 *Let $(\mathcal{A}, \mu, \alpha)$ be a Hom-associative algebra, $0 \neq p, q, s \in \mathcal{A}$ such that $p \overset{L}{|} q$ and $q \overset{L}{|} s$, that is, $q = p \cdot k_q$ and $s = q \cdot k_s$ for some nonzero $k_q, k_s \in \mathcal{A}$. If $k_s \in Im(\alpha)$, then $\alpha(p) \overset{L}{|} s$.*

Proof From the divisibility properties, we have $s = (p \cdot k_q) \cdot k_s$. If $k_s \in Im(\alpha)$, that is, there is $r \in \mathcal{A} \setminus \ker \alpha$ such that $k_s = \alpha(r)$, then $s = (p \cdot k_q) \cdot \alpha(r) = \alpha(p) \cdot (k_q \cdot r) \Rightarrow \alpha(p) \overset{L}{|} s$. □

Lemma 13.3 *Let $(\mathcal{A}, \mu, \alpha)$ be a Hom-associative algebra, $0 \neq p, q, s \in \mathcal{A}$ such that $p \overset{R}{|} q$ and $q \overset{R}{|} s$, that is, $q = k_q \cdot p$ and $s = k_s \cdot q$ for some nonzero $k_q, k_s \in \mathcal{A}$. If $k_s \in Im(\alpha)$, then $\alpha(p) \overset{R}{|} s$.*

Proof From the divisibility properties, we have $s = k_s \cdot (k_q \cdot p)$. If $k_s \in Im(\alpha)$, that is, there is $r \in \mathcal{A} \setminus \ker \alpha$ such that $k_s = \alpha(r)$, then $s = \alpha(r) \cdot (k_q \cdot p) = (r \cdot k_q) \cdot \alpha(p) \Rightarrow \alpha(p) \overset{R}{|} s$. □

The identity $s = \alpha(p) \cdot (k_q \cdot r)$ is heavily dependent on s. Particularly, if s is nonzero that forces $\alpha(p) \neq 0$ and $k_q \cdot r \neq 0$.

The first condition is equivalent to $p \notin \ker \alpha$. For the second, we consider the fact that k_q is not necessarily known. This is not a problem given that it is in the center of the Hom-associator, that is, it doesn't change after Hom-associativity. Hence, we impose the non-zero-product condition on r as follows: r cannot be a zero divisor of any nonzero $k_q \in \mathcal{A}$.

Proposition 13.7 *Let $(\mathcal{A}, \mu, \alpha)$ be a Hom-associative algebra. For all $q \in \mathcal{A}$, $p \in \mathcal{A} \setminus \ker \alpha$, the following statements hold.*

$$\text{For all right-regular } r \in \mathcal{A}, \text{ if } p \overset{L}{|} q, \text{ then } \alpha(p) \overset{L}{|} q\alpha(r). \tag{13.4}$$

$$\text{For all left-regular } r \in \mathcal{A}, \text{ if } p \overset{R}{|} q, \text{ then } \alpha(p) \overset{R}{|} \alpha(r)q. \tag{13.5}$$

Proof From $p \overset{L}{|} q$ we have $q = p \cdot k_q \Rightarrow q \cdot \alpha(r) = (p \cdot k_q) \cdot \alpha(r) = \alpha(p) \cdot (k_q \cdot r)$.

From $p \overset{R}{|} q$ we have $q = k_q \cdot p \Rightarrow \alpha(r) \cdot q = \alpha(r) \cdot (k_q \cdot p) = (r \cdot k_q) \cdot \alpha(p)$. □

Theorem 13.2 *Let $(\mathcal{A}, \mu, \alpha)$ be a Hom-associative algebra. The following statements hold for all nonzero $q \in \mathcal{A}$.*

$$\text{For all } p \in \mathcal{L}_0(\mathcal{A}), \text{ if } q \in Ann^R_{\mathcal{A}}(p) \text{ then } q\mathcal{A} \subseteq Ann^R_{\mathcal{A}}(\alpha(p)),$$
$$\text{For all } p \in \mathcal{R}_0(\mathcal{A}), \text{ if } q \in Ann^L_{\mathcal{A}}(p) \text{ then } \mathcal{A}q \subseteq Ann^L_{\mathcal{A}}(\alpha(p)).$$

Proof Let $q \in \mathcal{A}$ be an element nullified by p.

If $p \in Ann^L_{\mathcal{A}}(q)$, then $p \cdot q = 0$. By Hom-associativity, $\alpha(p) \cdot (q \cdot r) = 0$ for all $r \in \mathcal{A} \Rightarrow q \cdot r \in Ann^R_{\mathcal{A}}(\alpha(p))$ for all $r \in \mathcal{A} \Rightarrow q \cdot \mathcal{A} \subseteq Ann^R_{\mathcal{A}}(\alpha(p))$.

If $p \in Ann^R_{\mathcal{A}}(q)$, then $q \cdot p = 0$. By Hom-associativity, $(r \cdot q) \cdot \alpha(p) = 0$ for all $r \in \mathcal{A} \Rightarrow r \cdot q \in Ann^L_{\mathcal{A}}(\alpha(p))$ for all $r \in \mathcal{A} \Rightarrow \mathcal{A} \cdot q \subseteq Ann^L_{\mathcal{A}}(\alpha(p))$. □

Remark 13.4 Regularity on r is not required here, as $\alpha(p) \cdot (q \cdot r) = 0$. The zero on the right-hand side allows $q \cdot r$ to be zero in exchange of not being able to establish divisibility. If $\mathcal{A}$ has no zero divisors, then all elements are regular, that is, $Ann_{\mathcal{A}}(p) = \{0\}$ for all $p \in \mathcal{A}$ and Theorem 13.3 holds trivially.

Theorem 13.3 *Let $(\mathcal{A}, \mu, \alpha)$ be a Hom-associative algebra. If $\mathcal{A}$ has a right-regular element, then the set $\mathcal{L}_0(\mathcal{A})$ is invariant under α, that is $\alpha(\mathcal{L}_0(\mathcal{A})) \subseteq \mathcal{L}_0(\mathcal{A})$. If $\mathcal{A}$ has a left-regular element, then the set $\mathcal{R}_0(\mathcal{A})$ is invariant under α, that is $\alpha(\mathcal{R}_0(\mathcal{A})) \subseteq \mathcal{R}_0(\mathcal{A})$. If $\mathcal{L}_0(\mathcal{A}) = \mathcal{R}_0(\mathcal{A})$ (for example if $\mathcal{A}$ is commutative) then $\mathcal{L}_0(\mathcal{A}) = \mathcal{R}_0(\mathcal{A})$ is invariant under α, that is $p|0 \Rightarrow \alpha(p)|0$.*

Proof Let p be a zero divisor on $\mathcal{A}$, that is, there is $q \in \mathcal{A}$ such that either $p \cdot q = 0$ (left zero divisor) or $q \cdot p = 0$ (right zero divisor). If $p \in \mathcal{L}_0(\mathcal{A})$, that is, $p \overset{L}{|} 0$, then $p \cdot q = 0$. By Proposition 13.7, for right-regular $r \in \mathcal{A}$,

$$\alpha(p) \cdot (q \cdot r) = (p \cdot q) \cdot \alpha(r) = 0 \Rightarrow \alpha(p) \overset{L}{|} 0.$$

If $p \in \mathcal{R}_0(\mathcal{A})$, that is, $p \overset{R}{|} 0$, then $q \cdot p = 0$. By Proposition 13.7, for left-regular $r \in \mathcal{A}$,

$$(r \cdot q) \cdot \alpha(p) = \alpha(r) \cdot (q \cdot p) = 0 \Rightarrow \alpha(p) \overset{R}{|} 0.$$

If $\mathcal{A}$ is commutative, then these two conclusions are equivalent, and hence $\mathcal{L}_0(\mathcal{A}) = \mathcal{R}_0(\mathcal{A})$ is invariant under α, that is, $p|0 \Rightarrow \alpha(p)|0$. □

The reciprocal of this statement is naturally interesting. It takes, however, much stronger hypotheses to realize.

Proposition 13.8 *If $(\mathcal{A}, \mu, \alpha)$ is a Hom-associative algebra with α bijective and multiplicative, then the following statements hold:*

(i) *If $\mathcal{A}$ has a right-regular element, then $p \overset{L}{|} 0 \Leftrightarrow \alpha(p) \overset{L}{|} 0$ for all $p \in \mathcal{L}_0(\mathcal{A})$.*
(ii) *If $\mathcal{A}$ has a left-regular element, then $p \overset{R}{|} 0 \Leftrightarrow \alpha(p) \overset{R}{|} 0$ for all $p \in \mathcal{R}_0(\mathcal{A})$.*

Proof (i) Right implication is immediate by Theorem 13.3: if $\alpha(\mathcal{L}_0(\mathcal{A})) \subseteq \mathcal{L}_0(\mathcal{A})$, then for all $p \in \mathcal{L}_0(\mathcal{A})$ it holds that $\alpha(p) \in \mathcal{L}_0(\mathcal{A})$, that is $p \overset{L}{|} 0 \Rightarrow \alpha(p) \overset{L}{|} 0$. In order to prove the left implication, we use the properties of α in sequence:

$$0 = \alpha(p) \cdot k_0 \overset{surj}{=} \alpha(p) \cdot \alpha(\alpha^{-1}(k_0)) \overset{mult}{=} \alpha(p \cdot \alpha^{-1}(k_0)) \overset{inj}{\Rightarrow} p \cdot \alpha^{-1}(k_0) = 0.$$

It follows $\alpha(p) \overset{L}{|} 0 \Rightarrow p \overset{L}{|} 0$.
(ii) The right implication is immediate by Theorem 13.3: if $\alpha(\mathcal{R}_0(\mathcal{A})) \subseteq \mathcal{R}_0(\mathcal{A})$, then for all $p \in \mathcal{R}_0(\mathcal{A})$ it holds that $\alpha(p) \in \mathcal{R}_0(\mathcal{A})$, that is $p \overset{R}{|} 0 \Rightarrow \alpha(p) \overset{R}{|} 0$. In order to prove the left implication, we use the properties of α in sequence:

$$0 = k_0 \cdot \alpha(p) \overset{surj}{=} \alpha(\alpha^{-1}(k_0)) \cdot \alpha(p) \overset{mult}{=} \alpha(\alpha^{-1}(k_0) \cdot p) \overset{inj}{\Rightarrow} \alpha^{-1}(k_0) \cdot p = 0.$$

It follows $\alpha(p) \overset{R}{|} 0 \Rightarrow p \overset{R}{|} 0$. This completes the proof. □

Remark 13.5 If α is surjective and multiplicative but not injective, then $p \cdot \alpha^{-1}(k_0)$ under conditions of (i) (resp. $\alpha^{-1}(k_0) \cdot p$ under conditions of (ii)) is in $\ker \alpha$ for all elements in the preimage $\alpha^{-1}(k_0)$.

Divisibility by elements in $\alpha(\mathcal{A})$ is naturally essential in Hom-associative algebras, as they are involved in the fundamental identity. We introduce the following divisibility sequence that extends the traditional idea of *mutliplying elements consecutively to obtain the next term of the sequence* involving elements in $\alpha(\mathcal{A})$.

Proposition 13.9 *Let* $(\mathcal{A}, \mu, \alpha)$ *be a Hom-associative algebra. If* $0 \neq p_1, \ldots, p_n \in \mathcal{A}$ *are such that*

$$\alpha(p_{i-1}) \overset{L}{|} p_i, \text{ that is, } p_i = \alpha(p_{i-1}) \cdot k_{i-1} \text{ for all } i = 2, \ldots, n \text{ and } k_i \in \mathcal{A},$$

then, the elements p_i *can be expressed as*

$$p_i = (R_{k_{i-1}} \circ \alpha \circ \cdots \circ R_{k_1} \circ \alpha)(p_1).$$

Moreover, if α *is multiplicative they can be expressed as*

$$p_i = (R_{k_{i-1}} \circ \cdots \circ R_{\alpha^{i-j-1}(k_j)} \circ \cdots \circ R_{\alpha^{i-2}(k_1)})(\alpha^{i-1}(p_1)),$$

where $R_a : p \mapsto p \cdot a$ *is the right multiplication map.*

Proof We proceed by induction on n. On the trivial sequence, when $n = 2$, we have $p_2 = \alpha(p_1) \cdot k_1 = R_{k_1}(\alpha^1(p_1))$.

For the induction step, we use index m. In the non-multiplicative case,

$$\begin{aligned} p_{m+1} &= \alpha(p_m) \cdot k_m = (R_{k_m} \cdot \alpha)(p_m) = (R_{k_m} \circ \alpha)(R_{k_{m-1}} \circ \alpha \circ \cdots \circ R_{k_1} \cdot \alpha)(p_1) \\ &= (R_{k_m} \circ \alpha \circ \cdots \circ R_{k_1} \circ \alpha)(p_1). \end{aligned}$$

In the multiplicative case,

$$\begin{aligned} &p_{m+1} = \alpha(p_m) \cdot k_m = \alpha((R_{k_{m-1}} \circ \cdots \circ R_{\alpha^{m-j-1}(k_j)} \circ \cdots \circ R_{\alpha^{m-2}(k_1)})(\alpha^{m-1}(p_1)) \cdot k_m \\ &= R_{k_m}(\alpha((R_{k_{m-1}} \circ \cdots \circ R_{\alpha^{m-j-1}(k_j)} \circ \cdots \circ R_{\alpha^{m-2}(k_1)})(\alpha^{m-1}(p_1)))) \\ &\text{(using multiplicativity of } \alpha) \\ &= R_{k_m}((R_{\alpha(k_{m-1})} \circ \cdots \circ R_{\alpha^{m+1-j-1}(k_j)} \circ \cdots \circ R_{\alpha^{m+1-2}(k_1)})(\alpha^{m+1-1}(p_1))) \\ &= (R_{k_{m+1-1}} \circ R_{\alpha(k_{m+1-2})} \circ \cdots \circ R_{\alpha^{m+1-j-1}(k_j)} \circ \cdots \circ R_{\alpha^{m+1-2}(k_1)})(\alpha^{m+1-1}(p_1)). \end{aligned}$$

□

Proposition 13.10 *Let* $(\mathcal{A}, \mu, \alpha)$ *be a Hom-associative algebra with* α *multiplicative. Also, let* $0 \neq p_1, \ldots, p_n \in \mathcal{A}$ *be such that*

$$\alpha(p_{i-1}) \overset{R}{|} p_i, \text{ that is, } p_i = k_{i-1} \cdot \alpha(p_{i-1}) \text{ for all } i = 2, \ldots, n \text{ and } k_i \in \mathcal{A}.$$

Then, the elements p_i *can be expressed as*

$$p_i = (L_{k_{i-1}} \circ \alpha \circ \cdots \circ L_{k_1} \circ \alpha)(p_1). \tag{13.6}$$

Moreover, if α *is multiplicative they can be expressed as*

$$p_i = (L_{k_{i-1}} \circ \cdots \circ L_{\alpha^{i-j-1}(k_j)} \circ \cdots \circ L_{\alpha^{i-2}(k_1)})(\alpha^{i-1}(p_1)), \tag{13.7}$$

where $L_a : p \mapsto a \cdot p$ is the right multiplication map.

Proof We proceed by induction on n. If $n = 2$, then $p_2 = k_1 \cdot \alpha(p_1) = L_{k_1}(\alpha^1(p_1))$ in both cases.

For the induction step, we use index m. In the non-multiplicative case,

$$\begin{aligned} p_{m+1} &= k_m \cdot \alpha(p_m) = (L_{k_m} \cdot \alpha)(p_m) = (L_{k_m} \circ \alpha)(L_{k_{m-1}} \circ \alpha \circ \cdots \circ L_{k_1} \cdot \alpha)(p_1) \\ &= (L_{k_m} \circ \alpha \circ \cdots \circ L_{k_1} \circ \alpha)(p_1). \end{aligned}$$

In the multiplicative case,

$$\begin{aligned} &p_{m+1} = k_m \cdot \alpha(p_m) = k_m \cdot \alpha((L_{k_{m-1}} \circ \cdots \circ L_{\alpha^{m-j-1}(k_j)} \circ \cdots \circ L_{\alpha^{m-2}(k_1)})(\alpha^{m-1}(p_1)) \\ &= L_{k_m}(\alpha((L_{k_{m-1}} \circ \cdots \circ L_{\alpha^{m-j-1}(k_j)} \circ \cdots \circ L_{\alpha^{m-2}(k_1)})(\alpha^{m-1}(p_1)))) \\ &\text{(by multiplicativity of } \alpha) \\ &= L_{k_m}((L_{\alpha(k_{m-1})} \circ \cdots \circ L_{\alpha^{m+1-j-1}(k_j)} \circ \cdots \circ L_{\alpha^{m+1-2}(k_1)})(\alpha^{m+1-1}(p_1))) \\ &= (L_{k_{m+1-1}} \circ L_{\alpha(k_{m+1-2})} \circ \cdots \circ L_{\alpha^{m+1-j-1}(k_j)} \circ \cdots \circ L_{\alpha^{m+1-2}(k_1)})(\alpha^{m+1-1}(p_1)). \end{aligned}$$

□

13.3.1 Divisibility in Unital Hom-Associative Algebras

Consider now $(\mathcal{A}, \mu, \alpha)$ to be Hom-associative and one-sided unital, with the usual unitality conditions $q = 1_L \cdot q$ or $q = q \cdot 1_R$ for all $q \in \mathcal{A}$. These hard units offer a certain array of limitations in the Hom-associative setting, but they offset that with strong divisibility relations.

Corollary 13.1 *Let $(\mathcal{A}, \mu, \alpha)$ be a Hom-associative left-unital algebra with unity 1_L. For all $q \in \mathcal{A} \setminus \ker \alpha$ and all right-regular $r \in \mathcal{A}$ the following statements hold:*

(i) $\alpha(1_L) \overset{L}{|} q\alpha(r)$,

(ii) $\alpha(1_L) \overset{L}{|} \alpha(r)$,

(iii) $q \overset{L}{|} 1_L \Rightarrow \alpha(q) \overset{L}{|} \alpha(r)$.

If α is multiplicative then, for all $p \in \mathcal{A} \setminus \ker \alpha$, it holds that $p \overset{L}{|} q \Rightarrow \alpha(p) \overset{L}{|} \alpha(q)$.

Proof This corollary comes from applying Proposition 13.7, replacing (p, q, r) in (13.4) by different triples of elements:

(i) Replace (p, q, r) by $(1_L, q, r)$.

(ii) Replace (p, q, r) by $(1_L, 1_L, r)$. Here $\alpha(1_L) \overset{L}{|} (1_L \cdot \alpha(r)) = \alpha(r)$.

(iii) Replace (p, q, r) by $(q, 1_L, r)$. It follows that $\alpha(q) \overset{L}{|} (1_L \cdot \alpha(r)) = \alpha(r)$.
The last property is an immediate consequence of multiplicativity of α:

$$p \overset{L}{|} q \Rightarrow q = p \cdot k_q \Rightarrow \alpha(q) = \alpha(p \cdot k_q) = \alpha(p) \cdot \alpha(k_q) \Rightarrow \alpha(p) \overset{L}{|} \alpha(q).$$

□

Corollary 13.2 *Let* $(\mathcal{A}, \mu, \alpha)$ *be a Hom-associative right-unital algebra with unity* 1_R. *For all* $q \in \mathcal{A} \setminus \ker \alpha$ *and all left-regular* $r \in \mathcal{A}$ *the following statements hold:*

(i) $\alpha(1_R) \overset{R}{|} \alpha(r)q$,

(ii) $\alpha(1_R) \overset{R}{|} \alpha(r)$,

(iii) $q \overset{R}{|} 1_R \Rightarrow \alpha(q) \overset{R}{|} \alpha(r)$.

If α *is multiplicative then, for all* $p \in \mathcal{A} \setminus \ker \alpha$, *it holds that* $p \overset{R}{|} q \Rightarrow \alpha(p) \overset{R}{|} \alpha(q)$.

Proof This corollary comes from applying Proposition 13.7, replacing (p, q, r) in (13.5) by different triples of elements:
(i) Replace (p, q, r) by $(1_R, q, r)$.
(ii) Replace (p, q, r) by $(1_R, 1_R, r)$. Here $\alpha(1_R) \overset{R}{|} (\alpha(r) \cdot 1_R) = \alpha(r)$.
(iii) Replace (p, q, r) by $(q, 1_R, r)$. It follows that $\alpha(q) \overset{R}{|} (\alpha(r) \cdot 1_R) = \alpha(r)$.
The last property is an immediate consequence of multiplicativity of α:

$$p \overset{r}{|} q \Rightarrow q = k_q \cdot p \Rightarrow \alpha(q) = \alpha(k_q \cdot p) = \alpha(k_q) \cdot \alpha(p) \Rightarrow \alpha(p) \overset{R}{|} \alpha(q).$$

□

Remark 13.6 These results appear to mark a steep increase in zero divisors of $\alpha(1)$. Applying Hom-associativity to the triple $(1_L, 1_L, q)$ we obtain

$$(1_L \cdot 1_L) \cdot \alpha(q) = \alpha(1_L) \cdot (1_L \cdot q) \Rightarrow \alpha(q) = \alpha(1_L) \cdot q. \tag{13.8}$$

If $\ker \alpha$ is non-trivial, then all elements of it satisfy $\alpha(1_L) \cdot q = 0$, hence $\alpha(1_L) \overset{L}{|} 0$.

13.3.2 Divisibility in Hom-Unital Hom-Associative Algebras

A unital Hom-associative algebra $(\mathcal{A}, \mu, \alpha, 1)$ with α injective is associative as an algebra $(\mathcal{A}, \mu, 1)$ [3]. Under the action of α, hard unities verify the twisted unitality condition (13.8). We observe that $\alpha(1_L)$ acts as a *Hom*-unity of $(\mathcal{A}, \mu, \alpha)$. A Hom-algebra with such an element is (one-sided) *Hom-unital*.

Corollary 13.3 *Let* $(\mathcal{A}, \mu, \alpha)$ *be a Hom-associative left-Hom-unital algebra with Hom-unity* $1_L \in \mathcal{A} \setminus \ker \alpha$. *For all* $q \in \mathcal{A} \setminus \ker \alpha$ *and all right-regular* $r \in \mathcal{A}$,

(i) $1_L \overset{L}{|} q \Rightarrow \alpha(1_L) \overset{L}{|} q\alpha(r).$

(ii) $q \overset{L}{|} 1_L \Rightarrow \alpha(q) \overset{L}{|} 1_L\alpha(r).$

(iii) $q \overset{L}{|} 1_L \Rightarrow \alpha(q) \overset{L}{|} \alpha^2(r).$

If α is multiplicative, then

(iv) $q \overset{L}{|} 1_L \Rightarrow \alpha(q) \overset{L}{|} \alpha(1_L)\alpha(r),$

(v) $1_L \overset{L}{|} q \Rightarrow \alpha(1_L) \overset{L}{|} \alpha(kr),$

where $k \in \mathcal{A}$ such that $q = 1_L k = \alpha(k)$.

Proof This corollary comes from applying Proposition 13.7, replacing (p, q, r) in (13.4) by different triples of elements:

(i) Replace (p, q, r) by $(1_L, q, r)$.

(ii) Replace (p, q, r) by $(q, 1_L, r)$. Here $\alpha(q) \overset{L}{|} 1_L \cdot \alpha(r)$.

(iii) Property (iii) follows from (ii) since$1_L \cdot \alpha(r) = \alpha(r)$.

(iv) Property (iv) follows from (iii). Using multiplicativity of α it follows that $\alpha^2(r) = \alpha(1_L)\alpha(r) \Rightarrow \alpha(q) \overset{L}{|} \alpha(1_L) \cdot \alpha(r)$.

(v) Property (v) follows from $1_L \overset{L}{|} q \Rightarrow q = \alpha(k)$ for some k. Applying (i) we obtain $\alpha(1_L) \overset{L}{|} (\alpha(k) \cdot \alpha(r)) = \alpha(k \cdot r)$. □

Corollary 13.4 *Let $(\mathcal{A}, \mu, \alpha)$ be a Hom-associative right-Hom-unital algebra with right Hom-unity 1_R. For all $q \in \mathcal{A} \setminus \ker \alpha$ and all left-regular $r \in \mathcal{A}$,*

(i) $1_R \overset{R}{|} q \Rightarrow \alpha(1_R) \overset{L}{|} \alpha(r)q.$

(ii) $q \overset{R}{|} 1_R \Rightarrow \alpha(q) \overset{R}{|} \alpha(r)1_R.$

(iii) $q \overset{R}{|} 1_R \Rightarrow \alpha(q) \overset{R}{|} \alpha^2(r).$

If α is multiplicative, then

(iv) $q \overset{R}{|} 1_R \Rightarrow \alpha(q) \overset{R}{|} \alpha(r)\alpha(1_R),$

(v) $1_R \overset{R}{|} q \Rightarrow \alpha(1_R) \overset{R}{|} \alpha(rk),$

where $k \in \mathcal{A}$ such that $q = k1_R = \alpha(k)$.

Proof This corollary comes from applying Proposition 13.7, replacing (p, q, r) in (13.5) by different triples of elements:

(i) Replace (p, q, r) by $(1_R, q, r)$.

(ii) Replace (p, q, r) by $(q, 1_R, r)$. Here $\alpha(q) \overset{R}{|} \alpha(r) \cdot 1_R$.

(iii) Property (iii) follows from (ii) since $\alpha(r) \cdot 1_R = \alpha^2(r)$.

(iv) Property (iv) follows from (iii). Using multiplicativity of α it follows that $\alpha^2(r) = \alpha(r)\alpha(1_R) \Rightarrow \alpha(q) \overset{R}{|} \alpha(r) \cdot \alpha(1_R)$.

(v) Property (v) follows from $1_R \overset{R}{|} q \Rightarrow q = \alpha(k)$ for some k. Applying (i) we obtain $\alpha(1_R) \overset{R}{|} (\alpha(r) \cdot \alpha(k)) = \alpha(r \cdot k)$. □

13.4 Sandwich Twisted Derivations and Pivot Commutation

Across this section we explore (σ, τ)-derivations composed with pairs of linear maps. Composition with two linear maps can be concretized into multiplication by two elements of the algebra in more forgiving (particularly, commutative associative unital) algebras. In [2], Theorem 1.7, the authors establish that on the proper circumstances, every (σ, τ)-derivation is $\tau - \sigma$ multiplied with two elements of the algebra.

We explore relations (element-wise) between two maps σ, τ and a corresponding twisted derivation D. Some common results can be generalized under partial commutation with a single element. We call that element pivot and denote it g_0.

13.4.1 Approach to Relations

Building (σ, τ)-derivations based on algebra homomorphisms is the first strong property that one may consider. Lemma 1.4 of [4] establishes a strong connection between all three maps, in exchange of strong association and commutation. These conditions can be relaxed considering a softer quadratic relation between σ and τ up to an element $c \in \mathcal{A}$.

The following proposition is an extension of Proposition 13.1. It expands on properties of element c, replacing it by a more general element of $\mathcal{A}$.

Proposition 13.11 *Let $\mathcal{A}$ be an algebra, $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ be two linear maps and $c \in \mathcal{A}$ or $c \in \mathbb{F}$. Consider the following properties for some $f, g \in \mathcal{A}$:*

1) $c((\tau - \sigma)(f)\tau(g)) - (c(\tau - \sigma)(f))\tau(g)$
$= \sigma(f)(c(\tau - \sigma)(g)) - c(\sigma(f)(\tau - \sigma)(g))$
2) $c(\sigma(fg) - \sigma(f)\sigma(g)) = c(\tau(fg) - \tau(f)\tau(g))$.
3) $D : \mathcal{A} \to \mathcal{A}, f \mapsto D(f) = c(\tau - \sigma)(f)$ *is a (σ, τ)-derivation in the sense of Definition 13.7.*

If any of these properties hold, the other two are equivalent.

Proof We show $1 \wedge 2 \Rightarrow 3$, $1 \wedge 3 \Rightarrow 1$ and $2 \wedge 3 \Rightarrow 1$. The equivalences come from commutativity on $\wedge$ and the logical axiom $P \wedge Q \to R \Leftrightarrow P \to (Q \to R)$.

Firstly, let $D(f \cdot g) = c \cdot (\tau - \sigma)(fg)$. By definition, $(\tau - \sigma)(f \cdot g) = \tau(f \cdot g) - \sigma(f \cdot g)$, and thus

$$\begin{aligned}D(f\cdot g)&=c\cdot(\tau(f\cdot g)-\sigma(f\cdot g))\overset{2}{=}c\cdot(\tau(f)\cdot\tau(g)-\sigma(f)\cdot\sigma(g))\\&=c\cdot(\tau(f)\cdot\tau(g)-\sigma(f)\cdot\tau(g)+\sigma(f)\cdot\tau(g)-\sigma(f)\cdot\sigma(g))\\&=c\cdot((\tau-\sigma)(f)\cdot\tau(g))+c\cdot(\sigma(f)\cdot(\tau-\sigma)(g))\\&\overset{1}{=}(c\cdot(\tau-\sigma)(f))\cdot\tau(g)+\sigma(f)\cdot(c\cdot(\tau-\sigma)(g)).\end{aligned}$$

This proves $1\wedge 2\Rightarrow 3$.

In order to prove $1\wedge 3\Rightarrow 2$, we expand 3:

$$\begin{aligned}c\cdot(\tau-\sigma)(f\cdot g)&=(c\cdot(\tau-\sigma)(f))\cdot\tau(g)+\sigma(f)\cdot(c\cdot(\tau-\sigma)(g))\\&\overset{1}{=}c\cdot((\tau-\sigma)(f)\cdot\tau(g))+c\cdot(\sigma(f)\cdot(\tau-\sigma)(g))\\&=c\cdot(\tau(f)\cdot\tau(g)-\sigma(f)\cdot\tau(g)+\sigma(f)\cdot\tau(g)-\sigma(f)\cdot\sigma(g))\\&=c\cdot(\tau(f)\cdot\tau(g)-\sigma(f)\cdot\sigma(g)).\end{aligned}$$

Finally, we prove $2\wedge 3\Rightarrow 1$ by expanding the twisted Leibniz rule in 3:

$$\begin{aligned}(c\cdot(\tau-\sigma)(f))\cdot\tau(g)+\sigma(f)\cdot(c\cdot(\tau-\sigma)(g))&=c\cdot(\tau-\sigma)(fg)\\&\overset{2}{=}c\cdot(\tau(f)\cdot\tau(g)-\sigma(f)\cdot\sigma(g))\\&=c\cdot(\tau(f)\cdot\tau(g)-\sigma(f)\cdot\tau(g)+\sigma(f)\cdot\tau(g)-\sigma(f)\cdot\sigma(g))\\&=c\cdot((\tau-\sigma)(f)\cdot\tau(g))+c\cdot(\sigma(f)\cdot(\tau-\sigma)(g)),\end{aligned}$$

and by rearranging terms, 1 is reached.

Finally, from $1\wedge 2\Rightarrow 3$ and $1\wedge 3\Rightarrow 2$ we obtain $1\Rightarrow(2\Leftrightarrow 3)$ and if we add $2\wedge 3\Rightarrow 1$ both $2\Rightarrow(1\Leftrightarrow 3)$ and $3\Rightarrow(1\Leftrightarrow 2)$ hold true as potentially more convenient reformulations.

If $c\in\mathbb{F}$ (i.e. it is a scalar), $c\cdot(\sigma(f)\cdot(\tau-\sigma)(g))=\sigma(f)\cdot(c\cdot(\tau-\sigma)(g))$ comes by definition and thus this Proposition reduces to $2\Leftrightarrow 3$. □

Proposition 13.11 has several immediate ramifications in different algebras.

If $\mathcal{A}$ is associative, any element c commuting with elements of the form $\sigma(f)$ satisfies statement 1 immediately.

Corollary 13.5 *Let $\mathcal{A}$ be an associative algebra, $c\in\mathcal{A}$ such that $c\sigma(f)=\sigma(f)c$ for all $f\in\mathcal{A}$. Then, $D:f\to c(\tau-\sigma)(f)$ is a (σ,τ)-derivation over $\mathcal{A}$ if and only if, for all $f,g\in\mathcal{A}$, $c(\tau(fg)-\sigma(fg))=c(\tau(f)\tau(g)-\sigma(f)\sigma(g))$.*

If $\mathcal{A}$ is left-unital, statement 1 is immediately fulfilled by choosing $c=1_L$.

Corollary 13.6 *Let $\mathcal{A}$ be a left-unital algebra, $\sigma,\tau:\mathcal{A}\to\mathcal{A}$ two linear maps. Then $D:f\to(\tau-\sigma)(f)$ is a (σ,τ)-derivation over $\mathcal{A}$ if and only if, for all $f,g\in\mathcal{A}$, $\tau(fg)-\sigma(fg)=\tau(f)\tau(g)-\sigma(f)\sigma(g)$.*

This statement can be extended to arbitrary algebras by using a scalar c. Then, $2\Leftrightarrow 3$ in Proposition 13.11 becomes another generalization of [4, Lemma 1.4].

Corollary 13.7 *Let $\mathcal{A}$ be an algebra and $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ be two linear maps. The linear map $D : f \longmapsto (\tau - \sigma)(f)$ is a (σ, τ)-derivation of $\mathcal{A}$ iff, for all $f, g \in \mathcal{A}$, $\sigma(fg) - \sigma(f)\sigma(g) = \tau(fg) - \tau(f)\tau(g)$*

Remark 13.7 The reasoning used in unital algebras, using a unity to grant Property 1 in Proposition 13.11, does not yield the same results in Hom-unital algebras. Let 1_L be a left Hom-unity, that is, $\alpha(f) = 1_L f$ for all $f \in \mathcal{A}$. Let σ, τ be such linear maps that Property 1 holds, that is,

$$1_L \cdot ((\tau - \sigma)(f) \cdot \tau(g)) - (1_L \cdot (\tau - \sigma)(f)) \cdot \tau(g) \\ = \sigma(f) \cdot (1_L \cdot (\tau - \sigma)(g)) - 1_L \cdot (\sigma(f) \cdot (\tau - \sigma)(g)).$$

Using Hom-unitality we can express this condition as

$$\alpha((\tau - \sigma)(f) \cdot \tau(g)) - \alpha((\tau - \sigma)(f)) \cdot \tau(g) \\ = \sigma(f) \cdot \alpha((\tau - \sigma)(g)) - \alpha(\sigma(f) \cdot (\tau - \sigma)(g)),$$

or in terms of operators,

$$(\alpha \circ R_{\tau(g)} \circ (\tau - \sigma))(f) - (R_{\tau(g)} \circ \alpha \circ (\tau - \sigma))(f) \\ = (L_{\sigma(f)} \circ \alpha \circ (\tau - \sigma))(g) - (\alpha \circ L_{\sigma(f)} \circ (\tau - \sigma))(g).$$

For such α and such elements, $\alpha \circ (\tau - \sigma)$ is a (σ, τ)-derivation in the sense of Definition 13.7 if and only if $\alpha(\sigma(fg) - \sigma(f)\sigma(g)) = \alpha(\tau(fg) - \tau(f)\tau(g))$ by Proposition 13.11. In particular, this holds for those $f, g \in \mathcal{A}$ such that $L_{\sigma(f)}$ and $R_{\tau(g)}$ commute with α. This set of elements is interesting to consider in different algebras - it is a linear subspace, since Property 1 is linear in f and g. We study this phenomenon in Proposition 13.12.

These relations bound algebras and some subalgebras of more general algebras in which $\tau - \sigma$ plays an important role, up to an element $c \in \mathcal{A}$. Traditional papers relate derivations in the form $k(\tau - \sigma)$, where $k \in \mathcal{A}$ has certain form which is related to division and other factorization properties - and in certain settings it can be tracked to a product involving images by D and $\tau - \sigma$ of an element g_0. Said g_0 is chosen by the authors in [2] to be a GCD of $(\tau - \sigma)(\mathcal{A})$ in order to ensure that the quotient $\frac{(\tau-\sigma)(f)}{g_0}$ exists for all $f \in \mathcal{A}$. We follow that line, singling out minimal properties of g_0 in order to establish how deeply we can relate (σ, τ)-derivation operators based on the relations between them, σ, τ and g_0.

Lemma 13.4 *Let $\mathcal{A}$ be an algebra. Let $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ be two different linear maps, $D : \mathcal{A} \to \mathcal{A}$ be a (σ, τ)-derivation and $g_0 \in \mathcal{A}$. For all $f \in \mathcal{A}$ satisfying*

$$D([f, g_0]) = D(fg_0 - g_0 f) = 0, \tag{13.9}$$

$$[\sigma(g_0), D(f)] = 0, \tag{13.10}$$

it holds that

$$D(f)(\tau-\sigma)(g_0) = D(g_0)\tau(f) - \sigma(f)D(g_0). \tag{13.11}$$

For those f *that, in addition* (13.9) *and* (13.10)*, verify*

$$[\sigma(f), D(g_0)] = 0, \tag{13.12}$$

it holds that

$$D(f)(\tau-\sigma)(g_0) = D(g_0)(\tau-\sigma)(f). \tag{13.13}$$

For those f *that, in addition to* (13.9) *and* (13.10)*, verify*

$$[\tau(f), D(g_0)] = 0, \tag{13.14}$$

it holds that

$$D(f)(\tau-\sigma)(g_0) = (\tau-\sigma)(f)D(g_0). \tag{13.15}$$

Proof By applying (σ, τ)-Leibniz rule to the elements $D(fg_0)$ and $D(g_0 f)$ and subtracting the results we obtain:

$$\begin{aligned}
D(f\cdot g_0) &= D(f)\cdot\tau(g_0) + \sigma(f)\cdot D(g_0),\\
D(g_0\cdot f) &= D(g_0)\cdot\tau(f) + \sigma(g_0)\cdot D(f),\\
D(f\cdot g_0) - D(g_0\cdot f) &= D(f)\cdot\tau(g_0) - \sigma(g_0)\cdot D(f) + \sigma(f)\cdot D(g_0) - D(g_0)\cdot\tau(f)
\end{aligned}$$

which is (13.11) since $D(f\cdot g_0) - D(g_0\cdot f) = 0$. Applying (13.12) to this equality yields

$$0 = D(f\cdot g_0) - D(g_0\cdot f) \underset{(13.10)}{\overset{(13.12)}{=}} D(f)\cdot(\tau-\sigma)(g_0) - D(g_0)\cdot(\tau-\sigma)(f).$$

Similarly, applying (13.14) instead, yields (13.15) as follows

$$0 = D(f\cdot g_0) - D(g_0\cdot f) \underset{(13.10)}{\overset{(13.14)}{=}} D(f)\cdot(\tau-\sigma)(g_0) - (\tau-\sigma)(f)\cdot D(g_0).$$

□

Lemma 13.5 *Let* $\mathcal{A}$ *be an algebra. Let* $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ *be two different linear maps,* $D : \mathcal{A} \to \mathcal{A}$ *be a* (σ, τ)*-derivation and* $g_0 \in \mathcal{A}$*. For all* $f \in \mathcal{A}$ *satisfying*

$$D([f, g_0]) = D(fg_0 - g_0 f) = 0, \tag{13.16}$$

$$[\tau(g_0), D(f)] = 0, \tag{13.17}$$

it holds that

$$(\tau-\sigma)(g_0)D(f) = D(g_0)\tau(f) - \sigma(f)D(g_0). \tag{13.18}$$

If, in addition to (13.16) *and* (13.17)*,*

$$[\tau(f), D(g_0)] = 0, \tag{13.19}$$

then

$$(\tau - \sigma)(g_0)D(f) = (\tau - \sigma)(f)D(g_0). \tag{13.20}$$

Proof By applying (σ, τ)-Leibniz rule to the elements $D(fg_0)$ and $D(g_0f)$ and subtracting the results we obtain:

$$\begin{aligned}
D(f \cdot g_0) &= D(f) \cdot \tau(g_0) + \sigma(f) \cdot D(g_0),\\
D(g_0 \cdot f) &= D(g_0) \cdot \tau(f) + \sigma(g_0) \cdot D(f),\\
D(f \cdot g_0) - D(g_0 \cdot f) &= D(f) \cdot \tau(g_0) - \sigma(g_0) \cdot D(f) + \sigma(f) \cdot D(g_0) - D(g_0) \cdot \tau(f),
\end{aligned}$$

which is (13.18) since $D(f \cdot g_0) - D(g_0 \cdot f) = 0$. Applying (13.19) to this equality gives

$$0 = D(f \cdot g_0) - D(g_0 \cdot f) \underset{(13.17)}{\overset{(13.19)}{=}} (\tau - \sigma)(g_0) \cdot D(f) - (\tau - \sigma)(f) \cdot D(g_0).$$

□

Remark 13.8 Note that in commutative rings (thus, algebras) (13.13) and (13.20) are immediate, as all commutation conditions are trivially satisfied.

This lemma suggests relations between the images of σ and τ have more importance than they are usually given credit for. Existence of such a pivot element g_0 is without a doubt an important question - commutative algebras always have one (in fact, every element doubles as g_0), while in the non-commutative case we are tied to the two relations (13.10) and (13.12) that link the images of D and σ together with $[\mathcal{A}, g_0] \in \ker D$.

13.4.2 Sandwich Twisted Derivatives in Unital Algebras

One-sided unital algebras satisfy certain relations on (σ, τ)-derivation operators immediately. Existence of unities allows to define the corresponding one-sided inverse of elements in the algebra, and give these operators much more structure in exchange for tighter association relations involving a pivot element g_0 with invertible image by $\tau - \sigma$.

Lemma 13.6 *Let $\mathcal{A}$ be a right-unital algebra. Let $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ be two different linear maps, and $D : \mathcal{A} \to \mathcal{A}$ be a (σ, τ)-derivation. Let $g_0 \in \mathcal{A}$ be such that $(\tau - \sigma)(g_0)$ is right-invertible, with right-inverse $(\tau - \sigma)(g_0)_R^{-1}$. For all $f \in \mathcal{A}$ such that*

$$[D(f), (\tau - \sigma)(g_0), (\tau - \sigma)(g_0)_R^{-1}]_{as} = 0, \tag{13.21}$$

the action of D can be expressed as

$$D(f) = (D(g_0)\tau(f) - \sigma(f)D(g_0))(\tau - \sigma)(g_0)_R^{-1}.$$

For all $f \in \mathcal{A}$ that also satisfy (13.12), that is $[\sigma(f), D(g_0)] = 0$, the action of D can be expressed as

$$D(f) = (D(g_0)(\tau - \sigma)(f))(\tau - \sigma)(g_0)_R^{-1}.$$

For all $f \in \mathcal{A}$ that also satisfy (13.19), that is $[\tau(f), D(g_0)] = 0$, the action of D can be expressed as

$$D(f) = ((\tau - \sigma)(f)D(g_0))(\tau - \sigma)(g_0)_R^{-1}.$$

Proof If $(\tau - \sigma)(g_0)$ is right-invertible, then multiplying both sides of (13.11) with $(\tau - \sigma)(g_0)_R^{-1}$ yields

$$\begin{aligned}
&D(f) \cdot (\tau - \sigma)(g_0) = D(g_0) \cdot \tau(f) - \sigma(f) \cdot D(g_0) \\
&\Rightarrow (D(f) \cdot (\tau - \sigma)(g_0)) \cdot (\tau - \sigma)(g_0)_R^{-1} \\
&\qquad\qquad = (D(g_0) \cdot \tau(f) - \sigma(f) \cdot D(g_0)) \cdot (\tau - \sigma)(g_0)_R^{-1} \\
&\overset{13.21}{\Rightarrow} D(f) \cdot ((\tau - \sigma)(g_0) \cdot (\tau - \sigma)(g_0)_R^{-1}) \\
&\qquad\qquad = (D(g_0) \cdot \tau(f) - \sigma(f) \cdot D(g_0)) \cdot (\tau - \sigma)(g_0)_R^{-1} \\
&\Rightarrow D(f) = D(f) \cdot 1_R = (D(g_0) \cdot \tau(f) - \sigma(f) \cdot D(g_0)) \cdot (\tau - \sigma)(g_0)_R^{-1},
\end{aligned}$$

where 1_R is a right-unity of $\mathcal{A}$. If (13.12) holds, then

$$\begin{aligned}
&D(g_0) \cdot \tau(f) - \sigma(f) \cdot D(g_0) = D(g_0) \cdot (\tau - \sigma)(f) \\
&\quad \Rightarrow D(f) = (D(g_0) \cdot (\tau - \sigma)(f)) \cdot (\tau - \sigma)(g_0)_R^{-1}
\end{aligned}$$

and result follows. Similarly, if (13.19) holds, then

$$\begin{aligned}
&D(g_0) \cdot \tau(f) - \sigma(f) \cdot D(g_0) = (\tau - \sigma)(f) \cdot D(g_0) \\
&\quad \Rightarrow D(f) = ((\tau - \sigma)(f) \cdot D(g_0)) \cdot (\tau - \sigma)(g_0)_R^{-1}.
\end{aligned}$$

□

Corollary 13.8 *Let $\mathcal{A}$ be a right-unital associative algebra. Let $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ be two different linear maps, and $D : \mathcal{A} \to \mathcal{A}$ be a (σ, τ)-derivation. Let $g_0 \in \mathcal{A}$ be such that $(\tau - \sigma)(g_0)$ is right-invertible, with right-inverse $(\tau - \sigma)(g_0)_R^{-1}$. Then, for all $f \in \mathcal{A}$,*

$$D(f) = (D(g_0)\tau(f) - \sigma(f)D(g_0))(\tau - \sigma)(g_0)_R^{-1}.$$

The action of D can also be expressed as

1) $D(f) = D(g_0)(\tau - \sigma)(f)(\tau - \sigma)(g_0)_R^{-1}$ *for such $f \in \mathcal{A}$ that $[\sigma(f), D(g_0)] = 0$,*
2) $D(f) = (\tau - \sigma)(f)D(g_0)(\tau - \sigma)(g_0)_R^{-1}$ *for such $f \in \mathcal{A}$ that $[\tau(f), D(g_0)] = 0$.*

In Lemma 13.7, (13.23) and (13.24) are generalized commutation relations of the form $p(fg) = q(gf)$ for certain elements of $\mathcal{A}$ using two weights $p, q \in \mathcal{A}$. For central elements p and q in the nucleus of $\mathcal{A}$, conditions (13.25), (13.26), (13.27) and (13.28) are trivially satisfied.

Partial associativity is often overlooked, which motivates several traditional results that only work on associative algebras. We use relative nucleus notation (see Definition 13.1) for more precision when discussing commutation and association relations.

Lemma 13.7 *Let $\mathcal{A}$ be an algebra, $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ be two linear maps, $D : \mathcal{A} \to \mathcal{A}$ be (σ, τ)-derivation and $g_0, p, q \in \mathcal{A}$. Then, for all $f \in \mathcal{A}$ such that*

$$D \text{ commutes with operators } L_p \text{ and } L_q, \tag{13.22}$$
$$q(\sigma(g_0)D(f)) = p(D(f)\sigma(g_0)), \tag{13.23}$$
$$p(\sigma(f)D(g_0)) = q(D(g_0)\sigma(f)), \tag{13.24}$$
$$[p, D(f)] = 0, \tag{13.25}$$
$$p \in N(\mathcal{A})|_{D(f),\tau(g_0)} \cap N(\mathcal{A})|_{D(f),\sigma(g_0)}, \tag{13.26}$$
$$[q, D(f)] = 0, \tag{13.27}$$
$$q \in N(\mathcal{A})|_{\sigma(f),D(g_0)} \cap N(\mathcal{A})|_{\tau(f),D(g_0)}, \tag{13.28}$$
$$D(p(fg_0) - q(g_0 f)) = 0, \tag{13.29}$$

it holds that $D(f)(p(\tau - \sigma)(g_0)) = D(g_0)(q(\tau - \sigma)(f))$. If, moreover, $p(\tau - \sigma)(g_0)$ is right-invertible and

$$[D(f), p(\tau - \sigma)(g_0), (p(\tau - \sigma)(g_0))_R^{-1}]_{as} = 0, \tag{13.30}$$

then $D(f) = (D(g_0)(q(\tau - \sigma)(f)))(p(\tau - \sigma)(g_0))_R^{-1}$.

Proof Let $g_0 \in \mathcal{A}$. By conditions (13.22) and (13.29), for $f \in \mathcal{A}$,

$$D(p(f \cdot g_0)) = p \cdot D(fg_0) = p \cdot (D(f) \cdot \tau(g_0)) + p \cdot (\sigma(f) \cdot D(g_0)),$$
$$D(q(g_0 \cdot f)) = q \cdot D(g_0 f) = q \cdot (D(g_0) \cdot \tau(f)) + q \cdot (\sigma(g_0) \cdot D(f)),$$
$$0 = p \cdot (D(f) \cdot \tau(g_0)) - q \cdot (\sigma(g_0) \cdot D(f)) + p \cdot (\sigma(f) \cdot D(g_0)) - q \cdot (D(g_0) \cdot \tau(f)).$$

We use commutation relations (13.23) and (13.24) to set the terms in D as far left as possible,

$$0 = p \cdot (D(f) \cdot \tau(g_0)) - p \cdot (D(f) \cdot \sigma(g_0)) + q \cdot (D(g_0) \cdot \sigma(f)) - q \cdot (D(g_0) \cdot \tau(f)).$$

Now, apply commutation relations (13.25), (13.27) and association relations (13.26) and (13.28):

$$\begin{aligned}
&0 \underset{(13.26)}{\overset{(13.28)}{=}} (p \cdot D(f)) \cdot \tau(g_0) - (p \cdot D(f)) \cdot \sigma(g_0) + (q \cdot D(g_0)) \cdot \sigma(f) - (q \cdot D(g_0)) \cdot \tau(f),\\
&0 = (p \cdot D(f)) \cdot (\tau - \sigma)(g_0) + (q \cdot D(g_0)) \cdot (\sigma - \tau)(f)\\
&\Rightarrow (p \cdot D(f)) \cdot (\tau - \sigma)(g_0) = (q \cdot D(g_0)) \cdot (\tau - \sigma)(f)\\
&\underset{(13.26);(13.28)}{\overset{(13.25);(13.27)}{\Rightarrow}} \quad D(f) \cdot (p \cdot (\tau - \sigma)(g_0)) = D(g_0) \cdot (q \cdot (\tau - \sigma)(f)).
\end{aligned}$$

Finally, if $p(\tau - \sigma)(g_0)$ is right-invertible and condition (13.30) holds, then

$$D(f) = (D(g_0) \cdot (q(\tau - \sigma)(f))) \cdot (p(\tau - \sigma)(g_0))_R^{-1}.$$

□

Remark 13.9 If $p(\tau - \sigma)(g_0)$ is not right-invertible or (13.30) does not hold, the last step of the proof is not granted.

Corollary 13.9 *Let $\mathcal{A}$ be an associative algebra, $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ be two linear maps, $D : \mathcal{A} \to \mathcal{A}$ be (σ, τ)-derivation and $g_0, p, q \in \mathcal{A}$. Then, for all $f \in \mathcal{A}$ such that*

$$\begin{aligned}
&D \text{ commutes with operators } L_p \text{ and } L_q,\\
&q(\sigma(g_0)D(f)) = p(D(f)\sigma(g_0)),\\
&p(\sigma(f)D(g_0)) = q(D(g_0)\sigma(f)),\\
&[p, D(f)] = 0,\\
&[q, D(f)] = 0,\\
&D(p(fg_0) - q(g_0 f)) = 0,
\end{aligned}$$

it holds that $D(f)(p(\tau - \sigma)(g_0)) = D(g_0)(q(\tau - \sigma)(f))$. If, moreover, $p(\tau - \sigma)(g_0)$ is right-invertible with right-inverse $(\tau - \sigma)(g_0)_R^{-1} p_R^{-1}$, then the action of D can be expressed as $D(f) = D(g_0)q(\tau - \sigma)(f)(\tau - \sigma)(g_0)_R^{-1} p_R^{-1}$.

Lemma 13.8 *Let $\mathcal{A}$ be a right-unital non-associative domain. Let $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ be two different linear maps, $D : \mathcal{A} \to \mathcal{A}$ be a (σ, τ)-derivation and $q, p \in \mathcal{A}$. Also, let $g_0, g_1 \in \mathcal{A}$ be two elements such that $p(\tau - \sigma)(g_0)$ and $p(\tau - \sigma)(g_1)$ are right-invertible. Then, for all $f \in \mathcal{A}$ such that the following relations hold:*

$$D(f)(p(\tau-\sigma)(g_0)) = D(g_0)(q(\tau-\sigma)(f)), \tag{13.31}$$

$$[D(g_0), q(\tau-\sigma)(f), (p(\tau-\sigma)(g_0))_R^{-1}]_{as} = 0, \tag{13.32}$$

$$[D(g_0), q(\tau-\sigma)(f)(p(\tau-\sigma)(g_0))_R^{-1}, (\tau-\sigma)(g_1)]_{as} = 0, \tag{13.33}$$

$$[(p(\tau-\sigma)(g_0))_R^{-1}, q(\tau-\sigma)(f), p(\tau-\sigma)(g_1)]_{as} = 0, \tag{13.34}$$

$$[(p(\tau-\sigma)(g_0))_R^{-1}, q(\tau-\sigma)(g_1), q(\tau-\sigma)(f)]_{as} = 0, \tag{13.35}$$

$$[D(g_0), q(\tau-\sigma)(g_1)(p(\tau-\sigma)(g_0))_R^{-1}, q(\tau-\sigma)(f)]_{as} = 0, \tag{13.36}$$

$$[D(g_0), q(\tau-\sigma)(g_1), (p(\tau-\sigma)(g_0))_R^{-1}]_{as} = 0, \tag{13.37}$$

$$[q(\tau-\sigma)(f), (p(\tau-\sigma)(g_0))_R^{-1}] = 0, \tag{13.38}$$

$$(q(\tau-\sigma)(f))(p(\tau-\sigma)(g_1)) = (q(\tau-\sigma)(g_1))(q(\tau-\sigma)(f)), \tag{13.39}$$

it holds that $D(f)(p(\tau-\sigma)(g_1)) = D(g_1)(q(\tau-\sigma)(f))$.

Proof The proof is as follows:

$$
\begin{aligned}
& D(f)\cdot(p\cdot(\tau-\sigma)(g_1)) \\
& \underset{(13.31)}{\overset{(13.32)}{=}} ((D(g_0)\cdot(q\cdot(\tau-\sigma)(f)))\cdot((p\cdot(\tau-\sigma)(g_0))_R^{-1})\cdot(p\cdot(\tau-\sigma)(g_1)) \\
& \overset{(13.32)}{=} (D(g_0)\cdot((q\cdot(\tau-\sigma)(f))\cdot((p\cdot(\tau-\sigma)(g_0))_R^{-1})))\cdot(p\cdot(\tau-\sigma)(g_1)) \\
& \overset{(13.33)}{=} D(g_0)\cdot(((q\cdot(\tau-\sigma)(f))\cdot(p\cdot(\tau-\sigma)(g_0))_R^{-1}))\cdot(p\cdot(\tau-\sigma)(g_1))) \\
& \overset{(13.38)}{=} D(g_0)\cdot(((p\cdot(\tau-\sigma)(g_0))_R^{-1}\cdot(q\cdot(\tau-\sigma)(f)))\cdot(p\cdot(\tau-\sigma)(g_1))) \\
& \overset{(13.34)}{=} D(g_0)\cdot((p\cdot(\tau-\sigma)(g_0))_R^{-1}\cdot((q\cdot(\tau-\sigma)(f))\cdot(p\cdot(\tau-\sigma)(g_1)))) \\
& \overset{(13.39)}{=} D(g_0)\cdot((p\cdot(\tau-\sigma)(g_0))_R^{-1}\cdot((q\cdot(\tau-\sigma)(g_1))\cdot(q\cdot(\tau-\sigma)(f)))) \\
& \overset{(13.35)}{=} D(g_0)\cdot(((p\cdot(\tau-\sigma)(g_0))_R^{-1}\cdot(q\cdot(\tau-\sigma)(g_1)))\cdot(q\cdot(\tau-\sigma)(f))) \\
& \overset{(13.38)}{=} D(g_0)\cdot(((q\cdot(\tau-\sigma)(g_1))\cdot(p\cdot(\tau-\sigma)(g_0))_R^{-1})\cdot(q\cdot(\tau-\sigma)(f))) \\
& \overset{(13.36)}{=} (D(g_0)\cdot((q\cdot(\tau-\sigma)(g_1))\cdot(p\cdot(\tau-\sigma)(g_0))_R^{-1}))\cdot(q\cdot(\tau-\sigma)(f)) \\
& \overset{(13.37)}{=} ((D(g_0)\cdot(q\cdot(\tau-\sigma)(g_1)))\cdot(p\cdot(\tau-\sigma)(g_0))_R^{-1})\cdot(q\cdot(\tau-\sigma)(f)) \\
& = D(g_1)\cdot(q\cdot(\tau-\sigma)(f)).
\end{aligned}
$$

□

Let $p, q \in \mathcal{A}$, $D : \mathcal{A} \to \mathcal{A}$ be a linear operator and $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ two linear maps. Consider for $x, y \in \mathcal{A}$ the following conditions:

$$[p, (\tau-\sigma)(x), \tau(y)]_{as} = 0, \tag{13.40}$$
$$((p\cdot(\tau-\sigma)(x))\cdot\tau(y))\cdot q = ((p\cdot(\tau-\sigma)(x))\cdot q)\cdot\tau(y), \tag{13.41}$$
$$p\cdot(\sigma(x)\cdot(\tau-\sigma)(x)) = \sigma(x)\cdot(p\cdot(\tau-\sigma)(x)), \tag{13.42}$$
$$[\sigma(x), (p\cdot(\tau-\sigma)(y)), q]_{as} = 0, \tag{13.43}$$
$$(\tau-\sigma)([\mathcal{A},\mathcal{A}]) = 0, \tag{13.44}$$
$$[D(x), \sigma(y)] = 0, \tag{13.45}$$
$$[D(x), \tau(y)] = 0, \tag{13.46}$$
$$p\,(D(x)\cdot(\tau-\sigma)(y)) = D(x)\,(p\cdot(\tau-\sigma)(y))\,, \tag{13.47}$$
$$[p, (\tau-\sigma)(x), D(y)]_{as} = 0. \tag{13.48}$$

Theorem 13.4 *Let $\mathcal{A}$ be an algebra, p, q be elements of $\mathcal{A}$, $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ two linear maps such that, for all $f, g \in \mathcal{A}$,*

$$p(\sigma(fg) - \sigma(f)\sigma(g)) = p(\tau(fg) - \tau(f)\tau(g)) \tag{13.49}$$

If $D_{pq} : \mathcal{A} \to \mathcal{A}$ is the linear operator defined by $D_{pq} : f \mapsto (p\cdot(\tau-\sigma)(f))\cdot q$, then:

(i) *Under* (13.44)*, $D_{pq}([\mathcal{A},\mathcal{A}]) = 0$.*
(ii) *Under* (13.40)*,* (13.41)*,* (13.42) *and* (13.43)*, D_{pq} is a (σ,τ)-derivation.*
(iii) *Under (ii),* (13.44) *and* (13.45)*, $D_{pq}(f)(\tau-\sigma)(g) = D_{pq}(g)(\tau-\sigma)(f)$.*
(iv) *Under (ii),* (13.44) *and* (13.46)*, $(\tau-\sigma)(f)D_{pq}(g) = (\tau-\sigma)(g)D_{pq}(f)$.*
(v) *Under (iii) and* (13.47)*, $D_{pq}(f)(p(\tau-\sigma)(g)) = D_{pq}(g)(p(\tau-\sigma)(f))$.*
(vi) *Under (iv) and* (13.48)*, $(p(\tau-\sigma)(f))D_{pq}(g) = (p(\tau-\sigma)(g))D_{pq}(f)$.*

Proof (i) If $(\tau-\sigma)([f,g]) = 0$, $f, g \in \mathcal{A}$, then

$$D_{pq}([f,g]) = (p\cdot(\tau-\sigma)([f,g]))\cdot q = (p\cdot 0)\cdot q = 0 \quad\Rightarrow\quad D_{pq}([\mathcal{A},\mathcal{A}]) = 0.$$

(ii) We The linear operator D_{pq} is (σ,τ)-derivation, that is the twisted Leibniz rule holds, since

$$\begin{aligned}
D_{pq}(f\cdot g) &= (p\cdot(\tau-\sigma)(f\cdot g))\cdot q = (p\cdot(\tau(f\cdot g) - \sigma(f\cdot g)))\cdot q \\
&\overset{(13.49)}{=} (p\cdot(\tau(f)\cdot\tau(g) - \sigma(f)\cdot\sigma(g)))\cdot q \\
&= (p\cdot(\tau(f)\cdot\tau(g) - \sigma(f)\cdot\tau(g) + \sigma(f)\cdot\tau(g) - \sigma(f)\cdot\sigma(g)))\cdot q \\
&= (p\cdot((\tau-\sigma)(f)\cdot\tau(g)))\cdot q + (p\cdot(\sigma(f)\cdot(\tau-\sigma)(g)))\cdot q \\
&\overset{(13.40)}{=} ((p\cdot(\tau-\sigma)(f))\cdot\tau(g))\cdot q + (p\cdot(\sigma(f)\cdot(\tau-\sigma)(g)))\cdot q \\
&\overset{(13.41)}{=} ((p\cdot(\tau-\sigma)(f))\cdot q)\cdot\tau(g) + (p\cdot(\sigma(f)\cdot(\tau-\sigma)(g)))\cdot q \\
&\overset{(13.42)}{=} ((p\cdot(\tau-\sigma)(f))\cdot q)\cdot\tau(g) + (\sigma(f)\cdot(p\cdot(\tau-\sigma)(g)))\cdot q \\
&\overset{(13.43)}{=} D_{pq}(f)\cdot\tau(g) + \sigma(f)\cdot D_{pq}(g).
\end{aligned}$$

(iii) Statement (iii) follows from Lemma 13.4 because conditions (ii), (13.44) and (13.45) are a special case of the conditions of the Lemma.
(iv) Statement (iv) follows from Lemma 13.5 because conditions (ii), (13.44) and (13.46) are a special case of the conditions of the Lemma.
(v) We use (ii) and expand $D_{pq}(f \cdot g - g \cdot f)$ using linearity:

$$\begin{aligned}
D_{pq}(f \cdot g) &\overset{(ii)}{=} D_{pq}(f) \cdot \tau(g) + \sigma(f) \cdot D_{pq}(g),\\
D_{pq}(g \cdot f) &\overset{(ii)}{=} D_{pq}(g) \cdot \tau(f) + \sigma(g) \cdot D_{pq}(f),\\
0 \overset{(i)}{=} D_{pq}(f \cdot g - g \cdot f) &= D_{pq}(f \cdot g) - D_{pq}(g \cdot f)\\
&\overset{(13.45)}{=} D_{pq}(f) \cdot (\tau - \sigma)(g) + D_{pq}(g) \cdot (\sigma - \tau)(f),\\
D_{pq}(g) \cdot (\tau - \sigma)(f) &= D_{pq}(f) \cdot (\tau - \sigma)(g),\\
p \cdot (D_{pq}(g) \cdot (\tau - \sigma)(f)) &= p \cdot (D_{pq}(f) \cdot (\tau - \sigma)(g)),\\
&\text{(apply (13.47)to both sides)}\\
D_{pq}(g) \cdot (p \cdot (\tau - \sigma)(f)) &= D_{pq}(f) \cdot (p \cdot (\tau - \sigma)(g)).
\end{aligned}$$

(vi) Condition (13.48) is used in a similar way to prove (vi).

$$\begin{aligned}
0 \overset{(i)}{=} D_{pq}(f \cdot g - g \cdot f) &= D_{pq}(f \cdot g) - D_{pq}(g \cdot f)\\
&\overset{(13.46)}{=} (\tau - \sigma)(g) \cdot D_{pq}(f) + (\sigma - \tau)(f) \cdot D_{pq}(g),\\
(\tau - \sigma)(f) \cdot D_{pq}(g) &= (\tau - \sigma)(g) \cdot D_{pq}(f),\\
p \cdot ((\tau - \sigma)(f) \cdot D_{pq}(g)) &= p \cdot ((\tau - \sigma)(g) \cdot D_{pq}(f)),\\
&\text{(apply (13.48)to both sides)}\\
(p \cdot (\tau - \sigma)(f)) \cdot D_{pq}(g) &= (p \cdot (\tau - \sigma)(g)) \cdot D_{pq}(f).
\end{aligned}$$

□

Proposition 13.12 *Let $D : \mathcal{A} \to \mathcal{A}$ be (σ, τ)-derivation and $W : \mathcal{A} \to \mathcal{A}$ be a linear operator. The mapping $D_W : f \mapsto (W \circ D)(f)$ is a linear operator on $\mathcal{A}$ as the composition of the linear operators. Let $P \subseteq \mathcal{A}$ be the subset of all elements $f \in \mathcal{A}$ satisfying*

$$((W \circ L_{\sigma(f)} - L_{\sigma(f)} \circ W) \circ D)(\mathcal{A}) = \{0\}, \tag{13.50}$$
$$((W \circ R_{\tau(f)} - R_{\tau(f)} \circ W) \circ D)(\mathcal{A}) = \{0\}. \tag{13.51}$$

Then P is a linear subspace of $\mathcal{A}$, and the restriction $D_W : P \to \mathcal{A}$ is a (σ, τ)-derivation from P to $\mathcal{A}$ in the sense of Definition 13.7.

Proof The subset P is a linear subspace of $\mathcal{A}$ since using that the multiplication operators are linear in the factor and the linearity of W, (13.50), (13.51) yield that $f, g \in P \Rightarrow af + bg \in P$ for $a, b \in \mathbb{F}$, and thus that P is a linear subspace of $\mathcal{A}$.

If $f, g \in P$ satisfy (13.50) and (13.51) respectively, then

$$\begin{aligned}(13.51) \quad &\Leftrightarrow \quad W \circ R_{\tau(g)} \circ D(f) = R_{\tau(g)} \circ W \circ D(f),\\ (13.50) \quad &\Leftrightarrow \quad W \circ L_{\sigma(f)} \circ D(g) = L_{\sigma(f)} \circ W \circ D(g).\end{aligned}$$

This equalities can be rewritten as

$$\begin{aligned}W(D(f) \cdot \tau(g)) &= W(D(f)) \cdot \tau(g),\\ W(\sigma(f) \cdot D(g)) &= \sigma(f) \cdot W(D(g)).\end{aligned}$$

The sum of these two equations is

$$W(D(f) \cdot \tau(g)) + W(\sigma(f) \cdot D(g)) = (W(D(f))) \cdot \tau(g) + \sigma(f) \cdot W(D(g)).$$

Using linearity on the left hand side yields

$$\begin{aligned}W(D(fg)) &= W(D(f)) \cdot \tau(g) + \sigma(f) \cdot W(D(g))\\ &\Rightarrow D_W(fg) = D_W(f) \cdot \tau(g) + \sigma(f) \cdot D_W(g).\end{aligned}$$

□

Remark 13.10 If one would require $W \circ D$ to be only an additive map, this proposition would still hold for additive maps W.

Proposition 13.13 *Let $D : \mathcal{A} \to \mathcal{A}$ be a (σ, τ)-derivation, and $S, T : \mathcal{A} \to \mathcal{A}$ be linear operators on $\mathcal{A}$. The mapping $D_{TS} : f \mapsto (T \circ S \circ D)(f)$ is a linear operator on $\mathcal{A}$ as the composition of the linear operators. Let $P \subseteq \mathcal{A}$ be the subset of all elements $f \in \mathcal{A}$ satisfying*

$$(((T \circ S) \circ L_{\sigma(f)} - L_{\sigma(f)} \circ (T \circ S)) \circ D)(\mathcal{A}) = \{0\}, \quad (13.52)$$

$$(((T \circ S) \circ R_{\tau(f)} - R_{\tau(f)} \circ (T \circ S)) \circ D)(\mathcal{A}) = \{0\}. \quad (13.53)$$

Then, P is a linear subspace of $\mathcal{A}$ and the restriction $D_{TS} : P \to \mathcal{A}$ is a (σ, τ)-derivation from P to $\mathcal{A}$ in the sense of Definition 13.7.

Proof The subset P is a linear subspace of $\mathcal{A}$ since using that the multiplication operators are linear in the factor and the linearity of S and T, (13.52), (13.53) yield that $f, g \in P \Rightarrow af + bg \in P$ for $a, b \in \mathbb{F}$, and thus that P is a linear subspace of $\mathcal{A}$.

If $f, g \in P$ are the elements satisfying (13.52) and (13.53) respectively. Then,

$$\begin{aligned}(13.53) \quad &\Leftrightarrow \quad T \circ S \circ R_{\tau(g)} \circ D(f) = R_{\tau(g)} \circ T \circ S \circ D(f),\\ (13.52) \quad &\Leftrightarrow \quad T \circ S \circ L_{\sigma(f)} \circ D(g) = L_{\sigma(f)} \circ T \circ S \circ D(g).\end{aligned}$$

This equalities can be rewritten in the following way:

$$T(S(D(f)\cdot\tau(g))) = (T(S(D(f))))\cdot\tau(g),$$
$$T(S(\sigma(f)\cdot D(g))) = \sigma(f)\cdot T(S(D(g))).$$

The sum of these two equalities is

$$T(S(D(f)\cdot\tau(g))) + T(S(\sigma(f)\cdot D(g))) = (T(S(D(f))))\cdot\tau(g) + \sigma(f)\cdot T(S(D(g))).$$

Using linearity and twisted Leibniz rule for D on the left hand side of this equality yields

$$T(S(D(fg)) = (T(S(D(f))))\cdot\tau(g) + \sigma(f)\cdot T(S(D(g)))$$
$$\Rightarrow D_{TS}(fg) = D_{TS}(f)\cdot\tau(g) + \sigma(f)\cdot D_{TS}(g).$$

□

Corollary 13.10 *Let $P \subseteq \mathcal{A}$ be the subset of all $f \in \mathcal{A}$ such that $T\circ S$ commutes with $L_{\sigma(f)}$ and $R_{\tau(f)}$. Then P is a linear subspace of $\mathcal{A}$, and the restriction $D_{TS} = T\circ S\circ D : P \to \mathcal{A}$ is a (σ,τ)-derivation from P to $\mathcal{A}$ in the sense of Definition 13.7.*

Remark 13.11 Across this section it is essential that $P \subseteq \mathcal{A}$ is a linear subspace, because without associativity it cannot be granted that P is a subalgebra. This motivates the introduction of (σ,τ)-derivations from linear subspace into algebra, as we cannot grant that P is closed under product, but twisted Leibniz rule holds all across P.

Larson, Hartwig, Silvestrov [4] introduce a 1-dimensional $\mathcal{A}$-module of derivations on commutative associative UFDs, generated by the operator $\dfrac{k(\tau-\sigma)}{r}$. Provided $\tau-\sigma$ is a (σ,τ)-derivation, this can be expressed as $L_k\circ S_r\circ(\tau-\sigma)$, where L_k denotes left multiplication by k, and S_r denotes division by r or product by r^{-1}. Although a priori we do not have either unities or association as to grant that S_r is well-defined on any algebra, composition with two linear maps is interesting, especially in algebras with partial association relations.

We denote by L_p left multiplication by p and by R_q right multiplication by q. Consider the linear operators

$$D^L_{pq} : f \mapsto (R_q\circ L_p\circ D)(f) = (p\cdot D(f))\cdot q,$$
$$D^R_{pq} : f \mapsto (L_p\circ R_q\circ D)(f) = p\cdot(D(f)\cdot q).$$

Lemma 13.9 *If operators L_p and R_q commute, then $D^L_{pq} = D^R_{pq}$, that is, for all $f\in\mathcal{A}$,*

$$p\cdot(D(f)\cdot q) = (p\cdot D(f))\cdot q.$$

In terms of association relations, this can be written $[p, D(\mathcal{A}), q]_{as} = 0$.

Indeed, if L_p and R_q commute then $L_p \circ R_q \circ D = R_q \circ L_p \circ D \Rightarrow p \cdot (D(f) \cdot q) = (p \cdot D(f)) \cdot q$ for all $f \in \mathcal{A}$. It is still interesting to study each of them separately.

Corollary 13.11 (Left formulation) *Let $D : \mathcal{A} \to \mathcal{A}$ be a (σ, τ)-derivation. Define operator D^L_{pq} as the map that sends f to $(p \cdot D(f)) \cdot q$. Let $P \subseteq \mathcal{A}$ be the subspace of all $f \in \mathcal{A}$ such that:*

$$(((R_q \circ L_p) \circ L_{\sigma(f)} - L_{\sigma(f)} \circ (R_q \circ L_p)) \circ D)(\mathcal{A}) = \{0\},$$
$$(((R_q \circ L_p) \circ R_{\tau(f)} - R_{\tau(f)} \circ (R_q \circ L_p)) \circ D)(\mathcal{A}) = \{0\}.$$

Then, D^L_{pq} is a (σ, τ)-derivation from P to $\mathcal{A}$ in the sense of Definition 13.7.

Corollary 13.12 (Right formulation) *Let $D : \mathcal{A} \to \mathcal{A}$ be a (σ, τ)-derivation. Define operator D^R_{pq} as the map that sends f to $p \cdot (D(f) \cdot q)$. Let $P \subseteq \mathcal{A}$ be the subspace of all $f \in \mathcal{A}$ such that:*

$$(((L_p \circ R_q) \circ L_{\sigma(f)} - L_{\sigma(f)} \circ (L_p \circ R_q)) \circ D)(\mathcal{A}) = \{0\},$$
$$(((L_p \circ R_q) \circ R_{\tau(f)} - R_{\tau(f)} \circ (L_p \circ R_q)) \circ D)(\mathcal{A}) = \{0\}.$$

Then, D^R_{pq} is a (σ, τ)-derivation from P to $\mathcal{A}$ in the sense of Definition 13.7.

Corollary 13.13 *Let $D : \mathcal{A} \to \mathcal{A}$ be a (σ, τ)-derivation. If $R_{\tau(g)}$ and $L_{\sigma(f)}$ commute with $R_q \circ L_p$ (resp. $L_p \circ R_q$) for all $f, g \in \mathcal{A}$, then D^L_{pq} (resp. D^R_{pq}) is a (σ, τ)-derivation on $\mathcal{A}$.*

Corollary 13.14 *Let $D : \mathcal{A} \to \mathcal{A}$ be a (σ, τ)-derivation. Define operator D_{Lp} as the map that sends f to $(L_p \circ D)(f)$. Let $P \subseteq \mathcal{A}$ be the subspace of all elements $f \in \mathcal{A}$ such that*

$$((L_p \circ L_{\sigma(f)} - L_{\sigma(f)} \circ L_p) \circ D)(\mathcal{A}) = \{0\},$$
$$((L_p \circ R_{\tau(f)} - R_{\tau(f)} \circ L_p) \circ D)(\mathcal{A}) = \{0\}.$$

Then D_{Lp} is a (σ, τ)-derivation from P to $\mathcal{A}$ in the sense of Definition 13.7.

Corollary 13.15 *Let $D : \mathcal{A} \to \mathcal{A}$ be a (σ, τ)-derivation. Define operator D_{Rq} as the map that sends f to $(R_q \circ D)(f)$. Let $P \subseteq \mathcal{A}$ be the subspace of all elements $f \in \mathcal{A}$ such that*

$$((R_q \circ L_{\sigma(f)} - L_{\sigma(f)} \circ R_q) \circ D)(\mathcal{A}) = \{0\},$$
$$((R_q \circ R_{\tau(f)} - R_{\tau(f)} \circ R_q) \circ D)(\mathcal{A}) = \{0\}.$$

Then D_{Rq} is a (σ, τ)-derivation from P to $\mathcal{A}$ in the sense of Definition 13.7.

Corollary 13.16 *Let $D : \mathcal{A} \to \mathcal{A}$ be a (σ, τ)-derivation. If $R_{\tau(f)}$ and $L_{\sigma(f)}$ commute with R_q (resp. L_p) for all $f \in \mathcal{A}$, then $R_q \circ D$ (resp. $L_p \circ D$) is a (σ, τ)-derivation on $\mathcal{A}$.*

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Chapter 14
On Lie-Type Constructions over Twisted Derivations

Germán García Butenegro, Abdennour Kitouni, and Sergei Silvestrov

Abstract In this paper we examine interactions between (σ, τ)-derivations via commutator and consider new n-ary structures based on twisted derivation operators. We show that the sums of linear spaces of (σ^k, τ^l)-derivations and also of some of their subspaces, consisting of twisted derivations with some commutation relations with σ and τ, form Lie algebras, and moreover with the semigroup or group graded commutator product, yielding graded Lie algebras when the sum of the subspaces is direct. Furthermore, we extend these constructions of such Lie subalgebras spanned by twisted derivations of algebras to twisted derivations of n-ary algebras. Finally, we consider n-ary products defined by generalized Jacobian determinants based on (σ, τ)-derivations, and construct n-Hom-Lie algebras associated to the generalized Jacobian determinants based on twisted derivations extending some results of Filippov to (σ, τ)-derivations. We also establish commutation relations conditions for twisting maps and twisted derivations such that the generalised Jacobian determinant products yield (σ, τ, n)-Hom-Lie algebras, a new type of n-ary Hom-algebras different from n-Hom-Lie algebras in that the positions of twisting maps σ and τ are not fixed to positions of variables in n-ary products terms of the sum of defining identity as they were in Hom-Nambu-Filippov identity of n-Hom-Lie algebras.

Keywords Twisted derivation · Leibniz rule · Jacobian determinant · n-Hom-Lie algebra · Hom-algebra · Lie algebra

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14.1 Introduction

The area of Hom-algebra structures initiated in 2003 in [17] extends and connects many algebraic structures in mathematics and mathematical physics. Hom-Lie algebras and more general quasi-Hom-Lie algebras were introduced first in 2003 in [17] in connection to the general method for construction of deformations and discretizations of Lie algebras of vector fields based on twisted derivations satisfying twisted Leibniz rule. Central extensions and cocycle conditions for general quasi-Hom-Lie algebras and Hom-Lie algebras, generalizing in particular q-deformed Witt and Virasoro algebras, have been first considered in [17, 26] and for graded color quasi-Hom-Lie algebras in [38]. In [25, 29, 30, 35, 36] Quasi-Lie structure of σ-derivations has been further investigated. At the same time, in 2004–2005, general quasi-Lie and quasi-Leibniz algebras where introduced in [27] and color quasi-Lie and color quasi-Leibniz algebras where introduced in [28] generalizing and uniting in the same algebraic structure the Hom-Lie algebras and the quasi-Hom-Lie algebras, the color Hom-Lie algebras, quasi-Hom-Lie color algebras, quasi-Hom-Lie superalgebras and Hom-Lie superalgebras, as well as color quasi-Leibniz algebras, quasi-Leibniz superalgebras, quasi-Hom-Leibniz superalgebras and Hom-Leibniz algebras. Graded color quasi-Lie algebras of Witt type have been first considered in [37]. Hom-Lie admissible algebras, that is Hom-algebras consisting of an algebra and a linear map (homomorphism of linear space) such that the commutator bilinear product yields Hom-Lie algebra, have been considered first in 2006 in [33], where the Hom-associative algebras and more general G-Hom-associative algebras including the Hom-Vinberg algebras (Hom-left symmetric algebras), Hom-pre-Lie algebras (Hom-right symmetric algebras), and some other new Hom-algebra structures have been introduced and shown to be Hom-Lie admissible, in the sense that the operation of commutator as new product in these Hom-algebras structures yields Hom-Lie algebras. Furthermore, in [33], flexible Hom-algebras and Hom-algebra generalizations of derivations and of adjoint derivations maps have been introduced, and the Hom-Leibniz algebras appeared for the first time, as an important special subclass of quasi-Leibniz algebras introduced in more genera context of general quasi-Lie algebras in [27] following the standard Loday's conventions for Leibniz algebras (right Loday algebras) [12, 13, 31]. In [33], moreover the investigation of classification of finite-dimensional Hom-Lie algebras has been initiated with construction of families of the low-dimensional Hom-Lie algebras.

Ternary Lie algebras appeared in generalization of Hamiltonian mechanics by Nambu [34], the mathematical algebraic foundations of Nambu mechanics have been developed by Takhtajan in [39], Filippov, in [15] independently introduced and studied structure of n-Lie algebras and Kasymov [18] investigated their properties. Since these pioneering works the n-Lie algebras and their generalizations their constructions, properties, classifications, interplay with other algebraic structures and applications in geometry, analysis and mathematica physics. Hom-type generalization of n-ary algebras, such as n-Hom-Lie algebras and other n-ary Hom algebras of Lie type and associative type, were introduced in [7], by twisting the defining identities by a set of linear maps. The particular case, where all these maps are equal and are algebra morphisms has been considered and a way to generate examples of n-ary Hom-algebras from n-ary algebras of the same type have been described. Further properties, construction methods, examples, classification, representations, cohomology and central extensions of n-ary Hom-algebras have been considered in [4–6, 19–21, 23, 24, 40]. These generalizations include n-ary Hom-algebra structures generalizing the n-ary algebras of Lie type including n-ary Nambu algebras, n-ary Nambu-Lie algebras and n-ary Lie algebras, and n-ary algebras of associative type including n-ary totally associative and n-ary partially associative algebras. In [22], constructions of n-ary generalizations of BiHom-Lie algebras and BiHom-associative algebras have been considered. Generalized derivations of n-BiHom-Lie algebras have been studied in [9]. Generalized derivations of multiplicative n-ary Hom-Ω color algebras have been studied in [10]. Cohomology of Hom-Leibniz and n-ary Hom-Nambu-Lie superalgebras has been considered in [1] Generalized derivations and Rota-Baxter operators of n-ary Hom-Nambu superalgebras have been considered in [32]. A construction of 3-Hom-Lie algebras based on σ-derivation and involution has been studied in [2]. Multiplicative n-Hom-Lie color algebras have been considered in [8].

In this paper we examine interactions between (σ, τ)-derivations via commutator and consider new n-ary structures based on twisted derivation operators. Discretizations of derivatives depending on linear maps σ and τ often satisfy generalized twisted Leibniz rule, resulting in the linear space of (σ, τ)-derivations typically being not closed under commutator since the commutator of (σ, τ)-derivations is a (σ, τ)-derivations only under some special conditions on σ, τ and the (σ, τ)-derivations in the commutator. Thus, usually, the linear spaces of (σ, τ)-derivations are not Lie subalgebras of the Lie algebra of all linear maps with commutator product. We show however that the sums of linear spaces of (σ^k, τ^l)-derivations and also of some of their subspaces, consisting of twisted derivations with some commutation relations with σ and τ, form Lie algebras, and moreover with the semigroup or group graded commutator product, yielding graded Lie algebras when the sum of the subspaces is direct. Furthermore, we extend these constructions of such Lie subalgebras spanned by twisted derivations of algebras to twisted derivations of n-ary algebras. Finally, we consider n-ary products defined by generalized Jacobian determinants based on (σ, τ)-derivations, and construct n-Hom-Lie algebras associated to the generalized Jacobian determinants based on twisted derivations extending some results of Filippov in [16] to (σ, τ)-derivations. We also establish commutation relations conditions on twisting maps and twisted derivations such that the generalised Jacobian determinant products yield (σ, τ, n)-Hom-Lie algebras, a new type of n-ary Hom-algebras different from n-Hom-Lie algebras in that the positions of twisting maps σ and τ are not fixed to positions of variables in n-ary products terms of the sum of defining identity as they were in Hom-Nambu-Filippov identity of n-Hom-Lie algebras.

14.2 Definitions and Notations

An algebra over a field $\mathbb{F}$ is a pair $(\mathcal{A}, *)$ consisting of a vector space $\mathcal{A}$ over $\mathbb{F}$ with a bilinear binary operation $\mu : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, $\mu(x, y) \mapsto x * y$. Juxtaposition xy is often used for multiplication for convenience of notations, when it is clear which multiplication it stands for. For any algebra, the left multiplication operator $L_x : \mathcal{A} \to \mathcal{A},\ L_x(y) = xy$ and the right multiplication operator $R_y : \mathcal{A} \to \mathcal{A},\ R_y(x) = xy$ are linear operators.

An algebra $\mathcal{A}$ is *left unital* if there is an element $1_L \in \mathcal{A}$ (left unity) such that $a = 1_L \cdot a$ for all $a \in \mathcal{A}$. An algebra is *right unital* if there is an element $1_R \in \mathcal{A}$ (right unity) such that $a = a \cdot 1_R$ for all $a \in \mathcal{A}$, and *unital* if it is both left and right unital. An algebra $\mathcal{A}$ is called *associative* if $x(yz) = (xy)z$ (associativity) holds for all $x, y, z \in \mathcal{A}$. An algebra is called *non-associative* if $x(yz) \neq (xy)z$ for some elements in the algebra. If $xy = yx$ (commutativity) for all $x, y \in \mathcal{A}$, the algebra is called *commutative*, and it is called *non-commutative* if for some elements $xy \neq yx$. If $xy = -yx$ (skew-symmetry or anti-commutativity) for all $x, y \in \mathcal{A}$, the algebra is called *skew-symmetric* (or *anti-commutative*). Lie algebras are pairs $(\mathcal{A}, \langle\cdot,\cdot\rangle)$ consisting of a linear space $\mathcal{A}$ and a bilinear mapping (product,commonly referred as bracket) $\langle\cdot,\cdot\rangle : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, satisfying for all $x, y, z \in \mathcal{A}$,

$$\langle x, y\rangle = -\langle y, x\rangle \qquad \textbf{Skew-symmetry}$$

$$\sum_{\circlearrowleft(x,y,z)} \langle x, \langle y, z\rangle\rangle = \langle x, \langle y, z\rangle\rangle + \langle y, \langle z, x\rangle\rangle + \langle z, \langle x, y\rangle\rangle = 0. \qquad \textbf{Jacobi identity}$$

where $\sum\limits_{\circlearrowleft(x,y,z)}$ denotes the summation over cyclic permutations of (x, y, z).

In any algebra $(\mathcal{A}, *)$, the commutator defined by $[x, y] = [x, y]_- = xy - yx$ for any two elements $x, y \in \mathcal{A}$, is a bilinear map $[\cdot,\cdot] : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ defining a new algebra $(\mathcal{A}, [\cdot,\cdot])$ on the same vector space.

Algebra elements commute, $xy = yx$, if and only if commutator is zero, $[x, y] = [x, y]_- = xy - yx = 0$. The center $Z(\mathcal{A}) = Z(\mathcal{A}, *) = \{x \in \mathcal{A} \mid \forall y \in \mathcal{A} : xy = yx\}$, consisting of all those elements that commute with any element of an algebra $\mathcal{A}$, is a linear subspace of $\mathcal{A}$.

For any algebra, the commutator is skew-symmetric bilinear map, since $[x, y] = xy - yx = -(yx - xy) = -[y, x]$, and thus the new algebra $(\mathcal{A}, [\cdot,\cdot])$ is always a skew-symmetric algebra. If the algebra $(\mathcal{A}, *)$ is associative, then the new algebra $(\mathcal{A}, [\cdot,\cdot])$, with commutator bracket as multiplication, is a Lie algebra, that is the commutator on associative algebras satisfies not only skew-symmetry, but also the Jacobi identity of Lie algebras. *Lie admissible* algebras are those algebras for which the new algebra with commutator as product is a Lie algebra. So, in particular, all associative algebras are Lie admissible. There are many other classes of algebras which are Lie admissible.

If $[x, [y, z]] \neq [[x, y], z]$ for some elements in an algebra, then the commutator defines a non-associative product. For any elements,

$$[x, [y, z]] = x[y, z] - [y, z]x = x(yz - zy) - (yz - zy)x = x(yz) - x(zy) - (yz)x + (zy)x,$$
$$[[x, y], z] = [x, y]z - z[x, y] = (xy - yx)z - z(xy - yx) = (xy)z - (yx)z - z(xy) + z(yx),$$
$$[x, [y, z]] - [[x, y], z] = x(yz) - x(zy) - (yz)x + (zy)x - (xy)z + (yx)z + z(xy) - z(yx)$$
$$= x(yz) - (xy)z + (zy)x - z(yx) - x(zy) - (yz)x + (yx)z + z(xy)$$

(if the product is associative)

$$= y(xz) - y(zx) + (zx)y - (xz)y = y[x, z] - [x, z]y = [y, [x, z]].$$

Thus, in associative algebras, the commutator is associative if and only if $[\mathcal{A}, \mathcal{A}] \subseteq Z(\mathcal{A})$ where $Z(\mathcal{A}) = Z(\mathcal{A}, *)$ is center of $(\mathcal{A}, *)$. This is the case in nilpotent algebras of degree 3, where $[\mathcal{A}, \mathcal{A}] \subseteq Z(\mathcal{A}) \Rightarrow [[\mathcal{A}, \mathcal{A}], \mathcal{A}] = 0, [\mathcal{A}, [\mathcal{A}, \mathcal{A}]] = 0$.

The associator $[\cdot, \cdot, \cdot]_{as} : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, defined by $[x, y, z]_{as} = x(yz) - (xy)z$, is a trilinear mapping, thus also defining a ternary algebra structure on $\mathcal{A}$. The associator can be expressed using the commutator of the left and right multiplication operators L_x and R_z,

$$[x, y, z]_{as} = x(yz) - (xy)z = L_x(yz) - R_z(xy) = L_x(R_z(y)) - R_z(L_x(y)) = [L_x, R_z](y).$$

Elements associate if their associator is 0. The associative algebras are those algebras in which associator is identically 0 on all elements, or equivalently in which all left and right multiplication operators commute.

Definition 14.1 ([17, Definition 14]) A hom-Lie algebra $(\mathcal{A}, \alpha)$ is an algebra $\mathcal{A}$ together with a bilinear product $\langle \cdot, \cdot \rangle_{\mathcal{A}} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ (commonly referred as bracket) and a linear map $\alpha : \mathcal{A} \longrightarrow \mathcal{A}$, such that for all $x, y, z \in \mathcal{A}$,

$$\langle x, y \rangle_{\mathcal{A}} = -\langle y, x \rangle_{\mathcal{A}}, \qquad \textbf{Skew-symmetry} \qquad (14.1)$$
$$\sum_{\circlearrowleft (x,y,z)} \langle \alpha(x), \langle y, z \rangle_{\mathcal{A}} \rangle_{\mathcal{A}} = 0. \qquad \textbf{Hom-Jacobi identity} \qquad (14.2)$$

A natural n-ary generalization of Hom-Lie algebras is n-ary Hom-Lie algebras introduced first in [7]. The n-Hom-Lie algebras are an n-ary generalization of Hom-Lie algebras to n-ary algebras satisfying a generalisation of the Hom-algebra identity (14.2) involving n-ary product and $n - 1$ twisting linear maps. We will consider in this paper the special case of the n-ary Hom-Lie algebras with single twisting map (when all the $n - 1$ twisting linear maps are the same).

Definition 14.2 n-Ary Hom-Lie algebras or n-Hom-Lie algebras with one twisting map are triples $(\mathcal{A}, \mu, \alpha)$ consisting of an algebra $\mathcal{A}$ with a n-ary skew-symmetric bilinear product $\mu : \mathcal{A}^n \to \mathcal{A}$ and a linear map $\alpha : \mathcal{A} \to \mathcal{A}$ satisfying the following n-ary generalization of the Hom-Lie algebras Hom-Jacobi identity, for all $x_1, \ldots, x_n, y_2, \ldots, y_n \in \mathcal{A}$:

Hom-Nambu-Filippov identity

$$\mu(\mu(x_1,\ldots,x_n),\alpha(y_2),\ldots,\alpha(y_n)) = \\ = \sum_{i=1}^{n} \mu(\alpha(x_1),\ldots,\alpha(x_{i-1}),\mu(x_i,y_2,\ldots,y_n),\alpha(x_{i+1}),\ldots,\alpha(x_n)). \quad (14.3)$$

We also introduce more general class of n-ary algebras with two twisting maps. An interesting concrete class of such n-ary algebras will be constructed further on.

Definition 14.3 ((α, β, n)-Hom-Lie algebra) (α, β, n)-Hom-Lie algebras are quadruples $(\mathcal{A}, \mu, \alpha, \beta)$ consisting of an algebra $\mathcal{A}$ with a n-ary skew-symmetric bilinear product $\mu : \mathcal{A}^n \to \mathcal{A}$ and two linear maps $\alpha, \beta : \mathcal{A} \to \mathcal{A}$ such that, for all $x_1, \ldots, x_n, y_2, \ldots, y_n \in \mathcal{A}$:

$$\mu(\mu(x_1,\ldots,x_n),\beta(y_2),\ldots,\beta(y_n)) = \\ = \sum_{i=1}^{n} \mu(\alpha(x_1),\ldots,\alpha(x_{i-1}),\mu(x_i,y_2,\ldots,y_n),\beta(x_{i+1}),\ldots,\beta(x_n)). \quad (14.4)$$

Note that the defining identity of n-Hom-Lie algebras (14.3) is different from (14.4) in that in (14.3) the twisting maps are attached to the position of the elements in the product.

Definition 14.4 (*Leibniz's product rule*, [11]) A *derivation* is an $\mathbb{F}$-linear operator $D : \mathcal{A} \to \mathcal{A}$ such that $D(fg) = D(f)g + fD(g)$, for every $f, g \in \mathcal{A}$.

Definition 14.5 ((σ, τ)-derivations, [14], Definition 1.1) Let $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ be two linear maps. The $\mathbb{F}$-linear operator $D : \mathcal{A} \to \mathcal{A}$ is called (σ, τ)-derivation if it satisfies a generalized Leibniz's product rule ((σ, τ)-twisted Leibniz's product rule), for every $f, g \in \mathcal{A}$,

$$D(fg) = D(f)\tau(g) + \sigma(f)D(g).$$

If $\tau = id$, then D is referred to as σ-derivation.

Example 14.1 Derivations in the sense of Definition 14.4 are (σ, τ)-derivations with $\sigma = \tau = id$. Another relevant example of these twisted derivation operators underlying the foundations of q-analysis, are $D_q(f)(t) = \dfrac{f(qt) - f(t)}{qt - t}$ (the Jackson q-derivative) and the operator $M_t D_q(f)(t) = \dfrac{f(qt) - f(t)}{q - 1}$ which act on $\mathbb{C}[t, t^{-1}]$ or some other suitable function spaces, and satisfy the twisted Leibniz product rule

$$D(fg) = D(f)g + \sigma_q(f)D(g), \quad \sigma_q(f)(t) = f(qt).$$

A linear map $\alpha : \mathcal{A} \to \mathcal{A}$ is left-invertible if there exists $\beta : \mathcal{A} \to \mathcal{A}$ such that $\beta \circ \alpha = id_{\mathcal{A}}$. Then β is called left inverse of α and denoted α_L^{-1}. A linear map $\alpha : \mathcal{A} \to \mathcal{A}$ is right-invertible if there exists $\beta : \mathcal{A} \to \mathcal{A}$ such that $\alpha \circ \beta = id_{\mathcal{A}}$.

Then β is called right inverse of α and denoted α_R^{-1}. If $\alpha_L^{-1} = \alpha_R^{-1}$, α is said to be invertible and $\alpha^{-1} = \alpha_L^{-1} = \alpha_R^{-1}$ is called inverse of α.

An element $p \in \mathcal{A}$ in an algebra $\mathcal{A}$ is a *left zero divisor* in $\mathcal{A}$ if there is a nonzero element $k_0 \in \mathcal{A}$ such that $p \cdot k_0 = 0$, it is a *right zero divisor* in $\mathcal{A}$ if there is a nonzero element $k_0 \in \mathcal{A}$ such that $k_0 \cdot p = 0$ and it is a *zero divisor* in $\mathcal{A}$ if it is either a left or right zero divisor or both.

Throughout this article, the following notations are used unless specified otherwise.

- $[\cdot,\cdot,\cdot]_{as}$: Associator.
- $[\cdot,\cdot,\cdot]^{\alpha}_{as}$: Hom-associator.
- $[\cdot,\cdot]$: Commutator.
- $\sum_{\circlearrowleft}$: Cyclic sum (over cyclic permutation of variables of summation)
- $\mathfrak{L}(\mathcal{A})$: Space of linear maps on a linear space $\mathcal{A}$.
- $\mathcal{D}_{\sigma,\tau}(\mathcal{A})$: Space of (σ,τ)-derivations on $\mathcal{A}$.
- $\mathcal{D}^{(k)}_{\sigma,\tau}(\mathcal{A})$: Space of (σ^k,τ^k)-derivations on $\mathcal{A}$.
- $\mathcal{D}^{(k,l)}_{\sigma,\tau}(\mathcal{A})$: Space of (σ^k,τ^l)-derivations on $\mathcal{A}$.
- $\Delta_{\sigma,\tau}(\mathcal{A})$: Space of (σ,τ)-derivations on $\mathcal{A}$ commuting with σ and τ.
- $\Delta^{(k)}_{\sigma,\tau}(\mathcal{A})$: Space of (σ^k,τ^k)-derivations on $\mathcal{A}$ commuting with σ and τ.
- $\Delta^{(k,l)}_{\sigma,\tau}(\mathcal{A})$: Space of (σ^k,τ^l)-derivations on $\mathcal{A}$ commuting with σ and τ.
- $\nabla^{(k,l)}_{k_0 l_0}(\mathcal{A})$: Space of (σ^k,τ^l)-derivations on $\mathcal{A}$ commuting with σ^{k_0} and τ^{l_0}.
- $\lambda_k, \gamma_{jk}, \Gamma_{jk}$: Commutation factors.
- gcd : Greatest common divisor
- UFD : Unique Factorization Domain.

In the last part, we follow Filippov's notation, where operators act from the right. The image of element x by operator D will be denoted by xD. Sometimes for clarity of exposition, if no confusion arises, the product between elements of an algebra will be denoted by the dot "·".

14.3 Some Results on Twisted Derivations

The following proposition extends [14, Lemma 1.4] to arbitrary algebras.

Proposition 14.1 *If σ, τ are algebra endomorphisms of an algebra $\mathcal{A}$, and $c \in \mathcal{A}$ satisfies*

$$[c, \sigma(\mathcal{A})] = 0, \tag{14.5}$$
$$[c, \sigma(\mathcal{A}), \sigma(\mathcal{A})]_{as} = 0, \tag{14.6}$$
$$[\sigma(\mathcal{A}), c, \sigma(\mathcal{A})]_{as} = 0, \tag{14.7}$$
$$[c, \tau(\mathcal{A}), \tau(\mathcal{A})]_{as} = 0, \tag{14.8}$$
$$[\sigma(\mathcal{A}), c, \tau(\mathcal{A})]_{as} = 0, \tag{14.9}$$

then $D : \mathcal{A} \to \mathcal{A},\ f \mapsto D(f) = c(\tau(f) - \sigma(f))$ is a (σ,τ)-derivation on $\mathcal{A}$.

Proof For any $f, g \in \mathcal{A}$,

$$\begin{aligned}
&D(fg) = c(\tau(fg) - \sigma(fg)) = c(\tau(f)\tau(g) - \sigma(f)\sigma(g))\\
&= c(\tau(f)\tau(g)) - c(\sigma(f)\sigma(g)) \overset{14.6}{=} c(\tau(f)\tau(g)) - (c\sigma(f))\sigma(g)\\
&\overset{14.5}{=} c(\tau(f)\tau(g)) - (\sigma(f)c)\sigma(g) \overset{14.7}{=} c(\tau(f)\tau(g)) - \sigma(f)(c\sigma(g))\\
&\overset{14.8}{=} (c\tau(f))\tau(g) - \sigma(f)(c\sigma(g))\\
&= (c\tau(f))\tau(g) - (c\sigma(f))\tau(g) + (c\sigma(f))\tau(g) - \sigma(f)(c\sigma(g))\\
&\overset{14.5}{=} (c\tau(f))\tau(g) - (c\sigma(f))\tau(g) + (\sigma(f)c)\tau(g) - \sigma(f)(c\sigma(g))\\
&\overset{14.9}{=} (c\tau(f))\tau(g) - (c\sigma(f))\tau(g) + \sigma(f)(c\tau(g)) - \sigma(f)(c\sigma(g))\\
&= (c(\tau(f) - \sigma(f)))\tau(g) + \sigma(f)(c(\tau(g) - \sigma(g))) = D(f)\tau(g) + \sigma(f)D(g),
\end{aligned}$$

which completes the proof. □

The following proposition is straightforward extension of [14, Proposition 1.5, Corollary 1.6] to arbitrary algebras.

Proposition 14.2 *Let $\mathcal{A}$ be an algebra, and $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ be two linear maps on $\mathcal{A}$. Also, let D be a (σ, τ)-derivation and $\alpha : \mathcal{A} \to \mathcal{A}$ a homomorphism. Then,*

(i) *$\alpha \circ D$ is a $(\alpha \circ \sigma, \alpha \circ \tau)$-derivation.*
(ii) *$D \circ \alpha$ is a $(\sigma \circ \alpha, \tau \circ \alpha)$-derivation.*
(iii) *If σ is left-invertible, then $\sigma_L^{-1} \circ D$ is a $(id, \sigma_L^{-1} \circ \tau)$-derivation. If $\mathcal{A}$ is commutative, it is also a $(\sigma_L^{-1} \circ \tau)$-derivation.*
(iv) *If τ is left-invertible, then $\tau_L^{-1} \circ D$ is a $(\tau_L^{-1} \circ \sigma)$-derivation.*
(v) *If σ is a right-invertible homomorphism, then $D \circ \sigma_R^{-1}$ is a $(id, \tau \circ \sigma_R^{-1})$-derivation. If $\mathcal{A}$ is commutative, then $D \circ \sigma_R^{-1}$ is a $(\tau \circ \sigma_R^{-1})$-derivation.*
(vi) *If τ is a right-invertible homomorphism, then $D \circ \tau_R^{-1}$ is a $(\sigma \circ \tau_R^{-1})$-derivation.*

Lemma 14.1 *Let $\mathcal{A}$ be a commutative algebra, $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ be two linear maps and D be a (σ, τ)-derivation on $\mathcal{A}$. Then, for all $x \in ker(\tau - \sigma)$ and $y \in \mathcal{A}$,*

$$D(x)(\tau(y) - \sigma(y)) = 0.$$

Moreover, if $\mathcal{A}$ has no non-null zero divisors and $\sigma \neq \tau$, then

$$ker(\tau - \sigma) \subseteq ker(D).$$

Proof By commutativity of $\mathcal{A}$, as $D(0) = 0$ by linearity of D, for all $x, y \in \mathcal{A}$,

$$\begin{aligned}
0 = D(0) = D(xy - yx) &= D(xy) - D(yx)\\
&= D(x)\tau(y) + \sigma(x)D(y) - D(y)\tau(x) - \sigma(y)D(x)\\
&= D(x)(\tau - \sigma)(y) - D(y)(\tau - \sigma)(x),\\
\Rightarrow \quad D(x)(\tau - \sigma)(y) &= D(y)(\tau - \sigma)(x).
\end{aligned}$$

Hence, if $x \in \ker(\tau - \sigma)$, then $D(y)(\tau - \sigma)(x) = 0 \Rightarrow D(x)(\tau - \sigma)(y) = 0$ for all $y \in \mathcal{A}$. If $\sigma \neq \tau$, then $D(x)(\tau - \sigma)(y) = 0$ for $y \in \mathcal{A}$ such that $(\tau - \sigma)(y) \neq 0$. Since $\mathcal{A}$ has no non-null zero divisors, $D(x) = 0$ which means that $x \in \ker(D)$, that is $\ker(\tau - \sigma) \subseteq \ker(D)$. □

Theorem 14.1 ([14, Theorem 2.6]) *Let $\mathcal{A}$ be a commutative associative algebra with unity, and D be a (σ, τ)-derivation of $\mathcal{A}$ with linear maps $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ satisfying:*

$$\sigma \circ \tau = \tau \circ \sigma, \quad D \circ \sigma = \delta(\sigma \circ D), \quad D \circ \tau = \delta(\tau \circ D),$$

for some $\delta \in \mathcal{A}$. The bracket $\langle \cdot, \cdot \rangle_{\sigma,\tau}$ defined by

$$\langle f \cdot D, g \cdot D \rangle_{\sigma,\tau} = (\sigma(f)D(g) - \sigma(g)D(f)) \cdot D$$

is a well-defined product in $\mathcal{A}$. It satisfies skew-symmetry and a twisted Jacobi-like identity:

$$\sum_{\circlearrowleft(n,m,l)} \langle (\sigma + \tau)(d_n), \langle d_m, d_l \rangle_{\sigma,\tau} \rangle_{\sigma,\tau} = 0.$$

This bracket endows linear space $\mathcal{A} \cdot D$ with a Hom-Lie algebra structure $(\mathcal{A} \cdot D, \langle \cdot, \cdot \rangle, \overline{\sigma + \tau})$ where $\overline{\sigma + \tau}(a \cdot D) = (\sigma + \tau)(a) \cdot D$.

Using $\sigma + \tau$ instead of $\overline{\sigma + \tau}$ is an abuse of notation that is often overlooked. It is convenient to keep in mind that the latter is a natural extension of the former, but not the same map.

The following generalization of [17, Theorem 4] weakens requirement of twisting maps being algebra endomorphisms.

Proposition 14.3 *Let $\mathcal{A}$ be a commutative associative UFD, $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ different algebra morphisms on $\mathcal{A}$ satisfying, for some c in the center of $\mathcal{A}$,*

$$c(\sigma(fg) - \sigma(f)\sigma(g)) = c(\tau(fg) - \tau(f)\tau(g)).$$

Then $\mathcal{D}_{\sigma,\tau}(\mathcal{A})$ is free of rank 1 as an $\mathcal{A}$-module, and is generated by the following operator:

$$c\Delta = c\frac{(\tau - \sigma)}{r} : x \longmapsto c\frac{(\tau - \sigma)(x)}{r},$$

where $r = gcd((\tau - \sigma)(\mathcal{A}))$.

14.4 Composition of (σ, τ)-Derivations

Across this section, $\mathbb{Z}_{\geq 0}$ denotes the nonnegative integers. For a linear map α, $\alpha^0 = \mathrm{Id}$ is the identity map over algebra $\mathcal{A}$.

14.4.1 $\mathbb{Z}$-Grading Property of Commutator on (σ, τ)-Derivations

It is already well-known that the composition of (σ, τ)-derivations is a linear map on $\mathcal{A}$ but not necessarily another twisted derivation operator. Across this section we study the behaviour of the commutator over a wider space of twisted derivatives.

Definition 14.6 Let $(\alpha_1, \alpha_2, \alpha_3)$ and $(\beta_1, \beta_2, \beta_3)$ be two triples of linear maps over an algebra $\mathcal{A}$. These triples *cross-commute* if the following properties hold:

$$\alpha_1 \circ \beta_2 = \beta_2 \circ \alpha_1, \quad \beta_1 \circ \alpha_2 = \alpha_2 \circ \beta_1,$$
$$\alpha_1 \circ \beta_3 = \beta_3 \circ \alpha_1, \quad \beta_1 \circ \alpha_3 = \alpha_3 \circ \beta_1.$$

That is, α_1 commutes with β_2, β_3 and β_1 commutes with α_2, α_3.

Cross-commutation lies behind relevant symmetry properties when α_1 and β_1 are twisted derivations. The following proposition is direct extension of [14, Remark 1.3], stated there for associative algebras, to arbitrary not necessarily associative algebras.

Proposition 14.4 *Let $\mathcal{A}$ be an algebra. Also, let D be a (σ, τ)-derivation and D' be a (σ', τ')-derivation. If σ and σ' commute, and τ and τ' commute, then the commutator $[D, D'] = D \circ D' - D' \circ D$ is a $(\sigma \circ \sigma', \tau \circ \tau')$-derivation if and only if $T_1(f, g) - T_2(f, g) + S_1(f, g) - S_2(f, g) = 0$ for all $f, g \in \mathcal{A}$, where*

$$S_1(f, g) = (D \circ \sigma')(f)(\tau \circ D')(g), \qquad T_1(f, g) = (\sigma \circ D')(f)(D \circ \tau')(g),$$
$$S_2(f, g) = (\sigma' \circ D)(f)(D' \circ \tau)(g), \qquad T_2(f, g) = (D' \circ \sigma)(f)(\tau' \circ D)(g).$$

In particular, if σ and σ' commute, τ and τ' commute, and (D, σ, τ) and (D', σ', τ') cross-commute, then $[D, D']$ is a $(\sigma \circ \sigma', \tau \circ \tau')$-derivation.

Proof Compute the commutator of D and D' on an element $fg \in \mathcal{A}$ according to the twisted Leibniz rule:

$$\begin{aligned}
&(D\circ D')(fg) = D(D'(f)\tau'(g) + \sigma'(f)D'(g))\\
&= D(D'(f))\tau(\tau'(g)) + \sigma(D'(f))D(\tau'(g))\\
&\quad + D(\sigma'(f))\tau(D'(g)) + \sigma(\sigma'(f))D(D'(g))\\
&= (D\circ D')(f)(\tau\circ\tau')(g) + (\sigma\circ\sigma')(f)(D\circ D')(g) + (\sigma\circ D')(f)(D\circ\tau')(g)\\
&+ (D\circ\sigma')(f)(\tau\circ D')(g),\\
&(D'\circ D)(fg) = D'(D(f)\tau(g) + \sigma(f)D(g))\\
&= D'(D(f))\tau'(\tau(g)) + \sigma'(D(f))D'(\tau(g))\\
&\quad + D'(\sigma(f))\tau'(D(g)) + \sigma'(\sigma(f))D'(D(g))\\
&= (D'\circ D)(f)(\tau'\circ\tau)(g) + (\sigma'\circ\sigma)(f)(D'\circ D)(g) + (\sigma'\circ D)(f)(D'\circ\tau)(g)\\
&+ (D'\circ\sigma)(f)(\tau'\circ D)(g),
\end{aligned}$$

that is the operators $D\circ D'$ and $D'\circ D$ can be rewritten as follows:

$$\begin{aligned}
(D\circ D')(fg) &= (D\circ D')(f)(\tau\circ\tau')(g) + (\sigma\circ\sigma')(f)(D\circ D')(g)\\
&\qquad + T_1(f,g) + S_1(f,g),\\
(D'\circ D)(fg) &= (D'\circ D)(f)(\tau'\circ\tau)(g) + (\sigma'\circ\sigma)(f)(D'\circ D)(g)\\
&\qquad + S_2(f,g) + T_2(f,g).
\end{aligned}$$

Since σ commutes with σ' and τ commutes with τ',

$$\begin{aligned}
[D,D'](fg) &= (D\circ D' - D'\circ D)(fg)\\
&= [D,D'](f)(\tau\circ\tau')(g) + (\sigma\circ\sigma')(f)[D,D'](g)\\
&\qquad + T_1(f,g) - T_2(f,g) + S_1(f,g) - S_2(f,g).
\end{aligned}$$

So, as the commutator $[D, D']$ of linear maps is a linear map, it is a $(\sigma\circ\sigma', \tau\circ\tau')$-derivation if and only if $T_1(f,g) - T_2(f,g) + S_1(f,g) - S_2(f,g) = 0$ for all $f, g \in \mathcal{A}$. In particular, if (D,σ,τ) and (D',σ',τ') cross commute, then $T_1(f,g) - T_2(f,g) = 0$, $S_1(f,g) - S_2(f,g) = 0$, and so, $T_1(f,g) - T_2(f,g) + S_1(f,g) - S_2(f,g) = 0$, and for all $f, g \in \mathcal{A}$, $[D,D'](fg) = [D,D'](f)(\tau\circ\tau')(g) + (\sigma\circ\sigma')(f)[D,D'](g)$, which means that $[D,D']$ is a $(\sigma\circ\sigma', \tau\circ\tau')$-derivation. □

Corollary 14.1 *Let $\mathcal{A}$ be an algebra, and $D, D' \in \mathcal{D}_{\sigma,\tau}(\mathcal{A})$. If D and D' commute with σ and τ, then the commutator $[D,D'] = D\circ D' - D'\circ D$ is a (σ^2,τ^2)-derivation.*

This simplified version of cross-commutation is illustrative of the general behaviour of two twisted derivatives in relation to the twisting maps attached to each other.

As a first approach to this problem, consider that powers of σ and τ always commute as linear maps on $\mathcal{A}$. This limitation on the twisting maps taken into consideration introduces certain regularity in the commutator as will be seen, which opens the possibility of finding a *linear subspace* of pairwise cross-commuting (σ^k,τ^l)-derivations.

Definition 14.7 Let $\mathcal{A}$ be an algebra and $k \in \mathbb{Z}_{\geq 0}$. The linear space of (σ^k, τ^k)-derivations of $\mathcal{A}$ is denoted $\mathcal{D}^{(k)}_{\sigma,\tau}(\mathcal{A})$. If σ and τ are invertible, then the linear space of (σ^{-k}, τ^{-k})-derivations of $\mathcal{A}$ will be denoted $\mathcal{D}^{(-k)}_{\sigma,\tau}(\mathcal{A})$.

In the upcoming results, any (σ^k, τ^k)-derivation will be denoted $D^{(k)}$. Let $k, l \in \mathbb{Z}_{\geq 0}$ and consider two twisted derivations $D^{(k)}$ and $D^{(l)}$.

$$\begin{aligned}
&(D^{(k)} \circ D^{(l)})(fg) = D^{(k)}(D^{(l)}(f)\tau^l(g) + \sigma^l(f)D^{(l)}(g)) = D^{(k)}(D^{(l)}(f))\tau^k(\tau^l(g))\\
&+ \sigma^k(D^{(l)}(f))D^{(k)}(\tau^l(g)) + D^{(k)}(\sigma^l(f))\tau^k(D^{(l)}(g)) + \sigma^k(\sigma^l(f))D^{(k)}(D^{(l)}(g))\\
&= (D^{(k)} \circ D^{(l)})(f)\tau^{k+l}(g) + \sigma^{k+l}(f)(D^{(k)} \circ D^{(l)})(g) + (\sigma^k \circ D^{(l)})(f)(D^{(k)} \circ \tau^l)(g)\\
&+ (D^{(k)} \circ \sigma^l)(f)(\tau^k \circ D^{(l)})(g)\\
&(D^{(l)} \circ D^{(k)})(fg) = D^{(l)}(D^{(k)}(f)\tau^k(g) + \sigma^k(f)D^{(k)}(g)) = D^{(l)}(D^{(k)}(f))\tau^l(\tau^k(g))\\
&+ \sigma^l(D^{(k)}(f))D^{(l)}(\tau^k(g)) + D^{(l)}(\sigma^k(f))\tau^l(D^{(k)}(g)) + \sigma^l(\sigma^k(f))D^{(l)}(D^{(k)}(g))\\
&= (D^{(l)} \circ D^{(k)})(f)\tau^{l+k}(g) + \sigma^{l+k}(f)(D^{(l)} \circ D^{(k)})(g) + (\sigma^l \circ D^{(k)})(f)(D^{(l)} \circ \tau^k)(g)\\
&+ (D^{(l)} \circ \sigma^k)(f)(\tau^l \circ D^{(k)})(g)
\end{aligned}$$

Then, $D^{(k)} \circ D^{(l)}$, $D^{(l)} \circ D^{(k)}$ and $[D^{(k)}, D^{(l)}]$ can be rewritten as follows:

$$\begin{aligned}
(D^{(k)} \circ D^{(l)})(fg) &= (D^{(k)} \circ D^{(l)})(f)\tau^{k+l}(g) + \sigma^{k+l}(f)(D^{(k)} \circ D^{(l)})(g)\\
&\quad + S_{1,l,k}f, g) + S_{1,k,l}(f, g),\\
(D^{(l)} \circ D^{(k)})(fg) &= (D^{(l)} \circ D^{(k)})(f)\tau^{k+l}(g) + \sigma^{k+l}(f)(D^{(l)} \circ D^{(k)})(g)\\
&\quad + S_{2,k,l}(f, g) + S_{2,l,k}(f, g),\\
[D^{(k)}, D^{(l)}](fg) &= (D^{(k)} \circ D^{(l)} - D^{(l)} \circ D^{(k)})(fg)\\
&= [D^{(k)}, D^{(l)}](f)\tau^{k+l}(g) + \sigma^{k+l}(f)[D^{(k)}, D^{(l)}](g)\\
&\quad + S_{1,l,k}(f, g) - S_{2,l,k}(f, g) + S_{1,k,l}(f, g) - S_{2,k,l}(f, g),\\
\text{where}\quad & S_{1,k,l}(fg) = (D^{(k)} \circ \sigma^l)(f)(\tau^k \circ D^{(l)})(g),\\
& S_{2,k,l}(fg) = (\sigma^l \circ D^{(k)})(f)(D^{(l)} \circ \tau^k)(g).
\end{aligned}$$

We observe a similar phenomenon here: the two first terms of $[D^{(k)}, D^{(l)}](fg)$ look exactly like a $(\sigma^{k+l}, \tau^{k+l})$-Leibniz rule, while the rest terms separate it from $\mathcal{D}^{(k+l)}_{\sigma,\tau}(\mathcal{A})$. If $D^{(k)}$ commutes with σ^l, τ^l and $D^{(l)}$ commutes with σ^k and τ^k, that is if $(D^{(k)}, \sigma^k, \tau^k)$ and $(D^{(l)}, \sigma^l, \tau^l)$ cross-commute, then $S_{1,l,k}(f, g) = S_{2,l,k}(f, g)$ and $S_{1,k,l}(f, g) = S_{2,k,l}(f, g)$, the tail vanishes and the twisted Leibniz rule remains. We get thus the following statement on action of commutator on spaces $\mathcal{D}^{(j)}_{\sigma,\tau}(\mathcal{A})$.

Proposition 14.5 *Let $\mathcal{A}$ be an algebra. Let $D^{(j)} \in \mathcal{D}^{(j)}_{\sigma,\tau}(\mathcal{A})$, $j \in \mathbb{Z}_{\geq 0}$. Then, for any $k, l \in \mathbb{Z}_{\geq 0}$, the commutator $[D^{(k)}, D^{(l)}] : (D^{(k)} \circ D^{(l)} - D^{(l)} \circ D^{(k)})$ is a $(\sigma^{k+l}, \tau^{k+l})$-derivation if and only if, for all $f, g \in \mathcal{A}$,*

$$S_{1,l,k}(f,g) - S_{2,l,k}(f,g) + S_{1,k,l}(f,g) - S_{2,k,l}(f,g) = 0,$$
$$\textit{where} \quad \begin{aligned} S_{1,k,l}(f,g) &= (D^{(k)} \circ \sigma^l)(f)(\tau^k \circ D^{(l)})(g), \\ S_{2,k,l}(f,g) &= (\sigma^l \circ D^{(k)})(f)(D^{(l)} \circ \tau^k)(g). \end{aligned}$$

In particular, if the triples $(D^{(k)}, \sigma^k, \tau^k)$ *and* $(D^{(l)}, \sigma^l, \tau^l)$ *cross-commute, then* $[D^{(k)}, D^{(l)}]$ *is a* $(\sigma^{k+l}, \tau^{k+l})$*-derivation.*

Composition of maps in $\mathfrak{L}(\mathcal{A})$ is associative. This has important effects on commutation between maps.

Remark 14.1 If a linear map α is invertible and commutes with a linear map β on a linear space $\mathcal{A}$, then $\alpha^{-1} \circ \beta = \alpha^{-1} \circ \beta \circ \alpha \circ \alpha^{-1} = \alpha^{-1} \circ \alpha \circ \beta \circ \alpha^{-1} = \beta \circ \alpha^{-1}$, and thus, if α is invertible and β commutes with α, then it automatically commutes with α^{-1} and viceversa.

Lemma 14.2 *Let* $\alpha, \beta : \mathcal{A} \to \mathcal{A}$ *be commuting linear maps on a linear space* $\mathcal{A}$*. Then* α *commutes with* β^k *for all* $k \in \mathbb{Z}_{\geq 0}$*, and if moreover* β *is invertible, then* α *commutes also with* β^{-k} *for all* $k \in \mathbb{Z}_{\geq 0}$*. Also , if* α *is invertible, then* α^{-1} *commutes with* β^k *for all* $k \in \mathbb{Z}_{\geq 0}$*, and if both* α *and* β *are invertible, then* α^{-l} *commutes with* β^{-k} *for all* $k, l \in \mathbb{Z}_{\geq 0}$*.*

Proof We proceed by induction on $k \in \mathbb{Z}_{\geq 0}$. For $k = 0$ and $k = 1$, trivially $\alpha \circ \beta^0 = \alpha \circ \mathrm{Id} = \mathrm{Id} \circ \alpha = \beta^0 \circ \alpha$ and $\alpha \circ \beta = \beta \circ \alpha$ hold. By associativity of composition and commutativity of α and β, it holds that

$$\alpha \circ \beta^{k+1} = \alpha \circ (\beta^k \circ \beta) \overset{assoc.}{=} (\alpha \circ \beta^k) \circ \beta \overset{commut.}{=} (\beta^k \circ \alpha) \circ \beta$$
$$\overset{assoc.}{=} \beta^k \circ (\alpha \circ \beta) \overset{commut.}{=} \beta^k \circ (\beta \circ \alpha) \overset{assoc.}{=} (\beta^k \circ \beta) \circ \alpha = \beta^{k+1} \circ \alpha.$$

Combining this with Remark 14.1 completes the proof. □

The immediate consequence in terms of (σ, τ)-derivations is as follows.

Corollary 14.2 *A linear map* D *(and in particular, a* (σ, τ)*-derivation) on an algebra* $\mathcal{A}$ *that commutes with invertible maps* σ *and* τ*, commutes with all* σ^k *and* τ^k*,* $k \in \mathbb{Z}$*.*

This apparently naive observation has relevant consequences.

Corollary 14.3 *Let* $\mathcal{A}$ *be an algebra,* $k \in \mathbb{Z}_{\geq 0}$*,* $D^{(k)} \in \mathcal{D}^{(k)}_{\sigma,\tau}(\mathcal{A})$ *and* $D^{(l)} \in \mathcal{D}^{(l)}_{\sigma,\tau}(\mathcal{A})$*. If* $D^{(k)}$ *and* $D^{(l)}$ *commute with* σ *and* τ*, then* $D^{(k)}$ *and* $D^{(l)}$ *commute with* σ^j *and* τ^j *for all* $j \in \mathbb{Z}_{\geq 0}$*. Furthermore,* $(D^{(k)}, \sigma^k, \tau^k)$ *and* $(D^{(l)}, \sigma^l, \tau^l)$ *cross-commute. Their commutator* $[D^{(k)}, D^{(l)}]$ *is a* $(\sigma^{k+l}, \tau^{k+l})$*-derivation.*

Proof If $D^{(k)}$(resp. $D^{(l)}$) commutes with σ and τ, it forcefully commutes with σ^l and τ^l (resp. σ^k and τ^k) (Lemma 14.2). Cross-commutation is automatic, and conclusion follows from Proposition 14.5. □

Commutation with σ and τ is a baseline property behind this whole process. We introduce a new notation to indicate when a twisted derivative verifies these basic commutation relations.

Definition 14.8 We denote by $\Delta^{(k)}_{\sigma,\tau}(\mathcal{A})$ the $\mathbb{F}$-linear subspace of (σ^k, τ^k)-derivations that commute with σ and τ. Any operator in $\Delta^{(k)}_{\sigma,\tau}(\mathcal{A})$ will be denoted $\Delta^{(k)}$.

Proposition 14.6 *The space* $\mathcal{A}^{\Delta} = \sum\limits_{k,l\in\mathbb{Z}_{\geq 0}} \Delta^{(k)}_{\sigma,\tau}(\mathcal{A})$ *is* $\mathbb{F}$*-linear and closed under the commutator.*

Proof $\mathbb{F}$-linearity is granted by $\mathcal{A}^{\Delta}$ being a sum of $\mathbb{F}$-linear spaces. By Corollary 14.3, for $k, l \in \mathbb{Z}_{\geq 0}$, we have $[\Delta^{(k)}, \Delta^{(l)}] \in \mathcal{D}^{(k+l)}_{\sigma,\tau}(\mathcal{A})$. Commutation with σ and τ is proven using associativity of composition of maps in $\mathfrak{L}(\mathcal{A})$:

$$\begin{aligned}
&[\Delta^{(k)}, \Delta^{(l)}] \circ \sigma \\
&\quad = (\Delta^{(k)} \circ \Delta^{(l)}) \circ \sigma - (\Delta^{(l)} \circ \Delta^{(k)}) \circ \sigma = \Delta^{(k)} \circ (\Delta^{(l)} \circ \sigma) - \Delta^{(l)} \circ (\Delta^{(k)} \circ \sigma) \\
&\quad = \Delta^{(k)} \circ (\sigma \circ \Delta^{(l)}) - \Delta^{(l)} \circ (\sigma \circ \Delta^{(k)}) = (\Delta^{(k)} \circ \sigma) \circ \Delta^{(l)} - (\Delta^{(l)} \circ \sigma) \circ \Delta^{(k)} \\
&\quad = (\sigma \circ \Delta^{(k)}) \circ \Delta^{(l)} - (\sigma \circ \Delta^{(l)}) \circ \Delta^{(k)} = \sigma \circ (\Delta^{(k)} \circ \Delta^{(l)}) - \sigma \circ (\Delta^{(l)} \circ \Delta^{(k)}) \\
&\quad = \sigma \circ (\Delta^{(k)} \circ \Delta^{(l)} - \Delta^{(l)} \circ \Delta^{(k)}) = \sigma \circ [\Delta^{(k)}, \Delta^{(l)}], \\
&[\Delta^{(k)}, \Delta^{(l)}] \circ \tau \\
&\quad = (\Delta^{(k)} \circ \Delta^{(l)}) \circ \tau - (\Delta^{(l)} \circ \Delta^{(k)}) \circ \tau = \Delta^{(k)} \circ (\Delta^{(l)} \circ \tau) - \Delta^{(l)} \circ (\Delta^{(k)} \circ \tau) \\
&\quad = \Delta^{(k)} \circ (\tau \circ \Delta^{(l)}) - \Delta^{(l)} \circ (\tau \circ \Delta^{(k)}) = (\Delta^{(k)} \circ \tau) \circ \Delta^{(l)} - (\Delta^{(l)} \circ \tau) \circ \Delta^{(k)} \\
&\quad = (\tau \circ \Delta^{(k)}) \circ \Delta^{(l)} - (\tau \circ \Delta^{(l)}) \circ \Delta^{(k)} = \tau \circ (\Delta^{(k)} \circ \Delta^{(l)}) - \tau \circ (\Delta^{(l)} \circ \Delta^{(k)}) \\
&\quad = \tau \circ (\Delta^{(k)} \circ \Delta^{(l)} - \Delta^{(l)} \circ \Delta^{(k)}) = \tau \circ [\Delta^{(k)}, \Delta^{(l)}],
\end{aligned}$$

so indeed $[\Delta^{(k)}, \Delta^{(l)}] \in \Delta^{(k+l)}_{\sigma,\tau}(\mathcal{A})$. Finally, let $a_i \in \mathbb{F}$, $k_i \in \mathbb{Z}_{\geq 0}$, $\Delta^{(k_i)} \in \Delta^{(k_i)}_{\sigma,\tau}(\mathcal{A})$, and define $\Delta = \sum\limits_{i=1}^{n} a_i \Delta^{(k_i)}$. Then, for all $l \in \mathbb{Z}_{\geq 0}$,

$$[\Delta, \Delta^{(l)}] = [\sum_{i=1}^{n} a_i \Delta^{(k_i)}, \Delta^{(l)}] = \sum_{i=1}^{n} [a_i \Delta^{(k_i)}, \Delta^{(l)}] = \sum_{i=1}^{n} a_i [\Delta^{(k_i)}, \Delta^{(l)}] \in \mathcal{A}^{\Delta},$$

where bilinearity of the commutator is used to swap the sum out. □

The commutator is skew-symmetric by definition. If we consider $(\mathcal{A}^{\Delta}, +, [\cdot,\cdot])$ as a subalgebra of $\mathfrak{L}(\mathcal{A})$, composition is associative and thus the commutator satisfies the Jacobi identity. We apply these properties together with Proposition 14.6 to $\mathcal{A}^{\Delta}$.

Theorem 14.2 ($\mathbb{Z}_{\geq 0}$-graded Lie algebra of (σ, τ)-derivations) *Let* $\mathcal{A}$ *be an algebra, and* $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ *be two linear maps. Then,*

(i) *with the bilinear product* $[\cdot,\cdot]$ *defined by commutator*

$$[\Delta^{(k)}, \Delta^{(l)}] = \Delta^{(k)} \circ \Delta^{(l)} - \Delta^{(l)} \circ \Delta^{(k)}, \quad k, l \in \mathbb{Z}_{\geq 0},$$

$(\mathcal{A}^{\Delta} = \sum_{k,l \in \mathbb{Z}_{\geq 0}} \Delta^{(k)}_{\sigma,\tau}(\mathcal{A}), [\cdot,\cdot])$ *is a Lie algebra,*

(ii) *whenever the sum is direct,* $(\mathcal{A}^{\Delta} = \bigoplus_{k,l \in \mathbb{Z}_{\geq 0}} \Delta^{(k)}_{\sigma,\tau}(\mathcal{A}), [\cdot,\cdot])$ *is* $\mathbb{Z}_{\geq 0}$*-graded Lie subalgebra of* $\mathfrak{L}(\mathcal{A})$ *with grading* $g : \mathbb{Z}_{\geq 0} \longrightarrow \mathcal{A}^{\Delta},\ k \longmapsto \Delta^{(k)}_{\sigma,\tau}(\mathcal{A})$.

If σ and τ are invertible, it makes sense to speak of (σ^{-1}, τ^{-1})-derivations as well. In this case, the inductive result (14.2) appears naturally by considering $\alpha \in \{\sigma^{-1}, \tau^{-1}\}$, and Corollary 14.3 follows and we obtain a similar Lie subalgebra of $\mathfrak{L}(\mathcal{A})$ by considering the $\mathbb{F}$-linear subspace $\mathcal{D}^{(-k)}_{\sigma,\tau}(\mathcal{A})$ of (σ^{-k}, τ^{-k})-derivations of $\mathcal{A}$, with $\Delta^{(-k)}_{\sigma,\tau}(\mathcal{A})$ denoting the subspace of those commuting with σ^{-1} and τ^{-1}.

Theorem 14.3 ($\mathbb{Z}_{\geq 0}$-graded Lie algebra of (σ^{-1}, τ^{-1})-derivations) *Let* $\mathcal{A}$ *be an algebra, and* $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ *be invertible linear maps. Then,*

(i) *with the commutator bracket* $[\cdot,\cdot]$ *defined by*

$$[\Delta^{(-k)}, \Delta^{(-l)}] = \Delta^{(-k)} \circ \Delta^{(-l)} - \Delta^{(-l)} \circ \Delta^{(-k)}, \quad k, l \in \mathbb{Z}_{\geq 0},$$

$(\mathcal{A}^{-\Delta} = \sum_{k,l \in \mathbb{Z}_{\geq 0}} \Delta^{(-k)}_{\sigma,\tau}(\mathcal{A}), [\cdot,\cdot])$ *is a Lie algebra,*

(ii) *whenever the sum is direct,* $(\mathcal{A}^{-\Delta} = \bigoplus_{k,l \in \mathbb{Z}_{\geq 0}} \Delta^{(-k)}_{\sigma,\tau}(\mathcal{A}), [\cdot,\cdot])$ *is* $\mathbb{Z}_{\geq 0}$*-graded Lie subalgebra of* $\mathfrak{L}(\mathcal{A})$ *with grading* $g : \mathbb{Z}_{\geq 0} \longrightarrow \mathcal{A}^{\Delta},\ k \longmapsto \Delta^{(-k)}_{\sigma,\tau}(\mathcal{A})$.

Remark 14.1 hints about a similar interaction between (σ^k, τ^k)- and (σ^{-l}, τ^{-l})-derivations. Regular derivations correspond to the 0 case, and invertibility on *both* σ and τ allows to consider a wider $\mathcal{A}^{\Delta}$, which is graded by the whole group $\mathbb{Z}$ instead of the semigroup $\mathbb{Z}_{\geq 0}$ on each twisting map.

Theorem 14.4 ($\mathbb{Z}$-graded Lie algebra of (σ, τ)-derivations) *Let* $\mathcal{A}$ *be an algebra, and* $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ *be invertible linear maps. Then,*

(i) *with the bilinear product* $[\cdot,\cdot]$ *defined by commutator*

$$[\Delta^{(k)}, \Delta^{(l)}] = \Delta^{(k)} \circ \Delta^{(l)} - \Delta^{(l)} \circ \Delta^{(k)}, k \in \mathbb{Z},$$

$(\mathcal{A}^{\Delta} = \sum_{k \in \mathbb{Z}} \Delta^{(k)}_{\sigma,\tau}(\mathcal{A}), [\cdot,\cdot])$ *is a Lie algebra,*

(ii) *whenever the sum is direct,* $(\mathcal{A}^{\Delta} = \bigoplus_{k \in \mathbb{Z}} \Delta^{(k)}_{\sigma,\tau}(\mathcal{A}), [\cdot,\cdot])$ *is* $\mathbb{Z}$*-graded Lie algebra with grading* $g : \mathbb{Z} \longrightarrow \mathcal{A}^{\Delta},\ k \longmapsto \Delta^{(k)}_{\sigma,\tau}(\mathcal{A})$.

Proof Firstly, we need to prove that $\mathcal{A}^{\Delta}$ is closed under the commutator. Interaction between derivations twisted by positive powers of σ and τ are described in Proposition 14.6, and those twisted by negative powers are given in Theorem 14.3. So, let $k, l \in \mathbb{Z}_{\geq 0}$, $\Delta^{(k)} \in \Delta^{(k)}_{\sigma,\tau}(\mathcal{A})$, $\Delta^{(-l)} \in \Delta^{(-l)}_{\sigma,\tau}(\mathcal{A})$. Then,

$$
\begin{aligned}
(\Delta^{(k)} \circ \Delta^{(-l)})(fg) &= \Delta^{(k)}(\Delta^{(-l)}(f)\tau^{-l}(g) + \sigma^{-l}(f)\Delta^{(-l)}(g)) \\
&= \Delta^{(k)}(\Delta^{(-l)}(f))\tau^{k}(\tau^{-l}(g)) + \sigma^{k}(\Delta^{(-l)}(f))\Delta^{(k)}(\tau^{-l}(g)) \\
&+ \Delta^{(k)}(\sigma^{-l}(f))\tau^{k}(\Delta^{(-l)}(g)) + \sigma^{k}(\sigma^{-l}(f))\Delta^{(k)}(\Delta^{(-l)}(g)) \\
&= (\Delta^{(k)} \circ \Delta^{(-l)})(f)\tau^{k-l}(g) + \sigma^{k-l}(f)(\Delta^{(k)} \circ \Delta^{(-l)})(g) \\
&+ (\sigma^{k} \circ \Delta^{(-l)})(f)(\Delta^{(k)} \circ \tau^{-l})(g) + (\Delta^{(k)} \circ \sigma^{-l})(f)(\tau^{k} \circ \Delta^{(-l)})(g), \\
(\Delta^{(-l)} \circ \Delta^{(k)})(fg) &= \Delta^{(-l)}(\Delta^{(k)}(f)\tau^{k}(g) + \sigma^{k}(f)\Delta^{(k)}(g)) \\
&= \Delta^{(-l)}(\Delta^{(k)}(f))\tau^{-l}(\tau^{k}(g)) + \sigma^{-l}(\Delta^{(k)}(f))\Delta^{(-l)}(\tau^{k}(g)) \\
&+ \Delta^{(-l)}(\sigma^{k}(f))\tau^{-l}(\Delta^{(k)}(g)) + \sigma^{-l}(\sigma^{k}(f))\Delta^{(-l)}(\Delta^{(k)}(g)) \\
&= (\Delta^{(-l)} \circ \Delta^{(k)})(f)\tau^{-l+k}(g) + \sigma^{-l+k}(f)(\Delta^{(-l)} \circ \Delta^{(k)})(g) \\
&+ (\sigma^{-l} \circ \Delta^{(k)})(f)(\Delta^{(-l)} \circ \tau^{k})(g) + (\Delta^{(-l)} \circ \sigma^{k})(f)(\tau^{-l} \circ \Delta^{(k)})(g).
\end{aligned}
$$

Then, $D^{(k)} \circ D^{(-l)}$ and $D^{(-l)} \circ D^{(k)}$ and $[\Delta^{(k)}, \Delta^{(-l)}]$ can be rewritten as follows:

$$
\begin{aligned}
&(\Delta^{(k)} \circ \Delta^{(-l)})(fg) = (\Delta^{(k)} \circ \Delta^{(-l)})(f)\tau^{k-l}(g) + \sigma^{k-l}(f)(\Delta^{(k)} \circ \Delta^{(-l)})(g) \\
&+ S_{2,k,-l} + S_{1,k,-l}, \\
&(\Delta^{(-l)} \circ \Delta^{(k)})(fg) = (\Delta^{(-l)} \circ \Delta^{(k)})(f)\tau^{k-l}(g) + \sigma^{k-l}(f)(\Delta^{(-l)} \circ \Delta^{(k)})(g) \\
&+ S_{2,l,-k} + S_{1,-l,k}, \\
&[\Delta^{(k)}, \Delta^{(-l)}](fg) = (\Delta^{(k)} \circ \Delta^{(-l)} - \Delta^{(-l)} \circ \Delta^{(k)})(fg) \\
&= [\Delta^{(k)}, \Delta^{(-l)}](f)\tau^{k-l}(g) + \sigma^{k-l}(f)[\Delta^{(k)}, \Delta^{(-l)}](g) \\
&+ S_{2,k,-l} - S_{1,-l,k} + S_{1,k,-l} - S_{2,-l,k}, \\
&\quad \text{where} \quad \begin{array}{l} S_{1,k,-l}(f, g) = (\Delta^{(k)} \circ \sigma^{-l})(f)(\tau^{k} \circ \Delta^{(-l)})(g), \\ S_{2,-l,k}(f, g) = (\sigma^{k} \circ \Delta^{(-l)})(f)(\Delta^{(k)} \circ \tau^{-l})(g). \end{array}
\end{aligned}
$$

Now, if $\Delta^{(k)}$, $\Delta^{(-l)}$ commute with σ and τ, then they commute with all powers of σ and τ by Corollary 14.2, and then $T_1(f, g) - S_{1,-l,k}(f, g) = 0 = S_{1,k,-l}(f, g) - S_2(f, g)$, which implies that

$$[\Delta^{(k)}, \Delta^{(-l)}] \in D^{(k-l)}_{\sigma,\tau}(\mathcal{A}).$$

Replicating the last part of the proof of Proposition 14.6 yields $[\Delta^{(k)}, \Delta^{(-l)}] \in \Delta^{(k-l)}_{\sigma,\tau}(\mathcal{A})$. Also, $[\Delta, \Delta^{(-l)}] \in \mathcal{A}^{\Delta}$, where $\Delta = \sum_i a_i \Delta^{(k_i)}$ is a finite linear combination of the operators $\Delta^{(k_i)}$. This completes the proof. □

14.4.2 $\mathbb{Z}^2$-Grading on (σ, τ)-Derivations

The way the commutator interacts with powers of σ and τ to give place to new twisted derivation rules suggests that said powers are not necessarily related.

For $k, l \in \mathbb{Z}_{\geq 0}$, denote by $\mathcal{D}_{\sigma,\tau}^{(k,l)}(\mathcal{A})$ the $\mathbb{F}$-linear space of all (σ^k, τ^l)-derivations on $\mathcal{A}$. Let $k, l, m, r \in \mathbb{Z}_{\geq 0}$, and $D^{(k,l)} \in \mathcal{D}_{\sigma,\tau}^{(k,l)}(\mathcal{A})$ and $D^{(m,r)} \in \mathcal{D}_{\sigma,\tau}^{(m,r)}(\mathcal{A})$. Then,

$$\begin{aligned}
&(D^{(k,l)} \circ D^{(m,r)})(fg) = D^{(k,l)}(D^{(m,r)}(f)\tau^r(g) + \sigma^m(f)D^{(m,r)}(g))\\
&= D^{(k,l)}(D^{(m,r)}(f))\tau^l(\tau^r(g)) + \sigma^k(D^{(m,r)}(f))D^{(k,l)}(\tau^r(g))\\
&\quad + D^{(k,l)}(\sigma^m(f))\tau^l(D^{(m,r)}(g)) + \sigma^k(\sigma^m(f))D^{(k,l)}(D^{(m,r)}(g))\\
&= (D^{(k,l)} \circ D^{(m,r)})(f)\tau^{l+r}(g) + \sigma^{k+m}(f)(D^{(k,l)} \circ D^{(m,r)})(g)\\
&\quad + (D^{(k,l)} \circ \sigma^m)(f)(\tau^l \circ D^{(m,r)})(g) + (\sigma^k \circ D^{(m,r)})(f)(D^{(k,l)} \circ \tau^r)(g),\\
&(D^{(m,r)} \circ D^{(k,l)})(fg) = (D^{(m,r)} \circ D^{(k,l)})(f)\tau^{l+r}(g) + \sigma^{k+m}(f)(D^{(m,r)} \circ D^{(k,l)})(g)\\
&\quad + (D^{(m,r)} \circ \sigma^k)(f)(\tau^r \circ D^{(k,l)})(g) + (\sigma^m \circ D^{(k,l)})(f)(D^{(m,r)} \circ \tau^l)(g).
\end{aligned}$$

Then, $D^{(k,l)} \circ D^{(m,r)}$, $D^{(k,l)} \circ D^{(m,r)}$ and $[D^{(k,l)}, D^{(m,r)}]$ are rewritten as

$$\begin{aligned}
(D^{(k,l)} \circ D^{(m,r)})(fg) &= (D^{(k,l)} \circ D^{(m,r)})(f)\tau^{l+r}(g) + \sigma^{k+m}(f)(D^{(k,l)} \circ D^{(m,r)})(g)\\
&\quad + S_{1,k,l,m,r} + S_{2,k,l,m,r},\\
(D^{(m,r)} \circ D^{(k,l)})(fg) &= (D^{(m,r)} \circ D^{(k,l)})(f)\tau^{l+r}(g) + \sigma^{k+m}(f)(D^{(m,r)} \circ D^{(k,l)})(g)\\
&\quad + S_{1,m,r,k,l} + S_{2,m,r,k,l},\\
[D^{(k,l)}, D^{(m,r)}](fg) &= (D^{(k,l)} \circ D^{(m,r)} - D^{(m,r)} \circ D^{(k,l)})(fg)\\
&= [D^{(k,l)}, D^{(m,r)}](f)\tau^{l+r}(g) + \sigma^{k+m}(f)[D^{(k,l)}, D^{(m,r)}](g)\\
&\quad + S_{2,k,l,m,r} - S_{2,m,r,k,l} + S_{1,k,l,m,r} - S_{1,m,r,k,l},
\end{aligned}$$

$$\text{where} \quad \begin{aligned} S_{1,k,l,m,r} &= S_{1,k,l,m,r}(f, g) = (D^{(k,l)} \circ \sigma^m)(f)(\tau^l \circ D^{(m,r)})(g),\\ S_{2,k,l,m,r} &= S_{2,k,l,m,r}(f, g) = (\sigma^k \circ D^{(m,r)})(f)(D^{(k,l)} \circ \tau^r)(g).\end{aligned}$$

Hence, $[D^{(k,l)}, D^{(m,r)}]$ is a $(\sigma^{k+m}, \tau^{l+r})$-derivation if and only if the tail vanishes, $S_{2,k,l,m,r} - S_{2,m,r,k,l} + S_{1,k,l,m,r} - S_{1,m,r,k,l} = 0$. In particular,

$$\begin{aligned}
D^{(m,r)} \circ \sigma^k = \sigma^k \circ D^{(m,r)} \text{ and } D^{(k,l)} \circ \tau^r = \tau^r \circ D^{(k,l)} &\Rightarrow S_{2,k,l,m,r} - S_{2,m,r,k,l} = 0,\\
D^{(k,l)} \circ \sigma^m = \sigma^m \circ D^{(k,l)} \text{ and } D^{(m,r)} \circ \tau^l = \tau^l \circ D^{(m,r)} &\Rightarrow S_{1,k,l,m,r} - S_{1,m,r,k,l} = 0.
\end{aligned}$$

In other words, if the triples $\{D^{(k,l)}, \sigma^k, \tau^l\}$ and $\{D^{(m,r)}, \sigma^m, \tau^r\}$ cross-commute, both commutation relations hold and thus the commutator is another twisted derivation. We get thus the following statement on action of commutator on spaces of type $\mathcal{D}_{\sigma,\tau}^{(k,l)}(\mathcal{A})$.

Proposition 14.7 *Let $k, l, m, r \in \mathbb{Z}_{\geq 0}$, and $D^{(k,l)} \in \mathcal{D}^{(k,l)}_{\sigma,\tau}(\mathcal{A})$, $D^{(m,r)} \in \mathcal{D}^{(m,r)}_{\sigma,\tau}(\mathcal{A})$. Then, $[D^{(k,l)}, D^{(m,r)}]$ is a $(\sigma^{k+m}, \tau^{l+r})$-derivation if and only if, for all $f, g \in \mathcal{A}$,*

$$S_{2,k,l,m,r} - S_{2,m,r,k,l} + S_{1,k,l,m,r} - S_{1,m,r,k,l} = 0,$$
$$\text{where} \quad \begin{array}{l} S_{1,k,l,m,r} = S_{1,k,l,m,r}(f,g) = (D^{(k,l)} \circ \sigma^m)(f)(\tau^l \circ D^{(m,r)})(g), \\ S_{2,k,l,m,r} = S_{2,k,l,m,r}(f,g) = (\sigma^k \circ D^{(m,r)})(f)(D^{(k,l)} \circ \tau^r)(g). \end{array}$$

In particular, if the triples $\{D^{(k,l)}, \sigma^k, \tau^l\}$ and $\{D^{(m,r)}, \sigma^m, \tau^r\}$ cross-commute, then $[D^{(k,l)}, D^{(m,r)}]$ is a $(\sigma^{k+m}, \tau^{l+r})$-derivation.

According to Lemma 14.2, any linear map (and particularly, twisted derivations) that commutes with σ and τ automatically commutes with all their powers. This allows to introduce another Lie structure within $\mathfrak{L}(\mathcal{A})$. Denote by $\Delta^{(k,l)}_{\sigma,\tau}(\mathcal{A})$ the $\mathbb{F}$-linear subspace of the (σ^k, τ^l)-derivations commuting with σ and τ. Any operator in $\Delta^{(k,l)}_{\sigma,\tau}(\mathcal{A})$ will be denoted $\Delta^{(k,l)}$.

Theorem 14.5 ($\mathbb{Z}^2_{\geq 0}$-graded Lie algebra of (σ, τ)-derivations) *Let $\mathcal{A}$ be an algebra, and $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ be linear maps. Then,*

(i) *with the bilinear product $[\cdot,\cdot]$ defined for $\Delta^{(k,l)} \in \Delta^{(k,l)}_{\sigma,\tau}(\mathcal{A})$, $\Delta^{(m,r)} \in \Delta^{(m,r)}_{\sigma,\tau}(\mathcal{A})$ by commutator*

$$[\Delta^{(k,l)}, \Delta^{(m,r)}] = \Delta^{(k,l)} \circ \Delta^{(m,r)} - \Delta^{(k,l)} \circ \Delta^{(m,r)}, \quad k, l, m, r \in \mathbb{Z}_{\geq 0},$$

$(\mathcal{A}^{\Delta} = \sum_{k,l \in \mathbb{Z}_{\geq 0}} \Delta^{(k,l)}_{\sigma,\tau}(\mathcal{A}), [\cdot,\cdot])$ *is a Lie algebra;*

(ii) *when the sum is direct, $(\mathcal{A}^{\Delta} = \bigoplus_{k,l \in \mathbb{Z}_{\geq 0}} \Delta^{(k,l)}_{\sigma,\tau}(\mathcal{A}), [\cdot,\cdot])$ is $\mathbb{Z}^2_{\geq 0}$-graded Lie subalgebra of $\mathfrak{L}(\mathcal{A})$ with grading $g : \mathbb{Z}^2_{\geq 0} \to \mathcal{A}^{\Delta}$, $(k,l) \mapsto \Delta^{(k,l)}_{\sigma,\tau}(\mathcal{A})$.*

According to the commutation relations in Lemma 14.2, this theorem can naturally be extended to all integer powers of σ and τ if σ and τ are both invertible.

Theorem 14.6 ($\mathbb{Z}^2$-graded Lie algebra of (σ, τ)-derivations) *Let $\mathcal{A}$ be an algebra, $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ invertible linear maps. Then,*

(i) *with the bilinear product $[\cdot,\cdot]$ defined for $\Delta^{(k,l)} \in \Delta^{(k,l)}_{\sigma,\tau}(\mathcal{A})$ and $\Delta^{(m,r)} \in \Delta^{(m,r)}_{\sigma,\tau}(\mathcal{A})$ by commutator*

$$[\Delta^{(k,l)}, \Delta^{(m,r)}] = \Delta^{(k,l)} \circ \Delta^{(m,r)} - \Delta^{(k,l)} \circ \Delta^{(m,r)}, \quad k, l, m, r \in \mathbb{Z}_{\geq 0},$$

$(\mathcal{A}^{\Delta} = \sum_{(k,l) \in \mathbb{Z}^2} \Delta^{(k,l)}_{\sigma,\tau}(\mathcal{A}), [\cdot,\cdot])$ *is a Lie algebra,*

(ii) *whenever the sum is direct, $(\mathcal{A}^{\Delta} = \bigoplus_{(k,l) \in \mathbb{Z}^2} \Delta^{(k,l)}_{\sigma,\tau}(\mathcal{A}), [\cdot,\cdot])$ is a $\mathbb{Z}^2$-graded Lie subalgebra of $\mathfrak{L}(\mathcal{A})$, with grading $g : \mathbb{Z}^2 \longrightarrow \mathcal{A}^{\Delta}$, $(k,l) \mapsto \Delta^{(k,l)}_{\sigma,\tau}(\mathcal{A})$.*

Commutation with σ and τ is not the only relation that provides this type of construction of $\mathcal{A}^{\Delta}$. Let $k_0, l_0 > 0$, also let $\nabla_{k_0,l_0}^{(k,l)}(\mathcal{A})$ denote the space all (σ^k, τ^l)-derivations that commute with σ^{k_0} and τ^{l_0}. We denote by $\nabla_{k_0,l_0}^{(k,l)}$ elements of $\nabla_{k_0,l_0}^{(k,l)}(\mathcal{A})$.

Corollary 14.4 *Let $k_0, l_0 \in \mathbb{Z}_{\geq 0}$ be fixed, also let $k, l, m, r \in \mathbb{Z}_{\geq 0}$ such that k, m are multiples of k_0 and l, r are multiples of l_0. Let $\mathcal{A}$ be an algebra, and $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ be linear maps. Let $\nabla_{k_0,l_0}^{(k,l)}(\mathcal{A})$ be the space all (σ^k, τ^l)-derivations that commute with σ^{k_0} and τ^{l_0}. Then,*

(i) *with the bilinear product $[\cdot, \cdot]$, for $\nabla_{k_0,l_0}^{(k,l)} \in \nabla_{k_0,l_0}^{(k,l)}(\mathcal{A})$, $\nabla_{k_0,l_0}^{(m,r)} \in \nabla_{k_0,l_0}^{(m,r)}(\mathcal{A})$ defined by commutator*

$$[\nabla_{k_0,l_0}^{(k,l)}, \nabla_{k_0,l_0}^{(m,r)}] = \nabla_{k_0,l_0}^{(k,l)} \circ \nabla_{k_0,l_0}^{(m,r)} - \nabla_{k_0,l_0}^{(k,l)} \circ \nabla_{k_0,l_0}^{(m,r)},$$

$(\mathcal{A}^{\nabla} = \sum_{\substack{k \in k_0\mathbb{Z}_{\geq 0} \\ l \in l_0\mathbb{Z}_{\geq 0}}} \nabla_{k_0,l_0}^{(k,l)}(\mathcal{A}), [\cdot, \cdot])$ *is a Lie algebra,*

(ii) *whenever the sum is direct, $(\mathcal{A}^{\nabla} = \bigoplus_{\substack{k \in k_0\mathbb{Z}_{\geq 0} \\ l \in l_0\mathbb{Z}_{\geq 0}}} \nabla_{k_0,l_0}^{(k,l)}(\mathcal{A}), [\cdot, \cdot])$ is a $\mathbb{Z}_{\geq 0}^2$-graded Lie subalgebra of $\mathfrak{L}(\mathcal{A})$ with grading $g : \mathbb{Z}_{\geq 0}^2 \to \mathcal{A}^{\nabla}$, $(k, l) \mapsto \nabla_{k_0,l_0}^{(k,l)}(\mathcal{A})$.*

Proof Let $\hat{\sigma} = \sigma^{k_0}$ and $\hat{\tau} = \tau^{l_0}$. Then, $\nabla_{k_0,l_0}^{(kk_0,ll_0)}(\mathcal{A}) = \Delta_{\hat{\sigma}\hat{\tau}}^{(k,l)}(\mathcal{A})$. Construct the space $\mathcal{A}^{\nabla} = \sum_{(k,l) \in \mathbb{Z}_{\geq 0}} \Delta_{\hat{\sigma},\hat{\tau}}^{(k,l)}(\mathcal{A})$. Apply Theorem 14.5 to space $\mathcal{A}^{\nabla}$ and linear maps $\hat{\sigma}$ and $\hat{\tau}$. It follows that the commutator in $\mathcal{A}^{\nabla}$ is $\mathbb{Z}_{\geq 0}^2$-graded and, if the sum is direct, then $(\mathcal{A}^{\nabla}, [\cdot, \cdot])$ is a graded Lie subalgebra of $\mathfrak{L}(\mathcal{A})$. □

Corollary 14.5 *Let $k_0, l_0 \in \mathbb{Z}_{\geq 0}$ be fixed, also let $k, l, m, r \in \mathbb{Z}$ such that k, m are multiples of k_0 and l, r are multiples of l_0. Let $\mathcal{A}$ be an algebra, and $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ be linear maps. Let $\nabla_{k_0,l_0}^{(k,l)}(\mathcal{A})$ be the space all (σ^k, τ^l)-derivations that commute with σ^{k_0} and τ^{l_0}. Then,*

(i) *with the bilinear product $[\cdot, \cdot]$, for $\nabla_{k_0,l_0}^{(k,l)} \in \nabla_{k_0,l_0}^{(k,l)}(\mathcal{A})$, $\nabla_{k_0,l_0}^{(m,r)} \in \nabla_{k_0,l_0}^{(m,r)}(\mathcal{A})$, defined by commutator*

$$[\nabla_{k_0,l_0}^{(k,l)}, \nabla_{k_0,l_0}^{(m,r)}] = \nabla_{k_0,l_0}^{(k,l)} \circ \nabla_{k_0,l_0}^{(m,r)} - \nabla_{k_0,l_0}^{(k,l)} \circ \nabla_{k_0,l_0}^{(m,r)},$$

$(\mathcal{A}^{\nabla} = \sum_{\substack{k \in k_0\mathbb{Z} \\ l \in l_0\mathbb{Z}}} \nabla_{k_0,l_0}^{(k,l)}(\mathcal{A}), [\cdot, \cdot])$ *is a Lie algebra,*

(ii) *whenever the sum is direct, $(\mathcal{A}^{\nabla} = \bigoplus_{\substack{k \in k_0\mathbb{Z} \\ l \in l_0\mathbb{Z}}} \nabla_{k_0,l_0}^{(k,l)}(\mathcal{A}), [\cdot, \cdot])$ is a $\mathbb{Z}^2$-graded Lie subalgebra of $\mathfrak{L}(\mathcal{A})$ with grading $g : \mathbb{Z}^2 \to \mathcal{A}^{\nabla}$, $(k, l) \mapsto \nabla_{k_0,l_0}^{(k,l)}(\mathcal{A})$.*

When k, l, m, r are not multiples of k_0 and l_0 respectively, cross-commutation is lost and thus $\mathcal{A}^{\nabla}$ is no longer closed.

Proposition 14.8 *Let $\mathcal{A}$ be an algebra, $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ two linear maps, $k_0, l_0 \in \mathbb{Z}_{\geq 0}$ and fix $k, l \in \mathbb{Z}_{\geq 0}$. Then,*

$$\bigcap_{k_0, l_0 \in \mathbb{Z}_{\geq 0}} \nabla_{k_0, l_0}^{(k,l)}(\mathcal{A}) = \Delta_{\sigma,\tau}^{(k,l)}(\mathcal{A}).$$

Proof Observe that, for $k_0 = l_0 = 1$, $\nabla_{k_0,l_0}^{(k,l)}(\mathcal{A}) = \Delta_{\sigma,\tau}^{(k,l)}(\mathcal{A})$. If $\Delta_{\sigma,\tau}^{(k,l)}(\mathcal{A}) \subsetneq \bigcap_{k_0,l_0 \in \mathbb{Z}_{\geq 0}} \nabla_{k_0,l_0}^{(k,l)}(\mathcal{A})$ (strict inclusion), then the intersection contains an element not commuting with σ^1, τ^1. The element is not in $\Delta_{\sigma,\tau}^{(k,l)}(\mathcal{A}) = \nabla_{k_0,l_0}^{(k,l)}(\mathcal{A})$, so it is not in $\bigcap_{k_0,l_0 \in \mathbb{Z}_{\geq 0}} \nabla_{k_0,l_0}^{(k,l)}(\mathcal{A})$. This completes the proof. □

If σ and τ are invertible, define $\sigma^0 = \sigma \circ \sigma^{-1} = id_{\mathcal{A}}$, $\tau^0 = \tau \circ \tau^{-1} = id_{\mathcal{A}}$. The space of non-twisted derivatives can then be expressed as $\mathcal{D}_{\sigma,\tau}^{(0,0)}(\mathcal{A})$. For the sake of consistency, we adopt the same convention $\Delta_{\sigma,\tau}^{(0,0)}(\mathcal{A})$ for those that commute with σ and τ.

This construction accepts other gradings over commutative monoids, which vary with the properties of σ and τ.

Example 14.2 Let σ, τ be idempotent linear maps, that is, $\sigma^2 = \sigma$, $\tau^2 = \tau$. Then

$$\mathcal{A}^{\Delta} = \Delta_{\sigma,\tau}^{(0,0)}(\mathcal{A}) + \Delta_{\sigma,\tau}^{(1,0)}(\mathcal{A}) + \Delta_{\sigma,\tau}^{(0,1)}(\mathcal{A}) + \Delta_{\sigma,\tau}^{(1,1)}(\mathcal{A}).$$

This is graded by $G \times G$, where G is the commutative semigroup $(0, 1)$ with the following Cayley table:

+	0	1
0	0	1
1	1	1

. This grading brings an additional property: $[\Delta_{\sigma,\tau}^{(1,1)}(\mathcal{A}), \Delta_{\sigma,\tau}^{(1,1)}(\mathcal{A})] \subseteq \Delta_{\sigma,\tau}^{(1,1)}(\mathcal{A})$, that is, $\Delta_{\sigma,\tau}^{(1,1)}(\mathcal{A})$ itself is closed under the commutator, and so is a Lie subalgebra of $\mathcal{A}^{\Delta}$. Moreover, for $i, j \in \{0, 1\}$,

$$[\Delta_{\sigma,\tau}^{(i,j)}(\mathcal{A}), \Delta_{\sigma,\tau}^{(1,1)}(\mathcal{A})] \subseteq \Delta_{\sigma,\tau}^{(1,1)}(\mathcal{A}) \Rightarrow [\mathcal{A}^{\Delta}, \Delta_{\sigma,\tau}^{(1,1)}(\mathcal{A})] \subseteq \Delta_{\sigma,\tau}^{(1,1)}(\mathcal{A}),$$

that is, $\Delta_{\sigma,\tau}^{(1,1)}(\mathcal{A})$ is a left ideal of $\mathcal{A}^{\Delta}$. This property is two-sided: given the addition rule in G, it is immediate that, for any $i, j \in \{0, 1\}$, $[\Delta_{\sigma,\tau}^{(1,1)}(\mathcal{A}), \Delta_{\sigma,\tau}^{(i,j)}(\mathcal{A})] \subseteq \Delta_{\sigma,\tau}^{(1,1)}(\mathcal{A}) \Rightarrow [\Delta_{\sigma,\tau}^{(1,1)}(\mathcal{A}), \mathcal{A}^{\Delta}] \subseteq \Delta_{\sigma,\tau}^{(1,1)}(\mathcal{A})$, thus $\Delta_{\sigma,\tau}^{(1,1)}(\mathcal{A})$ is a right ideal of $\mathcal{A}^{\Delta}$.

Example 14.3 Let $m, n \in \mathbb{Z}_{\geq 0}$, σ be nilpotent of order n and τ be nilpotent of order m. Then $\mathcal{A}^{\Delta} = \sum_{k,l \in \mathbb{Z}_{\geq 0}} \Delta_{\sigma,\tau}^{(k,l)}(\mathcal{A})|_{\sigma^n = 0, \tau^m = 0} = \sum_{\substack{0 \leq k < n \\ 0 \leq l < m}} \Delta_{\sigma,\tau}^{(k,l)}(\mathcal{A})$. The commutator in $\mathcal{A}^{\Delta}$ is graded by $\mathbb{Z}_n \times \mathbb{Z}_m = (\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$.

14.4.3 $\mathbb{Z}^2$-Grading on (σ, τ)-Derivations over n-Ary Algebras

Definition 14.9 Let $\mathcal{A} = (A, \mu)$ be an n-ary algebra and σ, τ two linear maps. A linear map $D : A \to A$ is said to be a (σ, τ)-derivation of $\mathcal{A}$ if, for all $x_1, \dots, x_n \in A$,

$$D(\mu(x_1, \dots, x_n)) = \sum_{k=1}^{n} \mu(\sigma(x_1), \dots, \sigma(x_{k-1}), D(x_k), \tau(x_{k+1}), \dots, \tau(x_n)).$$

For $k, l \in \mathbb{Z}_{\geq 0}$, let $\mathcal{D}_{\sigma,\tau}^{(k,l)}(\mathcal{A})$ be the $\mathbb{F}$-linear space of all (σ^k, τ^l)-derivations on $\mathcal{A}$.

Proposition 14.9 *Let $\mathcal{A} = (A, \mu)$ be an n-ary algebra. Let $k, l, m, r \in \mathbb{Z}_{\geq 0}$. Let σ, τ be linear maps on A such that $\sigma^r \circ \tau^k = \tau^k \circ \sigma^r$ and $\sigma^m \circ \tau^l = \tau^l \circ \sigma^m$.*

The commutator $[D^{(k,l)}, D^{(m,r)}] = D^{(k,l)} \circ D^{(m,r)} - D^{(m,r)} \circ D^{(k,l)}$ is $(\sigma^{k+m}, \tau^{l+r})$-derivation of $\mathcal{A}$ if and only if for all $x_1, \dots, x_n \in \mathcal{A}$,

$$S_1(x_1, \dots, x_n) + T_1(x_1, \dots, x_n) - S_2(x_1, \dots, x_n) - T_2(x_1, \dots, x_n) = 0,$$

where

$$S_1(x_1, \dots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{i-1} \mu\big(\sigma^{k+m}(x_1), \dots, \sigma^{k+m}(x_{j-1}), D^{(k,l)} \circ \sigma^m(x_l),$$
$$\tau^l \circ \sigma^m(x_{j+1}), \dots, \tau^l \circ \sigma^m(x_{i-1}), \tau^l \circ D^{(m,r)}(x_i), \tau^{l+r}(x_{i+1}), \dots, \tau^{l+r}(x_n)\big),$$
$$T_1(x_1, \dots, x_n) = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \mu\big(\sigma^{k+m}(x_1), \dots, \sigma^{k+m}(x_{i-1}), \sigma^k \circ D^{(m,r)}(x_i),$$
$$\sigma^k \circ \tau^r(x_{i+1}), \dots, \sigma^k \circ \tau^r(x_{j-1}), D^{(k,l)} \circ \tau^r(x_j), \tau^{l+r}(x_{j+1}), \dots, \tau^{l+r}(x_n)\big),$$
$$S_2(x_1, \dots, x_n) = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \mu\big(\sigma^{k+m}(x_1), \dots, \sigma^{k+m}(x_{i-1}), \sigma^m \circ D^{(k,l)}(x_i),$$
$$\sigma^m \circ \tau^l(x_{i+1}), \dots, \sigma^m \circ \tau^l(x_{i-1}), D^{(m,r)} \circ \tau^l(x_j), \tau^{l+r}(x_{j+1}), \dots, \tau^{l+r}(x_n)\big),$$
$$T_2(x_1, \dots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{i-1} \mu\big(\sigma^{k+m}(x_1), \dots, \sigma^{k+m}(x_{j-1}), D^{(m,r)} \circ \sigma^k(x_j),$$
$$\tau^r \circ \sigma^k(x_{j+1}), \dots, \tau^r \circ \sigma^k(x_{i-1}), \tau^r \circ D^{(k,l)}(x_i), \tau^{l+r}(x_{i+1}), \dots, \tau^{l+r}(x_n)\big).$$

In particular, if $D^{(k,l)} \in \mathcal{D}_{\sigma,\tau}^{(k,l)}(\mathcal{A})$, $D^{(m,r)} \in \mathcal{D}_{\sigma,\tau}^{(m,r)}(\mathcal{A})$ is such that $\{D^{(k,l)}, \sigma^k, \tau^l\}$ and $\{D^{(m,r)}, \sigma^m, \tau^r\}$ cross-commute, that is, $D^{(k,l)}$ commutes with σ^m and τ^r, and $D^{(m,r)}$ commutes with σ^k and τ^l, then $[D^{(k,l)}, D^{(m,r)}] = D^{(k,l)} \circ D^{(m,r)} - D^{(m,r)} \circ D^{(k,l)}$ is $(\sigma^{k+m}, \tau^{l+r})$-derivation of $\mathcal{A}$.

Proof For $x_1, \dots, x_n \in A$,

$$D^{(k,l)} \circ D^{(m,r)}(\mu(x_1, \dots, x_n))$$

$$= D^{(k,l)}\left(\sum_{i=1}^{n}\mu(\sigma^m(x_1),\dots,\sigma^m(x_{i-1}),D^{(m,r)}(x_i),\tau^r(x_{i+1}),\dots,\tau^r(x_n))\right)$$

$$= \sum_{i=1}^{n} D^{(k,l)}\left(\mu(\sigma^m(x_1),\dots,\sigma^m(x_{i-1}),D^{(m,r)}(x_i),\tau^r(x_{i+1}),\dots,\tau^r(x_n))\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{i-1}\mu\big(\sigma^{k+m}(x_1),\dots,\sigma^{k+m}(x_{j-1}),D^{(k,l)}\circ\sigma^m(x_j),\tau^l\circ\sigma^m(x_{j+1}),$$
$$\dots,\tau^l\circ\sigma^m(x_{i-1}),\tau^l\circ D^{(m,r)}(x_i),\tau^{l+r}(x_{i+1}),\dots,\tau^{l+r}(x_n)\big)$$
$$+\sum_{i=1}^{n}\mu(\sigma^{k+m}(x_1),\dots,\sigma^{k+m}(x_{i-1}),D^{(k,l)}\circ D^{(m,r)}(x_k),\tau^{l+r}(x_{i+1}),\dots,\tau^{l+r}(x_n))$$
$$+\sum_{i=1}^{n}\sum_{j=i+1}^{n}\mu\big(\sigma^{k+m}(x_1),\dots,\sigma^{k+m}(x_{i-1}),\sigma^k\circ D^{(m,r)}(x_i),\sigma^k\circ\tau^r(x_{i+1}),$$
$$\dots,\sigma^k\circ\tau^r(x_{j-1}),D^{(k,l)}\circ\tau^r(x_j),\tau^{l+r}(x_{j+1}),\dots,\tau^{l+r}(x_n)\big),$$

$$D^{(m,r)}\circ D^{(k,l)}(\mu(x_1,\dots,x_n))$$
$$= D^{(m,r)}\left(\sum_{i=1}^{n}\mu(\sigma^k(x_1),\dots,\sigma^k(x_{i-1}),D^{(k,l)}(x_i),\tau^l(x_{i+1}),\dots,\tau^l(x_n))\right)$$
$$= \sum_{i=1}^{n} D^{(m,r)}\left(\mu(\sigma^k(x_1),\dots,\sigma^k(x_{i-1}),D^{(k,l)}(x_i),\tau^l(x_{i+1}),\dots,\tau^l(x_n))\right)$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{i-1}\mu\big(\sigma^{k+m}(x_1),\dots,\sigma^{k+m}(x_{j-1}),D^{(m,r)}\circ\sigma^k(x_j),\tau^r\circ\sigma^k(x_{j+1}),$$
$$\dots,\tau^r\circ\sigma^k(x_{i-1}),\tau^r\circ D^{(k,l)}(x_i),\tau^{l+r}(x_{i+1}),\dots,\tau^{l+r}(x_n)\big)$$
$$+\sum_{i=1}^{n}\mu(\sigma^{k+m}(x_1),\dots,\sigma^{k+m}(x_{i-1}),D^{(m,r)}\circ D^{(k,l)}(x_i),\tau^{l+r}(x_{i+1}),\dots,\tau^{l+r}(x_n))$$
$$+\sum_{i=1}^{n}\sum_{j=i+1}^{n}\mu\big(\sigma^{k+m}(x_1),\dots,\sigma^{k+m}(x_{i-1}),\sigma^m\circ D^{(k,l)}(x_i),\sigma^m\circ\tau^l(x_{i+1}),$$
$$\dots,\sigma^m\circ\tau^l(x_{j-1}),D^{(m,r)}\circ\tau^l(x_j),\tau^{l+r}(x_{j+1}),\dots,\tau^{l+r}(x_n)\big).$$

Then, $D^{(k,l)}\circ D^{(m,r)}$, $D^{(m,r)}\circ D^{(k,l)}$ and $[D^{(k,l)},D^{(m,r)}]$ can be written as follows:

$$D^{(k,l)}\circ D^{(m,r)}(\mu(x_1,\dots,x_n)) =$$
$$= \sum_{i=1}^{n}\mu(\sigma^{k+m}(x_1),\dots,\sigma^{k+m}(x_{i-1}),D^{(m,r)}\circ D^{(k,l)}(x_i),\tau^{l+r}(x_{i+1}),\dots,\tau^{l+r}(x_n))$$
$$+ S_1(x_1,\dots,x_n)+T_1(x_1,\dots,x_n),$$
$$D^{(m,r)}\circ D^{(k,l)}(\mu(x_1,\dots,x_n)) =$$
$$= \sum_{i=1}^{n}\mu(\sigma^{k+m}(x_1),\dots,\sigma^{k+m}(x_{k-1}),D^{(m,r)}\circ D^{(k,l)}(x_i),\tau^{l+r}(x_{i+1}),\dots,\tau^{l+r}(x_n))$$

$$
\begin{aligned}
&+ S_2(x_1, \ldots, x_n) + T_2(x_1, \ldots, x_n),\\
&[D^{(k,l)}, D^{(m,r)}](\mu(x_1, \ldots, x_n))\\
&= D^{(k,l)} \circ D^{(m,r)}(\mu(x_1, \ldots, x_n)) - D^{(m,r)} \circ D^{(k,l)}(\mu(x_1, \ldots, x_n))\\
&= \sum_{i=1}^{n} \mu(\sigma^{k+m}(x_1), \ldots, \sigma^{k+m}(x_{i-1}), D^{(k,l)} \circ D^{(m,r)}(x_i), \tau^{l+r}(x_{i+1}), \ldots, \tau^{l+r}(x_n))\\
&\quad - \sum_{i=1}^{n} \mu(\sigma^{k+m}(x_1), \ldots, \sigma^{k+m}(x_{i-1}), D^{(m,r)} \circ D^{(k,l)}(x_i), \tau^{l+r}(x_{i+1}), \ldots, \tau^{l+r}(x_n))\\
&\quad + S_1(x_1, \ldots, x_n) + T_1(x_1, \ldots, x_n) - S_2(x_1, \ldots, x_n) - T_2(x_1, \ldots, x_n)\\
&= \sum_{i=1}^{n} \mu(\sigma^{k+m}(x_1), \ldots, \sigma^{k+m}(x_{i-1}), [D^{(k,l)}, D^{(m,r)}](x_i), \tau^{l+r}(x_{i+1}), \ldots, \tau^{l+r}(x_n))\\
&\quad + (S_1(x_1, \ldots, x_n) - S_2(x_1, \ldots, x_n)) + (T_1(x_1, \ldots, x_n) - T_2(x_1, \ldots, x_n)).
\end{aligned}
$$

Thus, $[D^{(k,l)}, D^{(m,r)}]$ is $(\sigma^{k+m}, \tau^{l+r})$-derivation of $\mathcal{A}$ if and only if the tail vanishes

$$S_1(x_1, \ldots, x_n) - S_2(x_1, \ldots, x_n) + T_1(x_1, \ldots, x_n) - T_2(x_1, \ldots, x_n) = 0.$$

In particular, the cross-commutation condition yields

$$
\begin{aligned}
&T_1(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \mu\big(\sigma^{k+m}(x_1), \ldots, \sigma^{k+m}(x_{k-1}), \sigma^{k} \circ D^{(m,r)}(x_i),\\
&\quad \sigma^{k} \circ \tau^{r}(x_{i+1}), \ldots, \sigma^{k} \circ \tau^{r}(x_{j-1}), D^{(k,l)} \circ \tau^{r}(x_j), \tau^{l+r}(x_{j+1}), \ldots, \tau^{l+r}(x_n)\big)\\
&= \sum_{j=1}^{n} \sum_{i=1}^{j-1} \mu\big(\sigma^{k+m}(x_1), \ldots, \sigma^{k+m}(x_{i-1}), D^{(m,r)} \circ \sigma^{k}(x_i),\\
&\quad \tau^{r} \circ \sigma^{k}(x_{i+1}), \ldots, \tau^{r} \circ \sigma^{k}(x_{j-1}), D^{(k,l)} \circ \tau^{r}(x_j), \tau^{l+r}(x_{j+1}), \ldots, \tau^{l+r}(x_n)\big)\\
&= T_2(x_1, \ldots, x_n),
\end{aligned}
$$

and in the similar way, $S_1(x_1, \ldots, x_n) = S_2(x_1, \ldots, x_n)$, for all $x_1, \ldots, x_n \in A$, which implies that

$$(S_1(x_1, \ldots, x_n) - S_2(x_1, \ldots, x_n)) + (T_1(x_1, \ldots, x_n) - T_2(x_1, \ldots, x_n)) = 0,$$

and thus that $[D^{(k,l)}, D^{(m,r)}]$ is a $(\sigma^{k+m}, \tau^{l+r})$-derivation of $\mathcal{A}$. □

Similarly to the binary case, let $\Delta^{(k,l)}_{\sigma,\tau}(\mathcal{A})$ denote the space of (σ^k, τ^l)-derivations that commute with σ and τ. The corresponding twisted derivations will be written $\Delta^{(k,l)}$. If σ and τ commute, then we have commutation between their powers, which makes the commutator $[D^{(k,l)}, D^{(m,r)}]$ belong to $\mathcal{D}^{(k+m,l+r)}_{\sigma,\tau}(\mathcal{A})$ as seen in the calculations.

Proposition 14.10 *Let $\mathcal{A} = (A, \mu)$ be an n-ary algebra, $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ be two linear maps such that $\sigma \circ \tau = \tau \circ \sigma$. The linear space $\mathcal{A}^{\Delta} = \sum\limits_{k,l \in \mathbb{Z}_{\geq 0}} \Delta_{\sigma,\tau}^{(k,l)}(\mathcal{A})$ is closed under the commutator.*

Proof From $\sigma \circ \tau = \tau \circ \sigma$, and $\sigma^0 = \mathrm{Id}$, $\tau^0 = \mathrm{Id}$ it is immediate that $\sigma^i \circ \tau^j = \tau^j \circ \sigma^i$ for all $i, j \geq 0$. Two twisted derivations $\Delta^{(k,l)}, \Delta^{(m,r)}$ commuting with σ and τ, by Corollary 14.2, commute with all their powers. By Proposition 14.9, $[\Delta^{(k,l)}, \Delta^{(m,r)}] \in \mathcal{D}_{\sigma,\tau}^{(k+m,l+r)}(\mathcal{A})$. To prove commutation with σ and τ, we use the second argument in the proof of Proposition 14.6, and conclude $[\Delta^{(k,l)}, \Delta^{(m,r)}] \in \Delta_{\sigma,\tau}^{(k+m,l+r)}(\mathcal{A})$. Finally, for $a_i \in \mathbb{F}, k_i, l_i \in \mathbb{Z}_{\geq 0}$, $\Delta^{(k_i,l_i)} \in \Delta_{\sigma,\tau}^{(k_i l_i)}(\mathcal{A})$, $\Delta = \sum\limits_{i=1}^{n} a_i \Delta^{(k_i,l_i)}$ and $m, r \in \mathbb{Z}_{\geq 0}$,

$$
\begin{aligned}
[\Delta, \Delta^{(m,r)}] = [\sum_{i=1}^{n} a_i \Delta^{(k_i,l_i)}, \Delta^{(m,r)}] &= \sum_{i=1}^{n} [a_i \Delta^{(k_i,l_i)}, \Delta^{(m,r)}] \\
&= \sum_{i=1}^{n} a_i [\Delta^{(k_i,l_i)}, \Delta^{(m,r)}] \in \mathcal{A}^{\Delta},
\end{aligned}
$$

which completes the proof. □

Theorem 14.7 *($\mathbb{Z}_{\geq 0}^2$-graded Lie algebra of (σ, τ)-derivations) Let $\mathcal{A} = (A, \mu)$ be an n-ary algebra, and $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ two commuting linear maps. Then,*

(i) *with the bilinear product $[\cdot, \cdot]$, defined by commutator*

$$
[\Delta^{(k,l)}, \Delta^{(m,r)}] = \Delta^{(k,l)} \circ \Delta^{(m,r)} - \Delta^{(m,r)} \circ \Delta^{(k,l)}, \quad k, l, m, r \in \mathbb{Z}_{\geq 0},
$$

$(\mathcal{A}^{\Delta} = \sum\limits_{k,l \in \mathbb{Z}_{\geq 0}} \Delta_{\sigma,\tau}^{(k,l)}(\mathcal{A}), [\cdot, \cdot])$, is a Lie subalgebra of $\mathfrak{L}(\mathcal{A})$.

(ii) *when the sum is direct, $(\mathcal{A}^{\Delta} = \bigoplus\limits_{k,l \in \mathbb{Z}_{\geq 0}} \Delta_{\sigma,\tau}^{(k,l)}(\mathcal{A}), [\cdot, \cdot])$ is a $\mathbb{Z}_{\geq 0}^2$-graded Lie subalgebra of $\mathfrak{L}(\mathcal{A})$ with grading $g : \mathbb{Z}_{\geq 0}^2 \to \mathcal{A}^{\Delta}, (k, l) \mapsto \Delta_{\sigma,\tau}^{(k,l)}(\mathcal{A})$.*

If σ and τ are invertible, for (σ^k, τ^l)-derivations with $k, l \in \mathbb{Z}$, then

$$
[\Delta_{\sigma,\tau}^{(k,l)}(\mathcal{A}), \Delta_{\sigma,\tau}^{(m,r)}(\mathcal{A})] \subseteq \Delta_{\sigma,\tau}^{(k+m,l+r)}(\mathcal{A}), \; k, l, m, r \in \mathbb{Z}.
$$

Proposition 14.11 *Let $\mathcal{A} = (A, \mu)$ be an n-ary algebra, $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ be two commuting and invertible linear maps. The linear space $\mathcal{A}^{\Delta} = \sum\limits_{k,l \in \mathbb{Z}} \Delta_{\sigma,\tau}^{(k,l)}(\mathcal{A})$ is closed under commutator.*

Theorem 14.8 *($\mathbb{Z}^2$-graded Lie algebra of (σ, τ)-derivations) Let $\mathcal{A} = (A, \mu)$ be an n-ary algebra, and $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ two commuting linear maps. Then,*

(i) *with the bilinear product* $[\cdot,\cdot]$*, defined by commutator*

$$[\Delta^{(k,l)}, \Delta^{(m,r)}] = \Delta^{(k,l)} \circ \Delta^{(m,r)} - \Delta^{(m,r)} \circ \Delta^{(k,l)}, \quad k, l, m, r \in \mathbb{Z},$$

$(\mathcal{A}^{\Delta} = \sum_{k,l\in\mathbb{Z}} \Delta^{(k,l)}_{\sigma,\tau}(\mathcal{A}), [\cdot,\cdot])$*, is a Lie subalgebra of* $\mathfrak{L}(\mathcal{A})$*.*

(ii) *when the sum is direct,* $(\mathcal{A}^{\Delta} = \bigoplus_{k,l\in\mathbb{Z}} \Delta^{(k,l)}_{\sigma,\tau}(\mathcal{A}), [\cdot,\cdot])$ *is a* $\mathbb{Z}^2$*-graded Lie subalgebra of* $\mathfrak{L}(\mathcal{A})$ *with grading* $g : \mathbb{Z}^2 \to \mathcal{A}^{\Delta}$, $(k, l) \mapsto \Delta^{(k,l)}_{\sigma,\tau}(\mathcal{A})$.

14.5 Hom-Algebras of Generalized Jacobians for (σ, τ)-Derivations

Throughout this section, D and D_i represent (σ, τ)-derivations commuting with linear maps σ and τ on a commutative associative algebra $\mathcal{A}$. The product is denoted by the dot "$\cdot$", or by juxtaposition when no confusion is possible. We follow Filippov's notation [16], with operators acting from the right, xD represents the image of element x by map D. With these notations, (σ, τ)-Leibniz rule is expressed as

$$(x \cdot y)D = xD \cdot y\tau + x\sigma \cdot yD.$$

When we refer to D as a (σ, τ)-derivation of $\mathcal{A}$, we refer to this Leibniz rule with respect to the usual product in $\mathcal{A}$. We assume $\mathcal{A}$ to be commutative associative to ensure that it is possible to compute the Jacobian determinant in usual commutative way. Unless specified otherwise, we assume $\sigma \neq \tau$, and the D_s are different pairwise.

Definition 14.10 The Jacobian of n elements $x_1, \ldots, x_n \in \mathcal{A}$ is the determinant

$$|x_i D_j| = \begin{vmatrix} x_1 D_1 & \ldots & x_1 D_n \\ \vdots & \ddots & \vdots \\ x_n D_1 & \ldots & x_n D_n \end{vmatrix}.$$

The Jacobian is skew-symmetric with respect to x_i, $1 \leq i \leq n$

Define a skew-symmetric n-ary bracket bracket on $\mathcal{A}$ by $[x_1, \ldots, x_n] = |x_i D_j|$.

Lemma 14.3 *Let* $y_2, \ldots, y_n \in \mathcal{A}$*. The linear operator* $D : x \mapsto [x, y_2, \ldots, y_n]$ *is a* (σ, τ)*-derivation on* $\mathcal{A}$*.*

Proof For any $x_1, x_2, y_2, \ldots, y_n \in \mathcal{A}$,

$$
\begin{aligned}
[x_1x_2, y_2, \ldots, y_n] &= \begin{vmatrix} (x_1 \cdot x_2)D_1 & \ldots & (x_1 \cdot x_2)D_n \\ y_2 D_1 & \ldots & y_2 D_n \\ \vdots & & \vdots \\ y_n D_1 & \ldots & y_n D_n \end{vmatrix} \\
&= \begin{vmatrix} x_1 D_1 \cdot x_2\tau + x_1\sigma \cdot x_2 D_1 & \ldots & x_1 D_n \cdot x_2\tau + x_1\sigma \cdot x_2 D_n \\ y_2 D_1 & \ldots & y_2 D_n \\ \vdots & & \vdots \\ y_n D_1 & \ldots & y_n D_n \end{vmatrix} \\
&= \begin{vmatrix} x_1 D_1 \cdot x_2\tau & \ldots & x_1 D_n \cdot x_2\tau \\ y_2 D_1 & \ldots & y_2 D_n \\ \vdots & & \vdots \\ y_n D_1 & \ldots & y_n D_n \end{vmatrix} + \begin{vmatrix} x_1\sigma \cdot x_2 D_1 & \ldots & x_1\sigma \cdot x_2 D_n \\ y_2 D_1 & \ldots & y_2 D_n \\ \vdots & & \vdots \\ y_n D_1 & \ldots & y_n D_n \end{vmatrix} \\
&= \begin{vmatrix} x_1 D_1 & \ldots & x_1 D_n \\ y_2 D_1 & \ldots & y_2 D_n \\ \vdots & & \vdots \\ y_n D_1 & \ldots & y_n D_n \end{vmatrix} \cdot x_2\tau + x_1\sigma \begin{vmatrix} x_2 D_1 & \ldots & x_2 D_n \\ y_2 D_1 & \ldots & y_2 D_n \\ \vdots & & \vdots \\ y_n D_1 & \ldots & y_n D_n \end{vmatrix} \\
&= [x_1, y_2, \ldots, y_n] \cdot x_2\tau + x_1\sigma \cdot [x_1, y_2, \ldots, y_n]
\end{aligned}
$$

□

Lemma 14.4 (Leibniz rule on the Jacobian) *Let* $D \in \mathcal{D}_{\sigma,\tau}(\mathcal{A})$. *Then,*

$$
\begin{vmatrix} x_{11} & \ldots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \ldots & x_{nn} \end{vmatrix} D = \sum_{i=1}^{n} \begin{vmatrix} x_{11}\sigma & \ldots & x_{1n}\sigma \\ \vdots & & \vdots \\ x_{i-11}\sigma & \ldots & x_{i-1n}\sigma \\ x_{i1}D & \ldots & x_{in}D \\ x_{i+11}\tau & \ldots & x_{i+1n}\tau \\ \vdots & & \vdots \\ x_{n1}\tau & \ldots & x_{nn}\tau \end{vmatrix}. \tag{14.10}
$$

This lemma is a straightforward calculation from (σ, τ)-Leibniz rule of D. By considering $x_{ij} = x_i D_j$, and D commuting with all D_j, we obtain

$$[x_1, \dots, x_n]D = \sum_{i=1}^{n} \begin{vmatrix} x_1 D_1 \sigma & \dots & x_1 D_n \sigma \\ \vdots & & \vdots \\ x_{i-1} D_1 \sigma & \dots & x_{i-1} D_n \sigma \\ x_i D_1 D & \dots & x_i D_n D \\ x_{i+1} D_1 \tau & \dots & x_{i+1} D_n \tau \\ \vdots & & \vdots \\ x_n D_1 \tau & \dots & x_n D_n \tau \end{vmatrix} = \begin{vmatrix} x_1 \sigma D_1 & \dots & x_1 \sigma D_n \\ \vdots & & \vdots \\ x_{i-1} \sigma D_1 & \dots & x_{i-1} \sigma D_n \\ x_i D D_1 & \dots & x_i D D_n \\ x_{i+1} \tau D_1 & \dots & x_{i+1} \tau D_n \\ \vdots & & \vdots \\ x_n \tau D_1 & \dots & x_n \tau D_n \end{vmatrix} =$$

$$= \sum_{i=1}^{n} [x_1 \sigma, \dots, x_{i-1} \sigma, x_i D, x_{i+1} \tau, \dots, x_n \tau]$$

Proposition 14.12 *If $D \in D_{\sigma,\tau}(\mathcal{A})$ commutes with all D_j, then D is a (σ, τ)-derivation with respect to the Jacobian. The n-ary twisted Leibniz rule is given by*

$$[x_1, \dots, x_n]D = \sum_{i=1}^{n} [x_1 \sigma, \dots, x_{i-1} \sigma, x_i D, x_{i+1} \tau, \dots, x_n \tau].$$

Provided that the D_s commute, it is immediate that any D_j satisfies

$$\begin{aligned} &[x_i D_j, y_2 \tau, \dots, y_n \tau] \\ &= [x_i, y_2, \dots, y_n] D_j - \sum_{s=2}^{n} [x_i \sigma, y_2 \sigma, \dots, y_{s-1} \sigma, y_s D_j, y_{s+1} \tau, \dots, y_n \tau]. \end{aligned}$$

Filippov [16] uses the non-twisted version of these two properties together to prove that the Jacobian as an n-ary product induces a n-Lie algebra structure on $\mathcal{A}$. We generalize his procedure to try and find an n-ary Hom-Lie or quasi-Hom-Lie identity involving a n-tuple $\{D_1, \dots, D_n\}$ of (σ, τ)-derivations.

We apply Lemma 14.3 to the operator $D : x \to [x, y_2\tau, \dots, y_n\tau]$,

$$[[x_1,\dots,x_n], y_2\tau,\dots,y_n\tau] \overset{(14.4)}{=} \sum_{i=1}^{n} \begin{vmatrix} x_1\sigma D_1 & \dots & x_1\sigma D_n \\ \vdots & & \vdots \\ x_{i-1}\sigma D_1 & \dots & x_{i-1}\sigma D_n \\ [x_i D_1, y_2\tau,\dots,y_n\tau] & \dots & [x_i D_n, y_2\tau,\dots,y_n\tau] \\ x_{i+1}\tau D_1 & \dots & x_{i+1}\tau D_n \\ \vdots & & \vdots \\ x_n\tau D_1 & \dots & x_n\tau D_n \end{vmatrix} =$$

$$\sum_{i=1}^{n} \begin{vmatrix} x_1\sigma D_1 & \dots & x_1\sigma D_n \\ \vdots & & \vdots \\ x_{i-1}\sigma D_1 & \dots & x_{i-1}\sigma D_n \\ [x_i, y_2,\dots,y_n]D_1 - A_{i1} & \dots & [x_i, y_2,\dots,y_n]D_n - A_{in} \\ x_{i+1}\tau D_1 & \dots & x_{i+1}\tau D_n \\ \vdots & & \vdots \\ x_n\tau D_1 & \dots & x_n\tau D_n \end{vmatrix}$$

$$= \sum_{i=1}^{n} \begin{vmatrix} x_1\sigma D_1 & \dots & x_1\sigma D_n \\ \vdots & & \vdots \\ x_{i-1}\sigma D_1 & \dots & x_{i-1}\sigma D_n \\ [x_i, y_2,\dots,y_n]D_1 & \dots & [x_i, y_2,\dots,y_n]D_n \\ x_{i+1}\tau D_1 & \dots & x_{i+1}\tau D_n \\ \vdots & & \vdots \\ x_n\tau D_1 & \dots & x_n\tau D_n \end{vmatrix} - \sum_{i=1}^{n} \begin{vmatrix} x_1\sigma D_1 & \dots & x_1\sigma D_n \\ \vdots & & \vdots \\ x_{i-1}\sigma D_1 & \dots & x_{i-1}\sigma D_n \\ A_{i1} & \dots & A_{in} \\ x_{i+1}\tau D_1 & \dots & x_{i+1}\tau D_n \\ \vdots & & \vdots \\ x_n\tau D_1 & \dots & x_n\tau D_n \end{vmatrix} =$$

$$\sum_{i=1}^{n}[x_1\sigma,\dots,x_{i-1}\sigma,[x_i,y_2,\dots,y_n],x_{i+1}\tau,\dots,y_n\tau] - \sum_{i=1}^{n} \begin{vmatrix} x_1\sigma D_1 & \dots & x_1\sigma D_n \\ \vdots & & \vdots \\ x_{i-1}\sigma D_1 & \dots & x_{i-1}\sigma D_n \\ A_{i1} & \dots & A_{in} \\ x_{i+1}\tau D_1 & \dots & x_{i+1}\tau D_n \\ \vdots & & \vdots \\ x_n\tau D_1 & \dots & x_n\tau D_n \end{vmatrix},$$

where $A_{ij} = \sum_{s=2}^{n}[x_i\sigma, y_2\sigma,\dots,y_{s-1}\sigma, y_s D_j, y_{s+1}\tau,\dots,y_n\tau]$.

The first term looks like the RHS of a (σ,τ)-twisted Jacobi-type identity, and in the non-twisted case it is exactly a n-ary Filippov identity.

Proposition 14.13 *The following two identities are equivalent:*

$$[[x_1, \dots, x_n], y_2\tau, \dots, y_n\tau] = \sum_{i=1}^{n} [x_1\sigma, \dots, x_{i-1}\sigma, [x_i, y_2, \dots, y_n], x_{i+1}\tau, \dots, y_n\tau], \tag{14.11}$$

$$\sum_{i=1}^{n} \begin{vmatrix} x_1\sigma D_1 & \dots & x_1\sigma D_n \\ \vdots & & \vdots \\ x_{i-1}\sigma D_1 & \dots & x_{i-1}\sigma D_n \\ A_{i1} & \dots & A_{in} \\ x_{i+1}\tau D_1 & \dots & x_{i+1}\tau D_n \\ \vdots & & \vdots \\ x_n\tau D_1 & \dots & x_n\tau D_n \end{vmatrix} = 0, \tag{14.12}$$

where $A_{ij} = \sum_{s=2}^{n} [x_i\sigma, y_2\sigma, \dots, y_{s-1}\sigma, y_s D_j, y_{s+1}\tau, \dots, y_n\tau]$.

The asymmetry between instances of σ and τ will appear at different points of the argument on the sequel. Filippov's argument is based on the following theorem, featuring row-column substitutions on two given matrices.

Theorem 14.9 ([16], p. 576) *Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times n$ matrices with coefficients in $\mathcal{A}$. Then*

$$\sum_{i=1}^{n} \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{i-11} & \dots & a_{i-1n} \\ b_{i1} & \dots & b_{in} \\ a_{i+11} & \dots & a_{i+1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} = \sum_{j=1}^{n} \begin{vmatrix} a_{11} & \dots & a_{1j-1} & b_{1j} & a_{1j+1} & \dots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj-1} & b_{nj} & a_{nj+1} & \dots & a_{nn} \end{vmatrix}.$$

The proof to this result is a straightforward calculation: by taking minors on the ith rows on the LHS and on the jth columns on the RHS, each b_{ij} appears exactly once accompanied by the same minor of matrix A on both sides.

This result is fundamental to Filippov's proof. This translates into our work as two separate study cases depending on whether or not it can be used.

14.5.1 Concerning (σ, σ)-Derivations

First and foremost, consider the determinant (14.12). We transform it using elementary row properties of determinants to take the inner sum outside and obtain:

$$\sum_{\substack{s=2\\i=1}}^{n} \begin{vmatrix} x_1\sigma D_1 & \dots & x_1\sigma D_n \\ \vdots & & \vdots \\ x_{i-1}\sigma D_1 & \dots & x_{i-1}\sigma D_n \\ [x_i\sigma, y_2\sigma, \dots, y_{s-1}\sigma, y_s D_1, y_{s+1}\sigma, \dots, y_n\sigma] & \dots & [x_i\sigma, y_2\sigma, \dots, y_{s-1}\sigma, y_s D_n, y_{s+1}\sigma, \dots, y_n\sigma] \\ x_{i+1}\sigma D_1 & \dots & x_{i+1}\sigma D_n \\ \vdots & & \vdots \\ x_n\sigma D_1 & \dots & x_n\sigma D_n \end{vmatrix}$$

$$= \sum_{\substack{s=2\\i=1}}^{n} \begin{vmatrix} x_1\sigma D_1 & \dots & x_1\sigma D_n \\ \vdots & & \vdots \\ x_{i-1}\sigma D_1 & \dots & x_{i-1}\sigma D_n \\ \Delta_{i1} & \dots & \Delta_{in} \\ x_{i+1}\sigma D_1 & \dots & x_{i+1}\sigma D_n \\ \vdots & & \vdots \\ x_n\sigma D_1 & \dots & x_n\sigma D_n \end{vmatrix} = \sum_{s=2}^{n} \Delta_s\ , \quad \text{where} \quad \Delta_{ij} = \begin{vmatrix} x_i\sigma D_1 & \dots & x_i\sigma D_n \\ y_2\sigma D_1 & \dots & y_2\sigma D_n \\ \vdots & & \vdots \\ y_{s-1}\sigma D_1 & \dots & y_{s-1}\sigma D_n \\ y_s D_1 D_j & \dots & y_s D_n D_j \\ y_{s+1}\sigma D_1 & \dots & y_{s+1}\sigma D_n \\ \vdots & & \vdots \\ y_n\sigma D_1 & \dots & y_n\sigma D_n \end{vmatrix}.$$

We expand the Δ_{ij} as $n \times n$ determinants for a better visualization of further steps. Consider matrices $A = (x_i\sigma D_j)$ and $B = (\Delta_{ij})$. We apply Theorem 14.9 to A and B:

$$\Delta_s = \sum_{j=1}^{n} \begin{vmatrix} x_1\sigma D_1 & \dots & x_1\sigma D_{j-1} & \Delta_{1j} & x_1\sigma D_{j+1} & \dots & x_1\sigma D_n \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ x_n\sigma D_1 & \dots & x_n\sigma D_{j-1} & \Delta_{nj} & x_n\sigma D_{j+1} & \dots & x_n\sigma D_n \end{vmatrix}.$$

Expand now Δ_{ij} by minors using the row with y_s,

$$\Delta_{ij} = \sum_{k=1}^{n} (-1)^{s+k} y_s D_k D_j M_{ik},$$

where M_{ik} is the following subdeterminant

$$M_{ik} = \begin{vmatrix} x_i\sigma D_1 & \dots & x_i\sigma D_{k-1} & x_i\sigma D_{k+1} & \dots & x_i\sigma D_n \\ y_2\sigma D_1 & \dots & y_2\sigma D_{k-1} & y_2\sigma D_{k+1} & \dots & y_2\sigma D_n \\ \vdots & & \vdots & \vdots & & \vdots \\ y_{s-1}\sigma D_1 & \dots & y_{s-1}\sigma D_{k-1} & y_{s-1}\sigma D_{k+1} & \dots & y_{s-1}\sigma D_n \\ y_{s+1}\sigma D_1 & \dots & y_{s+1}\sigma D_{k-1} & y_{s+1}\sigma D_{k+1} & \dots & y_{s+1}\sigma D_n \\ \vdots & & \vdots & \vdots & & \vdots \\ y_n\sigma D_1 & \dots & y_n\sigma D_{k-1} & y_n\sigma D_{k+1} & \dots & y_n\sigma D_n \end{vmatrix}.$$

The Δ_s can thus be expressed as

$$\Delta_s = \sum_{j=1}^{n} \begin{vmatrix} x_1\sigma D_1 \dots x_1\sigma D_{j-1} \sum_{k=1}^{n}(-1)^{s+k} y_s D_k D_j M_{1k} \; x_1\sigma D_{j+1} \dots x_1\sigma D_n \\ \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \\ x_n\sigma D_1 \dots x_n\sigma D_{j-1} \sum_{k=1}^{n}(-1)^{s+k} y_s D_k D_j M_{nk} \; x_n\sigma D_{j+1} \dots x_n\sigma D_n \end{vmatrix}$$
$$= \sum_{j,k=1}^{n} (-1)^{s+k} y_s D_k D_j \begin{vmatrix} x_1\sigma D_1 \dots x_1\sigma D_{j-1} \; M_{1k} \; x_1\sigma D_{j+1} \dots x_1\sigma D_n \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ x_n\sigma D_1 \dots x_n\sigma D_{j-1} \; M_{nk} \; x_n\sigma D_{j+1} \dots x_n\sigma D_n \end{vmatrix}.$$

Now we look again at the M_{ik}: each of them has $x_i\sigma D_j$, $j \neq k$ in the first row. We expand by minors using that row,

$$M_{ik} = x_i\sigma D_1 N_{1k} - x_i\sigma D_2 N_{2k} + \dots + (-1)^{k-2} x_i\sigma D_{k-1} N_{k-1k} + \\ +(-1)^{k-1} x_i\sigma D_{kk+1} N_{kk+1} + \dots + (-1)^{n-2} x_i\sigma D_n N_{kn}.$$

Here the N_{kj}, $k \neq j$, represent the minor obtained after removing columns indexed k and j from Δ_{ij} (or column j from M_{ik}). Naturally, $N_{kj} = N_{jk}$ and thus we name these coefficients with increasing subindices.

Remark 14.2 The N_{kj} do not depend on the x_i.

Denote $x^{(i)} = \begin{pmatrix} x_i\sigma D_1 \\ \vdots \\ x_i\sigma D_n \end{pmatrix}$ and $M_k = \begin{pmatrix} M_{1k} \\ \vdots \\ M_{nk} \end{pmatrix}$. We can then express M_k as follows:

$$M_k = N_{1k} x^{(1)} + \dots + (-1)^{k-2} N_{k-1k} x^{(k-1)} \\ +(-1)^{k-1} N_{kk+1} x^{(k+1)} + \dots + (-1)^{n-2} N_{kn} x^{(n)}.$$

It is a linear combination of the columns of Δ_s.

$$\Delta_s = \sum_{j,k=1}^{n} y_s D_k D_j \cdot (-1)^{s+k} \left| x^{(1)} \dots x^{(j-1)} \; M_k \; x^{(j+1)} \dots x^{(n)} \right| =$$
$$= \sum_{j,k=1}^{n} y_s D_k D_j \cdot (-1)^{s+k} \Big| x^{(1)} \dots x^{(j-1)} \big(N_{1k} x^{(1)} + \dots + (-1)^{k-2} N_{k-1k} x^{(k-1)} +$$
$$+ (-1)^{k-1} N_{kk+1} x^{(k+1)} + \dots + (-1)^{n-2} N_{kn} x^{(n)} \big) x^{(j+1)} \dots x^{(n)} \Big|$$

By elementary properties of determinants, the coefficient of $y_s D_j D_j$, which we name Δ'_{jj}, is 0.

If $j < k$, we can rewrite the coefficient Δ'_{kj} of $y_s D_k D_j$ using elementary column manipulations to look like

$$\begin{aligned}&(-1)^{s+k}\left|x^{(1)} \dots x^{(j-1)} \left((-1)^{j-1} N_{jk} x^{(j)}\right) x^{(j+1)} \dots x^{(n)}\right| \\ &= (-1)^{s+k+j+1} N_{jk} \left|x^{(1)} \dots x^{(n)}\right|\end{aligned}$$

Conversely,

$$\begin{aligned}\Delta'_{jk} &= (-1)^{s+j}\left|x^{(1)} \dots x^{(k-1)} \left((-1)^{k} N_{jk} x^{(k)}\right) x^{(k+1)} \dots x^{(n)}\right| \\ &= (-1)^{s+j+k} N_{jk} \left|x^{(1)} \dots x^{(n)}\right|.\end{aligned}$$

Finally, since $D_k D_j = D_j D_k$, we have

$$y_s D_k D_j \cdot \Delta'_{kj} + y_s D_j D_k \cdot \Delta'_{jk} = y_s D_k D_j \cdot (\Delta'_{kj} + \Delta'_{jk}) = 0$$

for all $k \neq j$, and thus $\Delta_s = 0$.

Theorem 14.10 *Let $\mathcal{A}$ be a commutative associative algebra, $D_1, \dots, D_n$ pairwise commuting (σ, σ)-derivations. The skew-symmetric product $[x_1, \dots, x_n] = |x_i D_j|$ endows the space $\mathcal{A}$ with a n-Hom-Lie structure with twisting map σ and the following n-Hom-Jacobi identity:*

$$[[x_1, \dots, x_n], y_2\sigma, \dots, y_n\sigma] = \sum_{i=1}^{n} [x_1\sigma, \dots, x_{i-1}\sigma, [x_i, y_2, \dots, y_n], x_{i+1}\sigma, \dots, x_n\sigma].$$

14.5.2 When $\sigma \neq \tau$

We consider now the widest range of twisted derivation operators possible based on arbitrary linear maps σ and τ. We examine commutation relations between them, in the search to describe necessary conditions for certain regularity that generalizes what we have seen in both Filippov's proof and the previous case twisted by σ only.

Let each D_i be a (σ_i, τ_i)-derivation of $\mathcal{A}$. If we consider the adjoint action $x \mapsto [x, y_2, \dots, y_n]$ once more we get

$$[x_0 x_1, y_2, \dots, y_n] \overset{def}{=} \begin{vmatrix} (x_0 x_1) D_1 \dots (x_0 x_1) D_n \\ y_i D_j \end{vmatrix} =$$

$$\begin{vmatrix} x_0 D_1 \cdot x_1 \tau_1 + x_0 \sigma_1 \cdot x_1 D_1 \dots x_0 D_n \cdot x_1 \tau_n + x_0 \sigma_n \cdot x_1 D_n \\ y_i D_j \end{vmatrix} =$$

$$\begin{vmatrix} x_0 D_1 \cdot x_1 \tau_1 \dots x_0 D_n \cdot x_1 \tau_n \\ y_i D_j \end{vmatrix} + \begin{vmatrix} x_0 \sigma_1 \cdot x_1 D_1 \dots x_0 \sigma_n \cdot x_1 D_n \\ y_i D_j \end{vmatrix}.$$

This can split on a Leibniz-type rule if $x_1\tau_i = x_1\tau_j$ and $x_0\sigma_i = x_0\sigma_j$ for all i, j. This limits the scope of our research significantly.

Proposition 14.14 *The adjoint action $x \mapsto [x, y_2, \ldots, y_n]$ is a (σ, τ)-derivation on $\mathcal{A}$ if all D_i have the same twisting maps.*

14.5.3 With Equal Commutation Relations

On this section all D_i are considered to be (σ, τ)-derivations verifying the following commutation relations:

$$D_i\sigma = \sigma D_i \cdot \lambda_i,\ D_i\tau = \tau D_i \cdot \lambda_i,\ D_i D_k = D_k D_i \cdot \lambda_k \quad \text{if } i < k$$

for some invertible elements $\lambda_i \in \mathcal{A}$. Observe that $D_k D_i = D_i D_k \cdot \lambda_k^{-1}$ for $i > k$. We use these commutation relations to rewrite the n-ary generalized Leibniz rule for $[x_1, \ldots, x_n]D_j$ in the following way:

$$[x_1, \ldots, x_n]D_j = \sum_{i=1}^{n} \begin{vmatrix} x_1\sigma D_1 \cdot \lambda_1 & \ldots & x_1\sigma D_{j-1} \cdot \lambda_{j-1} & x_1\sigma D_j \cdot \lambda_j & x_1\sigma D_{j+1} \cdot \lambda_{j+1} & \ldots & x_1\sigma D_n \cdot \lambda_n \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ x_{i-1}\sigma D_1 \cdot \lambda_1 & \ldots & x_{i-1}\sigma D_{j-1} \cdot \lambda_{j-1} & x_{i-1}\sigma D_j \cdot \lambda_j & x_{i-1}\sigma D_{j+1} \cdot \lambda_{j+1} & \ldots & x_{i-1}\sigma D_n \cdot \lambda_n \\ x_i D_j D_1 \cdot \lambda_1 & \ldots & x_i D_j D_{j-1} \cdot \lambda_{j-1} & x_i D_j D_j & x_i D_j D_{j+1} \cdot \lambda_{j+1}^{-1} & \ldots & x_i D_j D_n \cdot \lambda_n^{-1} \\ x_{i+1}\tau D_1 \cdot \lambda_1 & \ldots & x_{i+1}\tau D_{j-1} \cdot \lambda_{j-1} & x_{i+1}\tau D_j \cdot \lambda_j & x_{i+1}\tau D_{j+1} \cdot \lambda_{j+1} & \ldots & x_{i+1}\tau D_n \cdot \lambda_n \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ x_n\tau D_1 \cdot \lambda_1 & \ldots & x_n\tau D_{j-1} \cdot \lambda_{j-1} & x_n\tau D_j \cdot \lambda_j & x_n\tau D_{j+1} \cdot \lambda_{j+1} & \ldots & x_n\tau D_n \cdot \lambda_n \end{vmatrix}. \tag{14.13}$$

Note that the elements $x_i D_j D_k$ are multiplied by λ_k^{-1} when $j < k$.

Definition 14.11 Consider I and J to be partitions of $\{1, \ldots, n\}$, $|I| = u$, $|I| + |J| = n$. We call by *u-Jacobian* the following operator

$$|\cdot|^{(J)} : \mathcal{A}^u \to \mathcal{A}, \quad x_{I_1}, \ldots, x_{I_u} \mapsto |x_i D_j|_{i\in I}^{j\notin J}.$$

We refer to each determinant in the sum (14.13) as Δ_{ij}. Using $u-$Jacobian notation, we can open them as follows:

$$\begin{aligned}\Delta_{ij} = &\sum_{k=1}^{j-1} x_i D_j D_k \lambda_k (-1)^{i+k} \cdot [x_1\sigma, \ldots, x_{i-1}\sigma, x_{i+1}\tau, \ldots, x_n\tau]^{(k)} \prod_{s\neq k} \lambda_s \\ &+ x_i D_j D_j (-1)^{i+j} \cdot [x_1\sigma, \ldots, x_{i-1}\sigma, x_{i+1}\tau, \ldots, x_n\tau]^{(j)} \prod_{s\neq j} \lambda_s \\ &+ \sum_{k=j+1}^{n} x_i D_j D_k \lambda_k^{-1} (-1)^{i+k} \cdot [x_1\sigma, \ldots, x_{i-1}\sigma, x_{i+1}\tau, \ldots, x_n\tau]^{(k)} \prod_{s\neq k} \lambda_s.\end{aligned}$$

Note that $\prod \lambda_s$ is in all cases a single element of $\mathcal{A}$ and thus included within each term of the corresponding sum. We can pile up the λ_s together with λ_k on the first $j-1$ terms, thus obtaining

$$\begin{aligned}\Delta_{ij} &= \sum_{k=1}^{j-1} x_i D_j D_k (-1)^{i+k} \cdot [x_1\sigma, \ldots, x_{i-1}\sigma, x_{i+1}\tau, \ldots, x_n\tau]^{(k)} \prod_{s=1}^{n} \lambda_s \\ &+ x_i D_j D_j (-1)^{i+j} \cdot [x_1\sigma, \ldots, x_{i-1}\sigma, x_{i+1}\tau, \ldots, x_n\tau]^{(j)} \prod_{s\neq j} \lambda_s \\ &+ \sum_{k=j+1}^{n} x_i D_j D_k \cdot \lambda_k^{-1} (-1)^{i+k} \cdot [x_1\sigma, \ldots, x_{i-1}\sigma, x_{i+1}\tau, \ldots, x_n\tau]^{(k)} \prod_{s\neq k} \lambda_s.\end{aligned}$$

This is, in general, pretty far away from a proper n-ary twisted Leibniz rule. If all λ_s are invertible, we can rewrite each Δ_{ij} a bit further.

$$\begin{aligned}\Delta_{ij} &= \sum_{k=1}^{j-1} x_i D_j D_k (-1)^{i+k} \cdot [x_1\sigma, \ldots, x_{i-1}\sigma, x_{i+1}\tau, \ldots, x_n\tau]^{(k)} \prod_{s=1}^{n} \lambda_s \\ &+ x_i D_j D_j (-1)^{i+j} \cdot [x_1\sigma, \ldots, x_{i-1}\sigma, x_{i+1}\tau, \ldots, x_n\tau]^{(j)} \left(\prod_{s\neq j} \lambda_s - \prod_{s=1}^{n} \lambda_s + \prod_{s=1}^{n} \lambda_s \right) \\ &+ \sum_{k=j+1}^{n} x_i D_j D_k (-1)^{i+k} \cdot [x_1\sigma, \ldots, x_{i-1}\sigma, x_{i+1}\tau, \ldots, x_n\tau]^{(k)} (\lambda_k^{-1} - \lambda_k + \lambda_k) \prod_{s\neq k} \lambda_s \\ &= \sum_{k=1}^{n} x_i D_j D_k (-1)^{i+k} \cdot [x_1\sigma, \ldots, x_{i-1}\sigma, x_{i+1}\tau, \ldots, x_n\tau]^{(k)} \prod_{s=1}^{n} \lambda_s \\ &+ x_i D_j D_j (-1)^{i+j} \cdot [x_1\sigma, \ldots, x_{i-1}\sigma, x_{i+1}\tau, \ldots, x_n\tau]^{(j)} \left(\prod_{s\neq j} \lambda_s - \prod_{s=1}^{n} \lambda_s \right) \\ &+ \sum_{k=j+1}^{n} x_i D_j D_k (-1)^{i+k} \cdot [x_1\sigma, \ldots, x_{i-1}\sigma, x_{i+1}\tau, \ldots, x_n\tau]^{(k)} \left((\lambda_k^{-1} - \lambda_k) \prod_{s\neq k} \lambda_s \right) \\ &= [x_1\sigma, \ldots, x_{i-1}\sigma, x_i D_j, x_{i+1}\tau, \ldots, x_n\tau] \prod_{s=1}^{n} \lambda_s \\ &+ x_i D_j D_j (-1)^{i+j} \cdot [x_1\sigma, \ldots, x_{i-1}\sigma, x_{i+1}\tau, \ldots, x_n\tau]^{(j)} \left(\prod_{s\neq j} \lambda_s - \prod_{s=1}^{n} \lambda_s \right) \\ &+ \sum_{k=j+1}^{n} x_i D_j D_k (-1)^{i+k} \cdot [x_1\sigma, \ldots, x_{i-1}\sigma, x_{i+1}\tau, \ldots, x_n\tau]^{(k)} \left((\lambda_k^{-1} - \lambda_k) \prod_{s\neq k} \lambda_s \right) \\ &= [x_1\sigma, \ldots, x_{i-1}\sigma, x_i D_j, x_{i+1}\tau, \ldots, x_n\tau] \prod_{s=1}^{n} \lambda_s + R_{ij1} + R_{ik2},\end{aligned}$$

where R_{ij1} and R_{ik2} represents the second and third sums in this expansion.

If $\lambda_s = 1$ for all s, then $R_{ij1} = 0$, $R_{ik2} = 0$ and $\prod_{s=1}^{n} \lambda_s = 1$. Hence,

$$[x_1, \ldots, x_n]D_j = \sum_{i=1}^{n} \Delta_{ij} = \sum_{i=1}^{n} [x_1\sigma, \ldots, x_{i-1}\sigma, x_i D_j, x_{i+1}\tau, \ldots, x_n\tau].$$

Theorem 14.11 *Let $\mathcal{A}$ be a commutative associative algebra, σ, τ two linear maps, D_s, $s \in \{1, \ldots, n\}$ pairwise commuting (σ, τ)-derivations that commute with σ and τ. Then each D_s is a (σ, τ)-derivation with respect to the Jacobian product, with the following Leibniz-type rule*

$$[x_1, \ldots, x_n]D_j = \sum_{i=1}^{n} \Delta_{ij} = \sum_{i=1}^{n} [x_1\sigma, \ldots, x_{i-1}\sigma, x_i D_j, x_{i+1}\tau, \ldots, x_n\tau].$$

Under softer commutation rules, we obtain Leibniz-type rules weighted by a product of commutation factors and deformed by an extra term that is quadratic in D_j:

(i) If $\lambda_k = -1$ for all $k \neq j$, then $R_{ik2} = 0$. Also, $\prod_{s \neq j} \lambda_s - \prod_{s=1}^{n} \lambda_s = 2(-1)^{n-1}$ and then

$$\begin{aligned}[x_1, \ldots, x_n]D_j = &\sum_{i=1}^{n} [x_1\sigma, \ldots, x_{i-1}\sigma, x_i D_j, x_{i+1}\tau, \ldots, x_n\tau] \cdot (-1)^{n-1} \\ &+ x_i D_j D_j [x_1\sigma, \ldots, x_{i-1}\sigma, x_{i+1}\tau, \ldots, x_n\tau]^{(j)} \cdot 2(-1)^{i+j+n-1}.\end{aligned}$$

(ii) More generally, if $\lambda_k = -1$ for all $k > j$ we have

$$\begin{aligned}[x_1, \ldots, x_n]D_j = &\sum_{i=1}^{n} [x_1\sigma, \ldots, x_{i-1}\sigma, x_i D_j, x_{i+1}\tau, \ldots, x_n\tau] \cdot (-1)^{n-j} \prod_{s=1}^{j-1} \lambda_s \\ &+ x_i D_j D_j [x_1\sigma, \ldots, x_{i-1}\sigma, x_{i+1}\tau, \ldots, x_n\tau]^{(j)} \cdot 2(-1)^{i+j+n-1}.\end{aligned}$$

Note that these apply for each D_j. This naive approach is very limited in terms of the commutation relations considered.

14.5.4 More General Commutation Relations

The commutation factor between the D_s may (and in general, is expected to) be different from the λ_s. We call γ_{jk} to such elements of $\mathcal{A}$ verifying $D_k D_j = D_j D_k \cdot$

γ_{jk}, no restrictions on j, k. Each $[x_1, \ldots, x_n]D_j$ expands into the sum of determinants below:

$$[x_1, \ldots, x_n]D_j = \sum_{i=1}^{n} \begin{vmatrix} x_1\sigma D_1\lambda_1 & \ldots & x_1\sigma D_{j-1}\lambda_{j-1} & x_1\sigma D_j\lambda_j & x_1\sigma D_{j+1}\lambda_{j+1} & \ldots & x_1\sigma D_n\lambda_n \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ x_{i-1}\sigma D_1\lambda_1 & \ldots & x_{i-1}\sigma D_{j-1}\lambda_{j-1} & x_{i-1}\sigma D_j\lambda_j & x_{i-1}\sigma D_{j+1}\lambda_{j+1} & \ldots & x_{i-1}\sigma D_n\lambda_n \\ x_i D_j D_1\gamma_{j1} & \ldots & x_i D_j D_{j-1}\gamma_{jj-1} & x_i D_j D_j & x_i D_j D_j\gamma_{jj+1} & \ldots & x_i D_j D_n\gamma_{jn} \\ x_{i+1}\tau D_1\lambda_1 & \ldots & x_{i+1}\tau D_{j-1}\lambda_{j-1} & x_{i+1}\tau D_j\lambda_j & x_{i+1}\tau D_{j+1}\lambda_{j+1} & \ldots & x_{i+1}\tau D_n\lambda_n \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ x_n\tau D_1\lambda_1 & \ldots & x_n\tau D_{j-1}\lambda_{j-1} & x_n\tau D_j\lambda_j & x_n\tau D_{j+1}\lambda_{j+1} & \ldots & x_n\tau D_n\lambda_n \end{vmatrix}.$$

Note that position jj of this matrix is the only element without a multiplying constant. This is due to D_j commuting with itself, that is, $\gamma_{jj} = 1$.

The Leibniz-type rule of the D_s is affected in the following way:

$$\Delta_{ij} = \sum_{k=1}^{n} x_i D_j D_k (-1)^{i+k} \cdot [x_1\sigma, \ldots, x_{i-1}\sigma, x_{i+1}\tau, \ldots, x_n\tau]^{(k)} \left(\gamma_{jk} \prod_{s \neq k} \lambda_s \right)$$

where the term in $D_j D_j$ is absorbed into the sum since $\gamma_{jj} = 1$.

The first term now cannot be expressed as an n-Jacobian because the constants $\Gamma_{jk} = \gamma_{jk} \prod_{s \neq k} \lambda_s$ vary with k. If all n of them are the same constant, which we name Γ_j, then this can be expressed in the form of an n-Jacobian.

For each γ_{jk} that is invertible, we can rewrite this restraint as $\prod_{s \neq k} \lambda_s = \Gamma_{jk}\gamma_{jk}^{-1}$.

Lemma 14.5 (Particular n-ary Leibniz-type rule for D_j) *If Γ_j is independent of k, then*

$$[x_1, \ldots, x_n]D_j = \sum_{i=1}^{n} \Delta_{ij} = \Gamma_j \cdot \sum_{i=1}^{n} [x_1\sigma, \ldots, x_{i-1}\sigma, x_i D_j, x_{i+1}\tau, \ldots, x_n\tau].$$

Remark 14.3 If Γ_{jk} are not independent of k, then

$$[x_1, \ldots, x_n]D_j = \sum_{i=1}^{n} \begin{vmatrix} x_1\sigma D_1 & \ldots & x_1\sigma D_{j-1} & x_1\sigma D_j & x_1\sigma D_{j+1} & \ldots & x_1\sigma D_n \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ x_{i-1}\sigma D_1 & \ldots & x_{i-1}\sigma D_{j-1} & x_{i-1}\sigma D_j & x_{i-1}\sigma D_{j+1} & \ldots & x_{i-1}\sigma D_n \\ x_i D_j D_1\Gamma_{j1} & \ldots & x_i D_j D_{j-1}\Gamma_{jj-1} & x_i D_j D_j\Gamma_{jj} & x_i D_j D_j\Gamma_{jj+1} & \ldots & x_i D_j D_n\Gamma_{jn} \\ x_{i+1}\tau D_1 & \ldots & x_{i+1}\tau D_{j-1} & x_{i+1}\tau D_j & x_{i+1}\tau D_{j+1} & \ldots & x_{i+1}\tau D_n \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ x_n\tau D_1 & \ldots & x_n\tau D_{j-1} & x_n\tau D_j & x_n\tau D_{j+1} & \ldots & x_n\tau D_n \end{vmatrix}$$

or expressed in the form of a sum of $(n-1)$-Jacobians,

$$[x_1, \dots, x_n]D_j = \sum_{i,k=1}^{n} x_i D_j D_k (-1)^{i+k} \cdot [x_1\sigma, \dots, x_{i-1}\sigma, x_{i+1}\tau, \dots, x_n\tau]^{(k)} \Gamma_{jk}.$$

The condition that Γ_j is independent of k relates the λ_s and the γ_{jk} very tightly. If we look at Γ_j, we have

$$\gamma_{jk} \prod_{s\neq k} \lambda_s \stackrel{def}{=} \prod_{s\neq j} \lambda_s,$$

since $\gamma_{jj} = 1$. This has the following important implication:

$$\gamma_{jk}\lambda_j \prod_{s\neq k,j} \lambda_s = \lambda_k \prod_{s\neq k,j} \lambda_s \Rightarrow (\gamma_{jk}\lambda_j - \lambda_k) \prod_{s\neq k,j} \lambda_s = 0.$$

Provided that $\mathcal{A}$ is a domain and λ_j is invertible, we have $\gamma_{jk}\lambda_j = \lambda_k \Rightarrow \gamma_{jk} = \lambda_k\lambda_j^{-1}$.

If we consider $l \neq j, k$, then

$$\gamma_{jk} \prod_{s\neq k} \lambda_s = \Gamma_j = \gamma_{jl} \prod_{s\neq l} \lambda_s,$$

$$\gamma_{jk}\lambda_l \prod_{s\neq k,l} \lambda_s = \gamma_{jl}\lambda_k \prod_{s\neq k,l} \lambda_s \Rightarrow (\gamma_{jk}\lambda_l - \gamma_{jl}\lambda_k) \prod_{s\neq k,l} \lambda_s = 0.$$

If $\mathcal{A}$ is a domain, $\gamma_{jk}\lambda_l = \gamma_{jl}\lambda_k$. If γ_{jl} and λ_l are invertible, $\gamma_{jk}\gamma_{jl}^{-1} = \lambda_k\lambda_l^{-1}$.

This is a very strong connection between σ, τ and the D_s in terms of commutation relations.

Proposition 14.15 *Let $\mathcal{A}$ be a commutative associative algebra, $\lambda_i \in \mathcal{A}$, σ and τ two linear maps, $D_1, \dots, D_n$ pairwise different (σ, τ)-derivations of $\mathcal{A}$ such that $D_i\sigma = \sigma D_i \cdot \lambda_i$ and $D_i\tau = \tau D_i \cdot \lambda_i$ for all i and $D_k D_j = D_j D_k \cdot \gamma_{jk}$, where $\gamma_{jk}\lambda_j = \lambda_k$ for $k \neq j$.*

Each D_j is a generalized (σ, τ)-derivation with respect to the Jacobian, with the following Leibniz-type rule:

$$[x_1, \dots, x_n]D_j = \Gamma_j \cdot \sum_{i=1}^{n} [x_1\sigma, \dots, x_{i-1}\sigma, x_i D_j, x_{i+1}\tau, \dots, x_n\tau],$$

where $\Gamma_j = \prod_{s\neq j} \lambda_s$.

The possibility can be considered where all D_s have the same Leibniz-type rule with the same weight Γ_j. Say $\Gamma_j = \Gamma$ for some constant Γ. In this case:

(i) $\prod_{s\neq j}\lambda_s = \prod_{s\neq k}\lambda_s$ for all $k \neq j$,

(ii) $\gamma_{jk}\prod_{s\neq k}\lambda_s = \gamma_{lk}\prod_{s\neq k}\lambda_s$ for all $l \neq j$.

The first condition implies $(\lambda_k - \lambda_j)\prod_{s\neq j,k}\lambda_s = 0$, and if $\mathcal{A}$ is a domain, $\lambda_k = \lambda_j$. The second condition implies $(\gamma_{jk} - \gamma_{lk})\prod_{s\neq k}\lambda_s = 0$, and if $\mathcal{A}$ is a domain, $\gamma_{jk} = \gamma_{lk}$. Immediately, $\gamma_{jk} = \gamma_{kk} = 1, k \neq j$. That is, all the D_s commute with D_j. This applies for all j, thus all the D_s commute. Denote by λ the commutation factor between σ and any D_i. We have $\Gamma = \lambda^{n-1}$. Consider now the first case where Γ_j does not depend on k and is invertible. Then, we can take one term out from the Leibniz-type rule sum and obtain the following expansions:

$$\begin{aligned}
&[x_1 D_j, x_2\tau, \dots, x_n\tau] \\
&\quad = \Gamma_j^{-1}[x_1, \dots, x_n]D_j - \sum_{i=2}^{n}[x_1\sigma, \dots, x_{i-1}\sigma, x_i D_j, x_{i+1}\tau, \dots, x_n\tau], \\
&[x_i D_j, y_2\tau, \dots, y_n\tau] \\
&\quad = \Gamma_j^{-1}[x_i, y_2 \dots, y_n]D_j - \sum_{s=2}^{n}[x_i\sigma, y_2\sigma \dots, y_{s-1}\sigma, y_s D_j, y_{s+1}\tau, \dots, y_n\tau].
\end{aligned}$$

We search for Jacobi-like identities and conditions for those to hold on as general a setting as possible:

$$\begin{aligned}
&[[x_1, \dots, x_n], y_2\tau, \dots, y_n\tau] = [|x_i D_j|, y_2\tau, \dots, y_n\tau] \\
&= \sum_{i=1}^{n}\begin{vmatrix} x_1 D_1\sigma & \dots & x_1 D_n\sigma \\ \vdots & & \vdots \\ [x_i D_1, y_2\tau, \dots, y_n\tau] & \dots & [x_i D_n, y_2\tau, \dots, y_n\tau] \\ \vdots & & \vdots \\ x_n D_1\tau & \dots & x_n D_n\tau \end{vmatrix} \\
&= \sum_{i=1}^{n}\begin{vmatrix} x_1\sigma D_1\lambda_1 & \dots & x_1\sigma D_n\lambda_n \\ \vdots & & \vdots \\ [x_i D_1, y_2\tau, \dots, y_n\tau] & \dots & [x_i D_n, y_2\tau, \dots, y_n\tau] \\ \vdots & & \vdots \\ x_n\tau D_1\lambda_1 & \dots & x_n\tau D_n\lambda_n \end{vmatrix} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}[x_i D_j, y_2\tau, \dots, y_n\tau](-1)^{i+j}\cdot[x_1\sigma, \dots, x_{i-1}\sigma, x_{i+1}\tau, \dots, x_n\tau]^{(j)}\Gamma_j.
\end{aligned}$$

With notation, extending Filippov's notation,

$$M_{ij} = [x_1\sigma, \ldots, x_{i-1}\sigma, x_{i+1}\tau, \ldots, x_n\tau]^{(j)} \cdot \Gamma_j,$$

this equality becomes

$$[[x_1, \ldots, x_n], y_2\tau, \ldots, y_n\tau] = \sum_{i,j=1}^{n} [x_i D_j, y_2\tau, \ldots, y_n\tau](-1)^{i+j} \cdot M_{ij}.$$

Applying the previous expansion of $[x_i D_j, y_2\tau, \ldots, y_n\tau]$ yields

$$[[x_1, \ldots, x_n], y_2\tau, \ldots, y_n\tau] = \sum_{i,j=1}^{n} [x_i, y_2 \ldots, y_n]D_j \cdot (-1)^{i+j} M_{ij}\Gamma_j^{-1}$$
$$- \sum_{i,j=1}^{n} \sum_{s=2}^{n} [x_i\sigma, y_2\sigma, \ldots, y_{s-1}\sigma, y_s D_j, y_{s+1}\tau \ldots, y_n\tau] \cdot (-1)^{i+j} M_{ij}.$$

We name the expressions in the RHS as R_1 and R_2 respectively. Firstly, observe that Γ_j^{-1} cancels with the Γ_j within each M_{ij}. Then R_1 expands as

$$\sum_{i=1}^{n} \begin{vmatrix} x_1\sigma D_1 & \ldots & x_1\sigma D_n \\ \vdots & & \vdots \\ [x_i, y_2, \ldots, y_n]D_1 & \ldots & [x_i, y_2, \ldots, y_n]D_n \\ \vdots & & \vdots \\ x_n\tau D_1 & \ldots & x_n\tau D_n \end{vmatrix}$$
$$= \sum_{i=1}^{n} [x_1\sigma, \ldots, x_{i-1}\sigma, [x_i, y_2, \ldots, y_n], x_{i+1}\tau, \ldots, x_n\tau].$$

Proposition 14.16 *Let R_2 be defined as above. If $R_2 = 0$, then*

$$[[x_1, \ldots, x_n], y_2\tau, \ldots, y_n\tau] = \sum_{i=1}^{n} [x_1\sigma, \ldots, x_{i-1}\sigma, [x_i, y_2, \ldots, y_n], x_{i+1}\tau, \ldots, x_n\tau].$$

Theorem 14.12 *Let $\mathcal{A}$ be a commutative associative algebra, $\lambda_i \in A$, σ and τ two linear maps, $D_1, \ldots, D_n$ pairwise different (σ, τ)-derivations of $\mathcal{A}$ such that*

$$D_i\sigma = \sigma D_i \cdot \lambda_i, \quad D_i\tau = \tau D_i \cdot \lambda_i, \quad \text{where} \quad 1 \le i \le n,$$
$$D_k D_j = D_j D_k \gamma_{jk}, \quad \text{where} \quad \gamma_{jk}\lambda_j = \lambda_k, \quad k \ne j.$$

Define the Jacobian $|\cdot|$ of n elements as $[x_1, \ldots, x_n] = |x_i D_j|$. Also, let $\Gamma_i = \prod_{s \ne i} \lambda_s$ and R_2 be the following sum of determinants:

$$\sum_{\substack{s=2\\ i=1}}^{n} \begin{vmatrix} x_1\sigma D_1 & \dots & x_1\sigma D_n \\ \vdots & & \vdots \\ [x_i\sigma, y_2\sigma, y_{s-1}\sigma, \dots, y_s D_1, y_{s+1}\tau, \dots, y_n\tau] & \dots & [x_i\sigma, y_2\sigma, y_{s-1}\sigma, \dots, y_s D_n, y_{s+1}\tau \dots, y_n\tau] \\ \vdots & & \vdots \\ x_n\tau D_1 & \dots & x_n\tau D_n \end{vmatrix} \Gamma_i.$$

If $R_2 = 0$, then $(\mathcal{A}, |\cdot|)$ has a structure of (σ, τ, n)-Hom-Lie algebra. The twisted Jacobi identity is given for all $x_1, \dots, x_n, y_2, \dots, y_n \in A$ by

$$[[x_1, \dots, x_n], y_2\tau, \dots, y_n\tau] = \sum_{i=1}^{n} [x_1\sigma, \dots, x_{i-1}\sigma, [x_i, y_2, \dots, y_n], x_{i+1}\tau, \dots, x_n\tau].$$

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Chapter 15
An Application of Twisted Group Rings in Secure Group Communications

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Abstract In this paper we introduce a group key management protocol for secure group communications in a non-commutative setting. To do so, we consider a group ring over the dihedral group with a twisted multiplication using a cocycle. The protocol is appropriate for the so-called post-quantum era and it is shown that the security of the initial key agreement is equivalent to the protocol given for just two communication parties, i.e., there is no information leakage as the number of users grows. Moreover we show that further rekeying messages provide forward and backward security, that means that no former or future user in a communication group can get information on previous or new future keys.

Keywords Secure communications · Group key management · Twisted group ring

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15.1 Introduction

In recent years, new hard problems have been proposed in public key cryptography, since those that we are using might be not secure soon. When two parties want to communicate through an insecure channel, they need to do a key agreement, which consist on agreeing on a secret shared key by exchanging information that does not compromise the common key.

The first widely used protocol that allows this to happen was proposed in 1976 by Diffie and Hellman [2], and works as follows:

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Let two users, Alice and Bob, who want to agree on a common key through an insecure channel. Let p a prime number, $\mathbb{Z}_p^*$ the multiplicative group of integers modulo p, and g a primitive root modulo p public.

(i) Alice chooses a secret integer a, and sends Bob $p_A = g^a (\text{mod } p)$.
(ii) Bob chooses a secret integer b, and sends Alice $p_B = g^b (\text{mod } p)$.
(iii) Alice computes $p_B^a (\text{mod } p)$, and Bob computes $p_A^b (\text{mod } p)$, so both obtain the same value, which is the secret shared key $K = g^{ab} (\text{mod } p)$.

Information shared does not compromise the shared key since the underlying problem an attacker would need to solve, the so-called Discrete Logarithm Problem (DLP) is believed to be hard. This key agreement can be seen as an example of this generalization by Maze et al. [9]:

Let S be a finite set, G an abelian semigroup, ϕ a $G-$action on S, and a public element $s \in S$.

(i) Alice chooses $a \in G$, and sends Bob $p_A = \phi(a, s)$.
(ii) Bob chooses $b \in G$, and sends Alice $p_B = \phi(b, s)$.
(iii) Alice computes $\phi(a, p_B)$, and Bob computes $\phi(b, p_A)$, so both obtain the secret shared key $K = \phi(a, \phi(b, s)) = \phi(b, \phi(a, s))$.

whose underlying problem is called the Semigroup Action Problem (SAP).

Semigroup Action Problem. Given a semigroup action ϕ of the group G on a set S and elements $x \in S$ and $y \in G$, find $g \in G$ such that $\phi(g, x) = y$.

In the context of SAP, we proposed in [4] a new setting, and some protocols. In our case, the platform is a twisted group ring, a new proposal in the context of group rings, that have also been recently used in cryptography in works like [3, 5–7]. And the action proposed is the two-sided multiplication in a twisted group ring, so the problem is a variation in the twisted case of the so-called Decomposition Problem (DP), which is a generalization of the Conjugate Search Problem (CSP).

Decomposition Problem. Given a group G, $(x, y) \in G \times G$ and $S \subset G$, the problem is to find $z_1, z_2 \in S$ such that $y = z_1 x z_2$.

A natural extension is how to extend this kind of schemes to more than two users. In the classic Diffie-Hellman protocol, a solution is proposed in [10]. And in the more case of SAP, this solution can be found in [8]. In both cases, it is shown that the extra information shared in the case of a n users key exchange does not imply information leakage for an attacker compared to the 2-users case.

Our aim in this work is to show that in our setting, that differs from those above given the non-commutativity of twisted group rings, and which could work better against problems that threat current communications, this is also true: the extra information shared between n users does not imply information leakage, so if the 2-users key exchange is computationally secure, then the extension to n users is also secure.

15.2 Algebraic Setting

In this section, twisted group rings are defined, and we also show some properties that make the key exchange possible.

Definition 15.1 Let K be a ring, G be a multiplicative group, and α be a cocycle in $U(K)$, the units of K. The group ring $K^{\alpha}G$ is defined to be the set of all finite sums of the form

$$\sum_{g_i \in G} r_i g_i,$$

where $r_i \in K$ and all but a finite number of r_i are zero.

The sum of two elements in $K^{\alpha}G$ is given by

$$\left(\sum_{g_i \in G} r_i g_i\right) + \left(\sum_{g_i \in G} s_i g_i\right) = \sum_{g_i \in G} (r_i + s_i) g_i.$$

The multiplication, which is twisted by a cocycle, is given by

$$\left(\sum_{g_i \in G} r_i g_i\right) \cdot \left(\sum_{g_i \in G} s_i g_i\right) = \sum_{g_i \in G} \left(\sum_{g_j g_k = g_i} r_j s_k \, \alpha(g_j, g_k)\right) g_i.$$

As an example, consider the finite field K, a primitive element t, and the dihedral group of $2m$ elements, $D_{2m} =< x, y : x^m = y^2 = 1, yx^a = x^{m-a} y >$. The group ring $R = K^{\alpha} D_{2m}$, where α is

$$\alpha : D_{2m} \times D_{2m} \to K^*$$

with $\alpha(x^i, x^j y^k) = 1$ and $\alpha(x^i y, x^j y^k) = t^j \;\; i, j = 1, \ldots, 2m-1$, is a twisted group ring.

Now we establish some useful properties that will allow us to make our key exchange possible.

Definition 15.2 Let $R = K^{\alpha} D_{2m}$, where t is the primitive root of unity that generates K and α is the cocycle defined above. Given $h \in R$,

$$h = \sum_{\substack{0 \le i \le m-1 \\ k=0,1}} r_i x^i y^k,$$

where $r_i \in K$ and $x, y \in D_{2m}$. We define $h^* \in K^\alpha D_{2m}$:

$$h^* = \sum_{\substack{0 \leq i \leq m-1 \\ k=0,1}} r_i t^{-i} x^i y^k,$$

where $r_i \in K$ and $x, y \in D_m$.

Note that $R = K^\alpha D_{2m}$ can be written as vector space as

$$R = R_1 \oplus R_2,$$

where $R_1 = KC_m$ and $R_2 = K^\alpha C_m y$, and C_m is a cyclic group of order m. In this context, we can define $A_j \leq R_j$ as

$$A_j = \left\{ \sum_{i=0}^{m-1} r_i x^i y^k \in R_j : r_i = r_{m-i} \right\}.$$

where $j = 1, 2$.

Proposition 15.1 *Given $h_1, h_2 \in R$,*

- *If $h_1, h_2 \in R_1$, then $h_1 h_2 = h_2 h_1$;*
- *If $h_1, h_2 \in A_2$, then $h_1 h_2^* = h_2 h_1^*$, and $h_1^* h_2 = h_2^* h_1$;*
- *If $h_1 \in A_1, h_2 \in A_2$, then $h_1 h_2 = h_2 h_1^*$.*

A proof of this proposition can be found in [4].

15.3 Key Management over Twisted Group Rings

In this section, we explain the protocols proposed in [4], over the twisted group ring $R = K^\alpha D_{2m}$ defined above.

Let $h \in R$ be a random public element. The key exchange between two users, Alice and Bob, is as follows:

(i) Alice selects a secret pair $s_A = (g_1, k_1)$, where $g_1 \in R_1, k_1 \in A_2 \leq R_2$.
(iii) Bob selects a secret pair $s_B = (g_2, k_2)$, where $g_2 \in R_1, k_2 \in A_2 \leq R_2$.
(iii) Alice sends Bob $p_A = g_1 h k_1$, and Bob sends Alice $p_B = g_2 h k_2$.
(iv) Alice computes $K_A = g_1 p_B k_1^*$, and Bob computes $K_B = g_2 p_A k_2^*$, and they get the same secret shared key.

This protocol works, it was shown in [4]. Let the underlying decisional problem be the following:

Let $R = K^\alpha D_{2m} = R_1 \oplus R_2$, $A_2 \leq R_2$, given $(h, g_1 h k_1, g_2 h k_2, r_1 h r_2)$, decide whether $(r_1, r_2) = (g_2 g_1, k_1 k_2^*)$ or not, where $h \in R$, $g_i, r_1 \in R_1$, $k_i \in A_2$, $r_2 \in A_1$.

It means that if someone breaks this problem, then the key exchange above can also be broken.

To define the general protocol for n users, let us define the action $\phi : (R_1 \times A_2) \times R \longrightarrow R$,

$$\phi(s_i, h) = g_i h k_i$$

where $s_i = (g_i, k_i)$. Note that

$$\phi(s_i \phi(s_j, h)) = \phi(s_i s_j, h)$$

We will sometimes write $\phi(s_i s_j, h)$ to refer to $\phi(s_i, \phi(s_j, h))$, to make some definitions more readable.

Let $h \in R$ be a random public element, and $h \in R = R_1 \oplus R_2$, described before. For $i = 1, \ldots, n$, user U_i has a secret pair $s_i = (g_i, k_i)$, where $g_i \in R_1$ and $k_i \in A_2 \leq R_2$. Let $\phi(s_i, h) = g_i h k_i$, 2-sided multiplication. We will denote $s_i^* = (g_i, k_i^*)$. The key establishment for n is as follows:

(i) For $i = 1, \ldots, n$, user U_i sends to user U_{i+1} the message

$$\{C_i^1, C_i^2, \ldots, C_i^{i+1}\},$$

where $C_1^1 = h$, $C_1^2 = g_1 h k_1$ and

- for $i > 1$ even, $C_i^j = \phi(s_i, C_{i-1}^j)$, when $j < i$, $C_i^i = C_{i-1}^i$, $C_i^{i+1} = \phi(s_i^*, C_{i-1}^i)$,
- for $i > 1$ odd, $C_i^j = \phi(s_i^*, C_{i-1}^j)$, when $j < i$, $C_i^i = C_{i-1}^i$, $C_i^{i+1} = \phi(s_i, C_{i-1}^i)$.

(ii) User U_n computes $\phi(s_n, C_{n-1}^n)$ if n is odd and $\phi(s_n^*, C_{n-1}^n)$ if n is even.

(iii) User U_n broadcasts

$$\{C_n^1, C_n^2, \ldots, C_n^n\}.$$

(iv) User U_i computes $\phi(s_i, C_n^i)$ if n is odd or $\phi(s_i^*, C_n^i)$ if n is even, and gets the shared key.

This protocol allows all users to obtain a common shared key, as shown in Proposition 3 of [4]. In this case, the underlying decisional problem is the following:

- *(n even) Let $R = K^\alpha D_{2m} = R_1 \oplus R_2$, $A_2 \leq R_2$, given $r_1 h r_2$, and*

  $$\left\{\phi(s_{i_1} s_{i_2}^* s_{i_3} \ldots s_{i_{m-2}}^* s_{i_{m-1}} s_{i_m}^*, h) : \{i_1, \ldots, i_m\} \subsetneq \{1, \ldots, n\}, m \in \{1, \ldots, n-1\}\right\}$$

  decide whether $(r_1, r_2) = (g_1 g_2 g_3 \ldots g_{n-1} g_n, k_1 k_2^* k_3 \ldots k_{n-1} k_n^*)$ or not, where $h \in R$, $g_i, r_1 \in R_1$, $k_i \in A_2$, $r_2 \in A_1$.
- *(n odd) Let $R = K^\alpha D_{2m} = R_1 \oplus R_2$, $A_2 \leq R_2$, given $r_1 h r_2$, and*

  $$\left\{\phi(s_{i_1} s_{i_2}^* s_{i_3} \ldots s_{i_{m-2}} s_{i_{m-1}}^* s_{i_m}, h) : \{i_1, \ldots, i_m\} \subsetneq \{1, \ldots, n\}, m \in \{1, \ldots, n-1\}\right\}$$

decide whether $(r_1, r_2) = (g_1 g_2 g_3 \dots g_{n-1} g_n, k_1 k_2^* k_3 \dots k_{n-1}^* k_n)$ *or not, where* $h \in R$, $g_i, r_1 \in R_1, k_i, r_2 \in A_2$.

We have described the so-called Initial Key Agreement (IKA), but another important process in group communication is key refreshment through the Auxiliary Key Agreement (AKA), which takes advantage of the information that was sent before to create a new key in a group when necessary, and is more computationally efficient than IKA. There exist three situations: the members of the group stay the same, a member leaves the group, or someone new joins it.

In the first situation, very user U_i has the information C_n^i received from the user U_n. The rekeying process can be carried out by any of them. We call this user U_c. He chooses a new element $\widetilde{s_c} = (\widetilde{g_c}, \widetilde{k_c})$, where $\widetilde{g_c} \in R_1$ and $\widetilde{k_c} \in A_2$. If n is odd, he changes his private key to $\widetilde{s_c}^* s_c$ and broadcasts the message

$$\{\phi(\widetilde{s_c}^*, C_n^1), \phi(\widetilde{s_c}^*, C_n^2), \dots, \phi(\widetilde{s_c}^*, C_n^{c-1}), C_n^c, \phi(\widetilde{s_c}^*, C_n^{c+1}), \dots, \phi(\widetilde{s_c}^*, C_n^n)\}.$$

If n is even, he changes his private key to $\widetilde{s_c} s_c^*$ and broadcasts the message

$$\{\phi(\widetilde{s_c}, C_n^1), \phi(\widetilde{s_c}, C_n^2), \dots, \phi(\widetilde{s_c}, C_n^{c-1}), C_n^c, \phi(\widetilde{s_c}, C_n^{c+1}), \dots, \phi(\widetilde{s_c}, C_n^n)\}.$$

Then every user recovers the common key using the private key s_i if n is even, and s_i^* if n is odd. A proof can be found in [4].

In the second case, when some user leaves the group, the corresponding position in the rekeying message is omitted.

In the last case, when a new user U_{n+1} joins the group, if n is odd, then U_c adds the element $\phi(\widetilde{s_c}, C_n^n)$ and sends the following to the new user:

$$\{\phi(\widetilde{s_c}, C_n^1), \phi(\widetilde{s_c}, C_n^2), \dots, \phi(\widetilde{s_c}, C_n^{c-1}), C_n^c, \phi(\widetilde{s_c}, C_n^{c+1}), \dots, \phi(\widetilde{s_c}, C_n^{n-1}), \phi(\widetilde{s_c}, C_n^n)\}.$$

If n is even, U_c adds the element $\phi(\widetilde{s_c}^*, C_n^n)$ and sends to U_{n+1} the following:

$$\{\phi(\widetilde{s_c}^*, C_n^1), \phi(\widetilde{s_c}^*, C_n^2), \dots, \phi(\widetilde{s_c}^*, C_n^{c-1}), C_n^c, \phi(\widetilde{s_c}^*, C_n^{c+1}), \dots, \phi(\widetilde{s_c}^*, C_n^{n-1}), \phi(\widetilde{s_c}^*, C_n^n)\}.$$

Finally, user U_{n+1} proceeds to step 3 of the group key protocol and sends the other users the information to obtain the shared key using their private keys.

15.4 Secure Group Key Management

In this section, we show that the extra information sent in the protocol of n users does not implies additional information leakage for an attacker respect to the 2-users case. For this purpose, we define the following random variables, choosing X randomly from $(R_1 \times A_2)^n$:

$$A_n = \Big(view(n, X), y\Big), \text{ for } y \in R \text{ randomly chosen.}$$

$$D_n = \begin{cases} \Big(view(n, X), \phi(s_n^* s_{n-1} s_{n-2}^* \dots s_3 s_2^* s_1, h), h\Big), & \text{if } n \text{ is even.} \\ \Big(view(n, X), \phi(s_n s_{n-1}^* s_{n-2} \dots s_3 s_2^* s_1, h)\Big), & \text{if } n \text{ is odd.} \end{cases}$$

where

- $view(n, X) :=$ the ordered set of all $\phi(s_{i_1} s_{i_2}^* s_{i_3} \dots s_{m-2}^* s_{m-1} s_m^*, h)$, for all proper subsets $\{i_1, \dots, i_m\}$ of $\{1, \dots, n\}$; $m \in \{1, \dots, n-1\}$.

when n is even, and

- $view(n, X) :=$ the ordered set of all $\phi(s_{i_1} s_{i_2}^* s_{i_3} \dots s_{m-2} s_{m-1}^* s_m, h)$, for all proper subsets $\{i_1, \dots, i_m\}$ of $\{1, \dots, n\}$; $m \in \{1, \dots, n-1\}$.

when n is odd.

Also note that $\phi(s_n^* s_{n-1} s_{n-2}^* \dots s_3 s_2^* s_1, h), h)$, or $\phi(s_n s_{n-1}^* s_{n-2} \dots s_3 s_2^* s_1, h)$, is the common secret key, is case n is even or odd respectively.

Let the relation $\sim$ be polynomial indistinguishability, as defined in [10]. In this context, it means that no polynomial-time algorithm can distinguish between a key and a random value with probability significantly greater than $\frac{1}{2}$.

Proposition 15.2 *The relation $\sim$ is an equivalence relation.*

A proof of this proposition can be found in [1]. Before we prove the main result, let us show that.

Lemma 15.1 *We can write $view(n, \{s_1, s_2\} \cup X)$, with $X = \{s_3, \dots, s_n\}$ as a permutation of*

$$V = \Big(view(n-1, \{s_1\} \cup X), \phi(s_n s_{n-1}^* \dots s_2, h), view(n-1, \{s_2\} \cup X),$$
$$\phi(s_n s_{n-1}^* s_{n-2} \dots s_3^* s_1, h), view(n-1, \{s_2^* s_1\} \cup X)\Big)$$

when n is even, and as a permutation of

$$V = \Big(view(n-1, \{s_1\} \cup X), \phi(s_n^* s_{n-1} s_{n-2}^* \dots s_2, h), view(n-1, \{s_2\} \cup X),$$
$$\phi(s_n^* s_{n-1} \dots s_3^* s_1, h), view(n-1, \{s_1 s_2^*\} \cup X)\Big)$$

when n is odd.

Proof Now we show that both sets are equal. First, we prove that $view(n, \{s_1, s_2\} \cup X) \subset V$: Let an element $a \in view(n, \{s_1, s_2\} \cup X)$:

- If n is even:

(i) If a contains $s_2^* s_1 (= s_1^* s_2)$, then it belongs to $view(n-1, \{s_2^* s_1\} \cup X) \subset V$.
(ii) If a does not contain s_1 (or s_1^*),

  – but it contains all the remaining elements, $s_2^{(*)}, \dots, s_n^{(*)}$, then it belongs to $\phi(s_n s_{n-1}^* \dots s_3^* s_2, h) \subset V$.
  – and if it does not contain all the remaining elements, then it belongs to $view(n-1, \{s_2\} \cup X) \subset V$.

(iii) If a does not contain s_2 (or s_2^*),

– but it contains all the remaining elements, $s_1^{(*)}, s_3^{(*)}, \ldots, s_n^{(*)}$, then it belongs to $\phi(s_n s_{n-1}^* \ldots s_3^* s_1, h) \subset V$.
– and if it does not contain all the remaining elements, then it belongs to $view(n-1, \{s_1\} \cup X) \subset V$.

(iv) Finally, if a does not contain s_1 neither s_2, it belongs to any of the following $view(n-1, \{s_1\} \cup X), view(n-1, \{s_2\} \cup X), view(n-1, \{s_1 s_2^*\} \subset V$.

- If n is odd:

(i) If a contains $s_2^* s_1 (= s_1^* s_2)$, then it belongs to $view(n-1, \{s_2^* s_1\} \cup X) \subset V$.
(ii) If a does not contain s_1 (or s_1^*),

– but it contains all the remaining elements, $s_2^{(*)}, \ldots, s_n^{(*)}$, then it belongs to $\phi(s_n^* s_{n-1} \ldots s_3^* s_2, h) \subset V$.
– and if it does not contain all the remaining elements, then it belongs to $view(n-1, \{s_2\} \cup X) \subset V$.

(iii) If a does not contain s_2 (or s_2^*),

– but it contains all the remaining elements, $s_1^{(*)}, s_3^{(*)}, \ldots, s_n^{(*)}$, then it belongs to $\phi(s_n^* s_{n-1} \ldots s_3^* s_1, h) \subset V$.
– and if it does not contain all the remaining elements, then it belongs to $view(n-1, \{s_1\} \cup X) \subset V$.

(iv) Finally, if a does not contain s_1 neither s_2, it belongs to any of the following $view(n-1, \{s_1\} \cup X), view(n-1, \{s_2\} \cup X), view(n-1, \{s_1 s_2^*\} \subset V$.

The reverse inclusion, $V \subset view(n, \{s_1, s_2\})$ is true since all the elements in V belong to

$$view(n, \{s_1, s_2\} \cup X)$$

by definition.

Let us finally prove, following the idea of [10], that if the 2-users underlying decisional problem is hard, then the n-users is hard as well, or equivalently:

Theorem 15.1 *For any $n > 2$, $A_2 \sim D_2$ implies that $A_n \sim D_n$.*

Proof We show this is true by induction on n. Assume that $A_2 \sim D_2$ and $A_i \sim D_i$, $i \in \{3, \ldots, n-1\}$. Thus, we have to show that $A_n \sim D_n$. We define the random variables B_n, C_n, and show that $A_n \sim B_n \sim C_n \sim D_n$, and since $\sim$ is a equivalence relation, by transitivity, this implies that $A_n \sim D_n$.

We split the proof in two cases:

(a) Assume n is even:
We redefine A_n, D_n using Lemma 15.1, and define B_n, C_n as follows:

- $A_n = \Big(view(n-1, \{s_1\} \cup X), \phi(s_n s_{n-1}^* \ldots s_2, h), view(n-1, \{s_2\} \cup X),$
$\phi(s_n s_{n-1}^* s_{n-2} \ldots s_3^* s_1, h), view(n-1, \{s_2^* s_1\} \cup X), y\Big)$
- $B_n = \Big(view(n-1, \{s_1\} \cup X), \phi(s_n s_{n-1}^* \ldots s_2, h), view(n-1, \{s_2\} \cup X),$
$\phi(s_n s_{n-1}^* s_{n-2} \ldots s_3^* s_1, h), view(n-1, \{c\} \cup X), y\Big)$
- $C_n = \Big(view(n-1, \{s_1\} \cup X), \phi(s_n s_{n-1}^* \ldots s_2, h), view(n-1, \{s_2\} \cup X),$
$\phi(s_n s_{n-1}^* s_{n-2} \ldots s_3^* s_1, h), view(n-1, \{c\} \cup X), \phi(s_n^* s_{n-1} \ldots s_4^* s_3 c, h)\Big)$
- $D_n = \Big(view(n-1, \{s_1\} \cup X), \phi(s_n s_{n-1}^* \ldots s_2, h), view(n-1, \{s_2\} \cup X),$
$\phi(s_n s_{n-1}^* s_{n-2} \ldots s_3^* s_1, h), view(n-1, \{s_2^* s_1\} \cup X), \phi(s_n^* s_{n-1} \ldots s_4^* s_3 s_2^* s_1, h)\Big)$

choosing $s_1, s_2 \in R_1 \times A_2$, $c \in R_1 \times A_1$; and $X \in (R_1 \times A_2)^{n-2}$, $y \in R_1 h A_1$ randomly. Note that only the last two components vary.

$\underline{A_2 \sim D_2 \Longrightarrow A_n \sim B_n}$

Suppose, for the sake of contradiction, that an adversary Eve distinguishes A_n and B_n. We produce an instance of $A_n \nsim B_n$ for Eve

$$\begin{aligned}
A_n &= \Big(view(n-1, \{s_1\} \cup X), \phi(s_n s_{n-1}^* \ldots s_2, h), view(n-1, \{s_2\} \cup X),\\
&\quad \phi(s_n s_{n-1}^* s_{n-2} \ldots s_3^* s_1, h), view(n-1, \{s_2^* s_1\} \cup X), y\Big)\\
&= \Big(\boldsymbol{g_1 h k_1}, \ldots, g_n g_{n-1} \ldots g_4 g_3 h k_3 k_4^* \ldots k_{n-1} k_n^*\\
&\quad , g_n g_{n-1} \ldots g_3 g_1 h k_1 k_3^* k_4 \ldots k_{n-1}^* k_n,\\
&\quad \boldsymbol{g_2 h k_2}, \ldots, g_{n-1} \ldots g_3 g_2 h k_2 k_3^* \ldots k_{n-2}^* k_{n-1},\\
&\quad g_n g_{n-1} \ldots g_3 g_2 h k_1 k_2^* k_4 \ldots k_{n-1}^* k_n,\\
&\quad \boldsymbol{g_2 g_1 h k_1 k_2^*}, \ldots, g_{n-1} g_{n-2} \ldots g_3 (g_2 g_1) h (k_1 k_2^*) k_3 \ldots k_{n-2}^* k_{n-1}, y\Big)\\
B_n &= \Big(view(n-1, \{s_1\} \cup X), \phi(s_n s_{n-1}^* \ldots s_2, h), view(n-1, \{s_2\} \cup X),\\
&\quad \phi(s_n s_{n-1}^* s_{n-2} \ldots s_3^* s_1, h), view(n-1, \{c\} \cup X), y\Big)\\
&= \Big(\boldsymbol{g_1 h k_1}, \ldots, g_n g_{n-1} \ldots g_4 g_3 h k_3 k_4^* \ldots k_{n-1} k_n^*,\\
&\quad g_n g_{n-1} \ldots g_3 g_1 h k_1 k_3^* k_4 \ldots k_{n-1}^* k_n,\\
&\quad \boldsymbol{g_2 h k_2}, \ldots, g_{n-1} \ldots g_3 g_2 h k_2 k_3^* \ldots k_{n-2}^* k_{n-1},\\
&\quad g_n g_{n-1} \ldots g_3 g_2 h k_1 k_2^* k_4 \ldots k_{n-1}^* k_n,\\
&\quad \boldsymbol{c_1 h c_2}, \ldots, g_{n-1} g_{n-2} \ldots g_3 (c_1) h (c_2) k_3 \ldots k_{n-2}^* k_{n-1}, y\Big)
\end{aligned}$$

if Eve distinguishes A_n and B_n, then in particular, she distinguishes $g_2g_1hk_1k_2^*$ from c_1hc_2 (given g_1hk_1 and g_2hk_2), which means that she distinguishes

$$\begin{aligned} A_2 &= \Big(view(2, \{s_1, s_2\}), y\Big) \\ &= \Big(g_1hk_1, g_2hk_2, y\Big) \\ D_2 &= \Big(view(2, \{s_1, s_2\}), \phi(s_2^*s_1, h)\Big) \\ &= \Big(g_1hk_1, g_2hk_2, g_2g_1hk_1k_2^*\Big) \end{aligned}$$

which contradicts our hypothesis.

<u>$A_{n-2} \sim D_{n-2} \Longrightarrow B_n \sim C_n$</u>

Suppose towards the sake of contradiction that an adversary Eve distinguishes B_n and C_n. We produce and instance of $B_n \nsim C_n$ for Eve

$$\begin{aligned} B_n = &\Big(view(n-1, \{s_1\} \cup X), \phi(s_n^*s_{n-1}\dots s_3^*s_2, h), view(n-1, \{s_2\} \cup X), \\ &\phi(s_n^*s_{n-1}\dots s_3^*s_1, h), view(n-1, \{c\} \cup X), y\Big) \\ = &\Big(g_1hk_1, \dots, g_ng_{n-1}\dots g_4g_3hk_3k_4^*\dots k_{n-1}k_n^*, \\ &g_ng_{n-1}\dots g_3g_1hk_1k_3^*k_4\dots k_{n-1}^*k_n, \\ &g_2hk_2, \dots, g_{n-1}\dots g_3g_2hk_2k_3^*\dots k_{n-2}^*k_{n-1}, \\ &g_ng_{n-1}\dots g_3g_2hk_1k_2^*k_4\dots k_{n-1}^*k_n, \\ &\boldsymbol{c_1hc_2, \dots, g_{n-1}\dots g_5g_4(g_3c_1)h(c_2k_3)k_4^*k_5\dots k_{n-2}k_{n-1}^*, y}\Big) \\ C_n = &\Big(view(n-1, \{s_1\} \cup X), \phi(s_n^*s_{n-1}\dots s_3^*s_2, h), view(n-1, \{s_2\} \cup X), \\ &\phi(s_ns_{n-1}^*\dots s_3^*s_1, h), view(n-1, \{c\} \cup X), \phi(s_ns_{n-1}^*\dots s_5s_4^*s_3c, h)\Big) \\ = &\Big(g_1hk_1, \dots, g_ng_{n-1}\dots g_4g_3hk_3k_4^*\dots k_{n-1}k_n^*, \\ &g_ng_{n-1}\dots g_3g_1hk_1k_3^*k_4\dots k_{n-1}^*k_n, \\ &g_2hk_2, \dots, g_{n-1}\dots g_3g_2hk_2k_3^*\dots k_{n-2}^*k_{n-1}, \\ &g_ng_{n-1}\dots g_3g_2hk_1k_2^*k_4\dots k_{n-1}^*k_n, \\ &\boldsymbol{c_1hc_2, \dots, g_{n-1}\dots g_5g_4(g_3c_1)h(c_2k_3)k_4^*k_5\dots k_{n-2}k_{n-1}^*}, \\ &\boldsymbol{g_n\dots g_4(g_3c_1)h(c_2k_3)k_4^*k_5\dots k_n}\Big) \end{aligned}$$

if Eve distinguishes B_n and C_n in polynomial time, in particular, she distinguishes y and $\phi(s_n^*s_{n-1}\dots s_4^*(s_3c), h)$ (given $view(n-1, \{c\} \cup X)$). Let

$$\Big((view(n-2, \{cs_3, s_4, s_5, \dots, s_{n-1}, s_n\}), y\Big)$$

be an instance of A_{n-2}, D_{n-2}:

$$\begin{aligned}
A_{n-2} &= \Big((view(n-2, \{s_3c, s_4, s_5, \dots, s_{n-1}, s_n\}), y\Big) \\
&= \Big((g_3c_1)h(c_2k_3), g_4hk_4, \dots, g_nhk_n, g_4(g_3c_1)h(c_2k_3)k_4^* \dots, \\
&\quad g_n(g_3c_1)h(c_2k_3)k_n^*, g_5g_4(g_3c_1)h(c_2k_3)k_4^*k_5, \dots, \\
&\quad g_ng_{n-1}\dots g_4g_3hk_3k_4^* \dots k_{n-1}k_n, y\Big) \\
D_{n-2} &= \Big(view(n-2, \{s_3c, s_4, s_5, \dots, s_{n-1}, s_n\}), \phi(s_n^*s_{n-1}\dots s_4^*(s_3c), h)\Big) \\
&= \Big((g_3c_1)h(c_2k_3), g_4hk_4, \dots, g_nhk_n, g_4(g_3c_1)h(c_2k_3)k_4^* \dots, \\
&\quad g_n(g_3c_1)h(c_2k_3)k_n^*, g_5g_4(g_3c_1)h(c_2k_3)k_4^*k_5, \dots, \\
&\quad g_ng_{n-1}\dots g_5g_4hk_4k_5^* \dots k_{n-1}k_n, g_ng_{n-1}\dots g_4(g_3c_1)h(c_2k_3)k_4^* \dots k_{n-1}k_n\Big)
\end{aligned}$$

since Eve can distinguish y and $\phi(s_n^*s_{n-1}\dots s_4^*(s_3c), h)$ given $view(n-1, \{c\} \cup X)$, then in particular she distinguishes y and $\phi(s_n^*s_{n-1}\dots s_4^*(s_3c), h)$ given

$$view(n-2, \{s_3c, s_4, s_5, \dots, s_{n-1}, s_n\}) \subset view(n-1, \{c\} \cup X),$$

and this means $A_{n-2} \nsim D_{n-2}$, but this contradicts our hypothesis.

$\underline{A_2 \sim D_2 \Longrightarrow C_n \sim D_n}$

Suppose, for the sake of contradiction, that an adversary Eve distinguishes C_n and D_n. We produce and instance of $C_n \nsim D_n$ for Eve

$$\begin{aligned}
C_n &= \Big(view(n-1, \{s_1\} \cup X), \phi(s_ns_{n-1}^* \dots s_2, h), view(n-1, \{s_2\} \cup X), \\
&\quad \phi(s_ns_{n-1}^*s_{n-2}\dots s_3^*s_1, h), view(n-1, \{c\} \cup X), \phi(s_n^*s_{n-1}\dots s_4^*s_3c, h)\Big) \\
&= \Big(g_1hk_1, \dots, g_ng_{n-1}\dots g_4g_3hk_3k_4^* \dots k_{n-1}k_n^*, \\
&\quad g_ng_{n-1}\dots g_3g_1hk_1k_3^*k_4 \dots k_{n-1}^*k_n, \\
&\quad g_2hk_2, \dots, g_{n-1}\dots g_3g_2hk_2k_3^* \dots k_{n-2}^*k_{n-1}, \\
&\quad g_ng_{n-1}\dots g_3g_2hk_1k_2^*k_4 \dots k_{n-1}^*k_n, \\
&\quad \boldsymbol{c_1hc_2}, \dots, g_{n-1}g_{n-2}\dots g_3c_1hc_2k_3 \dots k_{n-2}^*k_{n-1}, \\
&\quad g_ng_{n-1}\dots g_4g_3c_1hc_2k_3k_4^* \dots k_{n-1}k_n\Big) \\
D_n &= \Big(view(n-1, \{s_1\} \cup X), K(n-1, \{s_1\} \cup X), view(n-1, \{s_2\} \cup X), \\
&\quad K(n-1, \{s_2\} \cup X), view(n-1, \{s_2^*s_1\} \cup X), \phi(s_n^*s_{n-1}\dots s_4^*s_3s_2^*s_1, h)\Big) \\
&= \Big(g_1hk_1, \dots, g_ng_{n-1}\dots g_4g_3hk_3k_4^* \dots k_{n-1}k_n^*, \\
&\quad g_ng_{n-1}\dots g_3g_1hk_1k_3^*k_4 \dots k_{n-1}^*k_n, \\
&\quad g_2hk_2, \dots, g_{n-1}\dots g_3g_2hk_2k_3^* \dots k_{n-2}^*k_{n-1}, \\
&\quad g_ng_{n-1}\dots g_3g_2hk_1k_2^*k_4 \dots k_{n-1}^*k_n, \\
&\quad \boldsymbol{g_2g_1hk_1k_2^*}, \dots, g_{n-1}g_{n-2}\dots g_3(g_2g_1)h(k_1k_2^*)k_3 \dots k_{n-2}^*k_{n-1}, \\
&\quad g_ng_{n-1}\dots g_3(g_2g_1)h(k_1k_2^*)k_3 \dots k_{n-1}k_n^*\Big)
\end{aligned}$$

as in the first case, if Eve distinguishes A_n and B_n, then in particular, she distinguishes $g_2 g_1 h k_1 k_2^*$ from $c_1 h c_2$ (given $g_1 h k_1$ and $g_2 h k_2$), which means that she distinguishes

$$\begin{aligned} A_2 &= \Big(view(2, \{s_1, s_2\}), y\Big) \\ &= \Big(g_1 h k_1, g_2 h k_2, y\Big) \\ D_2 &= \Big(view(2, \{s_1, s_2\}), \phi(s_2^* s_1, h)\Big) \\ &= \Big(g_1 h k_1, g_2 h k_2, g_2 g_1 h k_1 k_2^*\Big) \end{aligned}$$

which contradicts our hypothesis.

(b) Similarly, if n is odd:
We redefine A_n, D_n using Lemma 15.1, and define B_n, C_n as follows:

- $A_n = \Big(view(n-1, \{s_1\} \cup X), \phi(s_n^* s_{n-1} \dots s_3^* s_2, h), view(n-1, \{s_2\} \cup X),$
  $\phi(s_n^* s_{n-1} \dots s_3^* s_1, h), view(n-1, \{s_2^* s_1\} \cup X), y\Big)$
- $B_n = \Big(view(n-1, \{s_1\} \cup X), \phi(s_n^* s_{n-1} \dots s_2, h), view(n-1, \{s_2\} \cup X),$
  $\phi(s_n^* s_{n-1} \dots s_3^* s_1, h), view(n-1, \{c\} \cup X), y\Big)$
- $C_n = \Big(view(n-1, \{s_1\} \cup X), \phi(s_n^* s_{n-1} \dots s_2, h), view(n-1, \{s_2\} \cup X),$
  $\phi(s_n^* s_{n-1} \dots s_3^* s_1, h), view(n-1, \{c\} \cup X), \phi(s_n s_{n-1}^* \dots s_5 s_4^* s_3 c, h)\Big)$
- $D_n = \Big(view(n-1, \{s_1\} \cup X), \phi(s_n^* s_{n-1} \dots s_2, h), view(n-1, \{s_2\} \cup X),$
  $\phi(s_n^* s_{n-1} \dots s_3^* s_1, h), view(n-1, \{s_2^* s_1\} \cup X), \phi(s_n s_{n-1}^* \dots s_5 s_4^* s_3 s_2^* s_1, h)\Big)$

choosing $s_1, s_2 \in R_1 \times A_2$, $c \in R_1 \times A_1$; and $X \in (R_1 \times A_2)^{n-2}$, $y \in R_1 h A_2$ randomly.

$\underline{A_2 \sim D_2 \Longrightarrow A_n \sim B_n}$.

Suppose towards the sake of contradiction that an adversary Eve distinguishes A_n and B_n. We produce an instance of $A_n \nsim B_n$ for Eve

$$
\begin{aligned}
A_n &= \Big(view(n-1, \{s_1\} \cup X), \phi(s_n^* s_{n-1} \dots s_2, h), view(n-1, \{s_2\} \cup X),\\
&\quad \phi(s_n^* s_{n-1} \dots s_3^* s_1, h), view(n-1, \{s_2^* s_1\} \cup X), y\Big)\\
&= \Big(\boldsymbol{g_1 h k_1}, \dots, g_n g_{n-1} \dots g_4 g_3 h k_3 k_4^* \dots k_{n-1} k_n^*,\\
&\quad g_n g_{n-1} \dots g_3 g_1 h k_1 k_3^* k_4 \dots k_{n-1} k_n^*,\\
&\quad \boldsymbol{g_2 h k_2}, \dots, g_{n-1} \dots g_3 g_2 h k_2 k_3^* \dots k_{n-2}^* k_{n-1},\\
&\quad g_n g_{n-1} \dots g_3 g_2 h k_1 k_2^* k_4 \dots k_{n-1} k_n^*,\\
&\quad \boldsymbol{g_2 g_1 h k_1 k_2^*}, \dots, g_{n-1} g_{n-2} \dots g_3 (g_2 g_1) h (k_1 k_2^*) k_3 \dots k_{n-2} k_{n-1}^*, y\Big)\\
B_n &= \Big(view(n-1, \{s_1\} \cup X), \phi(s_n^* s_{n-1} \dots s_2, h), view(n-1, \{s_2\} \cup X),\\
&\quad \phi(s_n^* s_{n-1} \dots s_3^* s_1, h), , view(n-1, \{c\} \cup X), y\Big)\\
&= \Big(\boldsymbol{g_1 h k_1}, \dots, g_n g_{n-1} \dots g_4 g_3 h k_3 k_4^* \dots k_{n-1} k_n^*,\\
&\quad g_n g_{n-1} \dots g_3 g_1 h k_1 k_3^* k_4 \dots k_{n-1} k_n^*,\\
&\quad \boldsymbol{g_2 h k_2}, \dots, g_{n-1} \dots g_3 g_2 h k_2 k_3^* \dots k_{n-2}^* k_{n-1},\\
&\quad g_n g_{n-1} \dots g_3 g_2 h k_1 k_2^* k_4 \dots k_{n-1} k_n^*,\\
&\quad \boldsymbol{c_1 h c_2}, \dots, g_{n-1} g_{n-2} \dots g_3 (c_1) h (c_2) k_3 \dots k_{n-2} k_{n-1}^*, y\Big)
\end{aligned}
$$

if Eve distinguishes A_n and B_n, then in particular, she distinguishes $g_2 g_1 h k_1 k_2^*$ from $c_1 h c_2$ (given $g_1 h k_1$ and $g_2 h k_2$), which means that she distinguishes

$$
\begin{aligned}
A_2 &= \Big(view(2, \{s_1, s_2\}), y\Big)\\
&= \Big(g_1 h k_1, g_2 h k_2, y\Big)\\
D_2 &= \Big(view(2, \{s_1, s_2\}), \phi(s_2^* s_1, h)\Big)\\
&= \Big(g_1 h k_1, g_2 h k_2, g_2 g_1 h k_1 k_2^*\Big)
\end{aligned}
$$

which contradicts our hypothesis.

$\underline{A_{n-2} \sim D_{n-2} \Longrightarrow B_n \sim C_n}$.

Suppose, for the sake of contradiction, that an adversary Eve distinguishes B_n and C_n. We produce and instance of $B_n \nsim C_n$ for Eve

$$\begin{aligned}
B_n &= \Big(view(n-1, \{s_1\} \cup X), \phi(s_n s_{n-1}^* \dots s_2, h), view(n-1, \{s_2\} \cup X),\\
&\quad \phi(s_n s_{n-1}^* s_{n-2} \dots s_3^* s_1, h), view(n-1, \{c\} \cup X), y\Big)\\
&= \Big(g_1 h k_1, \dots, g_n g_{n-1} \dots g_4 g_3 h k_3 k_4^* \dots k_{n-1} k_n^*,\\
&\quad g_n g_{n-1} \dots g_3 g_1 h k_1 k_3^* k_4 \dots k_{n-1}^* k_n,\\
&\quad g_2 h k_2, \dots, g_{n-1} \dots g_3 g_2 h k_2 k_3^* \dots k_{n-2}^* k_{n-1},\\
&\quad g_n g_{n-1} \dots g_3 g_2 h k_1 k_2^* k_4 \dots k_{n-1}^* k_n,\\
&\quad \boldsymbol{c_1 h c_2, \dots, g_{n-1} \dots g_5 g_4 (g_3 c_1) h (c_2 k_3) k_4^* k_5 \dots k_{n-2}^* k_{n-1}, y}\Big)\\
C_n &= \Big(view(n-1, \{s_1\} \cup X), \phi(s_n s_{n-1}^* \dots s_2, h), view(n-1, \{s_2\} \cup X),\\
&\quad \phi(s_n s_{n-1}^* s_{n-2} \dots s_3^* s_1, h), view(n-1, \{c\} \cup X), \phi(s_n s_{n-1}^* \dots s_4^* s_3 c, h)\Big)\\
&= \Big(g_1 h k_1, \dots, g_n g_{n-1} \dots g_4 g_3 h k_3 k_4^* \dots k_{n-1} k_n^*,\\
&\quad g_n g_{n-1} \dots g_3 g_1 h k_1 k_3^* k_4 \dots k_{n-1}^* k_n,\\
&\quad g_2 h k_2, \dots, g_{n-1} \dots g_3 g_2 h k_2 k_3^* \dots k_{n-2}^* k_{n-1},\\
&\quad g_n g_{n-1} \dots g_3 g_2 h k_1 k_2^* k_4 \dots k_{n-1}^* k_n,\\
&\quad \boldsymbol{c_1 h c_2, \dots, g_{n-1} \dots g_5 g_4 (g_3 c_1) h (c_2 k_3) k_4^* k_5 \dots k_{n-2}^* k_{n-1}},\\
&\quad \boldsymbol{g_n \dots g_4 (g_3 c_1) h (c_2 k_3) k_4^* k_5 \dots k_n^*}\Big)
\end{aligned}$$

if Eve distinguishes B_n and C_n in polynomial time, in particular, she distinguishes y and $\phi(s_n s_{n-1}^* \dots s_5 s_4^* (s_3 c), h)$ (given $view(n-1, \{c\} \cup X)$). Let

$$\Big((view(n-2, \{c s_3, s_4, s_5, \dots, s_{n-1}, s_n\}), y\Big)$$

be an instance of A_{n-2}, D_{n-2}:

$$\begin{aligned}
A_{n-2} &= \Big((view(n-2, \{s_3 c, s_4, s_5, \dots, s_{n-1}, s_n\}), y\Big)\\
&= \Big((g_3 c_1) h (c_2 k_3), g_4 h k_4, \dots, g_n h k_n,\\
&\quad g_4 (g_3 c_1) h (c_2 k_3) k_4^* \dots, g_n (g_3 c_1) h (c_2 k_3) k_n^*,\\
&\quad g_5 g_4 (g_3 c_1) h (c_2 k_3) k_4^* k_5, \dots, g_n g_{n-1} \dots g_4 g_3 h k_3 k_4^* \dots k_{n-1}^* k_n, y\Big)\\
D_{n-2} &= \Big(view(n-2, \{s_3 c, s_4, s_5, \dots, s_{n-1}, s_n\}), \phi(s_n s_{n-1}^* \dots s_5 s_4^* (s_3 c), h)\Big)\\
&= \Big((g_3 c_1) h (c_2 k_3), g_4 h k_4, \dots, g_n h k_n,\\
&\quad g_4 (g_3 c_1) h (c_2 k_3) k_4^* \dots, g_n (g_3 c_1) h (c_2 k_3) k_n^*,\\
&\quad g_5 g_4 (g_3 c_1) h (c_2 k_3) k_4^* k_5, \dots, g_n g_{n-1} \dots g_5 g_4 h k_4 k_5^* \dots k_{n-1}^* k_n,\\
&\quad g_n g_{n-1} \dots g_4 (g_3 c_1) h (c_2 k_3) k_4^* \dots k_{n-1}^* k_n\Big)
\end{aligned}$$

since Eve can distinguish y and $\phi(s_n s_{n-1}^* \dots s_5 s_4^* (s_3 c), h)$ given $view(n-1, \{c\} \cup X)$, then in particular she distinguishes y and $\phi(s_n^* s_{n-1} \dots s_4^* (s_3 c), h)$ given

$$view(n-2, \{s_3 c, s_4, s_5, \dots, s_{n-1}, s_n\}) \subset view(n-1, \{c\} \cup X),$$

and this means $A_{n-2} \nsim D_{n-2}$, but this contradicts our hypothesis.

$\underline{A_2 \sim D_2 \Longrightarrow C_n \sim D_n}$.

Suppose towards the sake of contradiction that an adversary Eve distinguishes C_n and D_n. We produce and instance of $C_n \nsim D_n$ for Eve

$$\begin{aligned}
C_n &= \Big(view(n-1, \{s_1\} \cup X), \phi(s_n s_{n-1}^* \ldots s_2, h), view(n-1, \{s_2\} \cup X),\\
&\quad \phi(s_n s_{n-1}^* s_{n-2} \ldots s_3^* s_1, h), view(n-1, \{c\} \cup X), \phi(s_n s_{n-1}^* \ldots s_4^* s_3 c, h)\Big)\\
&= \Big(g_1 h k_1, \ldots, g_n g_{n-1} \ldots g_4 g_3 h k_3 k_4^* \ldots k_{n-1} k_n^*,\\
&\quad g_n g_{n-1} \ldots g_3 g_1 h k_1 k_3^* k_4 \ldots k_{n-1}^* k_n,\\
&\quad g_2 h k_2, \ldots, g_{n-1} \ldots g_3 g_2 h k_2 k_3^* \ldots k_{n-2}^* k_{n-1},\\
&\quad g_n g_{n-1} \ldots g_3 g_2 h k_1 k_2^* k_4 \ldots k_{n-1}^* k_n,\\
&\quad \boldsymbol{c_1 h c_2}, \ldots, g_{n-1} g_{n-2} \ldots g_3 c_1 h c_2 k_3 \ldots k_{n-2}^* k_{n-1},\\
&\quad g_n g_{n-1} \ldots g_4 g_3 c_1 h c_2 k_3 k_4^* \ldots k_{n-1} k_n\Big)\\
D_n &= \Big(view(n-1, \{s_1\} \cup X), \phi(s_n s_{n-1}^* \ldots s_2, h), view(n-1, \{s_2\} \cup X),\\
&\quad \phi(s_n s_{n-1}^* s_{n-2} \ldots s_3^* s_1, h), view(n-1, \{s_2^* s_1\} \cup X),\\
&\quad \phi(s_n s_{n-1}^* \ldots s_4^* s_3 s_2^* s_1, h)\Big)\\
&= \Big(g_1 h k_1, \ldots, g_n g_{n-1} \ldots g_4 g_3 h k_3 k_4^* \ldots k_{n-1} k_n^*,\\
&\quad g_n g_{n-1} \ldots g_3 g_1 h k_1 k_3^* k_4 \ldots k_{n-1}^* k_n,\\
&\quad g_2 h k_2, \ldots, g_{n-1} \ldots g_3 g_2 h k_2 k_3^* \ldots k_{n-2}^* k_{n-1},\\
&\quad g_n g_{n-1} \ldots g_3 g_2 h k_1 k_2^* k_4 \ldots k_{n-1}^* k_n,\\
&\quad \boldsymbol{g_2 g_1 h k_1 k_2^*}, \ldots, g_{n-1} g_{n-2} \ldots g_3 (g_2 g_1) h (k_1 k_2^*) k_3 \ldots k_{n-2}^* k_{n-1},\\
&\quad g_n g_{n-1} \ldots g_3 (g_2 g_1) h (k_1 k_2^*) k_3 \ldots k_{n-1} k_n^*\Big)
\end{aligned}$$

as in the first case, if Eve distinguishes A_n and B_n, then in particular, she distinguishes $g_2 g_1 h k_1 k_2^*$ from $c_1 h c_2$ (given $g_1 h k_1$ and $g_2 h k_2$), which means that she distinguishes

$$\begin{aligned}
A_2 &= \Big(view(2, \{s_1, s_2\}), y\Big)\\
&= \Big(g_1 h k_1, g_2 h k_2, y\Big)\\
D_2 &= \Big(view(2, \{s_1, s_2\}), \phi(s_2^* s_1, h)\Big)\\
&= \Big(g_1 h k_1, g_2 h k_2, g_2 g_1 h k_1 k_2^*\Big)
\end{aligned}$$

which contradicts our hypothesis.

So in the Initial Key Agreement the n-users underlying decisional problem is as hard as the 2-users decisional problem. This is also true in the Auxiliary Key Agreement. We can say the protocol provides on forward and backward security, i.e. any former or future users cannot distinguish future or past distributed keys, as it is shown in the following result.

Corollary 15.1 *The AKA provides on forward and backward security.*

Proof Let Eve be a powerful adversary, that knows all the information of a past user or a future user. She would know a subset of $view(k, \varepsilon)$, where k is the number of current users, and ε the secret keys.

In the first case, when the members of the group stay the same, note that the key update adds a new secret key (and we consider it as a new user). Then we substitute n with $k = n + 1$, $\phi(s_n^* s_{n-1} \dots s_4^* s_3 s_2^* s_1, h)$ (or $\phi(s_n s_{n-1}^* \dots s_3 s_2^* s_1, h)$) with $\phi(\widetilde{s_c} s_n^* s_{n-1} \dots s_3 s_2^* s_1, h)$ (resp. $\phi(\widetilde{s_c}^* s_n s_{n-1}^* \dots s_3 s_2^* s_1, h)$) if n is even (if n is odd), and X with

$$\varepsilon = \{s_1, s_2, \dots, s_{c-1}, s_c, s_{c+1}, \dots, s_{n-1}, s_n, s_c'\}$$

in Theorem 15.1. It follows that

$$A_k = \Big(view(k, \varepsilon), y\Big), \text{ for } y \in R \text{ randomly chosen.}$$

$$D_k = \begin{cases} \Big(view(k, \varepsilon), \phi(\widetilde{s_c} s_n^* s_{n-1} \dots s_3 s_2^* s_1, h)\Big), & \text{if } k \text{ is odd.} \\ \Big(view(k, \varepsilon), \phi(\widetilde{s_c}^* s_n s_{n-1}^* \dots s_3 s_2^* s_1, h))\Big), & \text{if } k \text{ is even.} \end{cases}$$

and it still verifies that if $A_2 \sim D_2$, then $A_k \sim D_k$.

When a user leaves, the key update also adds a new secret key, so we replace n with $k = n + 1$ (the user left, but we suppose that Eve had access to the communications before that happened, and that private key is still part of the common secret key). The rest is the same, so we get again the first case, and the AKA benefits form the same security benefits in this case.

When a new users joins the group, we need to replace $k = n + 2$ (the new secret key and the key update), $\phi(s_n^* s_{n-1} \dots s_4^* s_3 s_2^* s_1, h)$ (or $\phi(s_n s_{n-1}^* \dots s_3 s_2^* s_1, h)$) with $\phi(s_{n+1}^* \widetilde{s_c} s_n^* s_{n-1} \dots s_3 s_2^* s_1, h)$ (resp. $\phi(s_{n+1} \widetilde{s_c}^* s_n s_{n-1}^* \dots s_3 s_2^* s_1, h)$) if n is even (if n is odd), and X with $\varepsilon = \{s_1, s_2, \dots, s_{n-1}, s_n, s_{n+1}, s_c'\}$ in Theorem 15.1. It follows that

$$A_k = \Big(view(k, \varepsilon), y\Big), \text{ for } y \in R \text{ randomly chosen.}$$

$$D_k = \begin{cases} \Big(view(k, \varepsilon), \phi(s_{n+1}^* \widetilde{s_c} s_n^* s_{n-1} \dots s_3 s_2^* s_1, h)\Big), & \text{if } k \text{ is even.} \\ \Big(view(k, \varepsilon), \phi(s_{n+1} \widetilde{s_c}^* s_n s_{n-1}^* \dots s_3 s_2^* s_1, h))\Big), & \text{if } k \text{ is odd.} \end{cases}$$

and it still verifies that if $A_2 \sim D_2$, then $A_k \sim D_k$, so the Auxiliary Key Agreement benefits from the same security properties.

Note that we could also consider D_k as

$$D_k = \begin{cases} \Big(view(k, \varepsilon), \phi(\widetilde{s_c}, K_p))\Big), & \text{if } k \text{ is odd.} \\ \Big(view(k, \varepsilon), \phi(\widetilde{s_c}^*, K_p))\Big), & \text{if } k \text{ is even.} \end{cases}$$

where K_p would be the previous key, when the number of users stay the same or someone left, and

$$D_k = \begin{cases} \Big(view(k, \varepsilon), \phi(s^*_{n+1}\widetilde{s_c}, K_p))\Big), & \text{if } k \text{ is even.} \\ \Big(view(k, \varepsilon), \phi(s_{n+1}\widetilde{s_c}^*, K_p))\Big), & \text{if } k \text{ is odd.} \end{cases}$$

when a new user joins the group.

Also note that in the key refresh, we consider $k = n + 1$ in the first two cases, but the set of secret keys are $\{s_1, s_2, \ldots, s_{c-1}, \widetilde{s_c}^* s_c, s_{c+1}, \ldots, s_{n-1}, s_n\}$ when n is odd, and

$$\{s_1, s_2, \ldots, s_{c-1}, \widetilde{s_c} s_c^*, s_{c+1}, \ldots, s_n\}$$

when n is even, i.e. the number of stored keys stay the same, and the private key of the user U_c is $\widetilde{s_c^*} s_c$ or $\widetilde{s_c} s_c^*$ depending on whether the number of users is even or odd. Finally when $k = n + 2$, the set of secret keys has just one new key, from the new user U_{n+1}, so it is $\{s_1, s_2, \ldots, s_{c-1}, \widetilde{s_c}^* s_c, s_{c+1}, \ldots, s_{n-1}, s_n, s_{n+1}\}$ when n is odd, and

$$\{s_1, s_2, \ldots, s_{c-1}, \widetilde{s_c} s_c^*, s_{c+1}, \ldots, s_n, s_{n+1}\}$$

when n is even.

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Chapter 16
Construction and Characterization of n-Ary Hom-Bialgebras and n-Ary Infinitesimal Hom-Bialgebras

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Abstract Constructions of n-ary bialgebras and n-ary infinitesimal bialgebras of associative type and their hom-analogs, generalizing the hom-bialgebras and infinitesimal hom-bialgebras are investigated. Main algebraic characteristics of n-ary totally, n-ary weak totally, n-ary partially and n-ary alternate partially associative algebras and bialgebras, and their hom-counterparts are described. Particular cases of ternary algebras are given as illustration.

Keywords Hom-associative algebras · Infinitesimal Hom-bialgebras · n-ary Hom-bialgebras of associative type · n-ary infinitesimal Hom-bialgebras of associative type

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16.1 Introduction

The n-ary algebraic structures and in particular ternary algebraic structures appeared more or less naturally in various domains of theoretical and mathematical physics and data processing. Theoretical physics progress involving n-ary algebraic structures in

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quantum mechanics including the Nambu mechanics and quantization in 1970th [44, 50], and in connection to Yang-Baxter equations [45, 46] gave an impulse to a significant development of investigations of n-ary algebras [16, 18, 35, 50, 51]. Further motivation from theoretical physics side come for the study of n-ary operations in string theory and M-branes [9], qauge theories, particle physics and supersymmetry [1, 23–27, 52]. The n-ary operations appeared first through cubic matrices which were introduced in the nineteenth century by Cayley, and again considered and generalized in [22, 47].

The n-ary algebras of associative type were studied by Lister, Loos, Myung and Carlsson (see [11, 36, 37, 43]). The n-ary operations of associative type lead to two principal classes of "associative" n-ary algebras, totally associative n-ary algebras and partially associative n-ary algebras. Also they admit some variants. The totally associative ternary algebras are also sometimes called associative triple systems.

The area of Hom-algebras was initiated in the work of Hartwig, Larsson and Silvestrov in [19], where the general quasi-deformations and discretizations of Lie algebras of vector fields using more general σ-derivations (twisted derivations) in place of ordinary derivations along with a general method for construction of deformations of Witt and Virasoro type algebras have been developed, motivated initially by specific examples of q-deformed Jacobi identities in the q-deformed (quantum) algebras in mathematical physics associated to q-difference operators and corresponding q-deformations of differential calculi [5, 12–15, 20, 31–34]. These q-deformed algebras, with q-deformed Jacobi identities associated to q-difference operators and corresponding q-deformations of differential calculi, serve as initial examples of more general quantum deformations of differential calculus and corresponding algebras obtained by replacing the usual derivation by general (σ, τ)-derivations, which satisfy modified Jacobi identities deformed by some linear maps in some special ways as was shown in [19, 28], where also general algebras satisfying such identities where introduced and called hom-Lie algebras and quasi-Hom-Lie algebras [19, 28]. Furthermore, the general quasi-Lie algebras, containing as subclasses the quasi-Hom-Lie algebras and Hom-Lie algebras, as well as general color quasi-Lie algebras, including also hom-Lie superalgebras and hom-Lie color algebras as subclasses, were introduced first in [19, 28–30, 48, 49]. With these works the area of Hom-algebras has started.

The hom-associative algebras play the role of associative algebras in the hom-Lie setting. They were introduced first in [39], where it is shown that the commutator bracket defined by the multiplication in a hom-associative algebra leads naturally to a hom-Lie algebra, that is that Hom-associative algebras are hom-Lie admissible. Also, in [39] hom-Lie-admissible algebras and more general G-hom-associative algebras with subclasses of hom-Vinberg and pre-hom-Lie algebras, generalizing to the twisted situation Lie-admissible algebras, G-associative algebras, Vinberg and pre-Lie algebras respectively, and shown that for these classes of algebras the operation of taking commutator leads to hom-Lie algebras as well. The adjoint functor from the category of hom-Lie algebras to the category of hom-associative algebras and the enveloping algebra were constructed in [59]. The fundamentals of the formal deformation theory and associated cohomology structures for hom-Lie algebras have been considered recently by the second and the third authors in [42]. Simultaneously, D. Yau has developed elements of homology for Hom-Lie algebras in [55]. In [40, 41], the theory of hom-coalgebras and related structures are developed. Further development could be found in [6–8, 10, 17, 38].

Infinitesimal bialgebras were introduced by Joni and Rota in [21] (under the name infinitesimal coalgebra). The current name is due to Aguiar, who developed a theory for them in a series of papers [2–4]. It turns out that infinitesimal bialgebras have connections with some other concepts such as Rota-Baxter operators, pre-Lie algebras, Lie bialgebras etc. Aguiar discovered a large class of examples of infinitesimal bialgebras, namely he showed that the path algebra of an arbitrary quiver carries a natural structure of infinitesimal bialgebra. In an analytical context, infinitesimal bialgebras have been used in [53] by Voiculescu in free probability theory.

The hom-analogue of infinitesimal bialgebras, called infinitesimal hom-bialgebras, was introduced and studied by Yau in [57]. He extended to the hom-context some of Aguiar's results.

In this paper, we will be concerned with n-ary totally and partially coalgebras, n-ary totally and partially bialgebras and infinitesimal n-ary totally and partially bialgebras of associative and hom-associative type.

This paper is organized as follows. In Sect. 16.2, we recall some basic definitions and make summary of the fundamental properties concerning hom-structures. In Sects. 16.3 and 16.4, we introduce and develop the notion of n-ary bialgebras of (hom)-associative type. First we construct partially and totally (hom)-coassociative n-ary coalgebras and discuss their main properties. Next, we describe a twist construction and discuss their dualization. In Sect. 16.5, we show that given a multiplicative infinitesimal (hom)-bialgebra, we can construct a multiplicative infinitesimal ternary (hom)-bialgebra) satisfying certain compatibility condition.

16.2 Basics and Notations

In the work, all vector spaces, tensor products, and linearity are considered over a field of characteristic zero $\mathbb{k}$, even if the general theory may hold for fields of other characteristics. For a vector space V, V^* denotes the dual space of V. For all $\xi_1, \dots, \xi_n \in V^*$ and $v_1, \dots, v_n \in V$, $\langle \xi_1 \otimes \cdots \otimes \xi_n, v_1 \otimes \cdots \otimes v_n \rangle = \langle \xi_1, v_1 \rangle \cdots \langle \xi_n, v_n \rangle$, where $\langle \xi_i, v_i \rangle = \xi_i(v_i),\ 1 \leq i \leq n$. Throughout this paper we refer to the standard one-to-one correspondence between linear maps $F : V_1 \otimes \cdots \otimes V_n \to W$ and multilinear maps $F : V_1 \times \cdots \times V_n \to W$ given by $F(v_1, \dots, v_n) = F(v_1 \otimes \cdots \otimes v_n)$, whenever the same notation is used for these maps. The map $\tau_{ij} : V^{\otimes^p} \to V^{\otimes^p}$ is defined by

$$\tau_{ij}(x_1 \otimes \ldots \otimes x_i \otimes \ldots \otimes x_j \ldots \otimes x_p) = x_1 \otimes \ldots \otimes x_j \otimes \ldots \otimes x_i \ldots \otimes x_p.$$

Sweedler's notation for the comultiplication, $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$ is used, and sometimes the multiplication is denoted by a dot for simplicity, when there is no confusion.

16.2.1 Hom-Associative Algebras

Definition 16.1 An associative algebra is a pair (A, μ) consisting of a $\Bbbk$-vector space A and a linear map $\mu : A \otimes A \to A$ (multiplication) satisfying

$$\mu \circ (Id \otimes \mu) = \mu \circ (\mu \otimes Id) . \tag{16.1}$$

The condition (16.1) is called associativity condition.

An associative algebra A is called unital if there exists a linear map $\eta : \Bbbk \to A$ such that

$$\mu \circ (\eta \otimes id_A) = \mu \circ (id_A \otimes \eta) = id_A . \tag{16.2}$$

The unit element is $1_A = \eta(1_\Bbbk)$. The associativity and unitality conditions (16.1) and (16.2) may be expressed by the following commutative diagrams:

$$\begin{array}{ccc} A \otimes A \otimes A & \stackrel{\mu \otimes Id}{\longrightarrow} & A \otimes A \\ Id \otimes \mu \downarrow & & \downarrow \mu \\ A \otimes A & \stackrel{\mu}{\longrightarrow} & A \end{array} \qquad \begin{array}{ccccc} \Bbbk \otimes A & \stackrel{\eta \otimes id_A}{\longrightarrow} & A \otimes A & \stackrel{id_A \otimes \eta}{\longleftarrow} & A \otimes \Bbbk \\ \cong & & \downarrow \mu & & \cong \\ A & \stackrel{id_A}{\longrightarrow} & A & \stackrel{id_A}{\longleftarrow} & A \end{array}$$

Let (A, μ) and (A', μ') be two associative algebras. A linear map $f : A \to A'$ is said to be a *hom-associative algebras morphism* if $\mu' \circ f^{\otimes 2} = f \circ \mu$.

Definition 16.2 ([39]) A hom-associative algebra is a triple (A, μ, α) consisting of a $\Bbbk$-vector space A, a linear map $\mu : A \otimes A \to A$ (multiplication) and a linear space homomorphism (a linear map) $\alpha : A \to A$ satisfying the hom-associativity condition

$$\mu \circ (\alpha \otimes \mu) = \mu \circ (\mu \otimes \alpha) . \tag{16.3}$$

If the mutliplicativity condition $\alpha \circ \mu = \mu \circ \alpha^{\otimes 2}$ holds, then the hom-associative algebra is called multiplicative. We assume in this paper, when not stated otherwise, that $\alpha \circ \mu = \mu \circ \alpha^{\otimes 2}$ holds.

A hom-associative algebra A is called unital if there exists a linear map $\eta : \Bbbk \to A$ such that $\alpha \circ \eta = \eta$ and

$$\mu \circ (\eta \otimes id_A) = \mu \circ (id_A \otimes \eta) = \alpha . \tag{16.4}$$

The unit element is $1_A = \eta(1_\Bbbk)$.

The hom-associativity and unitality conditions (16.3) and (16.4) can be expressed by the following commutative diagrams:

$$\begin{array}{ccc} A\otimes A\otimes A & \stackrel{\mu\otimes\alpha}{\longrightarrow} & A\otimes A \\ \alpha\otimes\mu\downarrow & & \downarrow\mu \\ A\otimes A & \stackrel{\mu}{\longrightarrow} & A \end{array} \qquad \begin{array}{ccccc} \Bbbk\otimes A & \stackrel{\eta\otimes id_A}{\longrightarrow} & A\otimes A & \stackrel{id_A\otimes\eta}{\longleftarrow} & A\otimes\Bbbk \\ \cong & & \downarrow\mu & & \cong \\ A & \stackrel{\alpha}{\longrightarrow} & A & \stackrel{\alpha}{\longleftarrow} & A \end{array}$$

Remark 16.1 1) We recover the classical associative algebra when the twisting map α is the identity map.
2) We have $\alpha\circ\eta=\eta$ then $\alpha(1_A)=1_A$ and $\mu(1_A\otimes 1_A)=1_A$.

For hom-associative algebras (A,μ,α) and (A',μ',α'), a linear map $f:A\to A'$ is called a *hom-associative algebra morphism* if $\mu'\circ f^{\otimes 2}=f\circ\mu,\ f\circ\alpha=\alpha'\circ f$. It is said to be a *weak morphism* if only the first condition holds. If, further, the hom-associative algebras are unital with respect to η and η', then, $f\circ\eta=\eta'$.

If $A=A'$, then the hom-associative algebras (resp. unital hom-associative algebras) are *isomorphic* if there exists a bijective linear map $f:A\to A$ such that

$$\mu=f^{-1}\circ\mu'\circ f^{\otimes 2},\alpha=f^{-1}\circ\alpha'\circ f,$$
$$(\text{resp. } \mu=f^{-1}\circ\mu'\circ f^{\otimes 2},\alpha=f^{-1}\circ\alpha'\circ f \quad and \quad \eta=f^{-1}\circ\eta').$$

Theorem 16.1 ([60]) *Every 2-dimensional multiplicative hom-associative algebra is isomorphic to one of the following pairwise non-isomorphic hom-associative algebras $(A,*,\alpha)$, where $*$ is the multiplication, α is the structure map, and $\{e_1,e_2\}$ is a basis of A:*

A_1^2: $e_1*e_1=-e_1,\quad e_1*e_2=e_2,\quad e_2*e_1=e_2,\quad e_2*e_2=e_1,$
$\alpha(e_1)=e_1,\quad \alpha(e_2)=-e_2;$
A_2^2: $e_1*e_1=e_1,\quad e_1*e_2=0,\quad e_2*e_1=0,\quad e_2*e_2=e_2,$
$\alpha(e_1)=e_1,\quad \alpha(e_2)=0;$
A_3^2: $e_1*e_1=e_1,\quad e_1*e_2=0,\quad e_2*e_1=0,\quad e_2*e_2=0,$
$\alpha(e_1)=e_1,\quad \alpha(e_2)=0;$
A_4^2: $e_1*e_1=e_1,\quad e_1*e_2=e_2,\quad e_2*e_1=e_2,\quad e_2*e_2=0,$
$\alpha(e_1)=e_1,\quad \alpha(e_2)=e_2;$
A_5^2: $e_1*e_1=e_1,\quad e_1*e_2=0,\quad e_2*e_1=0,\quad e_2*e_2=0,$
$\alpha(e_1)=0,\quad \alpha(e_2)=ke_2;$
A_6^2: $e_1*e_1=e_2,\quad e_1*e_2=0,\quad e_2*e_1=0,\quad e_2*e_2=0,$
$\alpha(e_1)=e_1,\quad \alpha(e_2)=e_2;$
A_7^2: $e_1*e_1=0,\quad e_1*e_2=ae_1,\quad e_2*e_1=be_1,\quad e_2*e_2=ce_1,$
$\alpha(e_1)=0,\quad \alpha(e_2)=e_1,$ *where* $a,b,c,k\in\mathbb{C}$;
A_8^2: $e_1*e_1=0,\quad e_1*e_2=e_1,\quad e_2*e_1=0,\quad e_2*e_2=e_1+e_2,$
$\alpha(e_1)=e_1,\quad \alpha(e_2)=e_1+e_2;$
A_9^2: $e_1*e_1=0,\quad e_1*e_2=0,\quad e_2*e_1=e_1,\quad e_2*e_2=e_1+e_2,$
$\alpha(e_1)=e_1,\quad \alpha(e_2)=e_1+e_2.$

Proposition 16.1 ([56]) *Let (A,μ,η) be a unital associative algebra and $\alpha: A\to A$ be a morphism of associative algebra, i.e. $\alpha\circ\mu=\mu\circ\alpha^{\otimes 2}$, and $\beta\circ\eta=\eta$. Then, $(A,\mu_\alpha=\alpha\circ\mu,\eta_\alpha=\alpha\circ\eta,\alpha)$ is a unital hom-associative algebra. Hence, denoting by α^n the n-fold composition of n copies of α, with $\alpha^0=id_A$, $\alpha^n\circ\mu=$*

$\mu \circ \left(\alpha^{\otimes 2}\right)^n$, then, $(A, \mu_{\alpha^n} = \alpha^n \circ \mu, \eta_{\alpha^n} = \alpha^n \circ \eta, \alpha^n)$ is a unital hom-associative algebra.

Remark 16.2 More generally, we can construct a hom-associative algebra starting from a hom-associative algebra and a weak endomorphism.

16.2.2 Hom-Associative Coalgebras

In the following, we recall the fundamental notion of coalgebras and hom-coalgebra, which is dual to that of a associative algebra and associative coalgebras, respectively.

Definition 16.3 A pair (A, Δ) is called associative coalgebra, where A is a $\mathbb{k}$-vector space, and $\Delta : A \longrightarrow A \otimes A$ is a linear map, satisfying the coassociativity condition,

$$(\Delta \otimes id_A) \circ \Delta = (id_A \otimes \Delta) \circ \Delta. \tag{16.5}$$

An associative coalgebra is said to be counital if there exists a linear map $\varepsilon : A \longrightarrow \mathbb{k}$ such that

$$(\varepsilon \otimes id_A) \circ \Delta = (id_A \otimes \varepsilon) \circ \Delta = id_A. \tag{16.6}$$

Conditions (16.5) and (16.6) are respectively equivalent to the following commutative diagrams:

$$\begin{array}{ccc} A & \stackrel{\Delta}{\longrightarrow} & A \otimes A \\ \Delta \downarrow & & \downarrow id_A \otimes \Delta \\ A \otimes A & \stackrel{\Delta \otimes id_A}{\longrightarrow} & A \otimes A \end{array} \qquad \begin{array}{ccccc} \mathbb{k} \otimes A & \stackrel{\varepsilon \otimes id_A}{\longleftarrow} & A \otimes A & \stackrel{id_A \otimes \varepsilon}{\longrightarrow} & A \otimes \mathbb{k} \\ \cong & & \uparrow \Delta & & \cong \\ A & \stackrel{id_A}{\longleftarrow} & A & \stackrel{id_A}{\longrightarrow} & A \end{array}$$

Let $(A, \Delta))$ and (A', Δ') be two associative coalgebras. A linear map $f : A \to A'$ is an *associative coalgebra morphism* if $f^{\otimes 2} \circ \Delta = \Delta' \circ f$.

Definition 16.4 A hom-associative coalgebra is a triple (A, Δ, β), where A is a $\mathbb{k}$-vector space, $\Delta : A \longrightarrow A \otimes A$ is a linear map, and $\beta : A \longrightarrow A$ is a homomorphism, satisfying the hom-associativity condition

$$(\Delta \otimes \beta) \circ \Delta = (\beta \otimes \Delta) \circ \Delta. \tag{16.7}$$

We assume, moreover, that $\Delta \circ \beta = \beta^{\otimes 2} \circ \Delta$.
A hom-associative coalgebra is said to be counital if there exists a linear map $\varepsilon : A \longrightarrow \mathbb{k}$ such that $\varepsilon \circ \beta = \beta$ and

$$(\varepsilon \otimes id_A) \circ \Delta = (id_A \otimes \varepsilon) \circ \Delta = \beta. \tag{16.8}$$

Conditions (16.7) and (16.8) are, respectively, equivalent to the following commutative diagrams:

$$\begin{array}{ccc} A & \stackrel{\Delta}{\longrightarrow} & A\otimes A \\ \Delta\downarrow & & \downarrow\beta\otimes\Delta \\ A\otimes A & \stackrel{\Delta\otimes\beta}{\longrightarrow} & A\otimes A \end{array} \qquad \begin{array}{ccccc} \Bbbk\otimes A & \stackrel{\varepsilon\otimes id_A}{\longleftarrow} & A\otimes A & \stackrel{id_A\otimes\varepsilon}{\longrightarrow} & A\otimes\Bbbk \\ \cong & & \uparrow\Delta & & \cong \\ A & \stackrel{\beta}{\longleftarrow} & A & \stackrel{\beta}{\longrightarrow} & A \end{array}$$

Remark 16.3 1) We recover the classical associative coalgebra when the twisting map β is the identity map.

2) Given a hom-associative coalgebra $A := (A, \Delta, \beta)$, we define the *coopposite* hom-associative coalgebra $A^{cop} := (A, \Delta^{cop}, \beta)$ to be the hom-associative coalgebra with the same underlying vector space as A and with comultiplication defined by $\Delta^{cop} = \tau_{A\otimes A} \circ \Delta$.

3) A hom-associative coalgebra (A, Δ, β) is *cocommutative* if and only if $\Delta^{cop} = \Delta$.

Proposition 16.2 *Let (A, Δ, ε) be an unital associative coalgebra and $\alpha : A \longrightarrow A$ be a morphism of associative coalgebra, i.e. $\Delta \circ \alpha = \alpha^{\otimes 2} \circ \Delta$ and $\varepsilon \circ \alpha = \varepsilon$. Then $(A, \Delta_\alpha = \Delta \circ \alpha, \varepsilon, \alpha)$ is an unital hom-associative coalgebra. Denoting by α^n the n-fold composition of n copies of β, with $\alpha^0 = id_A$, $\Delta \circ \alpha^n = \left(\alpha^{\otimes 2}\right)^n \circ \Delta$, then, $(A, \Delta_{\alpha^n} = \Delta \circ \alpha^n, \varepsilon, \alpha^n)$ is an unital hom-associative coalgebra.*

Let $(A, \Delta, \beta))$ and (A', Δ', β') be two hom-associative coalgebras. A linear map $f : A \to A'$ is a *hom-associative coalgebra morphism* if

$$f^{\otimes 2} \circ \Delta = \Delta' \circ f \qquad \textit{and} \quad f \circ \beta = \beta' \circ f.$$

It is said to be a *weak morphism* if only the first condition holds. If, furthermore, the hom-associative coalgebras admit counits ε and ε', we have, in addition, $\varepsilon = \varepsilon' \circ f$.

We say that a hom-associative coalgebra (A, Δ, β) is *isomorphic* to a hom-associative coalgebra (A', Δ', β'), if there exists a bijective hom-coalgebra morphism $f : A \longrightarrow A'$. We denote this by $A \cong A'$ when the context is clear, such that

$$\Delta' = f^{\otimes 2} \circ \Delta \circ f^{-1}, \quad \varepsilon' = \varepsilon \circ f^{-1} \ \textit{and} \quad \beta' = \beta \circ f^{-1}.$$

Theorem 16.2 ([41, Corollary 4.12]) *Let (A, μ, η, α) be a finite-dimensional unital hom-associative algebra, and A^* be the linear dual of A. We define the comultiplication by the composition*

$$\Delta : A^* \stackrel{\mu^*}{\longrightarrow} (A \otimes A)^* \stackrel{\rho^{-1}}{\longrightarrow} A^* \otimes A^*\ , \ \Delta = \rho^{-1}\mu^*,$$

and

$$\varepsilon : A^* \stackrel{\eta^*}{\longrightarrow} \Bbbk^* \stackrel{\psi}{\longrightarrow} \Bbbk,\ \varepsilon = \psi\eta^*,$$

where ψ *is the canonical isomorphism,* $\varepsilon(f) = f(1_A)$ *for* $f \in A^*$ *where* $1_A = \eta(1_{\Bbbk})$*, and the homomorphism* $\beta : A^* \longrightarrow A^*$, $\beta(h) = h \circ \alpha$*. Then,* $(A^*, \Delta, \varepsilon, \beta)$ *is a unital hom-coassociative coalgebra.*

16.2.3 Hom-Bialgebras and Infinitesimal Hom-Bialgebras

The notion of hom-bialgebra was introduced in [40, 41], see also [58].

Definition 16.5 A *bialgebra* is a tuple $(A, \mu, \eta, \Delta, \varepsilon)$ in which (A, μ, η) is a unital associative algebra, (A, Δ, ε) is a unital associative coalgebra, and the linear map Δ is a morphism of associative algebras, that is

$$\Delta \circ \mu = \mu^{\otimes 2} \circ \tau_{2,3} \circ \Delta^{\otimes 2}. \tag{16.9}$$

Definition 16.6 A *hom-bialgebra* is a tuple $(A, \mu, \eta, \Delta, \varepsilon, \alpha)$ in which (A, μ, η, α) is a unital hom-associative algebra, $(A, \Delta, \varepsilon, \alpha)$ is a unital hom-associative coalgebra satisfying the condition (16.9).

Remark 16.4 1) ([40]) In terms of elements, condition (16.9) could be expressed by the following identities:

$$\Delta(\mu(x \otimes y)) = \Delta(x) \cdot \Delta(y) = \sum_{(x)(y)} \mu(x_1 \otimes y_1) \otimes \mu(x_2 \otimes y_2),$$

where the dot "·" denotes the multiplication on tensor product.
2) Observe that a hom-bialgebra is neither associative algebra, nor associative coalgebra, unless $\alpha = id_A$, in which case we have a bialgebra.

A *morphism of hom-bialgebras* (resp. weak morphism of hom-bialgebras) is a morphism (resp. weak morphism) of hom-associative algebras and hom-associative coalgebras.

Definition 16.7 An *infinitesimal bialgebra* is a tuple $(A, \mu, \eta, \Delta, \varepsilon)$ in which (A, μ, η) is a unital associative algebra, (A, Δ, ε) is a unital associative coalgebra satisfying the following compatibility

$$\Delta \circ \mu(x, y) = (ad^L_\mu(x) \otimes id_A) \circ \Delta(y) + (id_A \otimes ad^R_\mu(y)) \circ \Delta(x), \tag{16.10}$$

where $x, y \in A$, and the maps $ad^L_\mu, ad^R_\mu : A \to End(A)$ are defined by $ad^L_\mu(x)(y) = \mu(x, y)$ and $ad^R_\mu(x)(y) = \mu(y, x)$.

Definition 16.8 An *infinitesimal hom-bialgebra* is a tuple $(A, \mu, \eta, , \Delta, \varepsilon, \alpha)$ in which (A, μ, η, α) is a unital hom-associative algebra, $(A, \Delta, \varepsilon, \alpha)$ is an unital hom-associative coalgebra satisfying the following compatibility

$$\Delta \circ \mu(x, y) = (ad_\mu^L(\alpha(x)) \otimes \alpha) \circ \Delta(y) + (\alpha \otimes ad_\mu^R(\alpha(y))) \circ \Delta(x), \ \forall\, x, y \in A. \tag{16.11}$$

The compatibility condition (16.11) can be written as

$$\Delta\mu = (\mu \otimes \alpha)(\alpha \otimes \Delta) + (\alpha \otimes \mu)(\Delta \otimes \alpha).$$

A *morphism of hom-bialgebras* (resp. weak morphism of hom-bialgebras) is a morphism (resp. weak morphism) of hom-associative algebras and hom-associative coalgebras.

Combining Propositions 16.1 and 16.2, we obtain the following:

Proposition 16.3 *Let $A = (A, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra (resp. infinitesimal bialgebra) and $\alpha : A \longrightarrow A$ be a bialgebra morphism, then $A_\alpha = (A, \mu_\alpha, \eta, \Delta_\alpha, \varepsilon, \alpha)$ is a hom-bialgebra. Hence, $(A, \mu_{\alpha^n}, \eta, \Delta_{\alpha^n}, \varepsilon, \alpha^n)$ is a hom-bialgebra (resp. infinitesimal hom-bialgebra).*

Proposition 16.4 *Let $A = (A, \mu, \eta, \Delta, \varepsilon, \alpha)$ be a finite-dimensional hom-bialgebra, (resp. infinitesimal hom-bialgebra). Then $A^* = (A^*, \Delta^*, \varepsilon^*, \mu^*, \eta^*, \alpha^*)$ which is a hom-bialgebra, (resp. infinitesimal hom-bialgebra), together with the hom-associative algebra structure, which is dual to the hom-coassociative coalgebra structure of A, and with the Hom-coassociative coalgebra structure which is dual to the hom-associative algebra structure of A, is a hom-bialgebra, (resp. infinitesimal hom-bialgebra), called dual hom-bialgebra of A, (resp. dual infinitesimal hom-bialgebra).*

16.3 *n*-Ary Bialgebras of Associative Type

In this section, we introduce the notion of *n*-ary bialgebras of associative and hom-associative type generalizing the associative and hom-associative bialgebras, and discuss their properties.

Definition 16.9 An *n*-ary totally associative algebra is a pair (A, μ) consisting of a vector space A and a linear map $\mu : A^{\otimes n} \to A$ satisfying

$$\begin{aligned}\mu \circ (\mu \otimes id_A \otimes \cdots \otimes id_A) &= \mu \circ (id_A \otimes \mu \otimes \cdots \otimes id_A)\\ &= \cdots = \mu \circ (id_A \otimes \cdots \otimes id_A \otimes \mu).\end{aligned} \tag{16.12}$$

In terms of elements, condition (16.12) could be re-expressed by the following identities, for all $x_1, \ldots, x_{2n-1} \in A$:

$$\begin{aligned}\mu(\mu(x_1, \ldots, x_n), x_{n+1}, \ldots, x_{2n-1}) &= \mu(x_1, \mu(x_2, \ldots, x_{n+1}), x_{n+2}, \ldots, x_{2n-1})\\ \ldots &= \mu(x_1, \ldots, x_i, \mu(x_{i+1}, \ldots, x_{i+n}), x_{i+1+n}, \ldots, x_{2n-1})\\ \ldots &= \mu(x_1, \ldots, x_{n-1}, \mu(x_n, \ldots, x_{2n-1}))\end{aligned} \tag{16.13}$$

For $n = 3$ the condition (16.13) can be written

$$\mu(\mu(x_1, x_2, x_3), x_4, x_5) = \mu(x_1, \mu(x_2, x_3, x_4), x_5) = \mu(x_1, x_2, \mu(x_3, x_4, x_5)).$$

Remark 16.5 An n-ary weak totally associative algebra is given by the identity

$$\mu(x_1, \ldots, x_{n-1}, \mu(x_n, \ldots, x_{2n-1})) = \mu(\mu(x_1, \ldots, x_n), x_{n+1}, \ldots, x_{2n-1}). \quad (16.14)$$

The proof of the following statement is straightforward.

Theorem 16.3 *Let (A, μ) be an associative algebra, then $(A, \widetilde{\mu} = \mu \circ (\mu \otimes id_A))$ is a ternary totally associative algebra.*

Definition 16.10 An n-ary partially associative algebra is a pair (A, μ) consisting of a vector space A and a linear map $\mu : A^{\otimes n} \to A$ satisfying

$$\begin{aligned} \mu \circ (\mu \otimes id_A \otimes \cdots \otimes id_A) + \mu \circ (id_A \otimes \mu \otimes \cdots \otimes id_A) \\ + \cdots + \mu \circ (id_A \otimes \cdots \otimes id_A \otimes \mu) = 0. \end{aligned} \quad (16.15)$$

In terms of elements, condition (16.15) could be reformulated as

$$\sum_{i=0}^{n-1} \mu(x_1, \ldots, x_i, \mu(x_{i+1}, \ldots, x_{i+n}), x_{i+1+n}, \ldots, x_{2n-1}) = 0, \quad (16.16)$$

where $x_1, \ldots, x_{2n-1} \in A$.

For the particular case $n = 3$, the condition (16.16) can be written as:

$$\mu(\mu(x_1, x_2, x_3), x_4, x_5) + \mu(x_1, \mu(x_2, x_3, x_4), x_5) + \mu(x_1, x_2, \mu(x_3, x_4, x_5)) = 0.$$

Definition 16.11 An n-ary alternate partially associative algebra is a pair (A, μ) consisting of a vector space A and a linear map $\mu : A^{\otimes n} \to A$ satisfying

$$\begin{aligned} \mu \circ (\mu \otimes id_A \otimes \cdots \otimes id_A) - \mu \circ (id_A \otimes \mu \otimes \cdots \otimes id_A) \\ + \cdots (-1)^{n-1} \mu \circ (id_A \otimes \cdots \otimes id_A \otimes \mu) = 0. \end{aligned} \quad (16.17)$$

In terms of elements, condition (16.17) could be translated into the identity

$$\sum_{i=0}^{n-1} (-1)^i \mu(x_1, \ldots, x_i, \mu(x_{i+1}, \ldots, x_{i+n}), x_{i+1+n}, \ldots, x_{2n-1}) = 0,$$

where $x_1, \ldots, x_{2n-1} \in A$.

For $n = 3$, it takes the form:

$$\mu(\mu(x_1, x_2, x_3), x_4, x_5) - \mu(x_1, \mu(x_2, x_3, x_4), x_5) + \mu(x_1, x_2, \mu(x_3, x_4, x_5)) = 0.$$

Remark 16.6 An n-ary totally associative algebra A, (resp. n-ary partially associative algebra, or n-ary alternate partially associative algebra) is called unital if there exists a linear map $\eta : \mathbb{k} \to A$ such that

$$\begin{aligned}\mu \circ (id_A \otimes \eta \cdots \otimes \eta) &= \mu \circ (\eta \otimes id_A \otimes \eta \cdots \otimes \eta) \\ &= \cdots = \mu \circ (\eta \otimes \cdots \otimes \eta \otimes id_A) = id_A.\end{aligned} \tag{16.18}$$

The unit element is $1_A = \eta\,(1_{\mathbb{k}})$.

The morphisms of n-ary algebras of associative type are defined as follows.

Definition 16.12 Let (A, μ) and (A', μ') be two n-ary totally associative algebras, (resp. n-ary partially associative algebras, or n-ary alternate partially associative algebras). A linear map $f : A \to A'$ is an n-ary totally associative algebra, (resp. n-ary partially associative algebra or n-ary alternate partially associative algebra), morphism if it satisfies

$$f(\mu(x_1, \ldots, x_n)) = \mu'(f(x_1), \ldots, f(x_n)),\ \forall\, x_1, \ldots, x_n \in A.$$

Definition 16.13 An n-ary totally associative coalgebra is a pair (A, Δ) consisting of a vector space A and a linear map $\Delta : A \to A^{\otimes n}$ satisfying

$$\begin{aligned}(\Delta \otimes id_A \otimes \cdots \otimes id_A) \circ \Delta &= (id_A \otimes \Delta \otimes \cdots \otimes id_A) \circ \Delta \\ &= \cdots = (id_A \otimes \cdots \otimes id_A \otimes \Delta) \circ \Delta.\end{aligned} \tag{16.19}$$

For $n = 3$ the condition (16.19) yields

$$(\Delta \otimes id_A \otimes id_A) \circ \Delta = (id_A \otimes \Delta \otimes id_A) \circ \Delta = (id_A \otimes id_A \otimes \Delta) \circ \Delta.$$

Remark 16.7 An n-ary weak totally associative coalgebra is given by the identity

$$(\Delta \otimes id_A \otimes \cdots \otimes id_A) \circ \Delta = (id_A \otimes \cdots \otimes id_A \otimes \Delta) \circ \Delta.$$

Theorem 16.4 *If (A, Δ) is an associative coalgebra, then $(A, \widetilde{\Delta} = (\Delta \otimes id_A) \circ \Delta)$ is a ternary totally associative coalgebra.*

Proof Since (A, Δ) is an associative coalgebra, we have

$$\begin{aligned}\left(\widetilde{\Delta} \otimes id_A \otimes id_A\right) \circ \widetilde{\Delta} &= \left((\Delta \otimes id_A) \circ \Delta \otimes id_A \otimes id_A\right) \circ (\Delta \otimes id_A) \circ \Delta \\ &= \left((id_A \otimes \Delta) \circ \Delta \otimes id_A \otimes id_A\right) \circ (\Delta \otimes id_A) \circ \Delta \\ &= \left((id_A \otimes \Delta \otimes id_A) \circ (\Delta \otimes id_A) \circ \Delta \otimes id_A\right) \circ \Delta \\ &= \left((id_A \otimes \Delta \otimes id_A) \circ (id_A \otimes \Delta) \circ \Delta \otimes id_A\right) \circ \Delta \\ &= \left(id_A \otimes (\Delta \otimes id_A) \circ \Delta \otimes id_A\right) \circ (\Delta \otimes id_A) \circ \Delta \\ &= \left(id_A \otimes \widetilde{\Delta} \otimes id_A\right) \circ \widetilde{\Delta}.\end{aligned}$$

Similarly, we can prove that $\left(\widetilde{\Delta} \otimes id_A \otimes id_A\right) \circ \widetilde{\Delta} = \left(id_A \otimes id_A \otimes \widetilde{\Delta}\right) \circ \widetilde{\Delta}$. □

Definition 16.14 An n-ary partially associative coalgebra is a pair (A, Δ) consisting of a vector space A and a linear map $\Delta : A \to A^{\otimes n}$ satisfying

$$(\Delta \otimes id_A \otimes \cdots \otimes id_A + id_A \otimes \Delta \otimes \cdots \otimes id_A + \cdots + id_A \otimes \cdots \otimes id_A \otimes \Delta) \circ \Delta = 0.$$

For $n = 3$, it reads as

$$(\Delta \otimes id_A \otimes id_A) \circ \Delta + (id_A \otimes \Delta \otimes id_A) \circ \Delta + (id_A \otimes id_A \otimes \Delta) \circ \Delta = 0.$$

Definition 16.15 An n-ary alternate partially associative coalgebra is a pair (A, Δ) consisting of a vector space A and a linear map $\Delta : A \to A^{\otimes n}$ satisfying

$$\begin{aligned}(\Delta \otimes id_A \otimes \cdots \otimes id_A - id_A \otimes \Delta \otimes \cdots \otimes id_A \\ + \cdots + (-1)^{n-1} id_A \otimes \cdots \otimes id_A \otimes \Delta) \circ \Delta = 0.\end{aligned}$$

For $n = 3$, it takes a simpler form:

$$(\Delta \otimes id_A \otimes id_A) \circ \Delta - (id_A \otimes \Delta \otimes id_A) \circ \Delta + (id_A \otimes id_A \otimes \Delta) \circ \Delta = 0.$$

Remark 16.8 An n-ary totally associative coalgebra A, (resp. n-ary partially associative coalgebra, or n-ary alternate partially coassociative algebra) is called unital if there exists a linear map $\varepsilon : A \to \mathbb{k}$ such that

$$\begin{aligned}(id_A \otimes \varepsilon \otimes \cdots \otimes \varepsilon) \circ \Delta &= (\varepsilon \otimes id_A \otimes \varepsilon \otimes \cdots \otimes \varepsilon) \circ \Delta \\ &= \cdots = (\varepsilon \otimes \cdots \otimes \varepsilon \otimes id_A) \circ \Delta = id_A.\end{aligned} \tag{16.20}$$

The unit element is $1_{\mathbb{k}} = \varepsilon\,(1_A)$.

The morphisms of n-ary coalgebras of associative type are defined as follows.

Definition 16.16 Let (A, Δ) and (A', Δ') be two n-ary totally coassociative algebras, (resp. n-ary partially associative coalgebras and n-ary alternate partially associative coalgebras). A linear map $f : A \to A'$ is an n-ary totally associative coalgebra, (resp. n-ary partially associative coalgebra, or n-ary alternate partially associative coalgebra) morphism if it satisfies

$$f^{\otimes n} \circ \Delta = \Delta' \circ f.$$

Definition 16.17 An *n-ary totally bialgebra* (resp. n-ary weak totally bialgebra, or *n-ary partially bialgebra*, or *n-ary alternate partially bialgebra*) is a quintuple $(A, \mu, \eta, \Delta, \varepsilon)$ in which (A, μ, η) is a unital n-ary totally associative algebra, (resp. n-ary weak totally algebra, or n-ary partially algebra, or n-ary alternate partially algebra), (A, Δ, ε) is a unital n-ary totally associative coalgebra, (resp. n-ary weak totally coalgebra, or n-ary partially coalgebra, or n-ary alternate partially coalgebra), satisfying the following compatibility condition for $x_1, \ldots, x_n \in A$:

$$\Delta \circ \mu(x_1, \ldots, x_n) = \textstyle\sum_{(x_1)\cdots(x_n)} \mu(x_1^{(1)}, \ldots, x_n^{(1)}) \otimes \mu(x_1^{(2)}, \ldots, x_n^{(2)}) \otimes \cdots \otimes \mu(x_1^{(n)}, \ldots, x_n^{(n)}), \quad (16.21)$$

where $\Delta(x_i) = \sum_{(x_i)} x_i^{(1)} \otimes \cdots \otimes x_i^{(n)}$.

The condition (16.21) can be written as

$$\Delta\mu = \mu^{\otimes n} \omega_n \Delta^{\otimes n}, \quad (16.22)$$

where $\omega : A^{\otimes n^2} \to A^{\otimes n^2}$ is given by the relation

$$\omega(x_1^{(1)} \otimes \cdots \otimes x_1^{(n)} \otimes x_2^{(1)} \otimes x_2^{(2)} \otimes \cdots \otimes x_n^{(n-1)} \otimes x_n^{(n)}) = x_1^{(1)} \otimes x_2^{(1)} \otimes \cdots \otimes x_n^{(1)} \otimes x_1^{(2)} \otimes \cdots \otimes x_{n-1}^{(n)} \otimes x_n^{(n)}.$$

The proof of the following statement is straightforward.

Theorem 16.5 *Let $(A, \mu, \eta, \Delta, \varepsilon)$ be a finite-dimensional unital n-ary bialgebra of associative type. Then $(A^*, \Delta^*, \varepsilon^*, \mu^*, \eta^*)$ is a unital n-ary bialgebra of associative type.*

Theorem 16.6 *Let (A, μ, Δ) be a bialgebra, then $(A, \widetilde{\mu}, \widetilde{\Delta})$ is a ternary totally associative bialgebra.*

Proof The compatibility condition (16.21) for a ternary case can be reduced to

$$\Delta\mu = \mu^{\otimes 3} \tau_{37} \tau_{68} \tau_{24} \Delta^{\otimes 3},$$

where τ_{ij} is defined by

$$\tau_{ij}(x_1 \otimes \cdots \otimes x_i \otimes \cdots \otimes x_j \otimes \cdots \otimes \cdots) = x_1 \otimes \cdots \otimes x_j \otimes \cdots \otimes x_i \otimes \cdots \otimes \cdots,$$

and $\tau_{ij}\tau_{kl} = \tau_{kl}\tau_{ij},\ \tau_{ij} = \tau_{ji},\ \tau_{ij}\tau_{ij} = id_A$ for $i, j \neq k, l$. Then,

$$\begin{aligned}
&\widetilde{\Delta}\widetilde{\mu} = (\Delta \otimes id_A)\Delta\mu(\mu \otimes id_A) = (\Delta \otimes id_A)\mu^{\otimes 2}\tau_{23}\Delta^{\otimes 2}(\mu \otimes id_A)\\
&= (\Delta\mu \otimes \mu)\tau_{23}(\Delta\mu \otimes \Delta) = \mu^{\otimes 3}\tau_{23}(\Delta^{\otimes 2} \otimes id_A^{\otimes 2})\tau_{23}(\mu^{\otimes 2} \otimes id_A^{\otimes 2})\tau_{23}\Delta^{\otimes 3}\\
&= \mu^{\otimes 3}\tau_{23}\tau_{45}\tau_{34}(\Delta \otimes id_A \otimes \Delta \otimes id_A)(\mu^{\otimes 2} \otimes id_A^{\otimes 2})\tau_{23}\Delta^{\otimes 3}\\
&= \mu^{\otimes 3}\tau_{23}\tau_{45}\tau_{34}(\Delta\mu \otimes \mu \otimes \Delta \otimes id_A)\tau_{23}\Delta^{\otimes 3}\\
&= \mu^{\otimes 3}\tau_{23}\tau_{45}\tau_{34}(\mu^{\otimes 3} \otimes id_A^{\otimes 3})\tau_{23}(\Delta^{\otimes 2} \otimes id_A^{\otimes 2} \otimes \Delta \otimes id_A)\tau_{23}\Delta^{\otimes 3}\\
&= \mu^{\otimes 3}\tau_{23}\tau_{45}\tau_{34}(\mu^{\otimes 3} \otimes id_A^{\otimes 3})\tau_{23}\tau_{45}\tau_{34}(\Delta \otimes id_A \otimes \Delta \otimes id_A \otimes \Delta \otimes id_A)\Delta^{\otimes 3}\\
&= \mu^{\otimes 3}(\mu \otimes id_A \otimes \mu \otimes id_A \otimes \mu \otimes id_A)\tau_{37}\tau_{68}\tau_{24}(\Delta \otimes id_A \otimes \Delta \otimes id_A \otimes \Delta \otimes id_A)\Delta^{\otimes 3}\\
&= \widetilde{\mu}^{\otimes 3}\tau_{37}\tau_{68}\tau_{24}\widetilde{\Delta}^{\otimes 3}.
\end{aligned}$$

□

16.4 *n*-Ary Bialgebras of Hom-Associative Type

In this section, we recall the definitions of n-ary totally hom-associative algebras and n-ary partially hom-associative algebras introduced in [8] (see also [54]) and introduce n-ary alternate partially hom-associative algebras. Then we generalize hom-associative coalgebras and hom-associative bialgebras to n-ary case.

Definition 16.18 An n-ary totally hom-associative algebra is a triple (A, μ, α) consisting of a vector space A, a linear map $\mu : A^{\otimes n} \to A$ and a family $\alpha = (\alpha_i)_{i=1,\dots,n-1}$ of linear maps $\alpha_i : A \to A$ satisfying

$$\begin{aligned}\mu \circ (\mu \otimes \alpha_1 \otimes \cdots \otimes \alpha_{n-1}) &= \mu \circ (\alpha_1 \otimes \mu \otimes \alpha_2 \otimes \cdots \otimes \alpha_{n-1})\\ &= \cdots = \mu \circ (\alpha_1 \otimes \cdots \otimes \alpha_{n-1} \otimes \mu).\end{aligned} \tag{16.23}$$

In terms of elements, condition (16.23) can be re-expressed by the following identities:

$$\begin{aligned}
&\mu(\mu(x_1, \dots, x_n), \alpha_1(x_{n+1}), \dots, \alpha_{n-1}(x_{2n-1}))\\
&= \mu(\alpha_1(x_1), \mu(x_2, \dots, x_{n+1}), \alpha_2(x_{n+2}), \dots, \alpha_{n-1}(x_{2n-1}))\\
&\cdots = \mu(\alpha_1(x_1), \dots, \alpha_i(x_i), \mu(x_{i+1}, \dots, x_{i+n}), \alpha_{i+1}(x_{i+1+n}), \dots, \alpha_{n-1}(x_{2n-1}))\\
&\cdots = \mu(\alpha_1(x_1), \dots, \alpha_{n-1}(x_{n-1}), \mu(x_n, \dots, x_{2n-1})),
\end{aligned} \tag{16.24}$$

where $x_1, \dots, x_{2n-1} \in A$.

For $n = 3$, the condition (16.24) can be transformed to a simpler expression as:

$$\mu(\mu(x_1, x_2, x_3), \alpha_1(x_4), \alpha_2(x_5)) = \mu(\alpha_1(x_1), \mu(x_2, x_3, x_4), \alpha_2(x_5)) \\ = \mu(\alpha_1(x_1), \alpha_2(x_2), \mu(x_3, x_4, x_5)).$$

Definition 16.19 An n-ary weak totally hom-associative algebra is given by the identity

$$\mu(\mu(x_1, \ldots, x_n), \alpha_1(x_{n+1}), \ldots, \alpha_{n-1}(x_{2n-1})) \\ = \mu(\alpha_1(x_1), \ldots, \alpha_{n-1}(x_{n-1}), \mu(x_n, \ldots, x_{2n-1})).$$

Remark 16.9 An n-ary (weak) totally hom-associative algebra is called multiplicative if $\alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = \alpha$ and $\alpha \circ \mu = \mu \circ \alpha^{\otimes n}$.

Theorem 16.7 *Let (A, μ, α) be a multiplicative hom-associative algebra, then the triple $(A, \widetilde{\mu} = \mu \circ (\mu \otimes \alpha), \alpha^2)$ is a ternary totally hom-associative algebra.*

Proof Since (A, μ, α) is a multiplicative hom-associative algebra,

$$\begin{aligned}\widetilde{\mu}(\widetilde{\mu}(x_1, x_2, x_3), \alpha^2(x_4), \alpha^2(x_5)) &= \mu(\mu(\mu(\mu(x_1, x_2), \alpha(x_3)), \alpha^2(x_4)), \alpha^3(x_5)) \\ &= \mu(\mu(\mu(\alpha(x_1), \mu(x_2, x_3)), \alpha^2(x_4)), \alpha^3(x_5)) \\ &= \mu(\mu(\alpha^2(x_1), \mu(\mu(x_2, x_3), \alpha(x_4))), \alpha^3(x_5)) \\ &= \mu(\mu(\alpha^2(x_1), \widetilde{\mu}(x_2, x_3, x_4)), \alpha^3(x_5)) \\ &= \widetilde{\mu}(\alpha^2(x_1), \widetilde{\mu}(x_2, x_3, x_4), \alpha^2(x_5)).\end{aligned}$$

Similarly, we can prove that

$$\widetilde{\mu}(\widetilde{\mu}(x_1, x_2, x_3), \alpha^2(x_4), \alpha^2(x_5)) = \widetilde{\mu}(\alpha^2(x_1), \alpha^2(x_2), \widetilde{\mu}(x_3, x_4, x_5)).$$ □

Definition 16.20 An n-ary partially hom-associative algebra is a triple (A, μ, α) consisting of a vector space A, a linear map $\mu : A^{\otimes n} \to A$ and a family $\alpha = (\alpha_i)_{i=1,\ldots,n-1}$ of linear maps $\alpha_i : A \to A$ satisfying

$$\mu \circ (\mu \otimes \alpha_1 \otimes \cdots \otimes \alpha_{n-1}) + \mu \circ (\alpha_1 \otimes \mu \otimes \alpha_2 \otimes \cdots \otimes \alpha_{n-1}) \\ + \cdots + \mu \circ (\alpha_1 \otimes \cdots \otimes \alpha_{n-1} \otimes \mu) = 0. \quad (16.25)$$

In terms of elements, the condition (16.25) becomes:

$$\sum_{i=0}^{n-1} \mu(\alpha_1(x_1), \ldots, \alpha_i(x_i),\ \mu(x_{i+1}, \ldots, x_{i+n}), \\ \alpha_{i+1}(x_{i+1+n}), \ldots, \alpha_{n-1}(x_{2n-1})) = 0,$$

where $x_1, \ldots, x_{2n-1} \in A$.

For $n = 3$, it simply reads:

$$\mu(\mu(x_1, x_2, x_3), \alpha_1(x_4), \alpha_2(x_5)) + \mu(\alpha_1(x_1), \mu(x_2, x_3, x_4), \alpha_2(x_5)) + \mu(\alpha_1(x_1), \alpha_2(x_2), \mu(x_3, x_4, x_5)) = 0.$$

Definition 16.21 An n-ary alternate partially hom-associative algebra (A, μ, α) is a triple consisting of a vector space A, a linear map $\mu : A^{\otimes n} \to A$ and a family $\alpha = (\alpha_i)_{i=1,\dots,n-1}$ of linear maps $\alpha_i : A \to A$ satisfying

$$\mu \circ (\mu \otimes \alpha_1 \otimes \cdots \otimes \alpha_{n-1}) - \mu \circ (\alpha_1 \otimes \mu \otimes \alpha_2 \otimes \cdots \otimes \alpha_{n-1}) + \cdots + (-1)^{n-1} \mu \circ (\alpha_1 \otimes \cdots \otimes \alpha_{n-1} \otimes \mu) = 0. \tag{16.26}$$

In terms of elements, condition (16.26) can be expressed by the following identities:

$$\sum_{i=0}^{n-1} (-1)^i \mu(\alpha_1(x_1), \dots, \alpha_i(x_i), \mu(x_{i+1}, \dots, x_{i+n}), \alpha_{i+1}(x_{i+1+n}), \dots, \alpha_{n-1}(x_{2n-1})) = 0,$$

where $x_1, \dots, x_{2n-1} \in A$.

For $n = 3$, we obtain the formula:

$$\mu(\mu(x_1, x_2, x_3), \alpha_1(x_4), \alpha_2(x_5)) - \mu(\alpha_1(x_1), \mu(x_2, x_3, x_4), \alpha_2(x_5)) + \mu(\alpha_1(x_1), \alpha_2(x_2), \mu(x_3, x_4, x_5)) = 0.$$

Remark 16.10 An n-ary (alternate) partially hom-associative algebra is called multiplicative if $\alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = \alpha$ and $\alpha \circ \mu = \mu \circ \alpha^{\otimes n}$.

Remark 16.11 An n-ary totally hom-associative algebra A, (resp. n-ary partially hom-associative algebra, or n-ary alternate partially hom-associative algebra), is called unital if there exists a linear map $\eta : \mathbb{k} \to A$ satisfying the condition (16.18) and $\eta\alpha_i = \eta$ for $i = 1, \dots, n-1$. The unit element is $1_A = \eta(1_{\mathbb{k}})$.

The morphisms of n-ary algebras of hom-associative type are defined as follows.

Definition 16.22 Let (A, μ, α) and (A', μ', α') be two n-ary totally hom-associative algebras, (resp. n-ary partially hom-associative algebras, or n-ary alternate partially hom-associative algebras). A linear map $f : A \to A'$ is an n-ary totally hom-associative algebra, (resp. n-ary partially hom-associative algebra, or n-ary alternate partially hom-associative algebra), morphism, if it satisfies

$$f(\mu(x_1, \dots, x_n)) = \mu'(f(x_1), \dots, f(x_n)) \text{ and } f \circ \alpha_i = \alpha'_i \circ f,$$

for all $x_1, \dots, x_n \in A$ and $i = 1, \cdots, n-1$. It is said to be a *weak morphism*, if only the first condition holds.

Definition 16.23 An n-ary totally hom-associative coalgebra is a triple (A, Δ, α) consisting of a vector space A, a linear map $\Delta : A \to A^{\otimes n}$ and a family $\alpha = (\alpha_i)_{i=1,\dots,n-1}$ of linear maps $\alpha_i : A \to A$ satisfying

$$\begin{aligned}(\Delta \otimes \alpha_1 \otimes \cdots \otimes \alpha_{n-1}) \circ \Delta &= (\alpha_1 \otimes \Delta \otimes \alpha_2 \otimes \cdots \otimes \alpha_{n-1}) \circ \Delta \\ &= \cdots = (\alpha_1 \otimes \cdots \otimes \alpha_{n-1} \otimes \Delta) \circ \Delta.\end{aligned} \tag{16.27}$$

For $n = 3$, the condition (16.27) gives

$$(\Delta \otimes \alpha_1 \otimes \alpha_2) \circ \Delta = (\alpha_1 \otimes \Delta \otimes \alpha_2) \circ \Delta = (\alpha_1 \otimes \alpha_2 \otimes \Delta) \circ \Delta.$$

Definition 16.24 An n-ary weak totally hom-associative coalgebra is given by the identity

$$(\Delta \otimes \alpha_1 \otimes \cdots \otimes \alpha_{n-1}) \circ \Delta = (\alpha_1 \otimes \cdots \otimes \alpha_{n-1} \otimes \Delta) \circ \Delta. \tag{16.28}$$

Remark 16.12 An n-ary (weak) totally hom-associative coalgebra is called multiplicative if $\alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = \alpha$ and $\alpha^{\otimes n} \circ \Delta = \Delta \circ \alpha$.

Theorem 16.8 *Let (A, Δ, α) be a multiplicative hom-associative coalgebra, then the triple $(A, \widetilde{\Delta} = (\Delta \otimes \alpha) \circ \Delta, \alpha^2)$ is a ternary totally hom-associative coalgebra.*

Proof Since (A, Δ, α) is a hom-associative coalgebra, we have

$$\begin{aligned}\left(\widetilde{\Delta} \otimes \alpha^2 \otimes \alpha^2\right) \circ \widetilde{\Delta} &= \left((\Delta \otimes \alpha) \circ \Delta \otimes \alpha^2 \otimes \alpha^2\right) \circ (\Delta \otimes \alpha) \circ \Delta \\ &= \left((\alpha \otimes \Delta) \circ \Delta \otimes \alpha^2 \otimes \alpha^2\right) \circ (\Delta \otimes \alpha) \circ \Delta \\ &= \left((\alpha \otimes \Delta \otimes \alpha) \circ (\Delta \otimes \alpha) \circ \Delta \otimes \alpha^3\right) \circ \Delta \\ &= \left((\alpha \otimes \Delta \otimes \alpha) \circ (\alpha \otimes \Delta) \circ \Delta \otimes \alpha^3\right) \circ \Delta \\ &= \left(\alpha^2 \otimes (\Delta \otimes \alpha) \circ \Delta \otimes \alpha^2\right) \circ (\Delta \otimes \alpha) \circ \Delta = \left(\alpha^2 \otimes \widetilde{\Delta} \otimes \alpha^2\right) \circ \widetilde{\Delta}.\end{aligned}$$

Similarly, we can prove that $\left(\widetilde{\Delta} \otimes \alpha^2 \otimes \alpha^2\right) \circ \widetilde{\Delta} = \left(\alpha^2 \otimes \alpha^2 \otimes \widetilde{\Delta}\right) \circ \widetilde{\Delta}$. □

Definition 16.25 An n-ary partially hom-associative coalgebra is a triple (A, Δ, α) consisting of a vector space A, a linear map $\Delta : A \to A^{\otimes n}$, and a family $\alpha = (\alpha_i)_{i=1,\dots,n-1}$ of linear maps $\alpha_i : A \to A$ satisfying

$$\begin{aligned}(\Delta \otimes \alpha_1 \otimes \cdots \otimes \alpha_{n-1} &+ \alpha_1 \otimes \Delta \otimes \alpha_2 \otimes \cdots \otimes \alpha_{n-1} \\ &+ \cdots + \alpha_1 \otimes \cdots \otimes \alpha_{n-1} \otimes \Delta) \circ \Delta = 0.\end{aligned} \tag{16.29}$$

For the particular $n = 3$, the condition (16.29) is reduced to

$$(\Delta \otimes \alpha_1 \otimes \alpha_2) \circ \Delta + (\alpha_1 \otimes \Delta \otimes \alpha_2) \circ \Delta + (\alpha_1 \otimes \alpha_2 \otimes \Delta) \circ \Delta = 0.$$

Definition 16.26 An n-ary alternate partially hom-associative coalgebra is a triple (A, Δ, α) consisting of a vector space A, a linear map $\Delta : A \to A^{\otimes n}$, and a family $\alpha = (\alpha_i)_{i=1,\dots,n-1}$ of linear maps $\alpha_i : A \to A$ satisfying

$$(\Delta \otimes \alpha_1 \otimes \cdots \otimes \alpha_{n-1} - \alpha_1 \otimes \Delta \otimes \alpha_2 \otimes \cdots \otimes \alpha_{n-1} \\ + \cdots + (-1)^{n-1}\alpha_1 \otimes \cdots \otimes \alpha_{n-1} \otimes \Delta) \circ \Delta = 0.$$

For $n = 3$, we get a simpler expression:

$$(\Delta \otimes \alpha_1 \otimes \alpha_2) \circ \Delta - (\alpha_1 \otimes \Delta \otimes \alpha_2) \circ \Delta + (\alpha_1 \otimes \alpha_2 \otimes \Delta) \circ \Delta = 0.$$

Remark 16.13 An n-ary (alternate) partially hom-associative coalgebra is called multiplicative if $\alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = \alpha$ and $\alpha^{\otimes n} \circ \Delta = \alpha \circ \Delta$.

Remark 16.14 An n-ary totally hom-associative coalgebra A, (resp. n-ary partially hom-associative coalgebra, or n-ary alternate partially hom-associative coalgebra), is called unital if there exists a linear map $\varepsilon : A \to \mathbb{k}$ satisfying the condition (16.20) and $\alpha_i \varepsilon = \varepsilon$ for $i = 1, \dots, n-1$. The unit element is $1_{\mathbb{k}} = \varepsilon(1_A)$.

The morphisms of n-ary coalgebras of hom-associative type are defined as follows.

Definition 16.27 Let (A, Δ, α) and (A', Δ', α') be two n-ary totally hom-associative coalgebras (resp. n-ary partially hom-associative coalgebras, or n-ary alternate partially hom-associative coalgebras). A linear map $f : A \to A'$ is an n-ary totally hom-associative coalgebra morphism (resp. n-ary partially hom-associative coalgebra, or n-ary alternate partially hom-associative coalgebra morphism) if for $i = 1, \cdots, n-1$,

$$f^{\otimes n} \circ \Delta = \Delta' \circ f, \qquad f \circ \alpha_i = \alpha'_i \circ f.$$

It is said to be a *weak morphism*, if only the first condition holds.

Theorem 16.9 *The triplet (A, Δ, α) defines a multiplicative n-ary totally hom-associative coalgebras (resp. n-ary partially hom-associative coalgebras, or n-ary alternate partially hom-associative coalgebras) if and only if $(A^*, \Delta^*, \alpha^*)$ is a multiplicative n-ary totally hom-associative algebras (resp. n-ary partially hom-associative algebras, or n-ary alternate partially hom-associative algebras), where*

$$\langle \Delta^*(\xi_1 \otimes \cdots \otimes \xi_n), x\rangle = \langle \xi_1 \otimes \cdots \otimes \xi_n, \Delta(x)\rangle \quad and \quad \langle \alpha^*(\xi_1), x\rangle = \langle \xi, \alpha(x)\rangle,$$

for all $\xi_1, \dots, \xi_n \in A^*$ and $x \in A$.

Proof Let $\xi_1, \dots, \xi_{2n-1} \in A^*$ and $x \in A$. Then,

$$\begin{aligned}
&\langle \Delta^* \circ (\Delta^* \otimes \alpha^* \otimes \cdots \otimes \alpha^*)(\xi_1 \otimes \cdots \otimes \xi_{2n-1}), x\rangle \\
&\quad = \langle \xi_1 \otimes \cdots \otimes \xi_{2n-1}, (\Delta \otimes \alpha \otimes \cdots \otimes \alpha) \circ \Delta(x)\rangle \\
&\quad = \langle \xi_1 \otimes \cdots \otimes \xi_{2n-1}, (\alpha \otimes \Delta \otimes \cdots \otimes \alpha) \circ \Delta(x)\rangle \\
&\quad = \langle \Delta^* \circ (\alpha^* \otimes \Delta * \otimes \cdots \otimes \alpha^*)(\xi_1 \otimes \cdots \otimes \xi_{2n-1}), x\rangle.
\end{aligned}$$

□

Definition 16.28 An *n-ary totally hom-bialgebra,* (resp. *n-ary weak totally hom-bialgebra*, or *n-ary partially hom-bialgebra,* or *n-ary alternate partially hom-bialgebra*), is a tuple $(A, \mu, \eta, \Delta, \varepsilon, \alpha)$ in which (A, μ, η, α) is a multiplicative unital n-ary totally hom-associative algebra, (resp. n-ary weak totally hom-associative algebra, or n-ary partially hom-associative algebra, or n-ary alternate partially hom-associative algebra), $(A, \Delta, \varepsilon, \alpha)$ is a multiplicative unital n-ary totally hom-associative coalgebra, (resp. n-ary weak totally hom-associative coalgebra, or n-ary partially hom-associative coalgebra, or n-ary alternate partially hom-associative coalgebra), satisfying the following compatibility condition

$$\Delta \circ \mu(x_1, \ldots, x_n) = \sum_{(x_1)\cdots(x_n)} \mu(x_1^{(1)}, \ldots, x_n^{(1)}) \otimes \mu(x_1^{(2)}, \ldots, x_n^{(2)}) \otimes \cdots \otimes \mu(x_1^{(n)}, \ldots, x_n^{(n)}), \tag{16.30}$$

where $x_1, \ldots, x_n \in A$ and $\Delta(x_i) = \sum_{(x_i)} x_i^{(1)} \otimes \cdots \otimes x_i^{(n)}$.

Theorem 16.10 *Let $(A, \mu, \eta, \Delta, \varepsilon, \alpha)$ be a multiplicative unital hom-associative bialgebra, then $(A, \widetilde{\mu}, \eta, \widetilde{\Delta}, \varepsilon, \alpha^2)$ is a multiplicative unital ternary totally hom-associative bialgebras.*

Proof According to Propositions 16.7 and 16.8, one needs only to show the compatibility condition (16.30) holds. Indeed, using the multiplicativity of α,

$$\begin{aligned}
\widetilde{\Delta}\widetilde{\mu} &= (\Delta \otimes \alpha)\Delta\mu(\mu \otimes \alpha) \\
&= (\Delta \otimes \alpha)\mu^{\otimes 2}\tau_{23}\Delta^{\otimes 2}(\mu \otimes \alpha) \\
&= (\Delta\mu \otimes \mu\alpha^{\otimes 2})\tau_{23}(\Delta\mu \otimes \alpha^{\otimes 2}\Delta) \\
&= \mu^{\otimes 3}\tau_{23}(\Delta^{\otimes 2} \otimes \alpha^{\otimes 2})\tau_{23}(\mu^{\otimes 2} \otimes \alpha^{\otimes 2})\tau_{23}\Delta^{\otimes 3} \\
&= \mu^{\otimes 3}\tau_{23}\tau_{45}\tau_{34}(\Delta \otimes \alpha \otimes \Delta \otimes \alpha)(\mu^{\otimes 2} \otimes \alpha^{\otimes 2})\tau_{23}\Delta^{\otimes 3} \\
&= \mu^{\otimes 3}\tau_{23}\tau_{45}\tau_{34}(\Delta\mu \otimes \mu\alpha^{\otimes 2} \otimes \alpha^{\otimes 2}\Delta \otimes \alpha^2)\tau_{23}\Delta^{\otimes 3} \\
&= \mu^{\otimes 3}\tau_{23}\tau_{45}\tau_{34}(\mu^{\otimes 3} \otimes \alpha^{\otimes 3})\tau_{23}(\Delta^{\otimes 2} \otimes \alpha^{\otimes 2} \otimes \Delta \otimes \alpha)\tau_{23}\Delta^{\otimes 3} \\
&= \mu^{\otimes 3}\tau_{23}\tau_{45}\tau_{34}(\mu^{\otimes 3} \otimes \alpha^{\otimes 3})\tau_{23}\tau_{45}\tau_{34}(\Delta \otimes \alpha \otimes \Delta \otimes \alpha \otimes \Delta \otimes \alpha)\Delta^{\otimes 3} \\
&= \mu^{\otimes 3}(\mu \otimes \alpha \otimes \mu \otimes \alpha \otimes \mu \otimes \alpha)\tau_{37}\tau_{68}\tau_{24}(\Delta \otimes \alpha \otimes \Delta \otimes \alpha \otimes \Delta \otimes \alpha)\Delta^{\otimes 3} \\
&= \widetilde{\mu}^{\otimes 3}\tau_{37}\tau_{68}\tau_{24}\widetilde{\Delta}^{\otimes 3}.
\end{aligned}$$

□

Proposition 16.5 *Let $A = (A, \mu, \eta, \Delta, \varepsilon)$ be an n-ary bialgebra of associative type, and $\alpha : A \longrightarrow A$ be an n-ary bialgebra morphism, then $A_\alpha = (A, \mu_\alpha, \eta, \Delta_\alpha, \varepsilon, \alpha)$ is an n-ary hom-bialgebra of associative type. Hence $(A, \mu_{\alpha^n}, \eta, \Delta_{\alpha^n}, \varepsilon, \alpha^n)$ is an n-ary hom-bialgebra of associative type.*

Proof First, we prove that (A, μ_α, α) is an n-ary hom-associative algebra. For all $x_1, \ldots, x_{2n-1} \in A$,

$$\begin{aligned}
&\mu_\alpha(\mu_\alpha(x_1, \ldots, x_n), \alpha(x_{n+1}), \ldots, \alpha(x_{2n-1}))\\
&\quad = \mu_\alpha(\alpha(\mu(x_1, \ldots, x_n)), \alpha(x_{n+1}), \ldots, \alpha(x_{2n-1}))\\
&\quad = \alpha(\mu(\alpha(\mu(x_1, \ldots, x_n)), \alpha(x_{n+1}), \ldots, \alpha(x_{2n-1})))\\
&\quad = \alpha(\mu(\mu(\alpha(x_1), \ldots, \alpha(x_n)), \alpha(x_{n+1}), \ldots, \alpha(x_{2n-1})))\\
&\quad = \alpha(\mu(\alpha(x_1), \mu(\alpha(x_2), \ldots, \alpha(x_{n+1})), \alpha(x_{n+2}), \ldots, \alpha(x_{2n-1})))\\
&\quad = \alpha(\mu(\alpha(x_1), \alpha(\mu(x_2, \ldots, x_{n+1})), \alpha(x_{n+2}), \ldots, \alpha(x_{2n-1})))\\
&\quad = \mu_\alpha(\alpha(x_1), \mu_\alpha(x_2, \ldots, x_{n+1}), \alpha(x_{n+2}), \ldots, \alpha(x_{2n-1})).
\end{aligned}$$

Similarly, we can prove that

$$\begin{aligned}
&\mu_\alpha(\mu_\alpha(x_1, \ldots, x_n), \alpha(x_{n+1}), \ldots, \alpha(x_{2n-1}))\\
&\cdots = \mu_\alpha(\alpha(x_1), \ldots, \alpha(x_i), \mu_\alpha(x_{i+1}, \ldots, x_{i+n}), \alpha(x_{i+1+n}), \ldots, \alpha(x_{2n-1}))\\
&\cdots = \mu_\alpha(\alpha(x_1), \ldots, \alpha(x_{n-1}), \mu_\alpha(x_n, \ldots, x_{2n-1})).
\end{aligned}$$

Next, we show that $(A, \Delta_\alpha, \alpha)$ satisfies (16.27). Indeed, using the fact that $\alpha^{\otimes n} \circ \Delta = \Delta \circ \alpha$, we have

$$\begin{aligned}
(\Delta_\alpha \otimes \alpha \otimes \cdots \otimes \alpha) \circ \Delta_\alpha &= ((\Delta \circ \alpha) \otimes \alpha \otimes \cdots \otimes \alpha) \circ \Delta \circ \alpha\\
&= ((\alpha^{\otimes n} \circ \Delta) \otimes \alpha \otimes \cdots \otimes \alpha) \circ \Delta \circ \alpha\\
&= ((\alpha^{\otimes 2n-1} \circ \Delta) \otimes id_A \otimes \cdots \otimes id_A) \circ \Delta \circ \alpha\\
&= \alpha^{\otimes 2n-1}(id_A \otimes \Delta \otimes id_A \otimes \cdots \otimes id_A) \circ \Delta \circ \alpha\\
&= (\alpha \otimes (\alpha^{\otimes n} \circ \Delta) \otimes \alpha \otimes \cdots \otimes \alpha) \circ \Delta \circ \alpha\\
&= (\alpha \otimes \Delta_\alpha \otimes \alpha \otimes \cdots \otimes \alpha) \circ \Delta_\alpha.
\end{aligned}$$

Similarly, we can show that

$$(\Delta_\alpha \otimes \alpha \otimes \cdots \otimes \alpha) \circ \Delta_\alpha = \cdots = (\alpha \otimes \cdots \otimes \alpha \otimes \Delta_\alpha) \circ \Delta_\alpha.$$

Now, it remains to prove the compatibility condition (16.21). From (16.22), the condition may be written as $\Delta_\alpha \mu_\alpha = \mu_\alpha^{\otimes n} \omega_n \Delta_\alpha^{\otimes n}$. This holds since

$$\begin{aligned}
\Delta_\alpha \mu_\alpha &= \Delta \circ \alpha \circ \alpha \circ \mu = \alpha^{\otimes n} \circ \Delta\mu \circ \alpha^{\otimes n}\\
&= \alpha^{\otimes n} \circ (\mu^{\otimes n} \omega_n \Delta^{\otimes n}) \circ \alpha^{\otimes n} = (\alpha \circ \mu)^{\otimes n} \omega_n (\Delta \circ \alpha)^{\otimes n} = \mu_\alpha^{\otimes n} \omega_n \Delta_\alpha^{\otimes n}.
\end{aligned}$$

□

16.5 From Infinitesimal (Hom)-Bialgebras to Ternary Infinitesimal (Hom)-Bialgebras

In this section, we construct a ternary infinitesimal (hom)-bialgebra starting from an infinitesimal (hom)-bialgebra with a necessary and sufficient condition.

16.5.1 From Infinitesimal Bialgebras to Ternary Infinitesimal Bialgebras

Definition 16.29 An *infinitesimal n-ary totally bialgebra*, (resp. *infinitesimal n-ary weak totally bialgebra*, or *infinitesimal n-ary partially bialgebra*, or *infinitesimal n-ary alternate partially bialgebra*), is a quintuple $(A, \mu, \eta, \Delta, \varepsilon)$ in which (A, μ, η) is a unital n-ary totally associative algebra, (resp. n-ary weak totally algebra, or n-ary partially algebra, or n-ary alternate partially algebra), (A, Δ, ε) is a unital n-ary totally associative coalgebra, (resp. n-ary weak totally coalgebra, or n-ary partially coalgebra, or n-ary alternate partially coalgebra), satisfying the following compatibility condition:

$$\begin{aligned}\Delta \circ \mu(x_1, \dots, x_n) = &(id_A \otimes \cdots \otimes id_A \otimes ad_\mu^{(1)}(x_2, \dots, x_n))\Delta(x_1)\\ &+ (id_A \otimes \cdots \otimes ad_\mu^{(2)}(x_1, x_3, \dots, x_n) \otimes id_A)\Delta(x_2)\\ &\cdots + (id_A \otimes \cdots \otimes ad_\mu^{(i)}(x_1, \dots, \widehat{x_i}, \dots, x_n) \otimes \cdots \otimes id_A)\Delta(x_i)\\ &\cdots + (ad_\mu^{(n)}(x_1, \dots, x_{n-1}) \otimes id_A \otimes \cdots \otimes id_A)\Delta(x_n),\end{aligned}$$

where $x_1, \dots, x_n \in A$ and

$$ad_\mu^{(i)}(x_1, \dots, x_{n-1})(y) = \mu(x_1, \dots, x_{i-1}, y, x_i, \dots, x_{n-1}).$$

Theorem 16.11 *Let $(A, \mu, \eta, \Delta, \varepsilon)$ be a finite-dimensional unital infinitisimal n-ary bialgebra of associative type. Then $(A^*, \Delta^*, \varepsilon^*, \mu^*, \eta^*)$ is a unital infinitesimal n-ary bialgebra of associative type.*

Proof Let $x_1, \dots, x_n \in A$ and $\xi_1, \dots, \xi_n \in A^*$, then we have

$$\begin{aligned}\langle \xi_1 \otimes \cdots \otimes \xi_n, \Delta\mu(x_1, \dots, x_n)\rangle &= \langle \Delta^*(\xi_1, \dots, \xi_n), \mu(x_1, \dots, x_n)\rangle\\ &= \langle \mu^*\Delta^*(\xi_1, \dots, \xi_n), x_1 \otimes \cdots \otimes x_n\rangle.\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\langle \xi_1 \otimes \cdots \otimes \xi_n, (id_A \otimes \cdots \otimes ad_\mu^{(i)}(x_1, \dots, \widehat{x_i}, \dots, x_n) \otimes \cdots \otimes id_A)\Delta(x_i)\rangle\\ &= \langle (id_{A^*} \otimes \cdots \otimes ad_{\mu^*}^{(n-i)}(\xi_1, \dots, \widehat{\xi_{n-i}}, \dots, \xi_n) \otimes \cdots \otimes id_{A^*})\Delta^*(\xi_{n-i}), x_1 \otimes \cdots \otimes x_n\rangle.\end{aligned}$$

□

Theorem 16.12 *Let (A, μ, Δ) be an infinitesimal bialgebra, then $(A, \widetilde{\mu}, \widetilde{\Delta})$ is an infinitesimal ternary totally hom-bialgebra if and only if*

$$\begin{aligned}&(id_A \otimes \widetilde{\mu} \otimes id_A)\tau_{12}\tau_{45}(id_A \otimes \widetilde{\Delta} \otimes id_A)\\ =&(\mu \otimes \mu \otimes id_A)(id_A \otimes \Delta \otimes \Delta) + (id_A \otimes \widetilde{\mu} \otimes id_A)(\Delta \otimes id_A \otimes \Delta)\\ +&(\mu \otimes id_A \otimes \mu)(id_A \otimes \widetilde{\Delta} \otimes id_A) + (id_A \otimes \mu \otimes \mu)(\Delta \otimes \Delta \otimes id_A). \qquad (16.31)\end{aligned}$$

Proof Let (A, μ, Δ) be an infinitesimal bialgebra. Then, the compatibility condition (16.10) can be written as $\Delta\mu = (\mu \otimes id_A)(id_A \otimes \Delta) + (id_A \otimes \mu)(\Delta \otimes id_A)$. The compatibility condition for an infinitesimal ternary totally hom-bialgebra (A, μ, Δ) can be written as

$$\begin{aligned}\Delta\mu = (\mu \otimes id_A^{\otimes 2})(id_A^{\otimes 2} \otimes \Delta) + (id_A \otimes \mu \otimes id_A)\tau_{12}\tau_{45}(id_A \otimes \Delta \otimes id_A)\\ + (id_A^{\otimes 2} \otimes \mu)(\Delta \otimes id_A^{\otimes 2}).\end{aligned}$$

Thus,

$$\begin{aligned}\widetilde{\Delta}\widetilde{\mu} &= (\Delta \otimes id_A)\Delta\mu(\mu \otimes id_A)\\ &= (\Delta \otimes id_A)(\mu \otimes id_A)(id_A \otimes \Delta)(\mu \otimes id_A)\\ &+ (\Delta \otimes id_A)(id_A \otimes \mu)(\Delta \otimes id_A)(\mu \otimes id_A)\\ &= (\Delta\mu \otimes id_A)(\mu \otimes id_A^{\otimes 2})(id_A^{\otimes 2} \otimes \Delta) + (id_A^{\otimes 2} \otimes \mu)(\Delta \otimes id_A^{\otimes 2})(\Delta\mu \otimes id_A)\\ &= (\mu \otimes id_A^{\otimes 2})(id_A \otimes \Delta \otimes id_A)(\mu \otimes id_A^{\otimes 2})(id_A^{\otimes 2} \otimes \Delta)\\ &+ (id_A \otimes \mu \otimes id_A)(\Delta \otimes id_A^{\otimes 2})(\mu \otimes id_A^{\otimes 2})(id_A^{\otimes 2} \otimes \Delta)\\ &+ (id_A^{\otimes 2} \otimes \mu)(\Delta \otimes id_A^{\otimes 2})(\mu \otimes id_A^{\otimes 2})(id_A \otimes \Delta \otimes id_A)\\ &+ (id_A^{\otimes 2} \otimes \mu)(\Delta \otimes id_A^{\otimes 2})(id_A \otimes \mu \otimes id_A)(\Delta \otimes id_A^{\otimes 2})\\ &= (\mu \otimes id_A^{\otimes 2})(\mu \otimes id_A^{\otimes 3})(id_A^{\otimes 2} \otimes \Delta \otimes id_A)(id_A^{\otimes 2} \otimes \Delta)\\ &+ (id_A \otimes \mu \otimes id_A)(\Delta\mu \otimes id_A^{\otimes 2})(id_A^{\otimes 2} \otimes \Delta)\\ &+ (id_A^{\otimes 2} \otimes \mu)(\Delta\mu \otimes id_A^{\otimes 2})(id_A \otimes \Delta \otimes id_A)\\ &+ (id_A^{\otimes 2} \otimes \mu)(id_A^{\otimes 2} \otimes \mu \otimes id_A)(\Delta \otimes id_A^{\otimes 3})(\Delta \otimes id_A^{\otimes 2})\\ &= (\widetilde{\mu} \otimes id_A^{\otimes 2})(id_A^{\otimes 2} \otimes \widetilde{\Delta})\\ &+ (id_A \otimes \mu \otimes id_A)(\Delta\mu \otimes id_A^{\otimes 2})(id_A^{\otimes 2} \otimes \Delta)\\ &+ (id_A^{\otimes 2} \otimes \mu)(\Delta\mu \otimes id_A^{\otimes 2})(id_A \otimes \Delta \otimes id_A)\\ &+ (id_A^{\otimes 2} \otimes \widetilde{\mu})(\widetilde{\Delta} \otimes id_A^{\otimes 2})\\ &= (\widetilde{\mu} \otimes id_A^{\otimes 2})(id_A^{\otimes 2} \otimes \widetilde{\Delta}) + (id_A^{\otimes 2} \otimes \widetilde{\mu})(\widetilde{\Delta} \otimes id_A^{\otimes 2})\\ &+ (id_A \otimes \mu \otimes id_A)(\mu \otimes id_A^{\otimes 3})(id_A \otimes \Delta \otimes id_A^{\otimes 2})(id_A^{\otimes 2} \otimes \Delta)\\ &+ (id_A \otimes \mu \otimes id_A)(id_A \otimes \mu \otimes id_A^{\otimes 2})(\Delta \otimes id_A^{\otimes 3})(id_A^{\otimes 2} \otimes \Delta)\\ &+ (id_A^{\otimes 2} \otimes \mu)(\mu \otimes id_A^{\otimes 3})(id_A \otimes \Delta \otimes id_A^{\otimes 2})(id_A \otimes \Delta \otimes id_A)\end{aligned}$$

$$
\begin{aligned}
&+ (id_A^{\otimes 2} \otimes \mu)(id_A \otimes \mu \otimes id_A^{\otimes 2})(\Delta \otimes id_A^{\otimes 3})(id_A \otimes \Delta \otimes id_A)\\
&= (\widetilde{\mu} \otimes id_A^{\otimes 2})(id_A^{\otimes 2} \otimes \widetilde{\Delta}) + (id_A^{\otimes 2} \otimes \widetilde{\mu})(\widetilde{\Delta} \otimes id_A^{\otimes 2})\\
&+ (\mu \otimes \mu \otimes id_A)(id_A \otimes \Delta \otimes \Delta) + (id_A \otimes \widetilde{\mu} \otimes id_A)(\Delta \otimes id_A \otimes \Delta)\\
&+ (\mu \otimes id_A \otimes \mu)(id_A \otimes \widetilde{\Delta} \otimes id_A) + (id_A \otimes \mu \otimes \mu)(\Delta \otimes \Delta \otimes id_A)\\
&= (\widetilde{\mu} \otimes id_A^{\otimes 2})(id_A^{\otimes 2} \otimes \widetilde{\Delta}) + (id_A^{\otimes 2} \otimes \widetilde{\mu})(\widetilde{\Delta} \otimes id_A^{\otimes 2})\\
&+ (id_A \otimes \widetilde{\mu} \otimes id_A)\tau_{12}\tau_{45}(id_A \otimes \widetilde{\Delta} \otimes id_A).
\end{aligned}
$$

□

Example 16.1 Let (A, μ, Δ) be an infinitesimal bialgebra with a basis $\{e_1, e_2\}$, given in [60], where $\mu : A \otimes A \to A$ and $\Delta : A \to A \otimes A$ are defined by

$$\mu(e_1 \otimes e_1) = e_1, \mu(e_i \otimes e_j) = 0, i, j = 1, 2, \quad (i, j) \neq (1, 1),$$

$$\Delta(e_1) = 0, \quad \Delta(e_2) = b_{22} e_2 \otimes e_2,$$

where b_{22} is a parameter in $\mathbb{K}$.
Using Theorem 16.12, we can construct an infinitesimal ternary totally associative bialgebra on A given by

$$
\begin{aligned}
&\widetilde{\mu} : A \otimes A \otimes A \to A, \quad \widetilde{\mu}(e_1 \otimes e_1 \otimes e_1) = e_1,\\
&\qquad\qquad\qquad\qquad\quad \widetilde{\mu}(e_i \otimes e_j \otimes e_k) = 0, i, j, k = 1, 2, \ (i, j, k) \neq (1, 1, 1),\\
&\widetilde{\Delta} : A \to A \otimes A \otimes A, \quad \widetilde{\Delta}(e_1) = 0, \quad \tilde{\Delta}(e_2) = b_{22}^2 e_2 \otimes e_2 \otimes e_2,
\end{aligned}
$$

if and only if (16.31) holds.

Example 16.2 We consider the 3-dimensional infinitisimal bialgebra given in [60] defined with respect to a basis $\{e_1, e_2, e_3\}$ by

$$
\begin{aligned}
&\mu(e_1 \otimes e_1) = e_1, \quad \mu(e_2 \otimes e_2) = \mu(e_2 \otimes e_3) = \mu(e_3 \otimes e_2) = \mu(e_3 \otimes e_3) = e_2 + e_3,\\
&\Delta(e_1) = 0, \quad \Delta(e_2) = -c_{22} e_2 \otimes e_2 - c_{23} e_2 \otimes e_3 - c_{32} e_3 \otimes e_2 - c_{33} e_3 \otimes e_3,\\
&\Delta(e_3) = c_{22} e_2 \otimes e_2 + c_{23} e_2 \otimes e_3 + c_{32} e_3 \otimes e_2 + c_{33} e_3 \otimes e_3,
\end{aligned}
$$

where $c_{22}, c_{23}, c_{32}, c_{33}$ are parameters in $\mathbb{K}$.

Then, according to Theorem 16.12, we obtain an infinitesimal ternary totally associative bialgebra defined by

$$
\begin{aligned}
&\widetilde{\mu} : A \otimes A \otimes A \to A, \quad \widetilde{\mu}(e_1 \otimes e_1 \otimes e_1) = e_1,\\
&\widetilde{\mu}(e_2 \otimes e_2 \otimes e_2) = \widetilde{\mu}(e_2 \otimes e_2 \otimes e_3) = \widetilde{\mu}(e_2 \otimes e_3 \otimes e_2)\\
&= \widetilde{\mu}(e_2 \otimes e_3 \otimes e_3) = \widetilde{\mu}(e_3 \otimes e_2 \otimes e_2)\\
&= \widetilde{\mu}(e_3 \otimes e_2 \otimes e_3) = \widetilde{\mu}(e_3 \otimes e_3 \otimes e_2) = \widetilde{\mu}(e_3 \otimes e_3 \otimes e_3) = 2(e_2 + e_3),\\
&\widetilde{\Delta} : A \to A \otimes A \otimes A, \qquad \widetilde{\Delta}(e_1) = 0,
\end{aligned}
$$

$$\widetilde{\Delta}(e_2) = (c_{22} - c_{32})(c_{22}e_2 \otimes e_2 \otimes e_2 + c_{23}e_2 \otimes e_3 \otimes e_2$$
$$+ c_{32}e_3 \otimes e_2 \otimes e_2 + c_{33}e_3 \otimes e_3 \otimes e_2) + (c_{23} - c_{33})(c_{22}e_2 \otimes e_2 \otimes e_3$$
$$+ c_{23}e_2 \otimes e_3 \otimes e_3 + c_{32}e_3 \otimes e_2 \otimes e_3 + c_{33}e_3 \otimes e_3 \otimes e_3),$$
$$\widetilde{\Delta}(e_3) = (c_{22} - c_{32})(-c_{22}e_2 \otimes e_2 \otimes e_2 - c_{23}e_2 \otimes e_3 \otimes e_2 - c_{32}e_3 \otimes e_2 \otimes e_2$$
$$- c_{33}e_3 \otimes e_3 \otimes e_2) + (c_{23} - c_{33})(-c_{22}e_2 \otimes e_2 \otimes e_3 - c_{23}e_2 \otimes e_3 \otimes e_3$$
$$- c_{32}e_3 \otimes e_2 \otimes e_3 - c_{33}e_3 \otimes e_3 \otimes e_3),$$

if and only if (16.31) holds.

16.5.2 From Infinitesimal Hom-Bialgebras to Ternary Infinitesimal Hom-Bialgebras

Definition 16.30 An *infinitesimal n-ary totally hom-bialgebra*, (resp. *infinitesimal n-ary weak totally hom-bialgebra*, or *infinitesimal n-ary partially hom-bialgebra*, or *infinitesimal n-ary alternate partially hom-bialgebra*), is a sextuple $(A, \mu, \eta, \Delta, \varepsilon, \alpha)$ in which (A, μ, η, α) is a unital n-ary totally hom-associative algebra, (resp. n-ary weak totally hom-associative algebra, or n-ary partially hom-associative algebra, or n-ary alternate partially hom-associative algebra), $(A, \Delta, \varepsilon, \alpha)$ is a unital n-ary totally hom-associative coalgebra, (resp. n-ary weak totally hom-associative coalgebra, or n-ary partially hom-associative coalgebra, or n-ary alternate partially hom-associative coalgebra), satisfying the following compatibility condition:

$$\Delta \circ \mu(x_1, \dots, x_n) = (\alpha_1 \otimes \cdots \otimes \alpha_{n-1} \otimes ad_\mu^{(1)}(\alpha_1(x_2), \dots, \alpha_{n-1}(x_n)))\Delta(x_1)$$
$$+ (\alpha_1 \otimes \cdots \otimes ad_\mu^{(2)}(\alpha_1(x_1), \alpha_2(x_3), \dots, \alpha_{n-1}(x_n)) \otimes \cdots \otimes \alpha_{n-1})\Delta(x_2)$$
$$\cdots + (ad_\mu^{(n)}(\alpha_1(x_1), \dots, \alpha_{n-1}(x_{n-1})) \otimes \alpha_1 \otimes \cdots \otimes \alpha_{n-1})\Delta(x_n), \tag{16.32}$$

where $x_1, \dots, x_n \in A$, and

$$ad_\mu^{(i)}(x_1, \dots, x_{n-1})(y) = \mu(x_1, \dots, x_{i-1}, y, x_i, \dots, x_{n-1}).$$

A morphism of infinitesimal n-ary hom-bialgebra is a linear map that commutes with the twisting maps, the multiplications, and the comultiplications.

Definition 16.31 An infinitesimal n-ary totally hom-bialgebra, (resp. infinitesimal n-ary weak totally hom-bialgebra, or infinitesimal n-ary partially hom-bialgebra, or infinitesimal n-ary alternate partially hom-bialgebra) (A, μ, Δ, α) is multiplicative if $(\alpha_i)_{1 \le i \le n-1}$ with $\alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = \alpha$ and satisfying

$$\alpha \circ \mu = \mu \circ \alpha^{\otimes n}, \Delta \circ \alpha = \alpha^{\otimes n} \circ \Delta,$$

$$\begin{aligned}\Delta \circ \mu(x_1, \ldots, x_n) = &(\alpha \otimes \cdots \otimes \alpha \otimes ad_\mu^{(1)}(\alpha(x_2), \ldots, \alpha(x_n)))\Delta(x_1)\\ &+ (\alpha \otimes \cdots \otimes ad_\mu^{(2)}(\alpha(x_1), \alpha(x_3) \otimes \alpha(x_n)) \otimes \cdots \otimes \alpha)\Delta(x_2)\\ &\cdots + (ad_\mu^{(n)}(\alpha(x_1), \ldots, \alpha(x_{n-1})) \otimes \alpha \otimes \cdots \otimes \alpha)\Delta(x_n),\end{aligned}$$

for any $x_1, \ldots, x_n \in A$.

Furthermore, if α is bijective then the infinitesimal n-ary totally hom-bialgebra, (resp. infinitesimal n-ary weak totally hom-bialgebra, or infinitesimal n-ary partially hom-bialgebra, or infinitesimal n-ary alternate partially hom-bialgebra) (A, μ, Δ, α) is called a regular infinitesimal n-ary totally hom-bialgebra, (resp. infinitesimal n-ary weak totally hom-bialgebra, or infinitesimal n-ary partially hom-bialgebra, or infinitesimal n-ary alternate partially hom-bialgebra).

In the case of multiplicative infinitesimal ternary totally associative hom-bialgebra (A, μ, Δ, α) can be written as:

$$\Delta\mu = (\mu \otimes \alpha^{\otimes 2})(\alpha^{\otimes 2} \otimes \Delta) + (\alpha \otimes \mu \otimes \alpha)\tau_{12}\tau_{45}(\alpha \otimes \Delta \otimes \alpha) + (\alpha^{\otimes 2} \otimes \mu)(\Delta \otimes \alpha^{\otimes 2}).$$

Example 16.3 1) Infinitesimal n-ary bialgebra is recovered if $\alpha_1 = \cdots = \alpha_{n-1} = id$.

2) An n-ary hom-associative algebra (A, μ, α) becomes an infinitesimal n-ary hom-bialgebra when equipped with the trivial comultiplication $\Delta = 0$. Likewise, an n-ary hom-coassociative coalgebra (A, Δ, α) becomes an infinitesimal n-ary hom-bialgebra when equipped with the trivial multiplication $\mu = 0$.

Proposition 16.6 *Let (A, μ, Δ, α) be a multiplicative infinitesimal n-ary hom-bialgebra, (resp. infinitesimal n-ary weak totally hom-bialgebra, or infinitesimal n-ary partially hom-bialgebra, or infinitesimal n-ary alternate partially hom-bialgebra). Then so are $(A, -\mu, \Delta, \alpha)$, $(A, \mu, -\Delta, \alpha)$.*

Proof By a direct computation, we obtain the results. □

Theorem 16.13 *Let (A, μ, Δ, α) be a multiplicative finite-dimensional infinitesimal n-ary hom-bialgebra, (resp. infinitesimal n-ary weak totally hom-bialgebra, or infinitesimal n-ary partially hom-bi-algebra, or infinitesimal n-ary alternate partially hom-bialgebra). Then so is $(A^*, \Delta^*, \mu^*, \alpha^*)$.*

Proof The same approach approved in the proof of Theorem 16.11. □

The following result shows that an infinitesimal n-ary hom-bialgebra deforms into an another infinitesimal n-ary hom-bialgebra along any self-morphism. It gives an efficient method for constructing Hom-Lie bialgebras.

Proposition 16.7 *Let (A, μ, Δ, α) be an infinitesimal n-ary hom-bialgebra and $\beta : A \to A$ be a morphism. Then $A_\beta = \big(A, \mu_\beta, \Delta_\beta, \alpha \circ \beta\big)$ is an infinitesimal n-ary hom-bialgebra, where $\mu_\beta = \beta \circ \mu$ and $\Delta_\beta = \Delta \circ \beta$.*

Proof One needs only to show that the compatibility condition (16.32) holds. Indeed,

$$\Delta_\beta \circ \mu_\beta(x_1, \dots, x_n) = \Delta \circ \beta \circ \beta \circ \mu = (\beta^2)^{\otimes n} \circ \Delta \circ \mu(x_1, \dots, x_n)$$

On the other hand, using the fact that $\Delta(x_i) = \sum_{(x_i)} x_i^{(1)} \otimes \cdots \otimes x_i^{(n)}$, for $x_1, \dots, x_n \in A$

$$\begin{aligned}
&(\beta \circ \alpha_1 \otimes \cdots \otimes \beta \circ \alpha_{n-1} \otimes ad^{(1)}_{\mu_\beta}(\beta \circ \alpha_1(x_2), \dots, \beta \circ \alpha_{n-1}(x_n)))\Delta_\beta(x_1)\\
&= (\beta \circ \alpha_1 \circ \beta(x_1^{(1)}) \otimes \cdots \otimes \beta \circ \alpha_{n-1} \circ \beta(x_1^{(n-1)}) \otimes \mu_\beta(\beta(x_1^{(n)}),\\
&\qquad\qquad \beta \circ \alpha_1(x_2), \dots, \beta \circ \alpha_{n-1}(x_n)))\\
&= (\beta^2 \circ \alpha_1(x_1^{(1)}) \otimes \cdots \otimes \beta^2 \circ \alpha_{n-1}(x_1^{(n-1)}) \otimes \mu(\beta^2(x_1^{(n)}),\\
&\qquad\qquad \beta^2 \circ \alpha_1(x_2), \dots, \beta^2 \circ \alpha_{n-1}(x_n)))\\
&= (\beta^2)^{\otimes n}(\alpha_1(x_1^{(1)}) \otimes \cdots \otimes \alpha_{n-1}(x_1^{(n-1)}) \otimes \mu(x_1^{(n)}, \alpha_1(x_2), \dots, \alpha_{n-1}(x_n)))\\
&= (\beta^2)^{\otimes n} \circ (\alpha_1 \otimes \cdots \otimes \alpha_{n-1} \otimes ad^{(1)}_\mu(\alpha_1(x_2), \dots, \alpha_{n-1}(x_n)))\Delta(x_1),\\
&(\beta \circ \alpha_1 \otimes \cdots \otimes ad^{(2)}_{\mu_\beta}(\beta \circ \alpha_1(x_1), \beta \circ \alpha_2(x_3), \dots, \beta \circ \alpha_{n-1}(x_n))\\
&\qquad\qquad \otimes \cdots \otimes \beta \circ \alpha_{n-1})\Delta(x_2)\\
&= (\beta \circ \alpha_1 \circ \beta(x_2^{(1)}) \otimes \cdots \otimes \mu_\beta(\beta \circ \alpha_1(x_1), \beta(x_2^{(2)}), \beta \circ \alpha_2(x_3), \dots, \beta \circ \alpha_{n-1}(x_n))\\
&\qquad\qquad \otimes \cdots \otimes \beta \circ \alpha_{n-1} \circ \beta(x_2^{(n)}))\\
&= (\beta^2 \circ \alpha_1(x_2^{(1)}) \otimes \cdots \otimes \mu(\beta^2 \circ \alpha_1(x_1), \beta^2(x_2^{(2)}), \beta^2 \circ \alpha_2(x_3), \dots, \beta^2 \circ \alpha_{n-1}(x_n))\\
&\qquad\qquad \otimes \cdots \otimes \beta^2 \circ \alpha_{n-1}(x_2^{(n)}))\\
&= (\beta^2)^{\otimes n}(\alpha_1(x_2^{(1)}) \otimes \cdots \otimes \mu(\alpha_1(x_1), x_2^{(2)}, \alpha_2(x_3), \dots, \alpha_{n-1}(x_n)) \otimes \cdots \otimes \alpha_{n-1}(x_2^{(n)}))\\
&= (\beta^2)^{\otimes n} \circ (\alpha_1 \otimes \cdots \otimes ad^{(2)}_\mu(\alpha_1(x_1), \alpha_2(x_3), \dots, \alpha_{n-1}(x_n)) \otimes \cdots \otimes \alpha_{n-1})\Delta(x_2),\\
&(ad^{(n)}_{\mu_\beta}(\beta \circ \alpha_1(x_1), \dots, \beta \circ \alpha_{n-1}(x_{n-1})) \otimes \beta \circ \alpha_1 \otimes \cdots \otimes \beta \circ \alpha_{n-1})\Delta_\beta(x_n)\\
&= (\mu_\beta(\beta \circ \alpha_1(x_1), \dots, \beta \circ \alpha_{n-1}(x_{n-1}), \beta(x_n^1)) \otimes \beta \circ \alpha_1 \circ \beta(x_n^{(2)})\\
&\qquad\qquad \otimes \cdots \otimes \beta \circ \alpha_{n-1} \circ \beta(x_n^{(n)}))\\
&= (\mu(\beta^2 \circ \alpha_1(x_1), \dots, \beta^2 \circ \alpha_{n-1}(x_{n-1}), \beta^2(x_n^1)) \otimes \beta^2 \circ \alpha_1(x_n^2)\\
&\qquad\qquad \otimes \cdots \otimes \beta^2 \circ \alpha_{n-1}(x_n^n))\\
&= (\beta^2)^{\otimes n}(\mu(\alpha_1(x_1), \dots, \alpha_{n-1}(x_{n-1}), x_n^1) \otimes \alpha_1(x_n^2) \otimes \cdots \otimes \alpha_{n-1}(x_n^n))\\
&= (\beta^2)^{\otimes n} \circ (ad^{(n)}_\mu(\alpha_1(x_1), \dots, \alpha_{n-1}(x_{n-1})) \otimes \alpha_1 \otimes \cdots \otimes \alpha_{n-1})\Delta(x_n).
\end{aligned}$$

□

Remark 16.15 Let (A, μ, Δ, α) be a regular multiplicative infinitesimal n-ary hom-bialgebra. Then $A_{\alpha^{-1}} = (A, \mu_{\alpha^{-1}}, \Delta_{\alpha^{-1}})$ is an infinitesimal n-ary bialgebra, where $\mu_{\alpha^{-1}} = \alpha^{-1} \circ \mu$ and $\Delta_{\alpha^{-1}} = \Delta \circ \alpha^{-1}$.

Theorem 16.14 *Let (A, μ, Δ, α) be an infinitesimal hom-bialgebra, then $(A, \widetilde{\mu}, \widetilde{\Delta}, \alpha^2)$ is an infinitesimal ternary totally associative hom- bialgebra if and only if*

$$
\begin{aligned}
&(\alpha^2 \otimes \widetilde{\mu} \otimes \alpha^2)\tau_{12}\tau_{45}(\alpha^2 \otimes \widetilde{\Delta} \otimes \alpha^2) \\
&= (\alpha\mu \otimes \alpha\mu \otimes \alpha^2)(\alpha^2 \otimes \Delta\alpha \otimes \Delta\alpha) + (\alpha^2 \otimes \widetilde{\mu} \otimes \alpha^2)(\Delta\alpha \otimes \alpha^2 \otimes \Delta\alpha) \\
&\quad + (\alpha\mu \otimes \alpha^2 \otimes \alpha\mu)(\alpha^2 \otimes \widetilde{\Delta} \otimes \alpha^2) + (\alpha^2 \otimes \alpha\mu \otimes \alpha\mu)(\Delta\alpha \otimes \Delta\alpha \otimes \alpha^2).
\end{aligned}
$$

Proof Let (A, μ, Δ, α) be an infinitesimal hom-bialgebra. Then

$$
\begin{aligned}
\widetilde{\Delta}\widetilde{\mu} &= (\Delta \otimes \alpha)\Delta\mu(\mu \otimes \alpha) \\
&= (\Delta \otimes \alpha)(\mu \otimes \alpha)(\alpha \otimes \Delta)(\mu \otimes \alpha) + (\Delta \otimes \alpha)(\alpha \otimes \mu)(\Delta \otimes \alpha)(\mu \otimes \alpha) \\
&= (\widetilde{\mu} \otimes (\alpha^{\otimes 2})^2)((\alpha^{\otimes 2})^2 \otimes \widetilde{\Delta}) \\
&\quad + (\alpha \otimes \mu \otimes \alpha)(\Delta\mu \otimes (\alpha^{\otimes 2})^2)(\alpha^{\otimes 2} \otimes \Delta) \\
&\quad + (\alpha^{\otimes 2} \otimes \mu)(\Delta\mu \otimes (\alpha^{\otimes 2})^2)(\alpha \otimes \Delta \otimes \alpha) \\
&\quad + ((\alpha^{\otimes 2})^2 \otimes \widetilde{\mu})(\widetilde{\Delta} \otimes (\alpha^{\otimes 2})^2) \\
&= (\widetilde{\mu} \otimes (\alpha^{\otimes 2})^2)((\alpha^{\otimes 2})^2 \otimes \widetilde{\Delta}) + ((\alpha^{\otimes 2})^2 \otimes \widetilde{\mu})(\widetilde{\Delta} \otimes (\alpha^{\otimes 2})^2) \\
&\quad + (\alpha\mu \otimes \alpha\mu \otimes \alpha^2)(\alpha^2 \otimes \Delta\alpha \otimes \Delta\alpha) + (\alpha^2 \otimes \widetilde{\mu} \otimes \alpha^2)(\Delta\alpha \otimes \alpha^2 \otimes \Delta\alpha) \\
&\quad + (\alpha\mu \otimes \alpha^2 \otimes \alpha\mu)(\alpha^2 \otimes \widetilde{\Delta} \otimes \alpha^2) + (\alpha^2 \otimes \alpha\mu \otimes \alpha\mu)(\Delta\alpha \otimes \Delta\alpha \otimes \alpha^2) \\
&= (\widetilde{\mu} \otimes (\alpha^{\otimes 2})^2)((\alpha^{\otimes 2})^2 \otimes \widetilde{\Delta}) + ((\alpha^{\otimes 2})^2 \otimes \widetilde{\mu})(\widetilde{\Delta} \otimes (\alpha^{\otimes 2})^2) \\
&\quad + (\alpha^2 \otimes \widetilde{\mu} \otimes \alpha^2)\tau_{12}\tau_{45}(\alpha^2 \otimes \widetilde{\Delta} \otimes \alpha^2).
\end{aligned}
$$

□

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Chapter 17
Network Rewriting Utility Description

Lars Hellström

Abstract This chapter describes the author's computer program for doing network rewriting calculations, in its capacity as a tool used for scientific exploration—more precisely to systematically discover non-obvious consequences of the axioms for various algebraic structures. In particular this program can cope with algebraic structures, such as bi- and Hopf algebras, that mix classical operations with co-operations.

Keywords Network rewriting · Hopf algebras · Type II hom-associativity

MSC2020 Classification 68W30 · 68V05 · 18M30 · 18-04

17.1 Introduction

Due to the asynchronous publication schedule, the results presented in my formal talk at the SPAS 2019 conference did some months later appear in my chapter [7] of the SPAS 2017 proceedings, so then what more was there on which I could report for 2019? Well, conferences consist not only of scheduled talks, but also of informal discussions between and after talks, that in some cases led to me promptly producing several dozen PDF pages of diagrammatic calculations based on axiom sets raised by other participants. *Where did this come from?* Obviously I must have a program which produces such calculations, and since said utility has not previously been described in the scientific literature, this might be as good a place as any to do so.

In spirit, the program in question is quite similar to the seminal Knuth–Bendix completion utility [13], but with the important difference that it works with *networks* (a kind of Directed Acyclic Graph) rather than treelike *terms* as the objects of rewriting. This permits exploring many novel algebraic theories, for example that of bi- and Hopf algebras, that defy even being formulated within the language of classical

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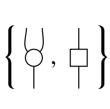
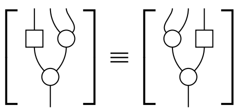
term rewriting due to the presence of co-operations such as Δ and ε. On the one hand this transition from trees to DAGs is a radical proposal that requires rebuilding from scratch most of the mathematical formula language, but on the other hand the overall structure of the computation is the familiar completion of a rewrite system, and the abstract theory of that remains applicable.

Without already getting into the technicalities, the purpose of the program can be characterised as *theory exploration*. It starts from three pieces of information:

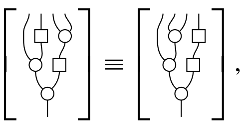
(i) a *signature* listing the operations (and co-operations) in the theory—for example a hom-algebra has one binary operation called 'multiplication' and one unary operation known as the 'hom'—which determines the language of expressions in the theory to explore,
(ii) the *axioms* of the theory, in the form of a set of given equalities $l = r$ between expressions in the theory,
(iii) an *ordering* of the expressions in the theory, which is used to orient equalities into rewrite rules—the smaller side in an equality is considered to be "simpler" as in 'more simplified'.

One intermediate operation the program can use this information for is to *reduce* arbitrary expressions in the theory by rewriting them to a "maximally simplified" *normal form*, by applying known equalities of expressions. A higher level operation is to seek new *nontrivial* equalities by constructing expressions that can be simplified in several different ways (*ambiguities*), and then checking whether both lead to the same normal form; if not, the program has discovered a lemma which it adds to its database of known equalities in the theory under consideration. New equalities give rise to new reductions and potentially new ambiguities, which in turn may produce new lemmas; this *completion* process need not terminate. Therefore the user would typically start the program, let it run for a while, and then inspect what lemmas have been discovered; there are several interfaces for this (Sect. 17.5.3). The results can be exported, likewise in several forms. The completion process can be stopped and resumed at a later time, should the user so desire.

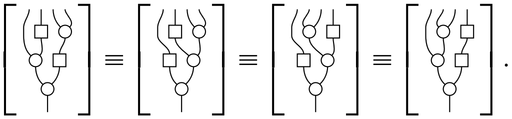
As a concrete example, if one wishes to explore the theory of hom-associative algebras using this program then one would feed it

the signature $\Omega =$  with axiom 

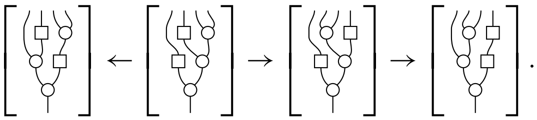
where however it should be observed that the exact representation of these things is a nontrivial matter (which we shall return to in Sects. 17.4.1 and 17.5.2). The first lemma proved is

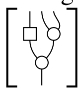
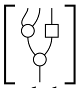


and the proof is merely



The program discovers this by first constructing the second of these steps as the site of an ambiguity—an expression which can be rewritten in two different ways—and then reducing either to a normal form:



The two outermost steps here are not equal, but they are equivalent modulo the given axiom despite both being normal forms with respect to the rewrite rule  → , so apparently their equivalence is a nontrivial consequence of the axiom, and thus worth turning into a lemma, which in turn gives rise to a new rewrite rule, that on the one hand changes which expressions are normal forms and on the other hand gets involved in additional ambiguities. Thus the cycle goes on.

Mathematically, the hardness of the results produced by the program varies depending on what kind of results these are. The individual lemmas are quite solid, as the program records all steps taken in deriving them and can convert these into an explicit proof. Results on the quotient by the given axioms typically require more manual work to be rigorously established; basic rewriting theory conditions such conclusions upon having a confluent rewrite system, which one would only know after running the completion procedure to termination, however for fundamental logical reasons (it would decide the Halting Problem) that is not always possible, and in practice it also can be infeasible (finishing may take way more time than we are prepared to wait, memory requirements may exceed what is available). None the less it can be possible to draw conclusions from incomplete calculations, for example if they suggest a verifiable conjecture about what the completed rewrite system would look like [6], or if it can be shown that sufficiently small normal forms will remain so forever since any lemmas which still remain to be discovered are all too large to have an effect on them [9].

Since this is a *tool description* rather than a traditional mathematics paper, its contribution to science lies not in theorems proved or results formally stated—there are technically some in the next section, although only as part of a greater example—but in discussing methods, approaches, and trade-offs. The primary audience for these discussions is expected to be another researcher seeking to reproduce this work—in whole or in part, for the same underlying model or for a different one, to the same end or to a different one—because *reproducibility is a scientific virtue* and supporting it is every researcher's responsibility. Concretely Sects. 17.4 and 17.5 contain a number of paragraphs labelled 'Topic' that each discusses one specific issue, aspect, or idea that went into the implementation of the completion utility. Having them discussed explicitly and separately provides for accurate citing and facilitates reviewing the greater body of work on network rewriting to which this chapter belongs.

This chapter is *not* a user's manual for the rewriting utility, even if it at times mentions details on how a user might accomplish various operations. The utility is not yet in such a polished form that one could hope to make use of it without knowledge of its internal composition. This chapter may however suffice for the more restricted purpose of decoding the records of computation created by the utility, which would be relevant if one wishes to archive such records, for example to satisfy requirements from funding agencies on Open Science Data.

Section 17.2 shows in more detail the SPAS conference problem alluded to above. Section 17.3 provides a brief characterisation of the completion utility as a software artefact. Section 17.4 discusses generic operations on networks whereas Sect. 17.5 deals specifically with combining those operations into a completion utility. Finally Sect. 17.6 explains where the completion utility source code can be found.

17.2 Example: The Conference Problem

The *Type II hom-associativity* identity can be written

$$\mu\Bigl(x, \alpha\bigl(\mu(y, z)\bigr)\Bigr) = \mu\Bigl(\alpha\bigl(\mu(x, y)\bigr), z\Bigr) \tag{17.1}$$

where μ denotes the multiplication operation and α denotes the hom (also known as twisting) map. Alexis Langlois-Rémillard was interested in combining this with two further axioms: that α is an involution

$$\alpha\bigl(\alpha(x)\bigr) = x \tag{17.2}$$

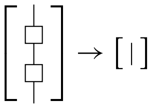
and that α is an antihomomorphism of the algebra

$$\alpha\bigl(\mu(x, y)\bigr) = \mu\bigl(\alpha(y), \alpha(x)\bigr). \tag{17.3}$$

When presented with such an unfamiliar set of axioms, it is quite a tough problem to imagine what an algebra satisfying them will be like; analogies sometimes work, and sometimes fail spectacularly. Examples are useful (and did often inspire the choice of axioms in the first place), but there is no guarantee that one's initial examples will be typical for the class of algebras satisfying a particular axiom system. A common research strategy is to just attempt to prove any property one can think of; along the way one usually encounters of something that holds, which may give insights into the general situation, but it is quite labour intensive.

In my case I happened to have another method that I could try, namely to explore this theory using my completion utility; arguably network rewriting is overkill here as there are no co-operations in sight, but computers are fast so the extra generality in the model does not hurt. What follows is (an excerpt from) a dump of the database of rules that the utility produced in exploring the theory implied by axioms (17.1), (17.2), and (17.3). The lemmas, proofs, and axioms below are all typeset completely from LaTeX code that the program generated, but comments have been added between them.

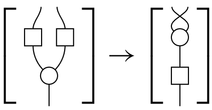
Axiom 17.1 (Involutive)



i.e., $\alpha(\alpha(x)) = x$.

By the processing order heuristic applied (see Topic 16), the simplest axiom is (17.2) about α being an involution, so that becomes the first one processed and the first one to give rise to a rewrite rule: double α elimination.

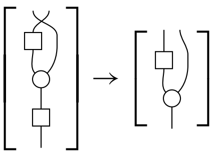
Axiom 17.2 (Anti-homomorphism)



i.e., $\mu(\alpha(x), \alpha(y)) = \alpha(\mu(y, x))$.

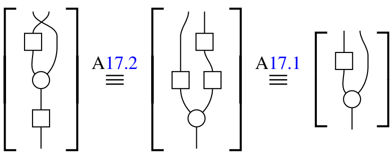
Likewise (17.3) is regarded as slightly smaller than (17.1), by virtue of its right hand side having one operation less, so that is what gets turned into a rule next.

Lemma 17.1

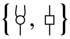


i.e., $\alpha(\mu(\alpha(y), x)) = \mu(\alpha(x), y)$.

Proof

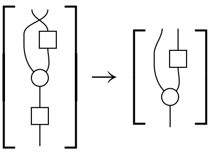


This is our first result: α both outside and inside a μ can be combined into a single α inside, if one also switches left and right factor. No great surprise, once one has seen the proof, but a consequence the program discovers on its own.

What are all these diagrams, though, from a formal mathematical perspective? Technically they are elements in the free PROP $\mathcal{P}$ generated by the hom-algebra signature  . We are interested in discovering which elements of this PROP are congruent ($\equiv$) modulo the given axioms. To that end, we are deriving rewrite rules $a \to b$ where a and b are congruent elements of the free PROP—this congruence being what needs proving—and direct the rewrite arrow towards the side that compares as the simpler of the two.

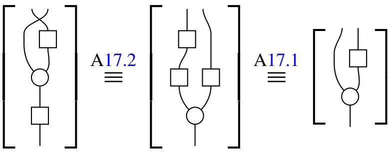
Because the signature only contains ordinary operations (having coarity 1), this free PROP $\mathcal{P}$ is the same thing as the free operad generated by those operation symbols. Moreover this makes it straightforward to additionally present the stated congruences as traditional algebraic identities, which is what the 'i.e.' clauses in the axioms and lemmas are doing. Technically, the formulae in these clauses are identities in algebras of the operad $\mathcal{P}/{\equiv}$; including these formulae here is purely a matter of presentation, as a service to readers who are more accustomed to reasoning at the algebra level.

Lemma 17.2



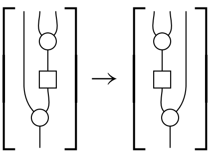
i.e., $\alpha(\mu(y, \alpha(x))) = \mu(x, \alpha(y))$.

Proof



Considering what Lemma 17.1 looks like, it is only to be expected that it has this sibling where α instead is applied to the right factor. Only after proving this does the program move on to (17.1), which a human mathematician probably would regard as the main identity in the system.

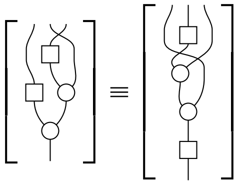
Axiom 17.3 (Type II hom-associative)



i.e., $\mu(x, \alpha(\mu(y, z))) = \mu(\alpha(\mu(x, y)), z)$.

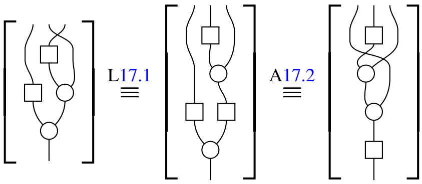
The third lemma proved does however not make use of that axiom; it is merely another consequence of just Axioms 17.1 and 17.2 that happened to be more complicated than the hom-associativity axiom.

Lemma 17.3



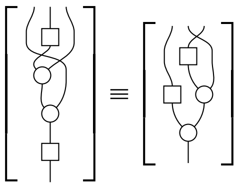
i.e., $\mu(\alpha(x), \mu(\alpha(z), y)) = \alpha(\mu(\mu(\alpha(y), z), x))$.

Proof



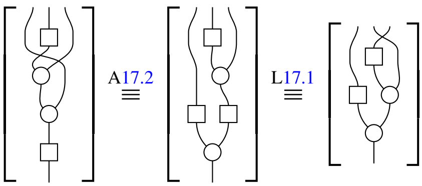
We see from the labels ‘L17.1’ and ‘A17.2’ that this uses Lemma 17.1 and Axiom 17.2, and Lemma 17.1 is in turn proved by combining Axioms 17.2 and 17.1.

Lemma 17.4



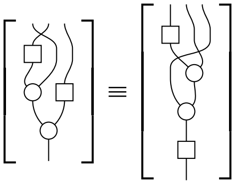
i.e., $\alpha(\mu(\mu(\alpha(y), z), x)) = \mu(\alpha(x), \mu(\alpha(z), y))$.

Proof



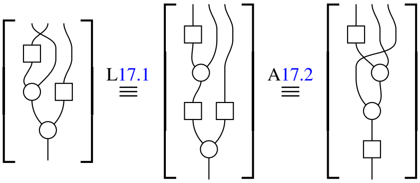
The astute reader notices that Lemma 17.4 is just Lemma 17.3 with left and right hand sides exchanged. The reason for that is that this list really is just a straight *dump* of what is in the database of rules computed by the program. As discussed in Sect. 17.5.1, if the user-supplied monomial order fails to orient a congruence—which we here can tell from Lemma 17.4 using ≡ rather than the directed → between the sides—then that congruence is given multiple entries in the database: one for each possible orientation, so that proofs may still make use of it in either direction. The raw dump contains both, but this excerpt rather constitutes a manual selection from the dump.

Lemma 17.5



i.e., $\mu(\mu(\alpha(y), x), \alpha(z)) = \alpha(\mu(z, \mu(\alpha(x), y)))$.

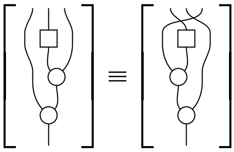
Proof



Lemma 17.5 resembles Lemma 17.3, particularly in view of what the proof employs, but it is not the same since the α's are in different positions in the expression. That is probably easier to see in the network than in the traditional formula, however.

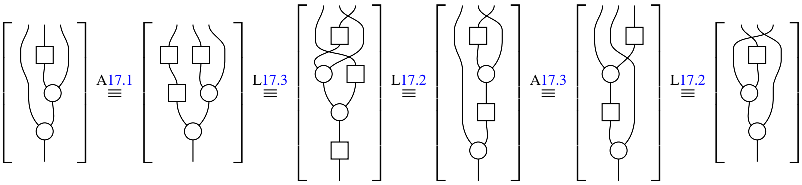
The dump then continues to prove another couple of siblings of these lemmas, but we shall skip ahead to the first that relies upon Axiom 17.3.

Lemma 17.6



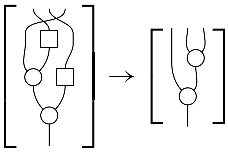
i.e., $\mu(x, \mu(\alpha(y), z)) = \mu(\mu(z, \alpha(x)), y)$.

Proof



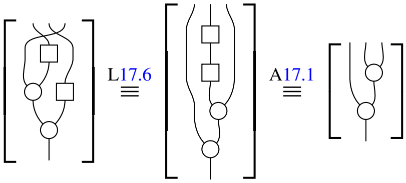
At first glance that congruence looks similar to Axiom 17.3, but here the α is inside both μ, and there is a cyclic permutation of the factors. Definitely an interesting result, and with a less trivial proof than the previous lemmas!

Lemma 17.7



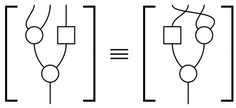
i.e., $\mu(\mu(z, \alpha(x)), \alpha(y)) = \mu(x, \mu(y, z))$.

Proof



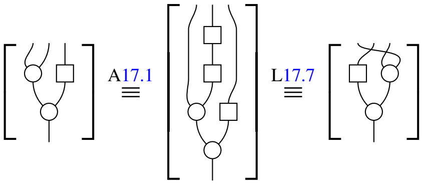
Our final lemma is almost the ordinary Type I hom-associativity, but again with this cyclic permutation of the factors.

Lemma 17.8



i.e., $\mu(\mu(x, y), \alpha(z)) = \mu(\alpha(y), \mu(z, x))$.

Proof

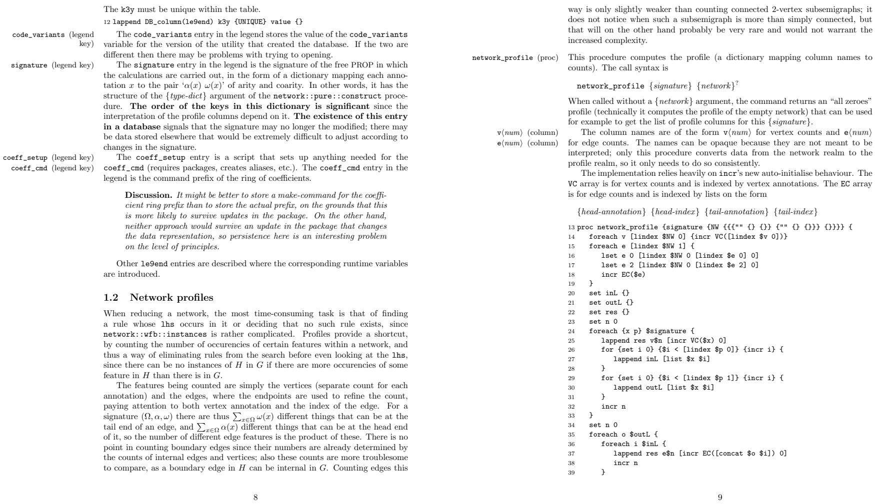


One may of course continue further, and indeed this exploration did; the records show only four seconds elapsed between adding Axiom 17.1 and deriving Lemma 17.8, so stopping at this particular point requires rather fast reflexes. Though the dump is not material for publication, one may certainly read on in it, looking for other interesting identities to shine a light on this class of algebras. But there is also reason for slightly adjusting the parameters of this exploration, and then having another go from scratch.

The fact that several interesting congruences, in particular Lemmas 17.6 and 17.8, end up unorientable suggest that we should come up with a better criterion for orienting them, as that will increase efficiency of the exploration and reduce the clutter produced. Mathematically defining appropriate orders is however not a trivial problem, and in this case one that seems to be compounded by the tendency of these congruences to permute the factors. We shall not delve into that issue here, but close this little theory exploration with the advice that the ordering of one's expressions is a matter that deserves our attention.

17.3 Program Composition

By the trivial lines-of-code metric, the completion utility in 2015 consisted of 11931 lines of Tcl code, but that is perhaps most useful as a datapoint if comparing several different pieces of software. For a party actually interested in *reading* the program,



The k3y must be unique within the table.

```
lappend DB_column(le9end) k3y {UNIQUE} va1ue {}
```

code_variants (legend key) The code_variants entry in the legend stores the value of the code_variants variable for the version of the utility that created the database. If the two are different then there may be problems with trying to opening.

signature (legend key) The signature entry in the legend is the signature of the free PROP in which the calculations are carried out, in the form of a dictionary mapping each annotation x to the pair '$\alpha(x)\ \omega(x)$' of arity and coarity. In other words, it has the structure of the {*type-dict*} argument of the network::pure::construct procedure. **The order of the keys in this dictionary is significant** since the interpretation of the profile columns depend on it. **The existence of this entry in a database** signals that the signature may no longer the modified; there may be data stored elsewhere that would be extremely difficult to adjust according to changes in the signature.

coeff_setup (legend key)
coeff_cmd (legend key) The coeff_setup entry is a script that sets up anything needed for the coeff_cmd (requires packages, creates aliases, etc.). The coeff_cmd entry in the legend is the command prefix of the ring of coefficients.

> **Discussion.** *It might be better to store a make-command for the coefficient ring prefix than to store the actual prefix, on the grounds that this is more likely to survive updates in the package. On the other hand, neither approach would survive an update in the package that changes the data representation, so persistence here is an interesting problem on the level of principles.*

Other le9end entries are described where the corresponding runtime variables are introduced.

1.2 Network profiles

When reducing a network, the most time-consuming task is that of finding a rule whose lhs occurs in it or deciding that no such rule exists, since network::wfb::instances is rather complicated. Profiles provide a shortcut, by counting the number of occurencies of certain features within a network, and thus a way of eliminating rules from the search before even looking at the lhs, since there can be no instances of H in G if there are more occurencies of some feature in H than there is in G.

The features being counted are simply the vertices (separate count for each annotation) and the edges, where the endpoints are used to refine the count, paying attention to both vertex annotation and the index of the edge. For a signature (Ω, α, ω) there are thus $\sum_{x\in\Omega}\omega(x)$ different things that can be at the tail end of an edge, and $\sum_{x\in\Omega}\alpha(x)$ different things that can be at the head end of it, so the number of different edge features is the product of these. There is no point in counting boundary edges since their numbers are already determined by the counts of internal edges and vertices; also these counts are more troublesome to compare, as a boundary edge in H can be internal in G. Counting edges this

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way is only slightly weaker than counting connected 2-vertex subsemigraphs; it does not notice when such a subsemigraph is more than simply connected, but that will on the other hand probably be very rare and would not warrant the increased complexity.

network_profile (proc) This procedure computes the profile (a dictionary mapping column names to counts). The call syntax is

network_profile {*signature*} {*network*}$^?$

When called without a {*network*} argument, the command returns an "all zeroes" profile (technically it computes the profile of the empty network) that can be used for example to get the list of profile columns for this {*signature*}.

v⟨*num*⟩ (column)
e⟨*num*⟩ (column) The column names are of the form v⟨*num*⟩ for vertex counts and e⟨*num*⟩ for edge counts. The names can be opaque because they are not meant to be interpreted; only this procedure converts data from the network realm to the profile realm, so it only needs to do so consistently.

The implementation relies heavily on incr's new auto-initialise behaviour. The VC array is for vertex counts and is indexed by vertex annotations. The EC array is for edge counts and is indexed by lists on the form

{*head-annotation*} {*head-index*} {*tail-annotation*} {*tail-index*}

```
proc network_profile {signature {NW {{{"" {} {}} {"" {} {}}} {}}}} {
   foreach v [lindex $NW 0] {incr VC([lindex $v 0])}
   foreach e [lindex $NW 1] {
      lset e 0 [lindex $NW 0 [lindex $e 0] 0]
      lset e 2 [lindex $NW 0 [lindex $e 2] 0]
      incr EC($e)
   }
   set inL {}
   set outL {}
   set res {}
   set n 0
   foreach {x p} $signature {
      lappend res v$n [incr VC($x) 0]
      for {set i 0} {$i < [lindex $p 0]} {incr i} {
         lappend inL [list $x $i]
      }
      for {set i 0} {$i < [lindex $p 1]} {incr i} {
         lappend outL [list $x $i]
      }
      incr n
   }
   set n 0
   foreach o $outL {
      foreach i $inL {
         lappend res e$n [incr EC([concat $o $i]) 0]
         incr n
      }
```

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Fig. 17.1 Pages 8–9 of cmplutil2.dtx

it is more relevant to know that it is a Literate [12] program written in the doc/docstrip tradition [15] (like the LaTeX kernel and most LaTeX packages), and that the sources therefore can be typeset straight off by LaTeX. At the time of writing, doing that to the two main source files produces documents of 212 and 149 pages, respectively; an excerpt is shown in Fig. 17.1. The above 11931 lines of code figure also includes code from various supporting packages of more generic utility. The typeset literate sources here could add another $154 + 36 + 83 + 41 = 314$ or so pages to the total (and there are additional bodies of code under development that could eventually become additional supporting packages), but understanding the workings of the completion utility would for the most part not require understanding the workings of those supporting packages. Conversely, rigorously understanding the operation of the utility as a whole probably does require familiarity with the underlying mathematical theory of network rewriting, which as presented in [3] adds another 188 pages to the reading list; that paper began as the opening section of the completion utility literate sources, but was split off since it grew a bit beyond the initial expectations.

17.3.1 Development History

The completion utility originated in work carried out during my postdoc year at the Mittag-Leffler institute (2003–04, Noncommutative Geometry), specifically the realisation that the *Network Algebra* of Ştefănescu [19] could be used as a formalism for expressions in Hopf algebras. While in hindsight not that spectacular as a find—shorthand diagrams had been known for decades to specialists in the field, even if often downplayed as "not really formulae"—the high degree of formality of Ştefănescu's networks made it clear that these could serve as building blocks for a universal-algebraic study of Hopf algebras (and other algebraic structures with co-operations). Would it for example be possible to start with just the axioms and by pure calculations (as opposed to by peeking at the group theory proof that $(ab)^{-1} = b^{-1}a^{-1}$) discover the fact that the antipode is an antihomomorphism? Initial attempts were hindered by the detail that networks, although excellent for showing the structure of expressions, are not as compact as traditional formulae; sometimes a mere four steps would fill up an entire sheet of paper! What to do if one does not want to be on a constant hunt for more area to write on? Erasable media? Well, the screen of a computer can be redrawn several times a second, so that's a neat way forward.

Version 0 of the program was written over the summer of 2004, and had at best an ambiguous aim. On the one hand it did perform completion of e.g. the Hopf algebra axioms, but on the other hand it is perhaps more fair to describe the rewrite operations on networks as its primary accomplishment, and the completion merely as a way of generating a stream of tasks on which to test these more basic operations. Notably the program ran at the speed of one rewrite step per second and would display each intermediate result as it did so, to give the user a chance to supervise what was being done and verify its correctness.

Operations that were implemented included searching for instances of one network as a subnetwork of another, replacing such an instance by a different network, testing whether two networks are equal/isomorphic, construction of ambiguities (enumerating all ways in which two networks can overlap), and dropping rules that become the larger part of an inclusion ambiguity. An operation not implemented in any rigorous way was that of comparing two networks to decide the orientation of a new rule—this decision was typically left to the user, making the completion semi-automated at best. As a curiosity, it can also be mentioned that the completion process was run through the windowing system event loop: each ambiguity would get its own toplevel window, calculations were carried out in the window that currently had focus, and once an ambiguity had been resolved the corresponding window would be closed (thus yielding focus to the next window).

About half of the program, and even more of the development effort, was spent on the graphical rendering of networks; graph drawing in general is a nontrivial problem, and networks have special requirements in that it is preferred that vertices connected to each other in a particular way also are positioned correspondingly. General graph drawing heuristics were tried but found to be mostly unsuitable, and in the end positions were found by running a sort of simulation where "tensions" in the edges would cause vertices to move to more appropriate positions. Since such vertex and edge positions were expensive to compute, they were being stored as part of the network data structure.

An outright mathematical flaw of Version 0 was that it for certain inputs would produce non-acyclic networks. This prompted the introduction of "feedbacks" in the formalism, but can also be viewed as the first sign of wrap ambiguities [4].

Version 1 was a thorough rewrite begun in 2006. It split the utility into one library (`network2`) of operations specifically on networks and a program (`cmplutil1`) whose aim was clearly that of completion. Work was still dispatched via the windowing system event loop as each step continued to be shown graphically as it was performed, but now the program actually could order networks according to a mathematically rigorous criterion.

In the library portion, the network datatype was split into several. The core datatype is that of a *pure* network which implements the formal mathematical concept [3, Definition 5.1] of a network, whereas the graphical information is moved to to a datatype of *rich* networks wrapped around the pure ones. In addition there is a new concept of *network with feedback* which combines a pure network with a nominal transferrence [3, Definition 6.14]; the latter is needed to ensure that rewrite operations preserve acyclicity.

Most of the old code for assigning positions to vertices and edges was scrapped, and a variety of new heuristics based on generating a layout were implemented; sensible (even if not always optimal) layouts could be generated as needed, so there was no longer any need for long-term storage of the graphical information with the networks. Old code for exporting presentations could instead be migrated to the new datatypes, thereby preserving all existing features.

Over time the library also grew to include rigorous operations for ordering networks, as appropriate mathematics was being developed; late in the development cycle it was discovered that much of this could be greatly simplified.

Version 2 became operational in 2009 and is a rewrite of the completion utility employing the same network operations library as version 1. Major novelties include:

- using an SQLite3 database for storing all rules and other aspects of the state of the completion,
- capable of running without a GUI,
- web server interface for viewing state of computations, and
- expressions can be formal linear combinations of networks.

It is primarily this version that is described in Sect. 17.5 below.

17.3.2 Implementation Language

Tcl [17] is a quite simple language, with several traits that are attractive from a mathematical rigour point of view, but since it is not one of the major languages for scientific computing or in computer science, it warrants being elaborated upon.

First and foremost, Tcl requires that every value has a well-defined serialisation—the so-called *string representation*—and that the semantics of a value are determined entirely by its string representation; this is a principle known as *everything is a string* (EIAS). This does not require that all values are stored as strings; ever since Tcl 8.0 (1997) values can also have an *internal representation* native to the computing hardware, and many values in a running Tcl program are born (created), live (accessed), and die (deallocated) without the string representation ever being realised, since it is sufficient that the *potential* to realise it always exists. EIAS does however imply that values are *immutable*—once created, guaranteed to permanently have the same string representation—since the outcome of a test whether a value is equal to a specific literal string constant must depend only on the string representation of the value, not on details of how it is represented in memory or what might have previously been done to it. This is an excellent match for the Platonic ideal of mathematical objects existing (at least *in potentia*) unchangeable and forever, beyond the material realm.

The above does not mean Tcl only has constants. A *variable* is a location where a value can be stored, and it is perfectly fine to change which value is stored in a variable. There are even primitive operations which modify the contents of a variable precisely in that they change a specific part (e.g. one element of a list) of the value this variable holds, and if the variable does not share the data structure storing that value with anything else (which it would do if that value is assigned also to a second variable) then this data structure can be modified in place; otherwise the modifying operation makes a shallow copy of the original value and modifies this copy before assigning it to the variable. These semantics of immutable values—that seemingly mutating operations really make shallow copies—are important for correctly interpreting some of the more intricate operations in the completion utility (in particular those that perform combinatorial searches); a naive port of these algorithms to languages with mutable value semantics is likely to malfunction.

Second, Tcl has a very simple syntax. A Tcl program, or *script*, consists of a sequence of *sentences*.[1] Sentences usually appear one per line (newline is a sentence separator), although one may put several on a single line by separating them with semicolons. A sentence is in turn a sequence of *words*, separated by whitespace. The first word of a sentence is the name of the *command* performed by that sentence, and the remaining words constitute the *arguments* of that command. What allows this to be a structured programming language is that words in turn may be anything, including entire scripts, if appropriately delimited; a control structure (e.g. for loop) is simply a command which expects some of its arguments to be entire scripts themselves. Words that begin with a left brace '`{`' and end with the matching right brace '`}`' denote the exact sequence of characters between (but not including) these two delimiters, regardless of whether these characters would include newlines, other whitespace, or semicolons; this makes it trivial to nest "blocks" of Tcl code as single words of sentences, to an arbitrary depth. A word without delimiters (a *bareword*) can contain whitespace and semicolons, but only if each special character is individually escaped by a backslash character '`\`'; this is unusual. Finally, a word beginning with a quote character '`"`' ends with the next (unescaped) quote character, and can likewise

[1] The official documentation uses the term 'command' for this, but also for the first word of a sentence; I'm using 'sentence' to avoid this ambiguity. Likewise what I call a *sentence prefix* later—a list of values meant to become the initial words of a sentence—is normally called a 'command prefix'.

contain whitespace (even newlines) and semicolons without individual escaping, but here nesting is impractical, so quote-delimited words are mostly used for text strings.

What gets passed to a command for processing is always the runtime value of a word. Brace-delimited words have their explicit constant values, but other words undergo *substitution* before they become a command argument; most commonly the entire value of a word comes from a single substitution, though in quote-delimited words it is not uncommon that constant and variable pieces are combined when forming the runtime value, e.g.

```
log message "Element $i is: [lindex $L $i]"
```

The first substitution form is *variable substitution*: a dollar sign `$` followed by the name of a variable is replaced by the current value of that variable. The second substitution form is *command substitution*: everything from a left bracket `[` up to the matching right bracket `]` is interpreted as a (sequence of) Tcl sentence(s), that gets evaluated, and the value returned by the command of the (final) sentence upon invocation is what becomes the runtime value of this bracket construction. As a somewhat silly example, the sentence

```
list 1 [set i 2] $i i [incr i] $i [incr i]
```

will have the argument list '`1 2 2 i 3 3 4`' (seven arguments) passed to the `list` command. The first word has `1` as explicit constant value. The second word is a command substitution, and the `set` command there primarily sets the variable `i` to `2` (a side-effect), but `set` also returns the new variable value, so this word has runtime value `2`. The third word `$i` is a straight variable substitution, and since we know `i` is now `2`, that will again be the runtime value also of this word. The fourth word `i` has nothing to trigger substitution, so its runtime value is just `i` (name of a variable, rather than its value). The fifth word `[incr i]` triggers a command substitution (uncharacteristically, again with a command `incr` whose primary purpose is to achieve a side-effect, namely to increment the value of the variable `i`), so the value of `i` steps up to `3` and this is also the runtime value of this word. The sixth word `$i` likewise reports this new `i` value of `3`, before another command substitution in the last word increments `i` again, to the final value of `4`.

Thus briefed on the language syntax, we are ready to analyse an example illustrating the first point about immutable values and shallow copying of internal representations. The script

```
set row [list]
for {set k 0} {$k<$n} {incr k} {
   lappend row 0
}
set Z [list]
for {set k 0} {$k<$n} {incr k} {
   lappend Z $row
}
```

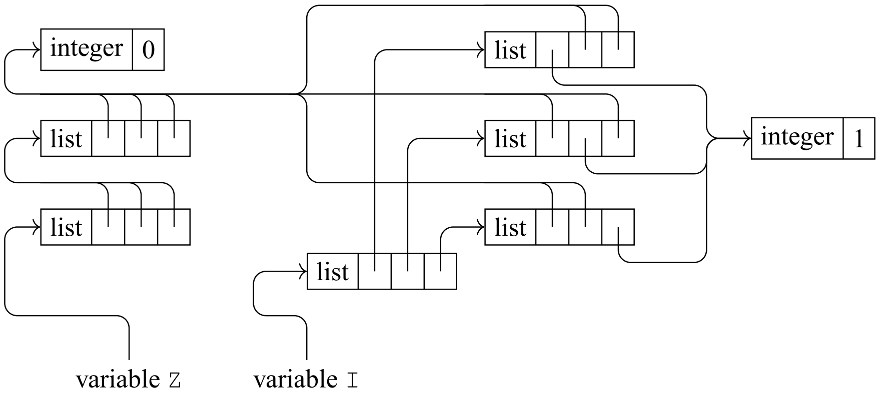


Fig. 17.2 Sharing of internal representations

sets variable Z to the $n \times n$ zero matrix (not in the easiest way possible, but in a way that follows common imperative style)—the first four lines make `row` a list of n zeroes by appending one 0 to this list at each iteration of the loop, and then the last four lines make Z a list of n such rows of zeroes by appending the value of `row` to Z at each iteration of the second loop. The generated data structure only has one 0 value (referenced n times), one row-of-n-zeroes value (referencing the zero n times, and referenced n times from the matrix value), and one matrix (list-of-lists) value (Fig. 17.2). However, if we were to make an identity matrix I from Z by changing the diagonal elements to 1s through the commands

```
set I $Z
for {set k 0} {$k<$n} {incr k} {
   lset I $k $k 1
}
```

(that `lset` does $I_{k,k} := 1$) then at each iteration of the loop the targeted row of I will be copied before its kth element is changed to 1, so the data structure storing the value of I ends up with n separate rows (as they need to be, since their values are distinct). There is still only one 0 value, but now it is referenced n^2 times ($n-1$ times by each of the n rows in I, and another n times by the only row in Z, which of course remains unchanged), and the 1 value is referenced n times (once by each row of I).

17.3.2.1 Syntax and Invariant Descriptions

Much of the practical syntax of Tcl ends up being syntaxes of individual commands, and accordingly the official language documentation [20] is organised as one man-page per command. Some of these (e.g. `if`, `try`) have rather complex clause-based syntaxes whereas others are more function-and-its-arguments (or verb-object-object-

object-… if you want to go natural-language grammatical), but common to all is that the division into words is given. Accordingly, it seems appropriate to stress when some unit constitutes a word, so in what follows the familiar $\langle bar \rangle$ notation for a metasyntactic variable is refined to $\{bar\}$ when this unit is additionally to be exactly one word in a Tcl sentence; the more generic $\langle bar \rangle$ would still be used for part of a word or a sequence of (zero or more) words.

Pragmatically, it is also convenient to make use of certain notations from regular expressions—such as the $^?$, *, and $^+$ quantifiers—to express repetition variation in command syntaxes. For example

list $\{element\}^*$
incr $\{variable\}$ $\{amount\}^?$

means the `list` command takes zero or more $\{element\}$s, whereas the `incr` command takes a $\{variable\}$ name and optionally also an $\{amount\}$ by which to increment it. With parentheses to group pieces in such syntax expressions, the syntax of `if` may be stated as

if $\{condition\}$ then$^?$ $\{script\}$ $\big($elseif $\{condition\}$ then$^?$ $\{script\}\big)^*$ $\big($else$^?$ $\{script\}\big)^?$

showing not only that the final `else` clause is optional, but also that it may be preceded by zero or more `elseif` clauses, and that the keywords `then` and `else` are optional.

These syntax conventions are in the completion utility sources applied not only to document individual commands, but also to document data structures, since the string representations of Tcl lists are parsed into elements according to the same rules as Tcl sentences are parsed into words; the main difference is that variable and command substitutions do not occur when parsing a list. Below it is the application of these conventions to lists and other data structures that is of more scientific interest, as data formats should be documented for posterity even if the programs that generated them may become obsolete.

To give an example, the string representation of a dictionary is

$\big(\{key\}\ \{value\}\big)^*$

so the same as a list with an even number of elements, alternatingly having the roles of key and value. As a matter of general computer science, a *dictionary* (associative array) is a sparse mapping of $\{key\}$s to $\{value\}$s. Tcl implements dictionaries with a hash table internal representation to achieve average $O(1)$ complexity for accessing individual elements; key equality is string equality.

Documentation may also need to state invariants and other properties of the data structures considered, and for this it rather becomes appropriate to mix Tcl code with ordinary mathematical formulae, when notation for some necessary operation exists only in one but not in the other. To signal what is what, Tcl elements of such mixed formulae will appear in `typewriter` font and typically be delimited by command substitution brackets `[]`, whereas standard mathematical elements use normal formula fonts (e.g. math italic). A trivial example is

$$\texttt{[lindex [list } a \ \ b \ \ c\texttt{] 1]} = b$$

where three values a, b, and c are given to the `list` command to build a list with those three as elements. Then this list is passed to the `lindex` command that goes on to extract the element in position 1 from this list; since Tcl indexing is 0-based, that will be the second element b. Finally at the top level of this mixed formula we assert mathematical equality $=$ of the value returned by that `lindex` and b. The standard mathematical formula language lacks good notation for this kind of construction and indexing operations—we rather prefer to give names to all elements we need to access—and programming languages conversely have little notation for stating claims, as programs are mainly about giving orders.

17.4 The Network Datatype Library

The informal specification for a network is that is should be like a term (in the logic sense, i.e., a formalised expression), except that the underlying combinatorial structure should be that of a directed acyclic graph (DAG) rather than a tree, because it needs to accommodate co-operations as well as ordinary operations. Formalising that does however require sorting out exactly what sense of 'tree' we will generalise.

The tree structure of a term can be defined as the tree which as vertices has the subterms of the term and links two vertices with an edge if one is a maximal proper subterm of the other, but for our purposes it is more convenient to take as vertices the *operations* (function symbols, including as nullary operations any named constants); this turns the tree into a kind of data-flow diagram (Fig. 17.3, middle). The reason this structure for an ordinary term will be a rooted tree is that each operation syntactically produces exactly one result, but that is not the case with co-operations; paths taken by data within the expression may branch as well as join, creating a general DAG rather than a tree. The problem of how to even interpret such combinatorial objects as expressions is nontrivial, but turns out to have a natural solution [3, Sect. 5].

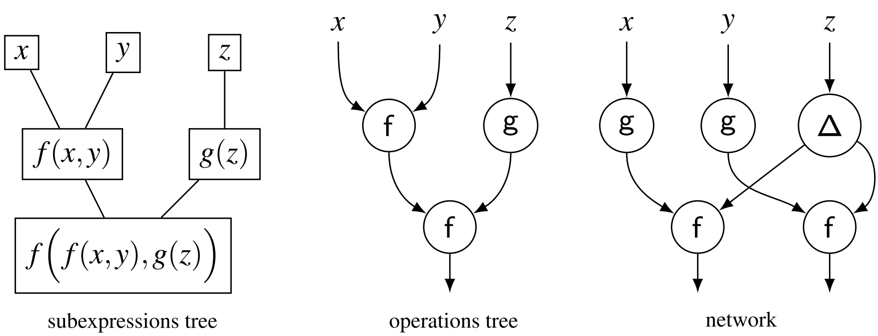


Fig. 17.3 Term trees and a network

It is important to note that already an operations tree needs more structure than just the tree graph to encode a term. For one thing, there are the operation symbols *decorating* the vertices, but more importantly the children of an operation are not interchangeable (unless that operation is commutative, but this is not a property that would be encoded into the syntax of a formula)—instead each incoming edge is attached to a separate *port* of the vertex, and the set of available ports depend on the operation symbol; for example a binary operation has one 'left operand' port and a separate 'right operand' port. In a network the set of outgoing ports similarly depends on the operation symbol, so that for example a vertex for the coproduct Δ has one incoming port but two outgoing ports. Finally networks are *open graphs* in that they can have external edges signifying output (results) from a network and input (arguments) to it. In comparison to traditional terms the incoming external edges assume the role of free variables, which has interesting consequences for the rewrite theory: rewrite linearity becomes a syntactic requirement (although having co-operations allows for effectively reintroducing nonlinearity via explicit rules) and unification is no longer a primitive operation.

17.4.1 Pure Networks

Mathematically, [3] defines a network as a tuple

$$G = (V, E, h, g, t, s, D)$$

where $\mathbb{N} \supset V \supseteq \{0, 1\}$ and $E \subset \mathbb{N}$ are the sets of vertices and edges of the underlying directed acyclic graph (V, E, h, t) in which $h, t \colon E \longrightarrow V$ are the functions mapping every edge e to its head $h(e)$ and tail $t(e)$ respectively. To turn this into an open graph, the two mandatory vertices 0 and 1 are fixed as representing the output and input respectively sides of a network; an edge e is outgoing external if it has $h(e) = 0$ and incoming external if it has $t(e) = 1$. Requiring that vertices and edges (or technically: the *labels* of vertices and edges) are all natural numbers have certain set-theoretic advantages for defining isomorphism classes of networks, which are technically the objects that are being rewritten. The two functions $g, s \colon E \longrightarrow \mathbb{Z}_{>0}$ map an edge (label) to the head and tail respectively *index*, which identify the ports to which the edge is attached. Finally D with a domain of $V \setminus \{0, 1\}$ is the function mapping inner vertices to the operation symbols by which these are decorated. Such a symbol x is given with an *arity* $\alpha(x)$ and a *coarity* $\omega(x)$ that are equal to the in-degree and out-degree respectively of any vertex decorated with x. One also speaks about arity $\alpha(G)$ and coarity $\omega(G)$ of an entire network; these are the numbers of inputs and outputs to the network G as a whole (thus technically the *out*-degree of 1 and *in*-degree of 0, respectively).

This mathematical concept is implemented in the network library as the datatype of *pure networks*. For ease of access, pure networks are implemented as heterogeneous nested lists, where some nesting levels act as records whereas others act as arrays. At the top level there is simply a pair

$\{vertices\}$ $\{edges\}$

where $\{vertices\}$ and $\{edges\}$ are both lists indexed by vertex or edge respectively label; the sets V and E are implicit in the lengths n and m of these lists, since the vertex and edge labels are the indices of elements therein: $V = \{0, \ldots, n-1\}$ and $E = \{0, \ldots, m-1\}$. The element at any index e in $\{edges\}$ is a record-like list of the following four integers

$\{h(e)\}$ $\{g(e) - 1\}$ $\{t(e)\}$ $\{s(e) - 1\}$

i.e., it specifies the head vertex label, head index, tail vertex label, and tail index; the only variation from the mathematical definition is that the indices are 0-based.

This leaves only the decoration map D to encode, but the $\{vertices\}$ also duplicates the incidence information from (h, g, t, s), to make traversing the network efficient. To that end, an element of $\{vertices\}$ is a record-like list

$\{decoration\}$ $\{out\text{-}edges\}$ $\{in\text{-}edges\}$

where $\{decoration\}$ is the value of D at that vertex (or an empty string for the external vertices 0 and 1), $\{out\text{-}edges\}$ is the list of labels of the outgoing edges in the order given by their tail-indices, and $\{in\text{-}edges\}$ similarly is the list of labels of the incoming edges in the order given by their head-indices. For every network G and every edge label e of that network, it is an invariant that

$$\texttt{[lindex } G \texttt{ 0 [lindex } G \texttt{ 1 } e \texttt{ 0] 2 [lindex } G \texttt{ 1 } e \texttt{ 1]]} =$$
$$e = \texttt{[lindex } G \texttt{ 0 [lindex } G \texttt{ 1 } e \texttt{ 2] 1 [lindex } G \texttt{ 1 } e \texttt{ 3]].}$$

A network  where the only operation vertex has a decoration '`m`' (for multiplication) can thus be encoded as the 69 characters string

```
{{{} {} 2} {{} {0 1} {}} {m 2 {1 0}}}
{{2 1 1 0} {2 0 1 1} {0 0 2 0}}
```

Though in some ways "wasteful" (for example using decimal digits to express numbers), this is actually quite compact in comparison to what it would take to realise the same graph structure with pointers on a contemporary architecture: a 64-bit pointer is 8 bytes, and with 3 edges each containing 2 pointers to vertices, times another 2 because the vertices have to point back, that would be a minimum of $8 \cdot 3 \cdot 2 \cdot 2 = 96$ bytes for just tying the DAG together. Section 17.5.2 explains how outright compression is used to achieve further savings for long-term storage of networks.

Topic 1 (Expressing networks) One practical problem that faces a user of the network library is how to construct any networks for it to operate on in the first place, as writing down by hand a string representation such as the above is not very convenient. The `network::pure::construct` procedure provides a more practical interface where the user provides a textual "scan-list" of the items in the network—lines in order top (input side) to bottom (output side), and within each line going left to right—that is easy to write down when looking at a drawing of the network, and with some practice possible to compose even without a drawn original. Inner vertex items are in the scan-list expressed as their decorations; one needs to first declare how many incoming and outgoing edges each vertex type has. Furthermore one may write '`.`' for an edge merely passing through a scan line, '`X`' for two edges crossing (or more generally $\langle k\rangle$`X`$\langle l\rangle$ for a crossing of k edges going down right and l edges going down left), and a newline (`\n` or `\r`) to mark the end of a scan line. This way, the above  network can be written as '`X \n m`' and the network on the right in Fig. 17.3 can be written '`g g Delta \n . X . \n f f`'.

An alternative interface for entering networks into the program, which would probably be more appealing to the beginner, could be a graphical point-and-click approach; the actual programming is not very difficult. However even in a mainly graphical interface the scan lines seem to be a valuable abstraction to maintain, as they lessen the amount of graphical details that the user would have to provide.

Topic 2 (Network isomorphism) Since the objects of algebraic interest are isomorphism classes of networks, the library must provide for deciding network isomorphism. It does so via the `network::pure::canonise` command, which given any network returns a canonical representative from the same isomorphism class; two networks are isomorphic if and only if their respective canonical representatives are equal as strings. Moreover the only way in which isomorphic networks may differ is in how the individual vertices and edges are labelled, so what the `canonise` operation concretely does is that it generates a canonical labelling of the network.

Canonical labelling of simple graphs is quite complicated due to their flexible nature where vertices and edges have no identity except as in relation to each other, but networks are far more rigid: since edges attach to separate ports of a vertex, the identities of all vertices in a component become fixed as soon as one fixes the identity of one vertex. In addition the 0 and 1 vertices of a network are already fixed, so in most networks that are encountered there is already some way of identifying everything; what remains is to turn that into a deterministic (not depending on labels in the input) labelling scheme. The scheme that is used is to do a breadth-first search through a graph containing both vertices and edges of the network, each vertex looking at incident edges in a deterministic order and likewise each edge looking at its endpoints in a deterministic order. The canonical labels are assigned sequentially in the order vertices and edges are encountered by this search, and the search starts with vertices 0 and 1 already enqueued.

Things get trickier for networks with components that are isolated from both the output and the input; rewrite systems often have rules that make these disappear, but during the intermediate steps they must still be canonically labelled. This is done by first computing a separate "simplified" canonical representation for each component, sorting the components by the string representation of this simplified representation, and finally assigning canonical labels component by component; the exact ordering of the components is irrelevant as long as it is deterministic, and equal components simply appear in sequence.

The simplified canonical representations are in turn obtained from breadth-first searches of the individual components. Here we have no canonical starting point for the search, so instead the search is done multiple times with different vertices as starting point, and whatever simplified representation happens to have the lexicographically smallest string representation is picked as the canonical one; since alternatives are considered only for the starting point, the algorithm as a whole remains comfortably polynomial. The `network::pure::canonise` command has a slight optimisation in that only vertices decorated by the lexicographically smallest symbol found in a component are considered as roots for the search in that component, but this can at best improve the constant factor of the asymptotic complexity.

Overall, canonical representations should probably not be expected to remain the same between different codebases; a programmer would be wise to treat any network as not canonical in the current process until it has been explicitly canonised. In particular, most library subroutines that produce networks will not bother to canonise them.

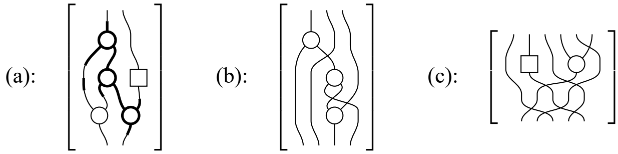
Case in point, the library provides a number of operations on networks that correspond to basic operations in the free PROP—the elements of the free PROP being isomorphism classes of networks, these network operations produce one representative from the equivalence class that is the result of applying the corresponding PROP operation to the equivalence class(es) of the argument(s). In the `network::pure` namespace we find

`composition` Serial composition $\circ$ of one or more networks.
`tensorprod` Parallel composition (tensor product) $\otimes$ of zero or more networks.
`permutation` Construct a permutation network, where all inputs are also outputs.
`left_action` Permute outputs of a network.
`right_action` Permute inputs of a network.
`substitute` Replace individual vertices of a network by other networks, gadget style; this corresponds to applying a PROP homomorphism that one gets from the universal property of a free PROP.

These are however only in rare cases immediately useful for rewriting. Instead rewriting is more easily expressed in terms of *surgery* on networks, where some edges are cut, the detached pieces removed, and other pieces spliced into their place.

Topic 3 (Regions in networks) A useful concept for describing surgical operations is that of a *region* in a network. This is by the library encoded as a list

{*vertices*} {*outputs*} {*inputs*}



Subfigure (b) is the selected subnetwork uncovered, showing also the relative order between its inputs and outputs; 9 denoting the second (index 1) part of edge 0, that is mostly parallel to the first part of the same edge. Subfigure (c) is instead what remains of G when the selected region has been detached; inner inputs and outputs (those that connected to the detached region) appear to the right of the original inputs and outputs from (a).

Fig. 17.4 **a** Region in a network. **b** subnetwork inside region. **c** remainder when region is detached

where $\{vertices\}$ is a list of vertex labels and the other two lists of (primarily) edge labels. Regions defining the ambiguity processed are recorded in the database by the completion utility, which is why their encoding should be documented. The concept in [3] that corresponds to regions is that of *strong embedding* of a network into another.

If for intuition viewing a network as a topological space, then a region is a (finitely presented) open subset thereof; in particular, if a vertex is in the region then also (at least parts of) the edges incident with that vertex must be included. Outputs and inputs of the network must however be listed explicitly, since the subnetwork selected by the region always has its outputs and inputs in some specific order. An edge not occurring in the $\{outputs\}$ or $\{inputs\}$ must either be wholly in the region and thus have both its endpoints in the $\{vertices\}$, or wholly outside the region and thus have neither of its endpoints in the $\{vertices\}$.

A complication that arises is that edges sometimes are decomposed into more than two pieces by the region boundary—a single edge may appear arbitrarily many times in the $\{outputs\}$ and $\{inputs\}$, if the intermediate pieces in the selected subnetwork count as edges going directly from input side to output side. To keep track of which part connects to what, the elements of $\{outputs\}$ and $\{inputs\}$ are in fact integers $e + mi$ where $e \in E$ is the edge label proper, $m = |E|$ is the total number of edges, and $i \in \mathbb{N}$ is an index to distinguish the separate pieces the region has of edge e; lower index values are closer to the tail of the edge. Figure 17.4 shows an example of a network with a region, that also exhibits a multipart edge.

The outright `replacement` of a region in one network G by a different network H is both in the library and in [3] decomposed into separate operations of first `detaching` the part of G which was inside the region and then joining the part K which remains up with the new piece H using an `annexation` operation $\rtimes$ to form a new network $K \rtimes H$ where all inputs of H are joined to the last few outputs of K and all outputs of H are joined to the last few inputs of K; in the "$\rtimes H$" part, edges are going ⤩↓. This last step in principle carries the risk of creating directed cycles in the result, but this risk is carefully managed; [3, Sec. 7.1] uses boolean matrices to keep track of exactly when such join operations are well-defined, and in the network library the concept of *network with feedback* (next subsection) serves the same purpose. In particular, the operations that compute regions typically take networks with feedback as operands.

It should be observed that replacement of regions within a network is a *more general* operation than rewriting by means of double pushouts. The reason for this is that the double pushout formalism produces left and right hand sides of a rewrite step as images of a common pattern network where a single vertex is mapped to the left and right hand sides of a rewrite rule, but regions are more general than what can be the image of a single vertex; an output of a region can connect back to an input of that region, whereas an output of a single vertex connecting back to an input of that vertex is immediately an acyclicity violation. Regions are said to be *convex* if there is no directed path which leaves the region and then returns to it; any region obtained as the image of a single vertex will be convex, and double pushout rewriting is thus restricted to making convex replacements. It turns out completion frequently derives nonconvex rewrite rules, even if starting from a purely convex set of axioms.

Topic 4 (Structural decomposition of networks) When networks are constructed through a sequence of replacements, there is an obvious risk that any governing principle for their structure—such as being produced through a sequence of serial and parallel compositions—is destroyed; therefore it becomes interesting to take a general network and seek a comprehensible decomposition of it. Unfortunately this seems to be a difficult problem, with no obvious solution. The network library contains a number of procedures attacking this decomposition problem, but most are dead code used for nothing. What exceptions there are (Topics 6 and 7) participate in the generation of graphical layouts for networks (Topic 12), and that is a major topic of its own.

Topic 5 (Monomial ordering of networks) Completion requires that the objects being rewritten can be compared, so that derived equalities can be oriented into rewrite rules. Similarly to Topic 4, the network library contains a body of procedures written as experiments in developing a useful ordering, but here the story has a happier ending in that a good solution was eventually discovered, even if it ended up not actually needing any of the code here.

The theory for ordering networks is explained in [3, Sect. 3]: first construct any sufficiently fine ordered PROP $\mathcal{P}$—a good choice is the biaffine PROP over any partially ordered cancellative semiring—then use the universal property of the free PROP to pull this order back to the networks. Practically one evaluates the networks one wishes to compare in the PROP $\mathcal{P}$, with some choice of value for each element in one's signature, and then compares the values of the networks as a whole. A suggested interface for implementations of PROPs [8] contains a `fuse` operation that is straightforward to use to that end.

As a matter of development history, this conceptually neat construction evolved from the idea of making lexicographic comparisons along all possible paths through the networks being compared. An ordering that worked for orienting the axioms of a bialgebra could be defined in that way, but for compatibility with composition one needed to first make sure that the networks being compared had the same number of paths for each combination of input and output; this is similar to how in word rewriting one needs to first compare by word length before one can make a lexicographic comparison. However in the bialgebra case it was then realised that already the path-counting stage (if slightly tweaked) would suffice for oriented all rules as desired [5, 6]; the biaffine PROP can be interpreted as precisely counting paths.

Topic 6 (Level decomposition of networks) A more modest goal compared to that of Topic 4 is to make a serial decomposition of a network b as $b_1 \circ \cdots \circ b_l$ where each b_i only employs $\otimes$ and permutations. In [3, Sec. 4–5] this was done to the end of proving that networks indeed may be interpreted as expressions for arbitrary PROPs, by putting every vertex in a level of its own, but that is often too drawn out to be comprehensible. Instead it is useful to have a decomposition into a minimal number of levels.

The vertical/serial aspect of a level decomposition boils down to what levels the vertices should be placed in, for which problem there again exists a number of implementations in the library. The current production procedure `network::pure::vertex_levels4` conceptually treats the whole situation as a linear programme with the vertical vertex positions as the variables, inequalities expressing that every edge has to have length at least 1, and an objective of minimising the sum of all edge lengths. In principle this linear programme is then solved using the simplex algorithm (with infinitesimal perturbations to avoid degenerate corners of the polytope), but in practice the state of the algorithm boils down to keeping track of the edges for a spanning tree in the network, since every edge corresponds to an inequality and the set of tight inequalities are what determines the current feasible point. In particular there is never a need to solve a general linear equation system.

The effect is similar to decomposing a network into scan lines as per Topic 1, but different in that it allows for arbitrary permutations *between* levels whereas the scan lines presentation only does permutations *within* levels. Doing permutations between levels can sometimes lead to messy situations: the network in Fig. 17.4c needed considerable tweaking of the layout (in particular the insertion of an extra level without vertices) to prevent certain edges from touching (to such an extent that it became unclear what connected to what).

Topic 7 (Order within network levels) Having assigned every vertex to a specific level, what remains for producing a full layout is to somehow order the items within each level. This is again a difficult problem with no obvious solution, and naive algorithms such as trying all possibilities end up with superexponential complexity. Moreover even the choice of an appropriate objective function for comparing different orders is not entirely trivial; several seemingly good ideas have turned out to produce aesthetically unpleasing results. Thus the library again contains a number of procedures to this end that have been found wanting and are not used.

The procedure `network::pure::ordered_graded_components` currently used in production works by having each level suggest hints on how items should be ordered in adjacent levels, compiling the hints received into a partial order, and then suggest new hints based on relations exhibited by that partial order; repeat until propagation dies down, make an arbitrary choice and repeat again until all level orders are total. The initial suggestions come primarily from the orders of incident edges at each vertex, e.g. a vertex with two inputs would suggest to the level above that the item connected to the first input is placed to the left of the item connected to the second input. Conflicts between suggestions are resolved simply by arbitrarily picking one of them, and never backtracking once a choice has been made. This works fairly well when there indeed is close to a consensus on what should go left and what should go right—with a slight reservation for the fact that it has a tendency to make large jumps in order at the boundary between two "zones of influence" rather than many small jumps to gradually switch between two conflicting opinions—but appears to produce less aesthetically pleasing results when there is much conflict; however it has not been systematically examined to what extent there even exists more pleasing layouts in those cases.

Aesthetically superior heuristics for network layouts would be a valuable addition, but the current ones do have the virtue of being quick enough to execute and good enough so far.

17.4.2 Networks with Feedback

A *network with feedback* is defined as a pair

{*network*} {*feedback-list*}

where in turn a {*feedback-list*} is a list (order irrelevant, so effectively a set) of pairs (i, j) encoding the sentiment that the output with index i of the {*network*} may be fed back into the input with index j of that same network. When looking for redexes at which to apply a rewrite rule to a network, one must take into account not only the {*network*} that constitutes the left hand side of the rule, but also whether the region into which this network gets embedded satisfies the dependency constraints under which that rule was derived, i.e., would it be possible to replace this single rule step by a sequence of more elementary rewrite steps while still preserving acyclicity of all intermediate networks? Those are exactly the constraints expressed by the {*feedback-list*}: one may not have a directed path from output i returning back to input j unless the pair (i, j) is in the {*feedback-list*}.

A related concept is that of the *transferrence* of a network G, which is the boolean $\omega(G) \times \alpha(G)$ matrix $\mathrm{Trf}(G)$ that has a 1 in position (i, j) if and only if there is a directed path in G from input j to output i. In [3, Sects. 7–8] it is proved that Trf may also be interpreted as a PROP homomorphism from the free PROP to the PROP $\mathbb{B}^{\bullet\times\bullet}$ of boolean matrices. Taking into account that the standard order of boolean matrices (the partial order doing elementwise comparisons) is compatible with this PROP structure, we even get that the free PROP is equipped with a $\mathbb{B}^{\bullet\times\bullet}$-filtration $\{\mathcal{F}_q\}_{q\in\mathbb{B}^{\bullet\times\bullet}}$ called the dependency filtration, defined by the condition that $[G] \in \mathcal{F}_q$ iff $\mathrm{Trf}(G) \leqslant q$. The *formal feedback* structure [3, Sect. 9] on the free PROP is that a variety of connect-output-back-to-input operations ($\rtimes$, $\bowtie$, and $\uparrow$) can be defined as *total* (not just partial) operations on appropriate components in this dependency filtration, and also that their codomains are again found in this filtration.

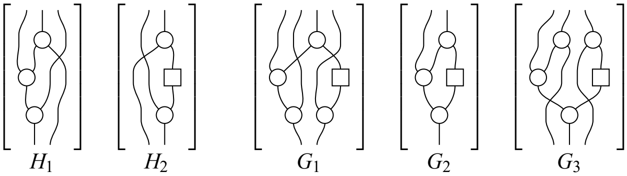
Topic 8 (Mathematical interpretation of feedbacks) Within the completion utility, the feedbacks primarily matter for operations that are looking for matches of (part of) one network with feedback to (part of) another network with feedback, by going through a search tree of ways to match the underlying pure networks. If during this search a path arises from an output to an input which is not listed among the feedbacks of this network, then the feedback constraints have been violated and the search backtracks. While the completion utility was being developed, this was pretty much the extent to which the feedbacks had any interpretation.

The theoretical foundations in [3, Sect. 10] are that the networks being rewritten live in a particular component of the dependency filtration $\{\mathcal{F}_q\}_{q\in\mathbb{B}^{\bullet\times\bullet}}$, and that rewrite rules do the same. Since the operation of feeding output i back to input j is well-defined for $\mu \in \mathcal{F}_q$ if and only if $q_{i,j} = 0$, the interpretation of the {*feedback-list*} became that this is a sparse encoding of this transferrence type matrix q: all elements are 1 except those whose positions (i, j) are given. Some later additions to the completion utility explicitly make this interpretation when exporting/displaying rewrite rules. It was however already in [3] observed that this model had certain problems, and in [4] it was suggested that some further refinement might be needed.

The present (2021) understanding is rather that the {*feedback-list*}s have the right amount of detail, but that the original theoretical understanding of them is slightly off. Rather than a pair (j, i) signalling that element (i, j) of an $\omega(G) \times \alpha(G)$ boolean matrix q is 0, they signal that element (j, i) of an $\alpha(G) \times \omega(G)$ boolean matrix p is 1, where p keeps track of how the context is allowed to connect things; rather than the constraint on $\mathrm{Trf}(G)$ being $\mathrm{Trf}(G) \leqslant q$, it should be that $p\,\mathrm{Trf}(G)$ (a product of two boolean matrices) is nilpotent. For {*feedback-list*}s with only one element these two interpretations are equivalent, but when there are more feedbacks the p interpretation becomes more flexible.

Topic 9 (Finding subnetwork instances) In order to do rewriting, it is necessary to check if one network (the rewrite rule left hand side) appears as a subnetwork of another network (that being rewritten), and if so in what region. The network library provides this functionality through the `instances` procedure in the `network::wfb` (With FeedBack) namespace. It works by searching through a tree of candidate ways of identifying vertices and edges of the subnetwork H with the supernetwork G, offering to halt the search when a certain number of matches (typically 1 or ∞) has been found.

The search tree is typically quite shallow, since identifying one vertex v of H with a vertex of G will determine how the entire component of v is embedded into G (or reveal that v would have to be embedded somewhere else). Subnetworks H with multiple components may however require several choices before an embedding is fixed, and H edges from input to output do in this case count as separate components, which contributes to the code complexity: there are two kinds of components to



Looking for ways of overlapping H_1 and H_2, all of G_1, G_2, and G_3 are valid possibilities; there are separate choices of whether to overlap the top vertices and whether to overlap the bottom vertices. The G_2 overlap does however require that H_1 and H_2 allow for feedbacks.

Fig. 17.5 Networks and overlaps

embed. Immutable value semantics do however make it trivial to cache a copy of any pre-choice state as something to return to after backtracking.

Topic 10 (Enumerating ambiguities) In order to do completion, it is necessary to enumerate all (critical) ambiguities—minimal networks that can be reduced in two different ways—and traditionally this is done by picking two rewrite rules and listing all the ways in which their left hand sides may be overlapped. The `ambiguities` procedure in the `network::wfb` namespace does precisely this: given two networks with feedback H_1 and H_2 it returns a list of networks with feedback G such that both H_1 and H_2 appear as subnetworks of G, and also the regions where they do this. In the terminology of [3], the list returned covers all *decisive* ambiguities, which are exactly the ones that it by the diamond lemma [3, Theorem 10.24] suffices to check.

The algorithm used in the `ambiguities` procedure resembles that of the `instances` procedure in that it explores a search tree of all possibilities for how to identify vertices and edges of the two networks, but here there is always also the possibility that something in one network is not identified with anything in the other, which increases the tree valency a bit. It is also not the case that components get fixed by only one choice each, because there is with networks no requirement that the intersection of H_1 and H_2 (as embedded into G) is connected; whenever there is a way of overlapping them in two places, there will also be two more ambiguities in which H_1 and H_2 overlap in just one of these places (see Fig. 17.5). This is different from the situation for trees/terms, which have the property that the common part in an overlap of two trees is itself a tree and thus connected.

Concretely the procedure begins with H_1 as G, then working its way through the vertices of H_2 while choosing which if any G vertex with which to identify it. If an edge incident with an identified vertex is inner in both H_1 and H_2 then the other endpoint of this edge is also enqueued for identification, whereas if the edge is inner in H_2 but not so in H_1 then proceeding requires making a choice: is the other endpoint a vertex in H_1 (and if so which one) or not? The general story is that edges that are inner in H_1 or H_2 will be mapped to inner edges of G, but inputs and outputs of H_1 or H_2 may also be mapped to inner edges of G, if feedbacks permit this and the other H_i has an inner edge there. The search tree for `ambiguities` is therefore not so shallow as the `instances` one, even if many side branches can quickly be disposed of as barren.

One further complication that is useful to observe is that the plain `ambiguities` procedure often returns networks G with inputs and outputs in an order that is awkward for graphical renderings. Therefore there is wrapper procedure `groomed_ambiguities` which additionally permutes the inputs and outputs of the produced network G, to reduce the number of crossings. Finally, it should be mentioned that both `instances` and `ambiguities` come with a suite of test cases, as getting these operations right indeed is somewhat nontrivial.

17.4.3 Rich Networks

The third "network" datatype employed in the library is the *rich network*, which is a dictionary of many different pieces of information about a network, primarily towards the end of drawing that network. Different operations on rich networks make use of different entries, and several operations are about computing suitable values for additional entries, given the data already present. The underlying pure network is kept in the `pure` entry, and its vertex and edge labels are abundant also in other entries—sometimes as elements, sometimes as indices.

The output/export formats presently supported are:

- As graphics on a Tk `canvas` [21], for interactive use in a graphical user interface. Version 1 of the completion utility also used this for dumping to PDF.
- As SVG [14], for use in webpages.
- As LaTeX code, using the PGF [22] package.

Exact capabilities varies between the formats, for example whether there is support for drawing feedbacks, for showing regions (and if so: how), and which appearance primitives are supported. The coordinate system used in rich network entries follow the conventions of the Tk canvas, i.e., the positive y axis points *down* and the length unit is "screen pixels" (although there is no requirement that coordinates are integers). The development over time has been towards preferring the SVG and LaTeX renderings, but primarily because those formats provide other mechanisms for making networks part of larger structures: terms in sums, steps in proofs; if drawing that much on a `canvas`, positioning several networks relative to each other (while not making lines overly long) is also quite a lot of work.

Topic 11 (Signature with appearances) For drawing networks, it is not sufficient that one knows what abstract symbol decorates a vertex; one must also associate this symbol with an *appearance*. In the library, these are encoded as lists

$\{\textit{vertex-items}\}\ \{\textit{output-offsets}\}\ \{\textit{input-offsets}\}\ \{\textit{size}\}^?$

where the $\{\textit{size}\}$ part is a late addition, only looked at by some layout operations. The $\{\textit{output-offsets}\}$ and $\{\textit{input-offsets}\}$ are lists indexed by tail and head index, respectively, and specify where and how edges should attach to the vertex. An element of these lists is itself a list with the structure

$$\{\textit{x-ofs}\}\ \{\textit{y-ofs}\}\ \bigl(\{\textit{dir-x}\}\ \{\textit{dir-y}\}\bigr)^{?}$$

where {*x-ofs*} and {*y-ofs*} are the offsets from the reference coordinates of the vertex to the graphical endpoint of the edge. The optional {*dir-x*} and {*dir-y*} are the components of a tangent vector for the edge pointing away from the vertex; the length is irrelevant. The default for direction is $(0, 1)$ (straight down) for outputs and $(0, -1)$ (straight up) for inputs.

The {*vertex-items*} is a list specifying one or more Tk canvas items that together make up the graphical representation of the vertex. It has the general structure

$$\bigl(\{\textit{item-type}\}\ \{\textit{coordinates-command}\}\ \{\textit{options}\}\bigr)^{+}$$

i.e., three list elements per graphical item. The {*item-type*} is straight off the type of the item; common values include `rectangle`, `oval`, `line`, `text`, and `polygon`. The {*options*} is a dictionary of options for this item; in for example a `text` item this is where the actual text string to display is encoded. Finally the {*coordinates-command*} is a sentence prefix that given the reference coordinates of the vertex calculates the coordinates expected by this item. A common but somewhat unintuitive choice for {*vertex-items*} list is

```
oval {square 8} {}
```

which makes a radius 8 circle with the reference point as midpoint; the coordinates specifying an `oval` (ellipse) are the coordinates for its bounding box, and '`square` *r*' computes the coordinates for a square with side $2r$. Full generality is provided by using

`offsets` {*pair*}*

as {*coordinates-command*}, in which case each {*pair*} gives the offsets from the reference point of one point of the `canvas` item coordinates.

A point-and-click interface for designing signatures would be an obvious feature from a usability perspective, but so far the purely scientific aspects have been given higher priority.

Topic 12 (Network layout generation) As mentioned, the problem of how to generate a graphical layout for networks has been an important one throughout the development of the program. The present production method goes through the following stages:

(i) Every vertex is assigned an integer level, as explained in Topic 6. Thus the network can be presented as being built up from discrete levels, where the items in a level are either vertices at that level or edges passing through that level. The extreme levels (abstractly home solely to vertices 0 and 1) are regarded as having only edges as items.

(ii) The network is also split into components (counting vertices 0 and 1 as included and implicitly connected). Within each component and level, the horizontal order of items is determined as explained in Topic 7.
(iii) Items are assigned reference positions, based on their nominal sizes and options for separation between items. All items in a level have the same y-coordinate, but differ in x-coordinate. Relative x-positions within a level and component are first frozen, then adjacent levels have their relative x-coordinates adjusted taking edges connecting them into account. Finally different components are placed side by side.

The result of that is stored as the `level-layout-Tk` entry in a rich network. Edges are straight vertical when passing through a level, but in general curves where they go between levels.

The drawing routines rather work with the pure coordinate data found in the `vpos-Tk` and `ecurve-Tk-tt` or `ecurve-Tk-raw` entries, allowing for separate development of layout generation data export. The two entries for edge curve coordinates have to do with which kind of curves is being used: piecewise quadratic (`tt`) or cubic (`raw`) polynomial parametrised curves. The latter are native in both SVG and PS/PDF, whereas the former have their most famous application in the TrueType font format.

Graphical networks exported as SVG are XML (sub)documents and as such not much for human eyes before rendering, but targeting LaTeX is another matter.

Topic 13 (Graphical data in LaTeX manuscripts) The traditional view in printing has been that anything which cannot be produced solely using type is some sort of image which has to be supplied separately from the manuscript, but in a paper showing calculations using networks that quickly becomes a version control nightmare—keeping networks in the same manuscript as the text and all its mathematical formulae is the way to go. TikZ [22] and other LaTeX packages have long since demonstrated that it is feasible to code vector graphics directly within a LaTeX manuscript, but generating TikZ code is not so practical: much of TikZ's power lies in figuring out coordinates so that the user does not have to, but in this case that work was already done.

Feature-wise, the underlying PGF package is much closer to what we want, but plain PGF code is quite voluminous—in part because the command names are long, but primarily because the *arguments* are long; for example, the command `\pgfpathqcurveto` takes six arguments, each of which is a length (so including a TeX unit) denoting an x- or y-coordinate. One such command easily fills a `.tex`-file line, meaning each network would be several screenfuls of code. That is too long if one wishes to keep track of a narrative where this network is merely a monomial. TikZ code can be more compact—units are not mandatory, the command names are shorter, and a single command often suffices for drawing one item—but it is designed to be written by humans rather than generated by a program, so converting low-level graphics to TikZ code is not as straightforward as it might seem.

Instead the network library includes a LaTeX package sdpgf which offers an even more compact representation of graphical networks, as a thin wrapper around PGF. The most radical innovation is that the new commands added use single spaces as argument delimiters; not only does this save one character per argument, but it also allows standard line breaking in text editors to act sensibly on these commands, filling a line with arguments as far as it goes and then wrapping any that exceed the desired right margin to the next line. Coordinates are all implicitly in a unit length specified at the top of the network (thus making it easy to tweak the scale of networks throughout the authoring process), and by choosing this wisely it is possible to make all coordinates integers (thus saving a character for the decimal point). Finally coordinates are typically expressed relative to the previous point (counting control points as points), again reducing the typical number of digits per coordinate from three to two or one. This allows

```
\begin{sdpgf}{0}{0}{60}{-124}{0.15pt}
  \m 41 -51 \C 20 20 -46 10 0 16 \S \m 19 -51 \C -20 20 46 10
  0 16 \S \m 30 -119 \L 0 41 \S \ov 14 -78 32 32 \S
\end{sdpgf}
```

to suffice as code for drawing the network  . Fitting several times that on a screen page is quite trivial.

The utility procedure network_as_LaTeX included in the network library sources is also worth mentioning: it takes a pure network as argument and generates LaTeX code to draw it. All networks in this paper were drawn by code generated that way.

17.5 The Completion Utility

The normal way of applying the completion utility is to use the standard LaTeX DocStrip [16] utility to create an "amalgamation" script with all required packages, the main program, and finally set-up of a specific completion problem; this is convenient if one wishes to make long runs of the completion utility, perhaps on a remote computer, as there is then only a single file to install and starting it from the command line is trivial. It is however also possible to generate a blank amalgamation without a problem set-up; the lines-of-code figure in Sect. 17.3 report on that arrangement. Accordingly, there is in the completion utility at present no graphical user interface for setting up a completion problem, only for monitoring its progress, inspecting results, and starting/stopping processing.

The state of the completion is for most part being kept in a database, primarily to ensure *persistence*—if we're willing to keep it running for a day or more, then we don't want to lose our work due to a power outage or system crash—but also to not have it severely limited by available RAM; those who have tried running a *large* Gröbner basis calculation within a standard computer algebra system often discover

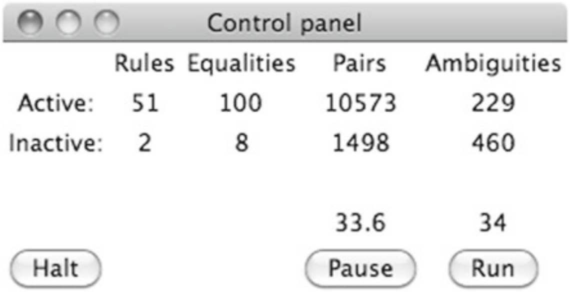


Fig. 17.6 Control panel window

that memory is the limiting factor much more than raw processing power. Having data written to disk could potentially be a bottleneck, but in practice databases are quite good at caching frequently needed data in memory, and definitely more sophisticated in their allocation of resources than any ad hoc solution we could hope to implement ourselves. The problem set-up part of an amalgamation script is typically written so that it either continues processing of the problem in an existing database, or creates a new database and enters the given completion problem into it.

The objects that undergo rewriting are formal linear combinations of networks (with a common set of feedbacks), wherein two networks count as the same monomial if they are isomorphic. The coefficients can be taken from any field (of which an implementation is available); it is a requirement in the algorithm that any nonzero coefficient has a multiplicative inverse. Congruences are stored as *rewrite rules* with just the leading monomial on the left hand side, whereas the right hand side is a general linear combination. A rewrite step $a \to b$ that manages to apply a rule $\mu \to \sum_{i=1}^{n} r_i \mu_i$ consists of finding the left hand side μ appearing as a subnetwork of some term $s\nu$ of a, then constructing networks $\nu_1, \dots, \nu_n$ by replacing this μ part of ν by each of $\mu_1, \dots, \mu_n$, and finally producing $b = a + s\bigl(-\nu + \sum_{i=1}^{n} r_i \nu_i\bigr)$. Technically these $\nu_i = \lambda \rtimes \mu_i$, where λ is the network that remains when one detaches (Fig. 17.4) the μ region from ν; in particular $\lambda \rtimes \mu = \nu$.

When run with a GUI, the completion utility has a control panel (Fig. 17.6) with three push-buttons ‘Halt’, ‘Pause’, and ‘Run’; when halted, then entire state of the computation performed so far is stored on file in the database and nothing is lost by quitting the utility, whereas when paused it may hold some intermediate results in RAM. Either way, computations may be resumed by pressing run. All other interface elements are for inspecting the current state of the completion; normally those are sufficiently responsive even if the completion is running, but the option of pausing exists to provide a way of ensuring that the user interface gets full attention.

17.5.1 Algorithms

Before getting into the detail choices made when implementing completion in this utility, it seems appropriate to recall how a basic completion algorithm works. There are two main tables: that of rewrite rules (the rewrite system) and that of ambiguities (also known as critical pairs, overlaps, etc.). The rewrite rules table constitutes the current approximation of the sought result, whereas the table of ambiguities constitutes a list of cases that must be checked before this current rewrite system can be declared complete; the ambiguities table is also a record of the outcomes of all these checks. Sometimes a check fails, and then this is overcome by adding a new rule to the rewrite system, but that also contributes new ambiguities to that table, so it may go "one step forward, two steps back." In the case of completion in commutative polynomial rings (classical Gröbner bases theory), it is well-known that this procedure must eventually terminate by Dickson's Lemma, but from noncommutative polynomial rings and up the completion of a rewrite system may indeed turn out to be infinite, in which case the procedure never terminates. It can however be proved that *if* every ambiguity is guaranteed to be processed within finite time *and* a finite completion exists *then* this completion procedure will eventually find it and thereafter terminate.

To allow for interaction with the completion utility, the completion procedure is run through the event loop, with new tasks scheduling themselves to run as soon as the process goes idle. Some tasks may potentially take a long time to complete, and are therefore split up over several subroutines which each return to the event loop upon completion; likewise high level loops are unrolled to only perform a limited number of iterations before returning to the event loop. The `completion_main_loop` maintains a stack of tasks to process, specifying both a subroutine to call and the data to pass to it, and the difference between a halt and a pause is that halts allow the main loop stack to become empty before stopping, whereas a pause may happen with data still on the main loop stack.

Topic 14 (Monomial order) An ordering of networks (monomials) is needed to identify the leading term in a linear combination, so that it can be taken as the left hand side of a rewrite rule. Such orderings are typically constructed by first comparing one parameter, then if that comes up equal comparing a second parameter, if both come up equal comparing a third parameter, and so on; in commutative Gröbner basis theory these "parameters" can all be taken to be different weightings of degree. To simplify defining such lexicographic orderings, the ordering is implemented by means of *comparison key*—effectively the list of values of these parameters, in the order that they are considered—that is computed for each network that needs to be compared: to compare two networks, the utility looks at their comparison keys.

A complication is that comparisons which take into account the graph structure of networks need to involve quantities that are more sophisticated than mere weighted vertex counts (which is what polynomial degree would correspond to); the simplest choice of quantity that does the trick may instead turn out to be a matrix. A related complication is that orderings of general networks typically have to be partial—allow for two networks to be incomparable—since *total* orderings that are compatible with the PROP operations are insensitive to the graph structure [3, pp. 31–32]. These matters are dealt with by two refinements of the comparison key mechanism.

First, each element of the comparison key has its own comparison command; this allows for using arbitrary data as comparison key elements. Second, the comparison key is logically divided into *blocks* for the lexicographic aspect: comparisons only advance to the next block if all elements in the current block compare equal, there is a strict inequality if at least one comparison in the current block comes up strict and the rest agree or say equal, and two networks come out incomparable if there are comparisons in the same block that come out strict in opposite directions. In the case of matrices, it is easier to dump all their elements as one block in the comparison key than it is to set up a custom command for comparing matrices. A consequence of this is however that the comparison keys tend to be quite long; having over 100 elements is not unusual.

The main subalgorithm in the completion procedure is that of *reducing* a general element (formal linear combination of networks) to what with respect to the current set of rewrite rules is a normal form. In principle that amounts to testing every network that appears against every rule in the system, for whether this rule can be applied to rewrite that network and if so do that, but in practice there are ways of lessening this workload. Those that have to do with selection of rules are treated in Sect. 17.5.2, but a more elementary matter is the order in which the networks are considered. Since the left hand side of a rewrite rule is always strictly larger in the monomial order than anything on the right hand side, it is a standard strategy to work in descending order of monomials; at the very least this avoids having to process the same monomial twice, and quite often it means some monomials disappear before we even get to them.

Topic 15 (Representation of linear combinations) The basic way to represent a formal linear combination of networks in Tcl is as a dictionary (hashmap), with canonised networks as keys and coefficients as values; in the typical case this allows for $O(1)$ access to individual terms. However, in the case that one wishes to represent a list of such formal linear combinations (for example the steps of a proof) and the same network is likely to appear in several list elements then it can be more compact to have one joint table associating each network with an index, and then represent the formal linear combinations as dictionaries with these indices as keys and again coefficients as values.

A downside of using a hashmap is that it places the terms in a completely arbitrary order, which means applying the standard rewriting strategy would require that we at each step look at all terms to determine which one is the largest; complexity-wise this nullifies any advantage we could have of $O(1)$ access to individual terms. Therefore one would for formal linear combinations of networks undergoing rewriting like to employ a different data structure, which keeps the terms sorted and preferably provides fast access to the largest term. The computer science literature knows at least two data structures that provide $O(\log n)$ access to arbitrary terms and $O(1)$ access to the leading term: threaded self-balancing trees and skip lists.

Self-balancing trees (of which there are a myriad of variants) are standard material in the computer science curriculum, but skip lists [18] tend to receive less attention; perhaps in part because they are probabilistic and thus attain their complexity bound on average rather than in worst case, but for us average complexity and implementation simplicity are what matters. The main reason for choosing skip lists in the completion utility is however that their search model is one of shrinking a closed interval rather than bisecting an open interval; one keeps track of both endpoints of the interval where a sought node is to be found. This is relevant because in the traditional analysis of these data structures it is typically assumed that comparing two keys is an atomic operation, but the comparison keys of networks are anything but atomic. When searching for a node in one of these skip lists, one can advance either to the next level in the data structure or to the next element of the comparison keys, meaning the length of the comparison key m and length of skip list n contribute roughly as $O(m + \log n)$ to the complexity, provided one keeps track of how many key elements are in fact equal throughout the current interval. If instead starting every comparison from the start of the key, the complexity would be more like $O(m \log n)$.

At each rewrite step, the monomial at the head of the skip list is popped off and processed. If no rewrite rule applies to it, then that term is added to a companion hashmap dictionary, whereas if a rule is found that applies then terms corresponding to the right hand side of this rule are added to the skip list formal linear combination. When the skip list is empty, the normal form can be found in the hashmap dictionary.

For each ambiguity, there is a formal linear combination of networks (corresponding to the S-polynomial of Gröbner basis theory) that should reduce to 0 for this ambiguity to be resolvable. If it does not, then the normal form is a new nontrivial congruence, which extends the table of rules. Left hand sides of new rules are tested against the left hand sides of existing rules, and if an old rule turns out to have a new one as subnetwork, then the status of that old rule is changed from *active* to *dropped*: it will no longer participate in neither reductions nor generation of new ambiguities, because anything that the old rule could reduce can equally well be reduced by the new rule. There will however be one final inclusion ambiguity between the old and the new rule, since the right hand side of the old need not correspond to the right hand side of the new.

Topic 16 (Ambiguity processing order) The basic strategy for processing ambiguities is to simply pick them in the order they are generated, since this ensures that every ambiguity is processed eventually, but it is well known in Gröbner basis theory that focusing on "small" ambiguities can drastically speed up overall runtime by letting important small rules be discovered faster. Therefore the ambiguities carry a heuristic "size" attribute, which determines the priority of an ambiguity in the queue of these. As long as the number of ambiguities below any particular size is bounded, eventual processing is still guaranteed.

One axis of available algorithm variants concern what to use as this size heuristic. The choice which has seen most use is the sum, over all terms in the ambiguity, of the squares of the orders (number of inner vertices) of the networks; for example the number 34 in Fig. 17.6 arises as $5^2 + 3^2$ for one term with an order 5 network and one term with an order 3 network.

The utility *does not* go through the right hand sides of old rules to check if a new rule could reduce these further—what in the Gröbner basis tradition would be known as maintaining a reduced basis. The reasoning here is primarily that such efforts would be spent on tinkering with congruences already discovered rather than seeking new ones, at best in the hope of speeding up future reductions, but in practice with a rather low yield in that regard. Moreover there is a definite risk that rules being examined for further reductions will later be dropped due to the discovery of a better rule; then any effort spent on additional reduction on their right hand sides is completely wasted.

Besides the active/dropped distinction, the utility also makes a distinction between *equalities* and proper *rules*, even though both have the same format and are stored in the same table. A proper rule is made from a congruence which has a unique maximal monomial, whereas discovered congruences with multiple maximal monomials (due to these being incomparable) give rise to one equality for each maximal monomial, where this monomial has been singled out as the left hand side. Equalities do not participate in reduction, but they participate in ambiguity generation just like proper rules; the purpose of this is to ensure that no information is lost, in the sense that all ways of combining known congruences to yield new ones will be explored, even if some of those known congruences are not orientable. Sideways deduction steps by way of an equality may be less efficient than the reduction steps performed by a proper rule—search-wise the sideways steps just try every direction possible, whereas the reduction steps follow a plan (only go down in the order)—but both are equally valid as steps in a mathematical deduction.

Equalities, like proper rules, will be dropped if their left hand sides become reducible by a new rule. As long as there are active equalities, the completion procedure will not have produced a confluent rewrite system, but if a complete system of proper rules exists then the completion procedure should eventually find it, possibly by using equalities as intermediate steps in the deduction of these rules. The task of designing an ordering which under which a confluent system of orientable rules exists is hard, and conveniently left to the user.

Topic 17 (Lazy ambiguity generation) Given the tables mentioned so far—that of rules and that of ambiguities—the obvious way of structuring the code is to generate all ambiguities involving a rule or equality as soon as that is added to the database; the sources describe this as the *eager* algorithm variant. Empirically this variant would even for rather small rule databases spend over 90% of its time generating ambiguities (effectively making lists of cases to explore later) and thus less than 10% of its time resolving ambiguities (actually proving stuff and discovering new lemmas). Besides the slow progress, this also carries a considerable risk of outright wasting effort, since if a rule or equality is dropped then the only additional ambiguity of it that we need to resolve is the inclusion with the new rule prompting the drop; the others become irrelevant. In theories where important identities exist which are not given as explicit axioms—for example $(ab)^{-1} = b^{-1}a^{-1}$ in group theory—the normal pattern for completion is that a large number of special cases are discovered before discovery of the simpler general case causes them to be dropped. For each of those special case rules not immediately involved in proving the general case rule, the effort spent on generating ambiguities will have been completely useless! Better then to no be so eager.

The lazier algorithm variant makes use of a third table of *pairs* of rules, which lists those unordered pairs of rules that have not yet been considered for ambiguity generation. Adding rows to this table is quick, and it is equally quick to drop all pairs containing a rule that is dropped. Delaying ambiguity generation pretty much reversed the percentages for how time was spent, so that with this non-eager algorithm variant reducing far outweighed ambiguity generation.

Being lazy does however complicate the matter of ambiguity processing order, since even a heuristic size cannot be known until the ambiguity is actually generated. The implementation is that also each pair comes with a value for the size heuristic, which is set at pair creation time based on statistics for ambiguities of the rules in question; at each processing step, the algorithm either picks an ambiguity and processes that, or picks a pair and generates any corresponding ambiguities, based on which table currently shows the lowest value for the size heuristic. In Fig. 17.6, the 33.6 for pairs versus 34 for ambiguities means the next thing picked will be a pair. That the ambiguities table has 229 active plus 460 fully processed ambiguities also means that the majority of the 1498 pairs so far fully processed did not give rise to any ambiguity; searching for a way of forming one still takes time, though.

For the sum of orders squared heuristic for ambiguity size, the unknown quantity that must be estimated is the cardinality of the overlap, i.e., the number of vertices which are common to the left hand sides of the two generating rules; in Fig. 17.5, G_1 and G_3 are cardinality 1 overlaps, whereas G_2 is a cardinality 2 overlap. The estimate used for overlap cardinality is simply the average, over all ambiguities so far generated that involve one of the rules, of that overlap cardinality. It may be argued that since the size heuristic (for two given rules) is a second degree polynomial of the overlap cardinality, an unbiased estimate of the sum of orders square heuristic should take into account also the second moment of the overlap cardinality stochastic variable; this would be quite easy, but at the time of writing the completion utility does not record enough information.

A more interesting possibility is to go beyond uniform averages, and instead base these size estimates on similarity with pairs already checked. In large rule tables, it turns out the left hand sides of rules have certain "active regions" often being involved in overlaps, and other regions which are not; for an ambiguity to arise, the active regions of two rules need to match (cf. active sites of molecules in biology). There is no a priori way of knowing what will constitute an active region, since this is an emergent property that depends on the population of rule left hand sides as a whole, but if networks $G_1, G_2, \ldots, G_m$ have the same overlap with network H_1, and in addition G_1 has such an overlap with networks $H_2, \ldots, H_n$, then one would expect that G_i for $i = 2, \ldots, m$ also has such an overlap with H_j for $j = 2, \ldots, n$. As the numbers of m and n of active rules with the same kind of active region increases, the number of overlaps that can be predicted from comparison with the response to a few test rules grows as mn; this holds promise of more accurate predictions than one can get from mere averages.

17.5.2 The Database

The completion utility stores its data in a standard relational database, accessed via the TDBC [11] interface. To date the only database engine employed has been SQLite [10], since the same-process architecture of SQLite simplifies deployment; otherwise it is traditional for database systems to follow a server–client architecture, where the server need not even run on the same hardware as the client.

Topic 18 (Parallelisation of completion procedure) Keeping most of the state for the completion procedure in a database suggests an easy route to parallelising the whole computation, namely to have several completion utility processes (workers) connect as clients to the same database server; the number of ambiguities or pairs processed per minute then scales linearly with the number of clients running, whereas scaling the database server is mostly a matter of choosing an appropriate engine. The completion utility has to date *not* been deployed in such a parallel manner—the most obvious piece of functionality missing is for clients to mark an ambiguity or pair as "checked out" before they start working on it, so that they don't merely duplicate the work of each other—but the fact that it all works through a database server means all low-level concurrency problems are automatically taken care of. (An example of a high-level concurrency problems would be what to do if two workers simultaneously derive the same rule, even though they worked on different ambiguities.)

The completion utility has been written to abstract the actual commands (statements) given to the database, so adapting it to a different database engine (potentially speaking a different dialect of SQL) should be straightforward. Based on advice to avoid English words as identifiers in the database, since any such word *might* potentially be claimed as a reserved word by some obscure SQL dialect, it has furthermore been a (possibly misguided) design decision to use `l33t` (leet) orthography for identifiers not otherwise containing a digit; thus the tables of rules and ambiguities are named `ru1es` and `am6iguities`, respectively. Both tables have one column `num6er` which constitutes the primary key and one column `st4te` which declares the current status (e.g. active/inactive, rule/equality) of the table row.

Content-wise the main columns of the rules table are `lhs` (left hand side of rule, a single network), `rhs` (right hand side of rule, a dictionary mapping networks to their coefficients in the linear combination), and `feed6acks` (the feedbacks, as explained in Sect. 17.4.2). For auditing there are columns `pr00f` (the number of the ambiguity providing the derivation of this rule), `wh3n1` (point in time at which this rule was derived), and `wh3n2` (point in time at which it was dropped, if no longer active). Statistics pertaining to a particular rule that are used for estimating ambiguity size are also kept in this table.

Topic 19 (Network profiles) The majority of columns in the rules table are however part of something called the *profile*, which is used to help with another problem: how does one find appropriate rules for reducing a network? It is highly desirable that a first screening is done already in the database, since the less the volume of data that needs to be transferred from database (potentially in slow storage, such as a disk) to the completion utility, the better. The profile contains detailed counts of various *features* that occur in the network, and any rule with a higher count of some feature than the network we seek to reduce can immediately be eliminated from consideration. While elementary, this form of condition is simple enough to be tested directly in SQL, and has empirically made it feasible to keep making progress even when the table of rules is quite large.

The first part of the profile is the vertex counts (columns $\mathtt{v}\langle k\rangle$), which for each vertex type in the signature counts how many vertices with that decoration there are in the network; this is analogous to keeping track of the multidegree of monomials in commutative Gröbner basis theory. The second part of the profile is the edge counts (columns $\mathtt{e}\langle k\rangle$), which similarly keep track of how many instances of each type of edge there are in the network; in this context, the *type* of an edge expresses not only the types of the vertices at the head and at the tail of the edge, but also to which ports of those vertices the edge connects. This is weaker than keeping track of the counts of all connected 2-vertex subnetworks (since sometimes a pair of vertices may be connected by more than one edge), but only marginally so. External edges of a network do not count in the profile, since the vertex at one end in that case is missing; whatever information could be had from adding up the possibilities for what it could be is already implicit in the vertex and edge counts already provided.

The column names deliberately do not attempt to encode the full data defining a vertex or edge type; instead their meaning depends on the signature, down to the order in which it lists the vertex types. It is not expected that network profiles can be decoded, only that they can be consistently computed.

There is an old algorithm variant providing profile columns also in the ambiguities table, to support using a kind of "Buchberger's Second Criterion" to avoid reducing some ambiguities; the gist of this criterion is that if the site of an ambiguity of rules s_1 and s_2 is acted upon also by a third rule s_3, then anything that can be derived from the (s_1, s_2) ambiguity also follows from combining the (s_1, s_3) and (s_2, s_3) ambiguities that would exist here, so reducing the (s_1, s_2) ambiguity is redundant. The eager aprofile variant of the algorithm would use this profile to screen for ambiguities that a newly added rule or equality would render redundant. The lazy algorithm variant instead performs this check when an ambiguity has been picked for reduction, which means we take (s_1, s_2) as given and search the rules table for s_3 rather than taking s_3 as given and search the larger ambiguities table for (s_1, s_2). Apart from making do with less stored data, this also seemed to progress faster, presumably for reasons similar to those discussed in Topic 17. Contrary to what the literature on commutative Gröbner bases suggest however, this Buchberger's Second Criterion only rarely seems to apply—possibly because matches essentially happen by chance, and the far more structured nature of networks compared to commutative power produces make such matches unlikely—so nowadays the normal configuration is to not even test it.

One group of columns in the ambiguities table characterise an ambiguity and remain fixed in each row after it is created; this includes `si7e` (the network that can be reduced in two different ways), `feed6acks` (its feedback patterns), the numbers `p1us` and `m1nus` of the two rules from which it was formed, and the respective regions `p1us2eg` and `m1nus2eg` that these rules would replace. The two rule numbers effectively determine what data was passed to `network::wfb::groomed_ambiguities`, and the rest are one match which that subroutine returned. A second group of columns is used to record the resolution of this ambiguity. The `re5olution` is the list of rewrite steps, starting with the difference between the result of applying the `p1us` rule to the `si7e` and the result of applying the `m1nus` rule to the `si7e`; this is the counterpart of the S-polynomial in Gröbner basis theory. The networks are stored in the separate `m0n0mials` list, whereas each element of the `re5solution` is a dictionary mapping the index of a network to its corresponding coefficient in this step, as mentioned in Topic 15. The `wh1ch` column is the list of rules (one per step of the `re5olution`) which have been applied, and the `wh3r3` column is a list stating to which monomial and to which region in that monomial the rule was applied. The reason the rule numbers are kept in a separate column is to facilitate a database query finding all ambiguities where a particular rule was used.

The by far largest of all columns tends to be the `m0m0mials`, which frequently holds several thousand characters per database row. There is an algorithm variant where there instead is a column `deflated_m0n0mials` that holds the Deflate [2] compression (a binary object) of the `m0n0mials` data; this provides a considerable reduction in the overall database size. It has not been examined how much the two steps of that compression—LZ77 sliding window dictionary and Huffman encoding—each contribute to this, but both have low hanging fruit to pick: there is definitely substring repetition (such as vertex types and list markup), and the set of characters used is for most part quite small (space, digits `0–9`, braces `{}`, and the letters found in vertex type names).

Topic 20 (Dense encoding of pairs) Even if pairs contain much less data than ambiguities, the fact that the number of pairs grows quadratically with the number of rules means the total `pa1rs` table after a while becomes a major factor in the overall database size. Asymptotically that is unavoidable, but within the bounds of realistic computational efforts substantial improvements are attainable through attention to how pairs are encoded. The `pa1rs` table is essentially a sparse encoding of a rather dense set of pairs of integers (even if there is also for each pair the estimated value of the selection heuristic), which means the overhead is considerable. The pairmap algorithm variant instead keeps most pairs in a `pa1rMap` table, where each pair is allocated just one bit in a binary object.

The idea is that pairs with a small value for the selection heuristic go into the sparse `pa1rs` table where this heuristic is explicit, whereas pairs with a large value for it go into the dense `pa1rMap` table which only records whether the pair is in there or not. The boundary between small and large is determined by a variable `pairs_top` which is raised whenever the `pa1rs` table is depleted; when that happens, the `pa1rMap` table is scanned, selection heuristics are recalculated from current overlap statistics, and anything falling below the new boundary is moved to the `pa1rs` table instead. This way the `pa1rs` table operates as a dynamic priority queue of pairs to process soon, whereas the bulk of pairs to process later are kept in compact storage.

Topic 21 (The use of SQL indices) A feature in SQL is that databases can have *indices*—essentially shadow tables which duplicate only some columns from a proper table, and are automatically updated whenever the proper table is updated—that facilitate fast access to particular data; database engines can sometimes answer queries using only the data in some index, rather than having to consult the table proper. As long as the indices can fit in RAM, it need not matter much that the table as a whole is large and has to be stored on disk. All of these observations are trivialities in the field of databases, but possibly unfamiliar to people coming from computer algebra, and thus worth mentioning as it really can make a huge difference. The completion utility has indices on the selection heuristics and state columns of ambiguities and pairs tables, and on the state and profile columns of the rules table, both of which need to be searched frequently. For inspecting the database, there are also indices on the columns saying *when* something (e.g. rule added, rule dropped) happened.

One final table in the database is `le9end`, which is essentially a dictionary (with one row per entry) of miscellaneous information describing the problem set-up. Data that may be recorded here include the `signature` and the coefficient ring that define the free PROP in which the calculations are carried out.

17.5.3 Inspecting Database Contents

When the aim is to discover new identities, it is essential that one can inspect and export the contents of the database, in some form that is comprehensible to a human reader (possibly after rendering by standard software). The completion utility offers a number of routes for this, even if the variety more reflects the development history and ideas about how material *could* be presented, than any coherent plan for covering all reasonable needs.

17.5.3.1 Interactive Introspection

The interactive methods of inspection look at one rule at a time, always presenting the current state of the database. They are suitable for monitoring the progress of the

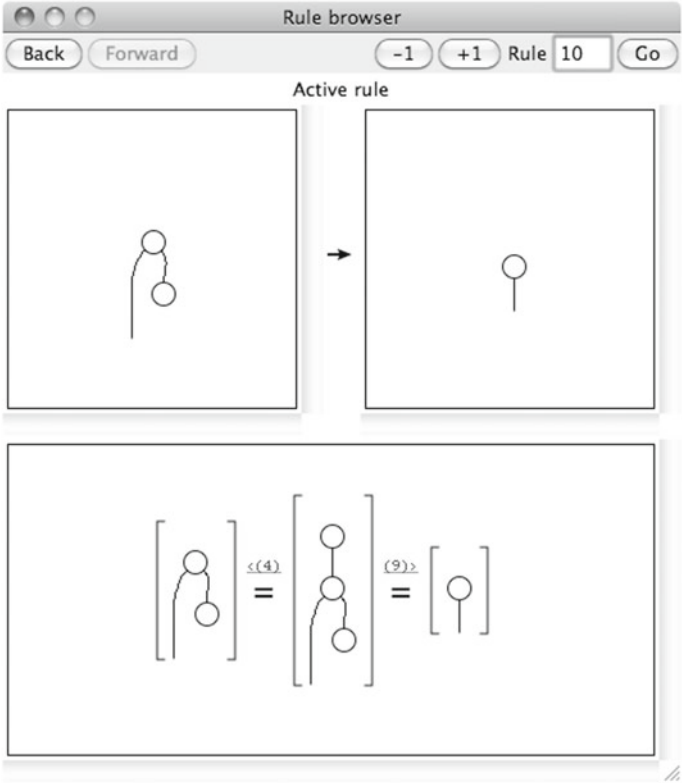


Fig. 17.7 The rule browser window

completion process, providing information on what kind of rules are being derived, which allows the user to make an informed decision on whether to stop the process or let it continue.

When running the completion utility with GUI, there is apart from the control panel window also a *rule browser* window, as shown in Fig. 17.7. This window has three resizable panes, each a separate `canvas` [21] widget. The top left pane shows the left hand side and feedbacks of a rule or equality, the top right pane shows the right hand side, and the bottom pane shows the derivation of this rule, or the word 'Axiom' if it one of the starting congruences for the completion. The material presented in a pane may be geometrically larger than what fits in that pane; in that case the user may grab the contents by holding down the mouse button and then drag them around. There is also a top line specifying the state of the item: proper rule or equality, active or dropped, and in the latter case when time it was dropped.

Topic 22 (Rule space navigation) Elementary controls provided for moving around in the rule space include jumping to a specific rule by number, as well as incrementing or decrementing the number of the current rule by one; more interesting is the feature of jumping to the rule used in a specific step of a proof. In the proof pane, each rewrite step carries an annotation giving the number of the rule (or equality) applied, together with an indication of in which direction it was applied, and this annotation text constitutes a link which when clicked takes the browser to that rule—this is extremely convenient when the rule applied is not a familiar one. Also convenient is that the rule browser remembers the path you have followed in your browsing, and has buttons also for going back and forward along that path, just like a web browser.

A significant limitation of the rule browser is however that it cannot show rules with more than one term on the right hand side, nor can it display coefficients that are $\neq 1$; if either would be at hand in a pane, then that pane will show no content. The reason for this limitation is the complexity of implementation: a coefficient can potentially be an element of an arbitrary field, which makes the problem of rendering it on a Tk `canvas` equivalent to the problem of rendering an arbitrary mathematical formula—certainly a worthy task, but not the one we primarily set out to address. In addition we encounter nontrivial layout problems even for coefficient fields where every element can be rendered as an integer, since having several terms per proof step makes it more interesting to line break the proofs for easy viewing, but according to which principles? Again, this is not our primary task, and for many interesting completion problems not even one that needs addressing, since all rules in fact have right hand sides with only one term and its coefficient is 1.

Still, these limitations are lifted in the alternative *web interface* for rule browsing. When this is active, the completion utility runs a tiny web server with a status page (Fig. 17.8a) and one page per rule in the database (Fig. 17.8b). These pages are XHTML with embedded SVG for the networks and (where necessary) MathML for coefficients, which is a classical combination of technologies; the formula rendering and layout problems are thus handed over to the user's web browser. Rather than drawing feedbacks, the web interface presents the transferrence type of a rule, as one of 'All' (all 1s matrix), 'None' (all 0s matrix), or an HTML table.

Unlike the GUI control panel, the web interface does not provide any controls for starting and stopping the completion process. This is for security reasons; whereas it would be possible to provide also that in a web interface, the overhead for authorisation of such remote commands is quite considerable. (Even a simple password mechanism for authentication would necessitate encrypting the connection, and the deployment complications this brings up are nontrivial.) Instead the standard set-up for a GUI-less version of the completion utility is that it quits in an orderly fashion when a specific child process terminates, which (since that child process just waits forever) happens when the user forcibly terminates it. Effectively this hands the authorisation problem over to the operating system, where it should anyway already have been resolved.

Even without control functionality, running a web server could be considered a security issue. The minimal embedded web server does not at present restrict access to being local, although operating system firewall settings may (and increasingly do) impose such restrictions. The embedded web server does have some resilience against denial-of-service attacks, but it would not stop an attacker from viewing each

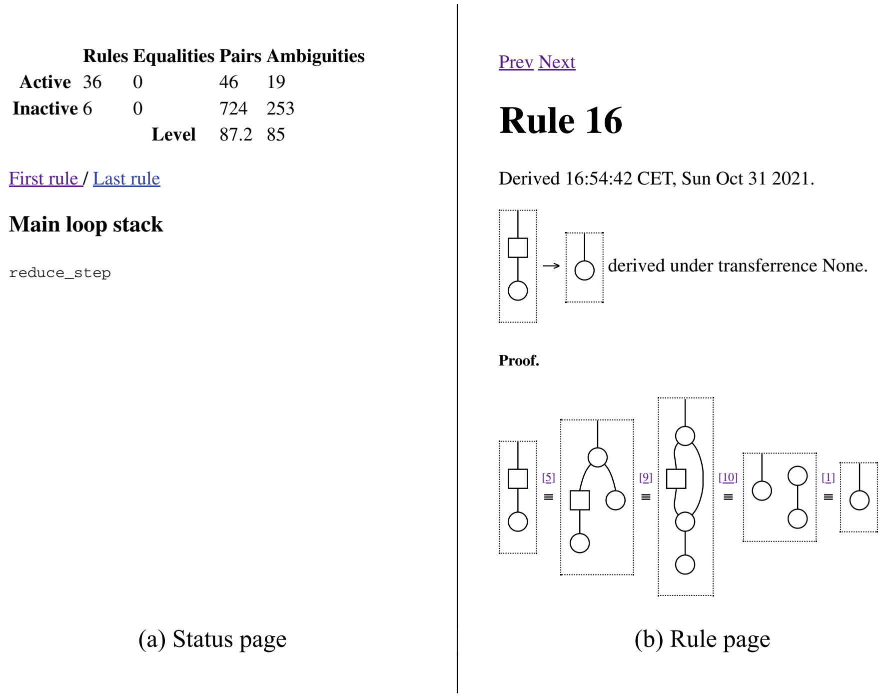


Fig. 17.8 Web interface

and every rule in the database. On the other hand science should be open, so for most probable users this might not be that big a deal.

17.5.3.2 Data Export

Compiling a database of rules implied by a given axiom system is of limited use if one cannot export its contents, to make use of them elsewhere, and realistically the main consumer within the foreseeable future will be a math paper. That means we need to export to LaTeX.

There is no graphical interface for exporting data; rather one should start the completion utility within an interactive shell and then type explicit commands to order the export. The `rule_as_LaTeX` command takes one rule number as argument and returns LaTeX code for that rule, whereas `database_as_LaTeX` takes the name of a file to create, which will be a LaTeX document with all the rules. `ambiguities_as_LaTeX` rather take a list of ambiguity numbers as first argument, and returns LaTeX code for their resolutions, regardless of whether those ended up prompting the creation of a new rule or not; this latter thing is what one would need for a proof of the completeness of a rewrite system.

All of these commands also take options to further configure the output. Most are simply passed on to the library routines for generating the graphical representation of the networks (Sect. 17.4.3), such as `-unit` and `-division` which determine the geometric size of the networks generated, but some control higher level aspects of the output generation. In particular `rule_as_LaTeX` and `database_as_LaTeX` have an option `-style` with supported values `equation` (default) and `theorem`. In the equation style, the export of a rule consists of one `equation` environment with the derivation of that rule; examples of this appear in Sect. 17.1. In the theorem style, the export of a rule is rather an entire lemma, where first equivalence of the left and right hand sides is asserted, and then the derivation is provided as its proof; examples of this appear in Sect. 17.2. Other options provide for configuring `\labels` assigned and environments used.

Topic 23 (Machine-readable export) Whereas an export to LaTeX may meet one's short-term needs, the long-term rather requires that data be exportable as machine-readable mathematical objects with full semantics preserved. Concretely that means having OpenMath [1] as an export format. Unfortunately that was not trivial, as much of the required vocabulary did not yet have a machine-readable formalisation; the `tensor2` content dictionary that provides for network notation encoding of expressions was only presented at the 2017 OpenMath workshop, long after main development of the completion utility version 2 was finished. A proper OpenMath export would in addition need symbols to express concepts coming from universal algebra and rewriting, that still awaits formalisation.

Yet the reader who examines the export-to-LaTeX mechanisms will discover that there is a noticeable amount of OpenMath in there; this is due to me already having an OM-to-LaTeX conversion codebase written, that could be leveraged by the completion utility. The intermediate OpenMath encodings of linear combinations of networks that are being generated in this export route will however express the networks as foreign objects with a pregenerated LaTeX encoding, and do not sensibly meet the requirement of encoding their data with semantics.

17.5.4 Included Completion Problems

The source for the completion utility also contains a number of "examples" of completion problems, which it would perhaps be more fair to describe as: the collection of completion problems that the author has coded but not gotten around to find a separate home for yet. They may certainly be read as examples of how one might code a problem of one's own, but several have accumulated a lot of DocStrip module guards which can make them hard to read; in that case it may help to also look at the stripped sources produced by TeXing `cmplutil2.ins`.

As mentioned in Sect. 17.1, these examples need to do three things: set the signature, enter the axioms, and define the ordering. A full explanation of how one does that would carry this text over into user's manual territory, but some of the key points are still worth stating.

First, there are two sides to specifying the signature: the appearance and the formal mathematical signature. By setting the `vertex_appearances` variable to a dictionary mapping the symbols to their appearances (as per Topic 11) one takes care of the former. This provide sufficient information also to compute the formal signature, but is not used to that end; it may well happen that examples specify an appearance for symbols not in the signature. Instead the formal signature is specified in the call that initialises the database file, since it determines how many columns there are in the profile. The command for this is `sqlite3_init`, and the signature is specified as a dictionary mapping each symbol x to the pair $\{\alpha(x)\ \omega(x)\}$ of its arity and coarity; the same kind of dictionary is used in calls to `network::pure::construct`.

On the same level as the signature is also the matter of what field the coefficients live in. This is specified by setting the variable `coefficient` to a sentence prefix implementing that field, but the included problems are typically fine with the default choice of integers modulo 32003 and therefore do nothing in this area. Changing the coefficient field might mean the introspection mechanisms need help generating a presentation of this data; the relevant commands have options related to this.

The second thing the included problems tend to do is to define the ordering, which is by far the most complicated of the three steps. Concretely, a completion problem set-up needs to define a command `set_cmpcmds` which gets called once for every ambiguity that gets picked. The function this command needs to perform is to set three entries of the `RS` (resolution state) array: `cmpkeycmd`, `cmpcmds`, and `cmpblocks`. The `cmpkeycmd` entry is a sentence prefix that takes a pure network as additional argument and returns the comparison key for this network. The `cmpcmds` entry is a list of sentence prefixes that compare individual elements of a comparison key, and the `cmpblocks` is a list of booleans that encode the block structure of the comparison keys. The exact structure of the comparison keys varies with the arity and coarity of the networks being treated, which is why these entries need to be reinitialised for every new ambiguity. Note that the problem set-ups often define not only `set_cmpcmds` but also one or several helper procedures for use in the `cmpkeycmd`.

The third thing done is to enter the axioms. This is most conveniently done using the `enter_congruence` procedure which takes as arguments the coefficients and networks of a formal linear combination that is to be held equivalent to 0; the networks are specified as scan-lists (Topic 1). Each congruence also has a list of feedbacks, but most axiom systems encountered in the literature have those lists all empty. An example of how such a block of commands to enter the congruences can look may be seen in Fig. 17.9.

Because opening an existing database file simply lets you resume processing from the state recorded in that database, many of these included problems have code that skip the axioms if there already are ambiguities in the database.

```
enter_congruence -short "Unit&counit" {} 1 {unit \n epsilon} -1 {}
enter_congruence -short "Left unit" {} 1 {unit . \n m} -1 {.}
enter_congruence -short "Right unit" {} 1 {. unit \n m} -1 {.}
enter_congruence -short "Associative" {} 1 {m . \n m} -1 {. m \n m}
enter_congruence -short "Left counit" {} 1 {Delta \n epsilon .} -1 {.}
enter_congruence -short "Right counit" {} 1 {Delta \n . epsilon} -1 {.}
enter_congruence -short "Coassociative" {}\
    1 {Delta \n . Delta} -1 {Delta \n Delta .}
enter_congruence -short "Unit&coproduct" {}\
    1 {unit \n Delta} -1 {unit unit}
enter_congruence -short "Product&counit" {}\
    1 {m \n epsilon} -1 {epsilon epsilon}
enter_congruence -short "Product&coproduct" {}\
    1 {m \n Delta} -1 {Delta Delta \n . X . \n m m}
enter_congruence -short "Left formal inverse" {}\
    1 {Delta \n S . \n m} -1 {epsilon \n unit}
enter_congruence -short "Right formal inverse" {}\
    1 {Delta \n . S \n m} -1 {epsilon \n unit}
```

Fig. 17.9 Entering the axioms of a Hopf algebra

17.6 Availability

All the source code can be found in the git repository at https://github.com/lars-hellstrom/algebra.

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Chapter 18
Double Constructions of BiHom-Frobenius Algebras

Mahouton Norbert Hounkonnou, Gbêvèwou Damien Houndedji, and Sergei Silvestrov

Abstract This paper addresses a Hom-associative algebra built as a direct sum of a given Hom-associative algebra $(\mathcal{A}, \cdot, \alpha)$ and its dual $(\mathcal{A}^*, \circ, \alpha^*)$, endowed with a non-degenerate symmetric bilinear form $\mathcal{B}$, where $\cdot$ and $\circ$ are the products defined on $\mathcal{A}$ and $\mathcal{A}^*$, respectively, and α and α^* stand for the corresponding algebra homomorphisms. Such a double construction, also called Hom-Frobenius algebra, is interpreted in terms of an infinitesimal Hom-bialgebra. The same procedure is applied to characterize the double construction of biHom-associative algebras, also called biHom-Frobenius algebra. Finally, a double construction of Hom-dendriform algebras, also called double construction of Connes cocycle or symplectic Hom-associative algebra, is performed. Besides, the concept of biHom-dendriform algebras is introduced and discussed. Their bimodules and matched pairs are also constructed, and related relevant properties.

Keywords Hom-associative algebra · BiHom-associative algebra · BiHom-Frobenius algebra · BiHom-dendriform algebra

MSC2020 Classification 17D30 · 17B61

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18.1 Introduction

The Hom-algebraic structures originated from quasi-deformations of Lie algebras of vector fields which gave rise to quasi-Lie algebras, defined as generalized Lie structures in which the skew-symmetry and Jacobi conditions are twisted. Hom-Lie algebras and more general quasi-Hom-Lie algebras where introduced first by Silvestrov and his students Hartwig and Larsson in [25], where the general quasi-deformations and discretizations of Lie algebras of vector fields using general twisted derivations, σ-derivations, and a general method for construction of deformations of Witt and Virasoro type algebras based on twisted derivations has been developed. The initial motivation came from examples of q-deformed Jacobi identities discovered in q-deformed versions and other discrete modifications of differential calculi and homological algebra, q-deformed Lie algebras and other algebras important in string theory, vertex models in conformal field theory, quantum mechanics and quantum field theory, such as the q-deformed Heisenberg algebras, q-deformed oscillator algebras, q-deformed Witt, q-deformed Virasoro algebras and related q-deformations of infinite-dimensional algebras [1, 15–21, 28, 29, 36–38].

Possibility of studying, within the same framework, q-deformations of Lie algebras and such well-known generalizations of Lie algebras as the color and super Lie algebras provided further general motivation for development of quasi-Lie algebras and subclasses of quasi-Hom-Lie algebras and Hom-Lie algebras. The general abstract quasi-Lie algebras and the subclasses of quasi-Hom-Lie algebras and Hom-Lie algebras, as well as their color (graded) counterparts, color (graded) quasi-Lie algebras, color (graded) quasi-Hom-Lie algebras and color (graded) Hom-Lie algebras, including in particular the super quasi-Lie algebras, super quasi-Hom-Lie algebras, and super Hom-Lie algebras, have been introduced in [25, 33–35, 49, 50]. In [42], Hom-associative algebras have been introduced. Hom-associative algebras is a generalization of the associative algebras with the associativity law twisted by a linear map. In [42], Hom-Lie admissible algebras generalizing Lie-admissible algebras, were introduced as Hom-algebras such that the commutator product, defined using the multiplication in a Hom-algebra, yields a Hom-Lie algebra, and Hom-associative algebras were shown to be Hom-Lie admissible. Moreover, in [42], more general G-Hom-associative algebras including Hom-associative algebras, Hom-Vinberg algebras (Hom-left symmetric algebras), Hom-pre-Lie algebras (Hom-right symmetric algebras), and some other Hom-algebra structures, generalizing G-associative algebras, Vinberg and pre-Lie algebras respectively, have been introduced and shown to be Hom-Lie admissible, meaning that for these classes of Hom-algebras, the operation of taking commutator leads to Hom-Lie algebras as well. Also, flexible Hom-algebras have been introduced, connections to Hom-algebra generalizations of derivations and of adjoint maps have been noticed, and some low-dimensional Hom-Lie algebras have been described. The enveloping algebras of Hom-Lie algebras were considered in [53] using combinatorial objects of weighted binary trees. In [27], for Hom-associative algebras and Hom-Lie algebras, the envelopment problem, operads, and the Diamond Lemma and Hilbert series for the Hom-associative operad and free algebra have been studied. Strong Hom-associativity yielding a confluent rewrite system and a basis for the free strongly hom-associative algebra has been considered in [26]. An explicit constructive way, based on free Hom-associative algebras with involutive twisting, was developed in [23] to obtain the universal enveloping algebras and Poincaré-Birkhoff-Witt type theorem for Hom-Lie algebras with involutive twisting map. Free Hom-associative color algebra on a Hom-module and enveloping algebra of color Hom-Lie algebras with involutive twisting and also with more general conditions on the powers of twisting map was constructed, and Poincaré-Birkhoff-Witt type theorem was obtained in [3, 4]. It is worth noticing here that, in the subclass of Hom-Lie algebras, the skew-symmetry is untwisted, whereas the Jacobi identity is twisted by a single linear map and contains three terms as in Lie algebras, reducing to ordinary Lie algebras when the twisting linear map is the identity map.

Hom-algebra structures include their classical counterparts and open new broad possibilities for deformations, extensions to Hom-algebra structures of representations, homology, cohomology and formal deformations, Hom-modules and hom-bimodules, Hom-Lie admissible Hom-coalgebras, Hom-coalgebras, Hom-Hopf algebras, Hom-bialgebras, L-modules, L-comodules and Hom-Lie quasi-bialgebras, n-ary generalizations of biHom-Lie algebras and biHom-associative algebras and generalized derivations, Rota-Baxter operators, Hom-dendriform color algebras, Rota-Baxter bisystems and covariant bialgebras, Rota-Baxter cosystems, coquasitriangular mixed bialgebras, coassociative Yang-Baxter pairs, coassociative Yang-Baxter equation and generalizations of Rota-Baxter systems and algebras, curved $\mathcal{O}$-operator systems and their connections with tridendriform systems and pre-Lie algebras [2, 6–12, 14, 24, 30, 32, 33, 39–41, 43–48, 51–55].

The notion of biHom-associative algebras was introduced in [22]. In fact, a biHom-associative algebra is a (nonassociative) algebra $\mathcal{A}$ endowed with two commuting multiplicative linear maps $\alpha, \beta : \mathcal{A} \to \mathcal{A}$ such that $\alpha(a)(bc) = (ab)\beta(c)$, for all $a, b, c \in \mathcal{A}$. This concept arose in the study of algebras in so-called group Hom-categories. In [22], the authors introduced biHom-Lie algebras (also by using the categorical approach) and biHom-bialgebras. They discussed these new structures by presenting some basic properties and constructions (representations, twisted tensor products, smash products, etc.).

A Frobenius algebra is an associative algebra equipped with a non-degenerate invariant bilinear form. This type of algebras also plays an important role in different areas of mathematics and physics, such as statistical models over two-dimensional graphs [13] and topological quantum field theory [31]. In [5], Bai described associative analogs of Drinfeld's double constructions for Frobenius algebras and for associative algebras equipped with non-degenerate Connes cocycles. We note that two different types of constructions are involved:

(i) the Drinfeld's double type constructions, from a Frobenius algebra or from an associative algebra equipped with a Connes cocyle; and
(ii) the Frobenius algebra obtained from anti-symmetric solution of associative Yang-Baxter equation and non-degenerate Connes cocycle obtained from a symmetric solution of a D-equation.

The aim of the present work is to establish the double constructions of biHom-Frobenius algebras and Hom-associative algebra equipped with a Connes cocyle, generalizing the double constructions of Frobenius algebras and Connes cocycle described in [5] by twisting the defining axioms by a certain twisting map. When the twisting map happens to be the identity map, one gets the ordinary algebraic structures. Furthermore, the bialgebras of related double constructions are built. We define the antisymmetric infinitesimal biHom-bialgebras and Hom-dendriform D-bialgebras.

The paper is organized as follows. In Sect. 18.2, we introduce the concepts of matched pairs of Hom-associative algebras and establish some relevant properties. In Sect. 18.3, we perform the double constructions of multiplicative Hom-Frobenius algebras and antisymmetric infinitesimal Hom-bialgebras. In Sect. 18.4, we define the bimodule of biHom-associative algebras, and achieve the double constructions of multiplicative biHom-Frobenius algebras and antisymmetric infinitesimal biHom-bialgebras. Section 18.5 deals with the double constructions of involutive symplectic Hom-associative algebras. Section 18.6 is devoted to the matched pairs of biHom-associative algebras and related important characteristics. In Sect. 18.7, we end with some concluding remarks.

18.2 Bimodules and Matched Pairs of Hom-associative Algebras

18.2.1 Bimodules of Hom-associative Algebras

Henceforth, when relevant, the multilinear maps $f: V_1 \times \cdots \times V_n \to W$ and linear maps $F: V_1 \otimes \cdots \otimes V_n \to W$, on finite tensor products of linear spaces are identified standardly via $F(v_1 \otimes \cdots \otimes v_n) = f(v_1, \ldots, v_n)$.

Definition 18.1 ([42]) A Hom-associative algebra is a triple $(\mathcal{A}, \cdot, \alpha)$ consisting of a linear space $\mathcal{A}$ over a field $\mathcal{K}$, $\mathcal{K}$-bilinear map $\cdot: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and a linear space map $\alpha : \mathcal{A} \to \mathcal{A}$ satisfying the Hom-associativity property:

$$\alpha(x) \cdot (y \cdot z) = (x \cdot y) \cdot \alpha(z).$$

If, in addition, α satisfies the multiplicativity property

$$\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y),$$

then $(\mathcal{A}, \cdot, \alpha)$ is said to be multiplicative.

Remark 18.1 If $\alpha = \mathrm{Id}_{\mathcal{A}}$, $(\mathcal{A}, \cdot, \mathrm{Id}_{A})$, simply denoted $(\mathcal{A}, \cdot)$, is an associative algebra.

Example 18.1 Let $\{e_1, e_2, e_3\}$ be a basis of a 3-dimensional vector space $\mathcal{A}$ over $\mathcal{K}$. The following multiplication $\cdot$ and map α on $\mathcal{A}$ define a Hom-associative algebra:

$$\begin{aligned}&e_1 \cdot e_1 = e_1,\ e_1 \cdot e_2 = e_2 \cdot e_1 = e_3,\\ &\alpha(e_1) = a_1 e_2 + a_2 e_3,\ \alpha(e_2) = b_1 e_2 + b_2 e_3, \alpha(e_3) = c_1 e_2 + c_2 e_3,\end{aligned}$$

where $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathcal{K}$.

Definition 18.2 A Hom-module is a pair (V, β), where V is a $\mathcal{K}$-vector space, and $\beta : V \to V$ is a linear map.

We will use in this article a definition of bimodule of a Hom-associative algebras including Hom-modules maps conditions (18.4), (18.5), while we note that there are also other definitions of Hom-modules and Hom-bimodules of Hom-associative algebras, for example the more general notions requiring only (18.1), (18.2) and (18.3), [7, 8, 24, 43–45, 52, 55].

In order to avoid, when necessary, the ambiguity of the general category endomorphisms notation $End(L)$ for endomorphisms of L as linear space, algebra or other structure, throughout this paper, we will use the notation $gl(L)$ for the set of all linear transformations on a linear space L, and viewing it context-dependent, as a linear space, as a associative algebra with usual associative composition product, as a Lie algebra of all linear transformations on L with the usual commutator product of the associative composition product (usual notation for Lie algebras), or as other structure type on the set $gl(L)$.

Definition 18.3 Let $(\mathcal{A}, \cdot, \alpha)$ be a Hom-associative algebra and let (V, β) be a Hom-module. Let $l, r : \mathcal{A} \to gl(V)$ be two linear maps. The quadruple (l, r, β, V) is called a bimodule of $\mathcal{A}$ if for all $x, y \in \mathcal{A}$, $v \in V$:

$$l(x \cdot y)\beta(v) = l(\alpha(x))l(y)v, \tag{18.1}$$

$$r(x \cdot y)\beta(v) = r(\alpha(y))r(x)v, \tag{18.2}$$

$$l(\alpha(x))r(y)v = r(\alpha(y))l(x)v, \tag{18.3}$$

$$\beta(l(x)v) = l(\alpha(x))\beta(v), \tag{18.4}$$

$$\beta(r(x)v) = r(\alpha(x))\beta(v). \tag{18.5}$$

Proposition 18.1 *Let $(\mathcal{A}, \cdot, \alpha)$ be a Hom-associative algebra and let (V, β) be a Hom-module. Let $l, r : \mathcal{A} \to gl(V)$ be two linear maps. The quadruple (l, r, β, V) satisfies a Hom-bimodule properties (18.1)–(18.3) of a Hom-associative algebra $(\mathcal{A}, \cdot, \alpha)$ if and only if the direct sum of vector spaces $\mathcal{A} \oplus V$ is a Hom-associative algebra with multiplication in $\mathcal{A} \oplus V$, defined for all $x_1, x_2 \in \mathcal{A}$, $v_1, v_2 \in V$, by*

$$\begin{aligned}(x_1 + v_1) * (x_2 + v_2) &= x_1 \cdot x_2 + (l(x_1)v_2 + r(x_2)v_1),\\ (\alpha \oplus \beta)(x_1 + v_1) &= \alpha(x_1) + \beta(v_1).\end{aligned} \tag{18.6}$$

Proof Let $v_1, v_2, v_3 \in V$ and $x_1, x_2, x_3 \in \mathcal{A}$. The left-hand side and right-hand side of Hom-associativity of $(\mathcal{A} \oplus V, *, \alpha \oplus \beta)$ are expended as follows:

$$\begin{aligned}
&((x_1 + v_1) * (x_2 + v_2)) * (\alpha \oplus \beta)(x_3 + v_3)\\
&\quad = ((x_1 + v_1) * (x_2 + v_2)) * (\alpha(x_3) + \beta(v_3))\\
&\quad = (x_1 \cdot x_2 + (l(x_1)v_2 + r(x_2)v_1)) * (\alpha(x_3) + \beta(v_3))\\
&\quad = (x_1 \cdot x_2) \cdot \alpha(x_3) + (l(x_1 \cdot x_2)\beta(v_3) + r(\alpha(x_3))(l(x_1)v_2 + r(x_2)v_1)))\\
&\quad = (x_1 \cdot x_2) \cdot \alpha(x_3) + (l(x_1 \cdot x_2)\beta(v_3) + r(\alpha(x_3))l(x_1)v_2 + r(\alpha(x_3))r(x_2)v_1)\\
&(\alpha \oplus \beta)(x_1 + v_1) * ((x_2 + v_2) * (x_3 + v_3))\\
&\quad = (\alpha(x_1) + \beta(v_1)) * ((x_2 + v_2) * (x_3 + v_3))\\
&\quad = (\alpha(x_1) + \beta(v_1)) * (x_2 \cdot x_3 + (l(x_2)v_3 + r(x_3)v_2))\\
&\quad = \alpha(x_1) \cdot (x_2 \cdot x_3) + l(\alpha(x_1))(l(x_2)v_3 + r(x_3)v_2)) + r(x_2 \cdot x_3)\beta(v_1)\\
&\quad = \alpha(x_1) \cdot (x_2 \cdot x_3) + (l(\alpha(x_1))l(x_2)v_3 + l(\alpha(x_1))r(x_3)v_2 + r(x_2 \cdot x_3)\beta(v_1))
\end{aligned}$$

These elements of $\mathcal{A} \oplus V$ are equal if and only if for all $x_1, x_2, x_3 \in \mathcal{A}$, $v_1, v_2, v_3 \in V$,

$$\begin{aligned}
&\alpha(x_1) \cdot (x_2 \cdot x_3) = (x_1 \cdot x_2) \cdot \alpha(x_3),\\
&l(x_1 \cdot x_2)\beta(v_3) + r(\alpha(x_3))l(x_1)v_2 + r(\alpha(x_3))r(x_2)v_1\\
&\qquad\qquad = l(\alpha(x_1))l(x_2)v_3 + l(\alpha(x_1))r(x_3)v_2 + r(x_2 \cdot x_3)\beta(v_1).
\end{aligned}$$

This holds if and only if the hom-associativity holds, and for each $j = 1, 2, 3$ the respective V terms involving $v_j \in V$ are equal. If the terms are equal, then the sums are equal. If the summs are equal, then the terms should be equal if one specifies all or two of v_1, v_2, v_3 to zero element of V and using that linear transformations map zero to zero. Since, for all $x_1, x_2, x_3 \in \mathcal{A}$,

$$\begin{aligned}
\text{Hom-associativity} \Leftrightarrow &\quad \alpha(x_1) \cdot (x_2 \cdot x_3) = (x_1 \cdot x_2) \cdot \alpha(x_3),\\
(18.1) \Leftrightarrow &\quad l(x_1 \cdot x_2)\beta(v_3) = l(\alpha(x_1))l(x_2)v_3,\\
(18.2) \Leftrightarrow &\quad l(\alpha(x_1))r(x_3)v_2 = r(\alpha(x_3))l(x_1)v_2,\\
(18.3) \Leftrightarrow &\quad r(\alpha(x_3))r(x_2)v_1 = r(x_2 \cdot x_3)\beta(v_1),
\end{aligned}$$

the proof is complete. □

We denote such a Hom-associative algebra $(\mathcal{A} \oplus V, *, \alpha + \beta)$, or $\mathcal{A} \times_{l,r,\alpha,\beta} V$.

Example 18.2 Let $(\mathcal{A}, \cdot, \alpha)$ be a multiplicative Hom-associative algebra. Let $L_{\cdot x}$ and $R_{\cdot x}$ denote the left and right multiplication operators, respectively, that is, $L_{\cdot x}(y) = x \cdot y$, $R_{\cdot x}(y) = y \cdot x$ for any $x, y \in \mathcal{A}$. Let $L_{\cdot} : \mathcal{A} \to gl(\mathcal{A})$ with $x \mapsto L_{\cdot x}$ and $R_{\cdot} : \mathcal{A} \to gl(\mathcal{A})$ with $x \mapsto R_{\cdot x}$ for every $x \in \mathcal{A}$, be two linear maps. Then, the triples $(L_{\cdot}, 0, \alpha)$, $(0, R_{\cdot}, \alpha)$ and $(L_{\cdot}, R_{\cdot}, \alpha)$ are bimodules of $(\mathcal{A}, \cdot, \alpha)$.

Proposition 18.2 *Let (l, r, β, V) be bimodule of a multiplicative Hom-associative algebra $(\mathcal{A}, \cdot, \alpha)$. Then, $(l \circ \alpha^n, r \circ \alpha^n, \beta, V)$ is a bimodule of $\mathcal{A}$ for any integer n.*

Proof We have

$$\begin{aligned} l \circ \alpha^n(x \cdot y)\beta(v) &= l(\alpha^n(x) \cdot \alpha^n(y))\beta(v) = l(\alpha(\alpha^n(x)))l(\alpha^n(y))v \\ &= l(\alpha^{n+1}(x))l(\alpha^n(y))v = l \circ \alpha^n(\alpha(x))l \circ \alpha^n(y)v. \end{aligned}$$

The other relations are established similarly. □

Example 18.3 Let $(\mathcal{A}, \cdot, \alpha)$ be a multiplicative Hom-associative algebra. Then, the quadruple $(L_{\cdot} \circ \alpha^n, R_{\cdot} \circ \alpha^n, \alpha, \mathcal{A})$ is a bimodule of $\mathcal{A}$ for any integer n.

Example 18.4 Let $(\mathcal{A}, \cdot, \alpha)$ be a multiplicative associative algebra, and $\beta : \mathcal{A} \to \mathcal{A}$ be a morphism. Then, $\mathcal{A}_\beta = (\mathcal{A}, \cdot_\beta = \beta \circ \cdot, \alpha_\beta = \beta \circ \alpha)$ is a multiplicative Hom-associative algebra. Hence $(L_{\cdot_\beta} \circ \alpha_\beta^n, R_{\cdot_\beta} \circ \alpha_\beta^n, \alpha_\beta, \mathcal{A})$ is a bimodule of $\mathcal{A}$ for any integer n.

18.2.2 Matched Pairs of Hom-associative Algebras

Theorem 18.1 *Let $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{B}, \circ, \beta)$ be two Hom-associative algebras. Suppose there are linear maps $l_\mathcal{A}, r_\mathcal{A} : \mathcal{A} \to gl(\mathcal{B})$ and $l_\mathcal{B}, r_\mathcal{B} : \mathcal{B} \to gl(\mathcal{A})$ such that the quadruple $(l_\mathcal{A}, r_\mathcal{A}, \beta, \mathcal{B})$ is a bimodule of $\mathcal{A}$, and $(l_\mathcal{B}, r_\mathcal{B}, \alpha, \mathcal{A})$ is a bimodule of $\mathcal{B}$, satisfying, for any $x, y \in \mathcal{A}$, $a, b \in \mathcal{B}$, the following conditions:*

$$l_\mathcal{A}(\alpha(x))(a \circ b) = l_\mathcal{A}(r_\mathcal{B}(a)x)\beta(b) + (l_\mathcal{A}(x)a) \circ \beta(b), \tag{18.7}$$

$$r_\mathcal{A}(\alpha(x))(a \circ b) = r_\mathcal{A}(l_\mathcal{B}(b)x)\beta(a) + \beta(a) \circ (r_\mathcal{A}(x)b), \tag{18.8}$$

$$l_\mathcal{B}(\beta(a))(x \cdot y) = l_\mathcal{B}(r_\mathcal{A}(x)a)\alpha(y) + (l_\mathcal{B}(a)x) \cdot \alpha(y), \tag{18.9}$$

$$r_\mathcal{B}(\beta(a))(x \cdot y) = r_\mathcal{B}(l_\mathcal{A}(y)a)\alpha(x) + \alpha(x) \cdot (r_\mathcal{B}(a)y), \tag{18.10}$$

$$\begin{aligned} l_\mathcal{A}(l_\mathcal{B}(a)x)\beta(b) + (r_\mathcal{A}(x)a) \circ \beta(b) - r_\mathcal{A}(r_\mathcal{B}(b)x)\beta(a) \\ -\beta(a) \circ (l_\mathcal{A}(x)b) = 0, \end{aligned} \tag{18.11}$$

$$\begin{aligned} l_\mathcal{B}(l_\mathcal{A}(x)a)\alpha(y) + (r_\mathcal{B}(a)x) \cdot \alpha(y) - r_\mathcal{B}(r_\mathcal{A}(y)a)\alpha(x) \\ -\alpha(x) \cdot (l_\mathcal{B}(a)y) = 0. \end{aligned} \tag{18.12}$$

Then, there is a Hom-associative algebra structure on the direct sum $\mathcal{A} \oplus \mathcal{B}$ of the underlying vector spaces of $\mathcal{A}$ and $\mathcal{B}$ given for all $x, y \in \mathcal{A}$, $a, b \in \mathcal{B}$ by

$$\begin{aligned} &(x + a) * (y + b) = (x \cdot y + l_\mathcal{B}(a)y + r_\mathcal{B}(b)x) + (a \circ b + l_\mathcal{A}(x)b + r_\mathcal{A}(y)a), \\ &(\alpha \oplus \beta)(x + a) = \alpha(x) + \beta(a). \end{aligned} \tag{18.13}$$

Proof Let $v_1, v_2, v_3 \in V$ and $x_1, x_2, x_3 \in \mathcal{A}$. Set

$$[(x_1 + v_1) * (x_2 + v_2)] * (\alpha(x_3) + \beta(v_3))$$
$$= (\alpha(x_1) + \beta(v_1)) * [(x_2 + v_2) * (x_3 + v_3)],$$

which is developed to obtain (18.7)-(18.12). Then, using the relations

$$\beta(l_{\mathcal{A}}(x)a) = l_{\mathcal{A}}(\alpha(x))\beta(a), \qquad \beta(r_{\mathcal{A}}(x)a) = r_{\mathcal{A}}(\alpha(x))\beta(a),$$
$$\alpha(l_{\mathcal{B}}(a)x) = l_{\mathcal{B}}(\beta(a))\alpha(x), \qquad \alpha(r_{\mathcal{B}}(a)x) = r_{\mathcal{B}}(\beta(a))\alpha(x),$$

we show that $*$ is a Hom-associative algebra. □

We denote this Hom-associative algebra by $(\mathcal{A} \bowtie \mathcal{B}, *, \alpha + \beta)$ or $\mathcal{A} \bowtie^{l_{\mathcal{A}}, r_{\mathcal{A}}, \beta}_{l_{\mathcal{B}}, r_{\mathcal{B}}, \alpha} \mathcal{B}$.

Definition 18.4 Let $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{B}, \circ, \beta)$ be two Hom-associative algebras. Suppose that there are linear maps $l_{\mathcal{A}}, r_{\mathcal{A}} : \mathcal{A} \to gl(\mathcal{B})$ and $l_{\mathcal{B}}, r_{\mathcal{B}} : \mathcal{B} \to gl(\mathcal{A})$ such that $(l_{\mathcal{A}}, r_{\mathcal{A}}, \beta)$ is a bimodule of $\mathcal{A}$ and $(l_{\mathcal{B}}, r_{\mathcal{B}}, \alpha)$ is a bimodule of $\mathcal{B}$. If the conditions (18.7)–(18.12) are satisfied, then, $(\mathcal{A}, \mathcal{B}, l_{\mathcal{A}}, r_{\mathcal{A}}, \beta, l_{\mathcal{B}}, r_{\mathcal{B}}, \alpha)$ is called a **matched pair of Hom-associative algebras**.

18.3 Double Constructions of Involutive Hom-Frobenius Algebras and Antisymmetric Infinitesimal Hom-bialgebras

In this section, we consider the multiplicative Hom-associative algebra $(\mathcal{A}, \cdot, \alpha)$ such that α is involutive, i.e., $\alpha^2 = \mathrm{Id}_{\mathcal{A}}$.

18.3.1 Double Constructions of Involutive Hom-Frobenius Algebras

Definition 18.5 Let V_1, V_2 be two vector spaces. For a linear map $\phi : V_1 \to V_2$, we denote the dual (linear) map by $\phi^* : V_2^* \to V_1^*$ given, for all $v \in V_1, u^* \in V_2^*$, by

$$\langle v, \phi^*(u^*)\rangle = \langle \phi(v), u^*\rangle.$$

Lemma 18.1 *Let (l, r, β, V) be a bimodule of a multiplicative Hom-associative algebra $(\mathcal{A}, \cdot, \alpha)$, and let $l^*, r^* : \mathcal{A} \to gl(V^*)$ be the linear maps given, for all $x \in \mathcal{A}$, $u^* \in V^*$, $v \in V$, by*

$$\langle l^*(x)u^*, v\rangle := \langle l(x)v, u^*\rangle, \langle r^*(x)u^*, v\rangle := \langle r(x)v, u^*\rangle.$$

Then

(i) (r^*, l^*, β^*, V^*) *is a bimodule of* $(\mathcal{A}, \cdot, \alpha)$;
(ii) $(r^*, 0, \beta^*, V^*)$ *and* $(0, l^*, \beta^*, V^*)$ *are also bimodules of* $\mathcal{A}$.

Proof (i) Let (l, r, β, V) be a bimodule of a multiplicative Hom-associative algebra $(\mathcal{A}, \cdot, \alpha)$. We show that (r^*, l^*, β^*, V^*) is a bimodule of $\mathcal{A}$. For all $x, y \in \mathcal{A}$ and $u^* \in V^*, v \in V$,
(i-1) the computation

$$\begin{aligned}\langle r^*(x \cdot y)\beta^*(u^*), v\rangle &= \langle \beta(r(x \cdot y)v), u^*\rangle = \langle r(\alpha(x \cdot y))\beta(v), u^*\rangle \\ &= \langle r(\alpha(x) \cdot \alpha(y))\beta(v), u^*\rangle = \langle r(\alpha^2(y))r(\alpha(x))v, u^*\rangle \\ &= \langle (r(y)r(\alpha(x)))^* u^*, v\rangle = \langle r^*(\alpha(x))r^*(y)u^*, v\rangle\end{aligned}$$

leads to $r^*(x \cdot y)\beta^*(u^*) = r^*(\alpha(x))r^*(y)u^*$;
(i-2) the computation

$$\begin{aligned}\langle l^*(x \cdot y)\beta^*(u^*), v\rangle &= \langle \beta(l(x \cdot y)(v)), u^*\rangle = \langle l(\alpha(x \cdot y))\beta(v), u^*\rangle \\ &= \langle l(\alpha(x) \cdot \alpha(y))\beta(v), u^*\rangle = \langle l(\alpha^2(x))l(\alpha(y))\beta(v), u^*\rangle \\ &= \langle (l(x)l(\alpha(y)))^* u^*, v\rangle = \langle l^*(\alpha(y))l^*(x)u^*, v\rangle\end{aligned}$$

gives $l^*(x \cdot y)\beta^*(u^*) = l^*(\alpha(y))l^*(x)u^*$;
(i-3) the computation

$$\begin{aligned}\langle r^*(\alpha(x))l^*(y)u^*, v\rangle &= \langle l(y)r(\alpha(x))v, u^*\rangle \\ &= \langle l(\alpha^2(y))r(\alpha(x))v, u^*\rangle = \langle (l \circ \alpha)(\alpha(y))(r \circ \alpha)(x))v, u^*\rangle \\ &= \langle r(\alpha^2(x))l(\alpha(y))v, u^*\rangle = \langle r(x)l(\alpha(y))v, u^*\rangle \\ &= \langle l^*(\alpha(y))r^*(x)u^*, v\rangle\end{aligned}$$

yields $r^*(\alpha(x))l^*(y)u^* = l^*(\alpha(y))r^*(x)u^*$.

$$\begin{aligned}\langle \beta^*(r^*(x))u^*, v\rangle &= \langle r(x)(\beta(v)), u^*\rangle = \langle r(\alpha^2(x))(\beta(v)), u^*\rangle \\ &= \langle (r \circ \alpha)(\alpha(x))(\beta(v)), u^*\rangle = \langle \beta(r(\alpha(x)))v, u^*\rangle \\ &= \langle r^*(\alpha(x))\beta^*(u^*), v\rangle.\end{aligned}$$

Then $\beta^*(r^*(x))u^* = r^*(\alpha(x))\beta^*(u^*)$. The equality $\beta^*(l^*(x))u^* = l^*(\alpha(x))\beta^*(u^*)$ can be shown in a similar way. Thus, (r^*, l^*, β^*, V^*) is a bimodule of $\mathcal{A}$.
(ii) Analogously, $(r^*, 0, \beta^*, V^*)$ and $(0, l^*, \beta^*, V^*)$ are bimodules of $\mathcal{A}$. □

Definition 18.6 Let $(\mathcal{A}, \cdot, \alpha)$ be a Hom-associative algebra, and $B : \mathcal{A} \times \mathcal{A} \to K$ be a bilinear form on $\mathcal{A}$. Then,

(i) B is said to be nondegenerate if

$$\mathcal{A}^{\perp} = \{x \in \mathcal{A} \mid B(y, x) = 0, \forall y \in A\} = 0;$$

(ii) B is said to be symmetric if

$$B(x, y) = B(y, x);$$

(iii) B is said to be α-invariant if

$$B(\alpha(x) \cdot \alpha(y), \alpha(z)) = B(\alpha(x), \alpha(y) \cdot \alpha(z)).$$

Definition 18.7 A Hom-Frobenius algebra is a Hom-associative algebra with a non-degenerate invariant bilinear form.

Definition 18.8 We call $(\mathcal{A}, \alpha, B)$ a **double construction of an involutive Hom-Frobenius algebra** associated to $(\mathcal{A}_1, \alpha_1)$ and $(\mathcal{A}_1^*, \alpha_1^*)$ if it satisfies the following conditions:

(i) $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_1^*$ as the direct sum of vector spaces;
(ii) $(\mathcal{A}_1, \alpha_1)$ and $(\mathcal{A}_1^*, \alpha_1^*)$ are Hom-associative subalgebras of $(\mathcal{A}, \alpha)$ with $\alpha = \alpha_1 \oplus \alpha_1^*$;
(iii) B is the natural non-degenerate $(\alpha_1 \oplus \alpha_1^*)$-invariant symmetric bilinear form on $\mathcal{A}_1 \oplus \mathcal{A}_1^*$ given, for all $x, y \in \mathcal{A}_1, a^*, b^* \in \mathcal{A}_1^*$, by

$$\begin{aligned} B(x + a^*, y + b^*) &= \langle x, b^* \rangle + \langle a^*, y \rangle, \\ B((\alpha + \alpha^*)(x + a^*), y + b^*) &= B(x + a^*, (\alpha + \alpha^*)(y + b^*)), \end{aligned} \qquad (18.14)$$

where $\langle , \rangle$ is the natural pairing between the vector space $\mathcal{A}_1$ and the dual vector space $\mathcal{A}_1^*$.

Let $(\mathcal{A}, \cdot, \alpha)$ be an involutive Hom-associative algebra. Suppose that there is an involutive Hom-associative algebra structure "$\circ$" on its dual space $\mathcal{A}^*$. We construct an involutive Hom-associative algebra structure on the direct sum $\mathcal{A} \oplus \mathcal{A}^*$ of the underlying vector spaces of $\mathcal{A}$ and $\mathcal{A}^*$ such that $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{A}^*, \circ, \alpha^*)$ are Hom-subalgebras, equipped with the non-degenerate $(\alpha_1 \oplus \alpha_1^*)$-invariant symmetric bilinear form on $\mathcal{A} \oplus \mathcal{A}^*$ given by (18.14). In other words, $(\mathcal{A} \oplus \mathcal{A}^*, \alpha \oplus \alpha^*, B)$ is an involutive symmetric Hom-associative algebra. Such a construction is called a double construction of an involutive Hom-Frobenius algebra associated to $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{A}^*, \circ, \alpha^*)$.

Theorem 18.2 *Let $(\mathcal{A}, \cdot, \alpha)$ be an involutive Hom-associative algebra. Suppose that there is an involutive Hom-associative algebra structure "$\circ$" on its dual space $\mathcal{A}^*$. Then, there is a double construction of an involutive Hom-Frobenius algebra associated to $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{A}^*, \circ, \alpha^*)$ if and only if $(\mathcal{A}, \mathcal{A}^*, R_\cdot^*, L_\cdot^*, \alpha^*, R_\circ^*, L_\circ^*, \alpha)$ is a matched pair of involutive Hom-associative algebras.*

Proof Let us consider the four maps defined, for $x, v, u \in \mathcal{A}$, x^*, v^* and $u^* \in \mathcal{A}^*$ by

$$\begin{aligned} L_\cdot^* &: \mathcal{A} \to gl(\mathcal{A}^*), & \langle L_\cdot^*(x)u^*, v\rangle &= \langle L_\cdot(x)v, u^*\rangle = \langle x \cdot v, u^*\rangle, \\ R_\cdot^* &: \mathcal{A} \to gl(\mathcal{A}^*), & \langle R_\cdot^*(x)u^*, v\rangle &= \langle R_\cdot(x)v, u^*\rangle = \langle v \cdot x, u^*\rangle, \\ R_\circ^* &: \mathcal{A}^* \to gl(\mathcal{A}), & \langle R_\circ^*(x^*)u, v^*\rangle &= \langle R_\circ(x^*)v^*, u\rangle = \langle v^* \circ x^*, u\rangle, \\ L_\circ^* &: \mathcal{A}^* \to gl(\mathcal{A}), & \langle L_\circ^*(x^*)u, v^*\rangle &= \langle L_\circ(x^*)v^*, u\rangle = \langle x^* \circ v^*, u\rangle. \end{aligned}$$

If $(\mathcal{A}, \mathcal{A}^*, R^*_{\cdot}, L^*_{\cdot}, \alpha^*, R^*_{\circ}, L^*_{\circ}, \alpha)$ is a matched pair of multiplicative Hom-associative algebras, then $(\mathcal{A} \bowtie \mathcal{A}^*, *, \alpha + \alpha^*)$ is a multiplicative Hom-associative algebra with the product $*$ given by (18.13), and the bilinear form $B(\cdot, \cdot)$ defined by (18.14) is $(\alpha \oplus \alpha^*)$-invariant, that is, for all $x, y \in \mathcal{A}^*$, $a^*, b^* \in \mathcal{A}^*$, and

$$(x + a^*) * (y + b^*) = (x \cdot y + l_B(a)y + r_B(b)x) + (a \circ b + l_{\mathcal{A}}(x)b + r_{\mathcal{A}}(y)a),$$

where $l_{\mathcal{A}} = R^*_{\cdot}, r_{\mathcal{A}} = L^*_{\cdot}, l_B = R^*_{\circ}, r_B = L^*_{\circ}$:

$$\begin{aligned} &B[(\alpha(x) + \alpha^*(a^*)) * (\alpha(y) + \alpha^*(b^*)), (\alpha(z) + \alpha^*(c^*))] \\ &\quad = B[\alpha(x) + \alpha^*(a^*), (\alpha(y) + \alpha^*(b^*)) * (\alpha(z) + \alpha^*(c^*))]. \end{aligned}$$

Indeed,

$$\begin{aligned} &B[(\alpha(x) + \alpha^*(a^*)) * (\alpha(y) + \alpha^*(b^*)), (\alpha(z) + \alpha^*(c^*))] \\ &\quad = B[(\alpha(x) \cdot \alpha(y) + l_{\mathcal{A}^*}(\alpha^*(a^*))\alpha(y) + r_{\mathcal{A}^*}(\alpha^*(b^*))\alpha(x)) + (\alpha^*(a^*) \circ \alpha^*(b^*) \\ &\qquad + l_{\mathcal{A}}(\alpha(x))\alpha^*(b^*) + r_{\mathcal{A}}(\alpha(y))\alpha^*(a^*)), \alpha(z) + \alpha^*(c^*)] \\ &\quad = \langle \alpha(x) \cdot \alpha(y), \alpha^*(c^*) \rangle + \langle \alpha^*(c^*) \circ \alpha^*(a^*), \alpha(y) \rangle + \langle \alpha^*(b^*) \circ \alpha^*(c^*), \alpha(x) \rangle \\ &\qquad + \langle \alpha^*(a^*) \circ \alpha^*(b^*), \alpha(z) \rangle + \langle \alpha(z) \cdot \alpha(x), \alpha^*(b^*) \rangle + \langle \alpha(y) \cdot \alpha(z), \alpha^*(a^*) \rangle \\ &B[\alpha(x) + \alpha^*(a^*), (\alpha(y) + \alpha^*(b^*)) * (\alpha(z) + \alpha^*(c^*))] \\ &\quad = B[\alpha(x) + \alpha^*(a^*), (\alpha(y) \cdot \alpha(z) + l_{\mathcal{A}^*}(\alpha^*(b^*))\alpha(z) + r_{\mathcal{A}^*}(\alpha^*(c^*))\alpha(y)) \\ &\qquad + (\alpha^*(b^*) \circ \alpha^*(c^*) + l_{\mathcal{A}}(\alpha(y))\alpha^*(c^*) + r_{\mathcal{A}}(\alpha(z))\alpha^*(b^*))] \\ &\quad = \langle \alpha(x), \alpha^*(b^*) \circ \alpha^*(c^*) \rangle + \langle \alpha^*(c^*), \alpha(x) \cdot \alpha(y) \rangle + \langle \alpha^*(b^*), \alpha(z) \cdot \alpha(x) \rangle \\ &\qquad + \langle \alpha(y) \cdot \alpha(z), \alpha^*(a^*) \rangle + \langle \alpha^*(a^*) \circ \alpha^*(b^*), \alpha(z) \rangle + \langle \alpha(c^*) \circ \alpha^*(a^*), \alpha(y) \rangle. \end{aligned}$$

Thus, B is well $(\alpha \oplus \alpha^*)$-invariant. Conversely, set for $x \in \mathcal{A}$, $a^* \in \mathcal{A}^*$,

$$x * a^* = l_{\mathcal{A}}(x)a^* + r_{\mathcal{A}^*}(a^*)x, \quad a^* * x = l_{\mathcal{A}^*}(a^*)x + r_{\mathcal{A}}(x)a^{\star}.$$

Then, $(\mathcal{A}, \mathcal{A}^*, R^*_{\cdot}, L^*_{\cdot}, \alpha^*, R^*_{\circ}, L^*_{\circ}, \alpha)$ is a matched pair of multiplicative Hom-associative algebras, since the double construction of the involutive Hom-Frobenius algebra associated to $(\mathcal{A}, \cdot, \alpha)$ and $(\mathcal{A}^*, \circ, \alpha^*)$ produces (18.7)–(18.12). □

Theorem 18.3 *Let $(\mathcal{A}, \cdot, \alpha)$ be an involutive Hom-associative algebra. Suppose that there is an involutive Hom-associative algebra structure "$\circ$" on its dual space $(\mathcal{A}^*, \alpha^*)$. Then,*

$$(\mathcal{A}, \mathcal{A}^*, R^*_{\cdot}, L^*_{\cdot}, \alpha^*, R^*_{\circ}, L^*_{\circ}, \alpha)$$

is a matched pair of involutive Hom-associative algebras if and only if, for any $x \in \mathcal{A}$ and $a^*, b^* \in \mathcal{A}^*$,

$$R^*_{\cdot}(\alpha(x))(a^* \circ b^*) = R^*_{\cdot}(L^*_{\circ}(a^*)x)\alpha^*(b^*) + (R^*_{\cdot}(x)a^*) \circ \alpha^*(b^*), \tag{18.15}$$

$$\begin{aligned} &R^*_{\cdot}(R^*_{\circ}(a^*)x)\alpha^*(b^*) + L^*_{\cdot}(x)a^* \circ \alpha^*(b^*) \\ &\quad = L^*_{\cdot}(L^*_{\circ}(b^*)x)\alpha^*(a^*) + \alpha^*(a^*) \circ (R^*_{\cdot}(x)b^*). \end{aligned} \tag{18.16}$$

Proof Obviously, (18.15) gives (18.7), and (18.16) reduces to (18.11) when $l_{\mathcal{A}} = R^*_{\cdot}$, $r_{\mathcal{A}} = L^*_{\cdot}$, $l_B = l_{\mathcal{A}^*} = R^*_{\circ}$ and $r_B = r_{\mathcal{A}^*} = L^*_{\circ}$. Now, we show that

$$(18.7) \Longleftrightarrow (18.8) \Longleftrightarrow (18.9) \Longleftrightarrow (18.10), \quad \text{and } (18.11) \Longleftrightarrow (18.12).$$

Suppose (18.15) and (18.16) are satisfied, and show that

$$\begin{aligned}
&L^*_{\cdot}(\alpha(x))(a^* \circ b^*) = L^*_{\cdot}(R^*_{\circ}(b^*)x)\alpha^*(a^*) + \alpha^*(a^*) \circ (L^*_{\cdot}(x)b^*),\\
&R^*_{\circ}(\alpha^*(a^*))(x \cdot y) = R^*_{\circ}(L^*_{\cdot}(x)a^*)\alpha(y) + (R^*_{\circ}(a)x) \cdot \alpha(y),\\
&L^*_{\circ}(\alpha^*(a^*))(x \cdot y) = L^*_{\circ}(R^*_{\cdot}(y)a^*)\alpha(x) + \alpha(x) \cdot (L^*_{\circ}(a^*)y),\\
&R^*_{\circ}(R^*_{\cdot}(x)a^*)\alpha(y) + (L^*_{\circ}(a^*)x) \cdot \alpha(y) - L^*_{\circ}(L_{\cdot}(y)a^*)\alpha(x) - \alpha(x) \cdot (R^*_{\circ}(a)y) = 0.
\end{aligned}$$

We have, for all $x, y \in \mathcal{A}$, $a^*, b^* \in \mathcal{A}^*$,

$$\begin{aligned}
\langle R^*_{\cdot}(x)a^*, y\rangle &= \langle L^*_{\cdot}(y)a^*, x\rangle = \langle y \cdot x, a^*\rangle,\\
\langle R^*_{\circ}(b^*)x, a^*\rangle &= \langle L^*_{\circ}(a^*)x, b^*\rangle = \langle a^* \circ b^*, x\rangle,\\
\alpha^*(R^*_{\cdot}(x)a^*) &= R^*_{\cdot}(\alpha(x))\alpha^*(a^*), \quad \alpha^*(L^*_{\cdot}(x)a^*) = L^*_{\cdot}(\alpha(x))\alpha^*(a^*),\\
\alpha(R^*_{\circ}(a^*)x) &= R^*_{\circ}(\alpha^*(a^*))\alpha(x), \quad \alpha(L^*_{\circ}(a^*)x) = L^*_{\circ}(\alpha^*(a^*))\alpha(x),
\end{aligned}$$

Set $\alpha(x) = z, \alpha(y) = t$, $\alpha^*(a^*) = c^*$ and $\alpha^*(b^*) = d^*$. Then,

(i) the statement (18.7) $\Longleftrightarrow$ (18.8) follows from

$$\begin{aligned}
\langle R^*_{\cdot}(\alpha(x))(a^* \circ b^*), y\rangle &= \langle y \cdot \alpha(x), a^* \circ b^*\rangle\\
&= \langle (L_{\cdot}(y) \circ \alpha)x, a^* \circ b^*\rangle\\
&= \langle x, \alpha^*(L^*_{\cdot}(y)(a^* \circ b^*))\rangle\\
&= \langle L^*_{\cdot}(\alpha(y))\alpha^*(a^* \circ b^*), x\rangle\\
&= \langle L^*_{\cdot}(\alpha(y))(\alpha^*(a^*) \circ \alpha^*(b^*)), x\rangle\\
&= \langle L^*_{\cdot}(\alpha(y))(c^* \circ d^*), x\rangle;\\
\langle R^*_{\cdot}(L^*_{\circ}(a^*)x)\alpha(b^*), y\rangle &= \langle y \cdot L^*_{\circ}(a^*)x, \alpha^*(b^*)\rangle\\
&= \langle L^*_{\cdot}(y)(\alpha^*(b^*)), L^*_{\circ}(a^*)x\rangle\\
&= \langle L^*_{\circ}(a^*)x, L^*_{\cdot}(y)(\alpha^*(b^*))\rangle\\
&= \langle a^* \circ (L^*_{\cdot}(y)(\alpha^*(b^*))), x\rangle\\
&= \langle \alpha^*(c^*) \circ (L^*_{\cdot}(y)(d^*)), x\rangle;\\
\langle (R^*_{\cdot}(x)a^*) \circ \alpha^*(b^*), y\rangle &= \langle R^*_{\circ}(\alpha^*(b^*))y, R^*_{\cdot}(x)a^*\rangle\\
&= \langle a^*, (R^*_{\circ}(\alpha^*(b^*))y) \cdot x\rangle\\
&= \langle L^*_{\cdot}[R^*_{\circ}(\alpha^*(b^*))y]a^*, x\rangle\\
&= \langle L^*_{\cdot}(R^*_{\circ}(d^*)y)\alpha^*(c^*), x\rangle;
\end{aligned}$$

(ii) the statement (18.8) $\Longleftrightarrow$ (18.9) follows from

$$\begin{aligned}
\langle L^*(\alpha(x))(a^* \circ b^*), y\rangle &= \langle a^* \circ b^*, \alpha(x) \cdot y\rangle\\
&= \langle R^*_{\circ}(b^*)(\alpha(x) \cdot y), a^*\rangle\\
&= \langle R^*_{\circ}(\alpha^*(d^*))(z \cdot y), a^*\rangle;
\end{aligned}$$

$$\begin{aligned}
\langle \alpha^*(a^*) \circ (L^*_\cdot(x)b^*), y\rangle &= \langle \alpha^*(a^*), R^*_\circ(L^*_\cdot(x)b^*)y\rangle \\
&= \langle a^*, \alpha[R^*_\circ(L^*_\cdot(x)b^*)y]\rangle \\
&= \langle a^*, R^*_\circ[\alpha^*(L^*_\cdot(x)b^*)]\alpha(y)\rangle \\
&= \langle a^*, R^*_\circ[L^*_\cdot(\alpha(x))\alpha^*(b^*)]\alpha(y)\rangle \\
&= \langle a^*, R^*_\circ(L^*_\cdot(z)d^*)\alpha(y)\rangle; \\
\langle L^*_\cdot(R^*_\circ(b^*)x)\alpha^*(a^*), y\rangle &= \langle (R^*_\circ(b^*)x) \circ y, \alpha^*(a^*)\rangle \\
&= \langle \alpha[(R^*_\circ(b^*)x) \circ y], a^*\rangle \\
&= \langle (R^*_\circ(\alpha^*(b^*))\alpha(x)) \circ \alpha(y), a^*\rangle \\
&= \langle R^*_\circ(d^*)z \cdot \alpha(y), a^*\rangle;
\end{aligned}$$

(iii) the statement (18.7) $\Longleftrightarrow$ (18.10) follows from

$$\begin{aligned}
\langle R^*(\alpha(x))(a^* \circ b^*), y\rangle &= \langle a^* \circ b^*, y \cdot \alpha(x)\rangle = \langle L_\cdot(a^*)b^*, y \cdot z\rangle \\
&= \langle L^*_\circ(a^*)(y \cdot z)\rangle = \langle L^*_\circ(\alpha^*(c^*))(y \cdot z)\rangle; \\
\langle (R^*_\cdot(x)a^*) \circ \alpha^*(b^*), y\rangle &= \langle \alpha^*(b^*), L^*_\cdot(R^*_\cdot(x)a^*)y\rangle = \langle b^*, \alpha^*[L^*_\cdot(R^*_\cdot(x)a^*)y]\rangle \\
&= \langle b^*, L^*_\cdot(R^*_\cdot(\alpha(x))\alpha^*(a^*))\alpha(y)\rangle \\
&= \langle b^*, L^*_\cdot(R^*_\cdot(z)c^*)\alpha(y)\rangle; \\
\langle R^*_\cdot(L^*_\circ(a^*)x)\alpha^*(b^*), y\rangle &= \langle y \cdot L^*_\circ(a^*)x, \alpha^*(b^*)\rangle = \langle \alpha(y) \cdot \alpha(L^*_\circ(a^*)x), b^*\rangle \\
&= \langle \alpha(y) \cdot L^*_\circ(\alpha^*(a^*))\alpha(x), b^*\rangle = \langle \alpha(y) \cdot L^*_\circ(c^*)z, b^*\rangle;
\end{aligned}$$

(iv) the statement (18.11) $\Longleftrightarrow$ (18.12) follows from

$$\begin{aligned}
\langle L^*_\cdot(L^*_\circ(b^*)x)\alpha^*(a^*), y\rangle &= \langle (L^*_\circ(b^*)x) \cdot y, \alpha^*(a^*)\rangle \\
&= \langle a^*, \alpha(L^*_\circ(b^*)x) \cdot \alpha(y)\rangle \\
&= \langle a^*, L^*_\circ(\alpha^*(b^*))\alpha(x) \cdot \alpha(y)\rangle \\
&= \langle a^*, L^*_\circ(d^*)z \cdot \alpha(y)\rangle; \\
\langle \alpha^*(a^*) \circ (R^*_\cdot(x)b^*), y\rangle &= \langle R^*_\circ(R^*_\circ(x)b^*)y, \alpha^*(a^*)\rangle \\
&= \langle \alpha^*(a^*) \circ (R^*_\cdot(x)b^*), y\rangle \\
&= \langle \alpha[R^*_\circ(R^*_\circ(x)b^*)y], a^*\rangle \\
&= \langle R^*_\circ[R^*_\circ(\alpha(x))\alpha^*(b^*)]\alpha(y), a^*\rangle \\
&= \langle R^*_\circ(R^*_\cdot(z)d^*)\alpha(y), a^*\rangle; \\
\langle (L^*_\cdot(x)a^*) \circ \alpha^*(b^*), y\rangle &= \langle R^*_\circ(\alpha^*(b^*))y, L^*_\cdot(x)a^*\rangle \\
&= \langle x \cdot (R^*_\circ(d^*)y), a^*\rangle \\
&= \langle \alpha(z) \cdot (R^*_\circ(d^*)y), a^*\rangle; \\
\langle R^*_\cdot(R^*_\circ(a^*)x)\alpha^*(b^*), y\rangle &= \langle y \cdot R^*_\circ(a^*)x, \alpha^*(b^*)\rangle \\
&= \langle \alpha^*(b^*), L_\cdot(y)(R^*_\circ(a^*)x)\rangle \\
&= \langle (L^*_\cdot(y)(d^*), R^*_\circ(a^*)x\rangle = \langle L^*_\cdot(y)d^* \circ a^*, x\rangle \\
&= \langle L^*_\circ(L^*_\cdot(y)d^*)x, a^*\rangle \\
&= \langle L^*_\circ(L^*_\cdot(y)d^*)\alpha(z), a^*\rangle
\end{aligned}$$

which completes the proof. □

18.3.2 Antisymmetric Infinitesimal Hom-bialgebras

Let $\mathcal{A}$ be a multiplicative Hom-associative algebra. Let $\sigma : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ be the exchange operator, that is $\sigma(x \otimes y) = y \otimes x$ for all $x, y \in \mathcal{A}$.

Proposition 18.3 *Let $(\mathcal{A}, \cdot, \alpha)$ be a multiplicative Hom-associative algebra. Then, $(\alpha \otimes L_{\cdot}, R_{\cdot} \otimes \alpha, \alpha \otimes \alpha, \mathcal{A} \otimes \mathcal{A})$ given, for any $x, a, b \in \mathcal{A}$, by*

$$\begin{aligned}(\alpha \otimes L)(x)(a \otimes b) = (\alpha \otimes L(x))(a \otimes b) = \alpha(a) \otimes x \cdot b,\\ (R_{\cdot} \otimes \alpha)(x)(a \otimes b) = (R(x) \otimes \alpha)(a \otimes b) = a \cdot x \otimes \alpha(b),\end{aligned}$$

is a bimodule of $\mathcal{A}$.

Proof Let $x, y, v_1, v_2 \in \mathcal{A}$.

(i) By formulas for the maps and hom-associativity,

$$\begin{aligned}(\alpha \otimes L_{\cdot})(x \cdot y)(\alpha \otimes \alpha)(v_1 \otimes v_2) &= [\alpha \otimes L_{\cdot}(x \cdot y)](\alpha(v_1) \otimes \alpha(v_2))\\ &= v_1 \otimes (x \cdot y) \cdot \alpha(v_2);\\ (\alpha \otimes L_{\cdot}(\alpha(x)))(\alpha \otimes L_{\cdot}(y))(v_1 \otimes v_2) &= (\alpha \otimes L_{\cdot}(\alpha(x)))(\alpha(v_1)) \otimes (y \cdot v_2)\\ &= v_1 \otimes \alpha(x) \cdot (y \cdot v_2)\end{aligned}$$

give $(\alpha \otimes L_{\cdot})(x \cdot y)(\alpha \otimes \alpha)(v_1 \otimes v_2) = (\alpha \otimes L(\alpha(x)))(\alpha \otimes L(y))(v_1 \otimes v_2)$.

(ii) By formulas for the maps and hom-associativity,

$$\begin{aligned}(R_{\cdot} \otimes \alpha)(x \cdot y)(\alpha \otimes \alpha)(v_1 \otimes v_2) &= (R_{\cdot}(x \cdot y) \otimes \alpha)(\alpha(v_1) \otimes \alpha(v_2))\\ &= \alpha(v_1) \cdot (x \cdot y) \otimes v_2\\ (R_{\cdot}(\alpha(y)) \otimes \alpha)(R_{\cdot}(x) \otimes \alpha)(v_1 \otimes v_2) &= (R_{\cdot}(\alpha(y))((v_1 \cdot x) \otimes \alpha(v_2))\\ &= (v_1 \cdot x) \cdot \alpha(y) \otimes v_2\end{aligned}$$

yield $(R_{\cdot} \otimes \alpha)(x \cdot y)(\alpha \otimes \alpha)(v_1 \otimes v_2) = (R_{\cdot}(\alpha(y)) \otimes \alpha)(R_{\cdot}(x) \otimes \alpha)(v_1 \otimes v_2)$.

(iii) By formulas for the maps,

$$\begin{aligned}(\alpha \otimes L_{\cdot}(\alpha(x)))(R_{\cdot}(y) \otimes \alpha)(v_1 \otimes v_2) &= (\alpha \otimes L_{\cdot}(\alpha(x)))(v_1 \cdot y \otimes \alpha(v_2))\\ &= \alpha(v_1) \cdot \alpha(y) \otimes \alpha(x) \cdot \alpha(v_2);\\ (R_{\cdot}(\alpha(y)) \otimes \alpha)(\alpha \otimes L_{\cdot}(x))(v_1 \otimes v_2) &= (R_{\cdot}(\alpha(y)) \otimes \alpha)(\alpha(v_1) \otimes x \cdot v_2)\\ &= \alpha(v_1) \cdot \alpha(y) \otimes \alpha(x) \cdot \alpha(v_2)\end{aligned}$$

give

$$(\alpha \otimes L_{\cdot}(\alpha(x)))(R_{\cdot}(y) \otimes \alpha)(v_1 \otimes v_2) = (R_{\cdot}(\alpha(y)) \otimes \alpha)(\alpha \otimes L_{\cdot}(x))(v_1 \otimes v_2).$$

(iv) By formulas for the maps,

$$\begin{aligned}&(\alpha \otimes \alpha)(\alpha \otimes L_{\cdot})(v_1 \otimes v_2) = (\alpha \otimes \alpha)(\alpha(v_1) \otimes x \cdot v_2) = v_1 \otimes \alpha(x) \cdot \alpha(v_2);\\ &(\alpha \otimes L_{\cdot}(\alpha(x)))(v_1 \otimes v_2) = (\alpha \otimes L_{\cdot}\alpha(x))(\alpha(v_1) \otimes \alpha(v_2)) = v_1 \otimes \alpha(x) \cdot \alpha(v_2)\end{aligned}$$

imply $(\alpha \otimes \alpha)(\alpha \otimes L_{\cdot})(v_1 \otimes v_2) = \alpha \otimes L_{\cdot}(\alpha(x)))(v_1 \otimes v_2)$.

(v) By formulas for the maps,

$$\begin{aligned}(\alpha \otimes \alpha)(R_{\cdot}\alpha(x))(v_1 \otimes v_2) &= (\alpha \otimes \alpha)(v_1 \cdot x \otimes \alpha(v_2)) \\ &= \alpha(v_1) \cdot \alpha(x) \otimes v_2; \\ (R_{\cdot}(\alpha(x)) \otimes \alpha)(\alpha \otimes \alpha)(v_1 \otimes v_2) &= (R_{\cdot}(\alpha(x)) \otimes \alpha)(\alpha(v_1) \otimes \alpha(v_2)) \\ &= \alpha(v_1) \cdot \alpha(x) \otimes v_2\end{aligned}$$

yield $(\alpha \otimes \alpha)(R_{\cdot}\alpha(x))(v_1 \otimes v_2) = (R_{\cdot}(\alpha(x)) \otimes \alpha)(\alpha \otimes \alpha)(v_1 \otimes v_2)$.

Hence, the proof is achieved. □

Remark 18.2 The quadruple $(L_{\cdot} \otimes \alpha, \alpha \otimes R_{\cdot}, \alpha \otimes \alpha, \mathcal{A} \otimes \mathcal{A})$ is also a bimodule of $\mathcal{A}$.

Theorem 18.4 *Let $(\mathcal{A}, \cdot, \alpha)$ be an involutive Hom-associative algebra. Suppose there is an involutive Hom-associative algebra structure "$\circ$" on its dual space $\mathcal{A}^*$ given by a linear map $\Delta^* : \mathcal{A}^* \otimes \mathcal{A}^* \to \mathcal{A}^*$. Then, $(\mathcal{A}, \mathcal{A}^*, R_{\cdot}^*, L_{\cdot}^*, \alpha^*, R_{\circ}^*, L_{\circ}^*, \alpha)$ is a matched pair of involutive Hom-associative algebras if and only if $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ satisfies the following conditions for all $x, y \in \mathcal{A}$:*

$$\Delta \circ \alpha(x \cdot y) = (\alpha \otimes L_{\cdot}(x)) \vartriangle (y) + (R_{\cdot}(y) \otimes \alpha) \vartriangle (x), \tag{18.17}$$

$$(L_{\cdot}(y) \otimes \alpha - \alpha \otimes R_{\cdot}(y))\Delta(x) + \sigma[(L_{\cdot}(x) \otimes \alpha - \alpha \otimes R_{\cdot}(x))\Delta(y)] = 0. \tag{18.18}$$

Proof Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathcal{A}$, and $\{e_1^*, \ldots, e_n^*\}$ be its dual basis. Then, set $e_i \cdot e_j = \sum_{k=1}^n c_{ij}^k e_k$ and $e_i^* \circ e_j^* = \sum_{k=1}^n f_{ij}^k e_k^*$. Hence, $\Delta(e_k) = \sum_{i,j=1}^n f_{ij}^k e_i \otimes e_j$, and

$$R_{\cdot}^*(e_i)e_j^* = \sum_{k=1}^n c_{ki}^j e_k^*, \; L_{\cdot}^*(e_i)e_j^* = \sum_{k=1}^n c_{ik}^j e_k^*, \; \alpha(e_i) = \sum_{q=1}^n b_q^i e_q,$$
$$R_{\circ}^*(e_i^*)e_j = \sum_{k=1}^n f_{ki}^j e_k, \; L_{\circ}^*(e_i^*)e_j = \sum_{k=1}^n f_{ik}^j e_k, \; \alpha^*(e_i^*) = \sum_{q=1}^n b_q^{*i} e_q^*.$$

We have $\langle \alpha^*(e_i^*), e_j \rangle = b_i^{*j} = \langle e_i^*, \alpha(e_j) \rangle = b_j^i$ which implies $b_i^{*j} = b_j^i$. From the identity $\alpha^2 = \mathrm{Id}$ and $\alpha(e_i) = \sum_{k=1}^n b_k^i e_k$, we get $\sum_{k=1}^n \sum_{l=1}^n b_k^i b_l^k e_l = \sum_{l=1}^n \delta_l^i e_l = e_i$, with $b_k^i b_l^k = \delta_l^i$. Hence, collecting the coefficient of $e_u \otimes e_v$ (for any i, j, k, m) yields

$$\begin{aligned}\Delta(\alpha(e_m) \cdot \alpha(e_i)) &= (\alpha \otimes L_{\cdot}(e_m))\Delta(e_i) + (R_{\cdot}(e_i) \otimes \alpha)\Delta(e_m) \\ &= (\alpha \otimes L(e_m))(\sum_{u,v=1}^n f_{uv}^i e_u \otimes e_v) + (R(e_i) \otimes \alpha)(\sum_{u,v=1}^n f_{uv}^m e_u \otimes e_v) \\ &= \sum_{u,v=1}^n f_{uv}^i \alpha(e_u) \otimes e_m \cdot e_v + \sum_{u,v=1}^n f_{uv}^m e_u \cdot e_i \otimes \alpha(e_v)\end{aligned}$$

$$= \sum_{u,v=1}^n f_{uv}^i (\sum_{j=1}^n b_j^u e_j) \otimes \left(\sum_{k=1}^n c_{mv}^k e_k\right) + \sum_{u,v=1}^n f_{uv}^m (\sum_{j=1}^n c_{ui}^j e_u) \otimes \left(\sum_{k=1}^n b_k^v e_k\right)$$

$$= \sum_{j,k,u,v=1}^{n} (f_{uv}^{i} b_{j}^{u} c_{mv}^{k} + f_{uv}^{m} c_{ui}^{j} b_{k}^{v}) e_j \otimes e_k;$$

$$\Delta \circ \alpha(e_m \cdot e_i) = \Delta \circ \alpha \left(\sum_{l=1}^{n} c_{mi}^{l} e_l \right) = \sum_{l=1}^{n} c_{mi}^{l} \Delta(\alpha(e_l))$$

$$= \sum_{l=1}^{n} c_{mi}^{l} \Delta \left(\sum_{q=1}^{n} b_{q}^{l} e_q \right) = \sum_{l,q,j,k=1}^{n} c_{mi}^{l} b_{q}^{l} f_{jk}^{q} e_j \otimes e_k,$$

since $\Delta \circ \alpha(e_m \cdot e_i) = (\alpha \otimes \alpha) \circ \Delta(e_m \cdot e_i)$. Then,

$$\sum_{l,u,v,j,k=1}^{n} c_{mi}^{l} f_{uv}^{l} b_{j}^{u} b_{k}^{v} e_j \otimes e_k = \sum_{l,q,j,k=1}^{n} c_{mi}^{l} b_{q}^{l} f_{jk}^{q} e_j \otimes e_k.$$

We obtain the relation

$$\sum_{q=1}^{n} c_{mi}^{l} b_{q}^{l} f_{jk}^{q} = \sum_{u,v=1}^{n} (f_{uv}^{i} b_{j}^{u} c_{mv}^{k} + f_{uv}^{m} c_{ui}^{j} b_{k}^{v})$$
$$\Updownarrow$$
$$\sum_{u,v=1}^{n} c_{mi}^{l} f_{uv}^{l} b_{j}^{u} b_{k}^{v} = \sum_{u,v=1}^{n} (f_{uv}^{i} b_{j}^{u} c_{mv}^{k} + f_{uv}^{m} c_{ui}^{j} b_{k}^{v}),$$

and the identity given by the coefficient of e_m^* in

$$\begin{aligned}
R_{\cdot}^{*}(\alpha(e_i))(e_j^* \circ e_k^*) &= R_{\cdot}^{*}(L_{\circ}^{*}(e_j^*)e_i)\alpha^*(e_k^*) + (R_{\cdot}^{*}(e_i)e_j^*) \circ \alpha^*(e_k^*). \\
&= R_{\cdot}^{*}\left(\sum_{u=1}^{n} f_{ju}^{i} e_u)\alpha^*(e_v^*) \right) + \left(\sum_{u=1}^{n} c_{ui}^{j} e_u^* \right) \circ \alpha^*(e_k^*) \\
&= \sum_{u=1}^{n} f_{ju}^{i} R_{\cdot}^{*}(e_u) \left(\sum_{v=1}^{n} b_{k}^{v} e_v^* \right) + \sum_{u=1}^{u} c_{ui}^{j} (e_u^* \circ \left(\sum_{v=1}^{n} b_{k}^{v} e_v^*) \right) \\
&= \sum_{u,v=1}^{n} f_{ju}^{i} b_{k}^{v} R_{\cdot}^{*}(e_u) e_v^* + \sum_{u,v=1}^{n} c_{ui}^{j} b_{k}^{v} (e_u^* \circ e_v^*) \\
&= \sum_{u,v=1}^{n} f_{ju}^{i} b_{k}^{v} \left(\sum_{m=1}^{n} c_{mu}^{v} e_m^* \right) + \sum_{u,v,m=1}^{n} c_{ui}^{j} b_{k}^{v} f_{uvp}^{m} e_m^* \\
&= \sum_{u,v,m=1}^{n} (f_{ju}^{i} b_{k}^{v} c_{mu}^{v} + c_{ui}^{j} b_{k}^{v} f_{uv}^{m}) e_m^*;
\end{aligned}$$

$$R_{\cdot}^{*}(\alpha(e_i))(e_j^* \circ e_k^*) = R_{\cdot}^{*}(\alpha(e_i))(\sum_{l=1}^{n} f_{jk}^{l} e_l^*) = \sum_{l=1}^{n} f_{jk}^{l} R_{\cdot}^{*} \left(\sum_{q=1}^{n} b_{q}^{i} e_q \right) (e_l^*) =$$
$$\sum_{l,q=1}^{n} f_{jv}^{l} b_{q}^{u} R_{\cdot}^{*}(e_q) e_l^* = \sum_{l,q=1}^{n} f_{jk}^{l} b_{q}^{i} \left(\sum_{m=1}^{n} c_{mq}^{l} e_m^* \right) = \sum_{l,q,m=1}^{n} f_{jk}^{l} b_{q}^{i} c_{mq}^{l} e_m^*.$$

Then, we arrive at

$$\sum_{q=1}^{n} f_{jk}^{l} b_{q}^{i} c_{mq}^{l} = \sum_{u,v=1}^{n} (f_{ju}^{i} b_{k}^{v} c_{mu}^{v} + c_{ui}^{j} b_{k}^{v} f_{uv}^{m})$$
$$\Updownarrow$$
$$\sum_{u,v=1}^{n} f_{uv}^{l} b_{j}^{u} b_{k}^{v} c_{mq}^{l} = \sum_{u,v=1}^{n} (f_{ju}^{i} b_{k}^{v} c_{mu}^{v} + c_{ui}^{j} b_{k}^{v} f_{uv}^{m}).$$

Thus, taking $c_{mi}^{l} = b_{q}^{i} c_{mq}^{l}$, $b_{q}^{l} f_{jk}^{q} = f_{jk}^{l}$, $f_{uv}^{i} b_{j}^{u} c_{mv}^{k} = f_{ju}^{i} c_{mu}^{v} b_{k}^{v}$, we obtain that (18.17) corresponds to (18.15). Similarly,

$$(L_{\cdot}(e_i) \otimes \alpha - \alpha \otimes R_{\cdot}(e_i))\Delta(e_m) + \sigma[(L_{\cdot}(e_m) \otimes \alpha - \alpha \otimes R_{\cdot}(e_m))\Delta(e_i)] = 0 \Leftrightarrow$$

$$\begin{array}{l}(L_{\cdot}(e_i) \otimes \alpha - \alpha \otimes R_{\cdot}(e_i))(\sum_{k,l=1}^{n} f_{lk}^{m} e_l \otimes e_k) \\ \quad +\sigma[(L_{\cdot}(e_m) \otimes \alpha - \alpha \otimes R_{\cdot}(e_m))(\sum_{k,l=1}^{n} f_{lk}^{i} e_l \otimes e_k)] = 0\end{array} \Leftrightarrow$$

$$\begin{array}{l}\sum_{k,l=1}^{n} f_{lk}^{m} (e_i \cdot e_l \otimes \alpha(e_k) - \alpha(e_l) \otimes e_k \cdot e_i) \\ \quad +\sigma[\sum_{k,l=1}^{n} f_{kl}^{i} (e_m \cdot e_l \otimes \alpha(e_k) - \alpha(e_l) \otimes e_k \cdot e_m)] = 0\end{array} \Leftrightarrow$$

$$\begin{array}{l}\sum_{k,l=1}^{n} f_{lk}^{m} ((\sum_{j=1}^{n} c_{il}^{j} e_j) \otimes (\sum_{p=1}^{n} b_{p}^{k} e_p) - (\sum_{p=1}^{n} b_{p}^{l} e_p) \otimes (\sum_{j=1}^{n} c_{ki}^{j} e_j)) \\ +\sigma[\sum_{j,l=1}^{n} f_{kl}^{i} ((\sum_{j=1}^{n} c_{ml}^{j} e_j) \otimes (\sum_{p=1}^{n} b_{p}^{k} e_p) - (\sum_{p=1}^{n} b_{p}^{l} e_p) \otimes (\sum_{j=1}^{n} c_{km}^{j} e_j))] = 0\end{array} \Leftrightarrow$$

$$\begin{array}{l}\sum_{k,l,j,p=1}^{n} (f_{lk}^{m} c_{il}^{j} d_{p}^{k} e_j \otimes e_p - f_{lk}^{m} c_{ki}^{j} b_{p}^{l} e_p \otimes e_j) \\ \quad +\sigma[\sum_{k,l,j,p=1}^{n} (f_{kl}^{i} c_{ml}^{j} b_{p}^{k} e_j \otimes e_p - f_{kl}^{i} c_{km}^{j} b_{p}^{l} e_p \otimes e_j)] = 0;\end{array}$$

$$\begin{array}{l}R_{\cdot}^{*}(R_{\circ}^{*}(e_j^{*})e_i)\alpha^{*}(e_k^{*}) + (L_{\cdot}^{*}(e_i)e_j^{*}) \circ \alpha^{*}(e_k^{*}) \\ \quad = L_{\cdot}^{*}(L_{\circ}^{*}(e_k^{*})e_i)\alpha^{*}(e_j^{*}) + \alpha^{*}(e_j^{*}) \circ (R_{\cdot}^{*}(e_i)e_k^{*})\end{array} \Leftrightarrow$$

$$\begin{array}{l}R_{\cdot}^{*}(\sum_{l=1}^{n} f_{lj}^{i} e_l)(\sum_{p=1}^{n} d_{p}^{k} e_p^{*}) + (\sum_{l=1}^{n} c_{il}^{j} e_l^{*}) \circ (\sum_{p=1}^{n} d_{p}^{k} e_p^{*}) \\ \quad = L_{\cdot}^{*}(\sum_{l=1}^{n} f_{kl}^{i} e_l)(\sum_{q=1}^{n} d_{q}^{j} e_q^{*}) + (\sum_{q=1}^{n} d_{q}^{j} e_q^{*}) \circ (\sum_{l=1}^{n} c_{li}^{k} e_l^{*})\end{array} \Leftrightarrow$$

$$\sum_{l,p=1}^{n} f_{lj}^{i} d_{p}^{k} R_{\cdot}^{*}(e_l) e_p^{*} + \sum_{l,p=1}^{n} c_{il}^{j} d_{p}^{k} e_l^{*} \circ e_p^{*} = \sum_{l,q=1}^{n} f_{kl}^{i} d_{q}^{j} L_{\cdot}^{*}(e_l) e_q^{*} + \sum_{q,l=1}^{n} d_{q}^{j} c_{li}^{k} e_q^{*} \circ e_l^{*} \Leftrightarrow$$

$$\sum_{l,p=1}^{n} f_{lj}^{i} d_{p}^{k} (\sum_{m=1}^{n} c_{ml}^{p} e_m^{*}) + \sum_{l,p,m=1}^{n} c_{il}^{j} d_{p}^{k} f_{lp}^{m} e_m^{*} = \sum_{l,q,m=1}^{n} f_{kl}^{i} d_{q}^{j} c_{lm}^{q} e_m^{*} + \sum_{q,l,m=1}^{n} d_{q}^{j} c_{li}^{k} f_{ql}^{m} e_m^{*}$$

$$\Leftrightarrow \sum_{l,m,p=1}^{n} (f_{lj}^{i} d_{p}^{k} c_{ml}^{p} + f_{lp}^{m} d_{p}^{k} c_{il}^{j}) e_m^{*} = \sum_{l,m,q=1}^{n} (f_{kl}^{i} d_{q}^{j} c_{lm}^{q} + f_{ql}^{m} d_{q}^{j} c_{li}^{k}) e_m^{*}.$$

Thus, we conclude that (18.18) corresponds to (18.16). □

Definition 18.9 Let $(\mathcal{A}, \cdot, \alpha)$ be an involutive Hom-associative algebra. An **anti-symmetric infinitesimal Hom-bialgebra** structure on $\mathcal{A}$ is a linear map $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ such that

(i) $\Delta^{*} : \mathcal{A}^{*} \otimes \mathcal{A}^{*} \to \mathcal{A}^{*}$ defines an involutive Hom-associative algebra structure on $\mathcal{A}^{*}$;
(ii) Δ satisfies (18.17) and (18.18).

We denote such an antisymmetric infinitesimal Hom-bialgebra by $(\mathcal{A}, \triangle, \alpha)$ or $(\mathcal{A}, \mathcal{A}^*, \alpha, \alpha^*)$.

Corollary 18.1 *Let* $(\mathcal{A}, \cdot, \alpha)$ *and* $(\mathcal{A}^*, \circ, \alpha^*)$ *be two involutive associative algebras. Then, the following conditions are equivalent*:

(i) *There is a double construction of an involutive Hom-Frobenius algebra associated to* $(\mathcal{A}, \cdot, \alpha)$ *and* $(\mathcal{A}^*, \circ, \alpha^*)$;
(ii) $(\mathcal{A}, \mathcal{A}^*, R^*_{\cdot}, L^*_{\cdot}, \alpha^* R^*_{\circ}, L^*_{\circ}, \alpha)$ *is a matched pair of involutive associative algebras*;
(iii) $(\mathcal{A}, \mathcal{A}^*, \alpha, \alpha^*)$ *is an antisymmetric infinitesimal Hom-bialgebra.*

Proof From Theorems 18.2 and 18.4, we have the equivalences. □

18.4 Double Constructions of Involutive BiHom-Frobenius Algebras

18.4.1 Bimodule and Matched Pair of BiHom-Associative Algebras

Definition 18.10 ([22]) A biHom-associative algebra is a quadruple $(\mathcal{A}, \cdot, \alpha, \beta)$ consisting of a linear space $\mathcal{A}$, $\mathcal{K}$-bilinear map $\cdot : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, linear maps $\alpha, \beta : \mathcal{A} \to \mathcal{A}$ satisfying, for all $x, y, z \in \mathcal{A}$, the following conditions:

$$\begin{array}{cl} \alpha \circ \beta = \beta \circ \alpha, & \text{(commutativity);} \\ \alpha(x \cdot y) = \alpha(x) \cdot \alpha(y), \quad \beta(x \cdot y) = \beta(x) \cdot \beta(y), & \text{(multiplicativity);} \\ \alpha(x) \cdot (y \cdot z) = (x \cdot y) \cdot \beta(z), & \text{(biHom-associativity).} \end{array}$$

Remark 18.3 If $\alpha = \beta$, then $(\mathcal{A}, \cdot, \alpha, \alpha)$ is a Hom-associative algebra.

Definition 18.11 A biHom-module is a triple (M, α, β), where M is a $\mathcal{K}$-vector space, and $\alpha, \beta : M \to M$ are two linear maps.

Definition 18.12 ([22]) Let $(A, \mu_{\mathcal{A}}, \alpha_{\mathcal{A}}, \beta_{\mathcal{A}})$ be a biHom-associative algebra. A left $\mathcal{A}$-module is a triple (M, α_M, β_M), where M is a linear space, $\alpha_M, \beta_M : M \to M$ are linear maps, with, in addition, another linear map: $\mathcal{A} \otimes M \to M, a \otimes m \mapsto a \cdot m$, such that, for all $a, a' \in \mathcal{A}, m \in M$:

$$\alpha_M \circ \beta_M = \beta_M \circ \alpha_M, \;\; \alpha_M(a \cdot m) = \alpha_{\mathcal{A}}(a) \cdot \alpha_M(m),$$
$$\beta_M(a \cdot m) = \beta_{\mathcal{A}}(a) \cdot \beta_M(m), \;\; \alpha_{\mathcal{A}}(a) \cdot (a' \cdot m) = (aa') \cdot \beta_M(m).$$

Let us give now the definition of bimodule of a biHom-associative algebra.

Definition 18.13 Let $(\mathcal{A}, \cdot, \alpha_1, \alpha_2)$ be a biHom-associative algebra, and let (V, β_1, β_2) be a biHom-module. Let $l, r : \mathcal{A} \to gl(V)$ be two linear maps. The quintuple $(l, r, \beta_1, \beta_2, V)$ is called a bimodule of $\mathcal{A}$ if, for all $x, y \in \mathcal{A}$, $v \in V$,

$$\begin{aligned} l(x \cdot y)\beta_1(v) &= l(\alpha_2(x))l(y)v, \quad r(x \cdot y)\beta_2(v) = r(\alpha_1(y))r(x)v, \\ l(\alpha_2(x))r(y)v &= r(\alpha_1(y))l(x)v, \\ \beta_1(l(x)v) &= l(\alpha_1(x))\beta_1(v), \quad \beta_1(r(x)v) = r(\alpha_1(x))\beta_1(v), \\ \beta_2(l(x)v) &= l(\alpha_2(x))\beta_2(v), \quad \beta_2(r(x)v) = r(\alpha_2(x))\beta_2(v). \end{aligned}$$

Proposition 18.4 *Let $(l, r, \beta_1, \beta_2, V)$ be a bimodule of a biHom-associative algebra $(\mathcal{A}, \cdot, \alpha_1, \alpha_2)$. Then, the direct sum $\mathcal{A} \oplus V$ of vector spaces is a biHom-associative algebra with multiplication in $\mathcal{A} \oplus V$ defined, for all $x_1, x_2 \in \mathcal{A}$, $v_1, v_2 \in V$, by*

$$\begin{aligned} &(x_1 + v_1) * (x_2 + v_2) = x_1 \cdot x_2 + (l(x_1)v_2 + r(x_2)v_1), \\ &(\alpha_1 \oplus \beta_1)(x_1 + v_1) = \alpha_1(x_1) + \beta_1(v_1), (\alpha_2 \oplus \beta_2)(x_1 + v_1) = \alpha_2(x_1) + \beta_2(v_1). \end{aligned}$$

Proof Let $v_1, v_2, v_3 \in V$ and $x_1, x_2, x_3 \in \mathcal{A}$. Setting and computing

$$\begin{aligned} &[(x_1 + v_1) * (x_2 + v_2)] * (\alpha_1(x_3) + \beta_1(v_3)) = \\ &\quad (\alpha_2(x_1) + \beta_2(v_1)) * [(x_2 + v_2) * (x_3 + v_3)], \end{aligned}$$

and similarly for the other relations, give the required conditions. □

We denote such a biHom-associative algebra by $(\mathcal{A} \oplus V, *, \alpha_1 + \beta_1, \alpha_2 + \beta_2)$, or $\mathcal{A} \times_{l,r,\alpha_1,\alpha_2,\beta_1,\beta_2} V$.

Example 18.5 Let $(\mathcal{A}, \cdot, \alpha, \beta)$ be a multiplicative biHom-associative algebra. Then, $(L_\cdot, 0, \alpha, \beta)$, $(0, R_\cdot, \alpha, \beta)$ and $(L_\cdot, R_\cdot, \alpha, \beta)$ are bimodules of $(\mathcal{A}, \cdot, \alpha, \beta)$.

Proposition 18.5 *Let $(l, r, \beta_1, \beta_2, V)$ be bimodule of a multiplicative biHom-associative algebra $(\mathcal{A}, \cdot, \alpha_1, \alpha_2)$. Then, $(l \circ \alpha_1^n, r \circ \alpha_2^n, \beta_1, \beta_2, V)$ is a bimodule of $\mathcal{A}$ for any integer n.*

Proof We have

$$\begin{aligned} (l \circ \alpha_1^n)(x \cdot y)\beta_1(v) &= l(\alpha_1^n(x) \cdot \alpha_1^n(y))\beta_1(v) = l(\alpha_2(\alpha_1^n(x)))l(\alpha_1^n(y))v \\ &= l(\alpha_1^n(\alpha_2(x)))l(\alpha_1^n(y))v = (l \circ \alpha_1^n)(\alpha_2(x))(l \circ \alpha_1^n)(y)v. \end{aligned}$$

Similarly, the other relations are established. □

Example 18.6 Let $(\mathcal{A}, \cdot, \alpha_1, \alpha_2)$ be a multiplicative biHom-associative algebra. Then, $(L_\cdot \circ \alpha_1^n, R_\cdot \circ \alpha_2^n, \alpha_1, \alpha_2, \mathcal{A})$ is a bimodule of $\mathcal{A}$ for any integer n.

Theorem 18.5 *Let $(\mathcal{A}, \cdot, \alpha_1, \alpha_2)$ and $(\mathcal{B}, \circ, \beta_1, \beta_2)$ be two biHom-associative algebras. Suppose there exist linear maps $l_{\mathcal{A}}, r_{\mathcal{A}} : \mathcal{A} \to gl(\mathcal{B})$, and $l_{\mathcal{B}}, r_{\mathcal{B}} : \mathcal{B} \to gl(\mathcal{A})$*

such that $(l_{\mathcal{A}}, r_{\mathcal{A}}, \beta_1, \beta_2, \mathcal{B})$ *is a bimodule of* $\mathcal{A}$*, and* $(l_{\mathcal{B}}, r_{\mathcal{B}}, \alpha_1, \alpha_2, \mathcal{A})$ *is a bimodule of* $\mathcal{B}$*, satisfying for any* $x, y \in \mathcal{A}$ *and* $a, b \in \mathcal{B}$*,*

$$l_{\mathcal{A}}(\alpha_2(x))(a \circ b) = l_{\mathcal{A}}(r_{\mathcal{B}}(a)x)\beta_1(b) + (l_{\mathcal{A}}(x)a) \circ \beta_1(b), \tag{18.19}$$

$$r_{\mathcal{A}}(\alpha_1(x))(a \circ b) = r_{\mathcal{A}}(l_{\mathcal{B}}(b)x)\beta_2(a) + \beta_2(a) \circ (r_{\mathcal{A}}(x)b), \tag{18.20}$$

$$l_{\mathcal{B}}(\beta_2(a))(x \cdot y) = l_{\mathcal{B}}(r_{\mathcal{A}}(x)a)\alpha_1(y) + (l_{\mathcal{B}}(a)x) \cdot \alpha_1(y), \tag{18.21}$$

$$r_{\mathcal{B}}(\beta_1(a))(x \cdot y) = r_{\mathcal{B}}(l_{\mathcal{A}}(y)a)\alpha_2(x) + \alpha_2(x) \cdot (r_{\mathcal{B}}(a)y), \tag{18.22}$$

$$\begin{aligned} &l_{\mathcal{A}}(l_{\mathcal{B}}(a)x)\beta_1(b) + (r_{\mathcal{A}}(x)a) \circ \beta_1(b) \\ &\qquad - r_{\mathcal{A}}(r_{\mathcal{B}}(b)x)\beta_2(a) - \beta_2(a) \circ (l_{\mathcal{A}}(x)b) = 0, \end{aligned} \tag{18.23}$$

$$\begin{aligned} &l_{\mathcal{B}}(l_{\mathcal{A}}(x)a)\alpha_1(y) + (r_{\mathcal{B}}(a)x) \cdot \alpha_1(y) \\ &\qquad - r_{\mathcal{B}}(r_{\mathcal{A}}(y)a)\alpha_2(x) - \alpha_2(x) \cdot (l_{\mathcal{B}}(a)y) = 0. \end{aligned} \tag{18.24}$$

Then, there is a biHom-associative algebra structure on the direct sum $\mathcal{A} \oplus \mathcal{B}$ *of the underlying vector spaces of* $\mathcal{A}$ *and* $\mathcal{B}$ *given, for all* $x, y \in \mathcal{A}$*,* $a, b \in \mathcal{B}$*, by*

$$\begin{aligned} &(x + a) * (y + b) = (x \cdot y + l_{\mathcal{B}}(a)y + r_{\mathcal{B}}(b)x) + (a \circ b + l_{\mathcal{A}}(x)b + r_{\mathcal{A}}(y)a), \\ &(\alpha_1 \oplus \beta_1)(x + a) = \alpha_1(x) + \beta_1(a), (\alpha_2 \oplus \beta_2)(x + a) = \alpha_2(x) + \beta_2(a). \end{aligned}$$

Proof Let $v_1, v_2, v_3 \in V$ and $x_1, x_2, x_3 \in \mathcal{A}$. Setting and computing

$$\begin{aligned} &[(x_1 + v_1) * (x_2 + v_2)] * (\alpha_1(x_3) + \beta_1(v_3)) = \\ &\quad (\alpha_2(x_1) + \beta_2(v_1)) * [(x_2 + v_2) * (x_3 + v_3)], \end{aligned}$$

we obtain (18.19)–(18.24). Then, using the relations

$$\begin{aligned} \beta_1(l_{\mathcal{A}}(x)a) &= l_{\mathcal{A}}(\alpha_1(x))\beta_1(a),\ \ \beta_1(r_{\mathcal{A}}(x)a) = r_{\mathcal{A}}(\alpha_1(x))\beta_1(a), \\ \beta_2(l_{\mathcal{A}}(x)a) &= l_{\mathcal{A}}(\alpha_2(x))\beta_2(a),\ \ \beta_2(r_{\mathcal{A}}(x)a) = r_{\mathcal{A}}(\alpha_2(x))\beta_2(a), \\ \alpha_1(l_{\mathcal{B}}(a)x) &= l_{\mathcal{B}}(\beta_1(a))\alpha_1(x),\ \ \alpha_1(r_{\mathcal{B}}(a)x) = r_{\mathcal{B}}(\beta_1(a))\alpha_1(x), \\ \alpha_2(l_{\mathcal{B}}(a)x) &= l_{\mathcal{B}}(\beta_2(a))\alpha_2(x),\ \ \alpha_2(r_{\mathcal{B}}(a)x) = r_{\mathcal{B}}(\beta_2(a))\alpha_2(x), \end{aligned}$$

yields that $*$ is a biHom-associative algebra structure. □

We use $(\mathcal{A} \bowtie \mathcal{B}, *, \alpha_1 + \beta_1, \alpha_2 + \beta_2)$ or $\mathcal{A} \bowtie^{l_{\mathcal{A}}, r_{\mathcal{A}}, \beta_1, \beta_2}_{l_{\mathcal{B}}, r_{\mathcal{B}}, \alpha_1, \alpha_2} \mathcal{B}$ to denote this biHom-associative algebra.

Definition 18.14 Let $(\mathcal{A}, \cdot, \alpha_1, \alpha_2)$ and $(\mathcal{B}, \circ, \beta_1, \beta_2)$ be two biHom-associative algebras. Suppose there exist linear maps $l_{\mathcal{A}}, r_{\mathcal{A}} : \mathcal{A} \to gl(\mathcal{B})$, and $l_{\mathcal{B}}, r_{\mathcal{B}} : \mathcal{B} \to gl(\mathcal{A})$ such that $(l_{\mathcal{A}}, r_{\mathcal{A}}, \beta_1, \beta_2)$ is a bimodule of $\mathcal{A}$, and $(l_{\mathcal{B}}, r_{\mathcal{B}}, \alpha_1, \alpha_2)$ is a bimodule of $\mathcal{B}$. Then, $(\mathcal{A}, \mathcal{B}, l_{\mathcal{A}}, r_{\mathcal{A}}, \beta_1, \beta_2, l_{\mathcal{B}}, r_{\mathcal{B}}, \alpha_1, \alpha_2)$ is called a **matched pair of biHom-associative algebras**, if the conditions (18.19)–(18.24) are satisfied.

18.4.2 Double Constructions of Involutive BiHom-Frobenius Algebras

Now, we consider the multiplicative biHom-associative algebra $(\mathcal{A}, \cdot, \alpha_1, \alpha_2)$ such that $\alpha_1 \circ \alpha_2 = \alpha_2 \circ \alpha_1 = \mathrm{Id}_{\mathcal{A}}$, that is, $\alpha_1^{-1} = \alpha_2$, and $\alpha_1^2 = \alpha_2^2 = \mathrm{Id}_A$.

Lemma 18.2 *Let $(l, r, \beta_1, \beta_2, V)$ be a bimodule of $(\mathcal{A}, \cdot, \alpha_1, \alpha_2)$. Then*

(i) $(r^*, l^*, \beta_2^*, \beta_1^*, V^*)$ *is a bimodule of* $(\mathcal{A}, \cdot, \alpha_1, \alpha_2)$;
(ii) $(r^*, 0, \beta_2^*, \beta_1^*, V^*)$ *and* $(0, l^*, \beta_2^*, \beta_1^*, V^*)$ *are also bimodules of* $\mathcal{A}$.

Proof (i) Let $(l, r, \beta_1, \beta_2, V)$ be a bimodule of an involutive biHom-associative algebra $(\mathcal{A}, \cdot, \alpha_1, \alpha_2)$. Show that $(r^*, l^*, \beta_2^*, \beta_1^*, V^*)$ is a bimodule of $\mathcal{A}$. Let $x, y \in \mathcal{A}$, $u^* \in V^*$, $v \in V$. Then,
(i-1) the following computation

$$\begin{aligned}\langle r^*(x \cdot y)\beta_2^*(u^*), v\rangle &= \langle \beta_2(r(x \cdot y)v), u^*\rangle = \langle r(\alpha_2(x \cdot y))\beta_2(v), u^*\rangle \\ &= \langle r(\alpha_2(x) \cdot \alpha_2(y))\beta_2(v), u^*\rangle = \langle r[\alpha_1(\alpha_2(y))]r(\alpha_2(x))v, u^*\rangle \\ &= \langle (r(y)r(\alpha_2(x)))^*u^*, v\rangle = \langle r^*(\alpha_2(x))r^*(y)u^*, v\rangle,\end{aligned}$$

leads to $r^*(x \cdot y)\beta_2^*(u^*) = r^*(\alpha_2(x))r^*(y)u^*$;
(i-2) the following computation

$$\begin{aligned}\langle l^*(x \cdot y)\beta_1^*(u^*), v\rangle &= \langle \beta_1(l(x \cdot y)(v)), u^*\rangle = \langle l(\alpha_1(x \cdot y))\beta_1(v), u^*\rangle \\ &= \langle l(\alpha_1(x) \cdot \alpha_1(y))\beta_1(v), u^*\rangle \\ &= \langle l[\alpha_2(\alpha_1(x))]l(\alpha_1(y))\beta(v), u^*\rangle \\ &= \langle (l(x)l(\alpha_1(y)))^*u^*, v\rangle = \langle l^*(\alpha_1(y))l^*(x)u^*, v\rangle\end{aligned}$$

gives $l^*(x \cdot y)\beta_1^*(u^*) = l^*(\alpha_1(y))l^*(x)u^*$;
(i-3) the following computation

$$\begin{aligned}\langle r^*(\alpha_2(x))l^*(y)u^*, v\rangle &= \langle l(y)r(\alpha_2(x))v, u^*\rangle = \langle (l \circ \alpha_1)(\alpha_2(y))(r \circ \alpha_2)(x))v, u^*\rangle \\ &= \langle (r \circ \alpha_2)(\alpha_1(x))(l \circ \alpha_1)(y)v, u^*\rangle = \langle r(x)l(\alpha_1(y))v, u^*\rangle \\ &= \langle l^*(\alpha_1(y))r^*(x)u^*, v\rangle\end{aligned}$$

yields $r^*(\alpha_2(x))l^*(y)u^* = l^*(\alpha_1(y))r^*(x)u^*$.
Furthermore,

$$\begin{aligned}\langle \beta_2^*(r^*(x))u^*, v\rangle &= \langle r(x)(\beta_2(v)), u^*\rangle = \langle (r \circ \alpha_1)(\alpha_2(x))(\beta_2(v)), u^*\rangle \\ &= \langle \beta_2(r(\alpha_1(x)))v, u^*\rangle = \langle r^*(\alpha_1(x))\beta_2^*(u^*), v\rangle.\end{aligned}$$

Hence, $\beta_2^*(r^*(x))u^* = r^*(\alpha_1(x))\beta_2^*(u^*)$. By analogy, we establish the other conditions. Hence, $(r^*, l^*, \beta_2^*, \beta_1^*, V^*)$ is a bimodule of $\mathcal{A}$.
(ii) Similarly, one can show that $(r^*, 0, \beta_2^*, \beta_1^*, V^*)$ and $(0, l^*, \beta_2^*, \beta_1^*, V^*)$ are bimodules of $\mathcal{A}$ as well. □

Definition 18.15 Let $(\mathcal{A}, \cdot, \alpha, \beta)$ be a biHom-associative algebra, and $B : \mathcal{A} \times \mathcal{A} \to K$ be a bilinear form on $\mathcal{A}$. B is said $\alpha\beta$-invariant if

$$B(\beta(x) \cdot \alpha(y), \alpha(z)) = B(\alpha(x), \beta(y) \cdot \alpha(z)). \tag{18.25}$$

Definition 18.16 A biHom-Frobenius algebra is a biHom-associative algebra with a non-degenerate invariant bilinear form.

Definition 18.17 We call $(\mathcal{A}, \alpha, \beta, B)$ a **double construction of an involutive biHom-Frobenius algebra** associated to $(\mathcal{A}_1, \alpha_1)$ and $(\mathcal{A}_1^*, \alpha_1^*)$ if it satisfies the conditions:

1) $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_1^*$ as the direct sum of vector spaces;
2) $(\mathcal{A}_1, \alpha_1, \alpha_2)$ and $(\mathcal{A}_1^*, \alpha_1^*, \alpha_2^*)$ are biHom-associative subalgebras of $(\mathcal{A}, \alpha)$ with $\alpha = \alpha_1 \oplus \alpha_1^*$ and $\beta = \alpha_2 \oplus \alpha_2^*$;
3) B is the natural non-degenerate $(\alpha_1 \oplus \alpha_1^*)(\alpha_2 \oplus \alpha_2^*)$-invariant symmetric bilinear form on $\mathcal{A}_1 \oplus \mathcal{A}_1^*$ given by

$$\begin{aligned} B(x + a^*, y + b^*) &= \langle x, b^* \rangle + \langle a^*, y \rangle, \\ B((\alpha_1 + \alpha_1^*)(x + a^*), y + b^*) &= B(x + a^*, (\alpha_1 + \alpha_1^*)(y + b^*)), \\ B((\alpha_2 + \alpha_2^*)(x + a^*), y + b^*) &= B(x + a^*, (\alpha_2 + \alpha_2^*)(y + b^*)) \end{aligned} \tag{18.26}$$

for all $x, y \in \mathcal{A}_1$, $a^*, b^* \in \mathcal{A}_1^*$, where $\langle , \rangle$ is the natural pairing between the vector space $\mathcal{A}_1$ and its dual space $\mathcal{A}_1^*$.

Let $(\mathcal{A}, \cdot, \alpha_1, \alpha_2)$ be an involutive biHom-associative algebra. Suppose there also exists an involutive biHom-associative algebra structure " $\circ$ " on its dual space $\mathcal{A}^*$. We construct an involutive biHom-associative algebra structure on the direct sum $\mathcal{A} \oplus \mathcal{A}^*$ of the underlying vector spaces of $\mathcal{A}$ and $\mathcal{A}^*$ such that $(\mathcal{A}, \cdot, \alpha_1, \alpha_2)$ and $(\mathcal{A}^*, \circ, \alpha_1^*, \alpha_1^*)$ are biHom-subalgebras, and the non-degenerate invariant symmetric bilinear form on $\mathcal{A} \oplus \mathcal{A}^*$ is given by (18.26). Hence, $(\mathcal{A} \oplus \mathcal{A}^*, \alpha_1 \oplus \alpha_1^*, \alpha_2 \oplus \alpha_2^*, B)$ is a symmetric multiplicative biHom-associative algebra. Such a construction is called a double construction of an involutive biHom-Frobenius algebra associated to $(\mathcal{A}, \cdot, \alpha_1, \alpha_2)$ and $(\mathcal{A}^*, \circ, \alpha_1^*, \alpha_2^*)$.

Theorem 18.6 *Let $(\mathcal{A}, \cdot, \alpha_1, \alpha_2)$ be an involutive biHom-associative algebra. Suppose there is an involutive biHom-associative algebra structure " $\circ$ " on its dual space $\mathcal{A}^*$. Then, there is a double construction of an involutive symmetric biHom-associative algebra associated to $(\mathcal{A}, \cdot, \alpha_1, \alpha_2)$ and $(\mathcal{A}^*, \circ, \alpha_1^*, \alpha_2^*)$ if and only if $(\mathcal{A}, \mathcal{A}^*, R_\cdot^*, L_\cdot^*, \alpha_2^*, \alpha_1^*, R_\circ^*, L_\circ^*, \alpha_2, \alpha_1)$ is a matched pair of involutive biHom-associative algebras.*

Proof By a similar proof as for Theorem 18.2, we obtain the results. Let us show that B is well $(\alpha_1 \oplus \alpha_1^*)(\alpha_2 \oplus \alpha_2^*)$-invariant. Let $x, y, z \in \mathcal{A}$ and $a^*, b^*, c^* \in \mathcal{A}^*$. We have

$$\begin{aligned}&\mathcal{B}[(\alpha_2(x)+\alpha_2^*(a^*))*(\alpha_1(y)+\alpha_1^*(b^*)),(\alpha_1(z)+\alpha_1^*(c^*))]\\&\quad=\langle\alpha_2(x)\cdot\alpha_1(y),\alpha_1^*(c^*)\rangle+\langle\alpha_1^*(c^*)\circ\alpha_2^*(a^*),\alpha_1(y)\rangle+\langle\alpha_1^*(b^*)\circ\alpha_1^*(c^*),\alpha_2(x)\rangle\\&\quad+\langle\alpha_2^*(a^*)\circ\alpha_1^*(b^*),\alpha_1(z)\rangle+\langle\alpha_1(z)\cdot\alpha_2(x),\alpha_1^*(b^*)\rangle+\langle\alpha_1(y)\cdot\alpha_1(z),\alpha_2^*(a^*)\rangle;\\&\mathcal{B}[\alpha_1(x)+\alpha_1^*(a^*),(\alpha_2(y)+\alpha_2^*(b^*))*(\alpha_1(z)+\alpha_1^*(c^*))]\\&\quad=\langle\alpha_1(x),\alpha_2^*(b^*)\circ\alpha_1^*(c^*)\rangle+\langle\alpha_1^*(c^*),\alpha_1(x)\cdot\alpha_2(y)\rangle+\langle\alpha_2^*(b^*),\alpha_1(z)\cdot\alpha_1(x)\rangle\\&\quad+\langle\alpha_2(y)\cdot\alpha_1(z),\alpha_1^*(a^*)\rangle+\langle\alpha_1^*(a^*)\circ\alpha_2^*(b^*),\alpha_1(z)\rangle+\langle\alpha_1(c^*)\circ\alpha_1^*(a^*),\alpha_2(y)\rangle.\end{aligned}$$

Using $\alpha_1^2=\alpha_2^2=\mathrm{Id}_{\mathcal{A}}$, $\alpha_1^{*2}=\alpha_2^{*2}=\mathrm{Id}_{\mathcal{A}^*}$, $\alpha_1=\alpha_2^{-1}$ and $\alpha_1^*=\alpha_2^{*-1}$, we obtain

$$\begin{aligned}&\mathcal{B}[(\alpha_2(x)+\alpha_2^*(a^*))*(\alpha_1(y)+\alpha_1^*(b^*)),(\alpha_1(z)+\alpha_1^*(c^*))]\\&=\mathcal{B}[\alpha_1(x)+\alpha_1^*(a^*),(\alpha_2(y)+\alpha_2^*(b^*))*(\alpha_1(z)+\alpha_1^*(c^*))]\\&=\langle x,b^*\circ c^*\rangle+\langle c^*,x\cdot y\rangle+\langle b^*,z\cdot x\rangle+\langle y\cdot z,a^*\rangle+\langle a^*\circ b^*,z\rangle+\langle c^*\circ a^*,y\rangle.\end{aligned}$$

This completes the proof. □

Theorem 18.7 *Let $(\mathcal{A},\cdot,\alpha_1,\alpha_2)$ be an involutive biHom-associative algebra. Suppose there exists an involutive biHom-associative algebra structure "$\circ$" on its dual space $(\mathcal{A}^*,\alpha_1^*,\alpha_2^*)$. Then, $(\mathcal{A},\mathcal{A}^*,R_\cdot^*,L_\cdot^*,\alpha_2^*,\alpha_1^*,R_\circ^*,L_\circ^*,\alpha_2,\alpha_1)$ is a matched pair of involutive biHom-associative algebras if and only if, for any $x\in\mathcal{A}$ and $a^*,b^*\in\mathcal{A}^*$,*

$$\begin{aligned}R_\cdot^*(\alpha_2(x))(a^*\circ b^*)&=R_\cdot^*(L_\circ^*(a^*)x)\alpha_2^*(b^*)+(R_\cdot^*(x)a^*)\circ\alpha_2^*(b^*),\\R_\cdot^*(R_\circ^*(a^*)x)\alpha_2^*(b^*)&+L_\cdot^*(x)a^*\circ\alpha_2^*(b^*)=\\&L_\cdot^*(L_\circ^*(b^*)x)\alpha_1^*(a^*)+\alpha_1^*(a^*)\circ(R_\cdot^*(x)b^*).\end{aligned}$$

Proof By a similar proof as for Theorem 18.3, and using the following valid relations

$$\begin{aligned}\alpha_2^*(R_\cdot^*(x)a^*)&=R_\cdot^*(\alpha_1(x))\alpha_2^*(a^*), &\alpha_2^*(L_\cdot^*(x)a^*)&=L_\cdot^*(\alpha_1(x))\alpha_2^*(a^*)\\\alpha_1^*(R_\cdot^*(x)a^*)&=R_\cdot^*(\alpha_2(x))\alpha_1^*(a^*), &\alpha_1^*(L_\cdot^*(x)a^*)&=L_\cdot^*(\alpha_2(x))\alpha_1^*(a^*)\\\alpha_2(R_\circ^*(a^*)x)&=R_\circ^*(\alpha_1^*(a^*))\alpha_2(x), &\alpha_2(L_\circ^*(a^*)x)&=L_\circ^*(\alpha_1^*(a^*))\alpha_2(x)\\\alpha_1(R_\circ^*(a^*)x)&=R_\circ^*(\alpha_2^*(a^*))\alpha_1(x), &\alpha_1(L_\circ^*(a^*)x)&=L_\circ^*(\alpha_2^*(a^*))\alpha_1(x),\end{aligned}$$

the equivalences

$$(18.19)\Longleftrightarrow(18.20)\Longleftrightarrow(18.21)\Longleftrightarrow(18.22),\quad\text{and}\quad(18.23)\Longleftrightarrow(18.24)$$

are obtained. □

Theorem 18.8 *Let $(\mathcal{A},\cdot,\alpha_1,\alpha_2)$ be an involutive biHom-associative algebra. Suppose there is an involutive biHom-associative algebra structure " $\circ$ " on its dual space $\mathcal{A}^*$ given by a linear map $\Delta^*:\mathcal{A}^*\otimes\mathcal{A}^*\to\mathcal{A}^*$. Then,*

$$(\mathcal{A},\mathcal{A}^*,R_\cdot^*,L_\cdot^*,\alpha_2^*,\alpha_1^*,R_\circ^*,L_\circ^*,\alpha_2,\alpha_1)$$

is a matched pair of involutive biHom-associative algebras if and only if $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ *satisfies the following two conditions for all* $x, y \in \mathcal{A}$:

$$\Delta \circ \alpha_2(x \cdot y) = (\alpha_2 \otimes L_{\cdot}(x)) \vartriangle (y) + (R_{\cdot}(y) \otimes \alpha_2) \vartriangle (x), \quad (18.27)$$

$$(L_{\cdot}(y) \otimes \alpha_2 - \alpha_1 \otimes R_{\cdot}(y))\Delta(x) + \sigma[(L_{\cdot}(x) \otimes \alpha_2 - \alpha_1 \otimes R_{\cdot}(x))\Delta(y)] = 0. \quad (18.28)$$

Proof This proof is simillar to that of Theorem 18.4. □

Definition 18.18 Let $(\mathcal{A}, \cdot, \alpha_1, \alpha_2)$ be an involutive biHom-associative algebra. An **antisymmetric infinitesimal biHom-bialgebra** structure on $\mathcal{A}$ is a linear map $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ such that

(a) $\Delta^* : \mathcal{A}^* \otimes \mathcal{A}^* \to \mathcal{A}^*$ defines an involutive biHom-associative algebra structure on $\mathcal{A}^*$;
(b) Δ satisfies (18.27) and (18.28).

We denote it by $(\mathcal{A}, \vartriangle, \alpha_1, \alpha_2)$ or $(\mathcal{A}, \mathcal{A}^*, \alpha_1, \alpha_2, \alpha_1^*, \alpha_2, \alpha_1^*)$.

Corollary 18.2 *Let* $(\mathcal{A}, \cdot, \alpha_1, \alpha_2)$ *and* $(\mathcal{A}^*, \circ, \alpha_1^*, \alpha_2^*)$ *be two biHom-associative algebras. Then, the following conditions are equivalent:*

1) *There is a double construction of an involutive biHom-Frobenius algebra associated to* $(\mathcal{A}, \cdot, \alpha_1, \alpha_2)$ *and* $(\mathcal{A}^*, \circ, \alpha_1^*, \alpha_2^*)$*;*
2) $(\mathcal{A}, \mathcal{A}^*, R_{\cdot}^*, L_{\cdot}^*, \alpha_2^*, \alpha_1^*, R_{\circ}^*, L_{\circ}^*, \alpha_2, \alpha_1)$ *is a matched pair of multiplicative biHom-associative algebras;*
3) $(\mathcal{A}, \mathcal{A}^*, \alpha_1, \alpha_2, \alpha_1^*, \alpha_2^*)$ *is an antisymmetric infinitesimal biHom-bialgebra.*

Proof From Theorems 18.6 and 18.8, we have the equivalences. □

18.5 Double Constructions of Involutive Symplectic Hom-associative Algebras

18.5.1 Hom-dendriform Algebras

Definition 18.19 A Hom-dendriform algebra is a quadruple $(\mathcal{A}, \prec, \succ, \alpha)$ consisting of a vector space $\mathcal{A}$ on which the operations $\prec, \succ: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, and $\alpha : \mathcal{A} \to \mathcal{A}$ are linear maps satisfying

$$\begin{aligned}(x \prec y) \prec \alpha(z) &= \alpha(x) \prec (y * z),\\ (x \succ y) \prec \alpha(z) &= \alpha(x) \succ (y \prec z),\\ \alpha(x) \succ (y \succ z) &= (x * y) \succ \alpha(z),\end{aligned}$$

where

$$x * y = x \prec y + x \succ y. \quad (18.29)$$

Definition 18.20 Let $(\mathcal{A}, \prec, \succ, \alpha)$ and $(\mathcal{A}', \prec', \succ', \alpha')$ be two Hom-dendriform algebras. A linear map $f : \mathcal{A} \to \mathcal{A}'$ is a Hom-dendriform algebra morphism if

$$\prec' \circ(f \otimes f) = f \circ \prec, \quad \succ' \circ(f \otimes f) = f \circ \succ, \quad f \circ \alpha = \alpha' \circ f.$$

Proposition 18.6 *Let $(\mathcal{A}, \prec, \succ, \alpha)$ be a Hom-dendriform algebra. Then, $(\mathcal{A}, *, \alpha)$ is a Hom-associative algebra.*

Proof For all $x, y, z \in \mathcal{A}$,

$$\begin{aligned}(x * y) * \alpha(z) &= (x \prec y) \prec \alpha(z) + (x \prec y) \succ \alpha(z) + (x \succ y) \prec \alpha(z) + (x \succ y) \succ \alpha(z)\\ &= (x \prec y) \prec \alpha(z) + (x \succ y) \prec \alpha(z) + (x * y) \succ \alpha(z)\\ &= \alpha(x) \prec (y * z) + \alpha(x) \succ (y \prec z) + \alpha(x) \succ (y \succ z)\\ &= \alpha(x) \prec (y * z) + \alpha(x) \succ (y * z) = \alpha(x) * (y * z).\end{aligned}$$

which completes the proof. □

We call $(\mathcal{A}, *, \alpha)$ the associated Hom-associative algebra of $(\mathcal{A}, \prec, \succ, \alpha)$, and $(\mathcal{A}, \succ, \prec, \alpha)$ is called a compatible Hom-dendriform algebra structure on the Hom-associative algebra $(\mathcal{A}, *, \alpha)$.

Let $(\mathcal{A}, \prec, \succ, \alpha)$ be a Hom-dendriform algebra. For any $x \in \mathcal{A}$, let $L_{\succ}(x)$, $R_{\succ}(x)$ and $L_{\prec}(x)$, $R_{\prec}(x)$ denote the left and right multiplication operators of $(\mathcal{A}, \prec)$ and $(\mathcal{A}, \succ)$, respectively, that is, for all $x, y \in \mathcal{A}$,

$$L_{\succ}(x)y = x \succ y, \quad R_{\succ}(x)y = y \succ x, \quad L_{\prec}(x)y = x \prec y, \quad L_{\prec}(x)y = y \prec x.$$

Moreover, let $L_{\succ}, R_{\succ}, L_{\prec}, R_{\prec} : \mathcal{A} \to gl(\mathcal{A})$ be respectively, the four linear maps $x \mapsto L_{\succ}(x)$, $x \mapsto R_{\succ}(x)$, $x \mapsto L_{\prec}(x)$, and $x \mapsto R_{\prec}(x)$.

Proposition 18.7 *The quadruple $(L_{\succ}, R_{\prec}, \alpha, \mathcal{A})$ is a bimodule of the associated Hom-associative algebra $(\mathcal{A}, *, \alpha)$.*

Proof For all $x, y, v \in \mathcal{A}$,

$$\begin{aligned}L_{\succ}(x * y)\alpha(v) &= (x * y) \succ \alpha(v) = \alpha(x) \succ (y \succ v) = L_{\succ}(\alpha(x))L_{\succ}(y)v,\\ R_{\prec}(x * y)\alpha(v) &= \alpha(v) \prec (x * y) = (v \prec x) \prec \alpha(y) = R_{\prec}(\alpha(y))R_{\prec}(x)v,\\ L_{\succ}(\alpha(x))R_{\prec}(y)v &= \alpha(x) \succ (v \prec y) = (x \succ v) \prec \alpha(y) = R_{\prec}(\alpha(y))L_{\succ}(x)v,\\ \alpha(L_{\succ}(x)v) &= \alpha(x \succ v) = \alpha(x) \succ \alpha(v) = L_{\succ}(\alpha(x))\alpha(v),\\ \alpha(R_{\prec}(x)v) &= \alpha(v \prec x) = \alpha(v) \prec \alpha(x) = R_{\prec}(\alpha(x))\alpha(v),\end{aligned}$$

which completes the proof. □

18.5.2 $\mathcal{O}$-operators and Hom-dendriform Algebras

Definition 18.21 Let $(\mathcal{A}, \cdot, \alpha)$ be a Hom-associative algebra, and (l, r, β, V) be a bimodule. Then, a linear map $T : V \to \mathcal{A}$ is called an **$\mathcal{O}$-operator associated** to (l, r, β, V), if T satisfies,

$$\alpha T = T\beta$$
$$T(u) \cdot T(v) = T(l(T(u))v + r(T(v))u) \text{ for all } u, v \in V$$

Example 18.7 Let $(\mathcal{A}, \cdot, \alpha)$ be a multiplicative Hom-associative algebra. Then, the identity map Id is an $\mathcal{O}$-operator associated to the bimodule $(L, 0, \alpha)$ or $(0, R, \alpha)$.

Example 18.8 Let $(\mathcal{A}, \cdot, \alpha)$ be a multiplicative Hom-associative algebra. A linear map $f : \mathcal{A} \to \mathcal{A}$ is called a **Rota-Baxter operator** on $\mathcal{A}$ of weight zero if f satisfies

$$f(x) \cdot f(y) = f(f(x) \cdot y + x \cdot f(y)) \text{ for all } x, y \in \mathcal{A}.$$

In fact, a Rota-Baxter operator on $\mathcal{A}$ is just an $\mathcal{O}$-operator associated to the bimodule (L, R, α).

Theorem 18.9 *Let $(\mathcal{A}, \cdot, \alpha)$ be a Hom-associative algebra, and (l, r, β, V) be a bimodule. Let $T : V \to \mathcal{A}$ be an $\mathcal{O}$-operator associated to (l, r, β, V). Then, there exists a Hom-dendriform algebra structure on V given, for all $u, v \in V$, by*

$$u \succ v = l(T(u))v, u \prec v = r(T(v))u.$$

So, there is an associated Hom-associative algebra structure on V given by the (18.29), and T is a homomorphism of Hom-associative algebras. Moreover, $T(V) = \{T(v) \setminus v \in V\} \subseteq \mathcal{A}$ is a Hom-associative subalgebra of $\mathcal{A}$, and there is an induced Hom-dendriform algebra structure on $T(V)$ given, for all $u, v \in V$, by

$$T(u) \succ T(v) = T(u \succ v), T(u) \prec T(v) = T(u \prec v).$$

Its corresponding associated Hom-associative algebra structure on $T(V)$ given by the (18.29) is just the Hom-associative subalgebra structure of $\mathcal{A}$, and T is a homomorphism of Hom-dendriform algebras.

Proof For any $x, y, z \in V$, we have

$$\begin{aligned}(x \succ y) \prec \beta(z) - \beta(x) \succ (y \prec z) &= l(T(x)y) \prec \beta(z) - \beta(x) \succ r(T(z)y)\\ &= r(T\beta(z))l(T(x))y - l(T\beta(x)y)r(T(z)y)\\ &= r(\alpha(T(z)))l(T(x))y - l(\alpha(T(x)))r(T(z))y = 0.\end{aligned}$$

The two other axioms are similarly checked. □

Corollary 18.3 *Let $(\mathcal{A}, *, \alpha)$ be a multiplicative Hom-associative algebra. Then, there is a compatible multiplicative Hom-dendriform algebra structure on $\mathcal{A}$ if and only if there exists an invertible $\mathcal{O}$-operator of $(\mathcal{A}, *, \alpha)$.*

Proof If T is an invertible $\mathcal{O}-$operator associated to a bimodule (l, r, β, V), then, the compatible multiplicative Hom-dendriform algebra structure on $\mathcal{A}$ is given by

$$x \succ y = T(l(x)T^{-1}(y)), x \prec y = T(r(y)T^{-1}(x)) \text{ for all } x, y \in \mathcal{A}.$$

Conversely, let $(\mathcal{A}, \succ, \prec, \alpha)$ be a Hom-dendriform algebra, and $(\mathcal{A}, *, \alpha)$ be the associated multiplicative Hom-associative algebra. Then, the identity map Id is an $\mathcal{O}-$operator associated to the bimodule $(L_\succ, R_\prec, \alpha)$ of $(\mathcal{A}, *, \alpha)$. □

18.5.3 Bimodules and Matched Pairs of Hom-dendriform Algebras

Definition 18.22 Let $(\mathcal{A}, \succ, \prec, \alpha)$ be a Hom-dendriform algebra, and V be a vector space. Let $l_\succ, r_\succ, l_\prec, r_\prec : \mathcal{A} \to gl(V)$ and $\beta : V \to V$ be linear maps. Then, the sextuple $(l_\succ, r_\succ, l_\prec, r_\prec, \beta, V)$ is called a **bimodule** of $\mathcal{A}$ if the following equations hold, for any $x, y \in \mathcal{A}$ and $v \in V$:

$$\begin{aligned}
&l_\prec(x \prec y)\beta(v) = l_\prec(\alpha(x))l_*(y)v, r_\prec(\alpha(x))l_\prec(y)v = l_\prec(\alpha(y))r_*(x)v,\\
&r_\prec(\alpha(y))r_\prec(y)v = r_\prec(x * y)\beta(v), l_\prec(x \succ y)\beta(v) = l_\succ(\alpha(x))l_\prec(y)v,\\
&r_\prec(\alpha(x))l_\succ(y)v = l_\succ(\alpha(y))r_\prec(x)v, r_\prec(\alpha(x))r_\succ(y)v = r_\succ(y \prec x)\beta(v),\\
&l_\succ(x * y)\beta(v) = l_\succ(\alpha(x))l_\succ(y)v, r_\succ(\alpha(x))l_*(y)v = l_\succ(\alpha(y))r_\succ(x)v,\\
&r_\succ(\alpha(x))r_*(y)v = r_\succ(y \succ x)\beta(v),\\
&\beta(l_\succ(x)v) = l_\succ(\alpha(x))\beta(v), \beta(l_\prec(x)v) = l_\prec(\alpha(x))\beta(v),\\
&\beta(r_\succ(x)v) = r_\succ(\alpha(x))\beta(v), \beta(r_\prec(x)v) = r_\prec(\alpha(x))\beta(v),
\end{aligned}$$

where $x * y = x \succ y + x \prec y,\ l_* = l_\succ + l_\prec,\ r_* = r_\succ + r_\prec$.

By a straightforward calculation, we obtain the following result.

Proposition 18.8 *Let $(l_\succ, r_\succ, l_\prec, r_\prec, \beta, V)$ be a bimodule of a Hom-dendriform algebra $(\mathcal{A}, \succ, \prec, \alpha)$. Then, there is a Hom-dendriform algebra structure on the direct sum $\mathcal{A} \oplus V$ of the underlying vector spaces of $\mathcal{A}$ and V given, for $x, y \in \mathcal{A}$, $u, v \in V$, by*

$$\begin{aligned}
(x + u) \succ (y + v) &= x \succ y + l_\succ(x)v + r_\succ(y)u,\\
(x + u) \prec (y + v) &= x \prec y + l_\prec(x)v + r_\prec(y)u,\\
(\alpha \oplus \beta)(x + u) &= \alpha(x) + \beta(u).
\end{aligned}$$

We denote this algebra by $\mathcal{A} \times_{l_\succ, r_\succ, l_\prec, r_\prec, \alpha, \beta} V$.

The following result is proved by a direct computation.

Proposition 18.9 *Let* $(l_{\succ}, r_{\succ}, l_{\prec}, r_{\prec}, \beta, V)$ *be a bimodule of a Hom-dendriform algebra* $(\mathcal{A}, \succ, \prec, \alpha)$. *Let* $(\mathcal{A}, *, \alpha)$ *be the associated Hom-associative algebra. Then, results:*

1) *Both* $(l_{\succ}, r_{\prec}, \beta, V)$ *and* $(l_{\succ} + l_{\prec}, r_{\succ} + r_{\prec}, \beta, V)$ *are bimodules of* $(\mathcal{A}, *, \beta)$;
2) *For any bimodule* (l, r, β, V) *of* $(\mathcal{A}, *, \alpha)$, $(l, 0, 0, r, \beta, V)$ *is a bimodule of* $(\mathcal{A}, \succ, \prec, \alpha)$;
3) *Both* $(l_{\succ} + l_{\prec}, 0, 0, r_{\succ} + r_{\prec}, \beta, V)$ *and* $(l_{\succ}, 0, 0, r_{\prec}, \beta, V)$ *are bimodules of* $(\mathcal{A}, \succ, \prec, \alpha)$;
4) *The dendriform algebras* $\mathcal{A} \times_{l_{\succ}, r_{\succ}, l_{\prec}, r_{\prec}, \alpha, \beta} V$ *and* $\mathcal{A} \times_{l_{\succ}+l_{\prec}, 0, 0, r_{\succ}+r_{\prec}, \alpha, \beta} V$ *have the same associated Hom-associative algebra* $\mathcal{A} \times_{l_{\succ}+l_{\prec}, r_{\succ}+r_{\prec}, \alpha, \beta} V$.

The proof of the following theorem is obtained similarly as for Theorem 18.1.

Theorem 18.10 *Let* $(\mathcal{A}, \succ_{\mathcal{A}}, \prec_{\mathcal{A}}, \alpha)$ *and* $(\mathcal{B}, \succ_{\mathcal{B}}, \prec_{\mathcal{B}}, \beta)$ *be two Hom-dendriform algebras. Suppose there are linear maps*

$$l_{\succ_{\mathcal{A}}}, r_{\succ_{\mathcal{A}}}, l_{\prec_{\mathcal{A}}}, r_{\prec_{\mathcal{A}}} : \mathcal{A} \to gl(\mathcal{B}), \quad l_{\succ_{\mathcal{B}}}, r_{\succ_{\mathcal{B}}}, l_{\prec_{\mathcal{B}}}, r_{\prec_{\mathcal{B}}} : \mathcal{B} \to gl(\mathcal{A}) \quad \textit{such that}$$

$(l_{\succ_{\mathcal{A}}}, r_{\succ_{\mathcal{A}}}, l_{\prec_{\mathcal{A}}}, r_{\prec_{\mathcal{A}}}, \beta, \mathcal{B})$ *is a bimodule of* $\mathcal{A}$ *and* $(l_{\succ_{\mathcal{B}}}, r_{\succ_{\mathcal{B}}}, l_{\prec_{\mathcal{B}}}, r_{\prec_{\mathcal{B}}}, \alpha, \mathcal{A})$ *is a bimodule of* $\mathcal{B}$, *satisfying for*

$$l_{\mathcal{A}} = l_{\succ_{\mathcal{A}}} + l_{\prec_{\mathcal{A}}}, r_{\mathcal{A}} = r_{\succ_{\mathcal{A}}} + r_{\prec_{\mathcal{A}}}, l_{\mathcal{B}} = l_{\succ_{\mathcal{B}}} + l_{\prec_{\mathcal{B}}}, r_{\mathcal{B}} = r_{\succ_{\mathcal{B}}} + r_{\prec_{\mathcal{B}}}$$

and any $x, y \in \mathcal{A}$, $a, b \in \mathcal{B}$, *the following relations:*

$$r_{\prec_{\mathcal{A}}}(\alpha(x))(a \prec_{\mathcal{B}} b) = \beta(a) \prec_{\mathcal{B}} (r_{\mathcal{A}}(x)b) + r_{\prec_{\mathcal{A}}}(l_{\mathcal{B}}(x)\beta(a)), \tag{18.30}$$

$$\begin{aligned} l_{\prec_{\mathcal{A}}}(l_{\prec_{\mathcal{B}}}(x))\beta(b) &+ (r_{\prec_{\mathcal{A}}}(x)a) \prec_{\mathcal{B}} \beta(b) = \\ &\beta(a) \prec_{\mathcal{B}} (l_{\prec_{\mathcal{A}}}(x)b) + r_{\prec_{\mathcal{A}}}(r_{\prec_{\mathcal{B}}}(b)x)\beta(a), \end{aligned} \tag{18.31}$$

$$l_{\prec_{\mathcal{A}}}(\alpha(x))(a *_{\mathcal{B}} b) = (l_{\prec_{\mathcal{A}}}(x)a) *_{\mathcal{B}} \beta(b) + l_{\prec_{\mathcal{A}}}(r_{\prec_{\mathcal{A}}}(a)x)\beta(b), \tag{18.32}$$

$$r_{\prec_{\mathcal{A}}}(\alpha(x))(a \succ_{\mathcal{B}} b) = r_{\succ_{\mathcal{A}}}(l_{\prec_{\mathcal{B}}}(b)x)\beta(a) + \beta(a) \succ_{\mathcal{B}} (r_{\prec_{\mathcal{A}}}(x)b), \tag{18.33}$$

$$\begin{aligned} l_{\prec_{\mathcal{A}}}(l_{\succ_{\mathcal{B}}}(a)x)\beta(b) &+ (r_{\succ_{\mathcal{A}}}(x)a) \prec_{\mathcal{B}} \beta(b) = \\ &\beta(a) \succ_{B} (l_{\prec_{\mathcal{A}}}(x)b) + r_{\succ_{\mathcal{A}}}(r_{\prec_{\mathcal{B}}}(b)x)\beta(a) \end{aligned} \tag{18.34}$$

$$l_{\succ_{\mathcal{A}}}(\alpha(x))(a \prec_{\mathcal{B}} b) = (l_{\succ_{\mathcal{A}}}(x)a) \prec_{\mathcal{B}} \beta(b) + l_{\prec_{\mathcal{A}}}(r_{\succ_{\mathcal{B}}}(a)x)\beta(b), \tag{18.35}$$

$$r_{\succ_{\mathcal{A}}}(\alpha(x))(a *_{\mathcal{B}} b) = \beta(a) \succ_{\mathcal{B}} (r_{\succ_{\mathcal{A}}}(x)b) + r_{\succ_{\mathcal{A}}}(l_{\succ_{\mathcal{B}}}(b)x)\beta(a), \tag{18.36}$$

$$\begin{aligned} \beta(a) \succ_{\mathcal{B}} (l_{\succ_{\mathcal{A}}}(x)b) &+ r_{\succ_{\mathcal{A}}}(r_{\succ_{\mathcal{B}}}(b)x)\beta(a) = \\ &l_{\succ_{\mathcal{A}}}(l_{\mathcal{B}}(a)x)\beta(b) + (r_{\mathcal{A}}(x)a) \succ_{\mathcal{B}} \beta(b), \end{aligned} \tag{18.37}$$

$$l_{\succ_{\mathcal{A}}}(\alpha(x))(a \succ_{\mathcal{B}} b) = (l_{\mathcal{A}}(x)a) \succ_{\mathcal{B}} \beta(b) + l_{\succ_{\mathcal{A}}}(r_{\mathcal{B}}(a)x)\beta(b), \tag{18.38}$$

$$r_{\prec_{\mathcal{B}}}(\beta(a))(x \prec_{\mathcal{A}} y) = \alpha(x) \prec_{\mathcal{A}} (r_{\mathcal{B}}(a)y) + r_{\prec_{\mathcal{B}}}(l_{\mathcal{A}}(y)a)\alpha(x), \tag{18.39}$$

$$\begin{aligned} l_{\prec_{B}}(l_{\prec_{\mathcal{A}}}(x)a)\alpha(y) &+ (r_{\prec_{\mathcal{B}}}(a)x) \prec_{\mathcal{A}} \alpha(y) = \\ &\alpha(x) \prec_{\mathcal{A}} (l_{\mathcal{B}}(a)y) + r_{\prec_{\mathcal{B}}}(r_{\mathcal{A}}(y)a)\alpha(x), \end{aligned} \tag{18.40}$$

$$l_{\prec_{\mathcal{B}}}(\beta(a))(x *_{\mathcal{A}} y) = (l_{\prec_{\mathcal{B}}}(a)x) \prec_{\mathcal{A}} \alpha(y) + l_{\prec_{\mathcal{B}}}(r_{\prec_{\mathcal{A}}}(x)a)\alpha(y), \tag{18.41}$$

$$r_{\prec_{\mathcal{B}}}(\beta(a))(x \succ_{\mathcal{A}} y) = r_{\succ_{\mathcal{B}}}(l_{\prec_{\mathcal{B}}}(y)a)\alpha(x) + \alpha(x) \succ_{\mathcal{A}} (r_{\prec_{\mathcal{B}}}(a)y), \tag{18.42}$$

$$l_{\prec_{\mathcal{B}}}(l_{\succ_{\mathcal{A}}}(x)a)\alpha(y) + (r_{\succ_{\mathcal{B}}}(a)x) \prec_{\mathcal{A}} \alpha(y) = \alpha(x) \succ_{\mathcal{A}} (l_{\prec_{\mathcal{B}}}(a)y) + r_{\succ_{\mathcal{B}}}(r_{\prec_{\mathcal{A}}}(y)a)\alpha(x), \quad (18.43)$$

$$l_{\succ_{\mathcal{B}}}(\beta(a))(x \prec_{\mathcal{A}} y) = (l_{\succ_{\mathcal{B}}}(a)x) \prec_{\mathcal{A}} \alpha(y) + l_{\prec_{\mathcal{B}}}(r_{\succ_{\mathcal{A}}}(x)a)\alpha(y), \quad (18.44)$$

$$r_{\succ_{\mathcal{B}}}(\beta(a))(x *_{\mathcal{A}} y) = \alpha(x) \succ_{\mathcal{A}} (r_{\succ_{\mathcal{B}}}(a)y) + r_{\succ_{\mathcal{B}}}(l_{\succ_{\mathcal{A}}}(y)a)\alpha(x), \quad (18.45)$$

$$\alpha(x) \succ_{\mathcal{A}} (l_{\succ_{\mathcal{B}}}(a)y) + r_{\succ_{\mathcal{B}}}(r_{\succ_{\mathcal{A}}}(y)a)\alpha(x) = l_{\succ_{\mathcal{B}}}(l_{\mathcal{A}}(x)a)\alpha(y) + (r_{\mathcal{B}}(a)x) \succ_{\mathcal{A}} \alpha(y), \quad (18.46)$$

$$l_{\succ_{\mathcal{B}}}(\beta(a))(x \succ_{\mathcal{A}} y) = (l_{\mathcal{B}}(a)x) \succ_{\mathcal{A}} \alpha(y) + l_{\succ_{\mathcal{B}}}(r_{\mathcal{A}}(x)a)\alpha(y). \quad (18.47)$$

Then, there is a Hom-dendriform algebra structure on the direct sum $\mathcal{A} \oplus \mathcal{B}$ of the underlying vector spaces of $\mathcal{A}$ and $\mathcal{B}$ given, for any $x, y \in \mathcal{A}$, $a, b \in \mathcal{B}$, by

$$\begin{aligned}(x + a) \succ (y + b) &= (x \succ_{\mathcal{A}} y + r_{\succ_{\mathcal{B}}}(b)x + l_{\succ_{\mathcal{B}}}(a)y) + \\ &\quad (l_{\succ_{\mathcal{A}}}(x)b + r_{\succ_{\mathcal{A}}}(y)a + a \succ_{\mathcal{B}} b), \\ (x + a) \prec (y + b) &= (x \prec_{\mathcal{A}} y + r_{\prec_{\mathcal{B}}}(b)x + l_{\prec_{\mathcal{B}}}(a)y) + \\ &\quad (l_{\prec_{\mathcal{A}}}(x)b + r_{\prec_{\mathcal{A}}}(y)a + a \prec_{\mathcal{B}} b), \\ (\alpha \oplus \beta)(x + a) &= \alpha(x) + \beta(a).\end{aligned}$$

Let $\mathcal{A} \bowtie^{l_{\succ_{\mathcal{A}}}, r_{\succ_{\mathcal{A}}}, l_{\prec_{\mathcal{A}}}, r_{\prec_{\mathcal{A}}}, \beta}_{l_{\succ_{\mathcal{B}}}, r_{\succ_{\mathcal{B}}}, l_{\prec_{\mathcal{B}}}, r_{\prec_{\mathcal{B}}}, \alpha} \mathcal{B}$ or simply $\mathcal{A} \bowtie \mathcal{B}$ denote this Hom-dendriform algebra.

Definition 18.23 Let $(\mathcal{A}, \succ_{\mathcal{A}}, \prec_{\mathcal{A}}, \alpha)$ and $(\mathcal{B}, \succ_{\mathcal{B}}, \prec_{\mathcal{B}}, \beta)$ be Hom-dendriform algebras. Suppose there are linear maps

$$l_{\succ_{\mathcal{A}}}, r_{\succ_{\mathcal{A}}}, l_{\prec_{\mathcal{A}}}, r_{\prec_{\mathcal{A}}} : \mathcal{A} \to gl(\mathcal{B}), \quad l_{\succ_{\mathcal{B}}}, r_{\succ_{\mathcal{B}}}, l_{\prec_{\mathcal{B}}}, r_{\prec_{\mathcal{B}}} : \mathcal{B} \to gl(\mathcal{A})$$

such that $(l_{\succ_{\mathcal{A}}}, r_{\succ_{\mathcal{A}}}, l_{\prec_{\mathcal{A}}}, r_{\prec_{\mathcal{A}}}, \beta)$ is a bimodule of $\mathcal{A}$, and $(l_{\succ_{\mathcal{B}}}, r_{\succ_{\mathcal{B}}}, l_{\prec_{\mathcal{B}}}, r_{\prec_{\mathcal{B}}}, \alpha)$ is a bimodule of $\mathcal{B}$. If (18.30)–(18.47) are satisfied, then

$$(\mathcal{A}, \mathcal{B}, l_{\succ_{\mathcal{A}}}, r_{\succ_{\mathcal{A}}}, l_{\prec_{\mathcal{A}}}, r_{\prec_{\mathcal{A}}}, \beta, l_{\succ_{\mathcal{B}}}, r_{\succ_{\mathcal{B}}}, l_{\prec_{\mathcal{B}}}, r_{\prec_{\mathcal{B}}}, \alpha).$$

is called a **matched pair of Hom-dendriform algebras**.

Corollary 18.4 *If the tuple $(\mathcal{A}, \mathcal{B}, l_{\succ_{\mathcal{A}}}, r_{\succ_{\mathcal{A}}}, l_{\prec_{\mathcal{A}}}, r_{\prec_{\mathcal{A}}}, \beta, l_{\succ_{\mathcal{B}}}, r_{\succ_{\mathcal{B}}}, l_{\prec_{\mathcal{B}}}, r_{\prec_{\mathcal{B}}}, \alpha)$ is a matched pair of Hom-dendriform algebras, then*

$$(\mathcal{A}, \mathcal{B}, l_{\succ_{\mathcal{A}}} + l_{\prec_{\mathcal{A}}}, r_{\succ_{\mathcal{A}}} + r_{\prec_{\mathcal{A}}}, l_{\succ_{\mathcal{B}}} + l_{\prec_{\mathcal{B}}}, r_{\succ_{\mathcal{B}}} + r_{\prec_{\mathcal{B}}}, \alpha + \beta)$$

is a matched pair of the associated Hom-associative algebras $(\mathcal{A}, *_{\mathcal{A}}, \alpha)$ and $(\mathcal{B}, *_{\mathcal{B}}, \beta)$.

Proof The associated Hom-associative algebra $(\mathcal{A} \bowtie \mathcal{B}, *, \alpha + \beta)$ is exactly the Hom-associative algebra obtained from the matched pair $(\mathcal{A}, \mathcal{B}, l_{\mathcal{A}}, r_{\mathcal{A}}, \beta, l_{\mathcal{B}}, r_{\mathcal{B}}, \alpha)$ of Hom-associative algebras, with

$$(x+a)*(y+b) = x *_{\mathcal{A}} y + l_{\mathcal{B}}(a)y + r_{\mathcal{B}}(b)x + a *_{\mathcal{B}} b + l_{\mathcal{A}}(x)b + r_{\mathcal{A}}(y)a,$$
$$(\alpha \oplus \beta)(x+a) = \alpha(x) + \beta(a)$$

for $x, y \in \mathcal{A}, a, b \in \mathcal{B}$, and $l_{\mathcal{A}} = l_{\succ_{\mathcal{A}}} + l_{\prec_{\mathcal{A}}}, r_{\mathcal{A}} = r_{\succ_{\mathcal{A}}} + r_{\prec_{\mathcal{A}}}, l_{\mathcal{B}} = l_{\succ_{\mathcal{B}}} + l_{\prec_{\mathcal{B}}}$, $r_{\mathcal{B}} = r_{\succ_{\mathcal{B}}} + r_{\prec_{\mathcal{B}}}$. □

18.5.4 Double Constructions of Involutive Symplectic Hom-associative Algebras

In this sequel, we suppose that α is involutive.

Proposition 18.10 *Suppose that $(l_\succ, r_\succ, l_\prec, r_\prec, \beta, V)$ is a bimodule of a Hom-dendriform algebra $(\mathcal{A}, \succ, \prec, \alpha)$, and let $(\mathcal{A}, *, \alpha)$ be the associated involutive Hom-associative algebra.*

1) *Let $l^*_\succ, r^*_\succ, l^*_\prec, r^*_\prec : \mathcal{A} \to gl(V^*)$ be the linear maps given by*

$$\langle l^*_\succ(x)a^*, y\rangle = \langle l_\succ(x)y, a^*\rangle, \langle r^*_\succ(x)a^*, y\rangle = \langle r_\succ(x)y, a^*\rangle,$$
$$\langle l^*_\prec(x)a^*, y\rangle = \langle l_\prec(x)y, a^*\rangle, \langle r^*_\prec(x)a^*, y\rangle = \langle r_\prec(x)y, a^*\rangle.$$

   Then, $(r^*_\succ + r^*_\prec, -l^*_\prec, -r^*_\succ, l^*_\succ + l^*_\prec, \beta^*, V^*)$ is a bimodule of $(\mathcal{A}, \succ, \prec, \alpha)$;
2) *Both $(r^*_\succ + r^*_\prec, 0, 0, l^*_\succ + l^*_\prec, \beta^*, V^*)$ and $(r^*_\prec, 0, 0, l^*_\succ, \beta^*, V^*)$ are bimodules of $(\mathcal{A}, *, \alpha)$;*
3) *Both $(r^*_\succ + r^*_\prec, l^*_\succ + l^*_\prec, \beta^*, V^*)$ and $(r^*_\prec, l^*_\succ, \beta^*, V^*)$ are bimodules of $(\mathcal{A}, *, \alpha)$;*
4) *The Hom-dendriform algebras $\mathcal{A} \times_{r^*_\succ + r^*_\prec, -l^*_\prec, -r^*_\succ, l^*_\succ + l^*_\prec, \alpha, \beta^*} V^*$ and $\mathcal{A} \times_{r^*_\prec, 0, 0, l^*_\succ, \alpha, \beta^*} V^*$ have the same Hom-associative algebra $\mathcal{A} \times_{r^*_\prec, l^*_\succ, \alpha, \beta^*} V^*$.*

Example 18.9 Let $(\mathcal{A}, \prec, \succ, \alpha)$ be an involutive Hom-dendriform algebra. Then,

$$(L_\succ, R_\succ, L_\prec, R_\prec, \alpha, \mathcal{A}), (L_\succ, 0, 0, R_\prec, \alpha, \mathcal{A}), (L_\succ + L_\prec, 0, 0, R_\succ + R_\prec, \alpha, \mathcal{A})$$

are bimodules of $(\mathcal{A}, \prec, \succ, \alpha)$. On the other hand,

$$(R^*_\succ + R^*_\prec, -L^*_\prec, -R^*_\succ, L^*_\succ + L^*_\prec, \alpha^*, \mathcal{A}^*), (R^*_\prec, 0, 0, L^*_\succ, \alpha^*, \mathcal{A}^*),$$
$$(R^*_\succ + R^*_\prec, 0, 0, L^*_\succ + L^*_\prec, \alpha^*, \mathcal{A}^*)$$

are bimodules of $(\mathcal{A}, \succ, \prec, \alpha)$ too. There are two compatible Hom-dendriform algebra structures,

$$\mathcal{A} \times_{R^*_\succ + R^*_\prec, -L^*_\prec, -R^*_\succ, L^*_\succ + L^*_\prec, \alpha, \alpha^*} \mathcal{A}^* \text{ and } \mathcal{A} \times_{R^*_\succ + R^*_\prec, 0, 0, L^*_\succ + L^*_\prec, \alpha, \alpha^*} \mathcal{A}^*,$$

on the same Hom-associative algebra $\mathcal{A} \times_{R^*_\prec, L^*_\succ, \alpha, \alpha^*} \mathcal{A}^*$.

Definition 18.24 Let $(\mathcal{A}, \alpha)$ be a Hom-associative algebra. We say that $(\mathcal{A}, \alpha, \omega)$ is a symplectic Hom-associative algebra if ω is a non-degenerate skew-symmetric bilinear form on $\mathcal{A}$ such that the following identity (invariance condition) is satisfied for all $x, y, z \in \mathcal{A}$:

$$\omega(\alpha(x)\alpha(y), \alpha(z)) + \omega(\alpha(y)\alpha(z), \alpha(x)) + \omega(\alpha(z)\alpha(x), \alpha(y)) = 0.$$

Theorem 18.11 *Let $(\mathcal{A}, *, \alpha)$ be an involutive Hom-associative algebra, and let ω be an α-invariant non-degenerate skew-symmetric bilinear form on $\mathcal{A}$. Then, there exists a compatible Hom-dendriform algebra structure $\succ, \prec$ on $(\mathcal{A}, \alpha)$ given by*

$$\begin{aligned} &\omega(x \succ y, z) = \omega(y, z * x), \quad \omega(x \succ y, z) = \omega(x, y * z) \\ &\textit{for all}\, x, y \in \mathcal{A}. \end{aligned} \tag{18.48}$$

Proof Define a linear map $T : \mathcal{A} \to \mathcal{A}^*$ by $\langle T(x), y\rangle = \omega(x, y)$ for all $x, y \in \mathcal{A}$. Then, T is invertible and T^{-1} is an $\mathcal{O}-$operator of the involutive Hom-associative algebra $(\mathcal{A}, *, \alpha)$ associated to the bimodule (R^*_*, L^*_*, α^*). By Corollary 18.3, there is a compatible Hom-dendriform algebra structure $\succ, \prec$ on $(\mathcal{A}, *)$ given by

$$x \succ y = T^{-1}R^*_*(x)T(y), \quad x \prec y = T^{-1}L^*_*(y)T(x)$$

for all $x, y \in \mathcal{A}$, which gives exactly (18.48). □

Definition 18.25 We call $(\mathcal{A}, \alpha, \mathcal{B})$ a **double construction of involutive symplectic Hom-associative algebra** associated to $(\mathcal{A}_1, \alpha_1)$ and $(\mathcal{A}_1^*, \alpha_1^*)$ if it satisfies the conditions:

1) $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_1^*$ as the direct sum of vector spaces;
2) $(\mathcal{A}_1, \alpha_1)$ and $(\mathcal{A}_1^*, \alpha_1^*)$ are Hom-associative subalgebras of $(\mathcal{A}, \alpha)$ with $\alpha = \alpha_1 \oplus \alpha_1^*$;
3) ω is the natural non-degenerate antisymmetric $(\alpha_1 \oplus \alpha_1^*)$-invariant bilinear form on $\mathcal{A}_1 \oplus \mathcal{A}_1^*$ given, for all $x, y \in \mathcal{A}_1, a^*, b^* \in \mathcal{A}_1^*$, by

$$\begin{aligned} \omega(x + a^*, y + b^*) &= -\langle x, b^*\rangle + \langle a^*, y\rangle, \\ \omega((\alpha + \alpha^*)(x + a^*), y + b^*) &= \omega(x + a^*, (\alpha + \alpha^*)(y + b^*)) \end{aligned} \tag{18.49}$$

where $\langle, \rangle$ is the natural pairing between the vector space $\mathcal{A}_1$ and its dual space $\mathcal{A}_1^*$.

Let $(\mathcal{A}, *_{\mathcal{A}}, \alpha)$ be an involutive Hom-associative algebra, and suppose that there is an involutive Hom-associative algebra structure $*_{\mathcal{A}^*}$ on its dual space $\mathcal{A}^*$. We construct an involutive symplectic Hom-associative algebra structure on the direct sum $\mathcal{A} \oplus \mathcal{A}^*$ of the underlying vector spaces of $\mathcal{A}$ and $\mathcal{A}^*$ such that both $\mathcal{A}$ and $\mathcal{A}^*$ are Hom-subalgebras, equipped with the natural non-degenerate antisymmetric $(\alpha_1 \oplus \alpha_1^*)$-invariant bilinear form on $\mathcal{A} \oplus \mathcal{A}^*$ given by (18.49).

Such a construction, called double construction of involutive symplectic Hom-associative algebras associated to $(\mathcal{A}, *_{\mathcal{A}}, \alpha)$ and $(\mathcal{A}^*, *_{\mathcal{A}^*}, \alpha^*)$, is denoted by $(T(\mathcal{A}) = \mathcal{A} \bowtie_{\alpha}^{\alpha^*} \mathcal{A}^*, \omega)$.

Corollary 18.5 *Let $(T(\mathcal{A}) = \mathcal{A} \bowtie_{\alpha}^{\alpha^*} \mathcal{A}^*, \omega)$ be a double construction of involutive symplectic Hom-associative algebras. Then, there exists a compatible involutive Hom-dendriform algebra structure $\succ, \prec$ on $(T(\mathcal{A}), \alpha \oplus \alpha^*)$ defined by (18.49). Moreover, $\mathcal{A}$ and $\mathcal{A}^*$, endowed with this product, are Hom-dendriform subalgebras.*

Proof The first part follows from Theorem 18.11. Let $x, y \in \mathcal{A}$. Set $x \succ y = a + b^*$, for $a \in \mathcal{A}, b^* \in \mathcal{A}^*$. Since $\mathcal{A}$ is a Hom-associative subalgebra of $(T(\mathcal{A}), \alpha \oplus \alpha^*)$, and $\omega(\mathcal{A}, \mathcal{A}) = \omega(\mathcal{A}^*, \mathcal{A}^*) = 0$, we have

$$\omega(b^*, \mathcal{A}^*) = \omega(b^*, \mathcal{A}) = \omega(x \succ y, \mathcal{A}) = \omega(y, \mathcal{A} * x) = 0.$$

Therefore, $b^* = 0$ due to the independence of ω. Hence, $x \succ y = a \in \mathcal{A}$. Similarly, $x \prec y \in \mathcal{A}$. Thus, $\mathcal{A}$ is a Hom-dendriform subalgebra of $T(\mathcal{A})$ with $\prec, \succ$. By symmetry of $\mathcal{A}$, $\mathcal{A}^*$ is also a Hom-dendriform subalgebra. □

Theorem 18.12 *Let $(\mathcal{A}, \succ_{\mathcal{A}}, \prec_{\mathcal{A}}, \alpha)$ be an involutive Hom-dendriform algebra, and $(\mathcal{A}, *_{\mathcal{A}}, \alpha)$ be the associated involutive Hom-associative algebra. Suppose there is an involutive Hom-dendriform algebra structure " $\succ_{\mathcal{A}^*}, \prec_{\mathcal{A}^*}, \alpha^*$" on its dual space $\mathcal{A}^*$, and $(\mathcal{A}^*, *_{\mathcal{A}^*}, \alpha^*)$ is the associated involutive Hom-associative algebra. Then, there exists a double construction of involutive symplectic Hom-associative algebras associated to $(\mathcal{A}, *_{\mathcal{A}}, \alpha)$ and $(\mathcal{A}, *_{\mathcal{A}^*}, \alpha^*)$ if and only if the octuple $(\mathcal{A}, \mathcal{A}^*, R^*_{\prec_{\mathcal{A}}}, L^*_{\succ_{\mathcal{A}}}, \alpha^*, R^*_{\prec_{\mathcal{A}^*}}, L^*_{\succ_{\mathcal{A}^*}}, \alpha)$ is a matched pair of Hom-associative algebras.*

Proof The conclusion can be obtained by a similar proof as in Theorem 18.2. Then, if

$$(\mathcal{A}, \mathcal{A}^*, R^*_{\prec_{\mathcal{A}}}, L^*_{\succ_{\mathcal{A}}}, \alpha^*, R^*_{\prec_{\mathcal{A}^*}}, L^*_{\succ_{\mathcal{A}^*}}, \alpha)$$

is a matched pair of the involutive Hom-associative algebras $(\mathcal{A}, *_{\mathcal{A}}, \alpha)$ and $(\mathcal{A}, *_{\mathcal{A}^*}, \alpha^*)$, it is straightforward to show that the bilinear form (18.49) is an $(\alpha_1 \oplus \alpha_1^*)$-invariant on the Hom-associative algebra $\mathcal{A} \bowtie^{R^*_{\prec_{\mathcal{A}}}, L^*_{\succ_{\mathcal{A}}}, \alpha^*}_{R^*_{\prec_{\mathcal{A}^*}}, L^*_{\succ_{\mathcal{A}^*}}, \alpha} \mathcal{A}^*$ given by

$$\begin{aligned}(x + a^*) *_{\mathcal{A} \oplus \mathcal{A}^*} (y + b^*) = (&x *_{\mathcal{A}} y + R^*_{\prec_{\mathcal{A}^*}}(a^*)y + L^*_{\succ_{\mathcal{A}^*}}(b^*)x) \\ &+ (a^* *_{\mathcal{A}^*} b^* + R^*_{\prec_{\mathcal{A}}}(x)b^* + L^*_{\succ_{\mathcal{A}}}(y)a^*).\end{aligned}$$

In fact, we have

$$
\begin{aligned}
&\omega[(\alpha(x)+\alpha^*(a^*)) *_{\mathcal{A}\oplus\mathcal{A}^*} (\alpha(y)+\alpha^*(b^*)), \alpha(z)+\alpha^*(c^*)] \\
&\quad +\omega[(\alpha(y)+\alpha^*(b^*)) *_{\mathcal{A}\oplus\mathcal{A}^*} (\alpha(z)+\alpha^*(c^*)), \alpha(x)+\alpha^*(a^*)] \\
&\quad +\omega[(\alpha(z)+\alpha^*(c^*)) *_{\mathcal{A}\oplus\mathcal{A}^*} (\alpha(x)+\alpha^*(a^*)), \alpha(y)+\alpha^*(b^*)] \\
&= -\langle \alpha(x) *_{\mathcal{A}} \alpha(y) + R^*_{\prec_{\mathcal{A}^*}}(\alpha^*(a^*))\alpha(y) + L^*_{\succ_{\mathcal{A}^*}}(\alpha^*(b^*))\alpha(x), \alpha^*(c^*)\rangle \\
&\quad +\langle \alpha^*(a^*) *_{\mathcal{A}^*} \alpha^*(b^*) + R^*_{\prec_{\mathcal{A}}}(\alpha(x))\alpha^*(b^*) + L^*_{\succ_{\mathcal{A}}}(\alpha(y))\alpha^*(a^*), \alpha(z)\rangle \\
&\quad -\langle \alpha(y) *_{\mathcal{A}} \alpha(z) + R^*_{\prec_{\mathcal{A}^*}}(\alpha^*(b^*))\alpha(z) + L^*_{\succ_{\mathcal{A}^*}}(\alpha^*(c^*))\alpha(y), \alpha^*(a^*)\rangle \\
&\quad +\langle \alpha^*(b^*) *_{\mathcal{A}^*} \alpha^*(c^*) + R^*_{\prec_{\mathcal{A}}}(\alpha^*(y))\alpha^*(c^*) + L^*_{\succ_{\mathcal{A}}}(\alpha(z))\alpha^*(b^*), \alpha(x)\rangle \\
&\quad -\langle \alpha(z) *_{\mathcal{A}} \alpha(x) + R^*_{\prec_{\mathcal{A}^*}}(\alpha^*(c^*))\alpha(x) + L^*_{\succ_{\mathcal{A}^*}}(\alpha^*(a^*))\alpha(z), \alpha^*(b^*)\rangle \\
&\quad +\langle \alpha^*(c^*) *_{\mathcal{A}^*} \alpha(a^*) + R^*_{\prec_{\mathcal{A}}}(\alpha(z))\alpha^*(a^*) + L^*_{\prec_{\mathcal{A}}}(\alpha(x))\alpha^*(c^*), \alpha(y)\rangle \\
&= -\langle \alpha(x) \prec_{\mathcal{A}} \alpha(y), \alpha^*(c^*)\rangle - \langle \alpha(x) \succ_{\mathcal{A}} \alpha(y), \alpha^*(c^*)\rangle \\
&\quad -\langle \alpha^*(c^*) \prec_{\mathcal{A}^*} \alpha^*(a^*), \alpha(y)\rangle \\
&\quad -\langle \alpha^*(b^*) \succ_{\mathcal{A}^*} \alpha^*(c^*), \alpha(x)\rangle + \langle \alpha^*(a^*) \prec_{\mathcal{A}^*} \alpha^*(b^*), \alpha(z)\rangle \\
&\quad +\langle \alpha^*(a^*) \succ_{\mathcal{A}^*} \alpha^*(b^*), \alpha(z)\rangle + \langle \alpha(z) \prec_{\mathcal{A}} \alpha(x), \alpha^*(b^*)\rangle \\
&\quad +\langle \alpha(y) \succ_{\mathcal{A}} \alpha(z), \alpha^*(a^*)\rangle \\
&\quad -\langle \alpha(y) \succ_{\mathcal{A}} \alpha(z), \alpha^*(a^*)\rangle - \langle \alpha(y) \prec_{\mathcal{A}} \alpha(z), \alpha^*(a^*)\rangle \\
&\quad -\langle \alpha^*(a^*) \prec_{\mathcal{A}^*} \alpha^*(b^*), \alpha(z)\rangle - \langle \alpha^*(a^*) \succ_{\mathcal{A}^*} \alpha^*(c^*), \alpha(y)\rangle \\
&\quad +\langle \alpha^*(b^*) \prec_{\mathcal{A}^*} \alpha^*(c^*), \alpha(x)\rangle \\
&\quad +\langle \alpha^*(b^*) \succ_{\mathcal{A}^*} \alpha^*(c^*), \alpha(x)\rangle + \langle \alpha(x) \prec_{\mathcal{A}} \alpha(y), \alpha(c^*)\rangle \\
&\quad +\langle \alpha(z) \succ_{\mathcal{A}} \alpha(x), \alpha^*(b^*)\rangle \\
&\quad -\langle \alpha(z) \prec_{\mathcal{A}} \alpha(x), \alpha^*(b^*)\rangle - \langle \alpha(z) \succ_{\mathcal{A}} \alpha(x), \alpha^*(b^*)\rangle \\
&\quad -\langle \alpha^*(b^*) \prec_{\mathcal{A}^*} \alpha^*(c^*), \alpha(x)\rangle - \langle \alpha^*(a^*) \succ_{\mathcal{A}^*} \alpha^*(b^*), \alpha(z)\rangle \\
&\quad +\langle \alpha^*(c^*) \prec_{\mathcal{A}^*} \alpha^*(a^*), \alpha(y)\rangle \\
&\quad +\langle \alpha^*(a^*) \succ_{\mathcal{A}^*} \alpha^*(c^*), \alpha(y)\rangle + \langle \alpha(y) \prec_{\mathcal{A}} \alpha(z), \alpha^*(a^*)\rangle \\
&\quad +\langle \alpha(x) \succ_{\mathcal{A}} \alpha(y), \alpha^*(c^*)\rangle = 0.
\end{aligned}
$$

Conversely, if there exists a double construction of involutive symplectic Hom-associative algebras associated to $(\mathcal{A}, *_{\mathcal{A}}, \alpha)$ and $(\mathcal{A}, *_{\mathcal{A}^*}, \alpha^*)$, then $(\mathcal{A}, \mathcal{A}^*, R^*_{\prec_{\mathcal{A}}}, L^*_{\succ_{\mathcal{A}}}, \alpha^*, R^*_{\prec_{\mathcal{A}^*}}, L^*_{\succ_{\mathcal{A}^*}}, \alpha)$ is a matched pair of the involutive Hom-associative algebras given by the following equations:

$$
\begin{aligned}
&R^*_{\prec_{\mathcal{A}}}(\alpha(x))(a^* *_{\mathcal{A}^*} b^*) = R^*_{\prec_{\mathcal{A}}}(L_{\prec_{\mathcal{A}^*}}(a^*)x)\alpha^*(b^*) + (R^*_{\prec_{\mathcal{A}}}(x)a^*) *_{\mathcal{A}^*} \alpha^*(b^*), \\
&L^*_{\succ_{\mathcal{A}}}(\alpha(x))(a^* *_{\mathcal{A}^*} b^*) = L^*_{\succ_{\mathcal{A}}}(R_{\prec_{\mathcal{A}}}(b^*)x)\alpha^*(a^*) + \alpha^*(a^*) *_{\mathcal{A}^*} (L^*_{\succ_{\mathcal{A}}}(x)b^*), \\
&R^*_{\prec_{\mathcal{A}^*}}(\alpha^*(a^*))(x *_{\mathcal{A}} y) = R^*_{\prec_{\mathcal{A}^*}}(L_{\succ_{\mathcal{A}}}(x)a^*)\alpha(y) + (R^*_{\prec_{\mathcal{A}^*}}(a^*)x) *_{\mathcal{A}} \alpha(y), \\
&L^*_{\succ_{\mathcal{A}^*}}(\alpha^*(a^*))(x *_{\mathcal{A}} y) = L^*_{\succ_{\mathcal{A}^*}}(R_{\prec_{\mathcal{A}}}(y)a^*)\alpha(x) + \alpha(x) *_{\mathcal{A}} (L^*_{\succ_{\mathcal{A}^*}}(a^*)y), \\
&R^*_{\prec_{\mathcal{A}}}(R^*_{\prec_{\mathcal{A}^*}}(a^*)x)\alpha^*(b^*) + (L^*_{\prec_{\mathcal{A}}}(x)a^*) *_{\mathcal{A}^*} \alpha^*(b^*) - \\
&\qquad L^*_{\succ_A}(L^*_{\succ_{\mathcal{A}^*}}(b^*)x)\alpha^*(a^*) - \alpha^*(a^*) *_{\mathcal{A}^*} (R^*_{\prec_{\mathcal{A}}}(x)b^*) = 0, \\
&R^*_{\prec_{\mathcal{A}}}(R^*_{\prec_{\mathcal{A}^*}}(x)a^*)\alpha(y) + (L^*_{\succ_{\mathcal{A}^*}}(a^*)x) *_A \alpha(y) - \\
&\qquad L^*_{\succ_{A^*}}(L^*_{\succ_A}(y)a^*)\alpha(x) - \alpha(x) *_A (R^*_{\prec_{A^*}}(a^*)y) = 0,
\end{aligned}
$$

since the operation $*_{A\oplus A^*}$ is Hom-associative. □

Corollary 18.6 *Let* $(\mathcal{A}, \succ, \prec, \alpha)$ *be an involutive Hom-dendriform algebra, and* $(R^*_{\prec}, L^*_{\succ}, \alpha^*)$ *be the bimodule of the associated involutive Hom-associative alge-*

bra $(\mathcal{A}, *, \alpha)$. *Then,* $(T(\mathcal{A}) = \mathcal{A} \times_{R^*_\prec, L^*_\succ, \alpha, \alpha^*} \mathcal{A}^*, \omega)$ *is a double construction of the involutive symplectic Hom-associative algebras.*

Theorem 18.13 *Let* $(\mathcal{A}, \succ_\mathcal{A}, \prec_\mathcal{A}, \alpha)$ *be an involutive Hom-dendriform algebra, and* $(\mathcal{A}, *_\mathcal{A}, \alpha)$ *be the associated involutive Hom-associative algebra. Suppose that there is an involutive Hom-dendriform algebra structure* $\succ_{\mathcal{A}^*}, \prec_{\mathcal{A}^*}, \alpha^*$ *on its dual space* $\mathcal{A}^*$, *and* $(\mathcal{A}^*, *_{\mathcal{A}^*}, \alpha^*)$ *is its associated involutive Hom-associative algebra. Then,* $(\mathcal{A}, \mathcal{A}^*, R^*_{\prec_\mathcal{A}}, L^*_{\succ_\mathcal{A}}, \alpha^*, R^*_{\prec\mathcal{A}^*}, L^*_{\succ_{\mathcal{A}^*}}, \alpha)$ *is a matched pair of involutive Hom-associative algebras if and only if*

$$(\mathcal{A}, \mathcal{A}^*, R^*_{\succ_\mathcal{A}} + R^*_{\prec_\mathcal{A}}, -L^*_{\prec_\mathcal{A}}, -R^*_{\succ_\mathcal{A}}, L^*_{\succ_\mathcal{A}} + L^*_{\prec_\mathcal{A}}, \alpha^*, R^*_{\succ_{\mathcal{A}^*}} + R^*_{\prec_{\mathcal{A}^*}}, \\ - L^*_{\prec_{\mathcal{A}^*}}, -R^*_{\succ_{\mathcal{A}^*}}, L^*_{\succ_{\mathcal{A}^*}} + L^*_{\prec_{\mathcal{A}^*}}, \alpha)$$

is a matched pair of involutive Hom-dendriform algebras.

Proof The necessary condition follows from Corollary 18.4. We need to prove the sufficient condition only. If $(\mathcal{A}, \mathcal{A}^*, R^*_{\prec_\mathcal{A}}, L^*_{\succ_\mathcal{A}}, \alpha^*, R^*_{\prec\mathcal{A}^*}, L^*_{\succ_{\mathcal{A}^*}}, \alpha)$ is a matched pair of involutive Hom-associative algebras, then $(\mathcal{A} \bowtie^{R^*_{\prec_\mathcal{A}}, L^*_{\succ_\mathcal{A}}, \alpha^*}_{R^*_{\prec_{\mathcal{A}^*}}, L^*_{\succ_{\mathcal{A}^*}}, \alpha} \mathcal{A}^*, \omega)$ is a double construction of involutive symplectic Hom-associative algebras. Hence, there exists a compatible involutive Hom-dendriform algebra structure on $\mathcal{A} \bowtie^{R^*_{\prec_\mathcal{A}}, L^*_{\succ_\mathcal{A}}, \alpha^*}_{R^*_{\prec_{\mathcal{A}^*}}, L^*_{\succ_{\mathcal{A}^*}}, \alpha} \mathcal{A}^*$ given by (18.48). By a simple and direct computation, we show that $\mathcal{A}$ and $\mathcal{A}^*$ are its subalgebras, and the other products are given, for any $x \in \mathcal{A}$, $a^* \in \mathcal{A}^*$, by

$$\begin{aligned} x \succ a^* &= (R^*_{\succ_\mathcal{A}} + R^*_{\prec_\mathcal{A}})(x)a^* - L^*_{\prec_{\mathcal{A}^*}} x, \\ x \prec a^* &= -R^*_{\succ_\mathcal{A}}(x)a^* + (L^*_{\succ_{\mathcal{A}^*}} + L^*_{\prec_{\mathcal{A}^*}})(a^*)x, \\ a^* \succ x &= (R^*_{\succ_{\mathcal{A}^*}} + R^*_{\prec_{\mathcal{A}^*}})(a^*)x - L^*_{\prec_\mathcal{A}}(x)a^*, \\ a^* \prec x &= -R^*_{\succ_{\mathcal{A}^*}}(a^*)x + (L^*_{\succ_\mathcal{A}} + L^*_{\prec_\mathcal{A}})(x)a^*. \end{aligned}$$

Hence,

$$(\mathcal{A}, \mathcal{A}^*, R^*_{\succ_\mathcal{A}} + R^*_{\prec_\mathcal{A}}, -L^*_{\prec_\mathcal{A}}, -R^*_{\succ_\mathcal{A}}, L^*_{\succ_\mathcal{A}} + L^*_{\prec_\mathcal{A}}, \alpha^*, R^*_{\succ_{\mathcal{A}^*}} + R^*_{\prec_{\mathcal{A}^*}}, \\ -L^*_{\prec_{\mathcal{A}^*}}, -R^*_{\succ_{\mathcal{A}^*}}, L^*_{\succ_{\mathcal{A}^*}} + L^*_{\prec_{\mathcal{A}^*}}, \alpha)$$

is a matched pair of involutive Hom-dendriform algebras. □

18.5.5 Hom-dendriform D-bialgebras

Theorem 18.14 *Let* $(\mathcal{A}, \succ_\mathcal{A}, \prec_\mathcal{A}, \alpha)$ *be an involutive Hom-dendriform algebra whose products are given by two linear maps* $\beta^*_\succ, \beta^*_\prec : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$. *Further, suppose that there is an involutive Hom-dendriform algebra structure* $\succ_{\mathcal{A}^*}, \prec_{\mathcal{A}^*}, \alpha^*$ *on its dual space* $\mathcal{A}^*$ *given by two linear maps* $\Delta^*_\succ, \Delta^*_\prec : \mathcal{A}^* \otimes \mathcal{A}^* \to \mathcal{A}^*$. *Then,*

$(\mathcal{A}, \mathcal{A}^*, R^*_{\prec_{\mathcal{A}}}, L^*_{\succ_{\mathcal{A}}}, \alpha^*, R^*_{\prec_{\mathcal{A}^*}}, L^*_{\succ_{\mathcal{A}^*}}, \alpha)$ is a matched pair of involutive Hom-associative algebras if and only if

$$\Delta_{\prec} \circ \alpha(x *_{\mathcal{A}} y) = (\alpha \otimes L_{\prec_{\mathcal{A}}}(x))\Delta_{\prec}(y) + (R_{\mathcal{A}}(y) \otimes \alpha)\Delta_{\prec}(y), \tag{18.50}$$

$$\Delta_{\succ} \circ \alpha(x *_{\mathcal{A}} y) = (\alpha \otimes L_{\prec_{\mathcal{A}}}(x))\Delta_{\succ}(y) + (R_{\prec_{A}}(y) \otimes \alpha)\Delta_{\succ}(y), \tag{18.51}$$

$$\beta_{\prec} \circ \alpha^*(a^* *_{\mathcal{A}^*} b^*) = (\alpha^* \otimes L_{\prec_{\mathcal{A}^*}}(a^*))\beta_{\prec}(b^*) + (R_{\mathcal{A}^*}(b^*) \otimes \alpha^*)\beta_{\prec}(a^*) \tag{18.52}$$

$$\beta_{\succ} \circ \alpha^*(a^* *_{\mathcal{A}^*} b^*) = (\alpha^* \otimes L_{\mathcal{A}^*}(a^*))\beta_{\succ}(b^*) + (R_{\prec_{\mathcal{A}^*}}(b^*) \otimes \alpha^*)\beta_{\succ}(a^*), \tag{18.53}$$

$$\begin{aligned}(L_{\mathcal{A}}(x) \otimes \alpha\ &-\alpha \otimes R_{\prec_{\mathcal{A}}}(x))\Delta_{\prec}(y)\\ &+\sigma[(L_{\succ_{\mathcal{A}}}(y) \otimes (-\alpha) \otimes R_{\mathcal{A}}(y))\Delta_{\prec}(y)] = 0,\end{aligned} \tag{18.54}$$

$$\begin{aligned}(L_{\mathcal{A}^*}(a^*) \otimes \alpha^*\ &-\alpha^* \otimes R_{\prec_{\mathcal{A}^*}}(a^*))\beta_{\prec}(b^*)\\ &+\sigma[(L_{\succ_{\mathcal{A}^*}}(b^*) \otimes (-\alpha^*) \otimes R_{\mathcal{A}^*}(b^*))\beta_{\succ}(a^*)] = 0\end{aligned} \tag{18.55}$$

hold for any $x, y \in \mathcal{A}$ and $a^*, b^* \in \mathcal{A}^*$, where

$$L_{\mathcal{A}} = L_{\succ_{\mathcal{A}}} + L_{\prec_{\mathcal{A}}}, \qquad R_{\mathcal{A}} = R_{\succ_{\mathcal{A}}} + R_{\prec_{\mathcal{A}}},$$
$$L_{\mathcal{A}^*} = L_{\succ_{\mathcal{A}^*}} + L_{\prec_{\mathcal{A}^*}}, \qquad R_{\mathcal{A}^*} = R_{\succ_{\mathcal{A}^*}} + R_{\prec_{\mathcal{A}^*}}.$$

Proof Let $\{e_1, \dots, e_n\}$ be a basis of $\mathcal{A}$, and $\{e_1^*, \dots, e_n^*\}$ be its dual basis. Set

$$e_i \succ_{\mathcal{A}} e_j = \sum_{k=1}^{n} a_{ij}^k e_k, \quad e_i \prec_{\mathcal{A}} e_j = \sum_{k=1}^{n} b_{ij}^k e_k, \quad \alpha(e_i) = \sum_{q=1}^{n} f_q^i e_q,$$
$$e_i^* \succ_{\mathcal{A}^*} e_j^* = \sum_{k=1}^{n} c_{ij}^k e_k^*, \quad e_i^* \prec_{\mathcal{A}^*} e_j^* = \sum_{k=1}^{n} d_{ij}^k e_k^*, \quad \alpha^*(e_i^*) = \sum_{q=1}^{n} f_q^{*i} e_q^*.$$

We have $\langle \alpha^*(e_i^*), e_j \rangle = f_i^{*j} = \langle e^*, \alpha(e_j) \rangle = f_j^i \quad \Rightarrow \quad f_i^{*j} = f_j^i,$

$$\beta_{\succ}(e_k^*) = \sum_{i,j=1}^{n} a_{ij}^k e_i^* \otimes e_j^*, \qquad \beta_{\prec}(e_k^*) = \sum_{i,j=1}^{n} b_{ij}^k e_i^* \otimes e_j^*,$$
$$\Delta_{\succ}(e_k) = \sum_{i,j=1}^{n} c_{ij}^k e_i \otimes e_j, \qquad \Delta_{\prec}(e_k) = \sum_{i,j}^{n} d_{ij}^k e_i \otimes e_j,$$
$$R^*_{\succ_{\mathcal{A}}}(e_i)e_j^* = \sum_{k=1}^{n} a_{ki}^j e_k^*, \qquad R^*_{\prec_{\mathcal{A}}}(e_i)e_j^* = \sum_{k=1}^{n} b_{ki}^j e_k^*,$$
$$R^*_{\succ_{\mathcal{A}^*}}(e_i^*)e_j = \sum_{k=1}^{n} c_{ki}^j e_k, \qquad R^*_{\prec_{\mathcal{A}^*}}(e_i^*)e_j = \sum_{k=1}^{n} d_{ki}^j e_k,$$

$$L^*_{\succ_\mathcal{A}}(e_i)e^*_j = \sum_{k=1}^n a^j_{ik}e^*_k, \qquad L^*_{\prec_\mathcal{A}}(e_i)e^*_j = \sum_{k=1}^n b^j_{ik}e^*_k,$$
$$L^*_{\succ_{\mathcal{A}^*}}(e^*_i)e_j = \sum_{k=1}^n c^j_{ik}e_k, \qquad L^*_{\prec_{\mathcal{A}^*}}(e^*_i)e_j = \sum_{k=1}^n d^j_{ik}e_k.$$

Therefore, the coefficient of e^*_l in

$$R^*_{\prec_\mathcal{A}}(\alpha(e_i))(e^*_j *_{\mathcal{A}^*} e^*_k) = R^*_{\prec_\mathcal{A}}(L_{\prec_{\mathcal{A}^*}}(e^*_j)e_i)\alpha^*(e^*_k) + (R^*_{\prec_\mathcal{A}}(e_i)e^*_j) *_{\mathcal{A}^*} \alpha^*(e^*_k)$$

gives the following relation for any i, j, l, k, q :

$$\sum_{m=1}^n f^i_q b^m_{lq}(c^m_{jk} + d^m_{jk}) = \sum_{m=1}^n f^k_q[c^i_{jm}b^q_{lm} + b^j_{mi}(c^l_{mq} + d^l_{mq})]. \tag{18.56}$$

In fact, we have

$$\begin{aligned}
&R^*_{\prec_\mathcal{A}}(\alpha(e_i))(e^*_j *_{\mathcal{A}^*} e^*_k) = R^*_{\prec_\mathcal{A}}(\alpha(e_i))(e^*_j \succ_{\mathcal{A}^*} e^*_k + e^*_j \prec_{\mathcal{A}^*} e^*_k)\\
&= R^*_{\prec_A}\left(\sum_{q=1}^n f^i_q e_q\right)\sum_{m=1}^n (c^m_{jk} + d^m_{jk})e^*_m = \sum_{m,q=1}^n f^i_q(c^m_{jk} + d^m_{jk})R^*_{\prec_\mathcal{A}}(e_q)e^*_m\\
&= \sum_{m,q=1}^n f^i_q(c^m_{jk} + d^m_{jk})\left(\sum_{l=1}^n b^m_{lq}\right)e^*_l = \sum_{l=1}^n\left[\sum_{m,q=1}^n f^i_q b^m_{lq}(c^m_{jk} + d^m_{jk})\right]e^*_l,\\
&R^*_{\prec_A}(L_{\prec_{\mathcal{A}^*}}(e^*_j)e_i)\alpha^*(e^*_k) = R^*_{\prec_\mathcal{A}}\left(\sum_{m=1}^n c^i_{jm}e_m\right)\left(\sum_{q=1}^n f^k_q e^*_q\right)\\
&= \sum_{m,q=1}^n c^i_{jm}f^k_q R^*_{\prec_\mathcal{A}}(e_m)e^*_q = \sum_{m,q=1}^n c^i_{jm}f^k_q\left(\sum_{l=1}^n b^q_{lm}e^*_l\right) = \sum_{l=1}^n(\sum_{m,q=1}^n c^i_{jm}f^k_q b^q_{lm})e^*_l,\\
&(R^*_{\prec_\mathcal{A}}(e_i)e^*_j) *_{\mathcal{A}^*} \alpha(e^*_k) = \left(\sum_{m=1}^n b^j_{mi}e^*_m\right) *_{\mathcal{A}^*}\left(\sum_{q=1}^n f^k_q e^*_q\right)\\
&= \sum_{m,q=1}^n f^k_q b^j_{mi}(e^*_m \succ_{A^*} e^*_q + e^*_m \prec_{A^*} e^*_q) = \sum_{m,q=1}^n f^k_q b^j_{mi}\left[\sum_{l=1}^n (c^l_{mq} + d^l_{mq})\right]e^*_l\\
&= \sum_{l=1}^n\left[\sum_{m,q=1}^n f^k_q b^j_{mi}(c^l_{mq} + d^l_{mq})\right]e^*_l,
\end{aligned}$$

giving (18.56). Also, the coefficient of $e^*_l \otimes e^*_i$ in

$$\begin{aligned}
&\beta_\prec \circ \alpha^*(e^*_j *_{\mathcal{A}^*} e^*_k) = (\alpha^* \otimes L_{\prec_{\mathcal{A}^*}}(e^*_j))\beta_\prec(e^*_k) + (R_{\mathcal{A}^*}(e^*_k) \otimes \alpha^*)\beta_\prec(e^*_j),\\
&\beta_\prec \circ \alpha^*(e^*_j *_{\mathcal{A}^*} e^*_k) = \beta_\prec \circ \alpha^*(e^*_j \succ_{\mathcal{A}^*} e^*_k + e^*_j \prec_{\mathcal{A}^*} e^*_k)
\end{aligned}$$

$$= \beta_{\prec} \circ \alpha^*[\sum_{m=1}^{n}(c_{jk}^m + d_{jk}^m)e_m^*] = \sum_{m=1}^{n}(c_{jk}^m + d_{jk}^m)\beta_{\prec} \circ \alpha^*(e_m^*)$$

$$= \sum_{m=1}^{n}(c_{jk}^m + d_{jk}^m)(\sum_{l,i,q=1}^{n} f_q^m b_{li}^q e_l^* \otimes e_i^*) = \sum_{l,i,q=1}^{n}[\sum_{m=1}^{n} f_q^m b_{li}^q (c_{jk}^m + d_{jk}^m)]e_l^* \otimes e_i^*,$$

$$(\alpha^* \otimes L_{\prec_{\mathcal{A}^*}}(e_j^*))\beta_{\prec}(e_k^*) = (\alpha^* \otimes L_{\prec_{\mathcal{A}^*}}(e_j^*))(\sum_{l,m=1}^{n} b_{lm}^k e_l^* \otimes e_m^*)$$

$$= \sum_{l,m=1}^{n} b_{lm}^k \alpha^*(e_l^*) \otimes (e_j^* \prec_{\mathcal{A}^*} e_m^*) = \sum_{l,m,q=1}^{n} b_{lm}^k f_q^l e_q^* \otimes (\sum_{i=1}^{n} c_{jm}^i e_i^*)$$

$$= \sum_{l,i,q=1}^{n} (\sum_{m=1}^{n} f_q^l b_{lm}^k c_{jm}^i)e_q^* \otimes e_i^*,$$

$$(R_{\mathcal{A}^*}(e_k^*) \otimes \alpha^*)\beta_{\prec}(e_j^*) = (R_{\mathcal{A}^*}(e_k^*) \otimes \alpha^*)(\sum_{m,i=1}^{n} b_{mi}^j e_m^* \otimes e_i^*)$$

$$= \sum_{m,i,q=1}^{n} f_q^i b_{mi}^j (e_m^* *_{\mathcal{A}^*} e_k^*) \otimes e_q^* = \sum_{m,i,q=1}^{n} f_q^i b_{mi}^j [(e_m^* \succ_{\mathcal{A}^*} e_k^* + e_m^* \prec_{\mathcal{A}^*} e_k^*) \otimes e_q^*]$$

$$= \sum_{m,i,q=1}^{n} f_q^i b_{mi}^j [(e_m^* \succ_{\mathcal{A}^*} e_k^*) \otimes e_q^* + (e_m^* \prec_{\mathcal{A}^*} e_k^*) \otimes e_q^*]$$

$$= \sum_{m,i,q=1}^{n} f_q^i b_{mi}^j [\sum_{l=1}^{n} c_{mk}^l e_l^* \otimes e_q^* + \sum_{l1}^{n} d_{mk}^l e_l^* \otimes e_q^*]$$

$$= \sum_{l,i,q=1}^{n} [\sum_{m=1}^{n} f_q^i b_{mi}^j (c_{mk}^l + d_{mk}^l)]e_l^* \otimes e_q^*$$

gives the relation

$$\sum_{m=1}^{n}[f_q^m b_{li}^q (c_{jk}^m + d_{jk}^m) + f_q^l b_{lm}^k c_{jm}^i] = \sum_{m=1}^{n}[f_q^i b_{mi}^j (c_{mk}^l + d_{mk}^l)]. \quad (18.57)$$

Thus, (18.56) corresponds to (18.57). Therefore, (18.50) ⇔ (18.52).

So, in the case $l_{\mathcal{A}} = R^*_{\prec_{\mathcal{A}}}, r_{\mathcal{A}} = L^*_{\succ_{\mathcal{A}}}, l_{\mathcal{B}} = l_{\mathcal{A}^*} = R^*_{\prec_{\mathcal{A}^*}}, r_{\mathcal{B}} = r_{\mathcal{A}^*} = L^*_{\succ_{\mathcal{A}^*}}$, we have (18.7) ⇔ (18.50) ⇔ (18.52). Similarly, in this situation,

(18.8) ⇔ (18.50) ⇔ (18.53), (18.9) ⇔ (18.50) ⇔ (18.50),
(18.10) ⇔ (18.50) ⇔ (18.51),
(18.11) ⇔ (18.50) ⇔ (18.55),
(18.12) ⇔ (18.50) ⇔ (18.54).

Therefore, the conclusion holds due to Theorem 18.1. □

Definition 18.26 Let $\mathcal{A}$ be a vector space. A **Hom-dendriform D-bialgebra** structure on $\mathcal{A}$ is a set of linear maps $(\Delta_{\prec}, \Delta_{\succ}, \alpha, \beta_{\prec}, \beta_{\succ}, \alpha^*)$, $\Delta_{\prec}, \Delta_{\succ} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, $\beta_{\prec}, \beta_{\succ} : \mathcal{A}^* \to \mathcal{A}^* \otimes \mathcal{A}^*$, $\alpha : \mathcal{A} \to \mathcal{A}$, $\alpha^* : \mathcal{A}^* \to \mathcal{A}^*$, such that

(a) $(\Delta^*_{\prec}, \Delta^*_{\succ}, \alpha^*) : \mathcal{A}^* \otimes \mathcal{A}^* \to \mathcal{A}^*$ defines a Hom-dendriform algebra structure $(\succ_{\mathcal{A}^*}, \prec_{\mathcal{A}^*}, \alpha^*)$ on $\mathcal{A}^*$;
(b) $(\beta^*_{\prec}, \beta^*_{\succ}, \alpha) : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ defines a Hom-dendriform algebra structure $(\succ_{\mathcal{A}}, \prec_{\mathcal{A}}, \alpha)$ on $\mathcal{A}$;
(c) Equations (18.50)–(18.55) are satisfied.

We denote it by $(\mathcal{A}, \mathcal{A}^*, \Delta_{\succ}, \Delta_{\prec}, \alpha, \beta_{\succ}, \beta_{\prec}, \alpha^*)$ or simply $(\mathcal{A}, \mathcal{A}^*, \alpha, \alpha^*)$.

Theorem 18.15 *Let $(\mathcal{A}, \prec_{\mathcal{A}}, \succ_{\mathcal{A}}, \alpha)$, $(\mathcal{A}^*, \prec_{\mathcal{A}^*}, \succ_{\mathcal{A}^*}, \alpha^*)$ be involutive Hom-dendriform algebras. Let $(\mathcal{A}, *_{\mathcal{A}}, \alpha)$ and $(\mathcal{A}^*, *_{\mathcal{A}^*}, \alpha^*)$ be the corresponding associated involutive Hom-associative algebras. Then, the following conditions are equivalent:*

(i) *There is a double construction of involutive symplectic Hom-associative algebras associated to $(\mathcal{A}, *_{\mathcal{A}}, \alpha)$ and $(\mathcal{A}^*, *_{\mathcal{A}^*}, \alpha^*)$;*
(ii) $(\mathcal{A}, \mathcal{A}^*, R^*_{\prec_{\mathcal{A}}}, L^*_{\succ_{\mathcal{A}}}, \alpha^*, R^*_{\prec_{\mathcal{A}^*}}, L^*_{\succ_{\mathcal{A}^*}}, \alpha)$ *is a matched pair of involutive Hom-associative algebras;*
(iii) $(\mathcal{A}, \mathcal{A}^*, R^*_{\succ_{\mathcal{A}}} + R^*_{\prec_{\mathcal{A}}}, -L^*_{\prec_{\mathcal{A}}}, -R^*_{\succ_{\mathcal{A}}}, L^*_{\succ_{\mathcal{A}}} + L^*_{\prec_{\mathcal{A}}}, \alpha^*, R^*_{\succ_{\mathcal{A}^*}} + R^*_{\prec_{\mathcal{A}^*}}, -L^*_{\prec_{\mathcal{A}^*}}, -R^*_{\succ_{\mathcal{A}^*}}, L^*_{\succ_{\mathcal{A}^*}} + L^*_{\prec_{\mathcal{A}^*}}, \alpha)$ *is a matched pair of involutive Hom-dendriform algebras;*
(iv) $(\mathcal{A}, \mathcal{A}^*, \alpha, \alpha^*)$ *is an involutive Hom-dendriform D−bialgebra.*

18.6 Matched Pairs of BiHom-Associative Algebras

18.6.1 Bihom-dendriform Algebras

Definition 18.27 A biHom-dendriform algebra is a quintuple $(\mathcal{A}, \prec, \succ, \alpha, \beta)$ consisting of a vector space $\mathcal{A}$ on which the operations $\prec, \succ: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and $\alpha, \beta : \mathcal{A} \to \mathcal{A}$ are linear maps satisfying, for all $x, y, z \in \mathcal{A}$ and $x * y = x \prec y + x \succ y$,

$$
\begin{aligned}
&\alpha \circ \beta = \beta \circ \alpha,\\
&\alpha(x \prec y) = \alpha(x) \prec \alpha(y), \alpha(x \succ y) = \alpha(x) \succ \alpha(y),\\
&\beta(x \prec y) = \beta(x) \prec \beta(y), \beta(x \succ y) = \beta(x) \succ \beta(y),\\
&(x \prec y) \prec \beta(z) = \alpha(x) \prec (y * z),\\
&(x \succ y) \prec \beta(z) = \alpha(x) \succ (y \prec z),\\
&\alpha(x) \succ (y \succ z) = (x * y) \succ \beta(z).
\end{aligned}
$$

Definition 18.28 Let $(\mathcal{A}, \prec, \succ, \alpha, \beta)$ and $(\mathcal{A}', \prec', \succ', \alpha', \beta')$ be biHom-dendriform algebras. A linear map $f : \mathcal{A} \to \mathcal{A}'$ is a biHom-dendriform algebra morphism if

$\prec' \circ (f \otimes f) = f \circ \prec$, $\succ' \circ (f \otimes f) = f \circ \succ$, $f \circ \alpha = \alpha' \circ f$ and $f \circ \beta = \beta' \circ f$.

Proposition 18.11 *Let* $(\mathcal{A}, \prec, \succ, \alpha, \beta)$ *be a biHom-dendriform algebra. Then,* $(\mathcal{A}, *, \alpha, \beta)$ *is a biHom-associative algebra.*

Proof We have, for all $x, y, z \in \mathcal{A}$,

$$\begin{aligned}
(x * y) * \beta(z) &= (x \prec y) \prec \beta(z) + (x \prec y) \succ \beta(z) + (x \succ y) \prec \beta(z) + (x \succ y) \succ \beta(z) \\
&= (x \prec y) \prec \beta(z) + (x \succ y) \prec \beta(z) + (x \prec y) \succ \beta(z) + (x \succ y) \succ \beta(z) \\
&= (x \prec y) \prec \beta(z) + (x \succ y) \prec \beta(z) + (x * y) \succ \beta(z) \\
&= \alpha(x) \prec (y * z) + \alpha(x) \succ (y \prec z) + \alpha(x) \succ (y \succ z) \\
&= \alpha(x) \prec (y * z) + \alpha(x) \succ (y * z) = \alpha(x) * (y * z), \\
\alpha(x * y) &= \alpha(x \succ y) + \alpha(x \prec y) = \alpha(x) \succ \alpha(y) + \alpha(x) \prec \alpha(y) = \alpha(x) * \alpha(y)
\end{aligned}$$

which completes the proof. □

We call $(\mathcal{A}, *, \alpha, \beta)$ the biHom-associative algebra of $(\mathcal{A}, \prec, \succ, \alpha, \beta)$, and $(\mathcal{A}, \succ, \prec, \alpha, \beta)$ is called a compatible biHom-dendriform algebra structure on the biHom-associative algebra $(\mathcal{A}, *, \alpha, \beta)$.

Proposition 18.12 *Let* $(\mathcal{A}, \prec, \succ, \alpha, \beta)$ *be a biHom-dendriform algebra. Suppose that* $(\mathcal{A}, *, \beta, \alpha)$ *is a biHom-associative algebra. Then,* $(L_\succ, R_\prec, \beta, \alpha, \mathcal{A})$ *is a bimodule of* $(\mathcal{A}, *, \beta, \alpha)$.

Proof For $x, y, v \in \mathcal{A}$, we have

$$\begin{aligned}
L_\succ(x * y)\beta(v) &= (x * y) \succ \beta(v) = \alpha(x) \succ (y \succ v) = L_\succ(\alpha(x))L_\succ(y)v, \\
R_\prec(x * y)\alpha(v) &= \alpha(v) \prec (x * y) = (v \prec x) \prec \beta(y) = R_\prec(\beta(y))R_\prec(x)v, \\
L_\succ(\alpha(x))R_\prec(y)v &= \alpha(x) \succ (v \prec y) = (x \succ v) \prec \beta(y) = R_\prec(\beta(y))L_\succ(x)v,
\end{aligned}$$

which completes the proof. □

Remark 18.4 If $(\mathcal{A}, \prec, \succ, \alpha, \beta)$ is a biHom-dendriform algebra, $(L_\succ, R_\prec, \alpha, \beta, \mathcal{A})$ is not a bimodule of associated biHom-associative algebra $(\mathcal{A}, *, \alpha, \beta)$.

Proposition 18.13 *Let* $(\mathcal{A}, \prec, \succ, \alpha, \beta)$ *be a biHom-dendriform algebra. If*

$$\alpha^2 = \beta^2 = \alpha \circ \beta = \beta \circ \alpha = \mathrm{Id},$$

then $(\mathcal{A}, \prec, \succ, \alpha, \beta) \cong (\mathcal{A}, \prec, \succ, \beta, \alpha)$.

Proof Let $x, y, z \in \mathcal{A}$. We have

$$\begin{aligned}
\alpha(x)(yz) &= (xy)\beta(z) \Leftrightarrow \\
\alpha((\alpha \circ \beta)(x))(yz) &= (xy)\beta((\beta \circ \alpha)(z)) \Leftrightarrow \\
\alpha^2(\beta(x))(yz) &= (xy)\beta^2(\alpha(z)) \Leftrightarrow \\
\beta(x)(yz) &= (xy)\alpha(z).
\end{aligned}$$

Then $(\mathcal{A}, \prec, \succ, \alpha, \beta) \cong (\mathcal{A}, \prec, \succ, \beta, \alpha)$. □

18.6.2 $\mathcal{O}$-operators and BiHom-Dendriform Algebras

Definition 18.29 Let $(\mathcal{A}, \cdot, \alpha_1, \alpha_2)$ be a biHom-associative algebra, and $(l, r, \beta_1, \beta_2, V)$ be a bimodule. A linear map $T : V \to \mathcal{A}$ is called an **$\mathcal{O}$-operator associated** to $(l, r, \beta_1, \beta_2, V)$, if T satisfies

$$\alpha_1 T = T\beta_2, \quad \alpha_2 T = T\beta_1,$$
$$T(u) \cdot T(v) = T(l(T(u))v + r(T(v))u) \text{ for all } u, v \in V.$$

Example 18.10 Let $(\mathcal{A}, \cdot, \alpha_1, \alpha_2)$ be a multiplicative biHom-associative algebra. Then, the identity map Id is an $\mathcal{O}$-operator associated to the bimodule $(L, 0, \alpha_1, \alpha_2)$ or $(0, R, \alpha_1, \alpha_2)$.

Example 18.11 Let $(\mathcal{A}, \cdot, \alpha, \beta)$ be a multiplicative biHom-associative algebra. A linear map $f : \mathcal{A} \to \mathcal{A}$ is called a **Rota-Baxter operator** on $\mathcal{A}$ of weight zero if f satisfies, for all $x, y \in \mathcal{A}$,

$$f \circ \alpha = \alpha \circ f, \; f \circ \beta = \beta \circ f \text{ and } f(x) \cdot f(y) = f(f(x) \cdot y + x \cdot f(y)).$$

A Rota-Baxter operator on $\mathcal{A}$ is just an $\mathcal{O}$-operator associated to the bimodule (L, R, α, β).

Theorem 18.16 *Let $(\mathcal{A}, \cdot, \alpha_1, \alpha_2)$ be a biHom-associative algebra, and $(l, r, \beta_1, \beta_2, V)$ be a bimodule. Let $T : V \to \mathcal{A}$ be an $\mathcal{O}$-operator associated to $(l, r, \beta_1, \beta_2, V)$. Then, there exists a biHom-dendriform algebra structure on V given, for all $u, v \in V$, by*

$$u \succ v = l(T(u))v, \quad u \prec v = r(T(v))u.$$

Hence, there is an associated biHom-associative algebra structure on V given by (18.29) structure, and T is a homomorphism of biHom-associative algebras. Moreover, $T(V) = \{T(v) \mid v \in V\} \subseteq \mathcal{A}$ is a biHom-associative subalgebra of $\mathcal{A}$, and there is an induced biHom-dendriform algebra structure on $T(V)$ given, for $u, v \in V$, by

$$T(u) \succ T(v) = T(u \succ v), \quad T(u) \prec T(v) = T(u \prec v).$$

Its corresponding associated biHom-associative algebra structure on $T(V)$ given by (18.29) structure is just the biHom-associative subalgebra structure of $\mathcal{A}$, and T is a homomorphism of biHom-dendriform algebras.

Proof For any $x, y, z \in V$, we have

$$\begin{aligned}(x \succ y) \prec \beta_2(z) - \beta_1(x) \succ (y \prec z) &= l(T(x)y) \prec \beta_2(z) - \beta_1(x) \succ r(T(z)y) \\ &= r(T\beta_2(z))l(T(x))y - l(T\beta_1(x)y)r(T(z)y) \\ &= r(\alpha_1(T(z)))l(T(x))y - l(\alpha_2(T(x)))r(T(z))y = 0.\end{aligned}$$

The other two axioms are checked in a similar way. □

Corollary 18.7 *Let* $(\mathcal{A}, *, \alpha, \beta)$ *be a multiplicative biHom-associative algebra. There is a compatible multiplicative biHom-dendriform algebra structure on* $\mathcal{A}$ *if and only if there exists an invertible* $\mathcal{O}$*-operator of* $(\mathcal{A}, *, \alpha, \beta)$.

Proof In fact, if the homomorphism T is an invertible $\mathcal{O}-$operator associated to a bimodule (l, r, α, β, V), then the compatible multiplicative biHom-dendriform algebra structure on $\mathcal{A}$ is given, for all $x, y \in \mathcal{A}$, by

$$x \succ y = T(l(x)T^{-1}(y)), x \prec y = T(r(y)T^{-1}(x)).$$

Conversely, let $(\mathcal{A}, \succ, \prec, \alpha, \beta)$ be a multiplicative biHom-dendriform algebra, and $(\mathcal{A}, *, \alpha, \beta)$ be its associated biHom-associative algebra. Then, the identity map Id is an $\mathcal{O}-$operator associated to the bimodule $(L_\succ, R_\prec, \alpha, \beta)$ of $(\mathcal{A}, *, \alpha, \beta)$. □

18.6.3 Bimodules and Matched Pairs of BiHom-Dendriform Algebras

Definition 18.30 Let $(\mathcal{A}, \succ, \prec, \alpha_1, \alpha_2)$ be a biHom-dendriform algebra, and V be a vector space. Let $l_\succ, r_\succ, l_\prec, r_\prec : \mathcal{A} \to gl(V)$, and $\beta_1, \beta_2 : V \to V$ be six linear maps. Then, $(l_\succ, r_\succ, l_\prec, r_\prec, \beta_1, \beta_2, V)$ is called a **bimodule** of $\mathcal{A}$ if the following equations hold for any $x, y \in \mathcal{A}$ and $v \in V$, with $x * y = x \succ y + x \prec y, l_* = l_\succ + l_\prec, r_* = r_\succ + r_\prec$:

$$\begin{aligned}
l_\prec(x \prec y)\beta_2(v) &= l_\prec(\alpha_1(x))l_*(y)v, r_\prec(\alpha_2(x))l_\prec(y)v = l_\prec(\alpha_1(y))r_*(x)v,\\
r_\prec(\alpha_2(y))r_\prec(y)v &= r_\prec(x * y)\beta_1(v), l_\prec(x \succ y)\beta_2(v) = l_\succ(\alpha_1(x))l_\prec(y)v,\\
r_\prec(\alpha_2(x))l_\succ(y)v &= l_\succ(\alpha_1(y))r_\prec(x)v, r_\prec(\alpha_2(x))r_\succ(y)v = r_\succ(y \prec x)\beta_1(v),\\
l_\succ(x * y)\beta_2(v) &= l_\succ(\alpha_1(x))l_\succ(y)v, r_\succ(\alpha_2(x))l_*(y)v = l_\succ(\alpha_1(y))r_\succ(x)v,\\
r_\succ(\alpha_2(x))r_*(y)v &= r_\succ(y \succ x)\beta_1(v).
\end{aligned}$$

Proposition 18.14 *Let* $(l_\succ, r_\succ, l_\prec, r_\prec, \beta_1, \beta_2, V)$ *be a bimodule of a biHom-dendriform algebra* $(\mathcal{A}, \succ, \prec, \alpha_1, \alpha_2)$. *Then, there exists a biHom-dendriform algebra structure on the direct sum* $\mathcal{A} \oplus V$ *of the underlying vector spaces of* $\mathcal{A}$ *and* V *given, for all* $x, y \in \mathcal{A}$ *and* $u, v \in V$, *by*

$$\begin{aligned}
(x + u) \succ (y + v) &= x \succ y + l_\succ(x)v + r_\succ(y)u,\\
(x + u) \prec (y + v) &= x \prec y + l_\prec(x)v + r_\prec(y)u.
\end{aligned}$$

We denote it by $\mathcal{A} \times_{l_\succ, r_\succ, l_\prec, r_\prec, \alpha_1, \alpha_2, \beta_1, \beta_2} V$.

Proof Let $v_1, v_2, v_3 \in V$ and $x_1, x_2, x_3 \in \mathcal{A}$. Setting and computing

$$\begin{aligned}
&[(x_1 + v_1) \prec (x_2 + v_2)] \prec (\alpha_2(x_3) + \beta_2(v_3)) = \\
&\quad (\alpha_1(x_1) + \beta_1(v_1)) \prec [(x_2 + v_2) * (x_3 + v_3)], \\
&[(x_1 + v_1) \succ (x_2 + v_2)] \prec (\alpha_2(x_3) + \beta_2(v_3)) = \\
&\quad (\alpha_1(x_1) + \beta_1(v_1)) \succ [(x_2 + v_2) \prec (x_3 + v_3)], \\
&[\alpha_1(x_1) + \beta_1(v_1)] \succ [(x_2) + v_2) \succ (x_3 + v_3)] = \\
&\quad [(x_1 + v_1) * (x_2 + v_2)] \succ (\alpha_2(x_3) + \beta_2(v_3)),
\end{aligned}$$

one obtains the conditions of the bimodule of a biHom-dendriform algebra, which completes the proof. □

Proposition 18.15 *Suppose that* $(l_\succ, r_\succ, l_\prec, r_\prec, \beta_1, \beta_2, V)$ *is a bimodule of a biHom-dendriform algebra* $(\mathcal{A}, \succ, \prec, \alpha_1, \alpha_2)$. *Then*

1) $(l_\succ, r_\prec, \beta_2, \beta_1, V)$ *and* $(l_\succ + l_\prec, r_\succ + r_\prec, \beta_1, \beta_2, V)$ *are bimodules of* $(\mathcal{A}, *, \alpha_2, \alpha_1)$;
2) *for any bimodule* $(l, r, \beta_1, \beta_2, V)$ *of* $(\mathcal{A}, *, \alpha_1, \alpha_2)$,

$$(l, 0, 0, r, \beta_2, \beta_1, V) \text{ is a bimodule of } (\mathcal{A}, \succ, \prec, \alpha_2, \alpha_1);$$

3) $(l_\succ + l_\prec, 0, 0, r_\succ + r_\prec, \beta_1, \beta_2, V)$ *and* $(l_\succ, 0, 0, r_\prec, \beta_1, \beta_2, V)$ *are bimodules of* $(\mathcal{A}, \succ, \prec, \alpha_1, \alpha_2)$;
4) *the biHom-dendriform algebras*

$$\mathcal{A} \times_{l_\succ, r_\succ, l_\prec, r_\prec, \alpha_1, \alpha_2, \beta_1, \beta_2} V \text{ and } \mathcal{A} \times_{l_\succ + l_\prec, 0, 0, r_\succ + r_\prec, \alpha_1, \alpha_2, \beta_1, \beta_2} V$$

have the same associated biHom-associative algebra

$$\mathcal{A} \times_{l_\succ + l_\prec, r_\succ + r_\prec, \alpha_1, \alpha_2, \beta_1, \beta_2} V.$$

The following theorem is proved in a similar way as for Theorem 18.1.

Theorem 18.17 *Let* $(\mathcal{A}, \succ_\mathcal{A}, \prec_\mathcal{A}, \alpha_1, \alpha_2)$ *and* $(\mathcal{B}, \succ_\mathcal{B}, \prec_\mathcal{B}, \beta_1, \beta_2)$ *be biHom-dendriform algebras. Suppose that there are linear maps* $l_{\succ_\mathcal{A}}, r_{\succ_\mathcal{A}}, l_{\prec_\mathcal{A}}, r_{\prec_\mathcal{A}} : \mathcal{A} \to gl(\mathcal{B})$, *and* $l_{\succ_\mathcal{B}}, r_{\succ_\mathcal{B}}, l_{\prec_\mathcal{B}}, r_{\prec_\mathcal{B}} : \mathcal{B} \to gl(\mathcal{A})$ *such that* $(l_{\succ_\mathcal{A}}, r_{\succ_\mathcal{A}}, l_{\prec_\mathcal{A}}, r_{\prec_\mathcal{A}}, \beta_1, \beta_2, \mathcal{B})$ *is a bimodule of* $\mathcal{A}$, *and* $(l_{\succ_\mathcal{B}}, r_{\succ_\mathcal{B}}, l_{\prec_\mathcal{B}}, r_{\prec_\mathcal{B}}, \alpha_1, \alpha_2, \mathcal{A})$ *is a bimodule of* $\mathcal{B}$, *satisfying for* $l_\mathcal{A} = l_{\succ_\mathcal{A}} + l_{\prec_\mathcal{A}}$, $r_\mathcal{A} = r_{\succ_\mathcal{A}} + r_{\prec_\mathcal{A}}$, $l_\mathcal{B} = l_{\succ_\mathcal{B}} + l_{\prec_\mathcal{B}}$, $r_\mathcal{B} = r_{\succ_\mathcal{B}} + r_{\prec_\mathcal{B}}$ *and all* $x, y \in \mathcal{A}, a, b \in \mathcal{B}$:

$$r_{\prec_{\mathcal{A}}}(\alpha_2(x))(a \prec_{\mathcal{B}} b) = \beta_1(a) \prec_{\mathcal{B}} (r_{\mathcal{A}}(x)b) + r_{\prec_{\mathcal{A}}}(l_{\mathcal{B}}(x)\beta_1(a)), \quad (18.58)$$

$$\begin{aligned} l_{\prec_{\mathcal{A}}}(l_{\prec_{\mathcal{B}}}(x))\beta_2(b) + (r_{\prec_{\mathcal{A}}}(x)a) \prec_{\mathcal{B}} \beta_2(b) = \\ \beta_1(a) \prec_{\mathcal{B}} (l_{\prec_{\mathcal{A}}}(x)b) + r_{\prec_{\mathcal{A}}}(r_{\prec_{\mathcal{B}}}(b)x)\beta_1(a), \end{aligned} \quad (18.59)$$

$$l_{\prec_{\mathcal{A}}}(\alpha_1(x))(a *_{\mathcal{B}} b) = (l_{\prec_{\mathcal{A}}}(x)a) *_{\mathcal{B}} \beta_2(b) + l_{\prec_{\mathcal{A}}}(r_{\prec_{\mathcal{A}}}(a)x)\beta_2(b), \quad (18.60)$$

$$r_{\prec_{\mathcal{A}}}(\alpha_2(x))(a \succ_{\mathcal{B}} b) = r_{\succ_{\mathcal{A}}}(l_{\prec_{\mathcal{B}}}(b)x)\beta_1(a) + \beta_1(a) \succ_{\mathcal{B}} (r_{\prec_{\mathcal{A}}}(x)b), \quad (18.61)$$

$$\begin{aligned} l_{\prec_{\mathcal{A}}}(l_{\succ_{\mathcal{B}}}(a)x)\beta_2(b) + (r_{\succ_{\mathcal{A}}}(x)a) \prec_{\mathcal{B}} \beta_2(b) = \\ \beta_1(a) \succ_{\mathcal{B}} (l_{\prec_{\mathcal{A}}}(x)b) + r_{\succ_{\mathcal{A}}}(r_{\prec_{\mathcal{B}}}(b)x)\beta_1(a) \end{aligned} \quad (18.62)$$

$$l_{\succ_{\mathcal{A}}}(\alpha_1(x))(a \prec_{\mathcal{B}} b) = (l_{\succ_{\mathcal{A}}}(x)a) \prec_{\mathcal{B}} \beta_2(b) + l_{\prec_{\mathcal{A}}}(r_{\succ_{\mathcal{B}}}(a)x)\beta_2(b), \quad (18.63)$$

$$r_{\succ_{\mathcal{A}}}(\alpha_2(x))(a *_{\mathcal{B}} b) = \beta_1(a) \succ_{\mathcal{B}} (r_{\succ_{\mathcal{A}}}(x)b) + r_{\succ_{\mathcal{A}}}(l_{\succ_{\mathcal{B}}}(b)x)\beta_1(a), \quad (18.64)$$

$$\begin{aligned} \beta_1(a) \succ_{\mathcal{B}} (l_{\succ_{\mathcal{A}}}(x)b) + r_{\succ_{\mathcal{A}}}(r_{\succ_{\mathcal{B}}}(b)x)\beta_1(a) = \\ l_{\succ_{\mathcal{A}}}(l_{\mathcal{B}}(a)x)\beta_2(b) + (r_{\mathcal{A}}(x)a) \succ_{\mathcal{B}} \beta_2(b), \end{aligned} \quad (18.65)$$

$$l_{\succ_{\mathcal{A}}}(\alpha_1(x))(a \succ_{\mathcal{B}} b) = (l_{\mathcal{A}}(x)a) \succ_{\mathcal{B}} \beta_2(b) + l_{\succ_{\mathcal{A}}}(r_{\mathcal{B}}(a)x)\beta_2(b), \quad (18.66)$$

$$r_{\prec_{\mathcal{B}}}(\beta_2(a))(x \prec_{\mathcal{A}} y) = \alpha_1(x) \prec_{\mathcal{A}} (r_{\mathcal{B}}(a)y) + r_{\prec_{\mathcal{B}}}(l_{\mathcal{A}}(y)a)\alpha_1(x), \quad (18.67)$$

$$\begin{aligned} l_{\prec_{\mathcal{B}}}(l_{\prec_{\mathcal{A}}}(x)a)\alpha_2(y) + (r_{\prec_{\mathcal{B}}}(a)x) \prec_{\mathcal{A}} \alpha_2(y) = \\ \alpha_1(x) \prec_{\mathcal{A}} (l_{\mathcal{B}}(a)y) + r_{\prec_{\mathcal{B}}}(r_{\mathcal{A}}(y)a)\alpha_1(x), \end{aligned} \quad (18.68)$$

$$l_{\prec_{\mathcal{B}}}(\beta_1(a))(x *_{\mathcal{A}} y) = (l_{\prec_{\mathcal{B}}}(a)x) \prec_{\mathcal{A}} \alpha_2(y) + l_{\prec_{\mathcal{B}}}(r_{\prec_{\mathcal{A}}}(x)a)\alpha_2(y), \quad (18.69)$$

$$r_{\prec_{\mathcal{B}}}(\beta_2(a))(x \succ_{\mathcal{A}} y) = r_{\succ_{\mathcal{B}}}(l_{\prec_{\mathcal{B}}}(y)a)\alpha_1(x) + \alpha_1(x) \succ_{\mathcal{A}} (r_{\prec_{\mathcal{B}}}(a)y), \quad (18.70)$$

$$\begin{aligned} l_{\prec_{\mathcal{B}}}(l_{\succ_{\mathcal{A}}}(x)a)\alpha_2(y) + (r_{\succ_{\mathcal{B}}}(a)x) \prec_{\mathcal{A}} \alpha_2(y) = \\ \alpha_1(x) \succ_{\mathcal{A}} (l_{\prec_{\mathcal{B}}}(a)y) + r_{\succ_{\mathcal{B}}}(r_{\prec_{\mathcal{A}}}(y)a)\alpha_1(x), \end{aligned} \quad (18.71)$$

$$l_{\succ_{\mathcal{B}}}(\beta_1(a))(x \prec_{\mathcal{A}} y) = (l_{\succ_{\mathcal{B}}}(a)x) \prec_{\mathcal{A}} \alpha_2(y) + l_{\prec_{\mathcal{B}}}(r_{\succ_{\mathcal{A}}}(x)a)\alpha_2(y), \quad (18.72)$$

$$r_{\succ_{\mathcal{B}}}(\beta_2(a))(x *_{\mathcal{A}} y) = \alpha_1(x) \succ_{\mathcal{A}} (r_{\succ_{\mathcal{B}}}(a)y) + r_{\succ_{\mathcal{B}}}(l_{\succ_{\mathcal{A}}}(y)a)\alpha_1(x), \quad (18.73)$$

$$\begin{aligned} \alpha_1(x) \succ_{\mathcal{A}} (l_{\succ_{\mathcal{B}}}(a)y) + r_{\succ_{\mathcal{B}}}(r_{\succ_{\mathcal{A}}}(y)a)\alpha_1(x) = \\ l_{\succ_{\mathcal{B}}}(l_{\mathcal{A}}(x)a)\alpha_2(y) + (r_B(a)x) \succ_{\mathcal{A}} \alpha_2(y), \end{aligned} \quad (18.74)$$

$$l_{\succ_{\mathcal{B}}}(\beta_1(a))(x \succ_{\mathcal{A}} y) = (l_{\mathcal{B}}(a)x) \succ_{\mathcal{A}} \alpha_2(y) + l_{\succ_{\mathcal{B}}}(r_{\mathcal{A}}(x)a)\alpha_2(y). \quad (18.75)$$

Then, there is a biHom-dendriform algebra structure on the direct sum $\mathcal{A} \oplus \mathcal{B}$ of the underlying vector spaces of $\mathcal{A}$ and $\mathcal{B}$ given, for any $x, y \in \mathcal{A}$ and $a, b \in \mathcal{B}$, by

$$\begin{aligned} (x + a) \succ (y + b) &= (x \succ_{\mathcal{A}} y + r_{\succ_{\mathcal{B}}}(b)x + l_{\succ_{\mathcal{B}}}(a)y) \\ &\quad + (l_{\succ_{\mathcal{A}}}(x)b + r_{\succ_{\mathcal{A}}}(y)a + a \succ_{\mathcal{B}} b), \\ (x + a) \prec (y + b) &= (x \prec_{\mathcal{A}} y + r_{\prec_{\mathcal{B}}}(b)x + l_{\prec_{\mathcal{B}}}(a)y) \\ &\quad + (l_{\prec_{\mathcal{A}}}(x)b + r_{\prec_{\mathcal{A}}}(y)a + a \prec_{\mathcal{B}} b). \end{aligned}$$

Let $\mathcal{A} \bowtie^{l_{\succ_{\mathcal{A}}}, r_{\succ_{\mathcal{A}}}, l_{\prec_{\mathcal{A}}}, r_{\prec_{\mathcal{A}}}, \beta_1, \beta_1}_{l_{\succ_{\mathcal{B}}}, r_{\succ_{\mathcal{B}}}, l_{\prec_{\mathcal{B}}}, r_{\prec_{\mathcal{B}}}, \alpha_1, \alpha_2} \mathcal{B}$ denote this biHom-dendriform algebra.

Definition 18.31 Let $(\mathcal{A}, \succ_{\mathcal{A}}, \prec_{\mathcal{A}}, \alpha_1, \alpha_2)$ and $(\mathcal{B}, \succ_{\mathcal{B}}, \prec_{\mathcal{B}}, \beta_1, \beta_2)$ be biHom-dendriform algebras. Suppose that there are linear maps $l_{\succ_{\mathcal{A}}}, r_{\succ_{\mathcal{A}}}, l_{\prec_{\mathcal{A}}}, r_{\prec_{\mathcal{A}}} : \mathcal{A} \to gl(\mathcal{B})$, and $l_{\succ_{\mathcal{B}}}, r_{\succ_{\mathcal{B}}}, l_{\prec_{\mathcal{B}}}, r_{\prec_{\mathcal{B}}} : \mathcal{B} \to gl(\mathcal{A})$ such that $(l_{\succ_{\mathcal{A}}}, r_{\succ_{\mathcal{A}}}, l_{\prec_{\mathcal{A}}}, r_{\prec_{\mathcal{A}}}, \beta_1, \beta_2)$ is a bimodule of $\mathcal{A}$, and $(l_{\succ_{\mathcal{B}}}, r_{\succ_{\mathcal{B}}}, l_{\prec_{\mathcal{B}}}, r_{\prec_{\mathcal{B}}}, \alpha_1, \alpha_2)$ is a bimodule of $\mathcal{B}$. If (18.58)–(18.75) hold, then

$$(\mathcal{A}, \mathcal{B}, l_{\succ_{\mathcal{A}}}, r_{\succ_{\mathcal{A}}}, l_{\prec_{\mathcal{A}}}, r_{\prec_{\mathcal{A}}}, \beta_1, \beta_2, l_{\succ_{\mathcal{B}}}, r_{\succ_{\mathcal{B}}}, l_{\prec_{\mathcal{B}}}, r_{\prec_{\mathcal{B}}}, \alpha_1, \alpha_2)$$

is called a **matched pair of biHom-dendriform algebras**.

Corollary 18.8 *Let* $(\mathcal{A}, \mathcal{B}, l_{\succ_{\mathcal{A}}}, r_{\succ_{\mathcal{A}}}, l_{\prec_{\mathcal{A}}}, r_{\prec_{\mathcal{A}}}, \beta_1, \beta_2, l_{\succ_{\mathcal{B}}}, r_{\succ_{\mathcal{B}}}, l_{\prec_{\mathcal{B}}}, r_{\prec_{\mathcal{B}}}, \alpha_1, \alpha_2)$ *be a matched pair of biHom-dendriform algebras. Then,*

$$(\mathcal{A}, \mathcal{B}, l_{\succ_{\mathcal{A}}} + l_{\prec_{\mathcal{A}}}, r_{\succ_{\mathcal{A}}} + r_{\prec_{\mathcal{A}}}, l_{\succ_{\mathcal{B}}} + l_{\prec_{\mathcal{B}}}, r_{\succ_{\mathcal{B}}} + r_{\prec_{\mathcal{B}}}, \alpha_1 + \beta_1, \alpha_2 + \beta_2)$$

is a matched pair of the associated biHom-associative algebras $(\mathcal{A}, *_{\mathcal{A}}, \alpha_1, \alpha_2)$ *and* $(\mathcal{B}, *_{\mathcal{B}}, \beta_1, \beta_2)$.

Proof The associated biHom-associative algebra $(\mathcal{A} \bowtie \mathcal{B}, *, \alpha_1 + \beta_1, \alpha_2 + \beta_2)$ is exactly the biHom-associative algebra obtained from the matched pair of biHom-associative algebras,

$$(\mathcal{A}, \mathcal{B}, l_{\mathcal{A}}, r_{\mathcal{A}}, \beta_1, \beta_2, l_{\mathcal{B}}, r_{\mathcal{B}}, \alpha_1, \alpha_2),$$

with $(x + a) * (y + b) = x *_{\mathcal{A}} y + l_{\mathcal{B}}(a)y + r_{\mathcal{B}}(b)x + a *_{\mathcal{B}} b + l_{\mathcal{A}}(x)b + r_{\mathcal{A}}(y)a$ for $x, y \in \mathcal{A}, a, b \in \mathcal{B}$, where $l_{\mathcal{A}} = l_{\succ_{\mathcal{A}}} + l_{\prec_{\mathcal{A}}}$, $r_{\mathcal{A}} = r_{\succ_{\mathcal{A}}} + r_{\prec_{\mathcal{A}}}$, $l_{\mathcal{B}} = l_{\succ_{\mathcal{B}}} + l_{\prec_{\mathcal{B}}}$ and $r_{\mathcal{B}} = r_{\succ_{\mathcal{B}}} + r_{\prec_{\mathcal{B}}}$. □

18.7 Concluding Remarks

In this work, we have constructed a biHom-associative algebra with a decomposition into direct sum of the underlying vector spaces of a biHom-associative algebra and its dual such that both of them are biHom-subalgebras, with either the natural symmetric bilinear form being invariant, or the natural antisymmetric bilinear form being a Connes cocycle. Then, we have performed the double constructions of biHom-Frobenius algebras and Connes cocycle, and provided the bialgebra structures.

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Chapter 19
On Classification of (n+1)-Dimensional n-Hom-Lie Algebras with Nilpotent Twisting Maps

Abdennour Kitouni and Sergei Silvestrov

Abstract The aim of this work is to study properties of n-Hom-Lie algebras in dimension $n+1$ allowing to explicitly find them and differentiate them, to eventually classify them. Some specific properties of $(n+1)$-dimensional n-Hom-Lie algebra such as nilpotence, solvability, center, ideals, derived series and central descending series are studied, the Hom-Nambu-Filippov identity for various classes of twisting maps in dimension $n+1$ is considered, and systems of equations corresponding to each case are described. All 4-dimensional 3-Hom-Lie algebras with some of the classes of twisting maps are computed in terms of structure constants as parameters and listed in the way emphasising the number of free parameters in each class, and also some detailed properties of the Hom-algebras are obtained.

Keywords Hom-algebra · n-Hom-Lie algebra

MSC 2020 Classification 17B61 · 17D30 · 17A40 · 17A42 · 17B30

19.1 Introduction

Hom-Lie algebras and more general quasi-Hom-Lie algebras where introduced first by Hartwig, Larsson and Silvestrov in [49], where the general quasi-deformations and discretizations of Lie algebras of vector fields using more general σ-derivations (twisted derivations) and a general method for construction of deformations of Witt and Virasoro type algebras based on twisted derivations have been developed, initially motivated by the q-deformed Jacobi identities observed for the q-deformed algebras in physics, along with q-deformed versions of homological algebra and

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discrete modifications of differential calculi [7, 33–37, 39, 42, 51, 53, 66–68]. The general abstract quasi-Lie algebras and the subclasses of quasi-Hom-Lie algebras and Hom-Lie algebras as well as their general colored (graded) counterparts have been introduced in [49, 60–62, 80]. Subsequently, various classes of Hom-Lie admissible algebras have been considered in [70]. In particular, in [70], the Hom-associative algebras have been introduced and shown to be Hom-Lie admissible, that is leading to Hom-Lie algebras using commutator map as new product, and in this sense constituting a natural generalization of associative algebras, as Lie admissible algebras leading to Lie algebras via commutator map as new product. In [70], moreover several other interesting classes of Hom-Lie admissible algebras generalizing some classes of non-associative algebras, as well as examples of finite-dimensional Hom-Lie algebras have been described. Since these pioneering works [49, 60–63, 70], Hom-algebra structures are very useful since Hom-algebra structures of a given type include their classical counterparts and open more possibilities for deformations, extensions of cohomological structures and representations. Hom-algebra structures have developed in a popular broad area with increasing number of publications in various directions (see for example [8, 29, 44, 59, 60, 64, 71–73, 76, 78, 79, 84, 85] and references therein).

Ternary Lie algebras appeared in generalization of Hamiltonian mechanics by Nambu [74]. Besides Nambu mechanics, n-Lie algebras revealed to have many applications in physics. The mathematical algebraic foundations of Nambu mechanics have been developed by Takhtajan in [81]. Filippov, in [47] independently introduced and studied structure of n-Lie algebras and Kasymov [54] investigated their properties. Properties of n-ary algebras, including solvability and nilpotency, were studied in [16, 21, 54]. Kasymov [54] pointed out that n-ary multiplication allows for several different definitions of solvability and nilpotency in n-Lie algebras, and studied their properties. Further properties, classification, and connections of n-ary algebras to other structures such as bialgebras, Yang-Baxter equation and Manin triples for 3-Lie algebras were studied in [15–22, 24, 25, 54]. The structure of 3-Lie superalgebras induced by Lie superalgebras, classification of 3-Lie superalgebras and application to constructions of B.R.S. algebras have been considered in [2–4]. Interesting constructions of ternary Lie superalgebras in connection to superspace extension of Nambu-Hamilton equation is considered in [5]. In [32], Leibniz n-algebras have been studied. The general cohomology theory for n-Lie algebras and Leibniz n-algebras was established in [38, 77, 82]. For more details of the theory and applications of n-Lie algebras, see [43] and references therein.

Classifications of n-ary or Hom generalizations of Lie algebras have been considered, either in very special cases or in low dimensions. The classification of n-Lie algebras of dimension up to $n+1$ over a field of characteristic $p \neq 2$ has been completed by Filippov [47] using the specific properties of $(n+1)$-dimensional n-Lie algebras that make it possible to represent their n-ary products by a square matrix in a similar way as bilinear forms, the number of cases obtained depends on the properties of the base field, the list is ordered by ascending dimension of the derived ideal, and among them, one nilpotent algebra, and a class of simple algebras which are all isomorphic in the case of an algebraically closed field, the remaining algebras are k-solvable for some $2 \leq k \leq n$ depending on the algebra. These simple algebras are proven to be the only simple finite-dimensional n-Lie algebras in [65]. The classification of $(n+1)$-dimensional n-Lie algebras over a field of characteristic 2 has been done by Bai, Wang, Xiao, and An [22] by finding and using a similar result in characteristic 2. Bai, Song and Zhang [21] classify the $(n+2)$-dimensional n-Lie algebras over an algebraically closed field of characteristic 0 using the fact that an $(n+2)$-dimensional n-Lie algebra has a subalgebra of codimension 1 if the dimension of its derived ideal is not 3, thus constructing most of the cases as extensions of the $(n+1)$-dimensional n-Lie algebras listed by Filippov. In [31], Cantarini and Kac classified all simple linearly compact n-Lie superalgebras, which turned out to be n-Lie algebras, by finding a bijective correspondence between said algebras and a special class of transitive $\mathbb{Z}$-graded Lie superalgebras, the list they obtained consists of four representatives, one of them is the $(n+1)$-dimensional vector product n-Lie algebra, and the remaining three are infinite-dimensional n-Lie algebras.

Classifications of n-Lie algebras in higher dimensions have only been studied in particular cases. Metric n-Lie algebras, that is n-Lie algebras equipped with a non-degenerate compatible bilinear form, have been considered and classified, first in dimension $n+2$ by Ren, Chen and Liang [75] and dimension $n+3$ by Geng, Ren and Chen [48], and then in dimensions $n+k$ for $2 \leq k \leq n+1$ by Bai, Wu and Chen [23]. The classification is based on the study of the Levi decomposition, the center and the isotropic ideals and properties around them. Another case that has been studied is the case of nilpotent n-Lie algebras, more specifically nilpotent n-Lie algebras of class 2. Eshrati, Saeedi and Darabi [45] classify $(n+3)$-dimensional nilpotent n-Lie algebras and $(n+4)$-dimensional nilpotent n-Lie algebras of class 2 using properties introduced in [40, 46]. Similarly, Hoseini, Saeedi and Darabi [50] classify $(n+5)$-dimensional nilpotent n-Lie algebras of class 2. In [52], Jamshidi, Saeedi and Darabi classify $(n+6)$-dimensional nilpotent n-Lie algebras of class 2 using the fact that such algebras factored by the span of a central element give $(n+5)$-dimensional nilpotent n-Lie algebras of class 2, which were classified before. Classification of other classes of nilpotent n-Lie algebras depending on dimension of multiplier has been considered in [41]. There has been a study of the classification of 3-dimensional 3-Hom-Lie algebras with diagonal twisting maps by Ataguema, Makhlouf and Silvestrov in [13].

Hom-type generalization of n-ary algebras, such as n-Hom-Lie algebras and other n-ary Hom algebras of Lie type and associative type, were introduced in [13], by twisting the defining identities by a set of linear maps. The particular case, where all these maps are equal and are algebra morphisms has been considered and a way to generate examples of n-ary Hom-algebras from n-ary algebras of the same type have been described. Further properties, construction methods, examples, representations, cohomology and central extensions of n-ary Hom-algebras have been considered in [9–12, 55, 56, 83, 86]. These generalizations include n-ary Hom-algebra structures generalizing the n-ary algebras of Lie type including n-ary Nambu algebras, n-ary Nambu-Lie algebras and n-ary Lie algebras, and n-ary algebras of associative type including n-ary totally associative and n-ary partially associative algebras. In [57], constructions of n-ary generalizations of BiHom-Lie algebras and BiHom-associative algebras have been considered. Generalized Derivations of n-BiHom-Lie algebras have been studied in [28]. Generalized derivations of multiplicative n-ary Hom-Ω color algebras have been studied in [30]. Cohomology of Hom-Leibniz and n-ary Hom-Nambu-Lie superalgebras has been considered in [1]. Generalized Derivations and Rota-Baxter Operators of n-ary Hom-Nambu Superalgebras have been considered in [69]. A construction of 3-Hom-Lie algebras based on σ-derivation and involution has been studied in [6]. Multiplicative n-Hom-Lie color algebras have been considered in [26].

In [11, 12], the construction of $(n+1)$-Lie algebras induced by n-Lie algebras using combination of n-ary multiplication with a trace, motivated by the work of Awata, Li, Minic and Yoneya [14] on the quantization of the Nambu brackets, was generalized using the brackets of general Hom-Lie or n-Hom-Lie algebras and trace-like linear forms satisfying conditions depending on the twisting linear maps defining the Hom-Lie or n-Hom-Lie algebras. In [27], a method was demonstrated of how to construct n-ary multiplications from the binary multiplication of a Hom-Lie algebra and a $(n-2)$-linear function satisfying certain compatibility conditions. Solvability and nilpotency for n-Hom-Lie algebras and $(n+1)$-Hom-Lie algebras induced by n-Hom-Lie algebras have been considered in [58].

n-Hom-Lie algebras are fundamentally different from n-Lie algebras especially when the twisting maps are not invertible or not diagonalizable. When the twisting maps are not invertible, the Hom-Nambu-Filippov identity becomes less restrictive since when elements of the kernel of the twisting maps are used, several terms or even the whole identity might vanish. Isomorphisms of Hom-algebras are also different from isomorphisms of algebras since they need to intertwine not only the multiplications but also the twisting maps. All of this make the classification problem different, interesting, rich and not simply following from the case of n-Lie algebras. In this work, we consider n-Hom-Lie algebras with a nilpotent twisting map α, which means in particular that α is not invertible.

The aim of this work is to study properties of n-Hom-Lie algebras in dimension $n+1$ allowing to explicitly find them and differentiate them, to eventually classify them. We also present lists of 4-dimensional 3-Hom-Lie algebras in various special cases of the twisting map and study their properties. In Sect. 19.2, the definition and basic properties of n-Hom-Lie algebras are presented. In Sect. 19.3, some specific properties of $(n+1)$-dimensional n-Hom-Lie algebra are studied. In Sect. 19.4, the Hom-Nambu-Filippov identity for various classes of twisting maps in dimension $n+1$ is considered, and systems of equations corresponding to each case are described. In Sect. 19.5, all 4-dimensional 3-Hom-Lie algebras with some of the classes of twisting maps are computed in terms of structure constants as parameters and listed in classes in the way emphasizing the number of free parameters in each class. Some detailed properties of the Hom-algebras are obtained.

19.2 Preliminaries

In this section, we present the basic definitions and properties of n-Hom-Lie algebras that we need for our study. Throughout this article, it is assumed that all vector spaces are over a field $\mathbb{K}$ of characteristic 0, and for any subset S of a vector space, $\langle S\rangle$ denotes the linear span of S. Hom-Lie algebras are a generalization of Lie algebras introduced in [49] while studying σ-derivations. The n-ary case was introduced in [13].

Definition 19.1 ([49, 70]) A Hom-Lie algebra $(A, [\cdot,\cdot], \alpha)$ is a vector space A together with a bilinear map $[\cdot,\cdot] : A \times A \to A$ and a linear map $\alpha : A \to A$ satisfying, for all $x, y, z \in A$,

$$[x, y] = -[y, x] \qquad \text{Skew-symmetry}$$
$$[\alpha(x), [y, z]] = [[x, y], \alpha(z)] + [\alpha(y), [x, z]] \qquad \text{Hom-Jacobi identity}$$

In Hom-Lie algebras, by skew-symmetry, the Hom-Jacobi identity is equivalent to

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0. \qquad \text{Hom-Jacobi identity in cyclic form}$$

Definition 19.2 ([49, 60]) Hom-Lie algebra morphisms from Hom-Lie algebra $\mathcal{A} = (A, [\cdot,\cdot]_{\mathcal{A}}, \alpha)$ to Hom-Lie algebra $\mathcal{B} = (B, [\cdot,\cdot]_{\mathcal{B}}, \beta)$ are linear maps $f : A \to B$ satisfying, for all $x, y \in A$,

$$f([x, y]_{\mathcal{A}}) = [f(x), f(y)]_{\mathcal{B}}, \tag{19.1}$$
$$f \circ \alpha = \beta \circ f. \tag{19.2}$$

Linear maps $f : A \to B$ satisfying only condition (19.1) are called weak morphisms of Hom-Lie algebras.

Definition 19.3 ([29, 70]) A Hom-Lie algebra $(A, [\cdot,\cdot], \alpha)$ is said to be multiplicative if α is an algebra morphism, and it is said to be regular if α is an isomorphism.

Definition 19.4 ([13]) An n-Hom-Lie algebra $(A, [\cdot,\dots,\cdot], \{\alpha_i\}_{1\le i\le n-1})$ is a vector space A together with a n-linear map $[\cdot,\dots,\cdot] : A^n \to A$ and $(n-1)$ linear maps $\alpha_i : A \to A$, $1 \le i \le n-1$ satisfying, for all $x_1, \dots, x_{n-1}, y_1, \dots, y_n \in A$, Skew-symmetry

$$[x_{\sigma(1)}, \dots, x_{\sigma(n)}] = sgn(\sigma)[x_1, \dots, x_n], \tag{19.3}$$

Hom-Nambu-Filippov identity

$$\Big[\alpha_1(x_1), \dots, \alpha_{n-1}(x_{n-1}), [y_1, \dots, y_n]\Big] = \tag{19.4}$$
$$\sum_{i=1}^{n}\Big[\alpha_1(y_1), \dots, \alpha_{i-1}(y_{i-1}), [x_1, \dots, x_{n-1}, y_i], \alpha_i(y_{i+1}), \dots, \alpha_{n-1}(y_n)\Big].$$

Remark 19.1 If $\alpha_i = \mathrm{Id}_A$ for all $1 \le i \le n-1$, then one gets an n-Lie algebra [47]. Therefore, the class of n-Lie algebras is included in the class of n-Hom-Lie algebras. For any vector space A, if $[x_1, \dots, x_n]_0 = 0$ for all $x_1, \dots, x_n \in A$ and any linear maps $\alpha_1, \dots, \alpha_{n-1}$, then $(A, [\cdot, \dots, \cdot]_0, \alpha_1, \dots, \alpha_{n-1})$ is an n-Hom-Lie algebra.

Lemma 19.1 *Let A be a vector space, let $[\cdot, \dots, \cdot]$ be an n-linear skew-symmetric map and let $\alpha_1, \dots, \alpha_{n-1}$ be linear maps on A. If the $(n-1)$-linear map*

$$(x_1, \dots, x_{n-1}) \mapsto \big[\alpha_1(x_1), \dots, \alpha_{n-1}(x_{n-1}), d\big]$$

is skew-symmetric for all $d \in [A, \dots, A]$, then the $(2n-1)$-linear map H, defined for all $x_1, \dots, x_{n-1}, y_1, \dots, y_n \in A$ by

$$H(x_1, \dots, x_{n-1}, y_1, \dots, y_n) = \Big[\alpha_1(x_1), \dots, \alpha_{n-1}(x_{n-1}), [y_1, \dots, y_n]\Big]$$
$$- \sum_{k=1}^{n}\Big[\alpha_1(y_1), \dots, \alpha_{k-1}(y_{k-1}), [x_1, \dots, x_{n-1}, y_k], \alpha_k(y_{k+1}), \dots, \alpha_{n-1}(y_n)\Big],$$

is skew-symmetric in its first $n-1$ arguments and in its last n arguments.

Proof Let $x_1, \dots, x_{n-1}, y_1, \dots, y_n \in A$. If $x_i = x_{i+1} = x$, then

$$H(x_1, \dots, x_{i-1}, x, x, x_{i+2}, \dots, x_{n-1}, y_1, \dots, y_n) =$$
$$\Big[\alpha_1(x_1), \dots, \alpha_{i-1}(x_{i-1}), \alpha_i(x), \alpha_{i+1}(x), \alpha_{i+2}(x_{i+2}), \dots, \alpha_{n-1}(x_{n-1}), [y_1, \dots, y_n]\Big]$$
$$- \sum_{k=1}^{n}\Big[\alpha_1(y_1), \dots, \alpha_{k-1}(y_{k-1}), [x_1, \dots, x_{i-1}, x, x, x_{i+2}, \dots, x_{n-1}, y_k],$$
$$\alpha_k(y_{k+1}), \dots, \alpha_{n-1}(y_n)\Big]$$
$$= \Big[\alpha_1(x_1), \dots, \alpha_{i-1}(x_{i-1}), \alpha_i(x), \alpha_{i+1}(x), \alpha_{i+2}(x_{i+2}), \dots, \alpha_{n-1}(x_{n-1}),$$
$$[y_1, \dots, y_n]\Big] = 0.$$

Now, if $y_i = y_{i+1} = y$, then

$$
\begin{aligned}
&H(x_1,\dots,x_{n-1},y_1,\dots,y_{i-1},y,y,y_{i+2},\dots,y_n)\\
&=\Big[\alpha_1(x_1),\dots,\alpha_{n-1}(x_{n-1}),[y_1,\dots,y_{i-1},y,y,y_{i+2},\dots,y_n]\Big]\\
&\quad-\sum_{k=1}^{i-1}\Big[\alpha_1(y_1),\dots,\alpha_{k-1}(y_{k-1}),[x_1,\dots,x_{n-1},y_k],\\
&\qquad\qquad\qquad\qquad\alpha_k(y_{k+1}),\dots,\alpha_{i-1}(y),\alpha_i(y),\dots,\alpha_{n-1}(y_n)\Big]\\
&\quad-\Big[\alpha_1(y_1),\dots,\alpha_{i-1}(y_{i-1}),[x_1,\dots,x_{n-1},y],\alpha_i(y),\dots,\alpha_{n-1}(y_n)\Big]\\
&\quad-\Big[\alpha_1(y_1),\dots,\alpha_i(y),[x_1,\dots,x_{n-1},y],\alpha_{i+1}(y_{i+2}),\dots,\alpha_{n-1}(y_n)\Big]\\
&\quad-\sum_{k=i+2}^{n}\Big[\alpha_1(y_1),\dots,\alpha_i(y),\alpha_{i+1}(y),\dots,\alpha_{k-1}(y_{k-1}),[x_1,\dots,x_{n-1},y_k],\\
&\qquad\qquad\qquad\qquad\alpha_k(y_{k+1}),\dots,\alpha_{n-1}(y_n)\Big]\\
&=-\sum_{k=1}^{i-1}\Big[\alpha_1(y_1),\dots,\alpha_{k-1}(y_{k-1}),[x_1,\dots,x_{n-1},y_k],\\
&\qquad\qquad\qquad\qquad\alpha_k(y_{k+1}),\dots,\alpha_{i-1}(y),\alpha_i(y),\dots,\alpha_{n-1}(y_n)\Big]\\
&\quad-\Big[\alpha_1(y_1),\dots,\alpha_{i-1}(y_{i-1}),[x_1,\dots,x_{n-1},y],\alpha_i(y),\dots,\alpha_{n-1}(y_n)\Big]\\
&\quad+\Big[\alpha_1(y_1),\dots,[x_1,\dots,x_{n-1},y],\alpha_i(y),\alpha_{i+1}(y_{i+2}),\dots,\alpha_{n-1}(y_n)\Big]\\
&\quad-\sum_{k=i+2}^{n}\Big[\alpha_1(y_1),\dots,\alpha_i(y),\alpha_{i+1}(y),\dots,\alpha_{k-1}(y_{k-1}),[x_1,\dots,x_{n-1},y_k],\\
&\qquad\qquad\qquad\qquad\alpha_k(y_{k+1}),\dots,\alpha_{n-1}(y_n)\Big]\\
&=0.\quad\text{(applying the hypothesis)}
\end{aligned}
$$

Therefore, the map H defined above is skew-symmetric in its first $(n-1)$ arguments and in its last n arguments. □

Proposition 19.1 *Let A be an n-dimensional vector space, and let $(e_i)_{1\le i\le n}$ be a basis of A. Any skew-symmetric n-linear map $[\cdot,\dots,\cdot]$ on A is fully defined by*

$$[e_1,\dots,e_n]=d\in A.$$

Let $\alpha_1,\dots,\alpha_{n-1}$ be linear maps on A. If the $(n-1)$-linear map

$$(x_1,\dots,x_{n-1})\mapsto\big[\alpha_1(x_1),\dots,\alpha_{n-1}(x_{n-1}),d\big]$$

is skew-symmetric, then $(A,[\cdot,\dots,\cdot],\alpha_1,\dots,\alpha_{n-1})$ is an n-Hom-Lie algebra.

Proof By Lemma 19.1, it is sufficient to prove the Hom-Nambu-Filippov identity for $n-1$ pairwise different basis elements in place of the x_i and n pairwise different basis elements in place of the y_j in identity (19.4). Since $\dim A = n$, we get

$$\begin{aligned}
&\Big[\alpha_1(e_1), \ldots, \alpha_{i-1}(e_{i-1}), \alpha_i(e_{i+1}), \ldots, \alpha_{n-1}(e_n), [e_1, \ldots, e_n]\Big] \\
&\quad - \sum_{j=1}^{n} \Big[\alpha_1(e_1), \ldots, \alpha_{j-1}(e_{j-1}), [e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n, e_j], \\
&\qquad\qquad \alpha_j(e_{j+1}), \ldots, \alpha_{n-1}(e_n)\Big] \\
&= \Big[\alpha_1(e_1), \ldots, \alpha_{i-1}(e_{i-1}), \alpha_i(e_{i+1}), \ldots, \alpha_{n-1}(e_n), [e_1, \ldots, e_n]\Big] \\
&\quad - \sum_{j=1, j\neq i}^{n} \Big[\alpha_1(e_1), \ldots, \alpha_{j-1}(e_{j-1}), [e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n, e_j], \\
&\qquad\qquad \alpha_j(e_{j+1}), \ldots, \alpha_{n-1}(e_n)\Big] \\
&\quad - \Big[\alpha_1(e_1), \ldots, \alpha_{i-1}(e_{i-1}), [e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n, e_i], \\
&\qquad\qquad \alpha_i(e_{i+1}), \ldots, \alpha_{n-1}(e_n)\Big] \\
&= \Big[\alpha_1(e_1), \ldots, \alpha_{i-1}(e_{i-1}), \alpha_i(e_{i+1}), \ldots, \alpha_{n-1}(e_n), [e_1, \ldots, e_n]\Big] \\
&\quad - (-1)^{n-i}\Big[\alpha_1(e_1), \ldots, \alpha_{i-1}(e_{i-1}), \alpha_i(e_{i+1}), \ldots, \alpha_{n-1}(e_n), \\
&\qquad\qquad (-1)^{n-i}[e_1, \ldots, e_n]\Big] = 0.
\end{aligned}$$

Thus, $(A, [\cdot, \ldots, \cdot], \alpha_1, \ldots, \alpha_{n-1})$ is an n-Hom-Lie algebra. □

Corollary 19.1 *Let A be an n-dimensional vector space, and $(e_i)_{1\leq i\leq n}$ a basis of A. Any skew-symmetric n-linear map $[\cdot, \ldots, \cdot]$ on A is fully defined by giving*

$$[e_1, \ldots, e_n] = d \in A.$$

For any linear map α on A, $(A, [\cdot, \ldots, \cdot], \alpha)$ is an n-Hom-Lie algebra.

Hom-algebras morphisms are linear maps preserving both the multiplication and the structure maps.

Definition 19.5 ([13, 86]) n-Hom-Lie algebra morphisms of n-Hom-Lie algebras

$$\mathcal{A} = (A, [\cdot, \ldots, \cdot]_{\mathcal{A}}, \{\alpha_i\}_{1\leq i\leq n-1}) \text{ and } \mathcal{B} = (B, [\cdot, \ldots, \cdot]_{\mathcal{B}}, \{\beta_i\}_{1\leq i\leq n-1})$$

are linear maps $f : A \to B$ satisfying, for all $x_1, \ldots, x_n \in A$,

$$f([x_1, \ldots, x_n]_{\mathcal{A}}) = [f(x_1), \ldots, f(x_n)]_{\mathcal{B}}, \tag{19.5}$$

$$f \circ \alpha_i = \beta_i \circ f, \quad \text{for all} \quad 1 \leq i \leq n-1. \tag{19.6}$$

Linear maps satisfying only condition (19.5) are called weak morphisms of n-Hom-Lie algebras.

The n-Hom-Lie algebras $(A, [\cdot, \ldots, \cdot], \{\alpha_i\}_{1\leq i\leq n-1})$ with $\alpha_1 = \cdots = \alpha_{n-1} = \alpha$ will be denoted by $(A, [\cdot, \ldots, \cdot], \alpha)$.

Definition 19.6 ([86]) An n-Hom-Lie algebra $(A, [\cdot, \ldots, \cdot], \alpha)$ is called multiplicative if α is an algebra morphism, and regular if α is an algebra isomorphism.

The following proposition, providing a way to construct an n-Hom-Lie algebra from an n-Lie algebra and an algebra morphism, was first introduced in the case of Lie algebras and then generalized to the n-ary case in [13]. A more general version of this theorem, given in [86], states that the category of n-Hom-Lie algebras is closed under twisting by weak morphisms.

Proposition 19.2 ([13, 86]) *Let $\beta : \mathcal{A} \to \mathcal{A}$ be a weak morphism of n-Hom-Lie algebra $\mathcal{A} = \big(A, [\cdot, \ldots, \cdot], \{\alpha_i\}_{1\leq i\leq n-1}\big)$, and multiplication $[\cdot, \ldots, \cdot]_\beta$ is given by $[x_1, \ldots, x_n]_\beta = \beta([x_1, \ldots, x_n])$. Then, $\big(A, [\cdot, \ldots, \cdot]_\beta, \{\beta \circ \alpha_i\}_{1\leq i\leq n-1}\big)$ is an n-Hom-Lie algebra. Moreover, if $(A, [\cdot, \ldots, \cdot], \alpha)$ is multiplicative and $\beta \circ \alpha = \alpha \circ \beta$, then $\big(A, [\cdot, \ldots, \cdot]_\beta, \beta \circ \alpha\big)$ is multiplicative.*

The following particular case of Proposition 19.2 is obtained if $\alpha = \mathbf{Id}_A$.

Corollary 19.2 *Let $(A, [\cdot, \ldots, \cdot])$ be an n-Lie algebra, $\beta : A \to A$ an algebra morphism, and $[\cdot, \ldots, \cdot]_\beta$ is defined by $[x_1, \ldots, x_n]_\beta = \beta([x_1, \ldots, x_n])$. Then, $\big(A, [\cdot, \ldots, \cdot]_\beta, \beta\big)$ is a multiplicative n-Hom-Lie algebra.*

Fundamental objects and basic algebra were first introduced for n-Lie algebras in [38] and generalized to n-Hom-Lie algebras in [9]. They allow to define actions and representations of these n-ary algebras.

Definition 19.7 ([9]) Let $(A, [\cdot, \ldots, \cdot], \alpha)$ be a multiplicative n-Hom-Lie algebra and let $L(A) = \wedge^{n-1}A$ be the $(n-1)$th exterior power of A. The elements of $L(A)$ are called fundamental objects.

For $X = x_1 \wedge \cdots \wedge x_{n-1}$, $Y = y_1 \wedge \cdots \wedge y_{n-1} \in L(A)$, we define:

- The map $\bar{\alpha} : \wedge^{n-1}A \to \wedge^{n-1}A$ by $\bar{\alpha}(X) = \alpha(x_1) \wedge \cdots \wedge \alpha(x_{n-1})$.
- The action of fundamental objects on A by:

$$\forall\, z \in A,\; X \cdot z = ad_X(z) = \big[x_1, \ldots, x_{n-1}, z\big].$$

- The multiplication (composition) of two fundamental objects by:

$$[X, Y]_\alpha = X \cdot_\alpha Y = \sum_{i=1}^{n-1} \alpha(y_1) \wedge \cdots \wedge X \cdot y_i \wedge \cdots \wedge \alpha(y_{n-1}).$$

We extend the preceding definitions to the entire space $L(A)$ by linearity.

Proposition 19.3 ([9]) *The space $L(A)$ equipped with the product $[\cdot,\cdot]_\alpha$ defined above is a Hom-Leibniz algebra. That is the product $[\cdot,\cdot]_\alpha$ satisfies the following identity:*

$$[\bar{\alpha}(X),[Y,Z]_\alpha]_\alpha = [[X,Y]_\alpha,\bar{\alpha}(Z)]_\alpha + [\bar{\alpha}(Y),[X,Z]_\alpha]_\alpha.$$

Definition 19.8 ([29, 70, 86]) An n-Hom-Lie subalgebra $\mathcal{B} = (B,[\cdot,\ldots,\cdot]_{\mathcal{B}}, \{\beta_i\}_{1\le i\le n-1})$ of an n-Hom-Lie algebra $\mathcal{A} = (A,[\cdot,\ldots,\cdot]_{\mathcal{A}},\{\alpha_i\}_{1\le i\le n-1})$ consists of a subspace B of A satisfying, for all $x_1,\ldots,x_n \in B$,

1) $\alpha_i(B) \subseteq B$ for all $1 \le i \le n-1$,
2) $[x_1,\ldots,x_n]_{\mathcal{A}} \in B$,

with the restricted from A multiplication $[\cdot,\ldots,\cdot]_{\mathcal{B}} = [\cdot,\ldots,\cdot]_{\mathcal{A}}$ and the twisting maps $\beta_i = \alpha_i$, $1 \le i \le n-1$ on B.

Definition 19.9 ([29, 70, 86]) Let $\mathcal{A} = (A,[\cdot,\ldots,\cdot],\{\alpha_i\}_{1\le i\le n-1})$ be any n-Hom-Lie algebra. An ideal of $\mathcal{A}$ is a subspace I of A obeying for all $x_1,\ldots,x_{n-1} \in A$, $y \in I$,

1) $\alpha_i(I) \subseteq I$ for all $1 \le i \le n-1$;
2) $[x_1,\ldots,x_{n-1},y] \in I$.

Definition 19.10 ([58]) Let $(A,[\cdot,\ldots,\cdot],\{\alpha_i\}_{1\le i\le n-1})$ be an n-Hom-Lie algebra, and let I be an ideal of A. For $2 \le k \le n$, the k-derived series of the ideal I is defined by

$$D_k^0(I) = I \text{ and } D_k^{p+1} = \Big[\underbrace{D_k^p(I),\ldots,D_k^p(I)}_{k},\underbrace{A,\ldots,A}_{n-k}\Big],$$

and the k-central descending series of I by

$$C_k^0(I) = I \text{ and } C_k^{p+1}(I) = \Big[C_k^p(I),\underbrace{I,\ldots,I}_{k-1},\underbrace{A,\ldots,A}_{n-k}\Big].$$

Definition 19.11 ([58]) Let $\mathcal{A} = (A,[\cdot,\ldots,\cdot],\{\alpha_i\}_{1\le i\le n-1})$ be an n-Hom-Lie algebra, and let I be an ideal of $\mathcal{A}$. For $2 \le k \le n$, the ideal I is said to be k-solvable (resp. k-nilpotent) if there exists $r \in \mathbb{N}$ such that $D_k^r(I) = \{0\}$ (resp. $C_k^r(I) = \{0\}$). In this case, the smallest $r \in \mathbb{N}$ obeying this condition is called the class of k-solvability (resp. the class of k-nilpotency) of I.

Lemma 19.2 ([58]) *For n-Hom-Lie algebras*

$$\mathcal{A} = (A,[\cdot,\ldots,\cdot]_A,\{\alpha_i\}_{1\le i\le n-1}) \text{ and } \mathcal{B} = (B,[\cdot,\ldots,\cdot]_B,\{\beta_i\}_{1\le i\le n-1}),$$

let $f : \mathcal{A} \to \mathcal{B}$ be a surjective n-Hom-Lie algebras morphism and I an ideal of $\mathcal{A}$. Then for all $r \in \mathbb{N}$ and $2 \le k \le n$:

$$f\left(D_k^r(I)\right) = D_k^r\left(f(I)\right) \text{ and } f\left(C_k^r(I)\right) = C_k^r\left(f(I)\right).$$

This lemma also implies that if two n-Hom-Lie algebras are isomorphic, then they would also have isomorphic members of the derived series and central descending series, which also means that if two algebras have a significant difference in the derived series or the central descending series, for example different dimensions of given corresponding members, then these algebras cannot be isomorphic.

19.3 Properties of (n+1)-Dimensional N-Hom-Lie Algebras

All n-ary skew-symmetric algebras of dimension less than n are abelian and thus satisfy the Hom-Nambu-Filippov identity for any set of twisting maps. Also, there is only one non-abelian n-dimensional n-ary skew-symmetric algebra, up to isomorphism, and it satisfies the Hom-Nambu-Filippov identity for any twisting map (See [47] and Corollary 19.1).

In all the following, we use the principle that an n-linear skew-symmetric multiplication satisfies the Hom-Nambu-Filippov identity if and only if the multiplication satisfies it on any given basis of the underlying vector space. Moreover, as the multiplication is skew-symmetric, it is sufficient to check it for $n-1$ pairwise different basis elements in place of $x_1, \dots, x_{n-1}$ and n pairwise different basis elements in place of the $y_1, \dots, y_n$ in identity (19.4), where the order has no importance, that is, sequences of the form $e_1, \dots, e_{j-1}, e_{j+1}, \dots, e_{k-1}, e_{k+1}, \dots, e_{n+1}$ in place of $x_1, \dots, x_{n-1}$ and $e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_{n+1}$ in place of $y_1, \dots, y_n$. We shall denote by $H_{i,j,k}$ the left-hand side minus the right-hand side of the Hom-Nambu-Filippov identity for this sequence of vectors of the considered basis.

Proposition 19.4 *The Hom-Nambu-Filippov identity* (19.4) *is satisfied if and only if it is satisfied on sequences of the form*

$$e_1, \dots, e_{j-1}, e_{j+1}, \dots, e_{k-1}, e_{k+1}, \dots, e_{n+1}, e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_{n+1}, \ (i < j < k).$$

Proof When $i = j$ or $i = k$, the Hom-Nambu-Filippov identity is satisfied (the calculations are analogous to the case of n-dimensional n-Hom-Lie algebras, see Corollary 19.1), and if i, j, k are in a different order, we get the same identity as for $i < j < k$ up to a potential -1 factor.

If $i = j$, we get

$$H_{j,j,k} = \Big[\alpha(e_1), \dots, \widehat{\alpha(e_j)}, \dots, \widehat{\alpha(e_k)}, \dots, \alpha(e_{n+1}), \big[e_1, \dots, \widehat{e_j}, \dots, e_{n+1}\big]\Big]$$
$$- \sum_{p=1;\, p\neq j}^{n+1} [\alpha(e_1), \dots, \widehat{\alpha(e_i)}, \dots, \big[e_1, \dots, \widehat{e_j}, \dots, \widehat{e_k}, \dots, e_{n+1}, e_p\big], \dots, \alpha(e_{n+1})]$$
$$= [\alpha(e_1), \dots, \widehat{\alpha(e_j)}, \dots, \widehat{\alpha(e_k)}, \dots, \alpha(e_{n+1}), \big[e_1, \dots, \widehat{e_j}, \dots, e_{n+1}\big]]$$
$$- \Big[\alpha(e_1), \dots, \widehat{\alpha(e_j)}, \dots, \alpha(e_{k-1}), [e_1, \dots, \widehat{e_j}, \dots, \widehat{e_k}, \dots, e_{n+1}, e_k], \alpha(e_{k+1}), \dots, \alpha(e_{n+1})\Big]$$
$$= [\alpha(e_1), \dots, \widehat{\alpha(e_j)}, \dots, \widehat{\alpha(e_k)}, \dots, \alpha(e_{n+1}), \big[e_1, \dots, \widehat{e_j}, \dots, e_{n+1}\big]]$$
$$- \Big[\alpha(e_1), \dots, \widehat{\alpha(e_j)}, \dots, \alpha(e_{k-1}), (-1)^{n-k+1}[e_1, \dots, \widehat{e_j}, \dots, e_{n+1}], \alpha(e_{k+1}), \dots, \alpha(e_{n+1})\Big]$$
$$= [\alpha(e_1), \dots, \widehat{\alpha(e_j)}, \dots, \widehat{\alpha(e_k)}, \dots, \alpha(e_{n+1}), \big[e_1, \dots, \widehat{e_j}, \dots, e_{n+1}\big]]$$
$$- (-1)^{k-1-n}\Big[\alpha(e_1), \dots, \widehat{\alpha(e_j)}, \dots, \alpha(e_{k-1}), \alpha(e_{k+1}), \dots, \alpha(e_{n+1}), (-1)^{n-k+1}[e_1, \dots, \widehat{e_j}, \dots, e_{n+1}]\Big]$$
$$= \alpha(e_1), \dots, \widehat{\alpha(e_j)}, \dots, \widehat{\alpha(e_k)}, \dots, \alpha(e_{n+1}), \big[e_1, \dots, \widehat{e_j}, \dots, e_{n+1}\big]$$
$$- \Big[\alpha(e_1), \dots, \widehat{\alpha(e_j)}, \dots, \alpha(e_{k-1}), \alpha(e_{k+1}), \dots, \alpha(e_{n+1}), \big[e_1, \dots, \widehat{e_j}, \dots, e_{n+1}\big]\Big] = 0.$$

If $i = k$, we have

$$H_{k,j,k} = \Big[\alpha(e_1), \dots, \widehat{\alpha(e_j)}, \dots, \widehat{\alpha(e_k)}, \dots, \alpha(e_{n+1}), \big[e_1, \dots, \widehat{e_k}, \dots, e_{n+1}\big]\Big]$$
$$- \sum_{p=1;\, p\neq k}^{n+1} \Big[\alpha(e_1), \dots, \widehat{\alpha(e_k)}, \dots, \big[e_1, \dots, \widehat{e_j}, \dots, \widehat{e_k}, \dots, e_{n+1}, e_p\big], \dots, \alpha(e_{n+1})\Big]$$
$$= \Big[\alpha(e_1), \dots, \widehat{\alpha(e_j)}, \dots, \widehat{\alpha(e_k)}, \dots, \alpha(e_{n+1}), \big[e_1, \dots, \widehat{e_k}, \dots, e_{n+1}\big]\Big]$$
$$- \Big[\alpha(e_1), \dots, \alpha(e_{j-1}), \big[e_1, \dots, \widehat{e_j}, \dots, \widehat{e_k}, \dots, e_{n+1}, e_j\big], \alpha(e_{j+1}), \dots, \widehat{\alpha(e_k)}, \dots, \alpha(e_{n+1})\Big]$$
$$= \Big[\alpha(e_1), \dots, \widehat{\alpha(e_j)}, \dots, \widehat{\alpha(e_k)}, \dots, \alpha(e_{n+1}), \big[e_1, \dots, \widehat{e_k}, \dots, e_{n+1}\big]\Big]$$
$$- \Big[\alpha(e_1), \dots, \alpha(e_{j-1}), (-1)^{n-j}\big[e_1, \dots, \widehat{e_k}, \dots, e_{n+1}\big], \alpha(e_{j+1}), \dots, \widehat{\alpha(e_k)}, \dots, \alpha(e_{n+1})\Big]$$
$$= \Big[\alpha(e_1), \dots, \widehat{\alpha(e_j)}, \dots, \widehat{\alpha(e_k)}, \dots, \alpha(e_{n+1}), \big[e_1, \dots, \widehat{e_k}, \dots, e_{n+1}\big]\Big]$$
$$- (-1)^{j-n}\Big[\alpha(e_1), \dots, \alpha(e_{j-1}), \alpha(e_{j+1}), \dots, \widehat{\alpha(e_k)}, \dots, \alpha(e_{n+1}), (-1)^{n-j}\big[e_1, \dots, \widehat{e_k}, \dots, e_{n+1}\big]\Big] = 0,$$

If i, j, k are in a different order, then

$$
\begin{aligned}
H_{j,i,k} &= \Big[\alpha(e_1),\ldots,\widehat{\alpha(e_i)},\ldots,\widehat{\alpha(e_k)},\ldots,\alpha(e_{n+1}),\big[e_1,\ldots,\widehat{e_j},\ldots,e_{n+1}\big]\Big] \\
&\quad - \sum_{p=1;\,p\neq j}^{n+1} \Big[\alpha(e_1),\ldots,\widehat{\alpha(e_j)},\ldots,\big[e_1,\ldots,\widehat{e_i},\ldots,\widehat{e_k},\ldots,e_{n+1},e_p\big],\ldots,\alpha(e_{n+1})\Big] \\
&= \Big[\alpha(e_1),\ldots,\widehat{\alpha(e_i)},\ldots,\widehat{\alpha(e_k)},\ldots,\alpha(e_{n+1}),\big[e_1,\ldots,\widehat{e_j},\ldots,e_{n+1}\big]\Big] \\
&\quad - \Big[\alpha(e_1),\ldots,\alpha(e_{i-1}),\big[e_1,\ldots,\widehat{e_i},\ldots,\widehat{e_k},\ldots,e_{n+1},e_i\big], \\
&\qquad\qquad \alpha(e_{i+1}),\ldots,\widehat{\alpha(e_j)},\ldots,\alpha(e_{n+1})\Big] \\
&\quad - \Big[\alpha(e_1),\ldots,\widehat{\alpha(e_j)},\ldots,\alpha(e_{k-1}),\big[e_1,\ldots,\widehat{e_i},\ldots,\widehat{e_k},\ldots,e_{n+1},e_k\big], \\
&\qquad\qquad \alpha(e_{k+1}),\ldots,\alpha(e_{n+1})\Big] \\
&= (-1)^{n-k+1}\Big[\alpha(e_1),\ldots,\widehat{\alpha(e_i)},\ldots,\alpha(e_{k-1}),\big[e_1,\ldots,\widehat{e_j},\ldots,e_{n+1}\big], \\
&\qquad\qquad \alpha(e_{k+1}),\ldots,\alpha(e_{n+1})\Big] \\
&\quad - (-1)^{j-1-i}\Big[\alpha(e_1),\ldots,\widehat{\alpha(e_i)},\ldots,\alpha(e_{j-1}),(-1)^{n-i}\big[e_1,\ldots,\widehat{e_k},\ldots,e_{n+1}\big], \\
&\qquad\qquad \alpha(e_{j+1}),\ldots,\alpha(e_{n+1})\Big] \\
&\quad - (-1)^{k-1-n}\Big[\alpha(e_1),\ldots,\widehat{\alpha(e_j)},\ldots,\alpha(e_{k-1}),\alpha(e_{k+1}),\ldots,\alpha(e_{n+1}), \\
&\qquad\qquad (-1)^{n-k+1}\big[e_1,\ldots,\widehat{e_i},\ldots,e_{n+1}\big]\Big] \\
&= (-1)^{n-k+1}\Big[\alpha(e_1),\ldots,\widehat{\alpha(e_i)},\ldots,\alpha(e_{k-1}), \\
&\qquad (-1)^{k-1-n}\big[e_1,\ldots,\widehat{e_j},\ldots,\widehat{e_k},\ldots,e_{n+1},e_k\big],\alpha(e_{k+1}),\ldots,\alpha(e_{n+1})\Big] \\
&\quad - (-1)^{j-1-i}\Big[\alpha(e_1),\ldots,\widehat{\alpha(e_i)},\ldots,\alpha(e_{j-1}), \\
&\qquad\qquad (-1)^{n-i}(-1)^{n-j}\big[e_1,\ldots,\widehat{e_j},\ldots,\widehat{e_k},\ldots,e_{n+1},e_j\big], \\
&\qquad\qquad \alpha(e_{j+1}),\ldots,\alpha(e_{n+1})\Big] \\
&\quad - (-1)^{k-1-n}\Big[\alpha(e_1),\ldots,\widehat{\alpha(e_j)},\ldots,\widehat{\alpha(e_k)},\ldots,\alpha(e_{n+1}), \\
&\qquad\qquad (-1)^{n-k+1}\big[e_1,\ldots,\widehat{e_i},\ldots,e_{n+1}\big]\Big] \\
&= -\Big[\alpha(e_1),\ldots,\widehat{\alpha(e_j)},\ldots,\widehat{\alpha(e_k)},\ldots,\alpha(e_{n+1}),\big[e_1,\ldots,\widehat{e_i},\ldots,e_{n+1}\big]\Big] \\
&\quad + \Big[\alpha(e_1),\ldots,\widehat{\alpha(e_i)},\ldots,\alpha(e_{j-1}),\big[e_1,\ldots,\widehat{e_j},\ldots,\widehat{e_k},\ldots,e_{n+1},e_j\big], \\
&\qquad\qquad \alpha(e_{j+1}),\ldots,\alpha(e_{n+1})\Big] \\
&\quad + \Big[\alpha(e_1),\ldots,\widehat{\alpha(e_i)},\ldots,\alpha(e_{k-1}),\big[e_1,\ldots,\widehat{e_j},\ldots,\widehat{e_k},\ldots,e_{n+1},e_k\big], \\
&\qquad\qquad \alpha(e_{k+1}),\ldots,\alpha(e_{n+1})\Big] \\
&= -\Big(\Big[\alpha(e_1),\ldots,\widehat{\alpha(e_j)},\ldots,\widehat{\alpha(e_k)},\ldots,\alpha(e_{n+1}),\big[e_1,\ldots,\widehat{e_i},\ldots,e_{n+1}\big]\Big] \\
&\quad - \Big[\alpha(e_1),\ldots,\widehat{\alpha(e_i)},\ldots,\alpha(e_{j-1}),\big[e_1,\ldots,\widehat{e_j},\ldots,\widehat{e_k},\ldots,e_{n+1},e_j\big], \\
&\qquad\qquad \alpha(e_{j+1}),\ldots,\alpha(e_{n+1})\Big] \\
&\quad - \Big[\alpha(e_1),\ldots,\widehat{\alpha(e_i)},\ldots,\alpha(e_{k-1}),\big[e_1,\ldots,\widehat{e_j},\ldots,\widehat{e_k},\ldots,e_{n+1},e_k\big], \\
&\qquad\qquad \alpha(e_{k+1}),\ldots,\alpha(e_{n+1})\Big]\Big) = -H_{i,j,k}.
\end{aligned}
$$

Therefore, $H_{j,i,k} = 0$ if and only if $H_{i,j,k} = 0$. Similarly,

$$
\begin{aligned}
H_{k,i,j} &= \Big[\alpha(e_1), \ldots, \widehat{\alpha(e_i)}, \ldots, \widehat{\alpha(e_j)}, \ldots, \alpha(e_{n+1}), \big[e_1, \ldots, \widehat{e_k}, \ldots, e_{n+1}\big]\Big] \\
&\quad - \sum_{p=1;\, p\neq k}^{n+1} \Big[\alpha(e_1), \ldots, \widehat{\alpha(e_k)}, \ldots, \big[e_1, \ldots, \widehat{e_i}, \ldots, \widehat{e_j}, \ldots, e_{n+1}, e_p\big], \ldots, \alpha(e_{n+1})\Big] \\
&= \Big[\alpha(e_1), \ldots, \widehat{\alpha(e_i)}, \ldots, \widehat{\alpha(e_j)}, \ldots, \alpha(e_{n+1}), \big[e_1, \ldots, \widehat{e_k}, \ldots, e_{n+1}\big]\Big] \\
&\quad - \Big[\alpha(e_1), \ldots, \alpha(e_{i-1}), \big[e_1, \ldots, \widehat{e_i}, \ldots, \widehat{e_j}, \ldots, e_{n+1}, e_i\big], \\
&\qquad\qquad \alpha(e_{i+1}), \ldots, \widehat{\alpha(e_k)}, \ldots, \alpha(e_{n+1})\Big] \\
&\quad - \Big[\alpha(e_1), \ldots, \alpha(e_{j-1}), \big[e_1, \ldots, \widehat{e_i}, \ldots, \widehat{e_j}, \ldots, e_{n+1}, e_j\big], \\
&\qquad\qquad \alpha(e_{j+1}), \ldots, \widehat{\alpha(e_k)}, \ldots, \alpha(e_{n+1})\Big] \\
&= (-1)^{n-j+1}\Big[\alpha(e_1), \ldots, \widehat{\alpha(e_i)}, \ldots, \alpha(e_{j-1}), \big[e_1, \ldots, \widehat{e_k}, \ldots, e_{n+1}\big], \\
&\qquad\qquad \alpha(e_{j+1}), \ldots, \alpha(e_{n+1})\Big] \\
&\quad - \Big[\alpha(e_1), \ldots, \alpha(e_{i-1}), (-1)^{n-i}\big[e_1, \ldots, \widehat{e_j}, \ldots, e_{n+1}\big], \\
&\qquad\qquad \alpha(e_{i+1}), \ldots, \widehat{\alpha(e_k)}, \ldots, \alpha(e_{n+1})\Big] \\
&\quad - \Big[\alpha(e_1), \ldots, \alpha(e_{j-1}), (-1)^{n-j+1}\big[e_1, \ldots, \widehat{e_i}, \ldots, e_{n+1}\big], \\
&\qquad\qquad \alpha(e_{j+1}), \ldots, \widehat{\alpha(e_k)}, \ldots, \alpha(e_{n+1})\Big] \\
&= (-1)^{n-j+1}\Big[\alpha(e_1), \ldots, \widehat{\alpha(e_i)}, \ldots, \alpha(e_{j-1}), \\
&\qquad\qquad (-1)^{j-n}\big[e_1, \ldots, \widehat{e_j}, \ldots, \widehat{e_k}, \ldots, e_{n+1}, e_j\big], \alpha(e_{j+1}), \ldots, \alpha(e_{n+1})\Big] \\
&\quad - (-1)^{k-1-i}\Big[\alpha(e_1), \ldots, \widehat{\alpha(e_i)}, \ldots, \alpha(e_{k-1}), (-1)^{n-i}\big[e_1, \ldots, \widehat{e_j}, \ldots, e_{n+1}\big], \\
&\qquad\qquad \alpha(e_{k+1}), \ldots, \alpha(e_{n+1})\Big] \\
&\quad - (-1)^{j-1-n}\Big[\alpha(e_1), \ldots, \widehat{\alpha(e_j)}, \ldots, \widehat{\alpha(e_k)}, \ldots, \alpha(e_{n+1}), \\
&\qquad\qquad (-1)^{n-j}\big[e_1, \ldots, \widehat{e_i}, \ldots, e_{n+1}\big]\Big] \\
&= (-1)\Big[\alpha(e_1), \ldots, \widehat{\alpha(e_i)}, \ldots, \alpha(e_{j-1}), \big[e_1, \ldots, \widehat{e_j}, \ldots, \widehat{e_k}, \ldots, e_{n+1}, e_j\big], \\
&\qquad\qquad \alpha(e_{j+1}), \ldots, \alpha(e_{n+1})\Big] \\
&\quad - (-1)^{n+k-1}\Big[\alpha(e_1), \ldots, \widehat{\alpha(e_i)}, \ldots, \alpha(e_{k-1}), \big[e_1, \ldots, \widehat{e_j}, \ldots, e_{n+1}\big], \\
&\qquad\qquad \alpha(e_{k+1}), \ldots, \alpha(e_{n+1})\Big] \\
&\quad - (-1)\Big[\alpha(e_1), \ldots, \widehat{\alpha(e_j)}, \ldots, \widehat{\alpha(e_k)}, \ldots, \alpha(e_{n+1}), \big[e_1, \ldots, \widehat{e_i}, \ldots, e_{n+1}\big]\Big] \\
&= (-1)\Big[\alpha(e_1), \ldots, \widehat{\alpha(e_i)}, \ldots, \alpha(e_{j-1}), \big[e_1, \ldots, \widehat{e_j}, \ldots, \widehat{e_k}, \ldots, e_{n+1}, e_j\big], \\
&\qquad\qquad \alpha(e_{j+1}), \ldots, \alpha(e_{n+1})\Big] \\
&\quad - (-1)^{n+k-1}\Big[\alpha(e_1), \ldots, \widehat{\alpha(e_i)}, \ldots, \alpha(e_{k-1}), \\
&\qquad\qquad (-1)^{k-1-n}\big[e_1, \ldots, \widehat{e_j}, \ldots, \widehat{e_k}, \ldots, e_{n+1}\big], \alpha(e_{k+1}), \ldots, \alpha(e_{n+1})\Big]
\end{aligned}
$$

$$
\begin{aligned}
&- (-1)\Big[\alpha(e_1), \ldots, \widehat{\alpha(e_j)}, \ldots, \widehat{\alpha(e_k)}, \ldots, \alpha(e_{n+1}), \big[e_1, \ldots, \widehat{e_i}, \ldots, e_{n+1}\big]\Big] \\
&= -\Big[\alpha(e_1), \ldots, \widehat{\alpha(e_i)}, \ldots, \alpha(e_{j-1}), \big[e_1, \ldots, \widehat{e_j}, \ldots, \widehat{e_k}, \ldots, e_{n+1}, e_j\big], \\
&\qquad\qquad \alpha(e_{j+1}), \ldots, \alpha(e_{n+1})\Big] \\
&\quad - \Big[\alpha(e_1), \ldots, \widehat{\alpha(e_i)}, \ldots, \alpha(e_{k-1}), \big[e_1, \ldots, \widehat{e_j}, \ldots, \widehat{e_k}, \ldots, e_{n+1}\big], \\
&\qquad\qquad \alpha(e_{k+1}), \ldots, \alpha(e_{n+1})\Big] \\
&\quad + \Big[\alpha(e_1), \ldots, \widehat{\alpha(e_j)}, \ldots, \widehat{\alpha(e_k)}, \ldots, \alpha(e_{n+1}), \big[e_1, \ldots, \widehat{e_i}, \ldots, e_{n+1}\big]\Big] \\
&= \Big[\alpha(e_1), \ldots, \widehat{\alpha(e_j)}, \ldots, \widehat{\alpha(e_k)}, \ldots, \alpha(e_{n+1}), \big[e_1, \ldots, \widehat{e_i}, \ldots, e_{n+1}\big]\Big] \\
&\quad - \Big[\alpha(e_1), \ldots, \widehat{\alpha(e_i)}, \ldots, \alpha(e_{k-1}), \big[e_1, \ldots, \widehat{e_j}, \ldots, \widehat{e_k}, \ldots, e_{n+1}\big], \\
&\qquad\qquad \alpha(e_{k+1}), \ldots, \alpha(e_{n+1})\Big] \\
&\quad - \Big[\alpha(e_1), \ldots, \widehat{\alpha(e_i)}, \ldots, \alpha(e_{j-1}), \big[e_1, \ldots, \widehat{e_j}, \ldots, \widehat{e_k}, \ldots, e_{n+1}, e_j\big], \\
&\qquad\qquad \alpha(e_{j+1}), \ldots, \alpha(e_{n+1})\Big] = H_{i,j,k}.
\end{aligned}
$$

Therefore, $H_{k,i,j} = 0$ if and only if $H_{i,j,k} = 0$. □

Let $(A, [\cdot, \ldots, \cdot], \alpha)$ be an n-ary skew-symmetric algebra of dimension $n+1$ with a linear map α. Given a linear basis $(e_i)_{1\le i\le n+1}$ of A, linear map α is fully determined by its matrix determined by action of α on the basis, and a skew-symmetric n-ary multi-linear multiplication is fully determined by $\big[e_1, \ldots, \widehat{e_i}, \ldots, e_{n+1}\big]$ for all $1 \le i \le n+1$ represented by a matrix B as follows:

$$
\begin{aligned}
\big[e_1, \ldots, \widehat{e_i}, \ldots, e_{n+1}\big] &= (-1)^{n+1+i} w_i, \\
w_i &= \sum_{p=1}^{n+1} b(p,i) e_p, \\
(w_1, \ldots, w_{n+1}) &= (e_1, \ldots, e_{n+1})B, \text{ for } B = (b(i,j))_{1\le i,j\le n+1}.
\end{aligned}
$$

The following result gives a characterization of isomorphisms of $(n+1)$-dimensional n-ary skew-symmetric Hom-algebras of the considered form, it is a generalization of [47, Theorem 2]. The cited result corresponds to the case $\alpha = \mathbf{Id}_A$.

Proposition 19.5 *Let $\mathcal{A}_1 = (A, [\cdot, \ldots, \cdot]_1, \alpha_1)$ and $\mathcal{A}_2 = (A, [\cdot, \ldots, \cdot]_2, \alpha_2)$ be $(n+1)$-dimensional n-ary skew-symmetric Hom-algebras represented by matrices $[\alpha_1]$, B_1 and $[\alpha_2]$, B_2 respectively. The Hom-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic if and only if there exists an invertible matrix T satisfying the following conditions:*

$$B_2 = \det(T)^{-1} T B_1 T^T,$$

$$[\alpha_2] = T[\alpha_1]T^{-1}.$$

Proof The first condition is the characterization of the isomorphism of two $(n+1)$-dimensional n-ary skew-symmetric algebras proved by Filippov in [47]. The second condition expresses the compatibility with the twisting maps. □

Proposition 19.6 *Let $(e_i)_{1\le i\le n+1}$ be a basis of A, let σ be an $n+1$ permutation, and let $B=(b_{i,j})_{1\le i,j\le n+1}$ be a matrix representing a skew-symmetric n-ary multiplication in this basis, then the matrix representing the same multiplication in the basis $(e_{\sigma(i)})_{1\le i\le n+1}$ is given by $sgn(\sigma)(b_{\sigma^{-1}(i),\sigma^{-1}(j)})_{1\le i,j\le n+1}$.*

Proof Let T_σ be the basis change matrix from the basis $(e_i)_{1\le i\le n+1}$ to the basis $(e'_i)_{1\le i\le n+1}=(e_{\sigma(i)})_{1\le i\le n+1}$, that is $e'_i=e_{\sigma(i)}$. Then, $T_\sigma=(t_{i,j})_{1\le i,j\le n+1}$ with $t_{i,j}=1$ if $i=\sigma(j)$ and $t_{i,j}=0$ otherwise. Let $T_\sigma BT_\sigma^T=C=(c_{i,j})$. Then

$$
\begin{aligned}
c_{i,j}&=\sum_{p=1}^{n+1}\left(\sum_{q=1}^{n+1}t_{i,q}b_{q,p}\right)t_{j,p}, &&\text{(the second sum is the } (i,p) \text{ entry of } T_\sigma B, \text{ and } t_{j,p} \text{ is the } (p,j) \text{ entry of } T_\sigma^T)\\
&=\sum_{p=1}^{n+1}\left(t_{i,\sigma^{-1}(i)}b_{\sigma^{-1}(i),p}\right)t_{j,p} &&(t_{i,j}=0 \iff i\neq\sigma(j)\iff j\neq\sigma^{-1}(i).)\\
&=\left(t_{i,\sigma^{-1}(i)}b_{\sigma^{-1}(i),\sigma^{-1}(j)}\right)t_{j,\sigma^{-1}(j)}=b_{\sigma^{-1}(i),\sigma^{-1}(j)}. &&(t_{i,\sigma^{-1}(i)}=t_{j,\sigma^{-1}(j)}=1)
\end{aligned}
$$

We also have, by the skew-symmetry of the determinant, that

$$\det(T_\sigma)=sgn(\sigma)\det(I_{n+1})=sgn(\sigma).$$

Hence, $B'=\det(T_\sigma)^{-1}T_\sigma BT_\sigma^T=sgn(\sigma)C=sgn(\sigma)\left(b_{\sigma^{-1}(i),\sigma^{-1}(j)}\right)_{1\le i,j\le n+1}$. □

Proposition 19.7 *If* $\dim\ker\alpha\ge 3$ *then the Hom-Nambu-Filippov identity is always satisfied.*

Proof Let $(e_i)_{1\le i\le n+1}$ be a basis of $\ker\alpha$ completed to be a basis of A. Then, for all $1\le i<j<k\le n+1$,

$$
\begin{aligned}
H_{i,j,k}&=\left[\alpha(e_1),\dots,\widehat{\alpha(e_j)},\dots,\widehat{\alpha(e_k)},\dots,\alpha(e_{n+1}),\left[e_1,\dots,\widehat{e_i},\dots,e_{n+1}\right]\right]\\
&\quad-\sum_{p=1;\,p\neq i}^{n+1}\left[\alpha(e_1),\dots,\widehat{\alpha(e_i)},\dots,\left[e_1,\dots,\widehat{e_j},\dots,\widehat{e_k},\dots,e_{n+1},e_p\right],\dots,\alpha(e_{n+1})\right]\\
&=\left[\alpha(e_1),\dots,\widehat{\alpha(e_j)},\dots,\widehat{\alpha(e_k)},\dots,\alpha(e_{n+1}),\left[e_1,\dots,\widehat{e_i},\dots,e_{n+1}\right]\right]\\
&\quad-\left[\alpha(e_1),\dots,\widehat{\alpha(e_i)},\dots,\alpha(e_{j-1}),\left[e_1,\dots,\widehat{e_j},\dots,\widehat{e_k},\dots,e_{n+1},e_j\right],\right.\\
&\qquad\qquad\left.\alpha(e_{j+1}),\dots,\alpha(e_{n+1})\right]\\
&\quad-\left[\alpha(e_1),\dots,\widehat{\alpha(e_i)},\dots,\alpha(e_{k-1}),\left[e_1,\dots,\widehat{e_j},\dots,\widehat{e_k},\dots,e_{n+1},e_k\right],\right.\\
&\qquad\qquad\left.\alpha(e_{k+1}),\dots,\alpha(e_{n+1})\right]=0.
\end{aligned}
$$

Since at least three of the elements $\alpha(e_1), \ldots, \alpha(e_{n+1})$ are zero, and in each of the brackets in the sum above, we remove only two, which means that there is always a zero in one of the entries of the bracket multiplication. □

Remark 19.2 From this proof, one can see that if $\dim \ker \alpha = 1$, and we consider a basis (e_i) of A where $\ker \alpha = \langle e_1 \rangle$ then $H_{i,j,k} = 0$ when none of i, j, k is equal to 1. Also, if $\dim \ker \alpha = 2$, and we consider a basis (e_i) of A where $\ker \alpha = \langle e_1, e_{j_0} \rangle$, then $H_{i,j,k} = 0$ when none of i, j, k is equal to 1 or none of them is equal to j_0.

Proposition 19.8 *Let* $(A, [\cdot, \ldots, \cdot], \alpha)$ *be an* $(n+1)$*-dimensional n-Hom-Lie algebra and let B be the matrix representing its multiplication, if* $\det(B) = 0$ *then A is n-solvable.*

Proof Let $(A, [\cdot, \ldots, \cdot], \alpha)$ be an $(n+1)$-dimensional n-Hom-Lie algebra and let B be the matrix representing its multiplication. The space $D_n^1(A) = [A, \ldots, A]$ is generated by $\{w_1, \ldots, w_{n+1}\}$, which means that $Rank(B) = \dim D_n^1(A)$.

If $Rank(B) \leq n$ or equivalently $\det(B) = 0$ then $D_n^1(A)$ has dimension at most n, then by skew-symmetry we get that $D_n^2(A)$ has dimension at most 1 and then $D_n^3(A) = 0$. □

Remark 19.3 For the whole algebra A, all the k-central descending series, for all $2 \leq k \leq n$, are equal, therefore all the notions of k-nilpotency, for $2 \leq k \leq n$, are equivalent.

Lemma 19.3 *Let* $(A, [\cdot, \ldots, \cdot], \alpha)$ *be an n-Hom-Lie algebra. If A is k-nilpotent, for any* $2 \leq k \leq n$*, then the center* $Z(A)$ *of A is not trivial.*

Proof Suppose that A is k-nilpotent and let $p \in \mathbb{N}$ such that $C_k^p(A) = 0$ and $C_k^{p-1}(A) \neq 0$. Then $C_k^p(A) = \left[C_k^{p-1}(A), A, \ldots, A\right] = 0$, that is

$$\forall\, x_1, \ldots, x_{n-1} \in A, \forall c \in C_k^{p-1}(A) : \left[c, x_1, \ldots, x_{n-1}\right] = 0.$$

Thus, $C_k^{p-1} \subseteq Z(A)$, and $Z(A) \neq 0$ since $C_k^{p-1}(A) \neq 0$. □

Proposition 19.9 *Let* $(A, [\cdot, \ldots, \cdot], \alpha)$ *be an* $(n+1)$*-dimensional n-Hom-Lie algebra. The algebra A is nilpotent and non abelian if and only if* $\dim Z(A) = 1$ *and* $[A, \ldots, A] = Z(A)$.

Proof Suppose that A is nilpotent, we know, by Lemma 19.3, that $Z(A) \neq \{0\}$ since A is nilpotent. If $\dim Z(A) > 1$ then A is abelian (take a basis of $Z(A)$ and complete it to be a basis of A, then for all i, $\left[e_1, \ldots, \widehat{e_i}, \ldots, e_{n+1}\right] = 0$ because at least one of the basis elements in the bracket multiplication is in $Z(A)$). If $\dim Z(A) = 1$ then $\dim [A, \ldots, A] = 1$ (take a basis such that $e_{n+1} \in Z(A)$, then only $[e_1, \ldots, e_n] \neq 0$). Let $C_k^{p-1}(A)$ be the last non-zero term of the k-central descending series of A, we have $C_k^{p-1}(A) \subseteq [A, \ldots, A]$ and $\dim C_k^{p-1}(A) \geq 1 = \dim [A, \ldots, A]$, which means that

$C_k^{p-1}(A) = [A, \ldots, A]$. We also have that $C_k^{p-1}(A) \subseteq Z(A)$ and $\dim C_k^{p-1}(A) \geq 1 = \dim Z(A)$, we conclude then that $Z(A) = C_k^{p-1}(A) = [A, \ldots, A]$.

Conversely, if $\dim Z(A) = 1$ and $[A, \ldots, A] = Z(A)$, then

$$C^1(A) = [A, \ldots, A] = Z(A),$$
$$C^2(A) = [Z(A), A, \ldots, A] = 0.$$

Thus, A is nilpotent. □

Proposition 19.10 *For an $(n+1)$–dimensional n-ary skew-symmetric Hom-algebra $(A, [\cdot, \ldots, \cdot], \alpha)$, and a basis $\{e_i\}$ of A,*

$$\begin{aligned}
H_{i,j,k} &= (-1)^{n+i+1} \sum_{p=1}^{n+1} \det\left(\alpha(e_1), \ldots, \widehat{\alpha(e_j)}, \ldots, \widehat{\alpha(e_k)}, \ldots, \alpha(e_{n+1}), w_i\right)_p w_p \\
&- (-1)^{n+j}(-1)^{n+k+1} \sum_{p=1}^{n+1} \det\Big(\alpha(e_1), \ldots, \widehat{\alpha(e_i)}, \ldots, \alpha(e_{j-1}), w_k, \\
&\qquad \alpha(e_{j+1}), \ldots, \alpha(e_{n+1})\Big)_p w_p \\
&- (-1)^{n+k+1}(-1)^{n+j+1} \sum_{p=1}^{n+1} \det\Big(\alpha(e_1), \ldots, \widehat{\alpha(e_i)}, \ldots, \alpha(e_{k-1}), w_j, \\
&\qquad \alpha(e_{k+1}), \ldots, \alpha(e_{n+1})\Big)_p w_p,
\end{aligned}$$

where, in the determinant, the vectors are taken as columns in the considered basis, and the subscript p means that we remove the pth row.

Proof By Proposition 19.4, it is enough to consider of $H_{i,j,k}$ for $1 \leq i < j < k \leq n+1$:

$$\begin{aligned}
H_{i,j,k} &= \left[\alpha(e_1), \ldots, \widehat{\alpha(e_j)}, \ldots, \widehat{\alpha(e_k)}, \ldots, \alpha(e_{n+1}), [e_1, \ldots, \widehat{e_i}, \ldots, e_{n+1}]\right] \\
&\quad - \sum_{p=1; p\neq i}^{n+1} \left[\alpha(e_1), \ldots, \widehat{\alpha(e_i)}, \ldots, [e_1, \ldots, \widehat{e_j}, \ldots, \widehat{e_k}, \ldots, e_{n+1}, e_p], \ldots, \alpha(e_{n+1})\right] \\
&= \left[\alpha(e_1), \ldots, \widehat{\alpha(e_j)}, \ldots, \widehat{\alpha(e_k)}, \ldots, \alpha(e_{n+1}), [e_1, \ldots, \widehat{e_i}, \ldots, e_{n+1}]\right] \\
&\quad - \Big[\alpha(e_1), \ldots, \widehat{\alpha(e_i)}, \ldots, \alpha(e_{j-1}), [e_1, \ldots, \widehat{e_j}, \ldots, \widehat{e_k}, \ldots, e_{n+1}, e_j], \\
&\qquad \alpha(e_{j+1}), \ldots, \alpha(e_{n+1})\Big] \\
&\quad - \Big[\alpha(e_1), \ldots, \widehat{\alpha(e_i)}, \ldots, \alpha(e_{k-1}), [e_1, \ldots, \widehat{e_j}, \ldots, \widehat{e_k}, \ldots, e_{n+1}, e_k], \\
&\qquad \alpha(e_{k+1}), \ldots, \alpha(e_{n+1})\Big] \\
&= \left[\alpha(e_1), \ldots, \widehat{\alpha(e_j)}, \ldots, \widehat{\alpha(e_k)}, \ldots, \alpha(e_{n+1}), [e_1, \ldots, \widehat{e_i}, \ldots, e_{n+1}]\right]
\end{aligned}$$

$$
\begin{aligned}
&-\Big[\alpha(e_1),\dots,\widehat{\alpha(e_i)},\dots,\alpha(e_{j-1}),(-1)^{n+j}[e_1,\dots,\widehat{e_k},\dots,e_{n+1}],\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\alpha(e_{j+1}),\dots,\alpha(e_{n+1})\Big]\\
&-\Big[\alpha(e_1),\dots,\widehat{\alpha(e_i)},\dots,\alpha(e_{k-1}),(-1)^{n+k+1}[e_1,\dots,\widehat{e_j},\dots,e_{n+1}],\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\alpha(e_{k+1}),\dots,\alpha(e_{n+1})\Big]\\
=&\Big[\alpha(e_1),\dots,\widehat{\alpha(e_j)},\dots,\widehat{\alpha(e_k)},\dots,\alpha(e_{n+1}),(-1)^{n+i+1}w_i\Big]\\
&-\Big[\alpha(e_1),\dots,\widehat{\alpha(e_i)},\dots,\alpha(e_{j-1}),(-1)^{n+j}(-1)^{n+k+1}w_k,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\alpha(e_{j+1}),\dots,\alpha(e_{n+1})\Big]\\
&-\Big[\alpha(e_1),\dots,\widehat{\alpha(e_i)},\dots,\alpha(e_{k-1}),(-1)^{n+k+1}(-1)^{n+j+1}w_j,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\alpha(e_{k+1}),\dots,\alpha(e_{n+1})\Big]\\
=&(-1)^{n+i+1}\sum_{p=1}^{n+1}\det\Big(\alpha(e_1),\dots,\widehat{\alpha(e_j)},\dots,\widehat{\alpha(e_k)},\dots,\alpha(e_{n+1}),w_i\Big)_p w_p\\
&-(-1)^{n+j}(-1)^{n+k+1}\sum_{p=1}^{n+1}\det\Big(\alpha(e_1),\dots,\widehat{\alpha(e_i)},\dots,\alpha(e_{j-1}),w_k,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\alpha(e_{j+1}),\dots,\alpha(e_{n+1})\Big)_p w_p\\
&-(-1)^{n+k+1}(-1)^{n+j+1}\sum_{p=1}^{n+1}\det\Big(\alpha(e_1),\dots,\widehat{\alpha(e_i)},\dots,\alpha(e_{k-1}),w_j,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\alpha(e_{k+1}),\dots,\alpha(e_{n+1})\Big)_p w_p.
\end{aligned}
$$

□

Corollary 19.3 *Let $(A,[\cdot,\dots,\cdot],\alpha)$ be an $(n+1)$–dimensional n-ary skew-symmetric Hom-algebra. If there is a basis (e_i) such that, in this basis, the matrices $[\alpha]_{(e_i)}$ and $B_{(e_i)}$ representing α and $[\cdot,\dots,\cdot]$ respectively are proportional, then $(A,[\cdot,\dots,\cdot],\alpha)$ is an n-Hom-Lie algebra.*

Proof If in a given basis we have $[\alpha]_{(e_i)}=\lambda B_{(e_i)}$ for some $\lambda\in\mathbb{K}$, then in this basis, all the determinants in the formula given in Proposition 19.10 are zero, thus the Hom-Nambu-Filippov identity is satisfied. □

19.4 The Hom-Nambu-Filippov Identity

In this section, we study the Hom-Nambu-Filippov identity for $(n+1)$-dimensional skew-symmetric algebras for various cases of the twisting map α. In all the following, let A be an $(n+1)$-dimensional vector space, let $[\cdot,\dots,\cdot]$ be an n-linear skew-symmetric map on A and let α be a linear map on A. We consider a basis $(e_i)_{1\le i\le n+1}$

of A where α is in its Jordan form. The order of the various Jordan blocks can be chosen, the difference that arises from changing the order of Jordan blocks of α is given by Proposition 19.6. We consider the matrix B defining $[\cdot, \dots, \cdot]$ in this basis as in (19.7). Proposition 19.11 generalizes [47, Equation 17] obtained for $\alpha = \mathbf{Id}_A$.

We suppose that the matrix of α is diagonal in the basis $(e_i)_{1 \leq i \leq n+1}$ with eigenvalues λ_i, $1 \leq i \leq n+1$.

Proposition 19.11 *If α is invertible and diagonalizable, and λ_i, $1 \leq i \leq n+1$ are its eigenvalues, then $[\cdot, \dots, \cdot]$ satisfies the Hom-Nambu-Filippov identity if and only if*

$$\begin{aligned}&\forall\, 1 \leq i < j < k \leq n+1 :\\ &(\lambda_i b_{j,i} - \lambda_j b_{i,j}) w_k + (\lambda_k b_{i,k} - \lambda_i b_{k,i}) w_j + (\lambda_j b_{k,j} - \lambda_k b_{j,k}) w_i = 0, \end{aligned} \quad (19.7)$$

that is

$$\begin{cases} (\lambda_1 b_{2,1} - \lambda_2 b_{1,2}) w_3 + (\lambda_3 b_{1,3} - \lambda_1 b_{3,1}) w_2 + (\lambda_2 b_{3,2} - \lambda_3 b_{2,3}) w_1 = 0 \\ (\lambda_1 b_{2,1} - \lambda_2 b_{1,2}) w_4 + (\lambda_4 b_{1,4} - \lambda_1 b_{4,1}) w_2 + (\lambda_2 b_{4,2} - \lambda_4 b_{2,4}) w_1 = 0 \\ (\lambda_1 b_{3,1} - \lambda_3 b_{1,3}) w_4 + (\lambda_4 b_{1,4} - \lambda_1 b_{4,1}) w_3 + (\lambda_3 b_{4,3} - \lambda_4 b_{3,4}) w_1 = 0 \\ (\lambda_2 b_{3,4} - \lambda_3 b_{2,3}) w_4 + (\lambda_4 b_{2,4} - \lambda_2 b_{4,2}) w_3 + (\lambda_3 b_{4,3} - \lambda_4 b_{3,4}) w_2 = 0 \\ \vdots \\ (\lambda_i b_{j,i} - \lambda_j b_{i,j}) w_k + (\lambda_k b_{i,k} - \lambda_i b_{k,i}) w_j + (\lambda_j b_{k,j} - \lambda_k b_{j,k}) w_i = 0 \end{cases}$$

which is also equivalent to the following system, obtained by using the coordinates in the basis $(e_i)_{1 \leq i \leq n+1}$,

$$\begin{aligned}&\forall\, 1 \leq i, j, k, p \leq n+1, \quad i < j < k :\\ &(\lambda_i b_{j,i} - \lambda_j b_{i,j}) b_{p,k} + (\lambda_k b_{i,k} - \lambda_i b_{k,i}) b_{p,j} + (\lambda_j b_{k,j} - \lambda_k b_{j,k}) b_{p,i} = 0. \end{aligned} \quad (19.8)$$

Proof Let $(e_i)_{1 \leq i \leq n+1}$ be a basis of A such that $\alpha(e_i) = \lambda_i e_i$ for all $1 \leq i \leq n+1$. By Proposition 19.4, the Hom-Nambu-Filippov identity is satisfied if and only if

$$H_{i,j,k} = 0, \quad \forall\ 1 \leq i < j < k \leq n+1.$$

Computation of $H_{i,j,k}$ gives

$$\begin{aligned} H_{i,j,k} &= \Big[\alpha(e_1), \dots, \widehat{\alpha(e_j)}, \dots, \widehat{\alpha(e_k)}, \dots, \alpha(e_{n+1}), \big[e_1, \dots, \widehat{e_i}, \dots, e_{n+1}\big]\Big] \\ &\quad - \sum_{p=1;\, p \neq i}^{n+1} \Big[\alpha(e_1), \dots, \widehat{\alpha(e_i)}, \dots, \big[e_1, \dots, \widehat{e_j}, \dots, \widehat{e_k}, \dots, e_{n+1}, e_p\big], \dots, \alpha(e_{n+1})\Big] \\ &= \Big[\alpha(e_1), \dots, \widehat{\alpha(e_j)}, \dots, \widehat{\alpha(e_k)}, \dots, \alpha(e_{n+1}), \big[e_1, \dots, \widehat{e_i}, \dots, e_{n+1}\big]\Big] \\ &\quad - \Big[\alpha(e_1), \dots, \widehat{\alpha(e_i)}, \dots, \alpha(e_{j-1}), \big[e_1, \dots, \widehat{e_j}, \dots, \widehat{e_k}, \dots, e_{n+1}, e_j\big], \\ &\qquad\qquad \alpha(e_{j+1}), \dots, \alpha(e_{n+1})\Big] \end{aligned}$$

$$
\begin{aligned}
&- \Big[\alpha(e_1), \ldots, \widehat{\alpha(e_i)}, \ldots, \alpha(e_{k-1}), \big[e_1, \ldots, \widehat{e_j}, \ldots, \widehat{e_k}, \ldots, e_{n+1}, e_k\big], \\
&\qquad\qquad\qquad\qquad\qquad\qquad \alpha(e_{k+1}), \ldots, \alpha(e_{n+1})\Big] \\
&= \Big[\alpha(e_1), \ldots, \widehat{\alpha(e_j)}, \ldots, \widehat{\alpha(e_k)}, \ldots, \alpha(e_{n+1}), \big[e_1, \ldots, \widehat{e_i}, \ldots, e_{n+1}\big]\Big] \\
&- \Big[\alpha(e_1), \ldots, \widehat{\alpha(e_i)}, \ldots, \alpha(e_{j-1}), (-1)^{n-j}\big[e_1, \ldots, \widehat{e_k}, \ldots, e_{n+1}\big], \\
&\qquad\qquad\qquad\qquad\qquad\qquad \alpha(e_{j+1}), \ldots, \alpha(e_{n+1})\Big] \\
&- \Big[\alpha(e_1), \ldots, \widehat{\alpha(e_i)}, \ldots, \alpha(e_{k-1}), (-1)^{n-k+1}\big[e_1, \ldots, \widehat{e_j}, \ldots, e_{n+1}\big], \\
&\qquad\qquad\qquad\qquad\qquad\qquad \alpha(e_{k+1}), \ldots, \alpha(e_{n+1})\Big] \\
&= \Big[\alpha(e_1), \ldots, \widehat{\alpha(e_j)}, \ldots, \widehat{\alpha(e_k)}, \ldots, \alpha(e_{n+1}), (-1)^{n+i+1} w_i\Big] \\
&- \Big[\alpha(e_1), \ldots, \widehat{\alpha(e_i)}, \ldots, \alpha(e_{j-1}), (-1)^{n-j}(-1)^{n+k+1} w_k, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \alpha(e_{j+1}), \ldots, \alpha(e_{n+1})\Big] \\
&- \Big[\alpha(e_1), \ldots, \widehat{\alpha(e_i)}, \ldots, \alpha(e_{k-1}), (-1)^{n-k+1}(-1)^{n+j+1} w_j, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \alpha(e_{k+1}), \ldots, \alpha(e_{n+1})\Big] \\
&= (-1)^{n+i+1}\Big(\Big[\lambda_1 e_1, \ldots, \widehat{\alpha(e_j)}, \ldots, \widehat{\alpha(e_k)}, \ldots, \lambda_{n+1} e_{n+1}, b_{j,i} e_j\Big] \\
&\qquad\qquad + \Big[\alpha(e_1), \ldots, \widehat{\alpha(e_j)}, \ldots, \widehat{\alpha(e_k)}, \ldots, \alpha(e_{n+1}), b_{k,i} e_k\Big]\Big) \\
&- (-1)^{-j+k+1}\Big(\Big[\lambda_1 e_1, \ldots, \widehat{\alpha(e_i)}, \ldots, \lambda_{j-1} e_{j-1}, b_{i,k} e_i, \lambda_{j+1} e_{j+1}, \ldots, \lambda_{n+1} e_{n+1}\Big] \\
&\qquad + \Big[\lambda_1 e_1, \ldots, \widehat{\alpha(e_i)}, \ldots, \lambda_{j-1} e_{j-1}, b_{j,k} e_j, \lambda_{j+1} e_{j+1}, \ldots, \lambda_{n+1} e_{n+1}\Big]\Big) \\
&- (-1)^{j-k}\Big(\Big[\lambda_1 e_1, \ldots, \widehat{\alpha(e_i)}, \ldots, \lambda_{k-1} e_{k-1}, b_{i,j} e_i, \lambda_{k+1} e_{k+1}, \ldots, \lambda_{n+1} e_{n+1}\Big] \\
&\qquad + \Big[\lambda_1 e_1, \ldots, \widehat{\alpha(e_i)}, \ldots, \lambda_{k-1} e_{k-1}, b_{k,j} e_k, \lambda_{k+1} e_{k+1}, \ldots, \lambda_{n+1} e_{n+1}\Big]\Big) \\
&= (-1)^{n+i+1}\Big(\Big[\lambda_1 e_1, \ldots, \widehat{\alpha(e_j)}, \ldots, \widehat{\alpha(e_k)}, \ldots, \lambda_{n+1} e_{n+1}, b_{j,i} e_j\Big] \\
&\qquad + \Big[\lambda_1 e_1, \ldots, \widehat{\alpha(e_j)}, \ldots, \widehat{\alpha(e_k)}, \ldots, \lambda_{n+1} e_{n+1}, b_{k,i} e_k\Big]\Big) \\
&- (-1)^{-j+k+1}\Big(\Big[\lambda_1 e_1, \ldots, \widehat{\alpha(e_i)}, \ldots, \lambda_{j-1} e_{j-1}, b_{i,k} e_i, \lambda_{j+1} e_{j+1}, \ldots, \lambda_{n+1} e_{n+1}\Big] \\
&\qquad + \Big[\lambda_1 e_1, \ldots, \widehat{\alpha(e_i)}, \ldots, \lambda_{j-1} e_{j-1}, b_{j,k} e_j, \lambda_{j+1} e_{j+1}, \ldots, \lambda_{n+1} e_{n+1}\Big]\Big) \\
&- (-1)^{j-k}\Big(\Big[\lambda_1 e_1, \ldots, \widehat{\alpha(e_i)}, \ldots, \lambda_{k-1} e_{k-1}, b_{i,j} e_i, \lambda_{k+1} e_{k+1}, \ldots, \lambda_{n+1} e_{n+1}\Big] \\
&\qquad + \Big[\lambda_1 e_1, \ldots, \widehat{\alpha(e_i)}, \ldots, \lambda_{k-1} e_{k-1}, b_{k,j} e_k, \lambda_{k+1} e_{k+1}, \ldots, \lambda_{n+1} e_{n+1}\Big]\Big) \\
&= (-1)^{n+i+1} \prod_{r=1; r\neq j,k}^{n+1} \lambda_r \\
&\Big(\big[e_1, \ldots, \widehat{e_j}, \ldots, \widehat{e_k}, \ldots, e_{n+1}, b_{j,i} e_j\big] + \big[e_1, \ldots, \widehat{e_j}, \ldots, \widehat{e_k}, \ldots, e_{n+1}, b_{k,i} e_k\big]\Big)
\end{aligned}
$$

$$
\begin{aligned}
&- (-1)^{-j+k+1} \prod_{r=1;r\neq i,j}^{n+1} \lambda_r \Big(\big[e_1, \ldots, \widehat{e_i}, \ldots, e_{j-1}, b_{i,k}e_i, e_{j+1}, \ldots, e_{n+1}\big] \\
&\qquad + \big[e_1, \ldots, \widehat{e_i}, \ldots, e_{j-1}, b_{j,k}e_j, e_{j+1}, \ldots, e_{n+1}\big] \Big) \\
&- \quad (-1)^{j-k} \prod_{r=1;r\neq i,k}^{n+1} \lambda_r \Big(\big[e_1, \ldots, \widehat{e_i}, \ldots, e_{k-1}, b_{i,j}e_i, e_{k+1}, \ldots, e_{n+1}\big] \\
&\qquad + \big[e_1, \ldots, \widehat{e_i}, \ldots, e_{k-1}, b_{k,j}e_k, e_{k+1}, \ldots, e_{n+1}\big] \Big) \\
&= (-1)^{n+i+1} \prod_{r=1;r\neq j,k}^{n+1} \lambda_r \\
&\quad \Big(b_{j,i}(-1)^{n-j}(-1)^{n+k+1} w_k + b_{k,i}(-1)^{n-k+1}(-1)^{n+j+1} w_j \Big) \\
&- (-1)^{-j+k+1} \prod_{r=1;r\neq i,j}^{n+1} \lambda_r \Big(b_{i,k}(-1)^{j-1-i}(-1)^{n+j+1} w_j + b_{j,k}(-1)^{n+i+1} w_i \Big) \\
&- (-1)^{j-k} \prod_{r=1;r\neq i,k}^{n+1} \lambda_r \Big(b_{i,j}(-1)^{k-1-i}(-1)^{n+k+1} w_k + b_{k,j}(-1)^{n+i+1} w_i \Big) \\
&= \prod_{r=1;r\neq i,j,k}^{n+1} \lambda_r \Big((-1)^{n+i+j+k} \lambda_i b_{j,i} w_k + (-1)^{n+i+j+k+1} \lambda_i b_{k,i} w_j \\
&\qquad - (-1)^{n+j+k+i+1} \lambda_k b_{i,k} w_j - (-1)^{n+i+j+k} \lambda_k b_{j,k} w_i \\
&\qquad - (-1)^{n+i+j+k} \lambda_j b_{i,j} w_k - (-1)^{n+i+j+k+1} \lambda_j b_{k,j} w_i \Big) \\
&= (-1)^{n+i+j+k} \prod_{r=1;r\neq i,j,k}^{n+1} \lambda_r \\
&\quad \Big(\lambda_i b_{j,i} w_k - \lambda_i b_{k,i} w_j + \lambda_k b_{i,k} w_j - \lambda_k b_{j,k} w_i - \lambda_j b_{i,j} w_k + \lambda_j b_{k,j} w_i \Big) \\
&= (-1)^{n+i+j+k} \prod_{r=1;r\neq i,j,k}^{n+1} \lambda_r \\
&\quad \Big((\lambda_i b_{j,i} - \lambda_j b_{i,j}) w_k + (\lambda_k b_{i,k} - \lambda_i b_{k,i}) w_j + (\lambda_j b_{k,j} - \lambda_k b_{j,k}) w_i \Big),
\end{aligned}
$$

and since $(-1)^{n+i+j+k} \prod_{r=1;r\neq i,j,k}^{n+1} \lambda_r \neq 0$, the equality $H_{i,j,k} = 0$ holds if and only if

$$
(\lambda_i b_{j,i} - \lambda_j b_{i,j}) w_k + (\lambda_k b_{i,k} - \lambda_i b_{k,i}) w_j + (\lambda_j b_{k,j} - \lambda_k b_{j,k}) w_i = 0.
$$

□

Moreover, the following particular case holds. It is a generalization of [47, Theorem 3], the mentioned result corresponds to $\alpha = \mathrm{Id}_A$.

Proposition 19.12 *If α is diagonalizable and invertible, and if $Rank(B) \geq 3$, then the Hom-Nambu-Filippov identity holds if and only if $B[\alpha]^T = [\alpha]B^T$ or equivalently that $B[\alpha]^T$ is symmetric. If λ_i, $1 \leq i \leq n+1$ are the eigenvalues of α, then in any basis where α is diagonal, this is equivalent to $\lambda_i b_{j,i} - \lambda_j b_{i,j} = 0, \forall 1 \leq i, j \leq n+1$, that is the matrix B takes the form*

$$\begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} & \dots & \dots & b_{1,n+1} \\ \frac{\lambda_2 b_{1,2}}{\lambda_1} & b_{2,2} & b_{2,3} & \dots & \dots & b_{2,n+1} \\ \frac{\lambda_3 b_{1,3}}{\lambda_1} & \frac{\lambda_3 b_{2,3}}{\lambda_2} & b_{3,3} & \dots & \dots & b_{3,n+1} \\ \vdots & \vdots & \vdots & \ddots & \dots & \vdots \\ \vdots & \vdots & \vdots & \dots & \ddots & \vdots \\ \frac{\lambda_{n+1} b_{1,n+1}}{\lambda_1} & \frac{\lambda_{n+1} b_{2,n+1}}{\lambda_2} & \frac{\lambda_{n+1} b_{3,n+1}}{\lambda_3} & \dots & \frac{\lambda_{n+1} b_{n,n+1}}{\lambda_n} & b_{n+1,n+1} \end{pmatrix}.$$

Proof If the $Rank(B) \geq 3$ then we get the following: For all i, j, there exists k such that w_k is not a linear combination of w_i and w_j, then we have two cases:
1) $\dim\langle w_i, w_j \rangle = 2$: In this case,

$$\begin{aligned} &(\lambda_i b_{j,i} - \lambda_j b_{i,j})w_k + (\lambda_k b_{i,k} - \lambda_i b_{k,i})w_j + (\lambda_j b_{k,j} - \lambda_k b_{j,k})w_i = 0 \\ &\implies (\lambda_i b_{j,i} - \lambda_j b_{i,j}) = 0, \quad (\lambda_k b_{i,k} - \lambda_i b_{k,i}) = 0, \quad (\lambda_j b_{k,j} - \lambda_k b_{j,k}) = 0. \end{aligned}$$

2) $\dim\langle w_i, w_j \rangle < 2$: In this case,

$$\begin{aligned} &(\lambda_i b_{j,i} - \lambda_j b_{i,j})w_k + (\lambda_k b_{i,k} - \lambda_i b_{k,i})w_j + (\lambda_j b_{k,j} - \lambda_k b_{j,k})w_i = 0 \\ &\implies (\lambda_i b_{j,i} - \lambda_j b_{i,j})w_k + (\mu(\lambda_k b_{i,k} - \lambda_i b_{k,i}) + (\lambda_j b_{k,j} - \lambda_k b_{j,k}))w_i = 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{where } \mu \in \mathbb{K}) \\ &\implies \lambda_i b_{j,i} - \lambda_j b_{i,j} = 0. \end{aligned}$$

That is, both cases lead to $\lambda_i b_{j,i} - \lambda_j b_{i,j} = 0$, for all $1 \leq i, j \leq n+1$. This can also be expressed as $B[\alpha]^T = [\alpha]B^T$, or $B[\alpha]^T = \left(B[\alpha]^T\right)^T$, that is $B[\alpha]^T$ is symmetric.

This last formula holds for any basis of A, given that α is diagonalizable. Let $(e'_i)_{1 \leq i \leq n+1}$ be another basis of A, in which $[\cdot, \dots, \cdot]$ and α would be represented by the matrix B' and $[\alpha]_{(e'_i)}$. Let P be the basis change matrix. Then,

$$\begin{aligned} B'[\alpha]^T_{(e'_i)} &= \left(\det(P)^{-1} P B P^T\right)\left(P[\alpha]P^{-1}\right)^T = \det(P)^{-1} P B P^T (P^{-1})^T [\alpha]^T P^T \\ &= \det(P)^{-1} P B[\alpha]^T P^T = \det(P)^{-1} P[\alpha] B^T P^T \\ &= \det(P)^{-1} P[\alpha](P^{-1}P) B^T P^T = \det(P)^{-1} P[\alpha]P^{-1}\left(P B^T P^T\right) \\ &= \left(P[\alpha]P^{-1}\right)\left(\det(P)^{-1} P B P^T\right)^T = [\alpha]_{(e'_i)} B'^T. \end{aligned}$$

We conclude that, under the conditions mentioned above, the Hom-Nambu-Filippov identity is satisfied if and only if the matrices $[\alpha]$ and B representing α and $[\cdot, \dots, \cdot]$ in any basis satisfy $B[\alpha]^T = [\alpha]B^T$. □

The case where all the eigenvalues of α are equal and non-zero is equivalent to the case where $\alpha = \mathrm{Id}_A$.

Proposition 19.13 *For* $\alpha = \lambda \mathrm{Id}_A$, $\lambda \neq 0$, *n-Hom-Lie algebras* $(A, [\cdot, \dots, \cdot], \alpha)$ *are n-Lie algebras.*

Proof For all $x_1, \dots, x_{n-1}, y_1, \dots, y_n \in A$,

$$\Big[\alpha(x_1), \dots, \alpha(x_{n-1}), [y_1, \dots, y_n]\Big] - \sum_{k=1}^{n} \Big[\alpha(y_1), \dots, \alpha(y_{k-1}), \big[x_1, \dots, x_{n-1}, y_k\big], \alpha(y_{k+1}), \dots, \alpha(y_n)\Big]$$

$$= \Big[\lambda x_1, \dots, \lambda x_{n-1}, [y_1, \dots, y_n]\Big] - \sum_{k=1}^{n} \Big[\lambda y_1, \dots, \lambda y_{k-1}, \big[x_1, \dots, x_{n-1}, y_k\big], \lambda y_{k+1}, \dots, \lambda y_n\Big]$$

$$= \lambda^{n-1}\bigg(\Big[x_1, \dots, x_{n-1}, [y_1, \dots, y_n]\Big] - \sum_{k=1}^{n} \Big[y_1, \dots, y_{k-1}, \big[x_1, \dots, x_{n-1}, y_k\big], y_{k+1}, \dots, y_n\Big]\bigg).$$

Thus, since $\lambda \neq 0$, the Hom-Nambu-Filippov identity holds if and only if the Nambu-Filippov identity holds. □

Suppose now that α is diagonalizable and non invertible. We shall study the cases where $\dim \ker \alpha = 1$ and $\dim \ker \alpha = 2$, as Proposition 19.7 treats the case of higher dimensions of $\ker \alpha$. Propositions 19.14 and 19.15 are the counterparts of [47, Equation 17] in the cases of diagonalizable non-invertible α with a kernel of dimension 1 and 2 respectively. As mentioned above, [47, Equation 17] is a special case of Proposition 19.11 which is separate from the following results which concern with non-invertible α.

Proposition 19.14 *If* $\dim \ker \alpha = 1$, *let* $\lambda_1 = 0$. *Then,* $[\cdot, \dots, \cdot]$ *obeys the Hom-Nambu-Filippov identity if and only if*

$$\lambda_k b_{1,k} w_j - \lambda_k b_{j,k} w_1 - \lambda_j b_{1,j} w_k + \lambda_j b_{k,j} w_1 = 0, \quad \forall 1 < j < k \leq n+1.$$

Proof If $\dim \ker \alpha = 1$, we choose a basis such that $\lambda_1 = 0$ and all other eigenvalues are non-zero. Then, $H_{i,j,k} = 0$ for $i, j, k \neq 1$ (Remark 19.2). In the remaining cases,

$$H_{1,j,k} = (-1)^{n+1+j+k} \prod_{r=2;r\neq j,k}^{n+1} \lambda_r \Big((\lambda_1 b_{j,1} - \lambda_j b_{1,j}) w_k + (\lambda_k b_{1,k} - \lambda_1 b_{k,1}) w_j + (\lambda_j b_{k,j} - \lambda_k b_{j,k}) w_1 \Big)$$

$$= (-1)^{n+1+j+k} \prod_{r=2;r\neq j,k}^{n+1} \lambda_r \Big(- \lambda_j b_{1,j} w_k + \lambda_k b_{1,k} w_j + (\lambda_j b_{k,j} - \lambda_k b_{j,k}) w_1 \Big).$$

Since $(-1)^{n+1+j+k} \prod_{r=2;r\neq j,k}^{n+1} \lambda_r \neq 0$, we get that $H_{1,j,k} = 0$ if and only if

$$\Big(- \lambda_j b_{1,j} w_k + \lambda_k b_{1,k} w_j + (\lambda_j b_{k,j} - \lambda_k b_{j,k}) w_1 \Big) = 0.$$

□

Proposition 19.15 *If* $\dim \ker \alpha = 2$, *let* $\lambda_1 = \lambda_2 = 0$. *The multiplication* $[\cdot, \ldots, \cdot]$ *satisfies the Hom-Nambu-Filippov identity if and only if* $b_{1,k} w_2 - b_{2,k} w_1 = 0$ *for all* $3 \leq k \leq n + 1$.

Proof If $\dim \ker \alpha = 2$, suppose that $\lambda_1 = \lambda_2 = 0$ and $\lambda_k \neq 0$ for $k \neq 1, 2$. Then, similarly, if $(i, j) \neq (1, 2)$ where $i < j < k$, then $H_{i,j,k} = 0$. In the remaining cases,

$$H_{1,2,k} = (-1)^{n+1+2+k} \prod_{r=3;r\neq k}^{n+1} \lambda_r \Big((\lambda_1 b_{2,1} - \lambda_2 b_{1,2}) w_k + (\lambda_k b_{1,k} - \lambda_1 b_{k,1}) w_2 + (\lambda_2 b_{k,2} - \lambda_k b_{2,k}) w_1 \Big)$$

$$= (-1)^{n+1+2+k} \prod_{r=3;r\neq k}^{n+1} \lambda_r \Big(\lambda_k (b_{1,k} w_2 - b_{2,k} w_1) \Big).$$

Since $(-1)^{n+1+2+k} \prod_{r=3;r\neq k}^{n+1} \lambda_r \neq 0$, the equality $H_{1,2,k} = 0$ holds if and only if

$$\lambda_k (b_{1,k} w_2 - b_{2,k} w_1) = 0.$$

□

Remark 19.4 A particularity of this case is that the non-zero eigenvalues do not appear in the equations, which means that if a given skew-symmetric multiplication satisfies the Hom-Nambu-Filippov identity for a diagonalizable α with kernel of dimension 2, it would satisfy it for any such a linear map.

Let us consider now the case where α is nilpotent. By Proposition 19.7, we only need to investigate the cases where $\dim \ker \alpha = 1$ and $\dim \ker \alpha = 2$. Propositions

19.16 and 19.17 are the counterparts of [47, Equation 17] in the cases of nilpotent α with a kernel of dimension 2 and 1 respectively. As mentioned above, [47, Equation 17] is a special case of Proposition 19.7 which is separate from the following cases. If $\dim\ker\alpha = 2$, then for a basis in which α is in Jordan form, $\ker f = \langle e_1, e_{i_0}\rangle$. For $j > i_0$ we get the following statement.

Proposition 19.16 (Case: $\dim\ker\alpha = 2$) *The multiplication* $[\cdot, \dots, \cdot]$ *obeys the Hom-Nambu-Filippov identity if and only if*

$$b_{i_0-1,j}b_{p,n+1} - b_{n+1,j}b_{p,i_0-1} = 0, \quad \forall 1 \le j, p \le n+1,\ j \ne 1,\ j \ne i_0.$$

Proof In this case, by Remark 19.2 and Proposition 19.4, Hom-Nambu-Filippov identity holds if and only if $H_{1,i_0,j} = 0$ for all $i_0 < j < n+1$:

$$\begin{aligned}
H_{1,i_0,j} &= \Big[\alpha(e_1), \dots, \widehat{\alpha(e_{i_0})}, \dots, \widehat{\alpha(e_j)}, \dots, \alpha(e_{n+1}), \big[e_2, \dots, e_{n+1}\big]\Big] \\
&\quad - \Big[\alpha(e_2), \dots, \alpha(e_{i_0-1}), \big[e_1, \dots, \widehat{e_{i_0}}, \dots, \widehat{e_j}, \dots, e_{n+1}, e_{i_0}\big], \\
&\qquad\qquad \alpha(e_{i_0+1}), \dots, \alpha(e_{n+1})\Big] \\
&\quad - \Big[\alpha(e_2), \dots, \alpha(e_{i_0}), \dots, \alpha(e_{j-1}) \big[e_1, \dots, \widehat{e_{i_0}}, \dots, \widehat{e_j}, \dots, e_{n+1}, e_j\big], \\
&\qquad\qquad \alpha(e_{j+1}), \dots, \alpha(e_{n+1})\Big] \\
&= \Big[0, \dots, \widehat{\alpha(e_{i_0})}, \dots, \widehat{\alpha(e_j)}, \dots, e_n, \big[e_2, \dots, e_{n+1}\big]\Big] \\
&\quad - \Big[e_1, \dots, e_{i_0-2}, \big[e_1, \dots, \widehat{e_{i_0}}, \dots, \widehat{e_j}, \dots, e_{n+1}, e_{i_0}\big], e_{i_0}, \dots, e_n\Big] \\
&\quad - \Big[\alpha(e_2), \dots, 0, \dots, \alpha(e_{j-1}) \big[e_1, \dots, \widehat{e_{i_0}}, \dots, \widehat{e_j}, \dots, e_{n+1}, e_j\big], \\
&\qquad\qquad \alpha(e_{j+1}), \dots, \alpha(e_{n+1})\Big] \\
&= -\Big[e_1, \dots, e_{i_0-2}, \big[e_1, \dots, \widehat{e_{i_0}}, \dots, \widehat{e_j}, \dots, e_{n+1}, e_{i_0}\big], e_{i_0}, \dots, e_n\Big] \\
&= -(-1)^{n-i_0}\Big[e_1, \dots, e_{i_0-2} \big[e_1, \dots, \widehat{e_j}, \dots, e_{n+1}\big], e_{i_0}, \dots, e_n\Big] \\
&= -(-1)^{n-i_0}(-1)^{n+j+1}[e_1, \dots, e_{i_0-2}, w_j, e_{i_0}, \dots, e_n] \\
&= -(-1)^{n-i_0}(-1)^{n+j+1}b_{i_0-1,j}[e_1, \dots, e_{i_0-2}, e_{i_0-1}, e_{i_0}, \dots, e_n] \\
&\quad - (-1)^{n-i_0}(-1)^{n+j+1}b_{n+1,j}[e_1, \dots, e_{i_0-2}, e_{n+1}, e_{i_0}, \dots, e_n] \\
&= -(-1)^{n-i_0}(-1)^{n+j+1}b_{i_0-1,j}\,[e_1, \dots, e_n] \\
&\quad - (-1)^{n-i_0}(-1)^{n+j+1}(-1)^{i_0-1-n}b_{n+1,j}\,\big[e_1, \dots, e_{i_0-2}, e_{i_0}, \dots, e_n, e_{n+1}\big] \\
&= -(-1)^{n-i_0}(-1)^{n+j+1}(-1)^{n+n+1+1}b_{i_0-1,j}w_{n+1}
\end{aligned}$$

$$
\begin{aligned}
&- (-1)^{n-i_0}(-1)^{n+j+1}(-1)^{i_0-1-n}(-1)^{n+i_0-1+1}b_{n+1,j}w_{i_0-1}\\
&= -(-1)^{n-i_0}(-1)^{n+j+1}\left(b_{i_0-1,j}w_{n+1} - b_{n+1,j}w_{i_0-1}\right).
\end{aligned}
$$

For $j < i_0$, we get the same result, up to a (-1) factor. This means that $H_{i,j,k} = 0$ if and only if $b_{i_0-1,j}w_{n+1} - b_{n+1,j}w_{i_0-1} = 0$ for all $1 \leq j \leq n+1$, $j \neq 1$, $j \neq i_0$, that is,

$$
\begin{cases}
b_{i_0-1,2}w_{n+1} - b_{n+1,2}w_{i_0-1} = 0\\
b_{i_0-1,3}w_{n+1} - b_{n+1,3}w_{i_0-1} = 0\\
\vdots\\
b_{i_0-1,i_0-1}w_{n+1} - b_{n+1,i_0-1}w_{i_0-1} = 0\\
b_{i_0-1,i_0+1}w_{n+1} - b_{n+1,i_0+1}w_{i_0-1} = 0\\
\vdots\\
b_{i_0-1,n+1}w_{n+1} - b_{n+1,n+1}w_{i_0-1} = 0.
\end{cases}
$$

Rewriting the above equations, using the coordinates in the basis $(e_i)_{1\leq i\leq n+1}$, yields the following system:

$$
b_{i_0-1,j}b_{p,n+1} - b_{n+1,j}b_{p,i_0-1} = 0, \quad \forall 1 \leq j, p \leq n+1, j \neq 1, j \neq i_0.
$$

□

We suppose now that α is nilpotent and $\dim\ker\alpha = 1$. For a basis in which α is in Jordan normal form, this yields $\ker\alpha = \langle e_1\rangle$, and we have the following statement.

Proposition 19.17 (Case: $\dim\ker\alpha = 1$) *The multiplication* $[\cdot, \ldots, \cdot]$ *obeys the Hom-Nambu-Filippov identity if and only if*

$$
\begin{aligned}
&\forall\ 1 \leq i, k, p \leq n+1, i < k :\\
&(b_{k-1,i} - b_{i-1,k})b_{p,n+1} - b_{n+1,i}b_{p,k-1} + b_{n+1,k}b_{p,i-1} = 0.
\end{aligned}
$$

Proof In this case, by Remark 19.2 and Proposition 19.4, Hom-Nambu-Filippov identity holds if and only if $H_{i,1,k} = 0$ for all $1 < i < k \leq n+1$. Thus,

$$
\begin{aligned}
H_{i,1,k} = &\left[\alpha(e_2), \ldots, \widehat{\alpha(e_k)}, \ldots, \alpha(e_{n+1}), \left[e_1, \ldots, \widehat{e_i}, \ldots, e_{n+1}\right]\right]\\
&- \left[\left[e_2, \ldots, \widehat{e_k}, \ldots, e_{n+1}, e_1\right], \alpha(e_2), \alpha(e_3), \ldots, \widehat{\alpha(e_i)}, \ldots, \alpha(e_{n+1})\right]\\
&- \Big[\alpha(e_1), \ldots, \alpha(e_{k-1}), \left[e_2, \ldots, \widehat{e_k}, \ldots, e_{n+1}, e_k\right],\\
&\qquad\qquad \alpha(e_{k+1}), \ldots, \widehat{\alpha(e_i)}, \ldots, \alpha(e_{n+1})\Big]\\
= &\left[\alpha(e_2), \ldots, \widehat{\alpha(e_k)}, \ldots, \alpha(e_{n+1}), \left[e_1, \ldots, \widehat{e_i}, \ldots, e_{n+1}\right]\right]\\
&- \left[\left[e_2, \ldots, \widehat{e_k}, \ldots, e_{n+1}, e_1\right], \alpha(e_2), \alpha(e_3), \ldots, \widehat{\alpha(e_i)}, \ldots, \alpha(e_{n+1})\right]\\
&- \Big[0, \ldots, \alpha(e_{k-1}), \left[e_2, \ldots, \widehat{e_k}, \ldots, e_{n+1}, e_k\right],
\end{aligned}
$$

$$\alpha(e_{k+1}),\ldots,\widehat{\alpha(e_i)},\ldots,\alpha(e_{n+1})\Big]$$

$$= \Big[e_1,\ldots,\widehat{e_{k-1}},\ldots,e_n,\big[e_1,\ldots,\widehat{e_i},\ldots,e_{n+1}\big]\Big]$$
$$- \Big[\big[e_2,\ldots,\widehat{e_k},\ldots,e_{n+1},e_1\big],e_1,e_2,\ldots,\widehat{e_{i-1}},\ldots,e_n\Big]$$
$$= [e_1,\ldots,\widehat{e_{k-1}},\ldots,e_n,(-1)^{n+1+i}w_i]$$
$$- \Big[(-1)^{n-1}\big[e_1,e_2,\ldots,\widehat{e_k},\ldots,e_{n+1}\big],e_1,e_2,\ldots,\widehat{e_{i-1}},\ldots,e_n\Big]$$
$$= [e_1,\ldots,\widehat{e_{k-1}},\ldots,e_n,(-1)^{n+1+i}w_i]$$
$$- [(-1)^{n-1}(-1)^{n+1+k}w_k,e_1,e_2,\ldots,\widehat{e_{i-1}},\ldots,e_n]$$
$$= [e_1,\ldots,\widehat{e_{k-1}},\ldots,e_n,(-1)^{n+1+i}b_{k-1,i}e_{k-1}]$$
$$+ [e_1,\ldots,\widehat{e_{k-1}},\ldots,e_n,(-1)^{n+1+i}b_{n+1,i}e_{n+1}]$$
$$- [(-1)^{n-1}(-1)^{n+1+k}b_{i-1,k}e_{i-1},e_1,e_2,\ldots,\widehat{e_{i-1}},\ldots,e_n]$$
$$- [(-1)^{n-1}(-1)^{n+1+k}b_{n+1,k}e_{n+1},e_1,e_2,\ldots,\widehat{e_{i-1}},\ldots,e_n]$$
$$= (-1)^{n+1+i}b_{k-1,i}(-1)^{n-k+1}\,[e_1,\ldots,e_n]$$
$$+ (-1)^{n+1+i}b_{n+1,i}\,\big[e_1,\ldots,\widehat{e_{k-1}},\ldots,e_{n+1}\big]$$
$$- (-1)^{n-1}(-1)^{n+1+k}(-1)^{1-i+1}b_{i-1,k}\,[e_1,\ldots,e_n]$$
$$- (-1)^{n-1}(-1)^{n+1+k}(-1)^{1-n}b_{n+1,k}\,\big[e_1,\ldots,\widehat{e_{i-1}},\ldots,e_{n+1}\big]$$
$$= (-1)^{n+1+i}b_{k-1,i}(-1)^{n-k+1}(-1)^{n+n+1+1}w_{n+1}$$
$$+ (-1)^{n+1+i}b_{n+1,i}(-1)^{n+k-1+1}w_{k-1}$$
$$- (-1)^{n-1}(-1)^{n+1+k}(-1)^{-i}b_{i-1,k}(-1)^{n+n+1+1}w_{n+1}$$
$$- (-1)^{n-1}(-1)^{n+1+k}(-1)^{1-n}b_{n+1,k}(-1)^{n+i-1+1}w_{i-1}$$
$$= (-1)^{i-k}b_{k-1,i}w_{n+1} + (-1)^{1+i+k}b_{n+1,i}w_{k-1}$$
$$- (-1)^{k-i}b_{i-1,k}w_{n+1} - (-1)^{k+1+i}b_{n+1,k}w_{i-1}$$
$$= \Big((-1)^{i-k}b_{k-1,i} - (-1)^{k-i}b_{i-1,k}\Big)\,w_{n+1}$$
$$+ (-1)^{1+i+k}b_{n+1,i}w_{k-1} - (-1)^{k+1+i}b_{n+1,k}w_{i-1}$$

yields that the Hom-Nambu-Filippov identity is satisfied if and only if

$$\forall\ 1 \le i < k \le n+1:$$
$$(b_{k-1,i} - b_{i-1,k})w_{n+1} - b_{n+1,i}w_{k-1} + b_{n+1,k}w_{i-1} = 0,$$

which, using the coordinates in the basis $(e_i)_{1\le i\le n+1}$, gives the following system:

$$\forall\ 1 \le i,k,p \le n+1,\ i < k:$$
$$(b_{k-1,i} - b_{i-1,k})b_{p,n+1} - b_{n+1,i}b_{p,k-1} + b_{n+1,k}b_{p,i-1} = 0.$$

□

Remark 19.5 Let us compare the polynomial equations obtained from the Nambu-Filippov identity and the Hom-Nambu-Filippov identity in dimension $n+1$ with various types of twisting maps:

Diagonalizable and invertible with eigenvalues $\{\lambda_i\}_{1\le i\le n+1}$:

$$\forall\, 1 \le i, j, k, p \le n+1;\ i < j < k:$$
$$(\lambda_i b_{j,i} - \lambda_j b_{i,j}) b_{p,k} + (\lambda_k b_{i,k} - \lambda_i b_{k,i}) b_{p,j} + (\lambda_j b_{k,j} - \lambda_k b_{j,k}) b_{p,i} = 0; \quad (19.9)$$

Diagonalizable with $\dim\ker\alpha = 1$ with eigenvalues $\{\lambda_i\}_{1\le i\le n+1}$:

$$\forall\, 1 < j < k \le n+1:$$
$$\lambda_k b_{1,k} w_j - \lambda_k b_{j,k} w_1 - \lambda_j b_{1,j} w_k + \lambda_j b_{k,j} w_1 = 0; \quad (19.10)$$

Diagonalizable with $\dim\ker\alpha = 2$ with eigenvalues $\{\lambda_i\}_{1\le i\le n+1}$:

$$\forall\, 3 \le k \le n+1: \quad b_{1,k} w_2 - b_{2,k} w_1 = 0; \quad (19.11)$$

Nilpotent with $\dim\ker\alpha = 1$:

$$\forall\, 1 \le i, k, p \le n+1, i < k:$$
$$(b_{k-1,i} - b_{i-1,k}) b_{p,n+1} - b_{n+1,i} b_{p,k-1} + b_{n+1,k} b_{p,i-1} = 0; \quad (19.12)$$

Nilpotent with $\dim\ker\alpha = 2$:

$$\forall\, 1 \le j, p \le n+1, j \ne 1, j \ne i_0: \quad b_{i_0-1,j} b_{p,n+1} - b_{n+1,j} b_{p,i_0-1} = 0. \quad (19.13)$$

These different cases are separate from each other, and the case of n-Lie algebras is the special case of (19.9) where all the λ_i are equal. Notice that the higher the dimension of $\ker\alpha$ the less equation we have and the less terms we have in each equation, that is, in these cases, the Hom-Nambu-Filippov identity is considerably less restrictive. Another difference from the case of n-Lie algebra are the isomorphisms, in Hom-algebras, an isomorphism intertwines the multiplications and the twisting maps, which leads to different, more restrictive isomorphism conditions.

19.5 Lists of 4-Dimensional 3-Hom-Lie Algebras

In this section, we present lists of 3-Hom-Lie algebras of dimension 4 obtained using the computer algebra software Mathematica for special cases of the twisting map α.

Let $(A, [\cdot,\cdot,\cdot], \alpha)$ be a 4-dimensional ternary Hom-algebra. The multiplication is fully defined by its structure constants $(c(i,j,k,l))_{1\le i,j,k,l\le 4}$ in a basis $(e_i)_{1\le i\le 4}$ as

$$[e_i, e_j, e_k] = \sum_{l=1}^{4} c(i,j,k,l)e_l,$$

and α is defined by its matrix in the same basis, $[\alpha] = \left(a_{i,j}\right)_{1\le i,j\le 4}$. The equations for the skew-symmetry are

$$\begin{aligned}&\forall\ 1 \le i,j,k,l \le 4 :\\ &c(i,j,k,l) = -c(j,i,k,l); \quad c(i,j,k,l) = -c(i,k,j,l).\end{aligned}$$

After the equations for skew-symmetry are solved, one can use them to simplify the equations for the Hom-Nambu-Filippov identity and finally solve the latter simplified system.

Each algebra shall be represented by a square matrix B such that

$$\begin{aligned}(w_1,\dots,w_{n+1}) &= (e_1,\dots,e_{n+1})B,\\ \text{where}\quad w_j &= (-1)^{n+j+1}\left[e_1,\dots,\widehat{e_j},\dots,e_{n+1}\right].\end{aligned}$$

In the tables given below, the listed algebras shall be denoted following this pattern: $d_{n,[\alpha],i}$, where d is the dimension, n the arity, $[\alpha]$ the matrix representing the twisting maps, by the name given right above each table, where in those names, N stands for nilpotent and D for diagonal. Finally, i is the number of the algebra in each table.

19.5.1 Nilpotent α with Kernel of Dimension 1

Every nilpotent linear map α with kernel of dimension 1 can be represented, in some basis, by the matrix $N(1) = \begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}$. With this matrix and basis we get the following list of algebras.

$4_{3,N(1),1}$	$\begin{pmatrix} -c(2,3,4,1) & c(1,3,4,1) & -c(1,2,4,1) & 0 \\ -c(2,3,4,2) & c(1,3,4,2) & -c(1,2,4,2) & 0 \\ -c(2,3,4,3) & c(1,3,4,3) & -c(1,2,4,3) & 0 \\ -c(2,3,4,4) & 0 & 0 & 0 \end{pmatrix}$
$4_{3,N(1),2}$	$\begin{pmatrix} -c(2,3,4,1) & c(1,3,4,1) & -c(1,2,4,1) & c(1,2,3,1) \\ -c(2,3,4,2) & -c(1,2,4,1) & -c(1,2,4,2) & c(1,2,3,2) \\ -c(2,3,4,3) & c(1,2,3,1) & c(1,2,3,2) & c(1,2,3,3) \\ -c(2,3,4,4) & 0 & 0 & 0 \end{pmatrix}$
$4_{3,N(1),3}$	$\begin{pmatrix} s_1 & s_5 & -c(1,2,4,1) & c(1,2,3,1) \\ s_2 & s_6 & s_{10} & c(1,2,3,2) \\ s_3 & s_7 & \frac{c(1,2,4,4)^2}{c(1,2,3,4)} + c(1,2,3,2) - c(1,3,4,4) & c(1,2,3,3) \\ s_4 & c(1,3,4,4) & -c(1,2,4,4) & c(1,2,3,4) \end{pmatrix}$ $s_1 = \frac{c(1,2,3,1)c(1,2,4,4)^3}{c(1,2,3,4)^3} + \frac{c(1,2,3,1)c(1,2,3,3)c(1,2,4,4)^2}{c(1,2,3,4)^3}$ $- \frac{c(1,2,3,1)c(1,3,4,4)c(1,2,4,4)}{c(1,2,3,4)^2} + \frac{c(1,2,3,1)c(1,2,3,2)c(1,2,4,4)}{c(1,2,3,4)^2}$ $- \frac{c(1,2,4,1)c(1,3,4,4)}{c(1,2,3,4)} - \frac{c(1,2,3,1)c(1,2,3,3)c(1,3,4,4)}{c(1,2,3,4)^2} + \frac{c(1,2,3,1)^2}{c(1,2,3,4)}$ $s_2 = -\frac{c(1,3,4,4)c(1,2,4,4)^3}{c(1,2,3,4)^3} + \frac{c(1,2,3,2)c(1,2,4,4)^3}{c(1,2,3,4)^3}$ $- \frac{c(1,2,3,3)c(1,3,4,4)c(1,2,4,4)^2}{c(1,2,3,4)^3} + \frac{c(1,2,3,2)c(1,2,3,3)c(1,2,4,4)^2}{c(1,2,3,4)^3}$ $+ \frac{2c(1,3,4,4)^2 c(1,2,4,4)}{c(1,2,3,4)^2} - \frac{3c(1,2,3,2)c(1,3,4,4)c(1,2,4,4)}{c(1,2,3,4)^2}$ $+ \frac{c(1,2,3,2)^2 c(1,2,4,4)}{c(1,2,3,4)^2} + \frac{c(1,2,3,3)c(1,3,4,4)^2}{c(1,2,3,4)^2} - \frac{c(1,2,3,1)c(1,3,4,4)}{c(1,2,3,4)}$ $= -\frac{c(1,2,3,2)c(1,2,3,3)c(1,3,4,4)}{c(1,2,3,4)^2} + \frac{c(1,2,3,1)c(1,2,3,2)}{c(1,2,3,4)}$ $- \frac{c(1,3,4,4)^3}{c(1,2,3,4)c(1,2,4,4)} + \frac{c(1,2,3,2)c(1,3,4,4)^2}{c(1,2,3,4)c(1,2,4,4)} + \frac{c(1,2,4,1)c(1,3,4,4)}{c(1,2,4,4)}$ $s_3 = \frac{c(1,2,3,3)c(1,2,4,4)^3}{c(1,2,3,4)^3} + \frac{c(1,3,4,4)c(1,2,4,4)^2}{c(1,2,3,4)^2} + \frac{c(1,2,3,3)^2 c(1,2,4,4)^2}{c(1,2,3,4)^3}$ $- \frac{c(1,2,3,3)c(1,3,4,4)c(1,2,4,4)}{c(1,2,3,4)^2} + \frac{c(1,2,3,2)c(1,2,3,3)c(1,2,4,4)}{c(1,2,3,4)^2}$ $- \frac{c(1,3,4,4)^2}{c(1,2,3,4)} + \frac{c(1,2,3,2)c(1,3,4,4)}{c(1,2,3,4)} - \frac{c(1,2,3,3)^2 c(1,3,4,4)}{c(1,2,3,4)^2}$ $+ \frac{c(1,2,3,1)c(1,2,3,3)}{c(1,2,3,4)}$ $s_4 = \frac{c(1,2,4,4)^3}{c(1,2,3,4)^2} + \frac{c(1,2,3,3)c(1,2,4,4)^2}{c(1,2,3,4)^2} - \frac{2c(1,3,4,4)c(1,2,4,4)}{c(1,2,3,4)}$ $+ \frac{c(1,2,3,2)c(1,2,4,4)}{c(1,2,3,4)} + c(1,2,3,1) - \frac{c(1,2,3,3)c(1,3,4,4)}{c(1,2,3,4)}$ $s_5 = -\frac{c(1,2,3,1)c(1,2,4,4)^2}{c(1,2,3,4)^2} + \frac{c(1,2,4,1)c(1,2,4,4)}{c(1,2,3,4)} + \frac{c(1,2,3,1)c(1,3,4,4)}{c(1,2,3,4)}$ $s_6 = \frac{c(1,2,4,4)^4}{c(1,2,3,4)^3} + \frac{c(1,2,3,3)c(1,2,4,4)^3}{c(1,2,3,4)^3}$ $- \frac{2c(1,3,4,4)c(1,2,4,4)^2}{c(1,2,3,4)^2} + \frac{c(1,2,3,2)c(1,2,4,4)^2}{c(1,2,3,4)^2}$ $- \frac{c(1,2,3,3)c(1,3,4,4)c(1,2,4,4)}{c(1,2,3,4)^2} + \frac{c(1,2,3,1)c(1,2,4,4)}{c(1,2,3,4)}$ $+ \frac{c(1,3,4,4)^2}{c(1,2,3,4)} - c(1,2,4,1)$ $s_7 = -\frac{c(1,2,4,4)^3}{c(1,2,3,4)^2} - \frac{c(1,2,3,3)c(1,2,4,4)^2}{c(1,2,3,4)^2} + \frac{c(1,3,4,4)c(1,2,4,4)}{c(1,2,3,4)}$ $- \frac{c(1,2,3,2)c(1,2,4,4)}{c(1,2,3,4)} + \frac{c(1,2,3,3)c(1,3,4,4)}{c(1,2,3,4)}$ $s_{10} = -\frac{c(1,2,4,4)^3}{c(1,2,3,4)^2} - \frac{c(1,2,3,3)c(1,2,4,4)^2}{c(1,2,3,4)^2} + \frac{2c(1,3,4,4)c(1,2,4,4)}{c(1,2,3,4)}$ $- \frac{2c(1,2,3,2)c(1,2,4,4)}{c(1,2,3,4)} - c(1,2,3,1) + \frac{c(1,2,3,3)c(1,3,4,4)}{c(1,2,3,4)}$ $- \frac{c(1,3,4,4)^2}{c(1,2,4,4)} + \frac{c(1,2,3,4)c(1,2,4,1)}{c(1,2,4,4)} + \frac{c(1,2,3,2)c(1,3,4,4)}{c(1,2,4,4)}$

$4_{3,N(1),4}$	$\begin{pmatrix} t_1 & \frac{c(1,2,3,1)c(1,3,4,4)}{c(1,2,3,4)} & \frac{c(1,2,3,2)c(1,3,4,4)}{c(1,2,3,4)} - \frac{c(1,3,4,4)^2}{c(1,2,3,4)} & c(1,2,3,1) \\ t_2 & \frac{c(1,2,3,2)c(1,3,4,4)}{c(1,2,3,4)} & -c(1,2,4,2) & c(1,2,3,2) \\ t_3 & \frac{c(1,2,3,3)c(1,3,4,4)}{c(1,2,3,4)} & c(1,2,3,2) - c(1,3,4,4) & c(1,2,3,3) \\ t_4 & c(1,3,4,4) & 0 & c(1,2,3,4) \end{pmatrix}$ $t_1 = -\frac{c(1,3,4,4)^3}{c(1,2,3,4)^2} + \frac{c(1,2,3,2)c(1,3,4,4)^2}{c(1,2,3,4)^2} - \frac{c(1,2,3,1)c(1,2,3,3)c(1,3,4,4)}{c(1,2,3,4)^2} + \frac{c(1,2,3,1)^2}{c(1,2,3,4)}$ $t_2 = -\frac{c(1,2,3,3)c(1,3,4,4)c(1,2,3,2)}{c(1,2,3,4)^2} + \frac{c(1,2,3,1)c(1,2,3,2)}{c(1,2,3,4)} - \frac{c(1,2,4,2)c(1,3,4,4)}{c(1,2,3,4)}$ $t_3 = -\frac{c(1,3,4,4)c(1,2,3,3)^2}{c(1,2,3,4)^2} + \frac{c(1,2,3,1)c(1,2,3,3)}{c(1,2,3,4)} - \frac{c(1,3,4,4)^2}{c(1,2,3,4)} + \frac{c(1,2,3,2)c(1,3,4,4)}{c(1,2,3,4)}$ $t_4 = c(1,2,3,1) - \frac{c(1,2,3,3)c(1,3,4,4)}{c(1,2,3,4)}$

19.5.2 Nilpotent α with Kernel of Dimension 2

For nilpotent α with kernel of dimension 2, two possibilities arise, depending on the dimension of the generalized eigenspaces of α and thus the size of Jordan blocks.

The first case is when we have one eigenspace of dimension 1 and one generalized eigenspace of dimension 3. The Jordan form of α in this case is given by $N(2) = \begin{pmatrix} 0&0&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{pmatrix}$. We get the following list of algebras.

$4_{3,N(2),1}$	$\begin{pmatrix} -c(2,3,4,1) & c(1,3,4,1) & 0 & 0 \\ -c(2,3,4,2) & c(1,3,4,2) & -c(1,2,4,2) & c(1,2,3,2) \\ -c(2,3,4,3) & c(1,3,4,3) & -c(1,2,4,3) & c(1,2,3,3) \\ -c(2,3,4,4) & c(1,3,4,4) & 0 & 0 \end{pmatrix}$
$4_{3,N(2),2}$	$\begin{pmatrix} -c(2,3,4,1) & c(1,3,4,1) & -c(1,2,4,1) & 0 \\ -c(2,3,4,2) & c(1,3,4,2) & -c(1,2,4,2) & 0 \\ -c(2,3,4,3) & c(1,3,4,3) & -c(1,2,4,3) & 0 \\ -c(2,3,4,4) & c(1,3,4,4) & 0 & 0 \end{pmatrix}$
$4_{3,N(2),3}$	$\begin{pmatrix} \frac{c(1,2,3,1)^2}{c(1,2,3,4)} & c(1,3,4,1) & -\frac{c(1,2,3,1)c(1,2,4,4)}{c(1,2,3,4)} & c(1,2,3,1) \\ \frac{c(1,2,3,1)c(1,2,3,2)}{c(1,2,3,4)} & c(1,3,4,2) & -c(1,2,4,2) & c(1,2,3,2) \\ \frac{c(1,2,3,1)c(1,2,3,3)}{c(1,2,3,4)} & c(1,3,4,3) & -c(1,2,4,3) & c(1,2,3,3) \\ c(1,2,3,1) & c(1,3,4,4) & -c(1,2,4,4) & c(1,2,3,4) \end{pmatrix}$

$4_{3,N(2),4}$	$\begin{pmatrix} 0 & c(1,3,4,1) & -c(1,2,4,1) & 0 \\ \frac{c(1,2,3,2)c(1,2,4,1)}{c(1,2,4,4)} & c(1,3,4,2) & -c(1,2,4,2) & c(1,2,3,2) \\ \frac{c(1,2,3,3)c(1,2,4,1)}{c(1,2,4,4)} & c(1,3,4,3) & -c(1,2,4,3) & c(1,2,3,3) \\ 0 & c(1,3,4,4) & -c(1,2,4,4) & 0 \end{pmatrix}$
$4_{3,N(2),5}$	$\begin{pmatrix} 0 & c(1,3,4,1) & 0 & 0 \\ 0 & c(1,3,4,2) & -c(1,2,4,2) & c(1,2,3,2) \\ 0 & c(1,3,4,3) & -c(1,2,4,3) & c(1,2,3,3) \\ 0 & c(1,3,4,4) & -c(1,2,4,4) & c(1,2,3,4) \end{pmatrix}$
$4_{3,N(2),6}$	$\begin{pmatrix} 0 & c(1,3,4,1) & -c(1,2,4,1) & 0 \\ 0 & c(1,3,4,2) & -c(1,2,4,2) & 0 \\ 0 & c(1,3,4,3) & -c(1,2,4,3) & 0 \\ 0 & c(1,3,4,4) & -c(1,2,4,4) & 0 \end{pmatrix}$

Remark 19.6 Note that if one would like to consider the basis where α is represented by $\begin{pmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$, moving from this basis to the one considered above is made simply by changing the order of the basis elements following the permutation $\sigma = \begin{pmatrix} 1&2&3&4 \\ 2&3&4&1 \end{pmatrix}$. To each algebra listed above corresponds an isomorphic algebra that can be obtained by applying Proposition 19.6.

The other case is when α has two generalized eigenspaces of dimension 2, in this case its Jordan form is given by the matrix $N'(2) = \begin{pmatrix} 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{pmatrix}$. We get the following list of algebras.

$4_{3,N'(2),1}$	$\begin{pmatrix} -c(2,3,4,1) & \frac{c(1,2,3,1)c(1,3,4,4)}{c(1,2,3,4)} & -c(1,2,4,1) & c(1,2,3,1) \\ -c(2,3,4,2) & \frac{c(1,3,4,4)^2}{c(1,2,3,4)} & -c(1,2,4,2) & c(1,3,4,4) \\ -c(2,3,4,3) & \frac{c(1,2,3,3)c(1,3,4,4)}{c(1,2,3,4)} & -c(1,2,4,3) & c(1,2,3,3) \\ -c(2,3,4,4) & c(1,3,4,4) & -c(1,2,4,4) & c(1,2,3,4) \end{pmatrix}$
$4_{3,N'(2),2}$	$\begin{pmatrix} -c(2,3,4,1) & c(1,3,4,1) & -c(1,2,4,1) & c(1,2,3,1) \\ -c(2,3,4,2) & 0 & -c(1,2,4,2) & 0 \\ -c(2,3,4,3) & c(1,3,4,3) & -c(1,2,4,3) & c(1,2,3,3) \\ -c(2,3,4,4) & 0 & -c(1,2,4,4) & 0 \end{pmatrix}$
$4_{3,N'(2),3}$	$\begin{pmatrix} -c(2,3,4,1) & c(1,3,4,1) & -c(1,2,4,1) & 0 \\ -c(2,3,4,2) & c(1,3,4,2) & -c(1,2,4,2) & 0 \\ -c(2,3,4,3) & c(1,3,4,3) & -c(1,2,4,3) & 0 \\ -c(2,3,4,4) & 0 & -c(1,2,4,4) & 0 \end{pmatrix}$
$4_{3,N'(2),4}$	$\begin{pmatrix} -c(2,3,4,1) & 0 & -c(1,2,4,1) & c(1,2,3,1) \\ -c(2,3,4,2) & 0 & -c(1,2,4,2) & 0 \\ -c(2,3,4,3) & 0 & -c(1,2,4,3) & c(1,2,3,3) \\ -c(2,3,4,4) & 0 & -c(1,2,4,4) & c(1,2,3,4) \end{pmatrix}$

19.5.3 Diagonalisable α with Kernel of Dimension 2

Any diagonalisable linear map α with kernel of dimension 2 can be represented, in some basis, by the matrix $D(2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}$, $\lambda_3, \lambda_4 \neq 0$. In such basis, we get the following list of algebras.

$4_{3,D(2),1}$	$\begin{pmatrix} -c(2,3,4,1) & c(1,3,4,1) & 0 & 0 \\ -c(2,3,4,2) & c(1,3,4,2) & 0 & 0 \\ -c(2,3,4,3) & c(1,3,4,3) & -c(1,2,4,3) & c(1,2,3,3) \\ -c(2,3,4,4) & c(1,3,4,4) & -c(1,2,4,4) & c(1,2,3,4) \end{pmatrix}$
$4_{3,D(2),2}$	$\begin{pmatrix} \frac{c(1,2,3,1)c(1,3,4,1)}{c(1,2,3,2)} & c(1,3,4,1) & -\frac{c(1,2,3,1)c(1,2,4,2)}{c(1,2,3,2)} & c(1,2,3,1) \\ \frac{c(1,2,3,1)c(1,3,4,2)}{c(1,2,3,2)} & c(1,3,4,2) & -c(1,2,4,2) & c(1,2,3,2) \\ \frac{c(1,2,3,1)c(1,3,4,3)}{c(1,2,3,2)} & c(1,3,4,3) & -c(1,2,4,3) & c(1,2,3,3) \\ \frac{c(1,2,3,1)c(1,3,4,4)}{c(1,2,3,2)} & c(1,3,4,4) & -c(1,2,4,4) & c(1,2,3,4) \end{pmatrix}$
$4_{3,D(2),3}$	$\begin{pmatrix} \frac{c(1,2,4,1)c(1,3,4,1)}{c(1,2,4,2)} & c(1,3,4,1) & -c(1,2,4,1) & 0 \\ \frac{c(1,2,4,1)c(1,3,4,2)}{c(1,2,4,2)} & c(1,3,4,2) & -c(1,2,4,2) & 0 \\ \frac{c(1,2,4,1)c(1,3,4,3)}{c(1,2,4,2)} & c(1,3,4,3) & -c(1,2,4,3) & c(1,2,3,3) \\ \frac{c(1,2,4,1)c(1,3,4,4)}{c(1,2,4,2)} & c(1,3,4,4) & -c(1,2,4,4) & c(1,2,3,4) \end{pmatrix}$
$4_{3,D(2),4}$	$\begin{pmatrix} -c(2,3,4,1) & 0 & -c(1,2,4,1) & c(1,2,3,1) \\ -c(2,3,4,2) & 0 & 0 & 0 \\ -c(2,3,4,3) & 0 & -c(1,2,4,3) & c(1,2,3,3) \\ -c(2,3,4,4) & 0 & -c(1,2,4,4) & c(1,2,3,4) \end{pmatrix}$
$4_{3,D(2),5}$	$\begin{pmatrix} 0 & 0 & -c(1,2,4,1) & c(1,2,3,1) \\ 0 & 0 & -c(1,2,4,2) & c(1,2,3,2) \\ 0 & 0 & -c(1,2,4,3) & c(1,2,3,3) \\ 0 & 0 & -c(1,2,4,4) & c(1,2,3,4) \end{pmatrix}$

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Chapter 20
On Classification of (n+1)-Dimensional n-Hom-Lie Algebras for n = 4, 5, 6 and Nilpotent Twisting Map with 2-Dimensional Kernel

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Abstract The aim of this work is to study properties of n-Hom-Lie algebras in dimension $n+1$ allowing to explicitely find them and differentiate them, to eventually classify them. Specifically, the n-Hom-Lie algebras in dimension $n+1$ for $n = 4, 5, 6$ and nilpotent α with 2-dimensional kernel are computed and some detailed properties of these algebras are obtained.

Keywords Hom-algebra · n-Hom-Lie algebra

MSC2020 Classification 17B61 · 17D30 · 17A40 · 17A42 · 17B30

20.1 Introduction

Hom-Lie algebras and more general quasi-Hom-Lie algebras where introduced first by Hartwig, Larsson and Silvestrov in [49], where the general quasi-deformations and discretizations of Lie algebras of vector fields using more general σ-derivations (twisted derivations) and a general method for construction of deformations of Witt and Virasoro type algebras based on twisted derivations have been developed, initially motivated by the q-deformed Jacobi identities observed for the q-deformed algebras in physics, along with q-deformed versions of homological algebra and discrete modifications of differential calculi [7, 33–38, 41, 51, 53, 67–69]. The general abstract quasi-Lie algebras and the subclasses of quasi-Hom-Lie algebras and Hom-Lie algebras as well as their general colored (graded) counterparts have been introduced in [49, 61–63, 81]. Subsequently, various classes of Hom-Lie admissible

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algebras have been considered in [71]. In particular, in [71], the Hom-associative algebras have been introduced and shown to be Hom-Lie admissible, that is leading to Hom-Lie algebras using commutator map as new product, and in this sense constituting a natural generalization of associative algebras, as Lie admissible algebras leading to Lie algebras via commutator map as new product. In [71], moreover several other interesting classes of Hom-Lie admissible algebras generalizing some classes of non-associative algebras, as well as examples of finite-dimensional Hom-Lie algebras have been described. Since these pioneering works [49, 61–64, 71], Hom-algebras turned out to be very useful since Hom-algebra structures of a given type include their classical counterparts and open more possibilities for deformations, extensions of cohomological structures and representations. Hom-algebra structures have developed in a popular broad area with increasing number of publications in various directions (see for example [8, 29, 44, 60, 61, 65, 72–74, 77, 79, 80, 85, 86] and references therein).

Ternary Lie algebras appeared in generalization of Hamiltonian mechanics by Nambu [75]. Besides Nambu mechanics, n-Lie algebras revealed to have many applications in physics. The mathematical algebraic foundations of Nambu mechanics have been developed by Takhtajan in [82]. Filippov, in [47] independently introduced and studied structure of n-Lie algebras and Kasymov [54] investigated their properties. Properties of n-ary algebras, including solvability and nilpotency, were studied in [16, 21, 54]. Kasymov [54] pointed out that n-ary multiplication allows for several different definitions of solvability and nilpotency in n-Lie algebras, and studied their properties. Further properties, classification, and connections of n-ary algebras to other structures such as bialgebras, Yang-Baxter equation and Manin triples for 3-Lie algebras were studied in [15–22, 24, 25, 54]. The structure of 3-Lie superalgebras induced by Lie superalgebras, classification of 3-Lie superalgebras and application to constructions of B.R.S. algebras have been considered in [2–4]. Interesting constructions of ternary Lie superalgebras in connection to superspace extension of Nambu-Hamilton equation is considered in [5]. In [32], Leibniz n-algebras have been studied. The general cohomology theory for n-Lie algebras and Leibniz n-algebras was established in [42, 78, 83]. For more details of the theory and applications of n-Lie algebras, see [43] and references therein.

Classifications of n-ary or Hom generalizations of Lie algebras have been considered, either in very special cases or in low dimensions. The classification of n-Lie algebras of dimension up to $n+1$ over a field of characteristic $p \neq 2$ has been completed by Filippov [47] using the specific properties of $(n+1)$-dimensional n-Lie algebras that make it possible to represent their bracket by a square matrix in a similar way as bilinear forms, the number of cases obtained depends on the properties of the base field, the list is ordered by ascending dimension of the derived ideal, and among them, one nilpotent algebra, and a class of simple algebras which are all isomorphic in the case of an algebraically closed field, the remaining algebras are k-solvable for some $2 \leq k \leq n$ depending on the algebra. These simple algebras are proven to be the only simple finite-dimensional n-Lie algebras in [66]. The classification of $(n+1)$-dimensional n-Lie algebras over a field of characteristic 2 has been done by Bai et al. [22] by finding and using a similar result in characteristic 2. Bai et al. [21]

classify the $(n+2)$-dimensional n-Lie algebras over an algebraically closed field of characteristic 0 using the fact that an $(n+2)$-dimensional n-Lie algebra has a subalgebra of codimension 1 if the dimension of its derived ideal is not 3, thus constructing most of the cases as extensions of the $(n+1)$-dimensional n-Lie algebras listed by Filippov. In [31], Cantarini and Kac classified all simple linearly compact n-Lie superalgebras, which turned out to be n-Lie algebras, by finding a bijective correspondence between said algebras and a special class of transitive $\mathbb{Z}$-graded Lie superalgebras, the list they obtained consists of four representatives, one of them is the $(n+1)$-dimensional vector product n-Lie algebra, and the remaining three are infinite-dimensional n-Lie algebras.

Classifications of n-Lie algebras in higher dimensions have only been studied in particular cases. Metric n-Lie algebras, that is n-Lie algebras equipped with a non-degenerate compatible bilinear form, have been considered and classified, first in dimension $n+2$ by Ren et al. [76] and dimension $n+3$ by Geng et al. [48], and then in dimensions $n+k$ for $2 \leq k \leq n+1$ by Bai et al. [23]. The classification is based on the study of the Levi decomposition, the center and the isotropic ideals and properties around them. Another case that has been studied is the case of nilpotent n-Lie algebras, more specifically nilpotent n-Lie algebras of class 2. Eshrati et al. [45] classify $(n+3)$-dimensional nilpotent n-Lie algebras and $(n+4)$-dimensional nilpotent n-Lie algebras of class 2 using properties introduced in [39, 46]. Similarly Hoseini et al. [50] classify $(n+5)$-dimensional nilpotent n-Lie algebras of class 2. In [52], Jamshidi, Saeedi and Darabi classify $(n+6)$-dimensional nilpotent n-Lie algebras of class 2 using the fact that such algebras factored by the span of a central element give $(n+5)$-dimensional nilpotent n-Lie algebras of class 2, which were classified before. Classification of other classes of nilpotent n-Lie algebras depending on dimension of multiplier has been considered in [40]. There has been a study of the classification of 3-dimensional 3-Hom-Lie algebras with diagonal twisting maps by Ataguema, Makhlouf and Silvestrov in [13].

Hom-type generalization of n-ary algebras, such as n-Hom-Lie algebras and other n-ary Hom algebras of Lie type and associative type, were introduced in [13], by twisting the defining identities by a set of linear maps. The particular case, where all these maps are equal and are algebra morphisms has been considered and a way to generate examples of n-ary Hom-algebras from n-ary algebras of the same type have been described. Further properties, construction methods, examples, representations, cohomology and central extensions of n-ary Hom-algebras have been considered in [9–12, 55, 56, 84, 87]. These generalizations include n-ary Hom-algebra structures generalizing the n-ary algebras of Lie type including n-ary Nambu algebras, n-ary Nambu-Lie algebras and n-ary Lie algebras, and n-ary algebras of associative type including n-ary totally associative and n-ary partially associative algebras. In [58], constructions of n-ary generalizations of BiHom-Lie algebras and BiHom-associative algebras have been considered. Generalized Derivations of n-BiHom-Lie algebras have been studied in [28]. Generalized derivations of multiplicative n-ary Hom-Ω color algebras have been studied in [30]. Cohomology of Hom-Leibniz and n-ary Hom-Nambu-Lie superalgebras has been considered in [1] Generalized Derivations and Rota-Baxter Operators of n-ary Hom-Nambu Superalgebras have been considered in [70]. A construction of 3-Hom-Lie algebras based on σ-derivation and involution has been studied in [6]. Multiplicative n-Hom-Lie color algebras have been considered in [26].

In [11, 12], the construction of $(n+1)$-Lie algebras induced by n-Lie algebras using combination of bracket multiplication with a trace, motivated by the work of Awata et al. [14] on the quantization of the Nambu brackets, was generalized using the brackets of general Hom-Lie algebras or n-Hom-Lie algebras and trace-like linear forms satisfying conditions depending on the twisting linear maps defining the Hom-Lie or n-Hom-Lie algebras. In [27], a method was demonstrated of how to construct n-ary multiplications from the binary multiplication of a Hom-Lie algebra and a $(n-2)$-linear function satisfying certain compatibility conditions. Solvability and nilpotency for n-Hom-Lie algebras and $(n+1)$-Hom-Lie algebras induced by n-Hom-Lie algebras have been considered in [57].

n-Hom-Lie algebras are fundamentally different from n-Lie algebras especially when the twisting maps are not invertible or not diagonalizable. When the twisting maps are not invertible, the Hom-Nambu-Filippov identity becomes less restrictive since when elements of the kernel of the twisting maps are used, several terms or even the whole identity might vanish. Isomorphisms of Hom-algebras are also different from isomorphisms of algebras since they need to intertwine not only the multiplications but also the twisting maps. All of this make the classification problem different, interesting, rich and not simply following from the case of n-Lie algebras. In this work, we consider n-Hom-Lie algebras with a nilpotent twisting map α, which means in particular that α is not invertible.

In [59], properties and classification of n-Hom-Lie algebras in dimension $n+1$ were considered, 4-dimensional 3-Hom-Lie algebras for various special cases of the twisting map have been computed in terms of structure constants as parameters and listed in classes in the way emphasising the number of free parameters in each class.

The aim of this article is to study n-Hom-Lie algebras in dimension $n+1$ for $4 \leq n \leq 6$ and nilpotent α with 2-dimensional kernel, give a list of algebras, study their properties and compare different dimensions. Section 20.2 contains basic definitions and some general properties of n-Hom-Lie algebras as well some specific properties of $(n+1)$-dimensional n-Hom-Lie algebras and the systems of equations corresponding to Hom-Nambu-Filippov identity. In Sect. 20.3, the n-Hom-Lie algebras in dimension $n+1$ for $n = 4, 5, 6$ and nilpotent α with 2-dimensional kernel are computed and some detailed properties of these algebras are obtained.

20.2 Definitions and Properties of n-Hom-Lie Algebras

In this section, we present the basic definitions and properties of n-Hom-Lie algebras needed for our study. Throughout this article, it is assumed that all vector spaces are over a field $\mathbb{K}$ of characteristic 0, and for any subset S of a vector space, $\langle S\rangle$ denotes the linear span of S. Hom-Lie algebras are a generalization of Lie algebras introduced in [49] while studying σ-derivations. The n-ary case was introduced in [13].

Definition 20.1 ([49, 71]) A Hom-Lie algebra $(A, [\cdot,\cdot], \alpha)$ is a vector space A together with a bilinear map $[\cdot,\cdot] : A\times A \to A$ and a linear map $\alpha : A \to A$ satisfying, for all $x, y, z \in A$,

$$[x, y] = -[y, x] \qquad \text{Skew-symmetry}$$
$$[\alpha(x), [y, z]] = [[x, y], \alpha(z)] + [\alpha(y), [x, z]] \qquad \text{Hom-Jacobi identity}$$

In Hom-Lie algebras, by skew-symmetry, the Hom-Jacobi identity is equivalent to

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0. \qquad \text{Hom-Jacobi identity in cyclic form}$$

Definition 20.2 ([49, 61]) Hom-Lie algebra morphisms from Hom-Lie algebra $\mathcal{A} = (A, [\cdot,\cdot]_{\mathcal{A}}, \alpha)$ to Hom-Lie algebra $\mathcal{B} = (B, [\cdot,\cdot]_{\mathcal{B}}, \beta)$ are linear maps $f : A \to B$ satisfying, for all $x, y \in A$,

$$f([x, y]_{\mathcal{A}}) = [f(x), f(y)]_{\mathcal{B}}, \tag{20.1}$$
$$f\circ\alpha = \beta\circ f. \tag{20.2}$$

Linear maps $f : A \to B$ satisfying only condition (20.1) are called weak morphisms of Hom-Lie algebras.

Definition 20.3 ([29, 71]) A Hom-Lie algebra $(A, [\cdot,\cdot], \alpha)$ is said to be multiplicative if α is an algebra morphism, and it is said to be regular if α is an isomorphism.

Definition 20.4 ([13]) An n-Hom-Lie algebra $(A, [\cdot,\dots,\cdot], \{\alpha_i\}_{1\le i\le n-1})$ is a vector space A together with a n-linear map $[\cdot,\dots,\cdot] : A^n \to A$ and $(n-1)$ linear maps $\alpha_i : A \to A$, $1 \le i \le n-1$ satisfying, for all $x_1,\dots,x_{n-1}, y_1,\dots,y_n \in A$,

Skew-symmetry

$$[x_{\sigma(1)},\dots,x_{\sigma(n)}] = sgn(\sigma)[x_1,\dots,x_n], \tag{20.3}$$

Hom-Nambu-Filippov identity

$$[\alpha_1(x_1),\dots,\alpha_{n-1}(x_{n-1}), [y_1,\dots,y_n]] = \tag{20.4}$$
$$\sum_{i=1}^{n}[\alpha_1(y_1),\dots,\alpha_{i-1}(y_{i-1}), [x_1,\dots,x_{n-1}, y_i], \alpha_i(y_{i+1}),\dots,\alpha_{n-1}(y_n)].$$

Remark 20.1 If $\alpha_i = Id_A$ for all $1 \leq i \leq n-1$, then one gets an n-Lie algebra [47]. Therefore, the class of n-Lie algebras is included in the class of n-Hom-Lie algebras. For any vector space A, if $[x_1, \ldots, x_n]_0 = 0$ for all $x_1, .., x_n \in A$ and any linear maps $\alpha_1, \ldots, \alpha_{n-1}$, then $(A, [\cdot, \ldots, \cdot]_0, \alpha_1, \ldots, \alpha_{n-1})$ is an n-Hom-Lie algebra.

Definition 20.5 ([13, 87]) n-Hom-Lie algebra morphisms of n-Hom-Lie algebras

$$\mathcal{A} = (A, [\cdot, \ldots, \cdot]_{\mathcal{A}}, \{\alpha_i\}_{1\leq i\leq n-1}) \text{ and } \mathcal{B} = (B, [\cdot, \ldots, \cdot]_{\mathcal{B}}, \{\beta_i\}_{1\leq i\leq n-1})$$

are linear maps $f : A \to B$ satisfying, for all $x_1, \ldots, x_n \in A$,

$$f([x_1, \ldots, x_n]_{\mathcal{A}}) = [f(x_1), \ldots, f(x_n)]_{\mathcal{B}}, \tag{20.5}$$

$$f \circ \alpha_i = \beta_i \circ f, \quad \text{for all} \quad 1 \leq i \leq n-1. \tag{20.6}$$

Linear maps satisfying only condition (20.5) are called weak morphisms of n-Hom-Lie algebras.

The n-Hom-Lie algebras $(A, [\cdot, \ldots, \cdot], \{\alpha_i\}_{1\leq i\leq n-1})$ with $\alpha_1 = \cdots = \alpha_{n-1} = \alpha$ will be denoted by $(A, [\cdot, \ldots, \cdot], \alpha)$.

Definition 20.6 ([87]) An n-Hom-Lie algebra $(A, [\cdot, \ldots, \cdot], \alpha)$ is called multiplicative if α is an algebra morphism, and regular if α is an algebra isomorphism.

The following proposition, providing a way to construct an n-Hom-Lie algebra from an n-Lie algebra and an algebra morphism, was first introduced in the case of Lie algebras and then generalized to the n-ary case in [13]. A more general version of this theorem, given in [87], states that the category of n-Hom-Lie algebras is closed under twisting by weak morphisms.

Proposition 20.1 ([13, 87]) *Let $\beta : \mathcal{A} \to \mathcal{A}$ be a weak morphism of n-Hom-Lie algebra $\mathcal{A} = \left(A, [\cdot, \ldots, \cdot], \{\alpha_i\}_{1\leq i\leq n-1}\right)$, and multiplication $[\cdot, \ldots, \cdot]_\beta$ is defined by $[x_1, \ldots, x_n]_\beta = \beta([x_1, \ldots, x_n])$. Then, $\left(A, [\cdot, \ldots, \cdot]_\beta, \{\beta \circ \alpha_i\}_{1\leq i\leq n-1}\right)$ is an n-Hom-Lie algebra. Moreover, if $(A, [\cdot, \ldots, \cdot], \alpha)$ is multiplicative and $\beta \circ \alpha = \alpha \circ \beta$, then $\left(A, [\cdot, \ldots, \cdot]_\beta, \beta \circ \alpha\right)$ is multiplicative.*

The following particular case of Proposition 20.1 is obtained if $\alpha = Id_A$.

Corollary 20.1 *Let $(A, [\cdot, \ldots, \cdot])$ be an n-Lie algebra, $\beta : A \to A$ an algebra morphism, and $[\cdot, \ldots, \cdot]_\beta$ is defined by $[x_1, \ldots, x_n]_\beta = \beta([x_1, \ldots, x_n])$. Then, $\left(A, [\cdot, \ldots, \cdot]_\beta, \beta\right)$ is a multiplicative n-Hom-Lie algebra.*

Fundamental objects and basic algebra were first introduced for n-Lie algebras in [42] and generalized to n-Hom-Lie algebras in [9].They allow to define actions and representations of these n-ary algebras.

Definition 20.7 ([9]) Let $(A, [\cdot, \ldots, \cdot], \alpha)$ be a multiplicative n-Hom-Lie algebra and let $L(A) = \wedge^{n-1} A$ be the $(n-1)$th exterior power of A. The elements of $L(A)$ are called fundamental objects.

For $X = x_1 \wedge \cdots \wedge x_{n-1}$, $Y = y_1 \wedge \cdots \wedge y_{n-1} \in L(A)$, we define:

- The map $\bar{\alpha} : \wedge^{n-1}A \to \wedge^{n-1}A$ by $\bar{\alpha}(X) = \alpha(x_1) \wedge \cdots \wedge \alpha(x_{n-1})$.
- The action of fundamental objects on A by

$$\forall z \in A,\ \ X \cdot z = ad_X(z) = \left[x_1, \dots, x_{n-1}, z\right].$$

- The multiplication (composition) of two fundamental objects by

$$[X, Y]_\alpha = X \cdot_\alpha Y = \sum_{i=1}^{n-1} \alpha(y_1) \wedge \cdots \wedge X \cdot y_i \wedge \cdots \wedge \alpha(y_{n-1}).$$

We extend the preceding definitions to the entire space $L(A)$ by linearity.

Proposition 20.2 ([9]) *The space $L(A)$ equipped with the bracket $[\cdot, \cdot]_\alpha$ defined above is a Hom-Leibniz algebra, that is the bracket $[\cdot, \cdot]_\alpha$ satisfies the following identity*

$$[\bar{\alpha}(X), [Y, Z]_\alpha]_\alpha = [[X, Y]_\alpha, \bar{\alpha}(Z)]_\alpha + [\bar{\alpha}(Y), [X, Z]_\alpha]_\alpha.$$

Definition 20.8 ([29, 71, 87]) An n-Hom-Lie subalgebra $\mathcal{B} = (B, [\cdot, \dots, \cdot]_{\mathcal{B}}, \beta_1, \dots, \beta_{n-1})$ of an n-Hom-Lie algebra $\mathcal{A} = (A, [\cdot, \dots, \cdot]_{\mathcal{A}}, \alpha_1, \dots, \alpha_{n-1})$ consists of a subspace B of A satisfying, for all $x_1, \dots, x_n \in B$,

(1) $\alpha_i(B) \subseteq B$ for all $1 \leq i \leq n-1$,
(2) $[x_1, \dots, x_n]_{\mathcal{A}} \in B$,

with the restricted from A multiplication $[\cdot, \dots, \cdot]_{\mathcal{B}} = [\cdot, \dots, \cdot]_{\mathcal{A}}$ and linear maps $\beta_i = \alpha_i$, $1 \leq i \leq n-1$ on B.

Definition 20.9 ([29, 71, 87]) For any n-Hom-Lie algebra $(A, [\cdot, \dots, \cdot], \alpha_1, \dots, \alpha_{n-1})$, an ideal is a subspace I of A satisfying, for all $x_1, \dots, x_{n-1} \in A$ and $y \in I$,

(1) $\alpha_i(I) \subseteq I$ for all $1 \leq i \leq n-1$;
(2) $[x_1, \dots, x_{n-1}, y] \in I$.

Definition 20.10 ([57]) Let $(A, [\cdot, \dots, \cdot], \alpha_1, \dots, \alpha_{n-1})$ be an n-Hom-Lie algebra, and let I be an ideal of A. For $2 \leq k \leq n$, we define the k-derived series of the ideal I by

$$D_k^0(I) = I \text{ and } D_k^{r+1} = \left[\underbrace{D_k^p(I), \dots, D_k^p(I)}_{k}, \underbrace{A, \dots, A}_{n-k}\right].$$

We define the k-central descending series of I by

$$C_k^0(I) = I \text{ and } C_k^{p+1}(I) = \left[C_k^p(I), \underbrace{I, \dots, I}_{k-1}, \underbrace{A, \dots, A}_{n-k}\right].$$

Definition 20.11 ([57]) Let $(A, [\cdot, \ldots, \cdot], \alpha_1, \ldots, \alpha_{n-1})$ be an n-Hom-Lie algebra, and let I be an ideal of A. For $2 \leq k \leq n$, the ideal I is said to be k-solvable (resp. k-nilpotent) if there exists $r \in \mathbb{N}$ such that $D_k^r(I) = \{0\}$ (resp. $C_k^r(I) = \{0\}$). in this case, the smallest $r \in \mathbb{N}$ satisfying this condition is called the class of k-solvability (resp. the class of k-nilpotence) of I.

Lemma 20.1 ([57]) *For any two n-Hom-Lie algebras $\mathcal{A} = (A, [\cdot, \ldots, \cdot]_A, \alpha_1, \ldots, \alpha_{n-1})$ and $\mathcal{B} = (B, [\cdot, \ldots, \cdot]_B, \beta_1, \ldots, \beta_{n-1})$, let $f : \mathcal{A} \to \mathcal{B}$ be a surjective n-Hom-Lie algebras morphism and I an ideal of $\mathcal{A}$. Then for all $r \in \mathbb{N}$ and $2 \leq k \leq n$,*

$$f\left(D_k^r(I)\right) = D_k^r\left(f(I)\right) \text{ and } f\left(C_k^r(I)\right) = C_k^r\left(f(I)\right).$$

This lemma also implies that if two n-Hom-Lie algebras are isomorphic, they would also have isomorphic members of the derived series and central descending series, which also means that if two algebras have a significant difference in the derived series or the central descending series, for example different dimensions of given corresponding members, then these algebras cannot be isomorphic.

Lemma 20.2 ([59]) *Let A be a vector space, let $[\cdot, \ldots, \cdot]$ be an n-linear skew-symmetric map and let $\alpha_1, \ldots, \alpha_{n-1}$ be linear maps on A. If the $(n-1)$-linear map*

$$(x_1, \ldots, x_{n-1}) \mapsto \left[\alpha_1(x_1), \ldots, \alpha_{n-1}(x_{n-1}), d\right]$$

is skew-symmetric for all $d \in [A, \ldots, A]$, then the $(2n-1)$-linear map H, defined by

$$\begin{aligned} H(x_1, \ldots, x_{n-1}, y_1, \ldots, y_n) &= [\alpha_1(x_1), \ldots, \alpha_{n-1}(x_{n-1}), [y_1, \ldots, y_n]] \\ &- \sum_{k=1}^{n} [\alpha_1(y_1), \ldots, \alpha_{k-1}(y_{k-1}), [x_1, \ldots, x_{n-1}, y_k], \alpha_k(y_{k+1}), \ldots, \alpha_{n-1}(y_n)], \end{aligned}$$

for all $x_1, \ldots, x_{n-1}, y_1, \ldots, y_n \in A$, is skew-symmetric in its first $n-1$ arguments and in its last n arguments.

Proposition 20.3 ([59]) *Let A be an n-dimensional vector space, and $(e_i)_{1 \leq i \leq n}$ a basis of A. Any skew-symmetric n-linear map $[\cdot, \ldots, \cdot]$ on A is fully defined by giving $[e_1, \ldots, e_n] = d \in A$. Let $\alpha_1, \ldots, \alpha_{n-1}$ be linear maps on A. If the $(n-1)$-linear map*

$$(x_1, \ldots, x_{n-1}) \mapsto \left[\alpha_1(x_1), \ldots, \alpha_{n-1}(x_{n-1}), d\right]$$

is skew-symmetric, then $(A, [\cdot, \ldots, \cdot], \alpha_1, \ldots, \alpha_{n-1})$ is an n-Hom-Lie algebra.

Corollary 20.2 ([59]) *Let A be an n-dimensional vector space, and $(e_i)_{1 \leq i \leq n}$ a basis of A. Any skew-symmetric n-linear map $[\cdot, \ldots, \cdot]$ on A is fully defined by giving $[e_1, \ldots, e_n] = d \in A$. For any linear map α on A, $(A, [\cdot, \ldots, \cdot], \alpha)$ is an n-Hom-Lie algebra.*

Let $(A, [\cdot, \ldots, \cdot], \alpha)$ be an n-ary skew-symmetric algebra of dimension $n+1$ with a linear map α. Given a linear basis $(e_i)_{1\le i\le n+1}$ of A, linear map α is fully determined by its matrix determined by action of α on the basis, and a skew-symmetric n-ary multi-linear bracket is fully determined by $\left[e_1, \ldots, \widehat{e_i}, \ldots, e_{n+1}\right]$ for all $1 \le i \le n+1$ represented by a matrix B as follows:

$$\begin{aligned}\left[e_1, \ldots, \widehat{e_i}, \ldots, e_{n+1}\right] &= (-1)^{n+1+i} w_i,\\ w_i &= \sum_{p=1}^{n+1} b(p,i)e_p,\\ (w_1, \ldots, w_{n+1}) &= (e_1, \ldots, e_{n+1})B, \quad \text{for} \quad B = (b(i,j))_{1\le i,j\le n+1}.\end{aligned} \tag{20.7}$$

Proposition 20.4 ([59]) *Let* $\mathcal{A}_1 = (A, [\cdot, \ldots, \cdot]_1, \alpha_1)$ *and* $\mathcal{A}_2 = (A, [\cdot, \ldots, \cdot]_2, \alpha_2)$ *be two* $(n+1)$*-dimensional* n*-ary skew-symmetric Hom-algebras represented by matrices* $[\alpha_1]$, B_1 *and* $[\alpha_2]$, B_2 *respectively. The Hom-algebras* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *are isomorphic if and only if there exists an invertible matrix* T *satisfying the following conditions:*

$$B_2 = \det(T)^{-1} T B_1 T^T,$$

$$[\alpha_2] = T[\alpha_1]T^{-1}.$$

Proposition 20.5 ([59]) *Let* $(e_i)_{1\le i\le n+1}$ *be a basis of* A*, let* σ *be an* $n+1$ *permutation, and let* $B = (b_{i,j})_{1\le i,j\le n+1}$ *be a matrix representing a skew-symmetric* n*-ary bracket in this basis, then the matrix representing the same bracket in the basis* $(e_{\sigma(i)})_{1\le i\le n+1}$ *is* $sgn(\sigma)(b_{\sigma^{-1}(i),\sigma^{-1}(j)})_{1\le i,j\le n+1}$.

Remark 20.2 ([59]) Let $(A, [\cdot, \ldots, \cdot], \alpha)$ be an $(n+1)$-dimensional n-Hom-Lie algebra and let B be the matrix representing its bracket. $D_n^1(A) = [A, \ldots, A]$ is generated by $\{w_1, \ldots, w_{n+1}\}$. Which means that $Rank(B) = \dim D_n^1(A)$.

If $Rank(B) \le n$ or equivalently $\det(B) = 0$ then $D_n^1(A)$ has dimension at most n, which means that $D_n^2(A)$ has dimension at most 1 and then $D_n^3(A) = 0$.

Remark 20.3 ([59]) For the whole algebra A, all the k-central descending series, for all $2 \le k \le n$, are equal, therefore all the notions of k-nilpotency, for all $2 \le k \le n$, are equivalent.

Proposition 20.6 ([59]) *Let* $(A, [\cdot, \ldots, \cdot], \alpha)$ *be an* n*-ary Hom-algebra,* $\dim A = n+1$, $[\cdot, \ldots, \cdot]$ *skew-symmetric,* α *nilpotent,* $\dim \ker \alpha = 2$ *and the bracket is represented by the matrix* $B = (b_{i,j})$ *in a basis where* α *is in Jordan normal form, as detailed above. The bracket* $[\cdot, \ldots, \cdot]$ *satisfies the Hom-Nambu-Filippov identity if and only if*

$$\begin{aligned}&\forall\ 1 \le j, p \le n+1,\ j \ne 1,\ j \ne i_0 :\\ &\quad b_{i_0-1,j} b_{p,n+1} - b_{n+1,j} b_{p,i_0-1} = 0,\end{aligned}$$

where i_0 *is such that* $\ker \alpha = \langle e_1, e_{i_0}\rangle$.

Remark 20.4 Let us compare the polynomial equations obtained from the Nambu-Filippov identity and the Hom-Nambu-Filippov identity in dimension $n+1$ with various types of twisting maps:
Diagonalizable and invertible with eigenvalues $\{\lambda_i\}_{1\leq i\leq n+1}$:

$$\forall\ 1 \leq i, j, k, p \leq n+1,\ i < j < k :\\ (\lambda_i b_{j,i} - \lambda_j b_{i,j}) b_{p,k} + (\lambda_k b_{i,k} - \lambda_i b_{k,i}) b_{p,j} + (\lambda_j b_{k,j} - \lambda_k b_{j,k}) b_{p,i} = 0; \quad (20.8)$$

Diagonalizable with $\dim\ker\alpha = 1$ with eigenvalues $\{\lambda_i\}_{1\leq i\leq n+1}$:

$$\forall\ 1 < j < k \leq n+1\\ \lambda_k b_{1,k} w_j - \lambda_k b_{j,k} w_1 - \lambda_j b_{1,j} w_k + \lambda_j b_{k,j} w_1 = 0; \quad (20.9)$$

Diagonalizable with $\dim\ker\alpha = 2$ with eigenvalues $\{\lambda_i\}_{1\leq i\leq n+1}$:

$$\forall\ 3 \leq k \leq n+1 : \quad b_{1,k} w_2 - b_{2,k} w_1 = 0; \quad (20.10)$$

Nilpotent with $\dim\ker\alpha = 1$:

$$\forall\ 1 \leq i, k, p \leq n+1,\ i < k :\\ (b_{k-1,i} - b_{i-1,k}) b_{p,n+1} - b_{n+1,i} b_{p,k-1} + b_{n+1,k} b_{p,i-1} = 0; \quad (20.11)$$

Nilpotent with $\dim\ker\alpha = 2$:

$$\forall\ 1 \leq j, p \leq n+1,\ j \neq 1,\ j \neq i_0 : \quad b_{i_0-1,j} b_{p,n+1} - b_{n+1,j} b_{p,i_0-1} = 0. \quad (20.12)$$

These different cases are separate from each other, and the case of n-Lie algebras is the special case of (20.8) where all the λ_i are equal. Notice that the higher the dimension of $\ker\alpha$ the less equation we have and the less terms we have in each equation, that is, in these cases, the Hom-Nambu-Filippov identity is considerably less restrictive. Another difference from the case of n-Lie algebra are the isomorphisms, in Hom-algebras, an isomorphism intertwines the multiplications and the twisting maps, which leads to different, more restrictive isomorphism conditions.

20.3 Lists of (n+1)-Dimensional n-Hom-Lie Algebras in Various Arities

Now, we present lists of $(n+1)$-dimensional n-Hom-Lie algebras with nilpotent structure map α with $\ker\alpha = 2$, for $4 \leq n \leq 6$. For an $(n+1)$-dimensional vector space with a basis (e_i), we represent a linear map α by its matrix in this bases, and a skew-symmetric n-ary multi-linear bracket by a matrix B defined by

$$\left[e_1, \ldots, \widehat{e_i}, \ldots, e_{n+1}\right] = (-1)^{n+1+i} w_i,$$
$$w_i = \sum_{p=1}^{n+1} b(p, i) e_p$$
$$(w_1, \ldots, w_{n+1}) = (e_1, \ldots, e_{n+1})B, \quad \text{for} \quad B = (b(i, j))_{1 \le i, j \le n+1}.$$

Remark 20.5 In the usual way, when one would work with structure constants of an n-Hom-Lie algebra (or any n-ary algebra), we would have

$$\left[e_{i_1}, \ldots, e_{i_n}\right] = \sum_{p=1}^{d} c^p_{i_1,\ldots,i_n} e_p,$$

which can be simplified, in the case of $(n+1)$-dimensional n-Hom-Lie algebras by skew-symmetry to

$$\left[e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_{n+1}\right] = \sum_{p=1}^{n+1} c^p_{1,\ldots,i-1,i+1,\ldots,n+1} e_p.$$

In computations, this notation becomes unpractical with increasing arity. In our notation,

$$b(i, p) = (-1)^{n+i+1} c^p_{1,\ldots,i-1,i+1,\ldots,n+1}.$$

If α is nilpotent with $\dim \ker \alpha = 2$, then it would have two Jordan blocks of size p and q such that $p + q = n + 1$. If (e_i) is a basis where α is in Jordan normal form, then $\ker \alpha = \langle e_1, e_{i_0} \rangle$. The Hom-Nambu-Filippov identity is equivalent, in this case, to the following system of equations:

$$\forall\, 1 \le j, p \le n+1, \;\; j \ne 1, \;\; j \ne i_0 :$$
$$b(i_0 - 1, j) b(p, n+1) - b(n+1, j) b(p, i_0 - 1) = 0.$$

We solve this system using the computer algebra system Mathematica for $n = 4, 5, 6$, and for each n, we take all the values of i_0 so that we get all the possible cases for the sizes of the Jordan blocks, that is $2 \le i_0 \le \frac{n}{2} + 1$. We get the following brackets, represented by matrices as previously explained (Tables 20.1, 20.2, 20.3, 20.4, 20.5, 20.6, 20.7 and 20.8).

Remark 20.6 For each n, the algebras in the same position in different tables are isomorphic as n-ary skew-symmetric algebras. One can show this using Proposition 20.5 for the transposition exchanging the values of $i_0 - 1$ for the corresponding tables. However, they cannot be isomorphic as n-Hom-Lie algebras since the matrices representing the twisting maps for each table are not similar.

Table 20.1 $n = 6,\ \dim = 7,\ \ker\alpha = \langle e_1, e_2\rangle,\ [\alpha] = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0 \end{pmatrix}$

$7_{6,N,2,2,1}$	$\begin{pmatrix} b(1,1) & b(1,2) & b(1,3) & b(1,4) & b(1,5) & b(1,6) & b(7,1)\\ b(2,1) & b(2,2) & b(2,3) & b(2,4) & b(2,5) & b(2,6) & \frac{b(2,1)b(7,1)}{b(1,1)}\\ b(3,1) & b(3,2) & b(3,3) & b(3,4) & b(3,5) & b(3,6) & \frac{b(3,1)b(7,1)}{b(1,1)}\\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & b(4,5) & b(4,6) & \frac{b(4,1)b(7,1)}{b(1,1)}\\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & b(5,6) & \frac{b(5,1)b(7,1)}{b(1,1)}\\ b(6,1) & b(6,2) & b(6,3) & b(6,4) & b(6,5) & b(6,6) & \frac{b(6,1)b(7,1)}{b(1,1)}\\ b(7,1) & \frac{b(1,2)b(7,1)}{b(1,1)} & \frac{b(1,3)b(7,1)}{b(1,1)} & \frac{b(1,4)b(7,1)}{b(1,1)} & \frac{b(1,5)b(7,1)}{b(1,1)} & \frac{b(1,6)b(7,1)}{b(1,1)} & \frac{b(7,1)^2}{b(1,1)} \end{pmatrix}$
$7_{6,N,2,2,2}$	$\begin{pmatrix} 0 & b(1,2) & b(1,3) & b(1,4) & b(1,5) & b(1,6) & 0\\ b(2,1) & b(2,2) & b(2,3) & b(2,4) & b(2,5) & b(2,6) & b(2,7)\\ b(3,1) & b(3,2) & b(3,3) & b(3,4) & b(3,5) & b(3,6) & \frac{b(2,7)b(3,1)}{b(2,1)}\\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & b(4,5) & b(4,6) & \frac{b(2,7)b(4,1)}{b(2,1)}\\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & b(5,6) & \frac{b(2,7)b(5,1)}{b(2,1)}\\ b(6,1) & b(6,2) & b(6,3) & b(6,4) & b(6,5) & b(6,6) & \frac{b(2,7)b(6,1)}{b(2,1)}\\ 0 & \frac{b(1,2)b(2,7)}{b(2,1)} & \frac{b(1,3)b(2,7)}{b(2,1)} & \frac{b(1,4)b(2,7)}{b(2,1)} & \frac{b(1,5)b(2,7)}{b(2,1)} & \frac{b(1,6)b(2,7)}{b(2,1)} & 0 \end{pmatrix}$
$7_{6,N,2,2,3}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & b(2,2) & b(2,3) & b(2,4) & b(2,5) & b(2,6) & b(2,7)\\ 0 & b(3,2) & b(3,3) & b(3,4) & b(3,5) & b(3,6) & b(3,7)\\ 0 & b(4,2) & b(4,3) & b(4,4) & b(4,5) & b(4,6) & b(4,7)\\ 0 & b(5,2) & b(5,3) & b(5,4) & b(5,5) & b(5,6) & b(5,7)\\ 0 & b(6,2) & b(6,3) & b(6,4) & b(6,5) & b(6,6) & b(6,7)\\ 0 & b(7,2) & b(7,3) & b(7,4) & b(7,5) & b(7,6) & b(7,7) \end{pmatrix}$
$7_{6,N,2,2,4}$	$\begin{pmatrix} 0 & b(1,2) & b(1,3) & b(1,4) & b(1,5) & b(1,6) & 0\\ 0 & b(2,2) & b(...2,3) & b(2,4) & b(2,5) & b(2,6) & 0\\ b(3,1) & b(3,2) & b(3,3) & b(3,4) & b(3,5) & b(3,6) & b(3,7)\\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & b(4,5) & b(4,6) & \frac{b(3,7)b(4,1)}{b(3,1)}\\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & b(5,6) & \frac{b(3,7)b(5,1)}{b(3,1)}\\ b(6,1) & b(6,2.) & b(6,3) & b(6,4) & b(6,5) & b(6,6) & \frac{b(3,7)b(6,1)}{b(3,1)}\\ 0 & \frac{b(1,2)b(3,7)}{b(3,1)} & \frac{b(1,3)b(3,7)}{b(3,1)} & \frac{b(1,4)b(3,7)}{b(3,1)} & \frac{b(1,5)b(3,7)}{b(3,1)} & \frac{b(1,6)b(3,7)}{b(3,1)} & 0 \end{pmatrix}$
$7_{6,N,2,2,5}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0\\ b(2,1) & b(2,2) & b(2,3) & b(2,4) & b(2,5) & b(2,6) & b(2,7)\\ b(3,1) & b(3,2) & b(3,3) & b(3,4) & b(3,5) & b(3,6) & b(3,7)\\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & b(4,5) & b(4,6) & b(4,7)\\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & b(5,6) & b(5,7)\\ b(6,1) & b(6,2) & b(6,3) & b(6,4) & b(6,5) & b(6,6) & b(6,7)\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

(continued)

Table 20.1 (continued)

$7_{6,N,2,2,6}$	$\begin{pmatrix} 0 & b(1,2) & b(1,3) & b(1,4) & b(1,5) & b(1,6) & 0 \\ 0 & b(2,2) & b(2,3) & b(2,4) & b(2,5) & b(2,6) & 0 \\ 0 & b(3,2) & b(3,3) & b(3,4) & b(3,5) & b(3,6) & 0 \\ 0 & b(4,2) & b(4,3) & b(4,4) & b(4,5) & b(4,6) & 0 \\ 0 & b(5,2) & b(5,3) & b(5,4) & b(5,5) & b(5,6) & 0 \\ 0 & b(6,2) & b(6,3) & b(6,4) & b(6,5) & b(6,6) & 0 \\ 0 & b(7,2) & b(7,3) & b(7,4) & b(7,5) & b(7,6) & 0 \end{pmatrix}$
$7_{6,N,2,2,7}$	$\begin{pmatrix} 0 & b(1,2) & b(1,3) & b(1,4) & b(1,5) & b(1,6) & 0 \\ 0 & b(2,2) & b(2,3) & b(2,4) & b(2,5) & b(2,6) & 0 \\ 0 & b(3,2) & b(3,3) & b(3,4) & b(3,5) & b(3,6) & 0 \\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & b(4,5) & b(4,6) & b(4,7) \\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & b(5,6) & \frac{b(4,7)b(5,1)}{b(4,1)} \\ b(6,1) & b(6,2) & b(6,3) & b(6,4) & b(6,5) & b(6,6) & \frac{b(4,7)b(6,1)}{b(4,1)} \\ 0 & \frac{b(1,2)b(4,7)}{b(4,1)} & \frac{b(1,3)b(4,7)}{b(4,1)} & \frac{b(1,4)b(4,7)}{b(4,1)} & \frac{b(1,5)b(4,7)}{b(4,1)} & \frac{b(1,6)b(4,7)}{b(4,1)} & 0 \end{pmatrix}$
$7_{6,N,2,2,8}$	$\begin{pmatrix} 0 & b(1,2) & b(1,3) & b(1,4) & b(1,5) & b(1,6) & 0 \\ 0 & b(2,2) & b(2,3) & b(2,4) & b(2,5) & b(2,6) & 0 \\ 0 & b(3,2) & b(3,3) & b(3,4) & b(3,5) & b(3,6) & 0 \\ 0 & b(4,2) & b(4,3) & b(4,4) & b(4,5) & b(4,6) & 0 \\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & b(5,6) & b(5,7) \\ b(6,1) & b(6,2) & b(6,3) & b(6,4) & b(6,5) & b(6,6) & \frac{b(5,7)b(6,1)}{b(5,1)} \\ 0 & \frac{b(1,2)b(5,7)}{b(5,1)} & \frac{b(1,3)b(5,7)}{b(5,1)} & \frac{b(1,4)b(5,7)}{b(5,1)} & \frac{b(1,5)b(5,7)}{b(5,1)} & \frac{b(1,6)b(5,7)}{b(5,1)} & 0 \end{pmatrix}$
$7_{6,N,2,2,9}$	$\begin{pmatrix} 0 & b(1,2) & b(1,3) & b(1,4) & b(1,5) & b(1,6) & 0 \\ 0 & b(2,2) & b(2,3) & b(2,4) & b(2,5) & b(2,6) & 0 \\ 0 & b(3,2) & b(3,3) & b(3,4) & b(3,5) & b(3,6) & 0 \\ 0 & b(4,2) & b(4,3) & b(4,4) & b(4,5) & b(4,6) & 0 \\ 0 & b(5,2) & b(5,3) & b(5,4) & b(5,5) & b(5,6) & 0 \\ b(6,1) & b(6,2) & b(6,3) & b(6,4) & b(6,5) & b(6,6) & b(6,7) \\ 0 & \frac{b(1,2)b(6,7)}{b(6,1)} & \frac{b(1,3)b(6,7)}{b(6,1)} & \frac{b(1,4)b(6,7)}{b(6,1)} & \frac{b(1,5)b(6,7)}{b(6,1)} & \frac{b(1,6)b(6,7)}{b(6,1)} & 0 \end{pmatrix}$

Table 20.2 $n = 6$, $\dim = 7$, $\ker\alpha = \langle e_1, e_3\rangle$, $[\alpha] = \begin{pmatrix} 0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0 \end{pmatrix}$

$7_{6,N,2,3,1}$	$\begin{pmatrix} b(1,1) & b(1,2) & b(1,3) & b(1,4) & b(1,5) & b(1,6) & \frac{b(1,2)b(7,2)}{b(2,2)} \\ b(2,1) & b(2,2) & b(2,3) & b(2,4) & b(2,5) & b(2,6) & b(7,2) \\ b(3,1) & b(3,2) & b(3,3) & b(3,4) & b(3,5) & b(3,6) & \frac{b(3,2)b(7,2)}{b(2,2)} \\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & b(4,5) & b(4,6) & \frac{b(4,2)b(7,2)}{b(2,2)} \\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & b(5,6) & \frac{b(5,2)b(7,2)}{b(2,2)} \\ b(6,1) & b(6,2) & b(6,3) & b(6,4) & b(6,5) & b(6,6) & \frac{b(6,2)b(7,2)}{b(2,2)} \\ \frac{b(2,1)b(7,2)}{b(2,2)} & b(7,2) & \frac{b(2,3)b(7,2)}{b(2,2)} & \frac{b(2,4)b(7,2)}{b(2,2)} & \frac{b(2,5)b(7,2)}{b(2,2)} & \frac{b(2,6)b(7,2)}{b(2,2)} & \frac{b(7,2)^2}{b(2,2)} \end{pmatrix}$
$7_{6,N,2,3,2}$	$\begin{pmatrix} b(1,1) & b(1,2) & b(1,3) & b(1,4) & b(1,5) & b(1,6) & b(1,7) \\ b(2,1) & 0 & b(2,3) & b(2,4) & b(2,5) & b(2,6) & 0 \\ b(3,1) & b(3,2) & b(3,3) & b(3,4) & b(3,5) & b(3,6) & \frac{b(1,7)b(3,2)}{b(1,2)} \\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & b(4,5) & b(4,6) & \frac{b(1,7)b(4,2)}{b(1,2)} \\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & b(5,6) & \frac{b(1,7)b(5,2)}{b(1,2)} \\ b(6,1) & b(6,2) & b(6,3) & b(6,4) & b(6,5) & b(6,6) & \frac{b(1,7)b(6,2)}{b(1,2)} \\ \frac{b(1,7)b(2,1)}{b(1,2)} & 0 & \frac{b(1,7)b(2,3)}{b(1,2)} & \frac{b(1,7)b(2,4)}{b(1,2)} & \frac{b(1,7)b(2,5)}{b(1,2)} & \frac{b(1,7)b(2,6)}{b(1,2)} & 0 \end{pmatrix}$
$7_{6,N,2,3,3}$	$\begin{pmatrix} b(1,1) & 0 & b(1,3) & b(1,4) & b(1,5) & b(1,6) & b(1,7) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b(3,1) & 0 & b(3,3) & b(3,4) & b(3,5) & b(3,6) & b(3,7) \\ b(4,1) & 0 & b(4,3) & b(4,4) & b(4,5) & b(4,6) & b(4,7) \\ b(5,1) & 0 & b(5,3) & b(5,4) & b(5,5) & b(5,6) & b(5,7) \\ b(6,1) & 0 & b(6,3) & b(6,4) & b(6,5) & b(6,6) & b(6,7) \\ b(7,1) & 0 & b(7,3) & b(7,4) & b(7,5) & b(7,6) & b(7,7) \end{pmatrix}$
$7_{6,N,2,3,4}$	$\begin{pmatrix} b(1,1) & 0 & b(1,3) & b(1,4) & b(1,5) & b(1,6) & 0 \\ b(2,1) & 0 & b(2,3) & b(2,4) & b(2,5) & b(2,6) & 0 \\ b(3,1) & b(3,2) & b(3,3) & b(3,4) & b(3,5) & b(3,6) & b(3,7) \\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & b(4,5) & b(4,6) & \frac{b(3,7)b(4,2)}{b(3,2)} \\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & b(5,6) & \frac{b(3,7)b(5,2)}{b(3,2)} \\ b(6,1) & b(6,2) & b(6,3) & b(6,4) & b(6,5) & b(6,6) & \frac{b(3,7)b(6,2)}{b(3,2)} \\ \frac{b(2,1)b(3,7)}{b(3,2)} & 0 & \frac{b(2,3)b(3,7)}{b(3,2)} & \frac{b(2,4)b(3,7)}{b(3,2)} & \frac{b(2,5)b(3,7)}{b(3,2)} & \frac{b(2,6)b(3,7)}{b(3,2)} & 0 \end{pmatrix}$
$7_{6,N,2,3,5}$	$\begin{pmatrix} b(1,1) & 0 & b(1,3) & b(1,4) & b(1,5) & b(1,6) & 0 \\ b(2,1) & 0 & b(2,3) & b(2,4) & b(2,5) & b(2,6) & 0 \\ b(3,1) & 0 & b(3,3) & b(3,4) & b(3,5) & b(3,6) & 0 \\ b(4,1) & 0 & b(4,3) & b(4,4) & b(4,5) & b(4,6) & 0 \\ b(5,1) & 0 & b(5,3) & b(5,4) & b(5,5) & b(5,6) & 0 \\ b(6,1) & 0 & b(6,3) & b(6,4) & b(6,5) & b(6,6) & 0 \\ b(7,1) & 0 & b(7,3) & b(7,4) & b(7,5) & b(7,6) & 0 \end{pmatrix}$

(continued)

Table 20.2 (continued)

$7_{6,N,2,3,6}$	$\begin{pmatrix} b(1,1) & b(1,2) & b(1,3) & b(1,4) & b(1,5) & b(1,6) & b(1,7) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b(3,1) & b(3,2) & b(3,3) & b(3,4) & b(3,5) & b(3,6) & b(3,7) \\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & b(4,5) & b(4,6) & b(4,7) \\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & b(5,6) & b(5,7) \\ b(6,1) & b(6,2) & b(6,3) & b(6,4) & b(6,5) & b(6,6) & b(6,7) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
$7_{6,N,2,3,7}$	$\begin{pmatrix} b(1,1) & 0 & b(1,3) & b(1,4) & b(1,5) & b(1,6) & 0 \\ b(2,1) & 0 & b(2,3) & b(2,4) & b(2,5) & b(2,6) & 0 \\ b(3,1) & 0 & b(3,3) & b(3,4) & b(3,5) & b(3,6) & 0 \\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & b(4,5) & b(4,6) & b(4,7) \\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & b(5,6) & \frac{b(4,7)b(5,2)}{b(4,2)} \\ b(6,1) & b(6,2) & b(6,3) & b(6,4) & b(6,5) & b(6,6) & \frac{b(4,7)b(6,2)}{b(4,2)} \\ \frac{b(2,1)b(4,7)}{b(4,2)} & 0 & \frac{b(2,3)b(4,7)}{b(4,2)} & \frac{b(2,4)b(4,7)}{b(4,2)} & \frac{b(2,5)b(4,7)}{b(4,2)} & \frac{b(2,6)b(4,7)}{b(4,2)} & 0 \end{pmatrix}$
$7_{6,N,2,3,8}$	$\begin{pmatrix} b(1,1) & 0 & b(1,3) & b(1,4) & b(1,5) & b(1,6) & 0 \\ b(2,1) & 0 & b(2,3) & b(2,4) & b(2,5) & b(2,6) & 0 \\ b(3,1) & 0 & b(3,3) & b(3,4) & b(3,5) & b(3,6) & 0 \\ b(4,1) & 0 & b(4,3) & b(4,4) & b(4,5) & b(4,6) & 0 \\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & b(5,6) & b(5,7) \\ b(6,1) & b(6,2) & b(6,3) & b(6,4) & b(6,5) & b(6,6) & \frac{b(5,7)b(6,2)}{b(5,2)} \\ \frac{b(2,1)b(5,7)}{b(5,2)} & 0 & \frac{b(2,3)b(5,7)}{b(5,2)} & \frac{b(2,4)b(5,7)}{b(5,2)} & \frac{b(2,5)b(5,7)}{b(5,2)} & \frac{b(2,6)b(5,7)}{b(5,2)} & 0 \end{pmatrix}$
$7_{6,N,2,3,9}$	$\begin{pmatrix} b(1,1) & 0 & b(1,3) & b(1,4) & b(1,5) & b(1,6) & 0 \\ b(2,1) & 0 & b(2,3) & b(2,4) & b(2,5) & b(2,6) & 0 \\ b(3,1) & 0 & b(3,3) & b(3,4) & b(3,5) & b(3,6) & 0 \\ b(4,1) & 0 & b(4,3) & b(4,4) & b(4,5) & b(4,6) & 0 \\ b(5,1) & 0 & b(5,3) & b(5,4) & b(5,5) & b(5,6) & 0 \\ b(6,1) & b(6,2) & b(6,3) & b(6,4) & b(6,5) & b(6,6) & b(6,7) \\ \frac{b(2,1)b(6,7)}{b(6,2)} & 0 & \frac{b(2,3)b(6,7)}{b(6,2)} & \frac{b(2,4)b(6,7)}{b(6,2)} & \frac{b(2,5)b(6,7)}{b(6,2)} & \frac{b(2,6)b(6,7)}{b(6,2)} & 0 \end{pmatrix}$

Table 20.3 $n = 6$, $\dim = 7$, $\ker\alpha = \langle e_1, e_4\rangle$, $[\alpha] = \begin{pmatrix} 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0 \end{pmatrix}$

$7_{6,N,2,4,1}$	$\begin{pmatrix} b(1,1)&b(1,2)&b(1,3)&b(1,4)&b(1,5)&b(1,6)&\frac{b(1,3)b(7,3)}{b(3,3)}\\ b(2,1)&b(2,2)&b(2,3)&b(2,4)&b(2,5)&b(2,6)&\frac{b(2,3)b(7,3)}{b(3,3)}\\ b(3,1)&b(3,2)&b(3,3)&b(3,4)&b(3,5)&b(3,6)&b(7,3)\\ b(4,1)&b(4,2)&b(4,3)&b(4,4)&b(4,5)&b(4,6)&\frac{b(4,3)b(7,3)}{b(3,3)}\\ b(5,1)&b(5,2)&b(5,3)&b(5,4)&b(5,5)&b(5,6)&\frac{b(5,3)b(7,3)}{b(3,3)}\\ b(6,1)&b(6,2)&b(6,3)&b(6,4)&b(6,5)&b(6,6)&\frac{b(6,3)b(7,3)}{b(3,3)}\\ \frac{b(3,1)b(7,3)}{b(3,3)}&\frac{b(3,2)b(7,3)}{b(3,3)}&b(7,3)&\frac{b(3,4)b(7,3)}{b(3,3)}&\frac{b(3,5)b(7,3)}{b(3,3)}&\frac{b(3,6)b(7,3)}{b(3,3)}&\frac{b(7,3)^2}{b(3,3)} \end{pmatrix}$
$7_{6,N,2,4,2}$	$\begin{pmatrix} b(1,1)&b(1,2)&b(1,3)&b(1,4)&b(1,5)&b(1,6)&b(1,7)\\ b(2,1)&b(2,2)&b(2,3)&b(2,4)&b(2,5)&b(2,6)&\frac{b(1,7)b(2,3)}{b(1,3)}\\ b(3,1)&b(3,2)&0&b(3,4)&b(3,5)&b(3,6)&0\\ b(4,1)&b(4,2)&b(4,3)&b(4,4)&b(4,5)&b(4,6)&\frac{b(1,7)b(4,3)}{b(1,3)}\\ b(5,1)&b(5,2)&b(5,3)&b(5,4)&b(5,5)&b(5,6)&\frac{b(1,7)b(5,3)}{b(1,3)}\\ b(6,1)&b(6,2)&b(6,3)&b(6,4)&b(6,5)&b(6,6)&\frac{b(1,7)b(6,3)}{b(1,3)}\\ \frac{b(1,7)b(3,1)}{b(1,3)}&\frac{b(1,7)b(3,2)}{b(1,3)}&0&\frac{b(1,7)b(3,4)}{b(1,3)}&\frac{b(1,7)b(3,5)}{b(1,3)}&\frac{b(1,7)b(3,6)}{b(1,3)}&0 \end{pmatrix}$
$7_{6,N,2,4,3}$	$\begin{pmatrix} b(1,1)&b(1,2)&0&b(1,4)&b(1,5)&b(1,6)&b(1,7)\\ b(2,1)&b(2,2)&0&b(2,4)&b(2,5)&b(2,6)&b(2,7)\\ 0&0&0&0&0&0&0\\ b(4,1)&b(4,2)&0&b(4,4)&b(4,5)&b(4,6)&b(4,7)\\ b(5,1)&b(5,2)&0&b(5,4)&b(5,5)&b(5,6)&b(5,7)\\ b(6,1)&b(6,2)&0&b(6,4)&b(6,5)&b(6,6)&b(6,7)\\ b(7,1)&b(7,2)&0&b(7,4)&b(7,5)&b(7,6)&b(7,7) \end{pmatrix}$
$7_{6,N,2,4,4}$	$\begin{pmatrix} b(1,1)&b(1,2)&0&b(1,4)&b(1,5)&b(1,6)&0\\ b(2,1)&b(2,2)&b(2,3)&b(2,4)&b(2,5)&b(2,6)&b(2,7)\\ b(3,1)&b(3,2)&0&b(3,4)&b(3,5)&b(3,6)&0\\ b(4,1)&b(4,2)&b(4,3)&b(4,4)&b(4,5)&b(4,6)&\frac{b(2,7)b(4,3)}{b(2,3)}\\ b(5,1)&b(5,2)&b(5,3)&b(5,4)&b(5,5)&b(5,6)&\frac{b(2,7)b(5,3)}{b(2,3)}\\ b(6,1)&b(6,2)&b(6,3)&b(6,4)&b(6,5)&b(6,6)&\frac{b(2,7)b(6,3)}{b(2,3)}\\ \frac{b(2,7)b(3,1)}{b(2,3)}&\frac{b(2,7)b(3,2)}{b(2,3)}&0&\frac{b(2,7)b(3,4)}{b(2,3)}&\frac{b(2,7)b(3,5)}{b(2,3)}&\frac{b(2,7)b(3,6)}{b(2,3)}&0 \end{pmatrix}$
$7_{6,N,2,4,5}$	$\begin{pmatrix} b(1,1)&b(1,2)&0&b(1,4)&b(1,5)&b(1,6)&0\\ b(2,1)&b(2,2)&0&b(2,4)&b(2,5)&b(2,6)&0\\ b(3,1)&b(3,2)&0&b(3,4)&b(3,5)&b(3,6)&0\\ b(4,1)&b(4,2)&0&b(4,4)&b(4,5)&b(4,6)&0\\ b(5,1)&b(5,2)&0&b(5,4)&b(5,5)&b(5,6)&0\\ b(6,1)&b(6,2)&0&b(6,4)&b(6,5)&b(6,6)&0\\ b(7,1)&b(7,2)&0&b(7,4)&b(7,5)&b(7,6)&0 \end{pmatrix}$

(continued)

Table 20.3 (continued)

$7_{6,N,2,4,6}$	$\begin{pmatrix} b(1,1) & b(1,2) & b(1,3) & b(1,4) & b(1,5) & b(1,6) & b(1,7) \\ b(2,1) & b(2,2) & b(2,3) & b(2,4) & b(2,5) & b(2,6) & b(2,7) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & b(4,5) & b(4,6) & b(4,7) \\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & b(5,6) & b(5,7) \\ b(6,1) & b(6,2) & b(6,3) & b(6,4) & b(6,5) & b(6,6) & b(6,7) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
$7_{6,N,2,4,7}$	$\begin{pmatrix} b(1,1) & b(1,2) & 0 & b(1,4) & b(1,5) & b(1,6) & 0 \\ b(2,1) & b(2,2) & 0 & b(2,4) & b(2,5) & b(2,6) & 0 \\ b(3,1) & b(3,2) & 0 & b(3,4) & b(3,5) & b(3,6) & 0 \\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & b(4,5) & b(4,6) & b(4,7) \\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & b(5,6) & \frac{b(4,7)b(5,3)}{b(4,3)} \\ b(6,1) & b(6,2) & b(6,3) & b(6,4) & b(6,5) & b(6,6) & \frac{b(4,7)b(6,3)}{b(4,3)} \\ \frac{b(3,1)b(4,7)}{b(4,3)} & \frac{b(3,2)b(4,7)}{b(4,3)} & 0 & \frac{b(3,4)b(4,7)}{b(4,3)} & \frac{b(3,5)b(4,7)}{b(4,3)} & \frac{b(3,6)b(4,7)}{b(4,3)} & 0 \end{pmatrix}$
$7_{6,N,2,4,8}$	$\begin{pmatrix} b(1,1) & b(1,2) & 0 & b(1,4) & b(1,5) & b(1,6) & 0 \\ b(2,1) & b(2,2) & 0 & b(2,4) & b(2,5) & b(2,6) & 0 \\ b(3,1) & b(3,2) & 0 & b(3,4) & b(3,5) & b(3,6) & 0 \\ b(4,1) & b(4,2) & 0 & b(4,4) & b(4,5) & b(4,6) & 0 \\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & b(5,6) & b(5,7) \\ b(6,1) & b(6,2) & b(6,3) & b(6,4) & b(6,5) & b(6,6) & \frac{b(5,7)b(6,3)}{b(5,3)} \\ \frac{b(3,1)b(5,7)}{b(5,3)} & \frac{b(3,2)b(5,7)}{b(5,3)} & 0 & \frac{b(3,4)b(5,7)}{b(5,3)} & \frac{b(3,5)b(5,7)}{b(5,3)} & \frac{b(3,6)b(5,7)}{b(5,3)} & 0 \end{pmatrix}$
$7_{6,N,2,4,9}$	$\begin{pmatrix} b(1,1) & b(1,2) & 0 & b(1,4) & b(1,5) & b(1,6) & 0 \\ b(2,1) & b(2,2) & 0 & b(2,4) & b(2,5) & b(2,6) & 0 \\ b(3,1) & b(3,2) & 0 & b(3,4) & b(3,5) & b(3,6) & 0 \\ b(4,1) & b(4,2) & 0 & b(4,4) & b(4,5) & b(4,6) & 0 \\ b(5,1) & b(5,2) & 0 & b(5,4) & b(5,5) & b(5,6) & 0 \\ b(6,1) & b(6,2) & b(6,3) & b(6,4) & b(6,5) & b(6,6) & b(6,7) \\ \frac{b(3,1)b(6,7)}{b(6,3)} & \frac{b(3,2)b(6,7)}{b(6,3)} & 0 & \frac{b(3,4)b(6,7)}{b(6,3)} & \frac{b(3,5)b(6,7)}{b(6,3)} & \frac{b(3,6)b(6,7)}{b(6,3)} & 0 \end{pmatrix}$

Table 20.4 $n = 5$, $\dim = 6$, $\ker\alpha = \langle e_1, e_2\rangle$, $[\alpha] = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \end{pmatrix}$

$6_{5,N,2,2,1}$	$\begin{pmatrix} b(1,1) & b(1,2) & b(1,3) & b(1,4) & b(1,5) & b(6,1) \\ b(2,1) & b(2,2) & b(2,3) & b(2,4) & b(2,5) & \frac{b(2,1)b(6,1)}{b(1,1)} \\ b(3,1) & b(3,2) & b(3,3) & b(3,4) & b(3,5) & \frac{b(3,1)b(6,1)}{b(1,1)} \\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & b(4,5) & \frac{b(4,1)b(6,1)}{b(1,1)} \\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & \frac{b(5,1)b(6,1)}{b(1,1)} \\ b(6,1) & \frac{b(1,2)b(6,1)}{b(1,1)} & \frac{b(1,3)b(6,1)}{b(1,1)} & \frac{b(1,4)b(6,1)}{b(1,1)} & \frac{b(1,5)b(6,1)}{b(1,1)} & \frac{b(6,1)^2}{b(1,1)} \end{pmatrix}$

(continued)

Table 20.4 (continued)

$6_{5,N,2,2,2}$	$\begin{pmatrix} 0 & b(1,2) & b(1,3) & b(1,4) & b(1,5) & 0 \\ b(2,1) & b(2,2) & b(2,3) & b(2,4) & b(2,5) & b(2,6) \\ b(3,1) & b(3,2) & b(3,3) & b(3,4) & b(3,5) & \frac{b(2,6)b(3,1)}{b(2,1)} \\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & b(4,5) & \frac{b(2,6)b(4,1)}{b(2,1)} \\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & \frac{b(2,6)b(5,1)}{b(2,1)} \\ 0 & \frac{b(1,2)b(2,6)}{b(2,1)} & \frac{b(1,3)b(2,6)}{b(2,1)} & \frac{b(1,4)b(2,6)}{b(2,1)} & \frac{b(1,5)b(2,6)}{b(2,1)} & 0 \end{pmatrix}$
$6_{5,N,2,2,3}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b(2,2) & b(2,3) & b(2,4) & b(2,5) & b(2,6) \\ 0 & b(3,2) & b(3,3) & b(3,4) & b(3,5) & b(3,6) \\ 0 & b(4,2) & b(4,3) & b(4,4) & b(4,5) & b(4,6) \\ 0 & b(5,2) & b(5,3) & b(5,4) & b(5,5) & b(5,6) \\ 0 & b(6,2) & b(6,3) & b(6,4) & b(6,5) & b(6,6) \end{pmatrix}$
$6_{5,N,2,2,4}$	$\begin{pmatrix} 0 & b(1,2) & b(1,3) & b(1,4) & b(1,5) & 0 \\ 0 & b(2,2) & b(2,3) & b(2,4) & b(2,5) & 0 \\ b(3,1) & b(3,2) & b(3,3) & b(3,4) & b(3,5) & b(3,6) \\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & b(4,5) & \frac{b(3,6)b(4,1)}{b(3,1)} \\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & \frac{b(3,6)b(5,1)}{b(3,1)} \\ 0 & \frac{b(1,2)b(3,6)}{b(3,1)} & \frac{b(1,3)b(3,6)}{b(3,1)} & \frac{b(1,4)b(3,6)}{b(3,1)} & \frac{b(1,5)b(3,6)}{b(3,1)} & 0 \end{pmatrix}$
$6_{5,N,2,2,5}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ b(2,1) & b(2,2) & b(2,3) & b(2,4) & b(2,5) & b(2,6) \\ b(3,1) & b(3,2) & b(3,3) & b(3,4) & b(3,5) & b(3,6) \\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & b(4,5) & b(4,6) \\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & b(5,6) \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
$6_{5,N,2,2,6}$	$\begin{pmatrix} 0 & b(1,2) & b(1,3) & b(1,4) & b(1,5) & 0 \\ 0 & b(2,2) & b(2,3) & b(2,4) & b(2,5) & 0 \\ 0 & b(3,2) & b(3,3) & b(3,4) & b(3,5) & 0 \\ 0 & b(4,2) & b(4,3) & b(4,4) & b(4,5) & 0 \\ 0 & b(5,2) & b(5,3) & b(5,4) & b(5,5) & 0 \\ 0 & b(6,2) & b(6,3) & b(6,4) & b(6,5) & 0 \end{pmatrix}$
$6_{5,N,2,2,7}$	$\begin{pmatrix} 0 & b(1,2) & b(1,3) & b(1,4) & b(1,5) & 0 \\ 0 & b(2,2) & b(2,3) & b(2,4) & b(2,5) & 0 \\ 0 & b(3,2) & b(3,3) & b(3,4) & b(3,5) & 0 \\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & b(4,5) & b(4,6) \\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & \frac{b(4,6)b(5,1)}{b(4,1)} \\ 0 & \frac{b(1,2)b(4,6)}{b(4,1)} & \frac{b(1,3)b(4,6)}{b(4,1)} & \frac{b(1,4)b(4,6)}{b(4,1)} & \frac{b(1,5)b(4,6)}{b(4,1)} & 0 \end{pmatrix}$
$6_{5,N,2,2,8}$	$\begin{pmatrix} 0 & b(1,2) & b(1,3) & b(1,4) & b(1,5) & 0 \\ 0 & b(2,2) & b(2,3) & b(2,4) & b(2,5) & 0 \\ 0 & b(3,2) & b(3,3) & b(3,4) & b(3,5) & 0 \\ 0 & b(4,2) & b(4,3) & b(4,4) & b(4,5) & 0 \\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & b(5,6) \\ 0 & \frac{b(1,2)b(5,6)}{b(5,1)} & \frac{b(1,3)b(5,6)}{b(5,1)} & \frac{b(1,4)b(5,6)}{b(5,1)} & \frac{b(1,5)b(5,6)}{b(5,1)} & 0 \end{pmatrix}$

Table 20.5 $n = 5$, $\dim = 6$, $\ker\alpha = \langle e_1, e_3\rangle$, $[\alpha] = \begin{pmatrix} 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{pmatrix}$

$6_{5,N,2,3,1}$	$\begin{pmatrix} b(1,1) & b(1,2) & b(1,3) & b(1,4) & b(1,5) & \frac{b(1,2)b(6,2)}{b(2,2)}\\ b(2,1) & b(2,2) & b(2,3) & b(2,4) & b(2,5) & b(6,2)\\ b(3,1) & b(3,2) & b(3,3) & b(3,4) & b(3,5) & \frac{b(3,2)b(6,2)}{b(2,2)}\\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & b(4,5) & \frac{b(4,2)b(6,2)}{b(2,2)}\\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & \frac{b(5,2)b(6,2)}{b(2,2)}\\ \frac{b(2,1)b(6,2)}{b(2,2)} & b(6,2) & \frac{b(2,3)b(6,2)}{b(2,2)} & \frac{b(2,4)b(6,2)}{b(2,2)} & \frac{b(2,5)b(6,2)}{b(2,2)} & \frac{b(6,2)^2}{b(2,2)} \end{pmatrix}$
$6_{5,N,2,3,2}$	$\begin{pmatrix} b(1,1) & b(1,2) & b(1,3) & b(1,4) & b(1,5) & b(1,6)\\ b(2,1) & 0 & b(2,3) & b(2,4) & b(2,5) & 0\\ b(3,1) & b(3,2) & b(3,3) & b(3,4) & b(3,5) & \frac{b(1,6)b(3,2)}{b(1,2)}\\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & b(4,5) & \frac{b(1,6)b(4,2)}{b(1,2)}\\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & \frac{b(1,6)b(5,2)}{b(1,2)}\\ \frac{b(1,6)b(2,1)}{b(1,2)} & 0 & \frac{b(1,6)b(2,3)}{b(1,2)} & \frac{b(1,6)b(2,4)}{b(1,2)} & \frac{b(1,6)b(2,5)}{b(1,2)} & 0 \end{pmatrix}$
$6_{5,N,2,3,3}$	$\begin{pmatrix} b(1,1) & 0 & b(1,3) & b(1,4) & b(1,5) & b(1,6)\\ 0 & 0 & 0 & 0 & 0 & 0\\ b(3,1) & 0 & b(3,3) & b(3,4) & b(3,5) & b(3,6)\\ b(4,1) & 0 & b(4,3) & b(4,4) & b(4,5) & b(4,6)\\ b(5,1) & 0 & b(5,3) & b(5,4) & b(5,5) & b(5,6)\\ b(6,1) & 0 & b(6,3) & b(6,4) & b(6,5) & b(6,6) \end{pmatrix}$
$6_{5,N,2,3,4}$	$\begin{pmatrix} b(1,1) & 0 & b(1,3) & b(1,4) & b(1,5) & 0\\ b(2,1) & 0 & b(2,3) & b(2,4) & b(2,5) & 0\\ b(3,1) & b(3,2) & b(3,3) & b(3,4) & b(3,5) & b(3,6)\\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & b(4,5) & \frac{b(3,6)b(4,2)}{b(3,2)}\\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & \frac{b(3,6)b(5,2)}{b(3,2)}\\ \frac{b(2,1)b(3,6)}{b(3,2)} & 0 & \frac{b(2,3)b(3,6)}{b(3,2)} & \frac{b(2,4)b(3,6)}{b(3,2)} & \frac{b(2,5)b(3,6)}{b(3,2)} & 0 \end{pmatrix}$
$6_{5,N,2,3,5}$	$\begin{pmatrix} b(1,1) & 0 & b(1,3) & b(1,4) & b(1,5) & 0\\ b(2,1) & 0 & b(2,3) & b(2,4) & b(2,5) & 0\\ b(3,1) & 0 & b(3,3) & b(3,4) & b(3,5) & 0\\ b(4,1) & 0 & b(4,3) & b(4,4) & b(4,5) & 0\\ b(5,1) & 0 & b(5,3) & b(5,4) & b(5,5) & 0\\ b(6,1) & 0 & b(6,3) & b(6,4) & b(6,5) & 0 \end{pmatrix}$

(continued)

Table 20.5 (continued)

$6_{5,N,2,3,6}$	$\begin{pmatrix} b(1,1) & b(1,2) & b(1,3) & b(1,4) & b(1,5) & b(1,6) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ b(3,1) & b(3,2) & b(3,3) & b(3,4) & b(3,5) & b(3,6) \\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & b(4,5) & b(4,6) \\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & b(5,6) \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
$6_{5,N,2,3,7}$	$\begin{pmatrix} b(1,1) & 0 & b(1,3) & b(1,4) & b(1,5) & 0 \\ b(2,1) & 0 & b(2,3) & b(2,4) & b(2,5) & 0 \\ b(3,1) & 0 & b(3,3) & b(3,4) & b(3,5) & 0 \\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & b(4,5) & b(4,6) \\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & \frac{b(4,6)b(5,2)}{b(4,2)} \\ \frac{b(2,1)b(4,6)}{b(4,2)} & 0 & \frac{b(2,3)b(4,6)}{b(4,2)} & \frac{b(2,4)b(4,6)}{b(4,2)} & \frac{b(2,5)b(4,6)}{b(4,2)} & 0 \end{pmatrix}$
$6_{5,N,2,3,8}$	$\begin{pmatrix} b(1,1) & 0 & b(1,3) & b(1,4) & b(1,5) & 0 \\ b(2,1) & 0 & b(2,3) & b(2,4) & b(2,5) & 0 \\ b(3,1) & 0 & b(3,3) & b(3,4) & b(3,5) & 0 \\ b(4,1) & 0 & b(4,3) & b(4,4) & b(4,5) & 0 \\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & b(5,6) \\ \frac{b(2,1)b(5,6)}{b(5,2)} & 0 & \frac{b(2,3)b(5,6)}{b(5,2)} & \frac{b(2,4)b(5,6)}{b(5,2)} & \frac{b(2,5)b(5,6)}{b(5,2)} & 0 \end{pmatrix}$

Table 20.6 $n = 5$, $\dim = 6$, $\ker \alpha = \langle e_1, e_4 \rangle$, $[\alpha] = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$6_{5,N,2,4,1}$	$\begin{pmatrix} b(1,1) & b(1,2) & b(1,3) & b(1,4) & b(1,5) & \frac{b(1,3)b(6,3)}{b(3,3)} \\ b(2,1) & b(2,2) & b(2,3) & b(2,4) & b(2,5) & \frac{b(2,3)b(6,3)}{b(3,3)} \\ b(3,1) & b(3,2) & b(3,3) & b(3,4) & b(3,5) & b(6,3) \\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & b(4,5) & \frac{b(4,3)b(6,3)}{b(3,3)} \\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & \frac{b(5,3)b(6,3)}{b(3,3)} \\ \frac{b(3,1)b(6,3)}{b(3,3)} & \frac{b(3,2)b(6,3)}{b(3,3)} & b(6,3) & \frac{b(3,4)b(6,3)}{b(3,3)} & \frac{b(3,5)b(6,3)}{b(3,3)} & \frac{b(6,3)^2}{b(3,3)} \end{pmatrix}$
$6_{5,N,2,4,2}$	$\begin{pmatrix} b(1,1) & b(1,2) & b(1,3) & b(1,4) & b(1,5) & b(1,6) \\ b(2,1) & b(2,2) & b(2,3) & b(2,4) & b(2,5) & \frac{b(1,6)b(2,3)}{b(1,3)} \\ b(3,1) & b(3,2) & 0 & b(3,4) & b(3,5) & 0 \\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & b(4,5) & \frac{b(1,6)b(4,3)}{b(1,3)} \\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & \frac{b(1,6)b(5,3)}{b(1,3)} \\ \frac{b(1,6)b(3,1)}{b(1,3)} & \frac{b(1,6)b(3,2)}{b(1,3)} & 0 & \frac{b(1,6)b(3,4)}{b(1,3)} & \frac{b(1,6)b(3,5)}{b(1,3)} & 0 \end{pmatrix}$

(continued)

Table 20.6 (continued)

$6_{5,N,2,4,3}$	$\begin{pmatrix} b(1,1) & b(1,2) & 0 & b(1,4) & b(1,5) & b(1,6) \\ b(2,1) & b(2,2) & 0 & b(2,4) & b(2,5) & b(2,6) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ b(4,1) & b(4,2) & 0 & b(4,4) & b(4,5) & b(4,6) \\ b(5,1) & b(5,2) & 0 & b(5,4) & b(5,5) & b(5,6) \\ b(6,1) & b(6,2) & 0 & b(6,4) & b(6,5) & b(6,6) \end{pmatrix}$
$6_{5,N,2,4,4}$	$\begin{pmatrix} b(1,1) & b(1,2) & 0 & b(1,4) & b(1,5) & 0 \\ b(2,1) & b(2,2) & b(2,3) & b(2,4) & b(2,5) & b(2,6) \\ b(3,1) & b(3,2) & 0 & b(3,4) & b(3,5) & 0 \\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & b(4,5) & \frac{b(2,6)b(4,3)}{b(2,3)} \\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & \frac{b(2,6)b(5,3)}{b(2,3)} \\ \frac{b(2,6)b(3,1)}{b(2,3)} & \frac{b(2,6)b(3,2)}{b(2,3)} & 0 & \frac{b(2,6)b(3,4)}{b(2,3)} & \frac{b(2,6)b(3,5)}{b(2,3)} & 0 \end{pmatrix}$
$6_{5,N,2,4,5}$	$\begin{pmatrix} b(1,1) & b(1,2) & 0 & b(1,4) & b(1,5) & 0 \\ b(2,1) & b(2,2) & 0 & b(2,4) & b(2,5) & 0 \\ b(3,1) & b(3,2) & 0 & b(3,4) & b(3,5) & 0 \\ b(4,1) & b(4,2) & 0 & b(4,4) & b(4,5) & 0 \\ b(5,1) & b(5,2) & 0 & b(5,4) & b(5,5) & 0 \\ b(6,1) & b(6,2) & 0 & b(6,4) & b(6,5) & 0 \end{pmatrix}$
$6_{5,N,2,4,6}$	$\begin{pmatrix} b(1,1) & b(1,2) & b(1,3) & b(1,4) & b(1,5) & b(1,6) \\ b(2,1) & b(2,2) & b(2,3) & b(2,4) & b(2,5) & b(2,6) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & b(4,5) & b(4,6) \\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & b(5,6) \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
$6_{5,N,2,4,7}$	$\begin{pmatrix} b(1,1) & b(1,2) & 0 & b(1,4) & b(1,5) & 0 \\ b(2,1) & b(2,2) & 0 & b(2,4) & b(2,5) & 0 \\ b(3,1) & b(3,2) & 0 & b(3,4) & b(3,5) & 0 \\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & b(4,5) & b(4,6) \\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & \frac{b(4,6)b(5,3)}{b(4,3)} \\ \frac{b(3,1)b(4,6)}{b(4,3)} & \frac{b(3,2)b(4,6)}{b(4,3)} & 0 & \frac{b(3,4)b(4,6)}{b(4,3)} & \frac{b(3,5)b(4,6)}{b(4,3)} & 0 \end{pmatrix}$
$6_{5,N,2,4,8}$	$\begin{pmatrix} b(1,1) & b(1,2) & 0 & b(1,4) & b(1,5) & 0 \\ b(2,1) & b(2,2) & 0 & b(2,4) & b(2,5) & 0 \\ b(3,1) & b(3,2) & 0 & b(3,4) & b(3,5) & 0 \\ b(4,1) & b(4,2) & 0 & b(4,4) & b(4,5) & 0 \\ b(5,1) & b(5,2) & b(5,3) & b(5,4) & b(5,5) & b(5,6) \\ \frac{b(3,1)b(5,6)}{b(5,3)} & \frac{b(3,2)b(5,6)}{b(5,3)} & 0 & \frac{b(3,4)b(5,6)}{b(5,3)} & \frac{b(3,5)b(5,6)}{b(5,3)} & 0 \end{pmatrix}$

Table 20.7 $n = 4,\ \dim = 5,\ \ker\alpha = \langle e_1, e_2\rangle,\ [\alpha] = \begin{pmatrix} 0&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}$

$5_{4,N,2,2,1}$	$\begin{pmatrix} b(1,1) & b(1,2) & b(1,3) & b(1,4) & b(5,1)\\ b(2,1) & b(2,2) & b(2,3) & b(2,4) & \frac{b(2,1)b(5,1)}{b(1,1)}\\ b(3,1) & b(3,2) & b(3,3) & b(3,4) & \frac{b(3,1)b(5,1)}{b(1,1)}\\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & \frac{b(4,1)b(5,1)}{b(1,1)}\\ b(5,1) & \frac{b(1,2)b(5,1)}{b(1,1)} & \frac{b(1,3)b(5,1)}{b(1,1)} & \frac{b(1,4)b(5,1)}{b(1,1)} & \frac{b(5,1)^2}{b(1,1)} \end{pmatrix}$
$5_{4,N,2,2,2}$	$\begin{pmatrix} 0 & b(1,2) & b(1,3) & b(1,4) & 0\\ b(2,1) & b(2,2) & b(2,3) & b(2,4) & b(2,5)\\ b(3,1) & b(3,2) & b(3,3) & b(3,4) & \frac{b(2,5)b(3,1)}{b(2,1)}\\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & \frac{b(2,5)b(4,1)}{b(2,1)}\\ 0 & \frac{b(1,2)b(2,5)}{b(2,1)} & \frac{b(1,3)b(2,5)}{b(2,1)} & \frac{b(1,4)b(2,5)}{b(2,1)} & 0 \end{pmatrix}$
$5_{4,N,2,2,3}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & b(2,2) & b(2,3) & b(2,4) & b(2,5)\\ 0 & b(3,2) & b(3,3) & b(3,4) & b(3,5)\\ 0 & b(4,2) & b(4,3) & b(4,4) & b(4,5)\\ 0 & b(5,2) & b(5,3) & b(5,4) & b(5,5) \end{pmatrix}$
$5_{4,N,2,2,4}$	$\begin{pmatrix} 0 & b(1,2) & b(1,3) & b(1,4) & 0\\ 0 & b(2,2) & b(2,3) & b(2,4) & 0\\ b(3,1) & b(3,2) & b(3,3) & b(3,4) & b(3,5)\\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & \frac{b(3,5)b(4,1)}{b(3,1)}\\ 0 & \frac{b(1,2)b(3,5)}{b(3,1)} & \frac{b(1,3)b(3,5)}{b(3,1)} & \frac{b(1,4)b(3,5)}{b(3,1)} & 0 \end{pmatrix}$
$5_{4,N,2,2,5}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0\\ b(2,1) & b(2,2) & b(2,3) & b(2,4) & b(2,5)\\ b(3,1) & b(3,2) & b(3,3) & b(3,4) & b(3,5)\\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & b(4,5)\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
$5_{4,N,2,2,6}$	$\begin{pmatrix} 0 & b(1,2) & b(1,3) & b(1,4) & 0\\ 0 & b(2,2) & b(2,3) & b(2,4) & 0\\ 0 & b(3,2) & b(3,3) & b(3,4) & 0\\ 0 & b(4,2) & b(4,3) & b(4,4) & 0\\ 0 & b(5,2) & b(5,3) & b(5,4) & 0 \end{pmatrix}$
$5_{4,N,2,2,7}$	$\begin{pmatrix} 0 & b(1,2) & b(1,3) & b(1,4) & 0\\ 0 & b(2,2) & b(2,3) & b(2,4) & 0\\ 0 & b(3,2) & b(3,3) & b(3,4) & 0\\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & b(4,5)\\ 0 & \frac{b(1,2)b(4,5)}{b(4,1)} & \frac{b(1,3)b(4,5)}{b(4,1)} & \frac{b(1,4)b(4,5)}{b(4,1)} & 0 \end{pmatrix}$

Table 20.8 $n = 4$, $\dim = 5$, $\ker\alpha = \langle e_1, e_3\rangle$, $[\alpha] = \begin{pmatrix} 0&1&0&0&0\\ 0&0&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}$

$5_{4,N,2,3,1}$	$\begin{pmatrix} b(1,1) & b(1,2) & b(1,3) & b(1,4) & \frac{b(1,2)b(5,2)}{b(2,2)}\\ b(2,1) & b(2,2) & b(2,3) & b(2,4) & b(5,2)\\ b(3,1) & b(3,2) & b(3,3) & b(3,4) & \frac{b(3,2)b(5,2)}{b(2,2)}\\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & \frac{b(4,2)b(5,2)}{b(2,2)}\\ \frac{b(2,1)b(5,2)}{b(2,2)} & b(5,2) & \frac{b(2,3)b(5,2)}{b(2,2)} & \frac{b(2,4)b(5,2)}{b(2,2)} & \frac{b(5,2)^2}{b(2,2)} \end{pmatrix}$
$5_{4,N,2,3,2}$	$\begin{pmatrix} b(1,1) & b(1,2) & b(1,3) & b(1,4) & b(1,5)\\ b(2,1) & 0 & b(2,3) & b(2,4) & 0\\ b(3,1) & b(3,2) & b(3,3) & b(3,4) & \frac{b(1,5)b(3,2)}{b(1,2)}\\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & \frac{b(1,5)b(4,2)}{b(1,2)}\\ \frac{b(1,5)b(2,1)}{b(1,2)} & 0 & \frac{b(1,5)b(2,3)}{b(1,2)} & \frac{b(1,5)b(2,4)}{b(1,2)} & 0 \end{pmatrix}$
$5_{4,N,2,3,3}$	$\begin{pmatrix} b(1,1) & 0 & b(1,3) & b(1,4) & b(1,5)\\ 0 & 0 & 0 & 0 & 0\\ b(3,1) & 0 & b(3,3) & b(3,4) & b(3,5)\\ b(4,1) & 0 & b(4,3) & b(4,4) & b(4,5)\\ b(5,1) & 0 & b(5,3) & b(5,4) & b(5,5) \end{pmatrix}$
$5_{4,N,2,3,4}$	$\begin{pmatrix} b(1,1) & 0 & b(1,3) & b(1,4) & 0\\ b(2,1) & 0 & b(2,3) & b(2,4) & 0\\ b(3,1) & b(3,2) & b(3,3) & b(3,4) & b(3,5)\\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & \frac{b(3,5)b(4,2)}{b(3,2)}\\ \frac{b(2,1)b(3,5)}{b(3,2)} & 0 & \frac{b(2,3)b(3,5)}{b(3,2)} & \frac{b(2,4)b(3,5)}{b(3,2)} & 0 \end{pmatrix}$
$5_{4,N,2,3,5}$	$\begin{pmatrix} b(1,1) & 0 & b(1,3) & b(1,4) & 0\\ b(2,1) & 0 & b(2,3) & b(2,4) & 0\\ b(3,1) & 0 & b(3,3) & b(3,4) & 0\\ b(4,1) & 0 & b(4,3) & b(4,4) & 0\\ b(5,1) & 0 & b(5,3) & b(5,4) & 0 \end{pmatrix}$
$5_{4,N,2,3,6}$	$\begin{pmatrix} b(1,1) & b(1,2) & b(1,3) & b(1,4) & b(1,5)\\ 0 & 0 & 0 & 0 & 0\\ b(3,1) & b(3,2) & b(3,3) & b(3,4) & b(3,5)\\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & b(4,5)\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
$5_{4,N,2,3,7}$	$\begin{pmatrix} b(1,1) & 0 & b(1,3) & b(1,4) & 0\\ b(2,1) & 0 & b(2,3) & b(2,4) & 0\\ b(3,1) & 0 & b(3,3) & b(3,4) & 0\\ b(4,1) & b(4,2) & b(4,3) & b(4,4) & b(4,5)\\ \frac{b(2,1)b(4,5)}{b(4,2)} & 0 & \frac{b(2,3)b(4,5)}{b(4,2)} & \frac{b(2,4)b(4,5)}{b(4,2)} & 0 \end{pmatrix}$

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Chapter 21
Deforming Algebras with Anti-involution via Twisted Associativity

Alexis Langlois-Rémillard

Abstract This contribution introduces a framework to study a deformation of algebras with anti-involution. Starting with the observation that twisting the multiplication of such an algebra by its anti-involution generates a Hom-associative algebra of type II, it formulates the adequate modules theory over these algebras, and shows that there is a faithful functor from the category of finite-dimensional left modules of algebras with involution to finite-dimensional right modules of Hom-associative algebras of type II.

Keywords Hom-associative algebra · Anti-involution · Deformation

MSC2020 Classification 16W10 · 17A36 · 17D30

21.1 Introduction

An associative algebra A over a commutative, associative and unital ring R is called an algebra with anti-involution if it admits an anti-automorphism ι that is its own inverse; it is then called an ι-algebra. One of the most common examples of such algebras is the algebra of complex square matrices with the conjugate-transpose as anti-involution; for analysts, C^*-algebras constitute the backbone of many investigations in functional analysis and operator theory [6]. We will be interested in finite-dimensional algebras with an anti-involution and consider them purely from the algebraic point of view. The presence of an anti-involution, not a guaranteed fact if the algebra is not commutative, leads to many interesting properties. For example, ideals of identities of a finitely generated algebra with involution coincide with ideals

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of identities for a finite dimensional algebra [24], and if an associative algebra with involution satisfies some polynomial identities, then the Grassmann envelope of the related superalgebra with superinvolution satisfies the same polynomial properties [1, 23].

The approach considered in this note concentrates on twisting the multiplication of the algebra and emphasise the rôle of the anti-involution. It goes along the spirit of Hom-structures, an interesting program of deformations of algebraic structures sparked by the introduction of Hom-Lie algebras by Hartwig, Larsson and Silvestrov [13]. In the following fifteen years, the notion has been extended to many other algebraic objects: Hom-associative algebras [20], Hom-Poisson algebras [19], Hom-Novikov algebras [31], Hom-Hopf algebras [18], Hom-Weyl algebras [3], Hom-quantum groups [28, 29, 32], etc. Many of the Hom-structures keep the properties their classical counterparts have; for example Hom-associative algebras are the universal enveloping algebras of the corresponding Hom-Lie algebras [26]. General programs for investigating Hom-structures via higher algebraic tools have also been proposed via PROPs [30], or via universal algebras and operads [14].

We will be interested in (a type of) Hom-associative algebras: an R-module with an R-linear map α that has a binary operation for which associativity is "twisted" by the map α. A simple construction by Yau [27] allows to deform any associative algebra with an endomorphism into a Hom-associative algebra. As in an ι-algebra, ι is an anti-endomorphism, this construction does not give a "classical" Hom-associative algebra but instead what Frégier and Gohr call a Hom-associative algebra of type II in their hierarchy [8].

The main results and highlights of this contribution are reviewed here. We give a functor from the category of ι-algebras to the category of Hom-associative algebras of type II (Proposition 21.4), construct a theory of (finite-dimensional) Hom-modules of Hom-associative algebras of type II, effectively proving that their category is abelian (Propositions 21.5–21.9); and give a faithful functor between the category of finite-dimensional left modules of a ι-algebra to the category of finite-dimensional right modules of the associated Hom-associative algebra of type II (Proposition 21.10).

The contribution is organised as follows. Section 21.2 presents basic definitions for ι-algebras and then quickly surveys cellular algebras. Section 21.3 first introduces the vocabulary of Hom-structures for comparison purposes and then proceed to construct a Hom-modules theory for Hom-associative algebras of type II and to give an example on a family of diagrammatic ι-algebras: the Temperley-Lieb algebras [15] viewed by their cellular basis [10]. Finally, Sect. 21.4 informally discusses an idea for the study of this type of structure in more generality via diagrammatic formalism, and gives an example of application by illustrating alternate proofs of some equivalences from Frégier's and Gohr's hierarchy [8].

21.2 Algebras with Involution

In this section, we present some vocabularies regarding ι-algebra and briefly introduce cellular algebras. For the remaining of the note, let R be an associative, unital and commutative ring.

Definition 21.1 Let A be an associative and unital R-algebra. Let $\iota : A \to A$ be an anti-morphism of algebras such that $\iota^2 = 1$. The pair (A, ι) is called an *algebra with anti-involution*[1] or an ι-algebra.

A short note is in order here. Usually the involution is a conjugate-linear map ($\iota(r \cdot a) = \overline{r} \cdot \iota(a)$) to mimic the conjugate-transpose operator of complex matrices. Of course, this only makes sense if R has a conjugation, for example if one works on $\mathbb{C}$, as for C^* algebras. For R-algebras, the custom is to only ask for a R-linear map (for example as in [24]).

To make it explicit, the anti-involution ι now satisfies, for any elements $a, b \in A$ and $r, s \in R$,

$$\iota(ab) = \iota(b)\iota(a); \quad \iota(\iota(a)) = a; \quad \iota(r \cdot a + s \cdot b) = r \cdot \iota(a) + s \cdot \iota(b). \tag{21.1}$$

Morphisms of ι-algebras are algebras morphisms commuting with the two anti-involutions.

Definition 21.2 For two ι-algebras (A_1, ι_1) and (A_2, ι_2), a morphism of algebras $\phi : A_1 \to A_2$ is called a *morphism of ι-algebras* if it commutes with the anti-involutions:

$$\phi \circ \iota_1 = \iota_2 \circ \phi. \tag{21.2}$$

Ideals of ι-algebras have the additional restriction of being fixed by the anti-involution.

Definition 21.3 An algebraic ideal J of A is an *ι-ideal* of the ι-algebra (A, ι) if $\iota(j) \in J$ for all $j \in J$.

For an ι-ideal J, the quotient algebra A/J is also an ι-algebra. Note that any ι-algebra A decomposes into symmetric ($\iota(s) = s$) and skew-symmetric ($\iota(t) = -t$) parts by taking A^+ to be the set of all elements $a + \iota(a)$, $a \in A$ and A^-, the set of all elements $a - \iota(a)$, $a \in A$. Furthermore, A^+ is a Jordan algebra when endowed with the anti-commutator $\{a, b\} = ab + ba$, and A^- is a Lie algebra with the commutator $[a, b] = ab - ba$.

We close the section by presenting a motivating example that will be used in the end of Sect. 21.3: cellular algebras. The notion of cellular algebras was introduced by Graham and Lehrer in their seminal paper [10]. In a rough statement, a cellular algebra is an associative finite-dimensional unital algebra that admits an anti-involution and

[1] It is common to call them algebra with involution [6, 25]. We added the prefix "anti" to avoid the confusion and emphasise the fact that the map ι is an anti-automorphism.

a basis stratifying the algebra according to a certain poset while behaving correctly under the anti-involution. Many important families of algebras admit an interesting cellular basis: Temperley-Lieb algebras, Brauer algebras, most Hecke algebras, etc. Once a cellular basis is exhibited for an algebra, the hard problems of exhibiting a complete family of simple modules and giving the composition multiplicities of the indecomposable projective reduce to linear algebra problems. We also present a basis-free definition based on a ring-theoretical framework given by König and Xi [16] that emphasises the importance of the anti-involution. A good review on the subject is the second chapter of Mathas's book on Iwahori-Hecke algebras [21].

Definition 21.4 (Graham and Lehrer [10]) Let R be a commutative associative unital ring. An associative R-algebra A is called *cellular* if it admits a cellular datum (Λ, M, C, ι) consisting of the following:

(i) a partially ordered set Λ and, for each $d \in \Lambda$, a finite set $M(d)$;
(ii) an injective map $C : \bigsqcup_{d\in\Lambda} M(d) \times M(d) \to A$ whose image is an R-basis of A;
(iii) an anti-involution $\iota : A \to A$ such that

$$\iota(C(s,t)) = C(t,s), \qquad \text{for all } s, t \in M(d); \tag{21.3}$$

(iv) if $d \in \Lambda$ and $s, t \in M(d)$, then for any $a \in A$,

$$aC(s,t) \equiv \sum_{s'\in M(d)} r_a(s',s)C(s',t) \mod A^{>d}, \tag{21.4}$$

where $A^{>d} = \langle C^e(p,q) \mid e > d \mid p,q \in M(e)\rangle_R$ and $r_a(s',s) \in R$.

The anti-involution ι, together with (21.4), yields the equation:

$$C(t,s)a^* \equiv \sum_{s'\in M(d)} r_a(s',s)C(t,s') \mod A^{>d}, \tag{21.5}$$

for all $s, t \in M(d)$ and $a \in A$.

To be completely transparent, there are two small differences that became customary (for example [5, 21]) between this definition and the one of Graham and Lehrer: first the partial order is reversed, as to better compare with quasi-hereditary algebras [4] and second, the poset Λ is not finite (but all of the sets $M(d)$ are finite). Notable generalizations and deformations of the notion of cellularity include: relative cellularity, where multiple partial orders grade the algebra [7]; affine cellularity, where the notion is extended to infinite dimensional algebras [11, 17], and almost cellular algebra, where the anti-involution is replaced by a special filtration of the algebra [12].

As a simple example, consider the polynomial algebra $\mathbb{C}[x]$. Let $\Lambda = \mathbb{N} = \{0, 1, \dots\}$, for $n \in \mathbb{N}$ let $M(n) := \{n\}$ and define $C : M \times M \to \mathbb{C}[x]$ by $n \times n \mapsto$

x^n. The algebra is commutative so take the (anti-)involution simply to be the identity. The image of C is obviously a basis of $\mathbb{C}[x]$. Axiom (21.3) is trivially satisfied because all the sets $M(n)$ are singletons. Axiom (21.4) simply states that multiplying two non-trivial polynomials will yield a polynomial of higher degree. It is one of the simplest examples of cellular structure.

The previous example downplays the importance of the anti-involution in the structure of cellular algebras. The equivalent basis-free definition of König and Xi highlights its key importance.

Definition 21.5 (König and Xi [16]) Let A be an R-algebra where R is a commutative Noetherian integral domain. Assume there is an anti-involution ι on A. A two-sided ideal J of A is called a *cell ideal* if:

(i) it is an ι-ideal ($\iota(J) = J$);
(ii) there exists a left ideal $\Delta \subset J$ such that Δ is finitely generated and free over R;
(iii) there is an isomorphism of A-bimodules $\psi : J \xrightarrow{\sim} \Delta \otimes_R \iota(\Delta)$ making the following diagram commutative:

$$\begin{array}{ccc} J & \xrightarrow{\psi} & \Delta \otimes_R \iota(\Delta) \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle x\otimes y \mapsto \iota(y)\otimes\iota(x)} \\ & \xrightarrow{\psi} & \Delta \otimes_R \iota(\Delta) \end{array} \cdot \tag{21.6}$$

The algebra A together with the anti-involution ι is called *cellular* if there is an R-module decomposition $A = J'_1 \oplus J'_2 \oplus \cdots \oplus J'_n$ with $\iota(J'_j) = J'_j$ for each j and such that setting $J_j = \bigoplus_{k=1}^{j} J'_k$ gives a chain of two-sided ideals of A

$$0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A \tag{21.7}$$

in which each quotient $J'_j = J_j/J_{j-1}$ is a cell ideal of the quotient A/J_{j-1} with respect to the restriction of ι on the quotient.

Proposition 21.1 (König and Xi [16]) *The two definitions of cellular algebra are equivalent.*

From the cellular basis, it is possible to construct a family of *cell modules*.[2] Each cell module C_d admits a symmetric and invariant bilinear form $\phi^d(-, -)$.

Define, for any cellular algebra, Λ^0 to be the subset of Λ in which the bilinear form just defined is not identically zero. The *radical* of the bilinear form ϕ^d is denoted R^d. As the form is invariant, it is a submodule of C_d. However, there is even more to it.

[2] Also often called *standard modules* to emphasise the links with Hecke algebras and quasi-hereditary algebras.

Proposition 21.2 (Graham and Lehrer, Proposition 3.2 and Theorem 3.4 [10]) *Let A be a cellular algebra over a field and $d \in \Lambda^0$. The radical R^d of the bilinear form ϕ^d is the Jacobson radical of C_d; the quotient $\mathsf{I}^d := \mathsf{C}_d/\mathsf{R}^d$ is absolutely irreducible and $\{\mathsf{I}^d \mid d \in \Lambda^0\}$ is a complete set of non-isomorphic (absolutely) irreducible modules.*

The main theorem links the decomposition factors of the indecomposable projective modules and the irreducible ones. Recall that $[M : I]$ is the composition multiplicity of the simple module I in the module M, that is the number of simple quotients isomorphic to I in the composition series of M. Denote by P^d the projective cover of I^d and let $\mathbf{D} = ([\mathsf{C}_d : \mathsf{I}^e])_{d\in\Lambda, e\in\Lambda^0}$ be the decomposition matrix of A and $\mathbf{C} = ([\mathsf{P}^d : \mathsf{I}^e])_{d,e\in\Lambda^0}$, its Cartan matrix.

Theorem 21.1 (Graham and Lehrer, Theorem 3.7 [10]) *The matrices* $\mathbf{C}$ *and* $\mathbf{D}$ *are related by* $\mathbf{C} = \mathbf{D}^t\mathbf{D}$.

Therefore, one can characterise the composition series of the indecomposable projective modules of a cellular algebra by simpler linear algebraic tools. A simple criterion for semisimplicity follows: should the radical of the bilinear form of each cell module be trivial, then the algebra is semisimple.

21.3 Hom-Structures

The notion of Hom-associative algebras comes from Makhlouf and Silvestrov [20] and the ten complete cases of possible Hom-associativity appearing from possible choices of parentheses and twisting map placement on the associativity equation $(ab)c = a(bc)$ were given later by Frégier and Gohr [8].

Two types of Hom-associativity are considered here. The first one is the "classical" Hom-associativity that is the parallel of universal enveloping algebras for Hom-Lie algebras. It is characterised by (21.8). In the language of Frégier and Gohr, it is called of type I_1. The content presented in this section is mostly standard and can be found in many references, see for example [3, 18, 20]. Only the part on representation theory is slightly less common and we defer to Bäck and Richter [2] for a careful overview of module theory in general.

The second section is devoted to Hom-associativity of type II and some constructions around them. The associativity deformation is given by (21.12). This subject was mainly studied (e.g. in [9]) in the context of unital multiplicative Hom-associative algebras. The setup employed here will be of weakly unital Hom-associative algebras with the twisting map an anti-involution, thus enabling a functorial construction (Proposition 21.4) akin to Yau's twisting principle (Proposition 21.3) for ι-algebras that also extends to a functor on modules (Proposition 21.10).

The last section studies a concrete example: Temperley-Lieb algebras. Employing its diagrammatic formulation gives a very natural example of the module theory employed here and we shall see that some of the structures coming from its cellularity are preserved by the functor in the semisimple case at least.

21.3.1 Review of Hom-Associative Results

Definition 21.6 A *Hom-associative algebra* over an associative, commutative and unital ring R is a triple $(A, \cdot, \alpha)$ consisting of an R-module A, an R-bilinear binary operation $\cdot : A \times A \to A$ and an R-linear map $\alpha : A \to A$ satisfying

$$\alpha(a) \cdot (b \cdot c) = (a \cdot b) \cdot \alpha(c) \tag{21.8}$$

for all $a, b, c \in A$.

The map α is referred to as the *twisting map*. When it is a homomorphism, the Hom-algebra is said to be *multiplicative*.

A Hom-associative R-algebra A is said to be *weakly left unital* if there exists $e_\ell \in A$ such that $e_\ell \cdot a = \alpha(a)$ for all $a \in A$; it is said to be *weakly right unital* if there exists $e_r \in A$ such that $a \cdot e_r = \alpha(a)$ for all $a \in A$, and it is deemed *weakly unital* if there exists $e \in A$ that is both a weak left unit and a weak right unit. Beware, the word *unital* is reserved for an algebra with a unit id, that is $x \cdot \mathrm{id} = \mathrm{id} \cdot x = x$, for any $x \in A$.

There is a canonical way, known as *Yau's twisting*, of defining a weakly unital Hom-associative algebra from a unital associative algebra.

Proposition 21.3 (Yau [27]) *Let A be an associative algebra with unit 1_A and $\alpha : A \to A$ be a endomorphism. Defining the operation $\star : A \times A \to A$ by $a \star b = \alpha(a \cdot b)$ gives a Hom-associative algebra $(A, \star, \alpha)$ with weak unit 1_A.*

Definition 21.7 Hom-algebras morphism between two Hom-associative algebras $(A_1, \cdot_1, \alpha_1)$ and $(A_2, \cdot_2, \alpha_2)$ is an R-modules morphism $f : A_1 \to A_2$ satisfying

$$f \circ \alpha_1(x) = \alpha_2 \circ f(x); \quad f(x) \cdot_2 f(y) = f(x \cdot_1 y) \tag{21.9}$$

for all $x, y \in A_1$.

With this definition, Yau's twisting extends to a functor from the category of associative algebras to the one of Hom-associative algebras [27].

Ideals of Hom-associative algebras must behave well under the twisting map.

Definition 21.8 A *Hom-ideal* is an algebraic ideal fixed by α. So for any element $a \in A$ and $j \in J$, the multiplication $a \cdot j \in J$ and $\alpha(j) \in J$.

We call a Hom-algebra *Hom-simple* if it has no non-trivial Hom-ideal.

If the algebra is simple, then it is Hom-simple, but the converse is not true as Hom-ideal are a stronger notion, which in turn makes the notion of Hom-simplicity weaker.

Now, we turn to some vocabulary of module theory for comparison purpose with the type II case (see [2] for more details).

Definition 21.9 Let $(A, \cdot, \alpha)$ be a Hom-associative R-algebra. A triple $(V, \cdot_V, \alpha_V)$ is said to be a *(left) Hom-module* if V is an R-module; the operation $\cdot_V : A \times V \to V$ is R-linear, and $\alpha_V : V \to V$ is an R-linear map such that

$$(a \cdot b) \cdot_V \alpha_V(v) = \alpha(a) \cdot_V (b \cdot_V v), \tag{21.10}$$

for all $a, b \in A$ and $v \in V$.

Definition 21.10 Let (V, α_V) and (U, α_U) be two Hom-A-modules. Let $\phi : U \to V$ be a linear map. It is a *morphism of Hom-A-modules* if it also respects

$$\phi(au) = a\phi(u), \quad \alpha_V(\phi(u)) = \phi(\alpha_U(u)). \tag{21.11}$$

A submodule N of M is a *Hom-submodule* if it is invariant under the map α_M. Many usual properties of modules hold: intersection, union, image and preimage under morphism, quotient, and the first, second and third isomorphism theorems, as shown in [2].

From this, we can define a *Hom-simple module* as a module with no non-trivial Hom-submodule and a *Hom-semisimple algebra* as a Hom-algebra that decomposes into a sum of Hom-simple modules when viewed as a Hom-module over itself.

In conclusion, remark that the (left) Hom-modules along with their morphisms form an abelian category [33].

21.3.2 Hom-Associativity of Type II

Hom-associativity of type II introduces a slight change in the order of deformations by the twisting map. Hygienic procedures are required to ensure the correct definitions for equivalent objects of the preceding section.

Definition 21.11 A *Hom-associative algebra of type II* over an associative, commutative and unital ring R is a triple $(A, \cdot, \alpha)$ consisting of an R-module A, an R-bilinear binary operation $\cdot : A \times A \to A$ and an R-linear map $\alpha : A \to A$ satisfying

$$x \cdot \alpha(y \cdot z) = \alpha(x \cdot y) \cdot z, \tag{21.12}$$

for all $x, y, z \in A$.

The map α is still referred to as the *twisting map*; the notion of weakly unitality stays the same, and when α is an anti-endomorphism, the Hom-algebra is said to be *anti-multiplicative*.

That Hom-associative algebras of type II are an interesting avenue to deform ι-algebras is illustrated by the following proposition.

Proposition 21.4 *Let A be a unital associative R-algebra and $\iota : A \to A$ be an anti-involution. Consider the binary operation $\circledast : A \times A \to A$ defined by the mapping $(x, y) \mapsto \iota(x \cdot y) = \iota(y) \cdot \iota(x)$. If $e \in A$ is the unit of A, then the triple $(A, \circledast, \iota)$ is an anti-multiplicative Hom-associative algebra of type II with weak unit e.*

Proof First, A is an R-module as it is an R-algebra and ι is an R-linear map. That (21.12) holds follows from simple algebraic manipulations. Let $x, y, z \in A$.

$$
\begin{aligned}
x \circledast \iota(y \circledast z) &= x \circledast \iota(\iota(y \cdot z)) && \text{definition of } \circledast \\
&= x \circledast (y \cdot z) && \iota \text{ is involutive} \\
&= \iota(x \cdot (y \cdot z)) && \text{definition of } \circledast \\
&= \iota((x \cdot y) \cdot z) && \text{associativity in } A \\
&= (x \cdot y) \circledast z && \text{definition of } \circledast \\
&= \iota^2(x \cdot y) \circledast z && \iota \text{ is involutive} \\
&= \iota(x \circledast y) \circledast z && \text{definition of } \circledast .
\end{aligned}
$$

It is thus a Hom-associative algebra of type II. It is weakly unital, for e being an unit in A implies

$$e \circledast x = \iota(e \cdot x) = \iota(x) = \iota(x \cdot e) = x \circledast e.$$ □

Before continuing, it is worth noting that Yau's construction (Proposition 21.3) does not work on anti-multiplicative Hom-associative algebra of type I_1. Indeed, if one deforms multiplication in an associative algebra A with anti-endomorphism α, one gets for $a, b, c \in A$

$$
\begin{aligned}
\alpha(a) \star (b \star c) &= \alpha(a) \star \alpha(b \cdot c) \\
&= \alpha(\alpha(a) \cdot \alpha(c) \cdot \alpha(b))
\end{aligned}
$$

on one hand, and

$$
\begin{aligned}
(a \star b) \star \alpha(c) &= \alpha(a \cdot b) \star \alpha(c) \\
&= \alpha(\alpha(b) \cdot \alpha(a) \cdot \alpha(c))
\end{aligned}
$$

on the other. When α is an anti-involution for example, this amounts to $b \cdot c \cdot a = c \cdot a \cdot b$, which does not hold generally if A is not commutative.

The notions of morphisms between Hom-associative algebras and of Hom-ideal have a direct equivalent for type II.

Definition 21.12 Let $(A_1, \cdot_1, \alpha_1)$ and $(A_2, \cdot_2, \alpha_2)$ be two Hom-associative algebras of type II. We call an R-linear map $\phi : A_1 \to A_2$ a *Hom-associative algebras morphism* if

$$\phi \circ \alpha_1(x) = \alpha_2 \circ \phi(x) \tag{21.13}$$

and

$$\phi(x) \cdot_2 \phi(y) = \phi(x \cdot_1 y). \tag{21.14}$$

It is a *Hom-associative algebras anti-morphism* if instead of the last equation, it respects

$$\phi(x \cdot_1 y) = \phi(y) \cdot_2 \phi(x). \tag{21.15}$$

Definition 21.13 An algebraic ideal J of a Hom-associative algebra $(A, \cdot, \alpha)$ is called a *left Hom-ideal* if it is fixed by α. So for any $j \in J$ and $a \in A$ it must be that $a \cdot j \in J$ and $\alpha(j) \in J$. A *right Hom-ideal* is an α-invariant right algebraic ideal and a *two-sided Hom-ideal* is both a left and a right Hom-ideal.

As in the type I_1 case, Proposition 21.4 extends to a functor F between the ι-algebras and Hom-associative algebras of type II. Indeed for two ι-algebras (A, ι_A) and (B, ι_B) a morphism $\phi : A \to B$ of ι-algebras becomes a morphism of Hom-associative algebras under F by making

$$\begin{aligned} F(\phi) : (A, \circledast_A, \iota_A) &\longrightarrow (B, \circledast_B, \iota_B) \\ a &\longmapsto \phi(a), \end{aligned} \tag{21.16}$$

because then as a morphism of ι-algebras, ϕ commutes with the anti-involutions and thus (21.13) amounts to (21.2). Finally, working out the operations shows that (21.14) is respected. Let $x, y \in A$

$$\begin{aligned} F(\phi)(x \circledast_A y) &= \phi(x \circledast_A y) \\ &= \phi(\iota_A(y))\phi(\iota_A(x)) \\ &= \iota_B(\phi(y))\iota_B(\phi(x)) \\ &= \phi(x) \circledast_B \phi(y) = F(\phi)(x) \circledast_B F(\phi)(y). \end{aligned}$$

One must express cautions while defining modules for type II Hom-algebras. Indeed, the interaction between (21.10) and (21.12) constrains a lot the possible modules: in particular, a Hom-algebra of type II would not be a module on itself if one would use Definition 21.9 because then it would be required that

$$(a \cdot b) \cdot \alpha(c) = \alpha(a) \cdot (b \cdot c),$$

so type I_1 Hom-associativity, which is not in general a consequence of type II Hom-associativity. As Frégier and Gohr remarked, this would hold if α was an abelian group morphism and the algebra unital [8] (we show this diagrammatically at the end of Sect. 21.4). We do not impose such restrictions and the two concepts are different in general.

Therefore it seems more appropriate to use a slightly different definition. The switch to right modules is to stay coherent with Proposition 21.10, obviously such results will also hold for left modules.

Definition 21.14 Let $(A, \cdot, \alpha)$ be a Hom-associative algebra of type II. The triple $(V, \cdot_V, \alpha_V)$ is said to be a (right) *Hom-module* if V is a R-module, there is an action $\cdot_V : A \times V \to V$ and an R-linear map $\alpha_V : V \to V$ that respect

$$\alpha_V(v \cdot_V b) \cdot_V a = v \cdot_V \alpha(a \cdot b). \tag{21.17}$$

In this way, any Hom-associative algebra of type II $(A, \cdot, \alpha)$ is also a Hom-module $(A, \cdot, \alpha)$ on itself. The associated concepts of submodule and morphism are defined below.

Definition 21.15 Consider a Hom-module $(V, \cdot_V, \alpha_V)$ of a Hom-associative algebra of type II $(A, \cdot, \alpha)$. An additive subgroup U of V is called a *Hom-submodule* if it is closed under the scalar multiplication of V and $\alpha_V(U) \subset U$.

Definition 21.16 Let $(V, \cdot_V, \alpha_V)$ and $(W, \cdot_W, \alpha_W)$ be two Hom-modules. Then an R-linear map $\phi : V \to W$ is called a *morphism of Hom-modules* if it also respects

$$\phi(v \cdot_V a) = \phi(v) \cdot_W a, \quad \alpha_W(\phi(v)) = \phi(\alpha_V(v)). \tag{21.18}$$

The following propositions prove that the category of (finite dimensional) Hom-modules is abelian, enabling the general results that go with it: isomorphism theorems, exact sequences, diagram-chasing, etc. There should be no surprise here as Hom-modules are at their core modules over a ring. We will not delve too deeply in these considerations, they are to be seen mostly as a safeguard to prevent abuse. Not a lot of changes appear in the proofs from type I_1 as can be seen by comparing what follows with the survey of Hom-modules theory in Bäck and Richter [2].

For the following, let $(V, \cdot_V, \alpha_V)$ and $(W, \cdot_W, \alpha_W)$ be two (right) Hom-modules of an Hom-associative algebra of type II $(A, \cdot, \alpha)$. They will be denoted respectively by the slight abuses of notation V, W, and A.

Proposition 21.5 *Let $\phi : V \to W$ be a morphism of Hom-modules, and let $V' \subset V$ and $W' \subset W$ be respectively a Hom-submodule of V and a Hom-submodule of W. The image $\phi(V')$ is a Hom-submodule of W and the preimage $\phi^{-1}(W')$ is a Hom-submodule of V.*

Proof That $\phi(V')$ and $\phi^{-1}(W')$ are subgroups of their respective space comes from the fact that ϕ is an R-linear map. Let $a \in A$ and $w \in \phi(V')$. Consider a preimage $v \in V'$ of w. Then, $w \cdot_W a = \phi(v) \cdot_W a = \phi(v \cdot_V a) \in \phi(V')$ and $\alpha_W(w) = \alpha_W(\phi(v)) = \phi(\alpha_V(v)) \in \phi(V')$ because ϕ is a morphism and V' is a Hom-submodule of V, and thus fixed by α_V.

For $u \in \phi^{-1}(W')$, there is an element $x \in W'$ such that $\phi(u) = x$. Acting by $a \in A$ on u stays in $\phi^{-1}(W')$ for $\phi(u \cdot_V a) = \phi(u) \cdot_W a = x \cdot_W a \in W'$ as W' is a Hom-submodule of W. Furthermore $\alpha_V(u)$ is in $\phi^{-1}(W')$ for $\phi(\alpha_V(u)) = \alpha_W(\phi(u)) \subset W'$. □

Proposition 21.6 *Any intersection of Hom-submodules is a Hom-submodule.*

Proof Let $\{U_i\}_{i\in I}$ be a set of Hom-submodules of V. Note $U = \bigcap_{i\in I} U_i$. If $U = \emptyset$, then it is trivially a Hom-submodule. Assume it is non-empty. It is an additive group. Let $a \in A$ and $u \in U$. Fix $i \in I$. Then $u \cdot a \in U_i$ and $\alpha_V(u) \in U_i$ because U_i is a submodule. As $i \in I$ is arbitrary, then $a \cdot u \in U$ and $\alpha_V(u) \in U$. □

Proposition 21.7 *A finite sum of Hom-submodules is a Hom-submodule.*

Proof Let $U_1, \ldots, U_k$ be Hom-submodules of V. Let $U = \sum_{i=1}^k U_i$. Point-wise sum and A-action turn it in a Hom-submodule. Indeed $\alpha_V(\sum_{i=1}^k u_i) = \sum_{i=1}^k \alpha_V(u_i) \in U$. □

Proposition 21.8 *Consider finitely many Hom-modules $U_1, \ldots, U_k$ of A. The set $U = \bigoplus_{i=1}^k U_i$ is a Hom-submodule with the action $\cdot : U \times A \to U$ given by $(u_1, \ldots, u_k) \cdot a = (u_1 \cdot_{U_1} a, \ldots, u_k \cdot_{U_k} a)$ and the action of $\alpha_U : U \to U$ given by $\alpha_U(u_1, \ldots, u_k) = (\alpha_{U_1}(u_1), \ldots, \alpha_{U_1}(u_k))$.*

Proof The only point that requires proving is the good interaction of $\cdot_U$ and α_U. Let $a, b \in A$ and $u = (u_1, \ldots, u_k) \in U$.

$$\begin{aligned} u \cdot_U \alpha(a \cdot b) &= (u_1 \cdot_{U_1} \alpha(a \cdot b), \ldots, u_k \cdot_{U_k} \alpha(a \cdot b)) \\ &= (\alpha_{U_1}(u_1 \cdot_{U_1} b) \cdot_{U_1} a, \ldots, \alpha_{U_k}(u_k \cdot_{U_k} b) \cdot_{U_k} a) \\ &= (\alpha_{U_1}(u_1 \cdot_{U_1} b), \ldots, \alpha_{U_k}(u_k \cdot_{U_k} b)) \cdot_U a \\ &= \alpha_U(u \cdot_U b) \cdot_U a. \end{aligned}$$

And thus (21.17) holds. The definition indicates clearly that U will be invariant under α_V as each U_i is a Hom-submodule. □

Proposition 21.9 *Let U be a Hom-submodule of V. The quotient V/U is a well-defined Hom-module under the action and the map given by*

$$\begin{aligned} \cdot_{V/U} : V/U \times A &\to V/U & \alpha_{V/U} : V/U &\to V/U \\ (v + U, a) &\mapsto v \cdot_{V/U} a + U, & v + U &\mapsto \alpha_V(v) + U. \end{aligned} \tag{21.19}$$

Proof To show that it is well-defined is the core of the proof, and the only one that shall be verified.

Take $v_1 + U$ and $v_2 + U$ to be any two elements in V/U of the same equivalence class, thus $v_1 - v_2 \in U$. Now for any $a \in A$, the elements $(v_1 + U) \cdot_{V/U} a$ and $(v_2 + U) \cdot_{V/U} a$ are of the same equivalence class, because $v_1 \cdot_V a - v_2 \cdot_V a = (v_1 - v_2) \cdot_V a \in U$ as $v_1 - v_2 \in U$. Likewise, $\alpha_{V/U}$ is a well-defined morphism, for $\alpha_V(v_1) - \alpha_V(v_2) = \alpha_V(v_1 - v_2) \in U$.

The rest follows simply. □

Therefore, there is a well defined modules theory for type II Hom-associative algebras. This gives some hopes that it would be possible to have similar results to Bäck and Richter [2, 3] up to some technicalities, and to the non-trivial verification that there exist Ore-extensions for type II algebras.

The point of the preceding results is the following proposition. It links the categories of modules of algebras with involution and Hom-associative algebras of type II using Proposition 21.4.

Proposition 21.10 *Let (A, ι) be an ι-algebra. There is a faithful functor F going from the category of left modules of (A, ι) to the category of right modules of $(A, \circledast, \iota)$ given on objects by*

$$\begin{aligned} F : {}_{A,\iota}\mathsf{Mod} &\longrightarrow \mathsf{Mod}_{A,\circledast,\iota} \\ M &\longmapsto (M, \cdot_M, \mathrm{id}), \end{aligned} \tag{21.20}$$

with the action $\cdot_M : M \times A \to M$ given by $m \cdot_M a = \iota(a)m$, and on morphisms by

$$\begin{aligned} F : \mathrm{Hom}_{(A,\iota)}(M, N) &\longrightarrow \mathrm{Hom}_{(A,\circledast,\iota)}(F(M), F(N)) \\ \phi : M \to N &\longmapsto F(\phi) : F(M) \to F(N), \end{aligned} \tag{21.21}$$

with $F(\phi)(m) = \phi(m)$.

Proof The functoriality of the proposed F must be verified. Let M be a left module of (A, ι). That $F(M)$ is a right module of the Hom-associative algebra $(A, \circledast, \iota)$, with the operation $\circledast : A \to A$ of Proposition 21.4, necessitates the respect of condition (21.17). Let $m \in M$ and $a, b \in A$. Taking $\alpha_A = \iota$ and $\alpha_V = \mathrm{id}$ results on one hand in

$$m \cdot_M \iota(a \circledast b) = m \cdot_M \iota^2(ab) = \iota^3(ab) \cdot m = \iota(b) \cdot (\iota(a) \cdot m)$$

and on the other hand in

$$\mathrm{id}(m \cdot_M a) \cdot_M b = (m \cdot_M a) \cdot_M b = (\iota(a) \cdot m) \cdot_M b = \iota(b) \cdot (\iota(a) \cdot m).$$

Thus $(M, \cdot_M, \mathrm{id})$ is a right Hom-module.

Let $\phi : M \to N$ be a morphism of left (A, ι)-modules. Then $F(\phi)$ is a morphism of Hom-modules because it respects (21.18). Let $a \in A$ and $m \in M$. Then

$$\begin{aligned} F(\phi)(m \cdot_M a) &= \phi(\iota(a) \cdot m) = \iota(a) \cdot \phi(m) = \phi(m) \cdot_N a, \\ F(\phi)(\mathrm{id}_M(m)) &= F(\phi)(m) = \mathrm{id}_N(F(\phi)(m)). \end{aligned}$$

The functor respects the composition of morphisms directly from its definition.

There is thus a well-defined functor F. It remains to prove that it is faithful. Let $\phi, \psi : M \to N$ be two morphisms of left (A, ι)-modules. If $F(\phi) = F(\psi)$, then for $m \in M$

$$F(\phi)(m) - F(\psi)(m) = \phi(m) - \psi(m) = 0.$$

Hence $\phi = \psi$ and the application $\phi \mapsto F(\phi)$ is injective, proving the faithfulness of F. □

From this proof, we have that the representation theory of Hom-associative algebras of type II contains a copy of the representation theory of ι-algebras.

21.3.3 Example: Temperley-Lieb Algebras

As an example of the past subsections, we will apply the functor of Proposition 21.4 on the Temperley-Lieb algebra $\mathsf{TL}_4(q+q^{-1})$ and study what happens to its representation theory. Temperley-Lieb algebras are well studied algebras (see the survey [22] for details and reference) that are useful in describing scaling limit for conformal field theories and in knot theory. There are two main ways to define them. First, they can be seen as a quotient of a Hecke algebra of type A: the Temperley-Lieb algebra of rank n and of parameter $q \in \mathbb{C}\backslash\{0\}$ is the associative $\mathbb{C}$-algebra generated by $n-1$ elements $e_1, \dots, e_{n-1}$, a unit id and the relations:

$$\mathrm{id} e_i = e_i \mathrm{id} = e_i; \quad e_i^2 = (q+q^{-1})e_i; \quad e_i e_j = e_j e_i, \ (|i-j| > 1); \tag{21.22}$$

$$e_i e_{i+1} e_i = e_i, \ (1 \leq i \leq n-1); \quad e_i e_{i-1} e_i = e_i, \ (2 \leq i \leq n-1). \tag{21.23}$$

Its dimension is given by the Catalan number

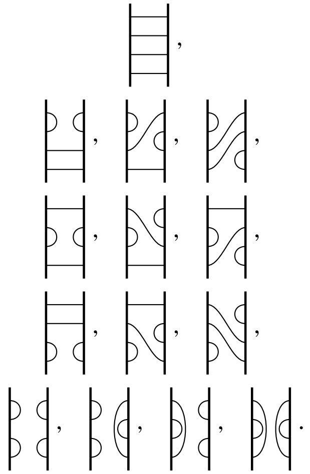
$$\dim \mathsf{TL}_n(q+q^{-1}) = C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{21.24}$$

The $C_4 = 14$ elements of $\mathsf{TL}_4(q+q^{-1})$ are given by:

$$\begin{gathered} \mathrm{id}, \\ e_1, \ e_1e_2, \ e_1e_2e_3, \\ e_2, \ e_2e_1, \ e_2e_3, \\ e_3, \ e_3e_2, \ e_3e_2e_1, \\ e_1e_3, \ e_1e_3e_2, \ e_2e_1e_3, \ e_2e_1e_3e_2. \end{gathered} \tag{21.25}$$

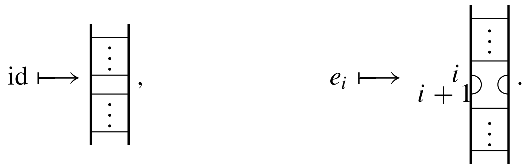
The other way to define the Temperley-Lieb algebra of rank n is via diagrammatic interpretation. A n-diagram is a diagram drawn in a rectangle with n points in its left side and n points in its right side all of the $2n$ linked together without crossing. Two diagrams are identified if they differ only by an isotopy. In this interpretation the elements of the algebra are formal $\mathbb{C}$-linear combinations of n-diagrams, and the multiplication is given by concatenation and replacing each of the created closed loops by a factor $q+q^{-1} = [2]_q$. It is an associative unital algebra.

The 14 diagrams giving a vector space basis of $\mathsf{TL}_4(q+q^{-1})$ are given below, ordered by the number of arcs on the same side (the order is the same as (21.25)):



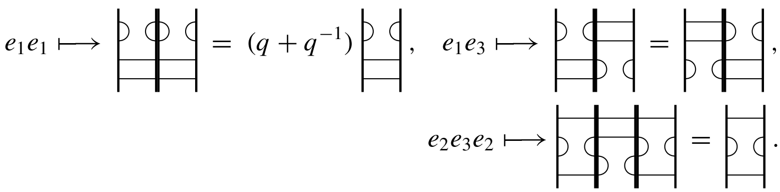
(21.26)

The identification between the two definitions follows from the morphism defined by

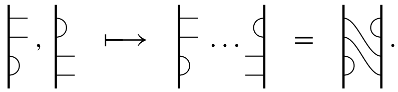


(21.27)

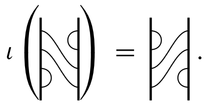
It is easy to see that the diagrammatic algebra respects the relations. For example, here is the verification of $e_1^2 = (q + q^{-1})e_1$, $e_1 e_3 = e_3 e_1$ and $e_2 e_3 e_2 = e_2$ in $\mathsf{TL}_4(q + q^{-1})$:



The advantage of this presentation is readily shown when exhibiting a cellular basis. For $\mathsf{TL}_4(q + q^{-1})$ take $\Lambda := \{0, 1, 2\}$, the number of arcs on the same side. For $d \in \Lambda$, let $M(d)$ be the set of left half-diagram with d arcs on the same side; the map C simply combines two half-diagrams with the same amount of arcs in the only way possible after flipping the second one. For example,

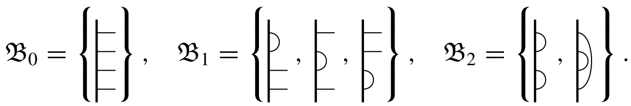


The anti-involution ι is simply the reflection of diagrams. It can also be defined as the only anti-endomorphism that leaves invariant the generators of the algebra. For example,

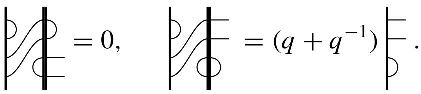


Constructing the diagrams from half-diagrams is an injective process and all possible cases are covered as Λ contains all the possible number of arcs, thus the image of C is a basis of $\mathsf{TL}_4(q+q^{-1})$. Axiom (21.3) is satisfied as flipping one diagram will indeed simply switch the place of the two half-diagrams, and axiom (21.4) amounts to the statement: "arcs can only be created, never destroyed."

There are three cell modules for $\mathsf{TL}_4(q+q^{-1})$: C_0, C_1 and C_2 with respective basis given by:



The action is also given by concatenation with the extra rules that whenever a new arc is created, the result is zero. For example,

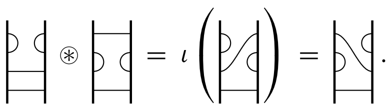


(21.28)

When q is not a root of unity, the Temperley-Lieb algebra is semisimple and decomposes as a module on itself, by the Wedderburn theorem, in the direct sum:

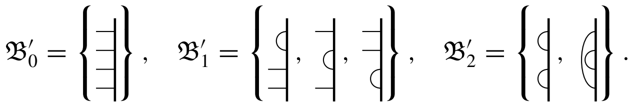
$$\mathsf{TL}_4(q+q^{-1}) = \bigoplus_{d\in\Lambda} \dim(\mathsf{C}_d)\mathsf{C}_d. \tag{21.29}$$

After applying Proposition 21.4, the new multiplication of the Hom-associative algebra of type II ($\mathsf{TL}_4(q+q^{-1})$, $\circledast$, ι) simply flips the result of the old. For example

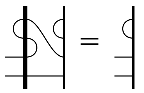


Interestingly, the action on module changes in a very natural way in this diagrammatic setting. Sending the left module C to a right Hom-module by $m \cdot a := \iota(a)m$ is portrayed in diagrammatic form simply by flipping the orientation of the half-diagram and keeping the natural action by concatenation.

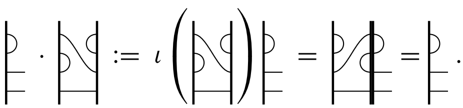
The new bases of the right cell modules C_0, C_1 and C_2 (with $\alpha_{\mathsf{C}} = \mathrm{id}_{\mathsf{C}}$) of the Hom-associative algebra of type II $(\mathsf{TL}_4(q+q^{-1}), \circledast, \iota)$ are given by



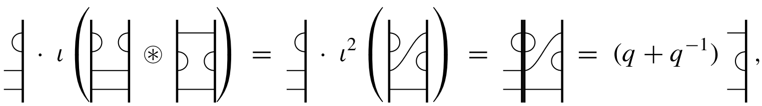
The action is given simply by concatenation diagrammatically, which amounts formally to the functor F of Proposition 21.10. For example,



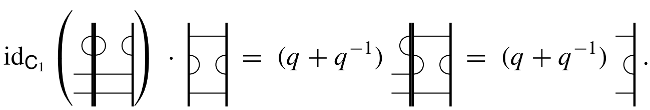
and formally by,



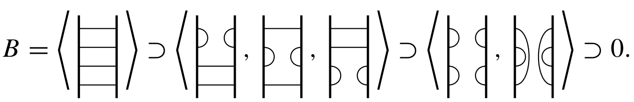
Equation 21.17 is also respected in this setting. For example, the right-hand side is



and the left-hand side is given by



The algebra $B = (\mathsf{TL}_4(q+q^{-1}), \circledast, \iota)$ keeps its cell filtration (21.3.3):



(21.30)

If q is not a root of unity, the algebra is Hom-semisimple. Indeed, variations on the arguments of [22] let one easily show that each cell module is cyclic and q not being a root of unity implies that each element of a cell module is a generator, the filtration (21.30) finishes the proof.

It is not surprising as for the Temperley-Lieb algebra, the process of going to Hom-associativity of type II is very similar on the representation theory level to considering the left cell modules as right modules: only one application of ι separates the two concepts.

Furthermore, for any cellular algebra, the cellular filtration is kept. Applying the faithful functor F to König's and Xi's definition of cellularity (Definition 21.5) results in the following definition.

Definition 21.17 Let $(A, \cdot, \alpha)$ be a Hom-associative algebra over an associative, commutative Noetherian integral domain R. Assume that there is an anti-involution ι in A. A two-sided Hom-ideal J of A is called a *Hom-cell ideal* if

(i) it is fixed by the anti-involution: $\iota(J) = J$;
(ii) there exists a Hom-module $\Delta \subset J$ such that Δ is finitely generated and free over R;
(iii) there is an isomorphism of Hom-bimodules $\psi : J \xrightarrow{\sim} \Delta \otimes_R \iota(\Delta)$.

$$\begin{array}{ccc} J & \xrightarrow{\psi} & \Delta \otimes_R \iota(\Delta) \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle x\otimes y \mapsto \iota(y)\otimes\iota(x)} \\ & \xrightarrow{\psi} & \Delta \otimes_R \iota(\Delta) \end{array} . \tag{21.31}$$

The algebra with the anti-involution ι is called *Hom-cellular* if there is a Hom-R-modules decomposition

$$A = J_1' \oplus J_2' \oplus \cdots \oplus J_n'$$

with $\iota(J_k') = J_k'$ for each k such that setting $J_k = \bigoplus_{l=1}^{k} J_l'$ gives a chain of Hom-A-ideals of A

$$0 = J_0 \subset J_1 \subset \cdots \subset J_n = A, \tag{21.32}$$

in which J_k' is a Hom-cell ideal for A/J_{k-1}.

Therefore we see that the functor F preserves the structure for some subfamilies of algebras with anti-involution. Of course, this short inquiry only presents arguments for the admittedly trivial case of semisimple cellular in which semisimplicity will be preserved by the faithfulness of the functor and the weaker notion of Hom-semisimplicity, but it hints that further investigation with weaker structures on algebra with anti-involution could preserve a sufficient amount of structure to be studied in the Hom-associativity of type II framework; that it could deform "enough" to open new applications is left for further studies.

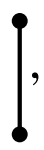
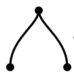
21.4 Discussion

In this independent short section, we will consider an avenue to systematise the study started here. It contains some ideas borrowed from Hellström, Makhlouf and Silvestrov [14] about universal algebras and from Yau [26]. Our main interest is the diagrammatic operadic approach that seems fruitful in considering the slight changes in the axiomatic rule that was exemplified here.

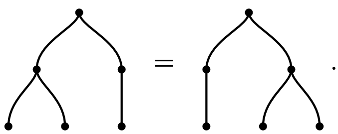
That type II associativity arose when one tried to deform a ι-algebra was an unexpected discovery. It justifies the construction of a module theory and consideration of this peculiar type of deformation. It is probably possible to do the same study for other types of Hom-associativity. The goal of this section is to hint at a more systematic way to do so.

We do not claim to offer the correct way of generalising these notions. We merely aim to present an interesting piece of material and show an application to Frégier and Gohr hierarchy [8] where this material proved useful to schematise the proofs.

Take a monoid M. Represent its identity map $\mathrm{id} : M \to M$ and its multiplication $\mu : M \times M \to M$ by the following diagrams:

$\mathrm{id} \longrightarrow$ , $\mu \longrightarrow$ . (21.33)

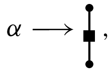
They are read from bottom to top. The diagram of μ takes two elements and gives back one. The identity condition is implicit here for one can deform the diagram as wanted as long as the topology is left unchanged. To add associativity, there must be a relation equivalent to $(\mu(\mu(a, b), c) = \mu(a, \mu(b, c))$. This is done by:

 (21.34)

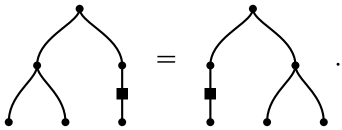
The system is simple for now. To consider Hom-associativity

$$\mu(\mu(a, b), \alpha(c)) = \mu(\alpha(a), \mu(b, c)),$$

one must add a diagram for the twisting map α and then use it to obtain a new condition to replace the associativity. It looks like

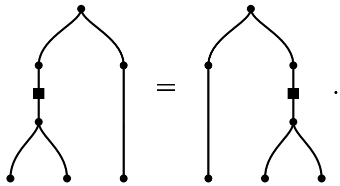


(21.35)



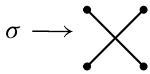
(21.36)

Hom-associativity of type II ($\mu(\alpha(\mu(a,b)),c) = \mu(a,\alpha(\mu(b,c)))$ is given by the following diagrams:



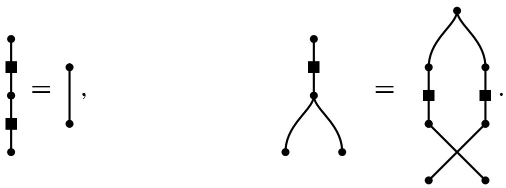
(21.37)

If now the map α is taken to be some anti-involution ι, one will need another application $\sigma : M \times M \to M \times M$ that switches two elements ($\sigma((a,b)) = (b,a)$):



(21.38)

and add two rules ($\iota \circ \iota = \mathrm{id}$ and $\iota(\mu(a,b)) = \mu(\iota(b),\iota(a))$):

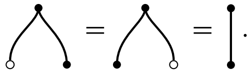


(21.39)

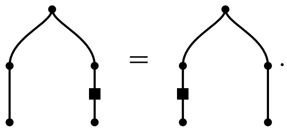
Multiple consequences can be derived from those last equations.[3] For example, proving the hierarchy of Frégier and Gohr amounts to diagrammatic consideration. As an example, the following prove that type I_1 and type II are equivalent only for unital Hom-associative algebras.

Indeed, unitality amounts in adding one distinguished element $\circ$ such that the following condition is present for the multiplication μ:

[3] Lars Hellström let a program he wrote for such investigation run for some hours and sent us back around one hundred lemmas. Unfortunately, the number of such lemmas is infinite and only when a previously known set of axiom is reached can a conclusion be drawn.

 (21.40)

With this, one gets from (21.36) (or from (21.37)) by placing the element $\circ$ in the second place

 (21.41)

It remains only to apply this new rule to (21.37) to obtain type I_1 Hom-associativity (or on (21.36) to obtain type II Hom-associativity).

It is possible to retrieve all their hierarchy in a similar fashion.

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Chapter 22
Admissible Hom-Novikov-Poisson and Hom-Gelfand-Dorfman Color Hom-Algebras

Ismail Laraiedh and Sergei Silvestrov

Abstract The main feature of color Hom-algebras is that the identities defining the structures are twisted by even linear maps. The purpose of this paper is to introduce and give some constructions of admissible Hom-Novikov-Poisson color Hom-algebras and Hom-Gelfand-Dorfman color Hom-algebras. Their bimodules and matched pairs are defined and the relevant properties and theorems are given. Also, the connections between Hom-Novikov-Poisson color Hom-algebras and Hom-Gelfand-Dorfman color Hom-algebras are proved. Furthermore, we show that the class of admissible Hom-Novikov-Poisson color Hom-algebras is closed under tensor product.

Keywords Hom-Novikov-Poisson color Hom-algebra · Hom-Gelfand-Dorfman color Hom-algebras

MSC 2020 Classification 17B61 · 17D30 · 17B63 · 16D20 · 17D25

22.1 Introduction

A Novikov algebra has a binary operation such that the associator is left-symmetric and that the right multiplication operators commute. Novikov algebras play a major role in the studies of Hamiltonian operators and Poisson brackets of hydrodynamic type [15, 28, 29, 32–34]. The left-symmetry of the associator implies that every

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Novikov algebra is Lie admissible, i.e., the commutator bracket $[x, y] = xy - yx$ gives it a Lie algebra structure.

Poisson algebras are used in many fields in mathematics and physics. In mathematics, Poisson algebras play a fundamental role in Poisson geometry [76], quantum groups [21, 27], and deformation of commutative associative algebras [35]. In physics, Poisson algebras are a major part of deformation quantization [43], Hamiltonian mechanics [4], and topological field theories [63]. Poisson-like structures are also used in the study of vertex operator algebras [31].

The theory of Hom-algebras has been initiated in [37, 54, 55] motivated by quasi-deformations of Lie algebras of vector fields, in particular q-deformations of Witt and Virasoro algebras. Hom-Lie algebras and more general quasi-Hom-Lie algebras were introduced first by Hartwig, Larsson and Silvestrov in [37] where a general approach to discretization of Lie algebras of vector fields using general twisted derivations (σ-derivations) and a general method for construction of deformations of Witt and Virasoro type algebras based on twisted derivations have been developed. The general quasi-Lie algebras, containing the quasi-Hom-Lie algebras and Hom-Lie algebras as subclasses, as well their graded color generalization, the color quasi-Lie algebras including color quasi-Hom-Lie algebras, color Hom-Lie algebras and their special subclasses the quasi-Hom-Lie superalgebras and Hom-Lie superalgebras, have been first introduced in [37, 53–56, 69]. Subsequently, various classes of Hom-Lie admissible algebras have been considered in [45]. In particular, in [45], the Hom-associative algebras have been introduced and shown to be Hom-Lie admissible, that is leading to Hom-Lie algebras using commutator map as new product, and in this sense constituting a natural generalization of associative algebras as Lie admissible algebras leading to Lie algebras using commutator map. Furthermore, in [45], more general G-Hom-associative algebras including Hom-associative algebras, Hom-Vinberg algebras (Hom-left symmetric algebras), Hom-pre-Lie algebras (Hom-right symmetric algebras), and some other Hom-algebra structures, generalizing G-associative algebras, Vinberg and pre-Lie algebras respectively, have been introduced and shown to be Hom-Lie admissible, meaning that for these classes of Hom-algebras, the operation of taking commutator leads to Hom-Lie algebras as well. Also, flexible Hom-algebras have been introduced, connections to Hom-algebra generalizations of derivations and of adjoint maps have been noticed, and some low-dimensional Hom-Lie algebras have been described. In Hom-algebra structures, defining algebra identities are twisted by linear maps. Since the pioneering works [37, 45, 53–56], Hom-algebra structures have developed in a popular broad area with increasing number of publications in various directions. Hom-algebra structures include their classical counterparts and open new broad possibilities for deformations, extensions to Hom-algebra structures of representations, homology, cohomology and formal deformations, Hom-modules and Hom-bimodules, Hom-Lie admissible Hom-coalgebras, Hom-coalgebras, Hom-bialgebras, Hom-Hopf algebras, L-modules, L-comodules and Hom-Lie quasi-bialgebras, n-ary generalizations of BiHom-Lie algebras and BiHom-associative algebras and generalized derivations, Rota-Baxter operators, Hom-dendriform color Hom-algebras, Rota-Baxter bisystems and covariant bialgebras, Rota-Baxter cosystems, coquasitriangular mixed bialgebras, coassociative Yang-Baxter pairs, coassociative Yang-Baxter equation and generalizations of Rota-Baxter systems and algebras, curved $\mathcal{O}$-operator systems and their connections with tridendriform systems and pre-Lie algebras, BiHom-algebras, BiHom-Frobenius algebras and double constructions, infinitesimal BiHom-bialgebras and Hom-dendriform D-bialgebras, Hom-algebras have been considered [2, 3, 6, 9, 10, 12–14, 16–18, 20, 25, 26, 30, 36, 38–42, 44, 46–49, 51–54, 57, 58, 60–62, 67, 68, 70, 72–75, 81–83, 86, 87].

In [84] the author initiated the study of a twisted generalization of Novikov algebras, called Hom-Novikov algebras. A Hom-Novikov algebra A has a binary operation $\cdot$ and a linear self-map α, and it satisfies some α-twisted versions of the defining identities of a Novikov algebra. In [84] several constructions of Hom-Novikov algebras were given and some low dimensional Hom-Novikov algebras were classified. Further, Hom-Poisson algebras were defined in [46] by Makhlouf and Silvestrov. It is shown in [46] that Hom-Poisson algebras play the same role in the deformation of commutative Hom-associative algebras as Poisson algebras do in the deformation of commutative associative algebras.

In this paper, we introduce and obtain some results on construction of admissible Hom-Novikov-Poisson color Hom-algebras and Hom-Gelfand-Dorfman color Hom-algebras. Their bimodules and matched pairs are defined and the relevant properties and theorems are obtained. We also show that the class of admissible Hom-Novikov-Poisson color Hom-algebras are closed under tensor product. In Sect. 22.2, we introduce the notions of bimodules and matched pairs of Hom-associative color Hom-algebras, Hom-Novikov color Hom-algebras and Hom-Lie color Hom-algebras in which we give some results and some examples. In Sect. 22.3, we establish the notions of admissible Hom-Novikov-Poisson color Hom-algebras and we give some explicit constructions. Their bimodule and matched pair are defined and their related relevant properties are also given. Finally, we show that the much larger class of admissible Hom-Novikov-Poisson color Hom-algebras is also closed under tensor products. In Sect. 22.4, we introduce the notions of Hom-Gelfand-Dorfman color Hom-algebras and we discuss some basic properties and examples of these objects. Moreover, we characterize the representations and matched pairs of Hom-Gelfand-Dorfman color Hom-algebras and provide some key constructions.

22.2 Preliminaries and Some Results

Throughout the article, we assume that all linear spaces are over an algebraically closed field $\mathbb{K}$ of characteristic 0, and denote by $\mathbb{K}^* = \mathbb{K}\backslash\{0\}$ the group of invertible elements of $\mathbb{K}$ with respect to the multiplication in $\mathbb{K}$, and by $\mathbb{N}^* = \{1, 2, 3, \dots\}$ the set of non-zero natural numbers.

In this section, we introduce the notions of bimodules and matched pairs of Hom-associative color Hom-algebras, Hom-Novikov color Hom-algebras and Hom-Lie color Hom-algebras in which we give some results and examples.

Let Γ be an abelian group. A linear space V is said to be Γ-graded, if there is a family $(V_\gamma)_{\gamma\in\Gamma}$ of vector subspace of V such that $V = \bigoplus_{\gamma\in\Gamma} V_\gamma$. An element $x \in V$ is said to be homogeneous of degree $\gamma \in \Gamma$ if $x \in V_\gamma$. In the sequel, we will denote the set of all the homogeneous elements of V by $\mathcal{H}(V)$, that is $\mathcal{H}(V) = \bigcup\limits_{\gamma\in\Gamma} V_\gamma$ As usual, we denote by $\overline{x}$ the degree of an element $x \in V$. Thus each homogeneous element $x \in V$ determines a unique group element $\overline{x} \in \Gamma$ by $x \in V_{\overline{x}}$. Thus, when no confusion occur, we can drop "$-$" in notation of degree for convenience of exposition.

Let $V = \bigoplus_{\gamma\in\Gamma} V_\gamma$ and $V' = \bigoplus_{\gamma\in\Gamma} V'_\gamma$ be two Γ-graded linear spaces. A linear mapping $f : V \longrightarrow V'$ is said to be homogeneous of degree $\upsilon \in \Gamma$ if $f(V_\gamma) \subseteq V'_{\gamma+\upsilon}$ for all $\gamma \in \Gamma$. If in addition f is homogeneous of degree zero, i.e. $f(V_\gamma) \subseteq V'_\gamma$ holds for any $\gamma \in \Gamma$, then f is said to be even.

An algebra $\mathcal{A}$ is said to be Γ-graded if its underlying linear space is Γ-graded, $\mathcal{A} = \bigoplus_{\gamma\in\Gamma} \mathcal{A}_\gamma$, and if, furthermore $\mathcal{A}_\gamma\mathcal{A}_{\gamma'} \subseteq \mathcal{A}_{\gamma+\gamma'}$, for all $\gamma, \gamma' \in \Gamma$. It is easy to see that if $\mathcal{A}$ has a unit element e, then $e \in \mathcal{A}_0$. A subalgebra of $\mathcal{A}$ is said to be Γ-graded if it is Γ-graded as a subspace of $\mathcal{A}$. Let $\mathcal{A}'$ be another Γ-graded algebra. A homomorphism $f : \mathcal{A} \longrightarrow \mathcal{A}'$ of Γ-graded algebras is by definition a homomorphism of the algebra $\mathcal{A}$ into the algebra $\mathcal{A}'$, which is, in addition an even mapping.

Definition 22.1 ([7, 22, 23, 50, 59, 64–66, 71]) Let $\mathbb{K}$ be a field and Γ be an abelian group. A map $\varepsilon : \Gamma \times \Gamma \to \mathbb{K}^*$ is called a commutation factor on Γ if the following identities hold, for all $a, b, c \in \Gamma$:

(i) $\varepsilon(a,b)\,\varepsilon(b,a) = 1$,
(ii) $\varepsilon(a, b+c) = \varepsilon(a,b)\,\varepsilon(a,c)$,
(iii) $\varepsilon(a+b, c) = \varepsilon(a,c)\,\varepsilon(b,c)$.

The definition above implies, in particular, the following relations

$$\varepsilon(a,0) = \varepsilon(0,a) = 1,\ \ \varepsilon(a,a) = \pm 1,\ \text{ for all } a \in \Gamma.$$

If x and x' are two homogeneous elements of degree Γ and Γ' respectively and ε is a skewsymmetric bicharacter, then we shorten the notation by writing $\varepsilon(x, x')$ instead of $\varepsilon(\Gamma, \Gamma')$ since the degree of every homogeneous element is unique.

Remark 22.1 Let A and V be two Γ-graded linear spaces such that

$$A \oplus V = \bigoplus_{\gamma\in\Gamma}(A \oplus V)_\gamma = \bigoplus_{\gamma\in\Gamma}(A_\gamma \oplus V_\gamma),$$

then, for all $X_1 = x_1 + v_1 \in A_{\gamma_1} \oplus V_{\gamma_1}$, $X_2 = x_2 + v_2 \in A_{\gamma_2} \oplus V_{\gamma_2}$ we have

$$\varepsilon(x_1, x_2) = \varepsilon(x_1, v_2) = \varepsilon(v_1, x_2) = \varepsilon(v_1, v_2) = \varepsilon(X_1, X_2).$$

Example 22.1 Some standard examples of skew-symmetric bicharacters are:

1) $\Gamma = \mathbb{Z}_2, \quad \varepsilon(i, j) = (-1)^{ij}$,
2) $\Gamma = \mathbb{Z}_2^n, \quad \varepsilon((\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n)) = (-1)^{\alpha_1\beta_1+\dots+\alpha_n\beta_n}$.
3) $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \varepsilon((i_1, i_2), (j_1, j_2)) = (-1)^{i_1 j_2 - i_2 j_1}$,
4) $\Gamma = \mathbb{Z} \times \mathbb{Z}, \quad \varepsilon((i_1, i_2), (j_1, j_2)) = (-1)^{(i_1+i_2)(j_1+j_2)}$.

Definition 22.2 A color Hom-algebra or a Hom-color algebra, $(A, \cdot, \varepsilon, \alpha)$ is a Γ-graded linear space A equipped with even bilinear multiplication $\cdot$, even twisting map α and commutation factor ε.

Definition 22.3 A derivation of degree $d \in \Gamma$ on a color Hom-algebra $(A, \cdot, \varepsilon, \alpha)$ is a linear map $D : A \to A$ such that for any $x, y \in \mathcal{H}(A)$,

$$D(x \cdot y) = D(x) \cdot y + \varepsilon(d, x)x \cdot D(y).$$

In particular, an even derivation $D : A \to A$ is a derivation of degree zero, that is, $D(x \cdot y) = D(x) \cdot y + x \cdot D(y)$ for all $x, y \in \mathcal{H}(A)$.

22.2.1 ε-Commutative Hom-associative color Hom-algebras

Definition 22.4 ([80]) A Hom-associative color Hom-algebra is a color Hom-algebra $(A, \cdot, \varepsilon, \alpha)$ satisfying for $x, y, z \in \mathcal{H}(A)$,

$$as_A(x, y, z) = \alpha(x) \cdot (y \cdot z) - (x \cdot y) \cdot \alpha(z) = 0. \quad \text{(Hom-associativity)} \quad (22.1)$$

If in addition, for any $x, y \in \mathcal{H}(A)$,

$$x \cdot y = \varepsilon(x, y)y \cdot x, \tag{22.2}$$

then $(A, \cdot, \varepsilon, \alpha)$ is said to be a ε-commutative Hom-associative color Hom-algebra.

Example 22.2 Let $A = A_0 \oplus A_1 =< e_1 > \oplus < e_2, e_3 >$ be a 3-dimensional superspace. Then A is a ε-commutative Hom-associative color Hom-algebra with

bicharacter: $\varepsilon(i, j) = (-1)^{ij}$,
multiplication: $e_1 \cdot e_2 = e_2 \cdot e_1 = -2e_3$,
even linear map $\alpha : A \to A : \ \alpha(e_1) = \sqrt{2}e_1, \alpha(e_2) = e_3 - e_2, \alpha(e_3) = e_3$.

In the following, we introduce the notion of bimodule of ε-commutative Hom-associative color Hom-algebra.

Definition 22.5 Let $(A, \cdot, \varepsilon, \alpha)$ be an ε-commutative Hom-associative color Hom-algebra, (V, β) be a pair consisting of Γ-graded linear space V and an even linear map $\beta : V \to V$, and $s : A \to End(V)$ be an even linear map. The triple (s, β, V) is called a bimodule of $(A, \cdot, \varepsilon, \alpha)$ if for all $x, y \in \mathcal{H}(A)$, $v \in \mathcal{H}(V)$,

$$s(x \cdot y)\beta(v) = s(\alpha(x))s(y)v. \tag{22.3}$$

Proposition 22.1 *If (s, β, V) is a bimodule of a ε-commutative Hom-associative color Hom-algebra $(A, \cdot, \varepsilon, \alpha)$, then the direct sum of Γ-graded linear spaces, $A \oplus V = \bigoplus_{\gamma\in\Gamma}(A \oplus V)_\gamma = \bigoplus_{\gamma\in\Gamma}(A_\gamma \oplus V_\gamma)$, is a ε-commutative Hom-associative color Hom-algebra with multiplication and twisting map in $A \oplus V$ given for all $X_1 = x_1 + v_1 \in A_{\gamma_1} \oplus V_{\gamma_1}$, $X_2 = x_2 + v_2 \in A_{\gamma_2} \oplus V_{\gamma_2}$ by*

$$\begin{aligned}(x_1 + v_1) * (x_2 + v_2) &= x_1 \cdot x_2 + \big(s(x_1)v_2 + \varepsilon(v_1, x_2)s(x_2)v_1\big),\\ (\alpha \oplus \beta)(x_1 + v_1) &= \alpha(x_1) + \beta(v_1).\end{aligned}$$

Proof We prove the commutativity and Hom-associativity in $A \oplus V$. For all elements $X_i = x_i + v_i \in A_{\gamma_i} \oplus V_{\gamma_i},\ i = 1, 2, 3,$

$$\begin{aligned}X_1 * X_2 &= (x_1 + v_1) * (x_2 + v_2) = x_1 \cdot x_2 + \big(s(x_1)v_2 + \varepsilon(v_1, x_2)s(x_2)v_1\big)\\
&= \varepsilon(x_1, x_2)x_2 \cdot x_1 + \big(\varepsilon(v_1, x_2)s(x_2)v_1 + s(x_1)v_2\big)\\
&= \varepsilon(X_1, X_2)x_2 \cdot x_1 + \big(\varepsilon(X_1, X_2)s(x_2)v_1 + s(x_1)v_2\big)\\
&= \varepsilon(X_1, X_2)\big(x_2 \cdot x_1 + s(x_2)v_1 + \varepsilon(X_2, X_1)s(x_1)v_2\big)\\
&= \varepsilon(X_1, X_2)\big(x_2 \cdot x_1 + s(x_2)v_1 + \varepsilon(v_2, x_1)s(x_1)v_2\big)\\
&= \varepsilon(X_1, X_2)(x_2 + v_2) * (x_1 + v_1) = \varepsilon(X_1, X_2)X_2 * X_1,\end{aligned}$$

$$\begin{aligned}&(X_1 * X_2) * (\alpha + \beta)X_3 - (\alpha + \beta)X_1 * (X_2 * X_3)\\
&\quad= \big((x_1 + v_1) * (x_2 + v_2)\big) * (\alpha + \beta)(x_3 + v_3)\\
&\qquad - (\alpha + \beta)(x_1 + v_1) * \big((x_2 + v_2) * (x_3 + v_3)\big)\\
&\quad= \big(x_1 \cdot x_2 + s(x_1)v_2 + \varepsilon(v_1, x_2)s(x_2)v_1\big) * (\alpha(x_3) + \beta(v_3))\\
&\qquad - (\alpha(x_1) + \beta(v_1)) * \big(x_2 \cdot x_3 + s(x_2)v_3 + \varepsilon(v_2, x_3)s(x_3)v_2\big)\\
&\quad= (x_1 \cdot x_2) \cdot \alpha(x_3) + s(x_1 \cdot x_2)\beta(v_3) + \varepsilon(x_1 + x_2, x_3)s(\alpha(x_3))s(x_1)v_2\\
&\qquad + \varepsilon(x_1 + x_2, x_3)\varepsilon(x_1, x_2)s(\alpha(x_3))s(x_2)v_1 - \alpha(x_1) \cdot (x_2 \cdot x_3)\\
&\qquad - s(\alpha(x_1))s(x_2)v_3 - \varepsilon(x_2, x_3)s(\alpha(x_1))s(x_3)v_2\\
&\qquad - \varepsilon(v_1, x_2 + x_3)s(x_2 \cdot x_3)\beta(v_1)\\
&\quad= \underbrace{\Big((x_1 \cdot x_2) \cdot \alpha(x_3) - \alpha(x_1) \cdot (x_2 \cdot x_3)\Big)}_{=0 \text{ by } (22.1)}\\
&\qquad + \underbrace{\Big(s(x_1 \cdot x_2)\beta(v_3) - s(\alpha(x_1))s(x_2)v_3\Big)}_{=0 \text{ by } (22.3)}\end{aligned}$$

$$+ \underbrace{\varepsilon(x_2, x_3)\Big(\varepsilon(x_1, x_3)s(\alpha(x_3))s(x_1)v_2 - s(\alpha(x_1))s(x_3)v_2\Big)}_{=0 \text{ by (22.2) and (22.3)}}$$
$$+ \underbrace{\varepsilon(x_1, x_2 + x_3)\Big(\varepsilon(x_2, x_3)s(\alpha(x_3))s(x_2)v_1 - s(x_2 \cdot x_3)\beta(v_1)\Big)}_{=0 \text{ by (22.2) and (22.3)}} = 0,$$

which completes the proof. □

The ε-commutative Hom-associative color Hom-algebra constructed in Proposition 22.1 is denoted by $(A \oplus V, *, \varepsilon, \alpha + \beta)$ or $A \ltimes_{s,\alpha,\beta} V$.

Example 22.3 If $(A, \cdot, \varepsilon, \alpha)$ is a ε-commutative Hom-associative color Hom-algebra, then (S, α, A) with $S(x)y = x \cdot y$ for all $x, y \in \mathcal{H}(A)$, is a bimodule of $(A, \cdot, \varepsilon, \alpha)$ called the regular bimodule of $(A, \cdot, \varepsilon, \alpha)$.

In the following, we introduce the notion of matched pair of ε-commutative Hom-associative color Hom-algebras.

Proposition 22.2 *Let $(A, \cdot_A, \varepsilon, \alpha)$ and $(B, \cdot_B, \varepsilon, \beta)$ be ε-commutative Hom-associative color Hom-algebras. Suppose that there are even linear maps $s_A : A \to End(B)$ and $s_B : B \to End(A)$ such that (s_A, β, B) is a bimodule of A, and (s_B, α, A) is a bimodule of B, satisfying, for any $x, y \in \mathcal{H}(A), a, b \in \mathcal{H}(B)$, the following conditions:*

$$\begin{aligned} &\varepsilon(b, x)\beta(a) \cdot_B (s_A(x)b) + \varepsilon(a, b + x)s_A(s_B(b)x)\beta(a) \\ &= \varepsilon(a + b, x)s_A(\alpha(x))(a \cdot_B b), \end{aligned} \tag{22.4}$$
$$\begin{aligned} &\beta(a) \cdot_B (s_A(x)b) + \varepsilon(a, x + b)\varepsilon(x, b)s_A(s_B(b)x)\beta(a) \\ &= \varepsilon(a, x)s_A(x)a \cdot_B \beta(b) + s_A(s_B(a)x)\beta(b), \end{aligned} \tag{22.5}$$
$$\begin{aligned} &\varepsilon(y, a)\alpha(x) \cdot_A (s_B(a)y) + \varepsilon(x, y + a)s_B(s_A(y)a)\alpha(x) \\ &= \varepsilon(x + y, a)s_B(\beta(a))(x \cdot_A y), \end{aligned} \tag{22.6}$$
$$\begin{aligned} &\alpha(x) \cdot_A (s_B(a)y) + \varepsilon(x, a + y)\varepsilon(a, y)s_B(s_A(y)a)\alpha(x) \\ &= \varepsilon(x, a)s_B(a)x \cdot_A \alpha(y) + s_B(s_A(x)a)\alpha(y). \end{aligned} \tag{22.7}$$

Then, $(A, B, l_A, r_A, \beta, l_B, r_B, \alpha)$ is called a matched pair of ε-commutative Hom-associative color Hom-algebras. In this case, there is a ε-commutative Hom-associative color Hom-algebra structure on the direct sum of the underlying Γ-graded linear spaces of A and B,

$$A \oplus B = \bigoplus_{\gamma \in \Gamma} (A \oplus B)_\gamma = \bigoplus_{\gamma \in \Gamma} (A_\gamma \oplus B_\gamma),$$

given for all $x + a \in A_{\gamma_1} \oplus B_{\gamma_1}, y + b \in A_{\gamma_2} \oplus B_{\gamma_2}$ by

$$(x+a)\cdot(y+b) = \big(x\cdot_A y + s_B(a)y + \varepsilon(x,b)s_B(b)x\big) + \big(a\cdot_B b + s_A(x)b + \varepsilon(a,y)s_A(y)a\big),$$
$$(\alpha\oplus\beta)(x+a) = \alpha(x)+\beta(a).$$

Proof Let $X = x + a \in A_{\gamma_1}\oplus B_{\gamma_1}$, $Y = y + b \in A_{\gamma_2}\oplus B_{\gamma_2}$, $Z = z + c \in A_{\gamma_3}\oplus B_{\gamma_3}$. First, we prove the commutativity condition:

$$\begin{aligned}
X\cdot Y - \varepsilon(X,Y)Y\cdot X &= (x+a)\cdot(y+b) - \varepsilon(X,Y)(y+b)\cdot(x+a)\\
&= x\cdot_A y + s_B(a)y + \varepsilon(x,b)s_B(b)x + a\cdot_B b + s_A(x)b + \varepsilon(a,y)s_A(y)a\\
&\quad - \varepsilon(X,Y)\big(y\cdot_A x + s_B(b)x + \varepsilon(y,a)s_B(a)y + b\cdot_B a + s_A(y)a + \varepsilon(b,x)s_A(x)b\big)\\
&= \big(x\cdot_A y - \varepsilon(X,Y)y\cdot_A x\big) + \big(a\cdot_B b - \varepsilon(X,Y)b\cdot_B a\big)\\
&\quad + \big(s_B(a)y - \varepsilon(X,Y)\varepsilon(y,a)s_B(a)y\big) + \big(\varepsilon(x,b)s_B(b)x - \varepsilon(X,Y)s_B(b)x\big)\\
&\quad + \big(s_A(x)b - \varepsilon(X,Y)\varepsilon(b,x)s_A(x)b\big) + \big(\varepsilon(a,y)s_A(y)a - \varepsilon(X,Y)s_A(y)a\big) = 0.
\end{aligned}$$
(using Remark 1 and (22.2))

Next, we prove the Hom-associativity condition:

$$\begin{aligned}
(X\cdot Y)\cdot(\alpha+\beta)Z &= \big((x+a)\cdot(y+b)\big)\cdot(\alpha+\beta)(z+c)
\end{aligned}$$
(using Remark 1)
$$\begin{aligned}
&= \Big(\big(x\cdot_A y + s_B(a)y + \varepsilon(x,y)s_B(b)x\big)\\
&\quad + \big(a\cdot_B b + s_A(x)b + \varepsilon(x,y)s_A(y)a\big)\Big)\cdot(\alpha(z)+\beta(c))\\
&= (x\cdot_A y)\cdot_A \alpha(z) + s_B(a)y\cdot_A \alpha(z) + \varepsilon(x,y)s_B(b)x\cdot_A \alpha(z)\\
&\quad + s_B(a\cdot_B b)\alpha(z) + s_B(s_A(x)b)\alpha(z) + \varepsilon(x,y)s_B(s_A(y)a)\alpha(z)\\
&\quad + \varepsilon(x+y,z)\Big(s_B(\beta(c))(x\cdot_A y) + s_B(\beta(c))s_B(a)y\\
&\quad + \varepsilon(x,y)s_B(\beta(c))s_B(b)x\Big)\\
&\quad + (a\cdot_B b)\cdot_B \beta(c) + s_A(x)b\cdot_B \beta(c) + \varepsilon(x,y)s_A(y)a\cdot_B \beta(c)\\
&\quad + s_A(x\cdot_A y)\beta(c) + s_A(s_B(a)y)\beta(c) + \varepsilon(x,y)s_A(s_B(b)x)\beta(c)\\
&\quad + \varepsilon(x+y,z)\Big(s_A(\alpha(z))(a\cdot_B b) + s_A(\alpha(z))s_A(x)b\\
&\quad + \varepsilon(x,y)s_A(\alpha(z))s_A(y)a\Big)
\end{aligned}$$
(using Remark 1)
$$\begin{aligned}
&= \Big((x\cdot_A y)\cdot_A \alpha(z) + s_B(a)y\cdot_A \alpha(z) + \varepsilon(x,y)s_B(b)x\cdot_A \alpha(z)\\
&\quad + s_B(a\cdot_B b)\alpha(z) + s_B(s_A(x)b)\alpha(z) + \varepsilon(x,y)s_B(s_A(y)a)\alpha(z)\\
&\quad + (a\cdot_B b)\cdot_B \beta(c) + s_A(x)b\cdot_B \beta(c) + \varepsilon(x,y)s_A(y)a\cdot_B \beta(c)\\
&\quad + s_A(x\cdot_A y)\beta(c) + s_A(s_B(a)y)\beta(c) + \varepsilon(x,y)s_A(s_B(b)x)\beta(c)\Big)
\end{aligned}$$

$$+ \varepsilon(x+y,z)\Big(s_B(\beta(c))(x \cdot_A y) + s_B(\beta(c))s_B(a)y$$
$$+ \varepsilon(x,y)s_B(\beta(c))s_B(b)x + s_A(\alpha(z))(a \cdot_B b) + s_A(\alpha(z))s_A(x)b$$
$$+ \varepsilon(x,y)s_A(\alpha(z))s_A(y)a\Big),$$

$$(\alpha+\beta)X \cdot (Y \cdot Z)$$
$$= (\alpha+\beta)(x+a) \cdot \big((y+b)\cdot(z+c)\big)$$
$$= (\alpha(x)+\beta(a)) \cdot \Big(\big(y \cdot_A z + s_B(b)z + \varepsilon(y,c)s_B(c)y\big)$$
$$+ \big(b \cdot_B c + s_B(y)c + \varepsilon(b,z)s_A(z)b\big)\Big)$$

(using Remark 1)

$$= \alpha(x) \cdot_A (y \cdot_A z) + \alpha(x) \cdot_A s_B(b)z + \varepsilon(y,z)\alpha(x) \cdot_A s_B(c)y$$
$$+ s_B(\beta(a))(y \cdot_A z) + s_B(\beta(a))s_B(b)z + \varepsilon(y,c)s_B(\beta(a))s_B(c)y$$
$$+ \varepsilon(x,y+z)\Big(s_B(b \cdot_B c)\alpha(x) + s_B(s_A(y)c)\alpha(x)$$
$$+ \varepsilon(b,z)s_B(s_A(z)b)\alpha(x)\Big) + \beta(a) \cdot_B (b \cdot_B c) + \beta(a) \cdot_B s_A(y)c$$
$$+ \varepsilon(b,z)\beta(a) \cdot_B s_A(z)b + s_A(\alpha(x))(b \cdot_B c) + s_A(\alpha(x))s_A(y)c$$
$$+ \varepsilon(y,z)s_A(\alpha(x))s_A(z)b + \varepsilon(x,y+z)\Big(s_A(y \cdot_A z)\beta(a)$$
$$+ s_A(s_B(b)z)\beta(a) + \varepsilon(y,z)s_A(s_B(c)y)\beta(a)\Big).$$

Using (22.4), (22.5), (22.6), (22.7) and that (s_A, β, B) and (s_B, α, A) are bimodules of $(A, \cdot_A, \varepsilon, \alpha)$ and $(B, \cdot_B, \varepsilon, \beta)$, respectively, we derive that $(A \oplus B, \cdot, \varepsilon, \alpha+\beta)$ is ε-commutative Hom-associative color Hom-algebra. This completes the proof. □

This ε-commutative Hom-associative color Hom-algebra, constructed in Proposition 22.2, is denoted by $(A \bowtie B, \cdot, \varepsilon, \alpha+\beta)$ or $A \bowtie_{s_B,\alpha}^{s_A,\beta} B$.

22.2.2 On Hom-Novikov Color Hom-algebras

Definition 22.6 ([8]) A color Hom-algebra $(A, \cdot, \varepsilon, \alpha)$ is called a Hom-Novikov color Hom-algebra if the following identities are satisfied for all $x, y, z \in \mathcal{H}(A)$:

$$(x \cdot y) \cdot \alpha(z) - \alpha(x) \cdot (y \cdot z) = \varepsilon(x,y)\big((y \cdot x) \cdot \alpha(z) - \alpha(y) \cdot (x \cdot z)\big), \quad (22.8)$$
$$(x \cdot y) \cdot \alpha(z) = \varepsilon(y,z)(x \cdot z) \cdot \alpha(y). \quad (22.9)$$

Remark 22.2 If $\alpha = id_A$ in Definition 22.6, we recover a Novikov color Hom-algebra. So, Novikov color Hom-algebras are a special case of Hom-Novikov color Hom-algebras when the twisting linear map is the identity map.

Example 22.4 (*Hom-Novikov color Hom-algebras*) Here are some examples of Hom-Novikov color Hom-algebras.

(1) Any ε-commutative Hom-associative color Hom-algebra is a Hom-Novikov color Hom-algebra.
(2) Let $A = A_0 \oplus A_1 =< e_1, e_2 > \oplus < e_3, e_4 >$ be a 4-dimensional superspace. Then A is a Hom-Novikov color Hom-algebra with

$$\begin{aligned}
&\text{bicharacter} \quad \varepsilon(i, j) = (-1)^{ij},\\
&\text{multiplication:} \quad e_1 \cdot e_1 = \lambda_1 e_2,\ e_1 \cdot e_3 = \lambda_2 e_4,\\
&\qquad\qquad\qquad e_3 \cdot e_3 = \lambda_3 e_2,\ e_3 \cdot e_1 = \lambda_4 e_4, \quad \lambda_i \in \mathbb{K}\\
&\text{even linear map}\, \alpha : A \to A :\\
&\alpha(e_1) = -e_1,\ \alpha(e_2) = e_1 - e_2,\ \alpha(e_3) = e_4,\ \alpha(e_4) = e_3 + 2e_4.
\end{aligned}$$

Proposition 22.3 ([8]) *Let $\mathcal{A} = (A, \cdot, \varepsilon)$ be a Novikov color Hom-algebra and $\alpha : \mathcal{A} \to \mathcal{A}$ be a Novikov color Hom-algebras morphism. Define $\cdot_\alpha : A \times A \to A$ for all $x, y \in \mathcal{H}(A)$, by $x \cdot_\alpha y = \alpha(x \cdot y)$. Then, $\mathcal{A}_\alpha = (A, \cdot_\alpha, \varepsilon, \alpha)$ is a Hom-Novikov color Hom-algebra called the α-twist or Yau twist of $(A, \cdot, \varepsilon)$.*

In the following we introduce the notions of bimodule and matched pair of Novikov color Hom-algebras.

Definition 22.7 Let $(A, \cdot, \varepsilon, \alpha)$ be a Hom-Novikov color Hom-algebra, (V, β) is a pair of Γ-graded linear space V and an even linear map $\beta : V \to V$. Let $l, r : A \to End(V)$ be two even linear maps. The quadruple (l, r, β, V) is called a bimodule of $(A, \cdot, \varepsilon, \alpha)$ if for all $x, y \in \mathcal{H}(A), v \in \mathcal{H}(V)$,

$$l(x \cdot y)\beta(v) - l(\alpha(x))l(y)v = \varepsilon(x, y)\big(l(y \cdot x)\beta(v) - l(\alpha(y))l(x)v\big), \quad (22.10)$$
$$r(\alpha(y))l(x)v - l(\alpha(x))r(y)v = \varepsilon(x, v)\big(r(\alpha(y))r(x)v - r(x \cdot y)\beta(v)\big), \quad (22.11)$$
$$r(\alpha(y))r(x)v - r(x \cdot y)\beta(v) = \varepsilon(v, x)\big(r(\alpha(y))l(x)v - l(\alpha(x))r(y)v\big), \quad (22.12)$$
$$l(x \cdot y)\beta(v) = \varepsilon(y, v)r(\alpha(y))l(x)v, \quad (22.13)$$
$$r(\alpha(y))l(x)v = \varepsilon(v, y)l(x \cdot y)\beta(v), \quad (22.14)$$
$$r(\alpha(y))r(x)v = \varepsilon(x, y)r(\alpha(x))r(y)v. \quad (22.15)$$

Proposition 22.4 *Let (l, r, β, V) is a bimodule of a Hom-Novikov color Hom-algebra $(A, \cdot, \varepsilon, \alpha)$. Then the direct sum of Γ-graded linear spaces,*

$$A \oplus V = \bigoplus_{\gamma\in\Gamma}(A \oplus V)_\gamma = \bigoplus_{\gamma\in\Gamma}(A_\gamma \oplus V_\gamma),$$

is a Hom-Novikov color Hom-algebra with multiplication in $A \oplus V$ such that, for all $X_1 = x_1 + v_1 \in A_{\gamma_1} \oplus V_{\gamma_1}$, $X_2 = x_2 + v_2 \in A_{\gamma_2} \oplus V_{\gamma_2}$,

$$(x_1 + v_1) * (x_2 + v_2) = x_1 \cdot x_2 + (l(x_1)v_2 + r(x_2)v_1),$$
$$(\alpha \oplus \beta)(x_1 + v_1) = \alpha(x_1) + \beta(v_1).$$

Proof We prove the axioms (22.8) and (22.9) in $A \oplus V$.
For all $X_i = x_i + v_i \in A_{\Gamma_i} \oplus V_{\Gamma_i},\ i \in \{1; 2; 3\}$,

$$\begin{aligned}
&(X_1 * X_2) * (\alpha + \beta)X_3 - (\alpha + \beta)X_1 * (X_2 * X_3)\\
&\quad - \varepsilon(X_1, X_2)\big((X_2 * X_1) * (\alpha + \beta)X_3 - (\alpha + \beta)X_2 * (X_1 * X_3)\big)\\
&= \big((x_1 + v_1) * (x_2 + v_2)\big) * (\alpha + \beta)(x_3 + v_3)\\
&\quad - (\alpha + \beta)(x_1 + v_1) * ((x_2 + v_2) * (x_3 + v_3))\\
&\quad - \varepsilon(X_1, X_2)\Big(\big((x_2 + v_2) * (x_1 + v_1)\big) * (\alpha + \beta)(x_3 + v_3)\\
&\quad - (\alpha + \beta)(x_2 + v_2) * \big((x_1 + v_1) * (x_3 + v_3)\big)\Big)\\
&= \big(x_1 \cdot x_2 + l(x_1)v_2 + r(x_2)v_1\big) * (\alpha(x_2) + \beta(v_3))\\
&\quad - (\alpha(x_1) + \beta(v_1))\big(x_2 \cdot x_3 + l(x_2)v_3 + r(x_3)v_2\big)\\
&\quad - \varepsilon(x_1, x_2)\Big(\big(x_2 \cdot x_1 + l(x_2)v_1 + r(x_1)v_2\big) * (\alpha(x_3) + \beta(v_3))\\
&\quad - (\alpha(x_2) + \beta(v_2)) * \big(x_1 \cdot x_3 + l(x_1)v_3 + r(x_3)v_1\big)\Big)\\
&= (x_1 \cdot x_2) \cdot \alpha(x_3) + l(x_1 \cdot x_2)\beta(v_3) + r(\alpha(x_3))l(x_1)v_2 + r(\alpha(x_3))r(x_2)v_1\\
&\quad - \alpha(x_1)(x_2 \cdot x_3) - l(\alpha(x_1))l(x_2)v_3 - l(\alpha(x_1))r(x_3)v_2 - r(x_2 \cdot x_3)\beta(v_1)\\
&\quad - \varepsilon(x_1, x_2)\Big((x_2 \cdot x_1) \cdot \alpha(x_3) + l(x_2 \cdot x_1)\beta(v_3) + r(\alpha(x_3))l(x_2)v_1\\
&\quad + r(\alpha(x_3))r(x_1)v_2 - \alpha(x_2)(x_1 \cdot x_3) - l(\alpha(x_2))l(x_1)v_3 - l(\alpha(x_2))r(x_3)v_1\\
&\qquad\qquad - r(x_1 \cdot x_3)\beta(v_2)\Big)\\
&= \underbrace{\Big((x_1 \cdot x_2) \cdot \alpha(x_3) - \alpha(x_1)(x_2 \cdot x_3) - \varepsilon(x_1, x_2)((x_2 \cdot x_1) \cdot \alpha(x_3) - \alpha(x_2) \cdot (x_1 \cdot x_3))\Big)}_{=0 \text{ by (22.8) in} A}\\
&\quad + \underbrace{\Big(l(x_1 \cdot x_2)\beta(v_3) - l(\alpha(x_1))l(x_2)v_3 - \varepsilon(x_1, x_2)(l(x_2 \cdot x_1)\beta(v_3) - l(\alpha(x_2))l(x_1)v_3)\Big)}_{=0 \text{ by (22.10)}}\\
&\quad + \underbrace{\Big(r(\alpha(x_3))l(x_1)v_2 - l(\alpha(x_1))r(x_3)v_2 - \varepsilon(x_1, v_2)(r(\alpha(x_3))r(x_1)v_2 - r(x_1 \cdot x_3)\beta(v_2))\Big)}_{=0 \text{ by (22.11)}}
\end{aligned}$$

$$+\underbrace{\Big(r(\alpha(x_3))r(x_2)v_1 - r(x_2\cdot x_3)\beta(v_1) - \varepsilon(v_1,x_2)(r(\alpha(x_3))l(x_2)v_1 - l(\alpha(x_2))r(x_3)v_1)\Big)}_{=0 \text{ by } (22.12)}$$

$$= 0,$$

$$\begin{aligned}
&(X_1 * X_2) * (\alpha+\beta)X_3 - \varepsilon(X_2,X_3)(X_1 * X_3) * (\alpha+\beta)X_2\\
&= \big((x_1+v_1)*(x_2+v_2)\big) * (\alpha+\beta)(x_3+v_3)\\
&\quad - \varepsilon(X_2,X_3)\big((x_1+v_1)*(x_3+v_3)\big) * (\alpha+\beta)(x_2+v_2)\\
&= \big(x_1\cdot x_2 + l(x_1)v_2 + r(x_2)v_1\big) * (\alpha(x_3)+\beta(v_3))\\
&\quad - \varepsilon(x_2,x_3)\big(x_1\cdot x_3 + l(x_1)v_3 + r(x_3)v_1\big) * (\alpha(x_2)+\beta(v_2))\\
&= (x_1\cdot x_2)\cdot\alpha(x_3) + l(x_1\cdot x_2)\beta(v_3) + r(\alpha(x_3))l(x_1)v_2 + r(\alpha(x_3))r(x_2)v_1\\
&- \varepsilon(x_2,x_3)((x_1\cdot x_3)\cdot\alpha(x_2) + l(x_1\cdot x_3)\beta(v_2)\\
&+ r(\alpha(x_2))l(x_1)v_3 + r(\alpha(x_2))r(x_3)v_1)\\
&= \underbrace{\Big((x_1\cdot x_2)\cdot\alpha(x_3) - (x_1\cdot x_3)\cdot\alpha(x_2)\Big)}_{=0 \text{ by } (22.9) \text{ in} A}\\
&\qquad + \underbrace{\Big(l(x_1\cdot x_2)\beta(v_3) - \varepsilon(x_2,x_3)r(\alpha(x_2))l(x_1)v_3\Big)}_{=0 \text{ by } (22.13)}\\
&\quad + \underbrace{\Big(r(\alpha(x_2))l(x_1)v_3 - \varepsilon(v_2,x_3)l(x_1\cdot x_3)\beta(v_2)\Big)}_{=0 \text{ by } (22.14)}\\
&\qquad + \underbrace{\Big(r(\alpha(x_3))r(x_2)v_1 - \varepsilon(x_2,x_3)r(\alpha(x_2))r(x_3)v_1\Big)}_{=0 \text{ by } (22.15)} = 0,
\end{aligned}$$

which completes the proof.

The Hom-Novikov color Hom-algebra constructed in Proposition 22.4 is denoted by $(A\oplus V, *, \varepsilon, \alpha+\beta)$ or $A\times_{l,r,\alpha,\beta} V$.

Example 22.5 If $(A,\cdot,\varepsilon,\alpha)$ be a Hom-Novikov color Hom-algebra, then (L, R, α, A) is a bimodule of $(A,\cdot,\varepsilon,\alpha)$, with $L(x)y = x\cdot y$ and $R(x)y = y\cdot x$ for all $x, y\in\mathcal{H}(A)$. It is called the regular bimodule of $(A,\cdot,\varepsilon,\alpha)$.

Proposition 22.5 *Let $(A,\cdot_A,\varepsilon,\alpha)$ and $(B,\cdot_B,\varepsilon,\beta)$ be two Hom-Novikov color Hom-algebras. Suppose there are even linear maps $l_A, r_A : A\to End(B)$ and $l_B, r_B : B\to End(A)$ such that the quadruple (l_A, r_A, β, B) is a bimodule of A, and (l_B, r_B, α, A) is a bimodule of B, satisfying, for any $x, y\in\mathcal{H}(A)$, $a, b\in\mathcal{H}(B)$,*

$$r_A(\alpha(x))(a \cdot_B b) - \beta(a) \cdot_B (r_A(x)b) - r_A(l_B(b)x)\beta(a)$$
$$= \varepsilon(a,b)\big(r_A(\alpha(x))(b \cdot_B a) - \beta(b) \cdot_B (r_A(x)a) - r_A(l_B(a)x)\beta(b)\big),$$

$$(r_A(x)a) \cdot_B \beta(b) + l_A(l_B(a)x)\beta(b) - \beta(a) \cdot_B (l_A(x)b) - r_A(r_B(b)x)\beta(a)$$
$$= \varepsilon(a,x)\big((l_A(x)a) \cdot_B \beta(b) + l_A(r_B(a)x)\beta(b) - l_A(\alpha(x))(a \cdot_B b)\big),$$

$$(l_A(x)a) \cdot_B \beta(b) - l_A(r_B(a)x)\beta(b) - l_A(\alpha(x))(a \cdot_B b)$$
$$= \varepsilon(x,a)\big((r_A(x)a) \cdot_B \beta(b) + l_A(l_B(a)x)\beta(b) - \beta(a) \cdot_B (l_A(x)b) - r_A(r_B(b)x)\beta(a)\big),$$

$$r_B(\beta(a))(x \cdot_A y) - \alpha(x) \cdot_A (r_B(a)y) - r_B(l_A(y)a)\alpha(x)$$
$$= \varepsilon(x,y)\big(r_B(\beta(a))(y \cdot_A x) - \alpha(y) \cdot_A (r_B(a)x) - r_B(l_A(x)a)\alpha(y)\big),$$

$$(r_B(a)x) \cdot_A \alpha(y) + l_B(l_A(x)a)\alpha(y) - \alpha(x) \cdot_A (l_B(a)y) - r_B(r_A(y)a)\alpha(x)$$
$$= \varepsilon(x,a)\big((l_B(a)x) \cdot_A \alpha(y) + l_B(r_A(x)a)\alpha(y) - l_B(\beta(a))(x \cdot_A y)\big),$$

$$(l_B(a)x) \cdot_A \alpha(y) - l_B(r_A(x)a)\alpha(y) - l_B(\beta(a))(x \cdot_A y)$$
$$= \varepsilon(a,x)\big((r_B(a)x) \cdot_A \alpha(y) + l_B(l_A(x)a)\alpha(y) - \alpha(x) \cdot_A (l_B(a)y) - r_B(r_A(y)a)\alpha(x)\big).$$

Then, $(A, B, l_A, r_A, \beta, l_B, r_B, \alpha)$ is called a matched pair of Hom-Novikov color Hom-algebras. In this case, there is a Hom-Novikov color Hom-algebra structure on the direct sum $A \oplus B = \bigoplus_{\Gamma\in\Gamma}(A \oplus B)_\Gamma = \bigoplus_{\Gamma\in\Gamma}(A_\Gamma \oplus B_\Gamma)$, of the underlying Γ-graded linear spaces of A and B given for all $x + a \in A_{\Gamma_1} \oplus B_{\Gamma_1}$, $y + b \in A_{\Gamma_2} \oplus B_{\Gamma_2}$ by

$$(x + a) \cdot (y + b) = \big(x \cdot_A y + l_B(a)y + r_B(b)x\big) + \big(a \cdot_B b + l_A(x)b + r_A(y)a\big),$$
$$(\alpha \oplus \beta)(x + a) = \alpha(x) + \beta(a).$$

We denote this Hom-Novikov color Hom-algebra either by $(A \bowtie B, \cdot, \varepsilon, \alpha + \beta)$ or $A \bowtie^{l_A,r_A,\beta}_{l_B,r_B,\alpha} B$.

22.2.3 On Hom-Lie color Hom-algebras

Definition 22.8 ([1, 19, 24, 55, 56, 69, 80]) A Hom-Lie color Hom-algebra is a quadruple $(A, [\cdot,\cdot], \varepsilon, \alpha)$ consisting of a Γ-graded vector space A, a bi-character ε, an even bilinear mapping $[\cdot,\cdot] : A \times A \to A$, (i.e. $[A_a, A_b] \subseteq A_{a+b}$, for all $a, b \in \Gamma$) and an even homomorphism $\alpha : A \to A$ such that for homogeneous elements $x, y, z \in A$,

$$[x, y] = -\varepsilon(x,y)[y,x], \qquad \varepsilon - \text{skew symmetry},$$
$$\circlearrowleft_{x,y,z} \varepsilon(z,x)[\alpha(x),[y,z]] = 0, \qquad \varepsilon - \text{Hom-Jacobi identity}$$

where $\circlearrowleft_{x,y,z}$ denotes the cyclic sum over (x, y, z).

Remark 22.3 Hom-Lie color Hom-algebras contain ordinary Lie color algebras, Lie superalgebras and Lie algebras, as well as Hom-Lie superalgebras and Hom-Lie algebras for specific choices of the twisting map, grading group and commutation factor.

(i) When $\alpha = id_A$, one recovers Lie color algebras, and in particular if the grading group is $\mathbb{Z}_2$ and the commutation factor is defined as $\varepsilon(i, j) = (-1)^{ij}$ for all $i, j \in \mathbb{Z}$, then one gets Lie superalgebras [7, 22, 23, 50, 59, 64–66].
(ii) When $\alpha = id_A$ and A is trivially graded, by the group with one element, we recover Lie algebras.
(iii) When A is trivially graded, while α is an arbitrary linear map, we recover Hom-Lie algebras [37, 45, 54–56], and if A is graded by the group of two elements $\mathbb{Z}_2$, while α is an arbitrary even linear map, and the commutation factor is defined as $\varepsilon(i, j) = (-1)^{ij}$ for all $i, j \in \mathbb{Z}_2$, then we get Hom-Lie superalgebras.

Proposition 22.6 ([8]) *Let $(A, \cdot, \varepsilon, \alpha)$ be a Hom-Novikov color Hom-algebra. Then, there exists a Hom-Lie color algebra structure on A given for all $x, y \in \mathcal{H}(A)$ by*

$$[x, y] = x \cdot y - \varepsilon(x, y) y \cdot x. \tag{22.16}$$

Example 22.6 Let $A = A_0 \oplus A_1 =< e_1 > \oplus < e_2, e_3 >$ be a 3-dimensional superspace. The quintuple $(A, \cdot, \varepsilon, \alpha)$ is a Hom-Novikov color Hom-algebra with

multiplication: $e_1 \cdot e_2 = e_3, \quad e_2 \cdot e_1 = -e_3,$

bicharacter: $\varepsilon(i, j) = (-1)^{ij},$

even linear map$\alpha : A \to A : \ \alpha(e_1) = -2e_1, \ \alpha(e_2) = e_3, \ \alpha(e_3) = e_2 - e_3.$

Therefore, by Proposition 22.6, $(A, [\cdot, \cdot], \varepsilon, \alpha)$ is a Hom-Lie color Hom-algebra with

$$[e_1, e_2] = -[e_2, e_1] = 2e_3.$$

Proposition 22.7 ([1]) *Let $\mathcal{A} = (A, [\cdot, \cdot], \varepsilon)$ be a Hom-Lie color Hom-algebra and $\alpha : \mathcal{A} \to \mathcal{A}$ be a Hom-Lie color Hom-algebras morphism. Define $[\cdot, \cdot]_\alpha : A \times A \to A$ for all $x, y \in A$, by $[x, y]_\alpha = \alpha([x, y])$. Then, $\mathcal{A}_\alpha = (A, [\cdot, \cdot]_\alpha, \varepsilon, \alpha)$ is a Hom-Lie color Hom-algebra called the α-twist or Yau twist of $(A, [\cdot, \cdot], \varepsilon)$.*

Definition 22.9 Let $(A, [\cdot, \cdot], \varepsilon, \alpha)$ be a Hom-Lie color Hom-algebra, (V, β) is a pair of Γ-graded linear space V and an even linear map $\beta : V \to V$. Let $\rho : A \to End(V)$ an even linear map. The triple (ρ, β, V) is called a representation of $(A, [\cdot, \cdot], \varepsilon, \alpha)$ if for all $x, y \in \mathcal{H}(A)$, $v \in \mathcal{H}(V)$,

$$\rho([x, y])\beta(v) = \rho(\alpha(x))\rho(y)v - \varepsilon(x, y)\rho(\alpha(y))\rho(x)v.$$

Proposition 22.8 *Let* (ρ, β, V) *is a representation of a Hom-Lie color Hom-algebra* $(A, [\cdot,\cdot], \varepsilon, \alpha)$*. Then the direct sum of* Γ*-graded linear spaces,*

$$A \oplus V = \bigoplus_{\gamma\in\Gamma}(A \oplus V)_\gamma = \bigoplus_{\gamma\in\Gamma}(A_\gamma \oplus V_\gamma),$$

is turned into a Hom-Lie color Hom-algebra by defining multiplication in $A \oplus V$ *for all* $X_1 = x_1 + v_1 \in A_{\gamma_1} \oplus V_{\gamma_1}$, $X_2 = x_2 + v_2 \in A_{\gamma_2} \oplus V_{\gamma_2}$ *by*

$$\begin{aligned}
&[x_1 + v_1, x_2 + v_2]' = [x_1, x_2] + \rho(x_1)v_2 - \varepsilon(v_1, x_2)\rho(x_2)v_1,\\
&(\alpha \oplus \beta)(x_1 + v_1) = \alpha(x_1) + \beta(v_1).
\end{aligned}$$

The Hom-Lie color Hom-algebra constructed in previous Proposition is denoted by $(A \oplus V, [\cdot,\cdot]', \varepsilon, \alpha + \beta)$ or $A \ltimes_{\rho,\alpha,\beta} V$.

Example 22.7 Let $(A, [\cdot,\cdot], \varepsilon, \alpha)$ be a Hom-Lie algebra. Then (ad, α, A) is a representation of $(A, [\cdot,\cdot], \varepsilon, \alpha)$, where $ad(x)y = [x, y]$ for all $x, y \in \mathcal{H}(A)$, called the adjoint representation of $(A, [\cdot,\cdot], \varepsilon, \alpha)$.

Now, we introduce the notion of matched pair of Hom-Lie color Hom-algebra

Proposition 22.9 *Suppose that* $(A, [\cdot,\cdot]_A, \varepsilon, \alpha)$ *and* $(B, [\cdot,\cdot]_B, \varepsilon, \beta)$ *are Hom-Lie color Hom-algebras, and there are even linear maps* $\rho_A : A \to End(B)$ *and* $\rho_B : B \to End(A)$ *such that* (ρ_A, β, B) *is a representation of* A *and* (ρ_B, α, A) *is a representation of* B *satisfying for any* $x, y \in \mathcal{H}(A),\ a, b \in \mathcal{H}(B)$,

$$\begin{aligned}
&\varepsilon(x, a)\big(\rho_A(\rho_B(a)x)\beta(b) - [\beta(a), \rho_A(x)b]_B\big) + \varepsilon(a + x, b)\big([\beta(b), \rho_A(x)a]_B\\
&\qquad - \rho_A(\rho_B(b)x)\beta(a)\big) + \rho_A(\alpha(x))([a, b]_B) = 0,
\end{aligned}$$

$$\begin{aligned}
&\varepsilon(a, x)\big(\rho_B(\rho_A(x)a)\alpha(y) - [\alpha(x), \rho_B(a)y]_A\big) + \varepsilon(x + a, y)\big([\alpha(y), \rho_B(a)x]_A\\
&\qquad - \rho_B(\rho_A(y)a)\alpha(x)\big) + \rho_B(\beta(a))([x, y]_A) = 0.
\end{aligned}$$

Then, $(A, B, \rho_A, \beta, \rho_B, \alpha)$ *is called a matched pair of Hom-Lie color Hom-algebras. In this case, there is a Hom-Lie color Hom-algebra structure on the linear space of the underlying* Γ*-graded linear spaces of* A *and* B,

$$A \oplus B = \bigoplus_{\gamma\in\Gamma}(A \oplus B)_\gamma = \bigoplus_{\gamma\in\Gamma}(A_\gamma \oplus B_\gamma),$$

given for all $x + a \in A_{\gamma_1} \oplus B_{\gamma_1}$, $y + b \in A_{\gamma_2} \oplus B_{\gamma_2}$ *by*

$$\begin{aligned}
&[x + a, y + b] = [x, y]_A + \rho_A(x)b - \varepsilon(a, y)\rho_A(y)a\\
&\qquad + [a, b]_B + \rho_B(a)y - \varepsilon(x, b)\rho_B(b)x,\\
&(\alpha \oplus \beta)(x + a) = \alpha(x) + \beta(a).
\end{aligned}$$

22.3 Admissible Hom-Novikov-Poisson Color Hom-algebras

In this section, we recall the main result of Hom-Novikov-Poisson color Hom-algebras in [5] and we introduce their notions of bimodules and matched pairs. Next, we introduce the definition of admissible Hom-Novikov-Poisson color Hom-algebras and we give some explicit constructions. Finally, we show that the much larger class of admissible Hom-Novikov-Poisson color Hom-algebras is also closed under tensor products.

22.3.1 Constructions and Bimodules of (Admissible) Hom-Novikov-Poisson Color Hom-Algebras

Definition 22.10 ([5]) Hom-Novikov-Poisson color Hom-algebras are quintuples $(A, \cdot, \diamond, \varepsilon, \alpha)$ consisting of an ε-commutative Hom-associative color Hom-algebra $(A, \cdot, \varepsilon, \alpha)$ and a Hom-Novikov color Hom-algebra $(A, \diamond, \varepsilon, \alpha)$ such that, for all $x, y, z \in \mathcal{H}(A)$,

$$(x \cdot y) \diamond \alpha(z) = \varepsilon(y, z)(x \diamond z) \cdot \alpha(y), \tag{22.17}$$

$$(x \diamond y) \cdot \alpha(z) - \alpha(x) \diamond (y \cdot z) = \varepsilon(x, y)\big((y \diamond x) \cdot \alpha(z) - \alpha(y) \diamond (x \cdot z)\big). \tag{22.18}$$

A Hom-Novikov-Poisson color Hom-algebra is called multiplicative if the linear map $\alpha : A \to A$ is multiplicative with respect to $\cdot$ and $\diamond$, that is, for all $x, y \in \mathcal{H}(A)$,

$$\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y), \quad \alpha(x \diamond y) = \alpha(x) \diamond \alpha(y).$$

Remark 22.4 Hom-Novikov-Poisson color Hom-algebras contain Novikov-Poisson color Hom-algebras, Hom-Novikov-Poisson algebras and Novikov-Poisson algebras for special choices of the twising map and grading group.

(i) When $\alpha = id$, we get Novikov-Poisson color Hom-algebra.
(ii) When $\Gamma = \{e\}$ and $\alpha \neq id$, we get Hom-Novikov-Poisson algebra [85].
(iii) When $\Gamma = \{e\}$ and $\alpha = id$, we get Novikov-Poisson algebra [77, 78].

Example 22.8 Let $A = A_0 \oplus A_1 =< e_1, e_2 > \oplus < e_3, e_4 >$ be a 4-dimensional superspace. Then $(A, \cdot, \diamond, \varepsilon, \alpha)$ is a Hom-Novikov-Poisson color Hom-algebra with

the bicharacter $\varepsilon(i, j) = (-1)^{ij}$,

the multiplications: $e_2 \cdot e_2 = \lambda_1 e_1, \quad e_2 \cdot e_4 = e_4 \cdot e_2 = \lambda_2 e_3, \quad \lambda_i \in \mathbb{K},$
$e_2 \diamond e_4 = \mu_2 e_3, \quad e_4 \diamond e_2 = \mu_3 e_3, \quad e_4 \diamond e_4 = \mu_4 e_1, \quad \mu_i \in \mathbb{K},$

even linear map $\alpha : A \to A :\ \alpha(e_1) = 2e_1, \quad \alpha(e_2) = e_2 - e_1,$
$\alpha(e_3) = -e_4, \quad \alpha(e_4) = e_3.$

Definition 22.11 Let $(A, \cdot, \diamond, \alpha)$ and $(A', \cdot', \diamond', \alpha')$ be Hom-Novikov-Poisson color Hom-algebras. A linear map of degree zero $f : A \to A'$ is a Hom-Novikov-Poisson color Hom-algebra morphism if

$$\cdot' \circ (f \otimes f) = f \circ \cdot, \quad \diamond' \circ (f \otimes f) = f \circ \diamond \text{ and } f \circ \alpha = \alpha' \circ f.$$

Theorem 22.1 ([5]) *Let $\mathcal{A} = (A, \cdot, \diamond, \varepsilon)$ be a Hom-Novikov-Poisson color Hom-algebra and $\alpha : \mathcal{A} \to \mathcal{A}$ be a Hom-Novikov-Poisson color Hom-algebras morphism. Define $\cdot_\alpha, \diamond_\alpha : A \times A \to A$ for all $x, y \in \mathcal{H}(A)$, by $x \cdot_\alpha y = \alpha(x \cdot y)$ and $x \diamond_\alpha y = \alpha(x \diamond y)$. Then, $\mathcal{A}_\alpha = (A_\alpha = A, \cdot_\alpha, \diamond_\alpha, \varepsilon, \alpha)$ is a Hom-Novikov-Poisson color Hom-algebra called the α-twist or Yau twist of $(A, \cdot, \diamond, \varepsilon)$.*

Definition 22.12 Let $(A, \cdot, \diamond, \varepsilon, \alpha)$ be a Hom-Novikov-Poisson color Hom-algebra. A bimodule of $(A, \cdot, \diamond, \varepsilon, \alpha)$ is a quintuple (s, l, r, β, V) such that (s, β, V) is a bimodule of the ε-commutative Hom-associative color Hom-algebra $(A, \cdot, \varepsilon, \alpha)$ and (l, r, β, V) is a bimodule of the Hom-Novikov color Hom-algebra $(A, \diamond, \varepsilon, \alpha)$ satisfying, for all $x, y \in \mathcal{H}(A)$, $v \in \mathcal{H}(V)$,

$$l(x \cdot y)\beta(v) = \varepsilon(x, y)s(\alpha(y))l(x)v, \tag{22.19}$$
$$r(\alpha(y))s(x)v = \varepsilon(v, y)s(x \diamond y)\beta(v), \tag{22.20}$$
$$r(\alpha(y))s(x)v = s(\alpha(x))r(y)v, \tag{22.21}$$
$$s(x \diamond y)\beta(v) - l(\alpha(x))s(y)v = \varepsilon(x, y)\big(s(y \diamond x)\beta(v) - l(\alpha(y))s(x)v\big), \tag{22.22}$$
$$\begin{aligned}\varepsilon(x + v, y)\big(s(\alpha(y))l(x)v - \varepsilon(x, v)s(\alpha(y))r(x)v\big) = {} & \varepsilon(v, y)l(\alpha(x))s(y)v \\ & -\varepsilon(x, v)r(x \cdot y)\beta(v).\end{aligned} \tag{22.23}$$

Proposition 22.10 *If $A \oplus V = \bigoplus_{\gamma\in\Gamma}(A \oplus V)_\gamma = \bigoplus_{\gamma\in\Gamma}(A_\gamma \oplus V_\gamma)$ is the direct sum of Γ-graded linear spaces, then $(A \oplus V, \cdot', \diamond', \varepsilon, \alpha + \beta)$ is a Hom-Novikov-Poisson color Hom-algebra, where $(A \oplus V, \cdot', \varepsilon, \alpha + \beta)$ is the semi-direct product ε-commutative Hom-associative color Hom-algebra $A \ltimes_{s,\alpha,\beta} V$ and $(A \oplus V, \diamond', \varepsilon, \alpha + \beta)$ is the semi-direct product Hom-Novikov color Hom-algebra $A \ltimes_{l,r,\alpha,\beta} V$.*

Proof Let $(A, \cdot, \diamond, \varepsilon, \alpha)$ be a Hom-Novikov-Poisson color Hom-algebra, and let (s, l, r, β, V) be a bimodule. By Propositions 22.1 and 22.4, $(A \oplus V, \cdot', \varepsilon, \alpha + \beta)$ is a ε-commutative Hom-associative color Hom-algebra, and $(A \oplus V, \diamond', \varepsilon, \alpha + \beta)$ is a Hom-Novikov color Hom-algebra. Now, we show that the compatibility conditions (22.17)–(22.18) are satisfied. For all $X_i = x_i + v_i \in A_{\gamma_i} \oplus V_{\gamma_i}$, $i = 1, 2, 3$ we have

$$\begin{aligned}
&(X_1 \cdot' X_2) \diamond' (\alpha + \beta)X_3 - \varepsilon(X_2, X_3)(X_1 \diamond' X_3) \cdot' (\alpha + \beta)X_2 \\
&\quad = \big((x_1 + v_1) \cdot' (x_2 + v_2)\big) \diamond' (\alpha + \beta)(x_3 + v_3) \\
&\qquad - \varepsilon(X_2, X_3)\big((x_1 + v_1) \diamond' (x_3 + v_3)\big) \cdot' (\alpha + \beta)(x_2 + v_2) \\
&\quad = \big(x_1 \cdot x_2 + s(x_1)v_2 + \varepsilon(v_1, x_2)s(x_2)v_1\big) \diamond' (\alpha(x_3) + \beta(v_3)) \\
&\qquad - \varepsilon(x_2, x_3)\big((x_1 \diamond x_3 + l(x_1)v_3 + r(x_3)v_1\big) \cdot' (\alpha(x_2) + \beta(v_2))
\end{aligned}$$

$$
\begin{aligned}
&= (x_1 \cdot x_3) \diamond \alpha(x_3) + l(x_1 \cdot x_2)\beta(v_3) + r(\alpha(x_3))s(x_1)v_2 \\
&\quad + \varepsilon(v_1, x_2)r(\alpha(x_3))s(x_2)v_1 - \varepsilon(x_2, x_3)\big(x_1 \diamond x_3) \cdot \alpha(x_2) + s(x_1 \diamond x_2)\beta(v_2)\big) \\
&\quad - \varepsilon(x_1, x_2)\big(s(\alpha(x_2))l(x_1)v_3 + s(\alpha(x_2))r(x_3)v_1\big) \\
&= \underbrace{\Big((x_1 \cdot x_2) \diamond \alpha(x_3) - \varepsilon(x_2, x_3)(x_1 \diamond x_3) \cdot \alpha(x_2)\Big)}_{=0 \text{ by } (22.17)} \\
&\quad + \underbrace{\Big(l(x_1 \cdot x_2)\beta(v_3) - \varepsilon(x_1, x_2)s(\alpha(x_2))l(x_1)v_3\Big)}_{=0 \text{ by } (22.19)} \\
&\quad + \underbrace{\Big(r(\alpha(x_3))s(x_1)v_2 - \varepsilon(x_2, x_3)s(x_1 \diamond x_3)\beta(v_2)\Big)}_{=0 \text{ by } (22.20) \text{ and Remark 1}} \\
&\quad + \underbrace{\varepsilon(x_1, x_2)\Big(r(\alpha(x_3))s(x_2)v_1 - s(\alpha(x_2))r(x_3)v_1\Big)}_{=0 \text{ by } (22.21)} = 0,
\end{aligned}
$$

$$
\begin{aligned}
&(X_1 \diamond' X_2) \cdot' (\alpha + \beta)X_3 - (\alpha + \beta)X_1 \diamond' (X_2 \cdot' X_3) \\
&- \varepsilon(X_1, X_2)\big((X_2 \diamond' X_1) \cdot' (\alpha + \beta)X_3 - (\alpha + \beta)X_2 \diamond' (X_1 \cdot' X_3)\big) \\
&\quad = \big((x_1 + v_1) \diamond' (x_2 + v_2)\big) \cdot' (\alpha + \beta)(x_3 + v_3) \\
&\quad\quad - (\alpha + \beta)(x_1 + v_1) \diamond' \big((x_2 + v_2) \cdot' (x_3 + v_3)\big) \\
&\quad\quad - \varepsilon(X_1, X_2)\big((x_2 + v_2) \diamond' (x_1 + v_1)\big) \cdot' (\alpha + \beta)(x_3 + v_3) \\
&\quad\quad - (\alpha + \beta)(x_2 + v_2) \diamond' \big((x_1 + v_1) \cdot' (x_3 + v_3)\big) \\
&\quad = (x_1 \diamond x_2)\alpha(x_3) + s(x_1 \diamond x_2)\beta(v_3) + \varepsilon(x_1 + x_2, v_3)s(\alpha(x_3))l(x_1)v_2 \\
&\quad\quad + \varepsilon(x_1 + x_2, v_3)s(\alpha(x_3))r(x_2)v_1 - \alpha(x_1) \diamond (x_2 \cdot x_3) \\
&\quad\quad - l(\alpha(x_1))s(x_2)v_3 + \varepsilon(v_2, x_3)l(\alpha(x_1))s(x_3)v_2 + r(x_2 \cdot x_3)\beta(v_1) \\
&\quad\quad - \varepsilon(x_1, x_2)\Big(x_2 \diamond x_1) \cdot \alpha(x_3) + s(x_2 \diamond x_1)\beta(v_3) + \varepsilon(x_1 + x_2, v_3)s(\alpha(x_3))l(x_2)v_1 \\
&\quad\quad + \varepsilon(x_1 + x_2, v_3)s(\alpha(x_3))r(x_1)v_2 - \alpha(x_2) \diamond (x_1 \cdot x_3) - l(\alpha(x_2))s(x_1)v_3 \\
&\quad\quad + \varepsilon(v_1, x_3)l(\alpha(x_2))s(x_3)v_1 + r(x_1 \cdot x_3)\beta(v_2)\Big) \\
&\quad = \underbrace{\Big((x_1 \diamond x_2) \cdot \alpha(x_3) - \alpha(x_2) \diamond (x_2 \cdot x_3)}_{} \\
&\quad\quad\quad\quad\quad\quad - \underbrace{\varepsilon(x_1, x_2)\big((x_2 \diamond x_1) \cdot \alpha(x_3) - \alpha(x_2) \diamond (x_1 \cdot x_3)\big)\Big)}_{=0 \text{ by } (22.18)} \\
&\quad\quad + \underbrace{\Big(s(x_1 \diamond x_2)\beta(v_3) - l(\alpha(x_1))s(x_2)v_3}_{}
\end{aligned}
$$

$$- \underbrace{\varepsilon(x_1, x_2)(s(x_2 \diamond x_1)\beta(v_3) - l(\alpha(x_2))s(x_1)v_3\Big)}_{=0 \text{ by } (22.22)}$$

$$+ \underbrace{\Big(\varepsilon(x_1 + x_2, v_3)\big(s(\alpha(x_3))l(x_1)v_2 - \varepsilon(x_1, x_2)s(\alpha(x_3))r(x_1)v_2\big)}$$

$$\underbrace{- \varepsilon(x_1, x_2)r(x_1 \cdot x_3)\beta(v_2) + \varepsilon(x_2, v_3)l(\alpha(x_1))s(x_3)v_2\Big)}_{=0 \text{ by } (22.23) \text{ and Remark 1}}$$

$$- \underbrace{\varepsilon(x_1, x_2)\Big(\varepsilon(x_1 + x_2, v_3)\big(s(\alpha(x_3))l(\alpha(x_2))v_1 - \varepsilon(x_2, x_1)s(\alpha(x_3))r(x_2)v_1\big)}$$

$$\underbrace{- \varepsilon(x_1, x_3)l(\alpha(x_2))s(x_3)v_1 + \varepsilon(x_2, x_1)r(x_2 \cdot x_3)\beta(v_1)\Big)}_{=0 \text{ by } (22.23) \text{ and Remark 1}} = 0.$$

Hence, $(A \oplus V, \cdot', [\cdot, \cdot]', \varepsilon, \alpha + \beta)$ is a Hom-Novikov-Poisson color Hom-algebra. □

The Hom-Novikov-Poisson color Hom-algebra constructed in previous Proposition is denoted by $A \ltimes_{s,l,r,\alpha,\beta} V$.

Example 22.9 Here are some important examples important bimodules of Hom-Novikov-PoissoncolorHom-algebras.

(1) Let $(A, \cdot, \diamond, \varepsilon, \alpha)$ be a Hom-Novikov-Poisson color Hom-algebra, and let for all $x, y \in \mathcal{H}(A)$, $S(x)y = x \cdot y = \varepsilon(x, y)y \cdot x$, $L(x)y = x \diamond y$ and $R(x, y) = y \diamond x$. Then (S, L, R, α, A) is a bimodule of $(A, \cdot, \diamond, \varepsilon, \alpha)$, called the regular bimodule of $(A, \cdot, \diamond, \varepsilon, \alpha)$.
(2) If $f : \mathcal{A}=(A, \cdot_1, \diamond_1, \varepsilon, \alpha) \longrightarrow (A', \cdot_2, \diamond_2, \varepsilon, \beta)$ is a morphism of Hom-Novikov-Poisson color Hom-algebras, then (s, l, r, β, A') is bimodule of $\mathcal{A}$ via f, that is, $s(x)y = f(x) \cdot_2 y,\ \ l(x)y = f(x) \diamond_2 y,\ \ r(x)y = y \diamond_2 f(x)$, for $(x, y) \in \mathcal{H}(A) \times \mathcal{H}(A')$.

Theorem 22.2 *Suppose that $\mathcal{A} = (A, \cdot_A, \diamond_A, \varepsilon, \alpha)$ and $\mathcal{B} = (B, \cdot_B, \diamond_B, \varepsilon, \beta)$ are two Hom-Novikov-Poisson color Hom-algebras, and there are such even linear maps $s_A, l_A, r_A : A \to End(B)$ and $s_B, l_B, r_B : B \to End(A)$ that $A \bowtie^{s_A,\beta}_{s_B,\alpha} B$ is a matched pair of ε-commutative Hom-associative color Hom-algebras, $A \bowtie^{l_A,r_A,\beta}_{l_B,r_B,\alpha} B$ is a matched pair of Hom-Novikov color Hom-algebras, and for all $x, y \in \mathcal{H}(A),\ a, b \in \mathcal{H}(B)$,*

$$r_A(\alpha(x))(a \cdot_B b) = \varepsilon(b, x)\big(r_A(x)a \cdot_B \beta(b) + s_A(l_B(a)x)\beta(b),$$

$$l_A(s_B(a)x)\beta(b) + \varepsilon(a, x)s(x)a \diamond_B \beta(b) \\ = \varepsilon(x, b)\big(\varepsilon(a + b, x)s(\alpha(x))(a \diamond_B b)\big),$$

$$\varepsilon(a, x)l_A(s_B(a)x)\beta(b) + s_A(x)a \diamond_B \beta(b) \\ = \varepsilon(a, b)\big(l_A(x)b \cdot_B \beta(a) + s_A(r_B(b)x)\beta(a)\big),$$

$$\varepsilon(a + b, x)s(\alpha(x))(a \diamond_B b) - \varepsilon(b, x)\beta(a) \diamond_B s_A(x)b - r_A(s_B(b)x)\beta(a) \\ = \varepsilon(a, b)\big(\varepsilon(a + b, x)s(\alpha(x))(b \diamond_B a) - \varepsilon(a, x)\beta(b) \diamond_B (s_A(x)a) \\ - r_A(s_B(a)x)\beta(b)\big),$$

$$r_A(x)b \cdot_B \beta(b) + s_A(l_B(a)x)\beta(b) \\ -\beta(a) \diamond_B (s_A(x)b) - \varepsilon(x, b)r_A(s_B(b)x)\beta(a) \\ = \varepsilon(a, x)\big(l_A(x)b \cdot_B \beta(b) - s_A(r_B(a)x)\beta(b) - l_A(\alpha(x))(a \cdot_B b)\big),$$

$$l_A(x)a \cdot_B \beta(b) + s_A(r_B(a)x)\beta(b) - l_A(\alpha(x))(a \cdot_B b) \\ = \varepsilon(x, a)\big(r_A(x)a \cdot_B \beta(b) + s_A(l_B(a)x)\beta(b) \\ - \beta(a) \diamond_B (s_A(x)b) - \varepsilon(x, b)r_A(s_B(b)x)\beta(a)\big),$$

$$r_B(\beta(a))(x \cdot_A y) = \varepsilon(y, a)\big(r_B(a)x \cdot_A \alpha(y) + s_B(l_A(x)a)\alpha(y),$$

$$l_B(s_A(x)a)\alpha(y) + \varepsilon(x, a)s(a)x \diamond_A \alpha(y) \\ = \varepsilon(a, y)\big(\varepsilon(x + y, a)s(\beta(a))(x \diamond_A y)\big),$$

$$\varepsilon(x, a)l_B(s_A(x)a)\alpha(y) + s_B(a)x \diamond_A \alpha(y) \\ = \varepsilon(x, y)\big(l_B(a)y \cdot_A \alpha(x) + s_B(r_A(y)a)\alpha(x)\big),$$

$$\varepsilon(x + y, a)s(\beta(a))(x \diamond_A y) - \varepsilon(y, a)\alpha(x) \diamond_A s_B(a)y - r_B(s_A(y)a)\alpha(x) \\ = \varepsilon(x, y)\big(\varepsilon(x + y, a)s(\beta(a))(y \diamond_A x) - \varepsilon(x, a)\alpha(y) \diamond_A (s_B(a)x) \\ - r_B(s_A(x)a)\alpha(y)\big),$$

$$r_B(a)y \cdot_A \alpha(y) + s_B(l_A(x)a)\alpha(y) - \alpha(x) \diamond_A (s_B(a)y) - \varepsilon(a, y)r_B(s_A(y)a)\alpha(x) \\ = \varepsilon(x, a)\big(l_B(a)y \cdot_A \alpha(y) - s_B(r_A(x)a)\alpha(y) - l_B(\beta(a))(x \cdot_A y)\big),$$

$$l_B(a)x \cdot_A \alpha(y) + s_B(r_A(x)a)\alpha(y) - l_B(\beta(a))(x \cdot_A y) \\ = \varepsilon(a, x)\big(r_B(a)x \cdot_A \alpha(y) + s_B(l_A(x)a)\alpha(y) \\ - \alpha(x) \diamond_A (s_B(a)y) - \varepsilon(a, y)r_B(s_A(y)a)\alpha(x)\big).$$

Then, $(A, B, s_A, l_A, r_A, \beta, s_B, l_B, r_B, \alpha)$ is called a matched pair of the Hom-Novikov-Poisson color Hom-algebras. In this case, on the direct sum $A \oplus B$ of the underlying linear spaces of $\mathcal{A}$ and $\mathcal{B}$, there is a Hom-Novikov-Poisson color Hom-algebra structure which is given for any $x + a \in A_{\Gamma_1} \oplus B_{\Gamma_1}$, $y + b \in A_{\Gamma_2} \oplus B_{\Gamma_2}$ by

$$(x + a) \cdot (y + b) = x \cdot_A y + (s_A(x)b + \varepsilon(a, y)s_A(y)a) \\ +a \cdot_B b + (s_B(a)y + \varepsilon(x, b)s_B(b)x),$$

$$(x + a) \diamond (y + b) = x \diamond_A y + (l_A(x)b + r_A(y)a) \\ +a \diamond_B b + (l_B(a)y + r_B(b)x).$$

Proof By Propositions 22.2 and 22.5, we deduce that $(A \oplus B, \cdot, \varepsilon, \alpha + \beta)$ is a ε-commutative Hom-associative color Hom-algebra and $(A \oplus B, \diamond, \alpha + \beta)$ is a Hom-

Novikov color Hom-algebra. Now, the rest, it is easy (in a similar way as for Proposition 22.2) to verify the compatibility conditions are satisfied. □

Definition 22.13 A transposed Hom-Poisson color Hom-algebra is defined as a quintuple $(A, \cdot, [\cdot,\cdot], \varepsilon, \alpha)$, where $(A, \cdot, \varepsilon, \alpha)$ is a ε-commutative Hom-associative color Hom-algebra and $(A, [\cdot,\cdot], \varepsilon, \alpha)$ is a Hom-Lie color Hom-algebra, satisfying the transposed Hom-ε-Leibniz identity for $x, y, z \in \mathcal{H}(A)$,

$$2\alpha(z)\cdot[x,y] = [z\cdot x, \alpha(y)] + \varepsilon(z,x)[\alpha(x), z\cdot y]. \tag{22.24}$$

Proposition 22.11 *Let $(A, \cdot, [\cdot,\cdot], \varepsilon, \alpha)$ be a multiplicative transposed Hom-Poisson color Hom-algebra. Then the following identities hold for all $h, x, y, z \in \mathcal{H}(A)$,*

$$\circlearrowleft_{x,y,z} \varepsilon(z,x)\alpha(x)\cdot[y,z] = 0, \tag{22.25}$$
$$\circlearrowleft_{x,y,z} \varepsilon(z,x)[\alpha(h)\cdot[x,y], \alpha^2(z)] = 0, \tag{22.26}$$
$$\circlearrowleft_{x,y,z} \varepsilon(z,x)[\alpha(h)\cdot\alpha(x), [\alpha(y),\alpha(z)]] = 0, \tag{22.27}$$
$$\circlearrowleft_{x,y,z} \varepsilon(z,x)[\alpha(h),\alpha(x)]\cdot[\alpha(y),\alpha(z)] = 0. \tag{22.28}$$

Proof **Proof of** (22.25)**:** Let $x, y, z \in \mathcal{H}(A)$. By the transposed Hom-$\varepsilon$-Leibniz identity,

$$\begin{aligned}\circlearrowleft_{x,y,z} \varepsilon(z,x)\Big(2\alpha(x)\cdot[y,z]\Big) &= \circlearrowleft_{x,y,z} \varepsilon(z,x)\Big([x\cdot y,\alpha(z)] + \varepsilon(x,y)[\alpha(y), x\cdot z]\Big)\\ &= \circlearrowleft_{x,y,z} \varepsilon(z,x)\Big([x\cdot y,\alpha(z)] + \varepsilon(x+y,z)[z\cdot x,\alpha(y)]\Big)\\ &= \circlearrowleft_{x,y,z} \Big(\varepsilon(z,x)[x\cdot y,\alpha(z)] - \varepsilon(y,z)[z\cdot x,\alpha(y)]\Big) = 0,\end{aligned}$$

which yields (22.25).

Proof of (22.26)**:** Let $x, y, z, h \in \mathcal{H}(A)$. First, by (22.24), we have

$$\begin{aligned}\circlearrowleft_{x,y,z} \varepsilon(z,x)\Big(2\alpha^2(h)[[x,y],\alpha(z)]\Big) = \circlearrowleft_{x,y,z} \varepsilon(z,x)\Big(&[\alpha(h),[x,y],\alpha^2(z)]\\ &+ \varepsilon(h,x+y)[\alpha([x,y]),\alpha(h\cdot z)]\Big).\end{aligned}$$

Applying the Hom-Jacobi identity of the above equality, we obtain

$$\begin{aligned}&\circlearrowleft_{x,y,z} \varepsilon(z,x)\Big([\alpha(h),[x,y],\alpha^2(z)]\Big)\\ &\quad+ \circlearrowleft_{x,y,z} \varepsilon(z,x)\Big(\varepsilon(h,x+y)[\alpha([x,y]),\alpha(h\cdot z)]\Big) = 0.\end{aligned} \tag{22.29}$$

Next, by the Hom-Jacobi identity, we have

$$\begin{aligned}&\varepsilon(h,x+y)[[\alpha(x),\alpha(y)],\alpha(h\cdot z)] + \varepsilon(x+y,z)[[h\cdot z,\alpha(x)],\alpha^2(y)]\\ &\qquad+ \varepsilon(h,y)\varepsilon(x,y+z)[[\alpha(y),h\cdot z],\alpha^2(x)] = 0,\end{aligned}$$

and by (22.24) we have

$$\varepsilon(h, y)\varepsilon(x, y+z)[[\alpha(y), h \cdot z], \alpha^2(x)] = 2\varepsilon(x, y+z)[\alpha(h) \cdot [y, z], \alpha^2(x)] \\ - \varepsilon(x, y+z)[[h \cdot y, \alpha(z)], \alpha^2(x)] = 0.$$

Thus we obtain

$$\varepsilon(h, x+y)[[\alpha(x), \alpha(y)], \alpha(h \cdot z)] + \varepsilon(x+y, z)[[h \cdot z, \alpha(x)], \alpha^2(y)] \\ + \varepsilon(x, y+z)\big(2[\alpha(h) \cdot [y, z], \alpha^2(x)] - \varepsilon(x, y+z)[[h \cdot y, \alpha(z)], \alpha^2(x)]\big) = 0.$$

Similarly, we have

$$\circlearrowleft_{x,y,z} \varepsilon(z, x)\Big(\varepsilon(h, x+y)[[\alpha(x), \alpha(y)], \alpha(h \cdot z)] + \varepsilon(x+y, z)[[h \cdot z, \alpha(x)], \alpha^2(y)] \\ + \varepsilon(x, y+z)\big(2[\alpha(h) \cdot [y, z], \alpha^2(x)] - \varepsilon(x, y+z)[[h \cdot y, \alpha(z)], \alpha^2(x)]\big)\Big) = 0.$$

Taking the above sum, we obtain

$$\circlearrowleft_{x,y,z} \varepsilon(z, x)\Big(\varepsilon(h, x+y)[[\alpha(x), \alpha(y)], \alpha(h \cdot z)]\Big) \\ + \circlearrowleft_{x,y,z} \varepsilon(z, x)\Big(2[\alpha(h) \cdot [x, y], \alpha^2(x)]\Big) = 0. \tag{22.30}$$

Finally, taking the difference between the two equations (22.29) and (22.30) we obtain (22.26).

Proof of (22.27)**:** Taking the difference between the two equations (22.26) and (22.29) we obtain (22.27)

Proof of (22.28)**:** Let $x, y, z, h \in \mathcal{H}(A)$. By (22.24) we have

$$\circlearrowleft_{x,y,z} \varepsilon(z, x)\Big(2\varepsilon(h, x+y)[\alpha(x), \alpha(y)] \cdot [\alpha(h), \alpha(z)]\Big) \\ = \circlearrowleft_{x,y,z} \varepsilon(z, x)\Big([\alpha(h) \cdot [x, y], \alpha^2(z)] + \varepsilon(x+y, z)[\alpha^2(h), \alpha(z) \cdot [x, y].]\Big).$$

Applying equations (22.25) and (22.26) to above equality, we obtain (22.28). □

Proposition 22.12 *Let* $(A, \cdot, \diamond, \varepsilon, \alpha)$ *be a Hom-Novikov-Poisson color Hom-algebra. With the bilinear multiplication* $[\cdot, \cdot] : A \times A \to A$ *such that for all* $x, y \in \mathcal{H}(A)$*,*

$$[x, y] = x \diamond y - \varepsilon(x, y) y \diamond x,$$

$(A, \cdot, [\cdot, \cdot], \varepsilon, \alpha)$ *is a Hom-transposed-Poisson color Hom-algebra.*

Proof By definition, we have $(A, \cdot, \varepsilon, \alpha)$ is a ε-commutative Hom-associative color Hom-algebra and by Proposition 22.6, $(A, [\cdot, \cdot], \varepsilon, \alpha)$ is a Hom-Lie color Hom-algebra. Now, we show that the ε-transposed Hom-Leibniz identity is satisfied. For any $x, y, z \in \mathcal{H}(A)$ we have

$$\varepsilon(z,x)[x\cdot z,\alpha(y)]+\varepsilon(z,x+y)[\alpha(x),y\cdot z]-2\alpha(z)\cdot[x,y]$$
$$=\varepsilon(z,x)\Big((x\cdot z)\diamond\alpha(y)-\varepsilon(x+z,y)\alpha(y)\diamond(x\cdot z)\Big)$$
$$+\varepsilon(z,x+y)\Big(\alpha(x)\diamond(y\cdot z)-\varepsilon(x,y+z)(y\cdot z)\diamond\alpha(x)\Big)-2\alpha(z)\cdot\Big(x\cdot y-\varepsilon(x,y)y\cdot x\Big)$$
(by (22.17) and (22.18))
$$=\varepsilon(z,x)(x\cdot z)\diamond\alpha(y)-\alpha(z)(x\diamond y)+\varepsilon(x,y)\alpha(z)\cdot(y\diamond x)$$
$$-\varepsilon(x+z,y)(y\cdot z)\diamond\alpha(x)-\Big(\varepsilon(z,x+y)(x\diamond y)\cdot\alpha(z)$$
$$-\varepsilon(x,y)(y\diamond x)\cdot\alpha(z)-\varepsilon(z,x+y)\alpha(x)\diamond(y\cdot z)+\varepsilon(x,y)\alpha(y)\diamond(x\cdot z)\Big)=0.$$

Hence the conclusion holds. □

Example 22.10 Let $A=A_0\oplus A_1=<e_1,e_2>\oplus<e_3,e_4>$ be a 4-dimensional superspace. Then $(A,\cdot,\diamond,\varepsilon,\alpha)$ is a Hom-Novikov-Poisson color Hom-algebra with

$$\begin{aligned}&\text{bicharacter: } \varepsilon(i,j)=(-1)^{ij},\\ &\text{the multiplications: } e_2\cdot e_2=e_1,\quad e_2\cdot e_4=e_4\cdot e_2=e_3\\ &\qquad e_2\diamond e_4=-e_3,\quad e_4\diamond e_2=e_3,\quad e_4\diamond e_4=2e_1,\\ &\text{even linear map } \alpha: A\to A:\ \alpha(e_1)=2e_1,\quad \alpha(e_2)=-e_2,\\ &\qquad \alpha(e_3)=-e_4,\quad \alpha(e_4)=e_3.\end{aligned}$$

Therefore, using the Proposition 22.12, $(A,\cdot,[\cdot,\cdot],\varepsilon,\alpha)$ is a transposed Hom-Poisson color Hom-algebra with

$$[e_2,e_4]=[e_4,e_2]=-2e_3,\quad [e_4,e_4]=4e_1.$$

Definition 22.14 ([11]) A *Hom-Poisson color Hom-algebra* is defined as a quintuple $(A,\cdot,[\cdot,\cdot],\varepsilon,\alpha)$ such that $(A,\cdot,\varepsilon,\alpha)$ is a ε-commutative Hom-associative color Hom-algebra, and $(A,[\cdot,\cdot],\varepsilon,\alpha)$ is a Hom-Lie color Hom-algebra, satisfying for all $x,y,z\in\mathcal{H}(A)$, the Hom-ε-Leibniz identity,

$$[\alpha(x),y\cdot z]=\varepsilon(x,y)\alpha(y)\cdot[x,z]+\varepsilon(x+y,z)\alpha(z)\cdot[x,y].\tag{22.31}$$

Condition (22.31), expressing the compatibility between the multiplication and the Poisson bracket, can be reformulated equivalently as

$$[x\cdot y,\alpha(z)]=\varepsilon(y,z)[x,z]\cdot\alpha(y)+\alpha(x)\cdot[y,z].\tag{22.32}$$

Definition 22.15 A Hom-Novikov-Poisson color Hom-algebra $(A,\cdot,\diamond,\varepsilon,\alpha)$ is called **admissible** if $(A,\cdot,[\cdot,\cdot],\varepsilon,\alpha)$ is a Hom-Poisson color Hom-algebra, with the bilinear multiplication $[\cdot,\cdot]:A\times A\to A$ such that for all $x,y\in\mathcal{H}(A)$,

$$[x,y]=x\diamond y-\varepsilon(x,y)y\diamond x.\tag{22.33}$$

Lemma 22.1 *Let* $(A, \cdot, \diamond, \varepsilon, \alpha)$ *be a Hom-Novikov-Poisson color Hom-algebra. Then for any* $x, y, z \in \mathcal{H}(A)$,

$$(x \cdot y) \diamond \alpha(z) = \alpha(x) \diamond (y \cdot z). \tag{22.34}$$

Proof For all $x, y, z \in \mathcal{H}(A)$,

$$\begin{aligned}
&((A, \cdot, \varepsilon, \alpha) \text{ is } \varepsilon - \text{commutative})\\
&(x \cdot y) \circ \alpha(z) = \varepsilon(x, y)(y \cdot x) \circ \alpha(z)\\
&\text{(by (22.17))}\\
&= \varepsilon(x, y + z)(y \circ z) \cdot \alpha(x)\\
&((A, \cdot, \varepsilon, \alpha) \text{ is } \varepsilon - \text{commutative})\\
&= \alpha(x) \cdot (y \circ z).
\end{aligned}$$

which completes the proof. □

The following result gives a necessary and sufficient condition under which a Hom-Novikov-Poisson color algebra is admissible.

Theorem 22.3 *Let* $(A, \cdot, \diamond, \varepsilon, \alpha)$ *be Hom-Novikov-Poisson color Hom-algebra. Then* A *is an admissible if and only if*

$$as^l_A(x, y, z) = (x \cdot y) \diamond \alpha(z) - \alpha(x) \diamond (y \cdot z) = 0. \tag{22.35}$$

Proof By definition, $(A, \cdot, \varepsilon, \alpha)$ is a ε-commutative Hom-associative color Hom-algebra and by Proposition 22.6, $(A, [\cdot, \cdot], \varepsilon, \alpha)$ is a Hom-Lie color Hom-algebra. Therefore,

$$\begin{aligned}
&\text{(using 22.16)}\\
&[\alpha(x), y \cdot z] = \alpha(x) \diamond (y \cdot z) - \varepsilon(x + y, z)(y \cdot z) \diamond \alpha(x)\\
&\text{(by (22.18) and Lemma 1)}\\
&= (x \diamond y) \cdot \alpha(z) - \varepsilon(x, y)(y \diamond x) \cdot \alpha(z) + \varepsilon(x, y)\alpha(y) \diamond (x \cdot z) - \varepsilon(x, y + z)\alpha(y) \cdot (z \diamond x),\\
&[x, y] \cdot \alpha(x) + \varepsilon(x, y)\alpha(y)[x, z]\\
&\text{(using 22.16)}\\
&= (x \diamond y) \cdot \alpha(z) - \varepsilon(x, y)(y \diamond x) \cdot \alpha(z) + \varepsilon(x, y)\alpha(y) \cdot (x \diamond z) - \varepsilon(x, y + z)\alpha(y) \cdot (z \diamond x)\\
&\text{(by Lemma 1)}\\
&= (x \diamond y) \cdot \alpha(z) - \varepsilon(x, y)(y \diamond x) \cdot \alpha(z) + \varepsilon(x, y)(y \cdot x) \diamond \alpha(z) - \varepsilon(x, y + z)\alpha(y) \cdot (z \diamond x).
\end{aligned}$$

Then A obeys the Hom-ε-Leibniz-identity if and only if $\alpha(y) \diamond (x \cdot z) = (y \cdot x) \diamond \alpha(z)$. □

Example 22.11 If $A = A_0 \oplus A_1 =< e_1, e_2 > \oplus < e_3, e_4 >$ is a 4-dimensional superspace, then $(A, \cdot, \diamond, \varepsilon, \alpha)$ is a Hom-Novikov-Poisson color Hom-algebra with bicharacter: $\varepsilon(i, j) = (-1)^{ij}$,

multiplications: $e_2 \cdot e_2 = \lambda_1 e_1, \quad e_2 \cdot e_4 = e_4 \cdot e_2 = \lambda_2 e_3, \quad \lambda_i \in \mathbb{K}$

$e_2 \diamond e_2 = \mu_1 e_1, \quad e_4 \diamond e_2 = \mu_2 e_3,$

$e_4 \diamond e_4 = \mu_3 e_1, \quad \mu_i \in \mathbb{K}$

even linear map $\alpha : A \to A$: $\alpha(e_1) = 2e_1 - e_2,\ \alpha(e_2) = e_1,$

$\alpha(e_3) = -e_4, \qquad \alpha(e_4) = e_3 - e_4.$

Then by Theorem 22.3, the Hom-Novikov-Poisson color Hom-algebra $(A, \cdot, \diamond, \varepsilon, \alpha)$ is admissible.

Theorem 22.4 *Let $\mathcal{A} = (A, \cdot, \diamond, \varepsilon)$ be an admissible Novikov-Poisson color Hom-algebra and $\alpha : \mathcal{A} \to \mathcal{A}$ be an admissible Novikov-Poisson color Hom-algebras morphism. Define $\cdot_\alpha, \diamond_\alpha : A \times A \to A$ by $x \cdot_\alpha y = \alpha(x \cdot y)$ and $x \diamond_\alpha y = \alpha(x \diamond y)$ for all $x, y \in \mathcal{H}(A)$. Then, $\mathcal{A}_\alpha = (A, \cdot_\alpha, \diamond_\alpha, \varepsilon, \alpha)$ is an admissible Hom-Novikov-Poisson color Hom-algebra called the α-twist or Yau twist of $(A, \cdot, \diamond, \varepsilon)$.*

Proof By Theorem 22.11, $\mathcal{A}_\alpha$ is a Hom-Novikov-Poisson color Hom-algebra. Moreover, the left Hom-associators in $\mathcal{A}$ and $\mathcal{A}_\alpha$ are related for all $x, y, z \in \mathcal{H}(A)$ by

$$as^l_{\mathcal{A}_\alpha}(x, y, z) = \alpha^2 as^l_{\mathcal{A}}(x, y, z).$$

Since $\mathcal{A}$ is left Hom-associative by Theorem 22.3, it follows that so is $\mathcal{A}_\alpha$. Therefore, by Theorem 22.3 again $\mathcal{A}_\alpha$ is admissible. □

Corollary 22.1 *If $\mathcal{A} = (A, \cdot, \diamond, \varepsilon, \alpha)$ is a multiplicative admissible Novikov-Poisson color algebra, then for any $n \in \mathbb{N}^*$,*

(i) *The nth derived admissible Hom-Novikov-Poisson color Hom-algebra of type 1 of $\mathcal{A}$ is defined by*

$$\mathcal{A}_1^n = (A, \cdot^{(n)} = \alpha^n \circ \cdot, \diamond^{(n)} = \alpha^n \circ \diamond, \varepsilon, \alpha^{n+1}).$$

(ii) *The nth derived admissible Hom-Novikov-Poisson color Hom-algebra of type 2 of A is defined by*

$$\mathcal{A}_2^n = (A, \cdot^{(2^n-1)} = \alpha^{2^n-1} \circ \cdot, \diamond^{(2^n-1)} = \alpha^{2^n-1} \circ \diamond, \varepsilon, \alpha^{2^n}).$$

Example 22.12 Let $A = A_0 \oplus A_1 =< e_1, e_2 > \oplus < e_3, e_4 >$ be a 4-dimensional superspace. There is a multiplicative admissible Hom-Novikov-Poisson color Hom-algebra $(A, \cdot, \diamond, \varepsilon, \alpha)$ with the bicharacter, $\varepsilon(i, j) = (-1)^{ij}$, and the multiplications tables for a basis $\{e_1, e_2, e_3, e_4\}$:

$\cdot$	e_1	e_2	e_3	e_4
e_1	0	0	0	0
e_2	0	e_1	0	$4e_3$
e_3	0	0	0	0
e_4	0	$4e_3$	0	0

$\diamond$	e_1	e_2	e_3	e_4
e_1	0	0	0	0
e_2	0	$4e_3$	0	$4e_3$
e_3	0	0	0	0
e_4	0	0	0	e_1

$$\alpha(e_1) = 4e_1,\ \alpha(e_2) = -2e_2,$$
$$\alpha(e_3) = e_3,\quad \alpha(e_4) = -2e_4.$$

Then there are admissible Hom-Novikov-Poisson color Hom-algebras $\mathcal{A}_1^n$ and $\mathcal{A}_2^n$ with multiplications tables respectively:

$\cdot^{(n)}$	e_1	e_2	e_3	e_4
e_1	0	0	0	0
e_2	0	$2^{2n}e_1$	0	$4e_3$
e_3	0	0	0	0
e_4	0	$4e_3$	0	0

$\diamond^{(n)}$	e_1	e_2	e_3	e_4
e_1	0	0	0	0
e_2	0	$4e_3$	0	$4e_3$
e_3	0	0	0	0
e_4	0	0	0	$2^{2n}e_1$

$$\alpha^{n+1}(e_1) = 4^{n+1}e_1,\ \alpha^{n+1}(e_2) = (-2)^{n+1}e_2,$$
$$\alpha^{n+1}(e_3) = e_3,\qquad \alpha^{n+1}(e_4) = (-2)^{n+1}e_4,$$

$\cdot^{(2^n-1)}$	e_1	e_2	e_3	e_4
e_1	0	0	0	0
e_2	0	$2^{2(2^n-1)}e_1$	0	$4e_3$
e_3	0	0	0	0
e_4	0	$4e_3$	0	0

$\diamond^{(2^n-1)}$	e_1	e_2	e_3	e_4
e_1	0	0	0	0
e_2	0	$4e_3$	0	$4e_3$
e_3	0	0	0	0
e_4	0	0	0	$2^{2(2^n-1)}e_1$

$$\alpha^{2^n}(e_1) = 4^{2^n}e_1,\ \alpha^{2^n}(e_2) = 2^{2^n}e_2,$$
$$\alpha^{2^n}(e_3) = e_3,\qquad \alpha^{2^n}(e_4) = 2^{2^n}e_4.$$

22.3.2 Tensor Products of Admissible Hom-Novikov-Poisson Color Hom-Algebras

Now, we show that the much larger class of admissible Hom-Novikov-Poisson color Hom-algebras is also closed under tensor products.

Theorem 22.5 *Let $(A_1, \cdot_1, \diamond_1, \varepsilon, \alpha_1)$ and $(A_2, \cdot_2, \diamond_2, \varepsilon, \alpha_2)$ be arbitrary admissible Hom-Novikov-Poisson color Hom-algebras, $A = A_1 \otimes A_2$, and operations $\alpha : A \to A$ and $\cdot, \diamond : A \otimes A \to A$ are defined, for all $x_i, y_i \in \mathcal{H}(A_i)$, $i \in \{1; 2\}$, by*

$$\alpha = \alpha_1 \otimes \alpha_2,$$
$$(x_1 \otimes x_2) \cdot (y_1 \otimes y_2) = \varepsilon(x_2, y_1)(x_1 \cdot_1 y_1) \otimes (x_2 \cdot_2 y_2),$$
$$(x_1 \otimes x_2) \diamond (y_1 \otimes y_2) = \varepsilon(x_2, y_1)\big((x_1 \diamond_1 y_1) \otimes (x_2 \cdot_2 y_2) + (x_1 \cdot_1 y_1) \otimes (x_2 \diamond_2 y_2)\big),$$

then $(A, \cdot, \diamond, \varepsilon, \alpha)$ is an admissible Hom-Novikov-Poisson color Hom-algebra.

Proof Pick $x = x_1 \otimes x_2$, $y = y_1 \otimes y_2$ and $z = z_1 \otimes z_2$ homogeneous elements in A.

Step 1: We show that $(A, \cdot, \varepsilon, \alpha)$ is ε-commutative Hom-associative-color Hom-algebra:

$$\begin{aligned}
x\cdot y &= (x_1\otimes x_2)\cdot(y_1\otimes y_2)\\
&= \varepsilon(x_2,y_1)(x_1\cdot_1 y_1)\otimes(x_2\cdot_2 y_2)\\
&= \varepsilon(x_2,y_1)\varepsilon(x_1,y_1)\varepsilon(x_2,y_2)(y_1\cdot_1 x_1)\otimes(y_2\cdot_2 x_2)\\
&= \varepsilon(x_1+x_2,y_1+y_2)\big(\varepsilon(y_2,x_1)(y_1\cdot_1 x_1)\otimes(y_2\cdot_2 x_2)\big)\\
&= \varepsilon(x_1+x_2,y_1+y_2)(y_1\otimes y_2)\cdot(x_1\otimes x_2)=\varepsilon(x,y)y\cdot x,\\
(x\cdot y)\cdot\alpha(z) &= ((x_1\otimes x_2)\cdot(y_1\otimes y_2))\cdot(\alpha_1\otimes\alpha_2)(z_1\otimes z_2)\\
&= \Big(\varepsilon(x_2,y_1)\cdot(x_1\cdot_1 y_1)\otimes(x_2\otimes_2 y_2)\Big)\cdot(\alpha_1\otimes\alpha_2)(z_1\otimes z_2)\\
&= \varepsilon(x_2+y_2,z_1)\varepsilon(x_2,y_1)\Big((x_1\cdot_1 y_1)\cdot_1\alpha_1(z_1)\otimes(x_2\cdot_2 y_2)\cdot_2\alpha_2(z_2)\Big)\\
&= \varepsilon(x_2,y_1+z_1)\varepsilon(y_2,z_1)\Big(\alpha_1(x_1)\cdot_1(y_1\cdot_1 z_1)\otimes\alpha_2(x_2)\cdot_2(y_2\cdot_2 z_2)\Big)\\
&= (\alpha_1\otimes\alpha_2)(x_1\otimes x_2)\cdot((y_1\otimes y_2)\cdot(z_1\otimes z_2))=\alpha(x)\cdot(y\cdot z).
\end{aligned}$$

Hence, $(A_1\otimes A_2,\cdot,\varepsilon,\alpha)$ is a ε-commutative Hom-associative color Hom-algebra.

Step 2: We show that $(A,\diamond,\varepsilon,\alpha)$ is Hom-Novikov-color Hom-algebra.

$$\begin{aligned}
&(x\diamond y)\diamond\alpha(z)-\alpha(x)\diamond(y\diamond z)-\varepsilon(x,y)\big((y\diamond x)\diamond\alpha(z)-\alpha(y)\diamond(x\diamond z)\big)\\
&=\big((x_1\otimes x_2)\diamond(y_1\otimes y_2)\big)\diamond\alpha(z_1\otimes z_2)-(\alpha_1\otimes\alpha_2)(x_1\otimes x_2)\diamond\big((y_1\otimes y_2)\diamond(z_1\otimes z_2)\big)\\
&\quad-\varepsilon(x_1+x_2,y_1+y_2)\Big(\big((y_1\otimes y_2)\diamond(x_1\otimes x_2)\big)\diamond(\alpha_1\otimes\alpha_2)(z_1\otimes z_2)\\
&\qquad\qquad-(\alpha_1\otimes\alpha_2)(y_1\otimes y_2)\diamond\big((x_1\otimes x_2)\diamond(z_1\otimes z_2)\big)\Big)\\
&=\varepsilon(x_2,y_1)\big((x_1\diamond_1 y_1)\otimes(x_2\cdot_2 y_2)+(x_1\cdot_1 y_1)\otimes(x_2\diamond_2 y_2)\big)\diamond(\alpha_1\otimes\alpha_2)(z_1\otimes z_2)\\
&\quad-\varepsilon(y_2,z_1)(\alpha_1\otimes\alpha_2)(x_1\otimes x_2)\diamond\big((y_1\diamond_1 z_1)\otimes(y_2\cdot_2 z_2)+(y_1\cdot_1 z_1)\otimes(y_2\diamond_2 z_2)\big)\\
&\quad-\varepsilon(x_1+x_2,y_1+y_2)\Big(\varepsilon(y_2,x_1)\big((y_1\diamond_1 x_1)\otimes(y_2\cdot_2 x_2)+(y_1\cdot_1 x_1)\otimes(y_2\diamond_2 x_2)\big)\diamond\\
&\qquad(\alpha_1\otimes\alpha_2)(z_1\otimes z_2)-\varepsilon(x_2,z_1)(\alpha_1\otimes\alpha_2)(y_1\otimes y_2)\diamond\big((x_1\diamond_1 z_1)\otimes(x_2\cdot_2 z_2)\\
&\qquad\qquad+(x_1\cdot_1 z_1)\otimes(x_2\diamond_2 z_2)\big)\Big)\\
&=\varepsilon(x_1+x_2,y_1+y_2)\times\\
&\Big[\underbrace{(x_1\diamond_1 y_1)\diamond_1\alpha_1(z_1)\otimes(x_2\cdot_2 y_2)\cdot_2\alpha_2(z_2)}_{A_1}\underbrace{+(x_1\diamond_1 y_1)\cdot_1\alpha_1(z_1)\otimes(x_2\cdot_2 y_2)\diamond_2\alpha_2(z_2)}_{A_2}\\
&\underbrace{+(x_1\cdot_1 y_1)\diamond_1\alpha_1(z_1)\otimes(x_2\diamond_2 y_2)\cdot_2\alpha_2(z_2)}_{A_3}\underbrace{+(x_1\cdot_1 y_1)\cdot_1\alpha_1(z_1)\otimes(x_2\diamond_2 y_2)\diamond_2\alpha_2(z_2)}_{A_4}\\
&\underbrace{-\alpha_1(x_1)\diamond_1(y_1\diamond_1 z_1)\otimes\alpha_2(x_2)\cdot_2(y_2\cdot_2 z_2)}_{A_5}\underbrace{-\alpha_1(x_1)\cdot_1(y_1\diamond_1 z_1)\otimes\alpha_2(x_2)\diamond_2(y_2\cdot_2 z_2)}_{A_6}\\
&\underbrace{-\alpha_1(x_1)\diamond_1(y_1\cdot_1 z_1)\otimes\alpha_2(x_2)\cdot_2(y_2\diamond_2 z_2)}_{A_7}\underbrace{-\alpha_1(x_1)\cdot_1(y_1\cdot_1 z_1)\otimes\alpha_2(x_2)\diamond_2(y_2\diamond_2 z_2)}_{A_8}\\
&\quad-\varepsilon(x_1,y_1)\varepsilon(x_2,y_2)\times\\
&\Big(\underbrace{(y_1\diamond_1 x_1)\diamond_1\alpha_1(z_1)\otimes(x_2\cdot_2 x_2)\cdot_2\alpha_2(z_2)}_{B_1}\underbrace{+(x_1\diamond_1 x_1)\cdot_1\alpha_1(z_1)\otimes(x_2\cdot_2 x_2)\diamond_2\alpha_2(z_2)}_{B_2}
\end{aligned}$$

$$\underbrace{+(x_1 \cdot_1 x_1) \diamond_1 \alpha_1(z_1) \otimes (x_2 \diamond_2 x_2) \cdot_2 \alpha_2(z_2)}_{B_3} \underbrace{+(x_1 \cdot_1 x_1) \cdot_1 \alpha_1(z_1) \otimes (x_2 \diamond_2 x_2) \diamond_2 \alpha_2(z_2)}_{B_4}$$
$$\underbrace{-\alpha_1(x_1) \diamond_1 (x_1 \diamond_1 z_1) \otimes \alpha_2(x_2) \cdot_2 (x_2 \cdot_2 z_2)}_{B_5} \underbrace{-\alpha_1(x_1) \cdot_1 (x_1 \diamond_1 z_1) \otimes \alpha_2(x_2) \diamond_2 (x_2 \cdot_2 z_2)}_{B_6}$$
$$\underbrace{-\alpha_1(x_1) \diamond_1 (x_1 \cdot_1 z_1) \otimes \alpha_2(x_2) \cdot_2 (x_2 \diamond_2 z_2)}_{B_7}$$
$$\underbrace{- \alpha_1(x_1) \cdot_1 (x_1 \cdot_1 z_1) \otimes \beta(x_2) \diamond_2 (x_2 \diamond_2 z_2)\Big)\Big]}_{B_8}.$$

Furthermore, we have

$$(A_1 + A_5) - \varepsilon(x_1, y_1)\varepsilon(x_2, y_2)(B_1 + B_5)$$
(by (22.1) and (22.2))
$$= \Big[\big((x_1 \diamond_1 y_1) \diamond_1 \alpha_1(z_1) - \alpha_1(x_1) \diamond_1 (y_1 \diamond_1 z_1)\big)$$
$$- \varepsilon(x_1, y_1)\big((y_1 \diamond_1 x_1) \diamond_1 \alpha_1(z_1) - \alpha_1(y_1) \diamond_1 (x_1 \diamond_1 z_1)\big)\Big] \otimes (x_2 \cdot_2 y_2) \cdot_2 \alpha_2(z_2)$$
(by (22.8)) $\quad = 0,$

$$(A_4 + A_8) - \varepsilon(x_1, y_1)\varepsilon(x_2, y_2)(B_4 + B_8)$$
(by (22.1) and (22.2))
$$= (x_1 \cdot_1 y_1) \cdot_1 \alpha_1(z_1) \otimes \Big[\big((x_2 \diamond_2 y_2) \diamond_2 \alpha_2(z_2)$$
$$- \alpha_2(x_2) \diamond_2 (y_2 \diamond_2 z_2)\big) - \varepsilon(x_2, y_2)\big((y_2 \diamond_2 x_2) \diamond_2 \alpha_2(z_2) - \alpha_2(y_2) \diamond_2 (x_2 \diamond_2 z_2)\big)\Big]$$
(by (22.8)) $\quad = 0,$

$$(A_2 + A_7) - \varepsilon(x_1, y_1)\varepsilon(x_2, y_2)(B_2 + B_7)$$
(by (22.1), (22.17) and (22.2))
$$= \Big[\big((x_1 \diamond_1 y_1) \cdot_1 \alpha_1(z_1) - \alpha_1(x_1) \diamond_1 (y_1 \cdot_1 z_1)\big)$$
$$- \varepsilon(x_1, y_1)\big((y_1 \diamond_1 x_1) \cdot_1 \alpha_1(z_1) - \alpha_1(y_1) \diamond_1 (x_1 \cdot_1 z_1)\big)\Big] \otimes (x_2 \cdot_2 y_2) \diamond_2 \alpha_2(z_2)$$
(by (22.18)) $\quad = 0,$

$$(A_3 + A_6) - \varepsilon(x_1, y_1)\varepsilon(x_2, y_2)(B_3 + B_6)$$
(by (22.1), (22.17) and (22.2))
$$= (x_1 \cdot_1 y_1) \diamond_1 \alpha_1(z_1) \otimes \Big[\big((x_2 \diamond_2 y_2) \cdot_2 \alpha_2(z_2)$$
$$- \alpha_2(x_2) \diamond_2 (y_2 \diamond_2 z_2)\big) - \varepsilon(x_2, y_2)\big((y_2 \diamond_2 x_2) \cdot_2 \alpha_2(z_2) - \alpha_2(y_2) \diamond_2 (x_2 \cdot_2 z_2)\big)\Big]$$
(by (22.18)) $\quad = 0.$

Then, we obtain

$$(x \diamond y) \diamond \alpha(z) - \alpha(x) \diamond (y \diamond z) - \varepsilon(x, y)\big((y \diamond x) \diamond \alpha(z) - \alpha(y) \diamond (x \diamond z)\big) = 0.$$

$$\begin{aligned}
&(x \diamond y) \diamond \alpha(z) - \varepsilon(y, z)\big((x \diamond z) \diamond \alpha(y)\\
&\quad = \big((x_1 \otimes x_2) \diamond (y_1 \otimes y_2)\big) \diamond \alpha(z_1 \otimes z_2)\\
&\qquad - \varepsilon(y_1 + y_2, z_1 + z_2)\Big(\big((x_1 \otimes x_2) \diamond (z_1 \otimes z_2)\big) \diamond (\alpha_1 \otimes \alpha_2)(y_1 \otimes y_2)\Big)\\
&\quad = \varepsilon(x_2, y_1)\Big((x_1 \diamond_1 y_1) \otimes (x_2 \cdot_2 y_2) + (x_1 \cdot_1 y_1) \otimes (x_2 \diamond_2 y_2)\Big) \diamond (\alpha_1 \otimes \alpha_2)(z_1 \otimes z_2)\\
&\quad - \varepsilon(y_1 + y_2, z_1 + z_2)\Big(\varepsilon(x_2, z_1)\big((x_1 \diamond_1 z_1) \otimes (x_2 \cdot_2 z_2) + (x_1 \cdot_1 z_1) \otimes (x_2 \diamond_2 z_2)\big)\diamond\\
&\qquad\qquad (\alpha_1 \otimes \alpha_2)(y_1 \otimes y_2)\Big)\\
&\quad = \varepsilon(x_2, y_1)\varepsilon(x_2 + y_2, z_1)\times\\
&\Big[\underbrace{(x_1 \diamond_1 y_1) \diamond_1 \alpha_1(z_1) \otimes (x_2 \cdot_2 y_2) \cdot_2 \alpha_2(z_2)}_{C_1} \underbrace{+(x_1 \diamond_1 y_1) \cdot_1 \alpha_1(z_1) \otimes (x_2 \cdot_2 y_2) \diamond_2 \alpha_2(z_2)}_{C_2}\\
&\underbrace{+(x_1 \cdot_1 y_1) \diamond_1 \alpha_1(z_1) \otimes (x_2 \diamond_2 y_2) \cdot_2 \alpha_2(z_2)}_{C_3}\\
&\qquad\qquad \underbrace{+(x_1 \cdot_1 y_1) \cdot_1 \alpha_1(z_1) \otimes (x_2 \diamond_2 y_2) \diamond_2 \alpha_2(z_2)}_{C_4}\Big]\\
&\qquad - \varepsilon(y_1 + y_2, z_1 + z_2)\varepsilon(x_2, z_1)\varepsilon(x_2 + z_2, y_1)\times\\
&\Big[\underbrace{(x_1 \diamond_1 z_1) \diamond_1 \alpha_1(y_1) \otimes (x_2 \cdot_2 z_2) \cdot_2 \alpha_2(y_2)}_{D_1}\\
&\qquad\qquad \underbrace{+(x_1 \diamond_1 z_1) \cdot_1 \alpha_1(y_1) \otimes (x_2 \cdot_2 z_2) \diamond_2 \alpha_2(y_2)}_{D_2}\\
&\underbrace{+(x_1 \cdot_1 z_1) \diamond_1 \alpha_1(y_1) \otimes (x_2 \diamond_2 z_2) \cdot_2 \alpha_2(y_2)}_{D_3}\\
&\qquad\qquad \underbrace{+(x_1 \cdot_1 z_1) \cdot_1 \alpha_1(y_1) \otimes (x_2 \diamond_2 z_2) \diamond_2 \alpha_2(y_2)}_{D_4}\Big].
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&\varepsilon(x_2, y_1)\varepsilon(x_2 + y_2, z_1)C_1 - \varepsilon(y_1 + y_2, z_1 + z_2)\varepsilon(x_2, z_1)\varepsilon(x_2 + z_2, y_1)D_1\\
&\text{(by (22.2))}\\
&\quad = \varepsilon(x_2, y_1 + y_2)\varepsilon(x_2 + y_2, z_1)\times\\
&\quad \big((x_1 \diamond_1 y_1) \diamond_1 \alpha_1(z_1)\big) \otimes (y_2 \cdot_2 x_2) \cdot_2 \alpha_2(z_2)\\
&\quad - \varepsilon(y_1 + y_2, z_1 + z_2)\varepsilon(x_2 + z_2, y_1 + y_2)\varepsilon(x_2, z_1)\times\\
&\quad \big((x_1 \diamond_1 z_1) \diamond_1 \alpha_1(y_1)\big) \otimes \alpha_2(y_2) \cdot_2 (x_2 \cdot_2 z_2)\\
&\text{(by (22.1))}\\
&\quad = \varepsilon(x_2, y_1 + y_2)\varepsilon(x_2 + y_2, z_1)\times\\
&\quad \Big[(x_1 \diamond_1 y_1) \diamond_1 \alpha_1(z_1) - \varepsilon(y_1, z_1)(x_1 \diamond z_1) \diamond_1 \alpha_1(y_1)\Big] \otimes \alpha_2(y_2) \cdot_2 (x_2 \cdot_2 z_2)\\
&\text{(by (22.9))} \quad = 0.
\end{aligned}$$

$$\varepsilon(x_2, y_1)\varepsilon(x_2 + y_2, z_1)C_4 - \varepsilon(y_1 + y_2, z_1 + z_2)\varepsilon(x_2, z_1)\varepsilon(x_2 + z_2, y_1)D_4$$

(by (22.2))

$$= \varepsilon(x_1 + x_2, y_1)\varepsilon(x_2 + y_2, z_1)\times$$
$$(y_1 \cdot_1 x_1) \cdot_1 \alpha_1(z_1) \otimes \big((x_2 \diamond_2 y_1) \diamond_1 \alpha_1(z_2)\big)$$
$$- \varepsilon(y_1 + y_2, z_1 + z_2)\varepsilon(x_2 + z_2, y_1)\varepsilon(x_1 + y_1, z_1)\times$$
$$\alpha_1(y_1) \cdot_1 (x_1 \cdot_2 1_2) \otimes \big((x_2 \diamond_2 z_2) \diamond_2 \alpha_2(y_2)\big)$$

(by (22.1))

$$= \varepsilon(x_1 + x_2, y_1)\varepsilon(x_2 + y_2, z_1)\times$$
$$(y_1 \cdot_1 x_1) \cdot_1 \alpha_1(z_1)\Big[(x_2 \diamond_2 y_2) \diamond_2 \alpha_2(z_2) - \varepsilon(y_2, z_2)(x_2 \diamond_2 z_2) \diamond_2 \alpha_2(y_2)\Big]$$

(by (22.9)) $\quad = 0.$

$$\varepsilon(x_2, y_1)\varepsilon(x_2 + y_2, z_1)C_2 - \varepsilon(y_1 + y_2, z_1 + z_2)\varepsilon(x_2, z_1)\varepsilon(x_2 + z_2, y_1)D_3$$

(by (22.17))

$$= \varepsilon(x_2, y_1)\varepsilon(x_2 + y_2, z_1)\Big[x_1 \diamond_1 y_1) \cdot_1 \alpha_1(z_1) \otimes (x_2 \cdot_2 y_2) \diamond_2 \alpha_2(z_2)$$
$$- (x_1 \diamond_1 y_1) \cdot_1 \alpha_1(z_1) \otimes (x_2 \cdot_2 y_2) \diamond_2 \alpha_2(z_2)\Big] = 0,$$

$$\varepsilon(x_2, y_1)\varepsilon(x_2 + y_2, z_1)C_3 - \varepsilon(y_1 + y_2, z_1 + z_2)\varepsilon(x_2, z_1)\varepsilon(x_2 + z_2, y_1)D_2$$

(by (22.17))

$$= \varepsilon(x_2, y_1)\varepsilon(x_2 + y_2, z_1)\Big[x_1 \cdot_1 y_1) \diamond_1 \alpha_1(z_1) \otimes (x_2 \diamond_2 y_2) \cdot_2 \alpha_2(z_2)$$
$$- (x_1 \cdot_1 y_1) \diamond_1 \alpha_1(z_1) \otimes (x_2 \diamond_2 y_2) \cdot_2 \alpha_2(z_2)\Big] = 0.$$

Then, we obtain $(x \diamond y) \diamond \alpha(z) - \varepsilon(y, z)\big((x \diamond z) \diamond \alpha(y) = 0$. Hence, $(A_1 \otimes A_2, \diamond, \varepsilon, \alpha)$ is a Hom-Novikov color Hom-algebra.

Step 3: We show that the compatibility conditions of Hom-Novikov-Poisson color Hom-algebras are satisfied

$$\begin{aligned}
&(x \cdot y) \diamond \alpha(z) - \varepsilon(y, z)(x \diamond z) \cdot \alpha(y)\\
&= \big((x_1 \otimes x_2) \cdot (y_1 \otimes y_2)\big) \diamond (\alpha_1 \otimes \alpha_2)(z_1 \otimes z_2)\\
&\quad - \varepsilon(y_1 + y_2, z_1 + z_2)\big((x_1 \otimes x_2) \diamond (z_1 \otimes z_2)\big) \cdot (\alpha_1 \otimes \alpha_2)(y_1 \otimes y_2)\\
&= \varepsilon(x_2, y_1)\big((x_1 \cdot_1 y_1) \otimes (x_2 \cdot_2 y_2)\big) \diamond (\alpha_1(z_1) \otimes \alpha_2(z_2))\\
&\quad - \varepsilon(y_1 + y_2, z_1 + z_2)\varepsilon(x_2, z_1\Big((x_1 \diamond_1 z_1) \otimes (x_2 \diamond_2 z_2)\\
&\quad + (x_1 \cdot_1 z_1) \otimes (x_2 \cdot_2 z_2)\Big) \cdot (\alpha_1 \otimes \alpha_2)(y_1 \otimes y_2)\\
&\quad = \varepsilon(x_2, y_1)\varepsilon(x_2 + y_2, z_1)\times\\
&\quad \Big(\underbrace{\big((x_1 \cdot_1 y_1) \diamond_1 \alpha_1(z_1)\big) \otimes \big((x_2 \cdot_2 y_2) \cdot_2 \alpha_2(z_2)\big)}_{E_1}
\end{aligned}$$

$$\underbrace{\big((x_1 \cdot_1 y_1) \cdot_1 \alpha_1(z_1)\big) \otimes \big((x_2 \cdot_2 y_2) \diamond_2 \alpha_2(z_2)\big)}_{E_2}\Big)$$
$$- \varepsilon(y_1 + y_2, z_1 + z_2)\varepsilon(x_2, z_1\varepsilon(x_2 + z_2, y_1)\times$$
$$\Big(\underbrace{\big(x_1 \diamond_1 z_1) \cdot_1 \alpha_1(y_1)\big) \otimes \big((x_2 \cdot_2 z_2) \cdot_2 \alpha_2(y_2)\big)}_{F_1}$$
$$+ \underbrace{\big((x_1 \cdot_1 z_1) \cdot_1 \alpha_1(y_1)\big) \otimes \big((x_2 \diamond_2 z_2) \cdot_2 \alpha_2(y_2)\big)}_{F_2}\Big).$$

Furthermore, we have

$$\varepsilon(x_2, y_1)\varepsilon(x_2 + y_2, z_1)E_1 - \varepsilon(y_1 + y_2, z_1 + z_2)\varepsilon(x_2, z_1)\varepsilon(x_2 + z_2, y_1)F_1$$

(by (22.2))

$$= \varepsilon(x_2, y_1)\varepsilon(x_2 + y_2, z_1)\varepsilon(x_2, y_2)\big((x_1 \cdot_1 y_1) \diamond_1 \alpha_1(z_1)\big) \otimes (y_2 \cdot_2 x_2) \cdot_2 \alpha_2(z_2)$$
$$- \varepsilon(y_1 + y_2, z_1 + z_2)\varepsilon(x_2, z_1)\varepsilon(x_2 + z_2, y_1 + y_2)\times$$
$$\big((x_1 \cdot_1 z_1) \cdot \alpha_1(y_1)\big) \otimes \big(\alpha_2(y_2) \cdot_2 (x_2 \cdot_2 z_2)\big)$$

(by (22.1))

$$= \varepsilon(x_2, y_1 + y_2)\varepsilon(x_2 + y_2, z_1)\times$$
$$\Big((x_1 \cdot_1 y_1) \diamond_1 \alpha_1(z_1) - \varepsilon(y_1, z_1)(x_1 \diamond z_1) \cdot_1 \alpha_1(y_1)\Big) \otimes \alpha_2(y_2) \cdot_2 (x_2 \cdot_2 z_2)$$

(by (22.17)) $= 0,$

$$\varepsilon(x_2, y_1)\varepsilon(x_2 + y_2, z_1)E_2 - \varepsilon(y_1 + y_2, z_1 + z_2)\varepsilon(x_2, z_1)\varepsilon(x_2 + z_2, y_1)F_2$$

(by (22.2))

$$= \varepsilon(x_2, y_1)\varepsilon(x_2 + y_2, z_1)\varepsilon(x_1 + y_1, z_1)\times$$
$$\big(\alpha_1(z_1) \cdot (x_1 \cdot y_1)\big) \otimes \big((x_2 \cdot_2 y_2) \diamond_2 \alpha_2(z_2)\big)$$
$$- \varepsilon(y_1 + y_2, z_1 + z_2)\varepsilon(x_1 + x_2, z_1)\varepsilon(x_2 + z_2, y_1)\times$$
$$\big((z_1 \cdot_1 x_1) \cdot_1 \alpha_1(y_1)\big) \otimes \big((x_2 \diamond_2 z_2) \diamond_2 \alpha_2(y_2)\big)$$

(by (22.1))

$$= \varepsilon(x_2, y_1)\varepsilon(x_1 + x_2 + y_1 + y_2, z_1)\times$$
$$\alpha_1(z_1) \cdot_1 (x_1 \cdot_1 y_1) \otimes \Big((x_2 \cdot_2 y_2) \diamond_2 \alpha_2(z_2) - \varepsilon(y_2, z_2)(x_2 \diamond_2 z_2) \cdot_2 \alpha_2(y_2)\Big)$$

(by (22.17)) $= 0.$

Then, $(x \cdot y) \diamond \alpha(z) - \varepsilon(y, z)(x \diamond z) \cdot \alpha(y) = 0$. Similarly,

$$(x \diamond y) \cdot \alpha(z) - \alpha(x) \diamond (y \cdot z) = \varepsilon(x, y)\big((y \diamond x) \cdot \alpha(z) - \alpha(y) \diamond (x \cdot z)\big).$$

Hence, $(A_1 \otimes A_2, \cdot, \diamond, \varepsilon, \alpha)$ is a Hom-Novikov-Poisson color Hom-algebra.

Step 4: We show that the Equation (22.35) is satisfied:

$$\alpha(x)\cdot(y\diamond z)=\alpha(x_1\otimes x_2)\cdot\varepsilon(y_2,z_1)\Big((y_1\diamond_1 z_1)\otimes(y_2\cdot_2 z_2)+(y_1\cdot_1 z_1)\otimes(y_2\diamond_2 z_2)\Big)$$
$$=\varepsilon(y_2,z_1)\varepsilon(x_2,y_1+z_1)\Big(\big(\alpha(x_1)\cdot_1(y_1\diamond_1 z_1)\big)\otimes\big(\alpha_2(x_2)\cdot_2(y_2\cdot_2 z_2)\big)$$
$$+\big(\alpha(x_1)\cdot_1(y_1\cdot_1 z_1)\big)\otimes\big(\alpha_2(x_2)\cdot_2(y_2\diamond z_2)\big)\Big),$$
$$\alpha(x)\diamond(y\cdot z)=(\alpha(x_1)\otimes\alpha_2(x_2))\diamond\big((y_1\otimes y_2)\cdot(z_1\otimes z_2)\big)$$
$$=(\alpha_1(x_1)\otimes\alpha_2(x_2))\diamond\Big(\varepsilon(y_2,z_1)(y_1\diamond_1 z_1)\otimes(y_2\cdot_2 z_2)\Big)$$
$$=\varepsilon(y_2,z_1)\varepsilon(x_2,y_1+z_1)\Big(\big(\alpha(x_1)\diamond_1(y_1\cdot_1 z_1)\big)\otimes\big(\alpha_2(x_2)\cdot_2(y_2\cdot_2 z_2)\big)$$
$$+\big(\alpha(x_1)\cdot_1(y_1\cdot_1 z_1)\big)\otimes\big(\alpha_2(x_2)\diamond_2(y_2\cdot_2 z_2)\big)\Big).$$

Now, using the Theorem 22.3 we conclude that $\alpha(x)\cdot(y\diamond z)=\alpha(x)\diamond(y\cdot z)$. Therefore $as^l_A(x,y,z)=0$ and hence, $(A=A_1\otimes A_2,\cdot,\diamond,\varepsilon,\alpha)$ is an admissible Hom-Novikov-Poisson color Hom-algebra. □

By taking in Theorem 22.5, $\alpha_1=id_{A_1}$ and $\alpha_2=id_{A_2}$, we have the following result.

Corollary 22.2 *Let $(A_1,\cdot_1,\diamond_1,\varepsilon)$ and $(A_2,\cdot_2,\diamond_2,\varepsilon)$ be admissible Novikov-Poisson color Hom-algebras and let $A=A_1\otimes A_2$. Define the operations $\cdot,\diamond:A\otimes A\to A$ by the following formulae for $x_i,y_i\in\mathcal{H}(A_i),\ i\in\{1;2\}$,*

$$(x_1\otimes x_2)\cdot(y_1\otimes y_2)=\varepsilon(x_2,y_1)(x_1\cdot_1 y_1)\otimes(x_2\cdot_2 y_2),$$
$$(x_1\otimes x_2)\diamond(y_1\otimes y_2)=\varepsilon(x_2,y_1)\big((x_1\diamond_1 y_1)\otimes(x_2\cdot_2 y_2)+(x_1\cdot_1 y_1)\otimes(x_2\diamond_2 y_2)\big).$$

Then $(A,\cdot,\diamond,\varepsilon)$ is an admissible Novikov-Poisson color Hom-algebra.

By taking in Theorem 22.5, $\Gamma=\{e\}$, we recover the following result

Corollary 22.3 ([85]) *Let $(A_1,\cdot_1,\diamond_1,\alpha_1)$ and $(A_2,\cdot_2,\diamond_2,\alpha_2)$ be admissible Hom-Novikov-Poisson algebras and let $A=A_1\otimes A_2$. Define the operations $\alpha:A\to A$ and $\cdot,\diamond:A\otimes A\to A$ by the following formulae for $x_i,y_i\in A_i,\ i\in\{1;2\}$,*

$$\alpha=\alpha_1\otimes\alpha_2,$$
$$(x_1\otimes x_2)\cdot(y_1\otimes y_2)=(x_1\cdot_1 y_1)\otimes(x_2\cdot_2 y_2),$$
$$(x_1\otimes x_2)\diamond(y_1\otimes y_2)=\big((x_1\diamond_1 y_1)\otimes(x_2\cdot_2 y_2)+(x_1\cdot_1 y_1)\otimes(x_2\diamond_2 y_2)\big).$$

Then $(A,\cdot,\diamond,\alpha)$ is an admissible Hom-Novikov-Poisson algebra.

By taking in Theorem 22.5, $\alpha_1=id_{A_1}$, $\alpha_2=id_{A_2}$ and $\Gamma=\{e\}$, we have the following result.

Corollary 22.4 ([77]) *Let $(A_1,\cdot_1,\diamond_1)$ and $(A_2,\cdot_2,\diamond_2)$ be admissible Novikov-Poisson algebras and let $A=A_1\otimes A_2$. Define the operations $\cdot,\diamond:A\otimes A\to A$ by the following formulae for $x_i,y_i\in A_i,\ i\in\{1;2\}$,*

$$(x_1\otimes x_2)\cdot(y_1\otimes y_2)=(x_1\cdot_1 y_1)\otimes(x_2\cdot_2 y_2),$$
$$(x_1\otimes x_2)\diamond(y_1\otimes y_2)=\big((x_1\diamond_1 y_1)\otimes(x_2\cdot_2 y_2)+(x_1\cdot_1 y_1)\otimes(x_2\diamond_2 y_2)\big).$$

Then $(A, \cdot, \diamond)$ *is an admissible Novikov-Poisson algebra.*

22.4 Hom-Gelfand-Dorfman Color Hom-algebras

In this section, our goals are to introduce Hom-Gelfand-Dorfman color Hom-algebras and to discuss some basic properties and examples of these objects. Moreover we characterize the representation of Hom-Gelfand-Dorfman color Hom-algebras and provide some key constructions.

Definition 22.16 A Gelfand-Dorfman color Hom-algebra is a quadruple $(A, \cdot, [\cdot,\cdot], \varepsilon)$ such that $(A, \cdot, \varepsilon)$ is a Novikov color Hom-algebra and $(A, [\cdot,\cdot], \varepsilon)$ is a Lie color Hom-algebra satisfying for all $x, y, z \in \mathcal{H}(A)$, the following compatibility condition:

$$y \cdot [x, z] = \varepsilon(y, x)[x, y \cdot z] - \varepsilon(x + y, z)[z, y \cdot x] + [y, x] \cdot z - \varepsilon(x, z)[y, z] \cdot x.$$

Definition 22.17 A Hom-Gelfand-Dorfman color Hom-algebra is defined as a quintuple $(A, \cdot, [\cdot,\cdot], \varepsilon, \alpha)$ such that $(A, \cdot, \varepsilon, \alpha)$ is a Hom-Novikov color Hom-algebra and $(A, [\cdot,\cdot], \varepsilon, \alpha)$ is a Hom-Lie color Hom-algebra satisfying for all $x, y, z \in \mathcal{H}(A)$, the following compatibility condition:

$$\alpha(y) \cdot [x, z] = \varepsilon(y, x)[\alpha(x), y \cdot z] - \varepsilon(x + y, z)[\alpha(z), y \cdot x] + [y, x] \cdot \alpha(z) - \varepsilon(x, z)[y, z] \cdot \alpha(x). \tag{22.36}$$

A Hom-Gelfand-Dorfman color Hom-algebra is called multiplicative if the even linear map $\alpha : A \to A$ is multiplicative with respect to $\cdot$ and $[\cdot,\cdot]$, that is, for all $x, y \in \mathcal{H}(A)$, $\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)$ and $\alpha([x, y]) = [\alpha(x), \alpha(y)]$.

Remark 22.5 Hom-Gelfand-Dorfman color Hom-algebras contain both the Gelfand-Dorfman algebras and the Hom-Gelfand-Dorfman Hom-algebras for special choices of grading group and the twisting map.

(i) When $\Gamma = \{e\}$ and $\alpha = id$, we get Gelfand-Dorfman algebra [34, 79].
(ii) When $\Gamma = \{e\}$ and $\alpha \neq id$, we get Hom-Gelfand-Dorfman Hom-algebra [88].

Example 22.13 Let $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ be an abelian group and A be a 4-dimensional Γ-graded linear space with one-dimensional homogeneous subspaces

$$A_{(0,0)} =< e_1 >, \quad A_{(0,1)} =< e_2 >, \quad A_{(1,0)} =< e_3 >, \quad A_{(1,1)} =< e_4 > .$$

Then $(A, \cdot, [\cdot,\cdot], \varepsilon, \alpha)$ is a Hom-Gelfand-Dorfman color Hom-algebra with

bicharacter: $\varepsilon\big((i_1,i_2),(j_1,j_2)\big)=(-1)^{i_1j_1+i_2j_2}$,
multiplication: $e_2\cdot e_3=\lambda_1 e_4,\quad e_3\cdot e_2=\lambda_2 e_4,\quad e_3\cdot e_3=\lambda_3 e_1,\quad \lambda_i\in\mathbb{K}$,
bracket: $[e_2,e_2]=\mu_1 e_1,\quad [e_3,e_2]=\mu_2 e_4,\quad \mu_i\in\mathbb{K}$,
even linear map $\alpha: A\to A$ given by $\alpha(e_1)=-e_1,\quad \alpha(e_2)=2e_2,$
$\alpha(e_3)=-2e_3,\quad \alpha(e_4)=e_4.$

Definition 22.18 For Hom-Gelfand-Dorfman color Hom-algebras $(A,\cdot,[\cdot,\cdot],\varepsilon,\alpha)$ and $(A',\cdot',[\cdot,\cdot]',\varepsilon',\alpha')$, a linear map of degree zero $f: A\to A'$ is a Hom-Gelfand-Dorfman color Hom-algebra morphism if

$$\cdot'\circ(f\otimes f)=f\circ\cdot,\quad [\cdot,\cdot]'\circ(f\otimes f)=f\circ[\cdot,\cdot],\quad f\circ\alpha=\alpha'\circ f.$$

Proposition 22.13 *Let $(A,\cdot,\varepsilon,\alpha)$ be a Hom-Novikov color Hom-algebra. For all $x,y\in\mathcal{H}(A)$, let $[x,y]=x\cdot y-\varepsilon(x,y)y\cdot x$. Then $(A,\cdot,[\cdot,\cdot],\varepsilon,\alpha)$ is a Hom-Gelfand-Dorfman color Hom-algebra.*

Proof Let $(A,\cdot,\varepsilon,\alpha)$ be a Hom-Novikov color Hom-algebra. By Proposition 22.6 $(A,[\cdot,\cdot],\varepsilon,\alpha)$ is a Hom-Lie color Hom-algebra. Now, we show that the compatibility condition (22.36) is satisfied. For any $x,y,z\in\mathcal{H}(A)$ we have

$$\begin{aligned}
&\alpha(y)\cdot[x,z]-\varepsilon(y,x)[\alpha(x),y\cdot z]+\varepsilon(x+y,z)[\alpha(z),y\cdot x]-[y,x]\cdot\alpha(z)\\
&\quad+\varepsilon(x,z)[y,z]\cdot\alpha(x)\\
&=\alpha(y)\big(x\cdot z-\varepsilon(x,y+z)(y\cdot z)\cdot\alpha(x)\big)+\varepsilon(x+y,z)\big(\alpha(z)\cdot(y\cdot x)\\
&\quad-\varepsilon(z,y+x)(y\cdot x)\cdot\alpha(z)\big)-\big(y\cdot x-\varepsilon(y,x)x\cdot y)\big)\cdot\alpha(z)+\varepsilon(x,z)\big(y\cdot z\\
&\quad-\varepsilon(y,z)z\cdot y\big)\cdot\alpha(x)\\
&=\alpha(y)\cdot(x\cdot z)-\varepsilon(x,z)\alpha(y)\cdot(z\cdot x)-\varepsilon(y,x)\alpha(x)\cdot(y\cdot z)+\varepsilon(x,z)(y\cdot z)\cdot\alpha(x)\\
&\quad+\varepsilon(x+y,z)\alpha(z)\cdot(y\cdot x)-(y\cdot x)\cdot\alpha(z)-(y\cdot x)\cdot\alpha(z)+\varepsilon(y,x)(x\cdot y)\cdot\alpha(z)\\
&\quad+\varepsilon(x,z)(y\cdot z)\cdot\alpha(x)-\varepsilon(x+y,z)(x\cdot y)\cdot\alpha(x)+\varepsilon(y,x)(x\cdot y)\cdot\alpha(z)\\
&=\underbrace{\Big(\alpha(y)\cdot(x\cdot z)-(y\cdot x)\cdot\alpha(z)-\varepsilon(y,x)\big(\alpha(x)\cdot(y\cdot z)-(x\cdot y)\cdot\alpha(z)\big)\Big)}_{=0\text{ by (22.8)}}\\
&\quad+\varepsilon(x,z)\underbrace{\Big((y\cdot z)\cdot\alpha(x)-\alpha(y)\cdot(z\cdot x)-\varepsilon(y,z)\big(\alpha(z)\cdot(y\cdot x)-(z\cdot y)\cdot\alpha(x)\big)}_{=0\text{ by (22.8)}}\\
&\qquad-\underbrace{\Big((y\cdot x)\cdot\alpha(z)-\varepsilon(x,z)(y\cdot z)\cdot\alpha(x)\Big)}_{=0\text{ by (22.9)}}=0.
\end{aligned}$$

Hence, $(A,\cdot,[\cdot,\cdot],\varepsilon,\alpha)$ is a Hom-Gelfand-Dorfman color Hom-algebra. □

Definition 22.19 If $(A,\cdot,[\cdot,\cdot],\varepsilon,\alpha)$ is a Hom-Gelfand-Dorfman color Hom-algebra, then a Γ-graded subspace H of A is called

(i) color Hom-subalgebra of $(A,\cdot,[\cdot,\cdot],\varepsilon,\alpha)$ if

$$\alpha(H) \subseteq H,\ H \cdot H \subseteq H,\ [H, H] \subseteq H,$$

(ii) color Hom-ideal of $(A, \cdot, [\cdot, \cdot], \varepsilon, \alpha)$ if

$$\alpha(H) \subseteq H,\ A \cdot H \subseteq H,\ H \cdot A \subseteq H,\ [A, H] \subseteq H.$$

The following statement follows from straightforward computation.

Proposition 22.14 *Let $(A, \cdot, [\cdot, \cdot], \varepsilon, \alpha)$ a Hom-Gelfand-Dorfman color Hom-algebra and I a color Hom-ideal of $(A, \cdot, [\cdot, \cdot], \varepsilon, \alpha)$. Then $(A/I, \bar{\cdot}, \{\cdot, \cdot\}, \varepsilon, \alpha)$ is a Hom-Gelfand-Dorfman color Hom-algebra where $\overline{x}\ \bar{\cdot}\ \overline{y} = \overline{x \cdot y}$, $\{\overline{x}, \overline{y}\} = \overline{[x, y]}$, $\overline{\alpha}(\overline{x}) = \overline{\alpha(x)}$ and $\varepsilon(\overline{x}, \overline{y}) = \varepsilon(x, y)$, for all $\overline{x}, \overline{y} \in \mathcal{H}(A/I)$.*

Proposition 22.15 *Any transposed Hom-Poisson color Hom-algebra is also a Hom-Gelfand-Dorfman color Hom-algebra.*

Proof Let $(A, \cdot, [\cdot, \cdot], \varepsilon, \alpha)$ be a transposed Hom-Poisson color Hom-algebra. By definition if $(A, \cdot, \varepsilon, \alpha)$ is a ε-commutative Hom-associative color Hom-algebra, then $(A, \cdot, \varepsilon, \alpha)$ is a Hom-Novikov color Hom-algebra and $(A, [\cdot, \cdot], \varepsilon, \alpha)$ is a Hom-Lie color Hom-algebra. Now, we show that the compatibility condition (22.36) is satisfied. For any $x, y, z \in \mathcal{H}(A)$,

$$\begin{aligned}
&\alpha(y) \cdot [x, z] - \varepsilon(y, x)[\alpha(x), y \cdot z] + \varepsilon(x + y, z)[\alpha(z), y \cdot x] \\
&\qquad\qquad\qquad\qquad - [y, x] \cdot \alpha(z) + \varepsilon(x, z)[y, z] \cdot \alpha(x) \\
&= \alpha(y) \cdot \big(x \cdot z - \varepsilon(x, z)z \cdot x\big) - \varepsilon(y, x)\big(\alpha(x) \cdot (y \cdot z) - \varepsilon(x, y + z)(y \cdot z) \cdot \alpha(x)\big) \\
&\quad - \varepsilon(x + y, z)\big(\alpha(z) \cdot (y \cdot x) - \varepsilon(z, y + x)(y \cdot x) \cdot \alpha(z)\big) \\
&\quad + \big(y \cdot x - \varepsilon(y, x)x \cdot y\big) \cdot \alpha(z) - \varepsilon(x, z)\big(y \cdot z - \varepsilon(y, z)z \cdot y\big) \cdot \alpha(x) \\
&= \alpha(y) \cdot (x \cdot z) - \varepsilon(x, z)\alpha(y) \cdot (z \cdot x) - \varepsilon(y, x)\alpha(x) \cdot (y \cdot z) + \varepsilon(x, z)(y \cdot z) \cdot \alpha(x) \\
&\quad + \varepsilon(x + y, z)\alpha(z) \cdot (y \cdot x) - (y \cdot x) \cdot \alpha(z) - (y \cdot x) \cdot \alpha(z) + \varepsilon(y, x)(x \cdot y) \cdot \alpha(z) \\
&\quad + \varepsilon(x, z)(y \cdot z) \cdot \alpha(x) - \varepsilon(x + y, z)(z \cdot y) \cdot \alpha(x) \\
&= \underbrace{\big(\alpha(y) \cdot (x \cdot z) - (y \cdot x) \cdot \alpha(z)\big)}_{=0 \text{ by } (22.1)} - \varepsilon(x, z) \underbrace{\big(\alpha(y) \cdot (z \cdot x) - (y \cdot z) \cdot \alpha(x)\big)}_{=0 \text{ by })22.1)} \\
&\quad - \varepsilon(y, x) \underbrace{\big(\alpha(x) \cdot (y \cdot z) - (x \cdot y) \cdot \alpha(z)\big)}_{=0 \text{ by } (22.1)} + \varepsilon(x + y, z) \underbrace{\big(\alpha(z) \cdot (y \cdot x) - (z \cdot y) \cdot \alpha(x)\big)}_{=0 \text{ by } (22.1)} \\
&\qquad\qquad\qquad\qquad + \underbrace{\big(\varepsilon(x, z)(y \cdot z) \cdot \alpha(x) - (y \cdot x) \cdot \alpha(z)\big)}_{=0 \text{ by } (22.1) \text{ and } (22.2)} = 0,
\end{aligned}$$

which completes the proof. □

Lemma 22.2 ([8]) *Let $(A, \cdot, \varepsilon, \alpha)$ be a ε-commutative Hom-associative color Hom-algebra with an even derivation D such that $\alpha \circ D = D \circ \alpha$. Define*

$$x \diamond y = x \cdot D(y), \tag{22.37}$$

for all $x, y \in \mathcal{H}(A)$. Then $(A, \diamond, \varepsilon, \alpha)$ is a Hom-Novikov color Hom-algebra.

Theorem 22.6 *Let* $(A, \cdot, [\cdot,\cdot], \varepsilon, \alpha)$ *be a Hom-Poisson color Hom-algebra with an even derivation* D *relative to the both products. Define a new operation* $\diamond$ *on* A *by*

$$x \diamond y = x \cdot D(y). \tag{22.38}$$

Then $(A, \diamond, [\cdot,\cdot], \varepsilon, \alpha)$ *is a Hom-Gelfand-Dorfman color Hom-algebra.*

Proof By Lemma 22.2, $(A, \diamond, \varepsilon, \alpha)$ is a Hom-Novikov color Hom-algebra and by definition of Hom-Poisson color Hom-algebra, $(A, [\cdot,\cdot], \varepsilon, \alpha)$ is a Hom-Lie color Hom-algebra. Now, we show that the compatibility condition (22.36) is satisfied. For any $x, y, z \in \mathcal{H}(A)$,

$$\begin{aligned}
&\alpha(y) \diamond [x,z] - \varepsilon(y,x)[\alpha(x), y \diamond z] + \varepsilon(x+y,z)[\alpha(z), y \diamond x]\\
&\quad - [y,x] \diamond \alpha(z) + \varepsilon(x,z)[y,z] \diamond \alpha(x)\\
&\text{(using (22.37))}\\
&\ = \alpha(y) \cdot D([x,z]) - \varepsilon(y,x)[\alpha(x), y \cdot D(z))] + \varepsilon(x+y,z)[\alpha(z), y \cdot D(x)]\\
&\quad - [y,x] \cdot D(\alpha(z)) + \varepsilon(x,z)[y,z] \cdot D(\alpha(x))\\
&(D \text{ is derivation})\\
&\ = \alpha(y) \cdot [D(x), z] + \alpha(y) \cdot [x, D(z)] - \varepsilon(y,x)[\alpha(x), y \cdot D(z)]\\
&\quad + \varepsilon(x+y,z)[\alpha(z), y \cdot D(x)] - [y,x] \cdot \alpha(D(x))\\
&\quad + \varepsilon(x,z)[y,z] \cdot \alpha(D(x))\\
&\ = -\varepsilon(y,x) \underbrace{\Big([\alpha(x), y \cdot D(z)] - \varepsilon(x,y)\alpha(y) \cdot [x, D(z)] - \varepsilon(y+x,z)\alpha(D(x)) \cdot [x,y]\Big)}_{=0 \text{ by } (22.32)}\\
&\quad + \varepsilon(x+y,z) \underbrace{\Big([\alpha(z), y \cdot D(x)] - \varepsilon(z,y)\alpha(y) \cdot [z, D(x)] - \alpha(D(x)) \cdot [z,y]\Big)}_{=0 \text{ by } (22.32)} = 0,
\end{aligned}$$

which completes the proof. □

Let us call a Hom-Gelfand-Dorfman color Hom-algebra $(A, \cdot, [\cdot,\cdot], \varepsilon, \alpha)$ is special if it can be embedded into a differential Hom-Poisson color Hom-algebra with operations $[\cdot,\cdot]$ and $\diamond$ given by (22.38).

Definition 22.20 A representation of a Hom-Gelfand-Dorfman color Hom-algebra $(A, \cdot, [\cdot,\cdot], \varepsilon, \alpha)$ is a quintuple (l, r, ρ, β, V) such that (l, r, β, V) is a bimodule of the Hom-Novikov color Hom-algebra $(A, \cdot, \varepsilon, \alpha)$ and (ρ, β, V) is a representation of the Hom-Lie color Hom-algebra $(A, [\cdot,\cdot], \varepsilon, \alpha)$ obeying, for $x, y \in \mathcal{H}(A)$, $v \in \mathcal{H}(V)$,

$$\begin{aligned}
l(\alpha(y))\rho(x)v &= \rho(y \cdot x)\beta(v) + \varepsilon(y,x)\rho(\alpha(x))l(y)v\\
&\quad - \varepsilon(x,v)r(\alpha(x))\rho(y)v + l([y,x])\beta(v),\\
r([x,y])\beta(v) &= \varepsilon(v,x)(\rho(\alpha(x))r(y)v - r(\alpha(y))\rho(x)v)\\
&\quad + \varepsilon(x+v,y)(r(\alpha(x))\rho(y)v - \rho(\alpha(y))r(x)v).
\end{aligned}$$

Proposition 22.16 *If* (l, r, ρ, β, V) *is a representation of a Hom-Gelfand-Dorfman color Hom-algebra* $(A, \cdot, [\cdot,\cdot], \varepsilon, \alpha)$*, then* $(A \oplus V, \cdot', [\cdot,\cdot]', \varepsilon, \alpha+\beta)$ *is a Hom-Gelfand-Dorfman color Hom-algebra, where* $(A \oplus V, \cdot', \varepsilon, \alpha+\beta)$ *is the semi-direct product Hom-Novikov color Hom-algebra* $A \ltimes_{l,r,\alpha,\beta} V$*, and* $(A \oplus V, [\cdot,\cdot]', \varepsilon, \alpha+\beta)$ *is the semi-direct product Hom-Lie color Hom-algebra* $A \ltimes_{\rho,\alpha,\beta} V$*.*

Proof Let (l, r, ρ, β, V) be a representation of a Hom-Gelfand-Dorfman color Hom-algebra $(A, \cdot, [\cdot,\cdot], \varepsilon, \alpha)$. By Propositions 22.4 and 22.8, $(A \oplus V, \cdot', \varepsilon, \alpha+\beta)$ is a Hom-Novikov color Hom-algebra, and $(A \oplus V, [\cdot,\cdot]', \varepsilon, \alpha+\beta)$ is a Hom-Lie color Hom-algebra respectively. Now, we show that the compatibility condition (22.36) is satisfied. For all $X_i = x_i + v_i \in A_{\gamma_i} \oplus V_{\gamma_i},\ i = 1, 2, 3,$

$$
\begin{aligned}
&(\alpha+\beta)(x_2+v_2) * [x_1+v_1, x_3+v_3]' \\
&- \varepsilon(x_2+v_2, x_1+v_1)[(\alpha+\beta)(x_1+v_1), (x_2+v_2)\cdot'(x_3+v_3)]' \\
&+ \varepsilon(x_1+x_2, x_3)[(\alpha+\beta)(x_3+v_3), (x_2+v_2)\cdot'(x_1+v_1)]' \\
&- [(x_2+v_2), (x_1+v_1)]' \cdot' (\alpha+\beta)(x_3+v_3) \\
&+ \varepsilon(x_1, x_3)[(x_2+v_2), (x_3+v_3)]' \cdot' (\alpha+\beta)(x_1+v_1) \\
&\quad = (\alpha(x_2)+\beta(v_2)) \cdot' ([x_1, x_3] + \rho(x_1)v_3 - \varepsilon(x_1, x_3)\rho(x_3)v_1) \\
&\qquad - \varepsilon(x_2, x_1)[\alpha(x_1)+\beta(v_1), x_2\cdot x_3 + l(x_2)v_3 + r(x_3)v_2]' \\
&\qquad + \varepsilon(x_1+x_2, x_3)[\alpha(x_3)+\beta(v_3), x_2\cdot x_1 + l(x_2)v_1 + r(x_1)v_2]' \\
&\qquad - ([x_2, x_1] + \rho(x_2)v_1 - \varepsilon(x_2, x_1)\rho(x_1)v_2) \cdot' (\alpha(x_3)+\beta(v_3)) \\
&\qquad + \varepsilon(x_1, x_3)([x_2, x_3] + \rho(x_2)v_3 - \varepsilon(v_2, x_3)\rho(x_3)v_2 \\
&\quad = \alpha(x_2)\cdot[x_1, x_3] + l(\alpha(x_2))\rho(x_1)v_3 \\
&\qquad - \varepsilon(x_1, x_3)l(\alpha(x_2))\rho(x_3)v_1 + r([x_1, x_3])\beta(v_2) \\
&\qquad - \varepsilon(x_2, x_1)\Big([\alpha(x_1), x_2\cdot x_3] + \rho(\alpha(x_1))l(x_2)v_3 \\
&\qquad + \rho(\alpha(x_1))r(x_3)v_2 - \varepsilon(x_1, x_2+x_3)\rho(x_2\cdot x_3)\beta(v_1)\Big) \\
&\qquad + \varepsilon(x_1+x_2, x_3)\Big([\alpha(x_3), x_2\cdot x_1] + \rho(\alpha(x_3))l(x_2)v_1 \\
&\qquad + \rho(\alpha(x_3))r(x_1)v_2 - \varepsilon(v_3, x_1+x_2)\rho(x_2\cdot x_1)\beta(v_3)\Big) \\
&\qquad - \Big([x_2, x_1]\cdot\alpha(x_3) + l([x_2, x_1])\beta(v_3) \\
&\qquad + r(\alpha(x_3))\rho(x_2)v_1 - \varepsilon(x_2, x_1)r(\alpha(x_3))\rho(x_1)v_2 \\
&\qquad + \varepsilon(x_1, x_3)\Big([x_2, x_3]\cdot\alpha(x_1) + l([x_2, x_3])\beta(v_1) \\
&\qquad + r(\alpha(x_1))\rho(x_2)v_3 - \varepsilon(x_2, x_3)r(\alpha(x_1))\rho(x_3)v_2 \\
&\quad = \alpha(x_2)\cdot[x_1, x_3] + l(\alpha(x_2))\rho(x_1)v_3 \\
&\qquad - \varepsilon(x_1, x_3)l(\alpha(x_2))\rho(x_3)v_1 + r([x_1, x_3])\beta(v_2) \\
&\qquad - \varepsilon(x_2, x_1)[\alpha(x_1), x_2\cdot x_3] - \varepsilon(x_2, x_1)\rho(\alpha(x_1))l(x_2)v_3 \\
&\qquad - [x_2, x_1]\cdot\alpha(x_3) - l([x_2, x_1])\beta(v_3)
\end{aligned}
$$

$$
\begin{aligned}
&- r(\alpha(x_3))\rho(x_2)v_1 + \varepsilon(x_2, x_1)r(\alpha(x_3))\rho(x_1)v_2\\
&+ \varepsilon(x_1, x_3)[x_2, x_3] \cdot \alpha(x_1) + \varepsilon(x_1, x_3)l([x_2, x_3])\beta(v_1)\\
&\varepsilon(x_1, x_3)r(\alpha(x_1))\rho(x_2)v_3 - \varepsilon(x_1 + x_2, x_3)r(\alpha(x_1))\rho(x_3)v_2\\
= &\Big(\alpha(x_2) \cdot [x_1, x_3] - \varepsilon(x_2, x_1)[\alpha(x_1), x_2 \cdot x_3]\\
&+ \varepsilon(x_1 + x_2, x_3)[\alpha(x_3), x_2 \cdot x_1] - [x_2, x_1] \cdot \alpha(x_3)\\
&+ \varepsilon(x_1, x_3)[x_2, x_3] \cdot \alpha(x_1)\Big)\\
&+ \Big(l(\alpha(x_2))\rho(x_1)v_3 - \rho(x_2 \cdot x_1)\beta(v_3) - \varepsilon(x_2, x_1)\rho(\alpha(x_1))l(x_2)v_3\\
&+ \varepsilon(x_1, x_3)r(\alpha(x_1))\rho(x_2)v_3 - l([x_2, x_1]\beta(v_3))\Big)\\
&- \varepsilon(x_1, x_3)\Big(l(\alpha(x_2))\rho(x_3)v_1 - \rho(x_2 \cdot x_3)\beta(v_1)\\
&- \varepsilon(x_2, x_3)\rho(\alpha(x_3))l(x_2)v_1 + \varepsilon(x_3, x_1)r(\alpha(x_3))\rho(x_2)v_1 - l([x_2, x_3])\beta(v_1)\Big)\\
&+ \Big(r([x_1, x_3])\beta(v_2) - \varepsilon(x_2, x_1)(\rho(\alpha(x_1))r(x_3)v_2 - r(\alpha(x_3))\rho(x_1)v_2)\\
&- \varepsilon(x_1 + x_2, x_3)(r(\alpha(x_1))\rho(x_3)v - \rho(\alpha(x_3))r(x_1)v_2)\Big) = 0.
\end{aligned}
$$

Thus, $(A \oplus V, \cdot', [\cdot, \cdot]', \varepsilon, \alpha + \beta)$ is a Hom-Gelfand-Dorfman color Hom-algebra. □

Example 22.14 Important examples of representations of Hom-Gelfand-Dorfman color Hom-algebras can be constructed as follows.

1) Let $(A, \cdot, [\cdot, \cdot], \varepsilon, \alpha)$ be a Hom-Gelfand-Dorfman color Hom-algebra. If

$$L(a)b = a \cdot b, \quad R(a)b = b \cdot a, \quad ad(a)b = [a, b] = -\varepsilon(a, b)[b, a],$$

for all $a, b \in \mathcal{H}(A)$, then (L, R, ad, α, A) is a representation of $(A, \cdot, [\cdot, \cdot], \varepsilon, \alpha)$.

2) If $f : \mathcal{A} = (A, \cdot_1, [\cdot, \cdot]_1, \varepsilon, \alpha) \to (A', \cdot_2, [\cdot, \cdot]_2, \varepsilon, \beta)$ is a morphism of Hom-Gelfand-Dorfman color Hom-algebras, then (l, r, ρ, β, A') becomes a representation of $\mathcal{A}$ via f, that is, for all $(x, y) \in \mathcal{H}(A) \times \mathcal{H}(A')$,

$$l(x)y = f(x) \cdot_2 y, \quad r(x)y = y \cdot_2 f(x), \quad \rho(x)y = [f(x), y]_2.$$

Theorem 22.7 *Let $\mathcal{A} = (A, \cdot_A, [\cdot, \cdot]_A, \varepsilon, \alpha)$ and $\mathcal{B} = (B, \cdot_B, [\cdot, \cdot]_B, \varepsilon, \beta)$ be Hom-Gelfand-Dorfman color Hom-algebras. Suppose that there are such even linear maps $l_A, r_A, \rho_A : A \to End(B)$ and $l_B, r_B, \rho_B : B \to End(A)$ that $A \bowtie^{\rho_A,\beta}_{\rho_B,\alpha} B$ is a matched pair of Hom-Lie color Hom-algebras, and $A \bowtie^{l_A,r_A,\beta}_{l_B,r_B,\alpha} B$ is a matched pair of Hom-Novikov color Hom-algebras, and for all $x, y \in \mathcal{H}(A)$, $a, b \in \mathcal{H}(B)$,*

$$\begin{aligned}&r_A(\rho_B(a)x)\beta(b) - r_A(\alpha(x))([b,a])\\&= \varepsilon(a,x)\big(\beta(b)\cdot_B \rho_A(x)a - \rho_A(l_B(b)x)\beta(a)\\&\quad - r_A(\rho_B(b)x)\beta(a)\big) - \varepsilon(a+b,x)\big(\rho_A(\alpha(x))(b\cdot_B a)\\&\quad - \rho_A(x)b\cdot_B \beta(a) + \varepsilon(b,a)[\beta(a), r_A(x)b]_B,\end{aligned}$$

$$\begin{aligned}&l_A(\alpha(x))([a,b]_B) - \rho_A(x)a\cdot_B \beta(b) - \rho_A(r_B(a)x)\beta(b)\\&= \varepsilon(x,a)\big([\beta(a), l_A(x)b]_B - r_A(\rho_B(a)x)\beta(b)\big)\\&\quad + \varepsilon(a,b)\big(\rho_A(r_B(b)x)\beta(a) - \rho_A(x)b\cdot_B \beta(a)\big)\\&\quad + \varepsilon(a+x,b)\big(l_A(\rho_B(b)x)\beta(a) - [\beta(b), l_A(x)a]_B\big),\end{aligned}$$

$$\begin{aligned}&r_B(\rho_A(x)a)\alpha(y) - r_B(\beta(a))([y,x])\\&= \varepsilon(x,a)\big(\alpha(y)\cdot_A \rho_B(a)x - \rho_B(l_A(y)a)\alpha(x)\\&\quad - r_B(\rho_A(y)a)\alpha(x)\big) - \varepsilon(x+y,a)\big(\rho_B(\beta(a))(y\cdot_A x)\\&\quad - \rho_B(a)y\cdot_A \alpha(x) + \varepsilon(y,x)[\alpha(x), r_B(a)y]_A,\end{aligned}$$

$$\begin{aligned}&l_B(\beta(a))([x,y]_A) - \rho_B(a)x\cdot_A \alpha(y) - \rho_B(r_A(x)a)\alpha(y)\\&= \varepsilon(a,x)\big([\alpha(x), l_B(a)y]_A - r_B(\rho_A(x)a)\beta(y)\big)\\&\quad + \varepsilon(x,y)\big(\rho_B(r_A(y)a)\alpha(x) - \rho_B(a)y\cdot_A \alpha(x)\big)\\&\quad + \varepsilon(a+x,y)\big(l_B(\rho_A(y)a)\alpha(x) - [\alpha(y), l_B(a)x]_A\big).\end{aligned}$$

Then, $(A, B, l_A, r_A, \rho_A, \beta, l_B, r_B, \rho_B, \alpha)$ is called a matched pair of the Hom-Gelfand-Dorfman color Hom-algebras. In this case, on the direct sum $A \oplus B$ of the underlying linear spaces of $\mathcal{A}$ and $\mathcal{B}$, there is a Hom-Gelfand-Dorfman color Hom-algebra structure which is given for any $x + a \in A_{\Gamma_1} \oplus B_{\Gamma_1}$, $y + b \in A_{\Gamma_2} \oplus B_{\Gamma_2}$ by

$$\begin{aligned}(x+a)\cdot(y+b) &= x\cdot_A y + (s_A(x)b + \varepsilon(a,y)s_A(y)a)\\&\quad + a\cdot_B b + (s_B(a)y + \varepsilon(x,b)s_B(b)x),\\ [x+a, y+b] &= [x,y]_A + (\rho_A(x)b - \rho_A(y)a)\\&\quad + [a,b]_B + (\rho_B(a)y - \rho_B(b)x).\end{aligned}$$

Proof By Proposition 22.5 and Proposition 22.9, $(A \oplus B, \cdot, \varepsilon, \alpha + \beta)$ is a Hom-Novikov color Hom-algebra and $(A \oplus B, [\cdot,\cdot], \alpha + \beta)$ is a Hom-Lie color Hom-algebra. It is easy to verify, in a similar way as for Proposition 22.2, that the compatibility condition is satisfied. □

Taking the color ε-commutator in a Hom-Novikov-Poisson color Hom-algebra, we obtain the following result.

Theorem 22.8 *If $(A, \cdot, \diamond, \varepsilon, \alpha)$ is a Hom-Novikov-Poisson color Hom-algebra, and for all $x, y \in \mathcal{H}(A)$,*

$$[x,y] = x\diamond y - \varepsilon(x,y)y\diamond x, \tag{22.39}$$

then $(A, \cdot, [\cdot,\cdot], \varepsilon, \alpha)$ is a Hom-Gelfand-Dorfman color Hom-algebra.

Proof By definition $(A, \cdot, \varepsilon, \alpha)$ is a ε-commutative Hom-associative color Hom-algebra. Then $(A, \cdot, \varepsilon, \alpha)$ is a Hom-Novikov color Hom-algebra. Moreover, by

Proposition 22.6, $(A, [\cdot,\cdot], \varepsilon, \alpha)$ is a Hom-Lie color Hom-algebra. Now, we show that the compatibility condition (22.36) is satisfied. For any $x, y, z \in \mathcal{H}(A)$,

$$
\begin{aligned}
&\alpha(y)\cdot[x,z]-\varepsilon(y,x)[\alpha(x),y\cdot z]+\varepsilon(x+y,z)[\alpha(z),y\cdot x]\\
&\quad -[y,x]\cdot\alpha(z)+\varepsilon(x,z)[y,z]\cdot\alpha(x)\\
&\text{(using (22.39))}\\
&=\alpha(y)\cdot\big(x\diamond z-\varepsilon(x,z)z\diamond x\big)-\varepsilon(y,x)\big(\alpha(x)\diamond(y\cdot z)\\
&\quad -\varepsilon(x,y+z)(y\cdot z)\diamond\alpha(x)\big)+\varepsilon(x+y,z)\big(\alpha(z)\diamond(y\cdot x)\\
&\quad -\varepsilon(z,y+x)(y\cdot x)\diamond\alpha(z)\big)-\big(y\diamond x-\varepsilon(y,x)x\diamond y\big)\cdot\alpha(z)\\
&\quad +\varepsilon(x,z)\big((y\diamond z-\varepsilon(y,z)z\diamond y)\cdot\alpha(x)\big)\\
&=\alpha(y)\cdot(x\diamond z)-\varepsilon(x,z)\alpha(y)\cdot(z\diamond x)-\varepsilon(y,x)\alpha(x)\diamond(y\cdot z)\\
&\quad +\varepsilon(x,z)(y\cdot z)\diamond\alpha(x)+\varepsilon(x+y,z)\alpha(z)\diamond(y\cdot x)\\
&\quad -(y\cdot x)\diamond\alpha(z)-(y\diamond x)\cdot\alpha(z)+\varepsilon(y,x)(x\diamond y)\cdot\alpha(z)\\
&\quad +\varepsilon(x,z)(y\diamond z)\cdot\alpha(x)-\varepsilon(x+y,z)(z\circ y)\cdot\alpha(x)\\
&=\varepsilon(y,x)\underbrace{\Big((x\diamond y)\cdot\alpha(z)-\alpha(x)\diamond(y\cdot z)-\varepsilon(x,y)\big((y\diamond x)\cdot\alpha(z)-\alpha(y)\cdot(x\diamond z)\big)\Big)}_{=0\text{ by }(22.18)}\\
&\quad -\varepsilon(x+y,z)\underbrace{\Big((z\diamond y)\cdot\alpha(x)-\alpha(z)\diamond(y\cdot x)-\varepsilon(z,y)\big((y\diamond z)\cdot\alpha(x)-\alpha(y)\diamond(z\cdot x)\big)\Big)}_{=0\text{ by }(22.18)}\\
&\quad -\underbrace{\Big((y\cdot x)\diamond\alpha(z)-\varepsilon(x,z)(y\diamond z)\cdot\alpha(x)\Big)}_{=0\text{ by }(22.17)}=0,
\end{aligned}
$$

which completes the proof. □

Example 22.15 Let $\Gamma=\mathbb{Z}_2\times\mathbb{Z}_2$ be an abelian group and A be a 4-dimensional Γ-graded linear space defined by $A_{(0,0)}=< e_1 >$, $A_{(0,1)}=< e_2 >$, $A_{(1,0)}=< e_3 >$ and $A_{(1,1)}=< e_4 >$. The quintuple $(A,\cdot,\diamond,\varepsilon,\alpha)$ is a Hom-Novikov-Poisson color Hom-algebra with

$$
\begin{aligned}
&\text{bicharacter: } \varepsilon\big((i_1,i_2),(j_1,j_2)\big)=(-1)^{i_1j_1+i_2j_2},\\
&\text{multiplication “}\cdot\text{”: } e_2\cdot e_3=e_3\cdot e_2=\mu e_4,\ \mu\in\mathbb{K},\\
&\text{multiplication “}\diamond\text{”: } e_2\diamond e_3=\lambda_1e_4,\ e_3\diamond e_2=\lambda_2e_4,\ e_3\diamond e_3=\lambda_3e_1,\ \lambda_i\in\mathbb{K},\\
&\text{even linear map } \alpha: A\to A:\ \alpha(e_1)=2e_1,\quad \alpha(e_2)=-e_2,\\
&\qquad\qquad\qquad\qquad\qquad\ \ \alpha(e_3)=-e_3,\quad \alpha(e_4)=-2e_4.
\end{aligned}
$$

Therefore, $(A,\cdot,[\cdot,\cdot],\varepsilon,\alpha)$ is a Hom-Gelfand-Dorfman color Hom-algebra with

$$[e_2,e_3]=-[e_3,e_2]=(\lambda_1-\lambda_2)e_4,\quad [e_3,e_3]=2\lambda_3e_3.$$

Lemma 22.3 ([5]) *Let $(A,\cdot,\varepsilon,\alpha)$ be a ε-commutative Hom-associative color Hom-algebra and D be an even derivation. With the bilinear operation $\diamond: A\times A\to A$,* *such that* $x \diamond y = x \cdot D(y)$, *for all* $x, y \in \mathcal{H}(A)$, $(A, \cdot, \diamond, \varepsilon, \alpha)$ *is a Hom-Novikov-Poisson color Hom-algebra.*

Combining Theorem 22.8 and Lemma 22.3 leads to the following corollary.

Corollary 22.5 *If* $(A, \cdot, \varepsilon, \alpha)$ *is* ε*-commutative Hom-associative color Hom-algebra and* D *is an even derivation, then* $(A, \cdot, [\cdot, \cdot], \varepsilon, \alpha)$ *is a Hom-Gelfand-Dorfman color Hom-algebra with* $[x, y] = x \cdot D(y) - \varepsilon(x, y)y \cdot D(x)$ *for* $x, y \in \mathcal{H}(A)$.

Next theorem provides a construction of the Hom-Gelfand-Dorfman color Hom-algebras from Gelfand-Dorfman color Hom-algebras and their morphisms.

Theorem 22.9 *Let* $\mathcal{A} = (A, \cdot, [\cdot, \cdot], \varepsilon)$ *be a Gelfand-Dorfman color Hom-algebra and* $\alpha : \mathcal{A} \to \mathcal{A}$ *be a Gelfand-Dorfman color Hom-algebras morphism. With the bilinear operations* $\cdot_\alpha$, $[\cdot, \cdot]_\alpha : A \times A \to A$ *such that for all* $x, y \in \mathcal{H}(A)$, $x \cdot_\alpha y = \alpha(x \cdot y)$ *and* $[x, y]_\alpha = \alpha([x, y])$, $\mathcal{A}_\alpha = (A_\alpha = A, \cdot_\alpha, [x, y]_\alpha, \varepsilon, \alpha)$ *is a Hom-Gelfand-Dorfman color Hom-algebra called the* α*-twist or Yau twist of* $(A, \cdot, [\cdot, \cdot], \varepsilon)$. *Moreover, assume that* $\mathcal{A}' = (A', \cdot', [\cdot, \cdot]', \varepsilon)$ *is another Gelfand-Dorfman color Hom-algebra and* $\alpha' : \mathcal{A}' \to \mathcal{A}'$ *is a Gelfand-Dorfman color Hom-algebras morphism. Let* $f : \mathcal{A} \to \mathcal{A}'$ *be a Hom-Gelfand-Dorfman color Hom-algebras morphism satisfying* $f \circ \alpha = \alpha' \circ f$. *Then,* $f : \mathcal{A}_\alpha \to \mathcal{A}'_\alpha$ *is a Hom-Gelfand-Dorfman color Hom-algebras morphism.*

Proof Being a Gelfand-Dorfman color Hom-algebras morphism, $\alpha : A \to A$ is an even linear map which is multiplicative with respect to $\cdot$ and $[\cdot, \cdot]$, that is, for all $x, y \in \mathcal{H}(A)$, $\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)$, $\alpha([x, y]) = [\alpha(x), \alpha(y)]$. The equality (22.36) in $\mathcal{A}_\alpha$ is proved as follows:

$$
\begin{aligned}
&\alpha(y) \cdot_\alpha [x, z]_\alpha = \alpha(y) \cdot_\alpha \alpha([x, z]) = \alpha(\alpha(y) \cdot \alpha([x, z]))\\
&(\alpha \text{ morphism})\\
&= \alpha^2(y) \cdot \alpha^2([x, z]) = \alpha^2(y) \cdot [\alpha^2(x), \alpha^2(z)]\\
&(\mathcal{A} \text{ is a Hom-G. D. color alg})\\
&= \varepsilon(y, x)[\alpha^2(x), \alpha^2(y) \cdot \alpha^2(z)] - \varepsilon(x + y, z)[\alpha^2(z), \alpha^2(y) \cdot \alpha^2(x)]\\
&\quad + [\alpha^2(y), \alpha^2(x)] \cdot \alpha^2(z) - \varepsilon(x, z)[\alpha^2(y), \alpha^2(z)] \cdot \alpha^2(x)\\
&(\alpha \text{ morphism})\\
&= \varepsilon(y, x)[\alpha^2(x), \alpha(\alpha(y) \cdot \alpha(z))] - \varepsilon(x + y, z)[\alpha^2(z), \alpha(\alpha(y), \alpha(x))]\\
&\quad + \alpha([\alpha(y), \alpha(x)]) \cdot \alpha^2(z) - \varepsilon(x, z)\alpha([\alpha(y), \alpha(z)] \cdot \alpha^2(x)\\
&= \varepsilon(y, x)[\alpha^2(x), \alpha(y \cdot_\alpha z)] - \varepsilon(x + y, z)[\alpha^2(z), \alpha(y \cdot_\alpha x)]\\
&\quad + \alpha([y, x]_\alpha \cdot \alpha^2(z) - \varepsilon(x, z)\alpha([y, z]_\alpha) \cdot \alpha^2(x)\\
&= \varepsilon(y, x)[\alpha(x), y \cdot_\alpha z]_\alpha - \varepsilon(x + y, z)[\alpha(z), y \cdot_\alpha x]_\alpha\\
&\quad + [y, x]_\alpha \cdot_\alpha \alpha(z) - \varepsilon(x, z)[y, z]_\alpha \cdot_\alpha \alpha(x).
\end{aligned}
$$

The second assertion follows from

$$f(x \cdot_\alpha y) = f(\alpha(x \cdot y)) = \alpha'(f(x \cdot y)) = \alpha'(f(x) \cdot' f(y)) = f(x) \cdot'_{\alpha'} f(y),$$
$$f([x, y]_\alpha) = f(\alpha([x, y])) = \alpha'(f([x, y])) = \alpha'([f(x), f(y)]') = [f(x), f(y)]'_{\alpha'},$$

which completes the proof. □

Corollary 22.6 *If $\mathcal{A} = (A, \cdot, [\cdot, \cdot], \varepsilon, \alpha)$ is a multiplicative Hom-Gelfand-Dorfman color algebra, then for any $n \in \mathbb{N}^*$,*

(i) *The* nth *derived Hom-Gelfand-Dorfman color Hom-algebra of type* 1 *of $\mathcal{A}$ is defined by*
$$\mathcal{A}_1^n = (A, \cdot^{(n)} = \alpha^n \circ \cdot, *^{(n)} = \alpha^n \circ [\cdot, \cdot], \varepsilon, \alpha^{n+1}).$$

(ii) *The* nth *derived Hom-Gelfand-Dorfman color Hom-algebra of type* 2 *of A is defined by*

$$\mathcal{A}_2^n = (A, \cdot^{(2^n-1)} = \alpha^{2^n-1} \circ \cdot, [\cdot, \cdot]^{(2^n-1)} = \alpha^{2^n-1} \circ [\cdot, \cdot], \varepsilon, \alpha^{2^n}).$$

Proof Apply Theorem 22.9 with $\alpha' = \alpha^n$ and $\alpha' = \alpha^{2^n-1}$ respectively. □

Example 22.16 Let $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ and A be a 4-dimensional Γ-graded linear space with $A_{(0,0)} =< e_1 >, A_{(0,1)} =< e_2 >, A_{(1,0)} =< e_3 >, A_{(1,1)} =< e_4 >$. Then, the quintuple $(A, \cdot, [\cdot, \cdot], \varepsilon, \alpha)$ is a multiplicative admissible Hom-Gelfand-Dorfman color Hom-algebra with the bicharacter $\varepsilon\big((i_1, i_2), (j_1, j_2)\big) = (-1)^{i_1 j_1 + i_2 j_2}$, and the multiplications tables for a basis $\{e_1, e_2, e_3, e_4\}$:

$\cdot$	e_1	e_2	e_3	e_4
e_1	0	0	0	0
e_2	0	0	$2e_4$	0
e_3	0	$2e_4$	e_1	0
e_4	0	0	0	0

$[\cdot, \cdot]$	e_1	e_2	e_3	e_4
e_1	0	0	0	0
e_2	0	0	$-2e_4$	0
e_3	0	$2e_4$	0	0
e_4	0	0	0	0

$$\alpha(e_1) = e_1, \quad \alpha(e_2) = -2e_2,$$
$$\alpha(e_3) = -e_3, \alpha(e_4) = 2e_4,$$

Then there are Hom-Gelfand-Dorfman color Hom-algebras $\mathcal{A}_1^n$ and $\mathcal{A}_2^n$ with multiplications tables respectively:

$\cdot^{(n)}$	e_1	e_2	e_3	e_4
e_1	0	0	0	0
e_2	0	0	$2^n e_4$	0
e_3	0	$2^n e_4$	e_1	0
e_4	0	0	0	0

$[\cdot, \cdot]^{(n)}$	e_1	e_2	e_3	e_4
e_1	0	0	0	0
e_2	0	0	$(-2)^n e_4$	0
e_3	0	$2^n e_4$	0	0
e_4	0	0	0	0

$$\alpha^{n+1}(e_1) = e_1, \qquad \alpha^{n+1}(e_2) = (-2)^{n+1} e_2,$$
$$\alpha^{n+1}(e_3) = (-1)^{n+1} e_3, \ \alpha^{n+1}(e_4) = 2^{n+1} e_4,$$

$\cdot^{(2^n-1)}$	e_1	e_2	e_3	e_4
e_1	0	0	0	0
e_2	0	0	$2^{2^n-1}e_4$	0
e_3	0	$2^{2^n-1}e_4$	e_1	0
e_4	0	0	0	0

$[\cdot,\cdot]^{(2^n-1)}$	e_1	e_2	e_3	e_4
e_1	0	0	0	0
e_2	0	0	$-2^{2^n-1}e_4$	0
e_3	0	$2^{2^n-1}e_4$	0	0
e_4	0	0	0	0

$$\alpha^{2^n}(e_1) = e_1,\ \alpha^{2^n}(e_2) = 2^{2^n}e_2,$$
$$\alpha^{2^n}(e_3) = e_3,\ \alpha^{2^n}(e_4) = 2^{2^n}e_4.$$

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Chapter 23
The Wishart Distribution on Symmetric Cones

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Abstract In this paper we discuss the extension of the Wishart probability distributions in higher dimension based on the boundary points of the symmetric cones in Jordan algebras. The symmetric cones form a basis for the construction of the degenerate and non-degenerate Wishart distributions in the field of $\mathrm{Herm}(m, \mathbb{C})$, $\mathrm{Herm}(m, \mathbb{H})$, $\mathrm{Herm}(3, \mathbb{O})$ that denotes respectively the Jordan algebra of all Hermitian matrices of size $m \times m$ with complex entries, the skew field $\mathbb{H}$ of quaternions, and the algebra $\mathbb{O}$ of octonions. This density is characterised by the Vandermonde determinant structure and the exponential weight that is dependent on the trace of the given matrix.

Keywords Vandermonde determinant · Jordan algebra · Symmetric cone · Wishart distribution

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23.1 Introduction

The maximal invariants that generally arise in statistical hypothesis testing as discussed in [20] are actually functions of eigenvalues of sample covariance matrices, see [1, 3, 14] and [25], among others. The normalised maximum likelihood estimate of the covariance matrix of a normal random vector has the *classical Wishart distribution*. From the algebraic point of view, the support of the Wishart distribution is the interior of the symmetric irreducible cone of nonnegative-definite matrices with real entries. There exist also *degenerate Wishart distributions* supported by subsets of the boundary of the above cone. This motivates us to study the problem of defining the Wishart distribution in other symmetric irreducible cones.

In a companion paper, we study the distributions of ordered eigenvalues of random matrices with values defined in the domain of classical symmetric cones coincides with the similar studies of random matrix theory which is fully discussed in [5, 6] and [23]. Thus, the applications of the joint eigenvalue distributions are not only limited to the hypothesis testing but also to many other problems for instance in quantum mechanics, principle component analysis, signal processing and many others, for details see [1, 15, 25–28] and [23].

23.1.1 Symmetric Cones and Jordan Algebra

Let E be a m-dimensional Euclidean space and let Ω be the set of all symmetric positive-definite $m \times m$ matrices. The set Ω has several interesting properties. To describe them, let GL(E) be the group of all invertible linear operators in E (GL stands for General Linear). Let Ω be a subset of E.

Definition 23.1 ([7]) The *automorphism group* $G(\Omega)$ of the set Ω is defined by

$$\mathrm{G}(\Omega) = \{\, g \in \mathrm{GL(E)} \colon g\Omega = \Omega \,\}.$$

Let Ω be again the set of all $m \times m$ positive-definite matrices. Let g be an invertible $m \times m$ matrix with positive determinant. The linear operator $\rho(g)$ acting in the linear space $\mathrm{Sym}(m, \mathbb{R})$ by

$$\rho(g)x = gxg^{\top}, \qquad x \in \mathrm{Sym}(m, \mathbb{R}),$$

leaves Ω invariant. We proved that the group $\mathrm{G} = \mathrm{GL}^{+}(m, \mathbb{R})$ is a subgroup of $\mathrm{G}(\Omega)$ (the upper index, $+$, stands for the positive determinant).

We write down a list of properties of the set Ω.

Theorem 23.1 *The set Ω as the following properties:*

(i) Ω *is a* cone*: $x \in \Omega$ and $\lambda > 0$ imply that $\lambda x \in \Omega$.*
(ii) Ω *is* convex*: $x, y \in \Omega$ and $\lambda \in [0, 1]$ imply that $\lambda x + (1 - \lambda)y \in \Omega$.*

(iii) *The open convex cone* Ω *is* self-dual*: we have* $\Omega^* = \Omega$*, where*

$$\Omega^* = \{\, y \in \mathrm{E}\colon (x|y) > 0 \quad \textit{for all} \quad x \in \overline{\Omega} \setminus \{0\}\,\}$$

is the open dual cone *of* Ω.

(iv) *The open cone* Ω *is* homogeneous*: for all* $x, y \in \Omega$ *there exists* $g \in \mathrm{G}(\Omega)$ *such that* $gx = y$.

(v) *Finally, the* symmetric *(self-dual and homogeneous) cone* Ω *is* irreducible*: there do not exist nontrivial subspaces* E_1, E_2 *and symmetric cones* $\Omega_1 \subset \mathrm{E}_1$, $\Omega_2 \subset \mathrm{E}_2$ *such that* $\mathrm{E} = \mathrm{E}_1 \oplus \mathrm{E}_2$ *and* $\Omega = \Omega_1 \times \Omega_2$.

Do there exists symmetric irreducible cones besides the cone $\Omega = \Pi_m(\mathbb{R})$ *of* $m \times m$ *positive-definite matrices with real entries? Can we define the Wishart distributions on all symmetric irreducible cones?*

To answer these questions, turn the linear space $\mathrm{E} = \mathrm{Sym}(m, \mathbb{R})$ into an *algebra*.

Definition 23.2 ([7]) A real linear space E is called an *algebra* if a bilinear mapping $(x, y) \mapsto x \circ y$ from $\mathrm{E} \times \mathrm{E}$ to E is defined. The above mapping is called a *product*.

Example 23.1 Let $\mathrm{E} = \mathrm{Sym}(m, \mathbb{R})$. Define the product by

$$x \circ y = \frac{1}{2}(xy + yx).$$

One can easily see that this product is indeed bilinear. Moreover, one can check the following properties of the introduced product.

$$x \circ y = y \circ x, \tag{23.1a}$$

$$x \circ (x^2 \circ y) = x^2 \circ (x \circ y), \tag{23.1b}$$

$$(x \circ u|v) = (u|x \circ v) \tag{23.1c}$$

for all x, y, u, and v in E.

Definition 23.3 An algebra $(\mathrm{E}, \circ)$ is called a *Jordan algebra* if it satisfies (23.1a) and (23.1b).

Definition 23.4 A Jordan algebra defined on a linear space E with an inner product is called *Euclidean* if it satisfies (23.1c).

Jordan algebras were introduced by P. Jordan, J. von Neumann, and E. Wigner in [16].

The Euclidean Jordan algebra $\mathrm{Sym}(m, \mathbb{R})$ has one more important property.

Definition 23.5 An nonempty subset I of a commutative (i.e., satisfying (23.1a)) algebra is called an *ideal* if $x \circ y \in \mathrm{I}$ as long as $x \in \mathrm{E}$ and $y \in \mathrm{I}$.

It is obvious that any algebra E contains at least two ideals: {0} and E. The above ideals are called *trivial*.

Table 23.1 Classification of simple Euclidean Jordan algebras

E	Ω	n	r	d
$\mathbb{R}^1$	$(0, \infty)$	1	1	0
$\mathbb{R}^1 \times \mathbb{R}^{m-1}$	Λ_m	m	2	$m-2$
$\mathrm{Sym}(m, \mathbb{R})$	$\Pi_m(\mathbb{R})$	$m(m+1)/2$	m	1
$\mathrm{Herm}(m, \mathbb{C})$	$\Pi_m(\mathbb{C})$	m^2	m	2
$\mathrm{Herm}(m, \mathbb{H})$	$\Pi_m(\mathbb{H})$	$m(2m-1)$	m	4
$\mathrm{Herm}(3, \mathbb{O})$	$\Pi_3(\mathbb{O})$	27	3	8

Definition 23.6 An algebra E is called *simple* if it does not contain nontrivial ideals.

The Euclidean Jordan algebra $\mathrm{Sym}(m, \mathbb{R})$ is simple, see [7, Theorem V.3.7].

The following result explains why we introduced Jordan algebras. It describes a one-to-one correspondence between irreducible symmetric cones and simple Euclidean Jordan algebras.

Theorem 23.2 *In a simple Euclidean Jordan algebra, the set Ω of squares of all invertible elements is an irreducible symmetric cone. Conversely, any irreducible symmetric cone is a set of squares of invertible elements of a certain simple Euclidean Jordan algebra.*

In Table 23.1 we introduce simple Euclidean Jordan algebras. This table is compiled by combining information from [7] and [21]. Simple Euclidean Jordan algebras have been classified by [16]. We explain the content of Table 23.1.

In Table 23.1, the symbol m runs over the set of all positive integers ≥ 3. In the first column, the algebra $\mathbb{R}^1 \times \mathbb{R}^{m-1}$ is called the *Lorentz algebra*. The product in this algebra has the form

$$(\lambda, u) \circ (\mu, v) = (\lambda\mu + (u|v), \lambda v + \mu u), \qquad \lambda, \mu \in \mathbb{R}^1, \quad u, v \in \mathbb{R}^{m-1}.$$

The corresponding cone, Λ_m, is called the *Lorentz cone*. It has the form

$$\Lambda_m = \{ (\lambda, u) \in \mathbb{R}^1 \times \mathbb{R}^{m-1} \colon \lambda^2 - (u|u) > 0, \lambda > 0 \}.$$

The symbol $\mathrm{Herm}(m, \mathbb{C})$ (resp. $\mathrm{Herm}(m, \mathbb{H})$, resp. $\mathrm{Herm}(3, \mathbb{O})$) denotes the Jordan algebra of all Hermitian matrices of size $m \times m$ with complex entries (resp. with entries in the skew field $\mathbb{H}$ of quaternions, resp. with entries in the algebra $\mathbb{O}$ of octonions). The scalar product in all of the above algebras has the form

$$(x|y) = \mathrm{Re}\,\mathrm{tr}(xy),$$

while the Jordan product is standard: $x \circ y = \frac{1}{2}(xy + yx)$. All the cones Ω are the sets of positive-definite matrices in the corresponding algebras. All algebras in Table 23.1

are pairwise non-isomorphic. In small dimensions, we have the following isomorphisms:

$$
\begin{aligned}
\mathrm{Sym}(1,\mathbb{R}) &\sim \mathrm{Herm}(1,\mathbb{C}) \sim \mathrm{Herm}(1,\mathbb{H}) \sim \mathrm{Herm}(1,\mathbb{O}) \sim \mathbb{R}^1,\\
\mathrm{Sym}(2,\mathbb{R}) &\sim \mathbb{R}^1 \oplus \mathbb{R}^2,\\
\mathrm{Herm}(2,\mathbb{C}) &\sim \mathbb{R}^1 \oplus \mathbb{R}^3,\\
\mathrm{Herm}(2,\mathbb{H}) &\sim \mathbb{R}^1 \oplus \mathbb{R}^5,\\
\mathrm{Herm}(2,\mathbb{O}) &\sim \mathbb{R}^1 \oplus \mathbb{R}^9.
\end{aligned}
\tag{23.2}
$$

In what follows, the symbol n always denotes the dimension of the real linear space E. All simple Euclidean Jordan algebras, except the Lorentz ones, will be called *matrix algebras*.

To explain the fourth column of Table 23.1, denote by e the identity element of the algebra E. For any $x \in \mathrm{E}$, put

$$
m(x) = \min\{k > 0 \colon (e, x, x^2, \dots, x^k) \quad \text{are linearly dependent}\}. \tag{23.3}
$$

The number $m(x)$ is bounded from above by n, the dimension of E.

Definition 23.7 The *rank* of a Jordan algebra E is given by

$$
r = \max\{m(x) \colon x \in \mathrm{E}\}.
$$

To explain the meaning of the last column of Table 23.1, we start from the following result, see [7].

Theorem 23.3 *Any simple Jordan algebra contains a* Jordan frame*, that is, the set* $\{c_1, \dots, c_r\}$ *such that*

(i) *its elements are* orthogonal*:* $c_i \circ c_j = 0$ *if* $i \neq j$*;*
(ii) *its elements are* idempotents*:* $c_i^2 = c_i$*;*
(iii) *its elements constitute a* resolution of identity*:* $c_1 + \dots + c_r = e$*.*

Denote by $\mathcal{L}(c_i)$ the linear operator in E acting by

$$
\mathcal{L}(c_i)x = c_i \circ x, \qquad x \in \mathrm{E}.
$$

By [7, Lemma IV.1.3], the linear operators $\mathcal{L}(c_i)$ and $\mathcal{L}(c_j)$ commute. Therefore, the admit a simultaneous diagonalisation. Let $\mathrm{E}_{ii} = \mathrm{E}(c_i, 1)$ be the one-dimensional eigenspace of the linear operator $\mathcal{L}(c_i)$ that corresponds to the eigenvalue 1. Let $\mathrm{E}(c_i, 1/2)$ be the eigenspace that corresponds to the eigenvalue $1/2$, and let

$$
\mathrm{E}_{ij} = \mathrm{E}(c_i, 1/2) \cap \mathrm{E}(c_j, 1/2).
$$

Theorem 23.4 ([7]) *The space* E *decomposes into the orthogonal direct sum*

$$\mathrm{E} = \bigoplus_{1 \le i \le j \le r} \mathrm{E}_{ij}. \tag{23.4}$$

The subspaces E_{ij} *with* $i \neq j$ *have the same dimension.*

Denote the above dimension by d. It follows that

$$n = r + d\frac{r(r-1)}{2}. \tag{23.5}$$

The number d is given in the last column of Table 23.1.

How to define a Wishart distribution on an irreducible symmetric cone Ω? First, we define the *determinant* and the *trace* of an element x of the corresponding Euclidean Jordan algebra E.

23.1.2 Trace, Determinant and Minimal Polynomials

Let $\mathbb{R}[\mathrm{X}]$ be the algebra of polynomials in one variable with real coefficients. It is well-known that any ideal in $\mathbb{R}[\mathrm{X}]$ is generated by a unique monic polynomial. In particular, for any $x \in \mathrm{E}$, the ideal

$$\mathcal{J}(x) = \{\, p \in \mathbb{R}[\mathrm{X}] \colon p(x) = 0 \,\}$$

is generated by a polynomial called the *minimal polynomial* of x. Its degree, $m(x)$, is determined by (23.3). An element x is called *regular* if $m(x) = r$. By [7, Proposition II.2.1], the set of regular elements is open and dense in E. There exist unique polynomials $a_1, a_2, \ldots, a_r$ such that the minimal polynomial of every regular element x is given by

$$f_x(\lambda) = \lambda^r - a_1(x)\lambda^{r-1} + a_2(x)\lambda^{r-2} + \cdots + (-1)^r a_r(x). \tag{23.6}$$

Moreover, the polynomial a_j is homogeneous of degree j.

Definition 23.8 The *trace* of x is $\operatorname{tr}(x) = a_1(x)$. The *determinant* of x is $\det(x) = a_r(x)$.

By definition, the Laplace transform of the Ω-valued Wishart random variable $\mathbf{Y}$ is defined on the set

$$\boldsymbol{\Sigma} - \Omega = \{\, \boldsymbol{\Sigma} - x \colon x \in \Omega \,\}$$

and is given by

$$\mathcal{L}_{\mathbf{Y}}(x) = (\det(e - \boldsymbol{\Sigma}^{-1}x))^{-\lambda}. \tag{23.7}$$

The result by [10] takes the form

Theorem 23.5 *The right hand side of* (23.7) *defines the Laplace transform of a random variable if and only if*

$$\lambda \in \Lambda = \left\{0, \frac{d}{2}, d, \dots, \frac{(r-1)d}{2}\right\} \cup \left(\frac{(r-1)d}{2}, \infty\right). \tag{23.8}$$

To write down the probability density of the Wishart distribution, we need to define the gamma function.

23.1.3 The Gamma Function of a Cone

Definition 23.9 The gamma function determined by the cone Ω is

$$\Gamma_\Omega(s) = \int_\Omega \exp(-\operatorname{tr}(x))(\det(x))^{s-n/r}\,\mathrm{d}x, \qquad \operatorname{Re} s > n/r - 1.$$

By [7, Corollary VII.1.3, part (i)], we have

$$\Gamma_\Omega(s) = (2\pi)^{(n-r)/2} \prod_{i=0}^{r-1} \Gamma(s - id/2). \tag{23.9}$$

Note that when $\Omega = \Pi_m(\mathbb{R})$, we have

$$\Gamma_\Omega(s) = 2^{(n-r)/2}\Gamma_m(s) \tag{23.10}$$

because of different parametrisations.

When $\lambda \in ((r-1)d/2, \infty)$, the Wishart distribution is supported by Ω and has probability density

$$f_{\mathbf{Y}}(x) = \frac{(\det(\boldsymbol{\Sigma}))^\lambda}{\Gamma_\Omega(\lambda)} \exp(-\operatorname{tr}(\boldsymbol{\Sigma} \circ x))(\det(x))^{\lambda-n/r}\mathbb{1}_{\boldsymbol{\Sigma}}(x). \tag{23.11}$$

For the case of $\Omega = \Pi_m(\mathbb{C})$, the Wishart distribution was studied by [11] and [17], for the case of $\Omega = \Pi_m(\mathbb{H})$ by [2], for the case of $\Omega = \Pi_3(\mathbb{O})$ by [8].

Recall that an *ensemble* is a joint distributions of finitely many real objects, see [18]. For example, when $\Omega = \Pi_m(\mathbb{R})$, the *classical Wishart ensemble* is the distribution of the ordered eigenvalues of the random Wishart matrix. How to define the Wishart ensemble in the general case? We need a theorem.

23.2 The Wishart Ensembles on Symmetric Cones

Theorem 23.6 ([7, Theorem III.1.2]) *For any $x \in E$ there exist a Jordan frame $\{c_1, \dots, c_r\}$ such that*

$$x = \sum_{i=1}^{r} \lambda_i c_i.$$

The numbers λ_i are uniquely determined by x. The polynomials $a_k(x)$ of (23.6) have the form

$$a_k(x) = \sum_{1 \le i_1 < \cdots < i_k \le r} \lambda_{i_1} \cdots \lambda_{i_k}, \qquad 1 \le k \le r.$$

In particular,

$$\operatorname{tr}(x) = \sum_{i=1}^{r} \lambda_i, \qquad \det(x) = \prod_{i=1}^{r} \lambda_i.$$

The numbers λ_i are called the *spectral eigenvalues* of x. When E consists of matrices with real or complex entries, they coincide with ordinary eigenvalues. For the case of $\mathrm{E} = \mathrm{Herm}(m, \mathbb{H})$ they coincide with *right* eigenvalues. For the case of $\mathrm{E} = \mathrm{Herm}(3, \mathbb{O})$ they do not coincide with ordinary eigenvalues. Indeed, [8] discovered matrices in $\mathrm{Herm}(3, \mathbb{O})$ whose eigenvalues are not real.

The distribution of the spectral eigenvalues of the Wishart matrix has been calculated by [22]. For the general case of an arbitrary shape parameter $\boldsymbol{\Sigma}$ the result includes a complicated integral.

Theorem 23.7 *When $\boldsymbol{\Sigma} = \mathbf{I}$, the probability density of the distribution of ordered spectral eigenvalues of x, $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_r$ is*

$$f_{\mathbf{Y}}(\lambda_1, \dots, \lambda_m) = c \left(\prod_{i=1}^{r} \lambda_i \right)^{\lambda - n/r} \prod_{1 \le i < j \le r} (\lambda_j - \lambda_i)^d \exp\left(-\sum_{i=1}^{r} \lambda_i \right). \tag{23.12}$$

where c is the normalizing constant.

Lemma 23.1 *The constant c in (23.12) is given by*

$$c = \frac{r![\Gamma(d/2)]^r (2\pi)^{n-r}}{\Gamma_\Omega(\lambda)\Gamma_\Omega(rd/2)}. \tag{23.13}$$

Proof To calculate the constant c, we use a version of the classical *Selberg integral*. The original integral is as follows, see [29]:

$$\int_{[0,1]^r} \prod_{1 \le i < j \le r} |x_j - x_i|^{2\gamma} \prod_{i=1}^{r} x_i^{\alpha-1}(1 - x_i)^{\beta-1} = \prod_{i=0}^{r-1} \frac{\Gamma(1 + (i+1)\gamma)\Gamma(\alpha + i\gamma)\Gamma(\beta + i\gamma)}{\Gamma(1+\gamma)\Gamma(\alpha + \beta + (r + i - 1)\gamma)}.$$

Following [9], order the integration variables, put $x_j = \lambda_j/L$, $\beta = L$, and take the limit as $L \to \infty$ in the Selberg integral. We obtain

$$\int_{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0} \left(\prod_{i=1}^{r} \lambda_i \right)^{\alpha - 1} \prod_{1 \leq i < j \leq r} (\lambda_i - \lambda_j)^{2\gamma} \exp\left(-\sum_{i=1}^{r} \lambda_i \right) \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_r$$
$$= \frac{1}{r!} \prod_{i=0}^{r-1} \frac{\Gamma(\alpha + i\gamma)\Gamma((i+1)\gamma)}{\Gamma(\gamma)}.$$

The coefficient c becomes

$$c = \frac{r![\Gamma(\gamma)]^r}{\prod_{i=0}^{r-1}[\Gamma(\alpha + i\gamma)\Gamma((i+1)\gamma)]}.$$

To transform the denominator, put $j = r - 1 - i$. We obtain

$$\prod_{i=0}^{r-1}[\Gamma(\alpha + i\gamma)\Gamma((i+1)\gamma)] = \prod_{j=0}^{r-1}[\Gamma(\alpha + (r-1)\gamma - j\gamma)\Gamma(r\gamma - j\gamma)]$$

Substitute the value of $\gamma = d/2$. Then, by (23.9)

$$\prod_{i=0}^{r-1}[\Gamma(\alpha + i\gamma)\Gamma((i+1)\gamma)] = \prod_{j=0}^{r-1}[\Gamma(\alpha + (r-1)d/2 - jd/2)\Gamma(rd/2 - jd/2)]$$
$$= \frac{\Gamma_\Omega(\alpha + (r-1)d/2)\Gamma_\Omega(rd/2)}{(2\pi)^{n-r}}.$$

Using the value of

$$\alpha = \lambda - n/r + 1 = \lambda - 1 - d\frac{r-1}{2} + 1 = \lambda - d\frac{r-1}{2},$$

where we applied (23.5), we have

$$\prod_{i=0}^{r-1}[\Gamma(\alpha + i\gamma)\Gamma((i+1)\gamma)] = \frac{\Gamma_\Omega(\lambda)\Gamma_\Omega(rd/2)}{(2\pi)^{n-r}}.$$

Finally,

$$c = \frac{r![\Gamma(d/2)]^r(2\pi)^{n-r}}{\Gamma_\Omega(\lambda)\Gamma_\Omega(rd/2)}.$$

In particular, when $\Omega = \Pi_m(\mathbb{R})$, we have $r = m$, $d = 1$, $n - r = m(m-1)/2$, and we have

$$c = \frac{m!\pi^{m/2}(2\pi)^{m(m-1)/2}}{2^{m(m-1)/2}\Gamma_m(\lambda)\Gamma_m(m/2)} = \frac{m!\pi^{m^2/2}}{\Gamma_m(\lambda)\Gamma_m(m/2)},$$

where we used (23.10).

Theorem 23.8 *Let λ be a real number that belongs to the interior of the Gindikin set* (23.8)*. The probability density of the distribution of the ordered spectral eigenvalues of the Wishart random variable with Laplace transform* (23.7) *is given by*

$$f(\lambda_1, \ldots, \lambda_r) = \frac{r![\Gamma(d/2)]^r(2\pi)^{n-r}}{\Gamma_\Omega(\lambda)\Gamma_\Omega(rd/2)} \prod_{i=1}^{r} \lambda_i^{\lambda - n/r} \prod_{1 \le i < j \le r} (\lambda_i - \lambda_j)^d \exp\left(-\sum_{i=1}^{r} \lambda_i\right).$$

Example 23.2 For the Lorentz cone Λ_m, we have $r = 2$, $d = m - 2$, $n = m$, and

$$\Gamma_\Omega(s) = (2\pi)^{(m-2)/2}\Gamma(s)\Gamma(s - 1/2).$$

The distribution of the spectral eigenvalues has the density

$$f(\lambda_1, \lambda_2) = \frac{2\sqrt{\pi}}{\Gamma(\lambda)\Gamma(\lambda - 1/2)}(\lambda_1\lambda_2)^{\lambda - m/2}(\lambda_2 - \lambda_1)^{m-2}\exp(-\lambda_1 - \lambda_2). \quad (23.14)$$

where $m \in \{3, 4, 6, 8, 10\}$ and $\lambda = \{2, 3, 4, 6, 8\}$.

For the cone $\mathrm{Herm}(3, \mathbb{C})$ we have $r = 3$, $d = 2$, $n = 9$, and

$$\Gamma_\Omega(s) = (2\pi)^3\Gamma(s)\Gamma(s - 1)\Gamma(s - 2).$$

The distribution of the spectral eigenvalues has the density

$$f(\lambda_1, \lambda_2, \lambda_3) = \frac{3(\lambda_1\lambda_2\lambda_3)^{\lambda-3}}{\Gamma(\lambda)\Gamma(\lambda - 1)\Gamma(\lambda - 2)} \prod_{1 \le i < j \le 3} (\lambda_j - \lambda_i)^2 \exp(-\lambda_1 - \lambda_2 - \lambda_3). \quad (23.15)$$

where $\lambda \ge 4$ since $\lambda = 3$ gives a special case.

For the cone $\mathrm{Herm}(3, \mathbb{H})$ we have $r = 3$, $d = 4$, $n = 15$, and

$$\Gamma_\Omega(s) = (2\pi)^6\Gamma(s)\Gamma(s - 2)\Gamma(s - 4).$$

The distribution of the spectral eigenvalues has the density

$$f(\lambda_1, \lambda_2, \lambda_3) = \frac{(\lambda_1\lambda_2\lambda_3)^{\lambda-5}}{120\Gamma(\lambda)\Gamma(\lambda - 2)\Gamma(\lambda - 4)} \prod_{1 \le i < j \le 3} (\lambda_j - \lambda_i)^4 \exp(-\lambda_1 - \lambda_2 - \lambda_3). \quad (23.16)$$

where $\lambda \ge 6$ since $\lambda = 5$ gives a special case.

For the cone Herm$(3, \mathbb{O})$ we have $r = 3$, $d = 8$, $n = 27$, and

$$\Gamma_\Omega(s) = (2\pi)^{12}\Gamma(s)\Gamma(s-4)\Gamma(s-8).$$

The distribution of the spectral eigenvalues has the density

$$f(\lambda_1, \lambda_2, \lambda_3) = \frac{6^3(\lambda_1\lambda_2\lambda_3)^{\lambda-9}}{11!7!\Gamma(\lambda)\Gamma(\lambda-4)\Gamma(\lambda-8)} \prod_{1\le i<j\le 3}(\lambda_j-\lambda_i)^8 \exp(-\lambda_1-\lambda_2-\lambda_3). \tag{23.17}$$

where $\lambda \ge 10$ since $\lambda = 9$ gives a special case.

More detailed discussions and demonstrations on the same cab be obtained in [24].

Next, we would like to prove a version of Theorem 23.8 for the case when λ belongs to the boundary of the Gindikin set. To do this, we first summarise the steps in proof of Theorem 23.8.

(i) Let μ_λ be the measure

$$\mathrm{d}\mu_\lambda(x) = \frac{1}{\Gamma_\Omega(\lambda)}(\det(x))^{\lambda-n/r}\mathbb{1}_\Omega(x)\,\mathrm{d}x,$$

where λ belongs to the interior of the Gindikin set. Its Laplace transform is

$$\mathcal{L}_{\mu_\lambda}(y) = (\det(-y))^{-\lambda}, \qquad y \in -\Omega. \tag{23.18}$$

(ii) The Wishart distribution is a member $\mathsf{P}_{\Sigma,\mu_\lambda}$ of the natural exponential family of the measure μ_λ given by

$$\mathrm{d}\mathsf{P}_{\boldsymbol{\Sigma},\mu_\lambda}(x) = \frac{(\det(\boldsymbol{\Sigma}))^\lambda}{\Gamma_m(\lambda)}\exp(-\operatorname{tr}(\boldsymbol{\Sigma}x))(\det x)^{\lambda-(m+1)/2}\mathbb{1}_\Omega(x)\,\mathrm{d}x. \tag{23.19}$$

(iii) We apply the results of [22] and calculate the constant c.

23.2.1 Lassalle Measure on Symmetric Cones and Probability Distribution

Let λ belongs to the *boundary* of the Gindikin set, that is, $\lambda = \ell d/2$, $0 \le \ell \le r-1$. The measure μ_λ with Laplace transform (23.18) was constructed by [19] and is called the *Lassale measure*. We describe a simplified construction of that measure due to [4].

First, the measure $\mu_{\ell d/2}$ is supported by the set

$$\partial_\ell \Omega = \{\, x \in \partial\Omega\colon\ \operatorname{rank}(x) = \ell \,\}.$$

We have $\partial_0\Omega = \{0\}$, and μ_0 is the probabilistic measure on E with $\mu_0(\{0\}) = 1$.

Let G be the connected component of identity of $\mathrm{G}(\Omega)$, the automorphism group of the cone Ω. Let K be the subgroup of G that fixes the identity e of the corresponding simple Euclidean Jordan algebra E:

$$\mathrm{K} = \{\, g \in G\colon ge = e \,\}.$$

Denote

$$u_\ell = c_1 + \cdots + c_\ell \in \Pi_\ell$$

and

$$\mathrm{E}_{u_\ell} = \{\, x \in \mathrm{E}\colon u_\ell \circ x = x \,\}.$$

By [7, Proposition IV.3.1], $x \in \partial_\ell\Omega$ if and only if there is $k \in \mathrm{K}$ such that $x \in k\Omega_\ell$, where Ω_ℓ is the symmetric cone of the simple Jordan algebra E_{u_ℓ}. Let M_ℓ be the stationary subgroup of the point u_ℓ, that is,

$$\mathrm{M}_\ell = \{\, k \in \mathrm{K}\colon ku_\ell = u_\ell \,\}.$$

Let $k\mathrm{M}_\ell = \{\, km\colon m \in \mathrm{M}_\ell \,\}$ be the *left coset of* M_ℓ *in* K *with respect to* k. Let Π_ℓ be the set of all rank ℓ idempotents in E, $1 \le \ell \le r-1$. Let c be a point in Π_ℓ. There is at least one $k \in \mathrm{K}$ with $ku_\ell = c$. For any element km of the coset $k\mathrm{M}_\ell$ we have $kmu_\ell = k(mu_\ell) = ku_\ell = c$. Conversely, if $k_1u_\ell = c$ for some $k_1 \in \mathrm{K}$, then $k^{-1}k_1u_\ell = k^{-1}(k_1u_\ell) = k^{-1}c = u_\ell$, that is, $k^{-1}k_1 = m \in \mathrm{K}_\ell$, or $k_1 = km$ which means that $k_1 \in k\mathrm{M}_\ell$. We established a one-to-one correspondence between Π_ℓ and the set $\mathrm{K}/\mathrm{M}_\ell$ of left cosets of M_ℓ in K.

Moreover, put

$$\mathrm{E}_c = \{\, x \in \mathrm{E}\colon c \circ x = x \,\}, \qquad c \in \Pi_\ell.$$

The symmetric cone of the simple Jordan algebra E_c is given by

$$\Omega_c = k\Omega_\ell,$$

where k is an arbitrary element of the left coset of M_ℓ in K that corresponds to c. Then $\partial_\ell\Omega$ is the union of nonintersecting sets Ω_c over all $c \in \Pi_\ell$. In other words, the set $\partial_\ell\Omega$ is stratified into strata. Each stratum is a rotated symmetric cone of a certain Euclidean Jordan algebra, and the strata are enumerated by the elements of the set Π_ℓ.

All the above mentioned groups are collected in Table 23.2. Notation for the groups in the last three columns is standard, see [12]. The content of the third and

Table 23.2 The groups associated to simple Euclidean Jordan algebras

E	Ω	G	K	M_ℓ
$\mathbb{R}^1$	$(0,\infty)$	$\mathbb{R}^+$	$\{1\}$	$\{1\}$
$\mathbb{R}^1\times\mathbb{R}^{m-1}$	Λ_m	$\mathrm{SO}_0(1,m-1)\times\mathbb{R}^+$	$\mathrm{SO}(m-1)$	$\mathrm{SO}(m-2)$
$\mathrm{Sym}(m,\mathbb{R})$	$\Pi_m(\mathbb{R})$	$\mathrm{SL}(m,\mathbb{R})\times\mathbb{R}^+$	$\mathrm{SO}(m)$	$\mathrm{SO}(\ell)\times\mathrm{SO}(m-\ell)$
$\mathrm{Herm}(m,\mathbb{C})$	$\Pi_m(\mathbb{C})$	$\mathrm{SL}(m,\mathbb{C})\times\mathbb{R}^+$	$\mathrm{SU}(m)$	$\mathrm{S}(\mathrm{U}(\ell)\times\mathrm{U}(m-\ell))$
$\mathrm{Herm}(m,\mathbb{H})$	$\Pi_m(\mathbb{H})$	$\mathrm{SU}^*(2m)\times\mathbb{R}^+$	$\mathrm{Sp}(m)$	$\mathrm{Sp}(\ell)\times\mathrm{Sp}(m-\ell)$
$\mathrm{Herm}(3,\mathbb{O})$	$\Pi_3(\mathbb{O})$	$\mathrm{E}_{6(-26)}\times\mathbb{R}^+$	$\mathrm{F}_{4(-52)}$	$\mathrm{Spin}(9)$

fourth column is adapted from [7], the content of the last column is based on the result by [13].

Example 23.3 Let $\Omega=\Lambda_m$. We have $r=2$, and the only possible value for ℓ is $\ell=1$. The set Π_1 is the sphere $\mathrm{S}^{m-2}=\mathrm{SO}(m-1)/\mathrm{SO}(m-2)$. The strata E_c are rank 1 simple Euclidean Jordan algebras, that is, the intervals $(0,\infty)$. The dimension of the set $\partial_1\Omega$ is $m-1+1=m-1$, as it should be for the boundary of the m-dimensional manifold Λ_m.

Example 23.4 Let Ω be a matrix cone, and let $\mathbb{F}$ be the corresponding division algebra. The set Π_ℓ is the *Grassmannian*

$$\mathrm{Gr}(\ell,\mathbb{F}^m)=\{x\in\Omega\colon x^2=x,\mathrm{tr}(x)=\ell\}.$$

Note that the Grassmannians $\mathrm{Gr}(\ell,\mathbb{F}^m)$ and $\mathrm{Gr}(m-\ell,\mathbb{F}^m)$ are homeomorphic under the map that maps an idempotent $c\in\mathrm{Gr}(\ell,\mathbb{F}^m)$ to the idempotent $e-c\in\mathrm{Gr}(r-\ell,\mathbb{F}^m)$.

If $\mathbb{F}\neq\mathbb{O}$, the set Π_ℓ can be described more classically as the set of all ℓ-dimensional linear subspaces of the (right) vector space $\mathbb{F}^m$. The above mentioned homeomorphism maps an ℓ-dimensional subspace $\mathrm{G}\in\mathrm{Gr}(\ell,\mathbb{F}^m)$ to its orthogonal complement $\mathrm{G}^\perp\in\mathrm{Gr}(m-\ell,\mathbb{F}^m)$. The strata E_c are either $\mathrm{Sym}(\ell,\mathbb{R})$, or $\mathrm{Herm}(\ell,\mathbb{C})$, or $\mathrm{Herm}(\ell,\mathbb{H})$, or $\mathrm{Herm}(\ell,\mathbb{O})$.

When $\Omega=\mathrm{Sym}(m,\mathbb{R})$, we have $\dim\mathrm{Sym}(\ell,\mathbb{R})=\ell(\ell+1)/2$ and $\dim\mathrm{Gr}(\ell,\mathbb{R}^m)=\ell(m-\ell)$. Then

$$\dim\partial_\ell\,\mathrm{Sym}(m,\mathbb{R})=\frac{\ell(\ell+1)}{2}+\ell(m-\ell)=\frac{\ell(2m-\ell+1)}{2}.$$

When $\Omega=\mathrm{Herm}(m,\mathbb{C})$, we have $\dim\mathrm{Herm}(\ell,\mathbb{C})=\ell^2$ and $\dim\mathrm{Gr}(\ell,\mathbb{C}^m)=2\ell(m-\ell)$. Then

$$\dim\partial_\ell\,\mathrm{Herm}(m,\mathbb{C})=\ell^2+2\ell(m-\ell)=\ell(2m-\ell).$$

When $\Omega=\mathrm{Herm}(m,\mathbb{H})$, we have $\dim\mathrm{Herm}(\ell,\mathbb{H})=\ell(2\ell-1)$ and $\dim\mathrm{Gr}(\ell,\mathbb{H}^m)=4\ell(m-\ell)$. Then

$$\dim \partial_\ell \operatorname{Herm}(m, \mathbb{H}) = \ell(2\ell - 1) + 4\ell(m - \ell) = \ell(4m - 2\ell - 1).$$

Finally, when $\Omega = \operatorname{Herm}(3, \mathbb{O})$, we have $\dim \operatorname{Herm}(2, \mathbb{O}) = 10$ and $\dim \operatorname{Gr}(2, \mathbb{O}^3) = 16$. Then

$$\dim \partial_2 \operatorname{Herm}(3, \mathbb{O}) = 26.$$

We also have $\dim \operatorname{Herm}(1, \mathbb{O}) = 1$ and $\dim \operatorname{Gr}(1, \mathbb{O}^3) = 16$. Then

$$\dim \partial_1 \operatorname{Herm}(3, \mathbb{O}) = 17.$$

A general result of measure theory (see, e.g., [4, Sect. 7]), says that there exists a unique probabilistic K-invariant measure on Π_ℓ, call it $\mathrm{d}c$. Fix a Jordan frame $\{c_1, \dots, c_r\}$. In the case of a matrix algebra, it is natural to choose c_ℓ as a $r \times r$ matrix whose matrix entries are all equal to 0 except the entry on the intersection of the ℓth row and the ℓth column, which is equal to 1. In the case of a Lorentz algebra, we choose

$$c_1 = (\frac{1}{2}, \frac{1}{2}, 0, \dots, 0)^\top, \qquad c_1 = (\frac{1}{2}, -\frac{1}{2}, 0, \dots, 0)^\top. \tag{23.20}$$

Definition 23.10 The *Lassalle measure* μ_0 is the probabilistic measure supported by the singleton $\{0\} \subset E$. The Lassalle measure $\mu_{\ell d/2}$, $1 \le \ell \le r - 1$, is given by

$$\mathrm{d}\mu_{\ell d/2}(\xi, c) = \frac{1}{\Gamma_{\Omega_{u_\ell}}(rd/2)} (\det(\xi + e - c))^{(r+1-\ell)d/2-1}\, \mathrm{d}\xi\, \mathrm{d}c, \qquad c \in \Pi_\ell, \quad \xi \in \Omega_c.$$

We need to prove that the Laplace transform of the introduced measure satisfies (23.18), that is,

$$\int_{\partial_\ell \Omega} \exp((x|y))\, \mathrm{d}\mu_{\ell d/2}(x) = (\det(-y))^{-\ell d/2}, \qquad x = (\xi, c) \in \partial_\ell \Omega, \quad y \in -\Omega.$$

First, we calculate the value of the Laplace transform at the point $y = -e$. By Fubini's theorem, we may integrate first with respect to $\mathrm{d}\xi$, then with respect to $\mathrm{d}c$. We have

$$L_{\mu_{\ell d/2}}(-e) = \frac{1}{\Gamma_{\Omega_{u_\ell}}(rd/2)} \int_{\Pi_\ell} \int_{\Omega_c} \exp(-(\xi|e))(\det(\xi + e - c))^{(r+1-\ell)d/2-1}\, \mathrm{d}\xi\, \mathrm{d}c.$$

Observe that for all $c \in \Pi_\ell$ and for all $\xi \in E_c$ we have $(\xi | e - c) = 0$. The integral becomes

$$L_{\mu_{\ell d/2}}(-e) = \frac{1}{\Gamma_{\Omega_{u_\ell}}(rd/2)} \int_{\Pi_\ell} \int_{\Omega_c} \exp(-(\xi|c))(\det(\xi + e - c))^{(r+1-\ell)d/2-1}\, \mathrm{d}\xi\, \mathrm{d}c.$$

To calculate the inner integral, we use [7, Corollary VII.1.3, part (ii)], that is,

$$\int_{\Omega} \exp(-(x|y))\mathrm{d}\mu_{\lambda}(x) = \Gamma_{\Omega}(\lambda)(\det(y))^{-\lambda},$$

and obtain

$$\begin{aligned}\int_{\Omega_c} \exp(-(\xi|c))(\det(\xi+e-c))^{(r+1-\ell)d/2-1}\,\mathrm{d}\xi &= \Gamma_{\Omega_{u_\ell}}(rd/2)(\det(c+e-c))^{-rd/2}\\ &= \Gamma_{\Omega_{u_\ell}}(rd/2).\end{aligned}$$

Let g be an arbitrary element of the group G with $g(-e) = y$. We have

$$\begin{aligned}L_{\mu_{\ell d/2}}(y) &= \frac{1}{\Gamma_{\Omega_{u_\ell}}(rd/2)} \int_{\Pi_\ell}\int_{\Omega_c} \exp((\xi|ge))(\det(\xi+e-c))^{(r+1-\ell)d/2-1}\,\mathrm{d}\xi\,\mathrm{d}c\\ &= \frac{1}{\Gamma_{\Omega_{u_\ell}}(rd/2)} \int_{\Pi_\ell}\int_{\Omega_c} \exp((g^\top\xi|e))(\det(\xi+e-c))^{(r+1-\ell)d/2-1}\,\mathrm{d}\xi\,\mathrm{d}c.\end{aligned}$$

According to [4], the Lassalle measure $\mu_{\ell d/2}$ has the following *semi-invariance* property:

$$\mathrm{d}\mu_{\ell d/2}(gx) = (\det(x))^{rd\ell/(2n)}\mathrm{d}\mu_{\ell d/2}(x), \qquad g \in \mathrm{G}.$$

Using this property, we obtain

$$L_{\mu_{\ell d/2}}(y) = \frac{1}{\Gamma_{\Omega_{u_\ell}}(rd/2)}\Gamma_{\Omega_{u_\ell}}(rd/2)(\det(-y))^{-\ell d/2} = (\det(-y))^{-\ell d/2},$$

as desired.

At the second step, we write down the natural exponential family of the Lassalle measure, using (23.19):

$$\mathrm{dP}_{\Sigma,\mu_{\ell d/2}}(x) = \frac{1}{\Gamma_{\Omega_{u_\ell}}(rd/2)}(\det(-\Sigma))^{\ell d/2}\exp((x|\Sigma))(\det(\xi+e-c))^{(r+1-\ell)d/2-1}\,\mathrm{d}\xi\,\mathrm{d}c,$$

where $x = (\xi, c) \in \partial_\ell\Omega$, $c \in K/M_\ell$, $\xi \in \Omega_c$, $\Sigma \in -\Omega_{u_\ell}$. Again, we would like to run Σ over Ω_{u_ℓ}, and we have

$$\mathrm{dP}_{\Sigma,\mu_{\ell d/2}}(\xi, c) = \frac{1}{\Gamma_{\Omega_{u_\ell}}(rd/2)}(\det(\Sigma))^{\ell d/2}\exp(-(\xi|\Sigma))(\det(\xi+e-c))^{(r+1-\ell)d/2-1}\,\mathrm{d}\xi\,\mathrm{d}c.$$

Example 23.5 Consider the Lassalle distributions on rank 2 cones. Put

$$\Omega = \Lambda_m = \{x \in \mathbb{R}^m : x_1 > 0, x_1^2 - x_2^2 - \cdots - x_m^2 > 0\}.$$

The set $\varPi_1$ is the sphere

$$\varPi_1 = \{ x \in \mathbb{R}^m : x_1 = 1/2, x_2^2 + \cdots + x_m^2 = 1/4 \},$$

and the determinant is

$$\det(x) = x_1^2 - x_2^2 - \cdots - x_m^2.$$

With our choice (23.20) we have $u_1 = (\frac{1}{2}, \frac{1}{2}, 0, \dots, 0)^\top$ and

$$\Omega_{u_1} = \{ \sigma u_1 : \sigma > 0 \}.$$

The measure $\mathsf{dP}_{\sigma,\mu_{1/2}}(\xi, c)$ is the product of the exponential distribution and the uniform distribution on the sphere $\varPi_1$:

$$\mathsf{dP}_{\sigma,\mu_{1/2}}(\xi, c) = \sigma \exp(-\sigma\xi)\, \mathrm{d}\xi\, \mathrm{d}c, \qquad \sigma > 0.$$

Example 23.6 Consider the Lassalle distributions on rank 3 cones $\varPi_3(\mathbb{F})$. When $\mathbb{F} = \mathbb{R}$, the sets $\varPi_1 = \varPi_2 = \mathrm{SO}(3)/\mathrm{O}(2)$ are real projective planes $\mathrm{P}_2(\mathbb{R})$:

$$\varPi_\ell = \{ \mathbf{C} = g u_\ell g^{-1} : g \in \mathrm{SO}(3) \}, \qquad \ell = 1, 2.$$

The cone Ω_{u_1} is the set of 3×3 matrices $\boldsymbol{\varXi}$ with $\xi_{11} = \xi > 0$ and all other entries equal to 0. The Wishart distribution has the form

$$\mathsf{dP}_{\sigma,\mu_{1/2}}(\boldsymbol{\varXi}, \mathbf{C}) = \frac{2\sqrt{\sigma}}{\sqrt{\pi}} \exp(-\sigma\xi) \sqrt{\det(\boldsymbol{\varXi} + \mathbf{I} - \mathbf{C})}\, \mathrm{d}\xi\, \mathrm{d}\mathbf{C}.$$

The cone Ω_{u_2} is the cone of the matrices of the form

$$\boldsymbol{\varXi} = \begin{pmatrix} \xi_{11} & \xi_{12} & 0 \\ \xi_{12} & \xi_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with $\xi_{11} > 0$ and $\xi_{11}\xi_{22} - \xi_{12}^2 > 0$. By (23.9), the gamma function of this cone has the form

$$\varGamma_{\Omega_{u_2}}(s) = (2\pi)^{1/2} \varGamma(s) \varGamma(s - 1/2).$$

In particular,

$$\varGamma_{\Omega_{u_2}}(3/2) = (2\pi)^{1/2} \frac{1}{2} \sqrt{\pi} = \frac{\pi}{\sqrt{2}}.$$

The Lassale distribution takes the form

$$\mathrm{d}\mathsf{P}_{\boldsymbol{\Sigma},\mu_1}(\boldsymbol{\Xi},\mathbf{C}) = \frac{\sqrt{2}\det(\boldsymbol{\Sigma})}{\pi}\exp(-\operatorname{tr}(\boldsymbol{\Sigma}\boldsymbol{\Xi}))\,\mathrm{d}\boldsymbol{\Xi}\,\mathrm{d}\mathbf{C},$$

where $\boldsymbol{\Sigma}$ is a 2×2 symmetric positive-definite matrix.

When $\mathbb{F} = \mathbb{C}$, the sets $\Pi_1 = \Pi_2 = \mathrm{SU}(3)/\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1))$ are complex projective planes $\mathrm{P}_2(\mathbb{C})$:

$$\Pi_\ell = \{\,\mathbf{C} = gu_\ell g^{-1}\colon g \in \mathrm{SU}(3)\,\}, \qquad \ell = 1, 2.$$

The cone Ω_{u_1} is the same as before. The Lassalle distribution has the form

$$\mathrm{d}\mathsf{P}_{\sigma,\mu_1}(\boldsymbol{\Xi},\mathbf{C}) = \frac{\sigma}{2}\exp(-\sigma\xi)(\det(\boldsymbol{\Xi} + \mathbf{I} - \mathbf{C}))^2\,\mathrm{d}\xi\,\mathrm{d}\mathbf{C}.$$

The cone Ω_{u_2} is the cone of Hermitian matrices of the form

$$\boldsymbol{\Xi} = \begin{pmatrix} \xi_{11} & \xi_{12} & 0 \\ \overline{\xi_{12}} & \xi_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{23.21}$$

with $\xi_{11} > 0$ and $\xi_{11}\xi_{22} - |\xi_{12}|^2 > 0$. By (23.9), the gamma function of this cone has the form

$$\Gamma_{\Omega_{u_2}}(s) = 2\pi\Gamma(s)\Gamma(s-1).$$

In particular,

$$\Gamma_{\Omega_{u_2}}(3) = 2\pi \cdot 2 = 4\pi.$$

The Lassalle distribution takes the form

$$\mathrm{d}\mathsf{P}_{\boldsymbol{\Sigma},\mu_2}(\boldsymbol{\Xi},\mathbf{C}) = \frac{(\det(\boldsymbol{\Sigma}))^2}{4\pi}\exp(-\operatorname{tr}(\boldsymbol{\Sigma}\boldsymbol{\Xi}))\det(\boldsymbol{\Xi} + \mathbf{I} - \mathbf{C})\,\mathrm{d}\boldsymbol{\Xi}\,\mathrm{d}\mathbf{C}.$$

When $\mathbb{F} = \mathbb{H}$, the sets $\Pi_1 = \Pi_2 = \mathrm{Sp}(3)/\mathrm{Sp}(2) \times \mathrm{Sp}(1)$ are quaternionic projective planes $\mathrm{P}_2(\mathbb{H})$:

$$\Pi_\ell = \{\,\mathbf{C} = gu_\ell g^{-1}\colon g \in \mathrm{Sp}(3)\,\}, \qquad \ell = 1, 2.$$

The cone Ω_{u_1} is the same as before. The Lassalle distribution has the form

$$\mathrm{d}\mathsf{P}_{\sigma,\mu_2}(\boldsymbol{\Xi},\mathbf{C}) = \frac{\sigma^2}{120}\exp(-\sigma\xi)(\det(\boldsymbol{\Xi} + \mathbf{I} - \mathbf{C}))^5\,\mathrm{d}\xi\,\mathrm{d}\mathbf{C}.$$

The cone Ω_{u_2} has the form (23.21), but this time $\xi_{12} \in \mathbb{H}$. By (23.9), the gamma function of this cone has the form

$$\Gamma_{\Omega_{u_2}}(s) = (2\pi)^2 \Gamma(s)\Gamma(s-2).$$

In particular,

$$\Gamma_{\Omega_{u_2}}(6) = (2\pi)^2 \cdot 5! \cdot 3! = 2880\pi^2.$$

The Lassalle distribution takes the form

$$\mathsf{dP}_{\boldsymbol{\Sigma},\mu_4}(\boldsymbol{\Xi}, \mathbf{C}) = \frac{(\det(\boldsymbol{\Sigma}))^4}{2880\pi^2} \exp(-\operatorname{tr}(\boldsymbol{\Sigma}\boldsymbol{\Xi}))(\det(\boldsymbol{\Xi} + \mathbf{I} - \mathbf{C}))^3 \,\mathrm{d}\boldsymbol{\Xi} \,\mathrm{d}\mathbf{C}.$$

Finally, when $\mathbb{F} = \mathbb{O}$, the sets $\Pi_1 = \Pi_2 = \mathrm{F}_{4(-52)}/\mathrm{Spin}(9)$ are octonionic projective planes $\mathrm{P}_2(\mathbb{O})$:

$$\Pi_\ell = \{\, \mathbf{C} = g u_\ell g^{-1} \colon g \in \mathrm{F}_{4(-52)} \,\}, \qquad \ell = 1, 2.$$

The cone Ω_{u_1} is the same as before. The Lassalle distribution has the form

$$\mathsf{dP}_{\sigma,\mu_4}(\boldsymbol{\Xi}, \mathbf{C}) = \frac{\sigma^4}{11!} \exp(-\sigma\xi)(\det(\boldsymbol{\Xi} + \mathbf{I} - \mathbf{C}))^{11} \,\mathrm{d}\xi \,\mathrm{d}\mathbf{C}.$$

The cone Ω_{u_2} has the form (23.21), but this time $\xi_{12} \in \mathbb{O}$. By (23.9), the gamma function of this cone has the form

$$\Gamma_{\Omega_{u_2}}(s) = (2\pi)^4 \Gamma(s)\Gamma(s-4).$$

In particular,

$$\Gamma_{\Omega_{u_2}}(12) = (2\pi)^4 \cdot 11! \cdot 7!.$$

The Lassalle distribution takes the form

$$\mathsf{dP}_{\boldsymbol{\Sigma},\mu_8}(\boldsymbol{\Xi}, \mathbf{C}) = \frac{(\det(\boldsymbol{\Sigma}))^8}{16 \cdot 7! \cdot 11! \pi^4} \exp(-\operatorname{tr}(\boldsymbol{\Sigma}\boldsymbol{\Xi}))(\det(\boldsymbol{\Xi} + \mathbf{I} - \mathbf{C}))^7 \,\mathrm{d}\boldsymbol{\Xi} \,\mathrm{d}\mathbf{C}.$$

At the third step, we have to prove a theorem similar to [22, Theorem 9]. Denote by n_ℓ the dimension of the algebra E_{u_ℓ}, and by d_ℓ the dimension of the subspaces E_{ij} in its decomposition (23.4). For example, when $\mathrm{E} = \mathrm{Herm}(3, \mathbb{O})$, we have $n_3 = 27$, $n_2 = 10$, $n_1 = 1$, $d_3 = 8$, $d_1 = d_2 = 1$. Let K^{u_ℓ} be the subgroup of the connected component of identity G of the group $\mathrm{G}(\Omega_{u_\ell})$ that fixes c:

$$\mathrm{K}^{u_\ell} = \{\, g \in G \colon g u_\ell = u_\ell \,\}.$$

23.2.2 Degenerate Wishart Ensembles on Symmetric Cones

Let $0 < \nu_1 \leq \nu_2 \leq \cdots \leq \nu_\ell$ be the spectral eigenvalues of $\boldsymbol{\Sigma}$.

Theorem 23.9 *The probability density of the nonzero ordered spectral eigenvalues of the degenerate Wishart distribution is given by*

$$f_{\boldsymbol{\Sigma}}(\lambda_{r-\ell+1}, \ldots, \lambda_\ell) = \frac{\ell![\Gamma(d_\ell/2)]^\ell (2\pi)^{n_\ell-\ell}(\det(\Sigma))^{\ell d/2}}{\Gamma_{\Omega_{u_\ell}}(\ell d_\ell/2)\Gamma_{\Omega_{u_\ell}}(rd/2)} \times \prod_{i=r-\ell+1}^{r} \lambda_i^{(r+1-\ell)d/2-1} \prod_{r-\ell+1\leq i<j\leq r} (\lambda_j - \lambda_i)^{d_\ell} \times \prod_{i=r-\ell+1}^{r} \prod_{j=1}^{\ell} \int_{K^{u_\ell}} \exp(-\lambda_i \nu_j (mc_i \mid c_j))\, \mathrm{d}m.$$

In particular, when $\boldsymbol{\Sigma} = e$, the above density is

$$f(\lambda_{r-\ell+1}, \ldots, \lambda_\ell) = \frac{\ell![\Gamma(d_\ell/2)]^\ell (2\pi)^{n_\ell-\ell}}{\Gamma_{\Omega_{u_\ell}}(\ell d_\ell/2)\Gamma_{\Omega_{u_\ell}}(rd/2)} \times \prod_{i=r-\ell+1}^{r} \lambda_i^{(r+1-\ell)d/2-1} \prod_{r-\ell+1\leq i<j\leq r} (\lambda_j - \lambda_i)^{d_\ell} \exp\left(-\sum_{i=1}^{\ell} \lambda_i\right). \tag{23.22}$$

Proof By [4, Theorem 2.7], if a function $\mathrm{F}\colon \partial_\ell\Omega \to \mathbb{R}$ is integrable with respect to the Lassalle measure $\mu_{\ell d/2}$, then we have

$$\int_{\partial_\ell\Omega} \mathrm{F}(\xi, c)\, \mathrm{d}\mu_{\ell d/2}(\xi, c) = \frac{1}{\Gamma_{\Omega_{u_\ell}}(rd/2)} \int_{\Pi_\ell} \int_{\Omega_c} \mathrm{F}(\xi, c)(\det(\xi + e - c))^{(r+1-\ell)d/2-1}\, \mathrm{d}\xi\, \mathrm{d}c. \tag{23.23}$$

Choose a Jordan frame in E_c as follows:

$$\tilde{c}_i = kc_i, \qquad r - \ell + 1 \leq i \leq r,$$

where k is an arbitrary element of K with $ke = c$. Let K^c be the subgroup of the connected component of identity G of the group $G(\Omega_c)$ that fixes c:

$$\mathrm{K}^c = \{ g \in \mathrm{G}\colon gc = c \}.$$

For any $x \in \mathrm{E}_c$ there is $m \in \mathrm{K}^c$ such that

$$x = m \sum_{i=r-\ell+1}^{r} \lambda_i(\tilde{c}_i),$$

where $\lambda_{r-\ell+1} \leq \cdots \leq \lambda_r$ are the ordered spectral eigenvalues of x. Denote

$$\mathrm{R}_+^c = \{\, \lambda = \lambda_{r-\ell+1}\tilde{c}_{r-\ell+1} + \cdots + \lambda_r \tilde{c}_r : \lambda_{r-\ell+1} < \cdots < \lambda_r \,\}.$$

By [7, Theorem VI.2.3], if a function $h_c \colon \mathrm{E}_c \to \mathbb{R}$ is integrable, then

$$\int_{\mathrm{E}_c} h_c(\xi)\,\mathrm{d}\xi = c_0 \int_{\mathrm{K}^c \times \mathrm{R}_+^c} h_c(m\lambda) \prod_{r-\ell+1 \leq i < j \leq \ell} (\lambda_j - \lambda_i)^{d_\ell}\,\mathrm{d}m\,\mathrm{d}\lambda_{r-\ell+1} \cdots \mathrm{d}\lambda_r, \tag{23.24}$$

where $\mathrm{d}m$ is the probabilistic invariant measure on K^c and where c_0 is a positive constant. To determine the value of this constant, we use [22]. It is proved there that for any simple Euclidean Jordan algebra E, the number $\frac{c_0}{\Gamma_\Omega(\lambda)}$ is equal to the constant (23.13). We apply this result to the algebra E_c with $\lambda = \ell d_\ell/2$ and obtain

$$c_0 = \frac{\ell![\Gamma(d_\ell/2)]^\ell (2\pi)^{n_\ell - \ell}}{\Gamma_{\Omega_{u_\ell}}(\ell d_\ell/2)}.$$

Equation (23.24) takes the form

$$\int_{\mathrm{E}_c} h_c(\xi)\,\mathrm{d}\xi = \frac{\ell![\Gamma(d_\ell/2)]^\ell (2\pi)^{n_\ell - \ell}}{\Gamma_{\Omega_{u_\ell}}(\ell d_\ell/2)} \int_{\mathrm{K}^c \times \mathrm{R}_+^c} h_c(m\lambda) \prod_{r-\ell+1 \leq i < j \leq \ell} (\lambda_j - \lambda_i)^{d_\ell}\,\mathrm{d}m\,\mathrm{d}\lambda_{r-\ell+1} \cdots \mathrm{d}\lambda_r.$$

Apply this formula to the function

$$h_c(\xi) = F(\xi, c)(\det(\xi + e - c))^{(r+1-\ell)d/2-1} \mathbb{1}_{\Omega_c}(\xi),$$

and substitute the result into (23.23). We obtain

$$\begin{aligned}\int_{\partial_\ell \Omega} \mathrm{F}(\xi, c)\,\mathrm{d}\mu_{\ell d/2}(\xi, c) &= \frac{\ell![\Gamma(d_\ell/2)]^\ell (2\pi)^{n_\ell - \ell}}{\Gamma_{\Omega_{u_\ell}}(\ell d_\ell/2)\Gamma_{\Omega_{u_\ell}}(rd/2)} \int_{\Pi_\ell} \int_{\mathrm{K}^c \times \mathrm{R}_+^c} \mathrm{F}(m\lambda, c)\\ &\quad \times (\det(m\lambda + e - c))^{(r+1-\ell)d/2-1} \mathbb{1}_{\Omega_c}(m\lambda) \prod_{r-\ell+1 \leq i < j \leq \ell} (\lambda_j - \lambda_i)^{d_\ell}\\ &\quad \times \mathrm{d}m\,\mathrm{d}\lambda_{r-\ell+1} \cdots \mathrm{d}\lambda_r\,\mathrm{d}c.\end{aligned}$$

We have

$$\det(m\lambda + e - c) = \lambda_{r-\ell+1} \cdots \lambda_r.$$

Therefore,

$$\begin{aligned}\int_{\partial_\ell \Omega} \mathrm{F}(\xi, c)\,\mathrm{d}\mu_{\ell d/2}(\xi, c) &= \frac{\ell![\Gamma(d_\ell/2)]^\ell (2\pi)^{n_\ell - \ell}}{\Gamma_{\Omega_{u_\ell}}(\ell d_\ell/2)\Gamma_{\Omega_{u_\ell}}(rd/2)} \int_{\Pi_\ell} \int_{K^c \times \mathrm{R}_+^c} F(m\lambda, c)\\ &\quad \times \prod_{i=r-\ell+1}^{r} \lambda_i^{(r+1-\ell)d/2-1} \mathbb{1}_{\Omega_c}(m\lambda) \prod_{r-\ell+1 \leq i < j \leq r} (\lambda_j - \lambda_i)^{d_\ell}\\ &\quad \times \mathrm{d}m\,\mathrm{d}\lambda_{r-\ell+1} \cdots \mathrm{d}\lambda_r\,\mathrm{d}c.\end{aligned}$$

The above formula can be understood as follows. Let $F(\xi, c)$ be the probability density of a $\partial_\ell\Omega$-valued random variable X with respect to the Lassalle measure $\mu_{\ell d/2}(\xi, c)$. Then, the probability density of the ordered spectral eigenvalues of the random variable X is

$$f_{\boldsymbol{\Sigma}}(\lambda_1, \ldots, \lambda_\ell) = \frac{\ell![\Gamma(d_\ell/2)]^\ell (2\pi)^{n_\ell-\ell}}{\Gamma_{\Omega_{u_\ell}}(\ell d_\ell/2)\Gamma_{\Omega_{u_\ell}}(rd/2)} \int_{\Pi_\ell}\int_{\mathrm{K}^c} F(m\lambda, c) \times \prod_{i=r-\ell+1}^{r} \lambda_i^{(r+1-\ell)d/2-1} \prod_{r-\ell+1\le i<j\le r} (\lambda_j - \lambda_i)^{d_\ell}\, \mathrm{d}m\, \mathrm{d}c.$$

In particular, for the Wishart density we have

$$\mathrm{F}(\xi, c) = (\det(\boldsymbol{\Sigma}))^{\ell d/2} \exp(-(\xi|\boldsymbol{\Sigma})),$$

and the probability density of the ordered spectral eigenvalues of the Wishart distribution becomes

$$f_{\boldsymbol{\Sigma}}(\lambda_1, \ldots, \lambda_\ell) = \frac{\ell![\Gamma(d_\ell/2)]^\ell (2\pi)^{n_\ell-\ell} (\det(\Sigma))^{\ell d/2}}{\Gamma_{\Omega_{u_\ell}}(\ell d_\ell/2)\Gamma_{\Omega_{u_\ell}}(rd/2)} \prod_{i=r-\ell+1}^{r} \lambda_i^{(r+1-\ell)d/2-1} \times \prod_{r-\ell+1\le i<j\le r} (\lambda_j - \lambda_i)^{d_\ell} \int_{\Pi_\ell}\int_{\mathrm{K}^c} \exp(-(m\lambda|\boldsymbol{\Sigma}))\, \mathrm{d}m\, \mathrm{d}c.$$

For the integral part, we have

$$\int_{\Pi_\ell}\int_{\mathrm{K}^c} \exp(-(m\lambda|\boldsymbol{\Sigma}))\, \mathrm{d}m\, \mathrm{d}c = \int_{\mathrm{K}^{u_\ell}} \exp(-(m\lambda|\boldsymbol{\Sigma}))\, \mathrm{d}m,$$

because the groups K^c and K^{u_ℓ} are isomorphic. Then we have

$$\int_{\mathrm{K}^{u_\ell}} \exp(-(m\lambda|\boldsymbol{\Sigma}))\, \mathrm{d}m = \int_{\mathrm{K}^{u_\ell}} \exp\left(-m \sum_{i=r-\ell+1}^{r} \lambda_i c_i \,\middle|\, m' \sum_{j=1}^{\ell} \nu_j(c_j)\right) \mathrm{d}m = \prod_{i=r-\ell+1}^{r} \prod_{j=1}^{\ell} \int_{K^{u_\ell}} \exp(-\lambda_i \nu_j (mc_i \mid c_j))\, \mathrm{d}m.$$

In particular, when $\boldsymbol{\Sigma} = e$, we obtain

$$\int_{\mathrm{K}^{u_\ell}} \exp(-(m\lambda|\Sigma))\, \mathrm{d}m = \int_{\mathrm{K}^{u_\ell}} \exp\left(-m \sum_{i=r-\ell+1}^{r} \lambda_i c_i \,\middle|\, e\right) \mathrm{d}m = \int_{\mathrm{K}^{u_\ell}} \exp\left(-\sum_{i=r-\ell+1}^{r} \lambda_i\right) \mathrm{d}m = \exp\left(-\sum_{i=r-\ell+1}^{r} \lambda_i\right).$$

Example 23.7 Assume $\ell = 2$ and E is a matrix algebra of rank $r = 3$. When $\mathbb{F} = \mathbb{R}$, we have $d_2 = 1$, $n_2 = 3$, $d = 1$, and

$$\Gamma_{\Omega_{u_2}}(s) = \sqrt{2\pi}\,\Gamma(s)\Gamma(s - 1/2).$$

The probability density of the distribution of the nonzero spectral eigenvalues of the degenerate Wishart matrix is

$$f(\lambda_2, \lambda_3) = \frac{4}{\sqrt{\pi}}(\lambda_3 - \lambda_2)\exp(-\lambda_2 - \lambda_3). \tag{23.25}$$

When $\mathbb{F} = \mathbb{C}$, we have $d_2 = 2$, $n_2 = 4$, $d = 2$, and

$$\Gamma_{\Omega_{u_2}}(s) = 2\pi\,\Gamma(s)\Gamma(s - 1/2).$$

The probability density of the distribution of the nonzero spectral eigenvalues of the degenerate Wishart matrix is

$$f(\lambda_2, \lambda_3) = \sqrt{\pi}\lambda_2\lambda_3(\lambda_3 - \lambda_2)^2\exp(-\lambda_2 - \lambda_3). \tag{23.26}$$

When $\mathbb{F} = \mathbb{H}$, we have $d_2 = 4$, $n_2 = 6$, $d = 4$, and

$$\Gamma_{\Omega_{u_2}}(s) = (2\pi)^2\Gamma(s)\Gamma(s - 1/2).$$

The probability density of the distribution of the nonzero spectral eigenvalues of the degenerate Wishart matrix is

$$f(\lambda_2, \lambda_3) = \frac{208}{15!!}(\lambda_2\lambda_3)^3(\lambda_3 - \lambda_2)^4\exp(-\lambda_2 - \lambda_3). \tag{23.27}$$

Finally, when $\mathbb{F} = \mathbb{O}$, we have $d_2 = 8$, $n_2 = 10$, $d = 8$, and

$$\Gamma_{\Omega_{u_2}}(s) = (2\pi)^4\Gamma(s)\Gamma(s - 1/2).$$

The probability density of the distribution of the nonzero spectral eigenvalues of the degenerate Wishart matrix is

$$f(\lambda_2, \lambda_3) = \frac{2^{11}}{11!21!!}(\lambda_2\lambda_3)^7(\lambda_3 - \lambda_2)^8\exp(-\lambda_2 - \lambda_3). \tag{23.28}$$

23.3 Conclusion

The Wishart probability distributions can be generalized in higher dimension based on the boundary points of the symmetric cones in Jordan algebras. This density is mainly characterised by the structure of the Vandermonde determinant and the exponential weight that is dependent on the trace of the given matrix. The symmetric cones especially the Gidinkin set form a suitable basis for the construction of the degenerate and non-degenerate Wishart distributions in the field of $\text{Herm}(m, \mathbb{C})$, $\text{Herm}(m, \mathbb{H})$, $\text{Herm}(3, \mathbb{O})$ denotes respectively the Jordan algebra of all Hermitian matrices of size $m \times m$ with complex entries, the skew field $\mathbb{H}$ of quaternions, and the algebra $\mathbb{O}$ of octonions.

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Chapter 24
Induced Ternary Hom-Nambu-Lie Algebras

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Abstract This study is concerned with induced ternary Hom-Nambu-Lie algebras from Hom-Lie algebras and their classification. The induced algebras are constructed from a class of Hom-Lie algebra with nilpotent linear map. The families of ternary Hom-Nambu-Lie algebras arising in this way of construction are classified for a given class of nilpotent linear maps. In addition, some results giving conditions on when morphisms of Hom-Lie algebras can still remain morphisms for the induced ternary Hom-Nambu-Lie algebras are given.

Keywords Hom-Nambu-Lie algebra · Hom-Lie algebra

MSC 2020 Classification 17B61 · 17D30 · 17A40 · 17A42

24.1 Introduction

Ternary Hom-Nambu-Lie algebras can be constructed from binary multiplications of a Hom-Lie algebra by introducing a trace function and an additional linear map satisfying certain compatibility conditions. Hom-Lie algebras were first introduced by Hartwig, Larsson and Silvestrov in [13] by studying some examples of deformed Lie algebras which arise from twisted discretizations of vector fields. Hom-Lie algebras

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are therefore generalisations of Lie algebras by having an additional twist α, a linear map. Therefore, in such a case we end up with a generalised Jacobi identity known as Hom-Jacobi identity for Hom-Lie algebras. If the linear map α is the identity map we end up with a Lie algebra. Other Hom-algebra structures are studied in [17].

The n-ary generalisation of Lie algebras, called n-Lie algebras or Nambu-Lie algebras, first appeared in relation of Nambu's generalization of Hamiltonian mechanics [18, 22], and independantly, as a purely algebraic generalisation of Lie algebra [11]. The generalisation was made regarding the Jacobi identity as the fact that the adjoint map is a derivation of the Lie bracket. They were afterwards widely studied and used in various applications (See for example [8, 10, 12, 14, 23]). Their Hom generalisation, called n-Hom-Lie algebras or n-ary Hom-Nambu-Lie algebras, was introduced in [5], together with other n-ary Hom-algebras. In the most general case, the defining identities of n-ary Hom-algebras are twisted by a set of $n-1$ linear maps. Properties of n-ary Hom-Nambu-Lie algebras were investigated in [1, 15, 16, 24]. In the same way as for binary algebras, an n-ary Hom-Nambu-Lie algebra becomes an n-ary Nambu-Lie algebra if all the twisting maps are the identity map.

The first occurence of ternary Nambu-Lie algebras induced by Lie algebras was in [6] while studying the quantisation of Nambu mechanics. Construction and studies on induced ternary Hom-Nambu Lie algebras from Hom-Lie algebras and even more generally from n-ary Hom-Nambu-Lie algebras to $(n+1)$-ary Hom-Nambu-Lie have been done in [2–5, 15, 24]. In the n-ary cases, the induction involves a generalised trace function. Some properties of the $(n+1)$-ary Nambu-Lie algebras induced by n-ary Nambu-Lie algebras were studied independantly in [7, 9].

In [19] and [20] classification of 3-dimensional Hom-Lie algebras with nilpotent linear map is presented. Moreover, Hom-Lie structures, that is, the space of possible endomorphisms that turn skew-symmetric algebras into a Hom-Lie algebra, have also been studied in [21]. A partial classification of 3-dimensional ternary Hom-Nambu-Lie algebras has also been provided in [5] where diagonal linear maps are considered.

In this paper, we will be concerned with induced ternary Hom-Nambu-Lie algebras from Hom-Lie algebras and their classification. In Sect. 24.2, we recall some needed definitions and results on Hom-Lie algebras and ternary Hom-Nambu-Lie algebras induced by Hom-Lie algebras. The induced algebras are constructed from a class of Hom-Lie algebras with nilpotent linear map α in Sect. 24.3. The families of ternary Hom-Nambu-Lie algebras arising in this construction are classified for a given class of nilpotent linear maps as the additional twisting map β, and conditions on when morphisms of Hom-Lie algebras can still remain morphisms for the induced ternary Hom-Nambu-Lie algebras are studied in Sect. 24.4.

24.2 Preliminaries

In this section we give definitions and some results that are used in this study. The vector spaces are defined over an algebraically closed field $\mathbb{K}$ of characteristic 0, and the notation $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$ is used.

Definition 24.1 A Hom-Lie algebra $(V, [\cdot,\cdot], \alpha)$ consists of a linear space V, a bilinear map $[\cdot,\cdot] : V \times V \to V$ and a linear map $\alpha : V \to V$ satisfying, for all $x, y, z \in V$,

$$[x, y] = -[y, x] \qquad \text{Skew-symmetry} \tag{24.1}$$

$$\sum_{\circlearrowleft(x,y,z)} [\alpha(x), [y, z]] = 0 \qquad \text{Hom-Jacobi identity} \tag{24.2}$$

where $\sum_{\circlearrowleft(x,y,z)} [\alpha(x), [y, z]] = [\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]]$.

Definition 24.2 A ternary Hom-Nambu-Lie algebra is a triple $(V, [\cdot,\cdot,\cdot], \tilde{\alpha})$ consisting of a linear space V, trilinear map $[\cdot,\cdot,\cdot] : V \times V \times V \longrightarrow V$ and a pair of linear maps $\tilde{\alpha} = (\alpha_1, \alpha_2)$ satisfying, for all $x_1, x_2, x_3, x_4, x_5 \in V$, $\sigma \in S_3$,

$$\text{Skew-symmetry:} \quad [x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}] = \mathrm{Sgn}(\sigma)[x_1, x_2, x_3], \tag{24.3}$$

$$\begin{aligned} \text{Hom-Nambu Identity:} \quad [\alpha_1(x_1), \alpha_2(x_2), [x_3, x_4, x_5]] &= [[x_1, x_2, x_3], \alpha_1(x_4), \alpha_2(x_5)] \\ &+ [\alpha_1(x_3), [x_1, x_2, x_4], \alpha_2(x_5)] \\ &+ [\alpha_1(x_3), \alpha_2(x_4), [x_1, x_2, x_5]] \end{aligned} \tag{24.4}$$

A procedure for constructing ternary Hom-Nambu-Lie algebras from Hom-Lie algebras is given in [2]. This involves a trace function satisfying certain compatibility conditions. We define trace functions and give the compatibility conditions that must be fulfilled when inducing Hom-Nambu Lie algebras from Hom-Lie algebras.

Definition 24.3 A linear map $\tau : V \to \mathbb{K}$ is called a trace function on $(V, [\cdot,\cdot])$ if $\tau([x, y]) = 0$ for all $x, y \in V$.

The induced ternary Hom-Nambu-Lie algebra is defined as follows:

Definition 24.4 Let $(V, [\cdot,\cdot])$ be a binary algebra and let $\tau : V \to \mathbb{K}$ be a linear map. The trilinear map $[\cdot,\cdot,\cdot]_\tau : V \times V \times V \to V$ is defined as

$$[x_1, x_2, x_3]_\tau = \tau(x_1)[x_2, x_3] + \tau(x_2)[x_3, x_1] + \tau(x_3)[x_1, x_2]. \tag{24.5}$$

It is proven in [2] that if the bilinear multiplication $[\cdot,\cdot]$ is skew-symmetric then the trilinear map $[\cdot,\cdot,\cdot]$ is skew-symmetric. Hence, this gives a way of constructing ternary Hom-Nambu-Lie algebras from Hom-Lie algebras.

Theorem 24.1 *Let* $(V, [\cdot, \cdot], \alpha)$ *be a Hom-Lie algebra and* $\beta : V \to V$ *be a linear map. Furthermore, assume that* τ *is a trace function on* V *satisfying, for all* $x, y \in V$,

$$\tau(\alpha(x))\tau(y) = \tau(x)\tau(\alpha(y)) \tag{24.6}$$

$$\tau(\beta(x))\tau(y) = \tau(x)\tau(\beta(y)) \tag{24.7}$$

$$\tau(\alpha(x))\beta(y) = \tau(\beta(x))\alpha(y). \tag{24.8}$$

Then $(V, [\cdot, \cdot, \cdot]_\tau, (\alpha, \beta))$ *is a Hom-Nambu-Lie algebra, induced by* $(V, [\cdot, \cdot], \alpha)$.

If α and β are identity maps, Theorem 24.1 gives the result for induced ternary Nambu-Lie algebra (See [2, 6]).

Corollary 24.1 *Let* $(V, [\cdot, \cdot])$ *be a Lie algebra and* $\tau : V \to \mathbb{K}$ *a trace function on* V. *Then* $(V, [\cdot, \cdot, \cdot]_\tau)$ *is a Nambu-Lie algebra.*

Definition 24.5 Let $(V, [\cdot, \cdot, \cdot], (\alpha, \beta))$ and $(V', [\cdot, \cdot, \cdot]', (\alpha', \beta'))$ be Hom-Nambu-Lie algebras. A linear map $\Phi : V \to V'$ is a ternary Hom-Nambu-Lie algebra homomorphism if it satisfies, for all $x, y, z \in V$,

$$\Phi([x, y, z]) = [\Phi(x), \Phi(y), \Phi(z)]' \tag{24.9}$$

$$\Phi \circ \alpha = \alpha' \circ \Phi, \qquad \Phi \circ \beta = \beta' \circ \Phi. \tag{24.10}$$

If Φ is bijective then it is a Hom-Nambu-Lie algebra isomorphism.

The following observation means that for the induced Hom-Nambu-Lie algebra induced from a Hom-Lie algebra to be non-abelian, τ needs to have a nontrivial $\ker \tau$.

Proposition 24.1 ([2]) *Let* $T = (V, [\cdot, \cdot, \cdot]_\tau, (\alpha, \beta))$ *be a Hom-Nambu-Lie algebra induced from a Hom-Lie algebra* $(V, [\cdot, \cdot], \alpha)$. *If* $\ker \tau = \{0\}$ *or* $\ker \tau = V$ *then* T *is abelian.*

Let V be a 3-dimensional space over $\mathbb{K}$ with basis elements $\{e_1, e_2, e_3\}$. In this paper, all linear maps $\alpha, \beta : V \to V$ are defined using the following convention:

$$\alpha(e_i) = \sum_{k=1}^{3} a_{ik} e_k \text{ and } \beta(e_i) = \sum_{k=1}^{3} \beta_{ik} e_k, \quad i = 1, 2, 3 \text{ and } a_{ik}, \beta_{ik} \in \mathbb{K}.$$

24.3 Induced Ternary Hom-Nambu Lie Algebras from Hom-Lie Algebras with Nilpotent Linear Endomorphism

In [19] families of 3-dimensional Hom-Lie algebras with nilpotent linear map α are given. In this section, we provide 3-dimensional Hom-Nambu-Lie algebras induced by Hom-Lie algebras with nilpotent α as given in [19]. The Hom-Lie algebras are defined with respect to a basis $\{e_1, e_2, e_3\}$.

We have the following equations from (24.6)–(24.8) together with the trace condition $\tau[e_i, e_j] = 0$:

$$\sum_{k=1}^{3} \Big(\tau(e_k)(a_{ik}\tau(e_j) - a_{jk}\tau(e_i))\Big) = 0, \quad i, j = 1, 2, 3 \tag{24.11}$$

$$\sum_{k=1}^{3} \Big(\tau(e_k)(\beta_{ik}\tau(e_j) - \beta_{jk}\tau(e_i))\Big) = 0, \quad i, j = 1, 2, 3 \tag{24.12}$$

$$\sum_{k=1}^{3} \Big(a_{ik}\tau(e_k)\beta_{jl} - \beta_{ik}\tau(e_k)a_{jl}\Big) = 0, \quad i, j, l = 1, 2, 3 \tag{24.13}$$

$$\sum_{k=1}^{3} C_{ij}^{k}\tau(e_k) = 0, \quad i, j = 1, 2, 3 \text{ and } C_{ij}^{k} = -C_{ji}^{k}. \tag{24.14}$$

By solving (24.11)–(24.14) for the Hom-Lie algebras with known Jordan forms of nilpotent linear map α and structure constants $\{C_{ij}^k\}_{i<j}$, we get the possible solutions for linear map β and trace function τ. From these solutions, we give all subfamilies of Hom-Lie algebras that can induce ternary Hom-Nambu-Lie algebras. Hom-Lie algebras which do not belong to such sub-families induce abelian ternary Hom-Nambu-Lie algebras with trivial trace functions as the only possible solutions. That is, $\tau(e_1) = \tau(e_2) = \tau(e_3) = 0$.

$(\mathcal{H}_i^3, \alpha_1)$

In this case, $\alpha_1(e_1) = e_2, \alpha_1(e_2) = 0, \alpha_1(e_3) = 0$, that is, $[\alpha_1] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

$(\mathcal{H}_1^3, \alpha_1)$

$[e_1, e_2] = C_{12}^1\, e_1 + C_{12}^2\, e_2 + C_{12}^3\, e_3$
$[e_1, e_3] = C_{13}^1\, e_1 + C_{13}^2\, e_2 + C_{13}^3\, e_3$
$[e_2, e_3] = C_{23}^2\, e_2$

The following subfamilies of $(\mathcal{H}_1^3, \alpha_1)$ with $C_{ij}^k \in \mathbb{K}$ induce a non-abelian ternary Hom-Nambu-Lie algebra with the indicated β and τ:

(i) For $b_1 \neq 0$ and $(C_{12}^1, C_{12}^2) \neq (0, 0)$,

$$\begin{aligned}[e_1, e_2] &= C_{12}^1\, e_1 + C_{12}^2\, e_2\\ [e_1, e_3] &= C_{13}^1\, e_1 + C_{13}^2\, e_2\\ [e_2, e_3] &= C_{23}^2\, e_2\end{aligned}$$

and the ternary bracket is defined as

$$T_{\alpha_1}^1 : [e_1, e_2, e_3] = b_1(C_{12}^1\, e_1 + C_{12}^2\, e_2), \tag{24.15}$$

$$[\beta] = \begin{pmatrix}\beta_{11} & \beta_{12} & 0\\ \beta_{21} & \beta_{22} & 0\\ \beta_{31} & \beta_{32} & 0\end{pmatrix} \text{ and } \tau(e_1) = 0,\ \tau(e_2) = 0,\ \tau(e_3) = b_1,\, b_1 \in \mathbb{K}^*.$$

(ii) For $b_2 \neq 0$ and $C_{23}^2 \neq 0$,

$$\begin{aligned}[e_1, e_2] &= C_{12}^2\, e_2 + C_{12}^3\, e_3\\ [e_1, e_3] &= C_{13}^2\, e_2 + C_{13}^3\, e_3\\ [e_2, e_3] &= C_{23}^2\, e_2\end{aligned}$$

and the ternary bracket is defined as

$$T_{\alpha_1}^2 : [e_1, e_2, e_3] = b_2 C_{23}^2\, e_2, \tag{24.16}$$

$$[\beta] = \begin{pmatrix}0 & \beta_{12} & \beta_{13}\\ 0 & \beta_{22} & \beta_{23}\\ 0 & \beta_{32} & \beta_{33}\end{pmatrix} \text{ and } \tau(e_1) = b_2,\ \tau(e_2) = 0,\ \tau(e_3) = 0,\, b_2 \in \mathbb{K}^*.$$

(iii) For $b_3 \neq 0$, $b_4 \neq 0$ and $(b_4 C_{12}^1, (b_3 C_{23}^2 + b_4 C_{12}^2), b_3 C_{12}^1) \neq (0, 0, 0)$, that is either $C_{12}^1 \neq 0$, or $C_{12}^1 = 0$ and $C_{12}^2 \neq \gamma C_{23}^2$:

$$\begin{aligned}[e_1, e_2] &= C_{12}^1\, e_1 + C_{12}^2\, e_2\ + \gamma\ C_{12}^1\, e_3\\ [e_1, e_3] &= C_{13}^1\, e_1 + C_{13}^2\, e_2\ + \gamma\ C_{13}^1\, e_3\\ [e_2, e_3] &= C_{23}^2\, e_2\end{aligned}$$

and the ternary bracket is defined as

$$T_{\alpha_1}^3 : [e_1, e_2, e_3] = b_4 C_{12}^1\, e_1 + (b_3 C_{23}^2 + b_4 C_{12}^2)\, e_2 - b_3 C_{12}^1\, e_3 \tag{24.17}$$

$$[\beta] = \begin{pmatrix} \beta_{11} & \beta_{12} & \gamma\beta_{11} \\ \beta_{21} & \beta_{22} & \gamma\beta_{21} \\ \beta_{31} & \beta_{32} & \gamma\beta_{31} \end{pmatrix}, \quad \begin{array}{c} \tau(e_1) = b_3,\ \tau(e_2) = 0,\ \tau(e_3) = b_4, \\ b_3, b_4 \in \mathbb{K}^*, \gamma = -\dfrac{b_3}{b_4}. \end{array}$$

$\underline{(\mathcal{H}_2^3, \alpha_1)}$

$[e_1, e_2] = C_{23}^3\, e_1 + \dfrac{C_{23}^2\, C_{23}^3}{C_{23}^1}\, e_2 + \dfrac{(C_{23}^3)^2}{C_{23}^1}\, e_3$
$[e_1, e_3] = C_{13}^1\, e_1 + C_{13}^2\, e_2 + C_{13}^3\, e_3$
$[e_2, e_3] = C_{23}^1\, e_1 + C_{23}^2\, e_2 + C_{23}^3\, e_3$ with $C_{23}^1 \neq 0$.

The following subfamily of $(\mathcal{H}_2^3, \alpha_1)$ with $C_{ij}^k \in \mathbb{K}$ induce an abelian ternary Hom-Nambu-Lie algebra with the indicated β and τ:

$$\begin{aligned} [e_1, e_2] &= \gamma\, C_{23}^1\, e_1 + \gamma\, C_{23}^2\, e_2 + (\gamma)^2\, C_{23}^1\, e_3 \\ [e_1, e_3] &= C_{13}^1\, e_1 + C_{13}^2\, e_2 + \gamma\, C_{13}^1\, e_3 \\ [e_2, e_3] &= C_{23}^1\, e_1 + C_{23}^2\, e_2 + \gamma\, C_{23}^1\, e_3 \end{aligned}$$

and the ternary bracket is defined as

$$T_{\alpha_1}^4 : [e_1, e_2, e_3] = 0,$$

$$[\beta] = \begin{pmatrix} \beta_{11} & \beta_{12} & \gamma\beta_{11} \\ \beta_{21} & \beta_{22} & \gamma\beta_{21} \\ \beta_{31} & \beta_{32} & \gamma\beta_{31} \end{pmatrix}, \quad \begin{array}{c} \tau(e_1) = b_5,\ \tau(e_2) = 0,\ \tau(e_3) = b_6, \\ b_5 \in \mathbb{K}, b_6 \in \mathbb{K}^*, \gamma = -\dfrac{b_5}{b_6} \end{array}.$$

$\underline{(\mathcal{H}_i^3, \alpha_3)}$

In this case, $\alpha_3(e_1) = e_2, \alpha_3(e_2) = e_3, \alpha_3(e_3) = 0$. The matrix form of α_3 is given by $[\alpha_3] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

$\underline{(\mathcal{H}_5^3, \alpha_3)}$

$[e_1, e_2] = C_{12}^1\, e_1 + C_{12}^2\, e_2 + C_{12}^3\, e_3$
$[e_1, e_3] = -C_{23}^3\, e_2 + C_{13}^3\, e_3$
$[e_2, e_3] = C_{23}^2\, e_2 + C_{23}^3\, e_3$

The following subfamily of $(\mathcal{H}_5^3, \alpha_3)$ with $C_{ij}^k \in \mathbb{K}$ induces a non-abelian ternary Hom-Nambu-Lie algebra with the indicated β and τ, for $b_7 \neq 0, (C_{23}^2, C_{23}^3) \neq (0, 0)$:

$$\begin{aligned} [e_1, e_2] &= C_{12}^2\, e_2 + C_{12}^3\, e_3 \\ [e_1, e_3] &= -C_{23}^3\, e_2 + C_{13}^3\, e_3 \\ [e_2, e_3] &= C_{23}^2\, e_2 + C_{23}^3\, e_3 \end{aligned}$$

and the ternary bracket is defined as

$$T^5_{\alpha_3} : [e_1, e_2, e_3] = b_7(C^2_{23}\, e_2 + C^3_{23}\, e_3) \tag{24.18}$$

$[\beta] = \begin{pmatrix} 0 & \beta_{12} & \beta_{13} \\ 0 & \beta_{22} & \beta_{23} \\ 0 & \beta_{32} & \beta_{33} \end{pmatrix}$, $\tau(e_1) = b_7, \tau(e_2) = 0, \tau(e_3) = 0, b_7 \in \mathbb{K}^*$.

$\underline{(\mathcal{H}^3_6, \alpha_3)}$

$[e_1, e_2] = C^1_{12}\, e_1 + C^2_{12}\, e_2 + C^3_{12}\, e_3$
$[e_1, e_3] = C^2_{13}\, e_2 + C^3_{13}\, e_3$
$[e_2, e_3] = 0,$ with $C^2_{13} \neq 0$.

The following subfamily of $(\mathcal{H}^3_6, \alpha_3)$ with $C^k_{ij} \in \mathbb{K}$ induces an abelian ternary Hom-Nambu-Lie algebra with the indicated β and τ:

$$\begin{aligned} [e_1, e_2] &= C^2_{12}\, e_2 + C^3_{12}\, e_3 \\ [e_1, e_3] &= C^2_{13}\, e_2 + C^3_{13}\, e_3 \\ [e_2, e_3] &= 0 \end{aligned}$$

and the ternary bracket is defined as

$$T^6_{\alpha_3} : [e_1, e_2, e_3] = 0,$$

$[\beta] = \begin{pmatrix} 0 & \beta_{12} & \beta_{13} \\ 0 & \beta_{22} & \beta_{23} \\ 0 & \beta_{32} & \beta_{33} \end{pmatrix}$, $\tau(e_1) = b_8,\ \tau(e_2) = 0,\ \tau(e_3) = 0, b_8 \in \mathbb{K}$.

$\underline{(\mathcal{H}^3_7, \alpha_3)}$

$$\begin{aligned} [e_1, e_2] = {} & \frac{(C^1_{13})^2 + C^2_{13}\, C^1_{23} + C^1_{23}\, C^3_{23}}{C^1_{23}}\, e_1 + \frac{C^1_{13}\, C^2_{13} + C^2_{13}\, C^2_{23} + C^2_{23}\, C^3_{23}}{C^1_{23}}\, e_2 \\ & + \frac{C^1_{13}\, C^3_{13} + C^2_{13}\, C^3_{23} + (C^2_{23})^2}{C^1_{23}}\, e_3 \end{aligned}$$

$[e_1, e_3] = C^1_{13}\, e_1 + C^2_{13}\, e_2 + C^3_{13}\, e_3$
$[e_2, e_3] = C^1_{23}\, e_1 + C^2_{23}\, e_2 + C^3_{23}\, e_3$ with $C^1_{23} \neq 0$.

The Hom-Lie algebras $(\mathcal{H}^3_7, \alpha_3)$ with $C^k_{ij} \in \mathbb{K}$ induce an abelian ternary-Hom-Nambu-Lie algebra

$$T^7_{\alpha_3} : [e_1, e_2, e_3] = 0,$$

$[\beta] = \begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix}$, $\tau(e_1) = 0,\ \tau(e_2) = 0,\ \tau(e_3) = 0$.

Referring to Proposition 24.1, as an example, the induced ternary Hom-Nambu-Lie algebra $T^7_{\alpha_3}$ is abelian. We see that $\ker(\tau) = Span\{e_1, e_2, e_3\} = V$.

24.4 Canonical Forms of Induced Ternary Hom-Nambu-Lie Algebras

In our classification problem, we consider a class of nilpotent non-zero β given as

$$\beta(e_1) = \beta_{12}\, e_2 + \beta_{13}\, e_3,\ \ \beta(e_2) = \beta_{23}\, e_3,\ \ \beta(e_3) = 0,$$

where $\beta_{12}, \beta_{23}, \beta_{23} \in \mathbb{K}$, not simultaneously zero. That is, $[\beta] = \begin{pmatrix} 0 & \beta_{12} & \beta_{13} \\ 0 & 0 & \beta_{23} \\ 0 & 0 & 0 \end{pmatrix}$.

We construct all canonical representatives of Hom-Nambu Lie algebras induced from Hom-Lie algebras with the given class of nilpotent linear endomorphisms.

Theorem 24.2 *Every three-dimensional ternary Hom-Nambu Lie algebra* $(V, [\cdot,\cdot,\cdot]_\tau, (\alpha_1, \beta))$ *induced by a Hom-Lie algebra with linear endomorphism* α_1 *is either abelian or isomorphic to one of the following ternary Hom-Nambu-Lie algebras*:

(i) $(\mathcal{N}_1, (\alpha_1, \beta_1))$
(ii) $(\mathcal{N}_2, (\alpha_1, \beta_i)),\ i = 1, 2, 3, 4$
(iii) $(\mathcal{N}_3, (\alpha_1, \beta_1))$

defined with respect to a basis $\{e_1, e_2, e_3\}$ *as follows*

$$\mathcal{N}_1 : [e_1, e_2, e_3] = e_1, \quad \mathcal{N}_2 : [e_1, e_2, e_3] = e_2, \quad \mathcal{N}_3 : [e_1, e_2, e_3] = e_1 + e_3$$

$$[\beta_1] = \begin{pmatrix} 0 & \beta_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ [\beta_2] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$[\beta_3] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ [\beta_4] = \begin{pmatrix} 0 & \beta_{12} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \textit{with } \beta_{12} \neq 0.$$

Proof Let $(T, [\cdot,\cdot,\cdot]_\tau, (\alpha_i, \beta))$ and $(T', [\cdot,\cdot,\cdot]'_{\tau'}, (\alpha_j, \beta'))$ be ternary Hom-Nambu-Lie algebras induced by two Hom-Lie algebras with the structure constants denoted by C_{ij}^k and D_{ij}^k respectively. We define Φ as an isomorphism of any two ternary Hom-Nambu-Lie algebras with basis $\{e_1, e_2, e_3\}$ and $\{f_1, f_2, f_3\}$ respectively, as

$$\Phi(e_i) = \sum_{k=1}^{3} \varphi_{ik} f_k,\ \ i = 1, 2, 3 \text{ and } \varphi_{ik} \in \mathbb{K}.$$

For each of the induced algebras we investigate the existence of isomorphism Φ such that conditions (24.9) and (24.10) hold and find all the isomorphism classes.

From (24.10), we first find Φ such that $\Phi \circ \alpha_1 = \alpha_1 \circ \Phi$. Since α_1 is in Jordan normal form, then $[\Phi] = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ 0 & \varphi_{11} & 0 \\ 0 & \varphi_{32} & \varphi_{33} \end{pmatrix}$. From $\Phi \circ \beta = \beta' \circ \Phi$ with $[\beta] = \begin{pmatrix} 0 & \beta_{12} & \beta_{13} \\ 0 & 0 & \beta_{23} \\ 0 & 0 & 0 \end{pmatrix}$ and $[\beta'] = \begin{pmatrix} 0 & \beta'_{12} & \beta'_{13} \\ 0 & 0 & \beta'_{23} \\ 0 & 0 & 0 \end{pmatrix}$, we get the following equations:

$$\beta'_{13}\varphi_{11} + \beta'_{23}\varphi_{12} = \beta_{13}\varphi_{33} \tag{24.19}$$

$$\beta_{12}\varphi_{11} + \beta_{13}\varphi_{32} = \beta'_{12}\varphi_{11} \tag{24.20}$$

$$\beta_{23}\varphi_{33} = \beta'_{23}\varphi_{11} \tag{24.21}$$

$$\beta_{23}\varphi_{32} = 0 \tag{24.22}$$

$$\beta'_{23}\varphi_{32} = 0. \tag{24.23}$$

Now $[\beta]$ can be partitioned to the following sub-classes with $\beta_{12}, \beta_{13}, \beta_{23} \neq 0$:

$$[\overline{\beta}_1]= \begin{pmatrix} 0 & \beta_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, [\overline{\beta}_2]= \begin{pmatrix} 0 & 0 & \beta_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, [\overline{\beta}_3] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \beta_{23} \\ 0 & 0 & 0 \end{pmatrix}, [\overline{\beta}_4] = \begin{pmatrix} 0 & \beta_{12} & \beta_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$[\overline{\beta}_5] = \begin{pmatrix} 0 & \beta_{12} & 0 \\ 0 & 0 & \beta_{23} \\ 0 & 0 & 0 \end{pmatrix}, [\overline{\beta}_6] = \begin{pmatrix} 0 & 0 & \beta_{13} \\ 0 & 0 & \beta_{23} \\ 0 & 0 & 0 \end{pmatrix}, [\overline{\beta}_7] = \begin{pmatrix} 0 & \beta_{12} & \beta_{13} \\ 0 & 0 & \beta_{23} \\ 0 & 0 & 0 \end{pmatrix}.$$

We now proceed to give the non-isomorphic classes of β. From equations (24.19) to (24.23) we have the following canonical representatives of β with $\beta_{12} \neq 0$:

$$[\beta_1] = \begin{pmatrix} 0 & \beta_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, [\beta_2] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, [\beta_3] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, [\beta_4] = \begin{pmatrix} 0 & \beta_{12} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The isomorphisms of β are shown in Table 24.1.

Table 24.1 Isomorphisms of β for ternary Hom-Nambu-Lie algebras with α_1

$[\overline{\beta}_2] \overset{\Phi_{\overline{2,4}}}{\cong} [\overline{\beta}_4] \overset{\Phi_{\overline{4},2}}{\cong} [\beta_2]$	$[\Phi_{\overline{2,4}}] = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ 0 & \varphi_{11} & 0 \\ 0 & \varphi_{32} & \varphi_{33} \end{pmatrix}$, $\varphi_{11} = \frac{\beta_{13}}{\beta'_{13}}\varphi_{33}$, $\varphi_{32} = \frac{\beta'_{12}}{\beta_{13}}\varphi_{11}$, $\varphi_{11}, \varphi_{33} \neq 0$, $\varphi_{12}, \varphi_{13} \in \mathbb{K}$ $[\Phi_{\overline{4},2}] = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ 0 & \varphi_{11} & 0 \\ 0 & \varphi_{32} & \varphi_{33} \end{pmatrix}$, $\varphi_{11} = \beta_{13}\varphi_{33}$, $\varphi_{32} = -\frac{\beta_{12}}{\beta_{13}}\varphi_{11}\varphi_{11}$, $\varphi_{33} \neq 0$, $\varphi_{12}, \varphi_{13} \in \mathbb{K}$
$[\overline{\beta}_3] \overset{\Phi_{\overline{3,6}}}{\cong} [\overline{\beta}_6] \overset{\Phi_{\overline{6},3}}{\cong} [\beta_3]$	$[\Phi_{\overline{3,6}}] = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ 0 & \varphi_{11} & 0 \\ 0 & 0 & \varphi_{33} \end{pmatrix}$, $\varphi_{11} = \frac{\beta_{23}}{\beta'_{23}}\varphi_{33}$, $\varphi_{12} = -\frac{\beta'_{13}}{\beta'_{23}}\varphi_{11}$, $\varphi_{11}, \varphi_{33} \neq 0$, $\varphi_{13} \in \mathbb{K}$ $[\Phi_{\overline{6},3}] = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ 0 & \varphi_{11} & 0 \\ 0 & 0 & \varphi_{33} \end{pmatrix}$, $\varphi_{11} = \beta_{23}\varphi_{33}$, $\varphi_{12} = \beta_{13}\varphi_{33}$ $\varphi_{11}, \varphi_{33} \neq 0, \varphi_{13} \in \mathbb{K}$
$[\overline{\beta}_5] \overset{\Phi_{\overline{5,7}}}{\cong} [\overline{\beta}_7] \overset{\Phi_{\overline{7},4}}{\cong} [\beta_4]$	$[\Phi_{\overline{5,7}}] = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ 0 & \varphi_{11} & 0 \\ 0 & 0 & \varphi_{33} \end{pmatrix}$, $\varphi_{11} = \frac{\beta_{23}}{\beta'_{23}}\varphi_{33}$, $\varphi_{12} = -\frac{\beta'_{13}}{\beta'_{23}}\varphi_{11}$, $\beta_{12} = \beta'_{12}$, $\varphi_{11}, \varphi_{33} \neq 0$, $\varphi_{13} \in \mathbb{K}$ $[\Phi_{\overline{7},4}] = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ 0 & \varphi_{11} & 0 \\ 0 & 0 & \varphi_{33} \end{pmatrix}$, $\varphi_{11} = \beta_{23}\varphi_{33}$, $\varphi_{12} = \beta_{13}\varphi_{33}$, $\beta_{12} = \beta'_{12}\varphi_{11}$, $\varphi_{33} \neq 0, \varphi_{13} \in \mathbb{K}$

The next part of the proof gives the canonical representatives of the ternary brackets of the induced algebras for each of the canonical representatives β_i, $1 \le i \le 4$.

$(T^1_{\alpha_1}, (\alpha_1, \beta_1))$ (See (24.15))

The ternary bracket is $T^1_{\alpha_1} : [e_1, e_2, e_3] = b_1(C^1_{12}\, e_1 + C^2_{12}\, e_2)$ with $C^1_{12} \neq 0$. From (24.10) and (24.9) for this case, we have the following equations:

$$b_1\varphi_{11}C^1_{12} = b'_1(\varphi_{11})^2\varphi_{33}D^1_{12} \quad (24.24)$$

$$b_1(\varphi_{12}C^1_{12} + \varphi_{11}C^2_{12}) = b'_1(\varphi_{11})^2\varphi_{33}D^2_{12} \quad (24.25)$$

$$b_1\varphi_{13}C^1_{12} = 0. \quad (24.26)$$

We have $(T^1_{\alpha_1}, (\alpha_1, \beta_1)) \overset{\Phi}{\cong} (\mathcal{N}_1, (\alpha_1, \beta_1))$ where $\mathcal{N}_1 : [e_1, e_2, e_3] = e_1$ and

$$[\Phi] = \begin{pmatrix} \varphi_{11} & \varphi_{12} & 0 \\ 0 & \varphi_{11} & 0 \\ 0 & \varphi_{32} & \varphi_{33} \end{pmatrix}, \quad \begin{array}{c} \varphi_{11}, \varphi_{33} \neq 0, \varphi_{12}, \varphi_{32} \in \mathbb{K}, \\ \varphi_{33} = \frac{b_1C^1_{12}}{\varphi_{11}}, \varphi_{12} = -\frac{C^2_{12}\varphi_{11}}{C^1_{12}}, b_1 \neq 0. \end{array}$$

$(T^2_{\alpha_1}, (\alpha_1, \beta_1))$ (See (24.16))

Here, the ternary bracket is given by $[e_1, e_2, e_3] = b_2C^2_{23}\, e_2$. Expanding (24.9) and (24.10) for this case, we end up with the following equation:

$$b_2\varphi_{11}C^2_{23} = b'_2(\varphi_{11})^2\varphi_{33}D^2_{23}. \quad (24.27)$$

We have $(T^2_{\alpha_1}, (\alpha_1, \beta_1)) \overset{\Phi}{\cong} (\mathcal{N}_2, (\alpha_1, \beta_1))$, where

$$\mathcal{N}_2 : [e_1, e_2, e_3] = e_2.$$

$$[\Phi] = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ 0 & \varphi_{11} & 0 \\ 0 & \varphi_{32} & \varphi_{33} \end{pmatrix}, \ \varphi_{11} \neq 0, \varphi_{12}, \varphi_{32}, \varphi_{13} \in \mathbb{K}, \varphi_{33} = \frac{b_2C^2_{23}}{\varphi_{11}}, b_2, C^2_{23} \in \mathbb{K}^*,$$

$(T^2_{\alpha_1}, (\alpha_1, \beta_j)),\ j = 2, 3, 4$

Expanding (24.9) and (24.10) for this case, we end up with the following equation:

$$b_2\varphi_{11}C^2_{23} = b'_2(\varphi_{11})^3D^2_{23}. \quad (24.28)$$

We have $(T^2_{\alpha_1}, (\alpha_1, \beta_2)) \overset{\Phi}{\cong} (\mathcal{N}_2, (\alpha_1, \beta_2))$ and

$$[\Phi] = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ 0 & \varphi_{11} & 0 \\ 0 & 0 & \varphi_{11} \end{pmatrix}, \ \varphi_{11} \neq 0, \varphi_{12}, \varphi_{13} \in \mathbb{K}, (\varphi_{11})^2 = b_2 C_{23}^2, b_2, C_{23}^2 \in \mathbb{K}^*.$$

Similarly, we have $(T^2_{\alpha_1}, (\alpha_1, \beta_3)) \overset{\Phi}{\cong} (\mathcal{N}_2, (\alpha_1, \beta_3))$ with $[\Phi]$ as above.

Also, $(T^2_{\alpha_1}, (\alpha_1, \beta_4)) \overset{\Phi}{\cong} (\mathcal{N}_2, (\alpha_1, \beta_4))$ with $[\Phi]$ as above, except that $\varphi_{12} = 0$.

$\underline{(T^3_{\alpha_1}, (\alpha_1, \beta_1))}$ (See (24.17))

Expanding (24.9) and (24.10) for this case, we end up with the following equations:

$$\omega_1 \varphi_{11} = \omega_1' (\varphi_{11})^2 \varphi_{33} \tag{24.29}$$

$$\omega_1 \varphi_{12} + \omega_2 \varphi_{11} + \omega_3 \varphi_{32} = \omega_2' (\varphi_{11})^2 \varphi_{33} \tag{24.30}$$

$$\omega_1 \varphi_{13} + \omega_3 \varphi_{33} = \omega_3' (\varphi_{11})^2 \varphi_{33} \tag{24.31}$$

where

$$\omega_1 = b_4 C_{12}^2, \quad \omega_2 = b_3 C_{23}^2 + b_4 C_{12}^2, \quad \omega_3 = -b_3 C_{12}^1$$

$$\omega_1' = b_4' D_{12}^2, \quad \omega_2' = b_3' D_{23}^2 + b_4' D_{12}^2, \quad \omega_3' = -b_3' D_{12}^1$$

We have the following canonical classes of $(T^3_{\alpha_1}, (\alpha_1, \beta_1))$:

$$(\mathcal{N}_2, (\alpha_1, \beta_1)), (\mathcal{N}_3, (\alpha_1, \beta_1))$$

$$\mathcal{N}_2 : [e_1, e_2, e_3] = e_2, \quad \mathcal{N}_3 : [e_1, e_2, e_3] = e_1 + e_3$$

and the isomorphism given by a matrix of the form

$$[\Phi] = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ 0 & \varphi_{11} & 0 \\ 0 & \varphi_{32} & \varphi_{33} \end{pmatrix}, \ \varphi_{11}, \varphi_{33} \neq 0, \varphi_{12}, \varphi_{13}, \varphi_{32} \in \mathbb{K}.$$

$\underline{(T^3_{\alpha_1}, (\alpha_1, \beta_1)) \overset{\Phi}{\cong} (\mathcal{N}_2, (\alpha_1, \beta_1))}$

Taking $D_{12}^1 = 0$, $D_{12}^2 = 1$, $D_{23}^2 = 0$, $b_4' = 1$, corresponding to $(\mathcal{N}_2, (\alpha_1, \beta_1))$, yields $\omega_1' = 1, \omega_2' = 1, \omega_3' = 0$, and

$$\omega_1 \varphi_{11} = (\varphi_{11})^2 \varphi_{33} \tag{24.32}$$

$$\omega_1 \varphi_{12} + \omega_2 \varphi_{11} + \omega_3 \varphi_{32} = (\varphi_{11})^2 \varphi_{33} \tag{24.33}$$

$$\omega_1 \varphi_{13} + \omega_3 \varphi_{33} = 0. \tag{24.34}$$

Since $\varphi_{11} \neq 0$ and $\varphi_{33} \neq 0$, it follows from (24.32) that $\omega_1 = b_4 C_{12}^2 \neq 0$, and

$$\begin{array}{rl} \varphi_{33} = \frac{\omega_1}{\varphi_{11}} & \\ \omega_1\varphi_{12} + \omega_2\varphi_{11} + \omega_3\varphi_{32} = \varphi_{11}\omega_1 & \Longrightarrow \\ \omega_1(\varphi_{13} + \frac{\omega_3}{\varphi_{11}}) = 0 & \end{array} \begin{array}{rl} \varphi_{13} = -\frac{\omega_3}{\varphi_{11}} = \frac{b_3C_{12}^1}{\varphi_{11}} \\ \varphi_{12} = \frac{\varphi_{11}(\omega_1-\omega_2)-\omega_3\varphi_{32}}{\omega_1} \\ = \frac{b_3}{b_4}\frac{\varphi_{11}C_{23}^2+\varphi_{32}C_{12}^1}{C_{12}^2}. \end{array}$$

Therefore the isomorphism Φ is given by

$$[\Phi] = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ 0 & \varphi_{11} & 0 \\ 0 & \varphi_{32} & \varphi_{33} \end{pmatrix}, \quad \begin{array}{c} \varphi_{11} \neq 0, \varphi_{33} = \frac{\omega_1}{\varphi_{11}}, \varphi_{12} = \frac{b_3}{b_4}\frac{\varphi_{11}C_{23}^2+\varphi_{32}C_{12}^1}{C_{12}^2}, \\ \varphi_{13} = \frac{b_3C_{12}^1}{\varphi_{11}}, \varphi_{32} \in \mathbb{K}, C_{12}^2 \neq 0, b_3 \neq 0, b_4 \neq 0. \end{array}$$

$(T_{\alpha_1}^3, (\alpha_1, \beta_1)) \overset{\Phi}{\cong} (\mathcal{N}_3, (\alpha_1, \beta_1))$

Taking $D_{12}^1=1$, $D_{12}^2=0$, $D_{23}^2 = 0$, $b_3' = -1$ $b_4' = 1$, corresponding to $(\mathcal{N}_3, (\alpha_1, \beta_1))$, yields $\omega_1' = 0$, $\omega_2' = 0$, $\omega_3' = 1$, and

$$\omega_1\varphi_{11} = 0 \tag{24.35}$$

$$\omega_1\varphi_{12} + \omega_2\varphi_{11} + \omega_3\varphi_{32} = 0 \tag{24.36}$$

$$\omega_1\varphi_{13} + \omega_3\varphi_{33} = (\varphi_{11})^2\varphi_{33}. \tag{24.37}$$

Since $\varphi_{11} \neq 0$ and $\varphi_{33} \neq 0$, we get that $\omega_1 = b_4C_{12}^2 = 0$ which means $C_{12}^2 = 0$, and that $\omega_3 = -b_3C_{12}^1 \neq 0$. Also

$$\begin{array}{rl} \omega_2\varphi_{11} + \omega_3\varphi_{32} = 0 & \\ \omega_3\varphi_{33} = (\varphi_{11})^2\varphi_{33} & \Longrightarrow \end{array} \begin{array}{rl} \varphi_{32} = -\frac{\omega_2\varphi_{11}}{\omega_3} = \frac{\varphi_{11}C_{23}^2}{C_{12}^1}, \\ (\varphi_{11})^2 = \omega_3 = -b_3C_{12}^1. \end{array}$$

Therefore the isomorphism Φ is given by

$$[\Phi] = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ 0 & \varphi_{11} & 0 \\ 0 & \varphi_{32} & \varphi_{33} \end{pmatrix}, \quad \begin{array}{c} (\varphi_{11})^2 = -b_3C_{12}^1, \varphi_{32} = \frac{\varphi_{11}C_{23}^2}{C_{12}^1}, \\ \varphi_{12}, \varphi_{13} \in \mathbb{K}, \varphi_{33} \neq 0, C_{12}^1 \neq 0, b_3 \neq 0. \end{array}$$

In addition, the induced ternary Hom-Nambu-Lie algebras with β commuting with β_2, β_3 and β_4 as given earlier are isomorphic to the corresponding canonical representatives of the Hom-Nambu Lie algebras with β_2, β_3 and β_4. □

Theorem 24.3 *Every three-dimensional ternary Hom-Nambu Lie algebra* $(V, [\cdot,\cdot,\cdot]_\tau, (\alpha_3, \beta))$ *induced by a Hom-Lie algebra with linear endomorphism* α_3 *is either abelian or isomorphic to one of the following ternary Hom-Nambu-Lie algebras with respect to a basis* $\{e_1, e_2, e_3\}$*:*

(i) $(\mathcal{N}_2, (\alpha_3, \beta_i))$, $i = 1, 2, 3$
(ii) $(\mathcal{N}_{4,\lambda}, (\alpha_3, \beta_i))$, $i = 1, 2, 3$
(iii) $(\mathcal{N}_{4,1}, (\alpha_3, \beta_k))$, $k = 4, 5, 6$

(iv) $(\mathcal{N}_5, (\alpha_3, \beta_j)),\ j = 1, \ldots, 6$

where

$$\mathcal{N}_2 : [e_1, e_2, e_3] = e_2, \quad \mathcal{N}_{4,\lambda} : [e_1, e_2, e_3] = e_2 + \lambda\, e_3,\ \lambda \in \mathbb{K}^*, \quad \mathcal{N}_5 : [e_1, e_2, e_3] = e_3$$

$$[\beta_1] = \begin{pmatrix} 0 & \beta_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad [\beta_2] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \beta_{23} \\ 0 & 0 & 0 \end{pmatrix}, \quad [\beta_3] = \begin{pmatrix} 0 & \beta_{12} & 0 \\ 0 & 0 & \beta_{23} \\ 0 & 0 & 0 \end{pmatrix}, \beta_{12} \neq \beta_{23},$$

$$[\beta_4] = \begin{pmatrix} 0 & \beta_{12} & 0 \\ 0 & 0 & \beta_{12} \\ 0 & 0 & 0 \end{pmatrix}, \quad [\beta_5] = \begin{pmatrix} 0 & 0 & \beta_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad [\beta_6] = \begin{pmatrix} 0 & \beta_{12} & \beta_{13} \\ 0 & 0 & \beta_{12} \\ 0 & 0 & 0 \end{pmatrix},$$

$\beta_{12}, \beta_{13}, \beta_{23} \neq 0.$

Proof We use the same notations as used in the previous proof of Theorem 24.2. From (24.10), we first find Φ such that $\Phi \circ \alpha_3 = \alpha_3 \circ \Phi$. Since α_3 is also in Jordan normal form, then $[\Phi] = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ 0 & \varphi_{11} & \varphi_{12} \\ 0 & 0 & \varphi_{11} \end{pmatrix}$. The condition $\Phi \circ \beta = \beta' \circ \Phi$ gives the following equations:

$$\beta_{13}\varphi_{11} + \beta_{12}\varphi_{12} = \beta'_{13}\varphi_{11} + \beta'_{23}\varphi_{12} \tag{24.38}$$
$$\beta_{12}\varphi_{11} = \beta'_{12}\varphi_{11} \tag{24.39}$$
$$\beta_{23}\varphi_{11} = \beta'_{23}\varphi_{11} \tag{24.40}$$

Now $[\beta]$ can be partitioned to the following sub-classes with $\beta_{12}, \beta_{13}, \beta_{23} \neq 0$.

$$[\overline{\beta}_1] = \begin{pmatrix} 0 & \beta_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, [\overline{\beta}_2] = \begin{pmatrix} 0 & 0 & \beta_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, [\overline{\beta}_3] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \beta_{23} \\ 0 & 0 & 0 \end{pmatrix}, [\overline{\beta}_4] = \begin{pmatrix} 0 & \beta_{12} & \beta_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$[\overline{\beta}_5] = \begin{pmatrix} 0 & \beta_{12} & 0 \\ 0 & 0 & \beta_{23} \\ 0 & 0 & 0 \end{pmatrix}, \quad [\overline{\beta}_6] = \begin{pmatrix} 0 & 0 & \beta_{13} \\ 0 & 0 & \beta_{23} \\ 0 & 0 & 0 \end{pmatrix}, \quad [\overline{\beta}_7] = \begin{pmatrix} 0 & \beta_{12} & \beta_{13} \\ 0 & 0 & \beta_{23} \\ 0 & 0 & 0 \end{pmatrix}.$$

We now proceed to find the non-isomorphic classes of β. From equations (24.38) to (24.40) we get the following canonical representatives of β with $\beta_{12}, \beta_{13}, \beta_{23} \neq 0$.

$$[\beta_1] = \begin{pmatrix} 0 & \beta_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad [\beta_2] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \beta_{23} \\ 0 & 0 & 0 \end{pmatrix}; \quad [\beta_3] = \begin{pmatrix} 0 & \beta_{12} & 0 \\ 0 & 0 & \beta_{23} \\ 0 & 0 & 0 \end{pmatrix}, \beta_{12} \neq \beta_{23};$$

$$[\beta_4] = \begin{pmatrix} 0 & \beta_{12} & 0 \\ 0 & 0 & \beta_{12} \\ 0 & 0 & 0 \end{pmatrix}; \quad [\beta_5] = \begin{pmatrix} 0 & 0 & \beta_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad [\beta_6] = \begin{pmatrix} 0 & \beta_{12} & \beta_{13} \\ 0 & 0 & \beta_{12} \\ 0 & 0 & 0 \end{pmatrix}.$$

The isomorphisms of β are given in Table 24.2.

Table 24.2 Isomorphisms of β for ternary Hom-Nambu-Lie algebras with α_3

$[\overline{\beta}_4] \overset{\Phi_{\overline{4,1}}}{\cong} [\overline{\beta}_1]$	$[\Phi_{\overline{4,1}}] = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ 0 & \varphi_{11} & \varphi_{12} \\ 0 & 0 & \varphi_{11} \end{pmatrix}$, $\varphi_{12} = -\frac{\beta_{13}}{\beta_{12}}\varphi_{11}, \beta_{12} = \beta'_{12}$, $\varphi_{11} \neq 0, \varphi_{13} \in \mathbb{K}$
$[\overline{\beta}_6] \overset{\Phi_{\overline{6,3}}}{\cong} [\overline{\beta}_3]$	$[\Phi_{\overline{6,3}}] = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ 0 & \varphi_{11} & \varphi_{12} \\ 0 & 0 & \varphi_{11} \end{pmatrix}$, $\varphi_{12} = \frac{\beta_{13}}{\beta'_{23}}\varphi_{11}, \beta_{23} = \beta'_{23}$, $\varphi_{11} \neq 0, \varphi_{13} \in \mathbb{K}$
$[\overline{\beta}_5] \overset{\Phi_{\overline{5,7}}}{\cong} [\overline{\beta}_7]$ iff $\beta_{12} \neq \beta_{23}$	$[\Phi_{\overline{5,7}}] = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ 0 & \varphi_{11} & \varphi_{12} \\ 0 & 0 & \varphi_{11} \end{pmatrix}$, $\varphi_{12} = \frac{\beta'_{13}}{(\beta_{12}-\beta_{23})}\varphi_{11}, \beta_{12} \neq \beta_{23}$, $\beta_{12} = \beta'_{12}, \beta_{23} = \beta'_{23}$, $\varphi_{11} \neq 0, \varphi_{13} \in \mathbb{K}$

In the next part of the proof we give the canonical representatives of the non-abelian ternary bracket $T^5_{\alpha_3}$ of the induced algebra for each of the β already obtained.

We recall $T^5_{\alpha_3} : [e_1, e_2, e_3] = b_7(C^2_{23}\, e_2 + C^3_{23}\, e_3)$. Expanding (24.9) and (24.10) for this case, we end up with the following equations:

$$b_7\varphi_{11}C^2_{23} = b'_7(\varphi_{11})^3 D^2_{23} \tag{24.41}$$

$$b_7\varphi_{12}C^2_{23} + b_7\varphi_{11}C^3_{23} = b'_7(\varphi_{11})^3 D^3_{23} \tag{24.42}$$

For β_i, $i = 1, 2, 3$, we have the following canonical representatives of $(T^5_{\alpha_3}, (\alpha_3, \beta_i))$:

$$(\mathcal{N}_2, (\alpha_3, \beta_i)), (\mathcal{N}_{4,\lambda}, (\alpha_3, \beta_i)), (\mathcal{N}_5, (\alpha_3, \beta_i))$$

$$\mathcal{N}_2 : [e_1, e_2, e_3] = e_2, \quad \mathcal{N}_{4,\lambda} : [e_1, e_2, e_3] = e_2 + \lambda\, e_3,\ \lambda \in \mathbb{K}^*, \quad \mathcal{N}_5 : [e_1, e_2, e_3] = e_3$$

and since the matrix $[\Phi] = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ 0 & \varphi_{11} & \varphi_{12} \\ 0 & 0 & \varphi_{11} \end{pmatrix}$ must satisfy $\Phi \circ \beta_i = \beta_i \circ \Phi$, $i = 1, 2, 3$, we get that $\varphi_{12} = 0$, and the isomorphisms are given by

$$[\Phi] = \begin{pmatrix} \varphi_{11} & 0 & \varphi_{13} \\ 0 & \varphi_{11} & 0 \\ 0 & 0 & \varphi_{11} \end{pmatrix}, \begin{array}{l} \mathcal{N}_2 : (\varphi_{11})^2 = b_7 C_{23}^2, C_{23}^2 \neq 0, C_{23}^3 = 0 \\ \mathcal{N}_{4,\lambda} : (\varphi_{11})^2 = b_7 C_{23}^2, \lambda C_{23}^2 = C_{23}^3 \neq 0 \\ \mathcal{N}_5 : (\varphi_{11})^2 = b_7 C_{23}^3, C_{23}^2 = 0, C_{23}^3 \neq 0 \end{array}, \varphi_{13} \in \mathbb{K}.$$

For $\beta_i, i = 4, 5, 6$ we have the following canonical representatives of $(T^5_{\alpha_3}, (\alpha_3, \beta_i))$:

$$(\mathcal{N}_{4,1}, (\alpha_3, \beta_i)), (\mathcal{N}_5, (\alpha_3, \beta_i))$$

$$\mathcal{N}_{4,1} : [e_1, e_2, e_3] = e_2 + e_3, \quad \mathcal{N}_5 : [e_1, e_2, e_3] = e_3$$

and the isomorphism given by

$$[\Phi] = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ 0 & \varphi_{11} & \varphi_{12} \\ 0 & 0 & \varphi_{11} \end{pmatrix}, \begin{array}{l} \mathcal{N}_{4,1} : (\varphi_{11})^2 = b_7 C_{23}^2, \varphi_{12} = \frac{\varphi_{11}(C_{23}^2 - C_{23}^3)}{C_{23}^2}, \\ \qquad C_{23}^2 \neq 0, \varphi_{13} \in \mathbb{K} \\ \mathcal{N}_5 : (\varphi_{11})^2 = b_7 C_{23}^3, C_{23}^2 = 0, C_{23}^3 \neq 0, \varphi_{12}, \varphi_{13} \in \mathbb{K} \end{array}.$$

In addition, ternary Hom-Nambu-Lie algebras with β commuting with β_1, β_2 and β_3 are isomorphic to the corresponding canonical representatives of the Hom-Nambu Lie algebras with β_1, β_2 and β_3 respectively. □

Remark 24.1 The parametric families of canonical representatives in Theorems 24.2 and 24.3 are isomorphic if and only if all the involved parameters are equal. This includes the parameters of β as well.

It is possible to have a Hom-Lie algebra morphism still be a morphism of the induced ternary Hom-Nambu-Lie algebras. More generally, there are morphisms of n-Hom-Lie algebras that still remain morphisms of induced $(n+1)$-Hom-Lie algebras. We first give the definition of an n-ary Hom-Nambu-Lie algebra.

Definition 24.6 An n-ary Hom-Nambu-Lie algebra is a vector space V together with an n-linear map $[\cdot, \ldots, \cdot] : V^n \to V$ and $(n-1)$ linear maps $\alpha_i : A \to A$, $1 \leq i \leq n-1$ satisfying, for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in V$, $\sigma \in S_n$,

Skew-symmetry: $[x_{\sigma(1)}, \ldots, x_{\sigma(n)}] = \mathrm{Sgn}(\sigma)[x_1, \ldots, x_n]$

Hom-Nambu Identity:

$$[\alpha_1(x_1), \alpha_{n-1}(x_{n-1}), [y_1, \ldots, y_n]]$$
$$= \sum_{i=1}^{n} [\alpha_1(y_1), \ldots, \alpha_{i-1}(y_{i-1}), [x_1, \ldots, x_{n-1}, y_i], \alpha_i(y_{i+1}), \ldots, \alpha_{n-1}(y_n)].$$

The following result is proved in [15].

Proposition 24.2 *Let* $(A, \mu, (\alpha_1, \ldots, \alpha_{n-1}))$ *and* $(A', \mu', (\beta_1, \ldots, \beta_{n-1}))$ *be* n*-Hom-Lie algebras. Let* τ *(resp.* τ'*) be a* μ*-trace (resp.* μ'*- trace) and* $\alpha_n : A \to A$ *(resp.* $\beta_n : A' \to A'$*) be linear maps. Set* $(A, \mu_\tau, (\alpha_1, \ldots, \alpha_n))$ *and* $(A', \mu'_{\tau'}, (\beta_1,$

$\ldots, \beta_n))$ to be the induced $(n+1)$-Hom-Lie algebras. Let $\Phi : A \to A'$ be an n-Hom-Lie algebra morphism satisfying $\tau' \circ \Phi = \tau$ and $\Phi \circ \alpha_n = \beta_n \circ \Phi$, then Φ is an $(n+1)$-Hom-Lie algebra morphism of the induced algebras.

In the next results in Propositions 24.3 and 24.4, we see that it is possible to have the Hom-Lie algebra morphism still be a morphism of the induced ternary Hom-Nambu-Lie algebras with $\tau' \circ \Phi = \tau$ not satisfied. However, this is possible having extra conditions added to conditions provided in Proposition 24.2.

Proposition 24.3 *Let $(A, \mu, (\alpha_1, \ldots, \alpha_{n-1}))$ and $(A', \mu', (\beta_1, \ldots, \beta_{n-1}))$ be n-Hom-Lie algebras. Let τ (resp. τ') be a μ-trace (resp. μ'-trace) and $\alpha_n : A \to A$ (resp. $\beta_n : A' \to A'$) be linear maps. Let $(A, \mu_\tau, (\alpha_1, \ldots, \alpha_n))$ and $(A', \mu'_{\tau'}, (\beta_1, \ldots, \beta_n))$ be the induced $(n+1)$-Hom-Lie algebras, and $\Phi : A \to A'$ be an n-Hom-Lie algebra morphism. Let $(e_i)_{1 \le i \le \dim A}$ be a basis of A. If for all i such that $\tau'(\Phi(e_i)) \ne \tau(e_i)$ we have $\mu(e_{j_1}, \ldots, e_{j_n}) \in \ker(\Phi)$ for all $j_t \ne i, t = 1, \ldots, n$, and $\Phi \circ \alpha_n = \beta_n \circ \Phi$, then Φ is an $(n+1)$-Hom-Lie algebra morphism of the induced algebras. In particular, if $\mu(e_{j_1}, \ldots, e_{j_n}) = 0$ or $\{e_{j_1}, \ldots, e_{j_n}\} \cap \ker(\Phi) \ne 0$ then the result will hold.*

Proof Let $e_{i_1}, \ldots, e_{i_p}$ be the basis elements with $\tau'(\Phi(e_{i_k})) \ne \tau(e_{i_k})$, for $1 \le k \le p$, and let $e_{j_1}, \ldots, e_{j_q}$ be the basis elements such that $\tau'(\Phi(e_{j_k})) = \tau(e_{j_k})$, for $1 \le k \le q$, with $p + q = n + 1$. Then,

$$
\begin{aligned}
&\Phi(\mu_\tau(e_{i_1}, \ldots, e_{i_p}, e_{j_1}, \ldots, e_{j_q}))\\
&= \sum_{k=1}^{p} (-1)^{k-1} \tau(e_{i_k}) \Phi(\mu(e_{i_1}, \ldots, \widehat{e_{i_k}}, \ldots, e_{i_p}, e_{j_1}, \ldots, e_{j_q}))\\
&+ \sum_{k=1}^{q} (-1)^{p+k-1} \tau(e_{j_k}) \Phi(\mu(e_{i_1}, \ldots, e_{i_p}, e_{j_1}, \ldots, \widehat{e_{j_k}}, \ldots, e_{j_q}))\\
&= \sum_{k=1}^{p} (-1)^{k-1} \tau(e_{i_k})\, 0\\
&+ \sum_{k=1}^{q} (-1)^{p+k-1} \tau'(\Phi(e_{j_k})) \mu'(\Phi(e_{i_1}), \ldots, \Phi(e_{i_p}), \Phi(e_{j_1}), \ldots, \widehat{\Phi(e_{j_k})}, \ldots, \Phi(e_{j_q}))\\
&= \sum_{k=1}^{p} (-1)^{k-1} \tau'(\Phi(e_{i_k}))\, 0\\
&+ \sum_{k=1}^{q} (-1)^{p+k-1} \tau'(\Phi(e_{j_k})) \mu'(\Phi(e_{i_1}), \ldots, \Phi(e_{i_p}), \Phi(e_{j_1}), \ldots, \widehat{\Phi(e_{j_k})}, \ldots, \Phi(e_{j_q}))\\
&= \sum_{k=1}^{p} (-1)^{k-1} \tau'(\Phi(e_{i_k})) \Phi(\mu(e_{i_1}, \ldots, \widehat{e_{i_k}}, \ldots, e_{i_p}, e_{j_1}, \ldots, e_{j_q}))\\
&+ \sum_{k=1}^{q} (-1)^{p+k-1} \tau'(\Phi(e_{j_k})) \mu'(\Phi(e_{i_1}), \ldots, \Phi(e_{i_p}), \Phi(e_{j_1}), \ldots, \widehat{\Phi(e_{j_k})}, \ldots, \Phi(e_{j_q}))\\
&= \mu'_{\tau'}(\Phi(e_{i_1}), \ldots, \Phi(e_{i_p}), \Phi(e_{j_1}), \ldots, \Phi(e_{j_q})),
\end{aligned}
$$

where $\widehat{e_{i_k}}$ means that e_{i_k} is omitted. □

Corollary 24.2 *Let (A, μ, α) and (A', μ', β) be Hom-Lie algebras. Let τ (resp. τ') be a μ-trace (resp. μ'-trace) and $\alpha' : A \to A$ (resp. $\beta' : A' \to A'$) a linear map. Set $(A, \mu_\tau, (\alpha, \alpha'))$ and $(A', \mu'_{\tau'}, (\beta, \beta'))$ to be induced ternary Hom-Nambu-Lie algebras. Let $\Phi : A \to A'$ be a Hom-Lie algebra morphism. If for all i such that $\tau(e_i) \neq \tau'(\Phi(e_i))$ we have $\mu(e_j, e_k) \in \ker(\Phi)$, for all $j, k \neq i$, with $\tau'(\Phi(e_j)) = \tau(e_j)$, $\tau'(\Phi(e_k)) = \tau(e_k)$ and $\Phi \circ \alpha' = \beta' \circ \Phi$, then Φ is an algebra morphism of the induced algebras. In particular, if $\mu(e_j, e_k) = 0$ or $\{e_j, e_k\} \cap \ker(\Phi) \neq 0$ then the result still holds.*

Let us denote by C the matrix of structure constants of the bilinear map of a 3-dimensional Hom-Lie algebras with basis $\{e_1, e_2, e_3\}$ and structure constants $\{C_{ij}^k\}_{i<j}, i, j, k = 1, 2, 3$:

$$C = \begin{pmatrix} C_{12}^1 & C_{12}^2 & C_{12}^3 \\ C_{13}^1 & C_{13}^2 & C_{13}^3 \\ C_{23}^1 & C_{23}^2 & C_{23}^3 \end{pmatrix}.$$

Proposition 24.4 *Let (A, μ, α) and (A', μ', β) be 3-dimensional Hom-Lie algebras and let C and D be the respective matrices of structure constants of μ and μ'. Let τ (resp. τ') be a μ-trace (resp. μ'-trace) and α' (resp. β') a linear map $\alpha' : A \to A$ (resp. $\beta' : A' \to A'$). Set $(A, \mu_\tau, (\alpha, \alpha'))$ and $(A', \mu'_{\tau'}, (\beta, \beta'))$ to be induced ternary Hom-Nambu-Lie algebras. Let $\Phi : A \to A'$ be a Hom-Lie algebra morphism which does not satisfy $\tau' \circ \Phi = \tau$.*

(i) *If Φ is a ternary Hom-Nambu-Lie algebra morphism of the induced algebras then* $\det C = 0$.
(ii) *If moreover Φ is a Hom-Lie algebra isomorphism then, if Φ is a ternary Hom-Nambu-Lie algebra isomorphism of the induced algebras then* $\det C = 0$ *and* $\det D = 0$.

Proof For the basis $\{e_1, e_2, e_3\}$ we have

$$\begin{aligned}
&\Phi(\mu_\tau(e_1, e_2, e_3)) = \tau(e_1)\Phi(\mu(e_2, e_3)) - \tau(e_2)\Phi(\mu(e_1, e_3)) + \tau(e_3)\Phi(\mu(e_1, e_2)) \\
&= \Phi\big(\tau(e_1)\mu(e_2, e_3) - \tau(e_2)\mu(e_1, e_3) + \tau(e_3)\mu(e_1, e_2)\big), \\
&\qquad\qquad (\text{Since } \tau(\Phi(e_i)) \in \mathbb{K} \text{ and } \Phi \text{ is linear}) \\
&\mu'_{\tau'}(\Phi(e_1), \Phi(e_2), \Phi(e_3)) = \tau'(\Phi(e_1))\mu'(\Phi(e_2), \Phi(e_3)) \\
&\qquad - \tau'(\Phi(e_2))\mu'(\Phi(e_1), \Phi(e_3)) + \tau'(\Phi(e_3))\mu'(\Phi(e_1), \Phi(e_2)) \\
&= \tau'(\Phi(e_1))\Phi(\mu(e_2, e_3)) - \tau'(\Phi(e_2))\Phi(\mu(e_1, e_3)) + \tau'(\Phi(e_3))\Phi(\mu(e_1, e_2)) \\
&= \Phi\big(\tau'(\Phi(e_1))\mu(e_2, e_3) - \tau'(\Phi(e_2))\mu(e_1, e_3) + \tau'(\Phi(e_3))\mu(e_1, e_2)\big). \\
&\qquad\qquad (\text{Since } \tau'(\Phi(e_i)) \in \mathbb{K} \text{ and } \Phi \text{ is linear})
\end{aligned}$$

If $\Phi(\mu_\tau(e_1, e_2, e_3)) = \mu'_{\tau'}(\Phi(e_1), \Phi(e_2), \Phi(e_3))$, then

$$\begin{aligned}
&\Phi(\mu_\tau(e_1, e_2, e_3)) - \mu'_{\tau'}(\Phi(e_1), \Phi(e_2), \Phi(e_3)) = 0\\
\Longrightarrow\ &\Phi[(\tau(e_1)\mu(e_2, e_3) - \tau(e_2)\mu(e_1, e_3) + \tau((e_3))\mu(e_1, e_2))-\\
&(\tau'(\Phi(e_1))\mu(e_2, e_3) - \tau'(\Phi(e_2))\mu(e_1, e_3) + \tau'(\Phi(e_3))\mu(e_1, e_2))] = 0\\
\Longrightarrow\ &(\tau(e_1) - \tau'(\Phi(e_1)))\mu(e_2, e_3) - (\tau(e_2) - \tau'(\Phi(e_2)))\mu(e_1, e_3)\\
&\qquad + (\tau((e_3)) - \tau'(\Phi(e_3)))\mu(e_1, e_2)) = 0.
\end{aligned}$$

Let $\sum_{\circlearrowleft(i,j,k)}$ denote the summation over the cyclic permutations on $\{i, j, k\}$.

Replacing $\mu(e_j, e_k) = \sum_{l=1}^{3} C_{jk}^l e_l$, we get the following equations:

$$\sum_{\circlearrowleft(i,j,k)} (\tau(e_i) - \tau'(\Phi(e_i)))C_{jk}^l = 0 \text{ for } l = 1, 2, 3$$

and $\{i, j, k\}$ ordered as $\{1, 2, 3\}$. Writing up the equations in matrix form:

$$C^T \begin{bmatrix} (\tau(e_3) - \tau'(\Phi(e_3))) \\ (\tau'(\Phi(e_2)) - \tau(e_2)) \\ (\tau(e_1) - \tau'(\Phi(e_1))) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Hence, $\tau \neq \tau' \circ \Phi \implies \det C = 0$.

Suppose now that, on top of the previous hypotheses, Φ is invertible, we show that the same holds for μ' with structure constants D_{ij}^k. Denote the matrix of such structure constants by

$$D = \begin{pmatrix} D_{12}^1 & D_{12}^2 & D_{12}^3 \\ D_{13}^1 & D_{13}^2 & D_{13}^3 \\ D_{23}^1 & D_{23}^2 & D_{23}^3 \end{pmatrix}.$$

Then by substitutions in order to eliminate μ instead, we get

$$\begin{aligned}
&\Phi(\mu_\tau(e_1, e_2, e_3)) = \tau(e_1)\Phi(\mu(e_2, e_3)) - \tau(e_2)\Phi(\mu(e_1, e_3)) + \tau(e_3)\Phi(\mu(e_1, e_2))\\
&= \tau(e_1)\mu'(\Phi(e_2), \Phi(e_3)) - \tau(e_2)\mu'(\Phi(e_1), \Phi(e_3)) + \tau(e_3)\mu'(\Phi(e_1), \Phi(e_2)),\\
&\mu'_{\tau'}(\Phi(e_1), \Phi(e_2), \Phi(e_3)) = \tau'(\Phi(e_1))\mu'(\Phi(e_2), \Phi(e_3))\\
&\qquad - \tau'(\Phi(e_2))\mu'(\Phi(e_1), \Phi(e_3)) + \tau'(\Phi(e_3))\mu'(\Phi(e_1), \Phi(e_2)).
\end{aligned}$$

If $\Phi(\mu_\tau(e_1, e_2, e_3)) = \mu'_{\tau'}(\Phi(e_1), \Phi(e_2), \Phi(e_3))$, then by similar computations as for C,

$$D^T P \begin{bmatrix} (\tau(e_3) - \tau'(\Phi(e_3))) \\ (\tau'(\Phi(e_2)) - \tau(e_2)) \\ (\tau(e_1) - \tau'(\Phi(e_1))) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$P = \begin{pmatrix} \varphi_{21}\varphi_{32} - \varphi_{22}\varphi_{31} & \varphi_{11}\varphi_{32} - \varphi_{12}\varphi_{31} & \varphi_{11}\varphi_{22} - \varphi_{12}\varphi_{21} \\ \varphi_{21}\varphi_{33} - \varphi_{23}\varphi_{31} & \varphi_{11}\varphi_{33} - \varphi_{13}\varphi_{31} & \varphi_{11}\varphi_{23} - \varphi_{13}\varphi_{21} \\ \varphi_{22}\varphi_{33} - \varphi_{23}\varphi_{32} & \varphi_{12}\varphi_{33} - \varphi_{13}\varphi_{32} & \varphi_{12}\varphi_{23} - \varphi_{13}\varphi_{22} \end{pmatrix}.$$

We have $\det P = (\det[\Phi])^2 \neq 0$. Hence $\tau \neq \tau' \circ \Phi \implies \det D = 0$. □

24.5 Some examples

Example 24.1 We give an example of Proposition 24.4. We take two Hom-Lie algebras that are isomorphic and show that their induced ternary Hom-Nambu-Lie algebras can be isomorphic with the condition $\tau = \tau' \circ \Phi$ not satisfied. Let the two Hom-Lie algebras be given as $(H, [\cdot,\cdot], \alpha_1)$ and $(H', [\cdot,\cdot]', \alpha_2)$, where $[\alpha_1] = \begin{pmatrix} 0&1&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}$, $[\alpha_2] = \begin{pmatrix} 0&0&0 \\ 0&0&1 \\ 0&0&0 \end{pmatrix}$, with the binary brackets given respectively as

$$\begin{aligned} [e_1, e_2] &= C_{12}^1\, e_1 + C_{12}^2\, e_2 & [f_1, f_2]' &= D_{12}^2\, f_2 + D_{12}^3\, f_3 \\ [e_1, e_3] &= C_{13}^1\, e_1 + C_{13}^2\, e_2 & [f_1, f_3]' &= D_{13}^3\, f_3 \\ [e_2, e_3] &= C_{23}^2\, e_2 & [f_2, f_3]' &= D_{23}^2\, f_2 + D_{23}^3\, f_3 \end{aligned}.$$

The Hom-Lie algebra isomorphism given by $\Phi(e_1) = f_2$, $\Phi(e_2) = f_3$, $\Phi(e_3) = f_1$, satisfies $\Phi \circ \alpha_1 = \alpha_2 \circ \Phi$. That is, $[\Phi] = \begin{pmatrix} 0&1&0 \\ 0&0&1 \\ 1&0&0 \end{pmatrix}$, with the following conditions on the structure constants of H and H':

$$C_{12}^1 = D_{23}^2,\ C_{12}^2 = D_{23}^3,\ C_{13}^1 = -D_{12}^2,\ C_{13}^2 = -D_{12}^3,\ C_{23}^2 = -D_{13}^3.$$

Let $(H, [\cdot,\cdot,\cdot]_\tau, (\alpha_1, \beta_1))$ and $(H', [\cdot,\cdot,\cdot]'_\tau, (\alpha_2, \beta_2))$ be the induced ternary Hom-Nambu Lie algebras. In general, the ternary brackets are given by

$$[e_1, e_2, e_3]_\tau = b_1(C_{12}^1\, e_1 + C_{12}^2\, e_2)$$

with $[\beta_1] = \begin{pmatrix} \beta_{11} & \beta_{12} & 0 \\ \beta_{21} & \beta_{22} & 0 \\ \beta_{31} & \beta_{32} & 0 \end{pmatrix}$ and $\tau(e_1) = 0,\ \tau(e_2) = 0,\ \tau(e_3) = b_1$, for all $b_1 \in \mathbb{K}^*$

and

$$[f_1, f_2, f_3]'_{\tau'} = b'_1(D_{23}^2\, f_2 + D_{23}^3\, f_3)$$

with $[\beta_2] = \begin{pmatrix} 0 & \beta_{12} & \beta_{13} \\ 0 & \beta_{22} & \beta_{23} \\ 0 & \beta_{32} & \beta_{33} \end{pmatrix}$ and $\tau'(f_1) = b'_1,\ \tau'(f_2) = 0,\ \tau'(f_3) = 0$, for all $b'_1 \in \mathbb{K}^*$.

Take $\beta_1 = \alpha_1$ and $\beta_2 = \alpha_2$. The induced Hom-Nambu Lie algebras can be isomorphic under this Φ with $\tau(e_1) = \tau'(\Phi(e_1)), \tau(e_2) = \tau'(\Phi(e_2))$ but $\tau(e_3) \neq \tau'(\Phi(e_3))$, when $b_1 \neq b_1'$. Moreover, if we define C and D as given before, we see that $\det C = \det D = 0$.

Example 24.2 We give examples of the two particular cases of Corollary 24.2. Let the two Hom-Lie algebras be given as $(H, [\cdot,\cdot], \alpha_1)$ and $(H', [\cdot,\cdot]', \alpha_2)$, where $[\alpha_1] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $[\alpha_2] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, with the binary brackets given respectively as

$$\begin{array}{ll} [e_1, e_2] = C_{12}^2\, e_2 & [f_1, f_2]' = D_{12}^3\, f_3 \\ [e_1, e_3] = C_{13}^2\, e_2 & [f_1, f_3]' = D_{13}^3\, f_3 \\ [e_2, e_3] = C_{23}^2\, e_2 & [f_2, f_3]' = D_{23}^3\, f_3 \end{array}.$$

The Hom-Lie algebra isomorphism given by $\Phi(e_1) = f_2, \Phi(e_2) = f_3, \Phi(e_3) = f_1$, satisfies $\Phi \circ \alpha_1 = \alpha_2 \circ \Phi$, with the following conditions on the structure constants of H and H':

$$C_{12}^2 = D_{23}^3,\ C_{13}^2 = -D_{12}^3,\ C_{23}^2 = -D_{13}^3$$

Let $(H, [\cdot,\cdot,\cdot]_\tau, (\alpha_1, \beta_1))$ and $(H', [\cdot,\cdot,\cdot]'_\tau, (\alpha_2, \beta_2))$ be the induced ternary Hom-Nambu Lie algebras. In general, the ternary brackets are given by

$$[e_1, e_2, e_3]_\tau = (b_3 C_{23}^2 + b_4 C_{12}^2)\, e_2$$

with $[\beta_1] = \begin{pmatrix} \beta_{11} & \beta_{12} & \lambda\beta_{11} \\ 0 & \beta_{22} & 0 \\ \beta_{31} & \beta_{32} & \lambda\beta_{31} \end{pmatrix}$ and $\tau(e_1) = b_3,\ \tau(e_2) = 0,\ \tau(e_3) = b_4, \lambda = -\dfrac{b_3}{b_4}$ for all $b_3, b_4 \in \mathbb{K}^*$, and

$$[f_1, f_2, f_3]'_{\tau'} = (b_3' D_{13}^3 + b_4' D_{23}^3)\, f_3$$

with $[\beta_2] = \begin{pmatrix} \beta_{11} & \gamma\beta_{11} & \beta_{13} \\ \beta_{21} & \gamma\beta_{21} & \beta_{23} \\ 0 & 0 & \beta_{31} \end{pmatrix}$, and $\tau'(f_1) = b_4',\ \tau'(f_2) = b_3',\ \tau'(f_3) = 0, \gamma = -\dfrac{b_4'}{b_3'}$ for all $b_3', b_4' \in \mathbb{K}^*$.

Take $\beta_1 = \alpha_1$ and $\beta_2 = \alpha_2, b_3 = b_3'$ and $b_4 \neq b_4'$. Suppose that $[e_1, e_2] = 0$. This means $C_{12}^2 = 0$, which leads to $D_{23}^3 = 0$. Then the induced Hom-Nambu Lie algebras can be isomorphic under this Φ with $\tau(e_1) = \tau'(\Phi(e_1)), \tau(e_2) = \tau'(\Phi(e_2))$ but $\tau(e_3) \neq \tau'(\Phi(e_3))$.

Now using this example, but with a different morphism defined by $\Phi(e_1) = f_3$, $\Phi(e_2) = 0, \Phi(e_3) = f_1$, that is, $e_2 \in \ker \Phi$. If we again take $\beta_1 = \alpha_1, \beta_2 = \alpha_2$ and let $b_4 = b_4'$, we have Φ a morphism of the induced ternary Hom-Nambu-Lie algebras with $\tau(e_1) \neq \tau'(\Phi(e_1)), \tau(e_2) = \tau'(\Phi(e_2))$ and $\tau(e_3) = \tau'(\Phi(e_3))$.

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Chapter 25
Commutants in Crossed Products for Piecewise Constant Function Algebras Related to Multiresolution Analysis

Sergei Silvestrov and Alex Behakanira Tumwesigye

Abstract In this paper we consider crossed product algebras of piecewise constant function algebras on the real line that arise in multiresolution analysis. Such algebras form an increasing sequence of algebras of functions on the real line. We derive conditions under which these algebras are invariant under a bijection on the real line, in which case we get an increasing sequence of crossed product algebras. We then give a comparison of commutants (centralizers) in a number of cases.

Keywords Crossed product algebra · Multiresolution analysis · Commutant

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25.1 Introduction

An important direction of investigation for any class of non-commutative algebras and rings, is the description of commutative subalgebras and commutative subrings. This is because such a description allows one to relate representation theory, non-commutative properties, graded structures, ideals and subalgebras, homological and other properties of non-commutative algebras to spectral theory, duality, algebraic geometry and topology naturally associated with commutative algebras. In representation theory, for example, semi-direct products or crossed products play a central role in the construction and classification of representations using the method of induced representations. When a non-commutative algebra is given, one looks for

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a subalgebra such that its representations can be studied and classified more easily and such that the whole algebra can be decomposed as a crossed product of this subalgebra by a suitable action.

When one has found a way to present a non-commutative algebra as a crossed product of a commutative subalgebra by some action on it, then it is important to know whether the subalgebra is maximal commutative, or if not, to find a maximal commutative subalgebra containing the given subalgebra. This maximality of a commutative subalgebra and related properties of the action are intimately related to the description and classification of representations of the non-commutative algebra.

Some work has been done in this direction [5, 10, 11] where the interplay between topological dynamics of the action on one hand and the algebraic property of the commutative subalgebra in the C^*-crossed product algebra $C(X) \rtimes \mathbb{Z}$ being maximal commutative on the other hand are considered. In [10], an explicit description of the (unique) maximal commutative subalgebra containing a subalgebra $\mathcal{A}$ of $\mathbb{C}^X$ is given. In [13], properties of commutative subrings and ideals in non-commutative algebraic crossed products by arbitrary groups are investigated and a description of the commutant of the base coefficient subring in the crossed product ring is given. More results on commutants in crossed products and dynamical systems can be found in [2, 7, 9] and the references therein.

In this article, we consider crossed product algebras for piecewise constant function algebras that form a multiresolution analysis in $L^2(\mathbb{R})$. Such multiresolution analysis is a basis for wavelet analysis and signal processing. Work in this direction can be found, for example, in [1, 3, 4, 6]. Commutants for the coefficient algebra in crossed products for piecewise constant function algebras have been studied in [7, 8], but in a different setting to the one here.

The paper is arranged as follows: After the introduction in Sect. 25.1, we give general definitions and preliminary notions about crossed product algebras in Sect. 25.2. In Sect. 25.3, we give an explicit description of the commutant $C(\mathcal{A}_0)$, of the algebra $\mathcal{A}_j$ of functions which are constant on intervals of the form $I_t = \big[2^{-j}t, 2^{-j}(t+1)\big)$ for every $t \in \mathbb{Z}$, in the crossed product algebra $\mathcal{A}_j \rtimes_{\tilde{\sigma}} \mathbb{Z}$. In Sect. 25.4, we give a comparison of commutants $C(\mathcal{A}_j),\ j \in \mathbb{Z}$ for the algebras $\mathcal{A}_j$ which arise in multiresolution in $L^2(\mathbb{R})$, in a number of cases.

25.2 Definitions and Preliminary Results

In this section we give general notions about algebraic crossed products. This section is based on [7, 10].

25.2.1 Algebraic Crossed Products

Let $\mathcal{A}$ be any commutative algebra. Using the notation in [10], we let $\phi : \mathcal{A} \to \mathcal{A}$ be any algebra automorphism on $\mathcal{A}$ and define

$$\mathcal{A} \rtimes_\phi \mathbb{Z} := \{f : \mathbb{Z} \to \mathcal{A} : f(n) = 0 \text{ except for a finite number of } n\}.$$

It has been proven in [10] that $\mathcal{A} \rtimes_\phi \mathbb{Z}$ is an associative $\mathbb{C}-$ algebra with respect to point-wise addition, scalar multiplication and multiplication defined by *twisted convolution*, $*$ as follows:

$$(f * g)(n) = \sum_{k\in\mathbb{Z}} f(k).\phi^k(g(n-k)),$$

where ϕ^k denotes the $k-$fold composition of ϕ with itself for positive k and we use the obvious definition for $k \leq 0$.

Definition 25.1 The algebra $\mathcal{A} \rtimes_\phi \mathbb{Z}$ defined above is called the crossed product algebra of $\mathcal{A}$ and $\mathbb{Z}$ under ϕ.

A useful and convenient way of working with $\mathcal{A} \rtimes_\phi \mathbb{Z}$, is to write elements $f, g \in \mathcal{A} \rtimes_\phi \mathbb{Z}$ in the form $f = \sum_{n\in\mathbb{Z}} f_n\delta^n$ and $g = \sum_{n\in\mathbb{Z}} g_m\delta^m$ where $f_n = f(n),\ g_m = g(m)$ and

$$\delta^n(k) = \begin{cases} 1, & \text{if } k = n \\ 0, & \text{if } k \neq n. \end{cases}$$

Then addition and scalar multiplication are canonically defined and multiplication is determined by the relation

$$(f_n\delta^n) * (g_m\delta^m) = f_n\phi^n(g_m)\delta^{n+m}$$

where $m, n \in \mathbb{Z}$ and $f_n, g_m \in \mathcal{A}$.

Definition 25.2 By the commutant (centralizer) $C(\mathcal{A})$ of $\mathcal{A}$ in $\mathcal{A} \rtimes_\phi \mathbb{Z}$ we mean

$$C(\mathcal{A}) := \{f \in \mathcal{A} \rtimes_\phi \mathbb{Z} : fg = gf \text{ for every } g \in \mathcal{A}\}.$$

It has been proven [10] that the commutant $\mathcal{A}'$ is commutative and thus, is the unique maximal commutative subalgebra containing $\mathcal{A}$.

25.2.2 Automorphisms Induced by Bijections

Now let X be any set and $\mathcal{A}$ an algebra of complex valued functions on X. Let $\sigma : X \to X$ be any bijection such that $\mathcal{A}$ is invariant under σ and σ^{-1}, that is for every $h \in \mathcal{A}$, $h \circ \sigma \in \mathcal{A}$ and $h \circ \sigma^{-1} \in \mathcal{A}$. Then (X, σ) is a discrete dynamical system and σ induces an automorphism $\tilde{\sigma} : \mathcal{A} \to \mathcal{A}$ defined by,

$$\tilde{\sigma}(f) = f \circ \sigma^{-1}. \tag{25.1}$$

Therefore we can consider the crossed product algebra $\mathcal{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$. Studies have been done on this algebraic crossed product and crossed product algebras in general, about maximal commutativity of $\mathcal{A}$ and other properties of the commutant of $\mathcal{A}$ in the crossed product algebra $\mathcal{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$ [7, 8, 10, 11, 13]. The following definition, which first appeared in [10], gives a description of sets which are crucial in the study of commutatnts this algebraic crossed product.

Definition 25.3 For any nonzero $n \in \mathbb{Z}$, we set

$$Sep^n_{\mathcal{A}}(X) := \left\{x \in X \mid \exists\, h \in \mathcal{A} \;:\; h(x) \neq \tilde{\sigma}^n(h)(x)\right\}.$$

The following theorem has been proven in [10].

Theorem 25.1 *The unique maximal commutative subalgebra of $\mathcal{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$ that contains $\mathcal{A}$ is precisely the set of elements*

$$C(\mathcal{A}) = \left\{ \sum_{n \in \mathbb{Z}} f_n \delta^n \;:\; (\textit{for all } n \in \mathbb{Z}),\; f_n|_{Sep^n_{\mathcal{A}}(X)} \equiv 0 \right\}.$$

25.3 Crossed Products for Piecewise Constant Function Algebras and Multiresolution Analysis

25.3.1 Crossed Products for Piecewise Constant Function Algebras

Let X be any set, J a countable set and $\mathbb{P} = \{X_j \;:\; j \in J\}$ be a partition of X, that is $X = \bigcup_{r \in J} X_r$ where $X_r \neq \emptyset\ \forall\, r \in J$ and $X_r \cap X_{r'} = \emptyset$ if $r \neq r'$.

Let $\mathcal{A}$ be the algebra of piecewise constant, complex-valued functions on X, that is

$$\mathcal{A} = \{h \in \mathbb{C}^X \;:\; \text{for every } j \in J \;:\; h(X_j) = \{c_j\}\}, \tag{25.2}$$

for some $c_j \in \mathbb{C}$ and let $\sigma : X \to X$ be a bijection on X. The lemma below [7, Lemma 3.1] gives the necessary and sufficient conditions under which $\mathcal{A}$ is invariant under σ.

Lemma 25.1 *Let $\sigma : X \to X$ be a bijection. Then the following are equivalent.*

(i) *The algebra $\mathcal{A}$ is invariant under σ and σ^{-1}.*
(ii) *For every $i \in J$ there exists $j \in J$ such that $\sigma(X_i) = X_j$.*

Proof We recall that the algebra $\mathcal{A}$ is invariant under σ if and only if for every $h \in \mathcal{A},\ h \circ \sigma \in \mathcal{A}$.

Obviously, if for every $i \in J$ there exists a unique $j \in J$ such that $\sigma(X_i) = X_j$, then

$$(h \circ \sigma)(X_i) = h(\sigma(X_i)) = h(X_j) = \{c_j\},$$

for some $c_j \in \mathbb{C}$. Therefore, $h \circ \sigma \in \mathcal{A}$.

Conversely, suppose $\mathcal{A}$ is invariant under σ but 2. does not hold. Let $x_1, x_2 \in X_j$ and $X_r, X_{r'} \in \mathbb{P}$ such that $\sigma(x_1) \in X_r$ and $\sigma(x_2) \in X_{r'}$. Let $h : X \to \mathbb{C}$ be the function defined by

$$h(x) = \begin{cases} 1 & \text{if } x \in X_r \\ 0 & \text{otherwise} \end{cases}$$

Then $h \in \mathcal{A}$. But $h \circ \sigma(x_1) = 1$ and $h \circ \sigma(x_2) = 0$. Thus $h \circ \sigma \notin \mathcal{A}$ which contradicts the assumption. A similar proof can be done to show that $\mathcal{A}$ is invariant under σ^{-1} if and only if condition 2. holds.

We let $\tilde{\sigma} : \mathcal{A} \to \mathcal{A}$ be the automorphism induced by σ, as defined by (25.1), and consider the crossed product algebra $\mathcal{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$. Some studies have been done regarding properties of maximal commutative subalgebras containing $\mathcal{A}$, in this crossed algebra [7, 8, 12]. In the next section we focus on particular piecewise constant algebra functions on the real line.

25.3.2 *Piecewise Constant Function Algebras Generated by the Haar Scaling Function*

In this section we let $X = \mathbb{R}$ and we consider special piecewise constant function algebras defined from the Haar scaling function. To this end we have the following.

Let $\phi : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\phi(x) := \begin{cases} 1, & \text{if } 0 \leqslant x < 1 \\ 0 & \text{otherwise} \end{cases},$$

and for every $j \in \mathbb{Z}$, let

$$\mathcal{A}_j := \left\{ h \in L^2(\mathbb{R}) \text{ such that } h(x) = \sum_{t \in \mathbb{Z}} a_t \phi(2^j x - t),\ \ a_t \in \mathbb{R} \right\}. \tag{25.3}$$

Then $\mathcal{A}_j$ consists of functions (step functions) which are constant on intervals of the form $I_t = \left[2^{-j}t, 2^{-j}(t+1)\right)$ for every $t \in \mathbb{Z}$. Note that for $j < 0$, the intervals I_t have length greater than 1. Clearly, $\mathcal{A}_j$ is an algebra of functions with respect to the pointwise operations of addition, scalar multiplication and multiplication.

Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a bijection such that $\mathcal{A}_j$ is invariant under σ and σ^{-1}. It follows from Lemma 25.1 that such a σ is a permutation of the intervals I_t. Let $\tilde{\sigma} : \mathcal{A}_j \to \mathcal{A}_j$ be the automorphism induced by σ, as given by (25.1), and consider the crossed product algebra $\mathcal{A}_j \rtimes_{\tilde{\sigma}} \mathbb{Z}$.

We write an element $f \in \mathcal{A}_j \rtimes_{\tilde{\sigma}} \mathbb{Z}$ as

$$f = \sum_{n\in\mathbb{Z}} f_n \delta^n,$$

where $f_n = f(n) \in \mathcal{A}_j$ for each $n \in \mathbb{Z}$ and $f_n = 0$ except for finitely many $n \in \mathbb{Z}$. Therefore, using the definition of $f \in \mathcal{A}_j$ as given by (25.3), we see that $f \in \mathcal{A}_j \rtimes_{\tilde{\sigma}} \mathbb{Z}$ can be written in the form

$$f(x, \delta) = \sum_{n\in\mathbb{Z}} \left(\sum_{t_n\in\mathbb{Z}} a_{t_n} \phi\left(2^j x - t_n\right) \right) \delta^n.$$

We would like to give an explicit description of the commutant (centralizer) $\mathcal{A}'_j$ of $\mathcal{A}_j$ in the crossed product algebra $\mathcal{A}_j \rtimes_{\tilde{\sigma}} \mathbb{Z}$.

From Theorem 25.1, this commutant (centralizer), $\mathcal{A}'_j$ in $\mathcal{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$ is precisely the set of elements

$$\mathcal{A}'_j = \left\{ \sum_{n\in\mathbb{Z}} f_n \delta^n \mid \text{for all } n \in \mathbb{Z} : \ f_n|_{Sep^n_{\mathcal{A}_j}(\mathbb{R})} \equiv 0 \right\}$$

where for each $n \in \mathbb{Z}$, $Sep^n_{\mathcal{A}_j}(\mathbb{R})$ follows from Definition 25.3.

In the following Theorem, we give the description of $Sep^n_{\mathcal{A}_j}(\mathbb{R})$ for each $n \in \mathbb{Z}$, where $\mathcal{A}_j$ is the algebra of functions defined by (25.3).

Theorem 25.2 *Let $\mathcal{A}_j$ the algebra piecewise constant functions on $\mathbb{R}$ as defined by* (25.3)*. Suppose $\sigma : \mathbb{R} \to \mathbb{R}$ is a bijection such that $\mathcal{A}_j$ is invariant under σ (and σ^{-1}) and $\tilde{\sigma} : \mathcal{A}_j \to \mathcal{A}_j$ is the automorphism on $\mathcal{A}_j$ induced by σ. Then for every $n \in \mathbb{Z}$,*

$$Sep^n_{\mathcal{A}_j}(\mathbb{R}) = \left(\bigcup_{k\nmid n} C_k \right) \cup C_\infty,$$

where For each $k \in \mathbb{Z}_{>0}$, C_k is given by

$$C_k := \{x \in \mathbb{R} \ : \ k \text{ is the smallest positive integer such that } x, \sigma^k(x) \in I_t \text{ for some } t \in \mathbb{Z}\} \tag{25.4}$$

and

$$C_\infty = \{x \in \mathbb{R} \ : \ (\nexists\, t \in \mathbb{Z}), \ : \ x, \sigma^k(x) \notin I_t, \ \forall\, k \ \geq 1\}. \tag{25.5}$$

Proof If $k \mid n$ then we can write $n = mk$ for some $m \in \mathbb{Z}$. If $x \in C_k$, then by definition of C_k, $x, \sigma^k(x) \in I_t$ for some $t \in \mathbb{Z}$. It follows that $x, \sigma^{-k}(x) \in I_u$ where, by invariance of $\mathcal{A}_j$, $\sigma(I_t) = I_u$. Therefore for every $h \in \mathcal{A}_j$,

$$\tilde{\sigma}^n(h)(x) = \tilde{\sigma}^{mk}(h)(x) = (h \circ \sigma^{-mk})(x) = h(\sigma^{-mk}(x)) = h(x),$$

since h is constant on I_u. It follows that, if $x \in C_k$, then $x \notin Sep^n_{\mathcal{A}_j}(\mathbb{R})$ for all $k \mid n$, and hence

$$Sep^n_{\mathcal{A}_j}(\mathbb{R}) \subseteq \left(\bigcup_{k \nmid n} C_k \right) \cup C_\infty.$$

On the other hand, if $k \nmid n$, then we can write $n = mk + q$ where $m, q \in \mathbb{Z}$ with $1 \le q < k$. If $x \in C_k$, then

$$\begin{aligned} \tilde{\sigma}^n(h)(x) &= \tilde{\sigma}^{mk+q}(h)(x) \\ &= (h \circ \sigma^{-(mk+q)})(x) \\ &= h(\sigma^{-mk-q}(x)) \\ &= \tilde{\sigma}^{-q}(h)(x) \\ &\neq h(x), \end{aligned}$$

since k is the smallest integer such that $x, \sigma^{-k}(x) \in I_u$ and $q < k$. Clearly, if $x \in C_\infty$, then $x \in Sep^n_{\mathcal{A}_j}(\mathbb{R})$ for any $n \in \mathbb{Z}$. Therefore

$$Sep^n_{\mathcal{A}_j}(\mathbb{R}) \supseteq \left(\bigcup_{k \nmid n} C_k \right) \cup C_\infty.$$

From the above theorem, the description of the maximal commutative subalgebra in $\mathcal{A}_j \rtimes_{\tilde{\sigma}} \mathbb{Z}$ containing $\mathcal{A}_j$ can be done as follows.

Theorem 25.3 *The centralizer of $\mathcal{A}_j$ in the crossed product algebra $\mathcal{A}_j \rtimes_{\tilde{\sigma}} \mathbb{Z}$ is given by*

$$\mathcal{A}'_j = \Bigg\{ f \in \mathcal{A}_j \rtimes_{\tilde{\sigma}} \mathbb{Z} \ : \ f(x,\delta) = \sum_{n \in \mathbb{Z}} \left(\sum_{t_n \in \mathbb{Z}} a_{t_n} \phi(2^j x - t_n) \right) \delta^n, \\ \text{such that } (\forall\, n \text{ such that } k \nmid n),\ a_{t_n} = 0 \Bigg\},$$

where the second sum is taken over all $x \in C_k$, and C_k is given by (25.4).

Proof Let $f = \sum_{n \in \mathbb{Z}} f_n \delta^n \in \mathcal{A}_j \rtimes_{\tilde{\sigma}} \mathbb{Z}$ be an element in $\mathcal{A}'_j$. Then, from Theorem 25.1, we have that for every $n \in \mathbb{Z}$, $f_n(x) = 0$ for every $x \in Sep^n_{\mathcal{A}_j}(\mathbb{R})$. Now, from Theorem 25.2, we have that for every $n \in \mathbb{Z}$,

$$Sep^n_{\mathcal{A}_j}(\mathbb{R}) = \left(\bigcup_{k \nmid n} C_k \right) \cup C_\infty, \tag{25.6}$$

where C_k and C_∞ are given by (25.4) and (25.5) respectively. Therefore, if $f = \sum\limits_{n\in\mathbb{Z}} f_n \delta^n \in \mathcal{A}'_j$ then $f_n = 0$ for all $k \nmid n \in \mathbb{Z}$. Therefore,

$$\mathcal{A}'_j = \left\{ \sum_{n\in\mathbb{Z}} f_n \delta^n \ : \ (\forall\, x \in C_k, \text{ where } k \nmid n),\ f_n(x) = 0 \right\},$$

And using the definition of $f_n \in \mathcal{A}_j$ as

$$f_n(x) = \sum_{t_n \in \mathbb{Z}} a_{t_n} \phi(2^j x - t_n)$$

with $a_{t_n} \in \mathbb{R}$, we have:

$$\mathcal{A}'_j = \Bigg\{ f \in \mathcal{A}_j \rtimes_{\tilde{\sigma}} \mathbb{Z} \ : \ f(x, \delta) = \sum_{n\in\mathbb{Z}} \left(\sum_{t_n\in\mathbb{Z}} a_{t_n} \phi(2^j x - t_n) \right) \delta^n,$$
$$\text{such that } (\forall\, n \text{ such that } k \nmid n),\ a_{t_n} = 0 \Big\},$$

where the second sum is taken over all $x \in C_k$.

25.4 A Comparison of Commutants in Nested Spaces

In the following section we give a comparison of commutants of the nested sequence of algebras $\cdots \mathcal{A}_{-2} \subset \mathcal{A}_{-1} \subset \mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots$, where for each $j \in \mathbb{Z}$, $\mathcal{A}_j$ is the collection of all square integrable functions which are constant on all 2^{-j} length intervals, as described by (25.3). We first derive conditions under which, starting with an algebra $\mathcal{A}_j$ $j \in \mathbb{Z}$, which is invariant under a bijection $\sigma : \mathbb{R} \to \mathbb{R}$, the algebras $\mathcal{A}_i$ are invariant under the same bijection $\sigma : \mathbb{R} \to \mathbb{R}$ for all $i \geqslant j$. This ensures that we have an increasing sequence of crossed products $\mathcal{A}_j \rtimes_{\tilde{\sigma}} \mathbb{Z} \subset \mathcal{A}_{j+1} \rtimes_{\tilde{\sigma}} \mathbb{Z} \subset \cdots$ for all $j \in \mathbb{Z}$, in which case the commutants form a decreasing sequence $\mathcal{A}'_j \supset \mathcal{A}'_{j+1}$ for all $j \in \mathbb{Z}$, [8]. We give the conditions in the following Lemma.

Lemma 25.2 *Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a bijection such that $\mathcal{A}_j$ is invariant under σ for some $j \in \mathbb{Z}$. Then $\mathcal{A}_i$ is invariant under σ for all integers $i \geqslant j$.*

Proof Since $\cdots \mathcal{A}_{-2} \subset \mathcal{A}_{-1} \subset \mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots$, it is enough to prove that invariance of $\mathcal{A}_j$ under a bijection $\sigma : \mathbb{R} \to \mathbb{R}$ implies invariance of $\mathcal{A}_{j+1}$ under σ. To this end, we have the following.

Suppose $\mathcal{A}_j$ is invariant under a bijection $\sigma : \mathbb{R} \to \mathbb{R}$. Then by Lemma 3.1, σ is a permutation of intervals of the form $I_t = \big[2^{-j}t, 2^{-j}(t+1)\big)$ for some $t \in \mathbb{Z}$. Note that in $\mathcal{A}_{j+1}$, each of the intervals I_t is divided into two half-intervals each of length $2^{-(j+1)}$. From [3], we also know that $\mathcal{A}_{j+1} = \mathcal{A}_j \oplus W_j$, where W_j is the subalgebra of $\mathcal{A}_{j+1}$ consisting of functions in $\mathcal{A}_{j+1}$ which take equal and opposite values on each half of every interval I_t, that is, $f \in W_j$ if and only if

$$f(x) = \begin{cases} a_t & 2^{-j}t \leqslant x < 2^{-j}(t + \frac{1}{2}) \\ -a_t & 2^{-j}(t + \frac{1}{2}) \leqslant x < 2^{-j}(t+1) \\ 0 & \text{otherwise} \end{cases}$$

for some $a_t \in \mathbb{R}$.

Now let $x \in I_t \subset \mathbb{R}$ for some $t \in \mathbb{Z}$ suppose $\sigma(I_t)$ is some interval, say, $I_s = \big[2^{-j}s, 2^{-j}(s+1)\big)$ for some $s \in \mathbb{Z}$. Let $g \in W_j$ such that

$$g(y) = \begin{cases} a_s & 2^{-j}s \leqslant y < 2^{-j}(s + \frac{1}{2}) \\ -a_s & 2^{-j}(s + \frac{1}{2}) \leqslant y < 2^{-j}(s+1) \\ 0 & \text{otherwise} \end{cases}.$$

Then

$$g \circ \sigma(x) = g(\sigma(x)) = \begin{cases} a_s & 2^{-j}s \leqslant \sigma(x) < 2^{-j}(s + \frac{1}{2}) \\ -a_s & 2^{-j}(s + \frac{1}{2}) \leqslant \sigma(x) < 2^{-j}(s+1) \\ 0 & \text{otherwise} \end{cases}.$$

Therefore $g \circ \sigma \in W_j$, and hence W_j is invariant under σ.

Now let $h \in \mathcal{A}_{j+1}$. Then $h = f + g$ for unique $f \in \mathcal{A}_j$ and $g \in W_j$. Therefore

$$h \circ \sigma = (f + g) \circ \sigma = (f \circ \sigma) + (g \circ \sigma) \in \mathcal{A}_{j+1},$$

since $f \circ \sigma \in \mathcal{A}_j$ and $g \in W_j$. Therefore $\mathcal{A}_{j+1}$ is invariant under σ.

Note that the conditions in Lemma 25.2 are sufficient but not necessary as illustrated in the following example.

Example 25.1 Let $\sigma : \mathbb{R} \to \mathbb{R}$ be the function defined as follows

$$\sigma(x) = \begin{cases} x + \frac{1}{2} & 0 \leqslant x < \frac{1}{2} \\ x + \frac{1}{2} & \frac{1}{2} \leqslant x < 1 \\ x + \frac{1}{2} & 1 \leqslant x < \frac{3}{2} \\ x - \frac{3}{2} & \frac{3}{2} \leqslant x < 2 \\ x & \text{otherwise.} \end{cases}$$

Then σ is a bijection such that $\mathcal{A}_1$ is invariant under σ but $\mathcal{A}_0$ is not invariant under σ.

Proof Let $f_0 \in \mathcal{A}_0$ be the function defined as

$$f_0(x) = \begin{cases} 1 & 1 \leqslant x < 2 \\ 0 & \text{otherwise} \end{cases}.$$

Then for all $x \in [0, 1)$ we have

$$\begin{aligned} (f_0 \circ \sigma)(x) &= f(\sigma(x)) \\ &= \begin{cases} 1 & 1 \leqslant \sigma(x) < 2 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \frac{1}{2} \leqslant x < 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Therefore $f_0 \circ \sigma \notin \mathcal{A}_0$.

We would like to give a comparison of the commutants $\mathcal{A}'_j$ and $\mathcal{A}'_{j+1}$ of $\mathcal{A}_j$ and $\mathcal{A}_{j+1}$ respectively in the crossed product algebra $\mathcal{A}_{j+1} \rtimes_{\tilde{\sigma}} \mathbb{Z}$, where $\sigma : \mathbb{R} \to \mathbb{R}$ is a bijection such that $\mathcal{A}_j$ and $\mathcal{A}_{j+1}$ are both invariant under σ and σ^{-1}. Observe that if σ satisfies the conditions of Lemma 25.2, then the crossed product $\mathcal{A}_j \rtimes_{\tilde{\sigma}} \mathbb{Z}$ is contained in $\mathcal{A}_{j+1} \rtimes_{\tilde{\sigma}} \mathbb{Z}$, and so we can compare the commutants.

By Theorem 25.1, the commutant $\mathcal{A}'_j$ is given by

$$\mathcal{A}'_j = \left\{ \sum_{n \in \mathbb{Z}} f_n \delta^n \mid \text{for all } n \in \mathbb{Z} : \ f_n|_{Sep^n_{\mathcal{A}_j}(\mathbb{R})} \equiv 0 \right\},$$

and $\mathcal{A}'_{j+1}$ is given by

$$\mathcal{A}'_{j+1} = \left\{ \sum_{n \in \mathbb{Z}} f_n \delta^n \mid \text{for all } n \in \mathbb{Z} : \ f_n|_{Sep^n_{\mathcal{A}'_{j+1}}(\mathbb{R})} \equiv 0 \right\}.$$

From the description of $\mathcal{A}'_j$ in Theorem 25.3 above, we see that only elements outside $Sep^n_{\mathcal{A}_j}(\mathbb{R})$ contribute something to the commutant. From (25.6), we observe that an element belongs to $Sep^n_{\mathcal{A}_j}(\mathbb{R})$ if and only if it belongs to C_k for all $k \nmid n$ where C_k is given by (25.4). Such C_k consists of intervals, say, $I_{t_1}, \cdots, I_{t_k}$ (of length 2^{-j} each) which, by invariance of $\mathcal{A}_j$ under σ, are mapped cyclically onto each other by σ. In $\mathcal{A}_{j+1}$ each of these interval is divided into two subintervals each of length $2^{-(j+1)}$ say, $I^1_{t_l} = \left[2^{-j} t_l, 2^{-j}(t_l + \frac{1}{2})\right)$ and $I^2_{t_l} = \left[2^{-j}(t_l + \frac{1}{2}), 2^{-j}(t_l + 1)\right)$ for $l = 1, 2, \cdots, k$. For the intervals $I_{t_1}, \cdots, I_{t_k} \subset C_k$ and for any positive integer n, we let

$$\tilde{C}_k := \big\{x \in I_{t_l} \ : \ k \text{ is the smallest positive integer such that} \\ x, \sigma^k(x) \in I_{t_l}^p \text{ for some } p \in \{1, 2\}\big\}. \tag{25.7}$$

Using this, we give a comparison of the commutants in the following Theorem.

Theorem 25.4 *Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a bijection such that $\mathcal{A}_j$ and $\mathcal{A}_{j+1}$ are both invariant under σ and σ^{-1}. Then,*

$$\mathcal{A}'_j \setminus \mathcal{A}'_{j+1} = \left\{ \sum_{n\in\mathbb{Z}} f_n \delta^n \ : \ (\forall\, n, k \textit{ such that } 2k \nmid n),\ \ f_n = 0 \textit{ on } \tilde{C}_{2k} \right\},$$

where $\tilde{C}_k$ is given by (25.7).

Proof From Theorem 25.3, we have that

$$\mathcal{A}'_j = \left\{ f \in \mathcal{A}_j \rtimes_{\tilde{\sigma}} \mathbb{Z} \ : \ f(x) = \sum_{n\in\mathbb{Z}} f_n \delta^n, \ : \ (\forall\, k \nmid n),\ \ f_n = 0 \text{ on } C_k \right\},$$

where C_k is given by (25.4). As observed earlier, each such C_k consists of $k-$integer intervals, say, $I_{t_1}, \cdots, I_{t_k}$ which are mapped cyclically onto each other by σ and each of which is divided into two subintervals of the form $I_{t_l}^1 = \big[2^{-j} t_l, 2^{-j}(t_l + \frac{1}{2})\big)$ and $I_{t_l}^2 = \big[2^{-j}(t_l + \frac{1}{2}), 2^{-j}(t_l + 1)\big)$ for $l = 1, 2, \cdots, k$ in $\mathcal{A}_{j+1}$. Invariance of $\mathcal{A}_{j+1}$ under σ (Lemma 25.1) implies that σ permutes these $2k$ subintervals. Therefore, each subinterval $I_{t_l}^{(p)}$, $l = 1, \cdots, k$, $p = 1, 2$ either belongs to $\tilde{C}_k$ or $\tilde{C}_{2k}$. It follows that, for every $n \in \mathbb{Z}$,

$$Sep^n_{\mathcal{A}_{j+1}}(\mathbb{R}) = \begin{cases} Sep^n_{\mathcal{A}_j}(\mathbb{R}) & \text{if } k \nmid n \\ Sep^n_{\mathcal{A}_j}(\mathbb{R}) \bigcup\limits_{n:2k\nmid n} \tilde{C}_k & \text{if } k \mid n \end{cases}.$$

Hence, the commutants $\mathcal{A}'_j$ and $\mathcal{A}'_{j+1}$ satisfy

$$\mathcal{A}'_j \setminus \mathcal{A}'_{j+1} = \left\{ \sum_{n\in\mathbb{Z}} f_n \delta^n \ : \ (\forall\, n, k \text{ such that } 2k \nmid n),\ \ f_n = 0 \text{ on } \tilde{C}_{2k} \right\},$$

25.4.1 Comparison of $C(\mathcal{A}_0)$ and $C(\mathcal{A}_j)$ for Some $j \in \mathbb{Z}_{>0}$

For each $j = 0, 1, \cdots$, let $\mathcal{A}_j$ be the collection of all square integrable functions which are constant on all intervals of length 2^{-j}, that is, $g(x) \in \mathcal{A}_j$ if and only if

$$g(x) = \sum_k a_k \phi(2^j x - k) = \begin{cases} a_k & \text{if } \frac{k}{2^j} \leqslant x \leqslant \frac{k+1}{2^j} \\ 0 & \text{otherwise} \end{cases}.$$

By Lemma 25.2, if $\sigma : \mathbb{R} \to \mathbb{R}$ is a bijection such that $\mathcal{A}_0$ is invariant under σ and σ^{-1}, then $\mathcal{A}_j$ is also invariant under σ for every positive $j \in \mathbb{Z}$. Since $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots$, if $\mathcal{A}_0$ is invariant under a bijection $\sigma : \mathbb{R} \to \mathbb{R}$ (and σ^{-1}) and $\tilde{\sigma}$ is the automorphism induced by σ, then

$$\mathcal{A}_0 \rtimes_{\tilde{\sigma}} \mathbb{Z} \subset \mathcal{A}_1 \rtimes_{\tilde{\sigma}} \mathbb{Z} \subset \cdots$$

and hence we can compare the commutants $\mathcal{A}_0'$ and $\mathcal{A}_j'$ for some $j \in \mathbb{Z}_{>0}$. By Theorem 25.1, the commutants $\mathcal{A}_0'$ and $\mathcal{A}_j'$ are given, respectively, by

$$\mathcal{A}_0' = \left\{ \sum_{n\in\mathbb{Z}} f_n \delta^n \mid \text{for all } n \in \mathbb{Z} : \; f_n|_{Sep^n_{\mathcal{A}_0}(\mathbb{R})} \equiv 0 \right\}$$

and

$$\mathcal{A}_j' = \left\{ \sum_{n\in\mathbb{Z}} f_n \delta^n \mid \text{for all } n \in \mathbb{Z} : \; f_n|_{Sep^n_{\mathcal{A}_j}(\mathbb{R})} \equiv 0 \right\}$$

As observed before, only the elements outside $Sep^n_{\mathcal{A}_0}(\mathbb{R})$ contribute something to the commutant and from (25.6), it follows that an element belongs to $Sep^n_{\mathcal{A}_0}(\mathbb{R})$ if and only if it belongs to C_k for all $k \nmid n$ or to C_∞, where C_k and C_∞ are given by (25.4) and (25.5), respectively. Such C_k consists of $k-$integer intervals, say, $I_{t_1}, \cdots, I_{t_k}$ which, by invariance of $\mathcal{A}_0$ under σ, are mapped cyclically onto each other by σ. In $\mathcal{A}_j$ each of the intervals $I_{t_l} = [t_l, t_l + 1)$, $l = 1, 2, \cdots, k$ is divided into 2^j subintervals of length 2^{-j}. That is,

$$I_{t_l} = [t_l, t_l + 1) = \bigcup_{p=1}^{2^j} \left[t_l + \frac{p-1}{2^j}, t_l + \frac{p}{2^j} \right).$$

Therefore, for each $l = 1, 2, \cdots, k$, we can write I_{t_l} as $I_{t_l} = \bigcup_{p=1}^{2^j} I^p_{t_l}$, where for each $p = 1, 2, \cdots, 2^j$, $I^p_{t_l} = \left[t_l + \frac{p-1}{2^j}, t_l + \frac{p}{2^j} \right)$. For the intervals $I_{t_l} \subset C_k$, let

$$\tilde{C}_k := \big\{ x \in I_{t_l} \;:\; k \text{ is the smallest positive integer such that } x, \sigma^k(x) \in I^p_{t_l} \text{ for some } p \in \{1, \cdots, 2^j\} \big\}. \tag{25.8}$$

We have the following theorem.

Theorem 25.5 *Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a bijection such that $\mathcal{A}_0$ and $\mathcal{A}_j$ are both invariant under σ and σ^{-1}. Then,*

$$\mathcal{A}'_0 \setminus \mathcal{A}'_j = \left\{ \sum_{n\in\mathbb{Z}} f_n \delta^n \ : \ (\forall\, n, k \text{ such that } 2^q k \nmid n, q = 1, \cdots, j),\ f_n = 0 \text{ on } \tilde{C}_{2^q k} \right\},$$

where $\tilde{C}_k$ is given by (25.8).

Proof We have seen from Theorem 25.3, that $C(\mathcal{A}_0)$ is given by

$$\mathcal{A}'_0 = \left\{ f \in \mathcal{A}_0 \rtimes_{\tilde{\sigma}} \mathbb{Z} \ : \ f(x) = \sum_{n\in\mathbb{Z}} f_n \delta^n, \ : \ (\forall\, k \nmid n),\ f_n = 0 \text{ on } C_k \right\},$$

where C_k is given by (25.4). Also, as has been observed, such C_k consists of $k-$integer intervals, $I_{t_l} = [t_l, t_l + 1),\ l = 1, 2, \cdots, k$ and $t_l \in \mathbb{Z}$, which are mapped cyclically onto each other by σ and each of which, in $\mathcal{A}_j$, is divided into 2^j subintervals of length 2^j. That is, $I_{t_l} = \bigcup_{p=1}^{2^j} I_{t_l}^p$, where for each $p = 1, 2, \cdots, 2^j$, $I_{t_l}^p = \left[t_l + \frac{p-1}{2^j}, t_l + \frac{p}{2^j}\right)$. Since $\mathcal{A}_0$ is invariant under σ, then by Lemma 25.2, $\mathcal{A}_1, \cdots, \mathcal{A}_j$ are all invariant under σ. Therefore, by Lemma 25.1, σ permutes these 2^j subintervals. Since the intervals $I_{t_l}, l = 1, \cdots, k$ belong to C_k, then we have the following.

- In $\mathcal{A}_1$, the subintervals $I_{t_l}^p = \left[t_l + \frac{p-1}{2}, t_l + \frac{p}{2}\right)$ either belong to $\tilde{C}_k$ or to $\tilde{C}_{2k}$.
- In $\mathcal{A}_2$, the subintervals $I_{t_l}^p = \left[t_l + \frac{p-1}{4}, t_l + \frac{p}{4}\right)$ either belong to $\tilde{C}_k$, $\tilde{C}_{2k}$ or to $\tilde{C}_{4k}$.
- In the same way, we observe that in $\mathcal{A}_j$, the subintervals $I_{t_l}^p = \left[t_l + \frac{p-1}{2^j}, t_l + \frac{p}{2^j}\right)$ either belong to one of $\tilde{C}_k, \tilde{C}_{2k}, \cdots, \tilde{C}_{2^j k}$.

From the fact that if $2^j k \mid n$, then $2^q k \mid n$ for all $q = 1, 2, \cdots, j$, it follows that, for every $n \in \mathbb{Z}$,

$$Sep^n_{\mathcal{A}_j}(\mathbb{R}) = \begin{cases} Sep^n_{\mathcal{A}_0}(\mathbb{R}) & \text{if } k \nmid n \\ Sep^n_{\mathcal{A}_0}(\mathbb{R}) \cup \left(\bigcup\limits_{q=1: 2^q k \nmid n}^{j} \tilde{C}_{2^q k} \right) & \text{if } k \mid n \end{cases}.$$

Hence, the commutants $\mathcal{A}'_0$ and $\mathcal{A}'_j$ satisfy

$$\mathcal{A}'_0 \setminus \mathcal{A}'_j = \left\{ \sum_{n\in\mathbb{Z}} f_n \delta^n \ : \ (\forall\, n, k \text{ such that } 2^q k \nmid n, q = 1, \cdots, j),\ f_n = 0 \text{ on } \tilde{C}_{2^q k} \right\}.$$

In the next example we give a comparison of commutants $\mathcal{A}_0'$ and $\mathcal{A}_1'$ in the crossed product $\mathcal{A}_1 \rtimes_{\tilde{\sigma}} \mathbb{Z}$ for a specific bijection $\sigma : \mathbb{R} \to \mathbb{R}$ under which both $\mathcal{A}_0$ and $\mathcal{A}_1$ are invariant.

Example 25.2 Let $\sigma : \mathbb{R} \to \mathbb{R}$ be the bijection defined by

$$\sigma(x) = \begin{cases} x+1 & 0 \leqslant x < 1 \\ x - \frac{1}{2} & 1 \leqslant x < \frac{3}{2} \\ x - \frac{3}{2} & \frac{3}{2} \leqslant x < 2 \\ x+1 & 2 \leqslant x < 3 \\ x-1 & 3 \leqslant x < 4 \\ x & \text{otherwise} \end{cases}$$

and consider the crossed product algebra $\mathcal{A}_0 \rtimes_{\tilde{\sigma}} \mathbb{Z}$, where $f \in \mathcal{A}_0 \rtimes_{\tilde{\sigma}} \mathbb{Z}$ is written in the form

$$f(x,\delta) = \sum_{n\in\mathbb{Z}} \left(\sum_{t_n \in \mathbb{Z}} a_{t_n} \phi(x - t_n) \right) \delta^n.$$

Then we have the following.

1. The commutant $\mathcal{A}_0'$ of $\mathcal{A}_0$ is given by

$$\mathcal{A}_0' = \left\{ \sum_{n\in\mathbb{Z}} \left(\sum_{t_n\in\mathbb{Z}} a_{t_n}\phi(x-t_n) \right) \delta^n \ , \ (\forall\, t \in \{0,1,2,3\}), a_{t_n} = 0 \text{ for all odd } n \right\}.$$

2. The comparison of the commutants $\mathcal{A}_0'$ and $\mathcal{A}_1'$ in the crossed product $\mathcal{A}_1 \rtimes_{\tilde{\sigma}} \mathbb{Z}$ is given by

$$\mathcal{A}_0' \setminus \mathcal{A}_1' = \left\{ \sum_{n\in\mathbb{Z}} f_n \delta^n \ : \ (\forall\, n \text{ such that } 4 \nmid n),\ f_n = 0 \text{ on } \bigcup_{p=1}^{2} \left(\bigcup_{l=0}^{1} I_t^p \right) \right\},$$

where for each $t \in \{0, 1, 2, 3\}$ and $p \in \{1, 2\}$, $I_t^p = \left[t + \frac{p-1}{2}, t + \frac{p}{2} \right)$.

Proof 1. If we denote integer intervals by $I_t = [t, t+1)$, we observe that $\sigma^2(I_t) = I_t$ for $t = 0, 1, 2, 3$ and $\sigma(I_t) = I_t$ for all other $t \in \mathbb{Z}$. Therefore, it follows from (25.4) that, for every $k \in \mathbb{Z}_{>0}$,

$$C_k = \begin{cases} \mathbb{R} \setminus \left(\bigcup_{t=0}^{3} I_t \right), & k = 1 \\ \bigcup_{t=0}^{3} I_t, & k = 2 \\ \emptyset, & k > 2 \end{cases}.$$

Therefore,

$$Sep^n_{\mathcal{A}_0}(\mathbb{R}) = \left\{ \bigcup_{k \nmid n} C_k \cup C_\infty \right\} = \begin{cases} \bigcup_{t=0}^{3} I_t & \text{if } n \text{ is odd} \\ \emptyset, & \text{if } n \text{ is even} \end{cases}.$$

From Theorem 25.1, we have

$$\begin{aligned} \mathcal{A}'_0 &= \left\{ \sum_{n\in\mathbb{Z}} f_n \delta^n \mid \text{for all } n \in \mathbb{Z}: \ f_n|_{Sep^n_{\mathcal{A}_0}(\mathbb{R})} \equiv 0 \right\} \\ &= \left\{ \sum_{n\in\mathbb{Z}} f_n \delta^n \ , \ (\forall\, n \text{ such that } 2 \nmid n),\, f_n = 0 \text{ on } \bigcup_{t=0}^{3} I_t \right\}. \end{aligned}$$

Using the definition of $f_n \in \mathcal{A}_0$ as given by (25.2), we have

$$\mathcal{A}'_0 = \left\{ \sum_{n\in\mathbb{Z}} \left(\sum_{t_n\in\mathbb{Z}} a_{t_n} \phi(x - t_n) \right) \delta^n \ , \ (\forall\, t \in \{0, 1, 2, 3\})\,, a_{t_n} = 0 \text{ for all odd } n \right\}.$$

2. Now consider the crossed product $\mathcal{A}_1 \rtimes_{\tilde{\sigma}} \mathbb{Z}$. We would like to explicitly determine the difference of the commutants $\mathcal{A}_0 \setminus \mathcal{A}_1$. Note that each of the intervals $I_t = [t, t+1)$, $t \in \{0, 1, 2, 3\}$ is divided into two subintervals $I_t^p = \left[t + \frac{p-1}{2}, t + \frac{p}{2}\right)$ $p \in \{1, 2\}$. Using (25.7) and the definition of σ, we see that, for every $n \in \mathbb{Z}$,

$$Sep^n_{\mathcal{A}_1}(\mathbb{R}) = \begin{cases} \bigcup_{p=1}^{2} \bigcup_{t=0}^{3} I_t^p & \text{if } 2 \nmid n \\ \left(\bigcup_{p=1}^{2} \bigcup_{t=0}^{1} I_t^p \right) & \text{if } 4 \nmid n \end{cases} = \begin{cases} Sep^n_{\mathcal{A}_0}(\mathbb{R}) & \text{if } 2 \nmid n \\ \left(\bigcup_{p=1}^{2} \bigcup_{t=0}^{1} I_t^p \right) & \text{if } 4 \nmid n \end{cases}.$$

Therefore the comparison of the commutants is given by

$$\mathcal{A}'_0 \setminus \mathcal{A}'_1 = \left\{ \sum_{n\in\mathbb{Z}} f_n \delta^n \ : \ (\forall\, n \text{ such that } 4 \nmid n), \ \ f_n = 0 \text{ on } \bigcup_{p=1}^{2} \left(\bigcup_{l=0}^{1} I_t^p \right) \right\}.$$

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Chapter 26
Constacyclic and Skew Constacyclic Codes Over a Finite Commutative Non-chain Ring

Om Prakash, Habibul Islam, and Ram Krishna Verma

Abstract For an odd prime p, this article studies the λ-constacyclic and skew λ-constacyclic codes of arbitrary length over the finite commutative non-chain ring $R = \mathbb{F}_{p^m}[u, v, w]/\langle u^2 - 1, v^2 - 1, w^2 - 1, uv - vu, vw - wv, wu - uw\rangle$, where λ is a unit in R. By using the decomposition method, we determine the structure of λ-constacyclic and skew λ-constacyclic codes. Also, the necessary and sufficient conditions of these codes to be self-dual are obtained. Further, it is shown that the Gray images of λ-constacyclic and skew λ-constacyclic codes of length n over R are quasi-twisted and skew quasi-twisted codes, respectively of length $8n$ and index 8 over $\mathbb{F}_{p^m}$. Finally, two non-trivial examples are given to validate the obtained results.

Keywords Constacyclic code · Skew constacyclic code · Gray map · Self-dual code

MSC2020 94B05 · 94B15 · 94B35 · 94B60

26.1 Introduction

The class of constacyclic codes is an important generalization of cyclic codes due to its efficient implementation using the shift register in engineering and technology. Also, it is a rich resource to produce better error-correcting codes. Researchers have inclined towards the study of cyclic and constacyclic codes over finite rings after the seminal work presented by Hammons et al. [13] in 1994. In 2006, Qian et al. [22] studied $(1 + u)$-constacyclic codes over $\mathbb{F}_2 + u\mathbb{F}_2$. Further, in 2009, cyclic and

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constacyclic codes of an arbitrary length over $\mathbb{F}_2 + u\mathbb{F}_2$ were considered in [1]. They proved that the Gray image of $(1+u)$-constacyclic code is a distance-preserving binary linear code. Later, some new optimal p-ary and binary linear codes have been introduced in [20, 21, 25].

On the other hand, in 2007, Boucher et al. [5] shown that better error-correcting codes can be found over non-commutative rings too. In fact, they introduced skew cyclic codes which are indeed a generalization of cyclic codes. They have constructed many new codes whose minimum distances are larger than the distances of previously known best codes [6, 7]. Thereafter, the study of linear codes in non-commutative set up has got huge attention among researchers. In 2011, Abualrub et al. [2] investigated structure of skew cyclic codes over $\mathbb{F}_2 + v\mathbb{F}_2$ while Siap et al. [23] studied skew cyclic codes of an arbitrary length over the finite fields. Further, in 2015, Gao [10] discussed linear codes (cyclic codes) over $\mathbb{F}_p + u\mathbb{F}_p + u^2\mathbb{F}_p$ by decomposition method. Later, some new studies based on decomposition come into the literature, we refer [3, 8, 9, 11, 12, 15–17, 19, 23, 24, 26].

Above all studies motivate to consider constacyclic and skew constacyclic codes over different finite non-chain rings. Thus, for any odd prime p, we study here constacyclic and skew constacyclic codes over finite non-chain ring

$$R = \mathbb{F}_{p^m}[u, v, w]/\langle u^2 - 1, v^2 - 1, w^2 - 1, uv - vu, vw - wv, wu - uw\rangle.$$

This paper completely determines the structure of these codes by the decomposition method. It is worth mentioning that for $m = 1$, recently authors [18] constructed some new quantum codes from cyclic codes over R. The presentation of the manuscript is organized as follows: In Sect. 26.2, we discuss basic setup and results for linear codes. Section 26.3 gives structure of constacyclic codes while Sect. 26.4 devotes to study of skew constacyclic codes. Section 26.5 includes Gray images of these codes and Sect. 26.6 concludes the article.

26.2 Basic Concepts and Results

For any odd prime p, let $\mathbb{F}_{p^m}$ be a finite field of order p^m with characteristic p and

$$R = \mathbb{F}_{p^m}[u, v, w]/\langle u^2 - 1, v^2 - 1, w^2 - 1, uv - vu, vw - wv, wu - uw\rangle.$$

Clearly, R is a finite commutative semi-local ring (with unity) of order p^{8m} and characteristic p. Recall that a *linear code* of length n over R is an R-submodule of R^n and its members are called *codewords*. Let $\mathcal{C}$ be a linear code of length n over R. For any $z_1 = (r_0, r_1, \ldots, r_{n-1})$, $z_2 = (r'_0, r'_1, \ldots, r'_{n-1}) \in R^n$, the Euclidean inner product is defined as $z_1 \cdot z_2 = \sum_{i=1}^{n-1} r_i r'_i$. The dual code of $\mathcal{C}$ is define as $\mathcal{C}^{\perp} = \{z_1 \in R^n \mid z_1 \cdot z_2 = 0,\ \forall\ z_2 \in \mathcal{C}\}$. The code $\mathcal{C}$ is said to be *self-orthogonal* if $\mathcal{C} \subseteq \mathcal{C}^{\perp}$ and *self-dual* if $\mathcal{C}^{\perp} = \mathcal{C}$. Let $\gamma \in \mathbb{F}_{p^m}$ such that $8\gamma \equiv 1 \pmod{p}$. Let

$$\begin{aligned}
\xi_1 &= \gamma[1 + u + v + w + uv + vw + uw + uvw],\\
\xi_2 &= \gamma[1 + u + v - w + uv - vw - uw - uvw],\\
\xi_3 &= \gamma[1 + u - v + w - uv - vw + uw - uvw],\\
\xi_4 &= \gamma[1 - u + v + w - uv + vw - uw - uvw],\\
\xi_5 &= \gamma[1 + u - v - w - uv + vw - uw + uvw],\\
\xi_6 &= \gamma[1 - u - v + w + uv - vw - uw + uvw],\\
\xi_7 &= \gamma[1 - u + v - w - uv - vw + uw + uvw],\\
\xi_8 &= \gamma[1 - u - v - w + uv + vw + uw - uvw].
\end{aligned}$$

Now, it is observed that $\sum_{i=1}^{8} \xi_i = 1$ and $\xi_i \xi_j = \begin{cases} \xi_i, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$. Thus, by the Chinese Remainder Theorem,

$$R = \xi_1 R \oplus \xi_2 R \oplus \cdots \oplus \xi_8 R \cong \xi_1 \mathbb{F}_{p^m} \oplus \xi_2 \mathbb{F}_{p^m} \oplus \cdots \oplus \xi_8 \mathbb{F}_{p^m}.$$

Hence, any element $r \in R$ can be uniquely expressed as

$$r = a_1 + ua_2 + va_3 + wa_4 + uva_5 + vwa_6 + uwa_7 + uvwa_8 = \sum_{i=1}^{8} \xi_i k_i,$$

where $k_i \in \mathbb{F}_{p^m}$ for $i = 1, 2, \ldots, 8$. In the rest discussion, ξ_i $(i = 1, 2, \ldots, 8)$ will represent the above mentioned primitive orthogonal idempotent elements of the ring R.

In the present section, we discuss results on linear codes over R which are useful to determine the structure of λ-constacyclic and skew λ-constacyclic codes in the subsequent sections. In this connection, first we define a Gray map $\phi : R \longrightarrow \mathbb{F}_{p^m}^8$ by

$$\phi(a_1 + ua_2 + va_3 + wa_4 + uva_5 + vwa_6 + uwa_7 + uvwa_8) = (\alpha_1, \alpha_2, \ldots, \alpha_8), \tag{26.1}$$

where

$$\begin{aligned}
\alpha_1 &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8,\\
\alpha_2 &= a_1 + a_2 + a_3 - a_4 + a_5 - a_6 - a_7 - a_8,\\
\alpha_3 &= a_1 + a_2 - a_3 + a_4 - a_5 - a_6 + a_7 - a_8,\\
\alpha_4 &= a_1 - a_2 + a_3 + a_4 - a_5 + a_6 - a_7 - a_8,\\
\alpha_5 &= a_1 + a_2 - a_3 - a_4 - a_5 + a_6 - a_7 + a_8,\\
\alpha_6 &= a_1 - a_2 - a_3 + a_4 + a_5 - a_6 - a_7 + a_8,\\
\alpha_7 &= a_1 - a_2 + a_3 - a_4 - a_5 - a_6 + a_7 + a_8,\\
\alpha_8 &= a_1 - a_2 - a_3 - a_4 + a_5 + a_6 + a_7 - a_8.
\end{aligned} \tag{26.2}$$

The map ϕ can be extended to R^n in the natural way. Throughout the article, α_i represents the value given by (26.2), for $i = 1, 2, 3, \ldots, 8$. The *Hamming* *weight* $w_H(c)$ is defined as the number of non-zero components of the codeword $c = (c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C}$ and the distance between two codewords is given by $d_H(c_1, c_2) = w_H(c_1 - c_2)$. The Hamming distance for a code $\mathcal{C}$ is defined by $d_H(\mathcal{C}) = \min\{d_H(c_1, c_2) \mid c_1 \neq c_2, \forall\, c_1, c_2 \in \mathcal{C}\}$. Also, the Gray weight of any element $r = a_1 + ua_2 + va_3 + wa_4 + uva_5 + vwa_6 + uwa_7 + uvwa_8 \in R$ is define as $w_G(r) = w_H(\phi(r)) = w_H(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8)$ and Gray weight for $r' = (r_0, r_1, \ldots, r_{n-1}) \in R^n$ is $w_G(r') = \sum_{i=0}^{n-1} w_G(r_i)$. Again, the Gray distance between any two codewords c_1, c_2 is define by $d_G(c_1, c_2) = w_G(c_1 - c_2)$ and Gray distance for the code $\mathcal{C}$ is $d_G(\mathcal{C}) = \min\{d_G(c_1, c_2) \mid c_1 \neq c_2, c_1, c_2 \in \mathcal{C}\}$.

Proposition 26.1 *The map ϕ defined in (26.1) is linear and distance preserving map from (R^n, d_G) to $(\mathbb{F}_{p^m}^{8n}, d_H)$.*

Proof Let $r_1, r_2 \in R^n$ and $\beta \in \mathbb{F}_{p^m}$. Then it is easy to see that $\phi(r_1 + r_2) = \phi(r_1) + \phi(r_2)$ and $\phi(\beta r_1) = \beta\phi(r_1)$. Therefore, ϕ is an $\mathbb{F}_{p^m}$-linear map. Moreover, $d_G(r_1, r_2) = w_G(r_1 - r_2) = w_H(\phi(r_1 - r_2)) = w_H(\phi(r_1) - \phi(r_2)) = d_H(\phi(r_1), \phi(r_2))$. Hence, ϕ is a distance preserving map.

Proposition 26.2 *Let $\mathcal{C}$ be an $[n, k, d_G]$ linear code. Then $\phi(\mathcal{C})$ is an $[8n, k, d_H]$ linear code where $d_G = d_H$.*

Proof By Proposition 26.1, $\phi(\mathcal{C})$ is a linear code of length $8n$. Also, ϕ is bijective and isometric. Therefore, $\phi(\mathcal{C})$ has the parameters $[8n, k, d_H]$ with $d_G = d_H$.

Proposition 26.3 *Let $\mathcal{C}$ be a linear code of length n over R. Then $\phi(\mathcal{C}^\perp) = (\phi(\mathcal{C}))^\perp$. Moreover, $\mathcal{C}$ is self-dual if and only if $\phi(\mathcal{C})$ is self-dual.*

Proof Let $r = (r_0, r_1, \ldots, r_{n-1}) \in \mathcal{C}$, $t = (t_0, t_1, \ldots, t_{n-1}) \in \mathcal{C}^\perp$, where

$$
\begin{aligned}
r_i &= a_i + ub_i + vc_i + wd_i + uve_i + vwf_i + uwg_i + uvwh_i,\\
t_i &= a'_i + ub'_i + vc'_i + wd'_i + uve'_i + vwf'_i + uwg'_i + uvwh'_i,\\
&a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i, a'_i, b'_i, c'_i, d'_i, e'_i, f'_i, g'_i, h'_i \in \mathbb{F}_{p^m}.
\end{aligned}
$$

for $0 \leq i \leq n-1$. Now, $r \cdot t = 0$ implies

$$
\begin{aligned}
&\sum_{i=0}^{n-1}(a_i a'_i + b_i b'_i + c_i c'_i + d_i d'_i + e_i e'_i + f_i f'_i + g_i g'_i + h_i h'_i) = 0;\\
&\sum_{i=0}^{n-1}(a_i b'_i + a'_i b_i + c_i e'_i + c'_i e_i + d_i g'_i + d'_i g_i + f_i h'_i + f'_i h_i) = 0;\\
&\sum_{i=0}^{n-1}(a_i c'_i + a'_i c_i + b_i e'_i + b'_i e_i + d_i f'_i + d'_i f_i + g_i h'_i + g'_i h_i) = 0;\\
&\sum_{i=0}^{n-1}(a_i d'_i + a'_i d_i + b_i g'_i + b'_i g_i + c_i f'_i + c'_i f_i + e_i h'_i + e'_i h_i) = 0;
\end{aligned}
$$

$$\sum_{i=0}^{n-1}(a_ie_i' + a_i'e_i + b_ic_i' + b_i'c_i + d_ih_i' + d_i'h_i + f_ig_i' + f_i'g_i) = 0;$$
$$\sum_{i=0}^{n-1}(a_if_i' + a_i'f_i + b_ih_i' + b_i'h_i + c_id_i' + c_i'd_i + e_ig_i' + e_i'e_i) = 0;$$
$$\sum_{i=0}^{n-1}(a_ig_i' + a_i'g_i + b_id_i' + b_i'd_i + c_ih_i' + c_i'h_i + e_if_i' + e_i'f_i) = 0;$$
$$\sum_{i=0}^{n-1}(a_ih_i' + a_i'h_i + b_if_i' + b_i'f_i + c_ig_i' + c_i'g_i + d_ie_i' + d_i'e_i) = 0.$$

Also, $\phi(r)\cdot\phi(t) = 8\sum_{i=0}^{n-1}(a_ia_i' + b_ib_i' + c_ic_i' + d_id_i' + e_ie_i' + f_if_i' + g_ig_i' + h_ih_i') = 0$. Therefore, $\phi(\mathcal{C}^{\perp}) \subseteq (\phi(\mathcal{C}))^{\perp}$. Since ϕ is bijection, $\mid \phi(\mathcal{C}^{\perp}) \mid = \mid (\phi(\mathcal{C}))^{\perp} \mid$. Hence, $\phi(\mathcal{C}^{\perp}) = (\phi(\mathcal{C}))^{\perp}$. Further, let $\mathcal{C}$ be self-dual. Then $\mathcal{C}^{\perp} = \mathcal{C}$, implies $\phi(\mathcal{C}^{\perp}) = \phi(\mathcal{C})$, and this implies $(\phi(\mathcal{C}))^{\perp} = \phi(\mathcal{C})$. Hence, $\phi(\mathcal{C})$ is self-dual. Converse follows similarly.

Let $\mathcal{C}$ be a linear code of length n over R, and α_i for $i = 1, 2, \ldots, 8$ given in (26.2). We denote

$$\mathcal{C}_1 = \{\alpha_1 \in \mathbb{F}_{p^m}^n \mid a_1 + ua_2 + va_3 + wa_4 + uva_5 + vwa_6 + uwa_7 + uvwa_8 \in \mathcal{C}\};$$
$$\mathcal{C}_2 = \{\alpha_2 \in \mathbb{F}_{p^m}^n \mid a_1 + ua_2 + va_3 + wa_4 + uva_5 + vwa_6 + uwa_7 + uvwa_8 \in \mathcal{C}\};$$
$$\mathcal{C}_3 = \{\alpha_3 \in \mathbb{F}_{p^m}^n \mid a_1 + ua_2 + va_3 + wa_4 + uva_5 + vwa_6 + uwa_7 + uvwa_8 \in \mathcal{C}\};$$
$$\mathcal{C}_4 = \{\alpha_4 \in \mathbb{F}_{p^m}^n \mid a_1 + ua_2 + va_3 + wa_4 + uva_5 + vwa_6 + uwa_7 + uvwa_8 \in \mathcal{C}\};$$
$$\mathcal{C}_5 = \{\alpha_5 \in \mathbb{F}_{p^m}^n \mid a_1 + ua_2 + va_3 + wa_4 + uva_5 + vwa_6 + uwa_7 + uvwa_8 \in \mathcal{C}\};$$
$$\mathcal{C}_6 = \{\alpha_6 \in \mathbb{F}_{p^m}^n \mid a_1 + ua_2 + va_3 + wa_4 + uva_5 + vwa_6 + uwa_7 + uvwa_8 \in \mathcal{C}\};$$
$$\mathcal{C}_7 = \{\alpha_7 \in \mathbb{F}_{p^m}^n \mid a_1 + ua_2 + va_3 + wa_4 + uva_5 + vwa_6 + uwa_7 + uvwa_8 \in \mathcal{C}\};$$
$$\mathcal{C}_8 = \{\alpha_8 \in \mathbb{F}_{p^m}^n \mid a_1 + ua_2 + va_3 + wa_4 + uva_5 + vwa_6 + uwa_7 + uvwa_8 \in \mathcal{C}\}.$$

Then $\mathcal{C}_i$ is a linear code of length n over $\mathbb{F}_{p^m}$, for $i = 1, 2, \ldots, 8$. Let A_i be a linear code over R for $i = 1, 2, \ldots, 8$. Consider $A_1 \oplus A_2 \oplus \cdots \oplus A_8 = \{a_1 + a_2 + \cdots + a_8 \mid a_i \in A_i \ \forall\, i\}$ and $A_1 \otimes A_2 \otimes \cdots \otimes A_8 = \{(a_1, a_2, \ldots, a_8) \mid a_i \in A_i \ \forall\, i\}$.

Theorem 26.1 *Let $\mathcal{C}$ be a linear code of length n over R. Then,*

$$\phi(\mathcal{C}) = \mathcal{C}_1 \otimes \mathcal{C}_2 \otimes \cdots \otimes \mathcal{C}_8, \ \textit{and} \mid \mathcal{C} \mid = \mid \mathcal{C}_1 \mid\mid \mathcal{C}_2 \mid \cdots \mid \mathcal{C}_8 \mid.$$

Proof Let $z = (a_0^1, a_1^1, \ldots, a_{n-1}^1, a_0^2, a_1^2, \ldots, a_{n-1}^2, \ldots, a_0^8, a_1^8, \ldots, a_{n-1}^8) \in \phi(\mathcal{C})$, and

$$
\begin{aligned}
r_i = \gamma[&(a_i^1 + a_i^2 + a_i^3 + a_i^4 + a_i^5 + a_i^6 + a_i^7 + a_i^8)+\\
&u(a_i^1 + a_i^2 + a_i^3 - a_i^4 + a_i^5 - a_i^6 - a_i^7 - a_i^8)+\\
&v(a_i^1 + a_i^2 - a_i^3 + a_i^4 - a_i^5 - a_i^6 + a_i^7 - a_i^8)+\\
&w(a_i^1 - a_i^2 + a_i^3 + a_i^4 - a_i^5 + a_i^6 - a_i^7 - a_i^8)+\\
&uv(a_i^1 + a_i^2 - a_i^3 - a_i^4 - a_i^5 + a_i^6 - a_i^7 + a_i^8)+\\
&vw(a_i^1 - a_i^2 - a_i^3 + a_i^4 + a_i^5 - a_i^6 - a_i^7 + a_i^8)+\\
&uw(a_i^1 - a_i^2 + a_i^3 - a_i^4 - a_i^5 - a_i^6 + a_i^7 + a_i^8)+\\
&uvw(a_i^1 - a_i^2 - a_i^3 - a_i^4 + a_i^5 + a_i^6 + a_i^7 - a_i^8)]
\end{aligned} \tag{26.3}
$$

for $i = 1, 2, \dots, n-1$. Since ϕ is bijective, $r = (r_0, r_1, \dots, r_{n-1}) \in \mathcal{C}$. From the definition of $\mathcal{C}_i$, we have $(a_0^i, a_1^i, \dots, a_{n-1}^i) \in \mathcal{C}_i$ for $i = 1, 2, \dots, 8$ and this implies $z \in \mathcal{C}_1 \otimes \mathcal{C}_2 \otimes \cdots \otimes \mathcal{C}_8$. Hence, $\phi(\mathcal{C}) \subseteq \mathcal{C}_1 \otimes \mathcal{C}_2 \otimes \cdots \otimes \mathcal{C}_8$. Conversely, let

$$z = (a_0^1, a_1^1, \dots, a_{n-1}^1, a_0^2, a_1^2, \dots, a_{n-1}^2, \dots, a_0^8, a_1^8, \dots, a_{n-1}^8) \in \mathcal{C}_1 \otimes \mathcal{C}_2 \otimes \cdots \otimes \mathcal{C}_8.$$

Then $a^i = (a_0^i, a_1^i, \dots, a_{n-1}^i) \in \mathcal{C}_i$ for $i = 1, 2, \dots, 8$. In order to show $z \in \phi(\mathcal{C})$, we have to find $z' = \sum_{i=1}^8 s_i \xi_i \in \mathcal{C}$ such that $\phi(z') = z$. Take $s_i = \sum_{j=1}^8 \xi_j t_{ij}$ where $t_{ii} = a^i$, $1 \leq i \leq 8$ and $t_{ij} \in \mathbb{F}_{p^m}$ for all i, j. Then $z' = \sum_{i=1}^8 \xi_i t_{ii} = \sum_{i=1}^8 \xi_i a^i$ and $\phi(z') = z$. Therefore, $\mathcal{C}_1 \otimes \mathcal{C}_2 \otimes \cdots \otimes \mathcal{C}_8 \subseteq \phi(\mathcal{C})$. Hence, $\mathcal{C}_1 \otimes \mathcal{C}_2 \otimes \cdots \otimes \mathcal{C}_8 = \phi(\mathcal{C})$. Moreover, ϕ being bijection, $\mid \mathcal{C} \mid = \mid \phi(\mathcal{C}) \mid$. Thus, $\mid \mathcal{C} \mid = \mid \mathcal{C}_1 \mid\mid \mathcal{C}_2 \mid \cdots \mid \mathcal{C}_8 \mid$.

Corollary 26.1 *Let M_i be a generator matrix of $\mathcal{C}_i$ for $i = 1, 2, \dots, 8$. Then the generator matrix for the code $\mathcal{C}$ is* $M = \begin{pmatrix} \xi_1 M_1 \\ \xi_2 M_2 \\ \vdots \\ \xi_8 M_8 \end{pmatrix}$.

Corollary 26.2 *If $\phi(\mathcal{C}) = \mathcal{C}_1 \otimes \mathcal{C}_2 \otimes \cdots \otimes \mathcal{C}_8$, then $\mathcal{C} = \xi_1 \mathcal{C}_1 \oplus \xi_2 \mathcal{C}_2 \oplus \cdots \oplus \xi_8 \mathcal{C}_8$.*

Corollary 26.3 *Suppose $\mathcal{C} = \xi_1 \mathcal{C}_1 \oplus \xi_2 \mathcal{C}_2 \oplus \cdots \oplus \xi_8 \mathcal{C}_8$ is a linear code of length n over R where $\mathcal{C}_i$ is an $[n, k_i, d_H(\mathcal{C}_i)]$ linear code over $\mathbb{F}_{p^m}$, then $\phi(\mathcal{C})$ is an* $[8n, \sum_{i=1}^8 k_i, \min\{d_H(\mathcal{C}_i) \mid i = 1, 2, \dots, 8\}]$ *linear code.*

Theorem 26.2 *Let $\mathcal{C} = \xi_1 \mathcal{C}_1 \oplus \xi_2 \mathcal{C}_2 \oplus \cdots \oplus \xi_8 \mathcal{C}_8$ be a linear code of length n over R. Then $\mathcal{C}^\perp = \xi_1 \mathcal{C}_1^\perp \oplus \xi_2 \mathcal{C}_2^\perp \oplus \cdots \oplus \xi_8 \mathcal{C}_8^\perp$. Moreover, $\mathcal{C}$ is self-dual if and only if $\mathcal{C}_i$ is self-dual for $i = 1, 2, \dots, 8$.*

Proof Let us consider α_i, $i = 1, 2, \dots, 8$, of (26.2). Let $\mathcal{D}_i = \{\alpha_i \in \mathbb{F}_{p^m}^n \mid a_1 + ua_2 + va_3 + wa_4 + uva_5 + vwa_6 + uwa_7 + uvwa_8 \in \mathcal{C}_i^\perp\}$. Then $\mathcal{C}^\perp$ is uniquely expressed as $\mathcal{C}^\perp = \xi_1 \mathcal{D}_1 \oplus \xi_2 \mathcal{D}_2 \oplus \cdots \oplus \xi_8 \mathcal{D}_8$. It is easy to see that $\mathcal{D}_1 \subseteq \mathcal{C}_1^\perp$. Let $s \in \mathcal{C}_1^\perp$. Then $s \cdot a_1 = 0$ for all $a_1 \in \mathcal{C}_1$. Let $z = \sum_{i=1}^8 \xi_i a_i \in \mathcal{C}$. Then $\xi_1 s z = \xi_1 a_1 s = 0$ and this implies $\xi_1 s \in \mathcal{C}^\perp$. By the unique representation of $\mathcal{C}^\perp$, we have $s \in \mathcal{D}_1$. Then $\mathcal{C}_1^\perp \subseteq \mathcal{D}_1$. Hence, $\mathcal{D}_1 = \mathcal{C}_1^\perp$. Similarly, we can show that $\mathcal{C}_i^\perp = \mathcal{D}_i$ for $i = 2, 3, \dots, 8$. Thus, $\mathcal{C}^\perp = \xi_1 \mathcal{C}_1^\perp \oplus \xi_2 \mathcal{C}_2^\perp \oplus \cdots \oplus \xi_8 \mathcal{C}_8^\perp$.

Moreover, $\mathcal{C}$ is self-dual, then $\mathcal{C}^{\perp} = \mathcal{C}$ implies

$$\xi_1\mathcal{C}_1 \oplus \xi_2\mathcal{C}_2 \oplus \cdots \oplus \xi_8\mathcal{C}_8 = \xi_1\mathcal{C}_1^{\perp} \oplus \xi_2\mathcal{C}_2^{\perp} \oplus \cdots \oplus \xi_8\mathcal{C}_8^{\perp},$$

that is $\mathcal{C}_i^{\perp} = \mathcal{C}_i$ for $i = 1, 2, \ldots, 8$. Converse follows similarly.

26.3 Constacyclic Codes Over *R*

In the present section, for a unit $\lambda \in R$, we determine the structure of λ-constacyclic codes by decomposing into constacyclic codes over $\mathbb{F}_{p^m}$ (Theorem 26.4). In this way, we prove that these codes and their dual are principally generated (Corollary 26.6 and Corollary 26.7, respectively).

Definition 26.1 Let λ be a unit in R. A linear code $\mathcal{C}$ of length n over R is said to be a λ-constacyclic code if for any $c = (c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C}$, we have $\tau_\lambda(c) = (\lambda c_{n-1}, c_0, \ldots, c_{n-2}) \in \mathcal{C}$. The operator τ_λ is known as the λ-constacyclic shift. Note that a constacyclic code is cyclic if $\lambda = 1$ and negacyclic if $\lambda = -1$.

Lemma 26.1 *Let $\lambda = \lambda_1 + u\lambda_2 + v\lambda_3 + w\lambda_4 + uv\lambda_5 + vw\lambda_6 + uw\lambda_7 + uvw\lambda_8 \in R$. Then λ is a unit in R if and only if*

$$\begin{aligned}
\delta_1 &= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8,\\
\delta_2 &= \lambda_1 + \lambda_2 + \lambda_3 - \lambda_4 + \lambda_5 - \lambda_6 - \lambda_7 - \lambda_8,\\
\delta_3 &= \lambda_1 + \lambda_2 - \lambda_3 + \lambda_4 - \lambda_5 - \lambda_6 + \lambda_7 - \lambda_8,\\
\delta_4 &= \lambda_1 - \lambda_2 + \lambda_3 + \lambda_4 - \lambda_5 + \lambda_6 - \lambda_7 - \lambda_8,\\
\delta_5 &= \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 - \lambda_5 + \lambda_6 - \lambda_7 + \lambda_8,\\
\delta_6 &= \lambda_1 - \lambda_2 - \lambda_3 + \lambda_4 + \lambda_5 - \lambda_6 - \lambda_7 + \lambda_8,\\
\delta_7 &= \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 - \lambda_5 - \lambda_6 + \lambda_7 + \lambda_8,\\
\delta_8 &= \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 - \lambda_8
\end{aligned}$$

are units in $\mathbb{F}_{p^m}$.

Proof Let δ_i be unit in $\mathbb{F}_{p^m}$ for $i = 1, 2, \ldots, 8$. Now, the representation of λ is $\lambda = \sum_{i=1}^{8} \xi_i\delta_i$. Let $\gamma = \sum_{i=1}^{8} \xi_i\delta_i^{-1}$. Then $\lambda\gamma = \sum_{i=1}^{8} \mu_i = 1$. Hence, λ is a unit in R.

Conversely, let $\lambda = \lambda_1 + u\lambda_2 + v\lambda_3 + w\lambda_4 + uv\lambda_5 + vw\lambda_6 + uw\lambda_7 + uvw\lambda_8 = \sum_{i=1}^{8} \xi_i\delta_i$ be a unit in R. Then there exists $\gamma = \sum_{i=1}^{8} \xi_i\gamma_i \in R$ such that $\lambda\gamma = 1$ where $\gamma_i \in \mathbb{F}_{p^m}^*$ for $i = 1, 2, \ldots, 8$. Also, $\lambda\gamma = 1$ gives us $\sum_{i=1}^{8} \xi_i\gamma_i\delta_i = 1$ which implies $\xi_i\gamma_i\delta_i = \xi_i$ and hence $\gamma_i\delta_i = 1$ for $i = 1, 2, 3, \ldots, 8$. Thus, δ_i is a unit in $\mathbb{F}_{p^m}$ for $i = 1, 2, 3, \ldots, 8$.

In the present section, we discuss λ-constacyclic codes over R for the unit $\lambda = \lambda_1 + u\lambda_2 + v\lambda_3 + w\lambda_4 + uv\lambda_5 + vw\lambda_6 + uw\lambda_7 + uvw\lambda_8 \in R$. Let $\mathcal{C}$ be a λ-constacyclic code of length n over R. We identify each codeword $c = (c_0, c_1, \dots, c_{n-1}) \in \mathcal{C}$ with a polynomial $c(x) \in R[x]/\langle x^n - \lambda\rangle$ under the correspondence $c = (c_0, c_1, \dots, c_{n-1}) \mapsto c(x) = (c_0 + c_1x + \cdots + c_{n-1}x^{n-1}) \bmod (x^n - \lambda)$. By this polynomial representation of $\mathcal{C}$, one can easily verify the next result.

Theorem 26.3 *Let $\mathcal{C}$ be a linear code of length n over R. Then $\mathcal{C}$ is a λ-constacyclic code if and only if it is an ideal of the ring $R[x]/\langle x^n - \lambda\rangle$.*

Theorem 26.4 *Let $\mathcal{C} = \xi_1\mathcal{C}_1 \oplus \xi_2\mathcal{C}_2 \oplus \cdots \oplus \xi_8\mathcal{C}_8$ be a linear code of length n over R. Then $\mathcal{C}$ is a λ-constacyclic code if and only if $\mathcal{C}_i$ is a δ_i-constacyclic code over $\mathbb{F}_{p^m}$, for $i = 1, 2, \dots, 8$ where δ_i's are defined in Lemma 26.1.*

Proof Let $\mathcal{C}$ be a λ-constacyclic code of length n over R. Let

$$a^i = (a_0^i, a_1^i, \dots, a_{n-1}^i) \in \mathcal{C}_i \text{ for } i = 1, 2, \dots, 8,$$

and for $j = 1, 2, \dots, n-1$,

$$\begin{aligned} r_j = \gamma[&(a_j^1 + a_j^2 + a_j^3 + a_j^4 + a_j^5 + a_j^6 + a_j^7 + a_j^8)+\\ &u(a_j^1 + a_j^2 + a_j^3 - a_j^4 + a_j^5 - a_j^6 - a_j^7 - a_j^8)+\\ &v(a_j^1 + a_j^2 - a_j^3 + a_j^4 - a_j^5 - a_j^6 + a_j^7 - a_j^8)+\\ &w(a_j^1 - a_j^2 + a_j^3 + a_j^4 - a_j^5 + a_j^6 - a_j^7 - a_j^8)+\\ &uv(a_j^1 + a_j^2 - a_j^3 - a_j^4 - a_j^5 + a_j^6 - a_j^7 + a_j^8)+\\ &vw(a_j^1 - a_j^2 - a_j^3 + a_j^4 + a_j^5 - a_j^6 - a_j^7 + a_j^8)+\\ &uw(a_j^1 - a_j^2 + a_j^3 - a_j^4 - a_j^5 - a_j^6 + a_j^7 + a_j^8)+\\ &uvw(a_j^1 - a_j^2 - a_j^3 - a_j^4 + a_j^5 + a_j^6 + a_j^7 - a_j^8)]. \end{aligned} \tag{26.4}$$

Then $r = (r_0, r_1, \dots, r_{n-1}) \in \mathcal{C}$. Also, $\tau_\lambda(r) = (\lambda r_{n-1}, r_0, \dots, r_{n-2}) \in \mathcal{C}$ where

$$\tau_\lambda(r) = \sum_{i=1}^{8} \xi_i \tau_{\delta_i}(a^i) \in \mathcal{C} = \xi_1\mathcal{C}_1 \oplus \xi_2\mathcal{C}_2 \oplus \cdots \oplus \xi_8\mathcal{C}_8.$$

Therefore, $\tau_{\delta_i}(a^i) \in \mathcal{C}_i$ for $i = 1, 2 \dots, 8$. Hence, $\mathcal{C}_i$ is a δ_i-constacyclic code over $\mathbb{F}_{p^m}$, for $i = 1, 2, \dots, 8$.
Conversely, let $\mathcal{C}_i$ be a δ_i-constacyclic code over $\mathbb{F}_{p^m}$, for $i = 1, 2, \dots, 8$. Let $r = (r_0, r_1, \dots, r_{n-1}) \in \mathcal{C}$ where r_j is given in (26.4) for $j = 0, 1, \dots, n-1$. Then $a^i = (a_0^i, a_1^i, \dots, a_{n-1}^i) \in \mathcal{C}_i$ for $i = 1, 2, \dots, 8$ and hence $\tau_{\delta_i}(a^i) \in \mathcal{C}_i$ for $i = 1, 2 \dots, 8$. Now, $\tau_\lambda(r) = \sum_{i=1}^{8} \xi_i \tau_{\delta_i}(a^i) \in \xi_1\mathcal{C}_1 \oplus \xi_2\mathcal{C}_2 \oplus \cdots \oplus \xi_8\mathcal{C}_8 = \mathcal{C}$. Thus, $\mathcal{C}$ is a λ-constacyclic code of length n over R.

Corollary 26.4 *Let $\mathcal{C} = \xi_1\mathcal{C}_1 \oplus \xi_2\mathcal{C}_2 \oplus \cdots \oplus \xi_8\mathcal{C}_8$ be a linear code of length n over R. Then $\mathcal{C}$ is a cyclic code if and only if $\mathcal{C}_i$ is a cyclic code over $\mathbb{F}_{p^m}$, for $i = 1, 2, \dots, 8$.*

Theorem 26.5 *Let $\mathcal{C} = \xi_1\mathcal{C}_1 \oplus \xi_2\mathcal{C}_2 \oplus \cdots \oplus \xi_8\mathcal{C}_8$ be a λ-constacyclic code of length n over R. Then there exists a polynomial $f(x) \in R[x]$ such that $\mathcal{C} = \langle f(x)\rangle$ and $f(x) \mid (x^n - \lambda)$.*

Proof Since $\mathcal{C} = \xi_1\mathcal{C}_1 \oplus \xi_2\mathcal{C}_2 \oplus \cdots \oplus \xi_8\mathcal{C}_8$ is a λ-constacyclic code of length n, by Theorem 26.4, $\mathcal{C}_i$ is a δ_i-constacyclic code of length n over $\mathbb{F}_{p^m}$. Let $\mathcal{C}_i = \langle f_i(x)\rangle$ where $f_i(x) \mid (x^n - \delta_i)$ for $i = 1, 2, \ldots, 8$. Then $\xi_1 f_1(x), \xi_2 f_2(x), \cdots \xi_8 f_8(x)$ are generators of $\mathcal{C}$. Let $f(x) = \sum_{i=1}^{8} \xi_i f_i(x)$. Then $\langle f(x)\rangle \subseteq \mathcal{C}$. Also, $f_i(x)\xi_i = f(x)\xi_i \in \langle f(x)\rangle$ for $i = 1, 2, \ldots, 8$, therefore, $\mathcal{C} \subseteq \langle f(x)\rangle$. Hence, $\mathcal{C} = \langle f(x)\rangle$.
Since $f_i(x) \mid (x^n - \delta_i)$, so there exists $h_i(x) \in \mathbb{F}_{p^m}[x]$ such that $(x^n - \delta_i) = f_i(x)h_i(x)$ for $i = 1, 2, \ldots, 8$. Now, $[\sum_{i=1}^{8} \xi_i h_i(x)] f(x) = \sum_{i=1}^{8} \xi_i f_i(x)h_i(x) = \sum_{i=1}^{8} \xi_i (x^n - \delta_i) = (x^n - \lambda)$. This shows that $f(x)$ is a factor of $(x^n - \lambda)$.

Corollary 26.5 *Let $\mathcal{C} = \xi_1\mathcal{C}_1 \oplus \xi_2\mathcal{C}_2 \oplus \cdots \oplus \xi_8\mathcal{C}_8$ be a cyclic code of length n over R. Then there exists a polynomial $f(x) \in R[x]$ such that $\mathcal{C} = \langle f(x)\rangle$ and $f(x) \mid (x^n - 1)$.*

Corollary 26.6 *Every ideal of $R[x]/\langle x^n - \lambda\rangle$ is principally generated.*

Corollary 26.7 *Let $\mathcal{C} = \xi_1\mathcal{C}_1 \oplus \xi_2\mathcal{C}_2 \oplus \cdots \oplus \xi_8\mathcal{C}_8$ be a λ-constacyclic code of length n over R and $f_i(x)$ be a generator of $\mathcal{C}_i$ such that $x^n - \delta_i = h_i(x) f_i(x)$ for $i = 1, 2, \ldots, 8$. Then*

(i) *the dual $\mathcal{C}^\perp = \xi_1\mathcal{C}_1^\perp \oplus \xi_2\mathcal{C}_2^\perp \oplus \cdots \oplus \xi_8\mathcal{C}_8^\perp$ is a λ^{-1}-constacyclic code over R.*
(ii) *$\mathcal{C}^\perp = \langle \sum_{i=1}^{8} \xi_i h_i^*(x)\rangle$ where $h_i^*(x)$ is the reciprocal polynomial of $h_i(x)$, that is, $h_i^*(x) = x^{\deg(h_i(x))} h_i(1/x)$ for $i = 1, 2, \ldots, 8$.*
(iii) *$\mid \mathcal{C}^\perp \mid = p^{m \sum_{i=1}^{8} \deg(f_i(x))}$.*
(iv) *$\mathcal{C}$ is a self-dual λ-constacyclic code if and only if $\mathcal{C}_i$ is a self-dual δ_i-constacyclic code over $\mathbb{F}_{p^m}$ for $i = 1, 2, \ldots, 8$.*

Proof
(i) Since $\mathcal{C} = \xi_1\mathcal{C}_1 \oplus \xi_2\mathcal{C}_2 \oplus \cdots \oplus \xi_8\mathcal{C}_8$ is a λ-constacyclic code of length n over R, so by Theorem 26.4, $\mathcal{C}_i$ is a δ_i-constacyclic code of length n over $\mathbb{F}_{p^m}$ for $i = 1, 2, \ldots, 8$. Therefore, $\mathcal{C}_i^\perp$ is a δ_i^{-1}-constacyclic code over $\mathbb{F}_{p^m}$. Hence, by Theorem 26.4, $\mathcal{C}^\perp = \xi_1\mathcal{C}_1^\perp \oplus \xi_2\mathcal{C}_2^\perp \oplus \cdots \oplus \xi_8\mathcal{C}_8^\perp$ is a λ^{-1}-constacyclic code over R.
(ii) Let $\mathcal{C}_i^\perp = \langle h_i^*(x)\rangle$ where $h_i^*(x) = x^{\deg(h_i(x))} h_i(1/x)$ for $i = 1, 2, \ldots, 8$. Then, by Theorem 26.5, $\mathcal{C}^\perp = \langle \sum_{i=1}^{8} \xi_i h_i^*(x)\rangle$ where $h_i^*(x) = x^{\deg(h_i(x))} h_i(1/x)$ for $i = 1, 2, \ldots, 8$.
(iii) $\mid \mathcal{C}^\perp \mid = \mid \mathcal{C}_1^\perp \mid\mid \mathcal{C}_2^\perp \mid \cdots \mid \mathcal{C}_8^\perp \mid$
$= (p^m)^{\deg(f_1(x))} \cdot (p^m)^{\deg(f_2(x))} \cdots (p^m)^{\deg(f_8(x))} = p^{m \sum_{i=1}^{8} \deg(f_i(x))}$.
(iv) It is obvious.

26.4 Skew Constacyclic Codes Over *R*

Analogous to constacyclic codes, here we discuss skew λ-constacyclic codes and prove these codes and their dual are principally generated ideals (Corollary 26.11 and Corollary 26.12, respectively). In this direction, first we define an automorphism $\theta : R \to R$ by

$$\theta(a_1 + ua_2 + va_3 + wa_4 + uva_5 + vwa_6 + uwa_7 + uvwa_8)$$
$$= a_1^{p^s} + ua_2^{p^s} + va_3^{p^s} + wa_4^{p^s} + uva_5^{p^s} + vwa_6^{p^s} + uwa_7^{p^s} + uvwa_8^{p^s},$$

where $a_i \in \mathbb{F}_{p^m}$ for $i = 1, 2, \ldots, 8$. The automorphism θ is known as *Frobineous automorphism*. Let $\eta = \frac{m}{s}$ be the order of the automorphism θ. Clearly, the subring $\mathbb{F}_{p^s}[u, v, w]/\langle u^2 - 1, v^2 - 1, w^2 - 1, uv - uv, vw - wv, wu - uw\rangle$ is invariant under the automorphism θ. Then the set of polynomials $R[x; \theta] = \{a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in R, \text{ for } 0 \leq i \leq n\}$ is a non-commutative ring under the usual addition of polynomials and multiplication of polynomials defined by $(ax^i)(bx^j) = a\theta(b)^i x^{i+j}$, known as a skew polynomial ring. It is clear that the set $R_{n,\lambda,\theta} = R[x; \theta]/\langle x^n - \lambda\rangle$ is no more ring unless $\langle x^n - \lambda\rangle$ is a two sided ideal of $R[x; \theta]$. However, we can consider it as a left $R[x; \theta]$-module under the left multiplication defined by $r(x)(g(x) + \langle x^n - \lambda\rangle) = r(x)g(x) + \langle x^n - \lambda\rangle$ where $r(x), g(x) \in R[x; \theta]$.

Definition 26.2 Let $\lambda = \lambda_1 + u\lambda_2 + v\lambda_3 + w\lambda_4 + uv\lambda_5 + vw\lambda_6 + uw\lambda_7 + uvw\lambda_8$ be a unit in R where $\lambda_i \in \mathbb{F}_{p^s}^*$. Then a linear code $\mathcal{C}$ of length n over R is said to be a skew λ-constacyclic code if and only if $\tau_{\lambda,\theta}(c) := (\theta(\lambda c_{n-1}), \theta(c_0), \ldots, \theta(c_{n-2})) \in \mathcal{C}$ whenever $c = (c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C}$. The operator $\tau_{\lambda,\theta}$ is known as the skew λ-constacyclic shift.

Let $\mathcal{C}$ be a skew λ-constacyclic code of length n over R. We identify each codeword $c = (c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C}$ with a polynomial $c(x) = c_0 + c_1 x + \cdots + c_{n-1}x^{n-1} \in R_{n,\lambda,\theta}$ under the correspondence $c = (c_0, c_1, \ldots, c_{n-1}) \mapsto c(x) = (c_0 + c_1 x + \cdots + c_{n-1}x^{n-1}) \bmod (x^n - \lambda)$. Therefore, under the above identification, we can consider a skew λ-constacyclic code $\mathcal{C}$ as a subset of both R^n and $R_{n,\lambda,\theta}$. Further, we have the following theorem.

Theorem 26.6 *Let $\mathcal{C}$ be a linear code of length n over R. Then $\mathcal{C}$ is a skew λ-constacyclic code if and only if it is a left $R[x; \theta]$-submodule of $R_{n,\lambda,\theta}$.*

Proof Straightforward.

Theorem 26.7 *Let β be a unit in $\mathbb{F}_{p^m}$ and $\mathcal{C}$ be a skew β-constacyclic code of length n over $\mathbb{F}_{p^m}$. Then $\mathcal{C}$ is a left $\mathbb{F}_{p^m}[x; \theta]$-submodule of $\mathbb{F}_{p^m}[x; \theta]/\langle x^n - \beta\rangle$ given by $\mathcal{C} = \langle f(x)\rangle$ where $f(x)$ is a right divisor of $(x^n - \beta)$ in $\mathbb{F}_{p^m}[x; \theta]$.*

Proof Combination of Lemma 2, Lemma 3 and Theorem 1 of [9].

Theorem 26.8 *Let $\mathcal{C} = \xi_1\mathcal{C}_1 \oplus \xi_2\mathcal{C}_2 \oplus \cdots \oplus \xi_8\mathcal{C}_8$ be a linear code of length n over R. Then $\mathcal{C}$ is a skew λ-constacyclic code if and only if $\mathcal{C}_i$ is a skew δ_i-constacyclic code of length n over $\mathbb{F}_{p^m}$, for $i = 1, 2, \ldots, 8$.*

Proof Let $\mathcal{C}$ be a skew λ-constacyclic code of length n over R. Let

$$a^i = (a_0^i, a_1^i, \ldots, a_{n-1}^i) \in \mathcal{C}_i \text{ for } i = 1, 2, \ldots, 8,$$

and for $j = 1, 2, \ldots, n-1$,

$$
\begin{aligned}
r_j = \gamma[(&a_j^1 + a_j^2 + a_j^3 + a_j^4 + a_j^5 + a_j^6 + a_j^7 + a_j^8)+ \\
&u(a_j^1 + a_j^2 + a_j^3 - a_j^4 + a_j^5 - a_j^6 - a_j^7 - a_j^8)+ \\
&v(a_j^1 + a_j^2 - a_j^3 + a_j^4 - a_j^5 - a_j^6 + a_j^7 - a_j^8)+ \\
&w(a_j^1 - a_j^2 + a_j^3 + a_j^4 - a_j^5 + a_j^6 - a_j^7 - a_j^8)+ \\
&uv(a_j^1 + a_j^2 - a_j^3 - a_j^4 - a_j^5 + a_j^6 - a_j^7 + a_j^8)+ \\
&vw(a_j^1 - a_j^2 - a_j^3 + a_j^4 + a_j^5 - a_j^6 - a_j^7 + a_j^8)+ \\
&uw(a_j^1 - a_j^2 + a_j^3 - a_j^4 - a_j^5 - a_j^6 + a_j^7 + a_j^8)+ \\
&uvw(a_j^1 - a_j^2 - a_j^3 - a_j^4 + a_j^5 + a_j^6 + a_j^7 - a_j^8)].
\end{aligned} \tag{26.5}
$$

Then $r = (r_0, r_1, \ldots, r_{n-1}) \in \mathcal{C}$. Thus, $\tau_{\lambda,\theta}(r) = (\theta(\lambda r_{n-1}), \theta(r_0), \ldots, \theta(r_{n-2})) \in \mathcal{C}$ where $\tau_{\lambda,\theta}(r) = \sum_{i=1}^{8} \xi_i \tau_{\delta_i,\theta}(a^i) \in \mathcal{C} = \xi_1\mathcal{C}_1 \oplus \xi_2\mathcal{C}_2 \oplus \cdots \oplus \xi_8\mathcal{C}_8$. Hence, $\tau_{\delta_i,\theta}(a^i) \in \mathcal{C}_i$ for $i = 1, 2, \ldots, 8$. Thus, $\mathcal{C}_i$ is a skew δ_i-constacyclic code over $\mathbb{F}_{p^m}$, for $i = 1, 2, \ldots, 8$.

Conversely, let $\mathcal{C}_i$ be a skew δ_i-constacyclic code over $\mathbb{F}_{p^m}$, for $i = 1, 2, \ldots, 8$ and $r = (r_0, r_1, \ldots, r_{n-1}) \in \mathcal{C}$ where r_j is given by (26.5) for $j = 0, 1, \ldots, n-1$. Then $a^i = (a_0^i, a_1^i, \ldots, a_{n-1}^i) \in \mathcal{C}_i$ for $i = 1, 2, \ldots, 8$ and hence $\tau_{\delta_i,\theta}(a^i) \in \mathcal{C}_i$ for $i = 1, 2 \ldots, 8$. Now, $\tau_{\lambda,\theta}(r) = \sum_{i=1}^{8} \xi_i \tau_{\delta_i,\theta}(a^i) \in \xi_1\mathcal{C}_1 \oplus \xi_2\mathcal{C}_2 \oplus \cdots \oplus \xi_8\mathcal{C}_8 = \mathcal{C}$. Thus, $\mathcal{C}$ is a skew λ-constacyclic code of length n over R.

Corollary 26.8 *Let $\mathcal{C} = \xi_1\mathcal{C}_1 \oplus \xi_2\mathcal{C}_2 \oplus \cdots \oplus \xi_8\mathcal{C}_8$ be a linear code of length n over R. Then $\mathcal{C}$ is a skew cyclic code if and only if $\mathcal{C}_i$ is a skew cyclic code of length n over $\mathbb{F}_{p^m}$, for $i = 1, 2, \ldots, 8$.*

Theorem 26.9 *Let $\mathcal{C} = \xi_1\mathcal{C}_1 \oplus \xi_2\mathcal{C}_2 \oplus \cdots \oplus \xi_8\mathcal{C}_8$ be a skew λ-constacyclic code of length n over R. Then there exists a polynomial $f(x) \in R[x; \theta]$ such that $\mathcal{C} = \langle f(x) \rangle$ and $f(x)$ is a right divisor of $(x^n - \lambda)$ in $R[x; \theta]$.*

Proof Since $\mathcal{C}$ is a skew λ-constacyclic code of length n over R, by Theorem 26.8, $\mathcal{C}_i$ is a skew δ_i-constacyclic code of length n over $\mathbb{F}_{p^m}$ for $i = 1, 2, \ldots, 8$. Also, by Theorem 26.7, we have $\mathcal{C}_i = \langle f_i(x) \rangle$ where $f_i(x)$ is a right divisor of $(x^n - \delta_i)$ in $\mathbb{F}_{p^m}[x; \theta]$ for $i = 1, 2, \ldots, 8$. Then $\xi_i f_i(x)$ is a generator of $\mathcal{C}$ for $i = 1, 2, \ldots, 8$. Let $f(x) = \sum_{i=1}^{8} \xi_i f_i(x)$. Then $\langle f(x) \rangle \subseteq \mathcal{C}$. On the other side, we have $\xi_i f_i(x) = \xi_i f(x) \in \langle f(x) \rangle$ for $i = 1, 2, \ldots, 8$. Hence, $\mathcal{C} \subseteq \langle f(x) \rangle$. Consequently, $\mathcal{C} = \langle f(x) \rangle$.
Since $f_i(x)$ is a right divisor of $(x^n - \delta_i)$ in $\mathbb{F}_{p^m}[x; \theta]$, so there exists $h_i(x) \in \mathbb{F}_{p^m}[x; \theta]$ such that $(x^n - \delta_i) = h_i(x) f_i(x)$ for $i = 1, 2, \ldots, 8$.
Now, $[\sum_{i=1}^{8} \xi_i h_i(x)] f(x) = \sum_{i=1}^{8} \xi_i h_i(x) f_i(x) = \sum_{i=1}^{8} \xi_i (x^n - \delta_i) = (x^n - \lambda)$. This proves that $f(x)$ is a right divisor of $(x^n - \lambda)$ in $R[x; \theta]$.

Corollary 26.9 *Let $\mathcal{C} = \xi_1\mathcal{C}_1 \oplus \xi_2\mathcal{C}_2 \oplus \cdots \oplus \xi_8\mathcal{C}_8$ be a skew cyclic code of length n over R. Then there exists a polynomial $f(x) \in R[x; \theta]$ such that $\mathcal{C} = \langle f(x) \rangle$ and $f(x)$ is a right divisor of $(x^n - 1)$ in $R[x; \theta]$.*

Corollary 26.10 *Let $\mathcal{C} = \xi_1\mathcal{C}_1 \oplus \xi_2\mathcal{C}_2 \oplus \cdots \oplus \xi_8\mathcal{C}_8$ be a skew cyclic code of length n over R. Then there exists a polynomial $f(x) \in R[x; \theta]$ such that $\mathcal{C} = \langle f(x) \rangle$ and $f(x)$ is a right divisor of $(x^n - 1)$ in $R[x; \theta]$.*

Corollary 26.11 *Every left submodule of $R[x;\theta]/\langle x^n-\lambda\rangle$ is principally generated.*

Corollary 26.12 *Let $\mathcal{C}=\xi_1\mathcal{C}_1\oplus\xi_2\mathcal{C}_2\oplus\cdots\oplus\xi_8\mathcal{C}_8$ be a skew λ-constacyclic code of length $n=\eta l$ (where l is some positive integer and η is the order of the automorphism θ) over R. Let $f_i(x)$ be a generator of $\mathcal{C}_i$ such that $x^n-\delta_i=h_i(x)f_i(x)$ in $\mathbb{F}_{p^m}[x;\theta]$ for $i=1,2,\ldots,8$. Then*

(i) *the dual $\mathcal{C}^{\perp}=\xi_1\mathcal{C}_1^{\perp}\oplus\xi_2\mathcal{C}_2^{\perp}\oplus\cdots\oplus\xi_8\mathcal{C}_8^{\perp}$ is a skew λ^{-1}-constacyclic code where $\mathcal{C}_i^{\perp}$ is the skew δ_i^{-1}-constacyclic code over $\mathbb{F}_{p^m}$ for $i=1,2,\ldots,8$.*
(ii) *$\mathcal{C}^{\perp}=\langle\sum_{i=1}^{8}\mu_i\tilde{h}_i(x)\rangle$ where $(x^n-\delta_i)=h_i(x)f_i(x)$, with*

$$f_i(x)=f_0^i+f_1^ix+\cdots+f_r^ix^r,\ h_i(x)=h_0^i+h_1^ix+\cdots+h_{n-r}^ix^{n-r},$$
$$\tilde{h}_i(x)=h_{n-r}^i+\theta(h_{n-r-1}^i)x+\cdots+\theta(h_0^i)x^{n-r}\ \textit{for}\ i=1,2,\ldots,8.$$

(iii) $|\,\mathcal{C}^{\perp}\,|=p^{m\sum_{i=1}^{4}\deg(f_i(x))}$.

Proof (i) Since λ is fixed by the automorphism θ, so by [19, Lemma 3.1], we have $\mathcal{C}^{\perp}$ is a skew λ^{-1}-constacyclic code. Also, we have $\lambda^{-1}=\sum_{i=1}^{8}\mu_i\delta_i^{-1}$. Therefore, by Theorem 26.8, $\mathcal{C}_i$ is a skew δ_i^{-1}-constacyclic code over $\mathbb{F}_{p^m}$ for $i=1,2,\ldots,8$.
(ii) From part (1), we have $\mathcal{C}^{\perp}=\xi_1\mathcal{C}_1^{\perp}\oplus\xi_2\mathcal{C}_2^{\perp}\oplus\cdots\oplus\xi_8\mathcal{C}_8^{\perp}$ is a skew λ^{-1}-constacyclic code where $\mathcal{C}_i^{\perp}$ is a skew δ_i^{-1}-constacyclic code over $\mathbb{F}_{p^m}$ for $i=1,2,\ldots,8$. Therefore, by [6, Theorem 4.4], we have $\mathcal{C}_i^{\perp}=\langle\tilde{h}_i(x)\rangle$ where $(x^n-\delta_i)=h_i(x)f_i(x)$, with $f_i(x)=f_0^i+f_1^ix+\cdots+f_r^ix^r$, $h_i(x)=h_0^i+h_1^ix+\cdots+h_{n-r}^ix^{n-r}$ and $\tilde{h}_i(x)=h_{n-r}^i+\theta(h_{n-r-1}^i)x+\cdots+\theta(h_0^i)x^{n-r}$ for $i=1,2,\ldots,8$. Hence, by Theorem 26.9, $\mathcal{C}^{\perp}=\langle\sum_{i=1}^{8}\mu_i\tilde{h}_i(x)\rangle$.
(iii) It is obvious.

26.5 Gray Images of Constacyclic Codes Over *R*

In this section we aim to discuss the $\mathbb{F}_{p^m}$-images of both λ-constacyclic and skew λ-constacyclic codes under the Gray map defined in (26.1). Theorem 26.10 and Theorem 26.11 are two main results which obtain quasi-twisted codes as the Gray images of these codes.

Definition 26.3 For some positive integers k and l, let $\mathcal{C}$ be a linear code of length $n=kl$ over R. The skew quasi-twisted shift operator $\pi_{\theta,l}:R^n\to R^n$ is define as

$$\pi_{\theta,l}(r)=(r_1\mid r_2\mid\cdots\mid r_l)=(\tau_{\delta_1,\theta}(r^1)\mid\tau_{\delta_2,\theta}(r^2)\mid\cdots\mid\tau_{\delta_8,\theta}(r^l)),$$

where $r^i\in R^k$ for $i=1,2,\ldots,l$ and $\tau_{\delta_i,\theta}$ is the skew δ_i-constacyclic shift. Then

(i) $\mathcal{C}$ is said to be a skew quasi-twisted code of length n and index l if $\pi_{\theta,l}(\mathcal{C})=\mathcal{C}$.

(ii) $\mathcal{C}$ is said to be a quasi-twisted code of length n and index l if $\pi_l(\mathcal{C}) = \mathcal{C}$, or in other words, skew quasi-twisted is quasi-twisted if θ is the identity automorphism on R.

Consider α_i given by (26.2). Also, we use the notations $\alpha_1^i = a_1^i + a_2^i + a_3^i + a_4^i + a_5^i + a_6^i + a_7^i + a_8^i, \alpha_2^i = a_1^i + a_2^i + a_3^i - a_4^i + a_5^i - a_6^i - a_7^i - a_8^i \quad (0 \le i \le n-1)$ and so on.

Lemma 26.2 *Let $\tau_{\lambda,\theta}$ be skew λ-constacyclic shift, $\pi_{\theta,8}$ be the skew quasi-twisted shift and ϕ be the Gray map defined in* (26.1)*. Then $\phi\tau_{\lambda,\theta} = \pi_{\theta,8}\phi$.*

Proof Let $r_i = a_1^i + ua_2^i + va_3^i + wa_4^i + uva_5^i + vwa_6^i + uwa_7^i + uvwa_8^i \in R$ for $i = 0, 1, \ldots, n-1$. Then $r = (r_0, r_1, \ldots, r_{n-1}) \in R^n$. Now

$$\begin{aligned}
\phi\tau_{\lambda,\theta}(r) &= \phi(\theta(\lambda r_{n-1}), \theta(r_0), \ldots, \theta(r_{n-2}))\\
&= (\delta_1\theta(\alpha_1^{n-1}), \theta(\alpha_1^0), \ldots, \theta(\alpha_1^{n-2}), \delta_2\theta(\alpha_2^{n-1}), \theta(\alpha_2^0), \ldots, \theta(\alpha_2^{n-2}),\\
&\quad \delta_3\theta(\alpha_3^{n-1}), \theta(\alpha_3^0), \ldots, \theta(\alpha_3^{n-2}), \delta_4\theta(\alpha_4^{n-1}), \theta(\alpha_4^0), \ldots, \theta(\alpha_4^{n-2}),\\
&\quad \delta_5\theta(\alpha_5^{n-1}), \theta(\alpha_5^0), \ldots, \theta(\alpha_5^{n-2}), \delta_6\theta(\alpha_6^{n-1}), \theta(\alpha_6^0), \ldots, \theta(\alpha_6^{n-2}),\\
&\quad \delta_7\theta(\alpha_7^{n-1}), \theta(\alpha_7^0), \ldots, \theta(\alpha_7^{n-2}), \delta_8\theta(\alpha_8^{n-1}), \theta(\alpha_8^0), \ldots, \theta(\alpha_8^{n-2})).
\end{aligned}$$

On the other side,

$$\begin{aligned}
\pi_{\theta,8}\phi(r) &= \pi_{\theta,8}(\alpha_1^0, \alpha_1^1, \ldots, \alpha_1^{n-1}, \alpha_2^0, \alpha_2^1, \ldots, \alpha_2^{n-1}, \alpha_3^0, \alpha_3^1, \ldots, \alpha_3^{n-1},\\
&\quad \alpha_4^0, \alpha_4^1, \ldots, \alpha_4^{n-1}, \alpha_5^0, \alpha_5^1, \ldots, \alpha_5^{n-1}, \alpha_6^0, \alpha_6^1, \ldots, \alpha_6^{n-1},\\
&\quad \alpha_7^0, \alpha_7^1, \ldots, \alpha_7^{n-1}, \alpha_8^0, \alpha_8^1, \ldots, \alpha_8^{n-1})\\
&= (\delta_1\theta(\alpha_1^{n-1}), \theta(\alpha_1^0), \ldots, \theta(\alpha_1^{n-2}), \delta_2\theta(\alpha_2^{n-1}), \theta(\alpha_2^0), \ldots, \theta(\alpha_2^{n-2}),\\
&\quad \delta_3\theta(\alpha_3^{n-1}), \theta(\alpha_3^0), \ldots, \theta(\alpha_3^{n-2}), \delta_4\theta(\alpha_4^{n-1}), \theta(\alpha_4^0), \ldots, \theta(\alpha_4^{n-2}),\\
&\quad \delta_5\theta(\alpha_5^{n-1}), \theta(\alpha_5^0), \ldots, \theta(\alpha_5^{n-2}), \delta_6\theta(\alpha_6^{n-1}), \theta(\alpha_6^0), \ldots, \theta(\alpha_6^{n-2}),\\
&\quad \delta_7\theta(\alpha_7^{n-1}), \theta(\alpha_7^0), \ldots, \theta(\alpha_7^{n-2}), \delta_8\theta(\alpha_8^{n-1}), \theta(\alpha_8^0), \ldots, \theta(\alpha_8^{n-2})).
\end{aligned}$$

Thus, $\phi\tau_{\lambda,\theta} = \pi_{\theta,8}\phi$.

Theorem 26.10 *Let $\mathcal{C}$ be a linear code of length n over R. Then $\mathcal{C}$ is a skew λ-constacyclic code if and only if its Gray image $\phi(\mathcal{C})$ is a skew quasi-twisted code of length $8n$ and index 8 over $\mathbb{F}_{p^m}$.*

Proof Let $\mathcal{C}$ be a skew λ-constacyclic code of length n over R. Then $\tau_{\lambda,\theta}(\mathcal{C}) = \mathcal{C}$. By Lemma 26.2, $\phi(\tau_{\lambda,\theta}(\mathcal{C})) = \phi(\mathcal{C}) = \pi_{\theta,8}(\phi(\mathcal{C}))$. Therefore, $\phi(\mathcal{C})$ is a skew quasi-twisted code of length $8n$ and index 8 over $\mathbb{F}_{p^m}$.

Conversely, let $\phi(\mathcal{C})$ be a skew quasi-twisted code of length $8n$ and index 8 over $\mathbb{F}_{p^m}$. Then $\pi_{\theta,8}(\phi(\mathcal{C})) = \phi(\mathcal{C})$. By Lemma 26.2, $\phi(\tau_{\lambda,\theta}(\mathcal{C})) = \pi_{\theta,8}(\phi(\mathcal{C})) = \phi(\mathcal{C})$. Since ϕ is injective, so $\tau_{\lambda,\theta}(\mathcal{C}) = \mathcal{C}$. Hence, $\mathcal{C}$ is a skew λ-constacyclic code of length n over R.

Lemma 26.3 *Let τ_λ be the λ-constacyclic shift, π_8 be the quasi-twisted shift and ϕ be the Gray map defined in (26.1). Then $\phi\tau_\lambda = \pi_8\phi$.*

Proof Same as the proof of Lemma 26.2.

Theorem 26.11 *Let $\mathcal{C}$ be a linear code of length n over R. Then $\mathcal{C}$ is a λ-constacyclic code if and only if its Gray image $\phi(\mathcal{C})$ is a quasi-twisted code of length $8n$ and index 8 over $\mathbb{F}_{p^m}$.*

Proof Using Lemma 26.3, it can be easily verified.

Now, to validate our obtained results we present two examples below.

Example 26.1 Let $\lambda = (1 + u + v + w + uv + vw + uw)$ be a unit in

$$R = \mathbb{F}_3[u, v, w]/\langle u^2 - 1, v^2 - 1, w^2 - 1, uv - vu, vw - wv, wu - uw\rangle.$$

Then $\delta_1 = \delta_2 = \delta_3 = \delta_4 = \delta_8 = 1$ and $\delta_5 = \delta_6 = \delta_7 = -1$. Now, in $\mathbb{F}_3[x]$, we have

$$x^8 - 1 = (x + 1)(x + 2)(x^2 + 1)(x^2 + x + 2)(x^2 + 2x + 2) = g_1g_2g_3g_4g_5;$$
$$x^8 + 1 = (x^4 + x^2 + 2)(x^4 + 2x^2 + 2) = l_1l_2.$$

Then $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}_3 = \mathcal{C}_4 = \mathcal{C}_8 = \langle g_3g_4\rangle = \langle x^4 + x^3 + x + 2\rangle$ is cyclic and $\mathcal{C}_5 = \mathcal{C}_6 = \mathcal{C}_7 = \langle l_2\rangle = \langle x^4 + 2x^2 + 2\rangle$ is a negacyclic code of length 8 over $\mathbb{F}_3$. Therefore, $\mathcal{C} = \langle(\xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_8)g_3g_4 + (\xi_5 + \xi_6 + \xi_7)l_2\rangle = \langle x^4 + 2(u + v + w + uv + vw + uw)(x^3 + x^2 + x) + x^3 + x + 2\rangle$ is a λ-constacyclic code of length 8 over R. Further, by Corollary 26.3, $\phi(\mathcal{C})$ is a [64, 32, 3] linear code over $\mathbb{F}_3$.

Example 26.2 Let $\lambda = (1 + u + v)$ be a unit in

$$R = \mathbb{F}_5[u, v, w]/\langle u^2 - 1, v^2 - 1, w^2 - 1, uv - vu, vw - wv, wu - uw\rangle.$$

Now,

$$\begin{aligned}
(x^{11} - 1) &= (x + 4)(x^5 + 2x^4 + 4x^3 + x^2 + x + 4)(x^5 + 4x^4 + 4x^3 + x^2 + 3x + 4)\\
&= g_1g_2g_3 \in \mathbb{F}_5[x],\\
(x^{11} + 1) &= (x + 1)(x^5 + x^4 + 4x^3 + 4x^2 + 3x + 1)(x^5 + 3x^4 + 4x^3 + 4x^2 + x + 1)\\
&= g'_1g'_2g'_3 \in \mathbb{F}_5[x],\\
(x^{11} - 3) &= (x + 3)(x^5 + 3x^4 + x^3 + 3x^2 + 3x + 3)(x^5 + 4x^4 + x^3 + 3x^2 + x + 3)\\
&= g''_1g''_2g''_3 \in \mathbb{F}_5[x].
\end{aligned}$$

Then $\mathcal{C}_1 = \mathcal{C}_2 = \langle g_2'' \rangle = \langle x^5 + 3x^4 + x^3 + 3x^2 + 3x + 3 \rangle$ is a 3-constacyclic code, $\mathcal{C}_6 = \mathcal{C}_8 = \langle g_2' \rangle = \langle x^5 + x^4 + 4x^3 + 4x^2 + 3x + 1 \rangle$ is a negacyclic code and $\mathcal{C}_3 = \mathcal{C}_4 = \mathcal{C}_5 = \mathcal{C}_7 = \langle g_2 \rangle = \langle x^5 + 2x^4 + 4x^3 + x^2 + x + 4 \rangle$ is a cyclic code of length 11 over $\mathbb{F}_5$. Therefore, $\mathcal{C} = \langle (\xi_1 + \xi_2) g_2'' + (\xi_3 + \xi_4 + \xi_5 + \xi_7) g_2 + (\xi_6 + \xi_8) g_2' \rangle = \langle x^5 + (2 + 3u + 3v)x^4 + (2 + 3u + 3v + 3uv)x^3 + (1 + u + v)x^2 + (2 + uv)x + 3 + 3u + 3v + 4uv \rangle$ is a $(1 + u + v)$-constacyclic code of length 11 over R. Hence, by Corollary 26.3, $\phi(\mathcal{C})$ is a [88, 48, 5] linear code over $\mathbb{F}_5$.

26.6 Conclusion

In this article, we investigate the structure of constacyclic and skew constacyclic codes over R. We obtain the necessary and sufficient conditions for self-dual λ-constacyclic codes over R. Along with other results, it is shown that constacyclic and skew constacyclic codes of an arbitrary length over R are principally generated. This work provides the open problem to obtain the quantum codes using λ-constacyclic codes over R in the future.

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Chapter 27
Wallis Type Formula and a Few Versions of the Number π in q-Calculus

Sladjana D. Marinković, Predrag M. Rajković, and Sergei Silvestrov

Abstract In this paper, we expose a geometrical interpretation of the q-Wallis formula. We construct plane regions which consist of rectangles whose edges' lengths are directly connected with factors in this formula. These regions are bounded by quarters of inside and outside circles from which we get estimates and conclusions about the number π_q.

Keywords q-Wallis formula · q-number · q-gamma function.

MSC 2020 Classification 33D05

27.1 Introduction

John Wallis [17] discovered the famous formula

$$\frac{\pi}{2} = \prod_{k=1}^{\infty} \frac{(2k)^2}{(2k-1)(2k+1)}, \tag{27.1}$$

in 1655 by the relation between the integral

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$$\int_0^1 \sqrt{1-x^2}\,dx,$$

and the area of the quarter of the unit circle $0 \le x \le 1, 0 \le y \le \sqrt{1-x^2}$. Here, he applied an original method of interpolation. This formula is directly connected with Brouncker's continued fraction (see [12]):

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \dots}}} = 1 + \mathbf{K}_{n=1}^{\infty} \frac{(2n-1)^2}{2}\ .$$

Later, Euler has discovered another proof of Wallis formula (27.1) based on the product development

$$\frac{\sin x}{x} = \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2\pi^2}\right), \tag{27.2}$$

for $x = \pi/2$. Also, Wallis formula is often connected with the evaluation of the definite integrals

$$\int_0^{\pi/2} \sin^n x\,dx \qquad (n \in \mathbb{N}).$$

Based on simple formula which L. Euler wrote in 1738:

$$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3},$$

and the property of the Fibonacci numbers

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n \quad \Leftrightarrow \quad \arctan F_{2n+1}^{-1} = \arctan F_{2n}^{-1} - \arctan F_{2n+2}^{-1},$$

D.H. Lehmer [14] has constructed the infinite expression for π in terms of the Fibonacci numbers:

$$\pi = 4\sum_{k=1}^{\infty} \arctan\left(\frac{1}{F_{2k+1}}\right).$$

K. Hayashi [9] gave one obvious geometrical proof in 1989. J. Sondow and H. Yi [15] used this idea to construct new infinite products of Wallis and Catalan type for π and e.

Wallis formula inspired a few mathematicians to avoid integration and prove it by the trigonometric identities as in [20], or by the special geometrical figures as in [18].

Here, we will use the previous idea to justify an approach to the number π in q-calculus.

27.2 On q-numbers

Throughout the whole paper, it will be assumed that $q \in (0, 1)$. The q-number $[a]_q$ defined in [7] is

$$[a]_q := \frac{1 - q^a}{1 - q}, \quad a \in \mathbb{R}. \tag{27.3}$$

The factorial of a positive integer number $[n]_q$ is given by

$$[0]_q! := 1, \quad [n]_q! := [n]_q[n-1]_q \cdots [1]_q, \qquad (n \in \mathbb{N}). \tag{27.4}$$

In particular,

$$\lim_{q \to 1-} [a]_q = a, \qquad \lim_{q \to 1-} [n]_q! = n!.$$

Let us recall that, for $a \in \mathbb{R}$, $\lfloor a \rfloor$ denotes the largest integer number less or equal to a and $\lceil a \rceil$ is the smallest integer number greater or equal to a.

Lemma 27.1 *For any $n \in \mathbb{N}$ and $0 < q < 1$, the following minimal and maximal relations hold:*

$$\min_{i \in \{0,1,\dots,n\}} \big([i]_q + [n-i]_q\big) = [n]_q, \tag{27.5}$$

$$\max_{i \in \{0,1,\dots,n\}} \big([i]_q + [n-i]_q\big) = [\lfloor n/2 \rfloor]_q + [\lceil n/2 \rceil]_q. \tag{27.6}$$

Proof Notice that

$$[i]_q + [n-i]_q = \frac{1 - q^i}{1 - q} + \frac{1 - q^{n-i}}{1 - q} = \frac{2 - q^i - q^{n-i}}{1 - q}.$$

Let us consider the function

$$f(x) = 2 - q^x - q^{n-x}, \qquad x \in [0, n].$$

Hence $f(x) \geq 0$ on $[0, n]$, and $f(0) = f(n) = 1 - q^n$. Its derivatives are

$$f'(x) = (q^{n-x} - q^x) \log q, \quad f''(x) = -(q^x + q^{n-x})(\log q)^2.$$

Since $f''(x) < 0$, the function $f(x)$ is the concave on $\mathbb{R}$. Hence

$$\min_{0 \leq x \leq n} f(x) = f(0) = f(n),$$

wherefrom the relation (27.5) follows.

Notice that the equation $f'(x) = 0$ has the unique solution $x = n/2$. Since $f(x)$ is a concave function on $[0, n]$ then we can conclude that

$$\max_{0 \le x \le n} f(x) = f(n/2).$$

If n is an even number, then $n/2 \in \mathbb{N}$ and it confirms (27.6). But, if n is an odd number, then $n/2 \notin \mathbb{N}$, the closest integer numbers are $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$. Since $f(n-x) = f(x)$ for all $x \in (0, n)$, taking the discrete values $x = i \in \{0, 1, \ldots, n\}$, we get the conclusion (27.6). □

Lemma 27.2 *The following estimates are true:*

$$0 < \frac{[n-1]_q [n+1]_q}{[n]_q^2} < 1, \qquad 1 < \frac{[n+1]_q}{[n]_q} \qquad (0 < q < 1; n \in \mathbb{N}). \tag{27.7}$$

Proof Since $0 \le (1-q)^2 = 1 + q^2 - 2q$, i.e., $-(1+q^2) \le -2q$, we have

$$\frac{[n-1]_q [n+1]_q}{[n]_q^2} = \frac{(1-q^{n-1})(1-q^{n+1})}{(1-q^n)^2} = \frac{1 + q^{2n} - q^{n-1}(1+q^2)}{1 + q^{2n} - 2q^n},$$

wherefrom the first statement follows. The second relation follows immediately from $0 < q^{n+1} < q^n$, i.e., $0 < 1 - q^n < 1 - q^{n+1}$.

27.3 On the q-Wallis Formula

Here, we will consider a q-analog of the number π defined by

$$\pi(q) = (1+q) \prod_{k=1}^{\infty} \frac{[2k]_q^2}{[2k-1]_q [2k+1]_q}. \tag{27.8}$$

Let us denote by

$$\pi_n(q) = (1+q) W_n(q), \quad \text{where} \quad W_n(q) = \prod_{k=1}^{n} \frac{[2k]_q^2}{[2k-1]_q [2k+1]_q}. \tag{27.9}$$

Lemma 27.3 *The sequence $\{W_n(q)\}$ is monotonically increasing, i.e.,*

$$W_n(q) \le W_{n+1}(q) \qquad (n \in \mathbb{N}). \tag{27.10}$$

Proof According to definition (27.9), obviously $W_n(q) > 0$ for every $q \in (0, 1)$ and $n \in \mathbb{N}$, and

$$W_n(q) = \frac{[2n]_q^2}{[2n-1]_q[2n+1]_q} W_{n-1}(q) \Leftrightarrow 0 < \frac{W_{n-1}(q)}{W_n(q)} = \frac{[2n-1]_q[2n+1]_q}{[2n]_q^2} < 1.$$

The last inequalities follow from the Lemma 2 and relation (27.7). □

The definition (27.8) is slightly different from the Gosper's q-version of the number π, here denoted with $\pi^{(G)}(q)$, given by (see [8] or [16])

$$\pi^{(G)}(q) = \lim_{n\uparrow\infty} \pi_n^{(G)}(q), \quad \text{where} \quad \pi_n^{(G)}(q) = 4\frac{1-q^{1/2}}{q^{1/4}} \cdot \frac{(q^2;q^2)_n^2}{(q;q^2)_n^2}. \tag{27.11}$$

These definitions are related by

$$\pi_n^{(G)}(q) = \frac{4(1-q^{2n+1})}{q^{1/4}(1+q^{1/2})(1+q)} \pi_n(q).$$

They both converge to the number π:

$$\lim_{q\to 1} \lim_{n\to\infty} \pi_n^{(G)}(q) = \lim_{q\to 1} \lim_{n\to\infty} \pi_n(q) = \pi.$$

Remark 27.1 Although the values of $\{\pi_n(q)\}$ and $\{\pi_n^{(G)}(q)\}$ are very close asymptotically, their expressions have influence in their numerical computing. Their values, computed in high precision for $n = 10000$ by *Wolfram Mathematica*, are shown in the Table 27.1.

Table 27.1 Values for $\{\pi_n(q)\}$ and $\{\pi_n^{(G)}(q)\}$

q	$\pi_{10000}(q)$	$\pi_{10000}^{(G)}(q)$
0.1	1.20008	5.89586
0.5	2.02122	3.75473
0.9	2.90827	3.22582
0.99	3.11805	3.1495
0.999	3.13924	3.14238

Let us introduce the additional sequence $\{S_n(q)\}$ by

$$S_0(q) = 0, \quad S_1(q) = 1, \qquad S_n(q) = \frac{[2n-1]_q}{[2n-2]_q} S_{n-1}(q) \quad (n \geq 2), \tag{27.12}$$

which is a monotonically increasing sequence. Also, consider the sequences $\{F_n(q)\}$ given by

$$F_1(q) = [2]_q, \qquad F_n(q) = \frac{[2n+1]_q}{[2n]_q} W_n(q) \qquad (n \geq 2). \tag{27.13}$$

Lemma 27.4 *The sequences $W_n(q)$ and $F_n(q)$ can be expressed via $\{S_n(q)\}$ as*

$$W_n(q) = \frac{[2n+1]_q}{S_{n+1}^2(q)}\ , \qquad F_n(q) = \frac{[2n]_q}{S_n^2(q)}\ . \tag{27.14}$$

Lemma 27.5 *The sequence $\{F_n(q)\}$ is monotonically decreasing, and related with $\{W_n(q)\}$ by*

$$W_n(q) \leq W_{n+1}(q) \leq F_{n+1}(q) \leq F_n(q) \qquad (n \in \mathbb{N}). \tag{27.15}$$

Proof According to definition (27.13), we can derive the recurrence relation

$$F_n(q) = \frac{[2n]_q[2n-2]_q}{[2n-1]_q^2} F_{n-1}(q).$$

The decreasing property follows from the Lemma 27.2 and relation (27.7). The last inequalities follow from the obvious relation $[2n+1]_q > [2n]_q$. □

Lemma 27.6 *The sequence $S_i(q)$ has the following bounds:*

$$\frac{[2i-1]_q}{W_{n-1}(q)} \leq S_i^2(q) \leq \frac{[2i]_q}{F_n(q)} \qquad (1 \leq i \leq n). \tag{27.16}$$

Proof Since $i \leq n-1$, then $W_i(q) \leq W_{n-1}(q)$, wherefrom

$$\frac{[2i-1]_q}{S_i^2(q)} = W_i(q) < W_{n-1}(q) \quad \Rightarrow \quad S_i^2(q) > \frac{[2i-1]_q}{W_{n-1}(q)}\ .$$

Similarly, for $i \leq n$, then $F_i(q) > F_n(q)$, wherefrom

$$F_i(q) = \frac{[2i]_q}{S_i^2(q)} > F_n(q) \quad \Rightarrow \quad S_i^2(q) < \frac{[2i]_q}{F_n(q)}\ .$$

□

Let the sequence $\{a_n(q)\}$ be defined by

$$a_n(q) = S_{n+1}(q) - S_n(q) = \left(\frac{[2n+1]_q}{[2n]_q} - 1\right) S_n(q) \qquad (n \in \mathbb{N}). \tag{27.17}$$

Lemma 27.7 *The sequence $\{a_n(q)\}$ has the properties:*

$$a_{n+1}(q) = q^2 \frac{[2n+1]_q}{[2n+2]_q^2} a_n(q),$$

and

$$[2i+2]_q a_{i+1}(q)a_j(q) + [2j+2]_q a_i(q)a_{j+1}(q) = q^2\big([2i+1]_q + [2j+1]_q\big)a_i(q)a_j(q).$$

27.4 The Geometrical Interpretation

We will consider the rectangles with horizontal and vertical sides (i.e., parallel to the x-axis and y-axis) whose diagonal endpoints are $T_{i,j}(q)$ and $T_{i+1,j+1}(q)$, where $T_{i,j}(q) = (S_i(q), S_j(q))$:

$$R_{i,j}(q) = \text{Rectangle}\left[T_{i,j}(q), T_{i+1,j+1}(q)\right].$$

Since they have the length $a_i(q)$ and width $a_j(q)$, they cover the area

$$\Delta R_{i,j}(q) = a_i(q)\cdot a_j(q).$$

Now, in the nth step, we consider the closed convex polygonal figure

$$P_n(q) = \bigcup_{i+j\leq n} R_{i,j}(q).$$

Example 27.1 According to definition, the rectangle $R_{0,0}$ is the unit square. Also $R_{i,i}$ are the squares with edge lengths equal to $a_i(q)$:

$$a_0(q) = 1, \quad a_1(q) = \frac{q^2}{1+q}\ , \quad a_2(q) = \frac{q^4(1+q+q^2)}{(1+q)^2(1+q^2)}\ , \ldots$$

But, when $i \neq j$, we have rectangles of different lengths and widths. For example, $R_{0,1}$ has the length 1 and width $a_1(q)$. The plane surfaces $P_1(q)$, $P_2(q)$ and $P_9(q)$ and quarter circles for the same $q = 0.8$ can be seen on Fig. 27.1a–c. Furthermore, the plane surfaces $P_3(q)$ and quarter circles for different q are shown on the Fig. 27.2a–c.

Remark 27.2 The quarter-circle of radius $r = S_{n+1}(q)$ does not give a lower bound for $q \in (q^*, 1]$, where $q^* \approx 0.9$. For $n = 2$, this can be seen on Fig. 27.3.

Remark 27.3 When $q \to 0$, the region $P_n(q)$ almost stays the square $R_{0,0}$. But, when $q \to 1$, the region $P_n(q)$ approaches to the quarter circle. It can be seen on Fig. 27.4.

The boundary rectangles of the region $P_n(q)$ are the rectangles which form the nth *layer* $L_n = \{R_{n,0},\ R_{n-1,1}, \ldots, R_{0,n}\}$. The *outer corners* of $P_n(q)$ are vertices on

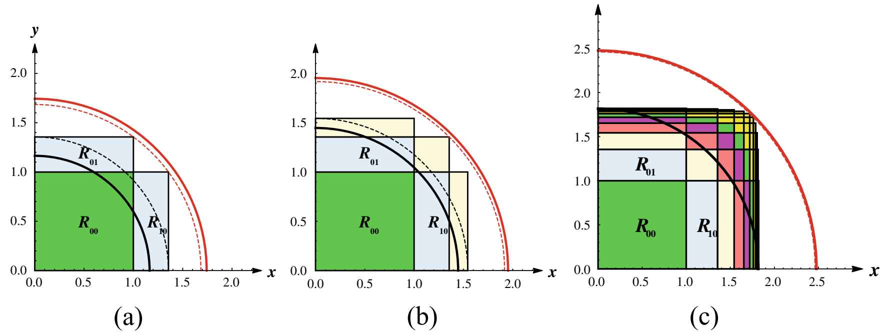


Fig. 27.1 The plane surfaces $P_1(q)$, $P_2(q)$ and $P_9(q)$ and quarter circles for $q = 0.8$

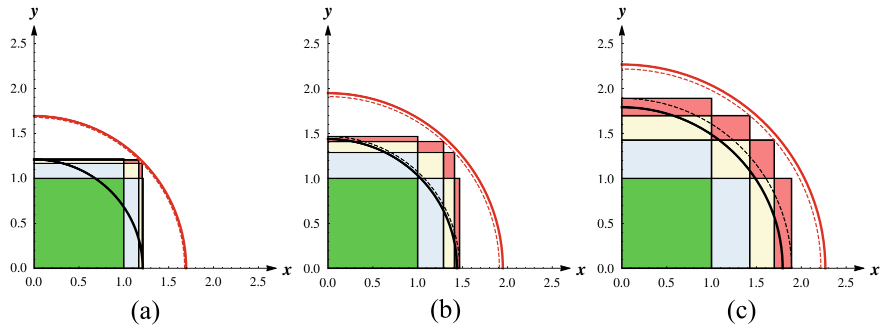
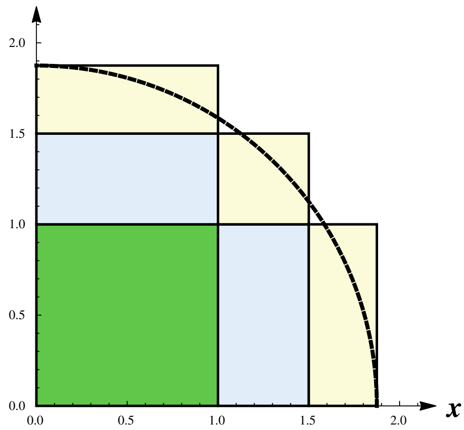


Fig. 27.2 The plane surfaces $P_3(q)$ and quarter circles for $q = 0.5(0.2)0.9$

Fig. 27.3 The plane surface $P_2(q)$ and quarter circle for $q = 1$



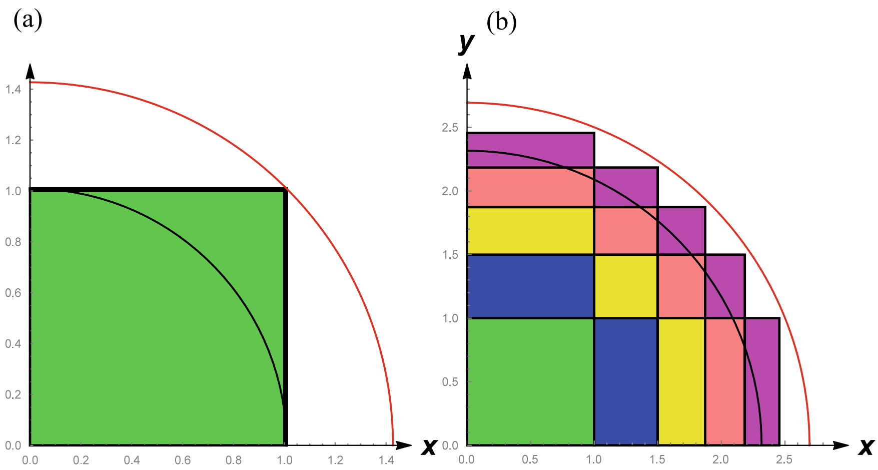


Fig. 27.4 The plane surface $P_5(q)$ for (**a**): $q = 0.1$ and for (**b**): $q = 0.999$

upper horizontal sides of the rectangles in L_n. The others are the *inner corners*. The radius of the inner quarter disc is

$$r_n(q) = \min_{\substack{0\le i,j\le n\\ i+j=n}} \overline{OT}_{i,j} = \min_{\substack{0\le i,j\le n\\ i+j=n}} \sqrt{S_i^2(q)+S_j^2(q)}.$$

Obviously,

$$\overline{OT}_{n,0} = \overline{OT}_{0,n} = S_n(q).$$

Since the inner corners of $P_n(q)$ satisfy (see Lemma 27.6)

$$S_i^2(q) + S_{n-i}^2(q) \ge \frac{[2i-1]_q + [2(n-i)-1]_q}{W_{n-1}} \qquad (i = 1, 2, \ldots, n-1).$$

Hence we have that

$$S_i^2(q) + S_{n-i}^2(q) \ge \min_{1\le j\le n-1} \frac{[j]_q + [(2n-2)-j]_q}{W_{n-1}} = \frac{[2n-2]_q}{W_{n-1}}\,.$$

Finally, by the relation (27.14), we find

$$r_0(q) = 1, \quad r_1(q) = \sqrt{2}, \quad r_n(q) = \sqrt{\frac{[2n-2]_q}{[2n-1]_q}}\,S_n(q) \quad (n = 2, 3, \ldots).$$

The radius of the disc centered at the origin which covers the region $P_n(q)$ is

$$\rho_n(q) = \max_{\substack{0 \le i, j \le n+1 \\ i+j=n+1}} \overline{OT}_{i,j}.$$

The distance of a point $T_{i,j}$ to the origin has the following upper bound

$$\overline{OT}_{i,j} = \sqrt{S_i^2(q) + S_j^2(q)} \le \sqrt{\frac{[2i]_q}{F_n(q)} + \frac{[2j]_q}{F_n(q)}} \le \sqrt{\frac{[2i]_q + [2j]_q}{F_n(q)}}\ .$$

Since $2i + 2j = 2(n+1)$, according to Lemma 27.1, we have

$$\rho_n(q) = \max_{\substack{0 \le i, j \le n+1 \\ i+j=n+1}} \overline{OT}_{i,j} = \sqrt{\frac{2[n+1]_q}{F_n(q)}}\ .$$

Using the connection between $F_n(q)$ and $S_n(q)$ given in Lemma 27.3, we can write

$$\rho_n(q) = \sqrt{\frac{2[n+1]_q}{[2n]_q}}\ S_n(q).$$

Therefore $P_n(q)$ contains a quarter circle of radius $r_n(q)$ and is contained in a quarter circle of radius $\rho_n(q)$:

$$\frac{\pi}{4} r_n^2(q) \le \Delta P_n(q) \le \frac{\pi}{4} \rho_n^2(q).$$

Theorem 27.1 *The following estimate is true for any* $q \in (0, 1)$*:*

$$\frac{\pi}{4} \frac{[2n-2]_q}{[2n-1]_q} \le \frac{\Delta P_n(q)}{S_n^2(q)} \le \frac{\pi}{2} \frac{[n+1]_q}{[2n]_q}\ . \tag{27.18}$$

Reminding on the formulas (27.9) and (27.14), we conclude that the following theorem is proven.

Theorem 27.2 *The following estimate is true for any* $q \in (0, 1)$*:*

$$\frac{[2n-2]_q[2n+1]_q}{2[2n-1]_q}\pi \le \Delta P_n(q)\pi_{n-1}(q) \le \frac{[n+1]_q[2n+1]_q}{[2n]_q}\pi. \tag{27.19}$$

We can write (27.19) in the form

$$\frac{[2n-2]_q[2n+1]_q}{2[2n-1]_q\ \Delta P_n(q)}\pi \le \pi_{n-1}(q) \le \frac{[n+1]_q[2n+1]_q}{[2n]_q\ \Delta P_n(q)}\pi. \tag{27.20}$$

Since $\lim\limits_{q\to 1^-} \Delta P_n(q) = n$, we have

$$\frac{(n-1)(2n+1)}{n(2n-1)}\pi \le \pi_{n-1}(1) \le \frac{(n+1)(2n+1)}{2n^2}\pi, \tag{27.21}$$

wherefrom

$$\lim_{n\to\infty}\lim_{q\to 1}\pi_n(q) = \pi. \tag{27.22}$$

Vice versa, from (27.19), we can get

$$\frac{\pi}{2(1-q)} \le \lim_{n\to\infty} \Delta P_n(q)\pi_{n-1}(q) \le \frac{\pi}{(1-q)}.$$

Hence

$$\frac{\pi}{2} \le \lim_{q\to 1}(1-q)\pi(q) \lim_{n\to\infty} \Delta P_n(q) \le \pi. \tag{27.23}$$

Remark 27.4 Notice that the formulas (27.22) and (27.23) do not lead to the same conclusion. This is due to the fact that the order of limits is crucially important. For example,

$$\lim_{n\to\infty}\lim_{q\to 1}\frac{[n]_q}{[2n]_q} = \frac{1}{2}, \qquad \lim_{q\to 1}\lim_{n\to\infty}\frac{[n]_q}{[2n]_q} = 1 \qquad (0 < q < 1,\ n \in \mathbb{N}).$$

27.5 The Number π_q from the q-gamma Function

The important role in q-calculus has q-Pochhammer symbol defined by

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{i=0}^{n-1}(1-aq^i) \quad \big(n \in \mathbb{N}\cup\{+\infty\}\big). \tag{27.24}$$

$$(a;q)_{-n} = \prod_{i=1}^{n}\frac{1}{1-aq^{-i}} \qquad \big(a \ne q, q^2, \ldots q^n;\ n \in \mathbb{N}\cup\{+\infty\}\big). \tag{27.25}$$

$$(a;q)_\lambda = \frac{(a;q)_\infty}{(aq^\lambda;q)_\infty} \qquad (|q| < 1,\ \lambda \in \mathbb{C})\,. \tag{27.26}$$

J. Thomae 1869. and F. H. Jackson 1904 defined q-gamma function as

$$\Gamma_q(z) = (q;q)_{z-1}\,(1-q)^{1-z} = \frac{(q;q)_\infty}{(q^z;q)_\infty}\,(1-q)^{1-z} \qquad (0 < q < 1,\ z \notin \mathbb{Z}^-). \tag{27.27}$$

The q-gamma function holds on the following properties:

$$\Gamma_q(z+1) = [z]_q\, \Gamma_q(z) \quad (z \in \mathbb{C} \setminus \mathbb{Z}^-), \qquad \Gamma_q(n+1) = [n]_q! \quad (n \in \mathbb{N}_0). \tag{27.28}$$

In special,

$$\lim_{q \to 1-} \Gamma_q(z) = \Gamma(z).$$

The exact q-Gauss multiplication formula can be find in [7] or [5]:

$$\Gamma_q(nx) \prod_{k=1}^{n-1} \Gamma_{q^n}\left(\frac{k}{n}\right) = [n]_q^{nx-1} \prod_{k=0}^{n-1} \Gamma_{q^n}\left(x + \frac{k}{n}\right) \quad (x > 0;\ n \in \mathbb{N}). \tag{27.29}$$

Equivalently, putting $z = nx$, it can be written in the form

$$\Gamma_q(z) \prod_{k=1}^{n-1} \Gamma_{q^n}\left(\frac{k}{n}\right) = [n]_q^{z-1} \prod_{k=0}^{n-1} \Gamma_{q^n}\left(\frac{z+k}{n}\right) \quad (z > 0;\ n \in \mathbb{N}). \tag{27.30}$$

The following theorem is formulated by W.S. Chung, T. Kim and T. Mansour in the paper [2] and proven, but only for $x = N$, where N is an integer number. Here are two versions of the complete proof.

Theorem 27.3 *For* $0 < |q| < 1$ *and* $x \in \mathbb{R}$ *the following is valid:*

$$\Gamma_q(x) = \lim_{n \to +\infty} \frac{[n]_q! [n]_q^x}{[x]_q [x+1]_q \cdots [x+n]_q} \qquad (x \notin \mathbb{Z}^-).$$

Proof According to the definitions (27.3) and (27.4), we have

$$\begin{aligned}
\lim_{n \to +\infty} \frac{[n]_q! [n]_q^x}{[x]_q [x+1]_q \cdots [x+n]_q} &= \lim_{n \to +\infty} \frac{\left(\dfrac{1-q^n}{1-q}\right)^x \displaystyle\prod_{j=1}^{n} \frac{1-q^j}{1-q}}{\displaystyle\prod_{j=0}^{n} \frac{1-q^{x+j}}{1-q}} \\
&= \lim_{n \to +\infty} \frac{\dfrac{(1-q^n)^x}{(1-q)^x} \dfrac{1-q}{1-q^{n+1}} \displaystyle\prod_{j=0}^{n} \frac{1-q^{1+j}}{1-q}}{\displaystyle\prod_{j=0}^{n} \frac{1-q^{x+j}}{1-q}} \\
&= \frac{1}{(1-q)^{x-1}} \lim_{n \to +\infty} \left((1-q^n)^x \frac{1}{1-q^{n+1}} \prod_{j=0}^{n} \frac{1-q^{1+j}}{1-q^{x+j}} \right).
\end{aligned}$$

Since $|q| < 1$, then

$$\lim_{n \to +\infty} (1-q^n)^x = \lim_{n \to +\infty} (1-q^{n+1}) = 1,$$

and

$$\lim_{n\to+\infty}\prod_{j=0}^{n}\frac{1-q^{1+j}}{1-q^{x+j}}=\prod_{j=0}^{\infty}\frac{1-q^{1+j}}{1-q^{x+j}}=\frac{(q;q)_\infty}{(q^x;q)_\infty}=(q;q)_{x-1}.$$

Therefore, the limiting value becomes

$$\lim_{n\to+\infty}\frac{[n]_q![n]_q^x}{[x]_q[x+1]_q\cdots[x+n]_q}=(1-q)^{1-x}(q;q)_{x-1}=\Gamma_q(x).$$

Proof (**The second proof**). Another proof is based on estimation [13]:

$$\Gamma_q(x)=a(q)(1-q)^{1/2-x}e^{\theta(x,q)\frac{q^x}{(1-q)(1-q^x)}}$$

where

$$a(q)=(q;q)_\infty(1-q)^{1/2};\qquad 0<\theta(x,q)<1.$$

Hence

$$\frac{\Gamma_q(n+1)}{\Gamma_q(x+n+1)}=\frac{a(q)(1-q)^{1/2-(n+1)}e^{\theta(n+1,q)\frac{q^{n+1}}{(1-q)(1-q^{n+1})}}}{a(q)(1-q)^{1/2-(x+n+1)}e^{\theta(x+n+1,q)\frac{q^{x+n+1}}{(1-q)(1-q^{x+n+1})}}},$$

i.e.,

$$\frac{\Gamma_q(n+1)}{\Gamma_q(x+n+1)}=(1-q)^xe^{\theta(n+1,q)\frac{q^{n+1}}{(1-q)(1-q^{n+1})}-\theta(x+n+1,q)\frac{q^{x+n+1}}{(1-q)(1-q^{x+n+1})}},$$

wherefrom

$$\lim_{n\to+\infty}\frac{\Gamma_q(n+1)}{\Gamma_q(x+n+1)}=(1-q)^x.$$

□

The exact q-Gauss multiplication formula can be found in [7] or [5]:

$$\Gamma_q(nx)\prod_{k=1}^{n-1}\Gamma_{q^n}\left(\frac{k}{n}\right)=[n]_q^{nx-1}\prod_{k=0}^{n-1}\Gamma_{q^n}\left(x+\frac{k}{n}\right)\quad(x>0;\ n\in\mathbb{N}).\tag{27.31}$$

Substituting $z=nx$, it can be written in the form

$$\Gamma_q(z)\prod_{k=1}^{n-1}\Gamma_{q^n}\left(\frac{k}{n}\right)=[n]_q^{z-1}\prod_{k=0}^{n-1}\Gamma_{q^n}\left(\frac{z+k}{n}\right)\quad(z>0;\ n\in\mathbb{N}).\tag{27.32}$$

Especially, for $n = 2$, it is

$$\Gamma_q(z)\Gamma_{q^2}\left(\frac{1}{2}\right) = [2]_q^{z-1}\Gamma_{q^2}\left(\frac{z}{2}\right)\Gamma_{q^2}\left(\frac{z+1}{2}\right). \tag{27.33}$$

Equivalently,

$$\Gamma_{\sqrt{q}}(2x)\Gamma_q\left(\frac{1}{2}\right) = [2]_{\sqrt{q}}^{2x-1}\Gamma_q(x)\Gamma_q\left(x+\frac{1}{2}\right). \tag{27.34}$$

Let

$$e_q(z) = \sum_{n=0}^{\infty}\frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty} \qquad (|z| < 1), \tag{27.35}$$

and

$$E_q(z) = \sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}}{(q;q)_n}z^n = (-z;q)_\infty \qquad (z \in \mathbb{R}). \tag{27.36}$$

It is known that $\Gamma_q(z)$ can be expressed by q-integral [11]

$$\int_0^b f(x)\,d_qx = (1-q)\sum_{n=0}^{\infty} f(bq^n)q^n, \tag{27.37}$$

in the next manner:

$$\Gamma_q(z) = \int_0^{\frac{1}{1-q}} x^{z-1}E_q\big(-q(1-q)x\big)\,d_qx. \tag{27.38}$$

The following theorem is inspired with the theorem [2] which has an error in formulation and proof. That is why we will repeat in the correct manner.

Theorem 27.4 *It is valid*

$$\Gamma_q(1/2) = \sqrt{1-q}\; e_q\big(\sqrt{q}\big)\; E_q(-q). \tag{27.39}$$

Proof From the definition (27.27) of the q-Gamma function, for $z = 1/2$, we have

$$\Gamma_q(1/2) = \frac{(q;q)_\infty}{(q^{1/2};q)_\infty}(1-q)^{1/2} = \sqrt{1-q}\frac{1}{(\sqrt{q};q)_\infty}(q;q)_\infty.$$

Here we recognize the functions e_q end E_q from (27.35) and (27.36).

Remark 27.5 Some authors use notation

$$\hat{e}_q(z) = e_q\big((1-q)z\big), \qquad \hat{E}_q(z) = E_q\big((1-q)z\big)$$

The previous theorem gets the form

$$\Gamma_q(1/2) = \sqrt{1-q}\, \hat{e}_q \left(\frac{\sqrt{q}}{1-q} \right) \hat{E}_q \left(\frac{-q}{1-q} \right).$$

The authors have proved in [2] that

$$\Gamma_q(x) = \frac{1}{[x]_q} \prod_{k=1}^{\infty} \left[1 + \frac{1}{k} \right]_{q^k}^{x} \left[1 + \frac{x}{k} \right]_{q^k}^{-1} .$$

It was used to define q-version of the number π:

$$\tilde{\pi}_q = \Gamma_q^2(1/2) = [2]_{\sqrt{q}} \cdot \prod_{k=1}^{\infty} \frac{[2k]_{\sqrt{q}}^2}{[2k-1]_{\sqrt{q}}[2k+1]_{\sqrt{q}}} . \tag{27.40}$$

This version of q-number π is directly connected with the Gosper's version by

$$\tilde{\pi}_q = q^{1/8} \frac{(1+q^{1/2})(1+q^{1/4})}{4} \pi_{q^{1/2}}^{(G)}. \tag{27.41}$$

Remark 27.6 At the beginning of page 1158 in [2], there is wrong connection between q-sine function and infinite product. Namely, Euler's infinite product function for the sine function (27.2) which can be rewritten in the form

$$\sin(\pi z) = \pi z \prod_{k=1}^{\infty} \left(1 + \frac{z}{k} \right) \left(1 - \frac{z}{k} \right), \tag{27.42}$$

was directly rewritten to q-function as

$$\widetilde{\sin}_q(\tilde{\pi}_q x) = \tilde{\pi}_q [x]_q \prod_{k=1}^{\infty} \left[1 + \frac{x}{k} \right]_{q^k} \left[1 - \frac{x}{k} \right]_{q^k} \tag{27.43}$$

Really, $\widetilde{\sin}_q(\tilde{\pi}_q/2) = 1$ and $\widetilde{\sin}_q(n\tilde{\pi}_q) = 0$ since the second factor becomes the zero when $x = k = n$. Of course, it is valid

$$\Gamma_q(z)\Gamma_q(1-z) = \frac{\tilde{\pi}_q}{\widetilde{\sin}_q(\tilde{\pi}_q z)} , \tag{27.44}$$

But, the function $\widehat{\sin}_q(\tilde{\pi}_q x)$ is a new version of q-sine function and *it is not the same* as

$$\widehat{\sin}_q(x) = \frac{\hat{e}^{ix} - \hat{e}^{-ix}}{2i} \qquad (i^2 = -1).$$

Equivalent definition of the gamma function introduced by Weierstrass is valid for all complex numbers z except the non-positive integers:

$$\Gamma(x) = \frac{1}{x\ e^{\gamma\ x}}\ \prod_{k=1}^{\infty} \frac{e^{x/k}}{1+\frac{x}{k}}\ ,$$

where

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right) \approx 0.577216.$$

Hence we can introduce

$$\gamma_q = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{[k]_q} - F_q(n)\right),$$

with

$$F_q(n) = \varphi\left(\frac{n}{n-1}; q\right), \quad \text{where} \quad \varphi(z;q) = -\sum_{k=1}^{\infty}\frac{q^k(1-q)}{1-q^k}(z;q)_{-k}.$$

Here, $(z;q)_{-k}$ should be computed by the definition (27.25). In the limit case, we get

$$\lim_{q\uparrow 1}(z;q)_{-k} = (1-z)^{-k}, \quad \lim_{q\uparrow 1}\varphi(z;q) = \ln\frac{z}{z-1} \quad (z>1),$$

and, finally,

$$\lim_{q\uparrow 1} F_q(n) = \ln n.$$

Remark 27.7 The authors of the paper [2], suggest a new version of the q-Gamma function

$$\widehat{\Gamma}_q(x) = \frac{1}{[x]_q e^{\gamma_q x}} \prod_{k=1}^{\infty}\frac{e^{x/[k]_q}}{\left[1+\frac{x}{k}\right]_{q^k}}\ ,$$

where

$$\tilde{\gamma}_q = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{[k]_q} - \ln[n]_q\right).$$

It is really

$$\lim_{q\uparrow 1}\widehat{\Gamma}_q(x) = \Gamma_(x), \qquad \lim_{q\uparrow 1}\tilde{\gamma}_q = \gamma.$$

But, it should be considered is it justified and useful, especially since the other conclusions on this page about $\widehat{\Gamma}_q(x+1)$ and $\tilde{\gamma}_q$ are wrong. Namely, the logarithm of the should be

$$\ln \widehat{\Gamma}_q(x+1) = \lim_{n\to+\infty} \left(\ln[n]_q! + x \ln[n]_q - \sum_{k=1}^{n} \ln[x+k]_q \right).$$

Also, they use the series

$$\sum_{k=1}^{\infty} \frac{1-q+q^k \ln q}{1-q^k} \ ,$$

as a convergent series. But, it is a divergent series since its general member is tending to $1-q$ for $q \in (0,1)$.

27.6 Other Versions of π_q

In the previous sections we deal with the Gosper's analog of π in the q-calculus, denoted by $\pi^{(G)}(q)$.

The second consideration which was implicitly leading to definition q-analogy of number π we noticed in the paper of R. Diaz and E. Pariguan [3]. Starting from the well known integral

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi},$$

they have considered

$$\int_{-\nu}^{\nu} E_{q^2}\big(-(qx)^2/[2]_q\big)\, d_q x = c(q) \qquad \Big(\nu = \frac{1}{\sqrt{1-q}}\Big),$$

where

$$E_{q^2}\big(-(qx)^2/[2]_q\big) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{[n]_q!}\ .$$

Really, they found that

$$\lim_{q\to 1-} c(q) = \sqrt{2\pi} \quad \Leftrightarrow \quad \frac{1}{2}\left(\lim_{q\to 1-} c(q)\right)^2 = \pi.$$

wherefrom it is reasonable to take

$$\pi_q^{(D)} = \frac{1}{2} c^2(q).$$

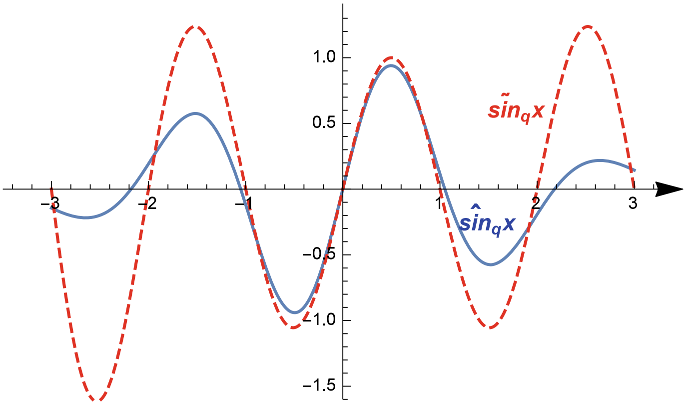


Fig. 27.5 The q-sine functions: full blue curve $\widetilde{\sin}_q(x)$ and dashed red curve $\widehat{\sin}_q(x)$ for $q = 0.9$

as q-analog of π. Hence

$$\pi_q^{(D)} = 2(1-q)\left(\sum_{m=0}^{\infty} \frac{(-1)^m q^{m(m+1)}}{(1-q^{2m+1})(1-q^2)^m [m]_{q^2}!}\right)^2. \tag{27.45}$$

This analog of π holds on the property of cognizable values of the moments

$$\frac{1}{2\sqrt{\pi_q^{(D)}}} \int_{-\nu}^{\nu} x^m E_{q^2}\Big(\frac{-(qx)^2}{[2]_q}\Big)\, d_q x = \begin{cases} [2n-1]_q!!, & m = \quad 2n, \\ \quad 0, & m = 2n-1, \end{cases} \quad (n \in \mathbb{N}). \tag{27.46}$$

which are q-analogs of those in the standard calculus.

27.7 Conclusion

In this paper, we presented an analog of π via the q-Wallis formula. Also, we gave a geometrical interpretation of it. Consideration of the areas of the introduced surfaces gave a few estimates and the number π in the limit case (Fig. 27.5).

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Chapter 28
On (λ, μ, γ)-Derivations of BiHom-Lie Algebras

Nejib Saadaoui and Sergei Silvestrov

Abstract In this paper, we generalize the results about generalized derivations of Lie algebras to the case of BiHom-Lie algebras. In particular we give the classification of generalized derivations of Heisenberg BiHom-Lie algebras. The definition of the generalized derivation depends on some parameters $(\lambda, \mu, \gamma) \in \mathbb{C}^3$. In particular for $(\lambda, \mu, \gamma) = (1, 1, 1)$, we obtain classical concept of derivation of BiHom-Lie algebra and for $(\lambda, \mu, \gamma) = (1, 1, 0)$ we obtain the centroid of BiHom-Lie algebra. We give classifications of 2-dimensional BiHom-Lie algebra, centroids and derivations of 2-dimensional BiHom-Lie algebras.

Keywords BiHom-Lie algebra · BiHom-Lie derivation · Derivation · Centroid

MSC2020 Classification 17D30 · 17B61

28.1 Introduction

The investigations of various quantum deformations or q-deformations of Lie algebras began a period of rapid expansion in 1980's stimulated by introduction of quantum groups motivated by applications to the quantum Yang-Baxter equation, quantum inverse scattering methods and constructions of the quantum deformations of universal enveloping algebras of semi-simple Lie algebras. Various q-deformed Lie algebras have appeared in physical contexts such as string theory, vertex models in conformal field theory, quantum mechanics and quantum field theory in the

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context of deformations of infinite-dimensional algebras, primarily the Heisenberg algebras, oscillator algebras and Witt and Virasoro algebras. In [8, 37–40, 43, 44, 46, 58, 59, 75–77], it was in particular discovered that in these q-deformations of Witt and Visaroro algebras and some related algebras, some interesting q-deformations of Jacobi identities, extending Jacobi identity for Lie algebras, are satisfied. This has been one of the initial motivations for the development of general quasi-deformations and discretizations of Lie algebras of vector fields using more general σ-derivations (twisted derivations) in [56].

Hom-Lie algebras and more general quasi-Hom-Lie algebras were introduced first by Hartwig, Larsson and Silvestrov [56], where the general quasi-deformations and discretizations of Lie algebras of vector fields using more general σ-derivations (twisted derivations) and a general method for construction of deformations of Witt and Virasoro type algebras based on twisted derivations have been developed, initially motivated by the q-deformed Jacobi identities observed for the q-deformed algebras in physics, along with q-deformed versions of homological algebra and discrete modifications of differential calculi. Hom-Lie algebras, Hom-Lie superalgebras, Hom-Lie color algebras and more general quasi-Lie algebras and color quasi-Lie algebras where introduced first in [70, 71, 98]. Quasi-Lie algebras and color quasi-Lie algebras encompass within the same algebraic framework the quasi-deformations and discretizations of Lie algebras of vector fields by σ-derivations obeying twisted Leibniz rule, and the well-known generalizations of Lie algebras such as color Lie algebras, the natural generalizations of Lie algebras and Lie superalgebras. In quasi-Lie algebras, the skew-symmetry and the Jacobi identity are twisted by deforming twisting linear maps, with the Jacobi identity in quasi-Lie and quasi-Hom-Lie algebras in general containing six twisted triple bracket terms. In Hom-Lie algebras, the bilinear product satisfies the non-twisted skew-symmetry property as in Lie algebras, and the Hom-Lie algebras Jacobi identity has three terms twisted by a single linear map, reducing to the Lie algebras Jacobi identity when the twisting linear map is the identity map. Hom-Lie admissible algebras have been considered first in [83], where in particular the Hom-associative algebras have been introduced and shown to be Hom-Lie admissible, that is leading to Hom-Lie algebras using commutator map as new product, and in this sense constituting a natural generalization of associative algebras as Lie admissible algebras. Since the pioneering works [56, 69–72, 83], Hom-algebra structures expanded into a popular area with increasing number of publications in various directions. Hom-algebra structures of a given type include their classical counterparts and open broad possibilities for deformations, Hom-algebra extensions of cohomological structures and representations, formal deformations of Hom-associative and Hom-Lie algebras, Hom-Lie admissible Hom-coalgebras, Hom-coalgebras, Hom-Hopf algebras [10, 34, 48, 69, 73, 84–86, 94, 95, 102, 104]. Hom-Lie algebras, Hom-Lie superalgebras and color Hom-Lie algebras and their n-ary generalizations have been further investigated in various aspects for example in [1, 7, 10–23, 26–29, 32–34, 36, 55, 65, 67, 68, 81–88, 91, 92, 94–100, 102–106, 108]. In [35], Hom-algebras has been considered from a category theory point of view, constructing a category on which algebras would be Hom-algebras. A generalization of this approach led to the discovery of

BiHom-algebras in [54], called BiHom-algebras because the defining identities are twisted by two morphisms instead of only one for Hom-algebras. BiHom-Frobenius algebras and double constructions have been investigated in [57].

Derivations and generalized derivations of different algebraic structures are an important subject of study in algebra and diverse areas. They appear in many fields of Mathematics and Physics. In particular, they appear in representation theory and cohomology theory among other areas. They have various applications relating algebra to geometry and allow the construction of new algebraic structures. There are many generalizations of derivations, for example, Leibniz derivations and δ-derivations of prime Lie and Malcev algebras and related n-ary algebras structures [45, 49–52, 63]. The properties and structure of generalized derivations algebras of a Lie algebra and their subalgebras and quasi-derivation algebras were systematically studied in [74], where it was proved for example that the quasi-derivation algebra of a Lie algebra can be embedded into the derivation algebra of a larger Lie algebra. Derivations and generalized derivations of n-ary algebras were considered in [90, 101] and it was demonstrated substantial differences in structures and properties of derivations on Lie algebras and on n-ary Lie algebras for $n > 2$. Generalized derivations of Lie superalgebras and Hom-Leibniz algebras have been considered in [107, 111]. Generalized derivations of Lie color algebras and n-ary (color) algebras have been studied in [41, 60–62, 64]. Generalized derivations of Lie triple systems have been considered in [109]. Generalized derivations of various kinds can be viewed as a generalization of δ-derivation. Quasi-Hom-Lie and Hom-Lie structures for σ-derivations and (σ, τ)-derivations have been considered in [48, 56, 72, 91, 92]. Graded q-differential algebras and applications to semi-commutative Galois extensions and reduced quantum plane and q-connection were studied in [4–6]. Generalized N-complexes coming from twisted derivations where considered in [73].

Generalizations of derivations in connection with extensions and enveloping algebras of Hom-Lie color algebras and Hom-Lie superalgebras have been considered in [18, 19, 28, 55]. Generalized derivations of multiplicative n-ary Hom-Ω color algebras have been studied in [31]. Derivations, L-modules, L-comodules and Hom-Lie quasi-bialgebras have been considered in [24, 25]. In [66], constructions of n-ary generalizations of BiHom-Lie algebras and BiHom-associative algebras have been investigated. Generalized derivations of n-BiHom-Lie algebras have been studied in [30]. Color Hom-algebra structures associated to Rota-Baxter operators have been considered in context of Hom-dendriform color algebras in [27]. Rota-Baxter bisystems and covariant bialgebras, Rota-Baxter cosystems, coquasitriangular mixed bialgebras, coassociative Yang-Baxter pairs, coassociative Yang-Baxter equation and generalizations of Rota-Baxter systems and algebras, curved $\mathcal{O}$-operator systems and their connections with (tri)dendriform systems and pre-Lie algebras have been considered in [78–80]. Generalisations of derivations are important for Hom-Gerstenhaber algebras, Hom-Lie algebroids and Hom-Lie-Rinehart algebras and Hom-Poisson homology [88]. It is well known that a derivation d of Lie algebra L is just a linear mapping on L such that

$$d([x, y]) = [d(x), y] + [x, d(y)] \tag{28.1}$$

for all $x,\ y \in L$. There were several non-equivalent ways generalizing this definition, for example:

1) The mapping $d \in End(L)$ is called a generalized derivation of L if there exist elements $d',\ d'' \in End(L)$ such that,

$$[d(x), y] + \left[x, d'(y)\right] = d''\left([x, y]\right)$$

for all $x,\ y \in L$, and we call $d \in End(L)$ a quasiderivation of L if there exists $d' \in End(L)$ such that

$$[d(x), y] + [x, d(y)] = d'\left([x, y]\right). \tag{28.2}$$

The centroid of L denoted as $\Gamma(L)$ is defined by

$$\Gamma(L) = \{d \in End(L) \mid d\left([x, y]\right) = [d(x), y] = [x, d(y)]\,,\ \forall x, y \in L\}.$$

(see for example [53]).

2) Given an arbitrary $\delta \in \mathbb{K}$, a δ-derivation of a Lie algebra L is defined to be a $\mathbb{K}$-linear mapping $d\colon L \to L$ satisfying the identity

$$d\left([x, y]\right) = \delta\,[d(x), y] + \delta\,[x, d(y)]$$

(see for example [50]). Observe that, any linear mapping in the centroid $\Gamma(L)$ is $\frac{1}{2}$-derivation of L.

3) We call a linear operator $d \in End(L)$ an (λ, μ, γ)-derivation of L if there exist $\lambda, \mu, \gamma \in \mathbb{K}$ such that for all $x, y \in L$

$$\lambda\, d\left([x, y]\right) = \mu\,[d(x), y] + \gamma\,[x, d(y)]\,.$$

(See for example [89]). Observe that, any linear mapping in the centroid $\Gamma(L)$ is a $(1, 1, 0)$-derivation of L.

In [94], the notion of α^k-derivation of Hom-Lie algebra, a generalization of derivation of Lie algebras (28.1), is considered. In [110] the authors extend the definition of type (28.1) of a generalized derivation of Lie algebras to Hom-Lie algebras. The definition of type (28.1) is extended to the BiHom-Lie case in [2]. In this article, we aim to discuss the version (28.2) of generalized derivations of BiHom-Lie algebras.

The paper is organized as follows. In Sect. 28.2, we recall some basic definitions and facts needed later for considerations and results in this article. In Sect. 28.3, we introduce (λ, μ, γ)-$\alpha^k\beta^l$-derivations and show their pertinent properties. Also, we classify the possible values of $\lambda, \mu, \gamma \in \mathbb{C}$ for a space $Der^{\lambda,\mu,\gamma}_{\alpha^k\beta^l}(G)$ of (λ, μ, γ)-$\alpha^k\beta^l$-derivations of regular BiHom-Lie algebra G. The previous classification is applied to Heisenberg BiHom-Lie algebra case. Next, we analyze each one of the following cases: $Der^{\delta,0,0}_{\alpha^k\beta^l}(G)$ with $\delta \in \{0, 1\}$, $Der^{\delta,1,0}_{\alpha^k\beta^l}(G)$, $Der^{\delta,1,1}_{\alpha^k\beta^l}(G)$, $Der^{1,1,-1}_{\alpha^k\beta^l}(G)$,

$Der^{0,1,-1}_{\alpha^k\beta^l}(G)$. In Sect. 28.4, we give a method to determine whether two different 2-dimensional multiplicative BiHom-Lie algebras are isomorphic or not, and then we obtain a complete classification of 2-dimensional multiplicative BiHom-Lie algebras up to isomorphism. In Sect. 28.5, we deal with the problem of description of centroids and derivations of 2-dimensional BiHom Lie algebras. Here we provide algorithms to find centroids and derivations by using an algebra software.

28.2 Definitions and Preliminary Results

Definition 28.1 ([42, 54]) A BiHom-Lie algebra over a field $\mathbb{K}$ is a 4-tuple $(L, [\cdot,\cdot], \alpha, \beta)$, where L is a $\mathbb{K}$-linear space, $\alpha\colon L \to L$, $\beta\colon L \to L$ and $[\cdot,\cdot]\colon L \times L \to L$ are linear maps, satisfying the following conditions, for all $x,\ y,\ z \in L$:

$$\alpha \circ \beta = \beta \circ \alpha, \tag{28.3}$$

$$[\beta(x), \alpha(y)] = -[\beta(y), \alpha(x)] \quad \text{(skew-symmetry)} \tag{28.4}$$

$$\begin{aligned}&\left[\beta^2(x), [\beta(y), \alpha(z)]\right] + \left[\beta^2(y), [\beta(z), \alpha(x)]\right]\\ &+ \left[\beta^2(z), [\beta(x), \alpha(y)]\right] = 0 \ \text{(BiHom-Jacobi identity)}.\end{aligned} \tag{28.5}$$

A Bihom-Lie algebra is called a multiplicative Bihom-Lie algebra if for any $x,\ y \in L$,

$$\alpha\left([x, y]\right) = [\alpha(x), \alpha(y)] \quad \text{and} \quad \beta\left([x, y]\right) = [\beta(x), \beta(y)]. \tag{28.6}$$

A BiHom-Lie algebra is called a regular BiHom-Lie algebra if $\alpha,\ \beta$ are bijective maps.

In general for n-dimensional case in terms of structure constants we have:

$$\begin{aligned}[e_i, e_j] &= \sum_{s=1}^{n} C_{ij}^s e_s,\\ \alpha(e_j) &= \sum_{s=1}^{n} a_{sj} e_s \qquad \text{and} \qquad \beta(e_j) = \sum_{s=1}^{n} b_{sj} e_s.\end{aligned} \tag{28.7}$$

Substituting (28.7) in the skew-symmetry identity (28.4) yields

$$\sum_{1\le p,q\le n} \left(b_{pi}a_{qj} + b_{pj}a_{qi}\right) C_{pq}^s = 0. \tag{28.8}$$

Substituting (28.7) in the BiHom-Jacobi identity (28.5) yields

$$\sum_{1\le p,q,s,l,s'\le n} \left(b_{s'i}b_{qj}a_{sk} + b_{s'j}b_{qk}a_{si} + b_{s'k}b_{qi}a_{sj}\right) b_{ps'} C_{qs}^l C_{pl}^r = 0. \tag{28.9}$$

Substituting (28.7) in the multiplicativity conditions (28.6) yields

$$\begin{aligned} \sum_{1\le k\le n} C_{ij}^k a_{sk} &= \sum_{1\le p,q\le n} a_{pi}a_{qj}C_{pq}^s \\ \sum_{1\le k\le n} C_{ij}^k b_{sk} &= \sum_{1\le p,q\le n} b_{pi}b_{qj}C_{pq}^s \end{aligned} \tag{28.10}$$

for all $i, j, k \in \{1, \dots, n\}$.

Definition 28.2 A morphism $f\colon (L, [\cdot,\cdot], \alpha, \beta) \to \big(L', [\cdot,\cdot]', \alpha', \beta'\big)$ of BiHom-Lie algebras is a linear map $f\colon L \to L'$ such that $\alpha' \circ f = f \circ \alpha$, $\beta' \circ f = f \circ \beta$ and

$$f([x, y]) = [f(x), f(y)]', \quad \forall x,\ y \in L. \tag{28.11}$$

In particular, BiHom-Lie algebras $(L, [\cdot,\cdot], \alpha, \beta)$ and $\big(L', [\cdot,\cdot]', \alpha', \beta'\big)$ are isomorphic if f is an isomorphism map.

Let $(L, [\cdot,\cdot], \alpha, \beta)$ be n-dimensional BiHom-Lie algebra with ordered basis $(e_1, \dots, e_n)$ and L' be n-dimensional vector spaces with ordered basis $(e'_1, \dots, e'_n)$. Let $f\colon L \to L'$ be an isomorphism map. Let $\alpha' = f\alpha f^{-1}$ and $\beta' = f\beta f^{-1}$. We set with respect to a basis $(e'_1, \dots, e'_n)$:

$$\begin{aligned} f(e_j) &= \sum_{i=1}^n f_{ij}e'_i, \\ [e'_i, e'_j]' &= \sum_{k=1}^n C'^k_{ij} e'_k,\ i, j \in \{1, \dots, n\}. \end{aligned}$$

Condition (28.11) translates to the following equation

$$\sum_{k=1}^n C_{ij}^k f_{sk} = \sum_{1\le p,q\le n} f_{pi} f_{qj} C'^s_{pq}, \quad i, j, s \in \{1, \dots, n\}. \tag{28.12}$$

Then, if the previous condition satisfied, L' is a BiHom-Lie algebra isomorphic to L.

Definition 28.3 ([2, 42]) Let $(L, [\cdot,\cdot], \alpha, \beta)$ be a BiHom-Lie algebra. A subspace $\mathfrak{h}$ of L is called a BiHom-Lie subalgebra of $(L, [\cdot,\cdot], \alpha, \beta)$ if $\alpha(\mathfrak{h}) \subseteq \mathfrak{h}$, $\beta(\mathfrak{h}) \subseteq \mathfrak{h}$ and $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$. In particular, a BiHom-Lie subalgebra $\mathfrak{h}$ is said to be an ideal of $(L, [\cdot,\cdot], \alpha, \beta)$ if $[\mathfrak{h}, L] \subseteq \mathfrak{h}$ and $[L, \mathfrak{h}] \subseteq \mathfrak{h}$.

If I is an ideal of $(L, [\cdot,\cdot], \alpha, \beta)$, then $\big(L/I, [\cdot,\cdot], \overline{\alpha}, \overline{\beta}\big)$, where $[\overline{x}, \overline{y}] = \overline{[x, y]}$, for all $\overline{x}$, $\overline{y} \in L/I$ and $\overline{\alpha}$, $\overline{\beta}\colon L/I \to L/I$ naturally induced by α and β, inherits a BiHom-Lie algebra structure, which is named quotient BiHom-Lie algebra.

In the following, we give some examples and applications of ideals of BiHom-Lie algebras.

Proposition 28.1 *If* $(L, [\cdot,\cdot], \alpha, \beta)$ *is a BiHom-Lie algebra, then* $I = \ker\alpha + \ker\beta$ *is an ideal of* L.

Proof By (28.3) we get $\alpha(I) \subseteq I$ and $\beta(I) \subseteq I$. By (28.6) we obtain $[I, L] \subseteq I$. □

Remark 28.1 If L is finite-dimensional and α (or β) is diagonalizable, then there exist a subspace G such that $L = I \oplus G$ and $\big(G, [\cdot,\cdot], \alpha_{/G}, \beta_{/G}\big)$ is a regular BiHom-Lie algebra.

Definition 28.4 Given a complex BiHom-Lie multiplicative algebra L, the center of L is given by $C(L) = \{x \in L \mid [x, y] = 0 \quad \forall y \in L\}$. The descending central series of a BiHom-Lie algebra L is given by the ideals

$$L^0 = L; \quad L^k = [L, L^{k-1}], \quad k \geq 1.$$

L is called nilpotent if $L^n = \{0\}$ for some $n \in \mathbb{N}$. If $L^{n-1} \neq \{0\}$, then L is said to be n-step nilpotent BiHom-Lie algebra. The derived series of a BiHom-Lie algebra L is given by the ideals $L^{(0)} = L,\ L^{(k)} = [L^{(k-1)}, L^{(k-1)}]$, $k \geq 1$. L is called solvable if $L^{(n)} = \{0\}$ for some $n \in \mathbb{N}$. If $L^{(n-1)} \neq \{0\}$, then L is said to be n-step solvable BiHom-Lie algebra.

Remark 28.2 The center $C(L)$ of L is not necessarily an ideal of L. If α and β are surjective then $C(L)$ is an ideal of L.

Definition 28.5 Let $(L, [\cdot,\cdot], \alpha, \beta)$ be a BiHom-Lie algebra. $(L, [\cdot,\cdot], \alpha, \beta)$ is called a simple BiHom-Lie algebra if $(L, [\cdot,\cdot], \alpha, \beta)$ has no proper ideals and is not abelian. $(L, [\cdot,\cdot], \alpha, \beta)$ is called a semisimple BiHom-Lie algebra if L is a direct sum of certain ideals.

Proposition 28.2 ([54]) *Let* $(L, [\cdot,\cdot]')$ *be an ordinary Lie algebra over a field* $\mathbb{K}$ *and let* $\alpha,\ \beta\colon L \to L$ *two commuting linear maps such that* $\alpha\big([a, b]'\big) = [\alpha(a), \alpha(b)]'$ *and* $\beta\big([a, b]'\big) = [\beta(a), \beta(b)]'$*, for all* $a,\ b \in L$. *Define the linear map* $[\cdot,\cdot]\colon L \times L \to L$, $[a, b] = [\alpha(a), \beta(b)]'$*, for all* $a,\ b \in L$. *Then* $L_{(\alpha,\beta)} := (L, [\cdot,\cdot], \alpha, \beta)$ *is a BiHom-Lie algebra, called the Yau twist of* $(L, [\cdot,\cdot]')$.

Example 28.1 (*Heisengberg BiHom-Lie algebras*) Let (X, Y, Z) a basis of a Heisenberg Lie algebra $(\mathfrak{h}_1, [\cdot,\cdot]')$ such that

$$[X, Y]' = Z,\ [X, Z]' = [Y, Z]' = 0.$$

Let $\begin{pmatrix} b & 0 & 0 \\ 0 & \frac{a}{b} & 0 \\ 0 & 0 & a \end{pmatrix}$ be the matrix of a linerar map $\alpha\colon \mathfrak{h}_1 \to \mathfrak{h}_1$ and let $\begin{pmatrix} y & 0 & 0 \\ 0 & \frac{x}{y} & 0 \\ 0 & 0 & x \end{pmatrix}$ be the matrix of a linerar map $\beta\colon \mathfrak{h}_1 \to \mathfrak{h}_1$ relative to a basis (X, Y, Z) of $\mathfrak{h}_1$. The Yau twist $H_{(\alpha,\beta)}$ of $\mathfrak{h}_1$ is called Heisengberg BiHom-Lie algebra. The bracket of Heisengberg BiHom-Lie algebra $(\mathfrak{h}_1, [\cdot,\cdot], \alpha, \beta)$ is gives by $[X, Y] = b\frac{x}{y}Z$, $[Y, X] = -y\frac{a}{b}Z$ and the other brother bracket are 0. For $m > 1$, we define Heisengberg BiHom-Lie algebra $(\mathfrak{h}_m, [\cdot,\cdot], \alpha, \beta)$ by

$$[X_i, Y_i] = b_i \frac{x}{y_i} Z, \quad [Y_i, X_i] = -y_i \frac{a}{b_i} Z \quad \forall i \in \{1, \ldots, m\}$$

and the other brackets are 0;

$$\alpha = diag(b_1, \ldots, b_m, \frac{a}{b_1}, \ldots, \frac{a}{b_m}, a),$$
$$\beta = diag(y_1, \ldots, y_m, \frac{x}{y_1}, \ldots, \frac{x}{y_m}, x).$$

Proposition 28.3 ([93]) *Let $(G, [\cdot,\cdot], \alpha, \beta)$ be a regular BiHom-Lie algebra. Define the bilinear map $[\cdot,\cdot]' \colon G \times G \to G$ by*

$$[x, y]' = [\alpha^{-1}(x), \beta^{-1}(y)],$$

for all $x,\ y \in G$. Then $\big(L, [\cdot,\cdot]'\big)$ is a Lie algebra, which we call it the induced Lie algebra of $(L, [\cdot,\cdot], \alpha, \beta)$.

Proposition 28.4 ([93]) *The induced Lie algebra of the multiplicative simple BiHom-Lie algebra is semisimple. There exist simple ideal L_1 and an integer $m \neq 2$ such that*

$$L = L_1 \oplus \alpha(L_1) \oplus \cdots \oplus \alpha^{m-1}(L_1) = L_1 \oplus \beta(L_1) \oplus \cdots \oplus \beta^{m-1}(L_1).$$

Proposition 28.5 *Any finite-dimensional multiplicative simple BiHom-Lie algebra is regular.*

Proof The statement holds since $\ker \alpha$ and $\ker \beta$ are ideals of the simple BiHom-Lie algebra $(L, [\cdot,\cdot], \alpha, \beta)$. □

Proposition 28.6 *A regular multiplicative BiHom-Lie algebra $(G, [\cdot,\cdot], \alpha, \beta)$ is nilpotent if and only if the induced Lie algebra $\big(G, [\cdot,\cdot]'\big)$ is nilpotent.*

Proposition 28.7 *A regular multiplicative BiHom-Lie algebra $(G, [\cdot,\cdot], \alpha, \beta)$ is solvable if and only if the induced Lie algebra $\big(G, [\cdot,\cdot]'\big)$ is solvable.*

Definition 28.6 A BiHom-Lie algebra L is said to be decomposable if it can be decomposed into the direct sum of two or more nonzero ideals. We say L is indecompasable if it is not decomposable.

Proposition 28.8 *Every decomposable 2-dimensional multiplicative BiHom-Lie algebra L is nonregular and it satisfies $L = [L, L] \oplus C(L)$.*

Throughout this work, $(L, [\cdot,\cdot], \alpha, \beta)$ denotes a multiplicative BiHom–Lie algebra over $\mathbb{C}$, $I = \ker \alpha + \ker \beta$, G a BiHom-Lie subalgebra of L satisfying $L = I \oplus G$ (if it exists), and $\Omega = \{f \in End(L) \mid f \circ \alpha = \alpha \circ f, f \circ \beta = \beta \circ f\}$.

28.3 Generalized Derivations of BiHom-Lie Algebras

Definition 28.7 ([42]) For any integer k, l, a linear map $D\colon L \to L$ is called an $\alpha^k\beta^l$-derivation of the BiHom-Lie algebra $(L, [\cdot,\cdot], \alpha, \beta)$, if $D \in \Omega$ and

$$D([x, y]) = [D(x), \alpha^k\beta^l(y)] + [\alpha^k\beta^l(x), D(y)],$$

for all $x, y \in L$. The set of all $\alpha^k\beta^l$-derivations of a BiHom-Lie algebra $(L, [\cdot,\cdot], \alpha, \beta)$ is denoted by $Der_{\alpha^k\beta^l}(L)$, and we denote by $Der(L)$ the vector space spanned by the set $\{d \in Der_{\alpha^k\beta^l}(L) \mid k, l \in \mathbb{N}\}$.

Definition 28.8 Let $(L, [\cdot,\cdot], \alpha, \beta)$ be a BiHom–Lie algebra and λ, μ, γ elements of $\mathbb{C}$. A linear map $d \in \Omega$ is a generalized $\alpha^k\beta^l$-derivation or a (λ, μ, γ)-$\alpha^k\beta^l$-derivation of L if for all $x, y \in L$ we have

$$\lambda\, d([x, y]) = \mu\, [d(x), \alpha^k\beta^l(y)] + \gamma\, [\alpha^k\beta^l(x), d(y)].$$

We denote the set of all (λ, μ, γ)-$\alpha^k\beta^l$-derivations by $Der^{(\lambda,\mu,\gamma)}_{\alpha^k\beta^l}(L)$ and $Der^{(\lambda,\mu,\gamma)}(L)$ the vector space spanned by $\{d \in Der^{(\lambda,\mu,\gamma)}_{\alpha^k\beta^l}(L) \mid k, l \in \mathbb{N}\}$.

Lemma 28.1 *For any $D \in Der^{(\lambda,\mu,\gamma)}_{\alpha^k\beta^l}(L)$ and $D' \in Der^{(\lambda',\mu',\gamma')}_{\alpha^s\beta^t}(L)$, their usual commutator defined by*

$$[D, D']' = D \circ D' - D' \circ D, \tag{28.13}$$

satisfies $[D, D']' \in Der^{(\lambda\lambda',\mu\mu',\gamma\gamma')}_{\alpha^{k+s}\beta^{l+t}}(L)$.

Proof The proof is similar to the one of $Der(L)$ in [42, Lemma 3.2]. □

Let us now classify the possible values of $\lambda, \mu, \gamma \in \mathbb{C}$ for a linear map $d\colon G \to G$ to be a (λ, μ, γ)-$\alpha^k\beta^l$-derivation of G

Lemma 28.2 *Let $(G, [\cdot,\cdot], \alpha, \beta)$ be a BiHom–Lie algebra such that the maps α and β are surjective. Let λ, μ, γ be elements of $\mathbb{C}$.*

1) *If $\lambda \neq 0$ and $\mu^2 \neq \gamma^2$. Then $Der^{(\lambda,\mu,\gamma)}_{\alpha^k\beta^l}(G) = Der^{(\frac{\lambda}{\mu+\gamma},1,0)}_{\alpha^k\beta^l}(G)$.*
2) *If $\lambda \neq 0$, $\mu \neq 0$ and $\gamma = -\mu$. Then*
   $Der^{(\lambda,\mu,\gamma)}_{\alpha^k\beta^l}(G) = Der^{(1,0,0)}_{\alpha^k\beta^l}(G) \cap Der^{(0,1,-1)}_{\alpha^k\beta^l}(G) = Der^{(1,1,-1)}_{\alpha^k\beta^l}(G)$.
3) *If $\lambda \neq 0$, $\mu = \gamma$ and $\mu \neq 0$. Then $Der^{(\lambda,\mu,\gamma)}_{\alpha^k\beta^l}(G) = Der^{(\frac{\lambda}{\mu},1,1)}_{\alpha^k\beta^l}(G)$.*
4) *If $\lambda \neq 0$, $\mu = \gamma = 0$. Then $Der^{(\lambda,\mu,\gamma)}_{\alpha^k\beta^l}(G) = Der^{(1,0,0)}_{\alpha^k\beta^l}(G)$.*
5) *If $\lambda = 0$ and $\mu^2 \neq \gamma^2$. Then $Der^{(\lambda,\mu,\gamma)}_{\alpha^k\beta^l}(G) = Der^{(0,1,0)}_{\alpha^k\beta^l}(G)$.*
6) *If $\lambda = 0$, $\mu \neq 0$ and $\mu = \gamma$. Then $Der^{(\lambda,\mu,\gamma)}_{\alpha^k\beta^l}(G) = Der^{(0,1,1)}_{\alpha^k\beta^l}(G)$.*
7) *If $\lambda = 0$ and $\mu = -\gamma$. Then $Der^{(\lambda,\mu,\gamma)}_{\alpha^k\beta^l}(G) = Der^{(0,1,-1)}_{\alpha^k\beta^l}(G)$.*

Proof Let $x,\ y \in G$. Since α and β are surjective, there exists $a,\ b \in G$ such that $x = \beta(a), y = \alpha(b)$. Suppose any $\lambda,\ \mu,\ \gamma \in \mathbb{C}$ are given. Then for $d \in Der^{(\lambda,\mu,\gamma)}_{\alpha^k\beta^l}(G)$ and arbitrary $a, b \in G$ we have

$$\lambda\, d([\beta(a), \alpha(b)]) = \mu\, [d(\beta(a)), \alpha^k\beta^l(\alpha(b))] + \gamma\, [\alpha^k\beta^l(\beta(a)), d(\alpha(b))]$$
$$\lambda\, d([\beta(b), \alpha(a)]) = \mu\, [d(\beta(b)), \alpha^k\beta^l(\alpha(a))] + \gamma\, [\alpha^k\beta^l(\beta(b)), d(\alpha(a))]$$

Thus, using $d \circ \alpha = \alpha \circ d, d \circ \beta = \beta \circ d, \alpha \circ \beta = \beta \circ \alpha$ and (28.4), we have

$$\lambda\, d([\beta(a), \alpha(b)]) = \mu\, [\beta(d(a)), \alpha^{k+1}\beta^l(b))] + \gamma\, [\alpha^k\beta^{l+1}(a), \alpha(d(b))]$$
$$\lambda\, d([\beta(b), \alpha(a)]) = -\mu\, [\alpha^k\beta^{l+1}(a), \alpha(d(b))] - \gamma\, [\beta(d(a)), \alpha^{k+1}\beta^l(b)]$$

By summing the two previous equalities we obtain

$$0 = (\mu - \gamma)\left([\beta(d(a)), \alpha^{k+1}\beta^l(b)] - [\alpha^k\beta^{l+1}(a), \alpha(d(b))]\right).$$

So, $(\mu - \gamma)\left([d(x), \alpha^k\beta^l(y)] - [\alpha^k\beta^l(x), d(y)]\right) = 0$. Therefore, for $\mu \neq \gamma$, $[d(x), \alpha^k\beta^l(y)] = [\alpha^k\beta^l(x), d(y)]$. Hence, applying $d \in Der^{(\lambda,\mu,\gamma)}_{\alpha^k\beta^l}(G)$ yields

$$\lambda d([x, y]) = (\mu + \gamma)[d(x), \alpha^k\beta^l(y)].$$

The rest of the proof is easily deduced. □

Theorem 28.1 *Let $(G, [\cdot,\cdot], \alpha, \beta)$ be a BiHom–Lie algebra such that the maps α and β are surjective. For any $\lambda, \mu, \gamma \in \mathbb{C}$ there exists $\delta \in \mathbb{C}$ such that the subspace $Der^{(\delta,\mu,\gamma)}_{\alpha^k\beta^l}(G)$ is equal to one of the four following subspaces:*

1) $Der^{(0,0,0)}_{\alpha^k\beta^l}(G)$,
2) $Der^{(1,0,0)}_{\alpha^k\beta^l}(G)$,
3) $Der^{(\delta,1,0)}_{\alpha^k\beta^l}(G)$,
4) $Der^{(\delta,1,1)}_{\alpha^k\beta^l}(G)$,
5) $Der^{(1,1,-1)}_{\alpha^k\beta^l}(G)$,
6) $Der^{(0,1,-1)}_{\alpha^k\beta^l}(G)$.

Example 28.2 Let H be a 3-dimensional Heisenberg BiHom-Lie algebra (Example 28.1).

$$Der^{(1,0,0)}_{\alpha^k\beta^l}(H) = \left\{ \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid d_1,\ d_2 \in \mathbb{C} \right\};$$
$$Der^{(\delta,1,0)}_{\alpha^k\beta^l}(H) = \left\{ \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_1 \frac{a^r x^l}{b^{2r} y^{2l}} & 0 \\ 0 & 0 & d_1 \frac{a^r x^l}{\delta b^r y^l} \end{pmatrix} \mid d_1 \in \mathbb{C} \right\};$$

$$Der^{(\delta,1,1)}_{\alpha^k\beta^l}(H) = \left\{ \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & \frac{d_2 b^{2r} y^{2l} + d_1 a^r x^l}{\delta b^r y^l} \end{pmatrix} \mid d_1, d_2 \in \mathbb{C} \right\};$$

$$Der^{(0,1,1)}_{\alpha^k\beta^l}(H) = \left\{ \begin{pmatrix} d_1 & 0 & 0 \\ 0 & -d_1 \frac{a^r x^l}{b^{2r} y^{2l}} & 0 \\ 0 & 0 & d_3 \end{pmatrix} \mid d_1, d_3 \in \mathbb{C} \right\};$$

$$Der^{(1,1,-1)}_{\alpha^k\beta^l}(H) = \left\{ \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_1 \frac{a^r x^l}{b^{2r} y^{2l}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid d_1 \in \mathbb{C} \right\};$$

$$Der^{(0,1,-1)}_{\alpha^k\beta^l}(H) = \left\{ \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_1 \frac{a^r x^l}{b^{2r} y^{2l}} & 0 \\ 0 & 0 & d_3 \end{pmatrix} \mid d_1, d_3 \in \mathbb{C} \right\}.$$

Next proposition allows to extend some results from [47] to BiHom-Lie case.

Proposition 28.9 *If $(G, [\cdot,\cdot], \alpha, \beta)$ is a regular multiplicative BiHom-Lie algebra, then any (λ, μ, γ)-$\alpha^0\beta^0$-derivation of $(G, [\cdot,\cdot], \alpha, \beta)$ is a (λ, μ, γ)-derivation of induced Lie algebra $\left(G, [\cdot,\cdot]'\right)$.*

Now we will discuss in detail the possible Theorem 28.1 for a finite-dimensional BiHom-Lie algebra L, and we give the connection between the generalized derivation of the type studied in [2] and the generalized derivations of the type studied in this work.

1. $\underline{Der^{(0,0,0)}_{\alpha^k\beta^l}(L) = \Omega}$. We have Ω is a Lie algebra, in which the Lie bracket is given by (28.13).
2. $\underline{Der^{(1,0,0)}_{\alpha^k\beta^l}(L) = \left\{d \in \Omega \mid d(L^2) = 0\right\}}$ and therefore its dimension is

$$dim Der^{(1,0,0)}_{\alpha^k\beta^l}(L) = codim L^2\, dim L.$$

   If the BiHom-Lie algebra L is simple, then $Der^{(1,0,0)}_{\alpha^0\beta^0}(L) = \{0\}$.
3. $\underline{Der^{(\delta,1,0)}_{\alpha^k\beta^l}(L)}$:

   (a) If $\delta = 0$, then $Der^{(0,1,0)}_{\alpha^k\beta^l}(L) = \left\{d \in \Omega \mid d(L) \subseteq C(\alpha^k\beta^l(L))\right\}$ where $C(\alpha^k\beta^l(L))$ is the centralizer of $\alpha^k\beta^l(L)$ given by

$$C(\alpha^k\beta^l(L)) = \left\{x \in L \mid [x, y] = 0; \forall y \in \alpha^k\beta^l(L)\right\}.$$

   Therefore, $\dim Der^{(0,1,0)}_{\alpha^k\beta^l}(L) = \dim L\ \dim C(\alpha^k\beta^l(L))$.
   If the BiHom-Lie algebra L is simple, then $Der^{(0,1,0)}_{\alpha^0\beta^0}(L) = \{0\}$.

   (b) If $\delta = 1$, then $Der^{(1,1,0)}_{\alpha^k\beta^l}(L)$ is the $\alpha^k\beta^l$-centroid of L denoted $\Gamma_{\alpha^k\beta^l}(L)$. We denote by $\Gamma(L)$ the vector space spanned by the set $\{d \in \Gamma_{\alpha^k\beta^l}(L) \mid k, l \in \mathbb{N}\}$.

(i) If $\phi \in \Gamma_{\alpha^k\beta^l}(G)$ and $d \in Der_{\alpha^s\beta^r}(G)$, then $\phi \circ d$ is a $\alpha^{k+s}\beta^{r+l}$-derivation of G.
(ii) $\Gamma_{\alpha^k\beta^l}(L) \cap Der_{\alpha^k\beta^l}(L) = CDer_{\alpha^k\beta^l}(G)$, where $CDer_{\alpha^k\beta^l}(G)$ is the set of $\alpha^k\beta^l$-central derivations defined by

$$CDer_{\alpha^k\beta^l}(G) = Der^{(1,0,0)}_{\alpha^k\beta^l}(G) \bigcap Der^{(0,1,0)}_{\alpha^k\beta^l}(G).$$

(iii) For any $d \in Der_{\alpha^k\beta^l}(G)$ and $\phi \in \Gamma_{\alpha^t\beta^s}(G)$ one has
- The composition $d \circ \phi$ is in $\Gamma_{\alpha^{k+t}\beta^{l+s}}(G)$ if and only if $\phi \circ d$ is a central derivation of L;
- The composition $d \circ \phi$ is a $\alpha^{k+t}\beta^{l+s}$-derivation of G if and only if $[d, \phi]'$ is a $\alpha^{k+t}\beta^{l+s}$-central derivation of G. (See [9] for the Leibniz case and [3] for the associative algebras case).

Suppose that L admits a generalized derivation $D \in Der^{(1,1,0)}_{\alpha^0\beta^0}(L)$. If $\lambda \in \sigma(D)$ is an eigenvalue of D, then the corresponding generalized eigenspace L_λ is an ideal of L. Moreover, the generalized eigenspace decomposition $L = \oplus_{\lambda\in\sigma(D)} L_\lambda$ is given in terms of ideals of L. Suppose that the BiHom-Lie algebra L is simple, then $Der^{(1,1,0)}_{\alpha^0\beta^0}(L)$ is the one–dimensional BiHom-Lie algebra containing multiples of the identity operator.

(c) For $\delta \notin \{0, 1\}$. Suppose that G is non-abelian. Then, by Propositions 28.6, 28.9 and [47, Proposition 2.19], the following statements are equivalent:
(i) G admits an invertible generalized derivation $D \in Der^{(\delta,1,0)}_{\alpha^0\beta^0}(G)$.
(ii) G is at most a 2-step nilpotent BiHom-Lie algebra.
(iii) G admits an invertible semisimple generalized derivation $D \in Der^{(\delta,1,0)}_{\alpha^0\beta^0}(G)$ with minimal polynomial $q(x) = (x - \delta^{-1})(x - 1)$.

4. $\underline{Der^{(\delta,1,1)}_{\alpha^k\beta^l}(L)}$:

(a) For $\delta = 0$.
We have a Lie algebra

$$Der^{(0,1,1)}(L) = \{d \in \Omega \mid \exists k, l \in \mathbb{N} : [d(x), \alpha^k\beta^l(y)] = -[\alpha^k\beta^l(x), d(y)], \forall x, y \in L\}.$$

If the BiHom-Lie algebra L is simple, then by Propositions 28.4, 28.9 and [47, Corollaries 2.10 and 2.11], we have $Der^{(0,1,1)}_{\alpha^0\beta^0}(L) = \{0\}$. If the simple BiHom-Lie algebra L admits an invertible generalized derivation $D \in Der^{(0,1,1)}_{\alpha^0\beta^0}(L)$, then L is solvable.

(b) For $\delta = 1$.
We get the Lie algebra of derivations of L: $Der^{(1,1,1)}_{\alpha^k\beta^k}(L) = Der_{\alpha^k\beta^k}(L)$ and $\big(Der(L), [\cdot,\cdot]'\big)$ is a Lie algebra ($Der(L)$ the vector space spanned by $\{d \in Der_{\alpha^k\beta^l}(L) \mid k, l \in \mathbb{N}\}$).

(c) For $\delta \notin \{-1, 0, 1, 2\}$.
When G admits an invertible semisimple generalized $D \in Der^{(\delta,1,1)}_{\alpha^0\beta^0}(G)$ by Propositions 28.7, 28.9 and [47, Proposition 2.8], G is at most a 3-step solvable BiHom-Lie algebra. When the invertible semisimple generalized D has only two different eigenvalues, by Propositions 28.6, 28.9 and [47, Lemma 2.2], G is at most a 2-step nilpotent BiHom-Lie algebra.

5. $\underline{Der^{(1,1,-1)}_{\alpha^k\beta^l}(G)}$: We have

$$\begin{aligned} Der^{(1,1,-1)}_{\alpha^k\beta^l}(G) &= Der^{(0,1,-1)}_{\alpha^k\beta^l}(G) \cap Der^{(1,0,0)}_{\alpha^k\beta^l}(G) \\ &= \left\{ d \in \Omega \mid d([x, y]) = 0 = [d(x), \alpha^k\beta^l(y)] = [\alpha^k\beta^l(x), d(y)] \right\}. \end{aligned}$$

Then $Der^{(1,1,-1)}_{\alpha^k\beta^l}(G)$ is the set of $\alpha^k\beta^l$-central derivations of G. Define the bilinear map

$$\mu \colon \Omega \times \Omega \to \Omega,\ \mu(f, g) = \frac{1}{2}(f \circ g + g \circ f). \tag{28.14}$$

Then $(CDer(G), \mu)$ is a Jordan algebra.

6. $\underline{Der^{(0,1,-1)}_{\alpha^k\beta^l}(L)}$: We have

$$Der^{(0,1,-1)}_{\alpha^k\beta^l}(G) = \left\{ d \in \Omega \mid [d(x), \alpha^k\beta^l(y)] = [\alpha^k\beta^l(x), d(y)] \right\}.$$

Then $Der^{(0,1,-1)}_{\alpha^k\beta^l}(L)$ is called $\alpha^k\beta^l$-quasi-centroid of L and denoted $QC_{\alpha^k\beta^l}(L)$. With the bilinear map μ defined in (28.14), we have that $\left(QC_{\alpha^k\beta^l}(G), \mu\right)$ is a Jordan algebra.

We end this section with a construction of a BiHom-Lie algebra from an extension of a Lie algebra L by a (b, a, a)-derivation of L.

Proposition 28.10 *Let $(L, [\cdot, \cdot]')$ be a Lie algebra and $D \in End(L)$ be a non-zero (b, a, a)-derivation and let $\alpha \colon L \oplus \mathbb{C}D \to L \oplus \mathbb{C}D$ and $\beta \colon L \oplus \mathbb{C}D \to L \oplus \mathbb{C}D$ defined respectively by $\alpha(x + \lambda D) = x + \lambda aD$ and $\beta(x + \lambda D) = x + \lambda b;\ x \in L,\ \lambda \in \mathbb{C}$. Let Define the bilinear map $[\cdot, \cdot] \colon L \oplus \mathbb{C}D \times L \oplus \mathbb{C}D \to L \oplus \mathbb{C}D$, $[x + \lambda d, y + \mu d] = [x, y]' - \mu bd(x) + \lambda ad(y)$. Then $(L \oplus \mathbb{C}D, [\cdot, \cdot], \alpha, \beta)$ is a BiHom-Lie algebra.*

28.4 Classification of Multiplicative 2-Dimensional BiHom-Lie Algebras

In this section, we aim to classify 2-dimensional non-trivial BiHom-Lie algebras. An n-dimensional multiplicative BiHom-Lie algebra is identified to its structure constants with respect to a fixed basis. It turns out that the axioms of multiplicative

BiHom-Lie algebra structure translate to a system of polynomial equations that define the algebraic variety of n-dimensional multiplicative BiHom-Lie algebra which is embedded into $\mathbb{K}^{n^3+2n^2}$. The classification requires to solve this algebraic system. The calculations are handled using a computer algebra system. For $n = 2$, we include in the following an outline of the computation.

1. Solving (28.3), we obtain the following solutions:

   1.1 $\alpha = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad \beta = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix};$

   1.2 $\alpha = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad \beta = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix};$

   1.3 $\alpha = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}, \quad \beta = \begin{pmatrix} x & z \\ 0 & x \end{pmatrix}.$

2. For each solution in 1, we provide a list of non-trivial 2-dimensional multiplicative BiHom-Lie algebras. We solve the system of (28.8), (28.9) and (28.10) such that

   (i) $a_{12} = a_{21} = 0$, $b_{12} = b_{21} = 0$, for 1.1

   (ii) $a_{12} = a_{21} = 0$, $a_{22} = a_{11}$, $b_{21} = 0$, $b_{12} = 1$, $b_{22} = b_{11}$, for 1.2

   (iii) $a_{21} = 0$, $a_{12} = 1$, $a_{22} = a_{11}$, $b_{21} = 0$, $b_{22} = b_{11}$, for 1.3.

3. Fix a BiHom-Lie algebra L in 2 and solve (28.12) such that C_{ij}^k are the structure constants corresponding to L and $f_{12} = f_{21} = 0$ (resp. $f_{11} = f_{22} = 0$) if $[L, L] \neq < e_2 >$ (resp. $[L, L] = < e_2 >$).

Therefore, we get the following result.

Proposition 28.11 *Every 2-dimensional multiplicative BiHom-Lie algebra is isomorphic to one of the following non-isomorphic BiHom-Lie algebras: each algebra is denoted by L_j^i where i is related to the couple (α, β), j is the number.*

$$L_1^1 : [e_1, e_1] = e_1, \quad [e_1, e_2] = e_1, \quad [e_2, e_1] = z_1 e_1, \quad [e_2, e_2] = 0,$$
$$\alpha(e_1) = 0, \quad \alpha(e_2) = b e_2, \quad \beta(e_1) = 0, \quad \beta(e_2) = y e_2.$$

$$L_2^1 : [e_1, e_1] = e_1, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = e_1, \quad [e_2, e_2] = 0,$$
$$\alpha(e_1) = 0, \quad \alpha(e_2) = b e_2, \quad \beta(e_1) = 0, \quad \beta(e_2) = y e_2.$$

$$L_3^1 : [e_1, e_1] = e_1, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = 0,$$
$$\alpha(e_1) = 0, \quad \alpha(e_2) = b e_2, \quad \beta(e_1) = 0, \quad \beta(e_2) = y e_2.$$

$$L_4^1 : [e_1, e_1] = 0, \quad [e_1, e_2] = e_1, \quad [e_2, e_1] = z_1 e_1, \quad [e_2, e_2] = 0,$$
$$\alpha(e_1) = 0, \quad \alpha(e_2) = b e_2, \quad \beta(e_1) = 0, \quad \beta(e_2) = y e_2.$$

$$L_5^1 : [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = e_1, \quad [e_2, e_2] = 0,$$
$$\alpha(e_1) = 0, \quad \alpha(e_2) = b e_2, \quad \beta(e_1) = 0, \quad \beta(e_2) = y e_2.$$

$$L_1^2 : [e_1, e_1] = e_1, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = 0,$$
$$\alpha(e_1) = e_1, \quad \alpha(e_2) = b e_2, \quad \beta(e_1) = 0, \quad \beta(e_2) = y e_2.$$

$L_1^3 : [e_1, e_1] = 0, \quad [e_1, e_2] = e_1, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = 0,$
$\alpha(e_1) = ae_1, \quad \alpha(e_2) = e_2, \quad \beta(e_1) = 0, \quad \beta(e_2) = ye_2.$

$L_1^4 : [e_1, e_1] = e_1, \quad [e_1, e_2] = e_1, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = 0,$
$\alpha(e_1) = e_1, \quad \alpha(e_2) = e_2, \quad \beta(e_1) = 0, \quad \beta(e_2) = ye_2.$

$L_1^5 : [e_1, e_1] = e_1, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = 0,$
$\alpha(e_1) = 0, \quad \alpha(e_2) = be_2, \quad \beta(e_1) = e_1, \quad \beta(e_2) = ye_2.$

$L_1^6 : [e_1, e_1] = e_1, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = e_1, \quad [e_2, e_2] = 0,$
$\alpha(e_1) = 0, \quad \alpha(e_2) = be_2, \quad \beta(e_1) = e_1, \quad \beta(e_2) = e_2.$

$L_1^7 : [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = e_1, \quad [e_2, e_2] = 0,$
$\alpha(e_1) = 0, \quad \alpha(e_2) = be_2, \quad \beta(e_1) = xe_1, \quad \beta(e_2) = e_2.$

$L_1^8 : [e_1, e_1] = 0, \quad [e_1, e_2] = e_1, \quad [e_2, e_1] = -\frac{x}{a}e_1, \quad [e_2, e_2] = 0,$
$\alpha(e_1) = ae_1, \quad \alpha(e_2) = e_2, \quad \beta(e_1) = xe_1, \quad \beta(e_2) = e_2.$

$L_1^9 : [e_1, e_1] = e_1, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = e_2,$
$\alpha(e_1) = e_1, \quad \alpha(e_2) = 0, \quad \beta(e_1) = 0, \quad \beta(e_2) = e_2.$

$L_1^{10} : [e_1, e_1] = 0, \quad [e_1, e_2] = e_1 + e_2, \quad [e_2, e_1] = -e_1 - e_2, \quad [e_2, e_2] = 0,$
$\alpha(e_1) = e_1, \quad \alpha(e_2) = e_2, \quad \beta(e_1) = e_1, \quad \beta(e_2) = e_2.$

$L_1^{11} : [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = e_1, \quad [e_2, e_2] = e_1.$
$\alpha(e_1) = e_1, \quad \alpha(e_2) = e_2, \quad \beta(e_1) = 0, \quad \beta(e_2) = ze_1.$

$L_2^{11} : [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = e_1, \quad [e_2, e_2] = 0.$
$\alpha(e_1) = e_1, \quad \alpha(e_2) = e_2, \quad \beta(e_1) = 0, \quad \beta(e_2) = e_1.$

$L_3^{11} : [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = e_1.$
$\alpha_{34}(e_1) = e_1, \quad \alpha(e_2) = e_2, \quad \beta(e_1) = 0, \quad \beta(e_2) = e_1.$

$L_1^{12} : [e_1, e_1] = 0, \quad [e_1, e_2] = e_1, \quad [e_2, e_1] = -e_1, \quad [e_2, e_2] = -e_1.$
$\alpha(e_1) = e_1, \quad \alpha(e_2) = e_2, \quad \beta(e_1) = e_1, \quad \beta(e_2) = e_1 + e_2.$

$L_1^{13} : [e_1, e_1] = 0, \quad [e_1, e_2] = e_1, \quad [e_2, e_1] = z_1 e_1, \quad [e_2, e_2] = t_1 e_1,$
$\alpha(e_1) = 0, \quad \alpha(e_2) = e_1, \quad \beta(e_1) = 0, \quad \beta(e_2) = e_1.$

$L_2^{13} : [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = e_1, \quad [e_2, e_2] = t_1 e_1,$
$\alpha(e_1) = 0, \quad \alpha(e_2) = e_1, \quad \beta(e_1) = 0, \quad \beta(e_2) = ze_1.$

$L_3^{13} : [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = e_1,$
$\alpha(e_1) = 0, \quad \alpha(e_2) = e_1, \quad \beta(e_1) = 0, \quad \beta(e_2) = ze_1.$

$L_1^{14} : [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = e_1,$
$\alpha(e_1) = e_1, \quad \alpha(e_2) = e_1 + e_2, \quad \beta(e_1) = 0, \quad \beta(e_2) = ze_1.$

$L_1^{15} : [e_1, e_1] = 0, \quad [e_1, e_2] = e_1, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = t_1 e_1,$
$\alpha(e_1) = 0, \quad \alpha(e_2) = e_1, \quad \beta(e_1) = e_1, \quad \beta(e_2) = e_2.$

$L_1^{16} : [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = e_1,$
$\alpha(e_1) = 0, \quad \alpha(e_2) = e_1, \quad \beta(e_1) = e_1, \quad \beta(e_2) = ze_1 + e_2.$

$L_1^{17} : [e_1, e_1] = 0, \quad [e_1, e_2] = e_1, \quad [e_2, e_1] = -e_1, \quad [e_2, e_2] = (1 - z)e_1,$
$\alpha(e_1) = e_1, \quad \alpha(e_2) = e_1 + e_2, \quad \beta(e_1) = e_1, \quad \beta(e_2) = ze_1 + e_2.$

Corollary 28.1 *Every decomposable* 2*-dimensional multiplicative BiHom-Lie algebra* L *is isomorphic to one of these* 4 *algebras:* L_3^1, L_1^2, L_1^5, L_1^9.

Remark 28.3 $< e_1 + e_2 >$ is an ideal of BiHom-Lie algebra L_1^{10}. For the others BiHom-Lie algebras $< e_1 >$ is an ideal of L_i^j. Hence, every 2-dimensional multiplicative BiHom-Lie algebra is not simple.

28.5 Centroids and Derivations of 2-Dimensional Multiplicative BiHom-Lie Algebras

Let $(L, [\cdot, \cdot], \alpha, \beta)$ be a n-dimensional multiplicative BiHom-Lie algebra. Let $\alpha^r\beta^l(e_j) = \sum_{k=1}^{n} m_{kj}e_k$. An element d of $Der_{\alpha^r\beta^l}^{(\delta,\mu,\gamma)}(L)$, being a linear transformation of the vector space L, is represented in a matrix form $(d_{ij})_{1\leq i,j\leq n}$ corresponding to $d(e_j) = \sum_{k=1}^{n} d_{kj}e_k$, for $j = 1, \dots, n$. According to the definition of the (δ, μ, γ)-$\alpha^r\beta^l$-derivation the entries d_{ij} of the matrix $(d_{ij})_{1\leq i,j\leq n}$ must satisfy the following systems $\mathcal{S}$ of equations:

$$\sum_{k=1}^{n} d_{ik}a_{kj} = \sum_{k=1}^{n} a_{ik}d_{kj}; \qquad \sum_{k=1}^{n} d_{ik}b_{kj} = \sum_{k=1}^{n} b_{ik}d_{kj};$$
$$\delta\sum_{k=1}^{n} c_{ij}^k d_{sk} - \mu\sum_{k=1}^{n}\sum_{l=1}^{n} d_{ki}m_{lj}c_{kl}^s - \gamma\sum_{k=1}^{n}\sum_{l=1}^{n} d_{lj}m_{ki}c_{kl}^s = 0,$$

where $(a_{ij})_{1\leq i,j\leq n}$ is the matrix of α, $(b_{ij})_{1\leq i,j\leq n}$ is the matrix of β and (c_{ij}^k) are the structure constants of L. First, let us give the following definitions:

Definition 28.9 A BiHom-Lie algebra is called characteristically nilpotent (denoted by CN) if the Lie algebra $Der_{\alpha^0\beta^0}(L)$ is nilpotent.

Definition 28.10 Let L be an indecomposable BiHom-Lie algebra. We say L is small if $\Gamma_{\alpha^0\beta^0}(L)$ is generated by central derivation and the scalars. The centroid of a decomposable BiHom-Lie algebra is small if the centroids of each indecomposable factor are small.

Now we apply the algorithms mention in the previous paragraph to centroid and derivation of 2-dimensional complex BiHom-Lie algebras. To find the centroids and derivations of 2-dimensional complex BiHom-Lie algebras we use the classification results from the previous section. The results are given in the following theorem. Moreover, we give the type of $\Gamma_{\alpha^r\beta^l}(L_i^j)$ and $Der_{\alpha^r\beta^l}(L_i^j)$ if $(r, l) = (0, 0)$.

Theorem 28.2

$$L_1^1 : [e_1, e_1] = e_1, \quad [e_1, e_2] = e_1, \quad [e_2, e_1] = z_1 e_1, \quad [e_2, e_2] = 0,$$
$$\alpha(e_1) = 0, \quad \alpha(e_2) = be_2, \quad \beta(e_1) = 0, \quad \beta(e_2) = ye_2.$$

$\alpha^r \beta^l$		$\Gamma_{\alpha^r\beta^l}(L_1^1)$	*Type of* $\Gamma_{\alpha^0\beta^0}(L_1^1)$	$Der_{\alpha^r\beta^l}(L_1^1)$	*CN*
$(r, l) = (0, 0)$	$z_1 = 0$	$\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$	*Not small*	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	*Yes*
$(r, l) = (0, 0)$	$z_1 \neq 0$	$\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}$	*Small*	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	*Yes*
$(r, l) \neq (0, 0)$		$\begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}$		$\begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix}$	

$$L_2^1 : [e_1, e_1] = e_1, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = e_1, \quad [e_2, e_2] = 0,$$
$$\alpha(e_1) = 0, \quad \alpha(e_2) = be_2, \quad \beta(e_1) = 0, \quad \beta(e_2) = ye_2.$$

$\alpha^r \beta^l$	$\Gamma_{\alpha^r\beta^l}(L_2^1)$	*Type of* $\Gamma_{\alpha^0\beta^0}(L_1^1)$	$Der_{\alpha^r\beta^l}(L_2^1)$	*CN*
$(r, l) = (0, 0)$	$\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}$	*Small*	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	*Yes*
$(r, l) \neq (0, 0)$	$\begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}$		$\begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix}$	

$$L_3^1 : [e_1, e_1] = e_1, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = 0,$$
$$\alpha(e_1) = 0, \quad \alpha(e_2) = be_2, \quad \beta(e_1) = 0, \quad \beta(e_2) = ye_2.$$

$\alpha^r \beta^l$	$\Gamma_{\alpha^r\beta^l}(L_3^1)$	*Type of* $\Gamma_{\alpha^0\beta^0}(L_3^1)$	$Der_{\alpha^r\beta^l}(L_3^1)$	*CN*
$(r, l) = (0, 0)$	$\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}$	*Not small*	$\begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix}$	*Yes*
$(r, l) \neq (0, 0)$	$\begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}$		$\begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix}$	

$$L_4^1 : [e_1, e_1] = 0, \quad [e_1, e_2] = e_1, \quad [e_2, e_1] = z_1 e_1, \quad [e_2, e_2] = 0,$$
$$\alpha(e_1) = 0, \quad \alpha(e_2) = be_2, \quad \beta(e_1) = 0, \quad \beta(e_2) = ye_2.$$

		$\Gamma_{\alpha^r\beta^l}(L_4^1)$	*Type of* $\Gamma_{\alpha^0\beta^0}(L_4^1)$	$Der_{\alpha^r\beta^l}(L_4^1)$	*CN*
$(r,l)=(0,0)$	$z_1=0$	$\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$	*Not small*	$\begin{pmatrix} d_1 & 0 \\ 0 & 0 \end{pmatrix}$	*Yes*
$(r,l)=(0,0)$	$z_1\neq 0$	$\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}$	*Small*	$\begin{pmatrix} d_1 & 0 \\ 0 & 0 \end{pmatrix}$	*Yes*
$(r,l)\neq(0,0)$	$z_1=0$; $b^ry^l=1$	$\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$		$\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$	
$(r,l)\neq(0,0)$	$b^ry^l\neq 1$	$\begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}$		$\begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix}$	
$(r,l)\neq(0,0)$	$z_1\neq 0$; $b^ry^l=1$	$\begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}$		$\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$	

$$\begin{aligned} L_5^1: [e_1,e_1]&=0, & [e_1,e_2]&=0, & [e_2,e_1]&=e_1, & [e_2,e_2]&=0,\\ \alpha(e_1)&=0, & \alpha(e_2)&=be_2, & \beta(e_1)&=0, & \beta(e_2)&=ye_2. \end{aligned}$$

		$\Gamma_{\alpha^r\beta^l}(L_5^1)$	*Type of* $\Gamma_{\alpha^0\beta^0}(L_5^1)$	$Der_{\alpha^r\beta^l}(L_5^1)$	*CN*
$(r,l)=(0,0)$		$\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}$	*Small*	$\begin{pmatrix} d_1 & 0 \\ 0 & 0 \end{pmatrix}$	*Yes*
$(r,l)\neq(0,0)$	$b^ry^l=1$	$\begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}$		$\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$	
$(r,l)\neq(0,0)$	$b^ry^l\neq 1$	$\begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}$		$\begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix}$	

$$\begin{aligned} L_1^2: [e_1,e_1]&=e_1, & [e_1,e_2]&=0, & [e_2,e_1]&=0, & [e_2,e_2]&=0,\\ \alpha(e_1)&=e_1, & \alpha(e_2)&=be_2, & \beta(e_1)&=0, & \beta(e_2)&=ye_2. \end{aligned}$$

	$\Gamma_{\alpha^r\beta^l}(L_1^2)$	*Type of* $\Gamma_{\alpha^0\beta^0}(L_1^2)$	$Der_{\alpha^r\beta^l}(L_1^2)$	*CN*
$l=0$	$\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$	*Not small*	$\begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix}$	*Yes*
$l\neq 0$	$\begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}$		$\begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix}$	

$$\begin{aligned} L_1^3: [e_1,e_1]&=0, & [e_1,e_2]&=e_1, & [e_2,e_1]&=0, & [e_2,e_2]&=0,\\ \alpha(e_1)&=ae_1, & \alpha(e_2)&=e_2, & \beta(e_1)&=0, & \beta(e_2)&=ye_2. \end{aligned}$$

		$\Gamma_{\alpha^r\beta^l}(L_1^3)$	*Type of* $\Gamma_{\alpha^0\beta^0}(L_1^3)$	$Der_{\alpha^r\beta^l}(L_1^3)$	*CN*
$l=0$		$\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$	*Not small*	$\begin{pmatrix} d_1 & 0 \\ 0 & 0 \end{pmatrix}$	*Yes*
$l\neq 0$	$y^l=1$	$\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$		$\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$	
$l\neq 0$	$y^l\neq 1$	$\begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}$		$\begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix}$	

$$\begin{aligned} L_1^4 : [e_1, e_1] &= e_1, \quad [e_1, e_2] = e_1, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = 0, \\ \alpha(e_1) &= e_1, \quad \alpha(e_2) = e_2, \quad \beta(e_1) = 0, \quad \beta(e_2) = ye_2. \end{aligned}$$

	$\Gamma_{\alpha^r\beta^l}(L_1^4)$	*Type of* $\Gamma_{\alpha^0\beta^0}(L_1^4)$	$Der_{\alpha^r\beta^l}(L_1^4)$	*CN*
$l=0$	$\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$	*Not small*	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	*Yes*
$l\neq 0$	$\begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}$		$\begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix}$	

$$\begin{aligned} L_1^5 : [e_1, e_1] &= e_1, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = 0, \\ \alpha(e_1) &= 0, \quad \alpha(e_2) = be_2, \quad \beta(e_1) = e_1, \quad \beta(e_2) = ye_2. \end{aligned}$$

	$\Gamma_{\alpha^r\beta^l}(L_1^5)$	*Type of* $\Gamma_{\alpha^0\beta^0}(L_1^5)$	$Der_{\alpha^r\beta^l}(L_1^5)$	*CN*
$r=0$	$\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$	*Not small*	$\begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix}$	
$r\neq 0$	$\begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}$		$\begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix}$	

$$\begin{aligned} L_1^6 : [e_1, e_1] &= e_1, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = e_1, \quad [e_2, e_2] = 0, \\ \alpha(e_1) &= 0, \quad \alpha(e_2) = be_2, \quad \beta(e_1) = e_1, \quad \beta(e_2) = e_2. \end{aligned}$$

	$\Gamma_{\alpha^r\beta^l}(L_1^6)$	*Type of* $\Gamma_{\alpha^0\beta^0}(L_1^6)$	$Der_{\alpha^r\beta^l}(L_1^6)$	*Yes*
$r=0$	$\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}$	*Small*	$\begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix}$	*Yes*
$r\neq 0$	$\begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}$		$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	

$$\begin{aligned} L_1^7 : [e_1, e_1] &= 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = e_1, \quad [e_2, e_2] = 0, \\ \alpha(e_1) &= 0, \quad \alpha(e_2) = be_2, \quad \beta(e_1) = xe_1,\ x\neq 0 \quad \beta(e_2) = e_2. \end{aligned}$$

		$\Gamma_{\alpha^r\beta^l}(L_1^7)$	*Type of* $\Gamma_{\alpha^0\beta^0}(L_1^7)$	$Der_{\alpha^r\beta^l}(L_1^7)$	*CN*
$r=0$		$\begin{pmatrix} c_1 & 0 \\ 0 & \frac{c_1}{x^l} \end{pmatrix}$	*Small*	$\begin{pmatrix} d_1 & 0 \\ 0 & 0 \end{pmatrix}$	*Yes*
$r\neq 0$	$x=1$	$\begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}$		$\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$	
$r\neq 0$	$x\neq 1$	$\begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}$		$\begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix}$	

$$L_1^8 : [e_1, e_1] = 0, \quad [e_1, e_2] = e_1, \quad [e_2, e_1] = -\frac{x}{a} e_1, \quad [e_2, e_2] = 0,$$
$$\alpha(e_1) = ae_1, \quad \alpha(e_2) = e_2, \quad \beta(e_1) = xe_1, \quad \beta(e_2) = e_2.$$

$\Gamma_{\alpha^r\beta^l}(L_1^8)$	*Type of* $\Gamma_{\alpha^0\beta^0}(L_1^8)$	$Der_{\alpha^r\beta^l}(L_1^8)$	*CN*
$\begin{pmatrix} c_1 & 0 \\ 0 & \frac{c_1}{a^r x^l} \end{pmatrix}$	*Small*	$\begin{pmatrix} d_1 & 0 \\ 0 & 0 \end{pmatrix}$	*Yes*

$$L_1^9 : [e_1, e_1] = e_1, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = e_2,$$
$$\alpha(e_1) = e_1, \quad \alpha(e_2) = 0, \quad \beta(e_1) = 0, \quad \beta(e_2) = e_2.$$

	$\Gamma_{\alpha^r\beta^l}(L_1^9)$	*Type of* $\Gamma_{\alpha^0\beta^0}(L_1^9)$	$Der_{\alpha^r\beta^l}(L_1^9)$	
$r=0,\ l=0$	$\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}$	*Not small*	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	*Yes*
$r=0,\ l\neq 0$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$		$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	
$r\neq 0$	$\begin{pmatrix} 0 & 0 \\ 0 & c_2 \end{pmatrix}$		$\begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix}$	

$$L_1^{10} : [e_1, e_1] = 0, \quad [e_1, e_2] = e_1 + e_2, \quad [e_2, e_1] = -e_1 - e_2, \quad [e_2, e_2] = 0,$$
$$\alpha(e_1) = e_1, \quad \alpha(e_2) = e_2, \quad \beta(e_1) = e_1, \quad \beta(e_2) = e_2.$$

$\Gamma_{\alpha^r\beta^l}(L_1^{10})$	*Type of* $\Gamma_{\alpha^0\beta^0}(L_1^{10})$	$Der_{\alpha^r\beta^l}(L_1^{10})$	*CN*
$\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}$	*Small*	$\begin{pmatrix} d_1 & d_2 \\ d_1 & d_2 \end{pmatrix}$	*No*

$$L_1^{11} : [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = e_1, \quad [e_2, e_2] = e_1.$$
$$\alpha(e_1) = e_1, \quad \alpha(e_2) = e_2, \quad \beta(e_1) = 0, \quad \beta(e_2) = ze_1.$$

	$\Gamma_{\alpha^r\beta^l}(L_1^{11})$	*Type of* $\Gamma_{\alpha^0\beta^0}(L_1^{11})$	$Der_{\alpha^r\beta^l}(L_1^{11})$	*CN*
$l=0$	$\begin{pmatrix} c_1 & c_2 \\ 0 & c_1 \end{pmatrix}$	*Not small*	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	*Yes*
$l \geq 1$	$\begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix}$		$\begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix}$	

$$L_2^{11} : [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = e_1, \quad [e_2, e_2] = 0.$$
$$\alpha_{34}(e_1) = e_1, \quad \alpha(e_2) = e_2, \quad \beta(e_1) = 0, \quad \beta(e_2) = e_1.$$

	$\Gamma_{\alpha^r\beta^l}(L_2^{11})$	*Type of* $\Gamma_{\alpha^0\beta^0}(L_2^{11})$	$Der_{\alpha^r\beta^l}(L_2^{11})$	*CN*
$l=0$	$\begin{pmatrix} c_1 & c_2 \\ 0 & c_1 \end{pmatrix}$	*Not small*	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	*Yes*
$l \geq 1$	$\begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix}$		$\begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix}$	

$$L_3^{11} : [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = e_1.$$
$$\alpha_{34}(e_1) = e_1, \quad \alpha(e_2) = e_2, \quad \beta(e_1) = 0, \quad \beta(e_2) = e_1.$$

	$\Gamma_{\alpha^r\beta^l}(L_3^{11})$	*Type of* $\Gamma_{\alpha^0\beta^0}(L_3^{11})$	$Der_{\alpha^r\beta^l}(L_3^{11})$	*CN*
$l=0$	$\begin{pmatrix} c_1 & c_2 \\ 0 & c_1 \end{pmatrix}$	*Small*	$\begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix}$	*Yes*
$l \geq 1$	$\begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix}$		$\begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix}$	

$$L_1^{12} : [e_1, e_1] = 0, \quad [e_1, e_2] = e_1, \quad [e_2, e_1] = -e_1, \quad [e_2, e_2] = -e_1.$$
$$\alpha(e_1) = e_1, \quad \alpha(e_2) = e_2, \quad \beta(e_1) = e_1, \quad \beta(e_2) = e_1 + e_2.$$

	$\Gamma_{\alpha^r\beta^l}(L_1^{12})$	*Type of* $\Gamma_{\alpha^0\beta^0}(L_1^{12})$	$Der_{\alpha^r\beta^l}(L_1^{12})$	*CN*
$l=0$	$\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}$	*Small*	$\begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix}$	*Yes*
$l \geq 1$	$\begin{pmatrix} c_1 & lc_1 \\ 0 & c_1 \end{pmatrix}$		$\begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix}$	

$$L_1^{13} : [e_1, e_1] = 0, \quad [e_1, e_2] = e_1, \quad [e_2, e_1] = z_1 e_1, \quad [e_2, e_2] = t_1 e_1,$$
$$\alpha(e_1) = 0, \quad \alpha(e_2) = e_1, \quad \beta(e_1) = 0, \quad \beta(e_2) = z e_1.$$

		$\Gamma_{\alpha^r\beta^l}(L_1^{13})$	*Type of* $\Gamma_{\alpha^0\beta^0}(L_1^{13})$	$Der_{\alpha^r\beta^l}(L_1^{13})$	*CN*
$r=l=0$	$z_1=-1$	$\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}$	*Small*	$\begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix}$	*Yes*
$r=l=0$	$z_1=0$	$\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}$	*Small*	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	*Yes*
$r=l=0$	$z_1\neq-1$	$\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}$	*Small*	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	*Yes*
$(r,l)\in\{(0,1),(1,0)\}$		$\begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix}$		$\begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix}$	
$r>1,\ l>1$		$\begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix}$		$\begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix}$	

$$
\begin{aligned}
L_2^{13}: [e_1,e_1]&=0, & [e_1,e_2]&=0, & [e_2,e_1]&=e_1, & [e_2,e_2]&=t_1e_1,\\
\alpha(e_1)&=0, & \alpha(e_2)&=e_1, & \beta(e_1)&=0, & \beta(e_2)&=ze_1.
\end{aligned}
$$

	$\Gamma_{\alpha^r\beta^l}(L_2^{13})$	*Type of* $\Gamma_{\alpha^0\beta^0}(L_2^{13})$	$Der_{\alpha^r\beta^l}(L_2^{13})$	*CN*
$r=l=0$	$\begin{pmatrix} c_1 & c_2 \\ 0 & c_1 \end{pmatrix}$	*Not small*	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	*Yes*
$(r,l)\in\{(0,1),(1,0)\}$	$\begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix}$		$\begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix}$	
$r>1,\ l>1$	$\begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix}$		$\begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix}$	

$$
\begin{aligned}
L_3^{13}: [e_1,e_1]&=0, & [e_1,e_2]&=0, & [e_2,e_1]&=0, & [e_2,e_2]&=e_1,\\
\alpha(e_1)&=0, & \alpha(e_2)&=e_1, & \beta(e_1)&=0, & \beta(e_2)&=ze_1.
\end{aligned}
$$

	$\Gamma_{\alpha^r\beta^l}(L_3^{13})$	*Type of* $\Gamma_{\alpha^0\beta^0}(L_3^{13})$	$Der_{\alpha^r\beta^l}(L_3^{13})$	*CN*
$r=l=0$	$\begin{pmatrix} c_1 & c_2 \\ 0 & c_1 \end{pmatrix}$	*Small*	$\begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix}$	*Yes*
$(r,l)\neq(0,0)$	$\begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix}$		$\begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix}$	

$$
\begin{aligned}
L_1^{14}: [e_1,e_1]&=0, & [e_1,e_2]&=0, & [e_2,e_1]&=0, & [e_2,e_2]&=e_1,\\
\alpha(e_1)&=e_1, & \alpha(e_2)&=e_1+e_2, & \beta(e_1)&=0, & \beta(e_2)&=ze_1.
\end{aligned}
$$

	$\Gamma_{\alpha^r\beta^l}(L_1^{14})$	*Type of* $\Gamma_{\alpha^0\beta^0}(L_1^{14})$	$Der_{\alpha^r\beta^l}(L_1^{14})$	*CN*
$r = l = 0$	$\begin{pmatrix} c_1 & c_2 \\ 0 & c_1 \end{pmatrix}$	*Not small*	$\begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix}$	*Yes*
$l \geq 1$	$\begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix}$		$\begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix}$	

$$L_1^{15} : [e_1, e_1] = 0, \quad [e_1, e_2] = e_1, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = t_1 e_1,$$
$$\alpha(e_1) = 0, \quad \alpha(e_2) = e_1, \quad \beta(e_1) = e_1, \quad \beta(e_2) = e_2.$$

	$\Gamma_{\alpha^r\beta^l}(L_1^{15})$	*Type of* $\Gamma_{\alpha^0\beta^0}(L_1^{15})$	$Der_{\alpha^r\beta^l}(L_1^{15})$	*CN*
$r = 0,\ l \geq 0$	$\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}$	*Small*	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	*Yes*
$r \geq 1$	$\begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix}$		$\begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix}$	

$$L_1^{16} : [e_1, e_1] = 0, \quad [e_1, e_2] = 0, \quad [e_2, e_1] = 0, \quad [e_2, e_2] = e_1,$$
$$\alpha(e_1) = 0, \quad \alpha(e_2) = e_1, \quad \beta(e_1) = e_1, \quad \beta(e_2) = ze_1 + e_2.$$

	$\Gamma_{\alpha^r\beta^l}(L_1^{16})$	*Type of* $\Gamma_{\alpha^0\beta^0}(L_1^{16})$	$Der_{\alpha^r\beta^l}(L_1^{16})$	*CN*
$r = 0,\ l \geq 0$	$\begin{pmatrix} c_1 & c_2 \\ 0 & c_1 \end{pmatrix}$	*Small*	$\begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix}$	*Yes*
$r \geq 1$	$\begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix}$		$\begin{pmatrix} 0 & d_2 \\ 0 & 0 \end{pmatrix}$	

$$L_1^{17} : [e_1, e_1] = 0, \quad [e_1, e_2] = e_1, \quad [e_2, e_1] = -e_1, \quad [e_2, e_2] = (1 - z)e_1,$$
$$\alpha(e_1) = e_1, \quad \alpha(e_2) = e_1 + e_2, \quad \beta(e_1) = e_1, \quad \beta(e_2) = ze_1 + e_2.$$

$\Gamma_{\alpha^r\beta^l}(L_1^{17})$	*Type of* $\Gamma_{\alpha^0\beta^0}(L_1^{17})$	$Der_{\alpha^r\beta^l}(L_1^{17})$	*CN*
$\begin{pmatrix} c_1 & (lz + r)c_1 \\ 0 & c_1 \end{pmatrix}$	*Small*	$\begin{pmatrix} 0 & c_2 \\ 0 & 0 \end{pmatrix}$	*Yes*

Corollary 28.2 *The following statements hold.*

i) *The dimensions of the centroids of* 2*-dimensional BiHom-Lie Algebras vary between one and two.*
ii) *Every* 2*-dimensional multiplicative BiHom-Lie algebra have a small centroid if and only if it isomorphic to one of the following BiHom-Lie algebras*:

$$L_1^1(z_1)(z_1 \neq 0),\ L_2^1,\ L_4^1(z_1)(z_1 \neq 0),\ L_5^1,$$
$$L_1^8,\ L_1^{10},\ L_3^{11},\ L_1^{12},\ L_1^{13},\ L_1^{15},\ L_1^{16},\ L_1^{17}.$$

iii) *The dimensions of the derivations of* 2*-dimensional BiHom-Lie algebras vary between zero and two.*

iv) *Every* 2*-dimensional multiplicative BiHom-Lie algebra is characteristically nilpotent if and only if it is not isomorphic to* L_1^{10}.

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Chapter 29
HNN-Extension of Involutive Multiplicative Hom-Lie Algebras

Sergei Silvestrov and Chia Zargeh

Abstract The construction of HNN-extensions of involutive Hom-associative algebras and involutive Hom-Lie algebras is described. Then, as an application of HNN-extension, by using the validity of Poincaré-Birkhoff-Witt theorem for involutive Hom-Lie algebras, we provide an embedding theorem.

Keywords HNN-extension · Hom-Lie algebra · Hom-associative algebra · Involution · Poincaré-Birkhoff-Witt theorem

MSC2020 Classification 17D30 · 17B61

29.1 Introduction

One of the most important constructions in combinatorial group theory is Higman-Neumann-Neumann extension (or HNN-extension, for short), which states that if A_1 and A_2 are isomorphic subgroups of a group G, then it is possible to find a group H containing G such that A_1 and A_2 are conjugate to each other in H and G is embeddable in H (see [45]). The HNN-extension of a group has a topological interpretation described in [28, 59], which is used as a motivation for its study. Spreading classical techniques in combinatorial group theory to other algebraic structures has shown outstanding capacities for solving problems in affine algebraic geometry, the theory of Lie algebras and mathematical physics. In this regard, HNN-extension of Lie algebras was constructed by Lichtman and Shirvani [58] and Wasserman [79] through different approaches. They used HNN-extension in order to give a new proof for

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Shirshov's theorem [75], namely, a Lie algebra of finite or countable dimension can be embedded into a 2-generator Lie algebra. Moreover, the idea of HNN-extension has been recently spread to Leibniz algebras in [51] and Lie superalgebras in [50], which are respectively, non-antisymmetric and natural generalization of Lie algebras.

In this paper we intend to introduce HNN-extension for the Hom-generalization of Lie algebras. Hom-Lie algebras and more general quasi-Hom-Lie algebras were introduced first by Hartwig, Larsson and Silvestrov in [43], where the general quasi-deformations and discretizations of Lie algebras of vector fields using more general σ-derivations (twisted derivations) and a general method for construction of deformations of Witt and Virasoro type algebras based on twisted derivations have been developed, initially motivated by the q-deformed Jacobi identities observed for the q-deformed algebras in physics, q-deformed versions of homological algebra and discrete modifications of differential calculi. Hom-Lie superalgebras, Hom-Lie color algebras and more general quasi-Lie algebras and color quasi-Lie algebras where introduced first in [54, 55, 76]. Quasi-Lie algebras and color quasi-Lie algebras encompass within the same algebraic framework the quasi-deformations and discretizations of Lie algebras of vector fields by σ-derivations obeying twisted Leibniz rule, and color Lie algebras, the well-known natural generalizations of Lie algebras and Lie superalgebras. In quasi-Lie algebras, the skew-symmetry and the Jacobi identity are twisted by deforming twisting linear maps, with the Jacobi identity in quasi-Lie and quasi-Hom-Lie algebras in general containing six twisted triple bracket terms. In Hom-Lie algebras, the bilinear product satisfies the non-twisted skew-symmetry property as in Lie algebras, and the Hom-Lie algebras Jacobi identity has three terms twisted by a single linear map, reducing to the Lie algebras Jacobi identity when the twisting linear map is the identity map. Hom-Lie admissible algebras have been considered first in [62], where in particular the Hom-associative algebras have been introduced and shown to be Hom-Lie admissible, leading to Hom-Lie algebras using commutator map as new product, and thus constituting a natural generalization of associative algebras as Lie admissible algebras. Since the pioneering works [43, 53–56, 62], Hom-algebra structures expanded into a popular area with increasing number of publications in various directions. Hom-algebra structures of a given type include their classical counterparts and open broad possibilities for deformations, Hom-algebra extensions of cohomological structures and representations, formal deformations of Hom-associative algebras and Hom-Lie algebras, Hom-Lie admissible Hom-coalgebras, Hom-coalgebras, Hom-Hopf algebras, Hom-Lie algebras, Hom-Lie superalgebras, color Hom-Lie algebras, BiHom-Lie algebras, BiHom-associative algebras, BiHom-Frobenius algebras and n-ary generalizations of Hom-algebra structures have been further investigated in various aspects for example in [1–27, 35, 36, 38–41, 44, 46–49, 52, 57, 60–74, 76–78, 80–84, 86–88].

Our approach for construction of the HNN-extension of Hom-generalization of Lie algebras is based on the corresponding construction for its envelope. Therefore, we concentrate on the study of HNN-extensions for involutive Hom-Lie algebras in which their universal enveloping algebras have been explicitly obtained in [42]. It is worth noting that there exists another approach provided in [80] for obtaining the universal enveloping algebra of a Hom-Lie algebra as a suitable quotient of the free Hom-nonassociative algebra through weighted trees, but the point of difficulty in the approach in [80] is the size of the weighted trees. Involutive Hom-Lie algebras have been constructed in [85], and the classical theory of enveloping algebras of Lie algebras was extended to an explicit construction of the free involutive Hom-associative algebra on a Hom-module in order to obtain the universal enveloping algebra [42]. This construction leads to a Poincare-Birkhoff-Witt theorem for the enveloping associative algebra of an involutive Hom-Lie algebra. This approach has been extended to the enveloping algebras for color Hom-Lie algebras in [11, 12]. Extensions of Hom-Lie superalgebras and Hom-Lie color algebras have been considered in [9, 13]. Hom-associative Ore extensions have been considered in [29–34]

The paper is organized as follows. In Sect. 29.2, we recall the preliminary concepts related to involutive Hom-associative algebras and involutive Hom-Lie algebras. In Sect. 29.3, we introduce the HNN-extension for involutive Hom-associative algebras. In Sect. 29.4, we construct the HNN-extension for involutive Hom-Lie algebras and provide an embedding theorem.

29.2 Involutive Hom-Algebras

In this section we recall necessary concepts related to involutive Hom-associative and involutive Hom-Lie algebras.

Definition 29.1 Let K be a field.

(a) Hom-module is a pair (V, α_V) consisting of a K-module V and a linear operator $\alpha_V : V \to V$.
(b) Hom-associative algebra is a triple $(A, *_A, \alpha_A)$ consisting of a K-module A, a linear map $*_A : A \otimes A \to A$, called the multiplication, and a linear operator $\alpha_A : A \to A$ satisfying the Hom-associativity

$$\alpha_A(x) *_A (y *_A z) = (x *_A y) *_A \alpha_A(z),$$

for all $x, y, z \in A$.
(c) Hom-associative algebra is said to be *multiplicative* if the linear map α is multiplicative in the sense of satisfying $\alpha_A(x *_A y) = \alpha_A(x) *_A \alpha_A(y)$ for all $x, y \in A$.
(d) Hom-associative algebra $(A, *_A, \alpha_A)$ (resp. Hom-module (V, α_V)) is said to be *involutive* if $\alpha_A^2 = id$ (resp. $\alpha_V^2 = id$).
(e) Let (V, α_V) and (W, α_W) be Hom-modules. A K-linear map $f : V \to W$ is called a morphism of Hom-modules if $f(\alpha_V(x)) = \alpha_W(f(x))$ for all $x \in V$.

(f) Let $(A, *_A, \alpha_A)$ and $(B, *_B, \alpha_B)$ be two Hom-associative algebras. A K-linear map $f : A \to B$ is a morphism of Hom-associative algebras if

$$f(x *_A y) = f(x) *_B f(y), \text{ and } f(\alpha_A(x)) = \alpha_B(f(x)),$$

for all $x, y \in A$.

(g) Let $(A, *_A, \alpha_A)$ be a Hom-associative algebra. A submodule $B \subseteq A$ is called a Hom-associative subalgebra of A if B is closed under the multiplication $*_A$ and $\alpha_A(B) \subseteq B$.

(h) Let $(A, *_A, \alpha_A)$ be a Hom-associative algebra. A submodule $I \subseteq A$ is called a Hom-ideal of A if $x *_A y \in I$, $y *_A x \in I$ for all $x \in I$, $y \in A$, and $\alpha_A(I) \subseteq I$.

Definition 29.2 For any non-negative integer k, a linear map $D : A \to A$ is called an α_A^k-derivation of involutive Hom-associative algebra $(A, *_A, \alpha_A)$, if

$$D \circ \alpha_A^k = \alpha_A^k \circ D,$$
$$D \circ (x *_A y) = D(x) *_A \alpha_A^k(y) + \alpha_A^k(x) *_A D(y).$$

Definition 29.3 Let (V, α_V) be an involutive Hom-module. A free involutive Hom-associative algebra on V is an involutive Hom-associative algebra $(F_{IHA}(V), *_F, \alpha_F)$ together with a morphism of Hom-modules $j_V : (V, \alpha_V) \to (F_{IHA}(V), \alpha_F)$ such that, for any involutive Hom-associative algebra $(A, *_A, \alpha_A)$ together with a morphism of Hom-modules $f : (V, \alpha_V) \to (A, \alpha_A)$, there is a unique morphism of Hom-associative algebras

$$f : (F_{IHA}(V), *_F, \alpha_F) \to (A, *_A, \alpha_A)$$

such that $f = f \circ j_V$.

Definition 29.4 A Hom-Lie algebra is a triple $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \beta)$ consisting of a vector space $\mathfrak{g}$, a skew-symmetric bilinear map (bracket) $[\cdot, \cdot]_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ and a linear map $\beta : \mathfrak{g} \to \mathfrak{g}$ satisfying the following Hom-Jacobi identity:

$$[\beta(u), [v, w]_{\mathfrak{g}}]_{\mathfrak{g}} + [\beta(v), [w, u]_{\mathfrak{g}}]_{\mathfrak{g}} + [\beta(w), [u, v]_{\mathfrak{g}}]_{\mathfrak{g}} = 0. \tag{29.1}$$

Hom-Lie algebra is called a *multiplicative* Hom-Lie algebra if β satisfies

$$\beta([u, v]_{\mathfrak{g}}) = [\beta(u), \beta(v)]_{\mathfrak{g}}. \tag{29.2}$$

A Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \beta)$ is called *involutive* if $\beta^2 = id_{\mathfrak{g}}$. Note that the classical Lie algebra can be recovered when $\beta = id_{\mathfrak{g}}$, with the identity (29.1) becoming the Jacobi identity for Lie algebras.

Definition 29.5 A morphism of Hom-Lie algebras

$$f : (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \beta_{\mathfrak{g}}) \to (\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}}, \beta_{\mathfrak{h}})$$

is a k-linear map $f : \mathfrak{g} \to \mathfrak{h}$ such that

$$f([x, y]_{\mathfrak{g}}) = [f(x), f(y)]_{\mathfrak{h}} \text{ and } f(\beta_{\mathfrak{g}}(x)) = \beta_{\mathfrak{h}}(f(x)) \text{ for all } x \in \mathfrak{g}.$$

Hom-associative algebras were introduced in [62], and shown to be Hom-Lie admissible, i.e. any Hom-associative algebra $(A, *_A, \alpha_A)$ yields a Hom-Lie algebra $(A, [\cdot, \cdot]_A, \beta_A)$ with $\beta_A = \alpha_A$ and $[x, y]_A = x *_A y - y *_A x$ for $x, y \in A$.

For simplicity, we will restrict our considerations to multiplicative Hom-Lie algebras and multiplicative Hom-associative algebras, meaning that the twisting map is not only linear, but also an endomorphism of the Hom-Lie algebra or Hom-associative algebra respectively. An interesting important problem is to understand completely the role of the multiplicatives restriction and extend the results and constructions from multiplicative to general, not necessarily multiplicative, Hom-Lie algebras and Hom-associative algebras.

Definition 29.6 ([42]) Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \beta)$ be a Hom-Lie algebra. A universal enveloping Hom-associative algebra of $\mathfrak{g}$ is a Hom-associative algebra $\mathfrak{U}_{\mathfrak{g}} = (\mathfrak{U}_{\mathfrak{g}}, *_{\mathfrak{g}}, \alpha_{\mathfrak{U}})$, together with a morphism $\phi_{\mathfrak{g}} : (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \beta) \to (\mathfrak{U}_{\mathfrak{g}}, [\cdot, \cdot]_{\mathfrak{U}_{\mathfrak{g}}}, \beta_{\mathfrak{U}_{\mathfrak{g}}})$ of Hom-Lie algebras, that satisfies the universal property.

The following lemma describes the universal property in the involutive case.

Lemma 29.1 ([42]) *Let* $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \beta_{\mathfrak{g}})$ *be an involutive multiplicative Hom-Lie algebra.*

(a) *Let* $(A, *_A, \alpha_A)$ *be a multiplicative Hom-associative algebra,*

$$f : (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \beta_{\mathfrak{g}}) \to (A, [\cdot, \cdot]_A, \beta_A)$$

be a morphism of Hom-Lie algebras, and B *be the multiplicative Hom-associative subalgbera of* A *generated by* $f(\mathfrak{g})$. *Then* B *is involutive.*

(b) *The universal enveloping multiplicative Hom-associative algebra* $(\mathfrak{U}_{\mathfrak{g}}, \phi_{\mathfrak{g}})$ *of* $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \beta_{\mathfrak{g}})$ *is involutive.*

(c) *In order to verify the universal property of* $(\mathfrak{U}_{\mathfrak{g}}, \phi_{\mathfrak{g}})$*, we only need to consider involutive multiplicative Hom-associative algebras* $A := (A, *_A, \alpha_A)$.

Definition 29.7 A linear subspace $\mathfrak{s} \subseteq \mathfrak{g}$ is called a Hom-Lie subalgebra of a Hom-Lie algebras $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \beta)$ if $\beta(\mathfrak{s}) \subseteq \mathfrak{s}$ and $\mathfrak{s}$ is closed under the bracket operation $[\cdot, \cdot]_{\mathfrak{g}}$:

$$\forall s_1, s_2 \in \mathfrak{s} : \quad [s_1, s_2]_{\mathfrak{g}} \in \mathfrak{s}.$$

Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \beta)$ be a multiplicative Hom-Lie algebra. For any nonnegative integer k,denote by β^k the k-times composition of β, i.e.

$$\beta^k = \beta \ldots \beta \ (k\text{-times}).$$

In particular, $\beta^0 = Id$ and $\beta^1 = \beta$.

Definition 29.8 For any nonnegative integer k, a linear map $d : \mathfrak{g} \to \mathfrak{g}$ is called a β^k-derivation of the involutive Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \beta)$, if

$$[d, \beta] = 0, \text{ that is, } d \circ \beta^k = \beta^k \circ d, \tag{29.3}$$

$$\forall u, v \in \mathfrak{g}: \quad d[u, v]_{\mathfrak{g}} = [d(u), \beta^k(v)]_{\mathfrak{g}} + [\beta^k(u), d(v)]_{\mathfrak{g}}. \tag{29.4}$$

Example 29.1 Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$ be an involutive multiplicative Hom-Lie algebra. For $x \in \mathfrak{g}$, let consider $\alpha(x) = x$, then $ad_x : \mathfrak{g} \to \mathfrak{g}$ defined by $ad_x(y) = [x, y]_{\mathfrak{g}}$ for all $y \in \mathfrak{g}$ is an α-derivation of $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$.

29.3 HNN-Extension of Involutive Hom-Associative Algebras

Let $(A, *_A, \alpha_A)$ be an involutive Hom-associative algebra over ring of integers. Let $(B_i, *_A, \alpha_{A|_{B_i}})$ $(i \in I)$ be a family of Hom-associative subalgebras of A as defined in Definition 29.1 (g), with injective morphisms $\theta_i : B_i \to A$, and for each $i \in I$, a θ_i-derivation $\delta_i : B_i \to A$ such that α_A commutes with θ_i and δ_i. The associated HNN-extension is presented as

$$H = \langle A, B_i, t_i, \delta_i, \theta_i : i \in I \rangle,$$

which is an involutive Hom-associative algebra $H := (A \cup \{t_i\}, *_H, \alpha_H)$ in such a way that $x *_H y = \alpha_H(x *_A y)$, where $\alpha_H(t_i) = t_i$ and $\alpha_H(a) = \alpha_A(a)$ along with a homomorphism $\phi : (A, *_A, \alpha_A) \to (H, *_H, \alpha_H)$ with the following conditions:

1. $t_i *_H (\phi(b)) - \phi(\theta_i(b)) *_H t_i = \phi(\delta_i(b))$ for all $b \in B_i$ and all $i \in I$.
2. Given any involutive Hom-associative algebra $(S, *_S, \alpha_S)$ with elements $\sigma_i \in S$ satisfying $\alpha_S(\sigma_i) = \sigma_i$, a morphism $f : (A, \alpha_A) \to (S, \alpha_S)$ such that $\sigma_i *_S \alpha_S(f(b)) - \alpha_S(f(b)) *_S \sigma_i = f(\delta_i(b))$ for all $b \in B_i$ and $i \in I$, there exists a unique morphism $\theta : (H, *_H, \alpha_H) \to (S, *_A, \alpha_A)$ such that $\theta(t_i) = \sigma_i$ and $\theta(\phi(a)) = f(a)$ for all $a \in A$.

Assume a single letter t in the condition (i) of construction of HNN-extension of involutive multiplicatve Hom-associative algebra. Since δ is an α_A-derivation,

$$\begin{aligned}
\delta(\alpha_A(b)) &= t *_H \alpha_A(b) - \alpha_A(b) *_H t \\
&= \alpha_H(t *_A \alpha_A(b)) - \alpha_H(\alpha_A(b) *_A t) \quad \text{(by definition of } *_H \text{)} \\
&= \alpha_H(t) *_A \alpha_A^2(b) - \alpha_A^2(b) *_A \alpha_H(t) \quad \text{(by Def. 1 (c), (d))} \\
&= t *_A b - b *_A t = \alpha_A(\delta(b)),
\end{aligned}$$

which implies that in the construction of HNN-extension for the case of involutive Hom-associative algebras, it is essential to consider the multiplicative property. It is worth pointing out that the second property of α-derivations in Definition 29.2 is straightforward by Hom-associativity.

A left Hom-B_i-module A/B_i is a Hom-module $(A/B_i, \alpha_{A/B_i})$ that comes equipped with a left B_i-action, $B_i \otimes A/B_i \to A/B_i$, with $b *_{A/B_i} (a + B_i) = (b *_A a) + B_i$ and $\alpha_{A/B_i} : A/B_i \to A/B_i$ with $\alpha_{A/B_i}(a + B_i) = \alpha_A(a) + B_i$, for all $b \in B_i$. Let X_i be a free basis of free left Hom-B_i-module A/B_i. We define a normal sequence as

$$(t_{i_1} *_A \alpha_A(x_1)) *_A (t_{i_2} *_A \alpha_A(x_2)) *_A \cdots *_A (t_{i_r} *_A \alpha_A(x_r)),$$

with $i_j \in I$ and $x_\alpha \in X_{i_j}$ for $1 \le \alpha \le r$. The set of all normal sequences is denoted by V.

Theorem 29.1 concerns the embeddability of involutive Hom-associative algebra into its HNN-extension. We follow the Lichtman and Shirvani's approach [58] in order to prove that.

Theorem 29.1 *Let $(A, *_A, \alpha_A)$ be an involutive Hom-associative algebra over ring of integers, B_i a family of Hom-associative subalgberas, with injective homomorphisms $\theta_i : B_i \to A$, a θ_i-derivations $\delta_i : B_i \to A$. Assume that A/B_i is a free left Hom-B_i-module for all i, and let (H, ϕ) be the corresponding HNN-extension as above. Then the map ϕ is an embedding of A into H.*

Proof Let us consider the free left Hom-A-module on the set of normal sequences, V, and denote it by

$$Q = (\oplus_{u\in V} Au, \alpha_Q), \quad \alpha_Q(u_1, \dots, u_r) = (\alpha_H(u_1), \dots, \alpha_H(u_r)).$$

Consider the morphism of (A, α_A) into S = $(\mathrm{End}_{\mathbb{Z}}(Q), \alpha_S)$ mapping $a \in A$ to left multiplication by a on every factor denoted by $a \mapsto \bar{a}$ and $\alpha_S = \alpha_A$. In the sequel, we need to define suitable $\sigma_i \in S$ for all $i \in I$. If $q \in Q$ is written as

$$\begin{aligned} q = \sum_{u\in V}\sum_{x\in X_i} (b_{x,u} *_{A/B} x) *_A u &= \sum_{u\in V}\sum_{x\in X_i} (b_{x,u} *_A \alpha_A(x)) *_A u \\ &= \sum_{u\in V}\sum_{x\in X_i} b_{x,u} *_A (\alpha_A(x) *_A u) \end{aligned}$$

for $b_{x,u} \in B_i$, define

$$\sigma_i(q) = \sum_{u\in V}\sum_{x\in X_i} (\theta_i(b_{x,u}) *_A ((t_i *_A \alpha_A(x)) *_A u) + \delta_i(b_{x,u}) *_A (\alpha_A(x) *_A u)).$$

We have $\sum_{x\in X_i} (\delta_i(b_{x,u}) *_A \alpha_A(x)) \in A$ and every $((t_i *_A \alpha_A(x)) *_A u) \in V$. For any element $b \in B_i$ $(i \in I)$, we recall that the left multiplication by b is denoted by $\bar{b}$, so we have

$$\sigma_i(\bar{b}(q)) = \sigma_i(\sum_{u\in V}\sum_{x\in X_i}((b *_B b_{x,u}) *_A (\alpha_A(x) *_A u)))$$
$$= \sum_{u,x}(\theta_i(b *_B b_{x,u}) *_A ((t_i *_A \alpha_A(x)) *_A u))$$
$$+ \sum_{u,x}(\delta_i(b *_B b_{x,u}) *_A (\alpha_A(x) *_A u)),$$

and

$$\overline{\theta_i(b)}(\sigma_i(q)) = \sum_i(\theta_i(b)) *_A (\sum_{u\in V}\sum_{x\in X_i}(\theta_i(b_{x,u}) *_A ((t_i *_A \alpha_A(x)) *_A u)))$$
$$+ \sum_i(\theta_i(b)) *_A (\sum_{u\in V}\sum_{x\in X_i}(\delta_i(b_{x,u}) *_A (\alpha_A(x) *_A u))).$$

Hence,

$$\sigma_i(\bar{b}(q)) - \overline{\theta_i(b)}(\sigma_i(q)) = \sum_{u,x}((\delta_i(b) *_A b_{x,u}) *_A (\alpha_A(x) *_A u)) = \overline{\delta_i(b)}(q).$$

Therefore, the property (2) implies that there exists $\theta : (H, *_H, \alpha_H) \to (S, *_S, \alpha_S)$ such that $\theta(t_i) = \sigma_i$ and $\theta(\phi(a)) = \bar{a}$ for all $a \in A$.

29.4 HNN-Extension of Involutive Hom-Lie Algebras

Let $(A, *_A, \alpha_A)$ be an arbitrary Hom-associative algebra, and let $(A, [\cdot,\cdot]_A, \beta_A)$ be the Hom-Lie algebra defined by

$$[x, y]_A = x *_A y - y *_A x,$$

and $\beta_A = \alpha_A$, for $x, y \in A$. If $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \beta_{\mathfrak{g}})$ is an involutive Hom-Lie algebra, then $(\mathfrak{U}_{\mathfrak{g}}, \phi_g)$ is called a universal enveloping Hom-associative algebra of $\mathfrak{g}$, if

$$\phi_{\mathfrak{g}} : (\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \beta_{\mathfrak{g}}) \to (\mathfrak{U}_{\mathfrak{g}}, [\cdot,\cdot]_{\mathfrak{U}_{\mathfrak{g}}}, \beta_{\mathfrak{U}_{\mathfrak{g}}})$$

is a homomorphism of Hom-Lie algebras,

$$\phi_{\mathfrak{g}}([x, y]_{\mathfrak{g}}) = [\phi_{\mathfrak{g}}(x), \phi_{\mathfrak{g}}(y)]_{\mathfrak{U}_{\mathfrak{g}}}, \qquad \phi_{\mathfrak{g}}(\beta_{\mathfrak{g}}(x)) = \beta_{\mathfrak{U}_{\mathfrak{g}}}(\phi_{\mathfrak{g}}(x)),$$

satisfying the following universal property: for any involutive Hom-associative algebra $A = (A, *_A, \alpha_A)$ and any Hom-Lie algebra morphism

$$\varepsilon : (\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \beta_{\mathfrak{g}}) \to (A, [\cdot,\cdot]_A, \beta_A),$$

there exists a unique morphism of Hom-associative algebras $\eta : \mathfrak{U}_{\mathfrak{g}} \to A$ such that $\eta\phi_{\mathfrak{g}} = \varepsilon$. For any involutive Hom-Lie algebra there exists a universal enveloping Hom-associative algebra, which is involutive and Poincare-Birkhoff-Witt theorem is valid for it. This shows that the map $\phi_{\mathfrak{g}}$ is injective, and we can say that every $\beta_{\mathfrak{g}}$-derivation of involutive Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \beta_{\mathfrak{g}})$ extends to $\beta_{\mathfrak{U}_{\mathfrak{g}}}$-derivation of $\mathfrak{U}_{\mathfrak{g}}$.

Definition 29.9 Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \beta_{\mathfrak{g}})$ be an involutive Hom-Lie algebra and $\mathfrak{s}$ be a subalgebra. Assume that $d : \mathfrak{s} \to \mathfrak{g}$ is a $\beta_{\mathfrak{g}}$-derivation. The associated HNN-extension is given by the following presentation

$$\mathfrak{h} := \langle \mathfrak{g}, t : d(s) = [t, s]_{\mathfrak{h}}, s \in \mathfrak{s} \rangle,$$

which is an involutive Hom-Lie algebra $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}}, \beta_{\mathfrak{h}})$ with $\beta_{\mathfrak{h}}(t) = t$, $\beta_{\mathfrak{h}}(g) = \beta_{\mathfrak{g}}(g)$ for $g \in \mathfrak{g}$. This means that the presentation of $\mathfrak{g}$ is augmented by adding a new generating symbol t, and for each $s \in \mathfrak{s}$, the relation $[t, s]_{\mathfrak{h}} = d(s)$ is added. We note that $[g_1, g_2]_{\mathfrak{h}} = [g_1, g_2]_{\mathfrak{g}}$, for all $g_1, g_2 \in \mathfrak{g}$.

Let assume that in the Definition 29.9, $\mathfrak{s} = \mathfrak{g}$, therefore, d is a $\beta_{\mathfrak{g}}$-derivation of $\mathfrak{g}$ and $\mathfrak{h}$ is then the semi-direct product of $\mathfrak{g}$ with a one-dimensional involutive Hom-Lie algebra which acts on $\mathfrak{g}$ via d. In order to make this special case more clear, we recall the concepts of Hom-action and semidirect product of Hom-Lie algebras in the sequel in accordance with [37].

Definition 29.10 Let $(\mathfrak{l}, \alpha_{\mathfrak{l}})$ and $(\mathfrak{m}, \alpha_{\mathfrak{m}})$ be Hom-Lie algebras. A Hom-action from $(\mathfrak{l}, \alpha_{\mathfrak{l}})$ on $(\mathfrak{m}, \alpha_{\mathfrak{m}})$ is expressed by a bilinear map

$$\sigma : \mathfrak{l} \otimes \mathfrak{m} \to \mathfrak{m}, \quad \sigma(x \otimes m) = x \cdot m$$

such that

(a) $[x, y] \cdot \alpha_{\mathfrak{m}}(m) = \alpha_{\mathfrak{l}}(x) \cdot (y \cdot m) - \alpha_{\mathfrak{l}}(y) \cdot (x \cdot m)$,
(b) $\alpha_{\mathfrak{l}}(x) \cdot [m, m'] = [x \cdot m, \alpha_{\mathfrak{m}}(m')] + [\alpha_{\mathfrak{m}}(m), x \cdot m']$,
(c) $\alpha_{\mathfrak{m}}(x \cdot m) = \alpha_{\mathfrak{l}}(x) \cdot \alpha_{\mathfrak{m}}(m)$,

for all $x, y \in \mathfrak{l}$ and $m, m' \in \mathfrak{m}$.

Definition 29.11 ([37]) Let $(\mathfrak{l}, \alpha_{\mathfrak{l}})$ and $(\mathfrak{m}, \alpha_{\mathfrak{m}})$ be Hom-Lie algebras with an action from $(\mathfrak{l}, \alpha_{\mathfrak{l}})$ on $(\mathfrak{m}, \alpha_{\mathfrak{m}})$. The semidirect product $(\mathfrak{m} \rtimes \mathfrak{l}, \tilde{\alpha})$ is the Hom-Lie algebra with underlying K-vector space $\mathfrak{m} \oplus \mathfrak{l}$, with bracket

$$[(m_1, x_1), (m_2, x_2)] = ([m_1, m_2] + x_1 \cdot m_2 - x_2 \cdot m_1, [x_1, x_2])$$

and endomorphism

$$\tilde{\alpha} : \mathfrak{m} \oplus \mathfrak{l} \to \mathfrak{m} \oplus \mathfrak{l}, \qquad \tilde{\alpha}(m, x) = (\alpha_{\mathfrak{m}}(m), \alpha_{\mathfrak{l}}(x))$$

for all $x, x_1, x_2 \in \mathfrak{l}$ and $m, m_1, m_2 \in \mathfrak{m}$.

If in the Definition 29.9 of HNN-extension of involutive Hom-Lie algebras, $\beta_{\mathfrak{g}}$-derivation map is defined on the whole involutive Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \beta_{\mathfrak{g}})$, then a semidirect product of one-dimensional involutive Hom-Lie algebra with $\mathfrak{g}$ with respect to $\beta_{\mathfrak{g}}$-derivation map will be obtained.

Theorem 29.2 *Any involutive Hom-Lie algebra embeds into its HNN-extension.*

Proof Let $(\mathfrak{U}_{\mathfrak{g}}, \phi_g)$ and $(\mathfrak{U}_{\mathfrak{s}}, \phi_s)$ be the universal enveloping Hom-associative algebras corresponding to, respectively, the involutive Hom-Lie algebra $\mathfrak{g}$ and its subalgebra $\mathfrak{s}$, which are involutive with respect to Lemma 29.1. Let $\mathfrak{h} = \langle \mathfrak{g}, t : d(s) = [t, s]_{\mathfrak{h}}, s \in \mathfrak{s} \rangle$ be the HNN-extension of involutive Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \beta_{\mathfrak{g}})$ as above. By extending d to a $\beta_{\mathfrak{U}_{\mathfrak{g}}}$-derivation of $\mathfrak{U}_{\mathfrak{g}}$ defined on $\mathfrak{U}_{\mathfrak{s}}$ we form the HNN-extension of involutive Hom-associative algebra $\mathfrak{U}_{\mathfrak{g}}$ which is denoted by $M = \langle \mathfrak{U}_{\mathfrak{g}}, \mathfrak{U}_{\mathfrak{s}}, t, \delta \rangle$. Let $(R, *_R, \alpha_R)$ be an arbitrary involutive Hom-associative algebra with a homomorphism of Hom-Lie algebras $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, \beta_{\mathfrak{h}}) \to (R, [\cdot,\cdot]_R, \beta_R)$. The restriction to $\mathfrak{g}$ extends to a homomorphism $\mathfrak{U}_{\mathfrak{g}} \to R$, which extends to a homomorphism $M \to R$, so we have $\mathfrak{U}_{\mathfrak{h}} \simeq M$. As $\mathfrak{U}_{\mathfrak{g}}/\mathfrak{U}_{\mathfrak{s}}$ is a free left Hom-$\mathfrak{U}_{\mathfrak{s}}$-module, Theorem 29.1 implies that $\mathfrak{U}_{\mathfrak{g}}$ is embedded into M, and so $\mathfrak{g}$ embeds into its HNN-extension.

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Chapter 30
Two-Sided Noncommutative Gröbner Basis on Quiver Algebras

Daniel K. Waweru and Damian M. Maingi

Abstract For a quiver Q, we define a path algebra KQ as a span of all the paths of positive length. We study left-sided (respective right-sided) ideals and their Gröbner bases. We introduce the two-sided ideals, a two-sided division algorithm for elements of KQ and study the two-sided Gröbner bases. We show that with the defined two-sided division algorithm and two-sided Buchberger's algorithm, we can find a finite or an infinite Gröbner basis for a two-sided ideal $I \subseteq KQ$ given a fixed admissible ordering.

Keywords Quiver algebras · Two-sided ideals · Two-sided division algorithm · Two-sided noncommutative Gröbner basis

MSC2020 Classification 16D25 · 16Z05 · 16Z10

30.1 Introduction

In 1986, Teo Mora published a paper [15] giving an algorithm for constructing a noncommutative Gröbner Basis. This work built upon the work of George Bergman in particular his diamond lemma for ring theory [4]. Mora's algorithm and the theory behind it, in many ways, give a noncommutative version of the Gröbner Basis theory as seen in the commutative case which states: given an initial set F generating an ideal I in a polynomial ring A, Gröbner basis theory uses F to find a basis G for I with the property that for any $f \in A$, division of f by G has a unique remainder.

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How we obtain that Gröbner Basis remains the same as in the commutative case where we add nonzero S-polynomials to an initial basis. The difference comes in the definition of an S-polynomial. The purpose of S-polynomial $S(f, g)$ for each pair of nonzero polynomials $f, g \in A$ is to ensure that any polynomial $h \in A$ reducible by both f and g has a unique remainder when divided by a set of polynomials containing both f and g. In the commutative case, there is only one way to divide h by f and g giving the reduction $(h - x_1 f)$ or $(h - x_2 g)$ respectively, where x_1 and x_2 are terms. Thus, there is only one S-polynomial for each pair of polynomials. However, in a noncommutative polynomial ring, a polynomial may divide another in many different ways. We do not have a fixed number of S-polynomials for each pair of polynomials in A. The number of S-polynomials depend on the number of overlaps between the leading monomials of f and g. One can therefore strengthen the division algorithm axiomatically to eliminate much ambiguity.

Section 30.2 will serve as an overview of Gröbner basis theory in a general polynomial ring theory, in preparation for Sects. 30.3 and 30.4 which introduce a path algebra and considers a one-sided noncommutative Gröbner basis in a path algebra respectively. Having established the foundations, Section 30.5 will present the concept of a two-sided noncommutative Gröbner basis for ideals in a path algebra, which is the main objective of the present work.

30.2 Noncommutative Gröbner Basis in Polynomial Ring

We start off by mentioning some of the rudiments to the concept of a noncommutative Gröbner basis over a noncommutative polynomial ring. Most of these deliberations may be found in [1, 7, 8, 15]. In this section $A = K[x_1, \ldots, x_n]$ is a noncommutative polynomial ring. Monomials of A are generated by alphabetical words of A over K. We denote the set of all monomials of A by M. We thus make the following important definitions.

Definition 30.1 A relation $\prec$ is said to be a noncommutative monomial ordering on set M if it satisfies

(i) $\prec$ is a total order on M.
(ii) $x \succeq 1, \forall x \in M$.
(iii) $x \succ y \Rightarrow wxz \succ wyz, \quad \forall x, y, w, z \in M$.

Definition 30.2 For $w, x, y, z \in M$, we say a monomial ordering $\prec$ is admissible when

(i) $x \prec y \Rightarrow xz \prec yz$.
(ii) $x \prec y \Rightarrow wx \prec wy$.
(iii) $x = yz \Rightarrow x \succ y$ and $x \succ z$.

Definition 30.3 Let $x, y \in M$. An $x - y$ overlap occurs when one can find factors $x = x_1 z$, $y = z y_1$ where $x \neq x_1$ and/ or $y \neq y_1$.

Different factorization in M gives different overlaps.

Definition 30.4 Every element $f \in A$ has a unique form $f = \sum_{i=1}^{n} a_i w_i, \quad a_i \in K, \quad w_i \in M$. We will denote all such monomials w_i appearing in f by $Mon(f)$. Furthermore, w is called the leading monomial of $f \in A$, denoted by $w = LM(f)$ if w occurs in f and $w \succeq m$ for all monomials $m \in Mon(f)$. The coefficient of $LM(f)$ in f is called the leading coefficient and is denoted by $LC(f)$. The leading term of f is denoted as $LT(f) = LC(f)LM(f)$. If $J \subset A$, then we define $LT(J) = \{LT(g) : g \in J\}$.

By convention, a polynomial will be written in descending order, with respect to a given monomial ordering, so that the leading term of the polynomial, (with associated leading coefficient and leading monomial), always comes first.

For nonzero polynomials $f, g \in A$, we say that f divides g if the leading term of f divides some term h in g, where $h = x_l LM(f) x_r$ and x_l and x_r are monomials. For noncommutative cases, the division algorithm is adapted to calculate s-polynomial as shown in Definition 30.6. Division removes an appropriate multiple of f from g in order to cancel off $LT(f)$ with the term involving h in g. We perform division as follows, $g - \frac{\lambda}{LC(f)} x_l f x_r = r$, where $\lambda \in K$ is to be chosen from $\{LC(x_l), LC(x_r)\}$.

Definition 30.5 For a set $F = \{f_1, \dots, f_s\}$, any $f \in A$ can be written in a form $f = u_{l1} f_1 v_{r1} + \cdots + u_{ls} f_s v_{rs} + r$, called the standard representation of f with respect to the set F, where $u_{li}, v_{ri}, r, f_i \in A$ and either $r = 0$ or r is a linear combination with coefficients in K of monomials which are not divisible by $LT(f_i)$ for all i. r is the remainder of f after dividing by F.

We denote by $r = Red_F(f)$ and call it a reduction of f with respect to the set F. Moreover, if $u_{li} f_i v_{ri} \neq 0$ then $LM(f) \succeq LM(u_{li} f_i v_{ri})$ for all i.

Definition 30.6 Let $f, g \in A$ and the leading monomials of f and g overlap such that $x_1 LM(f) y_1 = x_2 LM(g) y_2$, where $x_1, x_2, y_1, y_2 \in M$ are chosen so that at least one of x_1 and x_2 and at least one of y_1 and y_2 is equal to unit monomial. Then the S-polynomial associated with this overlap is given by

$$S(f, g) = \lambda_1 x_1 \cdot f \cdot y_1 - \lambda_2 x_2 \cdot g \cdot y_2$$

where $\lambda_1 = \frac{LC(x_1)}{LC(f)}$ when $x_1 \neq 1$ or $\lambda_1 = \frac{LC(y_1)}{LC(f)}$ when $y_1 \neq 1$ and $\lambda_2 = \frac{LC(x_2)}{LC(g)}$ when $x_2 \neq 1$ or $\lambda_2 = \frac{LC(y_2)}{LC(g)}$ when $y_2 \neq 1$.

Definition 30.7 Let I be an ideal of A and $\prec$ an admissible order on M. A subset G of I is called a Gröbner basis for I, if for every nonzero polynomial $f \in I$ there exist $g \in G$ such that $LM(g)$ divides $LM(f)$.

30.2.1 Mora's Algorithm

In commutative Gröbner basis theory, Buchberger's Algorithm [5] is used to compute the Gröbner basis. Dickson's Lemma and Hilbert's Basis Theorem assure termination of the algorithm for all possible inputs. Our next result is Mora's Algorithm which mimics Buchberger's Algorithm for noncommutative polynomial rings. However, there is no analogous Dickson's Lemma for noncommutative monomial ideals, hence, Mora's Algorithm does not terminates for all possible inputs.

Algorithm 30.1: Noncommutative Mora's Algorithm [15]

Input : A basis $F = \{f_1, \ldots, f_n\}$ for ideal I over a noncommutative polynomial ring $A = K[x_1, \ldots, x_n]$ and an admissible order $\prec$.
Output : A Gröbner basis $G = \{g_1, \ldots, g_t\}$ for I (In the case of termination).

Let $G = F$ and let $B = \emptyset$. For each pair $(g_i, g_j) \in G, i \leq j$, add an S-polynomial $S(g_i, g_j)$ to B for each overlap $x_1 LM(g_i) y_1 = x_2 LM(g_j) y_2$ between the leading monomials $LM(g_i)$ and $LM(g_j)$;
while $B \neq \emptyset$ **do**
  Remove the first entry s_1 from B. Set $s_1' = Red_G(s_1)$;
  if $s_1' \neq 0$ **then**
    Add s_1' to G and then for all $g_i \in G$ add all $S(g_i, g_j)$ to B;
  end
end
Return G.

It is indeed possible to have infinite Gröbner basis for some finitely generated ideal of $A = K[x_1, \ldots, x_n]$.

Proposition 30.1 *Not all noncommutative monomial ideals are finitely generated.*
Proof *Assume to the contrary that all noncommutative monomial ideals are finitely generated, and consider an ascending chain of such ideals* $J_1 \subset J_2 \subset \cdots$. *Then* $J = \cup J_i$ *is finitely generated and there is some* $d \geq 1$ *such that* $J_d = J_{d+1} = \cdots$. *For a counterexample, let* $A = K[x, y]$ *be a noncommutative polynomial ring, and define* J_i *for* $(i > 1)$ *to be the ideal in* A *generated by the set of monomials* $\{xyx, xy^2x, \ldots, xy^ix\}$. *Thus, we have an ascending chain of such ideals* $J_1 \subset J_2 \subset \cdots$. *However, because no member of this set is a multiple of any other member of the set, it is clear that there cannot be a* $d \geq 1$ *such that* $J_d = J_{d+1} = \cdots$, *because* $xy^{d+1}x \in J_{d+1}$ *and* $xy^{d+1}x \notin J_d$ *for all* $d > 1$.

30.3 Path Algebra

Now we are ready to study the core object of this paper, a noncommutative free associative algebras called path algebras. We begin by defining a path algebra.

Definition 30.8 A *quiver* is a quadruple $Q = (Q_0, Q_1, s, t)$ consisting of two sets and two maps. The sets Q_0, whose elements are called points or vertices, say $\{1, 2, 3, \ldots, n, \ldots\}$, and Q_1 whose elements are called arrows, say $\{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n, \ldots\}$. The two maps $s, t : Q_1 \longmapsto Q_0$ associates to each arrow $\alpha \in Q_1$ its source $s(\alpha) \in Q_0$ and its target $t(\alpha) \in Q_0$ respectively.

An arrow $\alpha \in Q_1$ with a source $s(\alpha) = 1$ and target $t(\alpha) = 2$ is usually denoted by $\alpha : 1 \longmapsto 2$. A path x, of length $l > 1$, with a source a and target b, is a sequence of arrows $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ such that $a = s(\alpha_1)$ and $b = t(\alpha_n)$ where $\alpha_k \in Q_1$ for all $1 \leq k \leq n$, and $t(\alpha_k) = s(\alpha_{k+1})$ for $1 \leq k < n$. Such a path x is denoted by $x = \alpha_1\alpha_2\alpha_3 \ldots \alpha_n$ and visualized as:

$$a = 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} \ldots \xrightarrow{\alpha_{n-1}} n \xrightarrow{\alpha_n} n + 1 = b.$$

The *length* of a path x, denoted by $l = l(x)$ is the number of arrows in it. An *arrow* $\alpha : 1 \longmapsto 2$ is a path of length 1. A *trivial* path denoted by v_i is a path of length zero associated with each vertex i. A path of length $l \geq 1$ is called a *cycle* whenever its source and target coincide. A *loop* is a cycle of $l = 1$. A quiver is said to be *acyclic* if it has no cycles. A quiver is said to be *finite* if Q_0 and Q_1 are both finite sets.

Definition 30.9 Let Q be a quiver and K an arbitrary field. The path algebra KQ of Q is the K-algebra whose underlying K-vector space has as its basis the set of all paths of length $l \geq 0$ in Q, and such that the product of two basis vectors namely $x = \alpha_1\alpha_2\alpha_3 \ldots \alpha_n$ and $y = \beta_1\beta_2\beta_3 \ldots \beta_k$ is defined by

$$xy = \begin{cases} \alpha_1\alpha_2\alpha_3 \ldots \alpha_n\beta_1\beta_2\beta_3 \ldots \beta_k, & \text{if} \quad s(y) = t(x) \\ 0, & \text{otherwise} \end{cases}$$

i.e the product xy is a concatenation or zero otherwise, so that $Q \bigcup \{0\}$ is closed under multiplication. Multiplication as defined above is also distributive K-linearly in $Q \bigcup \{0\}$. Addition in KQ is the usual K-vector space addition where Q is a K-basis for KQ.

The following two results, Remark 30.1 and Lemma 30.1, shows that KQ as defined in Definition 30.9 is indeed an associative algebra.

Remark 30.1 (Properties) [1]

(i) Let Q be finite. The set $\{v_1, v_2, v_3, \ldots, v_n\}$ of the trivial paths corresponding to the vertices $\{1, 2, \ldots, n\}$ is a complete set of primitive orthogonal idempotents. Thus $1 = v_1 + v_2 + v_3 + \cdots + v_n = \sum_{i=1}^{n} v_i$ is the called the identity element of KQ.

(ii) For each arrow $\alpha : 1 \longmapsto 2$ we have the following defining relations:

- $v_i^2 = v_i v_i = v_i$ for $i = 1, 2$.
- $v_1\alpha = \alpha$ and $v_2\alpha = 0$
- $\alpha v_2 = \alpha$ and $\alpha v_1 = 0$
- $v_1 v_2 = 0$.

(iii) Let Q denote the set of all paths of length $l \geq 0$, then the above product extend to all elements of KQ and there is a direct sum

$$KQ = KQ_1 \oplus KQ_2 \oplus \cdots \oplus KQ_i \oplus \ldots$$

Where KQ_i is subspace of KQ generated by the set Q_i, where Q_i is the set of all paths of length i, over K. Since the product of path of length n with path of length m is zero or a path of length $n + m$ then the above decomposition defines a grading on KQ. Hence, KQ is a graded K-algebra.

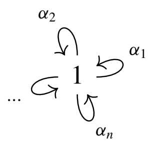
Lemma 30.1 *[1] Let Q be a quiver and KQ be its path algebra. Then*

(i) KQ is an associative algebra.
(ii) KQ has an identity element if and only if Q is finite.
(iii) KQ is finite dimensional if and only if Q is finite and acyclic.

Definition 30.10 An element $f \in KQ$: $(f = \sum \lambda_i x_i, \quad \lambda_i \in K)$, is a linear combination of paths $x_i \in Q$ over K. Elements of KQ will be called polynomials. The paths $x_i \in Q$ appearing in each polynomials will be called monomials. We shall denote by $Mon(f)$ the set of all monomials x_i appearing in the polynomial f.

Example 30.1 If Q consist of one vertex and n loops, $\alpha_1, \alpha_2 \ldots \alpha_n$, then $KQ \cong K[X_1, X_2, \ldots, X_n]$.

$Q =$



The isomorphism is induced by the K-linear maps

$$v_1 \longmapsto 1, \quad \alpha_1 \longmapsto X_1, \quad \alpha_2 \longmapsto X_2, \quad \ldots \quad \alpha_n \longmapsto X_n.$$

30.3.1 Basics to Noncommutative Gröbner Basis in a Path Algebra

For the rest of the paper, Q is taken to be finite. By convection we write a path $\alpha_1\alpha_2\alpha_3 \ldots \alpha_n$ from left to right such that $t(\alpha_i) = s(\alpha_{i+1})$. For path $x = \alpha_1\alpha_2\alpha_3 \ldots \alpha_n$ we denote its length $l(x) = n$. Ideals of KQ are the following:

(i) A subset L of KQ is called a left ideal if

a. $0 \in L$
b. $x + y \in L$ for all $x, y \in L$
c. $xy \in L$ for all $x \in KQ$ and all $y \in L$.

(ii) A subset R of KQ is called a right ideal if

(a) $0 \in R$
(b) $x + y \in R$ for all $x, y \in R$
(c) $xy \in R$ for all $x \in R$ and all $y \in KQ$.

(iii) A subset I of KQ is called a two-sided ideal or simply an ideal, if it is both a left and a right ideal.

In general an ideal I in the path algebra KQ has a Gröbner basis depending on the ordering of the paths in Q.

Proposition 30.2 *An ideal I in KQ with some path ordering has a Gröbner basis whenever the path ordering is admissible.*

Definition 30.11 (*Path Ordering*) By a path ordering we first arbitrary order the vertices $v_1 \prec v_2 \prec v_3 \prec \cdots \prec v_k$ and arbitrary order the arrows all larger than a given vertex say v_k as $v_k \prec \alpha_1 \prec \alpha_2 \prec \alpha_3 \prec \cdots \prec \alpha_r$. Then refer to the noncommutative ordering as defined in Definition 30.1.

Definition 30.12 A path order $\prec$ is said to be admissible order if

(i) Whenever $x \neq y$ either $x \prec y$ or $x \succ y$.
(ii) Every nonempty set of paths has a least element.
(iii) $x \prec y \Rightarrow xz \prec yz$, whenever $xz \neq 0$ and $yz \neq 0$.
(iv) Also $x \prec y \Rightarrow wx \prec wy$, whenever $wx \neq 0$ and $wy \neq 0$.
(v) $x = yz$ implies $x \succeq y$ and $x \succeq z$.

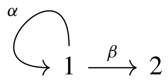
Remark 30.2 Conditions 1 through 3 makes $\prec$ a right admissible order. Condition 1, 2 and 4 make $\prec$ left admissible ordering whilst condition 2 say that an admissible ordering is a well ordering.

30.3.1.1 Constructing Admissible Path Ordering

In the following, we use appropriate monomial (path) ordering to construct admissible ordering for paths in KQ.

(i) Left Lexicographic order: Let $x = \alpha_1 \ldots \alpha_n$ and $y = \beta_1 \ldots \beta_m$ be paths. We say that x is less than y with respect to left lexicographic order and denote $x \prec_{llex} y$ if there exist a path z (otherwise we set $z = 1$), such that $x = z\alpha_k \ldots \alpha_n, \quad y =$

$z\beta_s \ldots \beta_m$ and $\alpha_k \prec \beta_s$. Left lexicographic order is not a left admissible ordering since it is not a well ordering. For example let

$Q =$ 

with $\alpha \prec \beta$. We have $(\alpha\beta \succ_{llex} \alpha^2\beta \succ_{llex} \alpha^3\beta \ldots)$. Then the subset $\{\alpha^n\beta : n \in \mathbb{N} - \{0\}\} \subset Q$ does not have a least element.

(ii) Length left lexicographic order: Let $x = \alpha_1 \ldots \alpha_n$ and $y = \beta_1 \ldots \beta_m$ be paths. We say that x is less than y with respect to length left lexicographic order and denote $x \prec_{Lex} y$ if $l(x) < l(y)$ or $l(x) = l(y)$ and $x \prec_{llex} y$. Length Left lexicographic order is a left admissible order.

(iii) Right lexicographic order: Let $x = \alpha_1 \ldots \alpha_n$ and $y = \beta_1 \ldots \beta_m$ be two paths in Q. We say that x is less than y with respect to right lexicographic order and denote $x \prec_{rlex} y$ if there exist a path z (otherwise we set $z = 1$), such that $x = \alpha_1 \ldots \alpha_k z, \quad y = \beta_1 \ldots \beta_s z$ and $\alpha_k \prec \beta_s$. This ordering is not a well ordering and hence not admissible.

(iv) Length right lexicographic order: Let $x = \alpha_1 \ldots \alpha_n$ and $y = \beta_1 \ldots \beta_m$ be paths. We say that x is less than y with respect to length right lexicographic order and denote $x \prec_{rLex} y$ if $l(x) < l(y)$ or $l(x) = l(y)$ and $x \prec_{rlex} y$. Length right lexicographic order is a right admissible order.

(v) Lexicographic Order: Order the arrows arbitrarily $\alpha_1 \prec \ldots \prec \alpha_m$ and also order the vertices. The vertices will be less than all paths of positive length. Let $x = \alpha_1 \ldots \alpha_n$ and $y = \beta_1 \ldots \beta_m$ be paths. We say that x is less than y with respect to lexicographic order and denote $x \prec_{lex} y$ if working left-to-right, the first (say $i - th$) arrow on which x and y differ is such that the $\alpha_i \prec \beta_i$ in the arrow ordering. This ordering is not admissible.

(vi) The total lexicographic order: Order the arrows arbitrarily $\alpha_1 \prec \ldots \prec \alpha_m$ and also order the vertices. The vertices will be less than all paths of positive length. Let $x, y \in Q$. We say that x is less than y with respect to total lexicographic order and denote $x \prec_{Tlex} y$, if there exists i such that $\forall j < i \quad \alpha_j$'s occurs in x and y the same number of times, and α_i occurs in x less than it occurs in y. If x and y have the same number of each arrow then $x \prec_{lex} y \Rightarrow x \prec_{Tlex} y$. This ordering is admissible.

With an admissible ordering, we can calculate the Gröbner basis for ideals in a path algebras. Calculating this basis consists of a series of division and reduction algorithms as summarized below.

Definition 30.13 Let $\prec$ be an admissible ordering and $A = KQ$. Then Definition 30.4 hold true for all $f \in A$. Moreover, if $x, y \in Q$, x left divide y if $y = wx$, and x right divide y if $y = xz$. So we shall say x divides y if $y = wxz$ for some paths $w, z \in Q$.

Definition 30.14 Let x and y be paths. An element $f \in KQ \setminus \{0\}$ is said to be uniform if there exist vertices u and v such that $f = uf = fv = ufv$.

Proposition 30.3 ([10]) *All elements of KQ are uniform.*

Proof $f = \sum_{i=1}^{n} \lambda_i x_i$ is uniform since for each monomial x_i, which is a sequence of arrows, has a source vertex say u_i and a target vertex say v_i and hence $x_i = u_i x_i v_i$. Therefore, f is sum if uniform elements $f = \sum_{i,j=1}^{n} u_i f v_j$.

Let H be a subset of KQ and $g \in KQ$. We say that g can be reduced by H if for some $x \in \text{Mon}(g)$ there exist $h \in H$ such that $LM(h)$ divides x, i.e $x = pLM(h)q$ for some monomials $p, q \in KQ$. The reduction of g by H is given by $g - \lambda phq$ where $h \in H$, $p, q \in Q$ and $\lambda \in K \setminus \{0\}$ such that $\lambda pLM(h)q$ is a term in g, λ is uniquely determined by $\lambda = \frac{LC(g)}{LC(h)}$. Moreover,

(i) A total reduction of g by H is an element resulting from a sequence of reductions that cannot be further reduced by H.
(ii) We say that an element $g \in KQ$ reduces to 0 by H if there is a total reduction of g by H which is 0. In general two total reductions need not be the same.
(iii) A set $H \subset KQ$ is said to be a reduced set if for all $g \in H$, g cannot be reduced by $H - \{g\}$.

30.4 One-Side Gröbner Bases in Path Algebra

Next, we introduce left and right division algorithms for polynomials in path algebras. These algorithms will be indirect entries in the respective left and right Buchberger's Algorithm which in turn produces respective left and right Gröbner basis. One-sided Gröbner basis and all corresponding one-sided division algorithms are given in this section.

30.4.1 Left Gröbner Bases in Path Algebra

Let L be a left ideal of KQ and $\prec$ a left admissible order. We say that a set $G_L \subset L$ is a left Gröbner basis for L with respect to $\prec$, if for all $f \in$ L $\setminus\{0\}$ there exist $g \in G_L$ such that $LM(g)$ left divides $LM(f)$. Equivalently, we say that a set $G_L \subset L$ is a left Gröbner basis for L with respect to a left admissible order $\prec$ if $\langle LM(G_L)\rangle = \langle LM(L)\rangle$.

Theorem 30.1 ([2]) *Let $\prec$ be a left admissible ordering and $S = \{f_1, \ldots, f_n\}$ be a set of nonzero polynomials in KQ. For $g \in KQ \setminus \{0\}$ there exist a unique determined expression $g = \sum_{i=1}^{n} g_i f_i + h$ where $h, g_1, \ldots, g_n \in KQ$ satisfying:*

A. *For any path p occurring in each g_i, $\quad t(p) = s(LM(f_i))$.*
B. *For $i > j$, no term $g_i LT(f_i)$ is left divisible by $LT(f_j)$.*
C. *No path in h is left divisible by $LM(f_i)$ for all $1 \leq i \leq n$.*

Remark 30.3 The expression $g = \sum_{i=1}^{n} g_i f_i + h$ in Theorem 30.1 is called the left standard representation of $g \in KQ$ with respect to the set S. Algorithm 30.2 gives as an output h, a remainder of g after left division by S. We denote by $LRed_S(g) = h$ the particular remainder of g produced by the division algorithm with respect to a fixed admissible ordering.

Algorithm 30.2: Left Division Algorithm

Input : g, $S = \{f_1, \ldots, f_n\}$, $f_i \in KQ \setminus \{0\}$ and left admissible order $\prec$ on KQ.

Output : $g_i, \ldots, g_n, h \in KQ$ such that $g = \sum_{i=1}^{n} g_i f_i + h$.

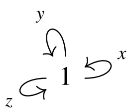
a. For any multiple v_{1i} of $LT(f_1)$ occurring in g with $(1 \leq i \leq r_1)$, find for each i a term h_{1i} such that $v_{1i} = h_{1i} LT(f_1)$. Afterwards do the same for any multiple v_{2i} of $LT(f_2)$ occurring in g such that $v_{2i} = h_{2i} LT(f_2)$ with $1 \leq i \leq r_2$. Continue in this way for any multiple v_{ki} of $LT(f_k)$ such that $v_{ki} = h_{ki} LT(f_k)$ with $1 \leq i \leq r_k$ and $k \in 3, \ldots, n$;

b. Write $g = \sum_{j=1}^{n} (\sum_{i=1}^{r_j} h_{ji}) LT(f_j) + h_1$ and set $g^1 = g - (\sum_{j=1}^{n} (\sum_{i=1}^{r_j} h_{ji}) f_j + h_1)$;

c. If $g^1 = 0$ then we are done and $g = \sum_{j=1}^{n} g_j LT(f_j) + h_1$ where $g_j = \sum_{i=1}^{r_j} h_{ji}$ and $h_1 = h$;

d. If $g^1 \neq 0$, go back to a and continue the process with $g = g^1$.

Example 30.2 Let Q be the quiver with one vertex and three loops over the field of rationals.

$Q =$ 

With a left length lexicographic ordering $z \prec y \prec x$. We find the standard representation of $g = zxxyz + xyxxy - xyz$ with respect to the set $\{f_1 = xyz - zy, f_2 = xxy - yx\}$. We first note that $LM(f_1) = xyz$ and $LM(f_2) = xxy$. Initializing we get $g = zxLM(f_1) + xyLM(f_2) - LM(f_1)$. We replace g by $g^1 = g - (zxf_1 + xyf_2 - f_1) = zxzy + xyyx + zy$. Neither $LM(f_1)$ and $Lm(f_2)$ left divides $zxzy + xyyx + zy$, so we set $h = zxzy + xyyx + zy$ and $zxzy + xyyx + zy$ is replaced by 0 and the algorithm stops. Thus, the standard representation of g is $g = zxf_1 + xyf_2 - f_1 + h$.

***Proof of Theorem* 30.1**

(i) Existence: First the algorithm removes any multiple of f_1 from g. Then removes any multiple of f_2 and continue in this way until any multiple of any of f_k has been removed. In this case if $g = \sum_{j=1}^{n} \sum_{i=1}^{r_j} h_{ji} LT(f_i) + h_1$ is the resulting standard representation of g, we have either $g^1 = g - (\sum_{j=1}^{n} \sum_{i=1}^{r_j} h_{ji}(f_i) + h_1) = 0$ or $LM(g^1) \prec LM(g)$. Since the path ordering $\prec$ is well ordering, by recursion the algorithm produces a standard representation for g^1, $g^1 = \sum_{j=1}^{n} \sum_{i=1}^{r_j} h^1_{ji}(f_i) + h^1$, satisfying conditions A, B and C. Thus $g = \sum_{j=1}^{n} \sum_{i=1}^{r_j} (h_{ji} + h^1_{ji})(f_i) + (h_1 + h^1)$ is a representation for g satisfying the conditions A, B and C.

(ii) Uniqueness: For $g \in L \setminus \{0\}$, let $g = g_1 f_1 + \cdots + g_n f_n + h$. Then the three conditions A, B and C implies that the terms $LT(g_i f_i) = LT(g_i)LT(f_i)$ and $LT(h)$ do not divide each other to the left. Otherwise these terms cancels with each other into zero polynomial. Therefore, the representation $g = \sum_{i=1}^{n} g_i f_i + h$ is unique.

(iii) Termination: The algorithm produces elements $g, g^1, g^2, \ldots, g^k$ so that at each k^{th} iteration $LM(g^{k+1}) \prec LM(g^k)$. Since $\prec$ is a well ordering, the algorithm terminates at some $g^k = 0$ satisfying the conditions of the theorem.

Given a finite generating set $S = \{f_1, \ldots, f_n\}$. For a left admissible order $\prec$, the following algorithm gives as an output $R_L = R_L(S)$, a left reduction of S.

Algorithm 30.3: Set Left Reduction Algorithm

Input : $S = \{f_1, \ldots, f_n\}$, $f_i \neq 0$, and a left admissible ordering $\prec$.
Output : R_L a left reduction of the set S.

a. $R_L = \emptyset$;

b. Find the maximal element f_k of S with respect to $\prec$, for $1 \leq k \leq n$;

c. Write $S = S - \{f_k\}$;

d. Do $f'_k = LRed_{S \cup R_L}(f_k)$;

e. If $f'_k \neq 0$ then $R_L = R_L \cup \{\frac{f'_k}{LM(f'_k)}\}$;

f. If $f'_k = 0$, Go back to a and continue with the process.

g. If $f_k \neq f'_k$ then $S = S \cup R_L$; Go back to a and continue with the process.

Proposition 30.4 ([2]) *Let $G = \{f_1, \ldots, f_n\} \subset KQ$ be a left Gröbner basis for the ideal*

$$L = \langle f_1, \ldots, f_n \rangle \subset KQ.$$

If $g = \sum_{i=1}^{n} g_i f_i + h$ is a left standard expression of $g \in KQ \setminus \{0\}$ then $g \in L$ if and only if $h = 0$.

Proof If $h = 0$ clearly $g \in L$. Conversely if $g \in L$ then $h \in L$ imply $LM(h) \in \langle LM(f_1), \ldots, LM(f_n) \rangle$ which is impossible by the Theorem 30.1.

Definition 30.15 (*Left S-Polynomial*) Let $f, g \in KQ \setminus \{0\}$ and $\prec$ be a left admissible ordering. Let p, q be paths such that $pLM(f) = qLM(g)$, the left S-polynomial $S_L(f, g)$ is defined as

$$S_L(f, g) = \frac{p}{LC(f)} \cdot f - \frac{q}{LC(g)} \cdot g.$$

Theorem 30.2 (Left Buchberger's Criterion) [2] *Let $f_1, \ldots, f_n \in KQ \setminus \{0\}$ and $\prec$ be a left admissible ordering. Let $S_L(f_i, f_j) = \sum_{k=1}^{n} g_k f_k + h_{ij}$ be a left a standard expression of $S_L(f_i, f_j)$ for each pair (i, j). Then $\{f_1, \ldots, f_n\}$ form a left Gröbner basis for $L = \langle f_1, \ldots, f_n \rangle$ if and only if all the remainders h_{ij} are zero.*

Algorithm 30.4: Left Buchberger's Algorithm

Input : $L = \langle f_1, \ldots, f_n \rangle \subset KQ$ and a left admissible order $\prec$.
Output : A reduced left Gröbner basis G_m for L.

a. $m = 0$; $G_0 = \emptyset$; $G_1 = R_L(\{f_1, \ldots, f_n\})$;

b. While $G_m \neq G_{m+1}$, $m = m + 1$;

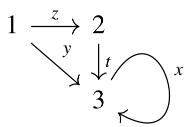
c. For all $g, h \in G_m$ find all $S_L(g, h) \neq 0$;

d. Write $G'_m = G'_m \cup \{S_L(g, h)\}$;

e. $G_{m+1} = R_L(G'_m)$.

30.4.2 Right Gröbner Basis in a Path Algebra

The right-sided division and reduction algorithms, in many ways, give a "right" version of the Gröbner basis theory as discussed in Sect. 30.4.1. This means that concepts from the previous section will have to be duplicated with a slight variant in the division as right-sided operation. Hence, it is omitted here, and we instead illustrate right Gröbner basis for an ideal in a path algebra using the following example.

Example 30.3 Let Q be the quiver 

Let $F = \{f_1 = ztx^3, f_2 = zt + y\}$ be a subset of KQ with respect to the right length lexicographic ordering $v_1 \prec v_2 \prec v_3 \prec t \prec z \prec y \prec x$. We note that $LM(f_1) = ztx^3$ and $LM(f_2) = zt$ and they only factor each other to the right in one way namely $LM(f_1)v_3 = LM(f_2)x^3$. Thus, we have one right S-polynomial $S_R(f_1, f_2) = f_1v_3 - f_2x^3 = -yx^3$. Neither $LM(f_1)$ nor $LM(f_2)$ right divide $-yx^3$ so we add $f_3 = -yx^3$ to F. Now every right S-polynomial reduces to zero by F. Thus $F = \{f_1, f_2, f_3\}$ is a right Gröbner basis for the ideal $R = \langle f_1, f_2 \rangle$.

30.5 Two-Sided Gröbner Bases

We say that a set $G \subset I$ is a Gröbner basis for I with respect to an admissible order $\prec$ if $\langle LM(G) \rangle = \langle LM(I) \rangle$.

Proposition 30.5 *If G is a Gröbner basis for the ideal I, then G is a generating set for the elements of I and also G reduces elements of I to* 0.

Proof Let KQ be a path algebra with an admissible ordering $\prec$. Let I be an ideal and let G be a Gröbner basis for I. Let $f_i \in I, i = 1, \ldots, n, \ldots$, for every $f_n \in I$ such that $f_n \neq 0 \;\; \exists \;\; g \in G$ such that $LM(g)$ divides $LM(f_n)$. Let $f_{n+1} = f_n - \frac{LC(f_n)}{LC(g)} xgy$ be a reduction of f_n by g. Then $LM(f_{n+1}) \prec LM(f_n)$. But $g, f_n \in I \implies f_{n+1} \in I$. Repeating this reduction on f_i to produce f_{i+1} yields a decreasing sequence $LM(f_1) \succ LM(f_2) \succ \ldots\ldots$, which terminates only if $f_n = 0$. Since $\prec$ is an admissible order, every set of paths has a least element hence the sequence must terminate at some $f_n = 0$.

30.5.1 Division Algorithms

Theorem 30.3 *Let $\prec$ be an admissible ordering and $S = \{f_1, \ldots, f_n\}$ be a set of non zero polynomials in KQ. For $g \in KQ \setminus \{0\}$ there exist a unique determined expression $g = \sum_{i=1}^{n} w_i f_i z_i + h$ where $h, w_1, \ldots, w_n, z_1 \ldots, z_n \in KQ$ satisfying:*

A3. For any path p occurring in each w_i, $t(p) = s(LM(f_i))$ and for any path q occurring in z_i, $t(LT(f_i)) = s(q)$.
B3. For $i > j$ no term $w_i LT(f_i) z_i$ is divisible by $LT(f_j)$.
C3. No path in h is divisible by $LM(f_i)$ for all $1 \leq i \leq n$.

Algorithm 30.5: Two-sided Division Algorithm

Input : $g, S = \{f_1, \ldots, f_n\}$ and an admissible order $\prec$ on elements of KQ.

Output : $w_1, \ldots, w_n, z_1, \ldots, z_n, h \in KQ$ such that $g = \sum_{i=1}^{n} w_i f_i z_i + h$.

a. For any multiple O_{1i} of $LT(f_i)$ occurring in g with $1 \leq i \leq r_1$, find for each i the terms u_{1i} and v_{1i} such that $O_{1i} = u_{1i} LT(f_1) v_{1i}$. Following this do the same for any multiple O_{2i} of $LT(f_2)$ occurring in g such that $O_{2i} = u_{2i} LT(f_2) v_{2i}$ with $1 \leq i \leq r_2$. Continue in this way for any multiple O_{ki} of f_k such that $O_{ki} = u_{ki} LT(f_k) v_{ki}$ with $1 \leq i \leq r_k$ and $k \in \{3, \ldots, n\}$;

b. Write $g = \sum_{j=1}^{n} \sum_{i=1}^{r_j} u_{ji} LT(f_j) v_{ji} + h_1$ and set $g^1 = g - (\sum_{j=1}^{n} \sum_{i=1}^{r_j} u_{ji} f_j v_{ji} + h_1)$;

c. If $g^1 = 0$ then we are done and $g = \sum_{j=1}^{n} w_j f_j z_j + h_1$ where $w_j = \sum_{i=1}^{r_j} u_{ji}, z_j = \sum_{i=1}^{r_j} v_{ji}$ and $h = h_1$;

d. If $g^1 \neq 0$, go back to a and proceed with $g = g^1$.

Let $Red_S(g) = h$ denote the particular total reduction of an element g by a set S produced by the Algorithm 30.5 with respect to a fixed admissible ordering.

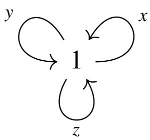
Proof (i) $\underline{Existence:}$ This algorithm finds a standard representation of g as follows. First it removes any multiple of f_1 in g. Afterwards removes any multiples of f_2. Continue in this way until any multiple of any $f_k, k \in \{3, 4, \ldots, n\}$ has been removed. Hence, if $g = \sum_{j=1}^{n}\sum_{i=1}^{r_j} u_{ji} LT(f_i) v_{ji} + h_1$ is the resulting representation of g then either $g^1 = g - (\sum_{j=1}^{n}\sum_{i=1}^{r_j} u_{ji}(f_i)v_{ji} + h_1)$ equal to zero and we are done, or $LM(g) \succ LM(g^1)$. Since $\prec$ is a well ordering then the algorithm finds a representation $g^1 = \sum_{j=1}^{n}\sum_{i=1}^{r_j} u^1_{ji} f_i v^1_{ji} + h^1$ satisfying conditions $A3$, $B3$ and $C3$ so that $g = \sum_{j=1}^{n}\sum_{i=1}^{r_j}(u^1_{ji} + u_{ji}) f_i (v^1_{ji} + v_{ji}) + (h^1 + h_1)$ is the standard representation of g satisfying conditions $A3$, $B3$ and $C3$.

(ii) $\underline{Uniqueness}$: Given g and conditions $A3$, $B3$ and $C3$, no term $LT(w_i f_i z_i)$ for all $1 \le i \le n$ divides $LT(h)$. Therefore, the algorithm produces a unique standard representation $g = \sum_{i=1}^{n} w_i f_i z_i + h$ where w_i or z_i may be unit monomials.

(iii) $\underline{Termination:}$ Note that the algorithm produces elements $g, g^1, g^2, \ldots, g^k$ such that at each k^{th} iteration $LM(g^k) \succ LM(g^{k+1})$ and the algorithm must terminate at some k where

$$g^k = \sum_{j=1}^{n}\sum_{i=1}^{r_j} u_{ji}(f_i)v_{ji} + h_k = 0$$

and every monomial occurring in the final h_k is not divisible by $LM(f_i)$, $1 \le i \le n$.

Example 30.4 Consider the quiver $Q =$ 

Let $\prec$ be the total lexicographic order with $x \succ y \succ z$. Let's divide $f = zxxyx$ by $\{f_1 = xy - x, f_2 = xx - xz\}$. Note that the $LM(f_1) = xy$ and $LM(f_2) = xx$. Beginning the Algorithm 30.5, we see that $zxxyx = (zx)LM(f_1)(x)$. Thus, $p_1 = zx, q_1 = x$ and we replace $zxxyx$ by $zxxyx - zx(f_1)x = zxxx$. Now $LM(f_1)$ does not divide $zxxx$. Continuing, $LM(f_2)$ does. There are two ways to divide $zxxx$ by xx and for the algorithm to be precise we must choose one. Say we choose the "left most" division, as based on our ordering going from left to right. Then $zxxx = z(LM(f_2))x$ and we let $p_2 = z, q_2 = x$ and replace $zxxx$ by $zxxx - z(f_2)x = zxzx$. Neither

$LM(f_1)$ nor $LM(f_2)$ divide $zxzx$ so we let $r = zxzx$ and $zxzx$ is replaced by 0 and the algorithm stops. We have $zxxyx = (zx)f_1(x) + (z)f_2(x) + zxzx$. The remainder is $zxzx$.

Given a finite generating set $S = \{f_1, \ldots, f_n\}$, an ideal $I \subset KQ$, and an admissible order $\prec$ the following algorithm gives as an output $R(S)$ a finite monic reduced generating set for I.

Algorithm 30.6: Set Reduction Algorithm

Input : $S = \{f_1, \ldots, f_n\}$, $f_i \neq 0$, and an admissible ordering $\prec$.

Output : $R = R(S)$ a reduction of elements of S.

a. $R = \emptyset$;

b. Find the maximal element f_k of S with respect to $\prec$;

c. Write $S = S - \{f_k\}$;

d. Do $f'_k = Red_{S \cup R}(f_k)$;

e. If $f'_k \neq 0$ then $R = R \bigcup \{\frac{f'_k}{LM(f'_k)}\}$;

f. If $f'_k = 0$, Go back to a and continue with the process.

g. If $f_k \neq f'_k$ then $S = S \cup R$; Go back to a and continue with the process.

Proposition 30.6 *Given an ideal I in KQ and an admissible order $\prec$, there is a unique Gröbner basis G such that G is a reduced set and the coefficient of the leading monomials of the polynomials in G are all* 1.

Proof Let KQ be a path algebra, I an ideal and $\prec$ an admissible order. Let G and G' be Gröbner bases for I. Suppose G and G' are both reduced monic sets. Since $G \subset I$, for every $g_1 \in G$ there exist $g' \in G'$ such that $LM(g')$ divides $LM(g_1)$. Also since $G' \subset I$ there exist $g_2 \in G$ such that $LM(g_2)$ divides $LM(g')$. Thus, $LM(g_2)$ divides $LM(g_1)$. But G is a reduced set hence we must have that $g_2 = g_1$ so that $LM(g_1) = LM(g') = LM(g_2)$. So there is a bijection correspondence between elements of G and the elements of G' with the same leading monomials. Thus, g' cannot be reduced by $G - \{g_1\}$. Hence $g' - g_1$ cannot be reduced by G, since $g' - g_1 \in I$. Thus $g' - g_1 = 0 \Longrightarrow g' = g_1$ hence $G' = G$.

We call the unique reduced monic Gröbner basis the reduced Gröbner basis for I. The reduced Gröbner basis G is minimal in the sense that for any other reduced Gröbner basis G' for the same ideal with the same admissible order, we have $LM(G') \subset LM(G)$.

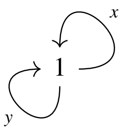
30.5.2 Two-Sided S-Polynomial

While noncommutative S-polynomials for each pair of polynomials $f, g \in KQ$ may be different due to different factorizations in the set of monomials in KQ, for one-sided case these S-polynomials are finitely many. However, we may have ambiguity while dealing with two-sided S-polynomials due to possible different choices of right and left factors of each overlap of $LM(f)$ and $LM(g)$. Therefore, a condition, namely $l(p) \leq l(LM(g))$ whenever $LM(f) \cdot p = q \cdot LM(g)$, is added to the definition of two-sided S-polynomial to eliminate such ambiguity.

Definition 30.16 Let $f, g \in KQ$ with an admissible order $\prec$ on elements of KQ. An $(f - g)$ overlap is said to occur if there are paths p and q of positive length such that $LM(f)p = qLM(g)$ where $l(p) \leq l(LM(g))$. Thus an f and g are said to have an overlap relation or a two-sided S-polynomial denoted by $S(f, g)$ and defined as

$$S(f, g, p, q) = \frac{1}{LC(f)} f \cdot p - \frac{1}{LC(g)} q \cdot g.$$

Remark 30.4 Given elements $f, g \in KQ$ such that $LM(f)p = qLM(g)$ where $l(p) \leq l(LM(g))$, monomials p and q will not necessarily be unique. Consequently, the same two elements f and g may still have multiple S-polynomials. In addition an element may have an S-polynomial with itself, i.e $S(f, f)$ will be a possible.

Example 30.5 Let Q be 

and $x \prec y$ with respect to the total lexicographic order. Let $f = 5yyxyx - 2xx$ and $g = xyxy - 7y$. We see that $LM(f) = yyxyx$ and $LM(g) = xyxy$. The following are the S-polynomials among f and g are:

$$\begin{aligned} S(f, g, y, yy) &= \tfrac{1}{5} fy - yyg = -\tfrac{2}{5}xxy + 7yyy \\ S(f, g, yxy, yyxy) &= \tfrac{1}{5} fy - yyg = -\tfrac{2}{5}xxyxy + 7yyxyy \\ S(g, g, xy, xy) &= gxy - xyg = -7yxy + 7xyy \end{aligned}$$

Lemma 30.2 (Bergman's Diamond, [4]) *Let G be a set of uniform elements that form a generating set for the ideal $I \subset KQ$, such that for all $g, g_1 \in G$, $LM(g) \nmid LM(g_1)$. If for each $f \in I$ and $g \in G$ every S-polynomial $S(f, g, p, q)$ is reduced to* 0 *by G, then G is a Gröbner basis for I.*

30.5.3 The Main Theorem

The beauty of Algorithm 30.5 in Theorem 30.3 is that its outputs are uniform elements of KQ. In this section we shall use Lemma 30.2 to greatly reduces calculations and ascertain finite Gröbner basis whenever Algorithm 30.7 terminates.

Theorem 30.4 *Given a path algebra KQ, an admissible order $\prec$ and a finite generating set*

$$\{f_1, f_2, \dots, f_m\}$$

for an ideal I the following algorithm gives a reduced Gröbner basis for I in the limit.

Algorithm 30.7: Two-sided Buchberger's Algorithm

Input : $I = \langle f_1, \dots, f_n \rangle$, $f_i \neq 0$ and an admissible order $\prec$.
Output : A reduced Gröbner basis G_m for I.

a. $m = 0$; $G_0 = \emptyset$; $G_1 = R(\{f_1, f_2, \dots, f_n\})$;

b. For $G_m \neq G_{m+1}$; $m = m + 1$;

c. For all pairs $(g_i, g_j) \in G_m$ and all $1 \leq i \leq j \leq n$, find $S(g_i, g_j, p, q) \neq 0$;

d. Do $G'_m = G_m \bigcup \{S(g_i, g_j, p, q)\}$;

e. $G_{m+1} = R(G'_m)$.

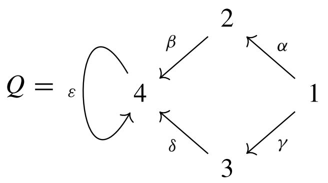
Let G_m be the output of the Algorithm 30.7. Thus, if this algorithm terminates on a given m^{th} iteration. The set G_m is a reduced Gröbner basis.

Proof (i) We first show by induction on m that at each m^{th} iteration, every S-polynomial has a standard representation $S(g_i, g_j, p, q) = \sum_{k=1}^{n} w_k f_k z_k + h_{ij}$. Consider $m = 1 : G_1 = R(\{f_1, f_2, \dots, f_n\})$. The algorithm produces $f = S(g_i, g_j, p, q) = \sum_{k=1}^{n} w_k f_k z_k + h_{ij}$ as a reduced of $S(g_i, g_j, p, q)$ with respect to $S \cup R$. If $h_{ij} \neq 0$, then $h_{ij} \in G_2$ and again f has a standard representation with respect to G_2. Suppose that the hypothesis hold true for m. We now prove for $m + 1$. If the algorithm terminates at $m + 1$ then $G_{m+2} = G_{m+1} = G_m$ and hence $f = S(g_i, g_j, p, q)$ has a standard representation with respect to $G_{m+2} = G_{m+1}$. If the algorithm does not terminate at $m + 1$, $G_{m+1} = R(G_m \cup \{S(g_i, g_j, p, q)\})$ so that the algorithm reduces $f = S(g_i, g_j, p, q)$ to h_{ij}. This ensures that f has a standard representation with respect to G_{m+2}. Hence, the hypothesis hold true for $m + 1$. By induction the statement hold true for all m.

(ii) We now show that the algorithm terminates at $m+1$ if and only if G_m is a finite Gröbner basis of I: If the algorithm terminates at some $m+1$ then all $S(g_i, g_j, p, q) = 0$ and $G_{m+1} = R(G_m) = G_m$, for G_m is a reduced set at every step. Since $\langle G_m \rangle = I$ then we conclude that G_m is a finite reduced Gröbner basis for I. Conversely if G_m is a finite reduced Gröbner basis of I, then $R(G_m) = G_m$ and for each pair $(g_i, g_j) \in G_m$, $f = S(g_i, g_j, p, q)$ is reduced to zero by G_m. Therefore, the algorithm terminates at G_{m+1}.

(iii) If the algorithm never terminates, Let $G = \cup_{m=1}^{\infty} G_m$, then for m sufficiently large every S-polynomial $S(g_i, g_j, p, q)$ has a standard representation with respect to $G_{m+1} \subset G$. Obviously $\langle G \rangle = I$ and hence G is an infinite Gröbner basis of I.

Example 30.6



with the total lexicographic ordering

$$v_1 \prec \cdots \prec v_4 \prec \varepsilon \prec \beta \prec \delta \prec \alpha \prec \gamma.$$

Let $f = \alpha\beta - \gamma\delta$, $g = \beta\varepsilon$ and $h = \varepsilon^3$. We see that $LM(f) = \gamma\delta$, $LM(g) = \beta\varepsilon$ and $LM(h) = \varepsilon^3$. $LM(f) \not| LM(h)$ and $LM(f) \not| LM(h)$. The only possible S-polynomial is $S(g, h, \varepsilon^2, \beta) = 0$. Thus, the set G=$\{f, g, h\}$ is the Gröbner basis since all the S-polynomial reduces to 0. On the other hand if we consider another admissible order $v_1 \prec \cdots \prec v_4 \prec \varepsilon \prec \beta \prec \delta \prec \gamma \prec \alpha$. We now see that $LM(f) = \alpha\beta$, $LM(g) = \beta\varepsilon$ and $LM(h) = \varepsilon^3$. In this case, the only S-polynomial possible is $S(f, g, \varepsilon, \alpha) = (\alpha\beta - \gamma\delta)\varepsilon - \alpha(\beta\varepsilon) = -\gamma\delta\varepsilon$. $LM(S(f, g, \varepsilon, \alpha)) = \gamma\delta\varepsilon \notin \langle LM(F), LM(g), LM(h) \rangle$, Thus, $G = \{f, g, h\}$ is not a Gröbner basis for $I = \langle G \rangle$. We add $r = \gamma\delta\varepsilon$ to G, and we set $G = \{f, g, h, r\}$. Therefore, $S(f, g, \varepsilon, \alpha) = r$ and there are no further possible S-polynomial relations. Thus $R(G) = G = \{f, g, h, r\}$ is a Gröbner basis for I.

In the work of [10], the author characterizes quivers whose path algebra has finite Gröbner basis. It follows that we can use the above procedures to exhaustively study finite Gröbner path algebras.

The Refs. [3, 6, 9, 11–14, 16] are recommended to the reader for further interesting relevant references on the topics considered in this work.

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