Chapter 7
Regular Vectors and Bi-Lipschitz Trivial Stratifications in O-Minimal Structures

Guillaume Valette

Contents

Abstract These notes deal with semialgebraic and subanalytic subsets of $\mathbb{R}^n$, and more generally with all the sets that are definable in a polynomially bounded o-minimal structure expanding $\mathbb{R}$, establishing existence of definably bi-Lipschitz trivial stratifications for these sets (Corollary 7.6.9). We start with basic definitions about o-minimal structures and Lipschitz geometry, and give a short survey of some historical results, such as existence of Mostowski's Lipschitz stratifications or the

G. Valette (✉)
Uniwersytet Jagielloński, Instytut Matematyki, Kraków, Poland
e-mail: guillaume.valette@im.uj.edu.pl

411

Preparation Theorem for definable functions, which gives us the opportunity to state some needed theorems as well as to describe the way the results presented in the second part fit in the landscape. Our stratification theorem (Corollary 7.6.9) is obtained as a byproduct of two foregoing results of the author that are proved in Sects. 7.5 and 7.6 respectively. The first one asserts that, given a family definable in an o-minimal structure, there is a regular vector, up to a definable family of bi-Lipschitz homeomorphisms. The second one is a bi-Lipschitz version of the famous Hardt's theorem. We give proofs of these two theorems that avoid the use of the real spectrum.

7.1 Introduction

The study of the Lipschitz geometry of singularities that arise in algebraic and analytic geometry began when T. Mostowski constructed stratifications of complex analytic sets that admit a Lipschitz version of Thom-Mather isotopy theorem [17]. This result was extended by A. Parusiński to real analytic geometry [22, 23], and his proof was then adapted to polynomially bounded o-minimal structures [18] (see also [12]). The Lipschitz geometry of singularities was investigated later independently by the author of the present paper [30–34] (see [35] for a complete expository), as well as by several other authors [1–4, 14, 15, 26] (among many others).

In [30], the author proved a bi-Lipschitz triviality theorem, which can be considered as a bi-Lipschitz version of Hardt's theorem (recalled in Theorem 7.6.3 below). These notes provide a new proof of this theorem and then derive existence of definably locally bi-Lipschitz trivial stratifications. We also include a short introduction to the theory of the Lipschitz geometry of sets that are definable in o-minimal structures, providing many necessary definitions, proofs, and references.

The existence of locally bi-Lipschitz trivial stratifications and the description of the aspect of singularities occurring in tame geometry (subanalytic, semialgebraic, or o-minimal) from the metric point of view that was achieved during the four last decades recently turned out to be valuable for applications to analysis of PDE's and geometric measure theory. In [33, 34], the author relied on it so as to compute the L^p cohomology of differential forms of bounded subanalytic manifolds, not necessarily compact. More recently, these techniques turned out to be useful to study the theory of currents [8], as well as to investigate the Sobolev spaces of these manifolds [11, 28, 29, 36], which is valuable for applications to the theory of PDE on domains with non Lipschitz boundary [37].

The main difficulty of the proof of the bi-Lipschitz version of Hardt's theorem [30] (see Theorem 7.6.3 below) is the "regular vector theorem" [30, Theorem 3.13] (see Theorem 7.3.2 and Corollary 7.3.4). This theorem asserts that, given a set $A \subset \mathbb{R}^n$ which is definable in an o-minimal structure, there is a definable bi-Lipschitz homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h(A)$ has a regular vector, which means that there is a vector such that the angle between it and all the tangent spaces to $h(A)$ (at its regular points) is bounded away from zero. We prove this result,

which does not require the structure to be polynomially bounded and which is of its own interest. The regular vector theorem was actually also the key ingredient of the Lipschitz conic structure theorem in [34], itself useful later to study Sobolev spaces of definable manifolds [28, 29, 36], and was of service to M. Czapla to show existence of triangulations inducing Whitney stratifications [7].

Quite often, especially if, like in the aforementioned applications of Lipschitz geometry, Lipschitz isotopies are necessary, one needs a regular vector not only for one single set but for a definable family of sets. In [30], this is obtained by applying the regular vector theorem to the generic fiber of a family, relying on the compactness of the Stone space of the Boolean algebra of definable sets (sometimes called the real spectrum). This has nevertheless the inconvenience to involve abstract material to which specialists of PDE or geometric measure theory may be unfamiliar, and to force to work with an o-minimal structure that expands an arbitrary real closed field, that may be non archimedean and totally disconnected, which is prone to generate technical complications.

In the present article, we give a parameterized version of the regular vector theorem (Theorem 7.3.2) on o-minimal structures following the proof given in [30], but relying only on very elementary methods. We also combine it with the techniques of [34] to prove a local version, with additional properties (Theorem 7.5.14), which was used in the latter article to prove the Lipschitz conic structure of subanalytic sets. We then provide a proof of the bi-Lipschitz version of Hardt's Theorem on polynomially bounded o-minimal structures (Theorem 7.6.3), and derive existence of definably bi-Lipschitz trivial stratifications (Corollary 7.6.9).

Some Notations and Definitions Throughout this article, m, n, j, and k will stand for integers. The origin of $\mathbb{R}^n$ will be denoted $0_{\mathbb{R}^n}$. When the ambient space will be obvious from the context, we will however omit the subscript $\mathbb{R}^n$. We write $e_1, \ldots, e_n$ for the canonical basis of $\mathbb{R}^n$. By $\mathscr{C}^k$ **mapping** on a set $X \subset \mathbb{R}^n$, we mean a mapping that extends to a $\mathscr{C}^k$ mapping on a neighborhood of X in $\mathbb{R}^n$.

We write $|x|$ for the euclidean norm and $d(x, y)$ for the euclidean distance (and the distance to a subset $P \subset \mathbb{R}^n$ will be denoted by $d(x, P)$). Given $x \in \mathbb{R}^n$ and $\varepsilon > 0$, we denote by $\mathbf{B}(x, \varepsilon)$ the open ball of radius ε centered at x (for the euclidean norm). The unit sphere of $\mathbb{R}^n$ centered at the origin is denoted $\mathbf{S}^{n-1}$. Given a subset A of $\mathbb{R}^n$, we respectively denote the closure and interior of A by $cl(A)$ and $int(A)$, and set $\delta A = cl(A) \setminus int(A)$.

A mapping $\xi : A \to \mathbb{R}^k$ is said to be **Lipschitz** if there is a constant L such that for all x and x' in A:

$$|\xi(x) - \xi(x')| \le L|x - x'|.$$

We say that ξ is L-**Lipschitz** if we wish to specify the constant. The smallest nonnegative number L having this property is called the **Lipschitz constant of** ξ and is denoted L_ξ. By convention, if A is empty then ξ is Lipschitz and $L_\xi = 0$. A mapping ξ is **bi-Lipschitz** if it is a homeomorphism onto its image such that ξ and ξ^{-1} are both Lipschitz.

Given two functions ζ and ξ on a set $A \subset \mathbb{R}^n$ with $\xi \leq \zeta$ we define the **closed band** $[\xi, \zeta]$ as the set:

$$[\xi, \zeta] := \{(x, y) \in A \times \mathbb{R} : \xi(x) \leq y \leq \zeta(x)\}.$$

The open and semi-open bands (ξ, ζ), $(\xi, \zeta]$, and $[\xi, \zeta)$, are then defined analogously.

Given a subset B of A, we write "$\xi \lesssim \zeta$ on B" when there is a positive constant C such that $\xi(x) \leq C\zeta(x)$ for all $x \in B$. We write "$\xi \sim \zeta$ on B" or "$\xi(x) \sim \zeta(x)$ for x in B" whenever both $\xi \lesssim \zeta$ and $\zeta \lesssim \xi$ hold on B.

7.2 O-Minimal Structures

A **structure** (expanding the field $(\mathbb{R}, +, .)$) is a family $\mathcal{S} = (\mathcal{S}_n)_{n \in \mathbb{N}}$ such that for each n the following properties hold

(1) $\mathcal{S}_n$ is a boolean algebra of subsets of $\mathbb{R}^n$,
(2) If $A \in \mathcal{S}_n$ then $\mathbb{R} \times A$ and $A \times \mathbb{R}$ belong to $\mathcal{S}_{n+1}$,
(3) $\mathcal{S}_n$ contains $\{x \in \mathbb{R}^n : P(x) = 0\}$, for all $P \in \mathbb{R}[X_1, \ldots, X_n]$,
(4) If $A \in \mathcal{S}_n$ then $\pi(A)$ belongs to $\mathcal{S}_{n-1}$, where $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the standard projection onto the first $(n-1)$ coordinates.

A structure $\mathcal{S}$ is said to be **o-minimal** if in addition:

(5) Any set $A \in \mathcal{S}_1$ is a finite union of intervals and points.

A set belonging to $\mathcal{S}_n$, for some n, is called a **definable set**, and a map whose graph is in some $\mathcal{S}_n$ is called a **definable map**.

A structure $\mathcal{S}$ is said to be **polynomially bounded** if for each definable function $f : \mathbb{R} \to \mathbb{R}$, there exists a positive number a and an $n \in \mathbb{N}$ such that $|f(x)| < x^n$ for all $x > a$.

We fix an o-minimal structure $\mathcal{S}$ for all this article. It will be assumed to be polynomially bounded in Sects. 7.2.2 and 7.6 only. For the other sections, this assumption is unnecessary.

We refer to [6, 9] for all the basic facts and definitions about o-minimal structures that we shall use all along this article, such as cell decompositions or curve selection lemma. We however recall a few definitions helpful to understand the statements of the theorems.

It is a fundamental feature of o-minimal structures that it is possible to construct a cell decomposition of $\mathbb{R}^n$ that is **compatible** with a given arbitrary finite collection of elements of $\mathcal{S}_n$, in the sense that the given sets are unions of cells of this decomposition (the word "adapted" is used in [6], instead of compatible).

The definition of cell decompositions being inductive on the dimension of the ambient space, it is obvious that if $\mathbb{C}$ is a cell decomposition of $\mathbb{R}^n$ and if $\pi : \mathbb{R}^n \to \mathbb{R}^k$ (with $k \leq n$) is the canonical projection, then $\{\pi(C) : C \in \mathbb{C}\}$ is a cell decomposition. We will denote it by $\pi(\mathbb{C})$.

Definition 7.2.1 A **stratification** of a subset of $\mathbb{R}^n$ is a finite partition of it into smooth submanifolds of $\mathbb{R}^n$, called **strata**. A **stratification is compatible** with a set if this set is the union of some strata. It is **definable** if so are the strata.

Remark 7.2.2 In the above definition, we write "smooth" without specifying the degree of smoothness. It is well-known that one can construct Whitney (b) or Verdier (w) regular definable $\mathscr{C}^k$ stratifications of any given definable set [27, 35], for every given k. When the structure has $\mathscr{C}^\infty$ cell decomposition, we can construct regular stratifications that have $\mathscr{C}^\infty$ strata. Definable $\mathscr{C}^k$ manifolds admit definable $\mathscr{C}^{k-1}$ tubular neighborhoods [6].

Definition 7.2.3 We say that $(A_t)_{t \in \mathbb{R}^m}$ is a **definable family of subsets of $\mathbb{R}^n$** if the set

$$A := \bigcup_{t \in \mathbb{R}^m} \{t\} \times A_t$$

is a definable subset of $\mathbb{R}^m \times \mathbb{R}^n$.

We will sometimes regard a definable subset $A \subset \mathbb{R}^m \times \mathbb{R}^n$ as a definable family, setting for $t \in \mathbb{R}^m$:

$$A_t := \{x \in \mathbb{R}^n : (t, x) \in A\}.$$

Given two definable families $A \subset \mathbb{R}^m \times \mathbb{R}^n$ and $B \subset \mathbb{R}^m \times \mathbb{R}^k$, we say that $F_t : A_t \to B_t, t \in \mathbb{R}^m$, is a **definable family of mappings** if the family of the graphs $(\Gamma_{F_t})_{t \in \mathbb{R}^m}$, is a definable family of subsets of $\mathbb{R}^{n+k}$. We will sometimes regard a function $f : A \to \mathbb{R}, A \in \mathscr{S}_{m+n}$, as a family of functions $f_t : A_t \to \mathbb{R}, t \in \mathbb{R}^m$, setting $f_t(x) := f(t, x)$.

A definable family of mappings $F_t : A_t \to B_t, t \in \mathbb{R}^m$ is **uniformly Lipschitz (resp. bi-Lipschitz)** if there exists a constant L such that F_t is L-Lipschitz (resp. F_t and F_t^{-1} are L-Lipschitz) for all $t \in \mathbb{R}^m$.

Given $B \in \mathscr{S}_m$ and A as above, we also define the **restriction of A to B**:

$$A_B := A \cap (B \times \mathbb{R}^n). \tag{7.1}$$

Define finally the *m*-**support** of A by

$$\operatorname{supp}_m(A) := \{t \in \mathbb{R}^m : A_t \neq \emptyset\}.$$

Definition 7.2.4 Let $A \in \mathscr{S}_{m+n}$. We will say that A is **definably topologically trivial along** $U \subset \mathbb{R}^m$ if there exist $t_0 \in U$ and a definable homeomorphism $H : U \times A_{t_0} \to A_U, (t, x) \mapsto (t, h_t(x))$. The mapping h is then called **a trivialization** of the set A along U.

The following theorem is sometimes called "definable Hardt's theorem", because it is the o-minimal counterpart of a theorem proved by R. Hardt about semialgebraic

families of sets [13]. In this theorem, by **definable partition of a set**, we mean a *finite* partition of it into definable sets.

Theorem 7.2.5 *[[6, Theorem 5.22]] Given $A \in \mathcal{S}_{m+n}$, there exists a definable partition of $\mathbb{R}^m$ such that A is definably topologically trivial along each element of this partition.*

Remark 7.2.6 We shall make use of the following immediate consequence of this theorem: given $A \in \mathcal{S}_{m+n}$, there is a definable partition $\mathscr{P}$ of $\mathbb{R}^m$ such that, for every $B \in \mathscr{P}$, E_t is connected for every connected component E of A_B and all $t \in B$.

Definition 7.2.7 Let $A \in \mathcal{S}_{m+n}$. We will say that A is **definably bi-Lipschitz trivial along** $U \subset \mathbb{R}^m$ if it is definably topologically trivial along this set, with a trivialization $h_t : A_{t_0} \to A_t$ which is bi-Lipschitz for every $t \in U$.

In Sect. 7.6, we show (Theorem 7.6.3) that definable bi-Lipschitz triviality holds up to a definable partition of the parameter space when the o-minimal structure is polynomially bounded (which is a necessary condition), giving a Lipschitz counterpart of Theorem 7.2.5.

Remark 7.2.8 In the above definition, the Lipschitz constant of h_t (or h_t^{-1}) is a function of t. This function is not required to be bounded or locally bounded, but, since it is a definable function, it must be continuous on the elements of a definable partition of U. As a matter of fact, possibly refining the partition provided by Theorem 7.6.3, we see that we could require the Lipschitz constants of the trivialization to be locally bounded on every element of this partition (see also Remark 7.6.7 on this issue).

7.2.1 Bi-Lipschitz Trivial Stratifications

In his pioneer's work [17], Mostowski constructed stratifications, called today *Mostowski's Lipschitz stratifications*, that are locally bi-Lipschitz trivial along the strata (see Theorem 7.2.10 below).

Definition 7.2.9 A stratification Σ of a set X is **locally bi-Lipschitz trivial** if for every $S \in \Sigma$, there are an open neighborhood V_S of S in X and a smooth retraction $\pi_S : V_S \to S$ such that every $x_0 \in S$ has an open neighborhood W in S for which there is a bi-Lipschitz homeomorphism

$$\Lambda : \pi_S^{-1}(W) \to \pi_S^{-1}(x_0) \times W,$$

satisfying:

(i) $\pi_S(\Lambda^{-1}(x, y)) = y$, for all $(x, y) \in \pi_S^{-1}(x_0) \times W$.
(ii) $\Sigma_{x_0} := \{\pi_S^{-1}(x_0) \cap Y : Y \in \Sigma\}$ is a stratification of $\pi_S^{-1}(x_0)$, and $\Lambda(\pi_S^{-1}(W) \cap Y) = (\pi_S^{-1}(x_0) \cap Y) \times W$, for all $Y \in \Sigma$.

When everything is definable (i.e. when so are X, Λ, π_S, and Σ), we say that Σ is **definably locally bi-Lipschitz trivial**.

We say that a vector field v on a subset of a set X stratified by a stratification Σ is **tangent to the strata**, when $v(x) \in T_x S$ for all $x \in S \in \Sigma$ (at which v is defined).

One of the main achievements of [17] can then be summarized as:

Theorem 7.2.10 *Every complex analytic set $X \subset \mathbb{C}^n$ admits a stratification having the following property: for every j, each locally Lipschitz vector field tangent to the strata on X^j (denoting the union of the strata of dimension not greater than j) can be extended to a locally Lipschitz vector field on X^{j+1} that is also tangent to the strata. This stratification is locally bi-Lipschitz trivial along the strata.*

The local bi-Lipschitz trivializations of Mostowski's Lipschitz stratifications are actually provided by the flow of a Lipschitz vector field tangent to the strata. More generally, the extension property of Lipschitz tangent vector fields that enjoy Mostowski's stratifications ensures a Lipschitz version of the famous Thom-Mather's First Isotopy Lemma.

Theorem 7.2.10 was extended to the real analytic and subanalytic categories by A. Parusiński [22, 23], and then to polynomially bounded o-minimal structures [18] by N. Nguyen together with the author of the present article (see also [12]). We construct in Sect. 7.6 some stratifications that are locally *definably* bi-Lipschitz trivial. We shall make use of the so-called preparation theorem for definable functions that we recall below, with a short survey on it (suggested by the referee).

7.2.2 The Preparation Theorem

This theorem was established in the subanalytic category by A. Parusiński in [23, 24] so as to construct Lipschitz stratifications for subanalytic sets. An independent proof was then given by J.-M. Lion and J.-P. Rolin [16], who also provided a generalization to the ln-exp structure (which is non polynomially bounded). It admits a version that holds on any polynomially bounded o-minimal structure, which we present now.

We thus assume for this subsection the structure to be polynomially bounded. We denote by $\mathscr{F}$ the valuation field of the structure, which is the subfield of $\mathbb{R}$ constituted by all the real numbers α for which the function $(0, +\infty) \ni x \mapsto x^\alpha \in \mathbb{R}$ is definable.

Theorem 7.2.11 *[10, Theorem 2.1] Given some definable functions $f_1, \ldots, f_l : \mathbb{R}^n \to \mathbb{R}$, there is a definable partition $\mathbb{C}$ of $\mathbb{R}^n$ such that for each set $S \in \mathbb{C}$, there are exponents $\alpha_1, \ldots, \alpha_l \in \mathscr{F}$, as well as definable functions $\theta, a_1, \ldots, a_l : \mathbb{R}^{n-1} \to \mathbb{R}$, satisfying $\Gamma_\theta \cap S = \emptyset$ and for $x = (\tilde{x}, x_n) \in S \subset \mathbb{R}^{n-1} \times \mathbb{R}$ and $i \in \{1, \ldots, l\}$:*

$$f_i(x) \sim |x_n - \theta(\tilde{x})|^{\alpha_i} a_i(\tilde{x}). \tag{7.2}$$

A function f_i having the property (7.2) is called **reduced** (in [16, 35]), and the function $U_i(x) := \frac{f_i(x)}{|x_n - \theta(\tilde{x})|^{\alpha_i} a_i(\tilde{x})}$ is called the **unit** of its reduction. The strength of Parusiński's original version of the Preparation Theorem [23, 24] lied in the very precise description of the units U_i, which could be written $\psi \circ \Phi$, with ψ analytic function on $cl(\Phi(S))$ and $\Phi = (\Phi_1, \ldots, \Phi_k)$ bounded mapping on S of the form $\Phi_j = |x_n - \theta(\tilde{x})|^{\beta_j} \cdot b_j(\tilde{x})$ (for each j), with b_j definable function and $\beta_j \in \mathscr{F}$. Theorem 7.2.11, although being much more general (it deals with an arbitrary polynomially bounded o-minimal structure), is thus a partial generalization, since it just provides units that are bounded away from zero and infinity (which is however satisfying for many purposes, see below). It was nevertheless established [18] that the units can be written as the composite of a definable Lipschitz function ψ with a mapping Φ as above. To the best knowledge of the author, it is not known whether this function ψ can always be chosen $\mathscr{C}^p$ with $p \geq 1$ (on $cl(\Phi(S))$, with Φ as above).

This precise description of the units is essential to construct Lipschitz tangent vector fields on stratified sets. Such vector fields lying in the tangent bundles of the strata, their constructions demand to have bounds for the derivative of the functions describing the strata. Nevertheless, since the present work, unlike [17, 18, 23], avoids constructing Lipschitz vector fields, we are able to spare this technical description of the units, which accounts for the fact that the far-reaching partial generalization of the Preparation Theorem given by Theorem 7.2.11 will be enough for our purpose.

The proof of Theorem 7.2.11 given in [10] relies on model theoretic principles, establishing a valuation property for definable functions which is of its own interest but which unfortunately goes beyond the scope of this survey. In the structure of semialgebraic sets [5], which is the smallest o-minimal structure (yet very valuable for applications), we however can give the following fairly short proof (taken from [30]):

Proof of Theorem 7.2.11 in the Semialgebraic Case As above, a function that satisfies (7.2) (for some semialgebraic functions a_i and θ, and $\alpha_i \in \mathbb{Q}$) will be called reduced.

We first show that the product of two reduced semialgebraic functions on a set S is reduced on every element of a semialgebraic partition of S. Take two semialgebraic functions f_1 and f_2 on a set S such that $f_1(\tilde{x}, x_n) \sim |x_n - \theta_1(\tilde{x})|^{\alpha_1} a_1(\tilde{x})$ and $f_2(\tilde{x}, x_n) \sim |x_n - \theta_2(\tilde{x})|^{\alpha_1} a_2(\tilde{x})$, for some semialgebraic functions $a_1, a_2, \theta_1, \theta_2$ and $\alpha_1, \alpha_2 \in \mathbb{Q}$. There is a semialgebraic partition $\mathscr{P}$ of S such that on each $D \in \mathscr{P}$, $(\theta_1 - \theta_2)$ has constant sign (positive, negative, or zero), and the functions $|x_n - \theta_i(\tilde{x})|, |\theta_i(\tilde{x}) - \theta_j(\tilde{x})|, i, j = 1, 2$, are comparable with each other (for $\leq$). Fix $D \in \mathscr{P}$. It is no loss of generality to assume $|x_n - \theta_2(\tilde{x})| \leq |x_n - \theta_1(\tilde{x})|$ for $(\tilde{x}, x_n) \in D$. We distinguish two cases.

Case 1: $|x_n - \theta_2| \leq |\theta_2 - \theta_1|$ on D. If $\theta_1 \equiv \theta_2$ on D then there is nothing to prove. Otherwise, either $(x_n - \theta_2)$ and $(\theta_2 - \theta_1)$ have the same sign on D or

$|\dfrac{x_n - \theta_2}{\theta_2 - \theta_1}| \le \dfrac{1}{2}$ on this set (since we assume $|x_n - \theta_2(\tilde{x})| \le |x_n - \theta_1(\tilde{x})|$). In either of these two possibilities:

$$|x_n - \theta_1| = |\theta_2 - \theta_1|(1 + \frac{x_n - \theta_2}{\theta_2 - \theta_1}) \sim |\theta_2 - \theta_1|,$$

which clearly yields that $f_1 \cdot f_2$ is reduced on D.

Case 2: $|\theta_2 - \theta_1| \le |x_n - \theta_2|$ on D. Writing now

$$x_n - \theta_1 = (x_n - \theta_2)(1 + \frac{\theta_2 - \theta_1}{x_n - \theta_2}),$$

it is easy to see that a similar argument applies (see [16] or [35, lemma 1.6.7] for more details, these tricks are actually taken from [16]) to show that $f_1 \cdot f_2$ is reduced on D.

Observe that this argument has also shown that a suitable refinement of the partition allows to assume that the reductions of two given functions involve the same θ. As we can always take a common refinement of finitely many given semialgebraic partitions, we therefore just have to show the theorem in the case of one single function f.

We start by proving the theorem in the case where f is a polynomial on a semialgebraic set S. In this case, there is a semialgebraic partition $\mathscr{P}$ of S such that on each $D \in \mathscr{P}$:

$$f(\tilde{x}, x_n) = a(\tilde{x})(x_n - \xi_1(\tilde{x})) \cdots (x_n - \xi_k(\tilde{x})) \cdot (x_n^2 + \zeta_1^2(\tilde{x})) \cdots (x_n^2 + \zeta_l^2(\tilde{x})), \qquad (7.3)$$

for some semialgebraic functions $\xi_1, \ldots, \xi_k, \zeta_1, \ldots, \zeta_l$ (a being the leading coefficient). Fix $D \in \mathscr{P}$. Partitioning D if necessary, we can assume x_n^2 and $\zeta_i^2(\tilde{x})$ to be comparable (for $\le$) on D for each i, which means that $(x_n^2 + \zeta_i^2(\tilde{x}))$ is $\sim$ on D to one of these two functions (for each i), and consequently, is reduced. By the above, since all the terms of the product displayed in the right-hand-side of (7.3) are reduced, so must be f.

We now prove the theorem for an arbitrary semialgebraic function $f : \mathbb{R}^n \to \mathbb{R}$. There is a semialgebraic partition $\mathscr{P}$ of $\mathbb{R}^n$ such that for every $S \in \mathscr{P}$, there is an $(n + 1)$-variable nonzero polynomial, say $P(x, y) = \sum_{i=0}^{d} a_i(x)y^i$, such that $P(x, f(x)) \equiv 0$ on S. Fix any $S \in \mathscr{P}$. Refining $\mathscr{P}$ if necessary, we can assume the functions $a_i(x)f(x)^i$ to be of constant sign and comparable with each other (for $\le$) on S. Let I (resp. J) denote the set of all the integers $i \le d$ such that $a_i(x)f(x)^i$ is nonnegative (resp. negative) on S, and let $i_0 \in I$ as well as $j_0 \in J$ be such that $a_{i_0}(x)f(x)^{i_0} = \max_{i \in I} a_i(x)f(x)^i$ and $-a_{j_0}(x)f(x)^{j_0} = \max_{j \in J} -a_j(x)f(x)^j$. We have on S:

$$a_{i_0}(x)f(x)^{i_0} \sim \sum_{i \in I} a_i(x)f(x)^i = -\sum_{j \in J} a_j(x)f(x)^j \sim -a_{j_0}(x)f(x)^{j_0},$$

and therefore (refining the partition, we can suppose that f has constant sign on S, we thus assume it to be positive)

$$f(x) \sim \left(\frac{-a_{j_0}(x)}{a_{i_0}(x)} \right)^{1/(i_0 - j_0)}. \tag{7.4}$$

As $a_{i_0}(x)$ and $a_{j_0}(x)$ are polynomials, by the above, refining $\mathscr{P}$ if necessary, we can assume that these are reduced functions. Clearly, $\frac{1}{a_{i_0}(x)}$ is then also reduced, and, since we have established that the product of two reduced functions can be reduced, so will be the rational fraction $a_{j_0}(x) \cdot \frac{1}{a_{i_0}(x)}$ on every $S \in \mathscr{P}$, after an extra refinement of the partition. By (7.4), this shows the desired fact. $\square$

7.3   The Regular Vector Theorem

We denote by $\mathbb{G}_k^n$ the Grassmannian of k-dimensional vector subspaces of $\mathbb{R}^n$, and we set $\mathbb{G}^n := \bigcup_{k=1}^{n} \mathbb{G}_k^n$ as well as $\mathbb{G}_*^n := \bigcup_{k=1}^{n-1} \mathbb{G}_k^n$.

Given a definable set $A \subset \mathbb{R}^n$, we denote by A_{reg} the set constituted by all the points of A at which this set is a $\mathscr{C}^1$ manifold (without boundary, of dimension $\dim A$ or smaller). Define $\tau(A)$ as the closure of the set of vector spaces that are tangent to A at a regular point, i.e.:

$$\tau(A) := cl(\{T_x A \in \mathbb{G}^n : x \in A_{reg}\}).$$

Given an element λ of $\mathbf{S}^{n-1}$ and a subset $Z \subset \mathbb{G}^n$ we set (caution, here Z is not a subset of $\mathbb{R}^n$):

$$d(\lambda, Z) := \inf\{d(\lambda, T) : T \in Z\},$$

with $d(\lambda, \emptyset) := +\infty$.

Definition 7.3.1 Let $A \in \mathscr{S}_n$. An element λ of $\mathbf{S}^{n-1}$ is said to be **regular for the set** A if there is $\alpha > 0$ such that:

$$d(\lambda, \tau(A)) \geq \alpha.$$

More generally, we say that $\lambda \in \mathbf{S}^{n-1}$ is **regular for** $A \in \mathscr{S}_{m+n}$ if there exists $\alpha > 0$ such that for any $t \in \mathbb{R}^m$:

$$d(\lambda, \tau(A_t)) \geq \alpha. \tag{7.5}$$

We then also say that λ is **regular for the family** $(A_t)_{t \in \mathbb{R}^m}$. A **subset** $C \subset \mathbf{S}^{n-1}$ **is regular** for a set $A \in \mathscr{S}_{m+n}$ if so are all the elements of $cl(C)$.

If $\lambda \in \mathbf{S}^{n-1}$ is regular for $A \in \mathcal{S}_{m+n}$, it is regular for $A_t \in \mathcal{S}_n$ for all $t \in \mathbb{R}^m$. But it is indeed even stronger since in (7.5), the angle between the vector λ and the tangent spaces to the fibers is required to be bounded below away from zero by a positive constant *independent of the parameter t*.

Regular vectors do not always exist, even if the considered set has empty interior (which is clearly a necessary condition), as it is shown by the simple example of a circle. Nevertheless, when the considered sets have empty interior, up to a definable bi-Lipschitz map, we can find such a vector:

Theorem 7.3.2 *Let $A \in \mathcal{S}_{m+n}$ such that A_t has empty interior for every $t \in \mathbb{R}^m$. There exists a uniformly bi-Lipschitz definable family of homeomorphisms $h_t :$ $\mathbb{R}^n \to \mathbb{R}^n$, $t \in \mathbb{R}^m$, such that e_n is regular for the family $(h_t(A_t))_{t \in \mathbb{R}^m}$.*

Remark 7.3.3 In the above theorem, the family h_t is not required to be Lipschitz nor continuous with respect to the parameter $t \in \mathbb{R}^m$. It is nevertheless continuous for generic parameters (see [6, Lemma 5.17] or [35, Proposition 2.4.9]). Moreover, using Proposition 7.6.5 below, one could see that, along the elements of a suitable partition of $\mathbb{R}^m$, h_t and h_t^{-1} may be required to be Lipschitz with respect to the parameters on compact sets.

In the case $m = 0$, we have the following immediate corollary which was proved in [30]:

Corollary 7.3.4 *Let $A \in \mathcal{S}_n$ be of empty interior. There exists a definable bi-Lipschitz homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that e_n is regular for $h(A)$.*

The example of a circle that we already mentioned points out the fact that it is not possible for the homeomorphism given by this corollary to be always smooth, even if so is A.

The proof of this theorem is given in Sect. 7.5.4. In order to motivate the material that we are going to introduce for this purpose in Sects. 7.4 and 7.5 (especially Definition 7.5.2 and Theorem 7.5.4), let us now give a brief outline of the construction of this homeomorphism (we assume $m = 0$ in the outline for simplicity), with explicit references to the key results and definitions.

It is actually easy to see that given $A \in \mathcal{S}_n$, there is a covering of $\mathbb{R}^n$ by finitely many sets, say $G_1, \ldots, G_l$, such that each $G_k \cap A$ has a regular vector λ_k. As a matter of fact, for each k, there is a linear automorphism h_k such that the vector e_n is regular for $h_k(G_k \cap A)$. The problem is that it is not easy to "paste" these "local embeddings" $h_1, \ldots, h_l$ into a bi-Lipschitz map $h : \mathbb{R}^n \to \mathbb{R}^n$. Somehow, the idea will be to define h on $\bigcup_{i=1}^{k} G_i$ inductively on k, by means of the h_i's, starting with $h = h_1$.

We introduce for this purpose an "induction machinery", called *regular systems of hypersurfaces* (Definition 7.5.2). Extending h from $\bigcup_{i=1}^{k} G_i$ to $\bigcup_{i=1}^{k+1} G_i$ somehow requires to change coordinates, and the transition map $h_{k+1} \circ h_k^{-1}$ can be interpreted as a turn from the direction λ_k to the direction λ_{k+1}. These turns could make it difficult to extend a bi-Lipschitz mapping to a bi-Lipschitz mapping for we might come back to our starting point. Working with a regular system of

hypersurfaces H makes it possible to turn "without turning back" (see (7.8) as well as (ii) of Definition 7.5.2), progressing in a zigzag but somehow always going toward the same Lipschitz upper half-space $G_b(H)$.

The main difficulty is therefore the proof of Theorem 7.5.4, which yields existence of a suitable regular system of hypersurfaces. In the proof of this theorem, the trick to avoid to "turn back" (in the sense of (ii) of Definition 7.5.2) is to choose λ_{k+1} in the same connected component as λ_k of the sets of all the regular vectors of the previous step (see Proposition 7.5.5). The key lemma on this issue is Lemma 7.5.10, which relies on the fact that the fiber $\tilde{\pi}_e^{-1}(\lambda)$, for each e and λ in $\mathbf{S}^{n-1}$ (see (7.9)), is a connected curve of length at least 2, which leaves enough space to choose our regular vector (see Remark 7.5.6).

The proof of Theorem 7.5.4 is splitted into four steps, and a more explicit outline of it is provided before the first step (see Sect. 7.5.3). Moreover, the second and third steps are preceded by a paragraph that motivates them and gives some more details on their proof.

Remark 7.3.5 Mostowski's work [17] involved establishing results about existence of regular projections (see also [14, 20, 21, 23]). Existence of a regular vector is closely related to existence of a regular projection but not completely equivalent [19, 20]. This is however not the main difference between Corollary 7.3.4 and the theorems of [17, 20, 21, 23]: Corollary 7.3.4 provides one single vector which is regular for the whole image of the considered set by some definable bi-Lipschitz mapping whereas the theorems of [17, 20, 21, 23] provide a finite set of projections such that, at each point of the considered set itself, at least one of them is regular.

7.4 A Few Lemmas on Lipschitz Geometry

7.4.1 Regular Vectors and Lipschitz Functions

Proving Theorem 7.3.2 will require to prove parameterized versions of all the lemmas and propositions of [30]. In [15], K. Kurdyka and A. Parusiński provided a parameterized version of the "L-regular cell decomposition theorem" [14], which enabled them to generalize their proof of Thom's gradient conjecture on o-minimal structures. We start with a result of [15] that will be useful for our purpose.

Given a and b in a definable connected set A, let:

$$d_A(a, b) := \inf\{length(\gamma) : \gamma : [0, 1] \to A, \ \mathscr{C}^0 \text{ definable arc joining } a \text{ and } b\}$$

(as definable arcs are piecewise $\mathscr{C}^1$, their length is well-defined). This defines a metric on A, generally referred as **the inner metric** of A.

To avoid any confusion, we will refer to the restriction to A of the euclidean metric as the **outer metric**. When A is smooth, a $\mathscr{C}^1$ function that has bounded derivative is Lipschitz with respect to the inner metric, but is not necessary Lipschitz

(w.r.t. the outer metric), these two metrics being not always equivalent. We however have the following result [15, Theorem 1.2]:

Theorem 7.4.1 *Every $A \in \mathcal{S}_{m+n}$ admits a definable partition into cells, such that for each $E \in \mathcal{P}$ and each $t \in \mathbb{R}^m$, the inner and outer metrics of E_t are equivalent. The constants of this equivalence just depend on n (and not on m and t).*

The techniques that we use in Sect. 7.4.2 to prove Proposition 7.4.13 are actually related to the main ideas of the proof of this theorem that is given in [15]. It is indeed possible to show the above theorem from the latter proposition, together with an induction on n. For more details we refer the reader to the latter article (see also [35, Chapter 3]).

Proposition 7.4.2 *Every definable Lipschitz function $\xi : A \to \mathbb{R}$, $A \in \mathcal{S}_n$, can be extended to an L_ξ-Lipschitz definable function $\tilde{\xi} : \mathbb{R}^n \to \mathbb{R}$.*

Proof Set $\widetilde{\xi}(q) := \inf\{\xi(p) + L_\xi |q - p| : p \in A\}$. By the quantifier elimination principle, it is a definable function. An easy computation shows that it is L_ξ-Lipschitz. □

Remark 7.4.3 Let $A \in \mathcal{S}_{m+n}$ and let a definable function $\xi : A \to \mathbb{R}$ be such that $\xi_t : A_t \to \mathbb{R}$ is a Lipschitz function for every $t \in \mathbb{R}^m$. The respective extensions $\tilde{\xi}_t$ of ξ_t, $t \in \mathbb{R}^m$ (with for instance $\tilde{\xi}_t \equiv 0$ if $t \notin \text{supp}_m A$), provided by the proof of the above proposition constitute a definable family of functions. We thus can extend definable families of Lipschitz functions to definable families of Lipschitz functions. This will be of service.

Lemma 7.4.4 *Let A and B in $\mathcal{S}_{n+m}$ with $B \subset A$. If $\lambda \in \mathbf{S}^{n-1}$ is regular for A, then it is regular for B.*

Proof Assume that $\lambda \in \mathbf{S}^{n-1}$ is not regular for B. It means that there is a sequence $((t_i, b_i))_{i \in \mathbb{N}}$, with $b_i \in B_{t_i,reg}$ such that $\tau := \lim T_{b_i} B_{t_i,reg}$ exists and contains λ. Choose for every i a Whitney (a) regular definable stratification of A_{t_i} (see for instance [5, 35] for the definition) compatible with B_{t_i} and $B_{t_i,reg}$ and denote by S_i the stratum containing b_i. Moving slightly b_i if necessary, we may assume that S_i is open in $B_{t_i,reg}$ (since $B_{t_i,reg}$ is open and dense in B_{t_i}), which entails that $T_{b_i} S_i = T_{b_i} B_{t_i,reg}$. As $A_{t_i,reg}$ is dense in A_{t_i}, for every $i \in \mathbb{N}$, we can find a_i in $A_{t_i,reg}$, which is close to b_i. Moreover, possibly extracting a sequence, we may assume that $\tau' := \lim T_{a_i} A_{t_i,reg}$ exists. If a_i is sufficiently close to b_i, by Whitney (a) condition, we deduce that $\tau' \supset \tau$, which contains λ. This yields that λ is not regular for A. □

Remark 7.4.5 It is worthy of notice that the proof of the above lemma has established that the corresponding number α (see (7.5)) can remain the same for B.

Given $\lambda \in \mathbf{S}^{n-1}$, we denote by $\pi_\lambda : \mathbb{R}^n \to N_\lambda$ the orthogonal projection onto the hyperplane N_λ normal to the vector λ, and by q_λ the coordinate of $q \in \mathbb{R}^n$ along λ, i.e. the number given by the euclidean inner product of q with λ.

Given $B \in \mathcal{S}_n$ and $\lambda \in \mathbf{S}^{n-1}$, with $B \subset N_\lambda$, as well as a function $\xi : B \to \mathbb{R}$, we set

$$\Gamma_\xi^\lambda := \{q \in \mathbb{R}^n : \pi_\lambda(q) \in B \quad \text{and} \quad q_\lambda = \xi(\pi_\lambda(q))\}, \tag{7.6}$$

and call this set **the graph of ξ for λ**.

Proposition 7.4.6 *The vector $\lambda \in \mathbf{S}^{n-1}$ is regular for the set $A \in \mathcal{S}_{m+n}$ if and only if there are finitely many uniformly Lipschitz definable families of functions $\xi_{i,t} : B_{i,t} \to \mathbb{R}$, $t \in \mathbb{R}^m$, with $B_i \subset \mathbb{R}^m \times N_\lambda$, $i = 1, \ldots, p$, such that for all $t \in \mathbb{R}^m$:*

$$A_t = \bigcup_{i=1}^{p} \Gamma_{\xi_{i,t}}^\lambda.$$

Proof As the "if" part is clear, we will focus on the converse. Up to an orthogonal linear mapping we can assume that $\lambda = e_n$. Let $A \in \mathcal{S}_{m+n}$. Take a cell decomposition compatible with A and let C be a cell included in A. This cell cannot be a band since e_n is regular for A (see Lemma 7.4.4). It is thus the graph of a $\mathscr{C}^1$ function $\xi : D \to \mathbb{R}$, with $D \in \mathcal{S}_{m+n-1}$, such that ξ_t has bounded first derivative (independently of t). It therefore must be uniformly Lipschitz with respect to the inner metric. It follows from Theorem 7.4.1 that there is a definable partition $\mathscr{P}$ of D such that for each $E \in \mathscr{P}$, the inner metric of E_t is equivalent to its outer metric for all $t \in \mathbb{R}^m$, with constants that just depend on n. The family of functions ξ induces a uniformly Lipschitz family of functions on every element of $\mathscr{P}$. $\square$

We finish this subsection with an elementary proposition that will be of service to prove Theorem 7.5.4. This proposition yields that we can replace a given collection of families of Lipschitz functions with an increasing collection of families of Lipschitz functions $\xi_{1,t} \leq \cdots \leq \xi_{k,t}$ in such a way that the union of the graphs is unchanged:

Proposition 7.4.7 *Let $f_{1,t}, \ldots, f_{k,t}$, $t \in \mathbb{R}^m$, be definable families of functions on N_λ, $\lambda \in \mathbf{S}^{n-1}$, and let $L \in \mathbb{R}$. Assume that for all $i \leq k$ and for all $t \in \mathbb{R}^m$, the function $f_{i,t}$ is L-Lipschitz. Then, there exist some definable families of functions $\xi_{1,t}, \ldots, \xi_{k,t}$ on N_λ such that for all $t \in \mathbb{R}^m$*

(i) $\xi_{i,t}$ is L-Lipschitz and all $i \leq k$.
(ii) $\bigcup_{i=1}^{k} \Gamma_{\xi_{i,t}}^\lambda = \bigcup_{i=1}^{k} \Gamma_{f_{i,t}}^\lambda$.
(iii) $\xi_{1,t} \leq \cdots \leq \xi_{k,t}$.

Proof Up to an orthogonal linear mapping with may assume $\lambda = e_n$. We are going to define inductively on j some definable integer valued functions $i_j : \mathbb{R}^m \times \mathbb{R}^{n-1} \to \mathbb{R}$, $j = 1, \ldots, k$ such that for every $t \in \mathbb{R}^m$ and $j \leq k$, the functions

$$\xi_{j,t}(x) := f_{i_j(t,x),t}(x) \tag{7.7}$$

are L-Lipschitz functions satisfying $\xi_{1,t} \leq \cdots \leq \xi_{k,t}$. Indeed, let $i_1(t, x) := \min\{i \leq k : f_{i,t}(x) = \min_{l \leq k} f_{l,t}(x)\}$. Then, assuming that $i_1, \ldots, i_{j-1}$ have been defined, let

$$i_j(t, x) := \min\{i \in I_j(t, x) : f_{i,t}(x) = \min_{l \in I_j(t,x)} f_{l,t}(x)\},$$

where $I_j(t, x)$ is the set constituted by the positive integers which are not greater than k and different from $i_1(t, x), \ldots, i_{j-1}(t, x)$. We clearly have $\xi_{1,t}(x) \leq \cdots \leq \xi_{k,t}(x)$ if $\xi_{j,t}(x)$ is defined as in (7.7).

Take a cell decomposition $\mathbb{C}$ of $\mathbb{R}^m \times \mathbb{R}^{n-1}$ such that the functions $(f_{j,t}(x) - f_{j',t}(x))$ have constant sign (positive, negative, or zero) on every cell and observe that, since the i_j's are constant on every cell, they are definable.

By construction, we have $\bigcup_{j=1}^k \Gamma_{\xi_{j,t}} = \bigcup_{j=1}^k \Gamma_{f_{j,t}}$, for all $t \in \mathbb{R}^m$, which entails that e_n is regular for the graphs of the families $\xi_{j,t}$, $t \in \mathbb{R}^m$. As a matter of fact, for each j, in order to show that $\xi_{j,t}$ is L-Lipschitz, it suffices to establish that the functions $\xi_{j|C_t}$, $C \in \mathbb{C}$, glue together into a continuous function on $\mathbb{R}^{n-1}$ for every t, which is left to the reader. $\qquad\square$

7.4.2 Finding Regular Directions

Lemma 7.4.8 *Given $\nu \in \mathbb{N}$, there exists $t_\nu > 0$ such that for any $P_1, \ldots, P_\nu$ in $\mathbb{G}_*^n$ there exists a vector $\lambda \in \mathbf{S}^{n-1}$ such that for any i:*

$$d(\lambda, P_i) > t_\nu.$$

Proof Given $P_1, \ldots, P_\nu$ in $\mathbb{G}_*^n$, let $\varphi(P_1, \ldots, P_\nu) := \sup_{\lambda \in \mathbf{S}^{n-1}} \min_{i \leq \nu} d(\lambda, P_i)$. Since the P_i's have positive codimension, φ is a positive function, which, since the Grassmannian is compact, must be bounded below away from zero. $\qquad\square$

The next lemma is a refinement of the just above lemma which says that the vector λ can be chosen among finitely many ones.

Lemma 7.4.9 *Given $\nu \in \mathbb{N}$, there exist $\lambda_1, \ldots, \lambda_N$ in $\mathbf{S}^{n-1}$ and $\alpha_\nu > 0$ such that for any $P_1, \ldots, P_\nu$ in $\mathbb{G}_*^n$ we may find $i \leq N$ such that for any $j \leq \nu$:*

$$d(\lambda_i, P_j) > \alpha_\nu.$$

Proof Let t_ν be the real number given by Lemma 7.4.8 and let $\lambda_1, \ldots, \lambda_N$ in $\mathbf{S}^{n-1}$ be such that $\bigcup_{i=1}^N \mathbf{B}(\lambda_i, \frac{t_\nu}{2}) \supset \mathbf{S}^{n-1}$. Suppose that there are $P_1, \ldots, P_\nu$ in $\mathbb{G}_*^n$ such

that for any $i \in \{1, \ldots N\}$ we have $d(\lambda_i, \bigcup_{j=1}^{v} P_j) \leq \frac{t_v}{2}$. This implies that any λ in $\mathbf{S}^{n-1}$ satisfies

$$d(\lambda, \bigcup_{j=1}^{v} P_j) < t_v,$$

contradicting Lemma 7.4.8. It is thus enough to set $\alpha_v := \frac{t_v}{2}$. $\square$

The next lemma will require a definition. We estimate the angle between two vector subspaces P and Q of $\mathbb{R}^n$ in the following way:

$$\angle(P, Q) = \sup\{d(\lambda, Q) : \lambda \text{ is a unit vector of } P\}.$$

This constitutes a metric on each $\mathbb{G}_k^n$, $k \leq n$.

Definition 7.4.10 Let $\alpha > 0$ and $Z \in \mathcal{S}_{m+n}$. We say that the family $(Z_t)_{t \in \mathbb{R}^m}$ is α-**flat** if:

$$\sup\{\angle(P, Q) : P, Q \in \bigcup_{t \in \mathbb{R}^m} \tau(Z_{t,reg})\} \leq \alpha.$$

We then also say that Z is (m, α)-**flat**. When $m = 0$, we say that Z is α-**flat**.

If P and Q are two vector subspaces of $\mathbb{R}^n$ satisfying $\dim P > \dim Q$ then $\angle(P, Q) = 1$. As a matter of fact, if Z is (m, α)-flat for some $\alpha < 1$, then Z_t must be of pure dimension for all t.

Remark 7.4.11 It follows from Lemma 7.4.9 that if $Z_{1,t}, \ldots, Z_{v,t}, t \in \mathbb{R}^m$, are α_v-flat definable families (where α_v is the constant provided by the latter lemma) of subsets of $\mathbb{R}^n$ of empty interiors then one of the λ_i's (that are also provided by the latter lemma) is regular for all these families.

Lemma 7.4.12 *Given $Z \in \mathcal{S}_{m+n}$ and $\alpha > 0$, we can find a finite partition of Z into (m, α)-flat sets.*

Proof Dividing Z into cells, we may assume that Z_t is a manifold for all $t \in \mathbb{R}^m$. We can cover the Grassmannian by finitely many balls of radius $\frac{\alpha}{2}$, which gives rise to a covering $U_1, \ldots, U_k$ of Z (via the family of mappings $Z_t \ni x \mapsto T_x Z_t$) by (m, α)-flat sets. $\square$

This leads us to the following result that originates in [30] and that will be of service in Sect. 7.6. It is closely related to the L-regular cell decompositions introduced and constructed in [14]. The difference is that we wish that the regular vector for δC can be chosen among finitely many ones. This result was then improved by W. Pawłucki [25] who has shown that we can require in addition $N = n$.

Proposition 7.4.13 *There exist $\lambda_1, \ldots, \lambda_N$ in $\mathbf{S}^{n-1}$ such that for any $A_1, \ldots, A_p$ in $\mathcal{S}_{m+n}$, there is a cell decomposition $\mathbb{C}$ of $\mathbb{R}^{m+n}$ compatible with all the A_k's and such that for each cell $C \in \mathbb{C}$ satisfying $\dim C_t = n$ (for all $t \in supp_m C$), we may find $\lambda_{j(C)}$, $1 \le j(C) \le N$, regular for the family $(\delta C_t)_{t \in \mathbb{R}^m}$.*

Proof According to Lemma 7.4.9 (see Remark 7.4.11 and Lemma 7.4.4) it is sufficient to prove by induction on n the following assertions: given $\alpha > 0$ and $A_1, \ldots, A_p$ in $\mathcal{S}_{m+n}$, there exists a cell decomposition of $\mathbb{R}^{m+n}$ compatible with $A_1, \ldots, A_p$ and such that for every cell $C \subset \mathbb{R}^{m+n}$ of this cell decomposition satisfying $\dim C_t = n$, $(\delta C_t)_{t \in \mathbb{R}^m}$ is included in the union of no more than $2n$ definable families of empty interior that are all α-flat.

For $n = 0$ this is clear. Fix $n \in \mathbb{N}$ nonzero, $\alpha > 0$, as well as $A_1, \ldots, A_p$ in $\mathcal{S}_{m+n}$. Taking a cell decomposition if necessary, we can assume that the A_i's are cells. Apply Lemma 7.4.12 to all the A_i's, and take a cell decomposition $\mathcal{D}$ of $\mathbb{R}^{m+n}$ compatible with all the elements of the obtained coverings. Applying then the induction hypothesis to the elements of $\pi_{e_{m+n}}(\mathcal{D})$, we get a refinement $\mathcal{D}'$ of $\pi_{e_{m+n}}(\mathcal{D})$.

Given a cell D of $\mathcal{D}'$, each A_i is above D, either the graph of a definable function, say $\xi_{i,D}$, or a band, say $(\xi_{i,D}, \xi'_{i,D})$, with $\xi_{i,D} < \xi'_{i,D}$ definable functions on D (or $\pm\infty$). Let $\mathbb{C}$ be the cell decomposition given by all the graphs $\Gamma_{\xi_{i,D}}$ and $\Gamma_{\xi'_{i,D}}$, $i \le p$, $D \in \mathcal{D}'$. To check that it has the required property, fix an open cell $C = (\xi_{i,D}, \xi'_{i,D})$, with $\xi_{i,D} < \xi'_{i,D}$ definable functions on an open cell D of $\mathcal{D}'$ (or $\pm\infty$). Since $\mathcal{D}'$ is compatible with the images under $\pi_{e_{m+n}}$ of the α-flat sets that cover the A_i's, the sets $\Gamma_{\xi_{i,D}}$ and $\Gamma_{\xi'_{i,D}}$ must be α-flat families, and since

$$\delta C_t \subset \left(\Gamma_{\xi_{i,D}}\right)_t \cup \left(\Gamma_{\xi'_{i,D}}\right)_t \cup \pi_{e_n}^{-1}(\delta D_t),$$

we see that the needed fact follows from the induction hypothesis. $\square$

Remark 7.4.14 We have proved a stronger statement: the distance between the regular vector $\lambda_{j(C)}$ and the tangent spaces to δC_t can be bounded below away from zero by a positive number depending only on n, and not on the A_k's. This is due to the fact that we apply Lemma 7.4.9 with $\nu = 2n$.

7.5 Regular Systems of Hypersurfaces

This section is entirely devoted to the proof of Theorem 7.3.2 which requires some material. We first introduce our machinery of regular systems of hypersurfaces.

Let $Z \in \mathcal{S}_n$ and $\lambda \in \mathbf{S}^{n-1}$, with $Z \subset N_\lambda$ (see Sect. 7.4.1 for N_λ). If $A \in \mathcal{S}_n$ is the graph of a function $\xi : Z \to \mathbb{R}$ for λ, we denote by $E(A, \lambda)$ the subset constituted by the points which lie "under the graph", i.e. we set:

$$E(A, \lambda) := \{q \in \pi_\lambda^{-1}(Z) : q_\lambda \le \xi(\pi_\lambda(q))\}. \tag{7.8}$$

Remark 7.5.1 If $A \in \mathscr{S}_{m+n}$ is such that A_t is the graph for λ of a function $\xi_t :$ $N_\lambda \to \mathbb{R}$ for every $t \in \mathbb{R}^m$, then $E(A_t, \lambda)$, $t \in \mathbb{R}^m$, is a definable family of sets of $\mathbb{R}^m \times \mathbb{R}^n$. Indeed, regarding λ as an element of $\mathbf{S}^{n+m-1}$ (i.e., identifying λ with $(0_{\mathbb{R}^m}, \lambda)$), $E(A, \lambda)$ is also well-defined and $E(A, \lambda)_t = E(A_t, \lambda)$, for all $t \in \mathbb{R}^m$.

7.5.1   Regular Systems of Hypersurfaces

Regular systems of hypersurfaces will help us to carry out constructions inductively on the dimension of the ambient space.

Definition 7.5.2 Let $B \in \mathscr{S}_m$. A **regular system of hypersurfaces** of $B \times \mathbb{R}^n$ (parametrized by B) is a finite collection $H = (H_k, \lambda_k)_{1 \le k \le b}$ with $b \in \mathbb{N}$, of definable subsets H_k of $B \times \mathbb{R}^n$ and elements $\lambda_k \in \mathbf{S}^{n-1}$ such that the following properties hold for each $k < b$ and every $t \in B$:

 (i) The sets $H_{k,t}$ and $H_{k+1,t}$ are the respective graphs for λ_k of two functions $\xi_{k,t} :$ $N_{\lambda_k} \to \mathbb{R}$ and $\xi'_{k,t} : N_{\lambda_k} \to \mathbb{R}$ such that $\xi_{k,t} \le \xi'_{k,t}$ and which are C-Lipschitz with $C \in \mathbb{R}$ independent of t.
 (ii) The following equality holds:

$$E(H_{k+1,t}, \lambda_k) = E(H_{k+1,t}, \lambda_{k+1}).$$

Let $A \in \mathscr{S}_{m+n}$. We say that H is **compatible** with A, if $A \subset \bigcup_{k=1}^b H_k$. An **extension** of H is a regular system of hypersurfaces (of $B \times \mathbb{R}^n$) compatible with the set $\bigcup_{k=1}^b H_k$.

Given a positive integer $k < b$, we set:

$$G_k(H) := E(H_{k+1}, \lambda_k) \setminus int(E(H_k, \lambda_k)).$$

We shall write $\Lambda_k(H)$ for the connected component of the set

$$\{\lambda \in \mathbf{S}^{n-1} : \lambda \text{ is regular for } H_k \cup H_{k+1}\}$$

that contains λ_k.

We will see (Proposition 7.5.5 below) that the set $G_k(H)$ may be defined using any $\lambda \in \Lambda_k(H)$ (instead of λ_k).

We will say that another regular system H' **coincides with H outside** $G_k(H)$ if for each j either $H'_j \subset G_k(H)$ or there exists j' such that $H'_j = H_{j'}$.

Given a regular system $H := (H_k, \lambda_k)_{k \le b}$ of $B \times \mathbb{R}^n$ and a definable set $B' \subset B$, we denote by $H_{B'}$ the regular system of hypersurfaces $(H_{k,B'}, \lambda_k)$ of $B' \times \mathbb{R}^n$, obtained by considering the sequence of the respective restrictions to B' of the H_k's (see (7.1)). We will call it the **restriction to B'** of the regular system H.

Remark 7.5.3 It is always possible to assume that the $G_k(H)_t$'s are of nonempty interior for some t. Indeed, if $int(G_k(H)_t) = \emptyset$ for all $t \in B$, then $H_k = H_{k+1}$ and in this case we may remove (H_k, λ_k) from the sequence.

Given a regular system of hypersurfaces (of $B \times \mathbb{R}^n$, $B \in \mathscr{S}_m$) H, it will be convenient to extend the notations in the following way. Set for any $t \in B$: $H_{0,t} := \{-\infty\}$ and $H_{b+1,t} := \{+\infty\}$. By convention, all the elements of $\mathbf{S}^{n-1}$ will be regular for these two sets. We will also consider that these two sets as the respective graphs of the two functions which take $-\infty$ and $+\infty$ as respective constant values. Define also $\lambda_0 := \lambda_1$, $\lambda_{b+1} := \lambda_b$, as well as $E(H_0, \lambda_0) := \emptyset$, $G_0(H) := E(H_1, \lambda_1)$, $G_b(H) := (B \times \mathbb{R}^n) \setminus int(E(H_b, \lambda_b))$, as well as $E(H_{b+1}, \lambda_{b+1}) := B \times \mathbb{R}^n$. Remark that now $B \times \mathbb{R}^n = \bigcup_{k=0}^b G_k(H)$.

Theorem 7.5.4 *Let $A \in \mathscr{S}_{m+n}$ be such that A_t has empty interior for all $t \in \mathbb{R}^m$. There exists a definable partition $\mathscr{P}$ of $\mathbb{R}^m$ such that for every $B \in \mathscr{P}$ there is a regular system of hypersurfaces of $B \times \mathbb{R}^n$ compatible with A_B.*

This theorem is the main ingredient of the proof of Theorem 7.3.2. The basic strategy of the proof of Theorem 7.5.4 (given in Sect. 7.5.3) relies on the following observation.

Proposition 7.5.5 *Let U be a connected subset of $\mathbf{S}^{n-1}$, $\lambda_0 \in U$, and let $\xi : N_{\lambda_0} \to \mathbb{R}$ be a continuous definable function. If U is regular for $X := \Gamma_\xi^{\lambda_0}$ then, for each $\lambda \in U$, the set X is the graph for λ of a function $\xi^\lambda : N_\lambda \to \mathbb{R}$. Moreover, $E(X, \lambda)$ is independent of $\lambda \in U$.*

Proof Let

$$C := \{\lambda \in U : \forall x \in N_\lambda, \quad card\, \pi_\lambda^{-1}(x) \cap X = 1\}.$$

We have to check that $C = U$. Let $\lambda \in C$ and set $r(\lambda) := d(\lambda, \tau(X))$.

We claim that $\mathbf{B}(\lambda, \frac{r(\lambda)}{3}) \cap U \subset C$. Pick $\lambda' \in \mathbf{B}(\lambda, \frac{r(\lambda)}{3}) \cap U$ different from λ and set $l' = \pi_\lambda(\lambda')$. We have to show that the line L generated by λ' and passing through any $x \in N_\lambda$ intersects X in exactly one point. The line L is the graph for λ of a function $\eta(x + tl') = \alpha \cdot t$ with $\alpha > \frac{2}{r(\lambda)}$ (we assume $\alpha > 0$ for simplicity).

Since $\lambda \in C$, the set X is the graph for λ of a function ξ^λ. By definition of $r(\lambda)$, $|d_x \xi^\lambda|$ (which exists almost everywhere) is bounded by $\frac{2}{r(\lambda)}$. It easily follows from the Mean Value Theorem that ξ^λ is $\frac{2}{r(\lambda)}$-Lipschitz.

This implies that for t positive large enough we will have $\eta(x + tl') \geq \xi^\lambda(x + tl')$ and $\eta(x - tl') \leq \xi^\lambda(x - tl')$ (since η is growing faster than ξ^λ). Thus, there is a point $q \in \pi_\lambda(L)$ such that $\xi^\lambda(q) = \eta(q)$, which implies that the line L cuts X. Uniqueness of the intersection point is clear from the fact that one function is growing faster than the other. This yields that $\mathbf{B}(\lambda, \frac{r(\lambda)}{3}) \cap U \subset C$.

We have shown that C is open in U. Let us now show that it is also closed in U. Consider $\lambda \in U$ and a continuous definable arc γ in C tending to λ. Since $r(\gamma(t))$ tends to $r(\lambda)$ which is nonzero, the ball $\mathbf{B}(\gamma(t), \frac{r(\gamma(t))}{3})$ contains λ for $t > 0$ small

enough. The closeness of C therefore follows from the fact that $\mathbf{B}(\gamma(t), \frac{r(\gamma(t))}{3}) \cap U \subset C$. As U is connected, this yields $U = C$.

It remains to check that $E(X, \lambda)$ is independent of $\lambda \in U$. It is the closure of one of the two connected components of the complement of X. The set X is the zero locus of the function $f(q) = q_{\lambda_0} - \xi(\pi_{\lambda_0}(q))$. Locally, at a smooth point q of X it is clear that $E(X, \lambda)$ is determined by the sign of $d_q f(\lambda)$. But as U is regular for X, this function never vanishes, and consequently $E(X, \lambda)$ is independent of $\lambda \in U$.

$\square$

Given $e \in \mathbf{S}^{n-1}$, we define a mapping $\widetilde{\pi}_e : \mathbf{S}^{n-1} \setminus \{\pm e\} \to \mathbf{S}^{n-1} \cap N_e$ by setting

$$\widetilde{\pi}_e(u) := \frac{\pi_e(u)}{|\pi_e(u)|}. \tag{7.9}$$

Remark 7.5.6 Let $e \in \mathbf{S}^{n-1}$ and suppose $\lambda_0 \in \mathbf{S}^{n-1} \cap N_e$ to be regular for a subset $A \subset N_e$. Since the elements of $\widetilde{\pi}_e^{-1}(\lambda_0)$ lie above the line generated by λ_0, for each positive real number a, the set

$$C := \widetilde{\pi}_e^{-1}(\lambda_0) \cap \{\lambda \in \mathbf{S}^{n-1} : d(\lambda, \{\pm e\}) \geq a\}$$

is regular for $\pi_e^{-1}(A)$. Moreover, by Proposition 7.5.5, if A is the graph of a Lipschitz function for λ_0 then $\pi_e^{-1}(A)$ is the graph of a Lipschitz function for each $\lambda \in C$. Furthermore, the latter proposition also entails that in this case we have for all $\lambda \in C$:

$$E(\pi_e^{-1}(A), \lambda) = \pi_e^{-1}(E(A, \lambda_0)).$$

7.5.2 Two Preliminary Lemmas

Proving Theorem 7.5.4 requires two preliminary lemmas on regular systems of hypersurfaces. The first one will make it possible for us to assume that the interiors of the $G_k(H)_t$'s are connected.

Lemma 7.5.7 *Let H be a regular system of hypersurfaces of $B \times \mathbb{R}^n$, $B \in \mathcal{S}_m$. There exists a definable partition $\mathcal{P}$ of B such that for every $B' \in \mathcal{P}$, we can find an extension $\widehat{H}$ of $H_{B'}$ such that the set $int(G_k(\widehat{H})_t)$ is connected for all $t \in B'$ and all k.*

Proof Let $1 \leq k \leq b-1$ and suppose that there is t for which $int(G_k(H)_t)$ is not connected. Applying Remark 7.2.6 to $int(G_k(H))$ provides a partition $\mathcal{P}$ of B. Let $B' \in \mathcal{P}$. Possibly replacing H with $H_{B'}$, we see that we can assume that the the property displayed in Remark 7.2.6 holds for $int(G_k(H))$.

Let $A_1, \ldots, A_\nu$ be the connected components of $int(G_k(H))$. Set $A'_i = \pi_{\lambda_k}(A_i)$ for $i \leq \nu$. For $t \in B'$, every fiber $A_{i,t}$ is of the form:

$$\{q \in A'_{i,t} \oplus \mathbb{R} \cdot \lambda_k \ : \ \xi_{k,t}(\pi_{\lambda_k}(q)) < q_{\lambda_k} < \xi'_{k,t}(\pi_{\lambda_k}(q))\}.$$

Clearly $\xi_{k,t} = \xi'_{k,t}$ on the boundary of $A'_{i,t}$. We thus may define some Lipschitz functions η_i, $1 \leq i \leq \nu - 1$, as follows. We set over $A'_{j,t}$, $\eta_{i,t} := \xi'_{k,t}$, when $1 \leq j \leq i$, and $\eta_{i,t} := \xi_{k,t}$ whenever $i < j$. Extend the function $\eta_{i,t}$ by setting $\eta_{i,t} := \xi_{k,t} = \xi'_{k,t}$ on $N_{\lambda_k} \setminus \pi_{\lambda_k}(int(G_k(H)))$.

Since $\eta_{1,t} \leq \cdots \leq \eta_{(\nu-1),t}$, it suffices to

- let $\widehat{H}_j := H_j$ and $\widehat{\lambda}_j := \lambda_j$ if $j \leq k$
- let $\widehat{H}_{j,t}$ be the graph of $\eta_{j-k,t}$ for λ_k (for every $t \in \mathbb{R}^m$) and $\widehat{\lambda}_j := \lambda_k$ for $k + 1 \leq j \leq k + \nu - 1$
- let $\widehat{H}_j := H_{j-\nu+1}$ and $\widehat{\lambda}_j := \lambda_{j-\nu+1}$ if $k + \nu \leq j \leq b + \nu - 1$.

This is clearly a regular system of hypersurfaces. Note that the $int(G_j(\widehat{H}))$, $k \leq j < k + \nu$, are the connected components of $int(G_k(H))$. $\qquad \square$

Lemma 7.5.8 *Let $H = (H_k, \lambda_k)_{1 \leq k \leq b}$ be a regular system of hypersurfaces of $B \times \mathbb{R}^n$, $B \in \mathscr{S}_m$, and let $j \in \{1, \ldots, b\}$. Let X be a definable subset of $G_j(H)$ and assume that λ_j is regular for X. Then H can be extended to a regular system of hypersurfaces H' compatible with X and which coincides with H outside $G_j(H)$.*

Proof By property (i) of Definition 7.5.2, for every parameter $t \in B$, the sets $H_{j,t}$ and $H_{j+1,t}$ are the respective graphs for λ_j of two functions $\xi_{j,t}$ and $\xi'_{j,t}$. By Propositions 7.4.6 and 7.4.2, the definable family X_t, $t \in B$, may be included in a finite number of graphs for λ_j of definable families of functions on N_{λ_j}, say $\theta_{1,t}, \ldots, \theta_{\nu,t}$, C-Lipschitz for every $t \in B$ with $C \in \mathbb{R}$ independent of t. Furthermore, by Proposition 7.4.7, these families of functions can be assumed to be totally ordered (for relation $\leq$), and satisfy $\xi_{j,t} \leq \theta_{i,t} \leq \xi'_{j,t}$, for every t. Now,

- let $H'_k := H_k$ and $\lambda'_k := \lambda_k$ whenever $1 \leq k \leq j$,
- let $H'_{k,t}$ be the graph of $\theta_{k-j,t}$ for λ_j and $\lambda'_k := \lambda_j$ for $j < k \leq j + \nu, t \in \mathbb{R}$,
- let $H'_k := H_{k-\nu}$ and $\lambda'_k := \lambda_{k-\nu}$, whenever $j + 1 + \nu \leq k \leq b + \nu$.

Properties (i) and (ii) of Definition 7.5.2 clearly hold by construction. $\qquad \square$

Remark 7.5.9 It will be of service that, in the proof of the above lemma, no extra regular vector was necessary, i.e. $\{\lambda_1, \ldots, \lambda_b\} = \{\lambda'_1, \ldots, \lambda'_{b+\nu}\}$.

7.5.3 Proof of Theorem 7.5.4

The proof is divided into four steps. The strategy is to rely on Lemma 7.5.8. The reader is invited to first glance at Step 4, which was deliberately made very short and sheds light on the reason why this lemma is helpful. The problem to get a

regular system of hypersufaces compatible with a set A using this lemma is that it requires to already have a regular system $H = (H_k, \lambda_k)_{1 \leq k \leq b}$ such that λ_j is regular for $A \cap G_j(H)$ (for all j). This fact is not granted by the system H provided by Step 1, which satisfies a slightly weaker property. We therefore shall provide (in Step 2) another system $\widehat{H}$ (see the paragraph just before Step 2 for more details on this issue) and then construct in Step 3 what can be considered as a "common refinement" of H and $\widehat{H}$, which will be satisfying to apply Lemma 7.5.8 in Step 4.

Let $A \in \mathcal{S}_{m+n}$ be such that A_t has empty interior for every $t \in \mathbb{R}^m$. Let also $\eta \in (0, 1]$ and $\lambda \in \mathbf{S}^{n-1}$. We are actually going to prove by induction on n that, given any such A, λ, and η, there exists a definable partition $\mathscr{P}$ of $\mathbb{R}^m$ such that for every $B \in \mathscr{P}$ we can find a regular system of hypersurfaces of $B \times \mathbb{R}^n$ compatible with A_B and such that all the λ_k's (see Definition 7.5.2) can be chosen in $\mathbf{B}(\lambda, \eta) \cap \mathbf{S}^{n-1}$.

Since the result is clear for $n = 1$, we take $n \geq 2$ and assume it to be true for $(n-1)$. Observe that it is enough to establish the claimed statement for arbitrarily small values of η. As explained just above, we split the induction step into 4 steps.

Step 1 There exists a definable partition $\mathscr{P}$ of $\mathbb{R}^m$ such that for every $B \in \mathscr{P}$, there is a regular system of hypersurfaces $H = (H_k, \lambda_k)_{1 \leq k \leq b}$ of $B \times \mathbb{R}^n$, with $\lambda_k \in \mathbf{S}^{n-1} \cap \mathbf{B}(\lambda, \frac{\eta}{2})$, such that for every k the set $int(G_k(H)) \cap A_B$ has a regular vector $\mu \in \mathbf{S}^{n-1} \setminus \mathbf{B}(\pm\lambda, \eta)$.

Proof of Step 1 Take $e \in \mathbf{S}^{n-1}$ such that $\pm e \notin \mathbf{B}(\lambda, \eta)$. By Lemma 7.4.12, for each $\sigma > 0$, there are finitely many $(m, \frac{\sigma}{2})$-flat sets $U_1, \ldots, U_\omega$ that cover A. Consider such a partition for $\sigma = t_\nu$, where t_ν is given by Lemma 7.4.8, with ν equal to the maximal number of connected components of $\pi_e^{-1}(x) \cap A_t$, $(t, x) \in \mathbb{R}^m \times N_e$. Changing η, we may assume that $\eta \leq \frac{t_\nu}{4}$.

Take a cell decomposition of $\mathbb{R}^m \times N_e$ (identify $\mathbb{R}^m \times N_e$ with $\mathbb{R}^m \times \mathbb{R}^{n-1}$) which is compatible with the $\pi_e(U_i)$, $i \leq \omega$, and denote by $(W_i)_{i \in I}$ the collection of the cells of this cell decomposition that are open (in $\mathbb{R}^m \times N_e$).

Since the set A_t has empty interior for each $t \in \mathbb{R}^m$, the set $A_t \cap \pi_e^{-1}(W_{i,t})$ is (for each $i \in I$ and t), above $W_{i,t}$, the union of at most ν (possibly 0) graphs for e of continuous functions (not necessarily Lipschitz).

Choose $\eta' > 0$ such that we have in $\mathbf{S}^{n-1} \cap N_e$:

$$\mathbf{B}(\widetilde{\pi}_e(\lambda), \eta') \subset \widetilde{\pi}_e(\mathbf{B}(\lambda, \frac{\eta}{2})). \tag{7.10}$$

Apply the induction hypothesis (identify $\mathbb{R}^m \times N_e$ with $\mathbb{R}^m \times \mathbb{R}^{n-1}$) to the families $(\delta W_{i,t})_{t \in \mathbb{R}^m}$ to get a partition $\mathscr{P}$. Fix $B \in \mathscr{P}$. There is a regular system of $B \times \mathbb{R}^{n-1}$, $\overline{H} = (\overline{H}_k, \overline{\lambda}_k)_{k \leq b}$, such that all the $\overline{\lambda}_k$'s belong to $\mathbf{B}(\widetilde{\pi}_e(\lambda), \eta')$.

By Lemma 7.5.7, up to a refinement of the partition, we may assume that each $int(G_k(\overline{H})_t)$ is connected for all $t \in B$. We may also assume these sets to be of nonempty interior for some t (see Remark 7.5.3). Up to an extra refinement, we may assume that it happens for all $t \in B$ (by Remark 7.2.6).

We claim that for each j and k and for every t, either $int(G_k(\overline{H})_t)$ is disjoint from $W_{j,t}$ or $int(G_k(\overline{H})_t) \subset W_{j,t}$. To see this, observe that, as $\overline{H}$ is compatible

with the $\delta W_{j,t}$'s, all the sets $W_{j,t} \cap int(G_k(\overline{H})_t)$ are open and of empty (topological) boundary in $int(G_k(\overline{H})_t)$, for each t. Hence, if nonempty, these are connected components of $int(G_k(\overline{H})_t)$. But, as $int(G_k(\overline{H})_t)$ is connected, this entails that $W_{j,t} \cap int(G_k(\overline{H})_t)$ is either the empty set or $int(G_k(\overline{H})_t)$ itself, as claimed.

As the $W_{i,t}$'s are disjoint from each other, for each k there is a unique i such that $int(G_k(H)_t) \subset W_{i,t}$. After a possible refinement of the partition $\mathscr{P}$, we can assume that for each k, $int(G_k(H)_t)$ meets the same $W_{i,t}$ for all t, i.e. that for every $B \in \mathscr{P}$, i depends only on k and not on $t \in B$.

We turn to define the regular system H claimed in Step 1. For $1 \le k \le b$, let:

$$H_k := \pi_e^{-1}(\overline{H}_k).$$

Since $\overline{\lambda}_k \in \mathbf{B}(\widetilde{\pi}_e(\lambda), \eta')$, by (7.10), we have $\overline{\lambda}_k \in \widetilde{\pi}_e(\mathbf{B}(\lambda, \frac{\eta}{2}))$. Choose some $\lambda_k \in \widetilde{\pi}_e^{-1}(\overline{\lambda}_k) \cap \mathbf{B}(\lambda, \frac{\eta}{2})$.

As $\lambda_k \in \mathbf{B}(\lambda, \frac{\eta}{2})$ for all k and neither e nor $-e$ belongs to $\mathbf{B}(\lambda, \eta)$ we have:

$$d(\lambda_k, \pm e) \ge \frac{\eta}{2}, \qquad \forall\, k \le b.$$

As a matter of fact, by Remark 7.5.6, as $\overline{H}$ fulfills conditions (i) and (ii) of Definition 7.5.2, these conditions are also fulfilled by $H := (H_k, \lambda_k)_{k \le b}$.

By Lemma 7.4.8 and our choice of σ, for all k, the set $A_B \cap int(G_k(H))$ is the union of finitely many definable sets having a common regular element $\mu \in \mathbf{S}^{n-1}$ (since we have seen that each $int(G_k(\overline{H})_t)$ is included in $W_{j,t}$ for some j independent of $t \in B$). Moving slightly μ, we may assume that $d(\mu, \pm\lambda) \ge \eta$ (we have assumed $\eta \le \frac{t_v}{4}$). This completes the proof of the first step. $\qquad\square$

The desired partition of $\mathbb{R}^m$ will be obtained after finitely many refinements of the partition $\mathscr{P}$. Clearly, it is enough to prove the result for all the sets A_B, $B \in \mathscr{P}$. We thus can fix B in $\mathscr{P}$ and identify A and A_B in the next steps below. For simplicity, we will not indicate either the partitions of the parameter space $\mathbb{R}^m$ resulting from the successive refinements of $\mathscr{P}$, working always with A (instead of A_B).

The flaw of the first step is that the regular vector μ that we get for $G_k(H) \cap A$ might not be in $\Lambda_k(H)$. If it belongs to this set, Proposition 7.5.5 and Lemma 7.5.8 suffice to conclude (see Step 4). Had the vector e (used in Step 1) been regular for A, we could have required $\mu \in \Lambda_k(H)$ in Step 1, using Lemma 7.5.10 below in the same way as we will do in Step 2 to construct $\widehat{H}$ by means of π_μ (see assumption (7.12)). One could therefore think that not much was achieved so far as we need a regular vector to carry out our construction and finding a regular vector is all our purpose. However, since we can focus on the sets $A \cap G_p(H)$, which all have a regular vector (provided by Step 1), by repeating in Step 2 the construction of the first step (replacing e with μ and making use of Lemma 7.5.10), we will get a system $(\widehat{H}_k, \widehat{\lambda}_k)_{k \le \widehat{b}}$ with $\widehat{\lambda}_k \in \Lambda_p(H)$ regular for $G_p(H) \cap G_k(\widehat{H}) \cap A$, for each fixed $p \le b$. It will then remain to find (in Step 3, see the paragraph before

Step 3 for more details) a common extension of H and $\widehat{H}$, obtained at Steps 1 and 2 respectively.

Step 2 Fix $p \leq b$. There exists a regular system of hypersurfaces $\widehat{H} = (\widehat{H}_k, \widehat{\lambda}_k)_{k \leq \widehat{b}}$ such that for every $k, \widehat{\lambda}_k \in \Lambda_p(H) \cap \mathbf{B}(\lambda, \eta)$ and is regular for $G_p(H) \cap G_k(\widehat{H}) \cap A$.

Proof of Step 2 Note that as λ_p is regular for the set $H_p \cup H_{p+1}$, there exists $r > 0$ such that $\mathbf{B}(\lambda_p, r)$ is regular for $H_p \cup H_{p+1}$. Taking r smaller if necessary, we may assume that $r \leq \frac{\eta}{4}$.

Let $r' > 0$ be such that we have in $\mathbf{S}^{n-1} \cap N_\mu$:

$$\mathbf{B}(\widetilde{\pi}_\mu(\lambda_p), r') \subset \widetilde{\pi}_\mu(\mathbf{B}(\lambda_p, \frac{r}{2})). \tag{7.11}$$

To complete the proof of Step 2, we need a lemma.

Lemma 7.5.10 *Let l in $\mathbf{S}^{n-1}$, $0 < r \leq 1$, and $\kappa \in \mathbb{N}$. Let C be a subset of $\mathbb{G}^n$ and $\mu \in \mathbf{S}^{n-1}$ such that*

$$d(\mu, C) > 0. \tag{7.12}$$

There exists $\alpha > 0$ such that for any $P_1, \ldots, P_\kappa$ in C and any $y \in \widetilde{\pi}_\mu(\mathbf{B}(l, \frac{r}{2}))$ there exists $\widehat{\lambda} \in \mathbf{B}(l, r) \cap \widetilde{\pi}_\mu^{-1}(y)$ such that:

$$d(\widehat{\lambda}, \bigcup_{i=1}^{\kappa} P_i) \geq \alpha.$$

The proof of this lemma is postponed. We first see why it is enough to carry out the proof of Step 2. Let $\kappa \geq 1$ be the maximal number of connected components of $A \cap G_p(H) \cap \pi_\mu^{-1}(x)$, $x \in N_\mu$. Applying this lemma with this integer κ, with $C = \bigcup_{t \in B} \tau(A_t \cap G_p(H)_t)$ and $l = \lambda_p$ (μ being given by Step 1), we get a positive constant α.

By Lemma 7.4.12, we can cover $(G_p(H)_t \cap A_t)_{t \in \mathbb{R}^m}$ by $\frac{\alpha}{2}$-flat families, say $U'_{1,t}, \ldots, U'_{\omega',t}, \omega' \in \mathbb{N}$. Take a cell decomposition $(W'_i)_{i \in I'}$ of $\mathbb{R}^m \times N_\mu$ (identify $\mathbb{R}^m \times N_\mu$ with $\mathbb{R}^m \times \mathbb{R}^{n-1}$) which is compatible with the $\pi_\mu(U'_i)$, $i \leq \omega'$.

For any $i \in I'$ the family $\pi_\mu^{-1}(W'_{i,t}) \cap G_p(H)_t \cap A_t$, $t \in B$, is thus included in the union of no more than κ $\frac{\alpha}{2}$-flat families.

Lemma 7.5.10 thus implies that for any $i \in I'$ and any $y \in \widetilde{\pi}_\mu(\mathbf{B}(\lambda_p, \frac{r}{2}))$, there exists $\widehat{\lambda} \in \mathbf{B}(\lambda_p, r) \cap \widetilde{\pi}_\mu^{-1}(y)$ such that for any $t \in B$:

$$d(\widehat{\lambda}, \tau(\pi_\mu^{-1}(W'_{i,t}) \cap G_p(H)_t \cap A_t)) \geq \frac{\alpha}{2}. \tag{7.13}$$

Apply the induction hypothesis to get a regular system of hypersurfaces $H'' = (H''_k, \lambda''_k)_{k \leq b''}$ of $B \times N_\mu$ (identify N_μ with $\mathbb{R}^{n-1}$, up to a refinement of the partition

$\mathscr{P}$) compatible with the $\delta W'_{i,t}$'s. Do it in such a way that all the λ''_k are elements of $\mathbf{B}(\widetilde{\pi}_\mu(\lambda_p), r')$ (where r' is given by (7.11)).

Define now:

$$\widehat{H}_{k,t} := \pi_\mu^{-1}(H''_{k,t}). \tag{7.14}$$

The compatibility with the families $\delta W'_{i,t}$ implies that every $int(G_k(H''))_t$ is included in $W'_{i,t}$ for some i (possibly refining the partition of the parameter space), by the same argument as the one we used in Step 1 for $G_k(H)$ and the partition $(W_i)_{i \in I}$.

As a matter of fact, according to (7.13) for $y = \lambda''_k$, we know that for every integer $k \leq b''$ there exists $\widehat{\lambda}_k \in \mathbf{B}(\lambda_p, r) \cap \widetilde{\pi}_\mu^{-1}(\lambda''_k)$ such that for any $t \in B$:

$$d\left(\widehat{\lambda}_k, \, \tau(\pi_\mu^{-1}(int(G_k(H'')_t)) \cap G_p(H)_t \cap A_t)\right) \geq \frac{\alpha}{2}. \tag{7.15}$$

Let us check that $\widehat{H} := (\widehat{H}_k, \widehat{\lambda}_k)_{k \leq \widehat{b}}$ (where $\widehat{b} := b''$) is the desired regular system of hypersurfaces. For this purpose, observe that, since neither μ nor $-\mu$ belongs to $\mathbf{B}(\lambda, \eta)$, we have for each k (recall that $r \leq \frac{\eta}{4}$ and $\lambda_p \in \mathbf{B}(\lambda, \frac{\eta}{2})$, as well as $\widehat{\lambda}_k \in \mathbf{B}(\lambda_p, r)$):

$$d(\widehat{\lambda}_k, \pm\mu) \geq r.$$

By Remark 7.5.6, as $\widehat{\lambda}_k \in \widetilde{\pi}_\mu^{-1}(\lambda''_k)$, this implies that the family $\widehat{H}$ fulfills the conditions of Definition 7.5.2.

Furthermore, as $\mathbf{B}(\lambda_p, r) \subset \mathbf{B}(\lambda, \eta)$ (since $r \leq \frac{\eta}{4}$ and $\lambda_p \in \mathbf{B}(\lambda, \frac{\eta}{2})$), all the $\widehat{\lambda}_k$'s belong to $\mathbf{B}(\lambda, \eta)$. Note also that as $\mathbf{B}(\lambda_p, r)$ is regular for $H_p \cup H_{p+1}$, the vector $\widehat{\lambda}_k$ belongs to $\Lambda_p(H)$. This completes the proof of the second step. □

The inconvenience of Step 2 (we would like to apply Lemma 7.5.8, see Step 4) is that the provided vector is regular for $A \cap G_p(H) \cap G_k(\widehat{H})$ (instead of $A \cap G_k(\widehat{H})$). If $\widehat{H}$ were an extension of the family H constructed in Step 1, this would be no problem since in this case we would have $G_k(\widehat{H}) \subset G_p(H)$ (or $int(G_k(\widehat{H})) \cap int(G_p(H)) = \emptyset$). Thus, we will have to find a common extension $\widetilde{H}$ of H and $\widehat{H}$ given by Steps 1 and 2 respectively. This is what is carried out in the proof of Step 3.

Step 3 Fix $p \leq b$. There exists an extension $\widetilde{H} = (\widetilde{H}_k, \widetilde{\lambda}_k)_{k \leq \widetilde{b}}$ of H which coincides with H outside $G_p(H)$, and such that for each k, $\widetilde{\lambda}_k$ belongs to $\mathbf{B}(\lambda, \eta)$ and is regular for $A \cap G_k(\widetilde{H}) \cap G_p(H)$.

Proof of Step 3 Let $\widehat{H}$ be the regular system given by Step 2 and let $k \leq \widehat{b}$ be an integer. Because $\widehat{\lambda}_k \in \Lambda_p(H)$, by Proposition 7.5.5, the sets H_p and H_{p+1} are respectively the graphs for $\widehat{\lambda}_k$ of two functions ζ_k and ζ'_k. Moreover, the set $\widehat{H}_k$ is also the graph for $\widehat{\lambda}_k$ of a function $\widehat{\xi}_k$. Define:

$$\theta_k := \min(\max(\zeta_k, \widehat{\xi}_k), \zeta'_k)$$

in order to get a function whose graph for $\widehat{\lambda}_k$ is included in $G_p(H)$. We now define the desired regular family $(\widetilde{H}_k, \widetilde{\lambda}_k)_{1 \leq k \leq \widetilde{b}}$ as follows:

- Let $\widetilde{H}_k := H_k$ and $\widetilde{\lambda}_k := \lambda_k$ if $k < p$.
- Let $\widetilde{H}_p := H_p$ and $\widetilde{\lambda}_p := \widehat{\lambda}_1$.
- Let $\widetilde{H}_k$ be the graph of θ_{k-p} for $\widehat{\lambda}_{k-p}$, and let $\widetilde{\lambda}_k := \widehat{\lambda}_{k-p}$, whenever $p + 1 \leq k \leq p + \widehat{b}$.
- And finally let $\widetilde{H}_k := H_{k-\widehat{b}}$ and $\widetilde{\lambda}_k := \lambda_{k-\widehat{b}}$ if $p + \widehat{b} + 1 \leq k \leq b + \widehat{b}$.

Let us check that properties (i) and (ii) of Definition 7.5.2 hold for the family $\widetilde{H}$. For $k < p - 1$, or $k \geq p + \widehat{b} + 1$, the result is clear since the family $\widetilde{H}$ is indeed the family H (after renumbering).

For $k = p - 1$, properties (i) and (ii) for $\widetilde{H}$ follow from (i) and (ii) for H and Proposition 7.5.5 since we have assumed $\widehat{\lambda}_1 \in \Lambda_p(H)$.

It remains to check (i) and (ii) for $\widetilde{H}_{k+p}$, with $0 \leq k \leq \widehat{b}$. We start with (i). By (i) for $\widehat{H}$, the set $\widehat{H}_{k+1}$ is the graph for $\widehat{\lambda}_k$ of a function $\widehat{\xi}'_k$ such that $\widehat{\xi}_k \leq \widehat{\xi}'_k$. Define now:

$$\theta'_k = \min(\max(\zeta_k, \widehat{\xi}'_k), \zeta'_k).$$

Claim The graph of θ'_k for $\widehat{\lambda}_k$ is that of θ_{k+1} for $\widehat{\lambda}_{k+1}$.

To see this, note that the graph of θ'_k (resp. θ_{k+1}) for $\widehat{\lambda}_k$ (resp. $\widehat{\lambda}_{k+1}$) matches with $\widehat{H}_{k+1}$ over $E(H_{p+1}, \widehat{\lambda}_k) \setminus E(H_p, \widehat{\lambda}_k)$ (resp. $\widehat{\lambda}_{k+1}$). But, by Proposition 7.5.5, the sets $E(H_p, l)$ and $E(H_{p+1}, l)$ do not depend on $l \in \Lambda_p(H)$. As $\widehat{\lambda}_k$ and $\widehat{\lambda}_{k+1}$ both belong to $\Lambda_p(H)$, this already shows that the two graphs involved in the above claim match over $int(G_p(H))$.

The graph of θ'_k (resp. θ_{k+1}) for $\widehat{\lambda}_k$ (resp. $\widehat{\lambda}_{k+1}$) is also constituted by the points of $H_p \setminus int(E(\widehat{H}_{k+1}, \widehat{\lambda}_k))$ (resp. $\widehat{\lambda}_{k+1}$) on the one hand and by the points of $H_{p+1} \cap E(\widehat{H}_{k+1}, \widehat{\lambda}_k)$ (resp. $\widehat{\lambda}_{k+1}$) on the other hand. By (ii) for $\widehat{H}$, the claim ensues.

This claim proves that $\widetilde{H}_{p+k+1}$, which is by definition the graph of θ_{k+1} for $\widehat{\lambda}_{k+1}$, is indeed also the graph of θ'_k for $\widehat{\lambda}_k$. Therefore, to check (i) (for $\widetilde{H}_{k+p}$, $k \leq \widehat{b}$), we just have to prove that $\theta_k \leq \theta'_k$. But, as $\widehat{\xi}_k \leq \widehat{\xi}'_k$, this comes down from the respective definitions of θ'_k and θ_k.

Let us check property (ii) for $\widetilde{H}_{k+p}$, for $k \leq \widehat{b}$. Observe first that if $k = \widehat{b}$, it is then a consequence of Proposition 7.5.5, since we have assumed that $\widehat{\lambda}_k$ belongs to $\Lambda_p(H)$.

Let now k be such that $0 \leq k \leq \widehat{b} - 1$. First note that by (ii) for $\widehat{H}$ we have:

$$E(\widehat{H}_{k+1}, \widehat{\lambda}_k) = E(\widehat{H}_{k+1}, \widehat{\lambda}_{k+1}).$$

But, $E(\widetilde{H}_{k+p+1}, \widehat{\lambda}_k)$ (resp. $\widehat{\lambda}_{k+1}$) is constituted by the points of $E(H_p, \widehat{\lambda}_k)$ (resp. $\widehat{\lambda}_{k+1}$) together with the points of $E(H_{p+1}, \widehat{\lambda}_k) \cap E(\widehat{H}_{k+1}, \widehat{\lambda}_k)$ (resp. $\widehat{\lambda}_{k+1}$). As $\widehat{\lambda}_{k+1}$ and $\widehat{\lambda}_k$ both belong to $\Lambda_p(H)$, this establishes (ii) for $\widetilde{H}$.

To complete the proof of Step 3, it remains to make sure that for every $k \leq \hat{b}$ the vector $\widetilde{\lambda}_{k+p}$ is regular for $G_{k+p}(\widetilde{H}) \cap G_p(H) \cap A$. By definition, we have $\widetilde{\lambda}_p = \widehat{\lambda}_1$, $\widetilde{\lambda}_{k+p} = \widehat{\lambda}_k$ for $1 \leq k \leq \widehat{b}$, and:

$$G_{k+p}(\widetilde{H}) \subset G_k(\widehat{H}) \cap G_p(H), \tag{7.16}$$

for each $0 \leq k \leq \widehat{b}$.

As for any k the vector $\widehat{\lambda}_k$ is regular for $A \cap G_k(\widehat{H}) \cap G_p(H)$ (see Step 2), this implies that for each $k \leq \widehat{b}$, the vector $\widetilde{\lambda}_{k+p}$ is regular for $A \cap G_{k+p}(\widetilde{H})$. This completes the proof of the third step. □

Step 4 There is a regular system of hypersurfaces $\check{H} = (\check{H}_k, \check{\lambda}_k)_{k \leq \check{b}}$ compatible with A and such that $\check{\lambda}_k \in \mathbf{B}(\lambda, \eta)$ for all k.

Proof of Step 4 By Lemma 7.5.8 (applied $(\widetilde{b} + 1)$ times to $\widetilde{H}$ of Step 3), we may extend $\widetilde{H}$ to a regular system $\check{H} = (\check{H}_k, \check{\lambda}_k)_{k \leq \check{b}}$ compatible with the set

$$G_p(H) \cap \bigcup_{k=0}^{\widetilde{b}} G_k(\widetilde{H}) \cap A = G_p(H) \cap A.$$

By Step 3, we have $\check{\lambda}_k \in \mathbf{B}(\lambda, \eta)$ for all k (see Remark 7.5.9). Since $\check{H}$ is an extension of H ($\widetilde{H}$ being itself an extension of H) which coincides with H outside $G_p(H)$, we may carry out this construction on all the $G_p(H)$'s successively. This provides the desired regular system. □

It remains to prove Lemma 7.5.10. The lemma below describes a property of $\widetilde{\pi}_\mu$ that we need for this purpose.

Lemma 7.5.11 *Let $\mu \in \mathbf{S}^{n-1}$, $T \in \mathbb{G}^n$, and $x \in T$. If $v \in \mathbf{S}^{n-1}$ is tangent at x to the curve $\widetilde{\pi}_\mu^{-1}(\widetilde{\pi}_\mu(x))$ then we have:*

$$d(\mu, T) \leq d(v, T).$$

Proof Let w be the vector of T which realizes $d(v, T)$. Remark that the vectors x, μ, and v are in the same two dimensional vector space. Moreover (x, v) is an orthonormal basis of this plane. Write $\mu = \alpha x + \beta v$ with $\alpha^2 + \beta^2 = 1$. Then, as x and w both belong to T we have

$$d(\mu, T) \leq |\mu - (\alpha x + \beta w)| = |\beta| \cdot |v - w| \leq d(v, T).$$

□

Proof of Lemma 7.5.10 We will work up to a ("projective") coordinate system of $\mathbf{S}^{n-1}$ defined as follows. Let U_i^+ (resp. U_i^-) denote

$$\{x \in \mathbf{S}^{n-1} : x_i \geq \epsilon\}$$

(resp. $x_i \leq -\epsilon$) with $\epsilon > 0$. Define then $h_i : U_i^+ \to \mathbb{R}^{n-1}$ (resp. $g_i : U_i^- \to \mathbb{R}^{n-1}$) by $h_i(x_1, \ldots, x_n) = (\frac{x_1}{x_i}, \ldots, \widehat{\frac{x_i}{x_i}}, \ldots, \frac{x_n}{x_i})$ (resp. $g_i(x_1, \ldots, x_n)$, here the $\widehat{}$ means the term is omitted). We can assume that $\mathbf{B}(l, r)$ entirely fits in one single U_i^+ or U_i^-, say U_i^+ (up to a linear change of coordinate of $\mathbb{R}^n$).

Through such a chart, the elements $\mathbf{S}^{n-1} \cap T$, $T \in C$, will be identified with their respective images, which are affine subspaces of $\mathbb{R}^{n-1}$. The set $\mathbf{S}^{n-1} \cap N_\mu$ also becomes an affine subspace Δ, and $\widetilde{\pi}_\mu$ an orthogonal projection along a line, say L. We denote by π this projection. By Lemma 7.5.11 and hypothesis (7.12), there exists $u > 0$ such that for any $T \in C$ (the angle between affine spaces is defined as the angle between the associated vector spaces):

$$\angle(L, T) \geq u. \tag{7.17}$$

We have to find $\alpha > 0$ such that for any $P_1, \ldots, P_\kappa$ in C and any $y \in \pi(h_i(\mathbf{B}(l, \frac{r}{2})))$ there exists $\widehat{\lambda} \in h_i(\mathbf{B}(l, r)) \cap \pi^{-1}(y)$ such that:

$$d(\widehat{\lambda}, \bigcup_{i=1}^{\kappa} P_i) \geq \alpha. \tag{7.18}$$

For any $y \in \pi(h_i(\mathbf{B}(l, \frac{r}{2})))$, the length of $\pi^{-1}(y) \cap h_i(\mathbf{B}(l, r))$ is bounded below away from zero by a strictly positive real number α_0 (h_i is bi-Lipschitz).

It is an easy exercise of elementary geometry to derive from (7.17) that for any $\alpha > 0$ the set $\{x \in \pi^{-1}(y) : d(x, T) \leq \alpha\}$ is a segment of length not greater than $\frac{2\alpha}{u}$, for all $T \in C$ and $y \in \Delta$.

Let $\alpha := \frac{\alpha_0 u}{4\kappa}$. By the preceding paragraph, we see that if (7.18) failed for some $y \in \pi(h_i(\mathbf{B}(l, \frac{r}{2})))$, we could cover $\pi^{-1}(y) \cap h_i(\mathbf{B}(l, r))$ by κ segments of length not greater than $\frac{\alpha_0}{2\kappa}$. This contradicts the fact that the length of $\pi^{-1}(y) \cap h_i(\mathbf{B}(l, r))$ is not less than α_0. $\qquad \square$

7.5.4 Proof of Theorem 7.3.2

By Theorem 7.5.4, there is a definable partition $\mathscr{P}$ of $\mathbb{R}^m$ such that for every $B \in \mathscr{P}$ there exists a regular system of hypersurfaces compatible with A_B. Fix $B \in \mathscr{P}$ and such a regular system of hypersurfaces $H = (H_k, \lambda_k)_{1 \leq k \leq b}$.

For each $t \in B$, we shall define the desired definable mapping h_t over $E(H_{k,t}, \lambda_k)$ by induction on k, in such a way that for all $t \in B$

$$h_t(E(H_{k,t}, \lambda_k)) = E(F_{k,t}, e_n),$$

where $F_{k,t}$ is the graph for e_n of $\eta_{k,t} : \mathbb{R}^{n-1} \to \mathbb{R}$, with $(\eta_{k,t})_{t \in B}$ uniformly Lipschitz definable family of functions.

For $k = 1$, choose an orthonormal basis of N_{λ_1} and set $h_t(q) := (x_{\lambda_1}, q_{\lambda_1})$, where x_{λ_1} stands for the coordinates of $\pi_{\lambda_1}(q)$ in this basis.

Let now $k \geq 1$. By (i) of Definition 7.5.2, for any $t \in B$ the sets $H_{k,t}$ and $H_{k+1,t}$ are the respective graphs for λ_k of two functions $\xi_{k,t}$ and $\xi'_{k,t}$. For $q \in E(H_{k+1,t}, \lambda_k) \setminus E(H_{k,t}, \lambda_k)$, extend h_t by defining $h_t(q)$ as the element:

$$h_t\big(\pi_{\lambda_k}(q) + \xi_{k,t}(\pi_{\lambda_k}(q)) \cdot \lambda_k\big) + \big(q_{\lambda_k} - \xi_{k,t}(\pi_{\lambda_k}(q))\big)e_n.$$

Thanks to the property (ii) of Definition 7.5.2, we have:

$$E(H_{k+1,t}, \lambda_{k+1}) = E(H_{k+1,t}, \lambda_k),$$

and hence h_t is actually defined over $E(H_{k+1,t}, \lambda_{k+1})$. Since $\xi_{k,t}$ is C-Lipschitz with C independent of t, the h_t's constitute a uniformly bi-Lipschitz family of homeomorphisms. Note also that the image of h_t is $E(F_{k+1,t}, e_n)$, where $F_{k+1,t}$ is the graph (for e_n) of the uniformly Lipschitz family of functions:

$$\eta_{k+1,t}(x) := \eta_{k,t}(x) + (\xi'_{k,t} - \xi_{k,t}) \circ \pi_{\lambda_k} \circ h_t^{-1}(x, \eta_{k,t}(x)),$$

for $(t, x) \in B \times \mathbb{R}^{n-1}$. This completes the induction step, giving h_t over $E(H_{b,t}, \lambda_b)$. To extend h_t to the whole of $\mathbb{R}^n$ do it similarly as in the case $k = 1$.

7.5.5 Regular Vectors and Set Germs

For R positive real number and $n > 1$ we set

$$\mathscr{C}_n(R) := \{(t, x) \in [0, +\infty) \times \mathbb{R}^{n-1} : |x| \leq Rt\}.$$

We also set $\mathscr{C}_1(R) := [0, +\infty)$.

Our purpose is to show Theorem 7.5.14, which is an improvement of Corollary 7.3.4 asserting that, in the case of germs of subsets of $\mathscr{C}_n(R)$, the provided homeomorphism may be required to preserve the first coordinate in the canonical basis. This fact is an essential ingredient of the Lipschitz conic structure theorem [34, 35], which recently proved very useful to study Sobolev spaces of bounded subanalytic domains [28, 29, 33, 34, 36].

Definition 7.5.12 Let $A, B \subset \mathbb{R}^n$. A definable map $h : A \to B$ is **vertical** if it preserves the first coordinate in the canonical basis of $\mathbb{R}^n$, i.e. if for any $t \in \mathbb{R}$, $\pi(h(t, x)) = t$, for all $x \in A_t$, where $\pi : \mathbb{R}^n \to \mathbb{R}$ is the orthogonal projection onto the first coordinate.

We start with a preliminary lemma which is of its own interest. We use the notation $f(t) \ll g(t)$ to express that $f(t) \leq g(t)\phi(t)$, for some function ϕ tending to zero as t goes to zero.

Lemma 7.5.13 *Let $h : (X, 0) \to (\mathscr{C}_n(R), 0)$ be a germ of vertical definable map, with $X \subset \mathscr{C}_n(R')$, for some R and R'. If $(h_t)_{t\in\mathbb{R}}$ is uniformly Lipschitz then h is a Lipschitz map germ.*

Proof Suppose that h fails to be Lipschitz. Then, by Curve Selection Lemma, we can find two definable arcs in X, say $p(t)$ and $q(t)$, tending to zero along which:

$$|p(t) - q(t)| \ll |h(p(t)) - h(q(t))|. \tag{7.19}$$

We may assume that $p(t)$ (and so $h(p(t))$) is parametrized by its first coordinate (since the first coordinate of $p(t)$ induces a homeomorphism from a right-hand-side neighborhood of zero in $\mathbb{R}$ onto a right-hand-side neighborhood of zero in $\mathbb{R}$), i.e. we may assume $p(t) = (t, p_2(t), \ldots, p_n(t))$.

As $p(t)$ and $h(p(t))$ are definable arcs in $\mathbb{C}_n(R')$ and $\mathbb{C}_n(R)$ respectively, we have:

$$|h(p(t)) - h(p(t'))| \sim |t - t'| \leq |p(t) - q(t)| \tag{7.20}$$

and

$$|p(t) - p(t')| \sim |t - t'| \leq |p(t) - q(t)|, \tag{7.21}$$

where t' denotes the first coordinate of $q(t)$.

Therefore, we can easily derive from (7.19), (7.20), and (7.21):

$$|h(p(t)) - h(q(t))| \sim |h(p(t')) - h(q(t))| \sim |p(t') - q(t)| \lesssim |p(t) - q(t)|,$$

in contradiction with (7.19).   □

Theorem 7.5.14 *Let X be the germ at 0 of a definable subset of $\mathscr{C}_n(R)$ (for some R) of empty interior. There exists a germ of vertical bi-Lipschitz definable homeomorphism (onto its image) $H : (\mathbb{C}_n(R), 0) \to (\mathbb{C}_n(R), 0)$ such that e_n is regular for $H(X)$.*

Proof We denote by $e_1, \ldots, e_n$ the canonical basis of $\mathbb{R}^n$ and by $e'_1, \ldots, e'_{n-1}$ the canonical basis of $\mathbb{R}^{n-1}$ (so that $e_n = (0, e'_{n-1})$). Apply Theorem 7.3.2 to X, regarded as a family of $\mathbb{R} \times \mathbb{R}^{n-1}$. This provides a uniformly bi-Lipschitz family of homeomorphisms $h_t : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$, $t \in (0, \varepsilon)$, such that e'_{n-1} is regular for the family $(h_t(X_t))_{t\in\mathbb{R}}$. Up to a family of translations, we may assume that $h(t, 0) \equiv 0$, which implies that $H : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, $(t, x) \mapsto (t, h_t(x))$, maps $\mathbb{C}_n(R)$ into $\mathbb{C}_n(R')$ for some R' (and up to a family of homothetic transformations

we may assume $R = R'$). By Lemma 7.5.13, the map-germ H is bi-Lipschitz near the origin.

We now have to check that e_n is regular for the germ of the definable set $Y :=$ $H(X)$. Suppose not. Then, by Curve Selection Lemma, there exists a definable arc $\gamma : [0, \epsilon] \to Y_{reg}$ with $\gamma(0) = 0$ and $e_n \in \tau := \lim_{t \to 0} T_{\gamma(t)} Y_{reg}$. But, as e'_{n-1} is regular for the family $(Y_t)_{t \in [0,\varepsilon]}$, we have $e'_{n-1} \notin \lim_{t \to 0} T_{\tilde{\gamma}(t)} Y_{\gamma_1(t)}$, if $\gamma(t) = (\gamma_1(t), \tilde{\gamma}(t)) \in \mathbb{R} \times \mathbb{R}^{n-1}$. This implies that:

$$\tau \cap N_{e_1} \neq \lim_{t \to 0} (T_{\gamma(t)} Y_{reg} \cap N_{e_1})$$

(since the latter does not contains the vector $e_n = (0, e'_{n-1})$ while the former does). Hence, τ cannot be transverse to N_{e_1} (since otherwise the intersection with the limit would be the limit of the intersection) which means that e_1 is orthogonal to τ. This implies that the limit vector $\lim_{t \to 0} \frac{\gamma(t)}{|\gamma(t)|} = \lim_{t \to 0} \frac{\gamma'(t)}{|\gamma'(t)|} \in \tau$ is orthogonal to e_1, from which we can conclude

$$\lim_{t \to 0} \frac{\gamma_1(t)}{|\gamma(t)|} = 0,$$

in contradiction with $\gamma(t) \in \mathbb{C}_n(R)$. $\square$

7.6 Definable Bi-Lipschitz Triviality in Polynomially Bounded O-Minimal Structures

The results of this section are valid *under the extra assumption that the structure is polynomially bounded*. It is not difficult to produce counterexamples to these results (except however to Proposition 7.6.5) as soon as this assumption fails. We are going to establish a bi-Lipschitz triviality theorem for definable families (Theorem 7.6.3), from which we will derive a stratification result (Corollary 7.6.9).

We start with a preliminary lemma. As in Sect. 7.2.2, we denote by $\mathscr{F}$ the valuation field of the structure, which is the subfield of $\mathbb{R}$ constituted by all the real numbers α for which the function $(0, +\infty) \ni x \mapsto x^\alpha \in \mathbb{R}$ is definable.

Lemma 7.6.1 *Let $\xi : \mathbb{R}^{m+n} \to \mathbb{R}$ be a definable nonnegative function. There exist some definable subsets of $\mathbb{R}^{m+n}$, say $W_1, \ldots, W_k$, and a definable partition $\mathscr{P}$ of $\mathbb{R}^{m+n}$ such that for any $V \in \mathscr{P}$ there are $\alpha_1, \ldots, \alpha_k$ in $\mathscr{F}$ such that for each $t \in \mathbb{R}^m$ we have on $V_t \subset \mathbb{R}^n$:*

$$\xi_t(x) \sim d(x, W_{1,t})^{\alpha_1} \cdots d(x, W_{k,t})^{\alpha_k}. \tag{7.22}$$

Proof We prove it by induction on n. For $n = 1$, the result follows from Theorem 7.2.11. Let $n \geq 2$ and assume that the proposition is true for $(n - 1)$. Let $\lambda_1, \ldots, \lambda_N$ be the elements of $\mathbf{S}^{n-1}$ given by Proposition 7.4.13.

For each i, applying the Preparation Theorem (Theorem 7.2.11) to $\xi \circ \Lambda_i$: $\mathbb{R}^{m+n} \to \mathbb{R}$, where Λ_i is an orthogonal linear mapping of $\mathbb{R}^{m+n}$ sending $(0_{\mathbb{R}^m}, e_n)$ onto $(0_{\mathbb{R}^m}, \lambda_i)$ and preserving the m first coordinates (below, we sometimes regard Λ_i as a transformation of $\mathbb{R}^n$), we obtain a partition of $\mathbb{R}^{m+n}$. The images of all the elements of this partition under the map Λ_i provide a new partition of $\mathbb{R}^{m+n}$, denoted by $\mathscr{P}_i$. Let $(V_j)_{j \in J}$ be a common refinement of the $\mathscr{P}_i$'s. Applying Proposition 7.4.13 to the finite family constituted by all the sets of the partition $(V_j)_{j \in J}$, we get a partition Σ of $\mathbb{R}^{m+n}$ into cells.

Let $E \in \Sigma$ be an open cell. By construction and Proposition 7.4.13, there is $i \le N$ such that λ_i is regular for δE. It means that e_n is regular for the family $(\Lambda_i^{-1}(\delta E)_t)_{t \in \mathbb{R}^m}$. Hence, it follows from Proposition 7.4.6 that there is a partition $\mathcal{Q}_E$ of $\Lambda_i^{-1}(cl(E))$ into cells, such that each element C is either the graph of a uniformly Lipschitz family of functions or a set of the form

$$C = \{(z, y) \in B \times \mathbb{R} : \eta_1(z) < y < \eta_2(z)\}, \tag{7.23}$$

with $B \in \mathcal{S}_{m+n-1}$ and $\eta_1 < \eta_2$ definable functions on B such that $(\eta_{1,t})_{t \in \mathbb{R}^m}$ and $(\eta_{2,t})_{t \in \mathbb{R}^m}$ are uniformly Lipschitz.

Observe that it suffices to show the desired statement for the restriction to each cell $C \in \mathcal{Q}_E$ of the family of functions $\xi_t \circ \Lambda_i$, $t \in \mathbb{R}^m$. For the elements C of the partition $\mathcal{Q}_E$ which are graphs of some uniformly Lipschitz family of functions, one may easily deduce the result from the induction hypothesis.

Fix thus a cell $C \subset \Lambda_i^{-1}(E)$ as in (7.23). There is j such that $C \subset \Lambda_i^{-1}(V_j)$. By construction, there are $r \in \mathcal{F}$ and some functions a and θ on the basis B of C such that for $x = (\tilde{x}, x_n) \in C_t$, $t \in \mathbb{R}^m$, we have:

$$\xi_t \circ \Lambda_i(x) \sim a_t(\tilde{x})|x_n - \theta_t(\tilde{x})|^r.$$

Thanks to the induction hypothesis we thus only have to check the result for the function $|x_n - \theta_t(\tilde{x})|$.

As $\Gamma_\theta \cap C = \emptyset$, we can assume that for every $(t, \tilde{x}) \in B$, either $\theta_t(\tilde{x}) \le \eta_{1,t}(\tilde{x})$ or $\theta_t(\tilde{x}) \ge \eta_{2,t}(\tilde{x})$. Assume for instance that $\theta_t(\tilde{x}) \le \eta_{1,t}(\tilde{x})$. Writing for $t \in \text{supp}_m(C)$ and $x = (\tilde{x}, x_n) \in C_t$:

$$x_n - \theta_t(\tilde{x}) = (x_n - \eta_{1,t}(\tilde{x})) + (\eta_{1,t}(\tilde{x}) - \theta_t(\tilde{x})),$$

we see that (up to a partition of C we may assume that the terms of the right-hand-side are comparable for the partial order relation $\le$) $|x_n - \theta_t(\tilde{x})|$ is $\sim$ either to $|x_n - \eta_{1,t}(\tilde{x})|$ or to $|\eta_{1,t}(\tilde{x}) - \theta_t(\tilde{x})|$.

For the latter functions, since they are $(n-1)$-variable functions, the desired result is a consequence of the induction hypothesis. Moreover, since $\eta_{1,t}$ is Lipschitz, $|x_n - \eta_{1,t}(\tilde{x})|$ is $\sim$ to the distance to the graph of $\eta_{1,t}$ for every t. This shows the result for the given cell C. $\qquad \square$

Remark 7.6.2 The constants of the equivalence in the above lemma depend on t. However, the family of exponents $\alpha_1, \ldots, \alpha_k$ just depends on $V \in \mathscr{P}$.

We recall that the structure is assumed to be polynomially bounded in this section.

Theorem 7.6.3 *Given $A \in \mathscr{S}_{m+n}$, there exists a definable partition of $\mathbb{R}^m$ such that A is definably bi-Lipschitz trivial along each element of this partition.*

Proof We prove the result by induction on n. We shall show that the trivialization H may be required to induce a trivialization of some given definable subsets of A.

Let $A \in \mathscr{S}_{m+n}$ and let $C_1, \ldots, C_k$ be some definable subsets of A. Apply Theorem 7.3.2 to the set $\{(t, x) : x \in \delta A_t \cup \bigcup_{i=1}^{k} \delta C_{i,t}\}$. This provides a definable family of bi-Lipschitz maps $G_t : \mathbb{R}^n \to \mathbb{R}^n$, $t \in \mathbb{R}^m$, such that e_n is regular for the families of sets $(\delta G_t(C_{i,t}))_{t \in \mathbb{R}^m}$, $i = 1, \ldots, k$, and $(\delta G_t(A_t))_{t \in \mathbb{R}^m}$.

As we can work up to a family of bi-Lipschitz maps, we will identify G_t with the identity map. By Propositions 7.4.2, 7.4.6, and 7.4.7, we can find some definable functions $\xi_1 \leq \cdots \leq \xi_s$ on $\mathbb{R}^{m+n-1}$, with $(\xi_{i,t})_{t \in \mathbb{R}^m}$ uniformly Lipschitz for all i, and a cell decomposition $\mathscr{D}$ of $\mathbb{R}^{m+n-1}$ such that A and the C_j's are unions of some graphs $\Gamma_{\xi_i|D}$, $i \in \{1, \ldots, s\}$, $D \in \mathscr{D}$, or bands $(\xi_{i|D}, \xi_{i+1|D})$, $i \in \{0, \ldots, s\}$, $D \in \mathscr{D}$ (where $\xi_0 \equiv -\infty$ and $\xi_{s+1} \equiv +\infty$).

Refining the cell decomposition $\mathscr{D}$ if necessary (without changing notations), we can assume it to be compatible with the zero loci of the functions $(\xi_{i+1} - \xi_i)$. By Lemma 7.6.1, up to an extra refinement of the cell decomposition, we can assume that there are finitely many definable subsets $W_1, \ldots, W_c$ of $\mathbb{R}^m \times \mathbb{R}^{n-1}$ such that on every cell we can find $r_1, \ldots, r_c$ in $\mathscr{F}$ such that for all $i = 1, \ldots, s - 1$ and any $t \in \mathbb{R}^m$:

$$\xi_{i+1,t}(x) - \xi_{i,t}(x) \sim d(x, W_{1,t})^{r_1} \cdots d(x, W_{c,t})^{r_c}. \tag{7.24}$$

Refining one more time the cell decomposition $\mathscr{D}$, we may assume that the W_i's are unions of cells.

Applying now the induction hypothesis to the cells of $\mathscr{D}$ provides a partition $\mathscr{P}$. Fix $B \in \mathscr{P}$ and let $H(t, x) = (t, h_t(x))$ denote the obtained trivialization of $B \times \mathbb{R}^{n-1}$ along B. We have $h_t(C_{t_0}) = C_t$ for some $t_0 \in B$ and for all $C \in \mathscr{D}$. We are going to lift the trivialization H to an trivialization of $B \times \mathbb{R}^n$.

Given a point $(t, x) \in B \times \mathbb{R}^{n-1}$ and $1 \leq i \leq s - 1$ let

$$\widetilde{H}(t, x, \nu\xi_{i,t_0}(x) + (1-\nu)\xi_{i+1,t_0}(x)) := (t, h_t(x), \nu\xi_{i,t}(h_t(x)) + (1-\nu)\xi_{i+1,t}(h_t(x)),$$

for all $\nu \in [0, 1]$. Set also for $\nu \in (0, \infty)$:

$$\widetilde{H}(t, x, \xi_{1,t_0}(x) - \nu) := (t, h_t(x), \xi_{1,t}(h_t(x)) - \nu),$$

as well as

$$\widetilde{H}(t, x, \xi_{s,t_0}(x) + \nu) := (t, h_t(x), \xi_{s,t}(h_t(x)) + \nu).$$

Because $\mathscr{D}$ is compatible with the zero loci of the functions $(\xi_{i+1} - \xi_i)$ and since the trivialization h was required to preserve the cells of $\mathscr{D}$, it is easily seen that $\widetilde{H}_t$ is a continuous mapping for each $t \in \mathbb{R}^m$. Observe also that, since the W_i's are unions of cells of $\mathscr{D}$, we have $h_t(W_{i,t_0}) = W_{i,t}$, for all i. Since h_t is bi-Lipschitz for every $t \in B$, we can derive from (7.24), that for each $t \in B$ we have:

$$(\xi_{i+1,t} - \xi_{i,t}) \circ h_t \sim (\xi_{i+1,t_0} - \xi_{i,t_0}).$$

This shows the bi-Lipschitzness of $\widetilde{H}_t$ on the sets $[\xi_{i,t|D_t}, \xi_{i+1,t|D_t}]$, $D \in \mathscr{D}$, $D \subset B$, $i < s$. The bi-Lipschitzness of $\widetilde{H}_t$ on the sets $(-\infty, \xi_{1,t|D_t})$ and $(\xi_{s,t|D_t}, +\infty)$ is clear since the $(\xi_{i,t})_{t \in B}$ are families of Lipschitz functions.

The continuity of H_t and H_t^{-1} with respect to t follows from a well-known fact, up to an extra refinement of the partition of the parameter space [6, Lemma 5.17 and Exercise 5.21].   □

Remark 7.6.4 We have proved a stronger statement since the trivialization is also defined on the ambient space $B \times \mathbb{R}^n$. We can also require the trivialization to preserve a finite number of given definable subfamilies of A.

In Theorem 7.6.3, the constructed trivialization h_t is Lipschitz for every t (see Definition 7.2.7). The Lipschitz condition may also be required to hold with respect to the parameter t on relatively compact sets, as it will be established by Proposition 7.6.6, which requires the following proposition.

Proposition 7.6.5 *Let $A \in \mathcal{S}_{m+n}$ and let $f_t : A_t \to \mathbb{R}$ be a definable family of functions. If f_t is Lipschitz for all $t \in \mathbb{R}^m$ then there exists a definable partition $\mathscr{P}$ of $\mathbb{R}^m$ such that for every $B \in \mathscr{P}$, $f : A \to \mathbb{R}$, $(t, x) \mapsto f_t(x)$ induces a Lipschitz function on $A \cap K$, for every compact subset K of $B \times \mathbb{R}^n$.*

Proof We prove the result by induction on m. The case $m = 0$ being vacuous, assume the result to be true for $(m - 1)$, $m \geq 1$. By Proposition 7.4.2 (see Remark 7.4.3), we may assume that $A = \mathbb{R}^{m+n}$. It is well-known that there is a definable partition $\mathscr{P}$ of the parameter space, such that f is continuous on every $B \times \mathbb{R}^n$, $B \in \mathscr{P}$ (again, see [6, Lemma 5.17 and Exercise 5.21]). Fix an element $B \in \mathscr{P}$ (we shall refine several times the partition $\mathscr{P}$).

We start with the (easier) case where $\dim B < m$. In this case, there is a partition of B such that every element of this partition has a regular vector (using for instance Remark 7.4.11), that, without loss of generality, we can assume to be $e_m \in \mathbf{S}^{m-1}$. Thanks to Proposition 7.4.6, it is therefore enough to deal with the case where B is the graph of a Lipschitz function, say $\xi : D \to \mathbb{R}$, $D \in \mathcal{S}_{m-1}$. The result in this case now follows from the induction hypothesis applied to the function $D \times \mathbb{R}^n \ni (t, x) \mapsto f(t, \xi(t), x)$.

We now address the case $\dim B = m$. The function $B \ni t \mapsto L_{f_t}$ being definable, partitioning B if necessary, we can assume this function to be continuous on this

set. In particular, it is bounded on compact subsets of B. Let Z be the set of points $q \in \Gamma_f$ for which there exists a sequence $q_k \in (\Gamma_f)_{reg}$ tending to q such that

$$(0_{\mathbb{R}^m}, e_{n+1}) \in \lim T_{q_k}(\Gamma_f)_{reg},$$

where e_{n+1} is the last vector of the canonical basis of $\mathbb{R}^{n+1}$. Let $\pi : \mathbb{R}^m \times \mathbb{R}^{n+1} \to \mathbb{R}^m$ denote the projection omitting the last $(n+1)$ coordinates. We claim that $\pi(Z)$ has dimension less than m.

Assume otherwise. Take a (w)-regular definable stratification of Γ_f compatible with Z and let $S \subset Z$ be a stratum such that $\pi(S)$ has dimension m. Let S' be the set of points of S at which $\pi_{|S}$ is a submersion. Since $\pi(S)$ is of dimension m, by Sard's Theorem, the set S' cannot be empty. Moreover, by definition of S', $T_q S'$ is transverse to $\{0_{\mathbb{R}^m}\} \times \mathbb{R}^{n+1}$ at any point q of S'.

Let $q \in S' \subset Z$. By definition of Z, there is a sequence q_k tending to q such that $(0_{\mathbb{R}^m}, e_{n+1}) \in \tau_q := \lim T_{q_k}(\Gamma_f)_{reg}$. The (w) condition ensures that $\tau_q \supset T_q S'$ (S' is a manifold for it is open in S). Consequently, τ_q is transverse to $\{0_{\mathbb{R}^m}\} \times \mathbb{R}^{n+1}$ as well.

But since L_{f_t} is locally bounded (it was assumed to be continuous), the vector e_{n+1} does not belong to $\lim T_{x_k} \Gamma_{f_{t_k}}$, if $q_k = (t_k, x_k)$ in $\mathbb{R}^m \times \mathbb{R}^{n+1}$, which means that

$$(\lim T_{q_k} \Gamma_f) \cap \{0_{\mathbb{R}^m}\} \times \mathbb{R}^{n+1} \neq \lim \left(T_{q_k} \Gamma_f \cap \{0_{\mathbb{R}^m}\} \times \mathbb{R}^{n+1} \right)$$

(since the latter does not contain the vector $(0_{\mathbb{R}^m}, e_{n+1})$ while the former does), and hence, that τ_q cannot be transverse to $\{0_{\mathbb{R}^m}\} \times \mathbb{R}^{n+1}$ (since otherwise the intersection with the limit would be the limit of the intersection). A contradiction.

This establishes that $\dim \pi(Z) < m$. Since we can refine $\mathscr{P}$ into a partition which is compatible with $\pi(Z)$, we thus see that we can suppose $B \subset \mathbb{R}^m \setminus \pi(Z)$ (we are dealing with the case $\dim B = m$).

For $(t, R) \in (\mathbb{R}^m \setminus \pi(Z)) \times [0, +\infty)$ set:

$$\varphi(t, R) := \sup\{\frac{\partial f}{\partial t}(t, x) : x \in \mathbf{B}(0_{\mathbb{R}^n}, R), f \text{ is } \mathscr{C}^1 \text{ at } x\}$$

(which is finite, by definition of Z, since L_{f_t} is bounded). As φ is definable, up to a partition of B, this function may be assumed to be continuous (and thus bounded on compact sets) for $R \geq \zeta(t)$, with $\zeta : B \to \mathbb{R}$ definable function. The function f therefore induces a function which is Lipschitz with respect to the inner metric on every compact set of $B \times \mathbb{R}^n$. By Theorem 7.4.1, up to an extra refinement partition, we can suppose that the inner metric and the outer metric of B are equivalent, which means that so are the inner and outer metrics of $B \times \mathbb{R}^n$, establishing that f is Lipschitz on every compact set of $B \times \mathbb{R}^n$. $\square$

As a matter of fact, the trivialization given by Theorem 7.6.3 may be required to satisfy the Lipschitz condition with respect to the parameters on compact sets:

Proposition 7.6.6 *Let $A \in \mathcal{S}_{m+n}$. Refining the partition provided by Theorem 7.6.3, we may obtain the following extra fact: for any element B of this partition, the trivialization $H : B \times A_{t_0} \to A_B$, $(t, x) \mapsto (t, h_t(x))$, induces a bi-Lipschitz mapping on $(B \times A_{t_0}) \cap K$, for every compact subset K of $B \times \mathbb{R}^n$.*

Proof This is a consequence of Theorem 7.6.3 and Proposition 7.6.5.   □

Remark 7.6.7 As explained in Remark 7.2.8, refining the partition if necessary, we can assume that the Lipschitz constants of h_t and h_t^{-1} are locally bounded on each element B of the partition. It is indeed worthy of notice that, thanks to Łojasiewicz inequality, we can show that $L_{h_t} \lesssim d(t, cl(B) \setminus B)^{-k}$, with $k \in \mathbb{N}$. This is an attractive feature of Theorem 7.6.3 since bi-Lipschitz trivializations that are obtained by integration of vector fields (like on Mostowski's Lipschitz stratifications, see Sect. 7.2.1) generally have Lipschitz constants that are bounded by functions of type $\exp\left(\frac{C}{d(t, cl(B) \setminus B)}\right)$, $C > 0$, (by Gronwall's inequality).

In the above proposition, the compactness assumption is essential, as shown by the following example.

Example 7.6.8 Consider the set $A = \{(t, x, y) \in \mathbb{R}^3 : y = tx\}$. This set is bi-Lipschitz trivial along $\mathbb{R}$. However, it is easy to check that we could not require a trivialization $H(t, x, y)$ to be bi-Lipschitz with respect the parameter t, even along a compact interval (i.e., we have to require that x and y also remain in a compact set in order to ensure Lipschitzness with respect to t).

The inconvenience of bi-Lipschitz triviality theorems that are provided by integration of Lipschitz vector fields, such as the bi-Lipschitz version of Thom-Mather First Isotopy Lemma that holds on Mostowski's Lipschitz stratifications [17, 23], is that they do not provide definable trivializations. Theorem 7.6.3 enables us to construct stratifications that are locally definably bi-Lipschitz trivial, which is valuable for applications [34].

Corollary 7.6.9 *Given a definable set X, we can find a definable stratification of this set which is locally definably bi-Lipschitz trivial. This stratification may be required to be compatible with finitely many given definable subsets of X.*

Proof This follows from standard arguments of construction of definable stratifications. Theorem 7.6.3 and Proposition 7.6.5 yield that local definable bi-Lipschitz triviality holds generically, which is sometimes rephrased by saying that it is a stratifying condition (see for instance [35, Proposition 2.7.5] for more details). We can require our stratification to satisfy Whitney's (a) condition (which is also a stratifying condition [5, 27, 35]), which yields that Σ_{x_0} (in (ii) of Definition 7.2.9) exclusively consists of manifolds.   □

References

1. L. Birbrair, A. Fernandes, V. Grandjean, Thick-thin decomposition for real definable isolated singularities, Indiana Univ. Math. J. 66 (2017), no. 2, 547–557.
2. L. Birbrair, A. Fernandes, V. Grandjean, D. O'Shea, Choking horns in Lipschitz geometry of complex algebraic varieties, Appendix by Walter D. Neumann. J. Geom. Anal. 24 (2014), no. 4, 1971–1981.
3. L. Birbrair, A. Fernandes, V. Grandjean, A. Gabrielov, Lipschitz contact equivalence of function germs in $\mathbb{R}^2$, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 17 (2017), no. 1, 81–92.
4. L. Birbrair, W. D. Neumann, A. Pichon, The thick-thin decomposition and the bilipschitz classification of normal surface singularities, Acta Math. 212 (2014), no. 2, 199–256.
5. J. Bochnak, M. Coste, M.-F. Roy, Géométrie algébrique réelle, Springer 1987.
6. M. Coste, An introduction to o-minimal geometry, Dip. Mat. Univ. Pisa, Dottorato di Ricerca in Matematica, Istituti Editoriali e Poligrafici Internazionali, Pisa 2000.
7. M. Czapla, Definable triangulations with regularity conditions, Geom. Topol. vol. 16 (2012), 2067–2095.
8. T. de Pauw, R. Hardt, Linear isoperimetric inequality for normal and integral currents in compact subanalytic sets, Journal of Singularities vol. 24 (2022), 145–168.
9. L. van den Dries, Tame topology and o-minimal structures, London Mathematical Society Lecture Note Series, 248. Cambridge University Press, Cambridge, 1998. x+180 pp.
10. L. van den Dries, P. Speissegger, O-minimal preparation theorems, Model theory and applications, 87–116, Quad. Mat., 11, Aracne, Rome, 2002.
11. S. Guillermou, G. Lebeau, A. Parusiński, P. Schapira, J.-P. Schneiders, Subanalytic sheaves and Sobolev spaces. Astérisque No. 383 (2016), xvi+120 pp.
12. I. Halupczok, Y. Yin, Lipschitz stratifications in power-bounded o-minimal fields, Journal of the European Mathematical Society, 20 (11). pp. 2717–2767.
13. R. M. Hardt, Semi-algebraic local-triviality in semi-algebraic mappings, Amer. J. Math. 102 (1980), no. 2, 291–302.
14. K. Kurdyka, On a subanalytic stratification satisfying a Whitney property with exponent 1. Real algebraic geometry (Rennes, 1991), 316–322, Lecture Notes in Math., 1524, Springer, Berlin, 1992.
15. K. Kurdyka, A. Parusiński, Quasi-convex decomposition in o-minimal structures. Application to the gradient conjecture, Advanced Studies in Pure Mathematics Vol. 43 (2006) 137 – 177.
16. J.-M. Lion, J.-P. Rolin, Théorème de préparation pour les fonctions logarithmico-exponentielles, Ann. Inst. Fourier 47 (1997), no. 3, 859–884.
17. T. Mostowski, Lipschitz Equisingularity, Dissertationes Math., CCXLIII, 1985, PWN, Warszawa.
18. N. Nguyen, G. Valette, Lipschitz stratifications in o-minimal structures, Ann. Sci. Ecole Norm. Sup. (4) 49, (2016), no. 2, 399–421.
19. N. Nguyen, Regular projection in o-minimal structures, Annals of Pure and Applied Logic, vol. 174, Issue 2,(2023), 103199.
20. M. Oudrane, Regular projections and regular covers in o-minimal structures, Ann. Polon. Math. 130 (2023), 63–83.
21. A. Parusiński, Regular projections for subanalytic sets, C. R. Acad. Sci. Paris Sér. I, Math. 307 (1988), 343–347.
22. A. Parusiński, Lipschitz stratification of real analytic sets, Singularities (Warsaw, 1985), 323–333, Banach Center Publ., 20, PWN, Warsaw, 1988.
23. A. Parusiński, Lipschitz stratification of subanalytic sets, Ann. Sci. Ecole Norm. Sup. (4) 27 (1994), no. 6, 661–696.
24. A. Parusiński, On the preparation theorem for subanalytic functions, New Developments in Singularity Theory (Cambridge, 2000), NATO Sci. Ser. II Math. Phys. Chem., vol 21, Kluwer Academic, Dordrecht (2001), pp. 192–215.

25. W. Pawłucki, Lipschitz cell decomposition in o-minimal structures, Illinois J. Math. 52 (2009), no. 3, 1045–1063.
26. M. A. S. Ruas, G. Valette, C^0 and bi-Lipschitz $\mathscr{K}$-equivalence, Math. Z., 269 (2011), 293–308.
27. Ta Lê Loi, Verdier and strict Thom stratifications in o-minimal structures, Illinois J. Math. 42 (1998), no. 2, 347-356.
28. A. Valette, G. Valette, Poincaré inequality on subanalytic sets, J. Geom. Anal. 31 (2021), 1046–10472.
29. A. Valette, G. Valette, Trace operators on bounded subanalytic manifolds, arXiv:2101.10701.
30. G. Valette, Lipschitz triangulations, Illinois J. Math. 49 (2005), issue 3, 953–979.
31. G. Valette, The link of the germ of a semi-algebraic metric space, Proc. Amer. Math. Soc. 135 (2007), no. 10, 3083–3090.
32. G. Valette, On metric types that are definable in an o-minimal structure, J. Symbolic Logic 73 (2008), no. 2, 439–447.
33. G. Valette, L^∞ cohomology is intersection cohomology, Adv. Math. 231 (2012), no. 3-4, 1818-1842.
34. G. Valette, Poincaré duality for L^p cohomology on subanalytic singular spaces, Math. Ann, 380 (2021), 789–823.
35. G. Valette, On subanalytic geometry, survey. Available at http://www2.im.uj.edu.pl/gkw/sub.pdf
36. G. Valette, On Sobolev spaces of bounded subanalytic manifolds, arXiv:2111.12338.
37. G. Valette, On the Laplace equation on bounded subanalytic manifolds, arXiv:2208.11931.