Chapter 11
Hilbert-Samuel Multiplicity and Finite Projections

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Contents

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Abstract In this (mainly) expository notes, we study the multiplicity of a local Noetherian ring $(B, \mathfrak{m})$ at an $\mathfrak{m}$-primary ideal I, paying special attention to the geometrical aspects of this notion. To this end, we will be considering suitably defined finite extensions $S \subset B$, with S regular. We will explore some applications like the explicit description of the equimultiple locus of an equidimensional variety, or the computation of the asymptotic Samuel function.

11.1 Introduction

When it comes to measure how bad a singularity is, the case of a hypersurface in the affine space can provide some intuition. Let us assume that k is a field, let $A = k[X_1, \dots, X_n]$ and let $f \in A$ be a non-zero polynomial defining a hypersurface $H \subset \mathbb{A}^n_k$. For a given point $\zeta \in H$, denote by $\mathfrak{p} \subset A$ the corresponding prime ideal. We can consider the *order of* f *at* $\mathfrak{p}$,

$$\nu_\zeta(f) := \max\{n : f \in \mathfrak{p}^n A_{\mathfrak{p}}\} \geqslant 1.$$

The hypersurface H is regular at ζ if $\nu_\zeta(f) = 1$, otherwise H is singular at ζ. In such case, $\nu_\zeta(f) \geqslant 2$, and we can think that the larger $\nu_\zeta(f)$ be, the more singular the point will be. It is quite natural to ask whether this measurement can be made directly at the local ring of the hypersurface at ζ. To address this question, let us first fix some notation. We will use $k(\mathfrak{p})$ to denote the residue field at $\mathfrak{p}$ and $\bar{\mathfrak{p}}^i$ to refer to $(\mathfrak{p}^i + \langle f\rangle/\langle f\rangle)$. Let $B = A/\langle f\rangle$, and set $B_{\bar{\mathfrak{p}}} = (A/\langle f\rangle)_{\bar{\mathfrak{p}}}$. Then the value $\nu_\zeta(f)$ has an impact on the dimensions of the $k(\mathfrak{p})$-vector spaces:

$$B_{\bar{\mathfrak{p}}}/\bar{\mathfrak{p}}B_{\bar{\mathfrak{p}}},\quad B_{\bar{\mathfrak{p}}}/\bar{\mathfrak{p}}^2B_{\bar{\mathfrak{p}}},\quad B_{\bar{\mathfrak{p}}}/\bar{\mathfrak{p}}^3B_{\bar{\mathfrak{p}}}, \dots,\quad B_{\bar{\mathfrak{p}}}/\bar{\mathfrak{p}}^\ell B_{\bar{\mathfrak{p}}}, \dots,$$

or equivalently, on the dimension of the $k(\mathfrak{p})$-vector spaces

$$B_{\bar{\mathfrak{p}}}/\bar{\mathfrak{p}}B_{\bar{\mathfrak{p}}},\quad \bar{\mathfrak{p}}B_{\bar{\mathfrak{p}}}/\bar{\mathfrak{p}}^2B_{\bar{\mathfrak{p}}},\quad \bar{\mathfrak{p}}^2B_{\bar{\mathfrak{p}}}/\bar{\mathfrak{p}}^3B_{\bar{\mathfrak{p}}}, \dots, \bar{\mathfrak{p}}^{\ell-1}B_{\bar{\mathfrak{p}}}/\bar{\mathfrak{p}}^\ell B_{\bar{\mathfrak{p}}}, \dots$$

Notice that the quotients of the later sequence correspond to the j-th degree pieces of the graded ring $\mathrm{Gr}_{\bar{\mathfrak{p}}B_{\bar{\mathfrak{p}}}}(B_{\bar{\mathfrak{p}}}) = \bigoplus_i \bar{\mathfrak{p}}^i B_{\bar{\mathfrak{p}}}/\bar{\mathfrak{p}}^{i+1}B_{\bar{\mathfrak{p}}}$ of the local ring $B_{\bar{\mathfrak{p}}} = \mathcal{O}_{H,\zeta}$.

Actually, the previous approach can be applied in a more general setting. We can assume, for instance, that B is the coordinate ring of an arbitrary algebraic variety defined over a field k, or even, just any local Noetherian ring. Then, for a prime $\mathfrak{p} \subset B$, *the multiplicity* B *at* $\mathfrak{p}$ comes in naturally as an invariant when trying to measure the growth of dimension of the graded pieces of the graded ring

$$\mathrm{Gr}_{\mathfrak{p}B_{\mathfrak{p}}}(B_{\mathfrak{p}}) = \bigoplus_{i \geqslant 0} \mathfrak{p}^i B_{\mathfrak{p}}/\mathfrak{p}^{i+1}B_{\mathfrak{p}}$$

as $k(\mathfrak{p})$-vector spaces, or equivalently, the growth of the lengths of the $B_{\mathfrak{p}}$-modules, $B_{\mathfrak{p}}/\mathfrak{p}^{i+1}B_{\mathfrak{p}}$, for $i = 0, 1, \ldots$.

To be more precise, this growth is encoded asymptotically by the so called *Hilbert-Samuel polynomial* of $B_{\mathfrak{p}}$ at $\mathfrak{p}$. This is a polynomial of degree $d = \dim(B_{\mathfrak{p}})$ and the *multiplicity* at $\mathfrak{p}$, $e_{B_{\mathfrak{p}}}$, is (up to some suitable factor) the leading coefficient of that polynomial.

But, what does the multiplicity tell us about the singularity at a given point? If $\mathfrak{p}$ is a regular point in $\mathrm{Spec}(B)$ then $\mathrm{Gr}_{\mathfrak{p}B_{\mathfrak{p}}}(B_{\mathfrak{p}})$ is isomorphic to a polynomial ring in d-variables with coefficients in $k(\mathfrak{p})$. In such case the Hilbert-Samuel polynomial can be easily computed and it can be checked that the multiplicity at $\mathfrak{p}$ equals one. Moreover, under some conditions, multiplicity one implies regularity. As another example, if $B = R/\langle f\rangle$, where R is a regular ring, then the multiplicity can be computed in terms of a local writing of the equation f at each point. More precisely, if the order of f at a prime $\mathfrak{p} \in \mathrm{Spec}(R)$ is m, then the multiplicity at the corresponding prime, $\mathfrak{p}/\langle f\rangle$, in B equals m. In general, however, there is no apparent algebraic method that brings clear geometric insight on the meaning of the multiplicity.

The purpose of these notes is to focus on the geometric aspects of the multiplicity. To fix ideas, assume that B is an equidimensional ring of finite type over a field k. Now suppose that we want to present B as a finite extension of a regular ring. To do so, we could start, for instance, by considering Noether's Normalization Lemma. This tells us that if the Krull dimension of B is d, then B is a finite extension of a polynomial ring in d variables with coefficients in k, $S = k[X_1, \ldots, X_d]$. Let $K(S)$ be the fraction field of S. Then it can be shown that the multiplicity of B at any prime $\mathfrak{p}$, $e_{B_{\mathfrak{p}}}(\mathfrak{p}B_{\mathfrak{p}})$, is bounded above by the generic rank of the extension $S \hookrightarrow B$, i.e.,

$$e_{B_{\mathfrak{p}}}(\mathfrak{p}B_{\mathfrak{p}}) \leqslant [K(S) \otimes_S B : K(S)].$$

Actually, the multiplicity can be defined at any $\mathfrak{p}$-primary ideal I, $e_{B_{\mathfrak{p}}}(IB_{\mathfrak{p}})$, and it can be equally shown that, for a finite extension with S regular the same inequality holds in this more general setting, if $I \cap S$ is a prime, i.e.,

$$e_{B_{\mathfrak{p}}}(IB_{\mathfrak{p}}) \leqslant [K(S) \otimes_S B : K(S)]. \tag{11.1}$$

Here, we will be interested in the study of finite extensions $S \subset B$ with S regular for which the equality in (11.1) holds, and then we will say that $S \subset B$ is *finite-transversal with respect to I*.

Finite-transversal extensions do not always exists, see Sect. 11.3.3. However they can be constructed at the completion of the local ring $(B_{\mathfrak{p}}, \mathfrak{p}B_{\mathfrak{p}})$, maybe after enlarging the residue field, see Sect. 11.3.4.

In [30] Villamayor pointed out that finite-transversal extensions with respect to a prime $\mathfrak{p}$ can always be constructed in a local étale neighborhood of $(B_{\mathfrak{p}}, \mathfrak{p}B_{\mathfrak{p}})$, when $B_{\mathfrak{p}}$ is essentially of finite type over a perfect field k. In [7, Appendix A] such construction is described in full detail. Here we will reproduce some of the arguments in [7] with a twofold purpose. On the one hand, to show that the same procedure can be used to construct finite-transversal morphisms with respect to any $\mathfrak{p}$-primary ideal I. On the other, we will follow the different parts of the proof to illustrate distinct aspects of this construction with a list of examples.

As it turns out, finite-transversal morphisms play a role in describing the top multiplicity locus of B, via the so called *presentations of the multiplicity*. Such description, given by Villamayor in [30], was presented to show a stronger result, namely, that it is possible to resolve the singularities of an algebraic variety using the multiplicity as the main invariant (this was a question posed by Hironaka in [16]). This approach to resolution will be discussed in Sect. 11.5.

In addition, finite-transversal morphisms appear as well as a tool for the computation of the asymptotic Samuel function. In [15], Hickel gives a procedure for the computation of the asymptotic Samuel function respect to an $\mathfrak{m}$-primary ideal I in an equicharacteristic local ring $(R, \mathfrak{m})$. To this end, he considers the completion $\hat{R}$, where he constructs a finite-transversal extension with respect to $I\hat{R}$. Here we will see that his arguments are equally valid if we can find a finite transversal projection in a suitably defined étale neighborhood of $(R, \mathfrak{m})$.

This manuscript is mostly expository and some of the statements presented here are slight variations of results in [30] and also in [15]. Precise references to these articles will be given along the paper.

These notes are organized as follows. The multiplicity at an ideal of a local Noetherian ring is treated in Sect. 11.2, and some known properties are described. Finite-transversal morphisms are defined and constructed in Sect. 11.3. Finally, applications are addressed in Sects. 11.4, 11.5 and 11.6.

Notation For a quotient of a polynomial ring with coefficients in a ring A, $B = A[X_1, \ldots, X_n]/J$, we will use lowercase, x_i, to denote the class of the variable X_i in B, for $i = 1, \ldots, n$. This notation will be used in the examples along the paper.

11.2 The Multiplity Function

Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, and consider the function:

$$\begin{aligned} \mathrm{HS}_{R,\mathfrak{m}} : \mathbb{N} &\to \mathbb{N} \\ \ell &\mapsto \dim_k(R/\mathfrak{m}^{\ell+1}). \end{aligned}$$

This is referred to as the *Hilbert-Samuel function of R at* $\mathfrak{m}$.

Theorem 11.2.1 *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. Then the Hilbert-Samuel function $HS_{R,\mathfrak{m}}$ is of polynomial type, i.e., there exists a polynomial $p_{R,\mathfrak{m}}(X) \in \mathbb{Q}[X]$ such that for $\ell \gg 0$,*

$$HS_{R,\mathfrak{m}}(\ell) = p_{R,\mathfrak{m}}(\ell).$$

In addition, the degree of $p_{R,\mathfrak{m}}(X)$ *equals* $d = \dim(R)$*, the Krull dimension of the ring* R*. Moreover,*

$$p_{R,\mathfrak{m}}(X) = e_R(\mathfrak{m})\frac{X^d}{d!} + \dots,$$

where $e_R(\mathfrak{m}) \in \mathbb{N}$*.*

For a proof see for instance [29, Theorem 11.1.3], where the theorem is stated in a much more general setting. We refer to $p_{R,\mathfrak{m}}(X)$ as the *Hilbert-Samuel polynomial of* R *with respect to* $\mathfrak{m}$, and we say that $e_R(\mathfrak{m}) \in \mathbb{N}$ is the *multiplicity of the local ring* R *at* $\mathfrak{m}$ or simply *the multiplicity of* R. Sometimes we write e_R to refer to $e_R(\mathfrak{m})$.

Example 11.2.2 If $(R, \mathfrak{m}, k)$ is a d dimensional regular local ring, then

$$\dim_k R/\mathfrak{m}^{\ell+1} = \binom{d+\ell}{d},$$

and

$$p_{R,\mathfrak{m}}(X) = \frac{X^d}{d!} + \dots.$$

Therefore, for a regular local ring, $e_R = 1$. The converse holds if we require R to be unmixed (i.e., formally equidimensional, that is, the $\mathfrak{m}$-adic completion of R is equidimensional).

Theorem 11.2.3 *[18, Theorem 40.6] A Noetherian local ring* $(R, \mathfrak{m})$ *is regular if and only if it is unmixed and* $e_R = 1$.

Remark 11.2.4 If $(R, \mathfrak{m}, k)$ is a regular local ring, $f \in R$ is a non-zero element, and $B = R/\langle f\rangle$, then

$$\nu_{\mathfrak{m}}(f) = e_B(\mathfrak{m}/\langle f\rangle),$$

see [29, Example 11.2.8]. Hence, in the hypersurface case the order of the defining equation at a given point equals the multiplicity at the point.

11.2.1 Multiplicity at $\mathfrak{m}$-Primary Ideals

Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of dimension d. As we will see in forthcoming sections, sometimes it is convenient to work with arbitrary $\mathfrak{m}$-primary ideals. If $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal, then, in the same way as we did before, the following Hilbert-Samuel function can be defined:

$$\begin{aligned} \mathrm{HS}_{R,\mathfrak{a}} : \mathbb{N} &\to \mathbb{N} \\ \ell &\mapsto \lambda_R(R/\mathfrak{a}^{\ell+1}), \end{aligned}$$

where λ_R denotes the length as R-module. An analogous of Theorem 11.2.1 holds, and there is a polynomial, *the Hilbert-Samuel polynomial of* R *at* $\mathfrak{a}$, $P_{R,\mathfrak{a}}(X)$, so that for $\ell \gg 0$,

$$\mathrm{HS}_{R,\mathfrak{a}}(\ell) = P_{\mathfrak{a}}(\ell),$$

and moreover,

$$P_{\mathfrak{a}}(X) = e_R(\mathfrak{a})\frac{X^d}{d!} + \dots,$$

with $e_R(\mathfrak{a}) \in \mathbb{N}$ (see [29, Theorem 11.1.3]). The positive integer $e_R(\mathfrak{a})$ is the *multiplicity of* R *at the* $\mathfrak{m}$*-primary ideal* $\mathfrak{a}$. From the definition it follows that

$$e_R(\mathfrak{a}) \geqslant e_R = e_R(\mathfrak{m}). \tag{11.2}$$

It is quite natural to ask under which conditions the previous inequality is an equality. First, recall that an element $r \in R$ belongs to the integral closure, $\overline{\mathfrak{a}}$, of $\mathfrak{a}$ if

$$r^{\ell} + a_1 r^{\ell-1} + \dots a_{\ell-1} r + a_{\ell} = 0$$

for some $\ell \in \mathbb{N}_{>0}$ and some $a_i \in \mathfrak{a}^i$, $i = 1, \dots, \ell$ (see [29, Section 1.1] for more details and properties). If $\mathfrak{a}$ and $\mathfrak{b}$ are two $\mathfrak{m}$-primary ideals with $\overline{\mathfrak{a}} = \overline{\mathfrak{b}}$, then $e_R(\mathfrak{a}) = e_R(\mathfrak{b})$ (see [29, Proposition 11.2.1]). What can be said if $e_R(\mathfrak{a}) = e_R(\mathfrak{b})$? The following theorem of Rees settles this question.

Theorem 11.2.5 *[24], [29, Theorem 11.3.1] Let* $(R, \mathfrak{m})$ *be a formally equidimensional Noetherian local ring and let* $\mathfrak{a} \subset \mathfrak{b}$ *be two* $\mathfrak{m}$*-primary ideals. Then* $\mathfrak{b} \subset \overline{\mathfrak{a}}$ *if and only if* $e_R(\mathfrak{a}) = e_R(\mathfrak{b})$.

In a Noetherian ring, if $\mathfrak{a} \subset \mathfrak{b}$ and $\mathfrak{b} \subset \overline{\mathfrak{a}}$ then $\mathfrak{a}$ is a *reduction of* $\mathfrak{b}$. See [29, Chapter 8] for further details.

There is a similar statement as that of Theorem 11.2.5 for ideals that are not primary to the maximal ideal of the local ring $(R, \mathfrak{m})$. But before stating that theorem we need another definition.

Definition 11.2.6 Let $(R, \mathfrak{m})$ be a local Notherian ring and let $I \subset R$ be an ideal. The *analytic spread of* I, $\ell(I)$, is defined as the Krull dimension of the ring $R[It]/\mathfrak{m}R[It]$, where t is an indeterminate. This is the same as the Krull dimension of the ring

$$\mathrm{Gr}_I(R) \otimes k(\mathfrak{m}) = \bigoplus_{i=0}^{\infty} I^i/(I^i\mathfrak{m}).$$

Observe that the analytic spread of I is the dimension of the fiber over $\mathfrak{m}$ of the blow up of R at I.

See [9] for the generalization of the notion of analytic spread to arbitrary filtrations of ideals and some advances about this invariant.

The following theorem of Böger generalizes Rees's Theorem.

Theorem 11.2.7 *[6], [29, Corollary 11.3.2] Let* $(R, \mathfrak{m})$ *be a Noetherian formally equidimensional local ring, and let* $I \subseteq J$ *be two ideals with* $\ell(I) = \mathrm{ht}(I)$. *Then* $J \subset \overline{I}$ *if and only if* $e_{R_\mathfrak{p}}(I_\mathfrak{p}) = e_{R_\mathfrak{p}}(J_\mathfrak{p})$ *for every prime ideal* $\mathfrak{p}$ *minimal over* I.

In the following lines we mention a couple of properties of the multiplicity: The additivity (Theorem 11.2.8) and the behavior under flat extensions Sect. 11.2.2.

Theorem 11.2.8 *[29, Theorem 11.2.4] Let* $(R, \mathfrak{m})$ *be a local Noetherian ring, let* $\mathfrak{a}$ *be an* $\mathfrak{m}$*-primary ideal, and let* $\mathscr{P}$ *be the set of minimal prime ideals* $\mathfrak{p}$ *of* R *such that* $\dim(R/\mathfrak{p}) = \dim R$. *Then*

$$e_R(\mathfrak{a}) = \sum_{\mathfrak{p}\in\mathscr{P}} e_{R/\mathfrak{p}}(\mathfrak{a}).$$

Geometrically, Theorem 11.2.8 says that the multiplicity at a point of an equidimensional algebraic variety is the sum of the multiplicities at each of the irreducible components containing the point. If the variety is not equidimensional, then the only additions to the multiplicity at a given point come from the irreducible components of maximum dimension that contain the point.

11.2.2 The Multiplicity and Flat Extensions

Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal and suppose that $(R, \mathfrak{m}, k) \to (R', \mathfrak{m}', k')$ is a flat extension of local Noetherian rings. Then it can be checked that

$$e_{R'}(\mathfrak{a}R') = e_R(\mathfrak{a}R) \cdot \lambda_{R'}(R'/\mathfrak{m}R'). \tag{11.3}$$

See [14, Chapter I, Proposition 5.1].

From there it follows that if $(R', \mathfrak{m}', k')$ is an étale extension of $(R, \mathfrak{m}, k)$ then $e_R(\mathfrak{m}) = e_{R'}(\mathfrak{m}')$. And the same equality holds if R' is the $\mathfrak{m}$-adic completion of R, i.e., $e_R(\mathfrak{m}) = e_{\hat{R}}(\hat{\mathfrak{m}})$.

11.2.3 Shortcuts for the Computation of the Multiplicity

We already saw how the multiplicity at a point of a hypersurface in a regular ring is related to the order of the ideal of definition. In general, there is no such straightforward procedure to compute this invariant. However, there are some cases in which the calculation becomes easier.

Suppose that $(R, \mathfrak{m}, k)$ is a Noetherian local ring of Krull dimension d, and let $\mathfrak{a}$ be an *ideal of definition of* R, that is, an $\mathfrak{m}$-primary ideal generated by d elements, $\mathfrak{a} = \langle a_1, \ldots, a_d \rangle$. Then:

$$e_R(\mathfrak{a}) \leqslant \lambda_R(R/\mathfrak{a}), \tag{11.4}$$

and the equality holds if and only if R is Cohen-Macaulay. This follows from considering the morphism of $R/\mathfrak{a}$-graded rings:

$$\begin{aligned} \varphi : (R/\mathfrak{a})[X_1, \ldots, X_d] &\to \mathrm{Gr}_{\mathfrak{a}}(R) = R/\mathfrak{a} \oplus \mathfrak{a}/\mathfrak{a}^2 \oplus \ldots \oplus \mathfrak{a}^n/\mathfrak{a}^{n+1} \oplus \ldots \\ X_i &\mapsto \bar{a}_i \in \mathfrak{a}/\mathfrak{a}^2. \end{aligned}$$

Since φ is surjective,

$$\lambda_R(\mathfrak{a}^j/\mathfrak{a}^{j+1}) \leqslant \lambda_R\left((R/\mathfrak{a})[X_1, \ldots, X_d]_j\right),$$

where $(R/\mathfrak{a})[X_1, \ldots, X_d]_j$ denotes the j-th degree piece of the graded $R/\mathfrak{a}$-algebra $(R/\mathfrak{a})[X_1, \ldots, X_d]$. Therefore,

$$\lambda_R(R/\mathfrak{a}^{j+1}) \leqslant \lambda_R(R/\mathfrak{a})\binom{j+d}{d}.$$

Notice that if R is Cohen-Macaulay then φ is an isomorphism and the equality in (11.4) holds. For a complete proof of this fact see [27, Theorem 19.3.11]. See also [29, Proposition 11.1.10].

The following statement gives a criterion for algorithmic computation of the multiplicity in the case of k-algebras of finite type, with k a field. For a given monomial order $>$ in a polynomial ring $k[X_1, \ldots, X_r]$ and for an ideal $I \subset k[X_1, \ldots, X_r]$, we use $L(I)$ to refer to the ideal generated by the leading monomials of the non-zero elements in I with respect to $>$.

Proposition 11.2.9 *[13, Proposition 5.5.7] Let $>$ be a local degree ordering on $k[\underline{X}] = k[X_1, \ldots, X_r]$ (that is, $deg(X^\alpha) > deg(X^\beta)$ implies $X^\alpha < X^\beta$). Let $I \subset \langle \underline{X} \rangle = \langle X_1, \ldots, X_r \rangle$ be an ideal and let $B = k[\underline{X}]_{\langle \underline{X} \rangle}/I$. Let $\mathfrak{m} = \langle \underline{X} \rangle/I$. Then*

$$HS_{B,\mathfrak{m}} = HS_{(k[\underline{X}]_{\langle \underline{X} \rangle}/L(I)),\langle \underline{X} \rangle}.$$

In particular, $k[\underline{X}]/I$ and $k[\underline{X}]_{\langle \underline{X} \rangle/L(I)}$ have the same multiplicity with respect to $\langle \underline{X} \rangle$.

11.2.4 Geometric Interpretation of the Multiplicity [14, Chapter I, pg. 15]

By examining the hypersurface case we can get some insight on the geometric meaning of the multiplicity. So let us consider a hypersurface $H \subset \mathbb{A}^n_k$ with defining ideal $\langle f \rangle \subset k[X_1, \ldots, X_n]$ and suppose that $\zeta \in H$ is the closed point with maximal ideal $\mathfrak{m} = \langle X_1, \ldots, X_n \rangle$. Assume that f is regular with respect to X_n, (i.e.,

$f(0, \ldots, 0, X_n) \neq 0$; this can always be assumed after a linear change of variables and a suitable finite extension of k). Then we can apply Weierstrass Preparation Theorem [13, Corollary 6.2.8] at the $\mathfrak{m}$-adic completion of $k[X_1, \ldots, X_n]$, and assume that, up to multiplication by a unit, f can be written as a polynomial in the variable X_n with coefficients in the ring $k[[X_1, \ldots, X_{n-1}]]$, i.e.,

$$f = X_n^r + a_1 X_n^{r-1} + \ldots + a_r \tag{11.5}$$

with $a_i \in k[[X_1, \ldots, X_{n-1}]]$. Actually, Weierstrass Preparation Theorem holds at étale neighborhood of the local ring at ζ (see [25, Theorem 6.7]), thus the expression (11.5) can also be seen as a polynomial in X_n with coefficients in some regular local ring S of dimension $n-1$, with $S[X_n]$ an étale extension of $k[X_1, \ldots, X_n]$.

Hence, after a convenient étale neighborhood of ζ is selected, we can assume that the coordinate ring of H is isomorphic to

$$B = S[X_n]/\langle X_n^r + a_1 X_n^{r-1} + \ldots + a_r \rangle.$$

Observe that B is a finite extension of S and if $K(S)$ is fraction field of S then the *generic rank* of the extension $S \subset B$ is given by $[B \otimes_S K(S) : K(S)] = r$.

Letting $\mathfrak{m}_B = \mathfrak{m}/\langle f \rangle$ we have that

$$e_B(\mathfrak{m}_B) = \nu_{\mathfrak{m}}(f) \leqslant r = [B \otimes_S K(S) : K(S)]. \tag{11.6}$$

From here it follows that, in a neighborhood of ζ, H cannot be finitely projected to a regular variety $Z = \mathrm{Spec}(S)$ with generic rank lower than $e_B(\mathfrak{m}_B)$.

Notice that, in the previous discussion, the maximal ideal $\mathfrak{m}_B \subset B$ dominates a maximal ideal $\mathfrak{m}_S \subset S$ and therefore $\mathfrak{m}_S B$ generates an $\mathfrak{m}_B$-primary ideal $I \subset B$. As we will see in the next section, the following inequality holds:

$$e_B(I) \leqslant [B \otimes_S K(S) : K(S)]. \tag{11.7}$$

In fact, we will see that the previous inequalities hold for the localization at a point of the affine coordinate ring B of an arbitrary variety over a field k. Recall that by Noether's Normalization Lemma the k-algebra B can be expressed as a finite extension of a polynomial ring over k.

However:

- it is not obvious that the generic rank of such an extension be bounded below by the maximum multiplicity of B, and
- it is not immediate either that one can find suitable finite projections to regular rings where the equality in (11.7) holds.

All these questions will be properly addressed in the next section.

11.3 Zariski's Multiplicity Formula

The starting point of this section is precisely the last discussion from the previous one. There, we were trying to understand the geometric meaning of the multiplicity of a local Noetherian ring $(R, \mathfrak{m})$ at an $\mathfrak{m}$-primary ideal $I \subset R$. To this end we want to consider finite extensions $S \subset R$ where S is a local regular ring. With this objective in mind, the goal of this section is twofold. On the one hand, we will see that under mild assumptions the inequality (11.7) holds. This will be a consequence of Zariski's multiplicity formula stated in Theorem 11.3.1 below. On the other hand, we will discuss the problem of finding a suitable finite extension $S \subset R$ so that the equality in (11.7) holds. This will lead us to the notion of *finite-transversal morphisms* which will be discussed in the second part of the section.

11.3.1 Zariski's Multiplicity Formula for Finite Projections

Our purpose is to study Zariski's multiplicity formula, which relates multiplicities in a finite extension of rings.

Let $A \subset B$ be a finite extension of rings. If A is local with maximal ideal $\mathfrak{M}$ then B is semi-local (see for instance [31, Th. 15, page 276]). Denote by $Q_1, \dots, Q_r$ the maximal ideals of B. Note that the set $\{Q_1, \dots, Q_r\}$ corresponds to the fiber over $\mathfrak{M}$ of the finite morphism,

$$\mathrm{Spec}(B) \to \mathrm{Spec}(A).$$

As we will see, Zariski's formula relates the multiplicity of A at $\mathfrak{M}$ to the multiplicities of the local rings B_{Q_i}, $i = 1, \dots, r$, at the extension of the ideal $\mathfrak{M}B_{Q_i}$.

Theorem 11.3.1 *[31, Theorem 24, page 297 and Corollary 1, page 299] With the previous assumptions and notation, suppose furthermore that $(A, \mathfrak{M})$ is a Noetherian local domain, that B is equidimensional, and that no non-zero element of A is a zero divisor in B. Denote by $K = K(A)$ the quotient field of A, and let $L = K \otimes_A B$. Let k_0 be the residue field of A, and let k_i be the residue field of B_{Q_i}, $i = 1, \dots, r$. Then:*

$$[L : K]e_A(\mathfrak{M}) = \sum_{i=1}^{r} [k_i : k_0] e_{B_{Q_i}}(\mathfrak{M}B_{Q_i}).$$

Note that from our assumptions $\dim(B_{Q_i}) = \dim(A) = d$ for all $i = 1, \dots, r$, and hence all the Hilbert polynomials that are involved in the formula have degree d.

Example 11.3.2 Let $A = k[Y]_{\langle Y\rangle}$ be the localization of the polynomial ring in one variable at the maximal ideal $\langle Y\rangle$. Here $\mathfrak{M} = \langle Y\rangle A$. Let $B = A[X]/\langle f\rangle$ where $f = X^a(X^2+1)^b + Y^c$ and $c > \max\{a, b\}$.

The extension $A \subset B$ is finite, and the generic rank is $[K(A)\otimes_A B : K(A)] = a + 2b = \deg_X(f)$. Since A is regular, we have that $e_A(\mathfrak{M}) = 1$.

Assume that $k = \mathbb{R}$. Then there are two maximal ideal ideals in B, Q_1 and Q_2, corresponding to $\langle x, y\rangle$ and $\langle x^2+1, y\rangle$, respectively. The residue field of B_{Q_1} is again $\mathbb{R}$ and the residue field of B_{Q_2} is $\mathbb{C}$. Hence Zariski's multiplicity formula is expanded as follows:

$$(a+2b)\cdot 1 = [L : K]e_A(\mathfrak{M}) =$$

$$= [k_1 : k_0]e_{B_{Q_1}}(\mathfrak{M}B_{Q_1}) + [k_2 : k_0]e_{B_{Q_2}}(\mathfrak{M}B_{Q_2}) = 1\cdot a + 2\cdot b.$$

Note that $\mathfrak{M}B_{Q_i}$ is a reduction of $Q_iB_{Q_i}$, for $i = 1, 2$, therefore $e_{B_{Q_i}}(\mathfrak{M}B_{Q_i}) = e_{B_{Q_i}}(Q_iB_{Q_i})$, and by Remark 11.2.4 $e_{B_{Q_i}}(Q_iB_{Q_i}) == \nu_{k[X,Y]_{\mathfrak{q}_i}}(f)$, where $\mathfrak{q}_i/\langle f\rangle = Q_i$.

If the ground field is $k = \mathbb{C}$ then B has three maximal ideals, Q_1, Q_2' and Q_2'', all whose residue fields are isomorphic to $\mathbb{C}$. In this case Zariski's multiplicity formula splits to

$$(a+2b)\cdot 1 = [L : K]e_A(\mathfrak{M}) =$$

$$= [k_1 : k_0]e_{B_{Q_1}}(\mathfrak{M}B_{Q_1}) + [k_2' : k_0]e_{B_{Q_2'}}(\mathfrak{M}B_{Q_2'}) + [k_2'' : k_0]e_{B_{Q_2''}}(\mathfrak{M}B_{Q_2''}) =$$

$$= 1\cdot a + 1\cdot b + 1\cdot b.$$

The multiplicities involved can be computed with the same argument as above.

11.3.2 Finite-Transversal Projections

A first consequence of Zariski's multiplicity formula (Theorem 11.3.1) is that we can generalize inequality (11.6) to the non hypersurface case, and even for a wider class of rings (not only those of finite type over a field k):

Assume that S is a regular local ring and that $S \subset B$ is a finite extension under the assumptions of Theorem 11.3.1. Then, for any prime ideal $P \in \operatorname{Spec}(B)$

$$e_{B_P}(PB_P) \leqslant [L : K],$$

where $K = K(S)$ and $L = K \otimes_S B$.

Finite extensions $S \subset B$ where the equality holds will be said to be *transversal* with respect to P. In fact we state the following more general definition:

Definition 11.3.3 Let $S \subset B$ be a finite extension of excellent Noetherian rings with S regular and B equidimensional. Suppose that no non-zero element of S is a zero divisor in B. We say that the projection $\text{Spec}(B) \to \text{Spec}(S)$ (or the extension $S \subset B$) is *finite-transversal with respect to* $P \in \text{Spec}(B)$ if

$$e_{B_P}(PB_P) = [L : K].$$

Let $I \subset B$ be a P-primary ideal. We say that the projection $\text{Spec}(B) \to \text{Spec}(S)$ (or the extension $S \subset B$) is *finite-transversal with respect to the ideal* I if

$$e_{B_P}(IB_P) = [L : K],$$

and $I \cap S$ is a prime ideal of S.

Note that the conditions in Definition 11.3.3 imply that the associated primes of B are exactly the minimal primes of B, $\text{Ass}(B) = \text{Min}(B)$.

A second consequence of Theorem 11.3.1 is the following equivalence, which gives a characterization of finite-transverval projections:

Proposition 11.3.4 *[30, Corollary 4.9] Let S be regular ring and let $S \subset B$ be a finite extension. Suppose that B is Noetherian, excellent and equidimensional and that the non-zero elements of S are non-zero divisors in B.*

Let $P \in \text{Spec}(B)$ be a prime ideal of B and $I \subset B$ be a P-primary ideal. Set $\mathfrak{p} = P \cap S$. The following are equivalent:

(i) $e_{B_P}(IB_P) = [L : K]$ *and* $I \cap S = \mathfrak{p}$ *is a prime ideal in S, i.e., the extension $S \subset B$ is finite-transversal w.r.t. I.*
(ii) The following three conditions hold:

  (i) P is the only prime of B dominating $\mathfrak{p}$,
  (ii) $k(P) = B_P/PB_P = S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}} = k(\mathfrak{p})$*,*
  (iii) $\mathfrak{p}B_P$ is a reduction of the ideal IB_P.

We will refer to (i)–(iii) as *Zariski's conditions* of the finite extension $S \subset B$ with respect to the P-primary ideal I.

Note that Proposition 11.3.4 is stated in [30, Corollary 4.9] for $I = P$ a prime ideal, but the generalization to primary ideals is straightforward.

Example 11.3.5 Let $B = k[X, Y]/\langle X^2 - Y^3\rangle$. We can consider two finite projections $\text{Spec}(B) \to \text{Spec}(S_i)$, $i = 1, 2$:

(a) $S_1 = k[Y] \subset B$, and
(b) $S_2 = k[X] \subset B$.

Let $P = \langle x, y\rangle \subset B$ and note that $e_{B_P}(PB_P) = 2$.

The finite extension in (a) is finite-transversal with respect to P, while the finite extension in (b) is not because the generic rank of $S_2 \subset B$ is 3. However, extension (b) is finite-transversal with respect to the P-primary ideal $\langle x\rangle B$.

On the other hand, note that almost any linear projection from $\text{Spec}(B)$ to a one dimensional regular linear subvariety of $\mathbb{A}^2_k$ is finite-transversal with respect to P.

11.3.3 Do Finite-Transversal Projections Exist in General?

Given a Noetherian, excellent and equidimensional ring B, a point $P \in \mathrm{Spec}(B)$ and a P-primary ideal I, we wonder if there exist a regular ring S and a finite-transversal projection w.r.t. I, $\mathrm{Spec}(B) \to \mathrm{Spec}(S)$.

The answer, in general, is negative, even for k-algebras of finite type over a field k. Note that if such projection exists then the ideal I has a reduction with $d = \dim(B)$ elements.

This last observation gives a necessary condition not always satisfied as the following example illustrates.

Example 11.3.6 Let $k = \mathbb{F}_2$ and let

$$B = k[X, Y]/\langle XY(X + Y)\rangle.$$

Then the ideal $\langle x, y\rangle \subset B$ has no reductions generated by one element, see [29, Example 8.3.2]. This means that for B and the maximal ideal $I = \langle x, y\rangle$, there does not exist a finite-transversal projection w.r.t. I.

11.3.4 Construction of Finite-Transversal Projections

If $(B, \mathfrak{m})$ is a local complete Noetherian, equidimensional ring containing an infinite coeficient field, such that $\mathrm{Ass}(B) = \mathrm{Min}(B)$, then the answer to (Sect. 11.3.3) is positive. Let $I \subset B$ be a $\mathfrak{m}$-primary ideal. Since the residue field is infinite, by Swanson and Huneke [29, Proposition 8.3.7] there exists a reduction of I generated by $d = \dim(B)$ elements, $x_1, \dots, x_d$. Choose a coefficient field $k' \subset B$ and set $S = k'[[x_1, \dots, x_d]] \subset B$. Note that since $x_1, \dots, x_d$ are analytically independent, S is a ring of power series in d variables. The extension $S \subset B$ is finite by Cohen [8, Theorem 8, page 68] and we conclude that the projection $\mathrm{Spec}(B) \to \mathrm{Spec}(S)$ is finite-transversal with respect to $\mathfrak{m}$. See also [15, Proof of Theorem 1.1].

If B is a k-algebra of finite type, then Noether's normalization Lemma seems to provide a possible approach to address Sect. 11.3.3. We could find a regular ring S and a finite extension $S \subset B$, but Noether's normalization is not enough to guarantee Zariski's conditions (i)–(iii) in Proposition 11.3.4.

However, (not necessarily finite) morphisms for which conditions (ii) and (iii) hold are not hard to construct. As we will see, this will be a consequence of applying Noether's normalization to the graded ring $\mathrm{Gr}_{IB_P}(B_P)$. This motivates the following definition, which is a weaker version of the notion of finite-transversal projection.

Definition 11.3.7 Let $S \subset B$ be a (possibly non-finite) extension of Noetherian rings, with S regular and B equidimensional. Let $P \in \mathrm{Spec}(B)$ be a prime ideal and let $I \subset B$ be a PB_P-primary ideal.

We say that the projection $\mathrm{Spec}(B) \to \mathrm{Spec}(S)$ is *local-transversal with respect to the ideal* I if

(i) $\mathfrak{p} = I \cap S$ is a prime ideal and P is an isolated point in the fiber over $\mathfrak{p}$,
(ii) $k(P) = B_P / P B_P = S_{\mathfrak{p}} / \mathfrak{p} S_{\mathfrak{p}} = k(\mathfrak{p})$,
(iii) $\mathfrak{p} B_P$ is a reduction of the ideal $I B_P$.

Given a k-algebra B of finite type and a P-primary ideal I, we want to show that, locally for the étale topology, there exists a projection to a regular ring S which is local-transversal w.r.t. to I. This is achieved by applying the following result.

Theorem 11.3.8 *[14, Th. 10.14, page 60] Let* $(B, \mathfrak{m})$ *be a Noetherian local ring and let* $\mathfrak{a}$ *be an ideal of* B*. If* $a_1, \ldots, a_s \in \mathfrak{a}$*, then the following conditions are equivalent:*

(i) $a_1, \ldots, a_s \in \mathfrak{a}$ *generate a reduction of* $\mathfrak{a}$*.*
(ii) The quotient ring

$$(\mathrm{Gr}_{\mathfrak{a}}(B) \otimes_B B/\mathfrak{m}) \big/ \langle \tilde{a}_1, \ldots, \tilde{a}_s \rangle$$

has Krull dimension zero, where $\tilde{a}$ *is the class of* a *in* $\mathfrak{a}/(\mathfrak{a}\mathfrak{m})$*.*

Proposition 11.3.9 *[7, Proposition 34.1] Let* B *an equidimensional* k*-algebra of finite type, where* k *is a perfect field, let* $\mathfrak{m} \subset B$ *be a maximal ideal, and let* I *be a* $\mathfrak{m}$*-primary ideal. Then there exists an étale extension* $\lambda : B \to B'$*, a maximal ideal* $\mathfrak{m}' \in \mathrm{Spec}(B')$ *dominating* $\mathfrak{m}$*, and a local-transversal projection w.r.t.* $I B'_{\mathfrak{m}'} \cap B'$*,* $\beta : S \to B'$*,*

$$\begin{array}{ccc} B & \xrightarrow{\lambda} & B' \\ & & \uparrow{\scriptstyle \beta} \\ & & S. \end{array}$$

Proposition 11.3.9 can be read as saying that, after an étale extension $B \to B'$, there exist a local-transversal projection w.r.t. IB', $S \subset B'$.

Proof First, we can assume that $\mathfrak{m}$ is a rational closed point in $\mathrm{Spec}(B)$. To do so, let $k_1 = B/\mathfrak{m}$ be the residue field of B at set $B_1 = B \otimes_k k_1$. Now choose a maximal ideal $\mathfrak{m}_1 \in \mathrm{Spec}(B_1)$ mapping to $\mathfrak{m}$, and replace B and $\mathfrak{m}$ by B_1 and $\mathfrak{m}_1$.

Next, we want to construct a reduction of $I B_{\mathfrak{m}_1}$ generated by $d = \dim((B_1)_{\mathfrak{m}_1})$ elements $y_1, \ldots y_d \in \mathfrak{m}_1$. Such a reduction exists, in the local ring, if the residue field is infinite, see [29, Proposition 8.3.7], but we can avoid this hypothesis enlarging k_1 if necessary. Set $R_1 = (B_1)_{\mathfrak{m}_1}$. Then the graded ring $\mathrm{Gr}_{IR_1}(R_1)$ is a k_1-algebra of finite type of dimension d (see [29, Proposition 5.1.6]). After considering a finite extension of the base field $k_1 \subset k_2$ (if needed) we can apply the graded version of Noether's normalization Lemma ([29, Theorem 4.2.3]): Set $B_2 = B_1 \otimes_{k_1} k_2$ and let $\mathfrak{m}_2 \in \mathrm{Spec}(B_2)$ be a maximal ideal mapping to $\mathfrak{m}_1$. If

$R_2 = (B_2)_{\mathfrak{m}_2}$, then there are degree one elements $\bar{y}_1, \dots, \bar{y}_d \in \mathrm{Gr}_{IR_2}(R_2)$ such that $k_2[\bar{y}_1, \dots, \bar{y}_d]$ is isomorphic to the polynomial ring of d variables and

$$k_2[\bar{y}_1, \dots, \bar{y}_d] \subset \mathrm{Gr}_{IR_2}(R_2) = \mathrm{Gr}_{I(B_2)_{\mathfrak{m}_2}}\left((B_2)_{\mathfrak{m}_2}\right)$$

is finite.

Choose representatives $y_1, \dots, y_d \in I\,(B_2)_{\mathfrak{m}_2}$ of $\bar{y}_1, \dots, \bar{y}_d$. By Theorem 11.3.8 we conclude that $y_1, \dots, y_d$ generate a reduction of $I\,(B_2)_{\mathfrak{m}_2}$. Select some $f \in B_2$ so that $y_1, \dots, y_d \in (B_2)_f$. Finally $B' = (B_2)_f$, $\mathfrak{m}' = \mathfrak{m}_2$, and $S_2 = k_2[y_1, \dots, y_d]$ give the required extension, local-transversal w.r.t. $IB'_{\mathfrak{m}'} \cap B'$. See [7, 34.3] for complete details. □

Remark 11.3.10 Note that after following the proof, the statement of Proposition 11.3.9 can reformulated as follows. There exists a finite extension $k \subset k'$, an element $f \in B \otimes_k k'$, and a maximal ideal $\mathfrak{m}' \subset B' = (B \otimes_k k')_f$, together with a smooth k'-algebra of finite type S and morphisms of finite type λ and β

$$\begin{array}{ccc} B & \xrightarrow{\lambda} & B' \\ & & \uparrow{\scriptstyle \beta} \\ & & S, \end{array}$$

such that the projection β is local-transversal w.r.t. $IB'_{\mathfrak{m}'}$. Moreover S can be chosen to be a polynomial ring with d variables over k'.

Example 11.3.11 Let $B = k[X, Y]/\langle h\rangle$, where $h = X^2(X^2 + 1) + Y^5$. Let $I = \mathfrak{m} = \langle x, y\rangle \subset B$.

Note that the graded ring of $B_{\mathfrak{m}}$ at $IB_{\mathfrak{m}}$ is $\mathrm{Gr}_{IB_{\mathfrak{m}}}(B_{\mathfrak{m}}) = k[X, Y]/\langle X^2\rangle$, and the ideal $\langle y\rangle B_{\mathfrak{m}}$ is a reduction of $IB_{\mathfrak{m}}$ by Theorem 11.3.8. If $S = k[Y]$, then $S \subset B$ is local-transversal w.r.t. I.

The extension $S \subset B$ is finite but the generic rank is 4 and multiplicity of $B_{\mathfrak{m}}$ is 2. This implies that $S \subset B$ is not finite-transversal w.r.t. I.

One could consider the localization at $f = X^2 + 1$ in order to have condition (i) in Proposition 11.3.4, but then $S \subset B_f$ is not a finite extension. However, as we will see in Example 11.3.15, if we consider a convenient étale extension of S then a finite-transversal projection can be constructed.

The next lemma guarantees that the local-transversal condition is stable under étale base changes.

Lemma 11.3.12 *[7, Corollary 34.6, Example 34.8] Assume that $S \subset B$ is local-transversal w.r.t. an $\mathfrak{m}$-primary ideal I. If $S \to C$ is an étale extension and $\mathfrak{m}' \subset B \otimes_S C$ dominates $\mathfrak{m}$, then*

$$C \to B \otimes_S C$$

is again local-transversal w.r.t. $I' = I(B \otimes_S C)_{\mathfrak{m}'} \cap (B \otimes_S C)$.

Theorem 11.3.13 *[7, Proposition 31.1] Let k be a perfect field and let B be an equidimensional k-algebra of finite type with $Ass(B) = Min(B)$. Let $\mathfrak{m} \in \mathrm{Spec}(B)$ be a maximal ideal and let I be an $\mathfrak{m}$-primary ideal. Then there exist k-algebras B' and S, morphisms of finite type λ and β,*

$$\begin{array}{ccc} B & \xrightarrow{\lambda} & B' \\ & & \uparrow \beta \\ & & S', \end{array}$$

and a maximal ideal $\mathfrak{m}' \in \mathrm{Spec}(B')$, such that

(i) *λ is an étale morphism, and $\mathfrak{m} = \mathfrak{m}' \cap B$,*
(ii) *S' is a regular ring,*
(iii) *β is finite-transversal w.r.t. $IB'_{\mathfrak{m}'}$.*

Moreover, if $S \subset B$ is a local-transversal projection w.r.t. I, then the extension $S' \subset B'$ can be obtained by pull-back of a suitable étale map $S \to S'$ and a localization at some $f \in B \otimes_S S'$,

$$\begin{array}{ccc} B & \xrightarrow{\lambda} & B' = (B \otimes_S S')_f \\ \uparrow & & \uparrow \beta \\ S & \longrightarrow & S'. \end{array}$$

Theorem 11.3.13 is sketched in [30, 6.11] when I is a maximal ideal and full details of the proof are given in [7, Appendix A]. We reproduce here that proof to check that it also holds for an arbitrary $\mathfrak{m}$-primary ideal and to illustrate such a construction with an example, see Example 11.3.15. One of the main ingredients of the proof is Zariski's Main Theorem:

Theorem 11.3.14 *[19, Theorem 1, page 41] Let $S \subset B$ be a ring extension, and assume that B is an S-algebra of finite type. Let $A \subset B$ be the integral closure of S in B. Let $P \in \mathrm{Spec}(B)$ be a prime ideal and set $\mathfrak{n} = P \cap S$. If P is an isolated point of the fiber over $\mathfrak{n}$ then there exists $f \in A$, $f \notin P$ such that $A_f = B_f$.*

In other words, if S is essentially of finite type over a field k, Theorem 11.3.14 is saying that the (non necessarily finite) S-algebra B is, locally at P, isomorphic to a localization of an algebra which is finite over S.

Proof of Theorem 11.3.13 By Theorem 11.3.9 we can assume that there exists a local-transversal projection $S \subset B$ w.r.t. I. In fact, it comes from the proof of Theorem 11.3.13 that S can be assumed to be a polynomial ring.

First, by Zariski's Main Theorem 11.3.14, there exists $f \in A$, such that $A_f = B_f$, where A is the integral closure of S in B. Set $\mathfrak{n} = I \cap S \in \mathrm{Spec}(S)$ and let

$\mathfrak{p}_1, \ldots, \mathfrak{p}_s \in \operatorname{Spec}(A)$ be all the maximal ideals dominating $\mathfrak{n}$. We may assume that $\mathfrak{m} \cap A = \mathfrak{p}_1$.

Moreover, choosing $g \in (\mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_s) \setminus \mathfrak{p}_1$, we have that $\mathfrak{m}B_{fg}$ is the only maximal ideal in B_{fg} dominating $\mathfrak{n}$.

This means that the extension $S \subset B_{fg}$ fulfills properties (i), (ii) and (iii) in Proposition 11.3.4, but the ring extension might not be finite. We will use Lemma 11.3.12 in order to prove that the Theorem 11.3.13 holds after an étale base change extension.

Consider the henselization $\tilde{S}$ of $(S)_{\mathfrak{n}}$ (see [25, Appendix C]), where $\mathfrak{n} = I \cap S$, and the diagram:

$$
\begin{array}{ccccccc}
\mathfrak{m} & B & \longrightarrow & B \otimes_S S_{\mathfrak{n}} & \longrightarrow & B' = B \otimes_S \tilde{S} & \mathfrak{m}' = \mathfrak{m}'_1, \ldots, \mathfrak{m}'_{t'} \\
 & \uparrow & & \uparrow & & \uparrow & \\
\mathfrak{p} = \mathfrak{p}_1, \ldots, \mathfrak{p}_s & A & \longrightarrow & A \otimes_S S_{\mathfrak{n}} & \longrightarrow & A' = A \otimes_S \tilde{S} & \mathfrak{p}' = \mathfrak{p}'_1, \ldots, \mathfrak{p}'_{s'} \\
 & \uparrow & & \uparrow & & \uparrow & \\
\mathfrak{n} & S & \longrightarrow & S_{\mathfrak{n}} & \longrightarrow & \tilde{S} & \mathfrak{n}',
\end{array}
$$

where $\mathfrak{n}'$ is the maximal ideal of the local ring $\tilde{S}$; note that A' (resp. B') is a semi-local ring and we are denoting by $\mathfrak{p}'_1, \ldots, \mathfrak{p}'_{s'}$ (resp. $\mathfrak{m}'_1, \ldots, \mathfrak{m}'_{t'}$) the maximal ideals dominating $\mathfrak{n}'$. Assume that $\mathfrak{p}'$ dominates $\mathfrak{p}$ and that $\mathfrak{m}'$ dominates $\mathfrak{m}$. Since $\tilde{S}$ is henselian we have that

$$A' = A'_{\mathfrak{p}'_1} \oplus \ldots \oplus A'_{\mathfrak{p}'_{s'}},$$

and each direct summand is finite over $\tilde{S}$. In particular $\tilde{S} \to A'_{\mathfrak{p}'}$ is finite.

By the choice of $g \in A$ it follows that $\mathfrak{p}$ is the only point in the fiber over $\mathfrak{n}$ of $S \to A_g$. Therefore by Bravo and Villamayor [7, Lemma 34.5] $\mathfrak{p}'$ is the only point over $\mathfrak{p}$ of $A_g \to A'_g$,

$$
\begin{array}{ccccccc}
\mathfrak{p} = \mathfrak{p}_1 & A_g & \longrightarrow & A_g \otimes_S S_{\mathfrak{n}} & \longrightarrow & A'_g = A_g \otimes_S \tilde{S} = A'_{\mathfrak{p}'} & \mathfrak{p}' = \mathfrak{p}'_1 \\
 & \uparrow & & \uparrow & & \uparrow & \\
\mathfrak{n} & S & \longrightarrow & S_{\mathfrak{n}} & \longrightarrow & \tilde{S} & \mathfrak{n}'.
\end{array}
$$

Since fg is invertible in $A'_{\mathfrak{p}'}$ then there exists an integral equation

$$
\begin{aligned}
&((fg)^{-1})^n + d_1((fg)^{-1})^{n-1} + \ldots + d_{n-1}(fg)^{-1} + d_n = 0, \\
&d_i \in \tilde{S}, \quad \forall i = 1, \ldots, n. \qquad (11.8)
\end{aligned}
$$

Now consider a local étale neighborhood $\tilde{E}$ of $S_{\mathfrak{n}}$ containing the elements d_i, for $i = 1, \ldots, n$.

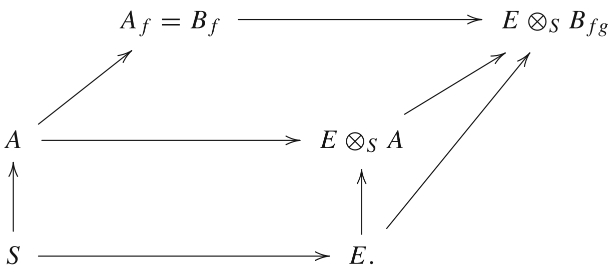
Observe that $\tilde{E} \subset \tilde{E} \otimes A_{fg}$ is a finite extension and there is a unique maximal ideal $\mathscr{P} \subset \tilde{E} \otimes A_{fg}$ dominating $\mathfrak{p}$. Therefore, $\tilde{E} \otimes A_{fg} = (\tilde{E} \otimes A)_{\mathscr{P}}$. And since $\tilde{E} \otimes A_{fg} \subset A'_{\mathfrak{p}'}$ is flat, the relation (11.8) also holds at $\tilde{E} \otimes A_{fg}$, and hence $\tilde{E} \otimes A_{fg}$ is finite over $\tilde{E}$.

Now note that,

$$\tilde{E} \otimes A_{fg} = \tilde{E} \otimes B_{fg} = \tilde{E} \otimes B_{\mathfrak{m}},$$

and therefore the extension $\tilde{E} \subset \tilde{E} \otimes B_{\mathfrak{m}}$ is finite-transversal w.r.t. $I' = I(\tilde{E} \otimes B_{\mathfrak{m}})$.

Finally observe that since $S_{\mathfrak{n}} \to \tilde{E}$ is local étale, then there exists an S-algebra of finite type E such that $S \to E$ is étale and such that $\tilde{E}$ is a localization of E (see [7, §32.4]). As consequence the finite extension is $E \to E \otimes_S B_{fg}$ is finite-transversal w.r.t. $I(E \otimes_S B_{fg})$. The following diagram summarizes the different extensions:



□

Example 11.3.15 Let us go back to Example 11.3.11 and let us assume now that the characteristic of k is different from 2. Let $\tilde{S} = k\{\{Y\}\}$ be the henselization of $k[Y]_{\langle Y \rangle}$. By Hensel's Lemma, the degree 4 polynomial $h = X^2(X^2+1)+Y^5$ factors as

$$h = h_1 \cdot h_2 = (X^2 + a_1 X + a_2)(X^2 + b_1 X + b_2) \tag{11.9}$$

where $a_1, a_2, b_1, b_2 \in \tilde{S}$ and such that $a_1, a_2, b_1, b_2 - 1 \in \langle Y \rangle$. A direct computation gives that

$$a_1 = 0, \quad b_1 = 0, \quad b_2 = 1 - a_2, \quad a_2^2 - a_2 + Y^5 = 0$$

Let $\alpha \in \tilde{S}$ be an element so that $\alpha^2 = \dfrac{1}{4} - Y^5$, for instance the power series

$$\alpha = \frac{1}{2}\sqrt{1 - 4Y^5} = \frac{1}{2} \sum_{i=0}^{\infty} \binom{1/2}{i} (-1)^i 4^i Y^{5i}.$$

Then $a_2 = \frac{1}{2} - \alpha$ and $b_2 = \frac{1}{2} + \alpha$ are solutions for the factorization (11.9).

Let $E = S[a_2, (2a_2 - 1)^{-1}]$ and note that $S \subset E$ is an étale extension. The extension

$$B \to B' = B \otimes_S E = E[X]/\langle X^2(X^2+1) + Y^5 \rangle$$

is also étale.

Let x be the class of X in B' and set $e = 1 + \frac{1}{1-2a_2}(x^2 + a_2)$. Note that $e^2 = e$, that the extension $E \subset B'_e$ is finite. Finally, we have that $E \subset B'_e$ is finite-transversal w.r.t. $\langle x, y \rangle B'_e$.

Another possibility is to consider an étale extension of B that contains a square root of $x^2 + 1$, see [7, §36] for further details.

11.4 Finite-Transversal Morphisms and Multiplicity

In this section we will focus on one of the main results of [30], stated below as Theorem 11.4.3. This theorem gives a procedure to describing the top multiplicity locus of a variety using finite-transversal projections. In this context, the notion of *algebraic presentations of a finite extension* plays a role (see Sect. 11.4.1). We discuss several applications and consequences of Theorem 11.4.3 in Sects. 11.4.2, 11.4.3 and 11.4.4. In addition, we present some results refining the number of generators needed for algebraic presentations in the context of finite-transversal morphisms (see Proposition 11.4.6). Finally, several examples are given in the hope that they help clarify some of the key ideas in the exposition.

We start with a generalization of a well known property of minimal polynomials for field extensions of the quotient field of an integrally closed domain. This result is essential in the exposition given in Sect. 11.4.1 which is a key step for understanding the statement of Theorem 11.4.3.

Proposition 11.4.1 *[30, Lemma 5.2] Let $S \subset B$ be a finite extension such that the non-zero elements of S are non-zero divisors in B. Assume that S is a regular ring and let $K = K(S)$ be the quotient field of S. Let $\theta \in B$ and let $f(Z) \in K[Z]$ be the monic polynomial of minimal degree such that $f(\theta) = 0$. If $S[\theta]$ denotes the S-subalgebra of B generated by θ, then*

(i) the coefficients of f are in S, $f(Z) \in S[Z]$, and
(ii) $S[\theta] \cong S[Z]/\langle f(Z) \rangle$.

Proof *(Comments on the Proof)* Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$ be the minimal primes of B. From the hypotheses we have that $\mathrm{Ass}(B) = \mathrm{Min}(B)$ and that $L = K \otimes_S B$ is the total ring of fractions of B. Then

$$L = L_1 \oplus \cdots \oplus L_m,$$

where L_i is a local Artinian ring for $i = 1, \ldots, m$. Note that the minimal primes of L, $Q_1, \ldots, Q_m$, are in one-to-one correspondence with $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$. Consider the following diagram,

$$\begin{array}{ccccccc}
S & \longrightarrow & B & \longrightarrow & & \longrightarrow & B/\mathfrak{q}_1 \oplus \cdots \oplus B/\mathfrak{q}_m \\
\downarrow & & \downarrow{\scriptstyle \alpha} & & & & \downarrow{\scriptstyle \beta} \\
K & \longrightarrow & K \otimes_S B = L & \xrightarrow{\cong} & L_1 \oplus \cdots \oplus L_m & \longrightarrow & K(B/\mathfrak{q}_1) \oplus \cdots \oplus K(B/\mathfrak{q}_m) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
K & \longrightarrow & K[\alpha(\theta)] & \xrightarrow{\cong} & K[\alpha(\theta)_1] \oplus \cdots \oplus K[\alpha(\theta)_m] & \longrightarrow & K[\bar{\theta}_1] \oplus \cdots \oplus K[\bar{\theta}_m],
\end{array}$$

where $\bar{\theta}_i \in K(B/\mathfrak{q}_i)$ is the class of $\alpha(\theta)_i \in L_i$.

Observe that $f(Z)$ has been chosen such that $f(\alpha(\theta)) = 0 \in L$. Let $g_i(Z)$ be the minimal polynomial of $\bar{\theta}_i$ over K, for $i = 1, \ldots, m$, and note that since S is normal, $g_i(Z) \in S[Z]$. Now we have that the irreducible factors of $f(Z)$ in $K[Z]$ are the $g_i(Z)$,

$$f(Z) = (g_1(Z))^{r_1} \cdots (g_m(Z))^{r_m}.$$

Hence $f(Z) \in S[Z]$.

Finally, since α is injective $f(\theta) = 0 \in B$. This gives a well defined morphism $S[Z]/\langle f(Z)\rangle \to S[\theta]$, which can be shown to be an isomorphism, see [30, page 342]. □

The following example illustrates the necessity of the hypothesis on the non-zero elements of S mapping to non-zero divisors in B.

Example 11.4.2 Let $S = k[X, Y]$ and let $B = k[X, Y, Z]/\langle (Z^2 + X^5)(Z + X^3), Y(Z^2 + X^5)\rangle$. The minimal primes of B are $\mathfrak{q}_1 = \langle z^2 + x^5\rangle$ and $\mathfrak{q}_2 = \langle y, z^2 + x^3\rangle$. We have that $K = K(S) = k(X, Y)$ and that $L = K \otimes_S B = K[Z]/\langle Z^2 + X^5\rangle$. The minimal polynomial of z over K is $f(Z) = Z^2 + X^5$. However $f(z)$ is not zero in B. In particular, $S[z] = B$ is not isomorphic to $S[Z]/\langle f(Z)\rangle$.

11.4.1 Algebraic Presentations of Finite Extensions

Let $S \subset B$ be a finite extension such that every non-zero element of S is not a zero-zivisor in B. Since the extension $S \subset B$ is finite, in particular of finite type, there are elements $\theta_1, \ldots, \theta_e \in B$ such that $B = S[\theta_1, \ldots, \theta_e]$. We will say that $S[\theta_1, \ldots, \theta_e]$ is an *algebraic presentation* of the extension $S \subset B$.

Let $f_i(Z_i) \in K[Z_i]$ be the polynomial of minimal degree such that $f_i(\theta_i)) = 0$, $i = 1, \ldots, e$. By Proposition 11.4.1, $f_i(Z_i) \in S[Z_i]$. Let $d_i = \deg(f_i(Z_i))$ be the degree of the polynomial $f_i(Z_i)$ for $i = 1, \ldots, e$. We may assume that all $d_i \geqslant 2$, since otherwise $\theta_i \in S$. We have a diagram

$$\begin{array}{ccccc} & \nearrow & B & \nwarrow & \\ S[\theta_1] \cong S[Z_1]/\langle f_1(Z_1)\rangle & & \uparrow \; \cdots \; \uparrow & & S[\theta_e] \cong S[Z_e]/\langle f_e(Z_e)\rangle \\ & \nwarrow & S. & \nearrow & \end{array} \tag{11.10}$$

We want to study such diagrams when $S \subset B$ is finite-transversal with respect to some prime P in B. This is the purpose of the next theorem.

Theorem 11.4.3 *[30, Proposition 5.7] Let B be an excellent and equidimensional ring, and let $S \subset B$ a finite extension such that every non-zero element of S is not a zero-zivisor in B. Fix an algebraic presentation $S[\theta_1, \ldots, \theta_e]$ of $S \subset B$. Let $\beta : \mathrm{Spec}(B) \to \mathrm{Spec}(S)$ and let $\beta_i : \mathrm{Spec}\,(S[Z_i]/\langle f_i(Z_i)\rangle) \to \mathrm{Spec}(S)$, for $i = 1, \ldots, e$. Suppose that the generic rank $n = [L : K] \geqslant 2$ and let $\mathfrak{p} \in \mathrm{Spec}(S)$. Then the following conditions are equivalent:*

- *The point $\mathfrak{p}$ is the image by β of a point of multiplicity n of* $\mathrm{Spec}(B)$.
- *For every $i = 1, \ldots, e$, the point $\mathfrak{p}$ is the image by β_i of a point of multiplicity d_i of* $\mathrm{Spec}\,(S[Z]/\langle f_i(Z)\rangle)$.

Theorem 11.4.3 has several interpretations and consequences, that we will describe in the next paragraphs.

11.4.2 Theorem 11.4.3 and Finite-Transversal Morphisms

With the notation and the hypotheses of the theorem, let $B_i = S[\theta_i] = S[Z_i]/\langle f_i(Z_i)\rangle$, for $i = 1, \ldots, e$. Then diagram (11.10) can be rewritten as

$$\begin{array}{ccccc} & \overset{\alpha_1}{\swarrow} & \mathrm{Spec}(B) & \overset{\alpha_e}{\searrow} & \\ \mathrm{Spec}(B_1) = \mathrm{Spec}\,(S[Z_1]/\langle f_1(Z_1)\rangle) & & \alpha_i\downarrow \; \cdots \; \downarrow\beta_i \quad \beta & & \mathrm{Spec}(B_e) = \mathrm{Spec}\,(S[Z_e]/\langle f_e(Z_e)\rangle) \\ & \underset{\beta_1}{\searrow} & \mathrm{Spec}(S). & \underset{\beta_e}{\swarrow} & \end{array} \tag{11.11}$$

Now observe that the theorem says that the projection $\beta : \mathrm{Spec}(B) \to \mathrm{Spec}(S)$ is finite-transversal w.r.t. $P \in \mathrm{Spec}(B)$ if and only if all the projections $\beta_i : \mathrm{Spec}(B_i) \to \mathrm{Spec}(S)$ are finite-transversal w.r.t. $P_i = \alpha_i(P)$.

11.4.3 Theorem 11.4.3 and an Explicit Description of the Top Multiplicity Locus of Spec(B)

Let $F_n \subset \mathrm{Spec}(B)$ be the set of points of multiplicity $n = [L : K]$, and assume that F_n is not empty. Then Theorem 11.4.3 gives us a procedure to describe F_n explicitly. To see this consider the following diagram

$$\begin{array}{ccccc} S[Z_1, \ldots, Z_e] & \longrightarrow & C = S[Z_1, \ldots, Z_e]/\langle f_1(Z_1), \ldots, f_e(Z_e)\rangle & \longrightarrow & B \\ & \nwarrow & \nwarrow & & \uparrow \\ & & & & S. \end{array} \tag{11.12}$$

Observe that there is a closed embedding of Spec(B) in the regular scheme Spec$(S[Z_1, \ldots, Z_e])$. Note that $\dim(B) = \dim(C)$ and that Spec(B) corresponds to the union of some irreducible components of Spec(C).

Let E_{d_i} be the set of points of multiplicity d_i of the hypersurface $\{f_i(Z_i) = 0\}$ in Spec$(S[Z_1, \ldots, Z_e])$, for $i = 1, \ldots, e$. The theorem says that

$$F_n = E_{d_1} \cap \cdots \cap E_{d_e}.$$

In other words, the top multiplicity locus of Spec(B) can be described as the top multiplicity locus of the complete intersection Spec(C). Note that the generic rank of $\beta' : \mathrm{Spec}(C) \to \mathrm{Spec}(S)$ is $d_1 \cdots d_e = \dim_K(C \otimes_S K)$ and that $n = \dim_K(B \otimes_S K)$ is the generic rank of $S \subset B$. Theorem 11.4.3 says that if $\mathfrak{p} \in \mathrm{Spec}(S)$ then the following assertions are equivalent

- The point $\mathfrak{p}$ is the image by β of a point of multiplicity n of Spec(B).
- The point $\mathfrak{p}$ is the image by β' of a point of multiplicity $d_1 \cdots d_e$ of Spec(C).

If S is a k-algebra of finite type, with k a perfect field, then $S[Z_1, \ldots, Z_e]$ is smooth over k, and the module of differential operators of order $\leqslant j$, $\mathrm{Diff}^j_k(S[Z_1, \ldots, Z_e])$, is free. Then we have that the closed set E_{d_i} is defined by the ideal

$$\left\langle \Delta(f_i(Z_i)) \mid \Delta \in \mathrm{Diff}_k^{d_i-1}(S[Z_1, \ldots, Z_e]) \right\rangle. \tag{11.13}$$

Actually, a similar description can also be given in a more general setting without assuming k to be perfect, see [2, Section 7] for details.

Finally, suppose we are give any equidimensional k-algebra of finite type B and let $P \in \mathrm{Spec}(B)$ be a maximal ideal with multiplicity greater than one. Then, by Theorem 11.3.13, and after an étale extension of B, we always may assume that we have a finite-transversal projection $S \subset B$ w.r.t. P, with S smooth over k. As a consequence, we can explicitly describe the top multiplicity locus of any B as above in a conveniently chosen étale extension.

Example 11.4.4 Let $V \subset \mathbb{A}^3_k = \text{Spec}(k[X, Y, Z])$ be the monomial curve defined by

$$X = t^3, \quad Y = t^4, \quad Z = t^5.$$

Then $V = \text{Spec}(B)$, where $B = k[X, Y, Z]/J$ and $J = \langle X^3 - YZ,\ X^2Y - Z^2,\ Y^2 - XZ\rangle$. Let $S = k[X]$. Then the extension $S \subset B$ is finite since $Y^3 - X^4,\ Z^3 - X^5 \in J$. Note that these integral relations can be obtained using standard bases, see [13, Proposition 3.1.5].

The generic rank of the extension is $3 = \dim_{k(X)}(B \otimes_{k[X]} k(X))$ and we have also that $e_{\mathfrak{m}}(B) = 3$, where $\mathfrak{m} = \langle x, y, z\rangle \subset B$. In this case, $S \subset B$ is a finite-transversal extension w.r.t. $\mathfrak{m}$, and $S[y, z]$ is an algebraic presentation of B.

The minimal polynomials (in the variable W) of y and z are, respectively $W^3 - x^4$ and $W^3 - x^5$. Set $C = k[X, Y, Z]/\langle Y^3 - X^4,\ Z^3 - X^5\rangle$. Note that Spec$(C)$ has two irreducible components of dimension one. The curve Spec(B) is one of the irreducible components of Spec(C). By Theorem 11.4.3, the locus of points of multiplicity 3 of Spec(B) (only the origin) coincides with the locus of points of multiplicity 9 of Spec(C).

In Example 11.4.4 the dimension of the ring $B_{\mathfrak{m}}$ is one and the embedding dimension is three. The algebraic presentation is generated by two elements, y and z, over S and any other algebraic presentation over a regular ring is generated by two elements at least.

In general, the difference of the embedding dimension and the dimension of the local ring, the so called *excess of embedding dimension*, is a lower bound for the number of generators of an algebraic presentation. However, this lower bound can always be achieved after localization, a fact that is proved in Proposition 11.4.6 below. First we need a technical result for finite extensions of local rings.

Lemma 11.4.5 *Let* $(S, \mathfrak{n})$ *and* $(B, \mathfrak{m})$ *be Noetherian local rings and suppose that* $S \subset B$ *is a finite extension, with* S *is regular. Assume that the residue fields are equal,* $S/\mathfrak{n} \cong B/\mathfrak{m} = k$*. If* $t = \dim_{k(\mathfrak{m})}(\mathfrak{m}/\mathfrak{m}^2) - \dim(B)$ *is the excess of embedding dimension of* B*, then there are elements* $\theta_1, \ldots, \theta_t \in \mathfrak{m}$ *such that*

$$B = S[\theta_1, \ldots, \theta_t].$$

Proof Write $B = S[\theta_1 \ldots, \theta_e]$. Since $S/\mathfrak{n} \cong B/\mathfrak{m}$ we can assume that $\theta_i \in \mathfrak{m}$ (here a translation by elements of S may be needed). We have that $\mathfrak{n}B + \langle\theta_1, \ldots, \theta_e\rangle = \mathfrak{m}$, hence $e \geqslant t$. After reordering the θ_i's, we may assume that $\mathfrak{m} = \mathfrak{n}B + \langle\theta_1, \ldots, \theta_t\rangle$.

Consider the S-module $N = S[\theta_1, \ldots, \theta_t] \subset B$. We claim that

$$N/\mathfrak{n}N = B/\mathfrak{n}B.$$

If the claim holds then by Nakayama's Lemma we have that $N = B$ as required.

To prove the claim, let $\bar{\theta}_i$ be the class of θ_i in $\bar{B} = B/\mathfrak{n}B$, for $i = 1, \dots, e$, and denote by $\overline{\mathfrak{m}} = \mathfrak{m}\bar{B}$ the maximal ideal of the local ring $\bar{B}$. We have that $\bar{\theta}_1, \dots \bar{\theta}_t$ generate $\overline{\mathfrak{m}}$ in $\bar{B}$.

Note that $\bar{B}$ is an Artinian local ring, hence complete. Since $\dim_k(\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2) = t$, by Cohen's structure Theorem, $\bar{B} \cong k[[Z_1, \dots, Z_t]]/J$ for some ideal J, where the classes of Z_i correspond to $\bar{\theta}_i$, $i = 1, \dots, t$. Now note that every $\bar{\theta}_i$ is nilpotent, hence for some integers α_i we have $Z_i^{\alpha_i} \in J$ and then

$$\bar{B} \cong k[Z_1, \dots, Z_t]/I$$

for some ideal $I \subset k[Z_1, \dots, Z_t]$. □

Proposition 11.4.6 *Let $S \subset B$ be a finite-transversal extension w.r.t. an $\mathfrak{m}$-primary ideal $I \subset B$, with $\mathfrak{m}$ a maximal ideal in B. Let $t = \dim_{k(\mathfrak{m})}(\mathfrak{m}/\mathfrak{m}^2) - \dim(B_\mathfrak{m})$ be the excess of embedding dimension of B at $\mathfrak{m}$. Then there are elements $\theta_1, \dots, \theta_t \in B$ and $g \in S$ such that*

$$B_g = S_g[\theta_1, \dots, \theta_t].$$

Proof As in Lemma 11.4.5, write $B = S[\theta_1 \dots, \theta_e]$, with $\theta_i \in \mathfrak{m}$ for $i = 1 \dots, e$. Let $\mathfrak{n} = \mathfrak{m} \cap S$. Then the extension $S_\mathfrak{n} \subset B \otimes_S S_\mathfrak{n}$ is finite. By condition (i) in Proposition 11.3.4 $B_\mathfrak{m} = B \otimes_S S_\mathfrak{n}$. Therefore $S_\mathfrak{n} \subset B_\mathfrak{m}$ is finite, and by Lemma 11.4.5, after reordering the θ_i, we have that

$$B_\mathfrak{m} = S_\mathfrak{n}[\theta_1, \dots, \theta_t].$$

Note that, for $j = t + 1, \dots, e$, θ_j is a polynomial in $\theta_1, \dots, \theta_t$ with coefficients in $S_\mathfrak{n}$. Hence there exists some $g \in S$ such that $\theta_j \in S_g[\theta_1, \dots, \theta_t]$, $j = t + 1, \dots, e$, and it follows that

$$B_g = S_g[\theta_1, \dots, \theta_t].$$

□

Example 11.4.7 Let $S = k[Y]$. In the polynomial ring $S[X_1, X_2]$ consider the ideal

$$J = \langle X_1^2 - Y^5,\ X_1 - X_2 - X_1 X_2(1 + X_1)\rangle,$$

and let $B = S[X_1, X_2]/J$. Note that $X_2^2(1 + Y^5 + Y^{10}) + 2Y^5 X_2 - Y^5 \in J$ which gives an integral relation of X_2 with coefficients in S_f, with $f = 1 + Y^5 + Y^{10}$. The extension $S_f \subset B_f$ is finite, and $S_f[x_1, x_2]$ is an algebraic presentation of $S_f \subset B_f$. We have that $S_f \subset B_f$ is finite-transversal w.r.t. $\mathfrak{m} = \langle y, x_1, x_2\rangle B_f$.

Note that the surface with affine ring $R = S[X_1, X_2]/\langle X_1 - X_2 - X_1 X_2\rangle$ is regular and $\mathrm{Spec}(B)$ is a curve in $\mathrm{Spec}(R)$. In this case the dimension of $B_\mathfrak{m}$ is one with embedding dimension two, since

$$\mathfrak{m}B_{\mathfrak{m}} = \langle y, x_1 \rangle B_{\mathfrak{m}} = \langle y, x_2 \rangle B_{\mathfrak{m}}.$$

Proposition 11.4.6 says that for some localization of S_f we have to be able to find an algebraic presentation of $S_f \subset B_f$ with only one generator, either x_1 or x_2.

Some Groebner basis computations show that

$$X_1(1 + Y^5) - X_2(1 + Y^5 + Y^{10}) - Y^5 \in J,$$

an then $B_f = S_f[x_1]$. On the other hand, if $g = (1 + Y^5)$ then $S_{fg} \subset B_{fg}$ is finite-transversal w.r.t. $\mathfrak{m}B_{fg}$ and $x_1 \in S_{fg}[x_2]$. Thus $B_{fg} = S_{fg}[x_2]$.

11.4.4 *Theorem 11.4.3 and Homeomorphic Copies of the Top Multiplicity Locus of* Spec(B)

Finally, there is a third main consequence of Theorem 11.4.3, whose meaning will be clarified in the Sect. 11.5, see Theorem 11.5.5 (iii) and part (iv), which is stated in Sect. 11.5.2.

Corollary 11.4.8 *[30, Corollary 5.9] Let $S \subset B$ be a finite extension of generic rank n, and suppose it is under the assumptions of Theorem 11.4.3. Assume that the set of points of multiplicity n of B, $F_n \subset \text{Spec}(B)$, is not empty. Then*

(i) Zariski's conditions hold for any $P \in F_n$:

- *(i) β is a set theoretical bijection between F_n and $\beta(F_n)$,*
- *(ii) if $P \in F_n$ and $\mathfrak{p} = \beta(P)$ then $k(P) = B_P/PB_P = S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}} = k(\mathfrak{p})$, and*
- *(iii) $\mathfrak{p}B_P$ is a reduction of PB_P.*

(ii) F_n is closed in Spec(B), and F_n is homeomorphic to $\beta(F_n)$.

(iii) $\beta(F_n) = $ Spec(S) if and only if $S = B_{\text{red}}$.

In other words, Corollary 11.4.8 says that when $S \subset B$ is finite transversal of generic rank n then we can see a homeomorphic image of F_n in Spec(S).

11.5 Finite-Transversal Morphisms and Resolution of Singularities

In the present section we discuss the central result in [30], stated as Theorem 11.5.5 below, and its applications to resolution of singularities. The key idea here is that the description of the top multiplicity locus of a variety given in Theorem 11.4.3 is stable after blowing up a regular equimultiple center, see Theorem 11.5.5. We will make these ideas more precise along the following paragraphs.

Resolution of Singularities

Now we go back to our discussion in the Introduction. There, we mentioned the role of the order function when measuring how singular a hypersurface $H \subset \mathbb{A}^n_k$ is. When the characteristic is zero, proving the existence of a resolution of singularities is a complex task, and yet, it somehow reduces to *considering the order of ideals*. In other words and very roughly speaking, to resolve singularities Hironaka faced two main problems:

- **Problem 1.** Given a sheaf of ideals J on a smooth scheme V and a positive integer b, prove that there exists a finite sequence of blow ups so that a *suitable transform of* J has maximum order below b (see Theorem 11.5.2 below).
- **Problem 2.** Prove that *improving the singularities* of an algebraic variety X by blow ups is equivalent to solving problem 1 (see Theorem 11.5.3 below and the discussion that follows).

Our purpose is to give a few hints on these ideas. In order to do so, we start with some definitions.

Pairs and Their Role in Resolution of Singularities

Definition 11.5.1 Let V be smooth scheme of finite type over a field k, let J be a sheaf of ideals on V and let $b \in \mathbb{Z}$ a positive integer.

- We refer to (J, b) as *a pair over* V.
- The *singular locus of* (J, b) is the closed subset of V,

$$\mathrm{Sing}\,(J, b) := \{\zeta \in V : \nu_{\zeta}(J) \geqslant b\}.$$

- A *permissible center* Y for (J, b) is a closed regular subset $Y \subset V$ such that $Y \subset \mathrm{Sing}\,(J, b)$.
- A permissible blow up of V is the blow up of V at a permissible center Y, $V \leftarrow V_1$.
- For a permissible blow up, $V \leftarrow V_1$, with exceptional divisor E_1, the transform of the pair (J, b) is the pair, (J_1, b), where

$$J\mathcal{O}_{V_1} = \mathcal{I}(E_1)^b \cdot J_1.$$

With the previous notation, a *resolution of a pair* is a sequence of permissible blow ups,

$$\begin{array}{ccccccc} V = V_0 & \leftarrow & V_1 & \leftarrow \dots \leftarrow & V_\ell \\ (J, b) = (J_0, b) & & (J_1, b) & \dots & (J_\ell, b), \end{array}$$

so that Sing $(J_\ell, b) = \emptyset$.

To be precise, an additional condition on the permissible centers needs to be asked: they need to have normal crossings with the exceptional divisors that successively appear in the sequence.

Theorem 11.5.2 *[16] If the characteristic of k is zero, a resolution of* (J, b) *exists.*

Why Pairs?

The previous statement might lead to more questions than answers:

(i) How is the theorem proven and why the hypothesis on the characteristic?
(ii) Is it really necessary and statement about general pairs? Is it not enough to resolve pairs (J, b) with b equal to the maximum order of J at V?
(iii) While it is clear what the pair for a hypersurface could be, it is not obvious how to proceed in the general case.

Regarding to question (1): Maximal contact

Theorem 11.5.2 is proven by induction on the dimension of V: the existence of a resolution of (J, b) follows from another theorem that basically says that a resolution of (J, b) can be achieved if we know how to resolve pairs in smooth $(n-1)$-dimensional schemes. And this does not hold in general over fields of positive characteristic. For those interested in a deeper understanding on the topic we refer to the so called theory of *maximal contact* (see [12], also [4]).

Regarding to question (2): An example

Let $H : \{z^2 + (y^3 - x^5) = 0\} \subset \mathbb{A}^3_k$, where k is a field of characteristic different from 2. If we want to find a resolution of singularities of H we can start by resolving the pair $(\langle z^2 + (y^3 - x^5)\rangle, 2)$ in $\mathbb{A}^3_k$. The *theory of maximal contact* would tell us that a finding a resolution of $(\langle z^2 + (y^3 - x^5)\rangle, 2)$ is equivalent to finding a resolution of the pair $(\langle (y^3 - x^5)\rangle, 2)$ in $\mathbb{A}^2_k$. Observe that the number $b = 2$ in the second pair is **not** the maximum order of the ideal $\langle (y^3 - x^5)\rangle$ in $\mathbb{A}^2_k$. Hence, a theorem of resolution of **general** pairs as Theorem 11.5.2 is needed.

Regarding to question (3): Presentations for the Hilbert-Samuel function

If $H \subset V$ is a hypersurface of maximum order m, then it is clear that a resolution of the pair $(\mathcal{I}(H), m)$ leads to a sequence of blow ups over H so that the strict transform of H has maximum order below m. And resolution follows by induction on the order.

For arbitrary varieties, Hironaka used *presentations of the Hilbert-Samuel function*. For a variety X we will use HS_X to refer to its Hilbert-Samuel function. This function satisfies a series of properties that make it suitable as an invariant to approach resolution (see [5]):

(A) HS_X is an upper semi-continuous function on X; we will denote by $\max \mathrm{HS}_X$ its maximum value on X, and by $\underline{\mathrm{Max}}\, \mathrm{HS}_X$ the closed set of X where this maximum is achieved;
(B) HS_X is constant on X if and only if X is regular;
(C) If $Y \subset \underline{\mathrm{Max}}\, \mathrm{HS}_X$ is a closed regular center and if $X \leftarrow X_1$ is the blow up at Y, then

$$\max \mathrm{HS}_X \geqslant \max \mathrm{HS}_{X_1}.$$

The key point here is that the closed set $\underline{\mathrm{Max}}\, \mathrm{HS}_X$ can be expressed as the singular locus of a pair. But not *any* pair will work. Actually we need a *suitably defined pair* whose resolution induces a sequence of blow ups over X that forces the maximum value of HS_X to go down. This is done via the so called *standard basis* (see [3]):

Theorem 11.5.3 *(Presentations for the Hilbert-Samuel function) At an étale neighborhood of each closed point $\xi \in \underline{Max}\, HS_X$, we can assume X to be locally embedded in a smooth scheme V where we can find elements $f_1, \dots, f_r \in \mathcal{O}_{V,\xi}$ such that:*

(i) $\mathcal{I}(X)_\xi = \langle f_1, \dots, f_r \rangle$;
(ii) Denoting by m_i the maximum order of the hypersuface $H_i = \{f_i = 0\}$, we have that

$$\underline{Max}\, HS_X = \bigcap_{i=1}^{r} \underline{Max}\, HS_{H_i} = \bigcap_{i=1}^{r} \{\zeta : \nu_\zeta(f_i) = m_i\};$$

(iii) If $Y \subset \underline{Max}\, HS_X$ is a closed regular center, if $V \leftarrow V_1$ is the blow up at Y, and

$$max\, HS_X = max\, HS_{X_1},$$

then

$$\underline{Max}\, HS_{X_1} = \bigcap_{i=1}^{r} \underline{Max}\, HS_{H_{i,1}}$$

where $H_{i,1}$ is the strict transform of H_i, $i = 1, \dots, r$.

A consequence of the theorem is that ***a pair*** (J, b) ***can be naturally attached to the previous data:***

Set $M := \prod_{i=1}^{r} m_i$ and $M_i = M/m_i$. Let $J := \langle f_1^{M/m_1}, \dots, f_r^{M/m_r} \rangle$. Then resolving the pair (J, M) leads to a sequences of blow ups,

$$X \leftarrow X_1 \leftarrow \dots \leftarrow X_\ell$$

so that

$$\max \mathrm{HS}_X = \max \mathrm{HS}_{X_1} = \ldots > \max \mathrm{HS}_{X_\ell}.$$

Remark 11.5.4 It is worthwhile to make two observations to regarding Hironaka's approach to resolution:

(a) Hironaka uses the Hilbert-Samuel function, which takes values in $\mathbb{N}^{\mathbb{N}}$. The multiplicity function also satisfies properties (A), (B) and (C) from above (see [11]), takes values on $\mathbb{N}$ and has a natural geometric interpretation. Hence, it is quite natural to ask whether it can replace the role of the Hilbert-Samuel function in the resolution process. This is a question posed by Hironaka in [16] and answered affirmatively by Villamayor in [30] (this follows from Theorem 11.5.5 below).

(b) The use of the Hilbert-Samuel function requires working with an embedding of X in some smooth scheme of dimension $n \geqslant d + 1$. This leads to the definition of convenient pairs some smooth n-dimensional scheme and then induction on resolution of pairs is applied in $n-1, n-2, \ldots,$-dimensions. As we will see, Villamayor's approach using the multiplicity simplifies this last step. More precisely, the problem of lowering the multiplicity of a d-dimensional variety is shown to be equivalent to the resolution of a pair in some smooth scheme of the same dimension d (at least when the characteristic is zero).

11.5.1 Regarding to Remark 11.5.4 (a): Presentations for the Multiplicity Function

We dedicate the following lines to Villamayor's approach to resolution using the multiplicity function as the main invariant. We start by fixing some notation. We will use Mult_X to refer to the multiplicity function on X, $\max \mathrm{Mult}_X$ for its maximum value on X and $\underline{\mathrm{Max}}\,\mathrm{Mult}_X$ for the closed set of points of X where this value is achieved.

Let X be an equidimensional variety with $\max \mathrm{Mult}_X > 1$. A *simplification of the multiplicity of* X is a finite sequence of blow ups

$$X = X_0 \leftarrow X_1 \leftarrow \ldots \leftarrow X_n$$

so that

$$\max \mathrm{Mult}_{X_0} = \max \mathrm{Mult}_{X_1} = \ldots = \max \mathrm{Mult}_{X_{n-1}} > \max \mathrm{Mult}_{X_n}.$$

As a corollary of Theorem 11.5.5 below, we get that it is possible to resolve singularities in characteristic zero via simplifications of the multiplicity. As Hironaka's Theorem 11.5.3, Villamayor's statement is also of local nature. Hence, we will assume that X is an affine variety. Note that the parts (i) and (ii) of the next theorem have been already stated and discussed in Theorem 11.4.3 and Sect. 11.4.3.

Theorem 11.5.5 *[30, §6, Theorem 6.8] (Presentations for the Multiplicity function) Let $X = Spec(B)$ be an affine equidimensional algebraic variety of dimension d defined over a perfect field k, and let $\xi \in \underline{Max}\, Mult_X$ be a closed point. Then, there is an étale neighborhood B' of B, mapping $\xi' \in \mathrm{Spec}(B')$ to ξ, a smooth k-algebra S together with a finite-transversal morphism at ξ', $\beta : \mathrm{Spec}(B') \to \mathrm{Spec}(S)$ so that if $B' = S[\theta_1, \ldots, \theta_e]$ and $f_i(Z_i) \in K(S)[Z_i]$ denote the minimum polynomial of θ_i over $K(S)$ for $i = 1, \ldots, e$, then $f_i(Z_i) \in S[Z_i]$ and there is a diagram:*

$$\begin{array}{ccccc} S[Z_1, \ldots, Z_e] & \longrightarrow & S[Z_1, \ldots, Z_e]/\langle f_1(Z_1), \ldots, f_e(Z_e)\rangle & \longrightarrow & B' \\ & & \uparrow & \nearrow & \\ & & S & & \end{array}$$

for which the following hold:

(i) Let $V = \mathrm{Spec}(S[Z_1, \ldots, Z_e])$, and let $\mathscr{I}(X)$ be the defining ideal of X at V. Then

$$\langle f_1, \ldots, f_e\rangle \subset \mathscr{I}(X);$$

(ii) Denoting by m_i the maximum order of the hypersuface $H_i = \{f_i = 0\} \subset V$, we have that

$$\underline{Max}\, Mult_X = \bigcap_{i=1}^{e} \underline{Max}\, Mult_{H_i} = \bigcap_{i=1}^{e} \{\zeta : \nu_\zeta(f_i) = m_i\};$$

(iii) Let $Y \subset \underline{Max}\, Mult_X$ be a closed regular center. Then $\beta(Y)$ is regular in $\mathrm{Spec}(S)$. Now, if $X \leftarrow X_1$ is the blow up at Y, and $\mathrm{Spec}(S) \leftarrow T_1$ is the blow up at $\beta(Y)$ then there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(B) & \longleftarrow & X_1 \\ \beta \downarrow & & \downarrow \beta_1 \\ \mathrm{Spec}(S) & \longleftarrow & T_1 \end{array} \qquad (11.14)$$

where the horizontal maps are blow ups and the vertical are finite morphisms. Moreover, if

$$max\, Mult_X = max\, Mult_{X_1},$$

then $\beta_1 : X_1 \to T_1$ *is finite-transversal w.r.t. any point in* $\underline{Max}\,Mult_{X_1}$ *and if* $V \leftarrow V_1$ *is the blow up at* Y*, then*

$$\underline{Max}\,Mult_{X_1} = \bigcap_{i=1}^{e} \underline{Max}\,Mult_{H_{i,1}}$$

where $H_{i,1}$ *is the strict transform of* H_i *in* V_1*, for* $i = 1, \ldots, e$.

A consequence of the theorem is that ***a pair*** (J, b) ***can be naturally attached to the previous data:***

Set $M := \prod_{i=1}^{e} m_i$ and $M_i = M/m_i$. Let $J := \langle f_1^{M/m_1}, \ldots, f_e^{M/m_e} \rangle$. Then resolving the pair (J, M) leads to a sequences of blow ups,

$$X \leftarrow X_1 \leftarrow \ldots \leftarrow X_\ell$$

so that

$$\max \mathrm{Mult}_X = \max \mathrm{Mult}_{X_1} = \ldots > \max \mathrm{Mult}_{X_\ell},$$

i.e., to a simplification of the multiplicity of X.

11.5.2 Regarding to Remark 11.5.4 (b): Resolution of Pairs in Dimension d = dim *X*

Notice that Villamayor's Presentations of the Multiplicity come equipped with a finite projection to dome d-dimensional smooth scheme. This finite-transversal projection has one additional property:

(iv) If $T \subset \beta(\underline{\mathrm{Max}}Mult_X) \subset \mathrm{Spec}(S)$ is a regular closed subscheme, then $\beta^{-1}(T)_{\mathrm{red}}$ is also regular and the simultaneous blow ups at T and $\beta^{-1}(T)_{\mathrm{red}}$ lead to a commutative diagram as (11.14) with the same properties as in Theorem 11.5.5 (iii).

When the characteristic of the base field is zero, there is a pair on $\mathrm{Spec}(S)$ (which is a d-dimensional scheme) whose resolution induces a resolution of the pair (J, M). Thus, a simplification of the multiplicity of X is directly achieved via the resolution of a d-dimensional pair, where $d = \dim X$, see [1, Chapter 7] for more details.

11.6 Finite-Transversal Morphisms and the Asymptotic Samuel Function

The *asymptotic Samuel function* was introduced by Samuel in [26] and afterwards studied by D. Rees [20–23]. Here we review the definition and some properties. For further details we refer the reader to [17] and [29]. Our purpose here is to prove

Theorem 11.6.8 which is a slightly modified version of a theorem of Hickel on the computation of the asymptotic Samuel function (see Theorem 11.6.7 for Hickel's statement).

Suppose A is a commutative ring with unit 1, and let $I \subset A$ be a proper ideal. For each $f \in A$ consider the value $\nu_I(f) = \sup\{\ell \in \mathbb{N} \cup \{\infty\} \mid f \in I^\ell\}$. Observe that for $f, g \in A$ we have $\nu_I(f+g) \geqslant \min\{\nu_I(f), \nu_I(g)\}$ and $\nu_I(f\cdot g) \geqslant \nu_I(f)+\nu_I(g)$. In particular, for $m \in \mathbb{N}$, $\nu_I(f^m) \geqslant m\nu_I(f)$ and the inequality could be strict. The asymptotic Samuel function is a normalized version of the ordinary function, which, as we will see, has a nicer behavior.

Definition 11.6.1 The *asymptotic Samuel function at* I, $\bar{\nu}_I : A \to \mathbb{R} \cup \{\infty\}$, is defined as:

$$\bar{\nu}_I(f) = \lim_{n\to\infty} \frac{\nu_I(f^n)}{n}, \qquad f \in A. \tag{11.15}$$

The limit (11.15) exists in $\mathbb{R}_{\geqslant 0} \cup \{\infty\}$ (see [17, Lemma 0.2.1]. When $(A, \mathfrak{m})$ is a local regular ring, $\overline{\nu}_{\mathfrak{m}} = \nu_{\mathfrak{m}}$. The next proposition summarizes some of the main properties of the asymptotic Samuel function.

Proposition 11.6.2 *[17, Corollary 0.2.6, Proposition 0.2.9] The function $\bar{\nu}_I$ is an order function, i.e., it satisfies the following properties:*

(i) $\bar{\nu}_I(f+g) \geqslant \min\{\bar{\nu}_I(f) + \bar{\nu}_I(g)\}$, *for all* $f, g \in A$,
(ii) $\bar{\nu}_I(f\cdot g) \geqslant \bar{\nu}_I(f) + \bar{\nu}_I(g)$, *for all* $f, g \in A$,
(iii) $\bar{\nu}_I(0) = \infty$ *and* $\bar{\nu}_I(1) = 0$.

Furthermore, for each $f \in A$ *and each* $r \in \mathbb{N}$*:*

(iv) $\bar{\nu}_I(f^r) = r\bar{\nu}_I(f)$;
(v) $\bar{\nu}_{I^r}(f) = \dfrac{1}{r}\bar{\nu}_I(f)$.

Note that if $f \in A$ is nilpotent then $\bar{\nu}_I(f) = \infty$.

Example 11.6.3 If $A = k[X, Y]/\langle X^2 + Y^3\rangle$ and if $\mathfrak{m} = \langle x, y\rangle \subset A$, then it can be checked that $\overline{\nu}_{\mathfrak{m}}(y) = 1$, while $\overline{\nu}_{\mathfrak{m}}(x) = 3/2$. However, if $A = k[X, Y, Z]/\langle X^2 + Y^2 + Z^3\rangle$, $\mathfrak{m} = \langle x, y, z\rangle$ and the characteristic is different from 2, then $\overline{\nu}_{\mathfrak{m}}(x) = \overline{\nu}_{\mathfrak{m}}(y) = \overline{\nu}_{\mathfrak{m}}(z) = 1$.

The Asymptotic Samuel Function on Noetherian Rings

When A is Noetherian, the number $\overline{\nu}_I(f)$ measures how deep the element f lies in the integral closure of powers of I:

Proposition 11.6.4 *[29, Corollary 6.9.1] Suppose A is Noetherian. Then for a proper ideal $I \subset A$ and every $a \in \mathbb{N}$,*

$$\overline{I^a} = \{f \in R \mid \bar{\nu}_I(f) \geqslant a\}.$$

Corollary 11.6.5 *Let A be a Noetherian ring and $I \subset A$ a proper ideal. If $f \in A$ then*

$$\bar{\nu}_I(f) \geqslant \frac{a}{b} \Longleftrightarrow f^b \in \overline{I^a}.$$

See also [10] for a generalization of the asymptotic Samuel function to arbitrary filtrations of ideals and properties.

The previous characterization of $\overline{\nu}_I$ leads to the following result that gives a valuative version of the function.

Theorem 11.6.6 *Let A be a Noetherian ring, and let $I \subset A$ be a proper ideal not contained in a minimal prime of A. Let $v_1, \ldots, v_s$ be a set of Rees valuations of the ideal I. If $f \in A$ then*

$$\bar{\nu}_I(f) = \min\left\{\frac{v_i(f)}{v_i(I)} \mid i = 1, \ldots, s\right\}.$$

Proof See [29, Lemma 10.1.5, Theorem 10.2.2] and [28, Proposition 2.2]. □

In particular, it follows from here that when A is a Noetherian ring, $\overline{\nu}_I(f)$ always takes values in $\mathbb{Q} \cup \{\infty\}$.

On a Explicit Formula for the Computation of the Asymptotic Samuel Function

In [15] M. Hickel presented a series of nice results regarding the asymptotic Samuel function. In particular, he proved the following theorem with an explicit method for its calculation:

Theorem 11.6.7 *[15, Theorem 2.1] Let $(R, \mathfrak{m}, k)$ be a complete local Noetherian domain of equal characteristic and Krull dimension d. Let $I \subset R$ be an $\mathfrak{m}$-primary ideal and suppose that I has a reduction generated by d elements, $J = \langle x_1, \ldots, x_d\rangle \subset R$. Let $r \in R$. Let $A := k[[x_1, \ldots, x_d]] \cong k[[X_1, \ldots, X_d]]$, let $\mathfrak{m}_A \subset A$ be the maximal ideal, and let*

$$p(Z) = Z^\ell + \sum_{i=1}^{\ell} a_i Z^{\ell-i}$$

be the minimal polynomial of r over $K(A)$. Then:

$$\overline{\nu}_I(r) = \min_{1 \leqslant i \leqslant \ell} \frac{\nu_{\mathfrak{m}_A}(a_i)}{i}.$$

The theorem gives a method for the explicit computation of the asymptotic Samuel function for a local Noetherian ring $(R, \mathfrak{m}, k)$ of equal characteristic, by passing to the completion and then reducing to the domain case. The existence of a reduction of the ideal I generated by d elements can be achieved by extending the residue field.

Following the arguments in Sect. 11.3, the previous result can be shown to hold in a suitable étale neighborhood of R, when R is equidimensional and an algebra of finite type over a perfect field k. Thus, under these additional hypotheses, the completion is not needed and the reduction to the domain case is avoided.

Theorem 11.6.8 *Let $(B, \mathfrak{m})$ be a Noetherian equicharacteristic, equidimensional local ring of Krull dimension d. Let $I \subset \mathfrak{m}$ be an $\mathfrak{m}$-primary ideal. Assume that there exists a local étale neighborhood $(B', \mathfrak{m}')$ of $(B, \mathfrak{m})$ with a finite-transversal morphism w.r.t. I, $S \subset B'$. Let $b \in B$. If*

$$p(Z) = Z^{\ell} + a_1 Z^{\ell-1} + \ldots + a_{\ell}$$

is the minimal polynomial of $b \in B'$ over the fraction field of S, $K(S)$, then $p(Z) \in S[Z]$ and

$$\overline{\nu}_I(b) = \min \left\{ \frac{\nu_{\mathfrak{m}_S}(a_i)}{i} : i = 1, \ldots, \ell \right\}, \qquad (11.16)$$

where $\mathfrak{m}_S = \mathfrak{m}' \cap S$.

Proof We follow the ideas of Hickel in the proof of [15, Theorem 2.1] to check that the result also holds in this different setting. To ease the notation let us assume that $B = B'$, and let $\mathfrak{m}_S = \mathfrak{m} \cap S$. Since the extension $S \subset B$ is assumed to be finite-transversal w.r.t. I we have that $\mathfrak{m}_S B$ is a reduction of I generated by d-elements. Let $b \in B$. Then $L = B \otimes_S K(S)$ is a finite extension of $K(S)$, although not necessarily a domain. Let

$$p(Z) = Z^{\ell} + a_1 Z^{\ell-1} + \ldots + a_{\ell} \in K(S)[Z]$$

be the minimal polynomial of b over $K(S)$. Observe that $p(Z)$ might not be irreducible over $K(S)$. By Proposition 11.4.1 $p(Z) \in S[Z]$ and the subring of $S[b] \subset B$ is isomorphic to $S[Z]/\langle p(Z) \rangle$. Thus $S[b]$ is free of rank ℓ over S. Taking the $\mathfrak{m}_S$-adic completion of S we have a commutative diagram,

$$\begin{array}{ccc} B & \longrightarrow & B \otimes_S \hat{S} = R \\ \uparrow & & \uparrow \\ S[b] & \longrightarrow & S[b] \otimes_S \hat{S} = T \\ \uparrow & & \uparrow \\ S & \longrightarrow & \hat{S} \cong \tilde{k}[[X_1, \ldots, X_d]] \end{array}$$

where the vertical maps are finite while the horizontal maps are faithfully flat and $\tilde{k}$ is the residue field of S (which is also the residue field of B). Since the rank of

$S[b] \otimes_S \hat{S}$ over $\widehat{S}$ is ℓ, $p(X)$ is also the minimal polynomial of $b \in S[b] \otimes_S \hat{S}$ over $\widehat{S}$. By Cohen [8, Theorem 8], $B \otimes_S \hat{S}$ is complete, and then the claim follows from [15, Theorem 2.1] if $B \otimes_S \hat{S}$ is a domain.

Otherwise, we will see that the same argument given in [15, Theorem 2.1] can be used in this case to prove a similar result. Set $J = \langle X_1, \dots, X_d \rangle \subset \hat{S}$, let $J_1 = \langle X_1, \dots, X_d \rangle T$ and let $J_2 = \langle X_1, \dots, X_d \rangle R$. Then J_2 is a reduction of IR. And we have that that

$$\overline{\nu}_I(b) = \overline{\nu}_{IR}(b) = \overline{\nu}_{J_2}(b) = \overline{\nu}_{J_1}(b),$$

where the last equality comes from the fact that $T \subset R$ is a finite extension and using Corollary 11.6.5. On the one hand, if $h(Z) = Z^m + c_1 Z^{m-1} + \dots + c_m \in \hat{S}[Z]$ is any monic polynomial with $q(b) = 0$ then

$$\overline{\nu}_{J_1}(b) \geqslant \min \left\{ \frac{\nu_{\mathfrak{m}_{\hat{S}}}(c_i)}{i} : i = 1, \dots, s \right\},$$

(see [15, pg. 1374]). To verify the equality (11.16), we follow the argument in the proof of [15, Theorem 2.1] to check that the hypothesis on T being a domain is not needed.

If b is a unit or nilpotent, then we are done. Suppose otherwise that $\overline{\nu}_I(b) = r/s > 0$, where $r, s \in \mathbb{N}_{>0}$ and consider the diagram:

$$\begin{array}{ccc}
S'[b^s] & \longrightarrow & S[b] \otimes_S \hat{S} = \hat{S}[Z]/\langle p(Z) \rangle = T \\
\uparrow & & \uparrow \\
S' = k[[X_1^r, \dots, X_n^r]] & \longrightarrow & \hat{S} \cong k[[X_1, \dots, X_n]]
\end{array}$$

Notice that all the maps are finite, and that the first vertical is under the assumptions of Proposition 11.4.1. Hence if $q(Z)$ is the minimal polynomial of b^s over $K(S')$, we have that $q(Z) \in S'[Z]$ and moreover, $S'[b^s] \cong S[Z]/\langle q(Z) \rangle$.

In addition, the conditions in Proposition 11.3.4 hold for the first vertical finite map:

(i) There is a unique maximal ideal $\mathfrak{m}'$ in $S'[b^s]$ dominating the maximal ideal $\mathfrak{m}_{S'}$ of S';
(ii) The residue fields at $\mathfrak{m}'$ and at $\mathfrak{m}_{S'}$ are the same, hence $\mathfrak{m}' = \mathfrak{m}_{S'} + \langle b^s \rangle$;
(iii) The expansion of the maximal ideal of S', $\mathfrak{m}_{S'}$ in $S'[b^s]$, generates a reduction of $\mathfrak{m}'$.

Hence, by Zariski's multiplicity formula for finite projections, the multiplicity of $S[b^s]$ at $\mathfrak{m}'$ is the same as the generic rank of the finite extension $S' \subset S'[b^s]$. In other words, $S' \subset S'[b^s]$ is finite-transversal w.r.t. $\mathfrak{m}'$. Therefore, the polynomial $q(Z) \in S'[Z]$ determines a hypersurface in $\mathrm{Spec}(S'[Z])$ whose multiplicity at

$\langle X_1^r, \ldots, X_d^r, Z\rangle$ is the same generic rank of the finite extension. Thus, if

$$q(Z) = Z^m + c_1 Z^{m-1} + \ldots + c_m$$

necessarily $c_i \in \langle X_1^r, \ldots, X_n^r\rangle^i S'$.

Next, since $q(Z^s)$ is a multiple of $p(Z)$, following word by word the proof of Hickel in [15, pgs. 1374-5], we get that

$$\overline{\nu}_J(b) \leqslant \min_i \left\{ \frac{\nu_{\mathfrak{m}_{\hat{S}}}(a_i)}{i} : i = 1, \ldots, \ell \right\}.$$

□

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