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Handbook of Geometry and Topology of Singularities IV

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Preface

Singularity theory dates back to the work of Newton, Leibniz, Cauchy, Lagrange and many others, although it only emerged as a field of mathematics in itself in the early 1960s, thanks to pioneering work by Thom, Zariski, Whitney, Hironaka, Milnor, Pham, Arnold et al.

Singularities are ubiquitous in mathematics, appearing naturally in a wide range of different areas of knowledge. Their scope is vast, their purpose is multifold. Its potential for applications in other areas of mathematics and of knowledge in general is unlimited, and so are its possible sources of inspiration. This theory is crucible where different types of mathematical problems interact and surprising connections are born. Just as mathematics interacts energetically with science in general, so does singularity theory with the rest of mathematics.

The downside is that before a researcher, or a student, can successfully detect and try to answer some interesting problem, he or she must become familiar with different subjects and their techniques, and the learning process is long. In most cases, various areas are involved, as for instance topology, geometry, differential equations and algebra. That makes this a fascinating area of mathematics. And that is also a reason why a handbook which presents in-depth and reader-friendly surveys of topics of singularity theory is useful.

This is the fourth volume of the *Handbook of the Geometry and Topology of Singularities*. By no means this collection pretends to be comprehensive, since the theory is vast. Yet, it has the intention of covering a wide scope of singularity theory, presenting in a clear and inspiring way, articles on various aspects of the theory and its interactions with other areas of mathematics. The authors are world experts; the various articles deal with both classical material and modern developments.

The first three volumes of this collection gathered foundational aspects of the theory, as well as some other important aspects. Some topics are studied in various chapters, and in some cases, also in more than one volume. The topics studied so far include:

– The combinatorics and topology of plane curves and surface singularities.
– The analytic classification of plane curve singularities and the existence of complex and real algebraic curves in the plane with prescribed singularities.

- Introductions to four of the classical methods for studying the topology and geometry of singular spaces, namely: resolution of singularities, deformation theory, stratifications and slicing the spaces *à la* Lefschetz.
- Milnor's fibration theorem for real and complex singularities, the monodromy, vanishing cycles and Lê numbers.
- Morse theory for stratified spaces and constructible sheaves.
- Simple Lie algebras and simple singularities.
- Limits of tangents to a complex analytic surface, a subject that originates in Whitney's work.
- Zariski's equisingularity and intersection homology.
- Mixed singularities, which are real analytic singularities with a rich structure that allows their study via complex geometry.
- Intersections of quadrics in $\mathbb{R}^n$, and their relation with holomorphic vector fields, toric geometry and moment-angle manifolds.
- Quasi-projective varieties, complements of plane curves and hypersurfaces in projective space.
- Singularities of mappings. Thom-Mather theory.
- The interplay between analytic and topological invariants of complex surface singularities and their relation with modern three-manifold invariants.
- Indices of vector fields on singular varieties and their relation with other invariants.
- Chern Class and Segre Class for singular varieties.
- Baum-Bott residues and localization in singularity theory.
- Mixed Hodge structures.
- Constructible sheaf complexes.

This Volume IV consists of 12 chapters. In Chap. 1, the authors look at limits of tangents spaces and Whitney stratifications. If X is a singular complex analytic space, then it has no tangent bundle and one cannot use in a direct way many classical and fundamental constructions. As explained in the introduction to that chapter, throughout the eighteenth century much work was done on singular curves and surfaces with the goal of generalizing Riemann's work understanding "conditions of adjunction". Then, John Semple [Proc. LMS 1954] introduced the space of limit directions of tangent spaces to an algebraic variety, which he called the *first derivate*. The construction can be naively explained by saying that one replaces the singular set by all limits of spaces tangent to the regular part. Semple's work passed unnoticed for a long time, and about 10 years later, independently, John Nash rediscovered the construction and this became known as the Nash modification, or blow up. It is just recently that credit is being given to Semple as well. The use and study of this construction is vast, and it has already appeared in several works in this handbook, as for instance in Mark Spivakovski's chapter in Volume 1, and in several works on characteristic classes in Volume III. It appears also in Chap. 6 of this Volume IV. There is an analogous construction where tangent spaces are replaced by tangent hyperplanes. This viewpoint is more suitable for using other techniques of algebraic geometry, as for instance intersection theory. Chapter 1 is devoted to these two constructions, their applications to stratification theory in the sense of Whitney and to a general Plücker type formula for projective varieties.

Chapter 2 surveys determinantal singularities. These varieties are spaces of matrices with a given upper bound on their ranks. These generalize the much studied class of complete intersections in several different aspects, and they exhibit interesting new phenomena such as, for instance, non-isolated singularities which are finitely determined, or smoothings with low connectivity. The chapter starts with the necessary algebraic background, and then continues by discussing the subtle interplay of unfoldings and deformations in this setting.

Chapters 3 and 4 concern the space of arcs in algebraic varieties. Roughly speaking, an arc is a very small portion of a curve on a scheme. The space of arcs and the space of m-jets have natural schemes structures with important properties. These spaces appeared in singularity theory for the first time in a short preprint in 1968 by John Nash, although these concepts somehow already appear in the work of Isaac Newton in the seventeenth century. If X is a singular complex analytic space and $Z \to X$ is a resolution, then one can construct infinitely many other resolutions of X by blowing up Z along regular loci. Nash wanted to codify the data which is common to all these resolutions, and he suggested that these data are hidden in the arc space. This led to what is nowadays known as the Nash problem. Chapter 3 is an introduction to the subject and it makes a remarkable bridge connecting this theory with birational geometry. Chapter 4 provides an overlook of the diverse aspects in the literature about the subject, complementing in several ways the existing literature.

Vector fields on a smooth manifolds and their local Poincaré-Hopf indices at the singular points play an important role in many different areas of mathematics. For singular varieties, the study of indices of a particular class of vector fields started in the 1960s with work by M. H. Schwartz [CRAS 1965] aimed towards extending Chern classes to singular varieties. With that same goal but a different viewpoint, MacPherson [Ann. Maths. 1975] used an index (the Euler obstruction) for a particular class of 1-forms on singular spaces. Seade [AMS Contemp. Math. 58, 1987] discovered an index of vector fields on smoothable normal complex Gorenstein surfaces germs and this gave rise to the so-called GSV-index of vector fields on complex ICIS germs. This was a markpoint in the study of indices of vector fields on singular varieties. In the 1990s, King and Trottman introduced another notion of index, much related to Schwartz' index, but their work was not published until some 20 years later [Proc. LMS 2014; in the meantime, the same notion was rediscovered independently by W. Ebeling and S. Gusein-Zade, and by M. Aguilar et al.]. This is known as the radial (or Schwartz) index. Then, as hinted by Arnold, Ebeling and Gusein-Zade began the study of indices of 1-forms. In Chap. 5, the authors survey the theory of indices of vector fields and 1-forms on singular varieties, a subject previously discussed with different viewpoints in the chapters by Brasselet and by Callejas et al. in Volume III of this handbook. The authors discuss also indices for appropriate collections of 1-forms, an interesting concept. Just as the index of a 1-form is morally linked with the Chern number defined by the top Chern class, so too the indices of collections of 1-forms are linked with other Chern numbers.

Chapter 6 is about the motivic Hirzebruch class for singular varieties and it complements various articles that appeared in Volume III on the theory of Chern classes for singular varieties. We recall that Hirzebruch used the Todd class of complex manifolds to prove a deep theorem that has as special cases:

(a) The theorem of Gauss–Bonnet
(b) A generalization of Riemann-Roch's theorem to higher dimensions and with cohomology in arbitrary holomorphic vector bundles
(c) The Thom-Hirzebruch signature theorem

These three theorems have been extended individually to singular varieties: via MacPherson's Chern class (mentioned above) in the first case, with the Baum-Fulton MacPherson's Todd class in the second case [Publ. Math. IHES 1975] and with Cappell-Shaneson's L-class [J. AMS 1991] in the latter case. In this chapter, the author discusses the motivic Hirzebruch class, which unifies these three classes.

Chapters 7–10 are about Lipschitz geometry in singularity theory, a subject that started with work by Pham and Teissier [CMI, Nice 1970]. Later, Mostowski [Rozprawy Mat. 1985] studied Lipschitz equisingularity and Lipschitz stratifications in analytic sets, a notion that grants the constancy of the Lipschitz type of the stratified set along each stratum. The existence of Lipschitz stratifications for analytic sets was established by Mostowski in 1989 in the complex case, and by Parusinsky in 1993 in the real setting. We refer to Parusinsky's paper in Volume II for an account on this subject, and to Chap. 7 in this Volume IV. This chapter deals with semialgebraic and subanalytic subsets of $\mathbb{R}^n$, and more generally with all the sets that are definable in a polynomially bounded o-minimal structure expanding $\mathbb{R}$. The chapter begins with basic definitions about o-minimal structures and Lipschitz geometry, and it gives a short survey of some historical results, such as existence of Mostowski's Lipschitz stratifications and the Preparation Theorem for definable functions. It then presents a stratification theorem and discusses related important results, including a bi-Lipschitz version of Hardt's theorem on polynomially bounded o-minimal structures.

Notice that given an analytic subset X of $\mathbb{R}^n$, we have two natural metrics on X: one is the metric induced from the ambient space; this is called the *outer metric*. The other is the *inner*, or length, metric, defined in the usual way in differential geometry, as the infimum of lengths of piecewise smooth curves connecting two given points. An embedding of X in $\mathbb{R}^n$ is normal if the two metrics are equivalent up to a bilipschitz homeomorphism. Chapter 8 presents basic results on the Lipschitz Geometry of germs. It reviews recent results related to the outer metric and to the ambient bi-Lipschitz classification of surface germs, explaining why the outer bi-Lipschitz classification is much harder than the inner classification. It also discusses relations with the theory of metric knots. Chapter 9 addresses the classical concept of multiplicity of singular points of complex algebraic sets (not necessarily complex curves). It approaches the nature of the multiplicity of singular points as a geometric invariant from the perspective of Zariski's Multiplicity Conjecture (1971). The chapter begins with a long introduction to the subject.

The study of Lipschitz normally embedded germs has attracted a lot of interest in the last decade, and this is the subject of Chap. 10. Here the authors discuss many general facts about Lipschitz normally embedded singularities, before moving their focus to some recent developments on criteria, examples and properties of such germs. The chapter concludes with a list of interesting open questions.

If X is a scheme of finite type over a perfect field k, as for instance $\mathbb{C}$, its multiplicity at each point x is the multiplicity of the local ring $\mathcal{O}_{X,x}$. This is a measure of how "bad", or perhaps "interesting", the singularity is. For instance, resolution of singularities of varieties over $\mathbb{C}$, and more generally, over fields of characteristic zero, can be proved by using the multiplicity as main invariant, as proved by O. Villamayor [Adv. Math. 2014]. In order to study the multiplicity, one may look at the Hilbert-Samuel function, which is defined for any local Noetherian ring. More precisely, for a prime $\mathfrak{p}$ in a Noetherian ring B, the multiplicity of B at $\mathfrak{p}$ springs when trying to measure the growth of dimension of the graded pieces of the graded ring

$$Gr_{\mathfrak{p}B_{\mathfrak{p}}}(B_{\mathfrak{p}}) = \bigoplus_{i\geq 0} \mathfrak{p}^i B_{\mathfrak{p}}/\mathfrak{p}^{i+1} B_{\mathfrak{p}}$$

as $k(\mathfrak{p})$-vector spaces. In fact this growth is encoded asymptotically by the so called Hilbert-Samuel polynomial of $B_{\mathfrak{p}}$ at $\mathfrak{p}$, which is a polynomial of degree $d = \dim(B_{\mathfrak{p}})$ and the multiplicity at $\mathfrak{p}$ is (up to some suitable factor) the leading coefficient of that polynomial. Chapter 11 is mostly expository and the authors pay special attention to the geometrical aspects of these notions. To this end, finite projections from $\mathrm{Spec}(B)$ to the spectrum of a regular ring S are studied. When the projections are "generic enough", then some applications are discussed, like the determination of the top multiplicity locus of $\mathrm{Spec}(B)$, or the computation of other invariants like the asymptotic Samuel function.

We close this volume with a chapter about the logarithmic comparison theorem and several results known as comparison theorems, in the line of Grothendieck's comparison theorem. We recall that if X is a complex analytic manifold, one has the classical de Rham complex of holomorphic forms on X. And if we have a divisor D in X, we have also the de Rham complex of meromorphic forms with logarithmic poles in D, a notion introduced by K. Saito and recalled in the text. In Chap. 12, the authors state and sketch the proof of the logarithmic comparison theorem (LCT) which says that for a locally quasihomogeneous free divisor $D \subset \mathbb{C}^n$, the complex of meromorphic differential forms with logarithmic poles along D can be used to calculate the cohomology of $\mathbb{C}^n - D$. It goes on to consider a range of related results in the theory of D-modules, including a characterization of the hypersurfaces for which the conclusion of LCT holds. The LCT owes its name to its analogy with Grothendieck's comparison theorem, which is made clear in a brief historical introduction. The opening section gives the necessary background on free divisors and logarithmic poles, and the background on D-module theory is given in Section 2. Section 3 deals with a D-module characterization of LCT for free divisors.

This handbook is addressed to graduate students and newcomers to the theory, as well as to specialists who can use it as a guidebook. It provides an accessible account of the state-of-the-art in several aspects of singularity theory, its frontiers and its interactions with other areas of research. This will continue with a Volume V that will focus on holomorphic foliations, an important subject on its own, with close connections with singularity theory and holomorphic vector fields, and a Volume VI with other important areas of singularity theory.

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Chapter 1
Limits of Tangents, Whitney Stratifications and a Plücker Type Formula

Lê Dũng Tráng and Bernard Teissier

Contents

Abstract Let X denote a purely d-dimensional reduced complex analytic space. If it has singularities, it has no tangent bundle, which makes many classical and fundamental constructions impossible directly. However, there is a unique proper

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map $\nu_X : NX \to X$ which has the property that it is an isomorphism over the non-singular part X^0 of X and the tangent bundle T_{X^0} lifted to NX by this isomorphism extends uniquely to a vector bundle on NX. For $x \in X$, the set-theoretical fiber $|\nu_X^{-1}(x)|$ is the set of limit directions of tangent spaces to X^0 at points approaching x. The space NX is reduced and equidimensional, but in general singular. If X is a closed analytic subspace of an open set U of $\mathbf{C}^N$, the space NX is a closed analytic subspace of $X \times \mathbf{G}(d, N)$, where $\mathbf{G}(d, N)$ denotes the Grassmannian of d-dimensional vector subspaces of $\mathbf{C}^N$. The rich geometry of the Grassmannian makes it complicated to study the geometry of the map ν_X using intersection theory. There is an analogous construction where tangent spaces are replaced by tangent hyperplanes, and the map ν_X is replaced by the conormal map $\kappa_X : C(X) \to X$, where $C(X)$ denotes the conormal space, which is a subspace of $X \times \check{\mathbf{P}}^{N-1}$, where $\check{\mathbf{P}}^{N-1}$ is the space of hyperplanes of $\mathbf{P}^N$, the dual projective space, so that the intersection theory is simpler. This paper is devoted to these two constructions, their applications to stratification theory in the sense of Whitney and to a general Plücker type formula for projective varieties.

1.1 Introduction

Let X denote a purely d-dimensional reduced subspace of an affine space $\mathbf{C}^N$ defined in an open subset by algebraic or analytic equations with coefficients in $\mathbf{C}$. The singular locus of X is usually defined as a point where "there is no tangent space" in the sense that the linear equations derived from the original equations of X do not define a unique linear subspace of dimension d. The direction of the tangent space at a non-singular point $x \in X$ is represented by a point in the Grassmannian $\mathbf{G}(d, N)$ of d-dimensional vector subspaces of $\mathbf{C}^N$. Thus, there is a map $\gamma : X^0 \to \mathbf{G}(d, N)$, where X^0 denotes the non-singular part of X, which is dense in X since X is reduced. This map is easily seen to be holomorphic, and algebraic if X is. It is called the Gauss map because a similar map was used by Gauss in his study of the curvature of differentiable surfaces, published in 1828.

Around the same time as Gauss, Poncelet, Bobillier, Plücker and others were studying the duality of plane projective curves. Here the motivations did not come from geodesy but rather from the interest in understanding the duals of known theorems and the problem of determining how many tangents can be drawn to a curve C of degree d from a general point in the plane. The plane projective duality which transforms a point in the projective plane $\mathbf{P}^2$ with homogeneous coordinates $(x : y : z)$ into a line in the dual plane simply by exchanging the roles of coefficients and variables in the equation $ax + by + cz = 0$ of lines going through the point $(x : y : z)$ shows that the number of tangents to C from a general point is the degree of the *dual curve* $\check{C} \subset \check{\mathbf{P}}^2$ consisting of the points of $\check{\mathbf{P}}^2$ representing the lines tangent to C. This degree is $d(d-1)$. Thus if $\check{C}$ was non-singular its dual could not be C as the geometry insists it should be, since $d(d-1)((d(d-1)-1) \neq d(d-1)$ unless $d = 2$. Thus $\check{C}$ has singularities and some points of C must represent limits of tangents to $\check{C}$ at non-singular points of $\check{C}$ tending to a singular point. This is perhaps one of the first occurrences of limits of tangent spaces.

Singular curves and surfaces were studied throughout the nineteenth century mostly[1] with the goal of generalizing Riemann's work, understanding "conditions of adjunction"† and more generally the behavior of differential forms and their integrals. It is perhaps not so surprising that it is only in 1954 that Semple introduced in [50] the space of limit directions of tangent spaces to an algebraic variety, which he called the *first derivate* in [50, §8]. It is the closure NX in $X \times \mathbf{G}(d, N)$ of the graph $NX^0 \subset X^0 \times \mathbf{G}(d, N)$ of the Gauss map. As a subspace of $X \times \mathbf{G}(d, N)$ it is endowed with a projection $\nu\colon NX \to X$ which is proper (since $\mathbf{G}(d, N)$ is compact) and is an isomorphism over X^0. The set-theoretic fiber $|\nu^{-1}(x) \subset \mathbf{G}(d, N)$ above a point $x \in X$ is the set of limit directions at x of tangent spaces at points of X^0 tending to x.

Semple also asked, in the last paragraph of his paper, whether iterating this construction would eventually resolve the singularities of X.

About 10 years after Semple, John Nash rediscovered the construction and the question and for a time the construction was called the *Nash blowing-up*, which explains the notation NX. Semple's paper is difficult to read and it is only after Monique Lejeune-Jalabert discovered his contribution that the map $\nu_X\colon NX \to X$ came to be called the *Semple-Nash modification*.

Also about 10 years after Semple, and after important preliminary work in the differentiable case by Whitney himself in 1957 and Thom in 1960 (see [63]), in 1965, Hassler Whitney published a study of possible definitions of limits of secants and tangents at a singular point of a complex analytic space, in which he introduced the fundamental notion of regular stratification, nowadays called Whitney stratifications. It is a locally finite partition of a complex analytic space into locally closed non-singular "strata" where each stratum has a "regular" behavior along the strata of its boundary. The definition of "regular" involves both limits of secants and limits of tangents for points tending to the boundary stratum. The definitions extend readily beyond the complex analytic case and in the hands of Thom, Mather, and others it became a most important conceptual and technical tool in the study of singularities of differentiable mappings, in particular when applied to infinite dimensional spaces such as jet spaces and function spaces.

Stratification theory in the large is the subject of David Trotman's contribution (see [63]) to the first volume of this Handbook. In this text we shall concentrate on the complex analytic case for both limits of tangent spaces and stratifications. We consider reduced equidimensional complex spaces and whenever we take the intersection of such a space with a non-singular subspace of some ambiant non-singular space, we endow it with its reduced structure.

[1] There are exceptions, for example in work of Cayley, Halphen, M. Noether, Salmon, H.J.S. Smith, often connected with generalizations of the Plücker formulas for curves and the study of linear systems and projective embeddings.

Although it can be read independently, this paper is in some ways a continuation of the paper [33] of Lê and Snoussi in Volume II of this Handbook. Also, a version of the content of Sect. 1.2 appears in [56, §3.9] under the name of Nash blowing up (which is more traditional) and a version of the content of Sect. 1.5 appears in [56, §3.3] in Volume I of this Handbook. Some of the topics exposed here can be found exposed in greater detail in [16] from which, with the permission of its authors and of the editors, we have copied some parts of this text.

1.2 Limits of Tangent Spaces: The Semple-Nash Modification

Let X be a reduced and equidimensional closed subspace of an open set $U \subset \mathbf{C}^N$. We denote by X^0 the set of non-singular points of X, which is open and dense in X, by d the dimension of X, and by $\mathbf{G}(d, N)$ the grassmannian of d-dimensional vector subspaces of $\mathbf{C}^N$. The Gauss map

$$\gamma_{X^0} : X^0 \to \mathbf{G}(d, N),\ \ x \mapsto [T_{X^0,x}] \in \mathbf{G}(d, N)$$

associates to every point of X^0 the direction of the tangent space to X at this point. Let us consider the graph $NX^0 \subset X^0 \times \mathbf{G}(d, N)$ of γ_{X^0}. It is a purely d-dimensional analytic subset of $X^0 \times \mathbf{G}(d, N)$ since it is isomorphic to X^0. The space of limits of (directions of) tangent spaces at points of X^0, the Semple-Nash modification of X, is the closure NX in $X \times \mathbf{G}(d, N)$ of NX^0. So we have to prove that it is a closed analytic subspace of $X \times \mathbf{G}(d, N)$. The singular locus $\mathrm{Sing}X = X \setminus X^0$ is a closed complex subspace of X, of dimension $\leq d - 1$. However, we cannot apply the Remmert-Stein theorem (see [38, Chap. IV, §6] or [1, Theorem 6]) to prove that NX is analytic because we have to extend $NX^0 \subset U \times \mathbf{G}(d, N)$ through $\mathrm{Sing}X \times \mathbf{G}(d, N)$ which is of dimension $> d$. The proofs in [65, Theorem 16.4] and [45, Theorem 1] build, using jacobian determinants, a system of equations for the closure $NX \subset U \times \mathbf{G}(d, N)$, thus proving its analyticity.

One has then to verify that the map $NX \to X$ is unique up to a unique X-isomorphism, independent of the immersion of X in an open set of an affine space.

Then for any reduced equidimensional complex space X the local Semple-Nash modifications will glue up into a unique proper map, the Semple-Nash modification $\nu_X : NX \to X$ (sometimes simplified to ν).

We note that since NX is a reduced equidimensional analytic space it makes sense to iterate the Semple-Nash modification: $N^2X = NNX, N^3X = NN^2X$, and so on.

We note that the pull-back by the second projection $\gamma_X : NX \to \mathbf{G}(d, N)$ of the tautological bundle on the grassmannian is a vector bundle on NX which extends the tangent bundle of $NX^0 \simeq X^0$.

There is another approach, based on Grothendieck's Grassmannian of a coherent module (see [19]) which shows directly the canonicity of the Semple-Nash modification.

Let X be a reduced equidimensional complex space and Ω^1_X its coherent module of differentials, which is locally free on X^0. It comes with a morphism of $\mathcal{O}_X$-modules $d_X\colon \mathcal{O}_X \to \Omega^1_X$, the differential, which cannot be confused with the dimension. Since the $\mathcal{O}_X$-module Ω^1_X is coherent, the symmetric algebra $\mathrm{Sym}_{\mathcal{O}_X}\Omega^1_X$ of the $\mathcal{O}_X$-module Ω^1_X is a graded $\mathcal{O}_X$-algebra locally of finite presentation and generated in degree one, and so corresponds to an analytic space $\mathrm{Specan}_X\mathrm{Sym}_{\mathcal{O}_X}\Omega^1_X$ over X.The fibers of the natural map

$$t\colon \mathrm{Specan}_X\mathrm{Sym}_{\mathcal{O}_X}\Omega^1_X \to X$$

are the Zariski tangent spaces $t^{-1}(x) = \mathrm{SpecSym}_{\mathbf{C}}(m_{X,x}/m^2_{X,x})^\vee$, where $\vee$ denotes the dual vector space over $\mathbf{C}$.

Since Ω^1_X is a coherent sheaf of $\mathcal{O}_X$-modules, $\mathrm{Specan}_X\mathrm{Sym}_{\mathcal{O}_X}\Omega^1_X$ is a complex analytic space. The sections $\partial\colon X \to \mathrm{Specan}_X\mathrm{Sym}_{\mathcal{O}_X}\Omega^1_X$ of the projection t correspond to elements of $\mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$, that is, derivations from $\mathcal{O}_X$ to $\mathcal{O}_X$. If X is non-singular $\mathrm{Specan}_X\mathrm{Sym}_{\mathcal{O}_X}\Omega^1_X$ is the tangent bundle to X and the sections ∂ are holomorphic vector fields on X.

Now Grothendieck has shown that for Ω^1_X, as indeed for any coherent $\mathcal{O}_X$-module, just as $t\colon \mathrm{Specan}_X\mathrm{Sym}_{\mathcal{O}_X}\Omega^1_X \to X$ is a relative vector space in the sense that its fibers are vector spaces, there is a relative grassmannian

$$g\colon \mathbf{G}_d(\Omega^1_X) \to X$$

whose fiber at $x \in X$ is the grassmannian of d-dimensional subspaces of the vector space $t^{-1}(x)$.

The defining property of the map g is that for any holomorphic map $h\colon W \to X$ it is equivalent to give, up to isomorphism, a locally free quotient of rank d of the $\mathcal{O}_W$-module $h^*\Omega^1_X$ and to give, up to isomorphism, a factorization of h through g.

Now a rank d locally free quotient of $h^*\Omega^1_X$ corresponds to a vector bundle over W with d-dimensional fibers which is contained in $\mathrm{Specan}_X\mathrm{Sym}_{\mathcal{O}_T}h^*\Omega^1_X$. That is exactly a family of analytically varying d-dimensional subspaces of the Zariski tangent spaces $t^{-1}(h(w))$ for $w \in W$.

In particular, the sheaf $g^*\Omega^1_X$ on $\mathbf{G}_d(\Omega^1_X)$ has a locally free quotient of rank d, which corresponds to the pull back of the tautological bundle on the grassmannian.

If one remembers that in analytic geometry limits can be obtained by moving along analytic arcs (curve selection lemma), we see that since any limit direction T of tangent spaces at a point $x \in X$ is a limit along germs of analytic arcs $h\colon (\mathrm{d}, 0) \to (X, x)$, it is the fiber over 0 of a locally free quotient of $h^*\Omega^1_X$ and so the arc lifts as $\tilde{h}\colon (\mathrm{d}, 0) \to (\mathbf{G}_d(\Omega^1_X), T)$, which (with a little work) defines a map $NX \to \mathbf{G}_d(\Omega^1_X)$ which one shows (with a little more work) to be an X-isomorphism.

The equivalence of this grassmannian construction with the Gauss map construction shows directly that the closure of the graph of the Gauss map is analytic, and that the result of the construction is unique up to a unique isomorphism.

Since the grassmannians embed into projective spaces, the map $NX \to X$ is locally projective and since it is locally bimeromorphic, it is locally on X the blowing-up (see Sect. 1.5 below for the definition) of a sheaf of ideals, a result proved explicitly by Nobile in [45, Theorem 1].

Examples

(i) Let $X \subset \mathbf{C}^4$ be the union of two planes meeting at the origin. Then $NX \to X$ maps the disjoint union of two 2-planes to X, each plane mapping isomorphically onto its image. It is a finite bimeromorphic map, and thus a resolution of singularities. If one follows the classical resolution algorithm, one blows up the intersection point. This again separates the two planes, but now the projection restricted to each of the separated planes is the blowing-up of a point, and is not finite.

(ii) Let $f(z_1, \ldots, z_N) = 0$ be an equation for a germ at the origin of a reduced hypersurface $(X, 0) \subset (\mathbf{C}^N, 0)$. The Semple-Nash modification is the blowing-up in X of the ideal generated by the partial derivatives of f. More generally, if X is a reduced complete intersection of dimension d in affine space $\mathbf{A}^N(\mathbf{C})$, then the blowing-up in X of the ideal generated by the $(N - d) \times (N - d)$ minors of the jacobian matrix of the equations is isomorphic to NX. For the general case, see [45].

The Semple-Nash modification has been used in the definition of characteristic classes for singular spaces (see [39] and Chapters 5–7 of Volume III of this Handbook), but we shall not go into this here. Much work has been devoted to understanding how the singularities of NX differ from those of X, and in particular to answer the question posed by Semple at the end of his paper and reiterated by Nash a decade later:

Does iterating the Semple-Nash modification resolve the singularities of X in finitely many steps?

In other words, given X as above, is there an integer k_0 such that $N^k X$ is non-singular for $k \geq k_0$?

It follows from the definition that if X is non-singular, we have $NX = X$. Nobile proved the converse in [45, Theorem 2]:

Theorem 1.2.1 (Nobile) *The Semple-Nash modification $\nu_X\colon NX \to X$ is an isomorphism if and only if X is non-singular.*

Nobile's original proof of this theorem is somewhat involved and relies on local parametric descriptions of a singular space and results of [65]. A different proof was proposed in [58, §2], based on the second construction of NX.

By definition, if $NX = X$, the module of differentials of X has a locally free quotient. The property of non-singularity being local we may assume after restricting to an open set $U \cap X$ of X that we have a surjection $\Omega^1_{U\cap X} \to \mathcal{O}^d_{U\cap X} \to 0$. Taking germs at $x \in U \cap X$, there is an element $h \in \mathcal{O}_{X,x}$ such that the differential $d_X h \in \Omega^1_{X,x}$ maps to $(1, 0 \ldots, 0) \in \mathcal{O}^d_{X,x}$, and thus a derivation D of $\mathcal{O}_{X,x}$ into itself such that $Dh = 1$. Since D is zero on $\mathbf{C} \subset \mathcal{O}_{X,x}$, we may assume that

$h \in m_{X,x}$. Geometrically, the derivation D corresponds to a holomorphic vector field on X not vanishing at x. Its integration (see [58, §2] for details) gives the germ (X, x) a product structure $(X, x) \simeq (X_1 \times \mathbf{C}, x)$, where $X_1 \subset X$ is the reduced equidimensional space defined by the ideal $h\mathcal{O}_{X,x}$ and satisfies $NX_1 = X_1$. The result follows by induction on the dimension.

This theorem has the important consequence that in order to prove the Semple-Nash conjecture, it suffices to prove that the sequence of the spaces N^kX eventually becomes stationary.

As already noted by Nobile, it implies immediately that if X is of dimension one N^kX is non-singular for large k. Since a curve has finitely many limit tangent lines at any point, the Semple-Nash modification of a curve is a finite bimeromorphic map, and thus dominated by the normalization. Since the normalization $\overline{\mathcal{O}_{X,x}}$, which is a resolution of singularities, is a finitely generated and thus a noetherian $\mathcal{O}_{X,x}$-module, there cannot be an infinite strictly increasing sequence of subalgebras finite over $\mathcal{O}_{X,x}$.

Apart from some special cases, the Semple-Nash conjecture is still open in dimensions ≥ 2. The best result is due to Spivakosky in [55], where he proves that iterating the operation of Semple-Nash modification followed by normalization eventually resolves the singularities of a surface. Spivakovsky's proof sheds light on the change of the dual graph of a minimal resolution when one passes from X to NX.

There are a number of other significant results for surfaces. For example Snoussi in [54] relates the planar components of the tangent cone to a surface to the singularities of its Semple-Nash transform and D. Duarte in [9] shows that iterating the Semple-Nash modification for toric surfaces has to stop in some charts.

In dimension ≥ 3 very little is known in general. The resolution problem is open even in the case of toric varieties, where in characteristic zero the Semple-Nash modification is the blowing up of a deceptively simple monomial ideal (see [17, §10]).

Indeed, apart from results of Vaquié in [64] concerning numerical invariants, and precise results for quasi-ordinary singularities (see [4] and [2]), there is no satisfactory description in general of the relation between the geometry of NX and that of X.

However there is another aspect of limits of tangent spaces which is rather well understood: as we shall see below, given $(X, 0) \subset (\mathbf{C}^N, 0)$, a hyperplane in $\mathbf{C}^N$ is said to be tangent to X^0 at a point if it contains the tangent space to X^0 at that point and a hyperplane through 0 is a limit of tangent hyperplanes at points of X^0 if and only if it contains a limit of tangent spaces to X^0.

When X is a hypersurface with isolated singularity it was shown in [57, Chap. II, §1, 1.6] that a hyperplane H through the singular point is *not* a limit of tangent hyperplanes if and only if the Milnor number $\mu(X \cap H)$ is minimal among the Milnor numbers of all intersections $X \cap H'$. Then it was shown in [59, Appendice] that the family of all sections $X \cap H$ where H is not a limit of tangent hyperplanes is equisingular in the sense of Whitney conditions (which we shall see below). These results were generalized, for normal surfaces by Snoussi in [52], for arbitrary reduced equidimensional germs by Gaffney in [13, Theorem 2.1, Corollary 2.4] and in a more topological framework by Tibăr in [62]; see also [53].

The result for isolated hypersurface singularities was used as part of a method to compute limits of tangent spaces in this case. See [44], and [46] for more methods of computation.

In the case where our singular germ $(X, 0)$ is the cone over a projective variety, there is an algebraic approach to the study of the Gauss map in [51]. and a geometric one in [31]. We shall come back to this in the paragraph on projective duality.

Given a flat map $\pi : X \to S$ where X is again reduced and equidimensional and say S is non-singular and the open set X^0 of points of X where the map π is smooth is dense in X, with $\dim. X/S = d$, one can define a *relative Semple-Nash modification* $\nu_\pi : N_\pi X \to X$ as $\mathrm{SpecanSym}_X \Omega^1_{X/S}$ where $\Omega^1_{X/S}$ is the sheaf of relative differentials. In a local presentation of π as the map induced by the first projection in an embedding $X \subset S \times \mathbf{C}^N$ it is the closure of the graph of the relative Gauss map $\gamma_{X^0/S} : X^0 \to \mathbf{G}(d, N)$ sending a point $x \in X^0$ to the direction of the tangent space to the fiber of π through x.

Example 1.2.2 Let $f : (\mathbf{C}^N, 0) \to (\mathbf{C}, 0)$ be a germ of holomorphic map. The relative Semple-Nash modification of $\mathbf{C}^N$ is the blowing up (see Sect. 1.5 below) of the ideal generated by the partial derivatives of f. It is a closed subspace of $\mathbf{C}^N \times \check{\mathbf{P}}^{N-1}$, of dimension N.

This construction is of course useful in the study of families of singularities but the geometry of grassmannians being much more complicated than the geometry of projective spaces, it is time to move to the study of tangent hyperplanes.

1.3 Limits of Tangent Hyperplanes: The Conormal Space

Whenever our reduced equidimensional singular space X is not locally a hypersurface in some $\mathbf{C}^N$, the tangent spaces belong to grassmannians instead of projective spaces, and the description of the Semple-Nash modification becomes more complicated, according to the complexity of describing algebraic subvarieties of grassmannians.

It is therefore natural to consider tangent hyperplanes instead of tangent spaces: a tangent hyperplane at a point of $X^0 \subset \mathbf{C}^N$ is a (direction of) hyperplane containing the tangent space to X^0 at that point. This is also the approach which allows the connection with duality of projective varieties, in the case where our singular germ $(X, 0)$ is the cone over a projective variety. Most importantly the spaces of limits of tangent hyperplanes to a singular subspace of a non-singular complex variety can be characterized by Lagrangian (or Legendrian) type conditions, a fact which has no direct equivalent for NX.[2] One must emphasize that, in contrast to the Semple-

[2] See, however, [18, Theorem 3.14] and [31, Theorem 14].

Nash modification, this constructions depends on a local or global embedding of our space X in a non-singular complex analytic variety M.

Let us begin with the case of a local embedding $X \subset \mathbf{C}^N$, where the directions of hyperplanes in $\mathbf{C}^N$ are parametrized by the projective space $\check{\mathbf{P}}^{N-1}$. At a non-singular point $x \in X^0$, by definition a tangent hyperplane is a hyperplane in the tangent space to $\mathbf{C}^N$ at x which contains the tangent space $T_{X^0,x}$. Tangent hyperplanes at a point $x \in X^0$ constitute a $\mathbf{P}^{N-d-1} \subset \mathbf{P}^{N-1}$. Thus we obtain a subspace $C(X^0) \subset X \times \check{\mathbf{P}}^{N-1}$ whose points are pairs (x, H) such that H is a tangent hyperplane at x. The **conormal space** $C(X)$ of $X \subset \mathbf{C}^N$ is the closure of $C(X^0)$ in $X \times \check{\mathbf{P}}^{N-1}$. By definition it is the set of pairs (x, H) such that H is a limit at x of tangent hyperplanes at points of X^0.

The natural map induced by the first projection is denoted by $\kappa_X : C(X) \to X$.

Again we have to show that this closure is a closed analytic subspace of $X \times \check{\mathbf{P}}^{N-1}$. Following [16, Section 3.3], we use a diagram relating the conormal space of $(X, 0) \subset (\mathbf{C}^N, 0)$ and its Semple-Nash modification.

It is convenient here to use the notation of projective duality of linear spaces.

Given a vector subspace $T \subset \mathbf{C}^N$ we denote by $\mathbf{P}T$ its projectivization, *i.e.*, the image of $T \setminus \{0\}$ by the projection $\mathbf{C}^N \setminus \{0\} \to \mathbf{P}^{N-1}$ and by $\check{T} \subset \check{\mathbf{P}}^{N-1}$ the projective dual of $\mathbf{P}T \subset \mathbf{P}^{N-1}$, which is a $\mathbf{P}^{N-d-1} \subset \check{\mathbf{P}}^{N-1}$, the set of all hyperplanes H of $\mathbf{P}^{N-1}$ containing $\mathbf{P}T$.

We denote by $\check{\Xi} \subset \mathbf{G}(d, N) \times \check{\mathbf{P}}^{N-1}$ the cotautological $\mathbf{P}^{N-d-1}$-bundle over $\mathbf{G}(d, N)$, that is $\check{\Xi} = \{(T, H) \mid T \in \mathbf{G}(d, N),\ H \in \check{T} \subset \check{\mathbf{P}}^{N-1}\}$, and consider the intersection

$$\begin{array}{ccc} E := (X \times \check{\Xi}) \cap (NX \times \check{\mathbf{P}}^{N-1}) & \hookrightarrow & X \times \mathbf{G}(d, N) \times \check{\mathbf{P}}^{N-1} \\ \downarrow p_1 & \searrow^{p_2} & \downarrow \\ NX & & X \times \check{\mathbf{P}}^{N-1} \end{array}$$

and the morphism p_2 induced on E by the projection onto $X \times \check{\mathbf{P}}^{N-1}$. We then have the following:

Proposition 1.3.1 *The set-theoretical image $p_2(E)$ of the morphism p_2 coincides with the conormal space of X in $\mathbf{C}^N$*

$$p_2(E) = C(X) \subset X \times \check{\mathbf{P}}^{N-1}.$$

It is a closed analytic subspace of dimension $N - 1$.

Proof If we define $E^0 = \{(x, T_{X,x}, H) \in E \mid x \in X^0, H \in \check{T}_{X,x}\}$, then by construction $E^0 = p_1^{-1}(\nu_X^{-1}(X^0))$, and $p_2(E^0) = C(X^0)$. Since the morphism p_2 is proper it is closed, which finishes the proof since E is a closed analytic subspace of $X \times \mathbf{G}(d, N) \times \check{\mathbf{P}}^{N-1}$ because $\check{\Xi}$ is a closed analytic (in fact algebraic) subspace of $\mathbf{G}(d, N) \times \check{\mathbf{P}}^{N-1}$ and NX is a closed analytic subspace in $X \times \mathbf{G}(d, N)$. The dimension of $C(X)$ is that of its open dense subset $C(X^0)$, which is $N-1$ because it maps to X^0 with fibers $\mathbf{P}^{N-d-1}$. □

Corollary 1.3.2 *A hyperplane $H \in \check{\mathbf{P}}^{N-1}$ is a limit of tangent hyperplanes to X at 0, i.e., $H \in \kappa_X^{-1}(0)$, if and only if there exists a d-plane $(0,T) \in \nu_X^{-1}(0)$ such that $T \subset H$.*

Proof Let $(0,T) \in \nu_X^{-1}(0)$ be a limit of tangent spaces to X at 0. By construction of E and Proposition 1.3.1, every hyperplane H containing T is in the fiber $\kappa_X^{-1}(0)$, and so is a limit at 0 of tangent hyperplanes to X^0.

On the other hand, by construction, for any hyperplane $H \in \kappa_X^{-1}(0)$ there is a sequence of points $\{(x_i, H_i)\}_{i\in\mathbf{N}}$ in $\kappa_X^{-1}(X^0)$ converging to $p = (0, H)$. Since the map p_2 is surjective, by definition of E, we have a sequence $(x_i, T_i, H_i) \in E^0$ with $T_i = T_{x_i}X^0 \subset H_i$. By compactness of Grassmannians and projective spaces, this sequence has to converge, up to taking a subsequence, to (x, T, H) with T a limit at x of tangent spaces to X. Since inclusion is a closed condition, we have $T \subset H$. □

Corollary 1.3.3 *The morphism $p_1 : E \to NX$ is a locally analytically trivial fiber bundle with fiber $\mathbf{P}^{N-d-1}$.*

Proof By definition of E, the fiber of the projection p_1 over a point $(x,T) \in NX$ is the set of all hyperplanes in $\mathbf{P}^{N-1}$ containing $\mathbf{P}T$. In fact, the tangent bundle T_{X^0}, lifted to NX by the isomorphism $NX^0 \simeq X^0$, extends to a fiber bundle over NX, called the Nash tangent bundle of X. It is the pull-back by γ_X of the tautological bundle of $G(d,N)$, and E is the total space of the $\mathbf{P}^{N-d-1}$-bundle of the projective duals of the projectivized fibers of the Nash bundle. □

Consider the diagram extracted from the diagram we have seen above:

$$\begin{array}{ccc} E & \xrightarrow{p_2} & C(X) \\ \Big\downarrow{\scriptstyle p_1} & & \Big\downarrow{\scriptstyle \kappa_X} \\ NX & \xrightarrow{\nu_X} & X \end{array}$$

Proposition 1.3.4 *The map $p_2\colon E \to C(X)$ is isomorphic to the blowing up in $C(X)$ of the lift $\mathcal{J}\mathcal{O}_{C(X)}$ to $C(X)$ by κ_X of an ideal $\mathcal{J}$ of $\mathcal{O}_X$ whose blowing up coincides with the map ν_X.*

Proof By construction, E is a closed subspace of $NX \times_X C(X)$. By definition of E, the map p_2 is an isomorphism over $C(X^0)$ since a tangent hyperplane at a nonsingular point contains only the tangent space at that point. Therefore the map $p_2\colon E \to C(X)$ is locally bimeromorphic. The lift by $\nu_X \circ p_1$ of the ideal $\mathcal{J}$ is invertible on E. By the universal property of blowing up, any map $W \to C(X)$ such that the lift to W from $C(X)$ of the ideal $\mathcal{J}\mathcal{O}_{C(X)}$ is invertible on W has to factor uniquely through NX and therefore through the fiber product $NX \times_X C(X)$. In particular the blowing-up of $\mathcal{J}\mathcal{O}_{C(X)}$ in $C(X)$ has to factor through a closed subspace of $NX \times_X C(X)$ and has to coincide with E since they coincide over X^0.[3] ◻

In general the fiber of p_2 over a point $(x, H) \in C(X)$ is the set of limit directions at x of tangent spaces to X that are contained in H. If X is a hypersurface, the conormal map coincides with the Semple-Nash modification. In general, the manner in which the geometric structure of the inclusion $\kappa_X^{-1}(x) \subset \check{\mathbf{P}}^{N-1}$ determines the set of limit positions of tangent spaces, *i.e.*, the fiber $\nu_X^{-1}(x)$ of the Semple-Nash modification, is not so simple: by Proposition 1.3.1 and its corollary, the points of $\nu_X^{-1}(x)$ correspond to *some of* the projective subspaces $\mathbf{P}^{N-d-1}$ of $\check{\mathbf{P}}^{N-1}$ contained in $\kappa_X^{-1}(x)$.

A linear subspace $\mathbf{P}^{N-d-1} \subset \kappa_X^{-1}(x) \subset \check{\mathbf{P}}^{N-1}$ is dual to a d-dimensional vector subspace $T \subset \mathbf{C}^N$. If T is not a limit at x of tangent spaces, then by Corollary 1.3.2 any hyperplane in this $\mathbf{P}^{N-d-1}$ must contain a limit at x of tangent spaces, but this limit cannot be constant. This provides a set-theoretic characterization of those $\mathbf{P}^{N-d-1} \subset \kappa_X^{-1}(x)$ which are dual to a limit at x of tangent spaces, in terms of the diagram we have seen above: they are those which are the image by p_2 of a fiber of p_1. In view of Proposition 1.3.4 this gives a geometric characterization, but we would prefer one solely in terms of the geometry of $C(X)$; see [16, Example 3.4].

Note also that given a limit of tangent spaces T at $x \in X$ and a general linear projection $p\colon \mathbf{C}^N \to \mathbf{C}^{d+1}$, the hyperplane $p(T)$ is a limit hyperplane at $p(x)$ for the hypersurface $p(X) \subset \mathbf{C}^{d+1}$. This follows from the fact that given $T \in \nu_X^{-1}(0)$ we can find an analytic arc in NX ending at T and whose image in X is outside of the inverse image by p of the singular locus of $p(X)$.

Definition 1.3.5 The map $\lambda_X\colon C(X) \to \check{\mathbf{P}}^{N-1}$ induced by the second projection $X \times \check{\mathbf{P}}^{N-1} \to \check{\mathbf{P}}^{N-1}$ is called the tangent hyperplane map. It is the analogue of the Gauss map. When there is no ambiguity it will be denoted by λ.

1.4 Some Symplectic Geometry

In order to describe this set of tangent hyperplanes, we are going to use the language of symplectic geometry and Lagrangian submanifolds. Let us start with a few definitions. This section is mostly taken from [16, Section 2.1].

Let M be any N-dimensional manifold, and let ω be a de Rham 2-form on M, that is, for each $x \in M$, the map

$$\omega_x : T_{M,x} \times T_{M,x} \to \mathbf{R}$$

[3] For the reader familiar with bimeromorphic geometry, as for example in [24], [3, Chap. 1, 1.5] and [25, §2], the map p_1 appears as the *strict transform* of the map κ by the blowing-up ν. Since p_1 is a $\mathbf{P}^{N-d-1}$-bundle by Corollary 1.3.3, the map ν is also the flattening map of κ: every blowing-up $t\colon T \to X$ of X such that the strict transform of κ by t is flat must factor uniquely through ν. In this sense κ determines ν.

is skew-symmetric bilinear on the tangent space to M at x, and ω_x varies smoothly with x. We say that ω is **symplectic** if it is closed and ω_x is non-degenerate for all $x \in M$. Non degeneracy means that the map which to $v \in T_{M,x}$ associates the homomorphism $w \mapsto \omega(v, w) \in \mathbf{R}$ is an isomorphism from $T_{M,x}$ to its dual. A **symplectic manifold** is a pair (M, ω), where M is a manifold and ω is a symplectic form. These definitions extend, replacing $\mathbf{R}$ by $\mathbf{C}$, to the case of a complex analytic manifold *i.e.*, nonsingular space.

For any manifold M, its cotangent bundle T^*M has a canonical symplectic structure as follows. Let

$$\pi : T^*M \longrightarrow M$$
$$p = (x, \xi) \longmapsto x,$$

where $\xi \in T^*_{M,x}$, be the natural projection. The **Liouville 1-form** α on T^*M may be defined pointwise by:

$$\alpha_p(v) = \xi\left(d\pi_p(v)\right), \quad \text{for } v \in T_{T^*M,p}.$$

Note that $d\pi_p$ maps $T_{T^*M,p}$ to $T_{M,x}$, so that α is well defined. The **canonical symplectic 2-form** ω on T^*M is defined as

$$\omega = -d\alpha.$$

And it is not hard to see that if $(U, x_1, \ldots, x_N)$ is a coordinate chart for M with associated cotangent coordinates $(T^*U, x_1, \ldots, x_N, \xi_1, \ldots, \xi_N)$, then locally:

$$\omega = \sum_{i=1}^{N} dx_i \wedge d\xi_i.$$

Definition 1.4.1 Let (M, ω) be a 2n-dimensional symplectic manifold. A submanifold Y of M is a **Lagrangian submanifold** if at each $y \in Y$, $T_{Y,y}$ is a Lagrangian subspace of $T_{M,y}$, *i.e.*, $\omega_y|_{T_{Y,y}} \equiv 0$ and $\dim. T_{Y,y} = \frac{1}{2}\dim. T_{M,y}$. Equivalently, if $i : Y \hookrightarrow M$ is the inclusion map, then Y is **Lagrangian** if and only if $i^*\omega = 0$ and $\dim. Y = \frac{1}{2}\dim. M$.

Let M be a nonsingular complex analytic space of even dimension equipped with a closed non degenerate 2-form ω. If $Y \subset M$ is a complex analytic subspace, which may have singularities, we say that it is a **Lagrangian subspace** of M if it is purely of dimension $\frac{1}{2}\dim. M$ and there is a dense nonsingular open subset of the corresponding reduced subspace which is a Lagrangian submanifold in the sense that ω vanishes on all pairs of vectors in the tangent space.

Example 1.4.2 The **zero section** of T^*M

$$X := \{(x, \xi) \in T^*M | \xi = 0 \text{ in } T^*_{M,x}\}$$

is an n-dimensional Lagrangian submanifold of T^*M.

Exercise 1.4.3 Let $f(z_1, \dots, z_N)$ be a holomorphic function on an open set $U \subset \mathbf{C}^N$. Consider the differential df as a section $df : U \to T^*U$ of the cotangent bundle. Verify that the image of this section is a Lagrangian submanifold of T^*U. Explain what it means. What is the image in U by the natural projection $T^*U \to U$ of the intersection of this image with the zero section?

1.4.1 The Conormal Space in General

Let now M be a complex analytic manifold of dimension N and $X \subset M$ be a possibly singular complex subspace of pure dimension d, and let as before $X^0 = X \setminus \mathrm{Sing}X$ be the nonsingular part of X, which is a submanifold of M.

Definition 1.4.4 Set

$$N^*_{X^0,x} = \{\xi \in T^*_{M,x} | \xi(v) = 0, \ \forall v \in T_{X^0,x}\};$$

this means that the hyperplane $\{\xi = 0\}$ contains the tangent space to X^0 at the point x.

The **conormal bundle** of X^0 is

$$T^*_{X^0}M = \{(x, \xi) \in T^*M | x \in X^0, \ \xi \in N^*_{X^0,x}\}.$$

Definition 1.4.5 A closed subvariety L of the cotangent space T^*M of a manifold M is said to be *conical* if it is left globally invariant by the homotheties on the fibers of the map $T^*M \to M$, described locally by $\rho.(x, \xi) = (x, \rho\xi)$, $\rho \in \mathbf{C}$.

Proposition 1.4.6 *Let $i : T^*_{X^0}M \hookrightarrow T^*M$ be the inclusion and let α be the Liouville 1-form in T^*M as before. Then $i^*\alpha = 0$. In particular the conormal bundle $T^*_{X^0}M$ is a conical Lagrangian submanifold of T^*M, and has dimension N.*

Proof See [8, Proposition 3.6]. □

In the same context we can define the **conormal space of X in M** as the closure $T^*_X M$ of $T^*_{X^0}M$ in T^*M, with the **conormal map** $\kappa_X : T^*_X M \to X$, induced by the natural projection $\pi : T^*M \to M$. The conormal space is of dimension N. It may be singular and by Proposition 1.4.6, α vanishes on every tangent vector at a nonsingular point, so it is by construction a Lagrangian subspace of T^*M.

The fiber $\kappa_X^{-1}(x)$ of the conormal map $\kappa_X : T^*_X M \to X$ above a point $x \in X$ consists, if $x \in X^0$, of the vector space $\mathbf{C}^{N-d}$ of all the equations of hyperplanes tangent to X at x, in the sense that they contain the tangent space $T_{X^0,x}$. If x is a singular point, the fiber consists of all equations of limits of hyperplanes tangent at nonsingular points of X tending to x.

Moreover, we can characterize those subvarieties of the cotangent space which are the conormal spaces of their images in M.

Proposition 1.4.7 (See [49, Chap. II, §10]) *Let M be a nonsingular analytic variety of dimension N and let L be a closed conical irreducible analytic subvariety of T^*M, also of dimension N. The following conditions are equivalent:*

1) *The variety L is the conormal space of its image in M.*
2) *The Liouville 1-form α vanishes on all tangent vectors to L at every nonsingular point of L.*
3) *The symplectic 2-form $\omega = -\mathrm{d}\alpha$ vanishes on every pair of tangent vectors to L at every nonsingular point of L.*

Since conormal varieties are conical we may as well projectivize with respect to vertical homotheties of T^*M and work in $\mathbf{P}T^*M$. This means that we consider hyperplanes and identify all linear equations defining the same hyperplane. In $\mathbf{P}T^*M$ it still makes sense to be Lagrangian since α is homogeneous by definition.[4]

Going back to our original problem we have $X \subset U$ where U is open in $\mathbf{C}^N$, so $T^*U = U \times \check{\mathbf{C}}^N$ and $\mathbf{P}T^*U = U \times \check{\mathbf{P}}^{N-1}$. So we have the **(projective) conormal space** $\kappa_X : C(X) \to X$ with $C(X) \subset X \times \check{\mathbf{P}}^{N-1}$, where $C(X)$ denotes the projectivization of the conormal space $T^*_X M$. Note that we have not changed the name of the map κ_X after projectivizing since there is no ambiguity, and that the dimension of $C(X)$ is $N-1$, which shows immediately that it depends on the embedding of X in an affine space.

When there is no ambiguity we shall often omit the subscript in κ_X. We have the following result showing that this projectivized conormal is the same as that of Sect. 1.3 :

Proposition 1.4.8 *Given a reduced closed complex analytic subspace X of an open set $U \subset \mathbf{C}^N$, the (projective) conormal space $C(X)$ is a closed, reduced, complex analytic subspace of $X \times \check{\mathbf{P}}^{N-1}$ of dimension $N-1$. For any $x \in X$ the fiber $|\kappa_X^{-1}(x)|$ is the set of limit positions at x of tangent hyperplanes at points of X^0. Its dimension is at most $N-2$.*

Proof These are classical facts. See [8, Chap. III] or [60, Chap. II, §4, Proposition 4.1, p. 379]. □

1.4.2 Conormal Spaces and Projective Duality

Let us assume for a moment that $V \subset \mathbf{P}^{N-1}$ is a projective algebraic variety. In the spirit of last section, let us take $M = \mathbf{P}^{N-1}$ with homogeneous coordinates

[4] In symplectic geometry it is called **Legendrian** with respect to the natural contact structure on $\mathbf{P}T^*M$.

$(z_1 : \ldots : z_N)$, and consider the dual projective space $\check{\mathbf{P}}^{N-1}$ with coordinates $(\xi_1 : \ldots : \xi_N)$; its points are the hyperplanes of $\mathbf{P}^{N-1}$ with equations $\sum_{i=1}^{N} z_i \xi_i = 0$.

Definition 1.4.9 Define the **incidence variety** $I \subset \mathbf{P}^{N-1} \times \check{\mathbf{P}}^{N-1}$ as the set of points satisfying:

$$\sum_{i=1}^{N} z_i \xi_i = 0,$$

where $(z_1 : \ldots : z_N; \xi_1 : \ldots : \xi_N) \in \mathbf{P}^{N-1} \times \check{\mathbf{P}}^{N-1}$

Lemma 1.4.10 (Kleiman; See [28, §4]) *The projectivized cotangent bundle of* $\mathbf{P}^{N-1}$ *is naturally isomorphic to the incidence variety* $I \subset \mathbf{P}^{N-1} \times \check{\mathbf{P}}^{N-1}$.

Proof Let us first take a look at the cotangent bundle of $\mathbf{P}^{N-1}$:

$$\pi : T^*\mathbf{P}^{N-1} \longrightarrow \mathbf{P}^{N-1}.$$

Remember that the fiber $\pi^{-1}(x)$ over a point x in $\mathbf{P}^{N-1}$ is by definition isomorphic to $\check{\mathbf{C}}^{N-1}$, the vector space of linear forms on $\mathbf{C}^{N-1}$. Recall that projectivizing the cotangent bundle means projectivizing the fibers, and so we get a map:

$$\Pi : \mathbf{P}T^*\mathbf{P}^{N-1} \longrightarrow \mathbf{P}^{N-1}$$

where the fiber is isomorphic to $\check{\mathbf{P}}^{N-2}$. So we can see a point of $\mathbf{P}T^*\mathbf{P}^{N-1}$ as a pair $(z, \xi) \in \mathbf{P}^{N-1} \times \check{\mathbf{P}}^{N-2}$. On the other hand, if we fix a point $z \in \mathbf{P}^{N-1}$, the equation defining the incidence variety I tells us that the set of points $(z, \xi) \in I$ is the set of hyperplanes of $\mathbf{P}^{N-1}$ that go through the point z, which we know is isomorphic to $\check{\mathbf{P}}^{N-2}$.

Now to explicitly define the map, take a chart $\mathbf{C}^{N-1} \times \left\{\check{\mathbf{C}}^{N-1} \setminus \{0\}\right\}$ of the manifold $T^*\mathbf{P}^{N-1} \setminus \{\text{zero section}\}$, where the $\mathbf{C}^{N-1}$ corresponds to a usual chart of $\mathbf{P}^{N-1}$ and $\check{\mathbf{C}}^{N-1}$ to its associated cotangent chart. Define the map:

$$\phi_i : \mathbf{C}^{N-1} \times \left\{\check{\mathbf{C}}^{N-1} \setminus \{0\}\right\} \longrightarrow \mathbf{P}^{N-2} \times \check{\mathbf{P}}^{N-2}$$

$$(z_1, \ldots, z_{N-1}; \xi_1, \ldots, \xi_{N-1}) \longmapsto \left(\varphi_i(z), (\xi_1 : \ldots : \xi_{i-1} : -\sum_{j=1}^{N-1 *_i} z_j \xi_j : \xi_{i+1} : \ldots : \xi_{N-1})\right)$$

where $\varphi_i(z) = (z_1 : \ldots : z_{i-1} : 1 : z_{i+1} : \ldots : z_{N-1})$ and the star means that the index i is excluded from the sum.

An easy calculation shows that ϕ_i is injective, has its image in the incidence variety I and is well defined on the projectivization $\mathbf{C}^{N-1} \times \check{\mathbf{P}}^{N-2}$. It is also clear, that varying i from 1 to $N-1$ we can reach any point in I. Thus, all we need to check now is that the ϕ_j's paste together to define a map. For this, the important thing is to remember that if φ_i and φ_j are charts of a manifold, and $h := \varphi_j^{-1}\varphi_i = (h_1, \ldots, h_{N-1})$ then the change of coordinates in the associated cotangent charts $\tilde{\varphi_i}$ and $\tilde{\varphi_j}$ is given by:

$$\begin{array}{ccc} & T^*M & \\ {\scriptstyle \tilde{\varphi_i}} \nearrow & & \searrow {\scriptstyle \tilde{\varphi_j}^{-1}} \\ \mathbf{C}^{N-1} \times \check{\mathbf{C}}^{N-1} & \xrightarrow{\quad h \quad} & \mathbf{C}^{N-1} \times \check{\mathbf{C}}^{N-1} \end{array}$$

$$(z_1, \ldots, z_{N-1}; \xi_1, \ldots, \xi_{N-1}) \longmapsto (h(z); (Dh^{-1}|_z)^T(\xi))$$

This end the proof. □

By Lemma 1.4.10 the incidence variety I inherits the Liouville 1-form α which is $\sum \xi_i dz_i$ in local coordinates) from its isomorphism with $\mathbf{P}T^*\mathbf{P}^{N-1}$. Exchanging $\mathbf{P}^{N-1}$ and $\check{\mathbf{P}}^{N-1}$, I is also isomorphic to $\mathbf{P}T^*\check{\mathbf{P}}^{N-1}$ so it also inherits the 1-form $\check{\alpha} := \sum z_i d\xi_i$ locally).

Lemma 1.4.11 (Kleiman; See [29, §4] and [30]) *Let I be the incidence variety as above. Then $\alpha + \check{\alpha} = 0$ on I.*

Proof Note that if the polynomial $\sum_{i=1}^N z_i\xi_i$ defined a function on $\mathbf{P}^{N-1} \times \check{\mathbf{P}}^{N-1}$, we would obtain the result by differentiating it. The idea of the proof is basically the same, it involves identifying the polynomial $\sum_{i=1}^N z_i\xi_i$ with a section of the line bundle $p^*O_{\mathbf{P}^{N-1}}(1) \otimes \check{p}^*O_{\check{\mathbf{P}}^{N-1}}(1)$ over I, where p and $\check{p}$ are the natural projections of I to $\mathbf{P}^{N-1}$ and $\check{\mathbf{P}}^{N-1}$ respectively and $O_{\mathbf{P}^{N-1}}(1)$ denotes the canonical line bundle, introducing the appropriate flat connection on this bundle, and differentiating. □

In particular, this lemma tells us that if at some point $z \in I$ we have that $\alpha = 0$, then $\check{\alpha} = 0$ too. Thus, a closed conical irreducible analytic subvariety of $T^*\mathbf{P}^{N-1}$ as in Proposition 1.4.7 is the conormal space of its image in $\mathbf{P}^{N-1}$ if and only if it is the conormal space of its image in $\check{\mathbf{P}}^{N-1}$. So we have $\mathbf{P}T^*_V\mathbf{P}^{N-1} \subset I \subset \mathbf{P}^{N-1} \times \check{\mathbf{P}}^{N-1}$ and the restriction of the two canonical projections:

$$\begin{array}{ccc} & \mathbf{P}T^*_V\mathbf{P}^{N-1} \subset I & \\ {\scriptstyle p} \swarrow & & \searrow {\scriptstyle \check{p}} \\ V \subset \mathbf{P}^{N-1} & & \check{\mathbf{P}}^{N-1} \supset \check{V} \end{array}$$

Definition 1.4.12 The **dual variety** $\check{V}$ of $V \subset \mathbf{P}^{N-1}$ is the image by the map $\check{p}$ of $\mathbf{P}T^*_V\mathbf{P}^{N-1} \subset I$ in $\check{\mathbf{P}}^{N-1}$. So by construction $\check{V}$ is the closure in $\check{\mathbf{P}}^{N-1}$ of the set of hyperplanes tangent to V^0.

We immediately get by symmetry that $\check{\check{V}} = V$. What is more, we see that establishing a projective duality is equivalent to finding a Lagrangian subvariety in I; its images in $\mathbf{P}^{N-1}$ and $\check{\mathbf{P}}^{N-1}$ are necessarily dual.

Lemma 1.4.13 *Let us assume that* $(X, 0) \subset (\mathbf{C}^N, 0)$ *is the cone over a projective algebraic variety* $V \subset \mathbf{P}^{N-1}$. *Let* $x \in X^0$ *be a nonsingular point of* X. *Then the tangent space* $T_{X^0,x}$, *contains the line* $\ell = \overline{0x}$ *joining* x *to the origin. Moreover, the tangent map at* x *to the projection* $\pi : X \setminus \{0\} \to V$ *induces an isomorphism* $T_{X^0,x}/\ell \simeq T_{V,\pi(x)}$.

Proof This is due to Euler's identity for a homogeneous polynomial of degree m:

$$m.f = \sum_{i=1}^{N} z_i \frac{\partial f}{\partial z_i}$$

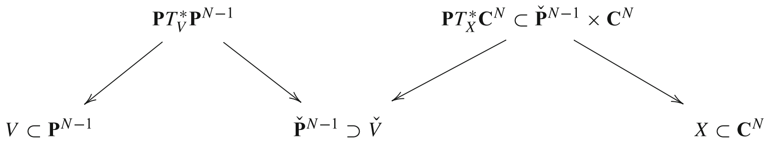
and the fact that if $\{f_1, \dots, f_r\}$ is a set of homogeneous polynomials defining X, then $T_{X^0,x}$ is the kernel of the matrix:

$$\begin{pmatrix} df_1 \\ \cdot \\ \cdot \\ df_r \end{pmatrix}$$

representing the differentials df_i in the basis $dz_1, \dots, dz_N$. □

It is also important to note that the tangent space to X^0 is constant along all non-singular points x of X in the same generating line since the partial derivatives are homogeneous as well, and contains the generating line. By Lemma 1.4.13, the quotient of this tangent space by the generating line is the tangent space to V at the point corresponding to the generating line.

So, $\mathbf{P}T^*_X\mathbf{C}^N$ has an image in $\check{\mathbf{P}}^{N-1}$ which is the projective dual of V.



The fiber over 0 of $\mathbf{P}T^*_X\mathbf{C}^N \to X$ is equal to $\check{V}$ as subvariety of $\check{\mathbf{P}}^{N-1}$: it is the set of limit positions at 0 of hyperplanes tangent to X^0.

For more information on projective duality, in addition to Kleiman's papers one can consult [61].

A *relative* version of the conormal space and of projective duality will play an important role in these notes. Useful references are [22, 29], [60, Chap. IV]. The relative conormal space is used in particular to define the *relative polar varieties*.

1.4.3 Polar Varieties and the Control of the Dimension of the Fibers of $\kappa_X : C(X) \to X$

The simplest measure of the complexity of the space of limits of tangent hyperplanes at a point $x \in X$ is the dimension of the fiber $\kappa_X^{-1}(x) \subset \check{\mathbf{P}}^{N-1}$. This dimension is the difference between $N-1$ and the maximum codimension of a linear subspace of $\check{\mathbf{P}}^{N-1}$ whose intersection with $\kappa^{-1}(x)$ is not empty. We are thus led to consider the subspaces $C(X) \cap (X \times L^{d-k})$ of $C(X)$, where $0 \leq k \leq d = \dim. X$ and L^{d-k} is a linear subspace of $\check{\mathbf{P}}^{N-1}$ of dimension $d-k$, dual to a vector subspace $D_{d-k+1} \subset \mathbf{C}^N$ of codimension $d-k+1$ in the sense that it is the space of directions of hyperplanes containing it. We remark that, with the notations introduced above, we have $C(X) \cap (X \times L^{d-k}) = \lambda^{-1}(L^{d-k})$.

The next proposition provides the relation between the geometry of $\kappa_X^{-1}(x) \subset \check{\mathbf{P}}^{N-1}$ as read by linear subspaces and geometrically defined subspaces of X, the *local polar varieties* of $X \subset \mathbf{C}^N$. These are defined as the closures in X of sets of critical points on X^0 of projections $X \to \mathbf{C}^{d-k+1}$ induced by general linear maps $\mathbf{C}^N \to \mathbf{C}^{d-k+1}$. They were originally defined in [34]. Recall the definition of the map λ in Definition 1.3.5.

Proposition 1.4.14 *For a sufficiently general D_{d-k+1}, the image $\kappa(\lambda^{-1}(L^{d-k}))$ is the closure in X of the set of points of X^0 which are critical for the projection $\pi|_{X^0} : X^0 \to \mathbf{C}^{d-k+1}$ induced by the projection $\mathbf{C}^N \to \mathbf{C}^{d-k+1}$ with kernel $D_{d-k+1} = (L^{d-k})\check{}$.*

Proof Note that $x \in X^0$ is critical for π if and only if the tangent map $d_x\pi : T_{X^0,x} \longrightarrow \mathbf{C}^{d-k+1}$ is not onto, which means dim. $\ker d_x\pi \geq k$ since $\dim T_{X^0,x} = d$, and $\ker d_x\pi = D_{d-k+1} \cap T_{X^0,x}$.

Note that the conormal space $C(X^0)$ of the nonsingular part of X is equal to $\kappa^{-1}(X^0)$ so by definition:

$$\lambda^{-1}(L^{d-k}) \cap C(X^0) = \{(x, H) \in C(X) | x \in X^0,\ H \in L^{d-k},\ T_{X^0,x} \subset H\}$$

equivalently:

$$\lambda^{-1}(L^{d-k}) \cap C(X^0) = \{(x, H), \in C(X) | x \in X^0,\ H \in (D_{d-k+1})\check{},\ H \in (T_{X^0,x})\check{}\ \}$$

thus $H \in (D_{d-k+1})\check{} \cap (T_{X^0,x})\check{}$, and from the equality $(D_{d-k+1})\check{} \cap (T_{X^0,x})\check{} = (D_{d-k+1} + T_{X^0,x})\check{}$ we deduce that the intersection is not empty if and only if $D_{d-k+1} + T_{X^0,x} \neq \mathbf{C}^N$, which implies that $\dim D_{d-k+1} \cap T_{X^0,x} \geq k$, and consequently $\kappa(H) = x$ is a critical point.

According to [60, Chap. IV, 1.3], there exists an open dense set U_k in the Grassmannian of $(N-d+k-1)$-planes of $\mathbf{C}^N$ such that if $D_{d-k+1} \in U_k$, the intersection $\lambda^{-1}(L^{d-k}) \cap C(X^0)$ is dense in $\lambda^{-1}(L^{d-k})$. So, for any $D_{d-k+1} \in U_k$, since κ is a proper map and thus closed, we have that $\kappa(\lambda^{-1}(L^{d-k})) =$

$\kappa\left(\overline{\lambda^{-1}(L^{d-k})\cap C(X^0)}\right) = \overline{\kappa(\lambda^{-1}(L^{d-k}))}$, which finishes the proof. See [60, Chap. IV, 4.1.1] for a complete proof of a more general statement. □

Remark 1.4.15 It is important to have in mind the following easily verifiable facts:

a) As we have seen before, the fiber $\kappa^{-1}(x)$ over a regular point $x \in X^0$ in the (projectivized) conormal space $C(X)$ is a $\mathbf{P}^{N-d-1}$, so by semicontinuity of fiber dimension we have that $\dim \kappa^{-1}(0) \geq N-d-1$.

b) For a general L^{d-k}, the intersection $C(X) \cap (X \times L^{d-k})$ is of pure dimension $N-1-N+d-k+1 = d-k$ if it is not empty.

   The proof of this is not immediate because we are working over an open neighborhood of a point $x \in X$, so we cannot assume that $C(X)$ is compact. However (see [60, Chap. IV]) we can take a Whitney stratification of $C(X)$ (these stratifications are explained below) such that the closed algebraic subset $\kappa^{-1}(0) \subset \check{\mathbf{P}}^{N-1}$, which is compact, is a union of strata. By general transversality theorems in algebraic geometry (see [28]) a sufficiently general L^{d-k} will be transversal to all the strata of $\kappa^{-1}(0)$ in $\check{\mathbf{P}}^{N-1}$ and then because of the Whitney conditions (see [63, section 4.9]) $\mathbf{C}^N \times L^{d-k}$ will be transversal in a neighborhood of $\kappa^{-1}(0)$ to all the strata of $C(X)$, which will imply in particular the statement on the dimension. Since κ is proper, the neighborhood of $\kappa^{-1}(0)$ can be taken to be the inverse image by κ of a neighborhood of 0 in X. The meaning of "general" in Proposition 1.4.14 is that of Kleiman's transversality theorem. Moreover, since $C(X)$ is a reduced equidimensional analytic space, for a general L^{d-k}, the intersection of $C(X)$ and $\mathbf{C}^N \times L^{d-k}$ in $\mathbf{C}^N \times \check{\mathbf{P}}^{N-1}$ is generically reduced and since according to our general rule we remove embedded components when intersecting with linear spaces, $\lambda^{-1}(L^{d-k})$ is a reduced equidimensional complex analytic space.

   Note that the existence of Whitney stratifications does not depend on the existence of polar varieties; see Theorem 1.8.3 below.

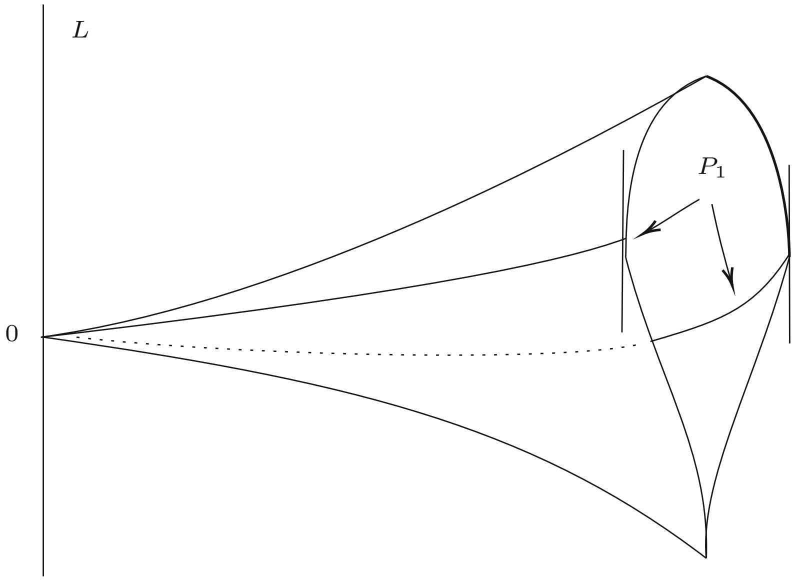
c) The fact that $\lambda^{-1}(L^{d-k}) \cap C(X^0)$ is dense in $\lambda^{-1}(L^{d-k})$ means that if a limit of tangent hyperplanes at points of X^0 contains D_{d-k+1}, it is a limit of tangent hyperplanes which also contain D_{d-k+1}. This equality holds because transversal intersections preserve the frontier condition; see [63, Theorem 4.2.15] or [7, Lemme 2.2.2], [60, Remarque 4.2.3].

d) Note that for a fixed L^{d-k}, the germ $(P_k(X; L^{d-k}), 0)$ is empty if and only if the intersection $\kappa^{-1}(0) \cap \lambda^{-1}(L^{d-k})$ is empty. From a) we know that $\dim \kappa^{-1}(0) = N-d-1+r$ with $r \geq 0$. Thus, by the same argument as in b), this implies that the polar variety $(P_k(X; L^{d-k}), 0)$ is not empty if and only if $\dim(\kappa^{-1}(0) \cap \lambda^{-1}(L^{d-k})) \geq 0$ and if and only if $r \geq k$.

Definition 1.4.16 With the notation and hypotheses of Proposition 1.4.14, for $0 \leq k \leq d-1$ the **local polar variety** is defined as:

$$P_k(X; L^{d-k}) = \kappa(\lambda^{-1}(L^{d-k})).$$

A priori, we have just defined local polar varieties set-theoretically, but since $\lambda^{-1}(L^{d-k})$ is empty or reduced and κ is a projective fibration over the smooth part of X we have the following result, for which a proof can be found in [60, Chap. IV, 1.3.2].



Proposition 1.4.17 *For a general linear subspace $L^{d-k} \subset \check{\mathbf{P}}^{N-1}$ and $0 \leq k \leq d$ the local polar variety $P_k(X; L^{d-k}) \subset X$ is a reduced closed analytic subspace of X, either of pure codimension k in X or empty.*

We have thus far defined a local polar variety that depends on both the choice of the embedding $(X, 0) \subset (\mathbf{C}^N, 0)$ and the choice of the general linear space D_{d-k+1}. However, an important information we will extract from these polar varieties is their multiplicities at 0, and these numbers are analytic invariants provided the linear spaces used to define them are general enough.

Proposition 1.4.18 (Teissier, See [60, Chap. IV, §3]) *Let $(X, 0) \subset (\mathbf{C}^N, 0)$ be as before, then for every $0 \leq k \leq d-1$ and a sufficiently general linear space $D_{d-k+1} \subset \mathbf{C}^N$ the multiplicity of the polar variety $P_k(X; L^{d-k})$ at 0 depends only on the analytic type of $(X, 0)$.*

Exercise 1.4.19 Let $0 \in Y \subset X \subset \mathbf{C}^N$ where Y is one dimensional and non-singular and X is d-dimensional. Show that the following conditions are equivalent:

(i) The germ of polar curve $(P_{d-1}(X; L^{d-k}), 0)$ is empty;
(ii) $\dim. \kappa^{-1}(0) < N-2$.

and imply:

A Zariski open and dense subset of the $\check{\mathbf{P}}^{N-2} \subset \check{\mathbf{P}}^{N-1}$ consisting of hyperplanes containing $T_{Y,0}$ is not contained in $\kappa^{-1}(0)$: a general hyperplane containing $T_{Y,0}$ is not a limit of tangent hyperplanes to X^0. Compare with Example 1.6.4 below.

1.4.4 Limits of Tangent Spaces and Bertini's Theorem

A very special but historically important case of Bertini's theorem states that given $(X, 0) \subset (\mathbf{C}^N, 0)$, for a sufficiently general hyperplane H through the origin, the singular locus of $H \cap X$ near 0 is set-theoretically the intersection with H of the singular locus of X. This means that near 0, the hyperplane H is transversal to the tangent spaces to X^0 at points of $X^0 \cap H$. However, a stronger result is true: *the hyperplane H is transversal to the limits as $x \to 0$ of tangent spaces to X at points $x \in X^0 \cap H$*. This is not a consequence of the usual transversality theorems since the limits move with H. It is a consequence of the fact that in the conormal map $\kappa\colon C(X) \to X$, since $C(X)$ is of dimension $N-1$, the dimension of $\kappa^{-1}(0)$ is at most $N-2$ so that a general hyperplane is not a limit of tangent hyperplanes to X and so cannot contain a limit of tangent spaces, according to Corollary 1.3.2.

But much more is true: Suppose that $(X, 0)$ is a germ of hypersurface defined by a holomorphic map $f\colon (\mathbf{C}^N, 0) \to (\mathbf{C}, 0)$, and we consider the tangent hyperplanes to the fibers $f^{-1}(t)$. Assume that f has an isolated critical point at the origin. Then, by Example 1.2.2, the set theoretical fiber of the relative Semple-Nash modification over 0 is the exceptional divisor: it is $\check{\mathbf{P}}^{N-1}$ which means that every hyperplane through the origin is a limit of tangent hyperplanes to the fibers of f. However it is true, without assuming that f has an isolated critical point at the origin, that a general hyperplane H *is transversal to the limits as $x \to 0$ of tangent hyperplanes to the fibers $f^{-1}(f(x))$ at points $x \in H$*.

It is a consequence of the idealistic Bertini theorem of [57, Proposition 2.7] (for hypersurfaces) and [60, section 2.2] for the general case. The statement implies that for a general hyperplane H the restriction to H of the jacobian ideal of X and the jacobial ideal of $X \cap H$ have the same integral closure as ideals in $\mathcal{O}_{X\cap H,0}$ while Bertini's theorem states that the radicals of their restrictions to X are equal. This equality of integral closures means that the restrictions to H of some jacobian determinants of the equations of X tend to 0 at least as fast as some others. The full form is more precise. It can be used to give an alternative proof of the existence of Whitney stratifications which we shall see below, and also to prove the transversality of local polar varieties to the kernel of the projection which defines them, even in the relative case. It even has applications to the theory of the maximum likelihood degree in mathematical statistics; see [43, Corollary 2.6].

1.5 Limits of Secants: The Blowing-Up

In this section we present the blowing up of a coherent sheaf of ideals in a way which is adapted to the construction of the normal/conormal diagram which is used in the study of Whitney conditions.

Let $\mathcal{I}$ be a coherent sheaf of ideals on X defining a closed analytic subspace $Y \subset X$. Let $U \subset X$ be an open set on which we have a presentation

$$\mathcal{O}_U^q \to \mathcal{O}_U^p \to \mathcal{I}|U \to 0.$$

We have thus a set of global generators $f_1, \ldots, f_p$ for $\mathcal{I}|U$. Consider the map $U \setminus Y \to \mathbf{P}^{p-1}$ defined by $x \mapsto (f_1(x) : \ldots : f_p(x))$, and its graph $E_Y(U \setminus Y) \subset (U \setminus Y) \times \mathbf{P}^{p-1}$. The closure $E_Y U$ of this graph in $U \times \mathbf{P}^{p-1}$ is a closed analytic subspace which, up to a unique isomorphism, depends only on $\mathcal{I}|U$.

To see this, consider the graded $\mathcal{O}_X$ algebra

$$P(\mathcal{I}) = \bigoplus_{n \in \mathbf{N}} \mathcal{I}^n,$$

which is locally finitely generated in degree one.

Because $\mathcal{I}$ is locally finitely presented, this algebra has also locally a finite presentation by an exact sequence of finitely generated graded $\mathcal{O}_U$ algebras and modules (see [3, Chap. 1, 1.3]).

$$0 \to \mathcal{K}_U \to \mathcal{O}_U[T_1, \ldots, T_p] \to P(\mathcal{I})|U \to 0,$$

where each T_j is mapped to $f_j \in \mathcal{I}|U$. The ideal $\mathcal{K}_U$ is generated by finitely many homogeneous polynomials in $T_1, \ldots, T_p$ which by definition generate all the algebraic homogeneous relations between $f_1, \ldots f_p$. The vanishing of these polynomials defines a closed subspace of $U \times \mathbf{P}^{p-1}$ which, by construction, is the closure $E_Y U$ of the graph we have just seen. One verifies that this subspace is independent of the choice of the generators $f_1, \ldots, f_p$ and so by uniqueness the local constructions glue up into a space E_Y over X, say

$$e_Y \colon E_Y X \to X$$

which is called the *blowing-up* of $\mathcal{I}$ (or Y) in X.

The construction we have just described is, when we give the subspace of $U \times \mathbf{P}^{p-1}$ its natural structure as a complex analytic space, the Projan $P(\mathcal{I})$ of the locally finitely presented graded $\mathcal{O}_X$-algebra $P(\mathcal{I})$.

The inverse image $e_Y^{-1}(Y)$ is the projan of the graded $\mathcal{O}_Y$-algebra

$$P(\mathcal{I}) \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I} = \bigoplus_{n \in \mathbf{N}} \mathcal{I}^n/\mathcal{I}^{n+1} = \mathcal{O}_Y \oplus \mathcal{I}/\mathcal{I}^2 \oplus \mathcal{I}^2/\mathcal{I}^3 \oplus \cdots$$

Besides the fact that the blowing-up is locally the closure of a graph, its essential feature is that $e_Y^{-1}(Y) \subset E_Y X$ is locally on $E_Y X$ defined by one equation which is not a zero divisor and is one of the generators of the pull-back $\mathscr{I}\mathscr{O}_{E_Y X}$ of the ideal $\mathscr{I}$. It is the *exceptional divisor* of the blowing-up. Indeed, in each affine chart V_j defined by $T_j \neq 0$ of $\mathbf{P}^{p-1}$ the T_i/T_j are coordinates, which implies that on the intersection of $E_Y X$ with $X \times V_j$ the functions f_i/f_j are regular and thus the ideal $(f_1 \circ e_Y, \ldots, f_p \circ e_Y)$, which is the restriction of $\mathscr{I}\mathscr{O}_{E_Y X}$ to the intersection of $E_Y X$ with $X \times V_j$, is principal and generated by $f_j \circ e_Y$.

The following universal property of blowing-up, which we state here in the complex analytic framework, is due to Hironaka (see [3, Lemma 1.3.1]):

Theorem 1.5.1 *A complex-analytic map $\pi : T \to X$ such that $\pi^{-1}(Y)$ is locally on T defined by a single equation which is not a zero divisor in the local rings of T factors uniquely through e_Y. This property characterizes the map e_Y.*

In what follows we shall consider the case where $Y \subset X \subset \mathbf{C}^N$, where $\mathbf{C}^N$ is endowed with coordinates $z_1, \ldots, z_N$ and Y is non-singular of dimension t. We may assume that the coordinates are adapted to Y in the sense that it is defined by the vanishing of coordinates $z_{t+1}, \ldots, z_N$ on $\mathbf{C}^N$. The map $X \setminus Y \to \mathbf{P}^{N-t-1}$ defined by $(z_1, \ldots, z_N) \mapsto (z_{t+1} : \ldots : z_N) \in \mathbf{P}^{N-t-1}$ can be deemed to associate to a point of $X \setminus Y$ the direction of the secant line joining this point to the point in Y with coordinates $z_1, \ldots, z_t$. The closure in $X \times \mathbf{P}^{N-t-1}$ of the graph of this map is the blowing up in X of the subspace Y. Although the secant lines clearly depend on the choice of coordinates, the blowing up does not.

A point of $E_Y X \subset X \times \mathbf{P}^{N-t-1}$ is therefore a pair $(x, [\ell])$ where if $x \in X \setminus Y$, $[\ell]$ is the direction of the secant line joining x to its linear projection on Y according to the coordinate system, and if $x \in Y$, the direction $[\ell]$ is a limit direction of such secant lines along a sequence of points of $X \setminus Y$ tending to x.

Denoting by $\mathscr{I}_Y$ the coherent sheaf of ideals defining $Y \subset X$, and by $\mathrm{gr}_{\mathscr{I}_Y}\mathscr{O}_X$ the graded $\mathscr{O}_Y$-algebra

$$\mathrm{gr}_{\mathscr{I}_Y}\mathscr{O}_X = \bigoplus_{n\in\mathbf{N}} \mathscr{I}_Y^n/\mathscr{I}_Y^{n+1},$$

the space $\mathrm{Specan}(\mathrm{gr}_{\mathscr{I}_Y}\mathscr{O}_X)$ with its natural mapping $\mathrm{Specan}(\mathrm{gr}_{\mathscr{I}_Y}\mathscr{O}_X) \to Y$ corresponding to the inclusion $\mathscr{O}_Y \subset \mathrm{gr}_{\mathscr{I}_Y}\mathscr{O}_X$ is called the *normal cone* of Y in X and usually denoted by $C_{X,Y} \to Y$. In the case where Y is a point, say $x \in X$, it is for historical reasons the *tangent cone* of X at x. If X is non-singular these notions coincide with the normal bundle of Y in X and the tangent space of X at y.

In the case where Y is a point $x \in X$, $\mathscr{I}_{\{x\}}$ corresponds to the maximal ideal $m_x \subset \mathscr{O}_{X,x}$ which is generated by the local coordinates $z_1, \ldots, z_N$. The multiplicity of the tangent cone at its vertex is the *multiplicity* of X at the point x. It is also the degree of the projective variety $e_x^{-1}(x) \subset \mathbf{P}^{N-1}$ associated to the tangent cone.

Remark 1.5.2 One may ask for the interpretation of the fiber of $e_Y : E_Y \to X$ at a point $y \in Y$. It is called the *analytic spread* of the ideal $\mathcal{I}$ at this point and plays an important role in detecting equimultiplicity of X along Y.

1.6 The Normal/Conormal Diagram

In this section we construct a space which, given a non-singular subspace $Y \subset X \subset \mathbf{C}^N$ and a local retraction $r : \mathbf{C}^N \to Y$ does for limit positions of pairs (ℓ, T) at a point $x \in X^0 \setminus Y$ of the direction of secant line $\overline{xr(x)}$ and a direction of tangent hyperplane $H \supset T_{X^0,x}$ what the conormal space and the blowing up of Y in X do separately.

With the help of the normal/conormal diagram and the polar varieties we will be able to obtain information on the limits of tangent spaces to X at 0, assuming that $(X, 0)$ is reduced and purely d-dimensional. This method is based on Whitney's lemma and the two results which follow it:

Lemma 1.6.1 (Whitney's Lemma for X^0) *Let $(X, 0)$ be a pure-dimensional germ of analytic subspace of $\mathbf{C}^N$, choose a representative X and let $\{x_n\} \subset X^0$ be a sequence of points tending to 0, such that*

$$\lim_{n\to\infty} [0x_n] = l \ \ \textit{and} \ \ \lim_{n\to\infty} T_{x_n} X = T.$$

Then $l \subset T$.

A stronger form of this lemma originally appeared in [65, Theorem 22.1], and you can also find a proof due to Hironaka in [32] and yet another below in assertion a) of Theorem 1.6.2.

Given $X \subset \mathbf{C}^N$ as above, consider the normal/conormal diagram

$$\begin{array}{ccccccc} X\times \mathbf{P}^{N-1}\times \check{\mathbf{P}}^{N-1} & \supset & E_0C(X) & \xrightarrow{\hat{e}_0} & C(X) & \hookrightarrow & X\times\check{\mathbf{P}}^{N-1} \\ \downarrow K & & \downarrow \kappa' \quad \searrow \xi & & \downarrow \kappa \quad \searrow \lambda & & \downarrow pr_2 \\ & & & & & & \check{\mathbf{P}}^{N-1} \\ X\times\mathbf{P}^{N-1} & \supset & E_0X & \xrightarrow{e_0} & X & & \end{array}$$

where e_0 is the blowing up of the point $0 \in X$, $\hat{e}_0$ is the blowing up of the subspace $\kappa^{-1}(0)$ and κ' is the map coming from the universal property of blowing ups applied to the map $\xi = \kappa \circ \hat{e}_0$.

Theorem 1.6.2 (Lê-Teissier, See [36, §2]) *In the normal/conormal diagram, consider the irreducible components D_j of the exceptional divisor $D = |\xi^{-1}(0)|$. Then we have:*

I) *The following hold*

(i) *Each $D_j \subset \mathbf{P}^{N-1} \times \check{\mathbf{P}}^{N-1}$ is contained in the incidence variety $I \subset \mathbf{P}^{N-1} \times \check{\mathbf{P}}^{N-1}$.*

(ii) *Each D_j is Lagrangian in I and therefore establishes a projective duality of its images:*

$$\begin{array}{ccc} D_j & \longrightarrow & W_j \subset \check{\mathbf{P}}^{N-1} \\ \downarrow & & \\ V_j \subset \mathbf{P}^{N-1} & & \end{array}$$

Note that, from commutativity of the diagram we obtain $\kappa^{-1}(0) = \bigcup_j W_j$, and $e_0^{-1}(0) = \bigcup_\alpha V_j$. It is important to notice that these expressions are not necessarily the irreducible decompositions of $\kappa^{-1}(0)$ and $e_0^{-1}(0)$ respectively, since there may be repetitions; it is the case for the surface of Example 1.6.4 below, where the dual of the tangent cone, a point in $\check{\mathbf{P}}^2$, is contained in the projective line dual to the exceptional tangent. However, it is true that they contain the respective irreducible decompositions.

In particular, note that if dim $V_{j_0} = d-1$, then the cone $O(V_{j_0}) \subset \mathbf{C}^N$ is an irreducible component of the tangent cone $C_{X,0}$ and its projective dual $W_{j_0} = \check{V}_{j_0}$ is contained in $\kappa^{-1}(0)$. That is, any tangent hyperplane to the tangent cone is a limit of tangent hyperplanes to X at 0. The converse is very far from true and we shall see more about this below.

II) *For any integer k, $0 \le k \le d-1$, and sufficiently general $L^{d-k} \subset \check{\mathbf{P}}^{N-1}$ the tangent cone $C_{P_k(X,L),0}$ of a non empty polar variety $P_k(X, L)$ at the origin consists of:*

- *The union of the cones $O(V_j)$ which are of dimension $d-k$ ($= \dim P_k(X, L)$).*
- *The polar varieties $P_\ell(O(V_j), L)$ of dimension $d-k$, for the projection p associated to L, of the cones $O(V_j)$, for j such that dim $O(V_j) = d-k+\ell$ for some $1 \le \ell \le k$.*

Note that $P_k(X, L)$ is not unique, since it varies with L, but we are saying that its tangent cone may have parts which do not vary with L. The V_α's are fixed, so the first part is the fixed part of $C_{P_k(X,L),0}$ because it is independent of L, the second part is the mobile part, since we are talking of polar varieties of certain cones, which by definition move with L (see [10]).

Proof The proof of **I)**, which can be found in [36, §2], is essentially a strengthening of Whitney's lemma (Lemma 1.6.1) using the normal/conormal diagram and the fact that the vanishing of a differential form (the symplectic form in our case) is a closed condition.

The proof of **II)**, a special case of [36, Proposition 2.2.1], is somewhat easier to explain geometrically:

Using our normal/conormal diagram, remember that we can obtain the blowing up $E_0(P_k(X,L))$ of the polar variety $P_k(X,L)$ by taking its strict transform under the morphism e_0, and as such we will get the projectivized tangent cone $\mathbf{P}C_{P_k(X,L),0}$ as the fiber over the origin.

The first step is to prove that set-theoretically the projectivized tangent cone can also be expressed as

$$|\mathbf{P}C_{P_k(X,L),0}| = \bigcup_j \kappa'(\hat{e}_0^{-1}(\lambda^{-1}(L)\cap W_j)) = \bigcup_j \kappa'(D_j \cap (\mathbf{P}^{N-1}\times L)).$$

Now recall that the intersection $P_k(X,L)\cap X^0$ is dense in $P_k(X,L)$, so for any point $(0,[l]) \in \mathbf{P}C_{P_k(X,L),0}$ there exists a sequence of points $\{x_n\}\subset X^0$ such that the directions of the secants $\overline{0x_n}$ converge to it. So, by definition of a polar variety, if $D_{d-k+1} = \check{L}$ and $T_n = T_{x_n}X^0$ then by Proposition 1.4.14 we know that $\dim T_n \cap D_{d-k+1} \geq k$ which is a closed condition. In particular if T is a limit of tangent spaces obtained from the sequence $\{T_n\}$, then $T\cap D_{d-k+1}\geq k$ also. But if this is the case, since the dimension of T is d, there exists a limit of tangent hyperplanes $H\in\kappa^{-1}(0)$ such that $T + D_{d-k+1}\subset H$ which is equivalent to $H\in\kappa^{-1}(0)\cap\lambda^{-1}(L)\neq\emptyset$. Therefore the point $(0,[l],H)$ is in $\bigcup_j \hat{e}_0^{-1}(\lambda^{-1}(L)\cap W_j)$, and so we have the inclusion:

$$|\mathbf{P}C_{P_k(X,L),0}| \subset \bigcup_j \kappa'(\hat{e}_0^{-1}(\lambda^{-1}(L)\cap W_j)).$$

For the other inclusion, recall that $\lambda^{-1}(L)\setminus\kappa^{-1}(0)$ is dense in $\lambda^{-1}(L)$ and so $\hat{e}_0^{-1}(\lambda^{-1}(L))$ is equal set theoretically to the closure in $E_0C(X)$ of $\hat{e}_0^{-1}(\lambda^{-1}(L)\setminus\kappa^{-1}(0))$. Then for any point $(0,[l],H)\in\hat{e}_0^{-1}(\lambda^{-1}(L)\cap\kappa^{-1}(0))$ there exists a sequence $\{(x_n,[x_n],H_n)\}$ in $\hat{e}_0^{-1}(\lambda^{-1}(L)\setminus\kappa^{-1}(0))$ converging to it. Now by commutativity of the diagram, we get that the sequence $\{(x_n,H_n)\}\subset\lambda^{-1}(L)$ and as such the sequence of points $\{x_n\}$ lies in the polar variety $P_k(X,L)$. This implies in particular, that the sequence $\{(x_n,[0x_n])\}$ is contained in $e_0^{-1}(P_k(X,L)\setminus\{0\})$ and the point $(0,[l])$ is in the projectivized tangent cone $|\mathbf{P}C_{P_k(X,L),0}|$.

The second and final step of the proof is to use that from *a)* and *b)* it follows that each $D_j\subset I\subset\mathbf{P}^{N-1}\times\check{\mathbf{P}}^{N-1}$ is the conormal space of V_j in $\mathbf{P}^{N-1}$, with the restriction of κ' to D_j being its conormal morphism.

Note that D_j is of dimension $N-2$, and since all the maps involved are just projections, we can take the cones over the V_j's and proceed as in Sect. 1.4.2. In this setting we get that since L is sufficiently general, by Proposition 1.4.14 and Definition 1.4.16:

- For the D_j's corresponding to cones $O(V_j)$ of dimension $d-k$ (= dim $P_k(X, L)$), the intersection $D_j \cap (\mathbf{P}^{N-1} \times L)$ is not empty and as such its image is a polar variety $P_0(O(V_j), L) = O(V_j)$ which is independent of L.
- For the D_j's corresponding to cones $O(V_j)$ of dimension $d - k + \ell$ for some $1 \leq \ell \leq k$, the intersection $D_j \cap (\mathbf{P}^{N-1} \times L)$ is either empty or of dimension $d - k$ and as such its image is a polar variety of dimension $d - k$, which is $P_\ell(O(V_j), L)$ and varies with L if it is not empty.

You can find a detailed proof of these results in [36, §2], [60, Chap. IV]. □

So for any reduced and purely d-dimensional complex analytic germ $(X, 0)$, we have a method to "compute", or rather describe, the set of limiting positions of tangent hyperplanes. Between parentheses are the types of computations involved:

1. For all integers k, $0 \leq k \leq d - 1$, compute the "general" polar varieties $P_k(X, L)$, leaving in the computation the coefficients of the equations of L as indeterminates. (Partial derivatives, Jacobian minors and residual ideals with respect to the Jacobian ideal);
2. Compute the tangent cones $C_{P_k(X,L),0}$ (computation of a standard basis with parameters);
3. Sort out those irreducible components of the tangent cone of each $P_k(X, L)$ which are independent of L (decomposition into irreducible components with parameters);
4. Take the projective duals of the corresponding projective varieties (Elimination).

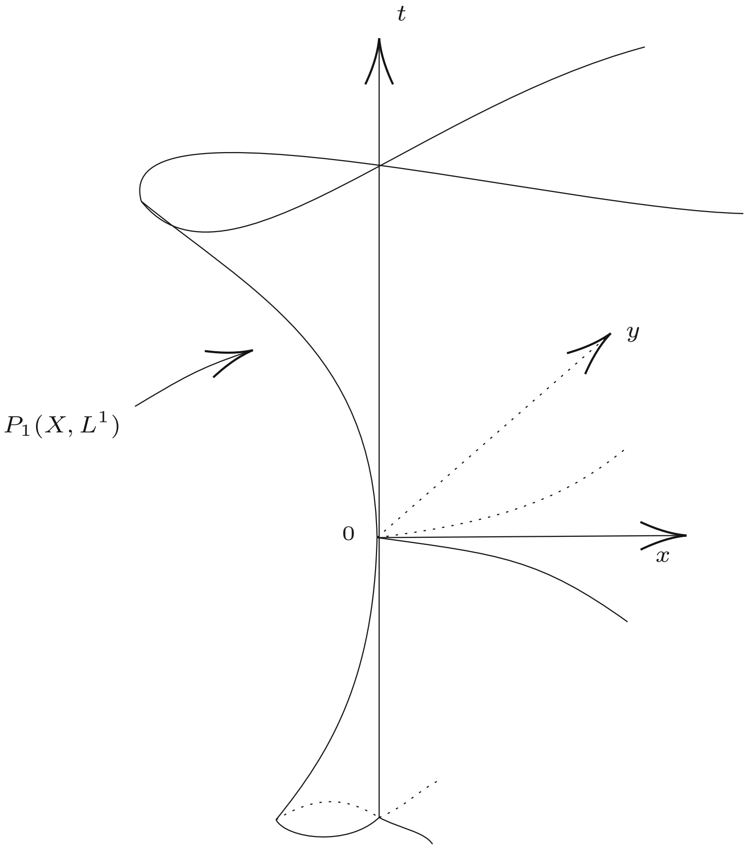
We have noticed, that among the V_j's, there are those which are irreducible components of Proj $C_{X,0}$ and those that are of lower dimension.

Definition 1.6.3 The cones $O(V_j)$'s such that

$$\dim.\, V_j < \dim.\, \mathrm{Proj}\; C_{X,0}$$

are called exceptional cones.

Example 1.6.4 Let $X := y^2 - x^3 - t^2x^2 = 0 \subset \mathbf{C}^3$, so $\dim X = 2$, and thus $k = 0, 1$. An easy calculation shows that the singular locus of X is the t-axis, and $m_0(X) = 2$.



Note that for $k = 0$, D_3 is just the origin in $\mathbf{C}^3$, so the projection

$$\pi : X^0 \to \mathbf{C}^3$$

with kernel D_3 is the restriction to X^0 of the identity map, which is of rank 2 and we get that the whole X^0 is the critical set of such a map. Thus,

$$P_0(X, L^2) = X.$$

For $k = 1$, D_2 is of dimension 1. So let us take for instance $D_2 = y$-axis, so we get the projection

$$\pi : X^0 \to \mathbf{C}^2 \quad (x, y, t) \mapsto (x, t),$$

and we obtain that the set of critical points of the projection is given by

$$P_1(X, L^1) = \begin{cases} x = -t^2 \\ y = 0 \end{cases}$$

If we had taken for D_2 the line $t = 0$, $\alpha x + \beta y = 0$, we would have found that the polar curve is a nonsingular component of the intersection of our surface with the surface $2\alpha y = \beta x(3x + 2t^2)$. For $\alpha \neq 0$ all these polar curves are tangent to the t-axis. As we shall see in the next subsection, this means that the t-axis is an "exceptional cone" in the tangent cone $y^2 = 0$ of our surface at the origin, and therefore all the 2-planes containing it are limits at the origin of tangent planes at nonsingular points of our surface.

1.6.1 Limits of Tangent Spaces of Quasi-Ordinary Hypersurfaces

Let $(X, 0)$ be a irreducible germ of complex analytic space of dimension d for which there exists a germ of finite morphism $\pi : (X, 0) \to (\mathbf{C}^d, 0)$ whose ramification set (that we often call the discriminant) is a hypersurface with only normal crossings singularities in $\mathbf{C}^d$. This type of singularity is called a quasi-ordinary singularity. In [4], C. Ban considered such singularities in the case where they are irreducible hypersurfaces. He gave a complete description of the limits of tangent spaces of X at 0 as follows:

A germ of an irreducible quasi-ordinary hypersurface in $\mathbf{C}^N$ can be parametrized in a Puiseux-like manner (see [14, 1.2.3]). If the quasi-ordinary projection is $(z_1, \ldots, z_{N-1}, z) \mapsto (z_1, \ldots, z_{N-1})$, then the hypersurface can be defined by the vanishing of a Weierstrass polynomial of degree n in z with coefficients in the maximal ideal of $\mathbf{C}\{z_1, \ldots, z_{N-1}\}$. The quasi-ordinary condition means that the discriminant of the Weierstrass polynomial with respect to z is, in suitable coordinates, the product of a unit and a monomial in $\mathbf{C}\{z_1, \ldots, z_{N-1}\}$. By the Abhyankar-Jung theorem (see [48]), it is parametrized by a convergent power series with rational exponents:

$$z = \zeta(z_1, \ldots, z_{N-1}) = \sum c_a z_1^{\frac{a_1}{n}} \ldots z_{N-1}^{\frac{a_{N-1}}{n}}, \text{ with } a = (a_1, \ldots, a_{N-1}).$$

Just as in the plane branch case, some of the rational exponents appearing in the series ζ, which are totally ordered for the product order, are closely related to the local topology of the hypersurface, and they are also called the characteristic exponents.

Let $z_1^{\frac{a_1}{n}} \ldots z_e^{\frac{a_e}{n}}$ be the monomial corresponding to the smallest characteristic exponent. Then Ban describes the collection of the irreducible components of the tangent cone and all the exceptional cones defined in Definition 1.6.3, which we have in [36] called the auréole of the singularity, as follows:

Theorem 1.6.5 (Ban) *The auréole of the quasi-ordinary irreducible hypersurface singularity parametrized as above consists of :*

(i) *If $n > a_1 + \cdots + a_e$, the following cones of C^N:*

$$C_I = \{(z_1, \ldots, z_{N-1}, z) | z_i = 0 \text{ for } i \in I \subset \{1, \ldots, e\} \text{ and } I \neq \emptyset\}$$

(ii) *If $n < a_1 + \cdots + a_e$, the following cones of C^N:*

$$C_I = \{(z_1, \ldots, z_{N-1}, z) | z = 0, \text{ and } z_i = 0 \text{ for } i \in I \subset \{1, \ldots, e\} \text{ such that}$$

$$n > \sum_{i \in I} a_i \text{ , or } I = \emptyset\}$$

(iii) *If $n = a_1 + \cdots + a_e$, the irreducible components of the tangent cone $C_{X,0}$.*

Thus, the characteristic monomials of a quasi-ordinary irreducible hypersurface determine its auréole, and in particular its exceptional cones, in all dimensions.

Remark 1.6.6

1) We repeat the remark on p. 567 of [36] to the effect that when $(X, 0)$ is analytically isomorphic to the germ at the vertex of a cone the polar varieties are themselves isomorphic to cones so that the families of tangent cones of polar varieties have no fixed components except when $k = 0$. Therefore in this case $(X, 0)$ has no exceptional cones.
2) The fact that the cone X over a nonsingular projective variety has no exceptional cones is thus related to the fact that the critical locus $P_1(X, 0)$ of the projection $\pi : X \to \mathbf{C}^d$, which is purely of codimension one in X if it is not empty, actually moves with the projection π; in the language of algebraic geometry, the ramification divisor of the projection is ample (see [66, Chap. I, cor. 2.14]) and even very ample (see [10]).
3) The dimension of $\kappa^{-1}(0)$ can be large for a singularity $(X, 0)$ which has no exceptional cones. This is the case for example if X is the cone over a projective variety of dimension $d - 1 < N - 2$ in $\mathbf{P}^{N-1}$ whose dual is a hypersurface.

1.7 The Relative Conormal

Let $f : X \to S$ be a morphism of reduced analytic spaces, with purely d-dimensional fibers and such that there exists a closed nowhere dense analytic space such that the restriction to its complement X^0 in X :

$$f|_{X^0} : X^0 \longrightarrow S$$

has all its fibers smooth. They are manifolds of dimension $d = \dim. X - \dim. S$. Let us assume furthermore that the map f is induced, via a closed embedding $X \subset Z$ by a smooth map $F: Z \to S$. This means that locally on Z the map F is analytically isomorphic to the first projection $S \times \mathbf{C}^N \to S$. Locally on X, this is always the case because we can embed the graph of f, which lies in $X \times S$, into $\mathbf{C}^N \times S$.

Let us denote by $\pi_F: T^*(Z/S) \to Z$ the relative cotangent bundle of Z/S, which is a fiber bundle whose fiber over a point $z \in Z$ is the dual $T^*_{Z/S,x}$ of the tangent vector space at z to the fiber $F^{-1}(F(z))$. For $x \in X^0$, denote by $X^0(x)$ the submanifold $f^{-1}(f(x)) \cap X^0$ of X^0. Using this submanifold we will build the **conormal space of** X **relative to** f, denoted by $T^*_{X/S}(Z/S)$, by setting

$$N^*_{X^0(x),x} = \{\xi \in T^*Z/S, x | \xi(v) = 0, \ \forall v \in T_{X^0(x),x}\}$$

and

$$T^*_{X^0/S}(Z/S) = \{(x, \xi) \in T^*(Z/S) |\ x \in X^0, \ \xi \in N^*_{X^0(x),x}\},$$

and finally taking the closure of $T^*_{X^0/S}(Z/S)$ in $T^*(Z/S)$, which is a complex analytic space $T^*_{X/S}(Z/S)$ by an argument similar to the one we saw in Proposition 1.3.1. Since X^0 is dense in X, this closure maps onto X by the natural projection $\pi_F: T^*(Z/S) \to Z$.

Now we can projectivize with respect to the homotheties on ξ, as in the case where S is a point, which we have seen above. We obtain the (projectivized) relative conormal space $C_f(X) \subset \mathbf{P}T^*(Z/S)$ (also denoted by $C(X/S)$), naturally endowed with a map

$$\kappa_f: C_f(X) \longrightarrow X.$$

We can assume that locally the map f is the restriction of the first projection to $X \subset S \times U$, where U is open in $\mathbf{C}^n$. Then we have $T^*(S \times U/S) = S \times U \times \check{\mathbf{C}}^n$ and $\mathbf{P}T^*(S \times U/S) = S \times U \times \check{\mathbf{P}}^{N-1}$. This gives an inclusion $C_f(X) \subset X \times \check{\mathbf{P}}^{N-1}$ such that κ_f is the restriction of the first projection, and a point of $C_f(X)$ is a pair (x, H), where x is a point of X and H is a limit direction at x of hyperplanes of $\mathbf{C}^N$ tangent to the fibers of the map f at points of X^0. By taking for S a point we recover the classical case studied above.

Definition 1.7.1 Given a smooth morphism $F: Z \to S$ as above, the projection to S of $Z = S \times U$, with U open in $\mathbf{C}^n$, we shall say that a reduced complex subspace $W \subset T^*(Z/S)$ is F-**Lagrangian** (or S-**Lagrangian** if there is no ambiguity on F) if the fibers of the composed map $q := (\pi_F \circ F)|W: W \to S$ are purely of dimension $n = \dim. Z - \dim. S$ and the differential ω_F of the relative Liouville differential form α_F on $\mathbf{C}^N \times \check{\mathbf{C}}^N$ vanishes on all pairs of tangent vectors at smooth points of the fibers of the map q.

With this definition it is not difficult to verify that $T^*_{X/S}(Z/S)$ is F-Lagrangian, and by abuse of language we will say the same of $C_f(X)$. But we have more:

Proposition 1.7.2 (Lê-Teissier, See [36], Proposition 1.2.6) *Let $F: Z \to S$ be a smooth complex analytic map with fibers of dimension n. Assume that S is reduced. Let $W \subset T^*(Z/S)$ be a reduced closed complex subspace and set as above $q = \pi_F \circ F|W: W \to S$. Assume that the dimension of the fibers of q over points of dense open analytic subsets U_i of the irreducible components S_i of S is n.*

(i) *If the Liouville form on $T^*_{F^{-1}(s)} = (\pi_F \circ F)^{-1}(s)$ vanishes on the tangent vectors at smooth points of the fibers $q^{-1}(s)$ for $s \in U_i$ and all the fibers of q are of dimension n, then the Liouville form vanishes on tangent vectors at smooth points of all fibers of q.*

(ii) *The following conditions are equivalent:*

- *The subspace $W \subset T^*(Z/S)$ is F-Lagrangian;*
- *The fibers of q, once reduced, are all purely of dimension n and there exists a dense open subset U of S such that for $s \in U$ the fiber $q^{-1}(s)$ is reduced and is a Lagrangian subvariety of $(\pi_F \circ F)^{-1}(s)$;*

  If moreover W is homogeneous with respect to homotheties on $T^*(Z/S)$, these conditions are equivalent to:
- *All fibers of q, once reduced, are purely of dimension n and each irreducible component W_j of W is equal to $T^*_{X_j/S}(Z/S)$, where $X_j = \pi_F(W_j)$.*

The essential content of this is that an equidimensional specialization of Lagrangian varieties is a union of irreducible Lagrangian varieties. For more details see [36] or [15, Chap. I].

1.8 Whitney Stratifications

1.8.1 Introduction

In this section we study Whitney stratifications of complex analytic spaces using the tools introduced in the preceding sections. For the history of the subject, including in real algebraic, real analytic, differentiable and definable geometry, we refer the reader to [63, §4.1] in Volume I of this Handbook. The complex analytic case has specific features which imply in particular that Whitney stratifications can be characterized by algebraic equimultiplicity conditions as well as topological equisingularity conditions, that they are also characterized by Lagrangian-type conditions for certain subspaces in auxiliary spaces, and finally that a complex analytic space has a canonical minimal Whitney stratification.

In his paper [65], Whitney gave a definition of a complex analytic stratification of a reduced complex analytic space X (see §18 of *loc.cit.*). The idea is to produce a locally finite decomposition $X = \bigsqcup_{\alpha \in A} S_\alpha$ of a reduced complex analytic space X

into disjoint non-singular locally closed subspaces called *strata* such that the "local geometry" of X is the same at all points of the same stratum. To achieve this he proposed two types of conditions:

- Topological/Analytic conditions: each stratum $S_\alpha \subset X$ is a non-singular analytic space, its closure $\overline{S_\beta}$ is a closed analytic subspace of X and the frontier $\overline{S_\beta} \setminus S_\beta$ is a union of strata
- Differential conditions: Consider a pair of strata (S_α, S_β) such that S_α is contained in the closure of S_β:

$$S_\alpha \subset \overline{S}_\beta$$

and consider a point $x \in S_\alpha$. We can assume that a neighborhood of x in X is a closed subset of an open subset U of an affine space $\mathbf{C}^N$. Now, consider a sequence x_n of points of $S_\beta \cap U$ which tends to x and a sequence y_n of points of $S_\alpha \cap U$ which also tends to x. By choosing good subsequences of (x_n) and (y_n), we may suppose that the limit of secant lines $\overline{x_n y_n}$ is ℓ and the limit of the tangents $T_{x_n} S_\beta$ is $\mathbf{T}$. Then one says that we have the Whitney condition for (S_α, S_β) at the point $x \in S_\alpha$, if for all sequences (x_n), (y_n), we have:

$$\ell \subset \mathbf{T}.$$

This is the same as condition b) of [63, Def. 4.2.1].

Note that the first condition is equivalent to: $S_\alpha \cap \overline{S_\beta} \neq \emptyset$ implies $S_\alpha \subset \overline{S}_\beta$. This is known as the *frontier condition*. .

Remark 1.8.1 Whitney's original definition had a condition a) stating that for sequences x_n as above, the limit $\mathbf{T}$ contains the tangent space $T_{Y,x}$ to Y at x. In fact condition b) implies a). See [63, 4.2, Exercise].

Definition 1.8.2 One says that a locally finite partition $X = \bigsqcup_{\alpha \in A} S_\alpha$ is a Whitney stratification if the topological/analytic conditions are satisfied by the collection of strata and the differential condition is satisfied for all pairs of strata (S_α, S_β) such that $S_\alpha \subset \overline{S}_\beta$ and all points $x \in S_\alpha$.

Theorem 1.8.3 (Whitney) *Any reduced complex analytic space admits Whitney stratifications.*

Proof For the original proof see [65, Theorem 19.2]. For a different proof see [60, Chap. III, Proposition 2.2.2]. □

Remark 1.8.4 As we mentioned in Lemma 1.6.1, Whitney discovered (see [65, Theorem 22.1]) that an analytic space is *asymptotically conical* near any of its points. This means that given $x \in X$, a sequence of points $x_n \in X$ tending to x, and a (limit of) tangent space(s) T_n at each x_n (or a limit of limits at x_n of tangent spaces at points of X^0 if the x_n are singular points), up to taking a subsequence, the limit ℓ of secant lines $\overline{x x_n}$ is contained in the limit $\mathbf{T}$ of the T_n. Dealing with the case where the x_n are singular points necessitates the existence of Whitney stratifications of X; that is why the theorem appears at the very end of Whitney's paper.

A consequence of this is that if we take a sufficiently small sphere $\mathbf{S}_\epsilon$, boundary of a ball $\mathbf{B}_\epsilon$ around x in $\mathbf{C}^N$, since it is transversal to the secants $\overline{xx_n}$ it has to be transversal to X^0 and in fact to all the strata S_α containing x in their closure. From this one deduces that $X \cap \mathbf{B}_\epsilon$ is homeomorphic to the (real) cone with vertex x over $X \cap \mathbf{S}_\epsilon$. This is the *local conicity theorem.*

The differential part of the Whitney conditions extends this to the case where the point $x \in X$ is extended to be the stratum $S_\alpha \subset \overline{S_\beta}$, where, as we may, we assume S_α to be a linear subspace of an ambient $\mathbf{C}^N$, so that $\overline{S_\beta}$ is asymptotically like a cone with vertex S_α. That is, the product of the (linear) S_α by a cone. The intuition then is that if we take a sufficiently small closed tubular neighborhood $\mathbf{T}_\epsilon$ of S_α in $\mathbf{C}^N$, then $\overline{S_\beta} \cap \mathbf{T}_\epsilon$ should be homeomorphic to the cone with vertex S_α over the intersection of $\overline{S_\beta}$ with the boundary of the tube. This ensures that at least topologically the local geometry of the $\overline{S_\beta}$ containing S_α is constant along S_α, and therefore also that of X.

This intuition turned out to be correct, and in fact more is true (see [47]), but the precise proofs, due to Thom and Mather, are far from easy; see [63].

Remark 1.8.5 In addition to the applications to the study of the topology of singular complex spaces, one must mention that complex Whitney stratifications play a key role in the theory of $\mathscr{D}$-modules (see [26, Chap. 6 and Appendix 2]) and constructible sheaves on complex spaces (see [42, Section 10.3.3]) and also in the theory of characteristic classes for singular complex varieties (see [5] and [6, Section 10]). They also play a key role in understanding the geometry of Plücker-type formulas as the reader will see at the end of this section.

1.8.2 Whitney Conditions and the Normal/Conormal Diagram

In order to simplify notations we consider a pair of strata $Y \subset X \subset \mathbf{C}^N$ in the neighborhood of $0 \in \mathbf{C}^N$, with Y linear of dimension t. They represent $S_\alpha \subset \overline{S_\beta} \subset \mathbf{C}^N$ with $X^0 = S_\beta$. Since we have to consider limits of secants starting in Y, we consider the following generalization of the normal/conormal diagram:

$$
\begin{array}{ccccccccc}
X \times \mathbf{P}^{N-t-1} \times \check{\mathbf{P}}^{N-1} & \supset & E_Y C(X) & \xrightarrow{\hat{e}_Y} & C(X) & \hookrightarrow & X \times \check{\mathbf{P}}^{N-1} & & \\
\downarrow K & & \downarrow \kappa' & \searrow \xi & \downarrow \kappa & \searrow \lambda & \downarrow pr_2 & & \\
 & & & & & & \check{\mathbf{P}}^{N-1} & & \\
X \times \mathbf{P}^{N-t-1} & \supset & E_Y X & \xrightarrow[e_Y]{} & X & & & &
\end{array}
$$

where now e_Y denotes the blowing-up of Y in X, which, as we remember from Sect. 1.5, builds limits of directions of secant lines $\overline{x\rho(x)}$ for $x \in X \setminus Y$ and some local retraction $\rho\colon \mathbf{C}^N \to Y$. Remember that $E_Y C(X)$ is the blowing up of the subspace $\kappa^{-1}(Y)$ in $C(X)$, and κ' is obtained from the universal property of the blowing up, with respect to $E_Y X$ and the map ξ. Just as in the case where $Y = \{0\}$, it is worth mentioning that $E_Y C(X)$ lives inside the fiber product $C(X) \times_X E_Y X \subset X \times \mathbf{P}^{N-t-1} \times \check{\mathbf{P}}^{N-1}$ and can be described in the following way: take the inverse image of $E_Y X \setminus e_Y^{-1}(Y)$ in $C(X) \times_X E_Y X$ and close it, thus obtaining κ' as the restriction of the second projection to this space.

Looking at the definitions, it is not difficult to prove that, if we consider the divisor:

$$D = |\xi^{-1}(Y)| \subset E_Y C(X), \quad D \subset Y \times \mathbf{P}^{N-t-1} \times \check{\mathbf{P}}^{N-1},$$

and denote by $\check{\mathbf{P}}^{N-t-1} \subset \check{\mathbf{P}}^{N-1}$ the space of hyperplanes containing $T_0 Y$:

- The pair (X^0, Y) satisfies Whitney's condition a) along Y (see Remark 1.8.1) if and only if we have the set theoretical equality $|C(X) \cap C(Y)| = |\kappa^{-1}(Y)|$. It satisfies Whitney's condition a) at 0 if and only if $|\xi^{-1}(0)| \subset \mathbf{P}^{N-t-1} \times \check{\mathbf{P}}^{N-t-1}$.

Note that we have the inclusion $C(X) \cap C(Y) \subset \kappa^{-1}(Y)$, so it all reduces to having the inclusion $|\kappa^{-1}(Y)| \subset C(Y)$, and since we have already seen that every limit of tangent hyperplanes H contains a limit of tangent spaces T, we are just saying that every limit of tangent hyperplanes to X at a point $y \in Y$, must be a tangent hyperplane to Y at y. Following this line of thought, satisfying condition a) at 0 is then equivalent to the inclusion $|\kappa^{-1}(0)| \subset \{0\} \times \check{\mathbf{P}}^{N-t-1}$ which implies $|\xi^{-1}(0)| \subset \mathbf{P}^{N-t-1} \times \check{\mathbf{P}}^{N-t-1}$.

- The pair (X^0, Y) satisfies Whitney's condition b) at 0 if and only if $|\xi^{-1}(0)|$ is contained in the incidence variety $I \subset \mathbf{P}^{N-t-1} \times \check{\mathbf{P}}^{N-t-1}$.

This is immediate from the relation between limits of tangent hyperplanes and limits of tangent spaces and the interpretation of $E_Y C(X)$ as the closure of the inverse image of $E_Y X \setminus e_Y^{-1}(Y)$ in $C(X) \times_X E_Y X$ since we are basically taking limits as $x \to Y$ of couples (l, H) where l is the direction in $\mathbf{P}^{N-t-1}$ of a secant line $\overline{yx}$ with $x \in X^0 \setminus Y$, $y = \rho(x) \in Y$, where ρ is some local retraction of the ambient space to the nonsingular subspace Y, and H is a tangent hyperplane to X at x. So, in order to verify the Whitney conditions, it is important to control the geometry of the projection $D \to Y$ of the divisor $D \subset E_Y C(X)$.

Remark 1.8.6 Although it is beyond the scope of these notes, we point out to the interested reader that there is an algebraic definition of the Whitney conditions for X^0 along $Y \subset X$ solely in terms of the ideals defining $C(X) \cap C(Y)$ and $\kappa^{-1}(Y)$ in $C(X)$. Indeed, the inclusion $C(X) \cap C(Y) \subset \kappa^{-1}(Y)$ follows from the fact that the sheaf of ideals $\mathcal{J}_{C(X) \cap C(Y)}$ defining $C(X) \cap C(Y)$ in $C(X)$ contains the sheaf of ideals $\mathcal{J}_{\kappa^{-1}(Y)}$ defining $\kappa^{-1}(Y)$, which is generated by the pull-back by κ of the equations of Y in X. What was said above means that condition a) is equivalent to the second inclusion in:

$$\mathcal{J}_{\kappa^{-1}(Y)} \subseteq \mathcal{J}_{C(X)\cap C(Y)} \subseteq \sqrt{\mathcal{J}_{\kappa^{-1}(Y)}}.$$

It is proved in [36, Proposition 1.3.8] that having both Whitney conditions is equivalent to having the second inclusion in:

$$\mathcal{J}_{\kappa^{-1}(Y)} \subseteq \mathcal{J}_{C(X)\cap C(Y)} \subseteq \overline{\mathcal{J}_{\kappa^{-1}(Y)}}\ ,$$

where the bar denotes the integral closure of the sheaf of ideals, which is contained in the radical and is in general much closer to the ideal than the radical. The second inclusion is an algebraic expression of the fact that locally near every point of the common zero set the modules of local generators of the ideal $\mathcal{J}_{C(X)\cap C(Y)}$ are bounded, up to a multiplicative constant depending only on the chosen neighborhood of the common zero set, by the supremum of the modules of generators of $\mathcal{J}_{\kappa^{-1}(Y)}$.

This result is used in [20] to produce an algorithm computing the Whitney stratification of a projective variety.

In the case where Y is a point x, the ideal defining $C(X) \cap C(\{x\})$ in $C(X)$ is just the pull-back by κ of the maximal ideal $m_{X,x}$, so it coincides with $\mathcal{J}_{\kappa^{-1}(x)}$ and Whitney's lemma for the smooth part X^0 follows.

Definition 1.8.7 Let $Y \subset X \subset \mathbf{C}^N$ as before. Then we say that the local polar variety $P_k(X; L^{d-k})$ is equimultiple along Y at a point $x \in Y$ if the map $y \mapsto m_y(P_k(X; L^{d-k}))$ is constant for $y \in Y$ in a neighborhood of x.

Note that this implies that if $(P_k(X; L^{d-k}), x) \neq \emptyset$, then $P_k(X; L^{d-k}) \supset Y$ in a neighborhood of x since the emptiness of a germ is equivalent to multiplicity zero.

We can now state the main theorem of this section, a complete proof of which can be found in [60, Chap. V, Thm. 1.2, p. 455].

Theorem 1.8.8 (Teissier; See Also [21] for Another Proof) *Given* $0 \in Y \subset X$ *as before, the following conditions are equivalent, where* ξ *is the diagonal map in the normal/conormal diagram above:*

1) The pair (X^0, Y) *satisfies Whitney's conditions at* 0.
2) The local polar varieties $P_k(X, L)$, $0 \le k \le d-1$, *are equimultiple along* Y *(at* 0*), for general* L.
3) dim. $\xi^{-1}(0) = N - t - 2$.

Note that since dim. $D = N - 2$, condition 3) is open and the theorem implies that (X^0, Y) satisfies Whitney's conditions at 0 if and only if it satisfies Whitney's conditions in a neighborhood of 0.

Note also that by analytic semicontinuity of fiber dimension (see [12, Chap. 3, 3.6] or [27, §49]), condition 3) is satisfied outside of a closed analytic subspace of Y, which shows that Whitney's conditions are a stratifying condition in the sense of [60, Chap. III, Definition 1.4].

Moreover, since a blowing up does not lower dimension, the condition dim. $\xi^{-1}(0) = N - t - 2$ implies dim. $\kappa^{-1}(0) \leq N - t - 2$. So that, in particular $\kappa^{-1}(0) \not\supset \check{\mathbf{P}}^{N-t-1}$, where $\check{\mathbf{P}}^{N-t-1}$ denotes as before the space of hyperplanes containing T_0Y. This tells us that *a general hyperplane containing T_0Y is not a limit of tangent hyperplanes to X*. This fact is crucial in the proof that Whitney conditions are equivalent to the equimultiplicity of polar varieties since it allows the start of an inductive process. In the actual proof of [60], one reduces to the case where dim. $Y = 1$ and shows by a geometric argument that the Whitney conditions imply that the polar curve has to be empty, which gives a bound on the dimension of $\kappa^{-1}(0)$. Conversely, the equimultiplicity condition on polar varieties gives bounds on the dimension of $\kappa^{-1}(0)$ by implying the emptiness of the polar curve and on the dimension of $e_Y^{-1}(0)$ by Hironaka's result, hence a bound on the dimension of $\xi^{-1}(0)$.

It should be noted that Hironaka had proved in [23, Corollary 6.2] that the Whitney conditions for X^0 along Y imply equimultiplicity of X along Y.

Finally, a consequence of the theorem is that *given a complex analytic space X, there is a unique minimal (coarsest) Whitney stratification*; any other Whitney stratification of X is obtained by adding strata inside the strata of the minimal one. A detailed explanation of how to construct this "canonical" Whitney stratification using Theorem 1.8.8, and the proof that this is in fact the coarsest one appears in [60, Chap. VI, §3]. The connected components of the strata of the minimal Whitney stratification give a minimal "Whitney stratification with connected strata"

1.8.3 The Whitney Conditions Are Lagrangian in Nature

Consider the irreducible components $D_j \subset Y \times \mathbf{P}^{N-t-1} \times \check{\mathbf{P}}^{N-1}$ of the divisor $D = |\xi^{-1}(Y)|$, that is $D = \bigcup_j D_j$, and their images:

$$V_j = \kappa'(D_j) \subset Y \times \mathbf{P}^{N-t-1},$$

$$W_j = \hat{e}_Y(D_j) \subset Y \times \check{\mathbf{P}}^{N-1}.$$

We have $\kappa_X^{-1}(Y) = \bigcup_j W_j$ and $e_Y^{-1}(Y) = \bigcup_j V_j$:

Theorem 1.8.9 (Lê-Teissier, See [36, Thm. 2.1.1]) *The equivalent statements of Theorem 1.8.8 are also equivalent to the following one.*

For each j, the irreducible divisor D_j is the relative conormal space of its image $V_j \subset \mathrm{Proj}_Y C_{X,Y} \subset Y \times \mathbf{P}^{N-t-1}$ under the first projection $Y \times \mathbf{P}^{N-t-1} \to Y$ restricted to V_j, and all the fibers of the restriction $\xi|D_j : D_j \to Y$ have the same dimension near 0.

In particular, Whitney's conditions are equivalent to the equidimensionality for $y \in Y$ of the fibers $D_j(y) = D_j \cap \xi^{-1}(y)$ of the map $D_j \to Y$, plus the fact that each D_j is contained in $Y \times I \subset Y \times \mathbf{P}^{N-t-1} \times \check{\mathbf{P}}^{N-t-1}$, where $\check{\mathbf{P}}^{N-t-1}$ is the space of hyperplanes containing the tangent space $T_{Y,0}$ and I is the incidence subvariety.

The new fact is that the contact form on $I \subset \mathbf{P}^{N-t-1} \times \check{\mathbf{P}}^{N-t-1}$ vanishes on the smooth points of $D_j(y)$ for $y \in Y$. This means that each D_j is Y-Lagrangian and is equivalent to a relative (or fiberwise) duality:

$$\begin{array}{ccc} D_j & \longrightarrow & W_j = Y\text{—dual of } V_j \subset Y \times \check{\mathbf{P}}^{N-t-1} \\ \downarrow & & \\ Y \times \mathbf{P}^{N-t-1} \supset V_j & & \end{array}$$

The proof uses that the Whitney conditions are stratifying in the sense of [60, Chap. III, Definition 1.4 and Proposition 2.2.2], and that Theorem 1.8.8 and the result of Remark 1.8.6 imply[5] that D_j is the conormal of its image over a dense open set of Y. The condition $\dim.\xi^{-1}(0) = N - t - 2$ then gives exactly what is needed, in view of Proposition 1.7.2, for D_j to be Y-Lagrangian.

Remark 1.8.10 As we have seen in Sect. 1.8.2, the original definition of the Whitney conditions, translates as the fact that $|\xi^{-1}(Y)|$ is in $Y \times \mathbf{P}^{N-t-1} \times \check{\mathbf{P}}^{N-t-1}$ and not just $Y \times \mathbf{P}^{N-t-1} \times \check{\mathbf{P}}^{N-1}$ (condition a) and moreover lies in the product $Y \times I$ of Y with the incidence variety $I \subset \mathbf{P}^{N-t-1} \times \check{\mathbf{P}}^{N-t-1}$ (condition b)). Theorem 1.8.9 shows that they are in fact of a Lagrangian, or Legendrian, nature. This explains their stability by general sections (by non singular subspaces containing Y) as proved in [60, Chap. V] and linear projections, as proved in [36, Théorème 2.2.4].

The condition $\dim.\kappa^{-1}(y) \leq N - t - 2$ which follows from $\dim.\xi^{-1}(y) = N - t - 2$ corresponds to the fact that a general hyperplane of $\mathbf{C}^N$ containing $T_{Y,y}$ is not a limit of tangent hyperplanes to X^0, which is an important consequence of the Whitney conditions as we have already noted.

1.9 The Multiplicities of Local Polar Varieties and a Plücker Type Formula

In this section we relate the multiplicities of the local polar varieties of the closures of strata, which are algebraic invariants of singularities which can be computed by intersection theory in the normal/conormal diagram at a point, with vanishing Euler characteristics associated to the strata of a Whitney stratification.

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[5] The proof of this in [36] uses a lemma, p. 559, whose proof is incorrect, but easy to correct. There is an unfortunate mixup in notation. One needs to prove that $\sum_{t+1}^{N} \xi_k dz_k = 0$ and use the fact that the same vector remains tangent after the homothety $\xi_k \mapsto \lambda\xi_k,\ t+1 \leq k \leq N$. Since we want to prove that L_1 is Y-Lagrangian, we must take $dy_i = 0$.

As we shall see, when applied to the cone over a projective variety $Z \subset \mathbf{P}^{N-1}$ this formula yields a general Plücker type formula expressing the degree of the dual variety $\check{Z} \subset \check{\mathbf{P}}^{N-1}$ of Z in terms of the Euler characteristics of the strata of the minimal Whitney stratification $(Z_\alpha)_{\alpha \in A}$ of Z and their sections by general linear subspaces of all dimensions, and the vanishing Euler-Poincaré characteristics associated to pairs of strata $Z_\alpha \subset \overline{Z_\beta}$.

Proposition 1.9.1 (Lê-Teissier, See [37, §3]) *Let $X = \bigsqcup_\alpha X_\alpha$ be a Whitney stratified complex analytic set of dimension d, with connected strata. Given $x \in X_\alpha$, choose a local embedding $(X, x) \subset (\mathbf{C}^N, 0)$. Set $d_\alpha = dim.\, X_\alpha$. For each integer $i \in [d_\alpha + 1, d]$ there exists a Zariski open dense subset $W_{\alpha,i}$ in the Grassmannian $G(N-i, N)$ and for each $L_i \in W_{\alpha,i}$ a semi-analytic subset E_{L_i} of the first quadrant of $\mathbf{R}^2$, of the form $\{(\epsilon, \eta) | 0 < \epsilon < \epsilon_0, 0 < \eta < \phi(\epsilon)\}$ with $\phi(\epsilon)$ a certain Puiseux series in ϵ, such that the homotopy type of the intersection $X \cap (L_i + t) \cap \mathbf{B}(0, \epsilon)$ for $t \in \mathbf{C}^N$ is independent of $L_i \in W_{\alpha,i}$ and (ϵ, t) provided that $(\epsilon, |t|) \in E_{L_i}$. Moreover, this homotopy type depends only on the stratified set X and not on the choice of $x \in X_\alpha$ or the local embedding. In particular the Euler-Poincaré characteristics $\chi_i(X, X_\alpha)$ of these homotopy types are invariants of the stratified analytic set X.*

Definition 1.9.2 The Euler-Poincaré characteristics $\chi_i(X, X_\alpha)$, for $i \in [d_\alpha + 1, d]$ are called the local vanishing Euler-Poincaré characteristics of X along X_α.

The independence of the point $x \in X_\alpha$ is a consequence of the local topological triviality of the closures of the Whitney strata along the strata of their boundaries (The Thom-Mather Theorem). We shall not go into this here. See [63, Theorem 4.2.17]. The connection between the local vanishing Euler characteristics and the multiplicities of polar varieties is expressed as follows:

Theorem 1.9.3 (Lê-Teissier, See [35, Théorème 6.1.9], [37, 4.11]) *With the conventions just stated, and for any Whitney stratified complex analytic set $X = \bigsqcup_\alpha X_\alpha \subset \mathbf{C}^N$, we have for $x \in X_\alpha$ the equality*

$$\chi_{d_\alpha+1}(X, X_\alpha) - \chi_{d_\alpha+2}(X, X_\alpha) = \sum_{d_\beta > d_\alpha} (-1)^{d_\beta - d_\alpha - 1} m_x(P_{d_\beta - d_\alpha - 1}(\overline{X_\beta}, x))(1 - \chi_{d_\beta+1}(X, X_\beta)),$$

where it is understood that $m_x(P_{d_\beta - d_\alpha - 1}(\overline{X_\beta}, x)) = 0$ if $x \notin P_{d_\beta - d_\alpha - 1}(\overline{X_\beta}, x)$.

It follows that given a Whitney stratified complex analytic set $X = \bigsqcup_\alpha X_\alpha$ with connected strata, it is equivalent to give the collections of multiplicities of the local polar varieties of the closures $\overline{X_\beta}$ of strata at the points of the strata X_α in their boundary and to give the collections of vanishing Euler-Poincaré characteristics $\chi_i(\overline{X_\beta}, X_\alpha)$. There is an invertible linear relation between the two sets.

Let us now consider the special case where X is the cone over a projective variety Z, which we assume not to be contained in a hyperplane. The dual variety $\check{Z}$ of Z was defined in Sect. 1.4.2. Remember that every complex analytic space, and in particular Z, has a minimal Whitney stratification. We shall use the following facts, with the notation of Proposition 1.9.1 and those introduced after Proposition 1.4.8:

Proposition 1.9.4 (See [16, Section 8]) *Let $Z \subset \mathbf{P}^{N-1}$ be a projective variety of dimension d.*

(i) *If $Z = \bigsqcup_\alpha Z_\alpha$ is a Whitney stratification of Z, denoting by $X_\alpha \subset \mathbf{C}^N$ the cone over Z_α, we have that $X = \{0\} \cup (\bigsqcup_\alpha X^*_\alpha)$, where $X^*_\alpha = X_\alpha \setminus \{0\}$, is a Whitney stratification of X. It may be that (Z_α) is the minimal Whitney stratification of V but $\{0\} \cup (\bigsqcup_\alpha X^*_\alpha)$ is not minimal, for example if Z is itself a cone.*
(ii) *If $L_i + t$ is an i-codimensional affine space in $\mathbf{C}^N$ it can be written as $L_{i-1} \cap (L_1 + t)$ with vector subspaces L_i and for general directions of L_i we have, denoting by $\mathbf{B}(0, \epsilon)$ the closed ball with center 0 and radius ϵ, for small ϵ and $0 < |t| \ll \epsilon$:*

$$\chi_i(X, \{0\}) := \chi(X \cap (L_i + t) \cap \mathbf{B}(0, \epsilon)) = \chi(Z \cap H_{i-1}) - \chi(Z \cap H_{i-1} \cap H_1),$$

where $H_i = \mathbf{P}L_i \subset \mathbf{P}^{N-1}$.
(iii) *For every stratum X^*_α of X, we have the equalities $\chi_i(X, X^*_\alpha) = \chi_i(Z, Z_\alpha)$.*
(iv) *If the dual $\check{Z} \subset \check{\mathbf{P}}^{N-1}$ is a hypersurface, its degree is equal to $m_0(P_d(X, 0))$, which is the number of non singular critical points of the restriction to Z of a general linear projection $\mathbf{P}^{N-1} \setminus L_2 \to \mathbf{P}^1$.*

Note that we will apply statements 2) and 3) not only to the cone X over Z but also to the cones $\overline{X_\beta}$ over the closed strata $\overline{Z_\beta}$.

If we now apply the Theorem 1.9.3, we see that, using Proposition 1.9.4, we can rewrite in this case the formula of Theorem 1.9.3 as a generalized Plücker formula for any d-dimensional projective variety $Z \subset \mathbf{P}^{N-1}$ whose dual is a hypersurface:

Proposition 1.9.5 (Teissier, See [60, §5]) *Given the projective variety $Z \subset \mathbf{P}^{N-1}$ equipped with a Whitney stratification $Z = \bigsqcup_{\alpha \in A} Z_\alpha$, denote by d_α the dimension of Z_α. We have, if the projective dual $\check{Z}$ is a hypersurface in $\check{\mathbf{P}}^{N-1}$:*

$$\begin{aligned}(-1)^d \deg \check{Z} &= \chi(Z) - 2\chi(Z \cap H_1) + \chi(Z \cap H_2) \\ &- \sum_{d_\alpha < d} (-1)^{d_\alpha} \deg_{N-2} P_{d_\alpha}(\overline{Z_\alpha})(1 - \chi_{d_\alpha+1}(Z, Z_\alpha)) ,\end{aligned}$$

where H_1, H_2 denote general linear subspaces of $\mathbf{P}^{N-1}$ of codimension 1 and 2 respectively, $\deg_{N-2} P_{d_\alpha}(\overline{Z_\alpha})$ is the number of nonsingular critical points of a general linear projection $\overline{Z_\alpha} \to \mathbf{P}^1$, which is the degree of $\check{\overline{Z_\alpha}}$ if it is a hypersurface and is set equal to zero otherwise. It is equal to 1 if $d_\alpha = 0$.

Here we remark that if $(Z_\alpha)_{\alpha\in A}$ is the minimal Whitney stratification of the projective variety $Z \subset \mathbf{P}^{N-1}$, and L is a general linear subspace in $\mathbf{P}^{N-1}$, the $Z_\alpha \cap L$ that are not empty constitute the minimal Whitney stratification of $Z \cap L$. See [60, Chap. III, Lemma 4.2.2] and use the fact that the minimal Whitney stratification is defined by equimultiplicity of polar varieties (see [60, Chap. VI, §3]) and that the multiplicity of polar varieties of dimension > 1 is preserved by general hyperplane sections as we saw before Theorem 1.9.3.

It is explained in [16, Section 8] that if the dual of Z is not a hypersurface, the dual of the intersection of Z with a general linear space of $\mathbf{P}^{N-1}$ of codimension $\delta(Z) = \mathrm{codim}_{\check{\mathbf{P}}^{N-1}} \check{Z} - 1$ is a hypersurface of the same degree as $\check{Z}$. Using this and an induction on the dimension by applying Proposition 1.9.4, possibly after general linear sections, to compute the degrees of the $\check{\overline{Z_\alpha}}$, we see that *we have proved the existence of a general formula to compute the degree of* $\check{Z}$ *from the Euler-Poincaré characteristics of the closed strata* $\overline{Z_\alpha}$ *and their general linear sections, and the vanishing Euler-Poincaré characteristics* $\chi_i(\overline{Z_\beta}, Z_\alpha)$. We shall not write this formula explicitly, only remark that it is linear in the Euler-Poincaré characteristics of the strata and their general linear sections, and polynomial of degree bounded by the depth (the integer d in [63, Definition 4.1.1]) of the stratification in the local vanishing Euler-Poincaré characteristics. *The degree of the variety* $\check{Z}$ *of all limit tangent hyperplanes to a projective variety* Z *depends explicitly on basic topological characters of its minimal Whitney stratification.*

For another interpretation of the Plücker formula and the relation to this one, see [11, 40, 41] and [16, §8].

Remark 1.9.6 We note that as we compute the degree of the dual $\check{Z}$, we also compute the degrees of the duals of the closures of at least some of the strata of the canonical Whitney stratification $Z = \bigsqcup_\alpha Z_\alpha$. This suggests the definition of the *total dual* of the projective variety Z: it is the union of the duals of the closures of the strata of its canonical Whitney stratification. For example if Z is the dual of a general non singular projective plane curve its total dual is the union of that curve, its bitangents and its tangents of inflexion, corresponding respectively to the nodes and cusps of Z. The total dual gives a tangentially exploded view of the singularities of Z.

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Chapter 2
Determinantal Singularities

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Contents

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Abstract We survey determinantal singularities, their deformations, and their topology. This class of singularities generalizes the well studied case of complete intersections in several different aspects, but exhibits a plethora of new phenomena such as for instance non-isolated singularities which are finitely determined, or smoothings with low connectivity; already the union of the coordinate axes in $(\mathbb{C}^3, 0)$ is determinantal, but not a complete intersection. We start with the algebraic background and then continue by discussing the subtle interplay of unfoldings and deformations in this setting, including a survey of the case of determinantal hypersurfaces, Cohen-Macaulay codimension 2 and Gorenstein codimension 3 singularities, and determinantal rational surface singularities. We conclude with a discussion of essential smoothings and provide an appendix listing known classifications of simple determinantal singularities.

2.1 Singularities of Matrices and Determinantal Varieties

Before this article can focus on geometric and topological properties of determinantal singularities, the first section starts by presenting background material from commutative algebra. This material is indispensible for understanding some common structural properties of this class. Already in the study of the much smaller class of complete intersection singularities, singled out by the algebraic property that their ideal possesses a set of generators which form a *regular sequence*,[1] a similar approach to the exposition of material has at times been used.

2.1.1 Determinantal Ideals

Let R be a commutative ring with unity. We write $R^{m \times n}$ for the space of $m \times n$-matrices with entries in R.

Definition 2.1.1 Let $A \in R^{m \times n}$ be a matrix and $I \subset R$ the ideal generated by the t-minors of A. Then we say that I is *determinantal of type* (m, n, t) *or determinantal of type* t *with matrix* A.

[1] A sequence of elements $a_1, \ldots, a_n \in R$ is called *weakly regular* on a module M if multiplication by a_{i+1} is injective on $M/\langle a_1, \ldots, a_i \rangle M$ for every $i < n$. It is called *regular* if, moreover, $M/\langle a_1, \ldots, a_n \rangle M \neq 0$, cf. [13, Definition 1.1.1].

Note that the same ideal can be determinantal in various different ways, that is for different matrices.

Example 2.1.2 The principal ideal generated by $f = xw - yz \in \mathbb{C}\{x, y, z, w\}$ is determinantal of type 1 with the 1×1-matrix (f) but also determinantal of type 2 with matrix

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

Another famous example is due to Pinkham [84]. He observed that the ideal $I \subseteq \mathbb{C}\{x_0, \ldots, x_4\}$ given by the 2-minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix}$$

can also be generated by the 2-minors of

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix}.$$

Hence, I is determinantal of type $(2, 4, 2)$ or of type $(3, 3, 2)$, depending on the choice of the matrix.

Later on, we will encounter cases of ideals which are determinantal in a unique way (cf. Theorem 2.3.16), but the examples given here show that in general, one really has to specify the matrix in order to describe the determinantal structure of a given ideal.

We now introduce some notation to keep the presentation readable in what follows. For a matrix $A \in R^{m\times n}$, we denote by $\langle A\rangle$ the ideal generated by the entries $a_{i,j}$ of A in R. Any such matrix A can be understood as a homomorphism of free modules $A\colon R^n \to R^m$ taking e_j to $\sum_{i=1}^m a_{i,j} \cdot f_i$, where $\{e_j\}_{j=1}^n$ is a free basis of R^n and $\{f_i\}_{i=1}^m$ of R^m. For any number t there is a natural induced morphism on the exterior powers[2] which we denote by

$$A^{\wedge t}\colon \bigwedge^t R^n \to \bigwedge^t R^m. \tag{2.1}$$

With the same free bases for R^m and R^n as before, the products $e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_t}$ with $0 < j_1 < j_2 < \cdots < j_t \leq n$ and $f_{i_1} \wedge \cdots \wedge f_{i_t}$ with $0 < i_1 < \cdots < i_t \leq m$ form free bases of the free modules $\bigwedge^t R^n$ and $\bigwedge^t R^m$ respectively. We will write

[2] For a discussion of the exterior algebra of a module, see e.g. [35, Appendix 2.3].

$\mathbf{j} = (j_1 < i_2 < \cdots < j_t)$ for the *ordered multiindex* of length $\#\mathbf{j} = t$ and $e_\mathbf{j} = e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_t}$ for corresponding generator. With this notation we find

$$\bigwedge^t R^n \to \bigwedge^t R^m, \quad e_\mathbf{j} \mapsto \sum_{\#\mathbf{i}=t} A^{\wedge t}_{\mathbf{i},\mathbf{j}} \cdot f_\mathbf{i}$$

where by $A^{\wedge t}_{\mathbf{i},\mathbf{j}}$ we denote the determinant of the $t \times t$-submatrix $A_{\mathbf{i},\mathbf{j}}$ of A specified by the selection of rows in $\mathbf{i}$ and columns in $\mathbf{j}$. Thus $(A^{\wedge t}_{\mathbf{i},\mathbf{j}})$ is the matrix for the induced homomorphism (2.1) on the exterior powers w.r.t. the chosen bases and the ideal of t-minors of A is nothing but $\langle A^{\wedge t}\rangle$.

Lemma 2.1.3 *Let R be a ring and $A \in R^{m\times n}$ a matrix. For any pair of invertible matrices $P \in \mathrm{GL}(m; R)$ and $Q \in \mathrm{GL}(n; R)$ and every number t one has*

$$\left\langle \left(P \cdot A \cdot Q^{-1}\right)^{\wedge t} \right\rangle = \langle A^{\wedge t}\rangle.$$

Proof This is immediate for the case of 1-minors, i.e. the ideal of entries of A. For the general case note that

$$\left(P \cdot A \cdot Q^{-1}\right)^{\wedge t} = P^{\wedge t} \cdot A^{\wedge t} \cdot \left(Q^{-1}\right)^{\wedge t}$$

where $P^{\wedge t} \in \mathrm{GL}(M; R)$ and $(Q^{-1})^{\wedge t} \in \mathrm{GL}(N; R)$ for $N = \binom{n}{t}$ and $M = \binom{m}{t}$. Since the t-minors of A are the entries of $A^{\wedge t}$, this reduces the problem to the case of 1-minors. □

Remark 2.1.4 As a consequence of Lemma 2.1.3 we note the following. Suppose one of the entries of the matrix A is a unit in R. Then there exist matrices P and Q as above such that

$$P \cdot A \cdot Q^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{A} \end{pmatrix}$$

with $\tilde{A}$ of size $(m-1) \times (n-1)$. It is now easy to see that the ideal of t-minors of A coincides with the ideal of $(t-1)$-minors of $\tilde{A}$. This allows for a *reduction of the defining matrix* A to $\tilde{A}$ for the determinantal ideals $\langle A^{\wedge t}\rangle = \langle \tilde{A}^{\wedge(t-1)}\rangle$. In the particular case where R is a *local* ring with maximal ideal $\mathfrak{m}$ this allows us to always reduce to the case of matrices $A \in \mathfrak{m}^{m\times n}$ and we will in the following always assume that the matrix A is of this form, unless specified otherwise.

Remark 2.1.5 In the following we will also consider *Pfaffian ideals* for skew symmetric matrices $A \in R_{\text{sk}}^{m\times m}$. Any such matrix with entries $a_{i,j} = -a_{j,i}$ can naturally be interpreted as an element of the second exterior power of a free module R^m in some generators $e_1, \ldots, e_m$, i.e.

$$A = \sum_{0<i<j\leq m} a_{i,j} \cdot e_i \wedge e_j \in \bigwedge^2 R^m.$$

In this setting, we can consider the exterior powers of A in the usual sense

$$A_{\text{sk}}^{\wedge s} = \underbrace{A \wedge A \wedge \cdots \wedge A}_{s \text{ times}} = \sum_{\#\mathbf{i}=2s} A_{\mathbf{i}}^{\wedge s} \cdot e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{2s}} \in \bigwedge^{2s} R^m \tag{2.2}$$

and we write $\langle A_{\text{sk}}^{\wedge s}\rangle$ for the ideal generated by the coefficients $A_{\mathbf{i}}^{\wedge s}$ in the expansion above. Note that in case $A \in R_{\text{sk}}^{2n\times 2n}$ is a skew symmetric matrix of even size, we indeed recover the Pfaffian of A as the coefficient in the top exterior power:

$$\operatorname{Pf} A = A_{(1,\ldots,2n)}^{\wedge n}.$$

More generally, the coefficient $A_{\mathbf{i}}^{\wedge s}$ is nothing but the Pfaffian of the skew-symmetric matrix obtained from A by selecting the even number of rows and columns specified by $\mathbf{i}$. The ideal $\langle A_{\text{sk}}^{\wedge s}\rangle$ thus coincides with the ideal of $2s$-Pfaffians of A as for example in [66].

The natural $\mathrm{GL}(m; R)$-operation on skew-symmetric matrices is given by

$$\mathrm{GL}(m; R) \times R_{\text{sk}}^{m\times m} \to R_{\text{sk}}^{m\times m}, \quad (S, A) \mapsto A' = S \cdot A \cdot S^T. \tag{2.3}$$

Regarding $S = (s_{k,j})_{k,j=1}^m$ as a change of basis in R^m with $e_j = \sum_{k=1}^m s_{k,j} \cdot f_k$ it is easy to see that (2.3) is compatible with the interpretation of A as a bi-vector, given that

$$\begin{aligned} \sum_{0<i<j\leq m} a_{i,j} \cdot e_i \wedge e_j &= \sum_{0<i<j\leq m} a_{i,j} \cdot \left(\sum_{0<k<l\leq m} (s_{k,i}s_{l,j} - s_{l,i}s_{k,j}) \cdot f_k \wedge f_l \right) \\ &= \sum_{0<k<l\leq m} a'_{k,l} \cdot f_k \wedge f_l. \end{aligned}$$

with $a'_{k,l}$ the entries of the matrix A' above.

We will in the following deliberately subsume the Pfaffian case under determinantal ideals in general, but indicate the differences whenever necessary.

2.1.2 Free Resolutions and Generic Perfection

One reason, why determinantal ideals received particular interest, is that in general they are *not* complete intersection ideals, but still provide sufficient additional structure for obtaining stronger results than in the general case, e.g. on the module of relations and on their free resolutions. For the remainder of this section, we will elaborate on the algebraic aspects in which determinantal ideals generalize complete intersections.

Let R be a commutative Noetherian ring and $I = \langle a_1, \ldots, a_n \rangle$ an ideal in I generated by n elements. By Krull's principal ideal theorem, we know that the height[3] of I can at most be n. This is the *expected height* or *expected codimension* for an ideal generated by n arbitrary elements. Observe that every such ideal is determinantal in a trivial way by setting $I = \langle A^{\wedge 1} \rangle$ for the $1 \times n$-matrix $A = (a_1, \ldots, a_n)$. For determinantal ideals in general, the following theorem establishes a similar bound. It is worth noting that this theorem has been proved for the special case of maximal minors by Macaulay [78] already in 1916, even before Krull established his principal ideal theorem in 1928.

Theorem 2.1.6 ([32, Theorem 3], cf. also [14, Theorem 2.1]) *Let R be a Noetherian ring and $A \in R^{m\times n}$ a matrix. If $\langle A^{\wedge t}\rangle \neq R$, then* $\operatorname{height}\langle A^{\wedge t}\rangle \leq (m-t+1)(n-t+1)$.

In analogy to the complete intersection case, we will refer to this bound on the height of a determinantal ideal as the *expected codimension* of the *determinantal ideal* $\langle A^{\wedge t}\rangle$. Similar bounds have been established in for determinantal ideals of symmetric matrices, see [68], where the expected codimension for the ideal of s-minors of an $n \times n$ matrix is $\frac{1}{2}(n-s+2)(n-s+1)$. For skew-symmetric matrices $A \in R_{\text{sk}}^{m\times m}$ the expected codimension of the the Pfaffian ideal $\langle A_{\text{sk}}^{\wedge s}\rangle$ is $\frac{1}{2}(m-2s+2)(m-2s+1)$, cf. [66, Theorem 17].

In what follows, all varieties and singularities will usually be embedded in a sufficiently "nice" (e.g. smooth) ambient space. On the algebraic side this corresponds to R having certain favourable properties; for example it can be a polynomial ring over a field, or a regular local ring. A reasonable and strictly less demanding assumption on R is to be Cohen-Macaulay:

Definition 2.1.7 (cf. [35, Section 18.2]) A Noetherian ring R is called a *Cohen-Macaulay* ring, if for every maximal ideal $\mathfrak{m}$ of R one has $\operatorname{grade}(\mathfrak{m}) = \operatorname{height}(\mathfrak{m})$.

Recall that the *grade* of an ideal $I \subset R$ can be defined as the maximal length of a regular R-sequence in I. In particular, regular local rings are Cohen-Macaulay (see e.g. [35, Section 18.5]) and polynomial rings over Cohen-Macaulay rings are again Cohen-Macaulay (see e.g. [35, Proposition 18.9]). In a Noetherian ring, the grade

[3] Depending on a textbook, $\operatorname{height} I = \inf\{\dim A_{\mathfrak{p}} \mid \mathfrak{p} \text{ prime containing } I\}$ is also called the codimension of I, alluding to the fact that e.g. for $I \subset k[x_1, \ldots, x_s]$ it is indeed the codimension of the variety $V(I) \subset k^s$.

of an ideal is bounded from above by the height of this ideal,[4] so the condition in the definition of a Cohen-Macaulay ring requires this estimate to be an equality for maximal ideals. It can be shown, that this already implies

$$\operatorname{height} I = \operatorname{grade} I \tag{2.4}$$

for *every* proper ideal $I \subset R$, cf. [35, Theorem 18.7]. If, additionally, the ring R is *local*, then the height of any proper ideal $I \neq R$ can actually be understood as codimension in the sense that one has

$$\operatorname{height} I + \dim R/I = \dim R, \tag{2.5}$$

see [13, Corollary 2.1.4].

In our context of a determinantal ideal I in a Cohen-Macaulay ring R, the grade (or height) is not the main focus of our interest, it is merely one ingredient to acquiring more information on free resolutions of R/I which are known to provide a wealth of subtle algebraic and geometric information. For instance, flatness of families can be checked using the first syzygy module, which is just the beginning of a free resolution (see [53, I 1.91]) , and for a Gorenstein ring R free resolutions allow computation of the dualizing module $\omega_{R/I}$ of R/I (cf.[13, Theorem 3.3.7 (b)]). On the other hand, explicitly computing a free resolution for a given ideal I can be an expensive task relying on the standard basis algorithm; knowing the general structure of a free resolution in advance is hence a precious advantage. For determinantal ideals, this is the case. The following theorem is the key to understanding how this arises.

Recall that a module M over a noetherian ring R is called *perfect*, if its *projective dimension*, i.e. the minimal length of a projective (or free) resolution of M, is equal to its *grade*.

Theorem 2.1.8 ([14], Theorem 3.5) *Let S be a Noetherian ring and M a perfect S-module of grade μ. Let R be a Noetherian S-algebra such that* $\operatorname{grade}_R(M \otimes_S R) \geq \mu$ *and $M \otimes_S R \neq 0$. Then $M \otimes_S R$ is perfect of grade μ and furthermore $K_\bullet \otimes_S R$ is a free resolution of $M \otimes_S R$ for every free resolution $K_\bullet$ of M of length μ.*

In a sufficiently nice setting, a known free resolution of an S-module M provides a free resolution of $M \otimes_S R$ for an S-algebra R. In our context the ring S will be a polynomial ring over $\mathbb{Z}$ or $\mathbb{Q}$ and the class of modules consists of those of the form $S/\langle Y^{\wedge s}\rangle$ for a matrix

$$Y = \begin{pmatrix} y_{1,1} & \cdots & y_{1,n} \\ \vdots & & \vdots \\ y_{m,1} & \cdots & y_{m,n} \end{pmatrix} \in S[y_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n].$$

[4] See e.g. [13, Proposition 1.2.14].

This is possible due to a result of Eagon and Hochster [58, Corollary 4] establishing the property of "generic perfection" for determinantal ideals:

Definition 2.1.9 ([14, Section 3.A]) A finitely generated $\mathbb{Z}[y]$-module M is called *generically perfect* if it is perfect and faithfully flat as a $\mathbb{Z}$-module. An ideal I is called generically perfect, if $\mathbb{Z}[y]/I$ is generically perfect.

Various different characterizations have been given for generically perfect ideals and modules, see [33], or [14, Proposition 3.2]. We will treat the two smallest examples, which are relevant to our setting, as an illustration and for later reference:

Example 2.1.10 (Complete Intersections and the Koszul Complex) Recall that the Koszul complex in n elements can be defined as follows. Let $y = y_1, \ldots, y_n$ be a set of indeterminates over the ring $\mathbb{Z}$ and F the free $\mathbb{Z}[y]$-module in n generators $e_1, \ldots, e_n$. Then exterior multiplication with the element $\theta = y_1 \cdot e_1 + y_2 \cdot e_2 + \cdots + y_n \cdot e_n \in F$ gives rise to an exact complex

$$\text{Kosz}\colon \quad 0 \longrightarrow \bigwedge^0 F \xrightarrow{\theta\wedge} \bigwedge^1 F \xrightarrow{\theta\wedge} \bigwedge^2 F \xrightarrow{\theta\wedge} \cdots$$

$$\cdots \xrightarrow{\theta\wedge} \bigwedge^{n-1} F \xrightarrow{\theta\wedge} \bigwedge^n F \xrightarrow{\varepsilon} \mathbb{Z}[y]/\langle y_1, \ldots, y_n\rangle \longrightarrow 0 \tag{2.6}$$

where ε takes the generator $e_1 \wedge e_2 \wedge \cdots \wedge e_n$ to $1 \in R$. For an arbitrary unital commutative ring R and n elements $a_1, \ldots, a_n \in R$, there is a unique structure of R as a $\mathbb{Z}[y]$-module substituting a_i for the variable y_i. Then the Koszul complex in the elements $a_1, \ldots, a_n$ can be written as

$$\text{Kosz}(a_1, \ldots, a_n; R) := \text{Kosz} \otimes_{\mathbb{Z}[y]} R. \tag{2.7}$$

If grade $\langle a_1, \ldots, a_n\rangle = n$, which is, in particular, the case if the a_i form a regular sequence, the *Koszul Complex* provides a free resolution of the quotient ring $R/\langle a_1, \ldots, a_n\rangle$, seen as an R-module (for textbook references see [13, Theorem 1.6.17] or [35, Corollary 17.5].

Example 2.1.11 (Perfect Ideals of Grade 2 *and the Hilbert-Burch Theorem)* Another famous and early example of generic perfection appears in the context of the Hilbert-Burch theorem, see [57] and [19], or [35] for a modern textbook account; the theorem is also stated explicitly as Theorem 2.3.16 in Sect. 2.3.4 below.

Consider the $(m+1) \times m$-matrix

$$Y = \begin{pmatrix} y_{1,1} & \cdots & y_{1,m} \\ \vdots & & \vdots \\ y_{m+1,1} & \cdots & y_{m+1,m} \end{pmatrix}.$$

The homogeneous ideal $I = \langle Y^{\wedge m} \rangle$ is a perfect ideal of grade 2 and a resolution of $\mathbb{Z}[y]/I$ is given by the complex

$$0 \longrightarrow \mathbb{Z}[y]^m \xrightarrow{Y} \mathbb{Z}[y]^{m+1} \xrightarrow{F} \mathbb{Z}[y] \xrightarrow{\varepsilon} \mathbb{Z}[y]/I \longrightarrow 0 \tag{2.8}$$

where F is the $1 \times (m+1)$-matrix with i-th entry equal to $(-1)^i$ times the i-th minor of Y. This can be proved directly using the results by Hilbert and Burch above, but it can also be regarded as a particular case of the Eagon-Northcott complex [32].

Now suppose that R is an arbitrary Noetherian ring and

$$A = \begin{pmatrix} a_{1,1} & \dots & a_{1,m} \\ \vdots & & \vdots \\ a_{m+1,1} & \dots & a_{m+1,m} \end{pmatrix}$$

a matrix with entries in R. Then the quotient ring with respect to the ideal $I = \langle A^{\wedge m} \rangle$ can be identified with

$$R/I \cong \mathbb{Z}[y]/\langle Y^{\wedge m} \rangle \otimes_{\mathbb{Z}[y]} R$$

via the map $\mathbb{Z}[y] \to R$, $y_{i,j} \mapsto a_{i,j}$. Whenever I has grade 2 in R, Theorem 2.1.8 assures that, moreover, the complex obtained from (2.8) by applying $- \otimes_{\mathbb{Z}[y]} R$ is in fact a free resolution of R/I.

Remark 2.1.12 While generic perfection of determinantal ideals has been established by Eagon and Hochster in [58, Corollary 4], explicit free resolutions of minimal length of $\langle Y^{\wedge t} \rangle$ have only been constructed in special cases over the ring $\mathbb{Z}[y]$, but are known for all values of m, n and t over the ring $\mathbb{Q}[y]$. Passing from $\mathbb{Z}[y]$ to $\mathbb{Q}[y]$ does not impose any severe restrictions in the setting of determinantal singularities, since the main focus is on complex algebraic and analytic spaces: All rings R in question will be of characteristic zero so that again any choice of elements $a_{i,j} \in R$ turns the ring into a $\mathbb{Q}[y]$-algebra by substitution of $y_{i,j}$ by $a_{i,j}$.

To conclude this brief discussion of facts from commutative algebra, we give a short (non-exhaustive) overview on results concerning the construction of resolutions of generic determinantal ideals $\langle Y^{\wedge t} \rangle$ over rings $k[y]$ for k being either $\mathbb{Z}$ or $\mathbb{Q}$. For a survey on such resolutions we refer the interested reader to [85].

As already mentioned, in the case of maximal minors $t = m \leq n$, a resolution of for $k = \mathbb{Z}$ is provided by the well known *Eagon-Northcott-complex* [32]. Another construction of this complex has been given by Buchsbaum in [15]. This complex is covered in various textbooks such as [35], or [14] and appears as a special case of the family of complexes described independently by Buchsbaum and Eisenbud [16] and Kirby [65]. The Hilbert-Burch theorem fits into this setting as the even more special case $n = m + 1$.

For submaximal minors of square matrices, i.e. for $t = m-1$, $m = n$ a resolution of $\mathbb{Z}[y]/\langle Y^{\wedge t} \rangle$ was found by Gulliksen and Negard [54]; the case of non-square matrices, i.e. $t = m-1$, $m \leq n$, has been treated a decade later by Akin, Buchsbaum, and Weyman in [1].

For the general case $t \leq m \leq n$, Lascoux provided a free resolution in [70], but only over the rationals (or, more generally, a field of characteristic zero). This is due to the fact that representation theory and the use of Schur-functors in his article, which made the change of coefficients necessary. The methods by Lascoux also work for symmetric matrices and for skew symmetric matrices with ideals generated by Pfaffians.

Resolutions of ideals of submaximal minors for symmetric matrices were also described by Józefiak in [60]. The interest in Gorenstein rings also led to the construction of free resolutions. While Gorenstein rings of codimension 1 and 2 are known to be complete intersections, those of codimension 3 are submaximal Pfaffians of skew-symmetric matrices (see [17] and for a discussion of a generic free resolution [61], for a slightly different perspective see also [108]). The further study of Gorenstein rings of higher codimension led to partial results, including (non-generic) resolutions, and is still a topic of active research in commutative algebra. A full description of the first syzygy module determinantal ideals for arbitrary $t \leq m \leq n$ over arbitrary unitary rings has been given by Kurano [67] and by Ma [77] solving a conjecture of Sharpe.

In what follows, we will refer to any free resolution of the quotient rings $k[y]/\langle Y^{\wedge t}\rangle$, for $k = \mathbb{Z}$ or $\mathbb{Q}$, as

$$K(m,n,t):\quad 0 \longrightarrow K_c \longrightarrow K_{c-1} \longrightarrow \dots \longrightarrow K_1 \longrightarrow K_0\ . \tag{2.9}$$

where c is the expected grade or codimension for the given values of m, n and t. In the case of symmetric or skew-symmetric matrices, we will occasionally write $K^{\mathrm{sym}}(m,t)$ and $K^{\mathrm{sk}}(m,t)$ for the resolutions of the quotients by the determinantal, respectively the Pfaffian ideals.

2.1.3 Determinantal Singularities and Their Deformations

Let again

$$Y = \begin{pmatrix} y_{1,1} & \cdots & y_{1,n} \\ \vdots & & \vdots \\ y_{m,1} & \cdots & y_{m,n} \end{pmatrix}$$

denote the matrix of $m \cdot n$ indeterminates but over a field k. The *generic determinantal varieties* $M^s_{m,n}(k)$ are defined as the vanishing loci of the ideals $\langle Y^{\wedge s}\rangle$ of s-minors of Y. It is evident that

$$M^s_{m,n}(k) = \left\{\varphi \in k^{m\times n} : \operatorname{rank}\varphi < s\right\}.$$

Now let R be a k-algebra and $A \in R^{m\times n}$ a matrix with entries $a_{i,j}$ in R. Then A gives rise to a homomorphism of rings $k[y] \to R$, $y_{i,j} \mapsto a_{i,j}$ which corresponds to a map $\operatorname{Spec} R \to k^{m\times n}$ on the geometric side. In accordance with Theorem 2.1.6 this suggests:

Definition 2.1.13 Let R be a Noetherian k-algebra and $A \in R^{m\times n}$ a matrix. The variety defined by the ideal $I = \langle A^{\wedge s}\rangle$ is a *determinantal variety of type s for the matrix A* if height I is equal to the expected codimension; that is $(m-s+1)(n-s+1)$ in the general case, $\frac{1}{2}(n-s+1)(n-s+2)$ for the ideal of s-minors of symmetric matrices, and $\frac{1}{2}(m-2s+2)(m-2s+1)$ for $2s$-Pfaffian ideals of $m \times m$-skew-symmetric matrices.

When R is Cohen-Macaulay and $I = \langle A^{\wedge s}\rangle$ defines a determinantal variety, then due to the equality of height and grade (2.4) Theorem 2.1.8 applies so that $K(m, n, s) \otimes_{k[y]} R$ is a free resolution of the module R/I.

We will in the following mostly be concerned with the case $k = \mathbb{C}$ and $R = \mathbb{C}\{x\} = \mathbb{C}\{x_1, \dots, x_p\}$, the ring of convergent power series at the origin in $\mathbb{C}^p$. A matrix $A \in \mathbb{C}\{x\}^{m\times n}$ will then be interpreted as a holomorphic map germ

$$A\colon (\mathbb{C}^p, 0) \to (\mathbb{C}^{m\times n}, 0).$$

By abuse of notation, let A denote a representative of this germ defined on some open neighbourhood U of the origin. We write $X^s_A \subset U$ for the complex analytic space defined by the sheaf of ideals $\langle A^{\wedge s}\rangle$ in $\mathscr{O}_U$. Then by construction one has an equality of sets

$$X^s_A = A^{-1}(M^s_{m,n}) \subset U$$

on the geometric side and an isomorphism

$$\mathbb{C}\{x\}/\langle A^{\wedge s}\rangle \cong \mathbb{C}[y]/\langle Y^{\wedge s}\rangle \otimes_{\mathbb{C}[y]} \mathbb{C}\{x\}$$

on the algebraic side, together with its sheafification on U.

Remark 2.1.14 For A as above let $\Gamma_A = \{(x, \varphi) \in U \times \mathbb{C}^{m\times n} : \varphi = A(x)\}$ be the *graph* of A. This is a subvariety of the product $U \times \mathbb{C}^{m\times n}$ defined by the $m \cdot n$ equations $h_{i,j} = y_{i,j} - a_{i,j}(x) \in \mathbb{C}\{x\}[y]$. A close inspection of the arguments of Eagon and Northcott show that $\langle A^{\wedge s}\rangle$ has expected codimension if and only if the equations $h_{i,j}$ form a regular sequence on the coordinate ring $\mathbb{C}\{x\}[y]/\langle Y^{\wedge s}\rangle$ of the product variety $(\mathbb{C}^p, 0) \times (M^s_{m,n}, 0)$.

It has been shown in [33, Proposition 2] that the homology of the complex $K(m, n, s) \otimes_{\mathbb{Z}[y]} \mathbb{C}\{x\}$ is isomorphic to the *Koszul homology*

$$H_i\left(K(m, n, s) \otimes_{\mathbb{Z}[y]} \mathbb{C}\{x\}\right) \cong H_i\left(\operatorname{Kosz}(h, \mathbb{C}\{x\}[y]/\langle Y^{\wedge s}\rangle)\right)$$

of the functions $h_{i,j}$ on the $\mathbb{C}\{x\}[y]/\langle Y^{\wedge s}\rangle$. We saw earlier in Example 2.1.10 that the Koszul homology vanishes when $h_{1,1}, \dots, h_{m,n}$ is a regular sequence on $\mathbb{C}\{x\}[y]/\langle Y^{\wedge s}\rangle$. Conversely, in a local ring, the Koszul homology can be used to measure the grade of an ideal: If the right hand side vanishes for all $i > 0$ one has grade $\langle h\rangle = m \cdot n$, cf. [13, Theorem 1.6.17]. Since $\mathbb{C}\{x\}[y]$ is a graded ring over a local ring, this implies that the $h_{1,1}, \dots, h_{m,n}$ have to be a regular sequence already, cf. [13, Corollary 1.6.19]. From this we see that $K(m, n, s) \otimes_{\mathbb{Z}[y]} \mathbb{C}\{x\}$ is a free resolution of $\mathbb{C}\{x\}/\langle A^{\wedge s}\rangle$ if and only if the equations $h_{i,j} = y_{i,j} - a_{i,j}(x)$ for the graph Γ_A form a regular sequence on $\mathbb{C}\{x\}[y]/\langle Y^{\wedge s}\rangle$. This allows us to always consider $(X_A^s, 0)$ as a complete intersection singularity in the singular, but yet well known ambient space $(\mathbb{C}^p, 0) \times (M_{m,n}^s, 0)$.

The interpretation of a matrix $A \in \mathbb{C}\{x\}^{m\times n}$ as a map germ is of particular interest when it comes to *deformations*[5] of the germ $(X_A^s, 0)$. Recall that for hypersurface and complete intersection singularities $(X, 0) \subset (\mathbb{C}^p, 0)$ of codimension m and a map $f\colon (\mathbb{C}^p, 0) \to (\mathbb{C}^m, 0)$ defining $(X, 0) = (f^{-1}(\{0\}), 0)$, an *unfolding* of f gives rise to a *deformation* of $(X, 0)$, cf. [20, Proposition 7.1.11]. The generic perfection of determinantal ideals is crucial for proving that, more generally, an unfolding of the matrix A gives rise to a deformation of the associated determinantal singularity, see Lemma 2.1.15. A schematic picture of such an induced deformation is given in Fig. 2.1. We briefly recall the notions involved; for a more thorough discussion of unfoldings and deformations, the reader is referred to [20, Sections 7.1 and 7.2].

An *unfolding* on k parameters of a map germ $f\colon (\mathbb{C}^p, 0) \to (\mathbb{C}^n, 0)$ is given by a map

$$F\colon (\mathbb{C}^p, 0) \times (\mathbb{C}^k, 0) \to (\mathbb{C}^n, 0) \times (\mathbb{C}^k, 0), \quad (x, t) \mapsto (F(x, t), t) = (f_t(x), t)$$

such that for $t = 0$ the germ f_0 coincides with f. In practice, an unfolding is nothing but a perturbation

$$F(x, t) = f(x) + t_1 \cdot g_1(x, t) + t_2 \cdot g_2(x, t) + \dots$$

of the original map germ f.

An (embedded) *deformation* of a complex analytic germ $(X, 0) \subset (\mathbb{C}^p, 0)$ over a germ $(T, 0)$ is given by a commutative diagram

$$\begin{array}{ccccc} (X,0) & \hookrightarrow & (\mathcal{X},0) & \hookrightarrow & (\mathbb{C}^p,0)\times(T,0) \\ \downarrow & & \downarrow \pi & \swarrow & \\ \{0\} & \hookrightarrow & (T,0) & & \end{array}$$

[5] See [20, Definition 7.1.1] for a definition of deformations of complex analytic germs.

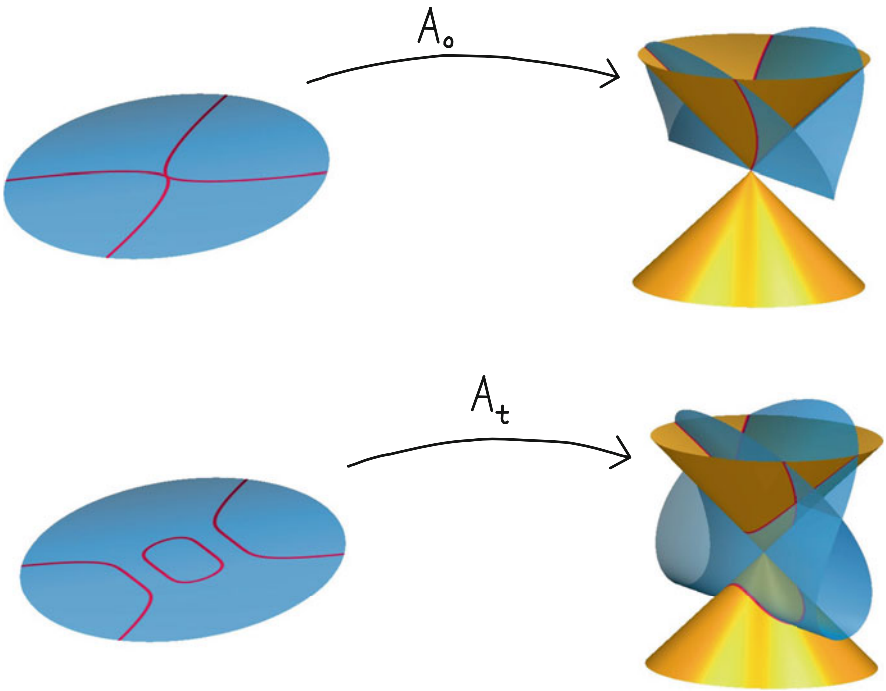


Fig. 2.1 A schematic picture of a matrix A_0, the intersection of its image with a singular variety $M^s_{m,n}$ in its codomain (pictured as a double cone) and the preimage of that variety $(X^s_A, 0)$. The second row shows a fiber $X^s_{\mathbf{A}}(t) = A_t^{-1}(M^s_{m,n})$ in the deformation of that singularity which is induced from an unfolding $\mathbf{A}(x, t) = A_t(x)$ of A_0

where $(\mathscr{X}, 0) \subset (\mathbb{C}^p, 0) \times (T, 0)$ is another complex analytic germ such that the projection π turns $\mathscr{O}_{\mathscr{X},0}$ into a *flat* $\mathscr{O}_{S,0}$-module so that the above diagram becomes a *flat family* with special fiber $(X, 0)$.

Flatness of a family is a technical algebraic criterion to assure that the fibers in a family vary nicely; for example, there can be no jumps in dimension of the fibers in a flat family, see [35]. When $f_1, \dots, f_n \in \mathbb{C}\{x\}$ are equations defining $(X, 0)$ as above, then an arbitrary perturbation of these defining equations given by functions

$$F_i(x, t) = f_i(x) + \sum_{j=1} t_j \cdot g_j(x, t) \in \mathbb{C}\{x, t\}, \quad i = 1, \dots, n,$$

in additional parameters $t = t_1, \dots, t_k$, will in general not lead to a flat family

$$(\{F_1 = \dots F_n = 0\}, 0) = (\mathscr{X}, 0) \xrightarrow{\pi} (\mathbb{C}^k, 0)$$

unless the f_i form a regular sequence, cf. Example 2.1.17 below.

A prominent characterization of flatness in the general case is the “Flatness by relations”, [20, Proposition 7.1.2]. Applied to the above situation it says that the family $\pi : (\mathscr{X}, 0) \to (\mathbb{C}^k, 0)$ is flat if and only if for every relation $r \in \mathbb{C}\{x\}^n$ among the f_i of the form

$$r_1 \cdot f_1 + r_2 \cdot f_2 + \cdots + r_n \cdot f_n = 0,$$

i.e. the *first syzygies* of the f_i, there exists a relation $R \in \mathbb{C}\{x, t\}^n$ of the F_i

$$R_1 \cdot F_1 + R_2 \cdot F_2 + \cdots + R_n \cdot F_n = 0$$

with R_i congruent to r_i modulo $\langle t \rangle$. In general, not every perturbation F_i of the f_i admits such a “lifting of relations”. For determinantal singularities, however, the inheritance of free resolutions granted by Theorem 2.1.8 assures that flatness by relations always applies:

Lemma 2.1.15 *Let $A \in \mathbb{C}\{x_1, \ldots, x_p\}^{m\times n}$ be a matrix defining a determinantal singularity $(X^s_A, 0) \subset (\mathbb{C}^p, 0)$ of type s. Then any unfolding*

$$(\mathbb{C}^p, 0) \times (\mathbb{C}^k, 0) \to (\mathbb{C}^{m\times n}, 0) \times (\mathbb{C}^k, 0), \quad (x, t) \mapsto (\mathbf{A}(x, t), t)$$

of A on k parameters induces a deformation

$$\begin{array}{ccccc} (X^s_A, 0) & \hookrightarrow & (\mathscr{X}^s_{\mathbf{A}}, 0) & \hookrightarrow & (\mathbb{C}^p, 0) \times (\mathbb{C}^k, 0) \\ \downarrow & & \downarrow \pi & \swarrow & \\ \{0\} & \hookrightarrow & (\mathbb{C}^k, 0) & & \end{array}$$

of the germ $(X^s_A, 0)$.

Proof The crucial point is to verify flatness of the family $(X^s_{\mathbf{A}}, 0) \xrightarrow{\pi} (\mathbb{C}^k, 0)$, where $(X^s_{\mathbf{A}}, 0) \subset (\mathbb{C}^p \times \mathbb{C}^k, 0)$ is the complex analytic germ defined by the ideal $\mathbf{I} = \langle \mathbf{A}^{\wedge s} \rangle$ in the ring $\mathbb{C}\{x_1, \ldots, x_p, t_1, \ldots, t_k\}$. Let $I = \langle A^{\wedge s} \rangle \subset \mathbb{C}\{x\}$ be the ideal defining $X^s_A \cong \mathscr{X}^s_{\mathbf{A}} \cap \{t = 0\}$. We have an inequality of dimensions

$$\dim \mathbb{C}\{x, t\}/\mathbf{I} \le \dim \mathbb{C}\{t\} + \dim \mathbb{C}\{x\}/I,$$

see [35, Theorem 10.10], where we identify $\mathbb{C}\{x, t\}/\mathbf{I} + \langle t \rangle \cong \mathbb{C}\{x\}/I$. Since $(X^s_A, 0) \subset (\mathbb{C}^p, 0)$ has expected codimension $c = (m - s + 1)(n - s + 1)$, this inequality entails that also $(\mathscr{X}^s_{\mathbf{A}}, 0)$ must have expected codimension c in $(\mathbb{C}^p, 0) \times (\mathbb{C}^k, 0)$. Due to the generic perfection of the determinantal ideals, a free resolution of $\mathbb{C}\{x, t\}/\mathbf{I}$ is given by $K_\bullet(m, n, s) \otimes_{\mathbf{A}} \mathbb{C}\{x, t\}$ according to Theorem 2.1.8. This resolution specializes to one of $\mathbb{C}\{x\}/I$ at $t = 0$ by using the same theorem again. This implies in particular flatness of the family by evoking its characterization via the relation lifting property (see e.g. [20, Proposition 7.1.2]) in the following way: A relation among elements of I (and $\mathbf{I}$ respectively) is an element of the corresponding first syzygy module, which can be read off as the image of the rightmost morphism in the given free resolution. As the resolution of $\mathbb{C}\{x,t\}/\mathbf{I}$ specializes to the one of $\mathbb{C}\{x\}/I$, clearly any element of the first syzygy module of I arises in this way, i.e. lifts to one of the first syzygy module of $\mathbf{I}$. □

Definition 2.1.16 Let $(X_A^s, 0) \subset (\mathbb{C}^p, 0)$ be a *determinantal* singularity of type s defined by a matrix $A \in \mathbb{C}\{x\}^{m\times n}$. Any deformation induced from an unfolding of A as in Lemma 2.1.15 is called a *determinantal deformation* of $(X_A^s, 0)$.

Conversely, we will say that a given deformation $(X,0) \hookrightarrow (\mathscr{X},0) \xrightarrow{\pi} (S,0)$ of an *arbitrary* singularity $(X,0) \subset (\mathbb{C}^p,0)$ is *determinantal for* A, if there exists an integer s and a matrix $A \in \mathbb{C}\{x\}^{m\times n}$ such that $(X,0) \cong (X_A^s,0)$ is determinantal for A of type s and an unfolding $\mathbf{A}\colon (\mathbb{C}^p,0)\times(\mathbb{C}^k,0) \to (\mathbb{C}^{m\times n},0)$ of A together with a commutative diagram

$$\begin{array}{ccccc}
(X,0) & \hookrightarrow & (\mathscr{X},0) & \longrightarrow & (\mathscr{X}_{\mathbf{A}}^s,0) \\
\downarrow & & \downarrow{\scriptstyle\pi} & & \downarrow \\
\{0\} & \hookrightarrow & (S,0) & \xrightarrow{\Psi} & (\mathbb{C}^k,0)
\end{array}$$

where $(\mathscr{X}_{\mathbf{A}}^s,0) \to (\mathbb{C}^k,0)$ is the family induced from $\mathbf{A}$ as in Lemma 2.1.15.

Example 2.1.17 Consider the three coordinate axes in $(\mathbb{C}^3,0)$, which form a determinantal singularity of type $s=2$ with matrix

$$A = \begin{pmatrix} x & 0 & z \\ 0 & y & z \end{pmatrix}.$$

The corresponding ideal is $\langle A^{\wedge 2}\rangle = \langle -yz, -xz, xy\rangle =: \langle f_1, f_2, f_3\rangle$. If we think of these generators as a map

$$f = (f_1, f_2, f_3) = A^{\wedge 2}\colon \mathbb{C}^3 \to \mathbb{C}^3,$$

then according to Sard's theorem [89], the set of points $v \in \mathbb{C}^3$ for which $f^{-1}(\{v\})$ is regular of complex codimension 3, is dense. Hence a generic perturbation of the f_i will "deform" the curve $(X_A^2, 0)$ to a collection of points.

Consider now the 1-parameter unfolding of X_A^s determined by the perturbed matrix

$$\mathbf{A} = \begin{pmatrix} x & t & z \\ 0 & y & z \end{pmatrix}$$

and the corresponding ideal $\langle \mathbf{A}^{\wedge 2} \rangle = \langle (t-y)z, -xz, xy \rangle \subset \mathbb{C}\{x, y, z, t\}$. This is a very specific perturbation of the generators given by

$$F_1 = f_1 + t \cdot z, \quad F_2 = f_2 + t \cdot 0, \quad F_3 = f_3 + t \cdot 0.$$

According to Lemma 2.1.15, the projection to the parameter t

$$(\mathscr{X}^s_{\mathbf{A}}, 0) \to (\mathbb{C}, 0), \quad (x, y, z, t) \mapsto t$$

is indeed a deformation of $(X^s_A, 0)$. This is because all relations among the f_i arise from the following two:

$$\begin{aligned} x \cdot f_1 + 0 \cdot f_2 + z \cdot f_3 = 0 \\ 0 \cdot f_1 + y \cdot f_2 + z \cdot f_3 = 0 \end{aligned}$$

and these lift to relations

$$\begin{aligned} x \cdot F_1 + t \cdot F_2 + z \cdot F_3 = 0 \\ 0 \cdot F_1 + y \cdot F_2 + z \cdot F_3 = 0 \end{aligned}$$

among the F_i.

Geometrically, this manifests itself in the fact that the fiber over $t \neq 0$ is indeed also of complex dimension 1 (Fig. 2.2). In this sense, Lemma 2.1.15 assures that of all possible perturbations of the generators f_i the very few ones which arise from perturbations of the matrix A are similarly well behaved.

In general, not every deformation of a given determinantal singularity is determinantal for the specific matrix and the interplay between unfoldings of matrices and the deformations of the associated singularities can be quite complicated. This is

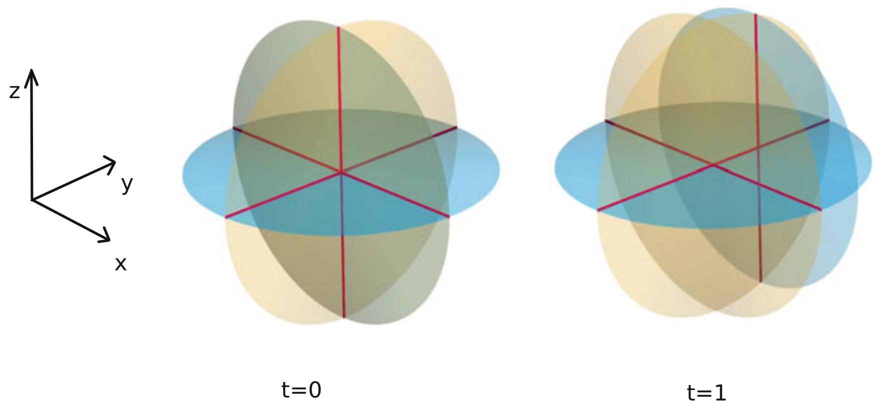


Fig. 2.2 A deformation of the three coordinate axis in $\mathbb{C}^3$

illustrated by several examples gathered in Sect. 2.3.2. For some particular classes of singularities such as complete intersections, Cohen-Macaulay singularities of codimension 2, or Gorenstein singularities of codimension 3, there are canonical choices of determinantal structures and the theory of unfoldings for the defining matrices indeed agrees with the deformation theory of the germs. These particular cases will be discussed in detail in Sect. 2.3.2. Finally, as we will report in Sect. 2.3.7, results of Buchweitz [18] and Svanes [94] imply that for a given determinantal singularity $(X_A^s, 0)$ which is not a hypersurface and which is *unobstructed*,[6] in fact every deformation of $(X_A^s, 0)$ is determinantal for the defining matrix A.

However, before we can present all these discussions in their full detail, we first need to develop the underlying notions of equivalence for unfoldings of matrices and the associated concepts of finite determinacy and versal unfoldings in Sect. 2.2.

2.1.4 Geometry of the Generic Determinantal Varieties

The preceding sections made it clear that the generic determinantal varieties $M_{m,n}^s$ play a fundamental role in the study of arbitrary determinantal varieties and singularities. As we shall see in what follows, this does not only apply to their algebraic, but also to their geometric and even their topological properties when working over the real, or the complex numbers. In this section we gather some auxiliary results on the geometry of the generic determinantal varieties $M_{m,n}^s(\mathbb{K}) \subset \mathbb{K}^{m \times n}$ where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. For some of the results, the reader can also make the obvious translations to other fields, if needed.

2.1.4.1 Resolution of Singularities for $M_{m,n}^s$

We start by observing that the generic determinantal varieties admit certain canonical resolutions of singularities

$$\begin{array}{ccccc} & & \tilde{M}_{m,n}^s & & \\ & \swarrow & \downarrow \tilde{\nu} & \searrow & \\ \hat{M}_{m,n}^s & & & & \check{M}_{m,n}^s \\ & \searrow^{\hat{\nu}} & & \swarrow_{\check{\nu}} & \\ & & M_{m,n}^s & & \end{array} \qquad (2.10)$$

---

[6] For the definition of obstructions in the context of deformation theory see e.g. [20, Section 7.1.5].

where

$$\begin{aligned}\hat{M}^s_{m,n} &= \left\{(\varphi, V) \in \mathbb{K}^{m\times n} \times \mathrm{Grass}(n-s+1,n) : V \subset \ker\varphi\right\} \\ &\cong \left\{(\varphi, V^\perp) \in \mathbb{K}^{m\times n} \times \mathrm{Grass}(s-1,n) : \mathrm{im}\,\varphi^\vee \subset V^\perp\right\}\end{aligned}$$

is the *Tjurina transform* with its projection $\hat{\nu}\colon (\varphi, W) \mapsto \varphi$,

$$\begin{aligned}\check{M}^s_{m,n} &= \left\{(\varphi, W) \in \mathbb{K}^{m\times n} \times \mathrm{Grass}(s-1,m) : \mathrm{im}\,\varphi \subset W\right\} \\ &\cong \left\{(\varphi, W^\perp) \in \mathbb{K}^{m\times n} \times \mathrm{Grass}(m-s+1,m)\colon W^\perp \subset \ker\varphi^\vee\right\}\end{aligned}$$

the *dual Tjurina transform* with projection $\check{\nu}$, and

$$\begin{aligned}\tilde{M}^s_{m,n} &= \hat{M}^s_{m,n} \times_{M^s_{m,n}} \check{M}^s_{m,n} \\ &= \{(\varphi, V, W) \in \mathbb{K}^{m\times n} \times \mathrm{Grass}(n-s+1,n) \times \mathrm{Grass}(s-1,m) : \\ &\qquad\qquad V \subset \ker\varphi,\ \mathrm{im}\,\varphi \subset W\}\end{aligned}$$

the *Nash transform* of $M^s_{m,n}$ with projection $\tilde{\nu}$.

Here we used the canonical isomorphisms of dual Grassmannians $\mathrm{Grass}(r,n) \cong \mathrm{Grass}(n-r,n)$ induced by the correspondence

$$V \mapsto V^\perp = \{f \in (\mathbb{K}^n)^\vee : f|_V = 0\}$$

with $(\mathbb{K}^n)^\vee = \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^n, \mathbb{K})$. The natural isomorphism

$$\mathrm{Hom}(\mathbb{K}^n, \mathbb{K}^m) \to \mathrm{Hom}\left((\mathbb{K}^n)^\vee, (\mathbb{K}^m)^\vee\right), \quad \varphi \mapsto \varphi^\vee,$$

which is given by transposition $\mathbb{K}^{m\times n} \xrightarrow{\cong} \mathbb{K}^{n\times m}$, $A \mapsto A^T$ in terms of matrices, takes $M^s_{m,n}$ into $M^s_{n,m}$. It is now easy to see that this identification extends to the Tjurina transforms and their duals so that $\hat{M}^s_{m,n} \cong M^s_{m,n} \times_{M^s_{n,m}} \check{M}^s_{n,m}$ and $\check{M}^s_{m,n} \cong M^s_{m,n} \times_{M^s_{n,m}} \hat{M}^s_{n,m}$.

Either one of the varieties $\hat{M}^s_{m,n}$, $\check{M}^s_{m,n}$ and $\tilde{M}^s_{m,n}$ is the total space of an algebraic vector bundle over the respective Grassmannians. For instance, if we let $0 \to S \to \mathscr{O}^n \to Q \to 0$ be the tautological sequence over $\mathrm{Grass}(n-s+1,n)$, then

$$\hat{M}^s_{m,n} \cong \left|\mathrm{Hom}(Q, \mathscr{O}^m)\right|,$$

where we write $|E|$ for taking the associated total space of a vector bundle E. Similar descriptions can be made for the dual Tjurina transform $\check{M}^s_{m,n}$ and the Nash transform. In particular, all of the three spaces $\hat{M}^s_{m,n}$, $\check{M}^s_{m,n}$, and $\tilde{M}^s_{m,n}$ are smooth.

Remark 2.1.18 That $\tilde{\nu}\colon \tilde{M}^s_{m,n} \to M^s_{m,n}$ is really the Nash modification in the sense of e.g. [20, Definition 3.9.2] was observed by Ebeling and Gusein-Zade in [34]. It is a well known fact that for any fixed rank r the tangent space to the stratum $V^r_{m,n}$ at a point φ is given by

$$T_\varphi V^r_{m,n} = \{\psi \in \mathbb{K}^{m\times n} : \psi(\ker\varphi) \subset \operatorname{im}\varphi\}.$$

It follows that the Gauss map $\gamma\colon V^r_{m,n} \to \mathrm{Grass}((m+n)r - r^2, mn)$ taking a point φ to its tangent space $T_\varphi V^r_{m,n} \subset T_\varphi \mathbb{K}^{m\times n}$ factors through the product

$$V^r_{m,n} \xrightarrow{\alpha} \mathrm{Grass}(n-r, n) \times \mathrm{Grass}(r, m) \xrightarrow{\beta} \mathrm{Grass}((m+n)r - r^2, mn), \qquad \gamma = \beta\circ\alpha$$

where $\alpha\colon \varphi \mapsto (\ker\varphi, \operatorname{im}\varphi)$ and $\beta\colon (V, W) \mapsto \mathrm{Hom}(\mathbb{K}^n/V, W) \subset \mathrm{Hom}(\mathbb{K}^n, \mathbb{K}^m)$, and consequently, the Nash blowup of $M^{r+1}_{m,n}$ can be performed using this product of Grassmannians.

For any integer r we let

$$V^r_{m,n} = \{\varphi \in \mathbb{K}^{m\times n} : \operatorname{rank}\varphi = r\}$$

be the set of matrices of fixed rank r. Whenever $\varphi \in M^s_{m,n}$ belongs to $V^{s-1}_{m,n}$, the spaces V and W in the definitions of $\hat{M}^s_{m,n}$, $\check{M}^s_{m,n}$ and $\tilde{M}^s_{m,n}$ above are uniquely determined by the kernel and the image of φ and vary algebraically with φ. Hence, either one of the three projections $\hat{\nu}$, $\check{\nu}$, and $\tilde{\nu}$ is a local isomorphism over the dense, open subset $V^{s-1}_{m,n} \subset M^s_{m,n}$. It is easy to see that $M^s_{m,n}$ is singular along the complement $M^{s-1}_{m,n}$ of $V^{s-1}_{m,n}$ in $M^s_{m,n}$ so that indeed either one of the three maps $\hat{\nu}$, $\check{\nu}$, and $\tilde{\nu}$ provides a resolution of singularities.[7]

2.1.4.2 The Rank Stratification

The sets $V^r_{m,n}$ of matrices of a fixed rank r form a complex algebraic *stratification* of $\mathbb{K}^{m\times n}$, that is a decomposition as a disjoint union of locally closed, complex algebraic submanifolds

$$\mathbb{K}^{m\times n} = \bigcup_{r=0}^{\min\{m,n\}} V^r_{m,n}. \tag{2.11}$$

[7] See for instance [20, Chapter 3] for a definition of resolution of singularities.

In particular, every $M^s_{m,n} = \bigcup_{r<s} V^r_{m,n}$ is a union of strata. We will in the following refer to this particular stratification of the space of matrices as the *rank stratification*. For an account on stratification theory see, for instance, [20, Chapter 4], or [45].

Left- and right-multiplication by invertible matrices does not change the rank of a matrix, so the action of the group $G = \mathrm{GL}(n; \mathbb{K}) \times \mathrm{GL}(m; \mathbb{K})$ given by

$$G \times \mathbb{K}^{m\times n} \to \mathbb{K}^{m\times n}, \quad ((P, Q), \varphi) \mapsto (P, Q) * \varphi = P \cdot \varphi \cdot Q^{-1} \tag{2.12}$$

preserves the rank stratification. Using this action, one can even construct local analytic trivializations: Let $\varphi \in \mathbb{K}^{m\times n}$ be an arbitrary matrix and r its rank so that $\varphi \in V^r_{m,n}$. By virtue of the G-action, we may assume that φ is of the block form

$$\varphi = \left(\begin{array}{c|c} \mathbf{1}_r & 0 \\ \hline 0 & 0 \end{array}\right) \in \mathbb{K}^{m\times n}$$

where by $\mathbf{1}_r$ we denote the $r \times r$ unit matrix. It is now easy to see that the map

$$\Phi\colon (\mathrm{GL}(r; \mathbb{K}), \mathbf{1}_r) \times (\mathbb{K}^{(m-r)\times r}, 0) \times (\mathbb{K}^{r\times(n-r)}, 0) \times (\mathbb{K}^{(m-r)\times(n-r)}, 0) \to \mathbb{K}^{m\times n},$$

taking a tuple (A, P, Q, N) to the block matrix

$$\left(\begin{array}{c|c} \mathbf{1}_r & 0 \\ \hline P\cdot A^{-1} & \mathbf{1}_{m-r} \end{array}\right) \cdot \left(\begin{array}{c|c} A & 0 \\ \hline 0 & N \end{array}\right) \cdot \left(\begin{array}{c|c} \mathbf{1}_r & A^{-1}\cdot Q \\ \hline 0 & \mathbf{1}_{n-r} \end{array}\right) = \left(\begin{array}{c|c} A & Q \\ \hline P & PA^{-1}Q + N \end{array}\right)$$

yields a local isomorphism $(\mathbb{K}^{(m+n)r-r^2}, 0) \xrightarrow{\cong} (V^r_{m,n}, \varphi)$ by setting $N = 0$ and restricting to the first three factors. The entries of $N \in \mathbb{K}^{(m-r)\times(n-r)}$ can be understood as normal coordinates to that stratum and using Lemma 2.1.3 we find that

$$\Phi\colon (\mathbb{K}^{(m+n)r-r^2}, 0) \times (M^{s-r}_{m-r,n-r}, 0) \xrightarrow{\cong} (M^s_{m,n}, \varphi) \tag{2.13}$$

is a stratification preserving isomorphism for every $s > r$. In particular, this local analytic triviality implies that the rank stratification satisfies Whitney's conditions (a) and (b), cf. [20, Chapter 4.2], so that it is in fact a *Whitney stratification* of the space of matrices.

2.1.4.3 Logarithmic Vector Fields in $\mathbb{C}^{m\times n}$

The G-action (2.12) on the space of matrices can also be exploited to construct so-called logarithmic vector fields, see [23] and [88], which will be important for the theory of unfoldings later on.

Let $U \subset \mathbb{C}^n$ be an open domain and $V \hookrightarrow U$ the closed embedding of a complex analytic subvariety. The module of logarithmic vector fields[8] for V at a point $q \in U$ is given by

$$\mathrm{Der}(-\log(V)) = \{\xi \in T_{\mathbb{C}^n,q} : \xi(I) \subset I\} \subset T_{\mathbb{C}^n,q}$$

where $I = I(V)$ denotes the ideal of functions vanishing on V. These form a coherent sheaf on U which coincides with the tangent sheaf T_U outside V. In the real case, a similar construction can be made whenever the ideal I is coherent, see [23]. We will restrict our exposition to the complex analytic setup.

The *logarithmic tangent space* to V at a point $q \in V$ is defined as the subspace

$$T_q^{\log} V = \mathrm{Der}(-\log(V))/(\mathrm{Der}(-\log(V)) \cap \mathfrak{m}_q T_{\mathbb{C}^n,q}) \subset T_{\mathbb{C}^n,q}/\mathfrak{m}_q T_{\mathbb{C}^n,q} = T_q\mathbb{C}^n \tag{2.14}$$

of vectors $v \in T_q\mathbb{C}^n$ which extend locally to logarithmic vector fields for V in $\mathbb{C}^n$. It is easy to check that whenever q is a smooth point of V then one has $T_q V = T_q^{\log} V$. At singular points very little is known about $T_q^{\log} V$ in general, but if we endow V with its *canonical* Whitney stratification as in [101], $V = \bigcup_{i=0}^m V^i$, then Damon and Mond have shown in [28, Proposition 3.11] that the logarithmic tangent space to V at a point q

$$T_q^{\log} V \subset T_q V^i \tag{2.15}$$

is contained in the tangent space of the stratum V^i containing q so that logarithmic vector fields are always tangent to the strata of the *canonical* Whitney stratification. However, one need not have equality in (2.15) and those strata V^i for which equality holds at all points are called *holonomic strata*.

The purpose of this section is to show:

Lemma 2.1.19 *The strata $V^r_{m,n} \subset M^s_{m,n} \subset \mathbb{C}^{m\times n}$ of the rank stratification of the generic determinantal varieties are holonomic for all values of $r < s \leq \min\{m, n\}$.*

To this end, let $\mathfrak{gl}_m \oplus \mathfrak{gl}_n \cong \mathbb{C}^{m\times m} \oplus \mathbb{C}^{n\times n}$ be the Lie-algebra of $G = \mathrm{GL}(m, \mathbb{C}) \times \mathrm{GL}(n; \mathbb{C})$. For any vector $(L, R) \in \mathfrak{gl}_m \oplus \mathfrak{gl}_n$ the exponential map gives rise to a holomorphic 1-parameter family $t \mapsto \exp(t \cdot (L, R))$ in G. This induces a family of stratification preserving automorphisms

$$\gamma : (\mathbb{C}, 0) \times \mathbb{C}^{m\times n} \to \mathbb{C}^{m\times n}, \quad (t, \varphi) \mapsto \gamma_t(\varphi) = \exp(t \cdot (L, R)) * \varphi.$$

via the G-action on $\mathbb{C}^{m\times n}$ and we denote by

$$\xi_{(L,R)}(\varphi) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} \exp(t \cdot (L, R)) * \varphi = L \cdot \varphi - \varphi \cdot R \tag{2.16}$$

[8] We use the notation $\mathrm{Der}(-\log(V))$ rather than $\mathrm{Der}\log(V)$ following [46].

the vector field on $\mathbb{C}^{m\times n}$ generated by that action. Here, we deliberately identified $T_\varphi\mathbb{C}^{m\times n} \cong \mathbb{C}^{m\times n}$ with the space of matrices again.

Proof of Lemma 2.1.19 We first remark that the rank stratification $M^s_{m,n} = \bigcup_{r<s} V^r_{m,n}$ always coincides with the canonical Whitney stratification constructed by Lê and Teissier.

The vector fields $\xi = \xi_{(L,R)}$ constructed above are logarithmic for $M^s_{m,n}$: If we let $\exp(t \cdot (L, R)) = (P_t, Q_t)$ be the associated 1-parameter family in G and $\gamma : (t, \varphi) \mapsto \gamma_t(\varphi)$ the induced family of automorphisms of $\mathbb{C}^{m\times n}$, then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(Y \circ \gamma_t)^{\wedge s} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(P_t \cdot Y \cdot Q_t^{-1}\right)^{\wedge s} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(P_t^{\wedge s} \cdot Y^{\wedge s} \cdot \left(Q_t^{-1}\right)^{\wedge s}\right) \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}t}P_t\right)^{\wedge s} \cdot Y^{\wedge s} \cdot (Q_t^{-1})^{\wedge s} + P_t^{\wedge s} \cdot Y^{\wedge s} \cdot \left(\frac{\mathrm{d}}{\mathrm{d}t}Q_t^{-1}\right)^{\wedge s}
\end{aligned}$$

and therefore, setting $t = 0$, clearly $\xi(Y^{\wedge s}_{\mathbf{i},\mathbf{j}}) \in \langle Y^{\wedge s}\rangle$ for every entry $Y^{\wedge s}_{\mathbf{i},\mathbf{j}}$ of $\langle Y^{\wedge s}\rangle$. The claim now follows from the observation that, since every stratum $V^r_{m,n}$ is a G-orbit in $\mathbb{C}^{m\times n}$, the linear map

$$\mathfrak{gl}_m \oplus \mathfrak{gl}_n \to T_\varphi V^r_{m,n}, \quad (L, R) \mapsto \xi_{(L,R)}(\varphi)$$

is surjective at every point $\varphi \in V^r_{m,n}$. □

Remark 2.1.20 Note that even though $T^{\log}_\varphi M^s_{m,n} = T_\varphi V^r_{m,n}$ for every $\varphi \in V^r_{m,n} \subset M^s_{m,n}$, the proof of Lemma 2.1.19 does *not* imply that the module $\mathrm{Der}(-\log(M^s_{m,n}))$ is generated by the vector fields $\xi_{(L,R)}$ in (2.16). This is true for the varieties $M^m_{m,m}$ of degenerate square, symmetric, and skew-symmetric matrices, cf. [11, 12], and [47], but false for the generic determinantal varieties $M^s_{m,n}$ defined by non-maximal minors for $s < \min\{m, n\}$: Considering $V^{s-1}_{m,n}$ as a stratum of $M^s_{m,n}$ only, we may extend any tangent vector field ζ to $V^{s-1}_{m,n}$ in an arbitrary way to a vector field on a neighborhood in $\mathbb{C}^{m\times n}$ while an extension as a linear combination of the $\xi_{(L,R)}$'s will necessarily be tangent to the orbits $V^t_{m,n}$ for $t \geq s$, as well. Thus, in general, the vector fields $\xi_{(L,R)}$ only generate a submodule of $\mathrm{Der}(-\log(M^s_{m,n}))$ for non-maximal s.

2.2 Unfoldings and Equivalence of Matrices

According to the previous section, the unfoldings of a matrix A determine the determinantal deformations of the associated singularities. In practice, unfoldings of map germs are far easier to handle and to classify than deformations; for instance there is no obstruction theory[9] to be taken into account. The paradigm of the following sections is to consider determinantal singularities as hybrid objects living in both the world of unfoldings of map germs and the world of deformations of space germs at the same time, and to consider unfoldings of matrices as a handle for studying deformations of the associated determinantal singularities. The appropriate framework for this was developed by Damon in [22] where he defines the notion of a "geometric subgroup" of the contact group $\mathscr{K}$. The natural notion of equivalence for germs of matrices —the so-called GL-equivalence defined below— leads to such a geometric subgroup as was observed in [56], [11, 12, 47] for square matrices and in [21] for the general case. In principal, all relevant theorems about finite determinacy, versality, etc. can be derived from that. For a general account on map germs, their unfoldings, finite determinacy and related techniques see for instance [109], or [82].

In this note we will give the specific statements for GL-equivalence of matrices in the complex analytic category with a focus on the explicit description of the infinitesimal theory. Many constructions and results can be carried over to the real analytic or even the real differentiable setup. Moreover, in [7] Belitskii and Kerner develop an analogous theory of unfoldings, equivalences, and finite determinacy in a purely algebraic fashion; particular applications to families of matrices can be found in [8] and [64] and various further preprints are available.

We start with the natural notion of equivalence of map germs in this context.

Definition 2.2.1 Two matrices $A, B \in \mathbb{C}\{x_1, \dots, x_p\}^{m\times n}$ are called GL-equivalent, if there exist matrices $P \in \mathrm{GL}(m; \mathbb{C}\{x\})$ and $Q \in \mathrm{GL}(n; \mathbb{C}\{x\})$ and a biholomorphism $\Phi\colon (\mathbb{C}^p, 0) \to (\mathbb{C}^p, 0)$ such that

$$P(x) \cdot \left(A \circ \Phi^{-1}(x)\right) \cdot (Q(x))^{-1} = B(x). \tag{2.17}$$

Note that in (2.17) we write $\Phi^{-1}(x)$ for the inverse of the map Φ applied to x while $(Q(x))^{-1}$ denotes the inverse of the matrix $Q(x)$ at that point.

Similarly, we say that A and B are SL-equivalent if P and Q only take values in the special linear groups. In the following, we will deliberately identify the space of matrices $\mathrm{GL}(m; \mathbb{C}\{x\})$ with the space of map germs $P\colon (\mathbb{C}^p, 0) \to (\mathrm{GL}(m; \mathbb{C}), P(0))$. Depending on the context, either one of the two notations has its advantages.

Remark 2.2.2 Note that in the above definition, neither A nor B are required to define a determinantal singularity. If they do, however, then it follows directly from Lemma 2.1.3 that any two associated determinantal singularities $(X^s_A, 0)$ and $(X^s_B, 0)$ are isomorphic as germs.

[9] cf. [20, Section 7.1.5].

Definition 2.2.3 Let $A \in \mathbb{C}\{x_1, \ldots, x_p\}^{m \times n}$ be a matrix. Two unfoldings of A on k parameters $t = t_1, \ldots, t_k$ given by $\mathbf{A}(x, t), \mathbf{B}(x, t) \in \mathbb{C}\{x, t\}^{m \times n}$ are called GL-equivalent if there exist unfoldings

$$
\begin{aligned}
(\mathbb{C}^p, 0) \times (\mathbb{C}^k, 0) \to (\mathrm{GL}(m, \mathbb{C}), \mathbf{1}_m), \qquad (x, t) \mapsto P_t(x) \\
(\mathbb{C}^p, 0) \times (\mathbb{C}^k, 0) \to (\mathrm{GL}(n, \mathbb{C}), \mathbf{1}_n), \qquad (x, t) \mapsto Q_t(x) \\
(\mathbb{C}^p, 0) \times (\mathbb{C}^k, 0) \to (\mathbb{C}^p, 0), \qquad (x, t) \mapsto \Phi_t(x)
\end{aligned}
$$

of the identities $P_0 = \mathbf{1}_m \in \mathrm{GL}(m; \mathbb{C}\{x\})$, $Q_0 = \mathbf{1}_n \in \mathrm{GL}(n; \mathbb{C}\{x\})$ and $\Phi_0 = \mathrm{Id}_{\mathbb{C}^p,0}$ such that

$$
\mathbf{B}(x, t) = P_t(x) \cdot \mathbf{A}(\Phi_t^{-1}(x), t) \cdot (Q_t(x))^{-1}
$$

An unfolding given by $\mathbf{A}(x, t)$ is GL-trivial (or just "trivial") if it is GL-equivalent (as unfoldings) to the map $\mathbf{B}(x, t) = (A(x), t)$.

Again, we obtain the notion of SL-equivalence of unfoldings by substituting SL for GL in the above definition.

Remark 2.2.4 Note that whenever A defines a determinantal singularity $(X_A^s, 0) \subset (\mathbb{C}^p, 0)$ of type s and $\mathbf{A}$ and $\mathbf{B}$ are two GL-equivalent unfoldings of A, the resulting flat families are isomorphic in the sense that there is a commutative diagram

$$
\begin{array}{ccc}
(\mathscr{X}_{\mathbf{A}}^s, 0) & \xrightarrow{\cong} & (\mathscr{X}_{\mathbf{B}}^s, 0) \\
& \searrow \quad \swarrow & \\
& (\mathbb{C}^k, 0). &
\end{array}
$$

In particular, a GL-trivial unfolding $\mathbf{A}$ of A gives rise to a product $(\mathscr{X}_{\mathbf{A}}^s, 0) \cong (X_A^s, 0) \times (\mathbb{C}^k, 0)$.

We wish to classify all unfoldings of a given matrix $A \in \mathbb{C}\{x\}^{m \times n}$ up to GL-equivalence, thereby also capturing all possible determinantal deformations of the associated singularities $(X_A^s, 0) \subset (\mathbb{C}^p, 0)$. While in the context of deformations of complex analytic germs this leads to the investigation of deformation functors over Artin rings developed by Schlessinger [92] and in particular the *first order deformations* (cf. [20, Section 7.1.4]), the common viewpoint for unfoldings of map germs is to consider the action of an (infinite dimensional) algebraic group on them. The particular group in question for GL-equivalence is the semi-direct product

$$
\mathscr{G} = (\mathrm{GL}(m; \mathbb{C}\{x\}) \times \mathrm{GL}(n; \mathbb{C}\{x\})) \rtimes \mathrm{Diff}(\mathbb{C}^p, 0) \tag{2.18}
$$

with composition defined by

$$(P, Q, \Phi) * (P', Q', \Psi) = \Big(P(x) \cdot P'(\Phi^{-1}(x)), Q(x) \cdot Q'(\Phi^{-1}(x)), \Psi(\Phi(x))\Big),$$

so as to be compatible with the left action (2.17) on the space of matrices $\mathbb{C}\{x\}^{m\times n}$.

As already mentioned in the introduction, the group $\mathscr{G}$ is a so called "geometric subgroup", a notion introduced by Damon in [22], of the *contact group* $\mathscr{K}$; see e.g. [21, Proposition 2.5.1]. This allows us to pursue a common path in the theory for unfoldings of map germs. The key object for further studies is the space $T^1_{\mathrm{GL}}(A)$ capturing the nontrivial *infinitesimal* unfoldings of a given matrix A up to GL-equivalence. We shall introduce it now.

For a given matrix $A \in \mathbb{C}\{x_1, \dots, x_p\}^{m\times n}$, the trivial unfoldings of A are those captured by the action of a 1-parameter family (P_t, Q_t, Φ_t) in $\mathscr{G}$ with $(P_0, Q_0, \Phi_0) = (\mathbf{1}_m, \mathbf{1}_n, \mathrm{Id}_{\mathbb{C}^p,0})$. Such unfoldings take the form

$$\mathbf{A}(x, t) = P_t(x) \cdot A(\Phi_t^{-1}(x)) \cdot (Q_t(x))^{-1}.$$

Differentiating with respect to t at $t = 0$ gives us the *infinitesimally* trivial unfoldings

$$\frac{\mathrm{d}P}{\mathrm{d}t}(0) \cdot A - A \cdot \frac{\mathrm{d}Q}{\mathrm{d}t}(0) - \sum_{i=1}^{p} \frac{\partial A}{\partial x_i} \cdot \frac{\mathrm{d}\Phi_i}{\mathrm{d}t}(0).$$

These generate the so-called *extended tangent space* (to the orbit) of A:

$$T_e\mathscr{G}(A) = \mathfrak{gl}_m(\mathbb{C}\{x\})\cdot A - A\cdot\mathfrak{gl}_n(\mathbb{C}\{x\}) + \left\langle \frac{\partial A}{\partial x_1}, \dots, \frac{\partial A}{\partial x_p} \right\rangle \subset \mathbb{C}\{x\}^{m\times n}, \tag{2.19}$$

cf. for instance [109, Part I, Section 1], or [21, Proposition 2.5.1]. We wrote $\mathfrak{gl}_m(\mathbb{C}\{x\})$ for the space $\mathbb{C}\{x\}^{m\times m}$ in which $\mathrm{d}P/\mathrm{d}t(0)$ lays, and vice versa for $\mathfrak{gl}_n(\mathbb{C}\{x\})$.

Geometrically, the extended tangent space $T_e\mathscr{G}(A)$ can be described as follows. Consider the pullback of the tangent bundle $A^*(T\mathbb{C}^{m\times n})$ of $\mathbb{C}^{m\times n}$ along A. The sheaf of sections in this bundle is a free $\mathbb{C}\{x\}$-module with stalk $(A^*T_{\mathbb{C}^{m\times n}})_0 \cong \mathbb{C}\{x\}^{m\times n}$ at the origin. Then the extended tangent space is the submodule generated by the pullback of the specific logarithmic vector fields (2.16),

$$\mathfrak{gl}_m(\mathbb{C}\{x\}) \cdot A - A \cdot \mathfrak{gl}_n(\mathbb{C}\{x\}),$$

and the image of the differential $\mathrm{d}A\colon T_{\mathbb{C}^p,0} \to A^*T_{\mathbb{C}^{m\times n},0}$ of A,

$$\left\langle \frac{\partial A}{\partial x_1}, \dots, \frac{\partial A}{\partial x_p} \right\rangle.$$

With $T_e\mathscr{G}(A) \subset \mathbb{C}\{x\}^{m\times n}$ being the trivial unfoldinds, the quotient by this submodule

$$T^1_{\mathrm{GL}}(A) := \mathbb{C}\{x\}^{m\times n} / T_e\mathscr{G}(A) \tag{2.20}$$

captures the *nontrivial* infinitesimal unfoldings of A up to GL-equivalence. We will in the following refer to the dimension of $T^1_{\mathrm{GL}}(A)$ over $\mathbb{C}$ as the GL-*codimension* of A:

$$\tau_{\mathrm{GL}}(A) = \dim_{\mathbb{C}} T^1_{\mathrm{GL}}(A). \tag{2.21}$$

Remark 2.2.5 Besides GL-equivalence some authors also consider other variants such as SL-equivalence, see for example [47], or [46].

SL-**equivalence:** For SL-equivalence, the group $\mathscr{S}$ is defined as in (2.18), only that the matrices P and Q are restricted to take values in the subgroup $\mathrm{SL}(m; \mathbb{C}\{x\})$ and $\mathrm{SL}(n; \mathbb{C}\{x\})$. Accordingly, for the extended tangent space of a matrix A one finds

$$T_e\mathscr{S}(A) = \mathfrak{sl}_m(\mathbb{C}\{x\}) \cdot A + A \cdot \mathfrak{sl}_n(\mathbb{C}\{x\}) + \left\langle \frac{\partial A}{\partial x_1}, \dots, \frac{\partial A}{\partial x_p} \right\rangle \subset \mathbb{C}\{x\}^{m\times n} \tag{2.22}$$

with $\mathfrak{sl}_m(\mathbb{C}\{x\})$ the set of trace-free matrices in $\mathbb{C}\{x\}^{m\times m}$ and vice versa for the other term. In this case, we speak of the SL-*codimension* of a matrix

$$\tau_{\mathrm{SL}}(A) = \dim_{\mathbb{C}} T^1_{\mathrm{SL}}(A).$$

Symmetric matrices: Adaptations of GL- and SL-equivalence can also be made for symmetric matrices $A \in \mathbb{C}\{x_1, \dots, x_p\}^{m\times m}_{\mathrm{sym}}$ where one usually considers the group

$$\mathscr{G}_{\mathrm{sym}} = \mathrm{GL}(m; \mathbb{C}\{x\}) \rtimes \mathrm{Diff}(\mathbb{C}^p, 0) \tag{2.23}$$

with composition

$$(P, \Phi) * (P', \Psi) = (P \cdot (P' \circ \Phi^{-1}), \Phi \circ \Psi)$$

and action on $\mathbb{C}\{x\}^{m\times m}_{\mathrm{sym}}$ given by

$$((P, \Phi), A) \mapsto P \cdot (A \circ \Phi^{-1}) \cdot P^T.$$

In this case, the extended tangent space of a matrix A is

$$T_e\mathscr{G}_{\mathrm{sym}}(A) = \left\langle M \cdot A + A \cdot M^T : M \in \mathfrak{gl}_m(\mathbb{C}\{x\}) \right\rangle + \left\langle \frac{\partial A}{\partial x_1}, \dots, \frac{\partial A}{\partial x_p} \right\rangle. \tag{2.24}$$

Skew symmetric matrices: For skew-symmetric matrices $A \in \mathbb{C}\{x\}_{\mathrm{sk}}^{m\times m}$ and their Pfaffian ideals it is customary to consider the same equivalences as for symmetric matrices. Then the extended tangent space for SL-equivalence, for example, reads

$$T_e\mathscr{S}_{\mathrm{sk}}(A) = \left\langle M \cdot A + A \cdot M^T \,:\, M \in \mathfrak{sl}_m(\mathbb{C}\{x\})\right\rangle + \left\langle \frac{\partial A}{\partial x_1}, \ldots, \frac{\partial A}{\partial x_p}\right\rangle \quad (2.25)$$

where again $\mathfrak{sl}_m(\mathbb{C}\{x\})$ denotes the traceless matrices.

2.2.1 Finite Determinacy

A natural question for map germs is whether or not they are finitely determined for a given notion of equivalence. Analogous to the case of holomorphic functions we define the k-jet of a matrix $A \in \mathbb{C}\{x\}^{m\times n}$ to be the Taylor expansion

$$j^k(A) = A(0) + \sum_{|\alpha|=1} \frac{x^\alpha}{\alpha!}\left(\frac{\partial}{\partial x}\right)^\alpha A|_{x=0} + \cdots + \sum_{|\alpha|=k} \frac{x^\alpha}{\alpha!}\left(\frac{\partial}{\partial x}\right)^\alpha A|_{x=0}$$

of the entries of A up to order k modulo $\mathfrak{m}^{k+1}\mathbb{C}\{x\}^{m\times n}$. As usual, $\alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbb{N}_0^p$ denotes a multi-index with $\alpha! = \alpha_1! \cdots \alpha_p!$, $|\alpha| = \alpha_1 + \cdots + \alpha_p$ and $(\partial/\partial x)^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_p)^{\alpha_p}$.

Definition 2.2.6 A matrix $A \in \mathbb{C}\{x_1, \ldots, x_p\}^{m\times n}$ is k-determined (for GL-equivalence) if for every other matrix B an equality of jets $j^k(A) = j^k(B)$ implies that B is GL-equivalent to A.

Straightforward adaptations can be given for the other groups discussed in Remark 2.2.5. We say that A is finitely GL-determined, if it is k-determined for some $k > 0$. In particular, any finitely GL-determined matrix is GL-equivalent to a matrix with polynomial entries.

The following explicit infinitesimal criterion for finite GL-determinacy of matrices has been given by Pereira in [21, Theorem 2.3.1]:

Theorem 2.2.7 *Let $A \in \mathbb{C}\{x_1, \ldots, x_p\}^{m\times n}$ be a matrix and k an integer such that*

$$\mathfrak{m}^{k+1}\mathbb{C}\{x\}^{m\times n} \subset \mathfrak{m}^2\left\langle \frac{\partial A}{\partial x_1}, \ldots, \frac{\partial A}{\partial x_p}\right\rangle + \mathfrak{m}\cdot\left(\mathfrak{gl}_m(\mathbb{C}\{x\})\cdot A + A\cdot \mathfrak{gl}_n(\mathbb{C}\{x\})\right).$$

Then A is k-determined for GL*-equivalence.*

It should be pointed out, that Pereira also covered the real analytic case and the above theorem is the adapted citation for holomorphic matrices. While this criterion is useful for explicit computations, there is also another, more geometric criterion

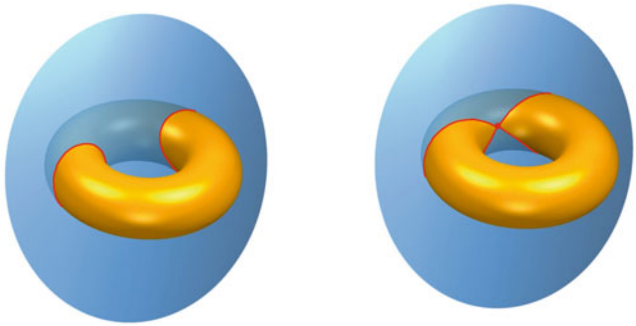


Fig. 2.3 A transverse and a non-transverse intersection (red) of a torus (yellow) in $\mathbb{R}^3$ with the immersion of a plane (blue)

based on the transversality of maps. This was observed by Bruce [11, Proposition 3.2] for the specific case of symmetric matrices and later carried out explicitly by Pereira [21, Theorem 2.4.1] for GL-equivalence for matrices of arbitrary size.

We briefly recall the classical notion of transversality[10] for smooth maps of manifolds. Let M and U be smooth manifolds and $V \subset M$ a locally closed submanifold. We say that a map $f: U \to M$ is *transverse* to V at a point $p \in U$, if either $f(p) \notin V$, or $f(p) \in V$ and the tangent spaces T_pU and $T_{f(p)}V$

$$\mathrm{d}f(p)T_pU + T_{f(p)}V = T_{f(p)}M \tag{2.26}$$

span the whole tangent space $T_{f(p)}M$ of the ambient manifold M; cf. [20, Definition 4.2.11]. We say that f is transverse to V on U if this holds at every point in U (see Fig. 2.3 for an illustration on transversality).

When $\bigcup_i V^i = M$ is a stratification of M into smooth, locally closed submanifolds we say that a map f as above is transverse to the stratification, if it is transverse to every stratum V^i. An illustration of this is given in Fig. 2.4.

With these notions at hand we now have [21, Theorem 2.4.1] (cf. also [11, Proposition 3.2] for the symmetric and [12, Proposition 3.2] for the arbitrary symmetric case):

Theorem 2.2.8 *Let* $A: (\mathbb{C}^p, 0) \to (\mathbb{C}^{m\times n}, A(0))$ *be a holomorphic map germ. Then* A *is finitely* GL*-determined if and only if it is transverse to the rank stratification on* $\mathbb{C}^{m\times n}$ *in a punctured neighborhood of the origin in* $\mathbb{C}^p$.

[10] Following Damon [25], we will also refer to this as *geometric transversality* in order to distinguish it from the *algebraic transversality* which will be introduced in the next section.

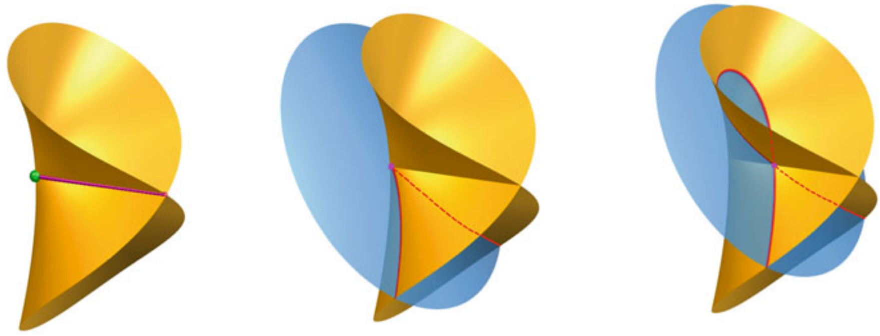


Fig. 2.4 The so-called "Whitney umbrella" $W \subset \mathbb{R}^3$ with its decomposition into three strata: The origin, the open half of a coordinate axis and the remainder. The other two pictures show the immersion of an affine plane D: In the middle picture, the intersection of D with W is not transverse at the origin. In the picture on the right hand side, D is transverse to W in a stratified sense, despite the fact that the intersection is not a smooth manifold

2.2.2 Versal Unfoldings

An unfolding F for a given map germ f is called *versal* , if every other unfolding F' of f can be written as a pullback from F up to the underlying notion of equivalence of map germs. For an account on these notions, see e.g. [109] or [82]. In the explicit case of matrices and GL-equivalence we can give the following

Definition 2.2.9 Let $A \in \mathbb{C}\{x_1, \dots, x_p\}^{m \times n}$ be a matrix. An unfolding

$$\mathbf{A}\colon (\mathbb{C}^p, 0) \times (\mathbb{C}^k, 0) \to (\mathbb{C}^{m \times n}, 0)$$

of A on k parameters t is called GL-versal, if for every other unfolding $\mathbf{B}$ of A on l parameters s there exists a holomorphic map germ $h\colon (\mathbb{C}^l, 0) \to (\mathbb{C}^k, 0)$ such that $\mathbf{B}$ is GL-equivalent as an unfolding (Definition 2.2.3) to the unfolding of A given by

$$A'(x, s) = A(x, h(s)).$$

An unfolding $\mathbf{A}$ of A is called GL-*miniversal* (or semi-universal), if it is versal and the number of parameters k is minimal among all versal unfoldings of A.

Versal unfoldings of matrices on a finite set of parameters as above do not necessarily exist. We saw earlier that the quotient $T^1_{\mathrm{GL}}(A)$ in (2.20) classifies all *infinitesimal unfoldings* up to GL-equivalence. If this space is of finite dimension, it can be used to construct miniversal unfoldings.

Theorem 2.2.10 *Let $A \in \mathbb{C}\{x_1, \ldots, x_p\}^{m\times n}$ be a matrix such that $T^1_{\mathrm{GL}}(A)$ has finite dimension τ as a $\mathbb{C}$-vector space. For any set of elements $B_1, \ldots, B_\tau \in \mathbb{C}\{x\}^{m\times n}$ reducing to a basis of $T^1_{\mathrm{GL}}(A)$, the unfolding on τ parameters given by*

$$\mathbf{A}(x,t) = A(x) + t_1 \cdot B_1(x) + \cdots + t_\tau \cdot B_\tau(x)$$

is miniversal.

In particular, we see that the minimal number of parameters of a miniversal unfolding is always equal to

$$\tau_{\mathrm{GL}}(A) := \dim_{\mathbb{C}} T^1_{\mathrm{GL}}(A).$$

We will in the following refer to this number as the GL-*Tjurina number* of A. Analogously, we will speak of the SL-Tjurina number in the context of SL-equivalence and likewise for the symmetric or skew-symmetric settings.

Theorem 2.2.10 is a special instance of the Unfolding Theorem of Damon [22, Theorem 9.3] for geometric subgroups of $\mathscr{K}$. As already remarked earlier, the theory developed by Damon allows one to also consider differentiable or real analytic setups. For another, explicit proof of Theorem 2.2.10 which does not rely on Damon's work, see [110, Theorem 1.4.10].

In parallel to the previous section, we also give a geometric criterion for the condition $\dim_{\mathbb{C}} T^1_{\mathrm{GL}}(A) < \infty$ to be satisfied, analogous to Theorem 2.2.8.

Definition 2.2.11 (Ebeling, Gusein-Zade [34]) Let $A\colon U \to \mathbb{C}^{m\times n}$ be a holomorphic map on some open subset $U \subset \mathbb{C}^p$. A point $x \in U$ is called *essentially nonsingular*, if A is transverse to the stratum $V^r_{m,n}$ containing $A(x)$.

Here $r = \operatorname{rank} A(x)$ is the rank of the matrix A at x. Note that the transversality of A to the stratum $V^r_{m,n}$ at an essentially non-singular point x and the Whitney-(a)-regularity of the rank stratification already imply that in a neighborhood of x the map A is stratified transversal to all strata $V^s_{m,n}$ for every $s \geq r$. In particular, the singularities $(A^{-1}(M^s_{m,n}), x) \subset (U, x)$ all have expected codimension.

Proposition 2.2.12 *Let $A\colon (\mathbb{C}^p, x) \to (\mathbb{C}^{m\times n}, A(x))$ be a holomorphic map germ. Then $T^1_{\mathrm{GL}}(A) = 0$ if and only if x is an essentially nonsingular point of A.*

We include a brief proof of Proposition 2.2.12 based on the notion of *algebraic transversality* due to Damon in [24] (see also [25]).

Suppose $f\colon U \to M$ is a holomorphic map of complex manifolds and $V \subset M$ is a subvariety of M with $I = I(V)$ the sheaf of ideals of functions vanishing on V. Then f is said to be *algebraically transverse* to V at a point $p \in U$, if

$$\mathrm{d}f(p)T_pU + T^{\log}_{f(p)}V = T_{f(p)}M \tag{2.27}$$

where $T^{\log}_q V$ is the logarithmic tangent space to V at the point q, see Sect. 2.1.4.3.

Proof of Proposition 2.2.12 In the setup of Proposition 2.2.12, and the generic determinantal varieties with $M^s_{m,n} \subset \mathbb{C}^{m\times n}$ in place of V and M we find

$$T^{\log}_{\varphi} M^s_{m,n} = T_{\varphi} V^r_{m,n}$$

for every $\varphi \in V^r_{m,n} \subset M^s_{m,n}$ according to the holonomicity of the rank stratification, Lemma 2.1.19. Consequently, a holomorphic map $A\colon U \to \mathbb{C}^{m\times n}$ is algebraically transversal to $M^s_{m,n}$ at a point $x \in U$ if and only if it is geometrically transversal. Moreover, we saw in the proof of Lemma 2.1.19 that the logarithmic tangent space $T^{\log}_{\varphi} M^s_{m,n}$ at the point $\varphi = A(x)$ is spanned by the specific vector fields $\xi = L \cdot \varphi - \varphi \cdot R$ introduced in (2.16). Using Nakayama's lemma, it is now easy to see from the geometric description of the extended tangent space $T_e\mathscr{G}(A)$ in (2.19) that the module $T^1_{\mathrm{GL}}(A) = A^*T_{\mathbb{C}^{m\times n}}/T_e\mathscr{G}(A)$ is zero at x if and only if A is transverse to $M^s_{m,n}$ (in either sense); the assertion follows. □

Corollary 2.2.13 *Let $A\colon (\mathbb{C}^p, 0) \to (\mathbb{C}^{m\times n}, A(0))$ be a holomorphic map germ. Then a miniversal unfolding of A exists if and only if A is transverse to the rank stratification of $\mathbb{C}^{m\times n}$ in a punctured neighborhood of the origin.*

Proof This follows directly from sheafification of the module $T^1_{\mathrm{GL}}(A)$, cf. [82, Section 4.5.1]. It has finite length if and only if it is supported only at $0 \in \mathbb{C}^p$ and according to Proposition 2.2.12 this is the case whenever A is transverse to the rank stratification off the origin. □

Example 2.2.14 Consider again the space curve singularity from Example 2.1.17 given by the union of the three coordinate axis in $(\mathbb{C}^3, 0)$. The defining matrix was

$$A = \begin{pmatrix} x & 0 & z \\ 0 & y & z \end{pmatrix}$$

and we will briefly indicate how to compute the space $T^1_{\mathrm{GL}}(A)$.

To shorten notation, let $R = \mathbb{C}\{x, y, z\}$. We consider the extended tangent space of A as a submodule $T_e\mathscr{G}(A) \subset R^{2\times 3} \cong R^6$ and denote by $E_{i,j}$ the generator of $R^{2\times 3}$ with 1 at the (i, j)-th entry and zeroes elsewhere. Then the generators of $T_e\mathscr{G}(A)$ in the last summand of (2.19) are

$$\frac{\partial A}{\partial x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \frac{\partial A}{\partial y} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \frac{\partial A}{\partial z} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

which allows us to reduce all multiples of $E_{1,1}$ and $E_{2,2}$ to zero and write every multiple of $E_{1,3}$ as a multiple of $E_{2,3}$ modulo $T_e\mathscr{G}(A)$. We may then proceed to show that for every one of the remaining generators $E_{i,j}$ of $R^{2\times 3}$ we have $\mathfrak{m}\cdot E_{i,j} \subset T_e\mathcal{G}(A)$ where $\mathfrak{m} = \langle x, y, z\rangle$. For instance for $E_{1,2}$ we find

$$x\cdot E_{1,2} = A\cdot\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},\ y\cdot E_{1,2} = \begin{pmatrix}-1&1\\0&0\end{pmatrix}\cdot A + x\frac{\partial A}{\partial x},\ z\cdot E_{1,2} = A\cdot\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix} - z\frac{\partial A}{\partial y}$$

and similarly for $E_{2,1}$

$$x\cdot E_{2,1} = \begin{pmatrix}0&0\\1&-1\end{pmatrix}\cdot A + y\frac{\partial A}{\partial y},\ y\cdot E_{2,1} = A\cdot\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix},\ z\cdot E_{2,1} = A\cdot\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix} - z\frac{\partial A}{\partial x}.$$

The generators of $\mathfrak{m}\cdot E_{2,3}$ are

$$x\cdot E_{2,3} = x\frac{\partial A}{\partial z} - A\cdot\begin{pmatrix}0&0&-1\\0&0&0\\0&0&0\end{pmatrix},\ y\cdot E_{2,3} = A\cdot\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},\ z\cdot E_{2,3} = \begin{pmatrix}0&0\\0&1\end{pmatrix}\cdot A - y\frac{\partial A}{\partial y}.$$

Further calculations yield that a $\mathbb{C}$-basis for $T^1_{\mathrm{GL}}(A)$ is indeed given by the matrices

$$\begin{pmatrix}0&1&0\\0&0&0\end{pmatrix},\quad \begin{pmatrix}0&0&0\\1&0&0\end{pmatrix},\quad \begin{pmatrix}0&0&0\\0&0&1\end{pmatrix}$$

so that we get a miniversal unfolding of A by setting

$$\mathbf{A}(x, y, z; t_1, t_2, t_3) = \begin{pmatrix}x&0&z\\0&y&z\end{pmatrix} + \begin{pmatrix}0&t_1&0\\t_2&0&t_3\end{pmatrix} \tag{2.28}$$

according to Theorem 2.2.10.

2.2.3 Discriminants of Matrices

Let $A\colon (\mathbb{C}^p, 0) \to (\mathbb{C}^{m\times n}, 0)$ be a matrix and $\mathbf{A}\colon (\mathbb{C}^p, 0)\times(\mathbb{C}^k, 0) \to (\mathbb{C}^{m\times n}, 0)$ an unfolding of A on k parameters $t_1, \ldots, t_k$. For such an unfolding, one can define the *relative*[11] space $T^1_{\mathrm{GL}}(\mathbf{A})$ of infinitesimal unfoldings of the fibers A_t as the quotient

[11] cf. e.g. [82, Definition 3.9].

of $\mathbb{C}\{x,t\}^{m\times n}$ by the submodule

$$T_e\mathscr{G}(\mathbf{A}) = \mathbb{C}\{x,t\}^{m\times m}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbb{C}\{x,t\}^{n\times n} + \left\langle \frac{\partial\mathbf{A}}{\partial x_1},\dots,\frac{\partial\mathbf{A}}{\partial x_p}\right\rangle \subset \mathbb{C}\{x,t\}^{m\times n}. \tag{2.29}$$

It is immediate from the definition, that $T^1_{\mathrm{GL}}(A) \cong T^1_{\mathrm{GL}}(\mathbf{A})/\langle t_1,\dots,t_k\rangle T^1_{\mathrm{GL}}(\mathbf{A})$. Using the Weierstrass finiteness theorem[12] it is easy to see that $T^1_{\mathrm{GL}}(A)$ is a finite $\mathbb{C}\{t\}$-module whenever $T^1_{\mathrm{GL}}(A)$ has finite dimension over $\mathbb{C}$.

Definition 2.2.15 Let $A \in \mathbb{C}\{x_1,\dots,x_t\}^{m\times n}$ be a matrix with $\dim_{\mathbb{C}} T^1_{\mathrm{GL}}(A) < \infty$. For an unfolding $\mathbf{A}$ of A on k parameters $t = (t_1,\dots,t_k)$, the *matrix discriminant* $(\Delta_{\mathbf{A}},0) \subset (\mathbb{C}^k,0)$ is the support of the $\mathbb{C}\{t\}$-module $T^1_{\mathrm{GL}}(A)$. If $\mathbf{A}$ is a miniversal unfolding, then we also speak of the matrix discriminant $(\Delta_A,0)$ of A rather than the discriminant of the unfolding $\mathbf{A}$.

Being the support of a finite analytic module, the matrix discriminant is a complex analytic set. In general we can decompose the discriminant into components

$$\Delta_{\mathbf{A}} = \bigcup_{0\le r<\min\{m,n\}} \Delta^r_{\mathbf{A}} \tag{2.30}$$

as follows. Let $\mathbf{A}\colon U\times T \to \mathbb{C}^{m\times n}$ be a representative of the unfolding and $t \in T \subset \mathbb{C}^k$ a fixed parameter. Then $t \in \Delta_{\mathbf{A}}$ if and only if there are points $x \in U$ for which $A_t\colon (U,x) \to (\mathbb{C}^{m\times n}, A_t(x))$ is not transverse to the rank stratification. To any such point we can associate the rank r of the stratum containing the critical value $A_t(x)$. Now $\Delta^r_{\mathbf{A}}$ is the component of $\Delta_{\mathbf{A}}$ whose *generic* fiber A_t has critical points of rank at most r. An example will be given below in Example 2.2.20.

Remark 2.2.16 It is easy to see that for either two miniversal unfoldings of A, the pullback maps in the parameter spaces take one matrix discriminant into the other. However, since these pullbacks are not uniquely determined, the matrix discriminant $(\Delta_A,0)$ of A exists and is unique, but in general only up to non-unique isomorphism.

Sheafifying allows us to pass from $A = A_0$ to a nearby map $A_t\colon U \to \mathbb{C}^{m\times n}$ defined on some suitable open subset $U \subset \mathbb{C}^p$. Then $t \in \Delta_A$ lays in the matrix discriminant if and only if there are points $x \in U$ at which A_t admits non-trivial unfoldings. Given Proposition 2.2.12, these are precisely the points at which A_t is not transversal to the rank stratification of the target space $\mathbb{C}^{m\times n}$. The Thom transversality theorem[13] assures that any differentiable map $f\colon X \to Y$ can be deformed to a map $\tilde{f}$ which is transversal to a given submanifold $Z \subset Y$. In the holomorphic setting, this problem is more delicate due to the rigidity of holomorphic

[12] See e.g. [53, Theorem 1.10].
[13] See [96].

mappings. It has been addressed by Trivedi in [102, Theorem 2.1] and [102, Theorem 3.1].

For complex analytic germs such as $A\colon (\mathbb{C}^p, 0) \to (\mathbb{C}^{m\times n}, 0)$ it also suffices to observe that the set of constant matrices $\varphi \in \mathbb{C}^{m\times n}$ for which the map $A + \varphi$ is transverse to the rank stratification in a neighborhood of the origin, is dense in $\mathbb{C}^{m\times n}$; cf. [102, Lemma 2.2]. As an immediate consequence we find:

Corollary 2.2.17 *Let A be as in Definition 2.2.15 and* $\mathbf{A}$ *a semi-universal unfolding of A (cf. Theorem 2.2.10) on $\tau = \dim_{\mathbb{C}} T^1_{\mathrm{GL}}(A)$ parameters. Then the matrix discriminant $(\Delta_A, 0) \subset (\mathbb{C}^\tau, 0)$ is a proper analytic subset of $(\mathbb{C}^\tau, 0)$.*

For a more detailed discussion, the reader may also consult [110, Section 2.2.1].

In the following, we will refer to any map

$$A_t\colon U \to \mathbb{C}^{m\times n} \tag{2.31}$$

which arises from an unfolding $\mathbf{A}$ of a finitely determined matrix A and with $t \notin \Delta_{\mathbf{A}}$ as a (topological) *stabilization* of A.

Remark 2.2.18 Using the results by Trivedi [102, Lemma 2.2], it is easy to see that stabilizations also exist for matrices A which are not finitely determined. In that case, however, a stabilization is not uniquely determined by the original matrix A, similar to the case of smoothings of non-isolated hypersurface singularities, see Example 2.5.5 below.

Example 2.2.19 Consider the miniversal unfolding (2.28) of the matrix A for the space curve singularity given by the three coordinate axis in $(\mathbb{C}^3, 0)$ from Examples 2.1.17 and 2.2.14:

$$\mathbf{A}(x, y, z; t_1, t_2, t_3) = \begin{pmatrix} x & 0 & z \\ 0 & y & z \end{pmatrix} + \begin{pmatrix} 0 & t_1 & 0 \\ t_2 & 0 & t_3 \end{pmatrix}$$

We claim that the t_1-axis in the parameter space $(\mathbb{C}^3, 0)$ is contained in the discriminant Δ_A.

Fix $t_2 = t_3 = 0$ and let $t_1 \neq 0$ be arbitrarily small. Then the perturbed map

$$A_{(t_1,0,0)}\colon 0 \mapsto \varphi := A_{(t_1,0,0)}(0) = \begin{pmatrix} 0 & t_1, & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

takes the origin to a matrix φ of rank 1. If we let $y_{i,j}$ be the coordinates for the space of matrices $\mathbb{C}^{2\times 3}$, then locally at the point φ the minor of the matrix Y obtained by deletion of the second column is easily seen to be a superfluous generator of the ideal $\langle Y^{\wedge 2}\rangle$. The generic determinantal variety $M^2_{2,3}$ is a smooth local complete intersection at φ with tangent space

$$T_\varphi M^2_{2,3} = \operatorname{span}\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right).$$

But the image of the differential of $A_{(t_1,0,0)}$ at the origin is spanned by the matrices

$$\mathrm{d}A_{(t_1,0,0)}(0)\,T_0\mathbb{C}^3 = \operatorname{span}\left(\frac{\partial A}{\partial x}, \frac{\partial A}{\partial y}, \frac{\partial A}{\partial z}\right)$$

which is clearly not transversal to $T_\varphi M^2_{2,3}$, since for $\lambda \neq 0$ none of the matrices $\lambda \cdot E_{2,1}$ is contained in the sum of the two subspaces.

This particular unfolding was already considered in Example 2.1.17 and we saw that the fibers $X^2_A(t_1, 0, 0)$ were not smooth. From the viewpoint taken in this example, this is merely a consequence of the non-transversality of the map $A_{(t_1,0,0)}$ at these points. This observation will be generalized in Lemma 2.3.4.

With the help of a computer algebra system it is easy to see that the full discriminant $(\Delta_A, 0) \subset (\mathbb{C}^3, 0)$ in the parameter space $(\mathbb{C}^3, 0)$ consists of the union of the three coordinate *hyperplanes* given by $t_1 \cdot t_2 \cdot t_3 = 0$. Hence, a simultaneous perturbation by $t_1 = t_2 = t_3 = t$ will lead to a smoothing of $(X^2_A, 0)$ (Fig. 2.5).

Example 2.2.20 The matrix discriminant is not always a hypersurface. For instance, the matrix

$$A = \begin{pmatrix} x & y & z \\ y & z & w \end{pmatrix}$$

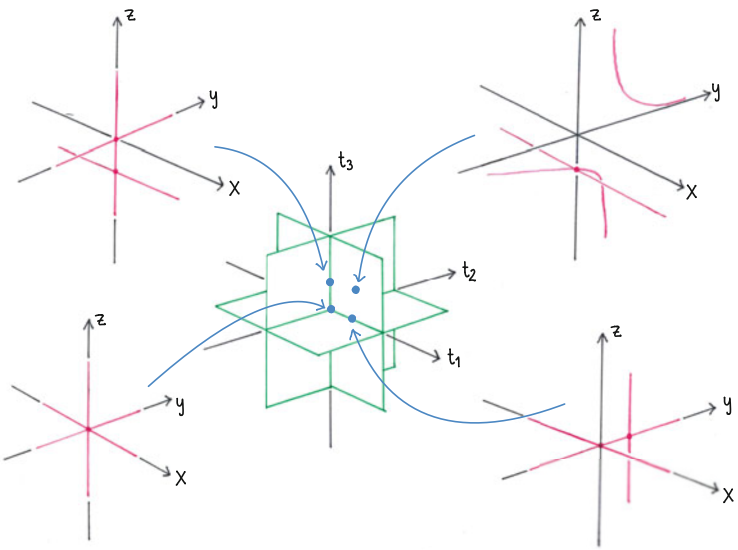


Fig. 2.5 The discriminant of the space curve singularity in $\mathbb{C}^3$ given by the union of the three coordinate axis

has a miniversal unfolding given by

$$\mathbf{A}(x, y, z, w; t_0, t_1) = \begin{pmatrix} x & y & z \\ y & z & w \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ t_0 & t_1 & 0 \end{pmatrix}.$$

Contrary to the previous example, the matrix discriminant Δ_A consists only of the point $\{0\} \subset \mathbb{C}^2$ in the parameter space.

This behaviour continues, as can be observed from the slightly more complicated matrix

$$B = \begin{pmatrix} x & y & z \\ y^2 & z & w \end{pmatrix}.$$

This is a member of the second series from the list of simple isolated Cohen-Macaulay codimension 2 surface singularities in Table 2.16. The miniversal unfolding can be realized on 3 parameters as

$$\mathbf{B}(x, y, z, w; t_0, t_1, t_2) = \begin{pmatrix} x & y & z \\ y^2 & z & w \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ t_0 + t_1 y & t_2 & 0 \end{pmatrix}.$$

The matrix discriminant decomposes as in (2.30) into components $\Delta_B = \Delta_B^0 \cup \Delta_B^1$. The first one

$$\Delta_B^0 = \{t_2 = t_0 = 0\}$$

is of codimension 2. Geometrically it is characterized by the fact that for a generic point $0 \neq t = (t_1, 0, 0) \in \Delta_B^0$ the map $B_t\colon (\mathbb{C}^4, 0) \to (\mathbb{C}^{2\times 3}, 0)$ GL-equivalent to the matrix A above. In particular, it is not transverse to the stratum $\{0\} = V_{2,3}^0$ of the rank stratification so that the origin is a critical point of rank $r = 0$ for B_t.

The other component

$$\Delta_B^1 = \left\{t_1^2 - 4t_0\right\}$$

is a divisor. For generic $t = \left(t_1^2/4, t_1, t_2\right) \in \Delta_B^1$ the map B_t has a non-transverse point at $(x, y, z, w) = (0, -t_1/2, -t_2, 0)$ whose image

$$B_t(0, -t_1/2, t_2, 0) = \begin{pmatrix} 0 & -t_1/2 & -t_2 \\ 0 & 0 & 0 \end{pmatrix}$$

is of rank 1. Note that now the full discriminant $(\Delta_B, 0)$ is of codimension 1, but not a divisor.

2.3 Essentially Isolated Determinantal Singularities and Their Deformations

In the preceding section we have discussed the singularities of matrices regarded as map germs. The geometric criteria for finite determinacy, Theorem 2.2.8, and the existence of miniversal unfoldings, Proposition 2.2.12, motivated the definition of an essentially nonsingular point, Definition 2.2.11, in a natural way. For the determinantal singularities associated to a matrix this leads to the following:

Definition 2.3.1 (Ebeling, Gusein-Zade [34]) A determinantal singularity $(X^s_A, 0) \subset (\mathbb{C}^p, 0)$ is called an *Essentially Isolated Determinantal Singularity* (EIDS) of type (m, n, s) if the defining matrix $A\colon (\mathbb{C}^p, 0) \to (\mathbb{C}^{m\times n}, 0)$ has only essentially non-singular points in a punctured neighborhood of the origin.

Remark 2.3.2 It follows directly from Proposition 2.2.12 and Theorem 2.2.8 that the defining matrix of an EIDS is finitely GL-determined. Note that in general any such matrix gives rise to several EIDS: one for every integer s satisfying $0 \le (m - s + 1)(n - s + 1) \le p$ so that the intersection of the image of A with the variety $M^s_{m,n}$ is generically nonempty.

An EIDS is *essentially isolated* in the sense that, in general, the space X^s_A has non-isolated singularities. However, due to the transversality condition in Definition 2.2.11 imposed on the map A, apart from the origin itself these singularities are locally products of the generic determinantal varieties with affine space:

Lemma 2.3.3 *Let $A \in \mathbb{C}\{x_1, \dots, x_p\}^{m\times n}$ be a matrix defining an EIDS $(X^s_A, 0) \subset (\mathbb{C}^p, 0)$. Then the preimages of the strata $V^r_A = A^{-1}(V^r_{m,n})$ with $r < s$ form a Whitney stratification of $X^s_A \setminus \{0\}$. Moreover, at any point $x \in V^r_A \subset X^s_A \setminus \{0\}$ one has an isomorphism*

$$(X^s_A, x) \cong (M^{s-r}_{m-r,n-r}, 0) \times (\mathbb{C}^{p-(m-r)(n-r)}, 0).$$

In particular, X^s_A has isolated singularity at the origin if and only if

$$p \le (m - s + 2)(n - s + 2) \tag{2.32}$$

and it is smoothable by a determinantal deformation if and only if this inequality is strict.

Proof The first part of this lemma is a basic application of stratification theory, see e.g. [45]. For the question on smoothability observe that in case $p = (m-s+2)(n-s+2)$, the space $(X^{s-1}_A, 0)$ is also an EIDS, but of dimension zero. An unfolding of A on a set of parameters t gives rise to a (flat) deformation of this singularity and the principle of conservation of number asserts that the total multiplicity of these points is preserved within the family. It is now easy to see that for $t \neq 0$ any of

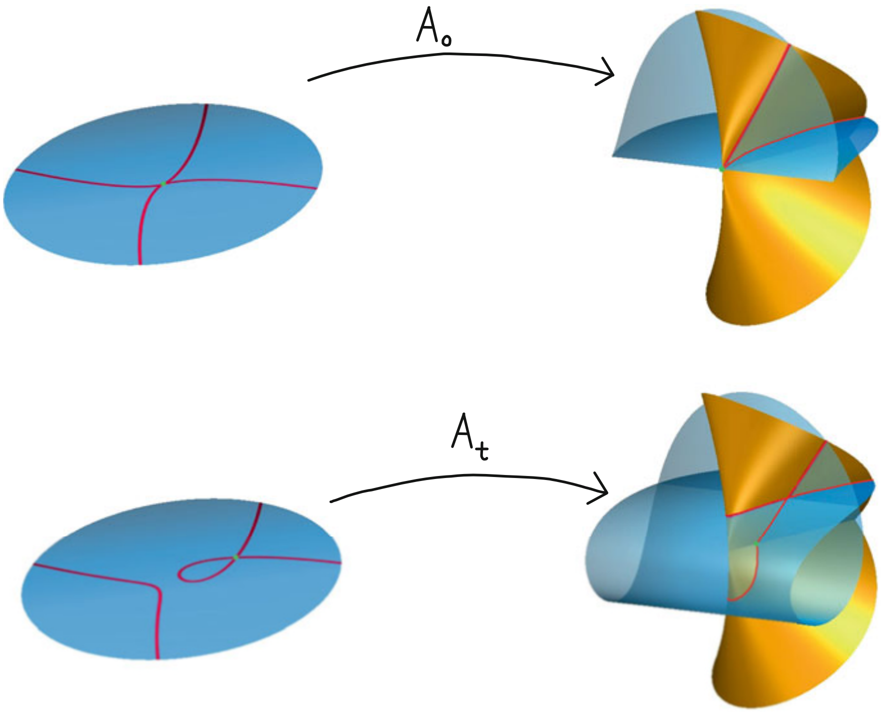


Fig. 2.6 A schematic picture of an isolated determinantal singularity $(X_A^s, 0)$ given as the preimage along the map $A = A_0$ of the generic determinantal variety $M_{m,n}^s$ with non-isolated singularities (drawn as a Whitney umbrella). The second row shows an essential smoothing $X_{\mathbf{A}}^s(t) = A_t^{-1}(M_{m,n}^s)$ which is still singular due to the unavoidable intersection of the image of A_t with the singularities of $M_{m,n}^s$

these points $x \in A_t^{-1}(M_{m,n}^{s-1}) \subset A_t^{-1}(M_{m,n}^s)$ must be a singular point of the fiber $A_t^{-1}(M_{m,n}^s)$. □

Along the same lines, it is easy to prove:

Lemma 2.3.4 *In the setup of Lemma 2.3.3 let $A_t : U \to \mathbb{C}^{m\times n}$ be a stabilization of A defined on some open neighborhood $U \subset \mathbb{C}^p$ of the origin. Then the preimages of the strata $V_A^r = A_t^{-1}(V_{m,n}^r)$ form a Whitney stratification of the essential smoothing M_A^s.*

An illustration of an essential smoothing as in Lemma 2.3.4 can be found in Fig. 2.6.

Remark 2.3.5 In this note we confined ourselves mostly to the complex analytic setup. However, the idea of essential smoothings works equally well in the purely algebraic setting. This was discussed by Laksov in [69] where he develops the notions of transversality, and determinantal deformations for affine determinantal schemes.

2.3.1 The Tjurina Transformation for EIDS

We need to introduce one technical tool that has proved to be very useful in the study of determinantal singularities: The Tjurina transform and the Tjurina transform in family for arbitrary EIDS. For the generic determinantal varieties $M^s_{m,n} \subset \mathbb{C}^{m\times n}$ the Tjurina transform has already been introduced in (2.10). For an arbitrary EIDS we give the following definition.

Definition 2.3.6 Let $(X^s_A, 0) \subset (\mathbb{C}^p, 0)$ be a determinantal singularity of type s given by a matrix $A \in \mathbb{C}\{x\}^{m\times n}$. We define the *Tjurina transform* $\hat{\nu}\colon \hat{X}^s_A \to X^s_A$ of X^s_A to be the fiber product

$$\begin{array}{ccc} X^s_A \times_{M^s_{m,n}} \hat{M}^s_{m,n} & \longrightarrow & \hat{M}^s_{m,n} \\ \downarrow \hat{\nu} & & \downarrow \hat{\nu} \\ X^s_A & \xrightarrow{A} & M^s_{m,n} \end{array} \tag{2.33}$$

where $\hat{M}^s_{m,n}$ is the Tjurina transform of the generic determinantal variety (2.10). The *strict Tjurina transform* $\overline{X}^s_A \subset \hat{X}^s_A$ is defined as the closure

$$\overline{X}^s_A = \overline{(A \circ \hat{\nu})^{-1}(V^{s-1}_{m,n})} \subset \hat{X}^s_A \tag{2.34}$$

of the open set over the matrices of rank $s - 1$.

Using the properties already described for $\hat{\nu}\colon \hat{M}^s_{m,n} \to M^s_{m,n}$, it is easy to see that $\hat{\nu}\colon \hat{X}^s_A \to X^s_A$ is an isomorphism outside the singular locus X^{s-1}_A. In many practical cases with matrices of small size compared to the dimension of $(X^s_A, 0)$, the Tjurina transform and the strict Tjurina transform coincide. For the general case, however, the Tjurina transform will have higher dimensional components in its exceptional set. While the strict Tjurina transform might be geometrically more intuitive, the definition in (2.33) has the advantage that it provides explicit equations to work with.

Definition 2.3.7 Let $(X^s_A, 0) \hookrightarrow (\mathscr{X}^s_{\mathbf{A}}, 0) \xrightarrow{\pi} (\mathbb{C}^k, 0)$ be a determinantal deformation of an EIDS induced from an unfolding $\mathbf{A}(x, t)$ of the defining matrix A. The *Tjurina transformation in family*

$$\begin{array}{ccccc} \hat{X}^s_A & \hookrightarrow & \widehat{\mathscr{X}^s_{\mathbf{A}}} & \longrightarrow & \hat{M}^s_{m,n} \\ \downarrow & & \downarrow \hat{\nu} & & \downarrow \\ X^s_A & \hookrightarrow & \mathscr{X}^s_A & \xrightarrow{\mathbf{A}} & M^t_{m,n} \\ \downarrow & & \downarrow \pi & & \\ \{0\} & \hookrightarrow & \mathbb{C}^k & & \end{array} \tag{2.35}$$

is obtained by applying the Tjurina transformation to the total space $(\mathscr{X}_A^s, 0)$ of the family.

It is a priori not clear whether the family $\pi \circ \hat{\nu} \colon \widehat{\mathscr{X}_{\mathbf{A}}^s} \to \mathbb{C}^k$ is well behaved (e.g. flat). In particular, the fibers of the family can not be expected to specialize to the strict Tjurina transform of $(X_A^s, 0)$. However, using Lemma 2.3.4 it is not difficult to see the following:

Proposition 2.3.8 *If the family in Definition 2.3.7 arises from a stabilization of A, then for suitable representatives and $t \notin \Delta_{\mathbf{A}}$ outside the discriminant of the deformation the restriction to the fiber*

$$\hat{\nu} \colon \hat{X}_{\mathbf{A}}^s(t) \to X_{\mathbf{A}}^s(t)$$

is a resolution of singularities of $X_{\mathbf{A}}^s(t)$. In particular, this is an isomorphism whenever $X_{\mathbf{A}}^s(t)$ is already smooth.

Remark 2.3.9 The same procedures can in principal also be applied to the dual Tjurina transform and the Nash transform described in (2.10). This has, for instance, been done by Ebeling and Gusein-Zade in [34] in order to construct other resolutions of singularities for essential smoothings as in Proposition 2.3.8.

Example 2.3.10 Let again $(X_A^2, 0) \subset (\mathbb{C}^3, 0)$ be the union of the three coordinate axis and $\pi \colon (\mathscr{X}_{\mathbf{A}}^2, 0) \to (\mathbb{C}, 0)$ its smoothing induced from the unfolding

$$\mathbf{A}(x, y, z; t) = \begin{pmatrix} x & 0 & z \\ 0 & y & z \end{pmatrix} + \begin{pmatrix} 0 & t & 0 \\ t & 0 & t \end{pmatrix}$$

as in the end of Example 2.2.19. The strict Tjurina transform $\overline{X}_A^2 = L_x \dot{\cup} L_y \dot{\cup} L_z$ consists of the three separated coordinate axis only. The original Tjurina transform $\hat{X}_A^2$, on the other hand, has as exceptional set $\hat{E} \cong \mathbb{P}^1$ a whole projective line as an additional component with the coordinate axis meeting $\hat{E}$ at the points 0, ∞, and 1, respectively (Fig. 2.7).

For $t \neq 0$ the Tjurina transformation in family provides an identification $\hat{\nu} \colon \hat{X}_{\mathbf{A}}^2(t) \xrightarrow{\cong} X_{\mathbf{A}}^2(t)$ of the smooth fibers. Observe that by construction, these fibers specialize to the central fiber $\hat{X}_A^2 = \hat{X}_{\mathbf{A}}^2(0)$ but not to the strict Tjurina transform of $(X_A^2, 0)$. Moreover, one can easily verify from explicit calculations, that the given family induces local, simultaneous smoothings of the three A_1-singularities of $\hat{X}_A^2$ at the intersection points of either L_x, L_y, and L_z with $\hat{E}$. In particular, the family $(\pi \circ \hat{\nu}) \colon (\widehat{\mathscr{X}_{\mathbf{A}}^2}, 0) \to (\mathbb{C}, 0)$ turns out to be flat.

For the *dual* Tjurina transform we find that

$$\check{X}_A^2 = \mathbb{P}^2 \cup L_x \cup L_y \cup L_z$$

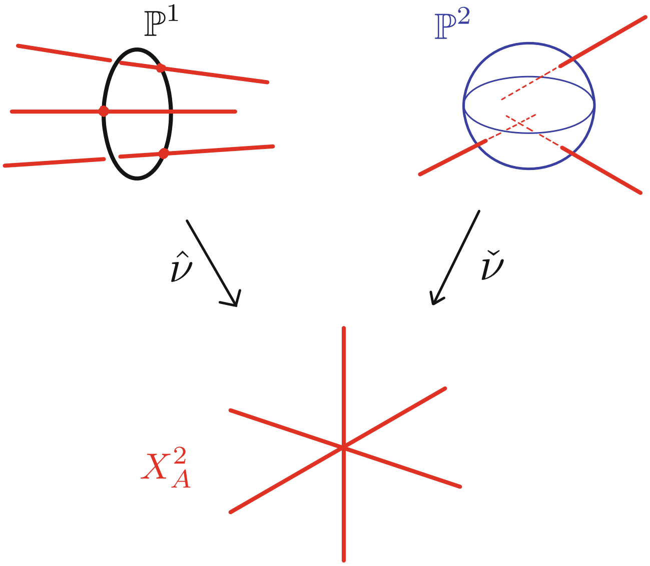


Fig. 2.7 The singularity $(X_A^2, 0)$ from Example 2.3.10 together with its Tjurina transform and its dual Tjurina transform

consists of an exceptional *plane* $\check{E} \cong \mathbb{P}^2$ with the components of the strict transform meeting $\check{E}$ at the points $(1 : 0 : 0)$, $(0 : 1 : 0)$, and $(0 : 0 : 1)$, respectively. Again, on the fibers over $t \neq 0$, the projection $\check{\nu}\colon \check{X}_{\mathbf{A}}^2(t) \to X_{\mathbf{A}}^2(t)$ is an isomorphism of curves with the fibers specializing to $\check{X}_{\mathbf{A}}^2(0) = \check{X}_A^2$; only that this time the family can not be flat, given the jump in dimensions of the fibers at $t = 0$.

2.3.2 Comparison of Unfoldings and Semi-universal Deformations

Given a finitely GL-determined matrix $A \in \mathbb{C}\{x_1, \ldots, x_p\}^{m \times n}$ together with its miniversal unfolding $\mathbf{A}$ on $k = \dim_{\mathbb{C}} T^1_{\mathrm{GL}}(A)$ parameters as, for instance, in Theorem 2.2.10, one can compare the unfoldings of A with the deformations of the associated EIDS $(X_A^s, 0)$ defined by A. To make this explicit, it is important that the germ $(X, 0) = (X_A^s, 0)$ given by the associated determinantal singularity has a *semi-universal deformation*. An excellent overview for this topic with further references for details can be found in [20, Chapter 7] and therefore we will restrict ourselves to briefly recalling the cornerstones of the theory.

Similar to the definition of versal unfoldings, a *versal deformation* of a complex analytic germ $(X, 0)$ is given by a flat family

$$\begin{array}{ccc} (X,0) & \hookrightarrow & (\mathscr{X},0) \\ \downarrow & & \downarrow \pi \\ \{0\} & \hookrightarrow & (S,0) \end{array} \tag{2.36}$$

such that any other deformation $(X, 0) \hookrightarrow (\mathscr{X}', 0) \to (S', 0)$ can be written as a pullback from $(\mathscr{X}, 0) \to (S, 0)$ via some comparison map $\Phi\colon (S', 0) \to (S, 0)$. A versal deformation is called *semi-universal*, if the differential of the comparison map Φ is uniquely determined by the particular family $(\mathscr{X}', 0) \to (S', 0)$. For a thourough discussion of these definitions we refer to [20, Definition 7.1.13].

This second condition on the differential of the comparison map is closely related to the space of *first order deformations* $T^1_{X,0}$, see [20, Section 7.1.4], which is central to the construction of semi-universal deformations. It can be defined as the set of isomorphism classes of deformations of $(X, 0)$ over the formal algebra $\mathbb{C}[t]/\langle t^2\rangle$, cf. [20, Definition 7.1.26] and it is the Zariski tangent space of the base $(S, 0)$ of a semi-universal deformation of $(X, 0)$—provided such a semi-universal deformation exists. One can explicitly compute the space $T^1_{X,0}$ from the ideal $I \in \mathbb{C}\{x\}$ for any given embedding $(X, 0) \subset (\mathbb{C}^p, 0)$, cf. [20, Proposition 7.1.33]: It appears in an exact sequence

$$T_{\mathbb{C}^p,0} \otimes_{\mathbb{C}\{x\}} \mathbb{C}\{x\}/I \xrightarrow{\ \beta\ } \mathrm{Hom}_{\mathbb{C}\{x\}}(I, \mathbb{C}\{x\}/I) \longrightarrow T^1_{X,0} \longrightarrow 0 \tag{2.37}$$

where $T_{\mathbb{C}^p,0}$ is the module of germs of holomorphic vector fields on $(\mathbb{C}^p, 0)$, $\mathrm{Hom}_{\mathbb{C}\{x\}}(I, \mathbb{C}\{x\}/I)$ is the *normal module* for the embedding of $(X, 0)$, and β is the map determined by

$$\beta\colon \frac{\partial}{\partial x_i} \mapsto \left(f \mapsto \frac{\partial f}{\partial x_i}\right)$$

for $i = 1, \dots, p$. This endows $T^1_{X,0}$ with the structure of a $\mathbb{C}\{x\}$-module and in particular, it is a $\mathbb{C}$-vector space.

The following theorem was proved by Grauert in [49], see also [20, Theorem 7.1.14] and [20, Remark 7.1.28].

Theorem 2.3.11 *Let* $(X, 0)$ *be a complex analytic singularity with* $T^1_{X,0}$ *of finite dimension. Then a semi-universal deformation of* $(X, 0)$ *exists.*

The construction of such semi-universal deformations is quite technical and in general much more complicated than the construction of miniversal unfoldings for map germs as for instance in Theorem 2.2.10. The general framework for this was developed by Schlessinger in his thesis [92]. For details of the general case, we refer to [20, Chapter 7], "Deformation and Smoothing of Singularities" by Greuel. For the specific purposes here, we only note the following, [20, Definition 7.1.38]:

Definition 2.3.12 A singularity $(X, 0)$ with finite dimensional $T^1_{X,0}$ is called *unobstructed* if it has a semi-universal deformation with a smooth base $(S, 0)$.

A criterion for a singularity $(X, 0)$ to be unobstructed is that the module $T^2_{X,0}$ is zero. This is another coherent analytic module associated to the singularity that can be computed explicitly from the ideal defining the singularity. Again, we refer to [20, Chapter 7] for details.

Suppose that $(X, 0) = (X^s_A, 0)$ is an EIDS with isolated singularity (cf. Lemma 2.3.3). Since every unfolding of A gives rise to a deformation of $(X^s_A, 0)$ due to Lemma 2.1.15, we obtain a flat family over the parameter space of the unfolding and some comparison map $\Phi\colon (\mathbb{C}^k, 0) \to (S, 0)$ to the base of the semi-universal deformation $\pi\colon (\mathscr{X}, 0) \to (S, 0)$ of $(X^s_A, 0)$ in the sense of Grauert's theorem. Altogether this forms the commutative diagram

$$\begin{array}{ccccc}
(X^s_A, 0) & \hookrightarrow & (\mathscr{X}^s_{\mathbf{A}}, 0) & \longrightarrow & (\mathscr{X}, 0) \\
\downarrow & & \downarrow \rho & & \downarrow \pi \\
\{0\} & \hookrightarrow & (\mathbb{C}^k, 0) & \xrightarrow{\Phi} & (S, 0).
\end{array} \tag{2.38}$$

Depending on the size of the matrix A and the size of the minors s, the map Φ can take very different forms. We first give a list of particular examples and then discuss some cases where more structural results are known.

Example 2.3.13

(i) Let $(X^2_A, 0) \subset (\mathbb{C}^3, 0)$ be the determinantal hypersurface singularity defined by the matrix

$$A = \begin{pmatrix} x & y \\ z & x \end{pmatrix}.$$

The ideal I of $(X^2_A, 0)$ is thus generated by the equation $f = x^2 - yz$ and we recognize the well-known A_1-surface singularity. A basis of $T^1_{\mathrm{GL}}(A)$ is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and hence if we let t be the deformation parameter in the semi-universal unfolding of A, then the induced deformation of the space germ $(X^2_A, 0)$ comes from a perturbation of f by $-t^2$.

The semi-universal deformation of $(X^2_A, 0)$ as a space germ on the other hand is given by the perturbation of f by a constant u. It follows that the comparison map (2.38) takes the form

$$\Phi : (\mathbb{C}, 0) \to (\mathbb{C}, 0), \quad t \mapsto u = t^2.$$

In other words: The base of the miniversal unfolding of A is a $2:1$ cover of the base of the semi-universal deformation of $(X_0, 0)$.

(ii) (Pinkham, [84]) Recall from Example 2.1.2 that the ideal $I \subset \mathbb{C}\{x_0, \ldots, x_4\}$ in Pinkham's example [84] was given as $I = \langle A^{\wedge 2}\rangle = \langle B^{\wedge 2}\rangle$ for the two matrices

$$A = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix}.$$

The germ $(X, 0) \subset (\mathbb{C}^5, 0)$ defined by I is the cone over the rational normal curve of degree 4 in $\mathbb{P}^4$. A direct computation of $T^1_{\mathrm{GL}}(A)$ and application of Theorem 2.2.10 yields that the unfolding of A given by

$$\mathbf{A}(x, t) = A(x) - \begin{pmatrix} 0 & 0 & 0 & 0 \\ t_1 & t_2 & t_3 & 0 \end{pmatrix} \tag{2.39}$$

on the parameters $t = (t_1, t_2, t_3)$ is miniversal. Similarly, one has a miniversal unfolding on a single parameter u for the symmetric matrix B given by

$$\mathbf{B}(x, u) = B(x) - \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ u & 0 & 0 \end{pmatrix}. \tag{2.40}$$

In [84] Pinkham shows explicitly that the semi-universal deformation of $(X, 0)$ as a complex analytic germ has a base $(S, 0) \subset (\mathbb{C}^4, 0)$ of the following form. Let t_1, t_2, t_3, u be the coordinates of $\mathbb{C}^4$. Then $(S, 0)$ consists of two components: The plane $H = \{u = 0\}$ and the line $L = \{t_1 = t_2 = t_3 = 0\}$. An explicit computation of $(S, 0)$ can also be found in [20, Example 7.1.41].

Indeed, the comparison map for the miniversal unfolding of A identifies the parameter space

$$\Phi_A : (\mathbb{C}^3, 0) \xrightarrow{\cong} (H, 0) \subset (S, 0), \quad t \mapsto (t, 0)$$

of $\mathbf{A}$ with the hyperplane $(H, 0)$ in $(S, 0)$ and similarly for B one has an isomorphism

$$\Phi_B : (\mathbb{C}, 0) \xrightarrow{\cong} (L, 0) \subset (S, 0), \quad u \mapsto (0, u).$$

(iii) Consider the A_1 threefold singularity in $(\mathbb{C}^4, 0)$ as a determinantal singularity of type (2, 2, 2) via the matrix

$$A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

There are no nontrivial unfoldings of this matrix. For the space germ on the other hand we find that the perturbation of $f = \det A$ by a constant is semi-universal so that we have $T^1_{X_0,0} \cong \mathbb{C}$. Therefore, the comparison map takes the form

$$\Phi : \{\mathrm{pt}\} \to (\mathbb{C}, 0).$$

(iv) This example is taken from Schaps [91] and it also appears in [18]. Explicit computations can be found in [38].

Let $(X_A^2, 0) \subset (\mathbb{C}^4, 0)$ be the union of the four coordinate axis. This is a determinantal singularity via any matrix

$$\begin{pmatrix} x_1 & \alpha \cdot x_2 & \beta \cdot x_3 & \gamma \cdot x_4 \\ 0 & x_2 & x_3 & x_4 \end{pmatrix}$$

for general values $\alpha, \beta, \gamma \in \mathbb{C}$. Using row and column operations and local coordinate changes, one can always bring this matrix to the form

$$A := \begin{pmatrix} x_1 & 0 & x_3 & \gamma' \cdot x_4 \\ 0 & x_2 & x_3 & x_4 \end{pmatrix}$$

with $\gamma' \notin \{0, 1\}$. One can show that the following matrices give a $\mathbb{C}$-basis of $T^1_{\mathrm{GL}}(A)$:

$$\begin{pmatrix} 0&0&0&0 \\ 1&0&0&0 \end{pmatrix}, \quad \begin{pmatrix} 0&1&0&0 \\ 0&0&0&0 \end{pmatrix}, \quad \begin{pmatrix} 0&0&1&0 \\ 0&0&0&0 \end{pmatrix},$$
$$\begin{pmatrix} 0&0&0&1 \\ 0&0&0&0 \end{pmatrix}, \quad \begin{pmatrix} 0&0&0&x_4 \\ 0&0&0&0 \end{pmatrix}.$$

Hence, the base of the miniversal unfolding of A is $(\mathbb{C}^5, 0)$. Let $t_1, \dots, t_5$ be the parameters associated to these matrices as in Theorem 2.2.10.

Computations of Rim[14] and independently of Buchweitz [18] have shown that the base $(S, 0)$ of the semi-universal deformation of $(X_A^2, 0)$ is isomorphic to the cone of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^3$ into $\mathbb{P}^7$ and thus also of

[14] The computations are attributed to Rim in [91] without further reference.

dimension 5. Consider the comparison map

$$\Phi : (\mathbb{C}^5, 0) \to (S, 0).$$

It is easy to see that the perturbation by t_5 alone does not change the ideal generated by the 2-minors of A in $\mathbb{C}\{x\}$: This is a non-trivial deformation of the map germ A which induces a trivial deformation of the underlying space germ! Accordingly, as the computations by first named author show, Φ is a contraction of the t_5-axis but a local diffeomorphism away from it; just as if Φ was a local chart in a resolution of singularities for $(S, 0)$.

2.3.3 Complete Intersections

It has been pointed out earlier that any (isolated) complete intersection singularity $(X, 0) \subset (\mathbb{C}^p, 0)$ of codimension c is determinantal of type 1 for some $1 \times c$-matrix $F = (f_1, \dots, f_c)$. For this class of singularities, the deformations of $(X, 0)$ coincide with the unfoldings of F up to GL-equivalence.

This starts with an explicit identification of the space of first order deformations $T^1_{X,0}$ of $(X, 0)$ as in (2.37) and the infinitesimal unfoldings $T^1_{\mathrm{GL}}(F)$ of F from (2.20).

Lemma 2.3.14 *Let $(X, 0) \subset (\mathbb{C}^p, 0)$ be a complete intersection singularity defined by a regular sequence $F = (f_1, \dots, f_c)$ in $\mathbb{C}\{x\}$. Then there is an explicit isomorphism $T^1_{\mathrm{GL}}(F) \cong T^1_{X,0}$ of the infinitesimal unfoldings of F considered as a $1 \times c$-matrix and the first order deformations of $(X, 0)$.*

The construction of this isomorphism builds on the description of $T^1_{X,0}$ for complete intersections as in [20, Remark 7.1.35].

Proof For a complete intersection ideal $I = \langle f_1, \dots, f_c\rangle$ in $\mathbb{C}\{x_1, \dots, x_p\}$ the resolution of $\mathbb{C}\{x\}/I$ by the Koszul complex (2.7) can be used to show that the normal module $\mathrm{Hom}_{\mathbb{C}\{x\}}(I, \mathbb{C}\{x\}/I)$ is a free $\mathbb{C}\{x\}/I$-module in generators $e_1, \dots, e_c$ which are dual to the f_i's. An element $g = g_1 \cdot e_1 + \dots + g_c \cdot e_c \in \mathrm{Hom}_{\mathbb{C}\{x\}}(I, \mathbb{C}\{x\}/I)$ corresponds to a formal deformation of $(X, 0)$ over $\mathbb{C}[t]/\langle t^2\rangle$ given by the c equations

$$\mathbf{F}(x, t) = F(x) + t \cdot \big(g_1(x) \; \dots \; g_c(x)\big) = 0$$

in the ring $\mathbb{C}\{x\}[t]/\langle t^2\rangle$. The obvious translation to unfoldings yields an isomorphism

$$\mathrm{Hom}_{\mathbb{C}\{x\}}(I, \mathbb{C}\{x\}/I) \xrightarrow{\cong} \mathbb{C}\{x\}^{1\times c}/\langle \mathbb{C}\{x\}^{1\times 1} \cdot F + F \cdot \mathbb{C}\{x\}^{c\times c}\rangle,$$
$$\sum_{i=1}^{c} g_i \cdot e_i \;\mapsto\; \big(g_1 \; \dots \; g_c\big)$$

where the right hand side is a quotient of $\mathbb{C}\{x\}^{1\times c}$ by a submodule of the extended tangent space (2.19) of F up to GL-equivalence. It is now easy to see that the remaining relations to be added, namely

$$\left\langle \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_p} \right\rangle$$

coincide with the image of the map β in (2.37). □

Once the space of first order deformations has been computed, the construction of a semi-universal deformation of an isolated complete intersection singularity $(X, 0) \subset (\mathbb{C}^p, 0)$ is straightforward, see [62, 99]; cf. also [20, Theorem 7.1.22]. First note that for a complete intersection singularity $T^1_{X,0}$ is finite dimensional if and only if $(X, 0)$ has isolated singularity. Now according to the above cited theorems, any set of elements $G_1, \dots, G_\tau \in \mathbb{C}\{x\}^{1\times c}$ reducing to a $\mathbb{C}$-basis of $T^1_{X,0}$ in the above description gives rise to a semi-universal deformation $(X, 0) \hookrightarrow (\mathscr{X}, 0) \xrightarrow{\pi} (\mathbb{C}^\tau, 0)$ where the germ $(\mathscr{X}, 0) \subset (\mathbb{C}^p, 0) \times (\mathbb{C}^\tau, 0)$ is defined by the equations

$$\mathbf{F}(x, t) = F + t_1 \cdot G_1 + \dots + t_\tau \cdot G_\tau = 0.$$

This turns out to be the same procedure as for the construction of the miniversal unfolding of $F \in \mathbb{C}\{x\}^{1\times c}$ described in Theorem 2.2.10.

Corollary 2.3.15 *For an isolated complete intersection singularity* $(X, 0) \subset (\mathbb{C}^p, 0)$ *defined by a regular sequence* $F = (f_1, \dots, f_c)$ *in* $\mathbb{C}\{x\}$ *the flat family (2.38) induced from a miniversal unfolding of* $F \in \mathbb{C}\{x\}^{1\times c}$ *is a semi-universal deformation of* $(X, 0)$.

In particular, the comparison map Φ in (2.38) is an isomorphism.

2.3.4 Cohen-Macaulay Codimension 2 Singularities

The particular interest in Cohen-Macaulay singularities of codimension 2 stems from the celebrated Hilbert-Burch theorem, see [57] and [19], or [35] for a modern textbook account. We reproduce the version found in [13, Theorem 1.4.17]:

Theorem 2.3.16 *Let* R *be a Noetherian ring and* $I \subset R$ *an ideal with a free resolution*

$$0 \longrightarrow R^m \xrightarrow{A} R^{m+1} \xrightarrow{f} R \longrightarrow R/I \longrightarrow 0. \tag{2.41}$$

Then there exists an R*-regular element* b *such that* $I = b \cdot \langle A^{\wedge m} \rangle$. *If* I *is projective, then* $I = \langle b \rangle$, *and if the projective dimension of* I *is* 1, *then* $\langle A^{\wedge m} \rangle$ *is perfect of grade two.*

Conversely, if $A\colon R^m \to R^{m+1}$ *is an* R*-linear map with* grade $\langle A^{\wedge m}\rangle \geq 2$*, then* $I = \langle A^{\wedge m}\rangle$ *has a free resolution as above.*

As was already discussed in Sect. 2.1, we may suppose that the underlying ring R is a regular local ring. This leads to the following

Corollary 2.3.17 *Suppose* $I \subset \mathbb{C}\{x_1, \ldots, x_p\}$ *is an ideal with* $\mathbb{C}\{x\}/I$ *Cohen-Macaulay of dimension* $p-2$*. Then* I *has a resolution of the form (2.41) for some matrix* $A \in \mathbb{C}\{x\}^{(m+1)\times m}$ *and* $I = \langle A^{\wedge m}\rangle$ *is generated by the maximal minors of* A*.*

In other words: Any Cohen-Macaulay singularity $(X, 0) \subset (\mathbb{C}^p, 0)$ of codimension 2 is determinantal for some $m \times (m+1)$-matrix in a canonical way. This differs drastically from the general case where one needs to specify the matrix in order to turn $(X, 0)$ into a determinantal singularity; cf. Pinkham's example in Example 2.3.13.

Proof of Corollary 2.3.17 Since $\mathbb{C}\{x\}$ is regular, the Auslander-Buchsbaum formula[15] implies that $\mathbb{C}\{x\}/I$ has a free $\mathbb{C}\{x\}$-resolution of length 2, which must be of the form

$$0 \longrightarrow \mathbb{C}\{x\}^m \xrightarrow{A} \mathbb{C}\{x\}^n \xrightarrow{f} \mathbb{C}\{x\} \longrightarrow \mathbb{C}\{x\}/I \longrightarrow 0$$

for some matrices A and f. The functor $\mathrm{Quot}(\mathbb{C}\{x\}) \otimes -$ is exact, taking $\mathbb{C}\{x\}/I$ to zero. Hence, applying it to the above sequence, we obtain a short exact sequence of $\mathrm{Quot}(\mathbb{C}\{x\})$-vector spaces and we see that necessarily $n = m + 1$. Thus, the Hilbert-Burch theorem applies and $I = b \cdot \langle A^{\wedge m}\rangle$ for some non-zerodivisor b.

This element b must in fact be a unit, for if it was not, there would necessarily be a primary component of I of height ≤ 1, contradicting the equidimensionality of the Cohen-Macaulay scheme $\mathbb{C}\{x\}/I$. We may therefore deliberately assume that the ideal $I = \langle A^{\wedge m}\rangle$ was generated by the maximal minors of A already. □

Schaps has observed that the Hilbert-Burch theorem can also be very well applied in the context of deformations. In [90] and [91] she pursues Schlessinger's approach to deformation theory [92] for affine algebraic determinantal schemes, in particular those which are Cohen-Macaulay of codimension 2. Her results can easily be adapted to the case of complex analytic singularities and, rephrasing them accordingly, she establishes the following, cf. [90, Corollary 1]:

Proposition 2.3.18 *Let* $(X, 0) \subset (\mathbb{C}^p, 0)$ *be Cohen-Macaulay of codimension 2, endowed with its canonical determinantal structure for some matrix* $A \in \mathbb{C}\{x\}^{m\times(m+1)}$*. Then a family* $(X, 0) \hookrightarrow (\mathscr{X}, 0) \xrightarrow{\pi} (\operatorname{Spec} B, 0)$ *over some Artinian ring* B *with special fiber* $(X, 0)$ *is flat, if and only if there exists a matrix* $\mathbf{A} \in (\mathbb{C}\{x\}\otimes_{\mathbb{C}} B)^{m\times(m+1)}$ *such that the germ* $(\mathscr{X}, 0)$ *is determinantal with matrix* $\mathbf{A}$*.*

[15] See e.g. [13, Theorem 1.3.3].

In other words: Any formal, infinitesimal deformation of a Cohen-Macaulay singularity of codimension 2 is determinantal for its canonical determinantal structure. In particular, this holds for the first order deformations which leads to an explicit description of the space $T^1_{X,0}$ in (2.37) in its "matrix form":

Corollary 2.3.19 *For $(X, 0) \subset (\mathbb{C}^p, 0)$ as in Proposition 2.3.18 one has a canonical isomorphism*

$$T^1_{\mathrm{GL}}(A) \cong T^1_{X,0}.$$

Proof This was explicitly carried out by the first named author in [37, Lemma 2.6] and [37, Lemma 2.7]. □

The deformation theory of Cohen-Macaulay codimension 2 singularities is unobstructed, cf. Definition 2.3.12, so there exists a semi-universal deformation for every such $(X, 0) \subset (\mathbb{C}^p, 0)$ with $\dim T^1_{X,0} = \tau < \infty$ over a smooth base $(\mathbb{C}^\tau, 0)$, cf. [20, Proposition 7.1.37]. Again, this semi-universal deformation can be derived from any $\mathbb{C}$-basis of $T^1_{X,0}$ in the same way as for isolated complete intersection singularities in the previous section. Using the explicit identification from Corollary 2.3.19 we find:

Corollary 2.3.20 *Let $(X, 0) \subset (\mathbb{C}^p, 0)$ be an isolated Cohen-Macaulay codimension 2 singularity with its canonical determinantal structure for a matrix $A \in \mathbb{C}\{x\}^{m\times(m+1)}$. Then the flat family (2.38) induced from a miniversal unfolding of A is a semi-universal deformation of $(X, 0)$.*

Again, the comparison map Φ in (2.38) can be chosen to be an isomorphism.

2.3.5 Gorenstein Singularities in Codimension 3

Similar to the case of Cohen-Macaulay codimension 2 singularities, Gorenstein singularities of codimension 3 are equipped with a canonical Pfaffian structure. This was established by Buchsbaum and Eisenbud in [17, Theorem 2.1]:

Theorem 2.3.21 *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let $n > 0$ be an integer and $A \in R^{(2n+1)\times(2n+1)}$ a matrix with entries in $\mathfrak{m}$. Suppose that the ideal $I = \langle A_{\mathrm{sk}}^{\wedge n}\rangle$ has grade 3. Then R/I is Gorenstein and I is minimally generated by $2n+1$ elements.*

Conversely, every ideal I of grade 3 with R/I Gorenstein arises this way.

They furthermore show that the free resolution of an ideal I as in Theorem 2.3.21 is given by the complex

$$0 \longrightarrow R \xrightarrow{f^T} R^{2n+1} \xrightarrow{A} R^{2n+1} \xrightarrow{f} R \longrightarrow R/I \longrightarrow 0 \tag{2.42}$$

where $f = (f_1, \ldots, f_{2n+1})$ is the $1 \times (2n + 1)$-matrix with the generators of $\langle A_{\mathrm{sk}}^{\wedge n} \rangle$ as entries. One may therefore follow the same arguments as in the Cohen-Macaulay codimension 2 case (cf. also the proof of Lemma 2.1.15) to show that every deformation of a Gorenstein singularity of codimension 3 is determinantal.

Waldi has used the results by Buchsbaum and Eisenbud to show that the deformation theory of Gorenstein ideals of codimension 3 is unobstructed, see [108, Satz 1]. Whenever such an ideal defines an isolated singularity, Waldi therefore finds [108, Satz 2]:

Theorem 2.3.22 *Let $(X, 0) \subset (k^p, 0)$ be an isolated algebraic Gorenstein singularity over an algebraically closed field k which is analytically irreducible and of codimension 3 with $p \leq 9$. Then the semi-universal deformation of $(X, 0)$ has a smooth base and the generic fiber of this deformation is also smooth.*

However, we should note that in this setting—at least to the author's knowledge—the interplay of $T^1_{\mathrm{GL}}(A)$ and $T^1_{X,0}$ has not yet been investigated. In particular, no classification of simple Gorenstein singularities in codimension 3 has been done as of this writing.

2.3.6 Rational Surface Singularities

Determinantal deformations also appear in the study of rational surface singularities. Recall that a singularity $(X, 0) \subset (\mathbb{C}^p, 0)$ of dimension $d \geq 2$ is called *rational* if there exists a *resolution of singularities*

$$\rho\colon (Z, E) \to (X, 0)$$

such that one has

$$R^i \rho_* \mathscr{O}_Z = \begin{cases} \mathscr{O}_X & \text{if } i = 0, \\ 0 & \text{otherwise} \end{cases} \tag{2.43}$$

for the higher direct images of the structure sheaf $\mathscr{O}_Z$. Here, Z is smooth, $E = \rho^{-1}(X_{\mathrm{sing}})$ is the preimage of the singular locus of X and ρ is an isomorphism outside E. We refer to [20, Chapter 3] for a discussion of resolutions of singularities.

For surface singularities, it is customary to require $\rho\colon (Z, E) \to (X, 0)$ to be a *good resolution*. This means that the set $E = \bigcup_i E_i$ is a *simple normal crossing divisor* with a decomposition into smooth, irreducible components E_i. To each component, one can assign two integers, the *genus* of E_i and its *self intersection* in Z. A good resolution is called *minimal* if there are no components $E_i \cong \mathbb{P}^1$ with self intersection -1 which meet only one or two other components of the exceptional set. Whenever such components occur in an arbitrary, good resolution, they can be blown down to a minimal good resolution. For a discussion of the existence and uniqueness of such minimal good resolutions for surfaces we refer to [20, Chapter 2] "The topology of surface singularities" by Michel.

The genera and intersection multiplicities of the components E_i of a good resolution can be summarized in its *dual graph*. Surface singularities are often described in terms of this resolution graph rather than by explicit equations; for instance, such dual graphs have already appeared in [20, Chapter 10] on "Finite dimensional Lie algebras in singularities". Note, however, that in general such a dual graph does not determine the singularity up to analytic isomorphism.[16]

Let $(X, 0)$ be a rational surface singularity, $(X, 0) \hookrightarrow (\mathscr{X}, 0) \xrightarrow{\pi} (S, 0)$ its semi-universal deformation, and suppose $\rho\colon Z \to X$ is a minimal good resolution of singularities for $(X, 0)$ as above. Artin has shown in [6, Theorem 3] that there exists a smooth space $(R, 0)$ parametrizing those deformations of Z that *blow down* to deformations of $(X, 0)$, i.e. the resolution ρ extends to a projection of the total space $\mathscr{Z}$ of the deformation of Z giving rise to another total space $\rho(\mathscr{Z})$ of a deformation of X. This provides a commutative diagram

$$\begin{array}{ccccc} Z & \hookrightarrow & \mathscr{Z} & & \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho} & & \\ X & \hookrightarrow & \rho(\mathscr{Z}) & \longrightarrow & \mathscr{X} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle\pi} \\ \{0\} & \hookrightarrow & R & \xrightarrow{\Phi} & S. \end{array} \tag{2.44}$$

where $\Phi\colon (R, 0) \to (S, 0)$ is the comparison map to the base of the semi-universal deformation. Artin has shown that Φ is finite and maps surjectively onto an irreducible component $(S', 0)$ of $(S, 0)$ which is now called the *Artin component* of the base $(S, 0)$.

Since the central fiber Z in (2.44) is smooth and proper over X, the family $\mathscr{Z} \to R$ is topologically trivial due to Ehresmann's Lemma (cf. Lemma 2.5.17 below). In particular, this entails that $\rho\colon Z_t \to X_t$ is a resolution of singularities for every fiber over $t \in R$ in a neighborhood of 0. For this reason, a diagram like (2.44) is also called a *resolution in family* or a *simultaneous resolution*.

Example 2.3.23 One instance of diagram (2.44) can be constructed for the A_1-singularity $(X, 0) \subset (\mathbb{C}^3, 0)$ which has already appeared in Example 2.3.13, (i) (Fig. 2.8). It is defined by the equation $f = x^2 - yz$, but can also be regarded as a

[16] Those normal surface singularities for which this is the case are called "taut", see [72].

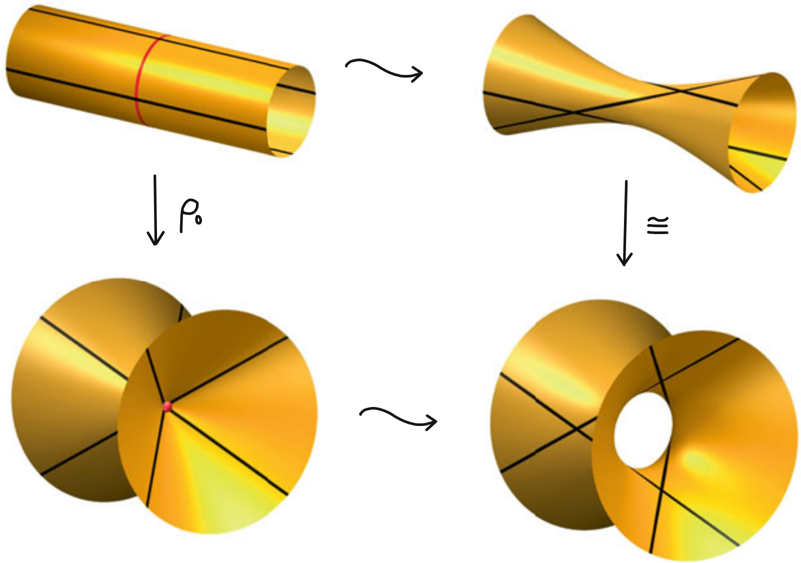


Fig. 2.8 A resolution in family for the A_1-surface singularity (and some linear sections) in the positions of the upper left square in (2.44). The deformation of the Tjurina transform does not change the topology, but forgets about the zero section (red) of the disc bundle. The projection of the smooth fibers is an intrinsic isomorphism despite the different embeddings

determinantal singularity of type 2 for the matrix

$$A = \begin{pmatrix} x & y \\ z & x \end{pmatrix}.$$

One can check that a resolution of singularities for $(X, 0) = (X_A^2, 0)$ is given by the Tjurina transform $\hat{X}_A^2 \subset \mathbb{C}^3 \times \mathbb{P}^1$. Furthermore, the Tjurina transformation in family for the deformation induced by the unfolding

$$\mathbf{A}(x, y, z; t) = A(x, y, z) - t \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

on a parameter t blows down to the deformation of $(X, 0)$ given by the perturbation $f - t^2$. This furnishes the left hand side of (2.44). As predicted by Artin's theorem, the comparison map $\Phi\colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ taking t to t^2 is a $2:1$-cover of the base of the semi-universal deformation of $(X, 0)$.

The previous example is also an instance of a general result on simultaneous resolutions for rational double points due to Brieskorn [10], which was independently proved by Tjurina [100]. We cite the summarized form from [104], cf. also [6, Theorem 2].

Theorem 2.3.24 *The versal deformation of a rational double point resolves simultaneously after a Galois base change.*

Note that Brieskorn's and Tjurina's constructions do not necessarily involve the choice of a matrix structure and Tjurina transformation in family as in Example 2.3.23 above. However, a resolution of singularities can at times be constructed using only Tjurina modifications; see [81] for further examples.

In [104] Wahl has addressed the question how to obtain equations and even free resolutions of the defining ideals for rational singularities, given their dual graphs. He shows in [104, Proposition 3.2]:

Proposition 2.3.25 *Suppose* $(X, 0) \subset (\mathbb{C}^p, 0)$ *is a rational surface singularity of embedding dimension* $p \geq 4$. *If* $(X, 0)$ *is determinantal, then it is determinantal of type* $(2, p-2, 2)$.

Using this result, he then obtains in [104, Theorem 3]:

Theorem 2.3.26 *Let* $(X, 0)$ *be a determinantal rational surface singularity of embedding dimension* p. *Then the dual graph of* $(X, 0)$ *consists of one* $-(p-1)$ *curve and (possibly) some* -2 *curves.*

The converse of this theorem was later established by Röhr [86] and by de Jong [31] (who also produces explicit matrices) so that indeed every rational surface singularity with this particular configuration in its dual graph is in fact determinantal.

Example 2.3.27 Consider the normal surface singularity $(X_A^2, 0) \subset (\mathbb{C}^4, 0)$ given by the matrix

$$A = \begin{pmatrix} x & y & z \\ y^2 & z & w \end{pmatrix}$$

from Example 2.2.20. Let $(u : v)$ be the homogeneous coordinates of $\mathbb{P}^1$ and $\hat{X}_A^2 \subset \mathbb{C}^4 \times \mathbb{P}^1$ the Tjurina transform of $(X_A^2, 0)$ defined by the equations

$$\begin{pmatrix} u & v \end{pmatrix} \cdot \begin{pmatrix} x & y & z \\ y^2 & z & w \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}.$$

Using the explicit equations it is easy to see that on the chart $\{u \neq 0\}$ the variables x, y and z can be eliminated so that $\hat{X}_A^2 \cap \{u \neq 0\}$ is smooth. In the other chart $\{v \neq 0\}$ we find a single A_1-hypersurface singularity at the origin, which is given by the equation

$$y^2 + x \cdot \frac{u}{v} = 0$$

after elimination of z and w. This singularity can be resolved by a single classical blowup so that the overall exceptional set is a normal crossing divisor.

In a subsequent article [105], Wahl then investigates the deformation theoretic behaviour of rational surface singularities and finds in [105, Theorem 3.2]:

Theorem 2.3.28 *For a determinantal rational surface singularity $(X, 0)$ of multiplicity $e \geq 3$ the Artin component $(S', 0)$ of deformations of $(X, 0)$ that admit a resolution in family consists precisely of the determinantal deformations.*

Note that for any rational surface singularity, the multiplicity e is equal to $p - 1$ where p is its embedding dimension, cf. [5, Corollary 6]. Wahl furthermore observes that for the determinantal deformations, the resolution in family always factors through the Tjurina transformation in family. In this context, the Tjurina transform $\hat{X}^2_A$ of $(X, 0)$ is obtained from the resolution (Z, E) by blowing down all the -2 configurations, cf. Theorem 2.3.26.

Example 2.3.29 We continue with Example 2.3.27. Given the A_1-hypersurface singularity in the chart $\{v \neq 0\}$ we already saw in Example 2.3.23 how to construct a resolution in family: Write the local equation for the singularity as the determinant of a 2×2-matrix

$$y^2 + \frac{u}{v} \cdot x = \det \begin{pmatrix} y & x \\ -\frac{v}{u} & y \end{pmatrix}$$

and resolve it by Tjurina modification rather than the standard blowup. One can check that the local resolution in family is compatible with the Tjurina transformation in family so that we indeed obtain a full resolution in family for any stabilization of the defining matrix A.

2.3.7 *Further References and Techniques*

2.3.7.1 Construction of "Versal Determinantal Deformations"

In view of the three particular cases of complete intersection, Cohen-Macaulay codimension 2, and Gorenstein singularities of codimension 3 discussed above and given the fact that the theory of unfoldings of map germs is in many respects much simpler than the theory of deformations of germs $(X, 0) \subset (\mathbb{C}^p, 0)$ in general, one might be tempted to develop a theory of "semi-universal determinantal deformations" for arbitrary determinantal singularities $(X^s_A, 0) \subset (\mathbb{C}^p, 0)$ by fixing the defining matrix $A \in \mathbb{C}\{x\}^{m \times n}$ and restricting to those deformations coming from unfoldings of A, i.e. the image of the comparison map Φ in (2.38). We already saw in Pinkham's example and, more generally, the discussion of the Artin component in Sect. 2.3.6 that Φ is not surjective in general. But for determinantal singularities it is nevertheless a natural question, whether the image of Φ can be reconstructed directly from the germ $(X^s_A, 0)$ and its fixed determinantal structure itself rather than from the unfoldings of the defining matrix A.

An attempt along these lines has been made by Schaps in [91] where she follows the classical approach to deformation theory due to Schlessinger [92] in order to define a functor of "A-determinantal deformations" for determinantal schemes with a fixed matrix A. The goal is then to prove that this functor has a pro-representable hull. However, one runs into technical difficulties caused by the fact in the general case, an infinitesimal determinantal deformation of $(X^s_A, 0)$ does not lift uniquely to an infinitesimal unfolding of A. One instance of this phenomenon can be found in Example 2.3.13, (iv) where we have a continuous modulus γ' of the defining matrix, the variation of which results in a trivial deformation of the associated determinantal singularity. This observation leads Schaps to define a "unique lifiting property":

Definition 2.3.30 Let $B' \to B$ be a surjection of Artin rings and $A \in B[x]$ a matrix with $x = x_1, \ldots, x_p$ a fixed set of variables. Then A is said to satisfy the unique lifting property, if for any two liftings $A_1, A_2 \in B'[x]$ of A, whose minors generate the same ideal, one has that A_1 is GL-equivalent to A_2.

Schaps then proceeds to show, [91, Proposition 1]:

Proposition 2.3.31 *Let $(X, 0) \subset (\mathbb{C}^p, 0)$ be an isolated determinantal singularity with a defining matrix $A \in \mathbb{C}\{x\}^{m\times n}$ satisfying the unique lifting property. Then the functor of A-determinantal deformations of $(X, 0)$ has a prorepresentable hull.*

2.3.7.2 Buchweitz' Criterion for Deformations to be Determinantal

The question as to when or under which conditions all deformations of a given singularity $(X^s_A, 0)$ are determinantal for the defining matrix A, has been addressed more generally by Buchweitz in his thesis in [18, Section 4.3] headed "Deformations d'un type donné et déploiements". Determinantal singularities appear as a special case in [18, Exemples 4.3.2 b)]. Before we can state the main theorem [18, Theorem 4.7.1] in its adapted version for determinantal singularities, we need to introduce some further mathematical notions.

Recall, that given a morphism of analytic local k-algebras $\varphi : R \longrightarrow S$ and an R-module M of finite type, φ is said to be transversal to M, if $\mathrm{Tor}_i^R(M, S)$ vanishes for all $i > 0$. For an R-module of finite type, the Auslander module $D(M)$ of M is defined as the cokernel of the dual of a free presentation of M. More precisely, let

$$F_1 \xrightarrow{\psi} F_0 \longrightarrow M \longrightarrow 0$$

be a free presentation of M, then dualizing

$$0 \longrightarrow M^\vee \longrightarrow F_0^\vee \xrightarrow{\psi^\vee} F_1^\vee \longrightarrow D(M) \longrightarrow 0$$

provides a presentation of $D(M)$. For an analytic space germ $(X, 0) \subset (\mathbb{C}^p, 0)$ with ideal $I_{X,0} \subseteq \mathbb{C}\{x\}$ the *Auslander module of* $(X, 0)$ is then the module $D(I_{X,0}/I^2_{X,0})$

whose homological properties are independent of the choice of embedding and free resolution, cf. [18, Section 4.5]. With this notation we have the following adaptation of [18, Theorem 4.7.1]:

Theorem 2.3.32 *Let* $(X, 0) = (X^s_A, 0) \subseteq (\mathbb{C}^p, 0)$ *be a determinantal singularity of type* s *defined by a matrix* $A \in \mathbb{C}\{x\}^{m\times n}$. *Then the following properties are equivalent:*

(i) *Each first order deformation of* $(X, 0)$ *is determinantal for* A.
(ii) *Each analytic deformation of* $(X, 0)$ *is determinantal for* A.
(iii) *The generic determinantal singularity* $(M^s_{m,n}, 0) \subset (\mathbb{C}^{m\times n}, 0)$ *is rigid and* $(X, 0)$ *is unobstructed.*
(iv) *The Auslander module of the generic determinantal singularity* $(M^s_{m,n}, 0)$ *is transversal to the map* $A : (X, 0) \longrightarrow (M^s_{m,n}, 0)$ *and* $(M^s_{m,n}, 0)$ *is rigid.*

If additionally the vector space dimension of $T^1_{X,0}$ *is finite, then this is also equivalent to the property that there is an unfolding as in Lemma 2.1.15 with smooth analytic base which induces a versal deformation of* $(X, 0)$.

Remark 2.3.33 The generic determinantal singularities $(M^s_{m,n}, 0) \subset (\mathbb{C}^{m\times n}, 0)$ are all rigid except for the generic determinantal hypersurface singularities $(M^m_{m,m}, 0)$ defined by $f = \det = 0$ (or $\mathrm{Pf} = 0$ for skew symmetric matrices). This was proved independently by Svanes [94] and Jähner [59]; see [14, Chapter 15.C] for a textbook account. Therefore, given characterization 3. in Theorem 2.3.32, the only condition on a given determinantal singularity $(X^s_A, 0)$ that really needs to be checked is the unobstructedness.

Remark 2.3.34 Theorem 2.3.32 suggests to review Corollary 2.3.15 for complete intersections and Corollary 2.3.20 for isolated Cohen-Macaulay codimension 2 singularities. Given the explicit identifications $T^1_{\mathrm{GL}}(A) \cong T^1_{X,0}$ from Lemma 2.3.14 and respectively Corollary 2.3.15, the coincidence of a semi-universal deformation of the determinantal singularity with the deformation induced from a miniversal unfolding of the defining matrix now follows directly from the implications $(i) \Rightarrow (iii) \Rightarrow (ii)$ in Theorem 2.3.32.

It is expected that the construction of a "matrix-T^1" as in Corollary 2.3.19 for ICMC2 singularities is possible in many more cases. For non-maximal minors, however, one can in general only expect surjective maps $T^1_{\mathrm{GL}}(A) \to T^1_{X^s_A,0}$ which have a non-trivial kernel. This can, for instance, be observed from Table 2.18 for the simple ICMC2 fourfold singularities given by matrices $A: (\mathbb{C}^6, 0) \to (\mathbb{C}^{2\times 3}, 0)$. In this case not only $(X^2_A, 0)$, but also the zero-dimensional complete intersections $(X^1_A, 0)$ are determinantal singularities. The GL-codimensions $\tau_{\mathrm{GL}}(A) = \dim_{\mathbb{C}} T^1_{\mathrm{GL}}(A)$ of the defining matrices and the Tjurina numbers $\tau(X^1_A, 0) = \dim_{\mathbb{C}} T^1_{X^1_A,0}$ of the singularities $(X^1_A, 0)$ are listed in the fourth and the fifth column, respectively. We find series for which these numbers coincide and others where $\tau(X^1_A, 0) \le \tau_{\mathrm{GL}}(A)$ with a strict inequality in general.

A geometric reason for this discrepancy for non-maximal minors is the following. The GL-equivalence of matrices is sensitive to the position of the image of the defining matrix A relative to all strata $V^r_{m,n}$ of the rank stratification. For non-maximal minors of size $s < \min\{m, n\}$ only the strata $V^r_{m,n}$ of $M^s_{m,n}$ with $r < s$ are relevant for the deformations of $(X^s_A, 0)$. Those inifitesimal unfoldings varying only the position of the image of A relative to higher dimensional strata will therefore be discarded by the comparison map $\Phi\colon T^1_{\mathrm{GL}}(A) \to T^1_{X^s_A,0}$.

2.4 Classification of Simple Singularities

Recall that a singularity is called *simple* if only a finite number of non-equivalent singularities appear in its versal family. Arnold gives a complete list of simple isolated hypersurface singularities of arbitrary dimension in his article [2] from 1972 and the classification of all simple isolated complete intersection singularities was completed by Giusti [44] in the mid 1980s. The question which singularities are simple is still not fully answered for determinantal singularities, but there exist classifications for symmetric square matrices by Bruce in [11], for square matrices by Bruce and Tari in [12], and for skew-symmetric ones by Haslinger in [56] (incomplete), as well as for isolated Cohen-Macaulay codimensions 2 singularities by the first named author and Neumer in [37] and [39]. Note that, in contrast to simple hypersurfaces or complete intersections, a simple determinantal singularity does not need to be smoothable or isolated, as there are rigid non-isolated determinantal singularities and any rigid singularity is simple for trivial reasons. We give a brief overview on known classification results, leaving the explicit tables to the appendix of this article.

2.4.1 Singularities of Square Matrices

In [12] Bruce and Tari classify all simple singularities for square matrices

$$A\colon (\mathbb{K}^p, 0) \to (\mathbb{K}^{m\times m}, 0)$$

up to GL-equivalence, where either $\mathbb{K} = \mathbb{R}$ and A smooth, or $\mathbb{K} = \mathbb{C}$ and A holomorphic, the treatement of the real case made possible by Damon's theory, [12, Remark 2.8]. In the Tables 2.7, 2.9, 2.10, and 2.11 the consideration of the real case occasionally leads to a $\pm$-sign with different singularities over $\mathbb{R}$. Over $\mathbb{C}$ these signs can be omitted.

Associated to the matrices A as above, Bruce and Tari also consider *determinantal hypersurface singularities*[17] (abbreviated by DHS) in the following)

$$(X_A^m, 0) = (\{\det A = 0\}, 0) \subset (\mathbb{C}^p, 0)$$

which are defined by the equation $f = \det A$. Moreover, they also consider the Tjurina transformation[18] and, as it turns out, the simple singularities of A are closely related to simple singularities of both, the determinantal hypersurface $(X_A^m, 0)$ and of its Tjurina transforms $\hat{X}_A^m$. Besides the simple isolated hypersurface singularities, the well known A-D-E-singularities that were classified by Arnold up to right- or $\mathscr{R}$-equivalence in [2], there also appear the simple singularities from the classification for functions on manifolds with boundary up to $\mathscr{R}_\delta$-equivalence from [4].

As usual, Bruce and Tari assume all matrices A to have entries in the maximal ideal $\mathfrak{m}$ of $\mathbb{C}\{x_1, \ldots, x_p\}$ so that the *corank* of the matrix $A(0)$, i.e. the codimension of its image, is equal to the size of the matrix m. A key handle to the classification is to also consider the corank of the differential

$$\mathrm{d}A(0)\colon T_0\mathbb{K}^p \to T_0\mathbb{K}^{m\times m}$$

of A at the origin.

With all this notation at hand, we can now reproduce their main results:

Theorem 2.4.1

(i) *When $p = 1$, all finitely* GL*-determined germs are simple and* GL*-equivalent to a germ of the form* $\mathrm{diag}(x^{\alpha_1}, x^{\alpha_2}, \ldots, x^{\alpha_m})$ *where $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_m$. This germ has* GL*-codimension* $\sum_{i=1}^m (2(m-i)+1)\alpha_i - 1$. *Its Tjurina transform is empty.*

(ii) *When the corank of the differential* $\mathrm{d}A(0)$ *is zero there is a normal form*

$$A\colon (\mathbb{K}^{m^2}, 0) \times (\mathbb{K}^l, 0) \to (\mathbb{K}^{m\times m}, 0), \quad (x, z) \mapsto A(x), \quad a_{i,j} = x_{i,j}.$$

where the variables $z_1, \ldots, z_l$ are redundant.

(iii) *When the corank of the differential is one, there are two cases.*

(a) A normal form $A\colon (\mathbb{K}^{m^2-1}, 0) \times (\mathbb{K}^l, 0) \to (\mathbb{K}^{m\times m}, 0)$ *given by*

$$\left(\sum_{i=2}^m x_{i,i} + f(z)\right) \cdot E_{1,1} + \sum_{(i,j)\neq(1,1)} x_{i,j} \cdot E_{i,j}$$

[17] In [12] these are called the *discriminants* of the matrix.

[18] Tjurina transforms are called *criminants* in [12].

where $E_{i,j}$ *is the matrix with a* 1 *in the* (i,j)*-th place and zeroes elsewhere,* $\{x_{i,j}\}_{(i,j)\neq(1,1)}$ *are the coordinates of the first factor* $\mathbb{K}^{m^2-1}$*, and* $f\colon(\mathbb{K}^l,0)\to(\mathbb{K},0)$ *is one of Arnold's* $\mathscr{R}$*-simple germs (see Table 2.3 in the appendix). The* GL*-codimension of* A *coincides with the Tjurina number of* f*.*

(b) A normal form A as above given by

$$\left(\sum_{i=2}^{m-1} x_{i,i} + f(x_{m,m},z)\right)\cdot E_{1,1} + \sum_{(i,j)\neq(1,1)} x_{i,j}\cdot E_{i,j}$$

where $f\colon(\mathbb{K},0)\times(\mathbb{K}^l,0)\to(\mathbb{K},0)$ *is one of Arnold's* $\mathscr{R}_\delta$*-simple germs of singularities of functions on manifolds with boundary (see Table 2.4). The* GL*-codimension of* A *coincides with the* $\mathscr{R}_\delta$*-codimension of* f*.*

In both cases, the Tjurina transforms of the singularities are smooth.

(iv) *When the corank of the differential is two, then* $m=3$ *and* $p=7$*. The simple matrices in this class can be derived from the symmetric ones*[19] *in Table 2.8 by addition of the matrix*

$$U=\begin{pmatrix}0 & u_{12} & u_{13}\\ -u_{12} & 0 & u_{23}\\ -u_{13} & -u_{23} & 0\end{pmatrix}.$$

The Tjurina transforms are all smooth.

(v) *When* $m=2$*, the simple germs that are not covered by the preceding items are given in Tables 2.9 and 2.10.*

(vi) *When* $m=3$ *the simple germs that are not covered by the preceding items are given in Table 2.7.*[20]

The following classification of simple *symmetric* matrices by Bruce [11, Theorem 1.1] precedes the above classification by one year. The setting is slightly different, since only *holomorphic* matrices

$$A\colon(\mathbb{C}^p,0)\to(\mathbb{C}^{m\times m}_{\mathrm{sym}},0)$$

are considered up to symmetric GL-equivalence, see Remark 2.2.5.

[19] Goryunov has informed us that there were mistakes in this part of the original classification in [12, Paragraph C, p 757]. The corrected description given here is taken from [46, Theorem 3.7].

[20] Again, the normal forms given in Table 2.7 are not the original ones found by Bruce and Tari. We were informed by Goryunov about a mistake in the original classification and the given table has the correct matrices up to GL-equivalence.

Theorem 2.4.2

(i) *When* $p = 1$*, all finitely* GL*-determined germs are simple and* GL*-equivalent to a germ of the form* $\operatorname{diag}(x^{\alpha_1}, x^{\alpha_2}, \dots, x^{\alpha_m})$ *where* $\alpha_1 \le \alpha_2 \le \dots \le \alpha_m$*. This germ has* GL*-codimension* $\sum_{i=1}^{m}((m-i)+1)\alpha_i - 1$.

(ii) *When the corank of the differential* $\mathrm{d}A(0)$ *is zero there is a normal form*

$$A\colon (\mathbb{C}^N, 0) \times (\mathbb{C}^l, 0) \to (\mathbb{C}^{m\times m}_{\mathrm{sym}}, 0), \quad (x, z) \mapsto A(x), \quad a_{i,j} = x_{i,j}.$$

where $N = \frac{m(m+1)}{2}$ *and the variables* $z_1, \dots, z_l$ *are redundant.*

(iii) *When the corank of the differential is one, there are two cases.*

(a) A normal form $A\colon (\mathbb{C}^{N-1}, 0) \times (\mathbb{C}^l, 0) \to (\mathrm{Sym}_m(\mathbb{C}), 0)$ *given by*

$$\left(\sum_{i=2}^{m} x_{i,i} + f(z)\right) \cdot E_{1,1} + \sum_{(i,j)\neq(1,1)} x_{i,j} \cdot E_{i,j}$$

where $E_{i,j}$ *is the matrix with a 1 in the* (i,j)*-th place and zeroes elsewhere,* $\{x_{i,j}\}_{(i,j)\neq(1,1)}$ *are the coordinates of the first factor* $\mathbb{C}^{N-1}$*, and* $f\colon (\mathbb{C}^l, 0) \to (\mathbb{C}, 0)$ *is one of Arnold's* $\mathcal{R}$*-simple germs. The* GL*-codimension of A coincides with the Tjurina number of* f*.*

(b) A normal form A as above given by

$$\left(\sum_{i=2}^{m-1} x_{i,i} + f(x_{m,m}, z)\right) \cdot E_{1,1} + \sum_{(i,j)\neq(1,1)} x_{i,j} \cdot E_{i,j}$$

where $f\colon (\mathbb{C}, 0) \times (\mathbb{C}^l, 0) \to (\mathbb{C}, 0)$ *is one of Arnold's* $\mathcal{R}_\delta$*-simple germs of singularities of functions on manifolds with boundary. The* GL*-codimension of A coincides with the* $\mathcal{R}_\delta$*-codimension of* f*.*

(iv) *When* $m = p = 2$ *the* GL*-simple germs are given in Table 2.12.*

(v) *If* $m = 3$*,* $p = 2$ *the* GL*-simple germs are listed in Table 2.7.*

(vi) *If* $m = 3$*,* $p = 4$ *the* GL*-simple germs are given in Table 2.8.*

For skew-symmetric square matrices, only a partial classification of simple singularities is known from the thesis of Haslinger [56].

Theorem 2.4.3 *Let* $A\colon (\mathbb{C}^p, 0) \to (\mathbb{C}^{m\times m}_{\mathrm{sk}}, 0)$ *be a germ of skew-symmetric matrices.*

1. *([56, Proposition 4.6.1]) When* $p = 1$*, then A is* $\mathcal{G}_{\mathrm{sk}}$*-equivalent to a matrix of the form*

$$\bigoplus_{j=1}^{m'} x^{k_j} \cdot I^{\mathrm{sk}}_{s_j}$$

for some $m' \leq m$ *and integers* $0 < k_1 < k_2 < \cdots < k_{m'}$ *where* $I_s^{\mathrm{sk}} = \sum_{i=1}^{s} \left(E_{2i-1,2i} - E_{2i,2i-1}\right)$ *is the skew symmetric analogue of the* $2s \times 2s$*-identity matrix and the direct sum stands for taking successive diagonal blocks.*

2. *([56, Lemma 4.6.2 (i)]) When* $m = 2$ *any two matrices*

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$$

are $\mathscr{G}_{\mathrm{sk}}$*-equivalent if and only if a and b are* $\mathscr{K}$*-equivalent. In particular, the* $\mathscr{G}_{\mathrm{sk}}$*-simple matrices are given by the classification of* $\mathscr{R}$*-simple functions* $(\mathbb{C}^p, 0) \to (\mathbb{C}, 0)$.

3. *([56, Lemma 4.6.2 (ii)]) When* $m = 3$ *then A is finitely* $\mathscr{G}_{\mathrm{sk}}$*-determined if and only if the corresponding map germ* $(\mathbb{C}^p, 0) \to (\mathbb{C}^3, 0)$ *is finitely* $\mathscr{K}$*-determined and two matrices are* $\mathscr{G}_{\mathrm{sk}}$*-equivalent if and only if this holds for the corresponding map germ. In particular, the* $\mathscr{G}_{\mathrm{sk}}$*-simple matrices are given by the* $\mathscr{K}$*-classification of simple germs* $(\mathbb{C}^p, 0) \to (\mathbb{C}^3, 0)$.
4. *([56, Theorem 6.1.1]) The* $\mathscr{G}_{\mathrm{sk}}$*-simple germs for* $p = 2$ *and* $m = 4$ *are listed in Table 2.13.*

Arnold's simple hypersurface singularities from the A-D-E-classification in Table 2.3 and the boundary singularities in Table 2.4 make numerous appearances among the simple matrices in Theorems 2.4.1, 2.4.2, and 2.4.3. While this is not really a surprise, the general theme of the interplay between the different lists is still far from being completely understood.

Remark 2.4.4 In [48] Goryunov and Zakalyukin relate the simple symmetric matrix singularities in two variables from Theorem 2.4.2 (Tables 2.12 and 2.7) to Arnold's simple hypersurfaces as follows.

As their name suggests, the A-D-E-singularities correspond to *Dynkin diagrams* which have already appeared in [20, Chapter 10] and [20, Chapter 8]. These diagrams encode the dual resolution graphs of the simple hypersurface singularities in dimension 2, see [20, Section 10.2.1], and, as suggested by the resolution in family for these singularities, Theorem 2.3.24, they can also be used to describe the vanishing homology and monodromy operations, see e.g. [20, Chapter 8].

Originally, the Dynkin diagrams classify the Weyl groups of the semi-simple Lie algebras. These are subgroups of the linear automorphisms of Euclidean space generated by reflections on hyperplanes. In a case-by-case analysis, Goryunov and Zakalyukin found that the simple symmetric matrices from Theorem 2.4.2 correspond uniquely to pairs $(X; Y)$ of a Weyl group X and a subgroup Y obtained by omitting certain reflections. This can be stated more precisely in terms of Dynkin diagrams: One obtains the *affine* Dynkin diagram of Y by deletion of either two 1-vertices for the corank 2 families (Table 2.12) or one 2-vertex for the corank 3 families (Table 2.7 from the affine Dynkin diagram of X. These pairs of Weyl groups are listed in the last columns of Tables 2.12 and 2.7, respectively.

Interestingly, the correspondence of simple symmetric matrices A with pairs of Weyl groups $(X; Y)$ is established via a study of the miniversal unfoldings of the matrix A and the associated hypersurface $f = \det A$. It turns out that for all the singularities in question one has $\mu = \mu_f = \tau_{\mathrm{GL}}(A)$, cf. Theorem 2.5.8 below. The space $\mathbb{C}^\mu$ can be regarded as the configuration space for the reflections in X and Y and then the bases of the respective miniversal unfoldings can be identified with $\mathbb{C}^{\tau_{\mathrm{GL}}(A)} \cong \mathbb{C}^\mu/X$ and $\mathbb{C}^{\mu_f} \cong \mathbb{C}^\mu/Y$, respectively so that the comparison map Φ from (2.38) completes the quotient maps to a diagram

$$\begin{array}{ccc} (\mathbb{C}^\mu, \mathscr{A}_X) & \xrightarrow{\ /Y\ } & (\mathbb{C}^\mu, \Sigma) \\ & {\scriptstyle /X}\searrow \quad \swarrow{\scriptstyle \Phi} & \\ & (\mathbb{C}^\mu, \Delta) & \end{array}$$

where $\mathscr{A}_X$ is the configuration of mirrors of X in $\mathbb{C}^\mu$, $\Sigma \subset \mathbb{C}^\mu$ is the discriminant of f, and $\Delta \subset \mathbb{C}^\mu$ the matrix discriminant of A, see [48, Corollary 3.10]. Then Φ is a finite covering of order $|X : Y|$, branched over the discriminant Δ of the function f.

Remark 2.4.5 Recently in [46], the investigation of the discriminants, bifurcation diagrams, and the monodromy of simple matrix singularities has been pushed further by Goryunov to also comprise those matrices A for which the associated hypersurface $f = \det(A)$ is not smoothable via determinantal deformations or has non-isolated singular locus (cf. Lemma 2.3.3). This has revealed a further correspondence of the simple symmetric/square matrices in Table 2.8 with $\mathscr{R}_{\mathrm{odd}}$-simple singularities of odd functions[21] on $(\mathbb{C}^2, 0)$ and $\mathscr{K}_{\mathrm{odd}}$-simple ICIS[22] in $(\mathbb{C}^3, 0)$. These associated odd functions and symmetric ICIS are listed[23] in the last two columns of Table 2.8. In this case, the correspondence is established by a natural identification of the *bifurcation diagrams* of the matrix with those of the respective odd functions and symmetric space curves, see [46, Proposition 6.7].

[21] A function $f\colon (\mathbb{C}^p, 0) \to (\mathbb{C}, 0)$ is *odd* if it changes sign under the central symmetry of $(\mathbb{C}^p, 0)$. These are classified up to right equivalence by diffeomorphisms of $(\mathbb{C}^p, 0)$ which preserve this symmetry.

[22] These are defined as centrally symmetric isolated complete intersection curves $(C, 0) \subset (\mathbb{C}^3, 0)$ and classified by the subgroup of $\mathscr{K}$ in which the group of diffeomorphisms $\mathscr{R}$ in the source is replaced by those preserving the symmetry.

[23] The notation used extends Giusti's list in Table 2.6 in a natural way (note that U_{11} and U_{13} are not simple if the symmetry condition is removed).

2.4.2 Cohen-Macaulay Codimension 2 Singularities

In [37] the first named author classified the simple isolated space curve singularities up to isomorphism of space germs. Later she extended this together with Neumer to all isolated Cohen-Macaulay codimension 2 singularities in [39]. By the Hilbert-Burch theorem (Theorem 2.3.16), this amounts to classifying singularities of matrices

$$A : (\mathbb{C}^p, 0) \longrightarrow (\mathbb{C}^{m\times(m+1)}, 0)$$

up to GL-equivalence.

Chronologically preceding the criteria for finite determinacy based on Damon's work, these articles use a weighted determinacy criterion followed by a matrix variant of the Arnold's rotating ruler method.

The main results can be summarized as follows

Theorem 2.4.6 *Isolated Cohen-Macaulay codimension* 2 *singularities with defining matrix* $A\colon (\mathbb{C}^p, 0) \to (\mathbb{C}^{m\times(m+1)}, 0)$ *exist only in the following cases:*

(i) *Fat points in* $(\mathbb{C}^2, 0)$*:* $p = 2$, $m \in \{1, 2\}$*; Giusti's list of complete intersections (*$m = 1$*) is included in Table 2.5, the one for* $m = 2$ *in Table 2.14.*
(ii) *Space curves in* $(\mathbb{C}^3, 0)$*:* $p = 3$, $m \in \{1, 2\}$*; the complete list can be found in Tables 2.6 and 2.15.*
(iii) *Normal surfaces in* $(\mathbb{C}^4, 0)$*:* $p = 4$, $m = 2$*; the list is in Table 2.16.*
(iv) *three-folds in* $(\mathbb{C}^5, 0)$*:* $p = 5$, $m = 2$*; see Table 2.17.*
(v) *four-folds in* $(\mathbb{C}^6, 0)$*:* $p = 6$, $m = 2$*; see Table 2.18.*

Concerning these lists, there are a few noteworthy observations. All singularities in the lists, except the four-folds, are smoothable, but only fat points and surfaces exhibit complete intersection singularities in their versal family, which are not of hypersurface type. For normal surfaces, the list reproduces the list of rational triple point singularities found by Tjurina in [98], where she proceeds by a completely different approach, cf. Sects. 2.3.6 and 2.5.3. For three-fold singularities there is one infinite series, which holds as one matrix entry precisely the equations of hypersurface singularities in 2 variables and also mimics their deformation behaviour (i.e. their adjacencies). For four-folds, we find the generic determinantal singularity of this type in the list and each of the other simple singularities is adjacent to it.

2.5 Stabilizations and the Topology of Essential Smoothings

In this section we will be concerned with *essential smoothings* of (essentially isolated) determinantal singularities. This is the generic object that a given EIDS can deform to using only determinantal deformations, see Definition 2.5.2 below. They can be regarded as a generalization of the Milnor fiber for isolated hypersurface and complete intersection singularities and they coincide with the latter whenever a complete intersection singularity defined by a regular sequence $F = (f_1, \dots, f_c)$ is considered as a determinantal singularity for the matrix $F \in \mathbb{C}\{x\}^{1\times c}$. For the construction and properties of Milnor fibers the reader may consult [20, Chapter 6].

Besides the topology of the essential smoothing itself, we will in the following also be interested in the interplay of topological invariants with analytic invariants arising from the deformation theory of the singularity. This is motivated from the classical results relating the Milnor with the Tjurina number for isolated hypersurface and complete intersection singularities. A treatment of this subject can be found in [20, Section 7.2.4].

To motivate this discussion, recall that the Milnor fiber

$$M_f := B_\varepsilon \cap f^{-1}(\{\delta\}), \quad 1 \gg \varepsilon \gg |\delta| > 0,$$

of an isolated hypersurface singularity $f\colon (\mathbb{C}^p, 0) \to (\mathbb{C}, 0)$ is homotopy-equivalent to a bouquet of spheres

$$M_f \cong_{\mathrm{ht}} \bigvee_{i=1}^{\mu(f)} S^{p-1} \tag{2.45}$$

of real dimension $p-1$. The number of these spheres is the *Milnor number*[24] $\mu(f)$ of f. This was first described by Milnor in [80] where he also shows that this number can be computed as the length

$$\mu = \mu(f) = \dim_{\mathbb{C}} \left(\mathbb{C}\{x_1, \dots, x_p\} / \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_p} \right\rangle \right) \tag{2.46}$$

of the so-called *Milnor algebra*, i.e. the quotient of $\mathbb{C}\{x\}$ by the *Jacobian ideal* $\mathrm{Jac}(f) = \langle \partial f/\partial x_1, \dots, \partial f/\partial x_p \rangle$, see [80] or [20, Theorem 6.5.3]. The latter space naturally coincides with the space $T^1_{\mathscr{R}}(f)$ of non-trivial unfoldings of f up to right- or $\mathscr{R}$-equivalence, see for instance [82, Definition 3.3], and therefore, the Milnor number of f is equal to its $\mathscr{R}$-codimension. This fundamental result already suggests a close connection between topological invariants of the smoothing and analytic invariants related to unfoldings and deformations.

If instead of the function f and its unfoldings, one considers the germ of the hypersurface $(X, 0) = (f^{-1}(\{0\}), 0) \subset (\mathbb{C}^p, 0)$ and its deformations, one is naturally lead to the *Tjurina module* $T^1_{X,0}$ which was already discussed earlier and which classifies the non-trivial first order deformations of the germ $(X, 0)$. For

[24] See e.g. [20, Definition 6.5.2].

isolated hypersurface singularities, this module becomes

$$T^1_{X,0} = \mathbb{C}\{x_1, \ldots, x_p\}/\left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_p}, f \right\rangle.$$

From this explicit form it is immediately clear that $T^1_{X,0}$ has finite dimension over $\mathbb{C}$ if the Milnor number $\mu(f) < \infty$ is finite. The converse is also true but not so obvious, see for instance [53, Lemma 2.3]. The length of the Tjurina module is called the *Tjurina number* of $(X, 0)$:

$$\tau = \dim_{\mathbb{C}} T^1_{X,0} \tag{2.47}$$

and one has the famous inequality

$$\mu \geq \tau \tag{2.48}$$

for isolated hypersurface singularities.

Suppose that f is weighted homogeneous of some degree $e = \deg_w f$ for weights $w_i = \deg_w x_i > 0$, i.e. f is a polynomial that can be written as a linear combination of monomials $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_p^{\alpha_p}$ which all satisfy

$$\deg_w x^\alpha = \sum_{i=1}^{p} \deg_w x_i^{\alpha_i} = w_1 \cdot \alpha_1 + w_2 \cdot \alpha_2 + \cdots + w_p \cdot \alpha_p = e.$$

Using the so-called *Euler vector field*

$$\theta_w = \sum_{i=1}^{p} w_i \cdot x_i \frac{\partial}{\partial x_i}$$

on $\mathbb{C}^p$ associated to these weights, it is easy to see that $e \cdot f = \theta_w(f)$. So in this case, the function f is already contained in the Jacobian ideal of f, and the Milnor algebra and the Tjurina module are isomorphic. It follows that $\mu = \tau$ if f is quasi-homogeneous.

The converse implication was proven by Saito in [87] in '71: Starting from f with $\mu = \tau$ he establishes the existence of a change of coordinates of $(\mathbb{C}^p, 0)$ such that f is quasi-homogeneous for some weights in this new coordinate system.

The colloquial form of this result

$$\text{“}\mu \geq \tau \text{ with equality iff } f \text{ is quasi-homogeneous”} \tag{2.49}$$

has been abundant in complex analytic singularity theory ever since and much effort has been invested in order to generalize it to isolated complete intersection singularities defined by maps $f: (\mathbb{C}^p, 0) \to (\mathbb{C}^c, 0)$.

The generalization of the bouquet decomposition (2.45) for ICIS is due to Hamm [55], which settles the definition of a Milnor number in this context. The Tjurina number is given by the length of $T^1_{X,0}$ as described in Sect. 2.3.3. With these definitions at hand, the journey went on for more than thirty years:

- The equality $\mu = \tau$ has been established in '80 for quasi-homogeneous ICIS of positive dimension[25] by Greuel in [52].
- The general inequality $\mu \geq \tau$ for ICIS of positive dimension was shown in '85 by Looijenga and Steenbrink in [76].
- In the same year, the full statement (2.49) has been proved for Gorenstein curve singularities by Greuel, Martin, and Pfister in [50],
- while Wahl proved (2.49) for ICIS of codimension 2 in [106].
- Finally, the part "$\mu = \tau$ implies quasi-homogeneity" was established in '02 by Vosegaard [103].

After the question as to whether or not (2.49) holds has been settled for isolated complete intersection singularities, it seems natural to ask to which extent this result can be generalized to arbitrary EIDS and we will report on what is known in this regard. For now, let us mention that in general, this question is wide open. For instance, one has the following conjecture by Wahl in [107]:

Conjecture 2.5.1 Let $(X, 0) \subset (\mathbb{C}^4, 0)$ be a normal surface singularity which is not a complete intersection. Then $\mu \geq \tau - 1$ with equality if and only if $(X, 0)$ is quasi-homogeneous.

These singularities fall into the category of isolated Cohen-Macaulay codimension 2 singularities discussed earlier and the precise definitions of the Milnor and Tjurina number will be given below. That quasi-homogeneity of $(X, 0)$ implies equality was already shown by Wahl in [107], but the converse implication is not settled as of this writing.[26]

2.5.1 Construction of Essential Smoothings

We briefly describe the construction of the essential smoothing of an EIDS, parallel to the well known Milnor-Lê-fibration for isolated hypersurface and complete intersection singularities, cf. [20]. Another description based on the transformation into a complete intersection on a singular ambient space, cf. Remark 2.1.14, will be given later in Corollary 2.5.24.

[25] Note that $\mu = \tau$ does not hold for quasi-homogeneous ICIS of dimension zero, see for instance Table 2.5.

[26] A counterexample given by the first named author in [38] turned out to be wrong.

Let $(X_A^s, 0) \subset (\mathbb{C}^p, 0)$ be an EIDS defined by a finitely GL-determined matrix $A \in \mathbb{C}\{x\}^{m\times n}$. Choose a miniversal unfolding $\mathbf{A}(x,t)$ on $\tau = \dim_{\mathbb{C}} T^1_{\mathrm{GL}}(A)$ parameters $t_1, \dots, t_\tau$ and let

$$\mathbf{A}\colon U \times T \to \mathbb{C}^{m\times n}$$

be a representative thereof defined on some product of open neighborhoods of the origin $U \subset \mathbb{C}^p$ and $T \subset \mathbb{C}^\tau$. As usual, let $\mathscr{X}_{\mathbf{A}}^s = \mathbf{A}^{-1}(M_{m,n}^s) \subset U \times T$ be the total space of the induced deformation $(X_A^s, 0) \hookrightarrow (\mathscr{X}_{\mathbf{A}}^s, 0) \xrightarrow{\pi} (T, 0)$ of the determinantal singularity.

Recall that, according to Lemma 2.3.3, any EIDS $(X_A^s, 0)$ has a canonical Whitney stratification by the strata $V_A^r = A^{-1}(V_{m,n}^r)$. This allows us to choose a *Milnor sphere*: For $\varepsilon_0 > 0$ sufficiently small, the intersection of the sphere $S_\varepsilon = \partial B_\varepsilon \subset U$ with the X_A^s is transverse for every $\varepsilon_0 \geq \varepsilon > 0$, see e.g. [20, Theorem 6.10.1]. The *real link* of $(X_A^s, 0)$ can then be defined as the transverse intersection

$$\mathscr{K}_A^s := S_\varepsilon \cap X_A^s. \tag{2.50}$$

Note that, since $(X_A^s, 0)$ has non-isolated singularities in general, the real link will also be singular, but endowed with a canonical Whitney regular stratification. Using Thom's first isotopy lemma[27] it is easy to see that due to the various transversalities, small determinantal deformations of X_A^s do not change the compact stratified space $\mathscr{K}_A^s$ up to homeomorphism. Hence, after shrinking T to some small disk $D_\delta \subset \mathbb{C}^\tau$ if necessary, the restriction of the projection

$$\pi : (S_\varepsilon \times T) \cap \mathscr{X}_A^s \cong_{\text{homeo}} \mathscr{K}_A^s \times T \to T \tag{2.51}$$

to the parameter space of the miniversal unfolding of A is a trivial topological fiber bundle. Here one makes essential use of the fact that $\mathscr{K}_A^s$ is compact.

Due to the characterization of transversality given in Proposition 2.2.12, the matrix discriminant $\Delta_A \subset T$ (Definition 2.2.15) consists of those parameters t for which $A_t\colon U \to \mathbb{C}^{m\times n}$ is not transversal to the rank stratification. It is easy to see that for $t \notin \Delta_A$, the projection

$$\pi : (B_\varepsilon \times (T \setminus \Delta_A)) \cap \mathscr{X}_A^s \to T \setminus \Delta_A \tag{2.52}$$

is a stratified submersion along every fiber $\pi^{-1}(\{t\}) = A_t^{-1}(M_{m,n}^s) \subset U$. By choice of the ball B_ε, these fibers are again compact and canonically stratified, so that π is proper. Another application of Thom's first isotopy lemma can now be made in

[27] See for instance [45].

order to show that (2.52) is a topological fiber bundle extending the fibration on the boundary (2.51) to the interior over $T \setminus \Delta_A$.

Definition 2.5.2 The essential smoothing of an EIDS $(X_A^s, 0) \subset (\mathbb{C}^p, 0)$ defined by a matrix $A \in \mathbb{C}\{x_1, \dots, x_p\}^{m \times n}$ is the generic fiber

$$M_A^s := A_t^{-1}(M_{m,n}^s)$$

of (2.52) over the base of the miniversal unfolding of A for $t \notin \Delta_A$ outside the matrix discriminant.

Example 2.5.3 (Smoothing of a Space Curve) We return to the study of the semi-universal deformation of the three coordinate axis in $(\mathbb{C}^3, 0)$ from Example 2.2.14:

$$\mathbf{A}(x, y, z; t_1, t_2, t_3) = \begin{pmatrix} x & 0 & z \\ 0 & y & z \end{pmatrix} + \begin{pmatrix} 0 & t_1 & 0 \\ t_2 & 0 & t_3 \end{pmatrix}.$$

It had already been discussed in Example 2.2.19 that the simultaneous perturbation by $t_1 = t_2 = t_3 = t$ leads to a smoothing M_A^2 of $(X_A^2, 0)$. Note that due to the homogeneity of the singularity, we may choose the Milnor ball B_ε arbitrarily large so that we can consider the whole affine varieties as suitable representatives of the singularity and its fibers in a deformation.

In order to determine the topological type of M_A^2 we can exploit the adjacencies of $(X_A^2, 0)$: First we deform only along the t_1-axis as in Example 2.1.17 to obtain a configuration of lines consisting of L'_x, L_y, and L_z, with L'_x and L_z meeting transversally in the point $(0, 0, t_1)$ and L_y and L_z at the origin, respectively. Since all these fibers for arbitrary t_1 sit over the discriminant Δ_A in a semi-universal deformation of $(X_A^2, 0)$, the (global) smoothing of this configuration must be diffeomorphic to M_A^2 (Fig. 2.9).

The topology of the local smoothings of the A_1-singularities at the intersection points is known: Over the complex numbers, a double cone is replaced by a tube bounding the two circles in its boundary. Now it is easy to see that in fact

$$M_A^2 \cong_{\mathrm{ht}} S^1 \vee S^1$$

is homotopy equivalent to a bouquet of two circles. In parallel to the definition of Milnor numbers for IHS and ICIS, one would say that the Milnor number for $(X_A^2, 0)$ is two in this case (Fig. 2.10).

Remark 2.5.4 Essential smoothings can also be defined for determinantal singularities $(X_A^s, 0) \subset (\mathbb{C}^p, 0)$ which are not EIDS; i.e. those for which the defining matrix $A \in \mathbb{C}\{x\}^{m \times n}$ is not finitely GL-determined. In the parallelism with hypersurface and complete intersection singularities, these correspond to non-isolated singularities and, as in the classical case, the essential smoothings are not uniquely determined by the singularity $(X_A^s, 0)$ itself anymore, but they can differ depending on the underlying unfolding of A.

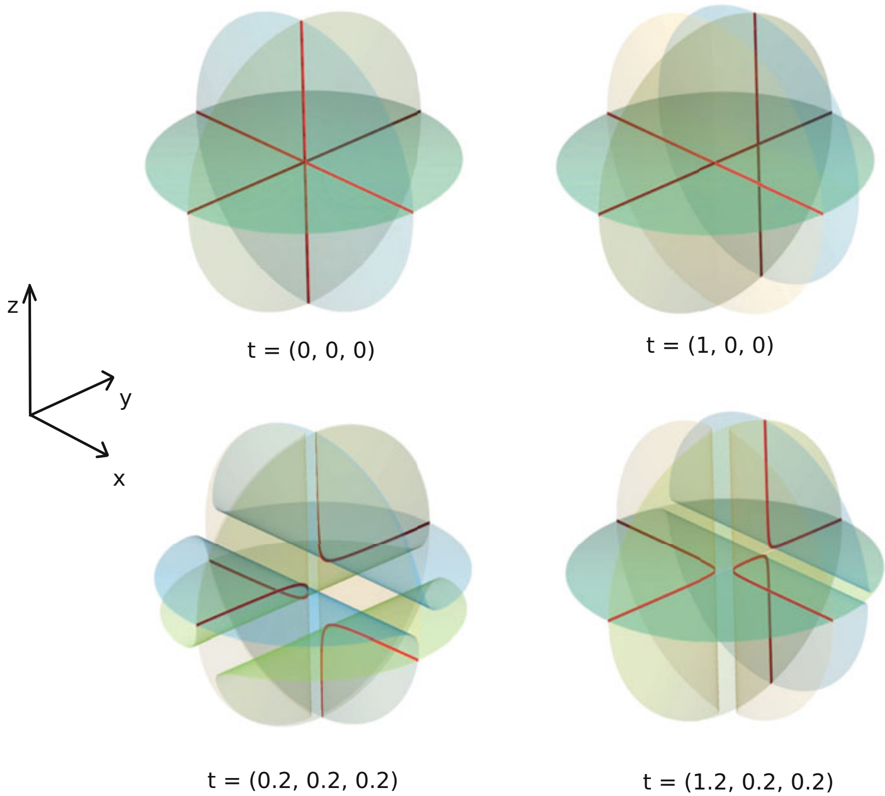


Fig. 2.9 Real picture of a deformation of the three coordinate axis in $(\mathbb{C}^3, 0)$ and its smoothing; the lower left picture shows a direct smoothing of the original singularity, while the lower right one arises from deforming the two adjacent A_1-hypersurface singularities from the upper right picture

To give a definition, let $\mathbf{A}\colon (\mathbb{C}^p, 0) \times (\mathbb{C}, 0) \to (\mathbb{C}^{m\times n}, 0)$ be a 1-parameter unfolding of A. We say that $\mathbf{A}$ is a *stabilization* of A if for some representative

$$\mathbf{A}\colon U \times T \to \mathbb{C}^{m\times n}$$

one has that $A_t\colon U \to \mathbb{C}^{m\times n}$ is transversal to the rank stratification for every $t \in T \setminus \{0\}$. The existence of such stabilizations can for example be derived from [102, Theorem 2.2 and Theorem 3.1]. In particular, the set of constant matrices $C \in \mathbb{C}^{m\times n}$ for which $\mathbf{A}(x, t) = A(x) + t \cdot C$ is a stabilization of A, is dense in $\mathbb{C}^{m\times n}$.

Given any suitable representative of a stabilization $\mathbf{A}$ of A as above, we can consider the total space of the induced deformation $\mathscr{X}^s_{\mathbf{A}} = \mathbf{A}^{-1}(M^s_{m,n}) \subset U \times T$ together with its projection $\pi : \mathscr{X}^s_{\mathbf{A}} \to T$ to the parameter space. General fibration

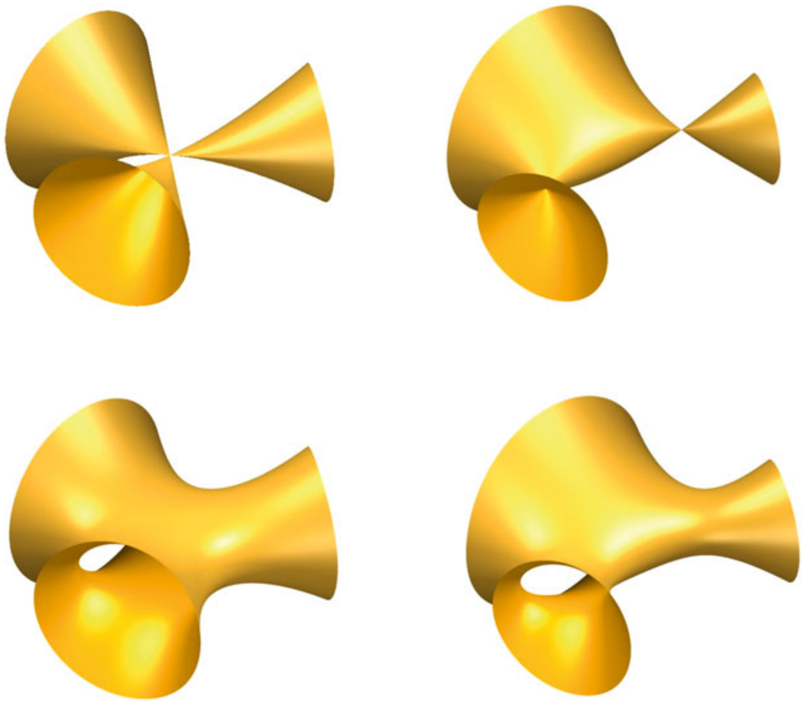


Fig. 2.10 The actual change in topology for the smoothings of the three coordinate axis in $(\mathbb{C}^3, 0)$, for a direct smoothing at the left hand side and passing through the adjacency with the two A_1-singularities on the right hand side

theorems[28] can be used to show that for a sufficiently small ball $B_\varepsilon \subset U$ of radius $\varepsilon > 0$ around the origin $0 \in U$ and some subsequently chosen, sufficiently small disc $D_\delta \subset T$, the projection

$$\pi : B_\varepsilon \times (D_\delta \setminus \{0\}) \cap \mathscr{X}^s_{\mathbf{A}} \to D_\delta \setminus \{0\} \tag{2.53}$$

is a topological fiber bundle with fiber

$$M^s_{\mathbf{A}} := B_\varepsilon \cap A_t^{-1}(M^s_{m,n}). \tag{2.54}$$

This is the *essential smoothing* of $(X^s_A, 0)$ defined by the stabilization $\mathbf{A}$.

It is easy to see using the properties of miniversal unfoldings that this notion of essential smoothing coincides with the previous one given for EIDS in case A is finitely GL-determined.

[28] See [74], cf. also [20, Theorem 6.10.3].

Note that the fibration (2.53) always exists for 1-parameter unfoldings, regardless of whether or not A_t is a stabilization for $t \neq 0$. However, we shall refer to the fiber of this fibration as an essential smoothing of $(X^s_A, 0)$ only if this is the case.

Example 2.5.5 Consider the space curve $(X^2_A, 0) \subset (\mathbb{C}^3, 0)$ given by the matrix

$$A = \begin{pmatrix} x & 0 & z \\ 0 & y & z^2 \end{pmatrix}.$$

Set theoretically this coincides with the union of the three coordinate axis discussed in several previous examples. But a primary decomposition of the ideal $I = \langle A^{\wedge 2} \rangle$ reveals that

$$I = \langle y, z^2 \rangle \cap \langle x, z \rangle \cap \langle x, y \rangle$$

so that X^2_A consists of the double x-axis in the x-z-plane and the y- and the z-axis. Consequently, the module $T^1_{\mathrm{GL}}(A)$ is indeed supported along the whole x-axis.

We describe two distinct unfoldings leading to topologically different smoothings of $(X^2_A, 0)$. The first one, illustrated by Fig. 2.11, is given by

$$\mathbf{A}(x, y, z; u) = \begin{pmatrix} x & 0 & z \\ 0 & y & z^2 - u^2 \end{pmatrix}.$$

For $u \neq 0$ the variety defined by the ideal $\langle A_u^{\wedge 2} \rangle$ in $\mathbb{C}^3$ consists of *four* lines with two of them being the now split parallels to the x-axis passing through the points $(0, 0, \pm u)$. A global smoothing of the three singular points of this variety is homotopy equivalent to a bouquet of three real spheres:

$$M^2_{\mathbf{A}} \cong_{\mathrm{ht}} S^1 \vee S^1 \vee S^1.$$

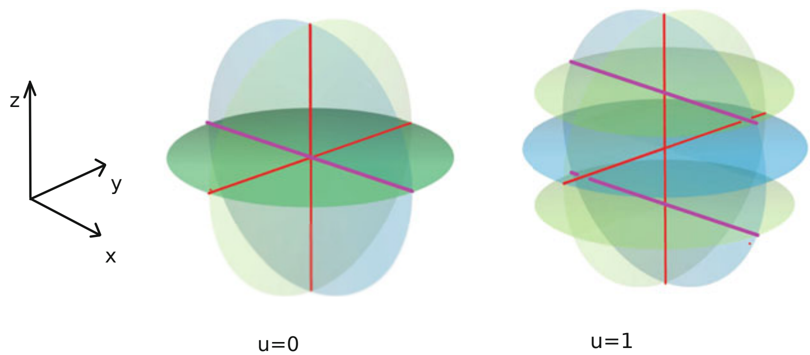


Fig. 2.11 The first deformation of a non-reduced space curve singularity (purple and green) with non-isolated singular locus (purple)

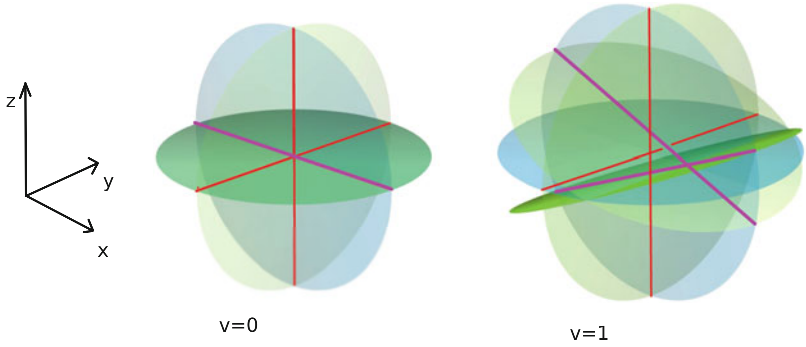


Fig. 2.12 The second deformation of a the same non-reduced space curve singularity

The other deformation that we wish to consider (see Fig. 2.12) is induced by the unfolding

$$\mathbf{B}(x, y, z; v) = \begin{pmatrix} x & 0 & z \\ 0 & y & z^2 - v^2(x - v)^2 \end{pmatrix}.$$

For this deformation, the variety defined by $\langle B_v^{\wedge 2}\rangle$ for $v \neq 0$ consists again of four lines, only that this time the double x-axis does not split into parallels, but opens up like a scissor which is pulled backwards at the same time:

$$\langle B_v^{\wedge 2}\rangle = \langle y, z - v(x - v)\rangle \cap \langle y, z + v(x - v)\rangle \cap \langle x, z\rangle \cap \langle x, y\rangle.$$

Thus for $v \neq 0$ the four lines meet transversally in pairs at four points in total. The global smoothing of this variety is then homotopy equivalent to a bouquet of *five* real spheres:

$$M_{\mathbf{B}}^2 \cong_{\mathrm{ht}} S^1 \vee S^1 \vee S^1 \vee S^1 \vee S^1.$$

Choosing appropriate combinations of either one of these two deformations with their respective global smoothings, it is easy to see that also the original singularity $(X_A^2, 0)$ can be deformed directly into $M_{\mathbf{A}}^2$, but also into $M_{\mathbf{B}}^2$. Therefore, a smoothing is not unique for this non-essentially isolated determinantal singularity.

2.5.2 Determinantal Hypersurfaces

Determinantal hypersurfaces have already appeared in the context of the classification of simple square matrices by Bruce and Tari in Sect. 2.4.1. For this section we will need this notion to also comprise the symmetric and the skew symmetric cases.

Definition 2.5.6 Let $A \in \mathbb{C}\{x_1, \dots, x_p\}^{m\times m}$ be a non-constant matrix, either arbitrary square, symmetric, or skew symmetric. Then the singularity $(X_A^m, 0) \subset (\mathbb{C}^p, 0)$ defined by the equation $f = \det A$ in the first two, or by $f = \operatorname{Pf} A$ in the third case, is called a *determinantal hypersurface singularity* with defining equation f.

Depending on the case, we will consider the matrix A up to either $\mathscr{G}$-, $\mathscr{G}_{\mathrm{sym}}$-, or $\mathscr{G}_{\mathrm{sk}}$-equivalence. Since the bound on the rank m is always equal to the size of the defining matrix A for determinantal hypersurface singularities, we will usually not mention m explicitly throughout this section and omit it from our notation. Note that, moreover, m is necessarily even in the skew-symmetric case since $\operatorname{Pf}(A) = 0$ for matrices of odd size.

2.5.2.1 The Singular Milnor Fiber

For determinantal hypersurfaces we always have two different deformation theories at hand: The deformations arising from unfoldings of f and the determinantal deformations induced from unfoldings of the matrix A. Correspondingly, there are two notions of "Milnor fiber" in this setup which differ in general. The first one is the classical Milnor fiber

$$M_f := B_\varepsilon(0) \cap f^{-1}(\{\delta\}), \tag{2.55}$$

$1 \gg \varepsilon \gg |\delta| > 0$ (see [80], cf. also [20, Chapter 6]), of the hypersurface singularity determined by f. The other one is the *essential smoothing* of $(X_A, 0)$ given by

$$M_A := B_\varepsilon(0) \cap A_t^{-1}(M_{m,m}^m) = B_\varepsilon(0) \cap \{\det A_t = 0\}, \tag{2.56}$$

$1 \gg \varepsilon \gg |t| > 0$ with the appropriate substitution for the skew-symmetric case. Depending on how large the dimension p is compared to the codimension of the singular locus of the set of degenerate matrices, these spaces either coincide or differ:

Lemma 2.5.7 *Let $(X_A, 0) \subset (\mathbb{C}^p, 0)$ be a determinantal hypersurface defined by a finitely determined square matrix $A \in \mathbb{C}\{x_1, \dots, x_p\}^{m\times m}$ which is either square, symmetric, or skew-symmetric. Depending on the case, let $c' = 4, 3,$ or 6 be the codimension of the singular locus of the respective set of degenerate matrices.*

When $p < c'$, the essential smoothing is in fact smooth and the manifolds $M_A \cong_{\mathrm{diff}} M_f$ are diffeomorphic.

When $p \geq c'$, the essential smoothing M_A is singular with singular locus of dimension $m^2 - 4$ and M_f is diffeomorphic to a global smoothing of M_A.

Proof In the first case when $p < c'$ this is clear since the smoothing M_f is unique. The second case must be split into two further subcases, namely $p = c'$ so that $f = \det A$ has *isolated* singularity, and $p > c'$ in which case f has *non-isolated singularities*.

When $p = c'$, there exists again a semi-universal deformation of the hypersurface germ $(X_A, 0) = (\{f = 0\}, 0)$ over some base $(\mathbb{C}^\tau, 0)$ with $\tau = \tau(X_A, 0)$ the Tjurina number of $(X_A, 0)$, and a comparison map $\Phi\colon (\mathbb{C}^{\tau_{\mathrm{GL}}(A)}, 0) \to (\mathbb{C}^{\tau(f)}, 0)$ as in the diagram (2.38) where $(\mathbb{C}^{\tau_{\mathrm{GL}}(A)}, 0)$ is the parameter space of a GL-miniversal unfolding $\mathbf{A}$ of A. But the image of Φ must be contained in the discriminant $(\Delta_f, 0) \subset (\mathbb{C}^{\tau(f)}, 0)$ of f since $(X_A, 0)$ does not admit any determinantal smoothing. Choosing an appropriate representative $\mathbf{A}\colon U \times T \to \mathbb{C}^{m\times m}$ of the GL-miniversal unfolding of A, it is easy to see that the generic fiber $M_A \subset U$ has only isolated singularities and hence posesses a unique smoothing. Since we can think of the fiber M_A as a fiber in the semi-universal deformation of $(X_A, 0)$ by virtue of Diagram (2.38), this smoothing of M_A must coincide with M_f as this is the only smooth nearby fiber of M_A in the semi-universal deformation $\pi\colon \mathscr{X} \to T'$ of $(X_A, 0)$.

In the case $p > c'$ there does not exist as semi-universal deformation of $(X_A, 0)$ anymore but one has the function $F = \det \mathbf{A}$ for a GL-miniversal unfolding $\mathbf{A}$ of A which assigns a canonical smoothing

$$B_\varepsilon \cap (\det A_t)^{-1}(\{\delta\}), \quad \varepsilon \gg |t| \gg |\delta| > 0$$

to every fiber $X_{\mathbf{A}}(t) = \{\det A_t = 0\}$ for $t \in \mathbb{C}^{\tau_{\mathrm{GL}}(A)}$ sufficiently small. Details for the treatment of this setup can for example be found in [93], see also [28, Proposition 4.5]. □

2.5.2.2 The Smoothable Case

The first case $p < c'$ has been studied by Goryunov and Mond in [47]. In this case the function $f\colon (\mathbb{C}^p, 0) \to (\mathbb{C}, 0)$ defines an isolated hypersurface singularity in the classical sense. Hence, on the topological side, the theory is in principal covered by Milnor's results [80]: The Milnor fiber

$$M_f \cong_{\mathrm{ht}} \bigvee_{i=1}^{\mu(f)} S^{p-1}$$

is homotopy equivalent to a bouquet of $\mu(f)$ spheres of real dimension $p-1$ where $\mu(f)$ is the Milnor number of the singularity defined by f.

For an arbitrary isolated hypersurface singularity, one would compare the Milnor with the Tjurina number of f. However, the classification of simple singularities of the square matrices in Table 2.10 and of the symmetric matrices in Table 2.7 shows an equality $\mu(f) = \tau_{\mathrm{GL}}(A)$ (resp. $\mu(f) = \tau_{\mathrm{GL}}^{\mathrm{sym}}(A)$) of the Milnor numbers with the GL-Tjurina numbers of the defining matrices. This peculiarity motivated Goryunov and Mond to closer inspections and they revealed in [47, Corollary 4.4] the following, more general phenomenon:

Theorem 2.5.8 *Let* $(X_A, 0) \subset (\mathbb{C}^p, 0)$ *be a smoothable determinantal hypersurface singularity for some matrix* $A \in \mathbb{C}\{x\}^{m\times m}$*, either symmetric, arbitray square, or skew symmetric with m even, with defining equation* f*. Then*

$$\tau_{\mathrm{SL}}^{\mathrm{sym}}(A) = \mu(f) \qquad \text{if } A \text{ is symmetric and } p = 2;$$
$$\tau_{\mathrm{SL}}^{\mathrm{sq}}(A) = \mu(f) \qquad \text{if } A \text{ is arbitrary square and } p = 3;$$
$$\tau_{\mathrm{SL}}^{\mathrm{sk}}(A) = \mu(f) \qquad \text{if } A \text{ is skew-symmetric and } p = 5.$$

For determinantal hypersurface singularities defined by quasi-homogeneous matrices (such as all the simple ones listed above), the GL- and the SL-Tjurina numbers coincide, see e.g. [46, Proposition 1.1]. In the general case, however, the SL-Tjurina numbers seem to be the prefered choice for comparison with topological invariants.

Remark 2.5.9 It is remarkable that the proof of Theorem 2.5.8 in [47] relies on an argument which is similar to the use of generic perfection introduced in Sect. 2.1.1, but where the bound on the grade in Theorem 2.1.8 is *not* attained.

More precisely, Goryunov and Mond consider the complex $K(m, m, m-1)$ from (2.9), (respectively $K^{\mathrm{sym}}(m, m-1)$, or $K^{\mathrm{sk}}\left(m, \frac{m}{2} - 1\right)$) over the ring $\mathbb{C}\{y\}$ of convergent power series at the origin $0 \in \mathbb{C}^{m\times m}$ in the target of A. These provide a resolution of the Jacobian ideal of the function det (resp. Pf) which is generated by

$$\frac{\partial \det}{\partial y_{i,j}} = (-1)^{i+j} Y^{\wedge m-1}_{\hat{i},\hat{j}}$$

where $\hat{i}$ denotes the multiindex $(1, 2, \dots, \hat{i}, \dots, m-1, m)$ obtained by deleting the i-th entry. The pullback $A^*K(m, m, m-1)$ of these complexes admit a morphism

$$\Phi\colon \operatorname{Kosz}\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_p}; \mathbb{C}\{x\}\right) \to A^*K(m, m, m-1)$$

lifting the natural map on the generators

$$\frac{\partial f}{\partial x_k} = \frac{\partial \det \circ A}{\partial x_k} \mapsto \sum_{(i,j)} \frac{\partial \det}{\partial y_{i,j}} \frac{\partial A_{i,j}}{\partial x_k} = \sum_{(i,j)} (-1)^{i+j} \frac{\partial A_{i,j}}{\partial x_k} A^{\wedge m-1}_{\hat{i},\hat{j}}$$

induced by the chain rule. Completing this to a short exact sequence of complexes via the mapping cone of Φ, the associated long exact sequence reads

$$\cdots \to H_1\left(\operatorname{Kosz}\left(\frac{\partial f}{\partial x}; \mathbb{C}\{x\}\right)\right) \to H_1\left(A^*K(m, m, m-1)\right) \to T^1_{\mathrm{SL}}(A) \to$$
$$\to \mathbb{C}\{x\}/\left\langle \frac{\partial f}{\partial x}\right\rangle \to H_0(A^*K(m, m, m-1)) \to 0,$$

cf. [47, Theorem 1.2] and [47, Corollary 1.3]. In case f has isolated singularity at the origin, the Koszul complex $\mathrm{Kosz}\left(\frac{\partial f}{\partial x}; \mathbb{C}\{x\}\right)$ is exact in degrees > 0 so that this restricts to an exact four-term complex. Writing β_i for the length of the module

$$H_i(A^* K(m, m, m-1)) = \mathrm{Tor}_i^{\mathbb{C}\{y\}}\left(\mathbb{C}\{x\}, \mathbb{C}\{y\}/\langle Y^{\wedge m-1}\rangle\right)$$

Goryunov and Mond obtain a more general formula

$$\tau_{\mathrm{SL}}(A) = \mu(f) - \beta_0 + \beta_1 \tag{2.57}$$

for arbitrary determinantal hypersurface singularities with *isolated* singularity.

When the dimension of the source p is equal to the expected codimension $c' = \mathrm{grade}\,\langle Y^{\wedge m-1}\rangle$ of the singular locus of the set of degenerate matrices in $\mathbb{C}^{m\times m}$, Theorem 2.1.8 applies: $\beta_0 = \dim_{\mathbb{C}} \mathbb{C}\{x\}/\langle A^{\wedge m-1}\rangle$ and $\beta_1 = 0$. This leads to Theorem 2.5.12 below.

For Theorem 2.5.8 observe that for a 1-parameter stabilization $\mathbf{A}(x, t)$ of A the complex $\mathbf{A}^* K(m, m, m-1)$ is exact. From the long exact sequence induced by multiplication with the unfolding parameter t on $\mathbf{A}^* K(m, m, m-1)$ one may then infer $\beta_1 = \beta_0$ which yields the desired result.

2.5.2.3 Isolated Hypersurface Singularities Without Determinantal Smoothing

Most of the simple singularities of square matrices do not admit determinantal smoothings, i.e. they are subsumed under the second case of Lemma 2.5.7; the essential smoothing M_A will be singular along the subspace M_A^{m-1} and in particular different from the Milnor fiber M_f. This raises the question for the correct notion of Milnor number for the determinantal hypersurface singularity in this setting. The answer is given by the following proposition which is based on a modified version of a theorem by Lê [73]:

Proposition 2.5.10 *Let $(X_A, 0) \subset (\mathbb{C}^p, 0)$ be an essentially isolated determinantal hypersurface singularity defined by a matrix $A \in \mathbb{C}\{x_1, \ldots, x_p\}$. Then the essential smoothing*

$$M_A \cong_{\mathrm{ht}} \bigvee_{i=1}^{\mu(A)} S^{p-1} \tag{2.58}$$

of $(X_A, 0)$ is homotopy equivalent to a bouquet of spheres of real dimension $p - 1$.

Proof We give a sketch of the proof; full details can be found in [73] and [93], cf. also [28]. For a suitable representative of a generic 1-parameter unfolding $\mathbf{A}: U \times T \to \mathbb{C}^{m\times m}$ of A, the *polar locus*

$$\Gamma = \{(x, t) \in U \times T : x \text{ is a critical point of } \det A_t\}$$

is a finite, branched covering over the parameter space T. Denote the fiber over $t \in T$ by Γ_t. By genericity, we may assume that for $t \neq 0$, the function $\det A_t$ has a complex Morse singularity at every point $x \in \Gamma_t$. Now we can use the real valued function $|\det A_t|$ as a (stratified) Morse function on the complement $B_\varepsilon \setminus M_A$ of M_A in a Milnor ball B_ε, to find that B_ε is obtained from the essential smoothing by attaching handles of Morse index p at the points of Γ_t. Since B_ε itself is contractible, the claim follows. □

Definition 2.5.11 The *singular Milnor number* of $(X_A, 0)$ is the number of spheres in the bouquet decomposition (2.58) of the essential smoothing.

When the number of variables p is equal to the codimension c' of the singular locus of the respective set of degenerate matrices, the determinantal hypersurface singularity $(X_A, 0)$ is still isolated, but its essential smoothing M_A will retain isolated singularities at some points $x_1, \dots, x_s \in M_A$. We are therefore in a boundary setting where we have *two* Milnor numbers at hand: The classical one, $\mu(f)$, for the isolated hypersurface singularity defined by $f = \det A$ (resp. $f = \operatorname{Pf} A$), and the singular Milnor number $\mu(A)$. In order to compare the two, let $\mathbf{A}(x, t)$ be an unfolding of the defining matrix A of $(X_A, 0)$ over some parameter space $T \subset \mathbb{C}^k$. As was already pointed out in the proof of Lemma 2.3.3, the (relative) singular locus of the fibers $X_{\mathbf{A}}(t)$ is again determinantal for the $(m-1)$-minors (resp. Pfaffians) of the defining matrices. The principle of conservation of number assures that the total multiplicity of the singular points of the fibers $X_{\mathbf{A}}(t)$ is preserved in any family and therefore equal to the multiplicity of the singular locus of the central fiber $e := \dim_{\mathbb{C}} \mathbb{C}\{x\}/\langle A^{\wedge m-1}\rangle$.

When the unfolding $\mathbf{A}(x, t)$ is sufficiently generic so that A_t is a stabilization for $t \neq 0$, we may assume that all these singular points $x_1, \dots, x_e \in M_A = X_{\mathbf{A}}(t)$ are Morse critical points of f on $\mathbb{C}^p$. Given that, according to Lemma 2.5.7, M_f is a global smoothing of M_A, it is then not difficult to see that on the topological side, M_A is homotopy equivalent to a suspension of exactly e spheres in the Milnor fiber M_f. Therefore, the number e measures the difference

$$e = \dim_{\mathbb{C}} \mathbb{C}\{x\}/\langle A^{\wedge m-1}\rangle = \mu(f) - \mu(A) \tag{2.59}$$

of the classical and the singular Milnor number when $p = 3$ in the symmetric, $p = 4$ in the arbitrary square, or $p = 6$ in the skew-symmetric case.

In these terms, Goryunov and Mond found that the SL-Tjurina number is equal to the *singular* Milnor number, cf. [47, Theorem 4.6] and [47, Corollary 4.2]:

Theorem 2.5.12 *In the same setting as Theorem 2.5.8 one has*

$$\tau_{\mathrm{SL}}^{\mathrm{sym}}(A) = \mu(f) - \dim_{\mathbb{C}} \mathbb{C}\{x\}/\langle A^{\wedge m-1}\rangle \quad \text{for } A \text{ symmetric and } p = 3;$$
$$\tau_{\mathrm{SL}}^{\mathrm{sq}}(A) = \mu(f) - \dim_{\mathbb{C}} \mathbb{C}\{x\}/\langle A^{\wedge m-1}\rangle \quad \text{for } A \text{ arbitrary square and } p = 4;$$
$$\tau_{\mathrm{SL}}^{\mathrm{sk}}(A) = \mu(f) - \dim_{\mathbb{C}} \mathbb{C}\{x\}/\langle A_{\mathrm{sk}}^{\wedge \frac{m}{2}-1}\rangle \quad \text{for } A \text{ skew-symmetric and } p = 6.$$

Note that in the skew-symmetric case we assume m to be even.

2.5.2.4 Determinantal Hypersurfaces with Non-isolated Singularities

For values of $p > c'$ in Lemma 2.5.7, the hypersurface singularities defined by f are non-isolated, so the classical Milnor number is no longer defined. In fact, Kato and Matsumoto have established a lower bound on the connectivity of the Milnor fiber in [63]: When s denotes the dimension of the critical locus of f, then M_f is only $(p - s - 2)$-connected and this bound is sharp in general. In this section we will survey recent results due to Goryunov [46] and Damon [26, 27], on both the singular Milnor fiber and its smoothing, respectively.

Starting with the singular Milnor fiber of a square matrix $A \in \mathbb{C}\{x_1, \ldots, x_p\}_{(*)}^{m\times m}$ with $(*)$ either sym, sq, or sk, Proposition 2.5.10 assures that despite the presumably low connectivity of the classical Milnor fiber due to Kato and Matsumoto, the essential smoothing M_A of $(X_A, 0) \subset (\mathbb{C}^p, 0)$ is again homotopy equivalent to a bouquet of real spheres of equal dimension $p - 1$. In particular, the singular Milnor fiber is defined.

The equality

$$\mu(A) = \tau_{\mathrm{SL}}(A) \tag{2.60}$$

(with $\tau_{\mathrm{SL}}(A)$ the appropriate Tjurina number in the symmetric, square, or skew-symmetric case) was established by Goryunov in [46] for a wider class of singularities comprising all the known simple singularities of square matrices with non-isolated singularities, [46, Corollary 6.3], i.e. those listed in Theorem 2.4.1, (iii) and (iv) (Table 2.8), Theorem 2.4.2, (iii) and (vi) (also Tables 2.8). In particular, he considers singularities of square matrices A whose differential $\mathrm{d}A(0)$ is of corank 1. Such singularities have already appeared in the classification by Bruce and Tari, Theorem 2.4.1, (iii). The same reasons leading to the distinction of the two cases a and b listed there yield the classes of matrices of the form (2.61) and (2.62) below.

Let $x_{i,j}$ be independent variables for $0 < i, j \le m$ with $(i, j) \ne (1, 1)$ and $z_1, \ldots, z_q$ an additional set of variables. Then for $m \times m$-matrices of the form

$$A = \left(g(z) - \sum_{i=2}^{m} x_{i,i}\right) \cdot E_{1,1} + \sum_{(i,j)\ne(1,1)} x_{i,j} \cdot E_{i,j} \tag{2.61}$$

where $g \in \mathbb{C}\{z_1, \ldots, z_q\}$ defines an isolated hypersurface singularity, Goryunov establishes in [46, Theorem 3.5] that $\mu(A) = \tau_{\mathrm{SL}}(A) = \mu(g)$ whenever the SL-Tjurina number is finite.

Similarly, for matrices of the form

$$A = \left(h(x_{2,2}, z) - \sum_{i=3}^{m} x_{i,i}\right) \cdot E_{1,1} + \sum_{(i,j)\neq(1,1)} x_{i,j} \cdot E_{i,j} \tag{2.62}$$

where $h \in \mathbb{C}\{x_{2,2}, z_1, \ldots, z_q\}$ defines a boundary singularity in the sense of Arnold [4], Goryunov establishes (2.60) in [46, Theorem 3.8], together with a further equality in the quasi-homogeneous case (cf. [46, Remark 3.9]) to the "boundary Milnor number" $\mu_\partial(h)$ for the boundary singularities. The analogous results also hold, with the appropriate adaptations, for symmetric and skew-symmetric matrices.

Given that no counterexamples to (2.60) have been encountered among the known simple singularities, Goryunov has conjectured:

Conjecture 2.5.13 ([46]) Let $A\colon (\mathbb{C}^p, 0) \to (\mathbb{C}^{m\times m}_{(*)}, 0)$ be a holomorphic map germ with $(*)$ either sq, sym, or sk with finite SL-codimension and the dimension p sufficiently large such that the associated hypersurface $(X^m_A, 0)$ has non-isolated singular locus. Then

$$\mu(A) = \tau_{\mathrm{SL}}(A).$$

We now turn to the study of smoothings of essentially isolated determinantal hypersurface singularities with non-isolated singularities. These have occupied central stage in Damon's study of prehomogeneous vector spaces and exceptional orbit varieties, [26] and [27].

First, he studies the Milnor fibration of the functions $f = \det$ in the arbitrary square and symmetric, and $f = \mathrm{Pf}$ in the skew-symmetric case on the space of matrices $\mathbb{C}^{m\times m}_{\mathrm{sq}}$, $\mathbb{C}^{m\times m}_{\mathrm{sym}}$, resp. $\mathbb{C}^{m\times m}_{\mathrm{sk}}$ with $m = 2n$ even. We shall refer to these as the *generic determinantal hypersurface singularities*. The homogeneity of all these singularities allows one to identify the local Milnor fibration at the origin with a fibration of affine manifolds

$$\begin{array}{ccc} F_m & \hookrightarrow & E_m \\ \downarrow & & \downarrow f \\ \{1\} & \hookrightarrow & S^1 \end{array} \tag{2.63}$$

where $S^1 \subset \mathbb{C}^*$ is the unitary subgroup. Damon observed in [26, Theorem 3.1] that the total spaces E_m and the fibers F_m are homotopy equivalent to "symmetric spaces" in the sense of Cartan. In particular, this allows for an explicit computation of their cohomology rings with coefficients in a field k of characteristic zero. The results are listed in Table 2.1.

Table 2.1 Milnor fibers of the generic determinantal hypersurfaces, their associated symmetric spaces, and cohomology rings according to Damon [26, Table 1]

Case	F_m	Symmetric space	Cohomology of F_m
sym, m odd	$\mathrm{SL}(m;\mathbb{C})/\mathrm{SO}(m;\mathbb{C})$	$\mathrm{SU}(m;\mathbb{C})/\mathrm{SO}(m;\mathbb{C})$	$\bigwedge k\langle e_5, e_9, \dots, e_{2m-1}\rangle$
sym, m even	$\mathrm{SL}(m;\mathbb{C})/\mathrm{SO}(m;\mathbb{C})$	$\mathrm{SU}(m;\mathbb{C})/\mathrm{SO}(m;\mathbb{C})$	$\{1, e_m\} \cdot \bigwedge k\langle e_5, e_9, \dots, e_{2m-3}\rangle$
sq	$\mathrm{SL}(m;\mathbb{C})$	$\mathrm{SU}(m;\mathbb{C})$	$\bigwedge k\langle e_3, e_5, \dots, e_{2m-1}\rangle$
sk, $m = 2n$ even	$\mathrm{SL}(m;\mathbb{C})/\mathrm{Sp}(n;\mathbb{C})$	$\mathrm{SU}(m;\mathbb{C})/\mathrm{Sp}(n;\mathbb{C})$	$\bigwedge k\langle e_5, e_9, \dots, e_{2m-3}\rangle$

Following Damon, we denote by $k\langle e_{i_1}, e_{i_2}, \dots\rangle$ the graded vector space generated by elements e_{i_l} which are homogeneous of degree i_l, respectively. By $\bigwedge M$ we denote the full exterior algebra of M and $\{1, e_m\} \cdot \bigwedge M$ denotes the free module generated by the elements 1 and e_m over $\bigwedge M$.

The results in Table 2.1 are also valid when replacing the coefficients k by $\mathbb{Z}$, except in the symmetric case where the cohomology has 2-torsion. In that case, the cohomology of F_m with coefficients in $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ is

$$H^\bullet(F_m; \mathbb{Z}_2) \cong H^\bullet(\mathrm{SU}(m;\mathbb{C})/\mathrm{SO}(m;\mathbb{R}); \mathbb{Z}_2) \cong \bigwedge \mathbb{Z}_2\langle s_2, s_3, \dots, s_m\rangle \qquad (2.64)$$

with s_j of degree j.

A "Schubert decomposition" of the global Milnor fibers F_m was presented by Damon in [27]. He shows that the associated *Schubert cycles* in the homology of F_m are dual to the generators in cohomology from Table 2.1 in many cases and it is conjectured that this always holds. The decomposition itself is too complicated to be reproduced here and we refer to [27] for details.

It is already clear from Table 2.1 that the smoothings of the generic determinantal hypersurface singularities are hardly ever homotopy equivalent to a bouquet of spheres of the same dimension. Damon also provides insight to the homotopy groups of the F_m in certain ranges in [26, Theorem 3.5] by considering the associated symmetric spaces as subspaces of the respective infinite dimensional symmetric spaces

$$\mathbb{SU} = \bigcup_{m=1}^{\infty} \mathrm{SU}(m;\mathbb{C}),$$

$$\mathbb{SU}/\mathbb{SO} = \bigcup_{m=1}^{\infty} \mathrm{SU}(m;\mathbb{C})/\mathrm{SO}(m;\mathbb{C}),$$

$$\mathbb{SU}/\mathbb{S}p = \bigcup_{n=1}^{\infty} \mathrm{SU}(2n;\mathbb{C})/\mathrm{Sp}(n;\mathbb{C}).$$

The homotopy groups of the spaces on the left hand side are known, see Table 2.2, and there are certain *stable ranges* in which these homotopy groups coincide with those of their respective subspaces:

Table 2.2 The stable homotopy groups of the infinite dimensional symmetric spaces. They are periodic of period 8 starting at $j = 2$

$j =$	0	1	2	3	4	5	6	7	8	9
$\pi_j(\mathbb{SU})$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$\pi_j(\mathbb{SU}/\mathbb{SO})$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$\pi_j(\mathbb{SU}/\mathbb{S}p)$	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$

Theorem 2.5.14 ([26]) *The homotopy groups of the Milnor fibers F_m in the different cases are:*

square matrices of size $m \times m$:

$$\pi_j(F_m) \cong \pi_j(\mathrm{SU}(m;\mathbb{C})) \cong \pi_j(\mathbb{SU}), \quad j < 2m;$$

symmetric matrices of size $m \times m$:

$$\pi_j(F_m) \cong \pi_j(\mathrm{SU(m;}\mathbb{C})/\mathrm{SO}(m;\mathbb{C})) \cong \pi_j(\mathbb{SU}/\mathbb{SO}), \quad j < m-1;$$

skew symmetric matrix of size $m \times m$ for $m = 2n$ even:

$$\pi_j(F_m) \cong \pi_j(\mathrm{SU}(2n;\mathbb{C})/\mathrm{Sp}(n;\mathbb{C})) \cong \pi_j(\mathbb{SU}/\mathbb{S}p), \quad j < 2m-2.$$

After studying the smoothings of the generic determinantal hypersurface singularities, Damon turns to the study of essentially isolated hypersurfaces given by map germs

$$A\colon (\mathbb{C}^p, 0) \to (\mathbb{C}^{m\times m}_{(*)}, 0)$$

and their smoothings $M_f = B_\varepsilon \cap (\det \circ A)^{-1}(\{\delta\})$, $1 \gg \varepsilon \gg |\delta| > 0$, where again $(*)$ denotes either sq, sym, or sk, and det is replaced by Pf in the latter case.

Suppose $A\colon U \to \mathbb{C}^{m\times m}$ is a suitable representative. Then by construction the map A restricts to

$$M_f \xrightarrow{A} F_m$$

where F_m is the Milnor fiber of the generic determinantal hypersurface singularity. In this setup, Damon gives the following definition:

Definition 2.5.15 The image $\mathscr{A}_{M^s_{m,n}}(A)$ of the pullback in cohomology $A^*\colon H^\bullet(F_m) \to H^\bullet(M_f)$ is the *characteristic cohomology* of M_f.

According to Damon, it is an open question to determine the structure of $H^\bullet(M_f)$ as a module over $H^\bullet(F_m)$. He notes that in the two extremal cases where either (i) the singularity $(X_A, 0)$ admits a determinantal smoothing, or (ii) when A is the germ of a submersion, one has

$$H^\bullet(M_f) \cong \mathscr{A}_{M^s_{m,n}}(A) \oplus k^\mu[p-1] \tag{2.65}$$

where k denotes the chosen ring of coefficients for cohomology (cf. Table 2.2 and (2.64)) and $[p-1]$ the shift in cohomological degree by $p-1$. He remarks that in case i) the characteristic cohomology $\mathscr{A}_{M^s_{m,n}}(A)$ consists of the degree-zero-part only, so that μ is in fact the classical Milnor number of the associated isolated hypersurface singularity and in case ii) the second summand is trivial. For all other cases he asks:

"How generally valid is (2.65) for matrix singularities of the three types?"

We will see in Theorem 2.5.23 below that similar decompositions for the cohomology can be observed for essential smoothings of EIDS defined by non-square matrices.

2.5.3 Isolated Cohen-Macaulay Codimension 2 Singularities

As was already discussed earlier, isolated Cohen-Macaulay codimension 2 singularities, which are not complete intersections, arise in a range of dimensions from 0 up to dimension 4. In either case, the miniversal unfolding of the defining matrix[29] $A \in \mathbb{C}\{x\}^{m\times(m+1)}$ induces a semi-universal deformation of the singularity $(X, 0) = (A^{-1}(M^m_{m,m+1}), 0)$ and there is a unique essential smoothing $M = M^m_A$ which is in fact smooth in dimensions $d = \dim(X, 0) \leq 3$.

The zero-dimensional case is rather trivial: Since determinantal singularities are Cohen-Macaulay, the multiplicity of a fat point is preserved under deformations. A stabilization of the defining matrix will therefore split any zero-dimensional determinantal singularity into a collection of finitely many simple points, the number of which is equal to the multiplicity of the singularity.

The 1-dimensional case is known as "space curve singularities". For curves, a lot of theory has been developed beyond the complete intersection case already, in particular for the study of their topology. We refer to [20, Section 7.2.6] for an account on smoothings of curves and deliberately exclude them from our further discussion here.

[29] Since the determinantal structure of a given ICMC2 singularity is unique, we will in what follows usually supress the matrix A from the notation and simply write $(X, 0)$ for the singularity, M for its (essential) smoothing, $\hat{X}$ for its Tjurina transform, etc.

2.5.3.1 From Simple ICMC2 Surfaces Towards Higher Dimensions

In dimension two, the isolated Cohen-Macaulay codimension 2 singularities are the normal surface singularities in $(\mathbb{C}^4, 0)$. Just like the curve case, normal surface singularities have been extensively studied. But even in the particular case of codimension 2 it is, for example, still unknown whether or not a smoothing of the singularity is always simply connected.

Similar to the case of determinantal hypersurfaces, the known results that we wish to present here for surfaces and threefolds are again mostly motivated by observations made for the lists of simple singularities in Tables 2.16 and 2.17. Starting with surfaces, we already noted that the list of simple isolated Cohen-Macaulay codimension 2 singularities coincides with the list of *rational triple points* that have already appeared in Sect. 2.3.6. The rational triple points were classified by Artin [6] in terms of the dual graphs of their resolution and explicit equations for their embeddings in $(\mathbb{C}^4, 0)$ have been given by Tjurina [98].

A resolution of these singularities can be constructed as follows. Tjurina has shown in [98] that any rational triple point $(X, 0)$ is determinantal of type $(2, 3, 2)$ in $(\mathbb{C}^4, 0)$ for some matrix $A \in \mathbb{C}\{x_1, \dots, x_4\}^{2\times 3}$ (cf. Proposition 2.3.25 and Theorem 2.3.26) and the matrix A has rank 1 along the smooth locus X_{reg} of $(X, 0)$. Blowing up the associated rational map

$$(X, 0) \dashrightarrow \mathbb{P}^1, \quad x \mapsto \operatorname{span} A(x)$$

then provides her with the *Tjurina transform* $\hat{X} \xrightarrow{\hat{\nu}} X$ which has at most A-D-E-singularities along its exceptional set $\hat{E} = \hat{\nu}^{-1}(\{0\})$. The A-D-E-singularities are known to be rational with explicitly given resolutions. It can then be shown that the resolution of singularities $Z \xrightarrow{\rho} X$ obtained from resolving the singular points of $\hat{X}$ in fact satisfies the requirements (2.43) for rationality.

This construction is of course not the original approach pursued by Tjurina, since she started with a configuration of exceptional divisors coming from a resolution of singularities in the first place. Just as in Theorem 2.3.28 by Wahl, she obtained the space $\hat{X}$ by *blowing down* the A-D-E configurations on (Z, E). The reason for the presentation given here is that the above construction is what generalizes to simple ICMC2 threefold singularities, as was shown by the authors in [40]. Applying the Tjurina transformation to simple ICMC2 singularities of higher dimension they obtain:

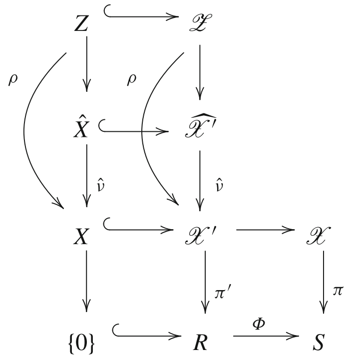
Theorem 2.5.16 ([40]) *All simple ICMC2 singularities $(X, 0)$ of dimension $d \geq 2$ have at most A-D-E-singularities in their Tjurina transform $(\hat{X}, \hat{E})$. Moreover, they admit a resolution of singularities $\rho\colon Z \to X$ factoring through $\hat{\nu}\colon \hat{X} \to X$ such that*

$$R^k \rho_* \mathscr{O}_Z = \begin{cases} \mathscr{O}_X & \text{if } k = 0, \\ 0 & \text{otherwise,} \end{cases}$$

i.e. they are rational.

Note that the exceptional set $\hat{E} = \hat{\nu}^{-1}(\{0\})$ is a $\mathbb{P}^1$ and therefore not a divisor in $\hat{X}$ in dimensions $d > 2$.

Interestingly, the proof of Theorem 2.5.16 in dimensions 3 and 4 is based on the compatibility of the Tjurina transformation with deformations for the simple singularities. For the rational triple points of surfaces this has already been noted (cf. Theorem 2.3.28) and exploited in order to construct a *resolution in family* factoring through the Tjurina transformation in family, i.e. a commutative diagram



(2.66)

where, as in (2.44), $Z \to X$ is a minimal good resolution of singularities, $\mathscr{Z} \to R$ a flat family deforming Z, $\mathscr{X}' = \rho(\mathscr{Z})$ the blowdown of the deformation of Z, $\widehat{\mathscr{X}'}$ the Tjurina transformation in family for the induced deformation of X, and $\mathscr{X} \to S$ a representative of the semi-universal deformation of $(X, 0)$. Given the results of Artin, Wahl, and de Jong summarized in Sect. 2.3.6, such resolutions in family can be constructed for all rational surface singularities which are determinantal and the family over the Artin component $\Phi(R) = S' \subset S$ is always a smoothing of $(X, 0)$.

Lemma 2.5.17 *Suppose* $(X, 0)$ *is a determinantal rational surface singularity and (2.66) a minimal good resolution in family. Then the generic fiber over the Artin component* $X_s = \pi^{-1}(\{s\})$ *is diffeomorphic to the central fiber* Z *of the resolution.*

Proof We may assume X to be embedded in some open domain $U \subset \mathbb{C}^p$ containing the origin. Let B_ε be a Milnor ball for the singularity of X at 0 and $\rho^{-1}(B_\varepsilon)$ its preimage in Z. Since $(R, 0)$ is smooth, we may replace R by a small disc $D_\delta \subset \mathbb{C}^k$ for some k and assume $\mathscr{X}'$ to be embedded in the product $U \times \mathbb{C}^k$. We then extend the Milnor ball to a tube $B_\varepsilon \times D_\delta$ and its preimage $\rho^{-1}(B_\varepsilon \times D_\delta)$ in the total space $\mathscr{Z}$.

Using Ehresmann's lemma and the transversality of the intersection $\partial B_\varepsilon \cap X$ we see that the deformation of X by $t \in D_\delta$ does not alter its boundary up to diffeomorphism for $1 \gg \delta > 0$ sufficiently small. Since the projection $\rho\colon Z \to X$ is an isomorphism off 0, the same holds for Z and its boundary $\rho^{-1}(\partial B_\varepsilon) \cap Z$ in the family $\mathscr{Z} \to D_\delta$. But by construction, the space $Z = (\pi' \circ \rho)^{-1}(\{0\})$ is already smooth and hence $Z \cap \rho^{-1}(B_\varepsilon)$ a smooth compact manifold with boundary. Another application of Ehresmann's lemma shows that, again for $\delta > 0$ sufficiently small, the family $\mathscr{X} \to D_\delta$ is a trivial fiber bundle.

Since the deformation of $(X, 0)$ over the Artin component $(S', 0)$ is a smoothing and Φ is finite and surjective onto $(S', 0)$, the fiber $X_s \cong \pi'^{-1}(\{t\})$ at a point $s = \Phi(t)$ is smooth for generic $t \in D_\delta$. But then the restriction of $\rho\colon Z_t \cap \rho^{-1}(B_\varepsilon) \to \pi'^{-1}(\{t\}) \cap B_\varepsilon$ must be an isomorphism for ρ was a resolution in family and the resolution in the central fiber was minimal. The assertion now follows from the identifications

$$X_s \cong_{\text{diff}} \pi'^{-1}(\{t\}) \cong_{\text{diff}} (\pi' \circ \rho)^{-1}(\{t\}) \cong_{\text{diff}} Z$$

up to diffeomorphism. □

Corollary 2.5.18 *The smoothing of a determinantal rational surface singularity over the Artin component is homotopy equivalent to a bouquet of* 2*-dimensional spheres.*

Proof Due to Laufer's criterion on rationality ([71, Theorem 4.2]), all components E_i of the exceptional set $E \subset Z$ in a good resolution of singularities must be smooth rational curves $E_i \cong \mathbb{P}^1 \cong S^2$ intersecting transversally. Since the dual graph of a rational singularity can have no cycles, the statement follows from the fact that $E \hookrightarrow Z$ is a deformation retract of an appropriately chosen representative of Z, cf. [75]. □

Remark 2.5.19 For an *arbitrary* ICMC2 surface singularity $(X, 0)$ not everything is known about the homotopy type of its smoothing M or even the homology groups with integer coefficients. Due to a result by Greuel and Steenbrink ([51], cf. [20, Theorem 7.2.16]) the smoothing is connected and furthermore its first Betti number vanishes ([51], [20, Theorem 7.2.18]). But for instance it is not known whether or not $\pi_1(M) = 0$ in general.

2.5.3.2 Vanishing Homology for ICMC2 Threefolds

For simple ICMC2 *threefold* singularities there is no full resolution in family. However, the Tjurina transformation in family applied to a smoothing of $(X, 0)$ provides us with a truncated version of the diagram (2.66): Just delete the top row $Z \hookrightarrow \mathscr{Z}$ and observe that due to the results by Schaps, Corollary 2.3.20, the map Φ and consequently also $\mathscr{X}' \to \mathscr{X}$ are isomorphisms.

Theorem 2.5.20 ([40]) *Let M be the smoothing of an ICMC2 threefold singularity* $(X, 0) \subset (\mathbb{C}^5, 0)$*, which has only isolated singularities in the Tjurina transform $\hat{X}$. Then the singularities of $\hat{X}$ are isolated complete intersection singularities and the homology groups with integer coefficients of M are*

$$H_0(M) = \mathbb{Z}, \quad H_1(M) = 0, \quad H_2(M) = \mathbb{Z}, \quad H_3(M) = \mathbb{Z}^r,$$

where $r \in \mathbb{N}_0$ *is the sum of the Milnor numbers* $r = \sum_{p \in \Sigma(\hat{X})} \mu(\hat{X}, p)$ *of the singularities in* $\hat{X}$ *if the defining matrix is of size* 2×3*, or* $r = \sum_{p \in \Sigma(\hat{X})} \mu(\hat{X}, p) - 1$ *otherwise.*

Proof We sketch a proof for the case of simple singularities. The full proof can be found in [40].

According to Theorem 2.5.16 the singularities in the Tjurina transform $\hat{X}$ of $(X, 0)$ are at most A-D-E-singularities. Choose a Milnor tube $B_\varepsilon \times D_\delta$ as in the proof of Corollary 2.5.18. One can show that a smoothing of $(X, 0)$ induces a smoothing of all singularities of $\hat{X}$ and that the projection $\rho\colon \hat{X}(t) \to X(t)$ is an isomorphism over smooth fibers $X(t) \cong_{\text{diff}} M$. Now the central fiber $\hat{X} = \hat{X}(0)$ of the Tjurina transform retracts onto its exceptional set $\hat{E} \cong \mathbb{P}^1 \cong S^2$ and the class of that sphere freely generates $H^2(\hat{X})$. Passing to the smooth fiber $\hat{X}(t) \cong_{\text{diff}} M$, one can show that this cycle survives while the vanishing cycles from the smoothings of the A-D-E-singularities in $\hat{X}$ freely generate $H_3(M)$. □

Remark 2.5.21 In [111] the second named author shows that the homology groups of the smoothing take the same form for all threefolds defined by matrices of size 2×3, i.e. also those with *non-isolated* singularities in the Tjurina transform. In that case, however, there is no formula for the computation of the rank r of the middle homology group.

Using the bouquet decomposition described in Theorem 2.5.23 below, it can even be shown that for matrices of size 2×3, the Milnor fibers of isolated threefold singularities are homotopy equivalent to a bouquet of one 2-sphere and a finite number of 3-spheres.

For ICMC2 threefold singularities defined by matrices of arbitrary size $m \times (m + 1)$, the second named author has recently announced in [113] that the second Betti number is always equal to 1, independent of the entries of the matrix. For the third Betti number, there is a lower bound depending only on m, cf. (2.75) and Theorem 2.5.23.

Remark 2.5.22 What is interesting in Theorem 2.5.20 is the non-vanishing of the second Betti number. It had already been observed by Damon and Pike in [29] that for some simple ICMC2 threefolds the reduced Euler characteristic of the smoothing was strictly positive. Since the first Betti number of M_A^m is known to be zero due to results by Greuel and Steenbrink,[30] this implied that at least $H_2(M_A^m)$ had to be nontrivial. The fact that there is more than one degree with nontrivial vanishing homology raises the question as to what is the correct notion of Milnor number for ICMC2 threefolds. As of this writing, this question has not been settled in a satisfactory way. On the one hand, the independence of the second Betti number from the defining matrix A suggests that $b_3 = \operatorname{rank} H_3(M_A^m)$ is a good candidate. On the other hand, given the Bouquet decomposition for essential smoothings,

[30] See [51], cf. also [20, Theorem 7.2.17].

Theorem 2.5.23, the numbers $\lambda(m-1, A)$ in (2.69) yield another candidate for a Milnor number, which does not necessarily agree with b_3, cf. Remark 2.5.25.

2.5.3.3 Comparison of Milnor and Tjurina Numbers

Coming back to the case of surface singularities again, there is Wahl's conjecture 2.5.1 which was already stated in the beginning of this section. For normal surface singularities, the first Betti number of any smoothing is zero due to the previously cited result of Greuel and Steenbrink, see [51], or also [20, Theorem 2.7.18]. Consequently, the Milnor number μ of the singularity is defined to be the second Betti number.

Since the "if"-part of Wahl's conjecture has been established already and all *simple* ICMC2 surface singularities are quasi-homogeneous, we see that in this case

$$\mu = b_2(M_A^2) = \tau(X_A^2, 0) - 1 = \tau_{\mathrm{GL}}(A) - 1 \tag{2.67}$$

where as usual $A \in \mathbb{C}\{x, y, z, w\}^{2\times 3}$ is the defining matrix of the singularity.

For the simple ICMC2 *threefolds* on the other hand, the authors observed a different behaviour in [40]: The Tjurina number seems to be rather unrelated to the topology of the smoothing. For instance, singularities defined by the matrices

$$\begin{pmatrix} w & y & x \\ z & w & y+v^k \end{pmatrix},$$

the so-called Π_k-family, have Tjurina number $\tau = 2k-1$. Their smoothings, however, are all homotopy equivalent to the sphere S^2, independent of k. More generally, they establish an equality

$$\tau(X, 0) = h^1(\hat{X}, T^0_{\hat{X}}) + \sum_{p\in\Sigma(\hat{X})} \tau(\hat{X}, p) \tag{2.68}$$

for all ICMC2 threefolds defined by matrices of size 2×3 with only isolated singularities in the Tjurina transform. Here $T^0_{\hat{X}}$ denotes the tangent sheaf of $\hat{X}$ and $\tau(\hat{X}, p)$ the local Tjurina numbers at the singularities $p \in \Sigma(\hat{X})$. These local Tjurina numbers can then be related to the local Milnor numbers and the topology of the smoothing. But the example given by the Π_k-family shows that the "correction term" given by $h^1(\hat{X}, T^0_{\hat{X}})$ can become arbitrary big.

2.5.4 Arbitrary EIDS and Some Further, Particular Cases

2.5.4.1 Bouquet Decomposition for Essential Smoothings

For arbitrary EIDS, the second named author has established a bouquet decomposition of the essential smoothing as an application of a result due to Tibăr [97] (see also [20, Theorem 6.10.6]) to determinantal singularities. Tibăr's bouquet theorem itself is based on the carousel construction exhibited in [20, Chapter 6].

Theorem 2.5.23 ([112]) *Let $(X_A^s, 0) \subset (\mathbb{C}^p, 0)$ be an EIDS of positive dimension and of type (m, n, s) defined by a matrix $A \in \mathbb{C}\{x_1, \ldots, x_p\}^{m\times n}$. Then the essential smoothing M_A^s of $(X_A^s, 0)$ is homotopy equivalent to*

$$M_A^s \cong_{\text{ht}} L_{m,n}^{s,p} \vee \bigvee_{0\le r<s} \bigvee_{i=1}^{\lambda(r,A)} S^{p-(m-r)(n-r)+1}(L_{m-r,n-r}^{s-r-1,(m-r)(n-r)-1}) \tag{2.69}$$

where $L_{m,n}^{s,k}$ is the intersection $L_{m,n}^{s,k} = H^k \cap M_{m,n}^s$ of a k-dimensional hyperplane H^k in general position off the origin with the generic determinantal variety.

In this formula, $S^k(\cdot)$ denotes the k-fold repeated suspension of a topological space with the convention that $S(\emptyset) = S^0$ is the 0-dimensional sphere, $S^0(X) = X$ for every X, and $S^k(X) = \emptyset$ for negative k. Note that $L_{m,n}^{s,mn-1}$ is nothing but the *complex link* of the generic determinantal variety $M_{m,n}^s$ at the origin, see e.g. [20, Section 5.9.3]. For more general values of p, the space $L_{m,n}^{s,p}$ is the essential smoothing of a linear EIDS determined by a generic linear map $\mathbb{C}^p \to \mathbb{C}^{m\times n}$.

Proof We give a rough outline of the proof in order to indicate its synopsis with the other methods. Recall from Remark 2.1.14 that $(X_A^s, 0)$ can be realized as a complete intersection on $(\mathbb{C}^p, 0) \times (M_{m,n}^s, 0)$ by considering the graph $\Gamma_A = \{(x, \varphi) \in \mathbb{C}^p \times \mathbb{C}^{m\times n} : \varphi = A(x)\}$ which is given by the $m \cdot n$ equations $h_{i,j} = y_{i,j} - a_{i,j}(x)$. Then

$$X_A^s \cong \Gamma_A \cap \mathbb{C}^p \times M_{m,n}^s \subset \mathbb{C}^p \times \mathbb{C}^{m\times n}$$

and these equations form a regular sequence on the coordinate ring $\mathbb{C}\{x\}[y]/\langle Y^{\wedge s}\rangle$ of the variety $(\mathbb{C}^p, 0) \times M_{m,n}^s$. Note that the latter variety is canonically Whitney stratified by the product of the rank stratification with $\mathbb{C}^p$. It is now not too difficult to see that the transversality conditions imposed on A off the origin translate to an appropriate notion of transversality of the functions $h_{i,j}$ that allows one to say that these equations define an isolated complete intersection singularity on $(\mathbb{C}^p, 0) \times (M_{m,n}^s, 0)$ in the *stratified sense*.

It is now easy to see that by construction the essential smoothing M^s_A coincides with the Milnor fiber[31] of this isolated complete intersection on $(\mathbb{C}^p, 0) \times (M^s_{m,n}, 0)$. The remainder of the proof now follows from Tibăr's decomposition theorem [97, Bouquet Theorem] and a modification of [97, Corollary 4.2]. □

Corollary 2.5.24 *The essential smoothing of an EIDS* $(X^s_A, 0) \subset (\mathbb{C}^p, 0)$ *is homeomorphic to the Milnor fiber of the complete intersection defined by* $y_{i,j} - a_{i,j}$ *for* $0 < i \leq m$, $0 < j \leq n$ *on* $(\mathbb{C}^p, 0) \times (M^s_{m,n}, 0)$.

In certain particular cases (see below), Theorem 2.5.23 leads to a full understanding of the essential smoothing M^s_A up to homotopy: The numbers $\lambda(r, A)$ can, in principal, be computed from the so-called Cerf-diagrams involved in the carousel construction, see e.g. [20, Theorem 6.6.6] and [20, Section 6.7]. What is more difficult, is the investigation of the complex links and, more generally, the spaces $L^{s,p}_{m,n}$ for the various values of m, n, s and p. These are the atomic building blocks for the topology of essential smoothings that can not be turned into singularities of complete intersections anymore.

The Euler characteristic of the complex links of generic determinantal varieties has been computed by Ebeling and Gusein-Zade in [34]. Without loss of generality, one may assume that $m \leq n$. Then the reduced Euler characteristic of the complex link of $M^s_{m,n}$ at the origin is

$$\overline{\chi}(L^{s,mn-1}_{m,n}) = (-1)^s \binom{m-1}{s-1}. \tag{2.70}$$

This formula has been used by Gaffney, Grulha, and Ruas to compute the *local Euler obstructions*[32] of the generic determinantal varieties in [42]. Again, for $m \leq n$ they find

$$\mathrm{Eu}(M^s_{m,n}, 0) = \binom{m}{s-1}. \tag{2.71}$$

This formula has then been generalized for the generic determinantal varieties over arbitrary fields by Zhang [114].

Both these invariants are of fundamental importance in the study of stratified Morse theory on determinantal varieties, cf. [20, Chapter 5], and [45]. In particular, they allow for the computation of the Euler characteristic of the essential smoothing M^s_A of an arbitrary EIDS $(X^s_A, 0)$ as in Theorem 2.5.23 (again assuming $m \leq n$)

$$\overline{\chi}(M^s_A) = \overline{\chi}(L^{s,p}_{m,n}) + \sum_{0 \leq r < s} (-1)^{p+s-r-(m-r)(n-r)} \cdot \binom{m-r-1}{s-r-2} \cdot \lambda(r, A) \tag{2.72}$$

up to the term $\overline{\chi}(L^{s,p}_{m,n})$, which is constant and independent of the specific matrix A.

[31] See e.g. [20, Theorem 6.10.3].

[32] The local Euler obstruction was introduced by MacPherson in [79].

Remark 2.5.25 In the spirit of Definition 2.5.15 for the "characteristic cohomology" for smoothings of determinantal hypersurface singularities due to Damon we can make the following definition for smoothings of EIDS defined by non-square matrices $A \in \mathbb{C}\{x_1, \ldots, x_p\}^{m\times n}$. If $(X^s_A, 0) \subset (\mathbb{C}^p, 0)$ is smoothable, then a stabilization

$$A_t : B_\varepsilon \to \mathbb{C}^{m\times n} \supset M^s_{m,n}$$

will not meet the singular locus $M^{s-1}_{m,n}$ of the generic determinantal variety $M^s_{m,n}$ so that the intersection $A_t(B_\varepsilon) \cap M^s_{m,n}$ will be completely contained in the stratum $V^{s-1}_{m,n}$. In analogy to Damon's definition we may set

$$\mathscr{A}_{M^s_{m,n}}(A) = A_t^*(H^\bullet(V^{s-1}_{m,n})) \subset H^\bullet(M^s_A)$$

to be the image of the pullback in cohomology of $A_t : M^s_A \to V^{s-1}_{m,n}$ to the essential smoothing $M^s_A = A_t^{-1}(M^s_{m,n})$. In the smoothable case the wedge sum in (2.69) simplifies to $r = s - 1$ so that

$$M^s_A \cong_{\mathrm{ht}} L^{s,p}_{m,n} \vee \bigvee_{i=1}^{\lambda} S^{p-(m-s+1)(n-s+1)}.$$

It has been announced in [113] that $\mathscr{A}_{M^s_{m,n}}(A)$ is precisely the contribution of $H^\bullet(L^{s,p}_{m,n})$ so that, given the analogy of the definitions, the number λ plays the rôle of μ in (2.65).

2.5.4.2 IDS of Maximal Minors and Their Newton Polyhedra

Extending the classical perspective of Newton polyhedra to IDS of maximal minors, Esterov obtains some results on isolated determinantal singularities and the topology of functions on it in terms of their *Newton polyhedra* in [36]. Recall that for a holomorphic germ $f = \sum_{\alpha\in\mathbb{N}_0^p} c_\alpha \cdot x^\alpha \in \mathbb{C}\{x_1, \ldots, x_p\}$ the Newton polyhedron $\Delta(f) \subset \mathbb{R}^p$ is defined as the convex hull of the set

$$\mathrm{Supp} f := \{\alpha \in \mathbb{N}_0^p : c_\alpha \neq 0\} \subset \mathbb{R}^p.$$

Conversely, for a given polyhedron $\Delta \subset \mathbb{R}^p$ we say that $f \in \mathbb{C}\{\Delta\}$ if $\mathrm{Supp}(f) \subset \Delta$. Then it is customary to compute invariants of such germs, such as for instance the topological Euler characteristic of their Milnor fibers, or the ζ-function of their monodromy, in terms of their Newton polyhedra. As can be expected for techniques, which focus on the occurring monomials and not their coefficients, these considerations require a genericity assumption on the setting and in particular on the coefficients involved. Considering a germ of a holomorphic matrix $A = (a_{ij}) \in \mathbb{C}\{x_1, \dots, x_p\}^{m,n}$, $m \leq n$, these conditions read:

(a) Unmixedness assumption: The Newton polyhedron of an entry $a_{i,j}$ is independent of the choice of the first index i.
(b) General position condition: For every choice of positive weights $w = (w_1, \dots, w_p)$ and any choice of a subset $I \subset \{1, 2, \dots, m\}$ of the rows of A, consider a new matrix $A' = \mathrm{in}_w(A_{I,\{1,\dots,n\}})$ whose entries are the *weighted initial forms*

$$\mathrm{in}_w\left(\sum_{\alpha \in \mathbb{N}_0^p} c_\alpha x^\alpha\right) = \sum_{\langle w,\alpha\rangle = d} c_\alpha x^\alpha, \quad d = \min\{\langle w, \alpha\rangle : c_\alpha \neq 0\}$$

of the entries of the chosen rows in I. Then the general position condition is satisfied, if for all such choices, the variety defined by the maximal minors of A' has expected codimension $(p - |I| + 1)$ at all points outside the coordinate hyperplanes.
(c) Strong general position condition: For every choice of positive weights, the maximal minors of the matrix $\mathrm{in}_w(A)$, whose entries are the weighted initial forms of the entries of A, define a determinantal variety which is *non-singular* outside the coordinate hyperplanes.

The unmixedness assumption can always be achieved by replacing the rows with sufficiently general $\mathbb{C}$-linear combinations of the rows. The conditions on (strong) general position, however, are indeed (open) conditions on the coefficients appearing in the entries of the matrix.

A crucial tool in Esterov's approach to computing algebraic and topological invariants of the singularities $(X_A^m, 0)$ defined by the *maximal* minors of A and the restrictions $f|(X_A^m, 0)$ to them via Newton polyhedra is the *mixed volume*, which is the unique symmetric function on p-tuples of bounded polyhedra

$$\mathrm{MV}\colon (\Delta_1, \dots, \Delta_p) \mapsto \mathrm{MV}(\Delta_1, \dots, \Delta_p) \in \mathbb{R}$$

which is multilinear with respect to scaling and the *Minkowski sum* of polyhedra and which takes the tuple $(\Delta, \dots, \Delta)$ to $\mathrm{Vol}_\mathrm{p}(\Delta)$ for any single bounded polyhedron $\Delta \subset \mathbb{R}^p$. For a polyhedron $\Delta \subset \mathbb{R}^p_{\geq 0}$, we denote the pair $(\mathbb{R}^p_{\geq 0}, \Delta)$ by $\tilde{\Delta}$. There is a definition for the *mixed volume* of *bounded pairs* of polyhedra which are *parallel* to a given cone, see [36, Definition 1.5]. We refer to [36] and the references there for the details of the above notions. Covering them here in all detail is beyond our scope.

Theorem 2.5.26 ([36, Theorem 1.9 and Theorem 1.12]) *Let $\Delta_0, \Delta_1, \dots, \Delta_n \subset \mathbb{R}^p_{\geq 0}$ be polyhedra with $\mathbb{R}^p_{\geq 0} \setminus \Delta_j$ bounded for every $j = 0, \dots, n$. Then the following hold:*

(i) *For all matrices $A \in \mathbb{C}\{x_1, \dots, x_p\}$ satisfying the unmixedness and the general position condition the multiplicity of $(X^m_A, 0)$ is equal to*

$$\sum_{0<j_0<\cdots<j_{n-m}\leq m} p! \cdot \mathrm{MV}(\tilde{\Delta}_{j_0}, \dots, \tilde{\Delta}_{j_{n-m}}, \underbrace{L, \dots, L}_{p-n+m-1})$$

where $L = (\mathbb{R}^p_{\geq 0}, \mathbb{R}^p_{\geq 0} \setminus S_p)$ is the complement of the standard simplex $S_p = \{x \in \mathbb{R}^p_{\geq 0} : \sum_{i=1}^p x_i \leq 1\}$.

(ii) *Whenever $p \leq 2(n - m + 2)$ and the complements of the polyhedra in $\mathbb{R}^p_{\geq 0}$ are bounded,then for all A in strong general position, the germ $(X^m_A, 0)$ is an isolated determinantal singularity.*

(iii) *Furthermore, in the setting of 2. for almost all functions[33] $f \in \mathbb{C}\{\Delta_0\}$, the Euler characteristic of the Milnor fiber of $f|(X^m_A, 0)$ equals*

$$\sum_{a_0\in\mathbb{N}} \sum_{I\subset\{1,\dots,p\}} \sum_{\{j_1,\dots,j_q\}\subset\{1,\dots,m\}} (-1)^{|I|+m+n} \binom{m+q-n-1}{|I|+q-a_0-2} \times$$

$$\times \left(\sum_{\substack{(a_{j_1},\dots,a_{j_q})\in\mathbb{N}^q,\\ a_{j_1}+\cdots+a_{j_q}=|I|-a_0}} |I|! \cdot \mathrm{MV}(\underbrace{\tilde{\Delta}^I_0, \dots, \tilde{\Delta}^I_0}_{a_0}, \underbrace{\tilde{\Delta}^I_{j_1}, \dots, \tilde{\Delta}^I_{j_1}}_{a_{j_1}}, \dots, \underbrace{\tilde{\Delta}^I_{j_q}, \dots, \tilde{\Delta}^I_{j_q}}_{a_{j_q}}) \right)$$

The approach pursued by Esterov differs strongly from anything else presented here. His interest focusses on "resultantal sets" of which ideals of maximal minors are a special case. In addition to the formulae above, he also obtains a way to compute the topological ζ-function of a function on a resultantal set. We refer readers interested in this approach to the original article [36].

2.5.4.3 Formulae for the Vanishing Euler Characteristic Using Polar Varieties

Various other methods have been developed in order to compute the Euler characteristics of essential smoothings, including the space $L^{s,p}_{m,n}$ itself. One of these methods is to study successive hyperplane sections of a given determinantal singularity

[33] See [36] for the precise condition which is again of the flavor of the general position condition.

$(X_A^s, 0)$ and its deformation, together with the associated *polar varieties*[34] and their multiplicities $m_i(X_A^s, 0)$ for $0 \leq i < \dim(X_A^s, 0) = d$. In addition to that, one needs the so-called m_d-multiplicity $m_d(X_A^s, 0)$ introduced by Gaffney in [41] to also capture the behaviour of the singularity in families. For a given singularity, this multiplicity measures the number of critical points of a generic linear form on the smooth locus of the nearby stable object.[35] To give a concise formula, we furthermore use the convention that $m_i(X, 0) = 0$ for negative i.

Theorem 2.5.27 *Let M_A^s be the essential smoothing of an EIDS $(X_A^s, 0) \subset (\mathbb{C}^p, 0)$ defined by a matrix $A \in \mathbb{C}\{x_1, \ldots, x_p\}^{m\times n}$ with $m \leq n$. Then*

$$\chi(M_A^s) = \sum_{0\leq r<s} \left(\sum_{j=0}^{d(r)} (-1)^{d(r)-j} m_{d(r)-j}(X_A^{r+1}, 0) \right) \cdot (-1)^{s-r-1} \binom{m-r}{s-r-1} \tag{2.73}$$

where $d(r) = \dim(X_A^{r+1}, 0) = p - (m-r)(n-r)$.

This theorem and its variants have appeared in several places such as [9, 21, 34, 42, 83], and [110].

Proof Again, we only sketch the proof to illustrate the related ideas. Details can be found in the various sources cited above.

Let $\mathbf{A}\colon U \times T \to \mathbb{C}^{m\times n}$ be a suitable representative of a 1-parameter unfolding of the defining matrix A on a parameter t such that A_t is a stabilization for $t \neq 0$ and choose a sequence of linear forms $l_1, \ldots, l_{d(s)}$ on the ambient space $\mathbb{C}^p$ of the singularity. We denote by D_j the hyperplane of codimension j defined by $l_1 = \cdots = l_j = 0$. If the linear forms have been chosen sufficiently general, then either one of the singularities $(X_A^s \cap D_j, 0) \subset (D_j, 0)$ is again an EIDS and the restriction of the unfolding $\mathbf{A}$ to D_j induces an essential smoothing thereof. Furthermore, for every $t \neq 0$ sufficiently small, every $r < s$, and every $j > 0$ the function l_{j+1} has only complex Morse singularities on the interior of the fiber

$$D_j \cap A_t^{-1}(M_{m,n}^{r+1}) \cong D_j' \cap A_t^{-1}(M_{m,n}^{r+1}) \cong D_j' \cap A_0^{-1}(M_{m,n}^{r+1}).$$

where D_j' denotes a hyperplane parallel to D_j off the origin. The above isomorphisms come from wiggeling either D_j or A_t and parallel transport of the associated fibers by virtue of Thom's isotopy lemma. The number of the Morse critical points is precisely $m_{d(r)-j}(X_A^{r+1}, 0)$, see [101]. Finally, for $j = 0$, the multiplicity

[34] Polar varieties for complex analytic germs were introduced in [101]. See also [95].

[35] Note that in order for $m_d(X_A^s, 0)$ to be an invariant of the singularity $(X_A^s, 0)$ itself rather than the given family, the nearby stable object needs to be uniquely determined by $(X_A^s, 0)$. This is guaranteed for EIDS by the existence of the miniversal unfolding and the uniqueness of the essential smoothing.

$m_{d(r)}(X_A^{r+1}, 0)$ counts the number of critical points of l_1 on the regular locus of the essential smoothing M_A^{r+1}, see [41].

By downwards induction on the codimension j of the hyperplanes one can now rebuild the original essential smoothing $M_A^s = A_t^{-1}(M_{m,n}^s)$ from its hyperplane sections starting with $j = d(s-1)$. The Morse critical points on the regular loci of the various hyperplane sections of $A_t^{-1}(M_{m,n}^{r+1})$ above are then *stratified* Morse critical points of the fiber $D_j \cap A_t^{-1}(M_{m,n}^s)$ located on the respective strata $D_j \cap A_t^{-1}(V_{m,n}^r)$. At any such point, the factor

$$(-1)^{s-r-1}\binom{m-r}{s-r-1} = 1 - \chi(L_{m-r,n-r}^{s-r,(m-r)(n-r)-1})$$

accounts for the *normal Morse datum*[36] associated to that point which is determined by the complex links of the generic determinantal varieties according to Lemma 2.3.3 and (2.13). □

Remark 2.5.28 The multiplicities $m_i(X_A^{r+1}, 0)$ appearing in (2.73) can in principal be computed by various different methods. For $i < d = \dim(X_A^{r+1}, 0)$ this is possible directly from their definitions in [101]. For the m_d-multiplicity, Gaffney and Ruas have given a formula in [43, Proposition 4.6] as the sum of the multiplicity of the pair of modules given by the Jacobian module and the normal module of the singularity and the intersection number of the image of the defining matrix with a certain polar variety of the generic determinantal variety.

Remark 2.5.29 In general, the hard task is to relate the multiplicities in (2.73) or the numbers $\lambda(r; A)$ in (2.69) to deformation theoretic invariants such as the GL-Tjurina number of the defining matrix. In the special cases described earlier, as for example in Theorem 2.5.20, one can, of course, also apply Theorem 2.5.23 or Theorem 2.5.27 and then compare $m_i(X_A^s, 0)$ and $\lambda(r; A)$ to the Milnor and Tjurina numbers of the singularities.

Yet another approach to the computation of the vanishing Euler characteristic has been pursued by Damon and Pike in [30] and [29]. For certain determinantal varieties (including the determinantal hypersurfaces and type $(2, 3, 2)$), they succeed to embed the associated generic determinantal variety in a special arrangement of so-called "H-holonomic divisors" W_i:

$$M_{m,n}^s \subset \bigcup_i W_i.$$

These *free completions* of $M_{m,n}^s$ are extracted from certain group representations on the space of matrices which are closely related to Cholesky decomposition.

[36] For the discussion of normal Morse data in complex stratified Morse theory, see [45].

A nonlinear section $A\colon (\mathbb{C}^p,0) \to (\mathbb{C}^{m\times n},0) \supset W$ of an H-holonomic divisor W which is algebraically transverse to W off the origin, gives rise to an "almost free divisor" $(W_A,0) = (A^{-1}(W),0) \subset (\mathbb{C}^p,0)$. Similar to the case of determinantal varieties, one considers the deformations of $(W_A,0)$ induced from unfoldings of A. Stabilizations A_t of A lead to the analogue of essential smoothings $A_t^{-1}(W)$ for $(W_A,0)$ and these spaces are homotopy equivalent to a bouquet of spheres for the same reasons as outlined in Lemma 2.5.7 for determinantal hypersurfaces—only that in this case the number μ of these spheres is equal to the $\mathscr{K}_{H,e}$-codimension of A ([29, Theorem 3.1]). This $\mathscr{K}_{H,e}$-equivalence for non-linear sections of H-holonomic free divisors is similar to the SL-equivalence of matrices discussed in Remark 2.2.5, see e.g. [47].

Using the additivity of the topological Euler characteristic on complex analytic sets, Damon and Pike can then infer the Euler characteristic of the essential smoothing of the determinantal singularity from its relative position in the stabilizations

$$M_A^s = A_t^{-1}(M_{m,n}^s) \subset \bigcup_i A_t^{-1}(W_i).$$

This machinery is constructed for the purpose of relating the vanishing Euler characteristic of essential smoothings to deformation theoretic invariants. For various specific configurations closed formulas are given for $\overline{\chi}(M_A^s)$ in terms of the various $\mathscr{K}_{H,e}$-codimensions, cf. e.g. [29, Theorem 8.1]. However, it is not clear how to determine such formulae in general and how the various $\mathscr{K}_{H,e}$-codimensions for the W_i relate to the GL- or SL-Tjurina numbers of the original matrix singularity.

2.5.4.4 Some Explicitly Known Complex Links

To conclude this section, we will summarize a few special cases in which the topology of the essential smoothings of generic linear EIDS, i.e. the basic building blocks in (2.69), is known.

The complex links of the degenerate square matrices have been described by Goryunov [46, Theorem 2.1]:

Theorem 2.5.30 *Let $L_{(*)}$ be the complex link of the generic determinantal hypersurface in $\mathbb{C}^{m\times m}_{(*)}$ where $(*)$ denotes either* sq, sym, *or* sk. *Then in every case $L_{(*)}$ is homotopy equivalent to a single sphere S^{N-2} where N equals either m^2, $\frac{1}{2}m(m+1)$, or $\frac{1}{2}m(m-1)$ with $m = 2n$ even, respectively. Depending on the case, these spheres can be chosen to be*

sq : *all degenerate Hermitian matrices in $\mathbb{C}^{m\times m}$ with trace* 1 *and all eigenvalues non-negative.*

sym : *all degenerate real matrices in $\mathbb{C}^{m\times m}_{\text{sym}}$ with trace* 1 *and all eigenvalues non-negative.*

sk : *all degenerate quaternionic matrices in $\mathbb{C}^{m\times m}_{\text{sk}}$ with skew trace* 1 *and all skew eigenvalues non-negative.*

The homogeneity of these particular singularities allow for scaling of these spaces to arbitrarily small sizes close to the origin. For the skew-symmetric matrices A of even size $m = 2n$ Goryunov has set the *skew trace* to be $\sum_{i=1}^{n} a_{2i-1,2i}$ and the skew eigenvalues λ_i the solutions of the equation

$$\mathrm{Pf}\left(A - \lambda \cdot \sum_{i=1}^{n} \left(E_{2i-1,2i} - E_{2i,2i-1}\right)\right) = 0.$$

For non-square matrices it is possible to determine the spaces $L^{s,p}_{m,n}$ up to homotopy for all values of n and p when $m = s = 2$, see [112, Section 4.1]:

$$L^{2,p}_{2,n} \cong_{\mathrm{ht}} \begin{cases} \{\mathrm{pt.}\} & \text{if } p \geq 2n, \\ S^2 & \text{if } n < p < 2n, \\ \bigvee_{i=1}^{n-1} S^1 & \text{if } p = n, \\ \{n \text{ points}\} & \text{if } p = n-1. \end{cases} \tag{2.74}$$

Recently, the Betti numbers and even the full cohomology groups with integer coefficients *below* the middle degree of *smooth* complex links of the generic determinantal varieties $(M^s_{m,n}, 0)$ have been computed for all values of (m, n, s) in [113]: We may suppose that $m \leq n$. Then the condition on the complex link $L^{s,k}_{m,n}$ to be smooth and non-empty restricts the admissible range for k to

$$(m - s + 1)(n - s + 1) \leq k < (m - s + 2)(n - s + 2)$$

and we have an exact sequence of graded $\mathbb{Z}$-modules

$$0 \to H^{\leq d}(\mathrm{Grass}(m - s + 1, m)) \to H^{\bullet}(L^{s,k}_{m,n}) \to Q \to 0 \tag{2.75}$$

where $d = \dim L^{s,k}_{m,n} = k - (m - s + 1)(n - s + 1)$ and $H^{\leq d}(\mathrm{Grass}(m - s + 1, m)) = \bigoplus_{i \leq d} H^i(\mathrm{Grass}(m - s + 1, m))$ denotes the *truncated* cohomology of the Grassmannian. Moreover, the quotient Q is concentrated in cohomological degree d.

Appendix: Lists of Simple Singularities

Arnold's Lists

For reader's convenience and to complement Theorems 2.4.1 and 2.4.2, we give the lists due to Arnold (see [3]) which are mentioned there. Note that the $\mathscr{R}$-simple germs in Table 2.3 are stated as plane curve singularities in cases D_k and E_k and as

Table 2.3 The $\mathscr{R}$-simple germs from [2]

Type	Normal form	μ	τ	
A_k	x^{k+1}	k	k	$k \geq 1$
D_k	$x^2y + y^{k-1}$	k	k	$k \geq 4$
E_6	$x^3 + y^4$	6	6	
E_7	$x^3 + xy^2$	7	7	
E_8	$x^3 + y^5$	8	8	

Table 2.4 The $\mathscr{R}_\delta$-simple germs from [4]. The boundary is given by $x = 0$

Name	Normal form	$\mu(f)$	$\mu(f\|\{x = 0\})$	$\mu_\partial(f)$	
B_k	$\pm x^k \pm y^2$	$k-1$	1	k	$k \geq 2$
C_k	$xy \pm y^k$	1	$k-1$	k	$k \geq 2$
F_4	$\pm x^2 + y^3$	2	2	4	

fat point singularities in the case A_k. This is the smallest dimension in which they occur, but they also exist as simple singularities in any higher dimension by stable equivalence (i.e. using the generalized Morse lemma).

A *boundary singularity* is given by a germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ together with a "boundary" specified by first coordinate $(\mathbb{C}^{n-1}, 0) = (\{x_1 = 0\}, 0) \subset (\mathbb{C}^n, 0)$. Associated to f we now have two Milnor fibers M_f and $M_{f|\{x_1=0\}}$ with a natural inclusion

$$M_{f|\{x_1=0\}} \subset M_f.$$

When both f and $f|\{x_1 = 0\}$ have isolated singularity, then the Milnor fibers are homotopy equivalent to bouquets of spheres of dimension $n-1$ and $n-2$, respectively. Arnold has shown in [4, Theorem 3] that also the factor space $M_f/M_{f|\{x_1=0\}}$ has the homotopy type of a bouquet of $\mu_\partial(f)$ spheres of dimension $n-1$ where $\mu_\partial(f)$ is the *boundary Milnor number*

$$\mu_\partial(f) = \dim_{\mathbb{C}} \mathbb{C}\{x_1, \ldots, x_n\}/\left\langle x_1 \cdot \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right\rangle.$$

All these Milnor numbers satisfy $\mu(f) + \mu(f|\{x_1 = 0\}) = \mu_\partial(f)$ (Table 2.4).

Complete Intersections

Giusti proved in [44] that, apart from hypersurfaces, simple complete intersections can only occur in two settings: fat points in the plane and curves in 3-space. He gave exhaustive lists of the simple singularities in these cases. For completeness of the ICMC2 case below, we also include these tables here (Tables 2.5 and 2.6):

Table 2.5 Simple ICIS fat point singularities

Type	Normal form	μ	τ	
A_k	$\langle y, x^{k+1}\rangle$	k	k	$k \geq 1$
$F^{q,r}_{q+r-1}$	$\langle xy, x^q + y^r\rangle$	$q+r-1$	$q+r$	$q,r \geq 2$
G_5	$\langle x^2, y^3\rangle$	5	7	
G_7	$\langle x^2, y^4\rangle$	7	10	
H_{q+3}	$\langle x^2 + y^q, xy^2\rangle$	$q+3$	$q+5$	$q \geq 3$
I_{2q-1}	$\langle x^2 + y^3, y^q\rangle$	$2q-1$	$2q+1$	$q \geq 4$
I_{2r+2}	$\langle x^2 + y^3, xy^r\rangle$	$2r+2$	$2r+4$	$r \geq 3$

Table 2.6 Simple ICIS space curve singularities

Type	Normal form	μ	τ	
S_{n+3}	$(x^2 + y^2 + z^n, yz)$	$n+3$	$n+3$	$n \geq 2$
T_7	$(x^2 + y^3 + z^3, yz)$	7	7	
T_8	$(x^2 + y^3 + z^4, yz)$	8	8	
T_9	$(x^2 + y^3 + z^5, yz)$	9	9	
U_7	$(x^2 + yz, xy + z^3)$	7	7	
U_8	$(x^2 + yz + z^3, xy)$	8	8	
U_9	$(x^2 + yz, xy + z^4)$	9	9	
W_8	$(x^2 + z^3, y^2 + xz)$	8	8	
W_9	$(x^2 + yz^2, y^2 + xz)$	9	9	
Z_9	$(x^2 + z^3, y^2 + z^3)$	9	9	
Z_{10}	$(x^2 + yz^2, y^2 + z^3)$	10	10	

Simple Square and Symmetric Matrices

We list the tables mentioned in the results of Bruce in [11] and Bruce and Tari [12] on simple determinantal singularities defined by symmetric and arbitrary square matrices. The original tables have been extended by several auxiliary results of related publications such as [48] and [46]. Also, we were informed by V. Goryunov about certain mistakes in the original classifications and we adopt his unified and corrected exposition from [46]. The notations were at times streamlined with those from the tables in the other sections.

All simple square matrices of size $m = 3$ in two variables turn out to be symmetric, whence Table 2.7 simultaneously states both cases. Note that in [11] Bruce considers the *strict Tjurina transform* and refers to it as the *criminant*. The list of simple square matrices of size $m = 3$ in seven variables is also very closely related to a list of simple symmetric matrices: It can be obtained from the simple symmetric matrices in four variables by adding the skew symmetric matrix U in three new variables as in Theorem 2.4.1 (iv). For square matrices the entries in Table 2.8 have to be understood in this sense.

Table 2.7 Simple singularities of symmetric and square matrices of size $m = 3$ in two variables

Normal form	associated hypersurface	strict Tjurina transform	GL-codim. as symm. mat.	GL-codim. as square mat.	pair of Weyl groups
$\begin{pmatrix} y^k & x & 0 \\ x & \pm y^l & 0 \\ 0 & 0 & y \end{pmatrix}$	D_{k+l+2}	$A_0 + A_0 + A_0$ for $k = l$ $A_0 + A_{\lvert l-k\rvert-1}$ for $k \neq l$	$k + l + 2$	$2k + l + 4$	$(D_{k+l+2}; D_{k+1} \oplus D_{l+1})$
$\begin{pmatrix} 0 & x & y \\ x & y & 0 \\ y & 0 & x^2 \end{pmatrix}$	E_6	A_0	6	9	$(E_6; A_5 \oplus A_1)$
$\begin{pmatrix} 0 & x & y \\ x & y & 0 \\ y & 0 & xy \end{pmatrix}$	E_7	A_1	7	10	$(E_7; D_6 \oplus A_1)$
$\begin{pmatrix} 0 & x & y \\ x & y & 0 \\ y & 0 & x^3 \end{pmatrix}$	E_8	A_2	8	11	$(E_8; E_7 \oplus A_1)$
$\begin{pmatrix} x & 0 & 0 \\ 0 & y & x \\ 0 & x & y^2 \end{pmatrix}$	E_7	$A_0 + A_0$	7	11	$(E_7; A_7)$
$\begin{pmatrix} x & 0 & y^2 \\ 0 & y & x \\ y^2 & x & 0 \end{pmatrix}$	E_8	A_0	8	12	$(E_8; D_8)$

Table 2.8 Simple singularities of symmetric matrices of size $m = 3$ in four variables (also providing simple singularities of square matrices of the same size in seven variables)

Name	Normal form	GL-codim. as symm. mat.	GL-codim. of ass. square mat.	Odd function	Ass. symm. ICIS
I_{k+1}, $k \geq 1$	$\begin{pmatrix} x & 0 & z \\ 0 & y+x^k & w \\ z & w & y \end{pmatrix}$	$k+1$	$k+1$	$D_{2k+2}/\mathbb{Z}_2$: $ac^2 + a^{2k+1}$	S_{2k+3}: $c^2 + 2bc + a^{2k}$ ab
II_4	$\begin{pmatrix} x & w^2 & y \\ w^2 & y & z \\ y & z & w \end{pmatrix}$	4	4	$E_8/\mathbb{Z}_2$: $b^3 + c^5$	U_9: $b^2 - ac + c^4$ $ab - c^4$
II_5	$\begin{pmatrix} x & 0 & y+w^2 \\ 0 & y & z \\ y+w^2 & z & w \end{pmatrix}$	5	5	$J_{10}/\mathbb{Z}_2$: $b^3 - bc^4$	U_{11}: $b^2 - ac + c^4$ ab
II_6	$\begin{pmatrix} x & w^3 & y \\ w^3 & y & z \\ y & z & w \end{pmatrix}$	6	6	$E_{12}/\mathbb{Z}_2$: $b^3 + c^7$	U_{13}: $b^2 - ac$ $ab - c^6$

Table 2.9 is dealing with curve singularities and hence the Tjurina transform needs to be considered with special care, as the (strict) Tjurina transform is of the same dimension as the exceptional locus (see Example 2.3.10). The strict Tjurina transforms are listed in the third column of this table. If the relative position of the exceptional locus w.r.t. the strict Tjurina transform is of interest, the singularity type of the reduced structure of the total Tjurina transform is stated in brackets. The naming of all singularities is according to the tables of simple plane curve and space curve singularities, i.e. Tables 2.3, 2.6, and 2.15. The only exception is the notation A_0 for a smooth branch meeting the exceptional locus transversally, which we adopted from the original table of Bruce and Tari (Tables 2.10, 2.11, and 2.12).

For instance the entry "A_0, A_{k-3}" in the third column of the second row indicates that there is an A_{k-3} singularity in the strict Tjurina transform and additionally a smooth branch, both of which meet the exceptional locus, but not in the same point. The entry "$E_6(1)$ (U_7)" in the fifth row indicates that the strict Tjurina transform is a space curve of type $E_6(1)$ from Table 2.15 (with parametrization (t^3, t^4, t^5)) and that it meets the exceptional locus to form an U_7 singularity from Giusti's list in Table 2.6.

Table 2.9 Simple singularities of square matrices in two variables of size $m = 2$, [12, Table 2]

Normal form	Hypersurface	Strict Tj. transf.	GL-codim.
$\begin{pmatrix} x & y^k \\ \pm y^l & x \end{pmatrix}$, $1 \le k \le l$	A_{k+l-1}	A_{l-k-1} for $k \neq l$ $2A_0$ for $k = l$,	$2k + l - 1$
$\begin{pmatrix} x & y \\ x^2 \pm y^k & 0 \end{pmatrix}$, $2 \le k$	D_{k+2}	A_0, A_{k-3}	$k + 3$
$\begin{pmatrix} x & x^2 \pm y^k \\ y & 0 \end{pmatrix}$, $2 \le k$	D_{k+2}	$A_{k-1} \vee L \quad (S_{k+3})$	$k + 3$
$\begin{pmatrix} x & y \\ y^3 & x^2 \end{pmatrix}$	E_6	A_0	7
$\begin{pmatrix} x & y^3 \\ y & x^2 \end{pmatrix}$	E_6	$E_6(1) \quad (U_7)$	7
$\begin{pmatrix} x & y \\ xy^2 & x^2 \end{pmatrix}$	E_7	A_1	8
$\begin{pmatrix} x & xy^2 \\ y & x^2 \end{pmatrix}$	E_7	$E_7(1) \quad (U_8)$	8
$\begin{pmatrix} x & y \\ y^4 & x^2 \end{pmatrix}$	E_8	A_2	9
$\begin{pmatrix} x & y^4 \\ y & x^2 \end{pmatrix}$	E_8	$E_8(1) \quad (U_9)$	9
$\begin{pmatrix} x & 0 \\ 0 & y^2 \pm x^k \end{pmatrix}$, $2 \le k$	D_{k+2}	A_0, A_{k-1}	$k + 4$
$\begin{pmatrix} x & 0 \\ 0 & xy + y^k \end{pmatrix}$, $3 \le k$	D_{2k}	A_0, A_1	$3k$
$\begin{pmatrix} x & y^k \\ \pm y^l & xy \end{pmatrix}$, $3 \le k \le l$	D_{k+l+1}	A_1 for $k = l$, $3A_0$ for $k + 1 = l$ $A_0 + A_{l-k-2}$ for $k + 1 < l$	$2k + l + 1$
$\begin{pmatrix} x & \pm y^l \\ y^k & xy \end{pmatrix}$, $3 \le k < l$	D_{k+l+1}	$A_{l-k} \vee L \quad (D_{l+k+3} \vee L)$	$2k + l + 1$
$\begin{pmatrix} x & y^2 \\ y^2 & x^2 \end{pmatrix}$	E_6	A_2	8
$\begin{pmatrix} x & y^2 \\ 0 & x^2 + y^3 \end{pmatrix}$	E_7	$2\ A_0$	9
$\begin{pmatrix} x & 0 \\ y^2 & x^2 + y^3 \end{pmatrix}$	E_7	$A_2 \vee L \quad (S_6)$	9
$\begin{pmatrix} x & 0 \\ 0 & x^2 + y^3 \end{pmatrix}$	E_7	A_0, A_2	10
$\begin{pmatrix} x & y^2 \\ y^3 & x^2 \end{pmatrix}$	E_8	A_0	10
$\begin{pmatrix} x & y^3 \\ y^2 & x^2 \end{pmatrix}$	E_8	$E_6(1) \quad (W_9)$	10

Table 2.10 Simple singularities of square matrices in three variables of size $m = 2$, [12, Table 3]

Normal form	Hypersurface	Tj. transf.	GL-codimension
$\begin{pmatrix} x & z^k \\ \pm z^l & y \end{pmatrix}$, $1 \leq k \leq l$	A_{k+l-1}	A_{k-1}, A_{l-1}	$k + l - 1$
$\begin{pmatrix} x & -y \\ y + z^k & x \end{pmatrix}$	A_{2k-1}	$2A_{k-1}$	$2k - 1$
$\begin{pmatrix} x & y \\ z^2 \pm y^k & x \end{pmatrix}$, $2 \leq k$	D_{k+2}	D_{k+1}	$k + 2$
$\begin{pmatrix} x & y \\ y^2 & x + z^2 \end{pmatrix}$	E_6	D_5	6
$\begin{pmatrix} x & y \\ y^2 + z^3 & x \end{pmatrix}$	E_7	E_6	7
$\begin{pmatrix} x & y \\ yz & x + z^k \end{pmatrix}$, $2 \leq k$	D_{2k+1}	A_{2k}	$2k + 1$
$\begin{pmatrix} x & y \\ yz + z^k & x \end{pmatrix}$, $3 \leq k$	D_{2k}	A_{2k-1}	$2k$

Table 2.11 Simple singularities of square matrices in two variables of size $m = 3$, [12, Table 4]

Normal form	Hypersurface	Tj. transf.	GL-codimension
$\begin{pmatrix} x & y^k & 0 \\ \pm y^l & x & 0 \\ 0 & 0 & y \end{pmatrix}$, $1 \leq k \leq l$	D_{k+l+2}	$2A_0$ for $l = k$ $A_0 + A_{l-k-1}$ for $l \neq k$	$2k + l + 4$
$\begin{pmatrix} x & y & 0 \\ 0 & x & y \\ y^2 & 0 & x \end{pmatrix}$	E_6	A_0	9
$\begin{pmatrix} x & y & 0 \\ 0 & x & y \\ xy & 0 & x \end{pmatrix}$	E_7	A_1	10
$\begin{pmatrix} x & y & 0 \\ y^2 & x & 0 \\ 0 & 0 & x \end{pmatrix}$	E_7	$2\ A_0$	11
$\begin{pmatrix} x & y & 0 \\ 0 & x & y^2 \\ y^2 & 0 & x \end{pmatrix}$	E_8	A_0	12

Table 2.12 Simple singularities of symmetric matrices of size $m = 2$ in two variables, [11, Theorem 1.1], extended according to [48, Table 1]

Normal form	Associated hypersurface	Tjurina transform	$\mathcal{K}$-type	GL-codim.	Pair of Weyl groups
$\begin{pmatrix} y^k & x \\ x & y^l \end{pmatrix}$, $k \geq 1, l \geq 2$	A_{k+l-1}	$A_0 + A_0$ for $k = l$ $A_{\lvert l-k \rvert -1}$ for $k \neq l$	A_{p-1}	$k+l-1$	$(A_{k+l-1};$ $A_{k-1} \oplus A_{l-1})$
$\begin{pmatrix} x & 0 \\ 0 & y^2+x^k \end{pmatrix}$, $k \geq 2$	D_{k+2}	$A_0 + A_{k-1}$	A_1	$k+2$	$(D_{k-2}; D_{k-3})$
$\begin{pmatrix} x & 0 \\ 0 & xy+y^k \end{pmatrix}$, $k \geq 2$	D_{2k}	$A_0 + A_1$	A_{k-1}	$2k$	$(D_{2k}; A_{2k-1})$
$\begin{pmatrix} x & y^k \\ y^k & xy \end{pmatrix}$, $k \geq 2$	D_{2k+1}	A_1	A_{k-1}	$2k+1$	$(D_{2k+1}; A_{2k})$
$\begin{pmatrix} x & y^2 \\ y^2 & x^2 \end{pmatrix}$	E_6	A_2	A_1	6	$(E_6; D_5)$
$\begin{pmatrix} x & 0 \\ 0 & x^2+y^3 \end{pmatrix}$	E_7	$A_0 + A_2$	A_2	7	$(E_7; E_6)$

2.5.5 Skew-Symmetric Matrices

For Haslinger's result [56], we only reproduce his table of simple singularities (see Table 2.13) given by skew-symmetric 4×4 matrices in 2 variables. All matrices in the table are of the block form

$$A = \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix}$$

for some 2×2-matrix B so that the associated hypersurface singularity is given by $f = \operatorname{Pf} A = \det B$.

Note that the classification of determinantal singularities of skew-symmetric matrices is incomplete and hence the list is only exhaustive for the given case, but does not exclude the existence of simple singularities for other values of m and p.

2.5.6 Cohen-Macaulay Codimension 2 Singularities

In this case the lists reproduced here (see Tables 2.14, 2.15, 2.16, 2.17 and 2.18) are extracted from the articles of Frühbis-Krüger [37] and of Frühbis-Krüger and Neumer [39]. Together with Giusti's lists of simple ICIS, these lists are complete for the simple isolated Cohen-Macaulay codimension 2 singularities:

Table 2.13 Simple singularities, skew-symmetric matrices, case m=4

Name	Matrix B		Hypersurface	$\mathcal{G}_{sk}$-codimension
B_{kl}	$\begin{pmatrix} x & y^k \\ y^l & x \end{pmatrix}$	$1 \leq k \leq l$	A_{k+l-1}	$4k + l - 1$
S_k	$\begin{pmatrix} x & xy \\ y & x^k \end{pmatrix}$	$k \geq 2$	D_{k+2}	$k + 5$
M_9	$\begin{pmatrix} x & y^3 \\ y & x^2 \end{pmatrix}$		E_6	9
M_{10}	$\begin{pmatrix} x & xy^2 \\ y & x^2 \end{pmatrix}$		E_7	10
M_{11}	$\begin{pmatrix} x & y^4 \\ y & x^2 \end{pmatrix}$		E_8	11
F_k	$\begin{pmatrix} x & 0 \\ 0 & y^2 + x^k \end{pmatrix}$	$k \geq 2$	D_{k+2}	$k + 8$
G_k	$\begin{pmatrix} x & 0 \\ 0 & xy + y^k \end{pmatrix}$	$k \geq 3$	D_{2k}	$5k$
H_{kl}	$\begin{pmatrix} x & y^k \\ y^l & xy \end{pmatrix}$	$2 \leq k \leq l$	D_{k+l+1}	$4k + l + 1$
T_{12}	$\begin{pmatrix} x & y^2 \\ y^2 & x^2 \end{pmatrix}$		E_6	12
T_{13}	$\begin{pmatrix} x & y^2 \\ 0 & x^2 + y^3 \end{pmatrix}$		E_8	14
T_{16}	$\begin{pmatrix} x & 0 \\ 0 & x^2 + y^3 \end{pmatrix}$		E_7	16

Table 2.14 Simple non-ICIS fat point singularities in the plane

Ξ_k	$\begin{pmatrix} x & y & 0 \\ 0 & x^k & y \end{pmatrix}$	$k + 3$	$k \geq 1$

The list of simple space curve singularities from [37] reads:

The list of simple ICMC2 surface singularities from [39] is given in Table 2.16 below. The Tjurina number τ is equal to both the GL-Tjurina number of the defining matrix $A \in \mathbb{C}\{x, y, z, w\}^{2\times 3}$, as well as the Tjurina number of the associated germ $(X_A^2, 0) \subset (\mathbb{C}^4, 0)$ according to Corollary 2.3.19. This list recovers the rational triple points from [98] and we give the corresponding name in the last column. The

Table 2.15 Simple non-ICIS space curve singularities

Type	Normal form	μ	τ
$A_{k-3} \vee L$ $k \geq 4$	$\begin{pmatrix} z & y & x^{k-3} \\ 0 & x & y \end{pmatrix}$	$k-2$	$k-1$
$E_6(1)$	$\begin{pmatrix} z & y & x^2 \\ x & z & y \end{pmatrix}$	4	5
$E_7(1)$	$\begin{pmatrix} z+x^2 & y & x \\ 0 & z & y \end{pmatrix}$	5	6
$E_8(1)$	$\begin{pmatrix} z & y & x^3 \\ x & z & y \end{pmatrix}$	6	7
$J_{2,0}(2)$	$\begin{pmatrix} z+x^2 & y & x^2 \\ 0 & z & y \end{pmatrix}$	6	7
$J_{2,1}(2)$	$\begin{pmatrix} z+x^2 & y & x^3 \\ 0 & z & y \end{pmatrix}$	7	8
$E_{12}(2)$	$\begin{pmatrix} z & y & x^3 \\ x^2 & z & y \end{pmatrix}$	8	9
$D_{k+4} \vee L$ $k \geq 0$	$\begin{pmatrix} z & 0 & x^{k+2}-y^2 \\ 0 & x & y \end{pmatrix}$	$k+5$	$k+6$
$E_6 \vee L$	$\begin{pmatrix} z & -y^2 & -x^3 \\ 0 & x & y \end{pmatrix}$	7	8
$E_7 \vee L$	$\begin{pmatrix} z & x^3-y^2 & 0 \\ 0 & x & y \end{pmatrix}$	8	9
$E_8 \vee L$	$\begin{pmatrix} z & -y^2 & -x^4 \\ 0 & x & y \end{pmatrix}$	9	10
S_6^*	$\begin{pmatrix} z & x & y \\ 0 & y & x^2-z^2 \end{pmatrix}$	6	7
T_7^*	$\begin{pmatrix} z & x & y \\ 0 & y & x^2-z^3 \end{pmatrix}$	7	8
U_7^*	$\begin{pmatrix} z & xy & x^2 \\ x & z & y \end{pmatrix}$	7	8
W_8^*	$\begin{pmatrix} z & y^2 & x^2 \\ x & z & y \end{pmatrix}$	8	9

Table 2.16 Simple normal surface singularities in $(\mathbb{C}^4, 0)$

Type	Normal form		τ	Name of triple point in [98]
$\Lambda_{1,1}$	$\begin{pmatrix} w & y & x \\ z & w & y \end{pmatrix}$		2	$A_{0,0,0}$
$\Lambda_{k,1}$	$\begin{pmatrix} w & y & x \\ z & w & y^k \end{pmatrix}$	$k \geq 2$	$k+1$	$A_{0,0,k-1}$
$\Lambda_{k,l}$	$\begin{pmatrix} w^l & y & x \\ z & w & y^k \end{pmatrix}$	$k \geq l \geq 2$	$k+l$	$A_{0,l-1,k-1}$
	$\begin{pmatrix} z & y & x \\ x & w & y^2+z^k \end{pmatrix}$	$k \geq 2$	$k+3$	$C_{k+1,0}$
	$\begin{pmatrix} z & y & x \\ x & w & yz+y^k w \end{pmatrix}$	$k \geq 1$	$2k+4$	$B_{2k+2,0}$
	$\begin{pmatrix} z & y & x \\ x & w & yz+y^k \end{pmatrix}$	$k \geq 3$	$2k+1$	$B_{2k-1,0}$
	$\begin{pmatrix} z & y & x \\ x & w & z^2+yw \end{pmatrix}$		7	D_0
	$\begin{pmatrix} z & y & x \\ x & w & z^2+y^3 \end{pmatrix}$		8	F_0
	$\begin{pmatrix} z & y+w^l & w^m \\ w^k & y & x \end{pmatrix}$	$k, l, m \geq 2$	$k+l+m-1$	$A_{k-1,l-1,m-1}$
	$\begin{pmatrix} z & y & x^l+w^2 \\ w^k & x & y \end{pmatrix}$	$k, l \geq 2$	$k+l+2$	$C_{l+1,k-1}$
	$\begin{pmatrix} z & y+w^l & xw \\ w^k & x & y \end{pmatrix}$	$k, l \geq 2$	$k+2l+1$	$B_{2l,k-1}$
	$\begin{pmatrix} z & y & xw+w^l \\ w^k & x & y \end{pmatrix}$	$k \geq 2, l \geq 3$	$k+2l$	$B_{2l+1,k-1}$
	$\begin{pmatrix} z & y+w^2 & x^2 \\ w^k & x & y \end{pmatrix}$	$k \geq 2$	$k+6$	D_{k-1}
	$\begin{pmatrix} z & y & x^2+w^3 \\ w^k & x & y \end{pmatrix}$	$k \geq 2$	$k+7$	F_{k-1}
	$\begin{pmatrix} z & y & xw+w^k \\ y & x & z \end{pmatrix}$		$3k+1$	H_{3k}
	$\begin{pmatrix} z & y & xw \\ y & x & z+w^k \end{pmatrix}$		$3k+2$	H_{3k+1}
	$\begin{pmatrix} z & y & xw \\ y+w^k & x & z \end{pmatrix}$		$3k+3$	H_{3k+2}

(continued)

Table 2.16 (continued)

Type	Normal form		τ	Name of triple point in [98]
	$\begin{pmatrix} z & y & w^2 \\ y & x & z+x^2 \end{pmatrix}$		8	
	$\begin{pmatrix} z & y & x^2 \\ y & x & z+w^2 \end{pmatrix}$		9	
	$\begin{pmatrix} z & y & x^3+w^2 \\ y & x & z \end{pmatrix}$		9	

last three entries of the list are the nameless sporadic members from [5] in the same order as there and in [98]. The Milnor numbers of the smoothings (i.e. the second Betti number of M_A^2) can be computed as $\mu = \tau - 1$ by virtue of the "if"-part of Wahl's conjecture 2.5.1.

For the simple ICMC2 threefold singularities we extend the list from [39] by the middle Betti number of the smoothing as in [40]. Recall that the second Betti number is always equal to 1, cf. Theorem 2.5.20.

Table 2.17 Simple three-fold singularities in $(\mathbb{C}^5, 0)$

Transpose of the presentation matrix A	Tjurina number τ	Singularities in Tj.-transf.	$b_3(M_A^2)$
$\begin{pmatrix} x & y & z \\ v & w & x \end{pmatrix}$	1	-	0
$\begin{pmatrix} x & y & z \\ v & w & x^{k+1}+y^2 \end{pmatrix}$	$k+2$	A_k	k
$\begin{pmatrix} x & y & z \\ v & w & xy^2+x^{k-1} \end{pmatrix}$	$k+2$	D_k	k
$\begin{pmatrix} x & y & z \\ v & w & x^3+y^4 \end{pmatrix}$	8	E_6	6
$\begin{pmatrix} x & y & z \\ v & w & x^3+xy^3 \end{pmatrix}$	9	E_7	7
$\begin{pmatrix} x & y & z \\ v & w & x^3+y^5 \end{pmatrix}$	10	E_8	8
$\begin{pmatrix} w & y & x \\ z & w & y+v^k \end{pmatrix}$	$2k-1$	-	0
$\begin{pmatrix} w & y & x \\ z & w & y^k+v^2 \end{pmatrix}$	$k+2$	A_{k-1}	$k-1$
$\begin{pmatrix} w & y & x \\ z & w & yv+v^k \end{pmatrix}$	$2k$	A_1	1
$\begin{pmatrix} w+v^k & y & x \\ z & w & yv \end{pmatrix}$	$2k+1$	A_1	1
$\begin{pmatrix} w+v^2 & y & x \\ z & w & y^2+v^k \end{pmatrix}$	$k+3$	A_{k-1}	$k-1$
$\begin{pmatrix} w & y & x \\ z & w & y^2+v^3 \end{pmatrix}$	7	A_2	2
$\begin{pmatrix} v^2+w^k & y & x \\ z & w & v^2+y^l \end{pmatrix}$	$k+l+1$	A_{k-1}, A_{l-1}	$k+l-2$
$\begin{pmatrix} v^2+w^k & y & x \\ z & w & yv \end{pmatrix}$	$k+4$	A_{k-1}, A_1	k
$\begin{pmatrix} v^2+w^k & y & x \\ z & w & y^2+v^l \end{pmatrix}$	$k+l+2$	A_{k-1}, A_{l-1}	$k+l-2$
$\begin{pmatrix} wv+v^k & y & x \\ z & w & yv+v^k \end{pmatrix}$	$2k+1$	A_1, A_1	2
$\begin{pmatrix} wv+v^k & y & x \\ z & w & yv \end{pmatrix}$	$2k+2$	A_1, A_1	2

(continued)

Table 2.17 (continued)

Transpose of the presentation matrix A	Tjurina number τ	Singularities in Tj.-transf.	$b_3(M_A^2)$
$\begin{pmatrix} wv+v^3 & y & x \\ z & w & y^2+v^3 \end{pmatrix}$	8	A_1, A_2	3
$\begin{pmatrix} wv & y & x \\ z & w & y^2+v^3 \end{pmatrix}$	9	A_1, A_2	3
$\begin{pmatrix} w^2+v^3 & y & x \\ z & w & y^2+v^3 \end{pmatrix}$	9	A_2, A_2	4
$\begin{pmatrix} z & y & x \\ x & w & v^2+y^2+z^k \end{pmatrix}$	$k+4$	D_{k+1}	$k+1$
$\begin{pmatrix} z & y & x \\ x & w & v^2+yz+y^k w \end{pmatrix}$	$2k+5$	A_{2k+2}	$2k+2$
$\begin{pmatrix} z & y & x \\ x & w & v^2+yz+y^{k+1} \end{pmatrix}$	$2k+4$	A_{2k+1}	$2k+1$
$\begin{pmatrix} z & y & x \\ x & w & v^2+yw+z^2 \end{pmatrix}$	8	D_5	5
$\begin{pmatrix} z & y & x \\ x & w & v^2+y^3+z^2 \end{pmatrix}$	9	E_6	6
$\begin{pmatrix} z & y & x+v^2 \\ x & w & vy+z^2 \end{pmatrix}$	7	D_3	3
$\begin{pmatrix} z & y & x+v^2 \\ x & w & vz+y^2 \end{pmatrix}$	8	A_4	4
$\begin{pmatrix} z & y & x+v^2 \\ x & w & z^2+y^2 \end{pmatrix}$	9	D_5	5

Table 2.18 Simple four-fold singularities in $(\mathbb{C}^6, 0)$

Type	Normal form A		$\tau_{GL}(A)$	$\tau(X_A^1, 0)$
Ω_1	$\begin{pmatrix} x & y & v \\ z & w & u \end{pmatrix}$		0	0
Ω_k	$\begin{pmatrix} x & y & v \\ z & w & x+u^k \end{pmatrix}$	$k \geq 2$	$k-1$	$k-1$
$A_k^\sharp$	$\begin{pmatrix} x & y & z \\ w & v & u^2+x^{k+1}+y^2 \end{pmatrix}$	$k \geq 1$	$k+2$	1
$D_k^\sharp$	$\begin{pmatrix} x & y & z \\ w & v & u^2+xy^2+x^{k-1} \end{pmatrix}$	$k \geq 4$	$k+2$	1
$E_6^\sharp$	$\begin{pmatrix} x & y & z \\ w & v & u^2+x^3+y^4 \end{pmatrix}$		8	1
$E_7^\sharp$	$\begin{pmatrix} x & y & z \\ w & v & u^2+x^3+xy^3 \end{pmatrix}$		9	1
$E_8^\sharp$	$\begin{pmatrix} x & y & z \\ w & v & u^2+x^3+y^5 \end{pmatrix}$		10	1
	$\begin{pmatrix} x & y & z \\ w & v & ux+y^k+u^l \end{pmatrix}$	$k \geq 2, l \geq 3$	$k+l-1$	$l-1$
	$\begin{pmatrix} x & y & z \\ w & v & x^2+y^2+u^3 \end{pmatrix}$		6	2
$F_{q,r}^\sharp$	$\begin{pmatrix} w & y & x \\ z & w+vu & y+v^q+u^r \end{pmatrix}$	$q, r \geq 2$	$q+r$	$q+r$
$G_5^\sharp$	$\begin{pmatrix} w & y & x \\ z & w+v^2 & y+u^3 \end{pmatrix}$		7	7
$G_7^\sharp$	$\begin{pmatrix} w & y & x \\ z & w+v^2 & y+u^4 \end{pmatrix}$		10	10
$H_{q+3}^\sharp$	$\begin{pmatrix} w & y & x \\ z & w+v^2+u^q & y+vu^2 \end{pmatrix}$	$q \geq 3$	$q+5$	$q+5$
$I_{2q-1}^\sharp$	$\begin{pmatrix} w & y & x \\ z & w+v^2+u^3 & y+u^q \end{pmatrix}$	$q \geq 4$	$2q+1$	$2q+1$

Table 2.18 (continued)

Type	Normal form A		$\tau_{\mathrm{GL}}(A)$	$\tau(X_A^1,0)$
$I^\sharp_{2r+2}$	$\begin{pmatrix} w & y & x \\ z & w+v^2+u^3 & y+vu^r \end{pmatrix}$	$r\geq 3$	$2r+4$	$2r+4$
	$\begin{pmatrix} w & y & x \\ z & w+v^{k_1}+u^{k_2} & y^l+uv \end{pmatrix}$	$k_1,k_2,l\geq 2$	k_1+k_2+l-1	k_1+k_2
	$\begin{pmatrix} w & y & x \\ z & w+v^2 & u^2+yv \end{pmatrix}$		6	4
	$\begin{pmatrix} w & y & x \\ z & w+uv & u^2+yv+v^k \end{pmatrix}$	$k\geq 3$	$k+4$	$k+2$
	$\begin{pmatrix} w & y & x \\ z & w+v^k & u^2+yv+v^3 \end{pmatrix}$	$k\geq 3$	$2k+2$	$2k+1$
	$\begin{pmatrix} w & y & x \\ z & w+uv^k & u^2+yv+v^3 \end{pmatrix}$	$k\geq 2$	$2k+5$	$2k+4$
	$\begin{pmatrix} w & y & x \\ z & w+v^3 & u^2+yv \end{pmatrix}$		9	7
	$\begin{pmatrix} w & y & x \\ z & w+v^k & u^2+y^2+v^3 \end{pmatrix}$	$k\geq 3$	$2k+3$	$2k+1$
	$\begin{pmatrix} w & y & x \\ z & w+uv^k & u^2+y^2+v^3 \end{pmatrix}$	$k\geq 2$	$2k+6$	$2k+4$

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Chapter 3
Singularities, the Space of Arcs and Applications to Birational Geometry

Shihoko Ishii

Contents

Abstract This paper is an introduction to the space of arcs and the space of jets of an algebraic variety. We also introduce the Nash problem on arc families, which makes a bridge between the theory of the space of arcs and the theory of birational geometry. We then focus on applications of the space of arcs to the theory of birational geometry and show the recent results.

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3.1 Introduction

3.1.1 Overview

Roughly speaking, an arc is a very small portion of a curve on a scheme and an m-jet is the approximation up to degree m of an arc. The space of arcs is the set of all arcs on a scheme and the space of m-jets is the set of all m-jets on a scheme. These spaces have the natural scheme structures and reflect the properties of the base scheme. The space of arcs plays the following roles:

1. a role to describe singularities of a variety (local problem);
2. a role to describe the global structure of a variety (global problem) and
3. the role as a differential algebra (algebraic problem).

These roles are based on understandings of the structure of the arc space, which is simultaneously developing with the study on 1–3. We should mention that these roles mutually interact and the works corresponding to them are not exclusively classified into one of the roles 1–3. We should also mention that the theory of the space of arcs/jets is still developing, so in the future, more roles will potentially appear.

The following is a brief history of the development of the space of arcs/jets whilst mentioning the roles 1–3 in each step.

3.1.2 Brief History

The space of arcs and the space of m-jets appeared for the first time in the short preprint in 1968 by John Forbes Nash. But according to an expert, Monique Lejeune-Jalabert, of Nash problem, the concepts, arcs and jets were already studied by Isaac Newton in seventeenth century. In his book " La Méthode des Fluxions et des Suites Infinies" Newton shows the method to express the x, y-coordinates of a plane curve by one parameter series which is the origin of an arc. Actually it is a natural question how to describe a curve by one parameter and it is not so mysterious to find this question in old literature. But we should have waited till the twentieth century for the concept "moduli space" consisting of all such parametrization.

In 1968, John Forbes Nash wrote a short preprint "Arc structures of singularities" in which he introduced the space of arcs. The preprint was not published at the beginning, but circulated in the world and was read by many people.

In 1995, the paper was eventually published as [81] in the issue of celebration of Nobel laureate Nash in Duke Mathematical Journal. Twenty seven years have passed since the paper was written. In those years Nash had suffered from mental disease, but later recovered miraculously, about which the reader can see in the book "A Beautiful Mind" [80].

Coming back to mathematics, the paper by Nash also posed a problem, so called "the Nash Problem". After the preprint was circulated around the world in 1968, the space of arcs in relation with the Nash problem is studied by many people, Bouvier, Gonzalez-Sprinberg, Hickel, Lejeune-Jalabert, Nobile, Reguera-Lopez and others (see, [12, 39, 44, 68–70, 82, 89]). The first direct answer to the Nash Problem is obtained by Ana Reguera ([89]) in 1995. This is an affirmative answer to the problem in the case of simple singularities on a surface. Then, by the contributions of many people, the problem was completely solved in 2013. It took 45 years after the problem was posed. [1, 10–13, 19, 20, 38, 39, 46, 54, 63, 70, 71, 74, 84–90, 92] The reader can see a more detailed history about the Nash Problem in Section 4. As a matter of fact, the problem is affirmatively solved for two-dimensional singularities by J.F.de Bobadilla and M.P.Pereira [11] and toric singularities of arbitrary dimension as well by S. Ishii and J. Kollár [46]. But otherwise, it was negatively solved by S. Ishii and Kollár for dimension greater than 3 and by T. De Fernex [19], J. Johnson and J. Kollár [63] for dimension 3. In spite of the fact that the answer is negative for many cases, the Nash Problem still holds great significance. The problem bridges the theory of arc space and the theory of birational geometry. So the Nash Problem plays an important role on (1) in a viewpoint of birational geometry.

A surprising step in this direction is made by M. Mustaţă in [77] at the beginning of the twenty-first century. He characterizes a locally complete intersection canonical singularity (a kind of singularity in birational geometry) by irreducibility of all the spaces of jets. After that there appear similar characterizations of singularities in birational geometry in terms of the space of arcs/jets (see Corollary 3.5.38). Some birational invariants ("mld" and "lct") of singularities are also interpreted in terms of the arc space. By making use of the interpretations some important results in birational geometry are obtained. One important point about these invariant is that the interpretations by the space of arcs/jets also work for the base field of positive characteristic. Comparing with the case of characteristic 0, algebraic geometry of positive characteristic is difficult to study, because some convenient properties do not hold in positive characteristic [60]. In such a situation, the interpretations by the space of arcs are expected to play significant roles. These things will be explained in Section 5.

Aside from the Nash Problem, a remarkable idea "motivic integration" on the space of arcs is introduced by Kontsevich [67] in 1995. He proved that birationally equivalent Calabi-Yau manifolds have the same Hodge numbers by making use of motivic integration. This is considered as the role (2) of the space of arcs. Unfortunately there is no written version of [67], however we can read the papers by J. Denef and F. Loeser [23–27] which describe their own developments of the theory of motivic integration including Kontsevich's original idea. Motivic integration leads the people to "motivic zeta function" on the arc space [23, 24] and also Batyrev's "stringy function" [7, 8]. These functions describe global and local structures of the variety, therefore these are considered as the roles (2) and also (1) of the space of arcs. Local theory of singularities in terms of zeta function is developed by Veys [93–95] and Veys and Zuniga-Galindo[96]. In this paper we do not step into motivic integration, since there are many good expository papers by A. Craw [17], W. Veys [97], F. Loeser [72].

The space of arcs/jets on an affine variety becomes an affine scheme and the coordinate ring of the space of arcs/jets has a canonical structure of differential algebra. From this viewpoint, the space of arcs/jets is studied by Arakawa and Moreau [4], Buium [15], and Kolchin [64]. These are the role (3) of the space of arcs.

Because of the limitation of the pages, the proofs are given only when the proof helps the understanding of new concepts. For statements for which we omit the proofs, we show the citations so that the reader can find the proofs.

3.1.3 The Goal of this Chapter

In this expository paper, we introduce the space of arcs/jets and show basic properties of the space of arcs/jets with a focus on (1) in a viewpoint of birational geometry.

For the reader not so familiar to birational geometry, we introduce basic notions in birational geometry in the fifth section.

The reader interested in the other roles is encouraged to see the references cited above.

Throughout this paper k is an algebraically closed field of arbitrary characteristic unless otherwise stated and a variety is an irreducile reduced separated scheme of finite type over k. The basic knowledge of algebraic geometry is based on [43] by Hartshorne.

3.2 Construction of the Space of Jets and the Space of Arcs

3.2.1 Construction of the Space of Jets

Definition 3.2.1 Let X be a scheme of finite type over k and $K \supset k$ a field extension. For $m \in \mathbf{N}$, a k-morphism $\operatorname{Spec} K[t]/(t^{m+1}) \to X$ is called an m-**jet** of X and a k-morphism $\operatorname{Spec} K[[t]] \to X$ is called an **arc** of X. We denote the unique point of $\operatorname{Spec} K[t]/(t^{m+1})$ by 0, while the closed point of $\operatorname{Spec} K[[t]]$ by 0 and the generic point by η.

Theorem 3.2.2 *Let X be a scheme of finite type over k. Let $\mathcal{S}ch/k$ be the category of k-schemes and $\mathcal{S}et$ the category of sets. Define a contravariant functor $F_m^X : \mathcal{S}ch/k \to \mathcal{S}et$ by*

$$F_m^X(Z) = \operatorname{Hom}_k(Z \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}), X)$$

for an object Z of $\mathcal{S}ch/k$. And for a morphism $f : Z \to Z'$ in $\mathcal{S}ch/k$, define $F_m^X(f)$:

$$\mathrm{Hom}_k(Z' \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[t]/(t^{m+1}), X) \to \mathrm{Hom}_k(Z \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[t]/(t^{m+1}), X)$$

by $\alpha' \mapsto \alpha' \circ (f \times 1)$.

Then, F_m^X is representable by a scheme X_m of finite type over k. This X_m is called the ***space of** m**-jets*** *of X or the m-**jet scheme** of X.*

Here, "F_m^X is representable by X_m" means that the functor F_m^X is naturally isomorphic (*i.e.,* there exists an invertible natural transformation) to the functor $\mathrm{Hom}_k(Z, X_m)$. In particular, for an object $Z \in \mathcal{S}ch/k$ the following bijection holds:

$$\mathrm{Hom}_k(Z, X_m) \simeq \mathrm{Hom}_k(Z \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[t]/(t^{m+1}), X). \tag{3.1}$$

The above theorem is proved in [14, p. 276]. In this paper, we prove this by a concrete construction of X_m for affine X and then patching them together for a general X. For our proof, we need some preparatory discussions.

Note 3.2.3 Let X be a k-scheme. Assume that F_m^X is representable by X_m for every $m \in \mathbf{N}$. Then, for $m < m'$, the canonical surjection $k[t]/(t^{m'+1}) \to k[t]/(t^{m+1})$ induces a morphism

$$\psi_{m',m} : X_{m'} \to X_m.$$

Indeed, the canonical surjection $k[t]/(t^{m'+1}) \to k[t]/(t^{m+1})$ induces a morphism

$$Z \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[t]/(t^{m'+1}) \leftarrow Z \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[t]/(t^{m+1}),$$

for an arbitrary k-scheme Z. Therefore we have a map

$$\mathrm{Hom}_k(Z \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[t]/(t^{m'+1}), X) \to \mathrm{Hom}_k(Z \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[t]/(t^{m+1}), X)$$

which gives the map by the bijection (3.1)

$$\mathrm{Hom}_k(Z, X_{m'}) \to \mathrm{Hom}_k(Z, X_m).$$

Take, in particular, $X_{m'}$ as Z,

$$\mathrm{Hom}_k(X_{m'}, X_{m'}) \to \mathrm{Hom}_k(X_{m'}, X_m)$$

then the image of $id_{X_{m'}} \in \mathrm{Hom}(X_{m'}, X_{m'})$ by this map gives the required morphism.

This morphism $\psi_{m',m}$ is called a **truncation morphism**. In particular for $m = 0$, $\psi_{m',0} : X_{m'} \to X$ is denoted by π_m. When we need to specify the scheme X, we denote it by π_m^X.

Actually $\psi_{m',m}$ "truncates" a power series in the following sense: A point α of $X_{m'}$ gives an m'-jet $\alpha : \operatorname{Spec} K[t]/(t^{m'+1}) \to X$, which corresponds to a ring homomorphism $\alpha^* : A \to K[t]/(t^{m'+1})$, where A is the affine coordinate ring of an affine neighborhood of the image of α. For every $f \in A$, let

$$\alpha^*(f) = a_0 + a_1 t + a_2 t^2 + \cdots + a_m t^m + \cdots + a_{m'} t^{m'},$$

then

$$(\psi_{m',m}(\alpha))^*(f) = a_0 + a_1 t + a_2 t^2 + \cdots + a_m t^m.$$

This fact can be seen by letting $Z = \{\alpha\}$ in the above discussion.

As we did already in the above argument, we denote the point of X_m corresponding to $\alpha : \operatorname{Spec} K[t]/(t^{m+1}) \to X$ by the same symbol α. Then, we should note that $\pi_m(\alpha) = \alpha(0)$, where in the left hand side we regard α as a point of X_m, while in the right hand side we regard it as a morphism $\operatorname{Spec} K[t]/(t^{m+1}) \to X$.

3.2.2 Morphisms of the Spaces of Jets

Proposition 3.2.4 *Let $f : X \to Y$ be a morphism of k-schemes of finite type. Assume that the functors F_m^X and F_m^Y are representable by X_m and Y_m, respectively. Then for every $m \in \mathbf{N}$ there is a canonical morphism $f_m : X_m \to Y_m$ such that the following diagram is commutative:*

$$\begin{array}{ccc} X_m & \xrightarrow{f_m} & Y_m \\ \pi_m^X \downarrow & & \downarrow \pi_m^Y \\ X & \xrightarrow{f} & Y \end{array}.$$

Proof Let $X_m \times \operatorname{Spec} k[t]/(t^{m+1}) \to X$ be the "universal family" of m-jets of X, *i.e.,* it corresponds to the identity map in $\operatorname{Hom}_k(X_m, X_m)$. By compositing this map and $f : X \to Y$, we obtain a morphism

$$X_m \times \operatorname{Spec} k[t]/(t^{m+1}) \to Y,$$

which gives a morphism $X_m \to Y_m$. Pointwise, this morphism maps an m-jet $\alpha \in X_m$ of X to the composite $f \circ \alpha$ which is an m-jet of Y. To see this, just take a point $\alpha \in X_m$ and see the image of $\{\alpha\} \times \operatorname{Spec} k[t]/(t^{m+1}) \to Y$. The commutativity of the diagram follows from this description. □

Proposition 3.2.5 *For k-schemes X and Y, assume that the functor F_m^X and F_m^Y are representable by X_m and Y_m, respectively. If $f : X \to Y$ is an étale morphism, then $X_m \simeq Y_m \times_Y X$, for every $m \in \mathbf{N}$.*

Proof By the above proposition we have a commutative diagram:

$$\begin{array}{ccc} X_m & \xrightarrow{f_m} & Y_m \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}.$$

It is sufficient to prove that for every commutative diagram:

$$\begin{array}{ccc} Z & \longrightarrow & Y_m \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array},$$

there is a unique morphism $Z \to X_m$ which is compatible with the projections to X and Y_m. By definition of Y_m, we are given the following commutative diagram:

$$\begin{array}{ccc} Z & \longrightarrow & Z \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}) \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y. \end{array}$$

As f is étale, there is a unique morphism $Z \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}) \to X$ which makes the two triangles commutative. This gives the required morphism:

$$Z \to X_m.$$

□

As a corollary of this proposition, we obtain the following lemma:

Lemma 3.2.6 *Let $U \subset X$ be an open subset of a k-scheme X. Assume the functors F_m^X and F_m^U are representable by X_m and U_m, respectively. Then, $U_m = (\pi_m^X)^{-1}(U)$.*

Proof of Theorem 3.2.2 Since a k-scheme X is separated, the intersection of two affine open subsets is again affine. Therefore, for an affine covering $\{U_i\}_i$ of a k-scheme X, if the functor $F_m^{U_i}$ is representable by $(U_i)_m$ for every i, then we can patch $(U_i)_m$'s together to obtain X_m by Lemma 3.2.6. Now, it is sufficient to prove the representability of F_m^X for affine X. Let X be Spec R, where we denote $R = k[x_1, \ldots, x_n]/(f_1, .., f_r)$. It is sufficient to prove the representability for an affine variety $Z = \operatorname{Spec} A$. Then, we obtain that

$$(3.2.2.1) \qquad \operatorname{Hom}(Z \times \operatorname{Spec} k[t]/(t^{m+1}), X) \simeq \operatorname{Hom}(R, A[t]/(t^{m+1}))$$

$$\simeq \left\{\varphi \in \operatorname{Hom}\left(k[x_1, .., x_n], A[t]/(t^{m+1})\right) \middle| \, \varphi(f_i) = 0 \text{ for } i = 1, .., r\right\}.$$

If we write $\varphi(x_j) = a_j^{(0)} + a_j^{(1)}t + a_j^{(2)}t^2 + .. + a_j^{(m)}t^m$ for $a_j^{(l)} \in A$, it follows that

$$\varphi(f_i) = F_i^{(0)}(a_j^{(l)}) + F_i^{(1)}(a_j^{(l)})t + .. + F_i^{(m)}(a_j^{(l)})t^m$$

for polynomials $F_i^{(s)}$ in $a_j^{(l)}$'s $(1 \le j \le n, 0 \le l \le s)$. Then the above set (3.2.2.1) is described as follows:

$$= \left\{\varphi \in \mathrm{Hom}\Big(k\Big[x_j, x_j^{(1)}, ., x_j^{(m)} \mid j = 1, ., n\Big], A\Big) \Big| \varphi(x_j^{(l)}) = a_j^{(l)},\ F_i^{(s)}(a_j^{(l)}) = 0\right\}$$

$$= \mathrm{Hom}\left(k\left[x_j, x_j^{(1)}, ., x_j^{(m)}\right] / (F_i^{(s)}(x_j^{(l)})), A\right).$$

If we define $X_m = \mathrm{Spec}\, k[x_j, x_j^{(1)}, ., x_j^{(m)}]/(F_i^{(s)}(x_j^{(l)}))$, the last set is bijective to

$$\mathrm{Hom}(Z, X_m).$$

This completes the proof of Theorem 3.2.2. □

Remark 3.2.7 The functor F_m^X is also representable even for k-scheme of non-finite type over k. The existence of the space of jets for wider class of schemes is presented in [98].

3.2.3 The Space of Arcs

Definition 3.2.8 The system $\{\psi_{m',m} : X_{m'} \to X_m\}_{m<m'}$ is a projective system. Let $X_\infty = \varprojlim_m X_m$ and call it the **space of arcs** of X or **arc space** of X. Note that X_∞ is not of finite type over k if $\dim X > 0$.

Remark 3.2.9 The reader may be afraid that the projective limit of the schemes $\varprojlim_m X_m$ may not exist. But in our case we need not to worry, since for an affine scheme $X = \mathrm{Spec}\, R$, the m-jet scheme $X_m = \mathrm{Spec}\, R_m$ is affine for every $m \in \mathbf{N}$. Here, the morphisms $\psi^*_{m',m} : R_m \to R_{m'}$ corresponding to $\psi_{m',m}$ are direct system. It is well known that there is a direct limit $R_\infty = \varinjlim_m R_m$ in the category of k-algebras. The affine scheme $\mathrm{Spec}\, R_\infty$ is our projective limit of X_m. For a general k-scheme X, we have only to patch affine pieces $\mathrm{Spec}\, R_\infty$.

Using the representability of F_m^X we obtain the following universal property of X_∞:

Proposition 3.2.10 *Let X be a scheme of finite type over k. Then for a k-algebra A we obtain:*

$$\mathrm{Hom}_k(\mathrm{Spec}\, A, X_\infty) \simeq \mathrm{Hom}_k(\mathrm{Spec}\, A[[t]], X).$$

Proof In case X is affine k-scheme $X = \operatorname{Spec} R$. Then by the representability of F_m^X we obtain an isomorphism of projective systems:

$$\begin{array}{ccccc} \downarrow & & \downarrow & & \downarrow \\ \operatorname{Hom}_k(\operatorname{Spec} A, X_m) & \simeq & \operatorname{Hom}_k(\operatorname{Spec} A[t]/(t^{m+1}), X) & \simeq & \operatorname{Hom}_k(R, A[t]/(t^{m+1})) \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Hom}_k(\operatorname{Spec} A, X_{m-1}) & \simeq & \operatorname{Hom}_k(\operatorname{Spec} A[t]/(t^m), X) & \simeq & \operatorname{Hom}_k(R, A[t]/(t^m)) \end{array}.$$

Then, we obtain an isomorphism of the projective limits:

$$\operatorname{Hom}_k(\operatorname{Spec} A, \varprojlim_m X_m) \simeq \operatorname{Hom}_k(R, A[[t]]),$$

which gives the required isomorphism for affine scheme X.

For a general X, see [9]. When we study singularities locally, we need only the affine case. □

Remark 3.2.11 Note that in general

$$A \otimes_k k[[t]] \not\simeq A[[t]] = \varinjlim_m A[t]/(t^{m+1}).$$

Indeed, for example, for $A = k[x]$, the ring $A[[t]]$ contains $\sum_{i=0}^{\infty} f_i(x)t^i$ such that $\deg f_i$ are unbounded, while $A \otimes_k k[[t]]$ does not contain such an element.

Now, consider the case $A = K$ for an extension field $K \supset k$, the bijection

$$\operatorname{Hom}_k(\operatorname{Spec} K, X_\infty) \simeq \operatorname{Hom}_k(\operatorname{Spec} K[[t]], X)$$

shows that a K-valued point of X_∞ is an arc $\operatorname{Spec} K[[t]] \to X$.

In [51, Proposition 2.13] the author sloppily stated Proposition 3.2.10 for every k-scheme Z instead of $\operatorname{Spec} A$. But actually the correct statement proved at this moment is in the form as Proposition 3.2.10.

Definition 3.2.12 Denote the canonical projection $X_\infty \to X_m$ induced from the surjection $k[[t]] \to k[t]/(t^{m+1})$ by ψ_m and the composite $\pi_m \circ \psi_m$ by π. When we need to specify the base space X, we write it by π^X.

A point $x \in X_\infty$ gives an arc $\alpha_x : \operatorname{Spec} K[[t]] \to X$ and $\pi(x) = \alpha_x(0)$, where K is the residue field at x. In the same way as in the case of m-jets, we denote both $x \in X_\infty$ and α_x by the same symbol α.

For every $m \in \mathbf{N}$, $\psi_m(X_\infty)$ is a constructible set, since $\psi_m(X_\infty) = \psi_{m',m}(X_{m'})$ for sufficiently big m' ([41]). We know that the image of a morphism of finite type is a constructible set.

Definition 3.2.13 Denote the canonical morphism $X \to X_m$ induced from the inclusion $k \hookrightarrow k[t]/(t^{m+1})$ $(m \in \mathbf{N} \cup \{\infty\})$ by σ_m. Here, we define $k[t]/(t^{m+1}) =$

$k[[t]]$ for $m = \infty$. As $k \hookrightarrow k[t]/(t^{m+1})$ is a section of the projection $k[t]/(t^{m+1}) \to k$, our morphism $\sigma_m : X \to X_m$ is a section of $\pi_m : X_m \to X$.

Let $x \in X$ be a point and $m \in \mathbf{N} \cup \{\infty\}$. Then the fiber scheme $\pi_m^{-1}(x)$ is denoted by $X_m(x)$.

For a point $x \in X$, let K be the residue field at x, then define

$$\sigma_m(x) : \operatorname{Spec} K[t]/(t^{m+1}) \to X$$

as the m-jet that factors through $\operatorname{Spec} K \to X$ whose image is x. Therefore, $\sigma_m(x)$ is the **constant** m**-jet** at x, this is denoted my x_m.

Example 3.2.14 Under the notation in the proof of Theorem 3.2.2, for $X = \mathbf{A}_k^n$, we have $(f_1, \ldots, f_r) = 0$. Therefore, it follows $X_m = \mathbf{A}_k^{n(m+1)}$ and the truncation morphism $\psi_{m',m} : X_{m'} \to X_m$ is the projection $\mathbf{A}_k^{n(m'+1)} = \mathbf{A}_k^{n(m+1)} \times \mathbf{A}_k^{n(m'-m)} \to \mathbf{A}_k^{n(m+1)}$.

Example 3.2.15 Let X be a non-singular variety of dimension n. Then for every $m \in \mathbf{N}$, the space of m-jets X_m is a non-singular variety of dimension $n(m+1)$ and the truncation morphism $\psi_{m',m} : X_{m'} \to X_m$ is a locally trivial fiber space with the fiber $\mathbf{A}_k^{(m'-m)n}$. Indeed, if X is non-singular, then at each point $x \in X$ there is an open neighborhood U_x such that we have an étale morphism $U_x \to \mathbf{A}_k^n$. By Proposition 3.2.5, it follows that $(U_x)_m \simeq U_x \times_{\mathbf{A}_k^n} \mathbf{A}_k^{n(m+1)} \simeq U_x \times_{\operatorname{Spec} k} \mathbf{A}_k^{mn}$. This shows that $\pi_m : X_m \to X$ is a locally trivial fiber space with the fiber $\mathbf{A}_k^{mn}$. For $m < m'$, we have$(U_x)_{m'} = (U_x)_m \times_{\operatorname{Spec} k} \mathbf{A}_k^{(m'-m)n}$ by the discussion above. Hence, $\psi_{m',m} : X_{m'} \to X_m$ is a locally trivial fiber space with the fiber $\mathbf{A}_k^{(m'-m)n}$.

Example 3.2.16 Let X be the hypersurface in $\mathbf{A}_k^3$ defined by the equation $f = xy + z^2 = 0$. We leave the calculation of X_1 to the reader and here we calculate X_2. The space of 2-jets X_2 is defined in $\mathbf{A}_k^9$ by the equations $xy + z^2 = x^{(1)}y + xy^{(1)} + 2zz^{(1)} = x^{(2)}y + x^{(1)}y^{(1)} + xy^{(2)} + z^{(1)}z^{(1)} + 2zz^{(2)} = 0$. We can prove that X_2 is irreducible and non-normal as follows: As an open subset $X \setminus \{0\}$ is non-singular, $\pi_2^{-1}(X \setminus \{0\})$ is 6-dimensional non-singular variety. On the other hand $\pi_2^{-1}(0)$ is a hypersurface in $\mathbf{A}_k^6$ defined by the equation $x^{(1)}y^{(1)} + z^{(1)}z^{(1)} = 0$, therefore its dimension is 5. As X_2 is defined by three equations, every irreducible component of X_2 has dimension greater than or equal to $9 - 3 = 6$. By this $\pi_2^{-1}(0)$ does not produce an irreducible component of X_2. Hence, X_2 is irreducible. On the other hand, by the Jacobian matrix, we can see that the singular locus of X_2 is $\pi_2^{-1}(0)$. This locus is of codimension 1 in X_2, which yields that X_2 is not normal. The origin is the unique singular point of X and is called an "A_1-singularity". Later on, in Corollary 3.5.38, we will have that X_m $(m \in \mathbf{N})$ are all irreducible.

Example 3.2.17 Let X be the plane curve defined by $x^2 - y^2 - x^3 = 0$ Then $\pi_1^{-1}(X \setminus \{0\}) \to X \setminus \{0\}$ is a locally trivial fiber space over $X \setminus \{0\}$ with the fiber $\mathbf{A}_k^1$, which shows that $\pi_1^{-1}(X \setminus \{0\})$ is of dimension 2. On the other hand, we have

$\pi_1^{-1}(0) \simeq \mathbf{A}_k^2$. Therefore X_1 consists of two irreducible components $\overline{\pi_1^{-1}(X \setminus \{0\})}$ and $\pi_1^{-1}(0)$.

Example 3.2.18 Consider the space of 1-jets for an arbitrary scheme X of finite type over k. For every closed point $x \in X$, the set of closed points of $\pi_1^{-1}(x)$ is the set of morphisms $\operatorname{Spec} k[t]/(t^2) \to X$ with the image x. This set is nothing but the Zariski tangent space of X at x. Therefore, $\pi_1 : X_1 \to X$ is regarded as the "tangent bunlde" of X.

Example 3.2.19 If $X = \mathbf{A}_k^n$, then $X_\infty = \operatorname{Spec} k[x_j, x_j^{(1)}, x_j^{(2)} \dots \mid j = 1, \dots, n]$ which is isomorphic to $\mathbf{A}_k^\infty = \operatorname{Spec} k[x_1, x_2, \dots, x_i, \dots]$. Here, we note that the set of closed points of $\mathbf{A}_k^\infty$ does not necessarily coincide with the set

$$k^\infty := \{(a_1, a_2, \dots) \mid a_i \in k\}$$

(see the following theorem).

Theorem 3.2.20 ([48], Proposition 2.10, 2.11) *Every closed point of $\mathbf{A}_k^\infty$ is a k-valued point if and only if k is an uncountable field.*

3.2.4 Thin and Fat Arcs

The concept "thin" in the following is first introduced in [33].

Definition 3.2.21 Let X be a variety over k. We say that an arc $\alpha : \operatorname{Spec} K[[t]] \to X$ is **thin** if α factors through a proper closed subvariety of X. An arc which is not thin is called a **fat arc**.

An irreducible subset C in X_∞ is called a **thin set** if C is contained in Z_∞ for a proper closed subvariety $Z \subset X$. An irreducible subset in X_∞ which is not thin is called a **fat set**.

In case an irreducible subset C has the generic point $\gamma \in C$ (*i.e.*, the closure $\overline{\{\gamma\}}$ contains C), C is a fat set if and only if γ is a fat arc.

The following holds by the definition and the valuative criterion of properness:

Proposition 3.2.22 ([49] Proposition 2.5) *Let X be a variety over k and α : $\operatorname{Spec} K[[t]] \to X$ an arc. Then, the following hold:*

(i) α is a fat arc if and only if the ring homomorphism $\alpha^* : \mathscr{O}_{X,\alpha(0)} \to K[[t]]$ induced from α is injective;
(ii) Assume that α is fat. For an arbitrary proper birational morphism $\varphi : Y \to X$, the arc α is lifted to Y.

Remark 3.2.23 A fat set in X_∞ for a variety X introduces a discrete valuation on the rational function field $K(X)$ of X (see Definition 3.5.22).

A Nash component (see section 4) is a fat set and the Nash map (see section 4) is just the correspondence to associate a fat set to the valuation induced from the fat set ([49]).

Example 3.2.24 One of typical examples of fat sets is an irreducible **cylinder** (*i.e.,* the pull back $\psi_m^{-1}(S)$ of a constructible set $S \subset X_m$) for a non-singular X. Actually, let C be an irreducible closed subset of X_m and take an m-jet α_m : $\operatorname{Spec} k[t]/(t^{m+1}) \to X$ in C, then, at a neighborhood of $x = \alpha_m(0) = \pi_m(\alpha_m)$, X is étale over $\mathbf{A}_k^n$. Therefore, we may assume that $X = \mathbf{A}_k^n$ and $x = 0$. Assume that $\psi_m^{-1}(\alpha_m)$ is thin, then it is contained in Z_∞ for some proper closed subset $Z \subset X$. The m-jet α_m corresponds to a ring homomorphism

$$\alpha_m^* : k[x_1, \ldots, x_n] \to k[t]/(t^{m+1}),\ \alpha_m^*(x_i) = \sum_{j=1}^{m} a_i^{(j)} t^j.$$

Let $x_i^{(j)}$ be an indeterminate for every $i = 1, \ldots, n$ and $j \geq m+1$. Let

$$\alpha^* : k[x_1, \ldots, x_n] \to k(x_i^{(j)} \mid i = 1, .., n, j \geq m+1)[[t]]$$

be an arc defined by

$$\alpha^*(x_i) = \sum_{j=1}^{m} a_i^{(j)} t^j + \sum_{j=m+1}^{\infty} x_i^{(j)} t^j.$$

Let $\alpha^*(f) = F_0(a_i^{(j)}, x_i^{(j)}) + F_1(a_i^{(j)}, x_i^{(j)})t + \cdots + F_\ell(a_i^{(j)}, x_i^{(j)})t^\ell + \cdots$ for $f \in I_Z$. Then, as $x_i^{(j)}$'s are indeterminates there is ℓ such that $F_\ell \neq 0$. Hence, we obtain $\alpha \in \psi_m^{-1}(C)$ such that $\alpha \notin Z_\infty$.

Example 3.2.25 ([21]) For a singular variety X, an irreducible cylinder is not necessarily fat. Indeed, let X be the Whitney Umbrella that is a hypersurface defined by $xy^2 - z^2 = 0$ in $\mathbf{A}_k^3$. For $m \geq 1$, let

$$\alpha_m^* : k[x, y, z]/(xy^2 - z^2) \to k[t]/(t^{m+1})$$

be the m-jet defined by $\alpha_m(x) = t, \alpha_m(y) = 0, \alpha_m(z) = 0$. Then, the cylinder $\psi_m^{-1}(\alpha_m)$ is contained in $\operatorname{Sing}(X)_\infty$, where $\operatorname{Sing}(X) = (y = z = 0)$. This is proved as follows: Let an arbitrary $\alpha \in \psi_m^{-1}(\alpha_m)$ be induced from

$$\alpha^* : k[x, y, z] \to k[[t]]$$

with

$$\alpha^*(x) = \sum_{j=1}^{\infty} a_j t^j,\ \ \alpha^*(y) = \sum_{j=1}^{\infty} b_j t^j,\ \ \alpha^*(z) = \sum_{j=1}^{\infty} c_j t^j,$$

where we note that $a_1 = 1$. Then, the condition $\alpha^*(xy^2 - z^2) = 0$ implies that the initial term of $\alpha^*(xy^2)$ and that of $\alpha^*(z^2)$ cancel each other. If $\alpha^*(y) \neq 0$, then the order of $\alpha^*(xy^2)$ is odd. On the other hand, if $\alpha^*(z) \neq 0$, the order of $\alpha^*(z^2)$ is even. Hence if $\alpha^*(y) \neq 0$ or $\alpha^*(z) \neq 0$, then the initial term of $\alpha^*(xy^2)$ and that of $\alpha^*(z^2)$ do not cancel each other. Therefore, $\alpha^*(y) = \alpha^*(z) = 0$, which shows that $\psi_m^{-1}(\alpha_m) \subset \mathrm{Sing}(X)_\infty$.

3.3 Properties of the Space of Arcs and the Space of Jets

3.3.1 Group Actions on the Space of Jets/Arcs

Note 3.3.1 Consider $G = \mathbf{A}_k^1 \setminus \{0\} = \operatorname{Spec} k[s, s^{-1}]$ as a multiplicative group scheme. Usually this group scheme is denoted by $\mathbf{G}_m$, but this symbol would conflict with the space of m-jets. Therefore we do not use the usual symbol in this paper. For $m \in \mathbf{N} \cup \{\infty\}$, the morphism $k[t]/(t^{m+1}) \to k[s, s^{-1}, t]/(t^{m+1})$ defined by $t \mapsto s \cdot t$ gives an action

$$\mu_m : G \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}) \to \operatorname{Spec} k[t]/(t^{m+1})$$

of G on $\operatorname{Spec} k[t]/(t^{m+1})$. Therefore, it gives an action

$$\mu_{Xm} : G \times_{\operatorname{Spec} k} X_m \to X_m$$

of G on X_m. As μ_m is extended to a morphism:

$$\overline{\mu}_m : \mathbf{A}_k^1 \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}) \to \operatorname{Spec} k[t]/(t^{m+1}),$$

we obtain the extension

$$\overline{\mu}_{Xm} : \mathbf{A}_k^1 \times_{\operatorname{Spec} k} X_m \to X_m$$

of μ_{Xm}.

Note that $\overline{\mu}_{Xm}(\{0\} \times \alpha) = x_m$, where x_m is the trivial m-jet on $x = \alpha(0) \in X$. Therefore, every orbit $\mu_{Xm}(G \times \{\alpha\})$ contains the trivial m-jet on $\alpha(0)$ in its closure.

Proposition 3.3.2 *For $m \in \mathbf{N} \cup \{\infty\}$, let $Z \subset X_m$ be a G-invariant closed subset. Then the image $\pi_m(Z)$ is closed in X. In particular the image $\pi_m(Z)$ of an irreducible component of $Z \subset X_m$ is closed in X.*

Proof Let $Z \subset X_m$ be a G-invariant closed subset. Then, we obtain:

$$\overline{\mu}_{Xm}(\mathbf{A}_k^1 \times Z) = Z.$$

On the other hand, $\overline{\mu}_{Xm}(\{0\} \times Z) = \sigma_m \circ \pi_m(Z)$ by Note 3.3.1. Therefore, as Z is closed, it follows that

$$Z \supset \overline{\sigma_m \circ \pi_m(Z)} \supset \sigma_m(\overline{\pi_m(Z)}),$$

which yields $\pi_m(Z) \supset \overline{\pi_m(Z)}$. □

Note 3.3.3 Let $G := \mathbf{A}^1 \setminus \{0\} = \operatorname{Spec} k[s, s^{-1}]$ be as above. As we have an action

$$\mu_{Xm} : G \times_{\operatorname{Spec} k} X_m \to X_m$$

of G on X_m, we have the $\mathcal{O}_X$-graded algebra $\oplus_{i\geq 0}\mathcal{R}_i$ with $\mathcal{R}_0 = \mathcal{O}_X$ such that

$$X_m = \operatorname{Spec} \oplus_{i\geq 0}\mathcal{R}_i.$$

Indeed, we can define

$$\mathcal{R}_i := \{f \in \mathcal{O}_{X_m} \mid \mu^*_{Xm}(f) = s^i \cdot f\}.$$

Lemma 3.3.4 ([56]) *For every $m \in \mathbf{N}$, the base scheme X is the categorical quotient of X_m by the action of G.*

Here, the definition of the categorical quotient is found in [76, Definition 0.5].

3.3.2 Morphisms of the Space of Jets/Arcs

Proposition 3.3.5 *Let $f : X \to Y$ be a morphism of k-schemes of finite type. Then there is a canonical morphism $f_\infty : X_\infty \to Y_\infty$ such that the following diagram is commutative:*

$$\begin{array}{ccc} X_\infty & \xrightarrow{f_\infty} & Y_\infty \\ \pi^X_m \downarrow & & \downarrow \pi^Y_m \\ X & \xrightarrow{f} & Y \end{array}.$$

Proof The morphism f_∞ is induced as the projective limit of f_m ($m \in \mathbf{N}$) (see Proposition 3.2.4). □

Proposition 3.3.6 *Let $f : X \to Y$ be a proper birational morphism of k-schemes of finite type such that $f|_{X\setminus W} : X \setminus W \simeq Y \setminus V$, where $W \subset X$ and $V \subset Y$ are closed. Then f_∞ gives a bijection*

$$X_\infty \setminus W_\infty \to Y_\infty \setminus V_\infty.$$

Proof Let $\alpha \in Y_\infty \setminus V_\infty$, then $\alpha(\eta) \in X \setminus V$. As $X \setminus W \simeq Y \setminus V$. We obtain the following commutative diagram:

$$\begin{array}{ccc} \operatorname{Spec} K((t)) & \rightarrow & Y \\ \downarrow & & \downarrow . \\ \operatorname{Spec} K[[t]] & \xrightarrow{\alpha} & X \end{array}$$

Then, as f is a proper morphism, by the valuative criteria of properness, there is a unique morphism $\tilde{\alpha} : \operatorname{Spec} K[[t]] \to Y$ such that $f \circ \tilde{\alpha} = \alpha$. This shows the bijectivity as required. □

Remark 3.3.7 The bijection above is not isomorphic in general. Actually the following is an example that $X_\infty \setminus W_\infty \to Y_\infty \setminus V_\infty$ is not isomorphic.

Let (Y, y) be a germ of isolated singularity and $f : X \to Y$ be a resolution of the singularity (Y, y). Let $W := f^{-1}(y)$ and $V := \{y\}$. Take a k-valued arc $\alpha \in (\pi^Y)^{-1}(y) \setminus V_\infty$ and let $\tilde{\alpha} \in X_\infty$ be the corresponding arc to α by the above bijective map. Then, by Grinberg and Kazhdan [42], the formal neighborhoods $(X_\infty)_{\tilde{\alpha}}$ of X_∞ at $\tilde{\alpha}$ and $(Y_\infty)_\alpha$ of Y_∞ at α are described as follows:

$$(X_\infty)_{\tilde{\alpha}} \simeq D^\infty, \quad \text{and} \quad (Y_\infty)_\alpha \simeq D^\infty \times Z_z,$$

where $D = \operatorname{Spf} k[[x]]$ and Z_z is the formal neighborhood of a scheme Z of finite type over k at a k-valued point $z \in Z$. In [28, Example], we can take Z singular at z, which implies that

$$(X_\infty)_{\tilde{\alpha}} \not\simeq (Y_\infty)_\alpha.$$

The following is the version for $m = \infty$ of Proposition 3.2.5:

Proposition 3.3.8 *If $f : X \to Y$ is an étale morphism, then*

$$X_\infty \simeq Y_\infty \times_Y X.$$

Proof As $\varprojlim_m (Y_m \times_Y X) = (\varprojlim_m Y_m) \times_Y X$, the case $m = \infty$ is reduced to the case $m < \infty$ which is proved in Proposition 3.2.5. □

Proposition 3.3.9 *There is a canonical isomorphism:*

$$(X \times_k Y)_m \simeq X_m \times_k Y_m,$$

for every $m \in \mathbf{N} \cup \{\infty\}$. Here, $\times_k$ means $\times_{\operatorname{Spec} k}$ for avoiding the bulky notation.

Proof For an arbitrary k-scheme Z,

$$\operatorname{Hom}_k(Z, X_m \times_k Y_m) \simeq \operatorname{Hom}_k(Z, X_m) \times \operatorname{Hom}_k(Z, Y_m),$$

and the right hand side is isomorphic to

$$\mathrm{Hom}_k(Z \times_k \operatorname{Spec} k[t]/(t^{m+1}), X) \times \mathrm{Hom}_k(Z \times_k \operatorname{Spec} k[t]/(t^{m+1}), Y)$$

$$\simeq \mathrm{Hom}_k(Z \times_k \operatorname{Spec} k[t]/(t^{m+1}), X \times_k Y).$$

$$\simeq \mathrm{Hom}_k(Z, (X \times_k Y)_m).$$

The case $m = \infty$ follows from this. □

Proposition 3.3.10 *Let $f : X \to Y$ be an open immersion (resp. closed immersion) of k-schemes of finite type. Then the induced morphism $f_m : X_m \to Y_m$ is also an open immersion (resp. closed immersion) for every $m \in \mathbf{N} \cup \{\infty\}$.*

Proof The open case follows from Lemma 3.2.5 and Proposition 3.3.8. For the closed case, we may assume that Y is affine. If Y is defined by f_i $(i = 1, ., r)$ in an affine space, then X is defined by f_i $(i = 1, ., r, ., u)$ with $r \le u$ in the same affine space. Then, Y_m is defined by $F_i^{(s)}$ $(i = 1, ., r,\ s \le m)$ and X_m is defined by $F_i^{(s)}$ $(i = 1, ., r, ., u,\ s \le m)$ in the corresponding affine space. This shows that X_m is a closed subscheme of Y_m. □

Remark 3.3.11 In the above proposition we see that the property open or closed immersion of the base spaces is inherited by the morphism of the space of jets and arcs. But some properties are not inherited. For example, surjectivity and closedness are not inherited.

Example 3.3.12 There is an example that $f : X \to Y$ is surjective and closed but $f_\infty : X_\infty \to Y_\infty$ is neither surjective nor closed. Let $X = \mathbf{A}^2_{\mathbf{C}}$ and $G = \langle \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{n-1} \end{pmatrix} \rangle$ be a finite cyclic subgroup in $\mathrm{GL}(2, \mathbf{C})$ acting on X, where $n \ge 2$ and ϵ is a primitive n-th root of unity. Let $Y = X/G$ be the quotient of X by the action of G. Then, it is well known that the singularity appeared in Y is A_{n-1}-singularity. Then the canonical projection $f : X \to Y$ is closed and surjective. We will see that these two properties are not inherited by $f_\infty : X_\infty \to Y_\infty$. Let p be the image $f(0) \in Y$. Then, by the commutativity

$$\begin{array}{ccc} X_\infty & \xrightarrow{f_\infty} & Y_\infty \\ \downarrow \pi^X & & \downarrow \pi^Y \\ X & \xrightarrow{f} & Y, \end{array}$$

we obtain $(\pi^X)^{-1}(0) = f_\infty^{-1} \circ (\pi^Y)^{-1}(p)$. Here, $(\pi^X)^{-1}(0)$ is irreducible, since X is non-singular. On the other hand $(\pi^Y)^{-1}(p)$ has $(n-1)$ irreducible components by Petrov [81] and Ishii and Kollár [46]. Therefore the morphism f_∞ is not surjective for $n \ge 3$. As $X \setminus \{0\} \to Y \setminus \{p\}$ is étale, The morphism

$$(X \setminus \{0\})_\infty \to (Y \setminus \{p\})_\infty$$

is also étale by Proposition 3.3.8. Since Y_∞ is irreducible by Corollary 3.5.38, f_∞ is dominant. Therefore, f_∞ is not closed.

Next we think of the irreducibility of the arc space or jet schemes. The following is proved in [64]. In [47] we gave another proof by using [46, Lemma 2.12] and a resolution of the singularities. Here we show a proof without a resolution.

3.3.3 The Structure of the Space of Jets/Arcs

Theorem 3.3.13 ([47, 64]) *If characteristic of k is zero, then the space of arcs of a variety X is irreducible.*

Proof By Ishii and J. Kollár [46, Lemma 2.12] we obtain the following:

(1) Given any arc $\phi : \operatorname{Spec} k'[[s]] \to X$, we construct an arc $\Phi : \operatorname{Spec} K[[s]] \to X$ such that $\phi \in \overline{\{\Phi\}}$ and $\Phi(\tilde{0}) = \Phi(\tilde{\eta}) = \phi(\eta)$, where η and $\tilde{\eta}$ are the generic points of $\operatorname{Spec} k'[[s]]$ and $\operatorname{Spec} K[[s]]$, respectively, while $\tilde{0}$ is the closed point of $\operatorname{Spec} K[[s]]$.
(2) We construct an arc Ψ such that $\Phi \in \overline{\{\Psi\}}$ and $\Psi(\tilde{\eta}) \in X \setminus \operatorname{Sing} X$.

Now for this Ψ we apply the procedure (1) again, then we obtain a new arc

$$\Psi' : \operatorname{Spec} K'[[s]] \to X$$

such that $\Psi \in \overline{\{\Psi'\}}$ and $\Psi'(\tilde{0}') = \Psi'(\tilde{\eta}') = \Psi(\tilde{\eta}) \in X \setminus \operatorname{Sing} X$, where $\tilde{0}'$ (resp. $\tilde{\eta}'$) is the closed point (resp. the generic point) of $\operatorname{Spec} K'[[s]]$. If we denote $\pi(\Psi') = \Psi'(\tilde{0}') = \lambda$, then as $\lambda \in X \setminus \operatorname{Sing} X$, it follows that

$$\Psi' \in \pi^{-1}(X \setminus \operatorname{Sing} X),$$

where the set of the right hand side is irreducible. This yields

$$\phi \in \overline{\pi^{-1}(X \setminus \operatorname{Sing} X)},$$

hence $X_\infty = \phi \in \overline{\pi^{-1}(X \setminus \operatorname{Sing} X)}$ which is irreducible. □

Example 3.3.14 ([46], Example 2.13) If the characteristic of k is $p > 0$, X_∞ is not necessarily irreducible. For example, the hypersurface X defined by $x^p - y^p z = 0$ has an irreducible component in $(\operatorname{Sing} X)_\infty$ which is not in the closure of $X_\infty \setminus (\operatorname{Sing} X)_\infty$.

Note that if the characteristic of k is 0, then every arc in $(\operatorname{Sing} X)_\infty$ lies in the closure of $X_\infty \setminus (\operatorname{Sing} X)_\infty$. But in our case $\operatorname{char} k = p > 0$, an arc $(x(t), 0, 0) \in (\operatorname{Sing} X)_\infty \cap \pi^{-1}((0, 0, 0))$ belongs to $\overline{X_\infty \setminus (\operatorname{Sing} X)_\infty}$ if and only if $x(t)$ has the form $x(t) = \sum_{j=1}^{\infty} a_{jp} t^{jp}$>

Example 3.3.15 ([48]) Let X be a toric variety over an algebraically closed field of arbitrary characteristic. Then, X_∞ is irreducible.

Next let us think of the space of m-jets. The space of m-jets of a variety is not necessarily irreducible even if the characteristic of k is zero (see Example 3.2.17).

The geometric structures of X and the space of arcs/jets affect each other.

Proposition 3.3.16 ([55]) *If X is smooth, then X_m is also smooth for every $m \in \mathbf{N}$. Conversely, if there is $m \in \mathbf{N}$, such that X_m is smooth, then X is smooth.*

Generally speaking, if X_m has property (P) for some $m \in \mathbf{N}$, then X has property (P) for many properties (P).

As the k-scheme X is the categorical quotient of X_m for every $m \in \mathbf{N}$ by the action of G (Lemma 3.3.4), we obtain by Mumford et al. [76] the following:

Proposition 3.3.17 ([56]) *The following is a list of the statements of the form X_m has (P) for an $m \in \mathbf{N}$, then X has (P).*

(i)	X_m *reduced*	$\Rightarrow$	X *reduced*
(ii)	X_m *connected*	$\Rightarrow$	X *connected*
(iii)	X_m *irreducible*	$\Rightarrow$	X *irreducible*
(iv)	X_m *locally integral*	$\Rightarrow$	X *locally integral*
(v)	X_m *locally integral* and normal	$\Rightarrow$	X *locally integral* and normal

Example 3.3.18 The converse of (i) does not hold in general. We give here an example in [40]. Let X be defined by $xy = 0$ in $\mathbf{A}^2_{\mathbf{C}}$. Then, X itself is reduced but X_m is not reduced for any $m \in \mathbf{N}$. Indeed, let I_m be the defining ideal of X_m in $(\mathbf{A}^2_{\mathbf{C}})_m$. Then I_m is a homogeneous ideal of $\mathbf{C}[x^{(0)}, y^{(0)}, x^{(1)}, y^{(1)}, \dots, x^{(m)}, y^{(m)}]$. The degree 0 part of I_m is generated by

$$x^{(0)}y^{(0)}$$

and the part of degree 1 is generated by

$$x^{(0)}y^{(1)} + x^{(1)}y^{(0)}$$

as $\mathbf{C}[x^{(0)}, y^{(0)}]$-modules. Then, $f := x^{(0)}y^{(1)} \notin I_m$, but $f^2 \in I_m$.

The paper [40] shows more general statement. Let I be a reduced monomial ideal on $\mathbf{A}_{\mathbf{C}^n}$, then I_m is not a monomial ideal in general but $\sqrt{I_m}$ is a monomial ideal for every $m \in \mathbf{N}$

Remark 3.3.19 About (ii), we have the converse statement: If X is connected, then X_m is connected for every $m \in \mathbf{N}$. This can be seen as follows: Let $P \in X_m$ be any point and let $x = \pi_m(P)$. Then, the orbit $O_G(P)$ of P by the action of G is irreducible and the closure $\overline{O_G(P)}$ contains $\sigma_m(x)$. Thus, every point of X_m is connected to the section $\sigma_m(X)$ by an irreducible curve. Since $\sigma_m(X) \simeq X$ is connected, X_m is connected.

Example 3.3.20 The converse of (iii) or the converse of (iv) do not hold in general. For example, let $X \subset \mathbf{A}^3_{\mathbf{C}}$ be a curve defined by $x^3 - y^2 = x^2 - z^3 = 0$. Then, the main component $\overline{\pi_m^{-1}(X_{reg})}$ of X_m has dimension $m + 1$. Here, X_{reg} is the open subset consisting of non-singular points of X. On the other hand, since $\pi_m^{-1}(0)$ is defined in $(\pi_{\mathbf{A}^3 m})^{-1}(0) = \mathbf{A}^{3m}_{\mathbf{C}}$ by $2m - 2$ equations, it follows that $\dim \pi_m^{-1}(0) \geq m + 2$. This shows that X_m is not irreducible for any $m \in \mathbf{N}$. As X_m is connected, it also shows that X_m is not locally integral for $m \in \mathbf{N}$.

Example 3.3.21 The converse of (v) does not hold in general. For example, let X be a normal surface defined by $x^2 + y^2 + z^2 = 0$ in $\mathbf{A}^3_{\mathbf{C}}$. It has an A_1-singularity at the origin. Then, X_m is irreducible by Mustaţă [77] but not normal for any $m \in \mathbf{N}$. Indeed, it is known that X_m is of dimension $2(m+1)$ for every $m \in \mathbf{N}$. On the other hand, we can see that $\dim \mathrm{Sing}(X_m) = \dim \pi_m^{-1}(0) = 2m + 1$, which shows that X_m is not normal.

Next we will think of further properties.

Theorem 3.3.22 ([56]) *If X_m is locally a complete intersection for an $m \in \mathbf{N}$, then X is also locally a complete intersection.*

Example 3.3.23 If X is locally a complete intersection, then X_m is not necessarily locally a complete intersection. Example 3.3.20 shows such an example.

Definition 3.3.24 Let X be a normal variety defined over k.

(i) If for a Weil divisor D on X there exists $r \in \mathbf{N}$ such that rD is a Cartier divisor, we call D a **Q-Cartier divisor** on X.
(ii) If every Weil divisor on X is **Q**-Cartier divisor, we say that X is **Q-factorial**.
(iii) If for a canonical divisor K_X of X there exists $r \in \mathbf{N}$ such that rK_X is a Cartier divisor, then we call X a **Q-Gorenstein** variety and the minimal such $r \in \mathbf{N}$ the **index** of X.

Remark 3.3.25 The property **Q**-Gorenstein plays an important role in birational geometry. Indeed, sometimes one needs to compare the canonical divisors K_X and K_Y of the varieties X and Y, respectively, in the situation that there exists a birational morphism $\varphi : Y \to X$. But the problem is how to compare them, because K_X and K_Y are on the different varieties and there is no canonical way to compare two divisors on different varieties. Here, if K_X is a Cartier divisor, then one can pull it back directly to get a Cartier divisor $\varphi^* K_X$ on Y and compare K_Y and $\varphi^* K_X$. A variety with **Q**-Cartier divisor K_X is called a **Q**-Gorenstein variety and studied in the Section 3.5.

Definition 3.3.26 Let D be a **Q**-Cartier Weil divisor on a normal variety X defined over k. Let $\varphi : Y \to X$ be a birational morphism. Let $r \in \mathbf{N}$ be such that rD is a Cartier divisor. Define $\varphi^* D \in \mathbf{Q} \otimes_{\mathbf{Z}} \mathrm{Div}(Y)$ as follows:

$$\varphi^* D := \frac{1}{r} \varphi^*(rD),$$

where note that $\varphi^*(rD)$ is well defined, as rD is a Cartier divisor. The **Q**-Cartier divisor $\varphi^* D$ is called the **pull-back** of D.

Theorem 3.3.27 ([56]) *If X_m is* **Q***-factorial for an $m \in \mathbf{N}$, then X is* **Q***-factorial*

Theorem 3.3.28 ([56]) *If X_m is* **Q***-Gorenstein of index r for an $m \in \mathbf{N}$, then X is* **Q***-Gorenstein of index $\leq r(m+1)$.*

In the following we show some results about singularities canonical, log-canonical, terminal, and log-terminal on the jet schemes. These notions will be introduced in Definition 3.5.2.

Theorem 3.3.29 ([56]) *Assume $\operatorname{char} k = 0$. If X_m has at worst canonical (resp. terminal, log-terminal) singularities for an $m \in \mathbf{N}$, then X has at worst canonical (resp. terminal, log-terminal) singularities.*

Theorem 3.3.30 ([56]) *Assume $\operatorname{char} k = 0$. If X_m has at worst log-canonical singularities for an $m \in \mathbf{N}$, then X has at worst log-terminal singularities.*

Theorem 3.3.31 ([56]) *Let $f : X \to Y$ be a morphism of k-schemes. If the induced morphism $f_m : X_m \to Y_m$ is flat for some $m \in \mathbf{N}$, then f is flat.*

Example 3.3.32 The converse of the theorem does not hold. Let $X \subset \mathbf{A}^3_{\mathbf{C}}$ be defined by the equation $t^d + x^d + y^d = 0$, with $d \geq 3$, then it is a normal surface with the singularity at the origin $0 = (0,0,0)$. Let $f : X \to Y = \mathbf{A}^1_{\mathbf{C}}$ be the first projection $(t, x, y) \mapsto t$. Then, as f is a surjective morphism from a reduced scheme to a non-singular curve, it is flat. However, for every $m \geq 2$ the induced morphism $f_m : X_m \to Y_m$ is non-flat. This is shown as follows: For every $m \in \mathbf{N}$, consider the commutative diagram:

$$\begin{array}{ccc} X_m & \xrightarrow{f_m} & Y_m \\ \pi^X_m \downarrow & & \downarrow \pi^Y_m \\ X & \xrightarrow{f} & Y \end{array}$$

As π^Y_m is smooth, it is sufficient to prove that $f \circ \pi^X_m$ is not flat for $m \geq 2$. Note that $(\pi^X_m)^{-1}(X \setminus \{0\})$ is irreducible and of dimension $2(m+1)$.

For $m < d$, $(\pi^X_m)^{-1}(0) = (\pi^{\mathbf{A}^3}_m)^{-1}(0) = \mathbf{A}^{3m}$. For $m \geq d$, as $(\pi^X_m)^{-1}(0)$ is defined by $m + 1 - d$ equations in $\mathbf{A}^{3m}$, it follows that

$$\dim(\pi^X_m)^{-1}(0) \geq 3m - (m+1) + d \geq 2(m+1).$$

If we assume that $m \geq 2$, in both cases above we have

$$\dim(f \circ \pi^X_m)^{-1}(0) \geq \dim(\pi^X_m)^{-1}(0) > 2m + 1 = \dim(f \circ \pi^X_m)^{-1}(t),$$

where $0 \neq t \in Y$. This yields that $f \circ \pi^X_m$ is not flat.

The structures of the space of arcs and the space of jets are determined by the base scheme. So, it is natural to ask whether the converse holds, *i.e.,* whether the space of arcs/jets determine the base scheme. This problem can be divided into the global case and the local case. First we discuss the global problem. This is again divided into two cases. The first one is posed under the additional assumption of existence of certain morphisms:

Proposition 3.3.33 *Let X and Y be two schemes over k and G as in Note 3.3.3. If there exists a G-equivariant isomorphism $X_m \xrightarrow{\sim} Y_m$ of m-jet schemes for some $m \in \mathbf{N} \cup \{\infty\}$, then there is an isomorphism $X \xrightarrow{\sim} Y$.*

Proof As X and Y are the categorical quotients of X_m and Y_m, respectively by the action of G (Lemma 3.3.4), the G-equivariant isomorphism of X_m and Y_m provides with the isomorphism of the categorical quotients. □

If there is a morphism $f : X \to Y$, the induced morphism $f_m : X_m \to Y_m$ is G-equivariant. Therefore, by the previous proposition and the universality of the categorical quotient, we obtain the following:

Corollary 3.3.34 *Let $f : X \to Y$ be a morphism of schemes over k. If the induced morphism $f_m : X_m \to Y_m$ is an isomorphism for some $m \in \mathbf{N} \cup \{\infty\}$, then the morphism f is an isomorphism.*

Remark 3.3.35 This corollary can be proved directly by using the fact that the morphism of the base spaces induces the morphism of the sections in the jet-schemes.

Now for the second case of global version, let us be just given an isomorphism of m-jet schemes and consider if it induces an isomorphism of base schemes. The following is a counterexample for this problem. We use the counterexample of the cancellation problem called Danielewski's example.

Theorem 3.3.36 ([53]) *Let X and Y be hypersurfaces in $\mathbf{A}^3_{\mathbf{C}}$ defined by $xz - y^2 + 1 = 0$ and $x^2z - y^2 + 1 = 0$, respectively. Then, $X \not\simeq Y$ but $X_m \simeq Y_m$ for every $m \in \mathbf{N} \cup \{\infty\}$.*

Now let us turn to the local problem. The following is the affirmative answer to the local problem assuming the existence of a morphism between the base schemes. Here, we note that the notation $X_m(x)$ in the following is defined in Definition 3.2.13.

Theorem 3.3.37 ([73]) *Let $f : (X, x) \to (Y, y)$ be a morphism of germs of a varieties. Assume that f induces isomorphisms $f_m : X_m(x) \simeq Y_m(y)$ for all $m \in \mathbf{N} \cup \{\infty\}$, then f is an isomorphism.*

Remark 3.3.38 Unlike the global version, only one isomorphism $f_m : X_m \simeq Y_m$ does not guarantee that f is isomorphic. Actually, for example the isomorphism $f_1 : X_1(x) \simeq Y_1(y)$ gives just that the Zariski tangent spaces of these singularities are isomorphic. One can see an example with isomorphic f_1 but not isomorphic f in the following:

Let $X \subset \mathbf{A}^2_{\mathbf{C}}$ be the closed subvariety defined by $x^2 - y^2 + x^3 = 0$. Then, the inclusion morphism $X \hookrightarrow \mathbf{A}^2_{\mathbf{C}} =: Y$ is not an isomorphism. But the induced morphism of the Zariski tangent spaces is an isomorphism.

The following is a modified version of local isomorphism problem.

Theorem 3.3.39 ([22, Proposition 4.12]) *Let $f : (X, x) \to (Y, y)$ be a morphism of germs of a varieties. Assume that f induces bijective morphisms $f_m : X_m(x) \to Y_m(y)$ for all $m \in \mathbf{N} \cup \{\infty\}$ (equivalently, f_m induces bijection $|X_m(x)| \to |Y_m(y)|$ of underlying spaces), then it follows that:*

(i) The morphism f is a closed immersion;
(ii) Let $X \hookrightarrow A$ be a closed immersion to a smooth variety A and let I_X and I_Y be the defining ideals of X and Y, respectively, in A. Then, $I_X \supset I_Y$ holds and I_X is integral over I_Y. Here, we note that by the isomorphism $f_1 : X_1(x) \simeq Y_1(y)$, which is viewed as an isomorphism of the Zariski tangent spaces, we can identify the ambient spaces of X and of Y.

Conversely, let $X \subset Y \subset A$ be closed subschemes with smooth A and $0 \in X$ a point. Assume that the defining ideal I_X of X is integral over the defining ideal I_Y of Y around 0. Then, we obtain the equalities $|X_m(0)| = |Y_m(0)|$ of underlying spaces for every $m \in \mathbf{N} \cup \{\infty\}$.

At the end of this section, we show a mysterious theorem by Grinberg and Kazhdan [42] about the formal neighborhood of a point of the arc space. This result is reproved in [28] in a simple way. This theorem is also used to construct the example in Remark 3.3.7 in this chapter.

Theorem 3.3.40 ([28, 41]) *Let X be a scheme of finite type over a filed k, and $\mathrm{Sing}\, X$ the singular locus. Let $\gamma \in X_\infty \setminus (\mathrm{Sing}\, X)_\infty$ be a k-valued point and $(X_\infty)_\gamma$ the formal neighborhood of γ. Denote the formal disk $\mathrm{Spf}(k[[t]])$ by D and the product of countably many copies of D by D^∞. Then, there exists a scheme $Y = Y(\gamma)$ of finite type over k and a k-valued point $y \in Y$, such that*

$$(X_\infty)_\gamma \simeq D^\infty \times Y_y,$$

where Y_y is the formal neighborhood of y in Y.

It is a very interesting problem to find the relationship between the singularity of (Y, y) and $(X, \gamma(0))$. Some people started to study this problem.

3.4 Introduction to the Nash Problem

In this section, we introduce the Nash problem. The author introduced the problem in the expository paper [51] in 2007. After that researches on this problem developed remarkably, so it seems a good timing to introduce the problem again and show the progress after 2007. In this section, we assume the existence of resolutions of singularities. It is sufficient to assume that the characteristic of k is zero.

3.4.1 Basics for the Statement for the Nash Problem

One of the most mysterious and fascinating problem in arc spaces is the Nash problem which was posed by Nash in his preprint in 1968. It is a question about the Nash components and the essential divisors. First we introduce the concept of essential divisors.

Definition 3.4.1 Let X be a variety, $g : X_1 \to X$ a proper birational morphism from a normal variety X_1 and $E \subset X_1$ an irreducible divisor. Let $f : X_2 \to X$ be another proper birational morphism from a normal variety X_2. The birational map $f^{-1} \circ g : X_1 \dashrightarrow X_2$ is defined on a (nonempty) open subset E^0 of E because, by Zariski's main theorem, the "fundamental locus" of a birational map between normal varieties is a closed subset of codimension ≥ 2. The closure of $(f^{-1} \circ g)(E^0)$ is called the **center** of E on X_2.

We say that E **appears** in f (or in X_2), if the center of E on X_2 is also a divisor. In this case the birational map $f^{-1} \circ g : X_1 \dashrightarrow X_2$ is a local isomorphism at the generic point of E and we denote the birational transform of E on X_2 again by E. For our purposes $E \subset X_1$ is identified with $E \subset X_2$. Such an equivalence class is called a **prime divisor over** X.

Let a prime divisor E over X appear on $g : X_1 \to X$. If g is not an isomorphism at the generic point of E, then we call E an **exceptional divisor** over X.

Definition 3.4.2 Let X be a variety over k and let $\operatorname{Sing} X$ be the singular locus of X. In this paper, by a **resolution** of the singularities of X we mean a proper, birational morphism $f : Y \to X$ with Y non-singular such that the restriction $Y \setminus f^{-1}(\operatorname{Sing} X) \to X \setminus \operatorname{Sing} X$ of f is an isomorphism.

A resolution $f : Y \to X$ whose fiber $f^{-1}(\operatorname{Sing} X)$ is of pure codimension one is called a **divisorial resolution**.

Definition 3.4.3 An exceptional divisor E over X is called an **essential divisor** over X if for every resolution $f : Y \to X$ the center of E on Y is an irreducible component of $f^{-1}(\operatorname{Sing} X)$.

For a given resolution $f : Y \to X$, the center of an essential divisor is called an **essential component** on Y.

Proposition 3.4.4 *Let $f : Y \to X$ be a resolution of the singularities of a variety X. The set*

$$\mathscr{E} = \mathscr{E}_{Y/X} = \left\{ \begin{array}{c} \textit{irreducible components of} f^{-1}(\operatorname{Sing} X) \\ \textit{which are centers of essential divisors over } X \end{array} \right\}$$

corresponds bijectively to the set of all essential divisors over X.

In particular, the set of essential divisors over X is a finite set.

Proof The map

$$\{\text{essential divisors over } X\} \to \mathscr{E}_{Y/X}, \quad E \mapsto \text{center of } E \text{ on } Y$$

is surjective by the definition of essential components. To prove the injectivity, take an essential component C and the blow up $Y' \to Y$ with the center C. Then, there is a unique divisor $E \subset Y'$ dominating C. Let $Y'' \to Y'$ be a resolution of the singularities of Y'. Then, E is the unique exceptional divisor on Y'' that dominates C. Therefore, every exceptional divisor over X with the center $C \subset Y$ has the center contained in E on a resolution Y'' of the singularities of X. Therefore, by the definition of essential divisor, this E is the unique essential divisor whose center on Y is C. □

C. Bouvier and G. Gonzalez-Sprinberg also introduce "essential divisors" and "essential components" in [12] and [13], but we should note that the definitions are different from ours. Nash problem is about our essential divisors and not about their "essential divisors". In order to avoid a confusion, we give different names to their "essential divisors" and "essential components" and clarify different points among them.

Definition 3.4.5 ([12, 13]) An exceptional divisor E over X is called a **BGS-essential divisor** over X if E appears in every resolution. An exceptional divisor E over X is called a **BGS-essential component** over X if the center of E on every resolution f of the singularity of X is an irreducible component of $f^{-1}(E')$, where E' is the center of E on X.

We will see how different they are from our essential divisors and essential components. First we see that they coincide for 2-dimensional case. To show this we need to introduce the concept minimal resolution.

Definition 3.4.6 A resolution $f : Y \to X$ of the singularities of X is called the **minimal resolution** if for any resolution $g : Y' \to X$, there is a unique morphism $Y' \to Y$ over X.

It is known that for a surface X the minimal resolution $f : Y \to X$ exists. It is characterized by the fact that Y has no exceptional curve of the first kind over X.

For higher dimensional variety X, the minimal resolution does not necessarily exist. For example, $X = \{xy - zw = 0\} \subset \mathbf{A}^4$ has two resolutions neither of which dominates the other. These two resolutions are obtained as follows: First take a blow-up $f : \widetilde{Y} \to X$ at the origin of X which has the unique singular point at the origin. Then, f is a resolution of the singularity of X and the exceptional divisor E of f is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. Here we have two contractions $g_1 : \widetilde{Y} \to Y_1$, $g_2 : \widetilde{Y} \to Y_2$ whose restrictions on E are the first projection $p_1 : E = \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$ and the second projection $p_2 : E = \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$, respectively. Then both Y_i's are non-singular, therefore $f_i : Y_i \to X$ $(i = 1, 2)$ are resolutions of the singularity of X. It is clear that there is no morphism between Y_1 and Y_2 over X.

Proposition 3.4.7 *If X is a surface, then each of the set of "essential divisors", "BGS-essential divisors" and "BGS-essential components" are bijective to the set of the components of the fiber $f^{-1}(\mathrm{Sing}\,X)$, where $f : Y \to X$ is the minimal resolution. These are also essential components on the minimal resolution.*

Remark 3.4.8 Four concepts "essential divisor", "essential component", "BGS-essential divisor" and "BGS-essential component" are mutually different in general.

First, our essential component is different from the others, because it is a closed subset on a specific resolution and the others are all equivalence classes of irreducible divisors.

Next, a BGS-essential divisor is different from a BGS-essential component or an essential divisor. Indeed, for $X = (xy - zw = 0) \subset \mathbf{A}_k^4$, the exceptional divisor obtained by a blow-up at the origin is the unique essential divisor and also the unique BGS-essential component, while there is no BGS-essential divisor, since X has a resolution whose exceptional set is $\mathbf{P}_k^1$, which is not a divisor.

Finally a BGS-essential component and an essential component are different. Indeed, consider a cone generated by $(0, 0, 1), (2, 0, 1), (1, 1, 1), (0, 1, 1)$ in $\mathbf{R}^3$. It is well known that a cone generated by integer points in a real Euclidean space defines an affine toric variety (see [36, 83] for basic notion of toric varieties). Let X be the affine toric variety defined by this cone. Then the canonical subdivision adding a one dimensional cone $\mathbf{R}_{\geq 0}(1, 0, 1)$ is a resolution of X. As the singular locus of X is of dimension one, there is no small resolution. Therefore, the divisor $D_{(1,0,1)}$ is the unique essential divisor, while $D_{(1,1,2)}$ and $D_{(2,1,2)}$ are BGS-essential components by the criterion [12, Theorem 2.3].

Definition 3.4.9 Let X be a variety and $\pi : X_\infty \to X$ the canonical projection. An irreducible component C of $\pi^{-1}(\mathrm{Sing}\,X)$ is called a **Nash component** if it contains an arc α such that $\alpha(\eta) \notin \mathrm{Sing}\,X$. This is equivalent to saying that $C \not\subset (\mathrm{Sing}\,X)_\infty$.

The following lemma is already quoted for the irreducibility of the space of arcs (Theorem 3.3.13).

Lemma 3.4.10 ([46]) *If the characteristic of the base field k is zero, then every irreducible component of $\pi^{-1}(\mathrm{Sing}\,X)$ is a Nash component.*

We note that for the positive characteristic case this lemma does not hold. Indeed, Example 3.3.14 is an example that $\pi^{-1}(\mathrm{Sing}\,X)$ has an irreducible component which is not a Nash component.

Let $f : Y \to X$ be a resolution of the singularities of X and E_l $(l = 1, .., r)$ the irreducible components of $f^{-1}(\mathrm{Sing}\,X)$. Now we are going to introduce a map $\mathcal{N}$ which is called the Nash map

$$\left\{ \begin{array}{c} \text{Nash components} \\ \text{of the space of arcs} \\ \text{of } X \end{array} \right\} \xrightarrow{\mathcal{N}} \left\{ \begin{array}{c} \text{essential} \\ \text{components} \\ \text{on } Y \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{essential} \\ \text{divisors} \\ \text{over } X \end{array} \right\}.$$

Note 3.4.11 (Construction of the Nash Map) The resolution $f : Y \to X$ induces a morphism $f_\infty : Y_\infty \to X_\infty$ of schemes. Let $\pi^Y : Y_\infty \to Y$ be the canonical projection. As Y is non-singular, $(\pi^Y)^{-1}(E_l)$ is irreducible for every l. Denote by $(\pi^Y)^{-1}(E_l)^o$ the open subset of $(\pi^Y)^{-1}(E_l)$ consisting of the points corresponding to arcs $\beta : \operatorname{Spec} K[[t]] \to Y$ such that $\beta(\eta) \notin f^{-1}(\operatorname{Sing} X)$. Let C_i $(i \in I)$ be the Nash components of X. Denote by C_i^o the open subset of C_i consisting of the points corresponding to arcs $\alpha : \operatorname{Spec} K[[t]] \to X$ such that $\alpha(\eta) \notin \operatorname{Sing} X$. As C_i is a Nash component, we have $C_i^o \neq \emptyset$. The restriction of f_∞ gives

$$f'_\infty : \bigcup_{l=1}^{r} (\pi^Y)^{-1}(E_l)^o \to \bigcup_{i \in I} C_i^o.$$

By Proposition 3.3.6, f'_∞ is surjective. Hence, for each $i \in I$ there is a unique l_i such that $1 \leq l_i \leq r$ and the generic point β_{l_i} of $(\pi^Y)^{-1}(E_{l_i})^o$ is mapped to the generic point α_i of C_i^o. By this correspondence $C_i \mapsto E_{l_i}$ we obtain a map

$$\mathcal{N} : \left\{ \begin{array}{c} \text{Nash components} \\ \text{of the space of arcs} \\ \text{through } \operatorname{Sing} X \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{irreducible} \\ \text{components} \\ \text{of } f^{-1}(\operatorname{Sing} X) \end{array} \right\}.$$

Lemma 3.4.12 *The map $\mathcal{N}$ is an injective map to the subset consisting of the essential components on Y.*

Proof Let $\mathcal{N}(C_i) = E_{l_i}$. Denote the generic point of C_i by α_i and the generic point of $(\pi^Y)^{-1}(E_l)$ by β_l. If $E_{l_i} = E_{l_j}$ for $i \neq j$, then $\alpha_i = f'_\infty(\beta_{l_i}) = f'_\infty(\beta_{l_j}) = \alpha_j$, a contradiction. This gives the injectivity of $\mathcal{N}$

To prove that the $\{E_{l_i} : i \in I\}$ are essential components on Y, let $Y' \to X$ be another resolution and $\tilde{Y} \to X$ a divisorial resolution which factors through both Y and Y'. Let $E'_{l_i} \subset Y'$ and $\tilde{E}_{l_i} \subset \tilde{Y}$ be the irreducible components of the exceptional sets corresponding to C_i. Then, we can see that E_{l_i} and E'_{l_i} are the image of $\tilde{E}_{l_i}$. This shows that $\tilde{E}_{l_i}$ is an essential divisor over X and therefore E_{l_i} is an essential component on Y. □

Problem 3.4.13 Is the Nash map

$$\left\{ \begin{array}{c} \text{Nash components} \\ \text{of the space of arcs} \\ \text{through } \operatorname{Sing} X \end{array} \right\} \xrightarrow{\mathcal{N}} \left\{ \begin{array}{c} \text{essential} \\ \text{components} \\ \text{on } Y \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{essential} \\ \text{divisors} \\ \text{over } X \end{array} \right\}.$$

bijective?

3.4.2 History of the Nash Problem

Here we will see the results for this problem according to time line. The first affirmative result for the Nash problem was given by Nash himself.

Theorem 3.4.14 ([81]) *The Nash problem is affirmatively answered for an A_n-singularity ($n \in \mathbf{N}$), where an A_n-singularity is the hypersurface singularity defined by $xy - z^{n+1} = 0$ in $\mathbf{A}_k^3$.*

It is difficult to realize the essential divisors for higher dimensional case, but for two-dimensional case the essential divisors are just the exceptional divisors on the minimal resolution. So, the people thought that surface case is the easiest for the problem and many people studied the problem for surface case. The first concrete result after Nash's paper is the following:

Theorem 3.4.15 ([89]) *The Nash problem is affirmatively answered for a minimal surface singularity. Here, a minimal surface singularity means a rational surface singularity with the reduced fundamental cycle. (A rational singularity is defined in Definition 3.5.6 in the next section.) The fundamental cycle is introduced by M. Artin (see [5] for the definition).*

Theorem 3.4.16 ([70, 88, 90]) *The Nash problem is affirmatively answered for a sandwiched surface singularity and D_n-singularity for $n > 4$. Here, a sandwiched surface singularity means the formal neighborhood of a singular point on a surface obtained by blowing up a complete ideal in the local ring of a closed point on a non-singular algebraic surface. A complete ideal is defined by O. Zariski and Samuel (see [99], Vol II, Appendix 4), but the idea of a sandwiched singularity is that it is a singularity which is birationally sandwiched by non-singular surfaces.*

These are results on rational surface singularities, the following gives affirmative answer for some non-rational surface singularities:

Theorem 3.4.17 ([86]) *The Nash problem is affirmatively answered for a normal surface singularities with the reduced fiber E of the singular point on the minimal resolution such that $E \cdot E_i < 0$ for every irreducible component E_i of E.*

This result is generalized to a wider class of surface singularities in [74]. We omit the statement, since it is not simple.

The following results are for arbitrary dimension.

Theorem 3.4.18 ([46]) *The Nash problem is affirmatively answered for a toric singularity of arbitrary dimension.*

When we say just"toric variety", we always assume normality of the variety. There is a notion "not-necessarily normal toric variety" and an even wider class "pretoric variety" that now we define.

Definition 3.4.19 A variety X is called a **pretoric variety** if

(1) there are a toric variety Z with the torus T' and a finite morphism $\rho : X \to Z$ étale on T',
(2) for the normalization $\nu : \overline{X} \to X$, $\overline{X}$ is a toric variety with the torus T and the composite $\rho \circ \nu : \overline{X} \to Z$ is the equivariant quotient morphism by the group N'/N, where N and N' are the lattice on which the fans of $\overline{X}$ and Z, respectively, are defined, and
(3) the subset $\nu^{-1}(\mathrm{Sing}\, X)$ is an invariant closed set on $\overline{X}$.

We will see two typical examples of a pretoric variety.

Note 3.4.20 ([37]) Here, we introduce a not-necessarily normal affine toric variety. A not-necessarily normal affine toric variety is of the form $X_\Gamma = \mathrm{Spec}\, \mathbf{C}[\Gamma]$, where $\Gamma \subset M = \mathbf{Z}^n$ is a finitely generated semigroup with 0 and Γ generates the abelian group M. Then, the torus $T = \mathrm{Spec}\, \mathbf{C}[M]$ acts on X_Γ. Denote by $K(\Gamma) \subset M_{\mathbf{R}}$, the convex cone which is the convex hull of Γ and by $\overline{\Gamma}$ the intersection $K(\Gamma) \cap M$. Then, $X_{\overline{\Gamma}}$ is a normal toric variety and the inclusion $\mathbf{C}[\Gamma] \hookrightarrow \mathbf{C}[\overline{\Gamma}]$ induces the equivariant normalization $X_{\overline{\Gamma}} \to X_\Gamma$.

Example 3.4.21 A not-necessarily normal toric variety is a pretoric variety. This is proved as follows: Let $X = \mathrm{Spec}\, \mathbf{C}[\Gamma]$ be a not-necessarily normal toric variety of dimension n. Let $\sigma \subset N_{\mathbf{R}}$ be the cone such that $\sigma^\vee = K(\Gamma)$ under the notation as above. Let $\overline{X} = \mathrm{Spec}\, \mathbf{C}[\sigma^\vee \cap M]$ be the normalization of X. Subdivide $\sigma^\vee$ into simplicial cones without adding any 1-dimensional cones. Let $\tau_1, \tau_2, .., \tau_s$ be the n-dimensional simplicial cones which are obtained by this subdivision. We can take generators $e_1^{(i)}, .., e_n^{(i)}$ of τ_i in Γ. Define $M_i = \oplus_{j=1}^n \mathbf{Z} e_j^{(i)}$, then M_i is a subgroup of M of finite index. Let M' be the intersection $\bigcap_{i=1}^s M_i$. Then, M' is a subgroup of M of finite index. It follows that $\sigma^\vee \cap M' \subset \Gamma$. Indeed, an arbitrary element $u \in \sigma^\vee \cap M'$ is contained in $\tau_i \cap M_i$ for some i. Then, by the definition of M_i, we have that $u = \sum_{j=1}^n a_j e_j^{(i)}$ with $a_j \in \mathbf{Z}_{\geq 0}$. As $e_j^{(i)}$'s are in Γ, it follows that $u \in \Gamma$. By this inclusion $\sigma^\vee \cap M' \subset \Gamma$ we obtain a finite morphism $\rho : X \to Z = \mathrm{Spec}\, \mathbf{C}[\sigma^\vee \cap M']$. The other conditions for a pretoric variety follow immediately.

The following is an example of a pretoric variety without a toric action.

Example 3.4.22 Let $\overline{X}$ be $\mathrm{Spec}\, \mathbf{C}[x, y]$ and X be $\mathrm{Spec}\, \mathbf{C}[x, y^3, y^4]$, then X is a non-normal toric variety with the normalization $\nu : \overline{X} \to X$. Therefore we have a diagram $\overline{X} \xrightarrow{\nu} X \xrightarrow{\rho} Z$ as in Definition 3.4.19. Here, $Z = \mathrm{Spec}\, \mathbf{C}[x, y^{12}]$ is constructed according to the previous example. Let X_0 be $\mathrm{Spec}\, \mathbf{C}[x, y+y^2, y^3, y^4]$, then X_0 is a pretoric variety with the diagram: $\overline{X} \to X_0 \to Z$. By the definition, X_0 does not admit a toric action.

Theorem 3.4.23 ([49]) *The Nash problem is affirmatively answered for a pretoric variety of arbitrary dimension.*

Theorem 3.4.24 ([1]) *The Nash problem is affirmatively answered for non-rational quasi-rational hypersurface singularities of arbitrary dimension.*

We have a notion of the local Nash problem which is a slight modification of the Nash problem ([50]).

Theorem 3.4.25 ([50]) *The local Nash problem hold true for quasi-ordinary singularities. Here, a quasi-ordinary singularity is a hypersurface singularity which is a finite cover over a non-singular variety with the normal crossing branch locus. We note that a quasi-ordinary singularity is not necessarily normal and its normalization is toric.*

The paper [87] by Plénat and Popescu-Pampu gives the affirmative answer to the Nash problem for a certain class of higher dimensional non-toric singularities.

So far we have seen the affirmative answers. But there are negative examples given in [46] by Ishii and Kollár .

Example 3.4.26 ([46]) Let X be a hypersurface defined by $x_1^3+x_2^3+x_3^3+x_4^3+x_5^6 = 0$ in $\mathbf{A}^5_{\mathbf{C}}$. Then the number of the Nash components is one, while the number of the essential divisors is two. Therefore the Nash map is not bijective.

By the above example we can construct counter examples to the Nash problem for any dimension greater than 3 by making the product with $\mathbf{A}^n_k$ for $n \geq 1$. Therefore at that moment of the paper, the unsolved case for Nash problem was only 2- and 3-dimensional cases. Then, J. F. Bobadilla and M.P.Pereira proved the affirmative answer for 2-dimensional case.

Theorem 3.4.27 ([11]) *The Nash problem is affirmatively answered for surfaces.*

This result is based on the topological observation by Bobadilla as follows:

Proposition 3.4.28 ([10]) *Nash problem for surface singularities depends only on the topological type.*

Later on, algebraic proof of the Nash problem for surface is given by De Fernex and Docampo as a corollary of their main theorem:

Theorem 3.4.29 ([20]) *Let $\varphi : Y \to X$ is a terminal model which means proper birational morphism from Y with at worst terminal singularities and φ-nef canonical divisor of Y. Then, the irreducible exceptional divisors on Y are in the images of the Nash map. In particular, irreducible exceptional curves on the minimal resolution of a surface are in the image of the Nash map.*

Here, we should note that the minimal resolution of a surface singularity is the terminal model.

The first 3-dimensional negative example for the Nash problem is given by De Fernex.

Example 3.4.30 ([19]) The singularity of 3-dimensional hypersurface in $\mathbf{A}^4_{\mathbf{C}}$ defined by

$$(x_2^2 + x_3^2)x_4 + x_1^3 + x_2^3 + x_3^3 + x_4^5 + x_4^6 = 0$$

has one Nash component and has two essential divisors. Thus the Nash map is not bijective.

The following is a bit more systematic example for the negative answer to the Nash problem for threefolds obtained by Johnson and Kollár:

Example 3.4.31 ([63]) For the singularities on $X(m) := (xy - z^2 + u^m = 0) \subset \mathbf{A}^4_{\mathbf{C}}$ the Nash map is not surjective for odd $m \geq 5$ but surjective for even m and $m = 3$. Thus the simplest example where the Nash map is not bijective is

$$(xy - z^2 + u^5 = 0) \subset \mathbf{A}^4_{\mathbf{C}}.$$

Now we can formulate a new version of the Nash problem:

Problem 3.4.32

(i) Characterize the image of the Nash map.
(ii) Characterize the singularities for which the Nash problem is affirmative.

Related to these problems, we have one characterization of the image of the Nash map given by Reguera [91]. To formulate her result, we introduce the concept "wedge" which is also used in [11].

Definition 3.4.33 Let X be a k-scheme. Let $K \supset k$ be a field extension. A **K-wedge** of X is a k-morphism $\gamma : \operatorname{Spec} K[[\lambda, t]] \to X$. A K-wedge γ can be identified to a $K[[\lambda]]$-point on X_∞. Denote by 0 and η the closed point and the generic point of $\operatorname{Spec} K[[\lambda]]$, respectively. We call the image $\gamma(0) \in X_\infty$ the **special arc** of γ and call the image$\gamma(\eta) \in X_\infty$ the **generic arc** of γ.

Theorem 3.4.34 ([91]) *Let E be an essential divisor over X and $f : Y \to X$ a resolution of the singularities of X on which E appears. Let $\alpha \in X_\infty$ be the generic point of $f_\infty(\pi^Y)^{-1}(E)$ and $k(\alpha)$ the residue field of α. Then the following conditions are equivalent:*

(i) E belongs to the image of the Nash map;
(ii) For any resolution of the singularities $g : Y' \to X$ and for any field extension K of $k(\alpha)$, any K-wedge γ on X whose special arc is α and whose generic arc belongs to $(\pi^X)^{-1}(\operatorname{Sing} X)$, lifts to Y';
(iii) There exists a resolution of the singularities $g : Y' \to X$ satisfying condition (ii).

As an application of this theorem, we obtain Theorem 3.4.16.

There are some notions "the Nash problem for a pair (X, Z)" consisting of a variety X and a closed subset Z (see [38, 85]).

3.5 Applications to Birational Geometry

3.5.1 Overview of Birational Geometry in Connection with the Space of Arcs

Birational geometry is the study properties of varieties which do not change under birational maps. In this viewpoint we identify varieties which are birationally equivalent each other. In each equivalence class, is there a "good" representative? We think that smaller variety is better, where we say X is smaller than Y if there is a proper birational morphism $Y \to X$.

"Find a minimal variety (called a minimal model) in the equivalence class."

This is one of the most important problems in birational geometry so called "Minimal Model Problem". In dimension one and two, it is classically well known that there are smooth minimal models in an equivalence class. But in higher dimensional case, it is known that we cannot have such a model by the example following Definition 3.4.6. So we need to reformulate the Minimal Model Problem allowing mild singularities. In this way, mild singularities (terminal, log terminal, canonical, log canonical, see Definition 3.5.2 below) allowable in minimal models appeared around 1980. Minimal Model Problem was solved in dimension three in the most basic form by S. Mori [75]. Then, the problem is generalized to several variants. By the work [6], a large part of the problems for arbitrary dimensional case in characteristic 0 is solved. However in its most general setting the problem is still open and the main point of the problem is reduced to certain behaviors of "the minimal log discrepancy", an invariant of a singularity. The research of this direction is still going on and the author thinks that it is good for the reader to know what is known and what is not.

In this section, we discuss about the expression of this invariant by the space of arcs and obtain one of the required behavior for Minimal Model Problem for a special case. We also obtain the characterization of the mild singularities by the space of jets.

3.5.2 Basics in Birational Geometry

Henceforth, we always assume that X is normal and $\mathbf{Q}$-Gorenstein variety (see Definition 3.3.24). The reader who would like to study this direction closely, please refer to [35] or [65]. A typical example of $\mathbf{Q}$-Gorenstein variety is a variety of locally a complete intersection. Here, the condition that X is "locally a complete intersection" means that at each point of X there is an affine open neighborhood embedded into a smooth affine variety with codimension c and defined by exactly c equations in the smooth variety. In particular, a hypersurface is an example of locally a complete intersection.

First we define the log discrepancy for a pair $(X, \mathfrak{a}^e)$ consisting of a normal **Q**-Gorenstein variety X and a coherent multi-ideal sheaf $\mathfrak{a} \subset \mathcal{O}_X$ with a real exponent e, which means

$$\mathfrak{a}^e = \mathfrak{a}_1^{e_1} \cdots \mathfrak{a}_s^{e_s}, \quad e = (e_1, \ldots, e_s) \in \mathbf{R}_{>0}^s$$

where $\mathfrak{a}_i \subset \mathcal{O}_X$ are non-zero coherent ideal sheaves.

As we assume that X is **Q**-Gorenstein, for a morphism

$$\varphi : Y \to X$$

the pull-back $\varphi^* K_X$ is always defined and becomes a **Q**-Cartier divisor on Y again (see, Definition 3.3.26).

Definition 3.5.1 Let E be a prime divisor over a normal **Q**-Gorenstein variety X. Then we define **log discrepancy** $k_E + 1 \in \mathbf{Z}$ of X at E as follows:

$$k_E + 1 := \mathrm{ord}_E(K_Y - \varphi^* K_X) + 1,$$

where $\varphi : Y \to X$ is a birational morphism such that Y is normal, E appears on Y and ord_E means the coefficient of the divisor at E.

Log discrepancy of a pair $(X, \mathfrak{a}^e)$ consisting of a normal **Q**-Gorenstein variety X and multi-ideal sheaf $\mathfrak{a}^e$ with a real exponent at E is defined as follows:

$$a(E; X, \mathfrak{a}^e) := k_E + 1 - \sum_{i=1}^{s} e_i \cdot v_E(\mathfrak{a}_i),$$

where v_E is the valuation defined by E.

Definition 3.5.2 We say that a pair $(X, \mathfrak{a}^e)$ is terminal / canonical / log terminal / log canonical at a point $x \in X$ if

$$\inf \left\{ a(E; X, \mathfrak{a}^e) \,\middle|\, \begin{array}{l} E : \text{exceptional prime divisor over } X \\ \text{with center containing } x \end{array} \right\} > 1 / \geq 1 / > 0 / \geq 0,$$

respectively.

We say that X has terminal / canonical / log terminal / log canonical singularities, if $(X, \mathcal{O}_X)$ is terminal / canonical / log terminal / log canonical, respectively, at every point of X.

By the definition, the following implications are clear:
terminal $\Rightarrow$ canonical $\Rightarrow$ log terminal $\Rightarrow$ log canonical.

One can see that if X is smooth and $\mathfrak{a} = \mathcal{O}_X$, then for every exceptional prime divisor E over X, we have $a(E; X, \mathcal{O}_X) \geq N := \dim X$. Therefore we also obtain

smooth $\Rightarrow$] terminal

According to the definition, in order to decide whether the pair is terminal or so one should check all prime divisors with the center containing x. However, if there is a "log resolution" for the pair, then we can decide by checking only finite number of exceptional prime divisors.

Definition 3.5.3 Let $(X, \mathfrak{a}^e)$ be as above. A morphism $\varphi : Y \to X$ is called a **log resolution** of $(X, \mathfrak{a}^e)$, if the following hold:

(i) φ is a proper birational morphism from a non-singular variety Y;
(ii) the ideals $\mathfrak{a}_i \cdot \mathcal{O}_Y$ are all locally principal on Y;
(iii) the union of all exceptional sets and the divisors defined by $\mathfrak{a}_i \cdot \mathcal{O}_Y$ is set theoretically a divisor with normal crossings.

Proposition 3.5.4 ([32, Proposition 7.2]) *Let X be a normal and locally a complete intersection variety defined over an algebraically closed field k of arbitrary characteristic. Assume there exists a log resolution $\varphi : Y \to X$ of a pair $(X, \mathfrak{a}^e)$.*

If $a(E_i; X, \mathfrak{a}^e) > 1 \;/\; \geq 1 \;/\; > 0 \;/\; \geq 0$, for every exceptional divisor E_i on Y with the center containing x, then $(X, \mathfrak{a}^e)$ is terminal / canonical / log terminal / log canonical at x, respectively.

Remark 3.5.5 At present, existence of log resolutions is known when the base field k is of characteristic 0 (by Hironaka [45], see also [66]) or $\dim X \leq 3$ (by Abhyankar [2, 3] and Cossart-Piltant [16]).

By using a resolution of the singularities $\varphi : Y \to X$ we have another important and popular notion of a singularity.

Definition 3.5.6 We say that a variety X has **rational singularity** at $x \in X$ if the following hold:

(i) X is normal;
(ii) X has a resolution of the singularities $\varphi : Y \to X$ and the vanishing $R^j\varphi_*\mathcal{O}_Y = 0$ holds for every $j \geq 1$ in a neighborhood of x.

Rational singularities do not affect the cohomologies between X and the smooth variety Y. So, a rational singularity is considered as a singularity close to a smooth point. It is well known that the singularities appearing on a toric variety are rational. It is natural to ask the relation of a rational singularity and the other classes of singularities defined above.

Proposition 3.5.7 ([34, 62]) *Assume the base field k is of characteristic 0. If $(X, \mathfrak{a}^e)$ is log terminal at $x \in X$, then the singularity (X, x) is rational.*

Definition 3.5.8 The **minimal log discrepancy** for a pair $(X, \mathfrak{a}^e)$ at a point $x \in X$ and at a proper closed subset $W \subset X$ is defined as follows:

(i) When $\dim X \geq 2$,

$$\mathrm{mld}(x; X, \mathfrak{a}^e) = \inf\{a(E; X, \mathfrak{a}^e) \mid E : \text{ prime divisor with the center at } x\},$$
$$\mathrm{mld}(W; X, \mathfrak{a}^e) = \inf\{a(E; X, \mathfrak{a}^e) \mid E : \text{ prime divisor with the center in } W\}.$$

(ii) When $\dim X = 1$, define $\mathrm{mld}(x; X, \mathfrak{a}^e)$ and $\mathrm{mld}(W; X, \mathfrak{a}^e)$ by the same definitions as above if the right hand sides of the above definition are non-negative and otherwise define $\mathrm{mld}(W; X, \mathfrak{a}^e) = -\infty$.

Here, we remark that either $\mathrm{mld}(x; X, \mathfrak{a}^e) \geq 0$ or $\mathrm{mld}(x; X, \mathfrak{a}^e) = -\infty$ holds in any dimension.

Proposition 3.5.9 *Let $(X, \mathfrak{a}^e)$ be a pair as above and $x \in X$ a point. If the pair is terminal / canonical / log terminal / log canonical at x, then*

$$\mathrm{mld}(x; X, \mathfrak{a}^e) > 1/ \geq 1/ > 0/ \geq 0.$$

Conversely, if $\mathrm{mld}(x; X, \mathfrak{a}^e) \geq 0$*, then the pair is log canonical at x. But for the other cases the converse does not hold in general.*

Example 3.5.10 Let $X = \mathbf{A}_k^3$, $\{x, y, z\}$ a coordinate system on $\mathbf{A}_k^3$ and $\mathfrak{a} := (x \cdot y)$. Then,

$$\mathrm{mld}(0; X, \mathfrak{a}) = 1 > 0,$$

but $(X, \mathfrak{a})$ is not log terminal at the origin 0 because the exceptional divisor E obtained by the blow up by the prime ideal (x, y) has the log discrepancy

$$a(E; X, \mathfrak{a}) = k_E + 1 - v_E(x \cdot y) = 1 + 1 - 2 = 0.$$

A modified pair $(X, \mathfrak{a}^e)$ $(1/2 < e < 1)$ from the above gives an example that has

$$\mathrm{mld}(0; X, \mathfrak{a}^e) > 1,$$

but $(X, \mathfrak{a}^e)$ is not terminal because for a prime divisor E as above has the log discrepancy

$$a(E; X, \mathfrak{a}^e) = k_E + 1 - e \cdot v_E(x \cdot y) = 1 + 1 - 2e < 1.$$

Definition 3.5.11 Let E be a prime divisor over X with the center at x. We say that E **computes** $\mathrm{mld}(x; X, \mathfrak{a}^e)$ if

$$a(E, X, \mathfrak{a}^e) = \begin{cases} \mathrm{mld}(x; X, \mathfrak{a}^e) \\ \text{or} \\ \text{negative} \end{cases}$$

Remark 3.5.12 If there exists a log resolution factored through the blow up at x, then there exists a prime divisor computing mld for the pair. Therefore, if $\mathrm{char} k = 0$, then such a prime divisor always exists.

If all e_i are rational numbers, then the set of log discrepancies is discrete, which implies the infimum is minimum or $-\infty$ and therefore there exists a prime divisor computing mld.

Definition 3.5.13 For a pair $(X, \mathfrak{a}^e)$ we define the **log canonical threshold** at $x \in X$ as follows:

$$\mathrm{lct}_x(X, \mathfrak{a}^e) = \sup\left\{c \in \mathbf{R}_{>0} \mid (X, \mathfrak{a}^{ec}) \text{ is log canonical at } x\right\},$$

where $\mathfrak{a}^{ec} = \mathfrak{a}_1^{e_1 c} \cdots \mathfrak{a}_s^{e_s c}$

Remark 3.5.14 For a pair $(X, \mathfrak{a}^e)$ and a point $x \in X$ the $\mathrm{lct}_x(X, \mathfrak{a}^e)$ is obtained as follows:

(i) $\mathrm{lct}_x(X, \mathfrak{a}^e) = \inf\left\{ \frac{k_E+1}{\sum e_i \cdot v_E(\mathfrak{a}_i)} \;\middle|\; \begin{array}{l} E : \text{prime divisor with} \\ \text{the center containing } x \end{array} \right\}.$

(ii) If $\varphi : Y \to X$ is a log resolution of $(X, \mathfrak{a}^e)$ and E_j $(j = 1, \ldots, m)$ are prime divisors on Y with the center containing x, which are either exceptional or in the support of $\mathfrak{a}_i \cdot \mathcal{O}_Y$'s. Then it follows that:

$$\mathrm{lct}_x(X, \mathfrak{a}^e) = \min_{j=1,\ldots,m} \left\{ \frac{k_{E_j} + 1}{\sum e_i \cdot v_{E_j}(\mathfrak{a}_i)} \right\}.$$

Note 3.5.15 Roughly speaking, a generalized MMP is in the form as follows:

"In the birational equivalence class of pairs $(X, \mathfrak{a}^e)$ with singularities of type **(P)**, does there exist a minimal model $(X_0, \mathfrak{a}_0^e)$ with the singularities of the same type?"

Here, **(P)** is the representative of "terminal", "log terminal", "canonical", "log canonical". Note that in MMP singularities are studied under a general setting but in this paper we restrict our attention to locally complete intersection case,

In order to get a minimal model, one strategy, called Minimal Model *Program*, is established around 1990 and the successful cases of the problem so far all follow from this program.

This program to get a minimal model, roughly speaking, goes as follows:

(i) If a pair with the singularities of type **(P)** is a minimal model, then there is nothing to do anymore.

(ii) If a pair is not a minimal model, then we do

(C) contract of extremal ray, which is to construct a certain proper birational morphism $X \to X'$ to obtain a new pair $(X', \mathfrak{a}'^e)$.

Assume the new pair has the singularities of the same type. If the new pair $(X', \mathfrak{a}'^e)$ is a minimal model, we stop. Otherwise continue the process;*i.e.,* go to (1) above and follow the instruction.

Assume the new pair $(X', \mathfrak{a}'^e)$ does not have singularities of the same type, then we do the following:

(F) make a birational map called a flip $X \dashrightarrow X''$ to get a new pair $(X'', \mathfrak{a}''^e)$ instead of the contraction.

Assume the new pair $(X'', \mathfrak{a}''^e)$ is a minimal model, then we stop.

Otherwise continue the process: *i.e.,* go to (1) above and follow the instruction. In this way we carry out: step (C) or step (F). If the procedure stops at some stage, then it means that we get a minimal model. It is known that the possible number of steps (C) is limited, but that of (F) is not obvious. V. Shokurov proved that if the following two conjectures (ACC Conjecture and LSC Conjecture) hold, then the possible number of steps (F) is finite.

See [35] for more detailed information about Minimal Model Program.

Conjecture 3.5.16 (ACC Conjecture) Let $J \subset \mathbf{R}_{\geq 0}$ be a DCC set. (I.e., there is no infinite strictly decreasing sequence in J). Then the following set satisfies ACC (*i.e.,* there is no infinite strictly increasing sequence).

$$M(N, J) := \{\mathrm{mld}(x; X, \mathfrak{a}^e) \mid \dim X = N, \mathfrak{a} : \text{ideal}, \ e_i \in J\}.$$

Conjecture 3.5.17 (LSC Conjecture) For a pair $(X, \mathfrak{a}^e)$ the following map is lower semi continuous (LSC):

$$X \to \mathbf{R} \cup \{-\infty\}, \quad x \mapsto \mathrm{mld}(x; X, \mathfrak{a}^e),$$

i.e., for every $r \in \mathbf{R}_{\geq 0}$ the set $\{x \in X \mid \mathrm{mld}(x; X, \mathfrak{a}^e) > r\}$ is an open subset of X.

Conjecture 3.5.18 (MN Conjecture) For N and e, there exists a number $\ell_{N,e}$ which depends on N and e, such that for every pair $(X, \mathfrak{a}^e)$ and a point $x \in X$ ($\dim X = N$) there exists a prime divisor E computing $\mathrm{mld}(x; X, \mathfrak{a}^e)$ and satisfying $k_E \leq \ell_{N,e}$.

MN Conjecture is Mustaţă-Nakamura's conjecture posed by them in [79] and proved for special cases (surfaces and monomial ideals on arbitrary dimensional affine space).

They prove the relation of the conjecture and ACC Conjecture as follows:

Theorem 3.5.19 (Theorem 1.5, [79]) *Fix a point $x \in X$ on a variety X with "mild" singularities such that the assertion in MN Conjecture holds for (X, x). Then, for every fixed DCC set J, the following set satisfies ACC:*

$$M(J; X, x) := \{\mathrm{mld}(x; X, \mathfrak{a}^e) \mid \mathfrak{a} : \textit{multi-ideal with exponents } e_i \in J\}.$$

For the precise meaning of "mild singularities", the reader can see in [79].

Remark 3.5.20 In ACC Conjecture and also in MN Conjecture, X and x may vary. But even for fixed $x \in X$, the problem is not easy. These conjectures appeared motivated by MMP, but the problems themselves are interesting from the point of view of singularity theory. So these are studied under various conditions and in such a situation the space of arcs contributed quite a bit. We will see it in the following subsection.

3.5.3 Log Discrepancies via the Spaces of Arcs

Let E be a prime divisor over a normal locally complete intersection variety X. In this subsection we will express the log discrepancy $k_E + 1$ of X at E in terms of the space of arcs of X. We assume that X is just a variety over k unless otherwise stated. First we prepare the notion of the contact loci of an ideal in the space of arcs.

Definition 3.5.21 ([33]) For an affine variety X and an ideal $\mathfrak{a} \subset \mathcal{O}_X$, we define

$$\mathrm{Cont}^m(\mathfrak{a}) = \{\alpha \in X_\infty \mid \mathrm{ord}_\alpha(\mathfrak{a}) = m\}$$

and

$$\mathrm{Cont}^{\geq m}(\mathfrak{a}) = \{\alpha \in X_\infty \mid \mathrm{ord}_\alpha(\mathfrak{a}) \geq m\},$$

where the order ord_α is defined by $\alpha \in X_\infty$ as follows:

$$\mathrm{ord}_\alpha(\mathfrak{a}) := \mathrm{ord}_t\, \alpha^*(\mathfrak{a}) := \min\{\mathrm{ord}_t\, \alpha^*(f) \mid f \in \mathfrak{a}\}.$$

Here, $\alpha^* : \mathcal{O}_X \to k[[t]]$ is the ring homomorphism corresponding to α.

These subsets are called **contact loci** of the ideal $\mathfrak{a}$. The subset $\mathrm{Cont}^{\geq m}(\mathfrak{a})$ is closed and $\mathrm{Cont}^m(\mathfrak{a})$ is locally closed. Indeed, let $Z \subset X$ be the closed subscheme defined by the ideal $\mathfrak{a} \subset \mathcal{O}_X$, then, by the definitions we have;

$$\mathrm{Cont}^{\geq m}(\mathfrak{a}) = \psi_{m-1}^{-1}(Z_{m-1}),$$

$$\mathrm{Cont}^m(\mathfrak{a}) = \mathrm{Cont}^{\geq m}(\mathfrak{a}) \setminus \mathrm{Cont}^{\geq m+1}(\mathfrak{a}),$$

which implies that the former subset is closed and the latter subset is locally closed. One can also see that both are cylinders.

In Definition 3.2.21, we introduced the concepts "thin" and "fat" for an arc and also for an irreducible subset on the space of arcs.

Definition 3.5.22 Let $\alpha : \mathrm{Spec}\, K[[t]] \to X$ be a fat arc of a variety X and $\alpha^* : \mathcal{O}_{X,\alpha(0)} \to K[[t]]$ the local homomorphism induced from α. Here, $\alpha(0) \in X$ is the image of the closed point $0 \in \mathrm{Spec}\, K[[t]]$ by α. By the definition of a fat arc, α^* is injective, therefore it is extended to the homomorphism of fields $\alpha^* : K(X) \to K((t))$, where $K(X)$ is the rational function field of X. Define a function $v_\alpha : K(X) \setminus \{0\} \to \mathbf{Z}$ by

$$v_\alpha(f) = \mathrm{ord}_t\, \alpha^*(f).$$

Then, v_α is a discrete valuation of $K(X)$. We call it the **valuation corresponding to** α.

Definition 3.5.23 A valuation v on the rational function field $K(X)$ of a variety X is called a **divisorial valuation** over X if $v = q \cdot v_E$ for some $q \in \mathbf{N}$ and a divisor E over X. The center of a divisor E is called the **center** of the valuation $v = q \cdot v_E$. A fat arc α of X is called a **divisorial arc** if v_α is a divisorial valuation over X. A fat set is called a **divisorial set** if the generic point is a divisorial arc.

Proposition 3.5.24 ([21], [60, Corollary 3.26]) *Let $\alpha \in X_\infty$ be the generic point of an irreducible fat component of a contact locus* $\mathrm{Cont}^m(\mathfrak{a})$ *or of a cylinder* $\psi_m^{-1}(S)$ *($S \subset X_m$ locally closed). Then α is a divisorial arc.*

We will think of the converse implication.

Definition 3.5.25 ([52]) For a divisorial valuation v over a variety X, define the **maximal divisorial set** corresponding to v as follows:

$$C_X(v) := \overline{\{\alpha \in X_\infty \mid \alpha : \text{fat and, } v_\alpha = v\}},$$

where $\overline{\{\ \}}$ is the Zariski closure in X_∞.

Proposition 3.5.26 *Let E be a prime divisor over X and $\varphi : Y \to X$ a birational morphism on which E appears. Let $\eta \in E$ be the generic point. Let $\tilde{\alpha} \in Y_\infty$ be the generic point of $(\pi^Y)^{-1}(\eta)$, where $\pi^Y : Y_\infty \to Y$ is the canonical projection. Then,*

$$C_X(v_E) = \overline{\varphi_\infty(\tilde{\alpha})}.$$

More generally for $q \in \mathbf{N}$, let $\eta_{q-1} \in E_{q-1}$ be the generic point of the space of $(q-1)$-jets of E. Let $\tilde{\alpha}_{q-1}$ be the generic point of $(\psi^Y_{q-1})^{-1}(\eta_{q-1})$ Then,

$$C_X(q \cdot v_E) = \overline{\varphi_\infty(\tilde{\alpha}_{q-1})}.$$

Proof The statements of the proposition follows from

$$C_X(q \cdot v_E) = \overline{\varphi_\infty(\mathrm{Cont}^q(E_0))}$$

where $E_0 \subset E$ is the open dense subset consisting of points $p \in E$ such that E and Y are both smooth at p ([52, Proposition 3.4]). □

The following is a kind of converse of Proposition 3.5.24:

Proposition 3.5.27 ([21, 60]) *Let X be a variety over an algebraically closed field of arbitrary characteristic. For every divisorial valuation v over X the maximal divisorial set is an irreducible fat component of a contact locus and, in particular, of a cylinder.*

As the arc space X_∞ of a variety X of dimension > 0 is a scheme of infinite dimension over k, codimension of a closed subscheme of X_∞ is not defined in general. But for subscheme of special type we can define the codimension whose important role is describing invariants of singularities on X.

Let X be an arbitrary variety over an algebraically closed field k, and let $n = \dim X$. Let $\mathcal{J}_X \subset \mathcal{O}_X$ be the Jacobian ideal sheaf of X. In a local affine chart this ideal is defined as follows:

Restrict X to an affine chart, and embed it in some $\mathbf{A}_k^d$, so that it is defined by a set of equations

$$f_1(u_1, \dots, u_d) = \cdots = f_r(u_1, \dots, u_d) = 0.$$

Then $\mathcal{J}_X$ is locally defined, in this chart, by the $d-n$ minors of the Jacobian matrix $(\partial f_j/\partial u_i)$. Let $S \subset X$ be subscheme defined by $\mathcal{J}_X$. Note that S is supported exactly over the singular locus of X.

We decompose

$$X_\infty \setminus S_\infty = \bigsqcup_{e=0}^{\infty} X_\infty^e, \quad \text{where} \quad X_\infty^e := \{\gamma \in X_\infty \mid \mathrm{ord}_\gamma(\mathcal{J}_X) = e\},$$

and let $X_{m,\infty} := \psi_m(X_\infty)$ and $X_{m,\infty}^e := \psi_m(X_\infty^e)$, where $\psi_m \colon X_\infty \to X_m$ is the truncation map. Also, let

$$X_\infty^{\le e} := \{\gamma \in X_\infty \mid \mathrm{ord}_\gamma(\mathcal{J}_X) \le e\} \text{ and } X_{m,\infty}^{\le e} := \psi_m(X_\infty^{\le e}).$$

We will need the following geometric lemma on the fibers of the truncation maps. A weaker version of this property was proven by Denef and Loeser in [24, Lemma 4.1]; the sharper stated here is taken from [32, Proposition 4.1].

Lemma 3.5.28 ([24, 32, 60]) *For $m \ge e$, the morphism $X_{m+1,\infty}^e \to X_{m,\infty}^e$ is a piecewise trivial fibration with fibers isomorphic to $\mathbf{A}^n$.*

Proposition 3.5.29 ([21, 60]) *For an irreducible component C of a cylinder in X_∞ such that $C \not\subset \mathrm{Sing}(X)_\infty$, then there exists e such that*

$$C_m^{\le e} := \psi_m(C) \cap X_{m,\infty}^{\le e}$$

is a nonempty open subset of $\psi_m(C)$ and the codimension of $C_m^{\le e}$ inside $X_{m,\infty}^{\le e}$ stabilizes for $m \gg e$.

Then we define

$$\mathrm{codim}(C, X_\infty) := \mathrm{codim}(C_m^{\le e}, X_{m,\infty}^{\le e}) \quad \text{for} \quad m \gg e.$$

Remark 3.5.30 The codimension of defined above is not the codimension in the usual sense. Let C be as above and $s = \mathrm{codim}(C, X_\infty)$ the codimension as defined above. Let r be the maximal length of a sequence $C = C_0 \subset C_1 \subset \cdots \subset C_r = X_\infty$ of strictly increasing irreducible closed subsets of X_∞, then we have the inequality

$$r \leq s.$$

The inequality can be seen as follows: from the strictly increasing sequence,

$$C = C_0 \subset C_1 \subset \cdots \subset C_r$$

of irreducible closed subsets of X_∞, we have the sequence

$$\overline{\psi_m(C)} = \overline{\psi_m(C_0)} \subset \overline{\psi_m(C_1)} \subset \cdots \subset \overline{\psi_m(C_r)}$$

for $m \gg 0$, since $C_i = \varprojlim \overline{\psi_m(C_i)}$.

The inequality $s \leq r$ can be a strict inequality, see for instance [59, Example 2.8]. The published version of the paper [21] contains a wrong statement

$$\text{“}s = r\text{”}$$

in Remark 3.3. The corrected remark is contained in the uploaded version arXiv:math/0701867.

Definition 3.5.31 Let E be a prime divisor over X, then the **Mather discrepancy** $\hat{k}_E \in \mathbf{Z}_{\geq 0}$ and the **Jacobian discrepancy** $j_E \in \mathbf{Z}_{\geq 0}$ are defined as follows:

Let $\varphi : Y \to X$ be a proper birational morphism from a normal variety Y such that E appears on Y. Then, there is a canonical $\mathcal{O}_Y$-homomorphism

$$\varphi^*(\wedge^n \Omega_X) \to \wedge^n \Omega_Y = \mathcal{O}_Y(K_Y)$$

on the smooth locus of Y, where n is the dimension of X. Denote the image of the homomorphism above by $Im \subset \mathcal{O}_Y(K_Y)$. Then

$$Im = \mathcal{J}\mathcal{O}_Y(K_Y)$$

for an ideal sheaf $\mathcal{J}$ in a neighborhood of the generic point $\eta \in E$, because $\eta \in Y$ is a smooth point and therefore $\mathcal{O}_Y(K_Y)$ is invertible. Define

$$\hat{k}_E := v_E(\mathcal{J}) \quad \text{and} \quad j_E := v_E(\mathcal{J}_X).$$

We call $\hat{k}_E - j_E$ the **Mather-Jacobian discrepancy** of X at the prime divisor E.

If X is non-singular, then $\wedge^n \Omega_X = \mathcal{O}_X(K_X)$, and $Im = \varphi^* \mathcal{O}_X(K_X)$. Therefore by Definition 3.5.1, we obtain

$$\hat{k}_E = k_E$$

for every prime divisor E over X.

Proposition 3.5.32 *Let E be a prime divisor over X. If X is locally a complete intersection, then*

$$k_E = \widehat{k}_E - j_E.$$

In particular, if X is smooth, then $k_E = \hat{k}_E$.

Proof As X is locally a complete intersection, we have

$$\wedge^n \Omega_X = \mathcal{J}_X \cdot \mathcal{O}_X(K_X).$$

(See for example, Proposition 9.1 in [32]). Therefore, by pulling back of this equality onto a normal Y by the birational morphism $\varphi : Y \to X$ where the exceptional prime divisor E appears, we obtain

$$k_E = \widehat{k}_E - j_E,$$

which yields the required equality. □

There are some researches studying singularities in terms of invariants, say Mather discrepancy or Mather-Jacobian discrepancy, which are involving $\hat{k}_E$ or $\hat{k}_E - j_E$ (see for example [18, 29, 53, 58]). The infimum of these is well described in terms of the space of arcs, and because of that we have "Inversion of Adjunction" for these invariants. However, there have some differences from k_E for general **Q**-Gorenstein variety which we do not step into in this paper. For a variety of locally a complete intersection, by virtue of Proposition 3.5.32, we have a description of infimum of log discrepancies in terms of the space of arcs (see Theorem 3.5.34).

Proposition 3.5.33 ([21, 60]) *Let E be a prime divisor over a variety X defined over an algebraically closed field k of arbitrary characteristic and $q \in \mathbf{N}$, then for the divisorial valuation $q \cdot v_E$ we have*

$$\operatorname{codim}(C_X(q \cdot v_E), X_\infty) = q(\hat{k}_E + 1).$$

By making use of this description, we obtain the interpretation of mld and lct by the space of arcs. In following discussions we will denote the symbol

$$\operatorname{Cont}^{w_1}(\mathfrak{a}_1) \cap \cdots \cap \operatorname{Cont}^{w_s}(\mathfrak{a}_s) \text{ by } \operatorname{Cont}^{w}(\mathfrak{a}).$$

Similarly, denote

$$\operatorname{Cont}^{\geq w_1}(\mathfrak{a}_1) \cap \cdots \cap \operatorname{Cont}^{\geq w_s}(\mathfrak{a}_s) \text{ by } \operatorname{Cont}^{\geq w}(\mathfrak{a}).$$

Here, $w = (w_1, \ldots, w_s)$.

Theorem 3.5.34 ([30, 32, 60]) *Let k be an algebraically closed field of arbitrary characteristic. Let X be a normal and locally complete intersection variety defined* *over k and* $\mathfrak{a}^e = \mathfrak{a}_1^{e_1} \cdots \mathfrak{a}_s^{e_s}$ *a multi-ideal with real exponents* $e = (e_1, \dots, e_s)$. *For a pair* $(X, \mathfrak{a}^e)$ *the* mld *is described in terms of the arc space as follows:*

$$\begin{aligned}&\mathrm{mld}(x; X, \mathfrak{a}^e)\\ &= \inf_{v, w_i \in \mathbf{Z}_{\geq 0}} \left\{\mathrm{codim}\left(\mathrm{Cont}^{w}(\mathfrak{a}) \cap \mathrm{Cont}^{v}(\mathcal{J}_X) \cap \pi^{-1}(x), X_\infty\right) - v - \sum_i e_i w_i\right\}.\end{aligned}$$

$$= \inf_{v, w_i \in \mathbf{Z}_{\geq 0}} \left\{\mathrm{codim}\left(\mathrm{Cont}^{\geq w}(\mathfrak{a}) \cap \mathrm{Cont}^{\geq v}(\mathcal{J}_X) \cap \pi^{-1}(x), X_\infty\right) - v - \sum_i e_i w_i\right\}.$$

In particular, if X is smooth, then we have the following:

$$\mathrm{mld}(x; X, \mathfrak{a}^e) = \inf_{v, w_i \in \mathbf{Z}_{\geq 0}} \left\{\mathrm{codim}\left(\mathrm{Cont}^{w}(\mathfrak{a}) \cap \pi^{-1}(x), X_\infty\right) - \sum_i e_i w_i\right\}$$

$$= \inf_{v, w_i \in \mathbf{Z}_{\geq 0}} \left\{\mathrm{codim}\left(\mathrm{Cont}^{\geq w}(\mathfrak{a}) \cap \pi^{-1}(x), X_\infty\right) - \sum_i e_i w_i\right\}.$$

We have the same expression of $\mathrm{mld}(W; X, \mathfrak{a}^e)$ *for a proper closed subset* $W \subset X$ *with replacing* $\pi^{-1}(x)$ *by* $\pi^{-1}(W)$ *in the right hand sides of the equalities above.*

Theorem 3.5.35 ([78, 100]) *Let X be a smooth variety defined over k and* $\mathfrak{a}^e$ *a multi-ideal on X with real exponents e. For a point* $x \in X$ *and a cylinder* $C \subset X_\infty$, *we define*

$$\begin{aligned}&\mathrm{codim}_x(C, X_\infty)\\ &\quad := \min\{\mathrm{codim}\, T \mid T : \textit{irreducible component of } C \textit{ with } x \in \overline{\pi(T)}\}.\end{aligned}$$

For a pair $(X, \mathfrak{a}^e)$ *the* lct *is described in terms of the arc space as follows:*

$$\mathrm{lct}_x(X, \mathfrak{a}^e) = \inf_{w \in \mathbf{Z}^s_{\geq 0}} \left\{\frac{\mathrm{codim}_x(\mathrm{Cont}^{w}(\mathfrak{a}), X_\infty)}{\sum e_i w_i}\right\}.$$

$$= \inf_{w \in \mathbf{Z}^s_{\geq 0}} \left\{\frac{\mathrm{codim}_x(\mathrm{Cont}^{\geq w}(\mathfrak{a}), X_\infty)}{\sum e_i w_i}\right\}.$$

Proof These formulae are essentially proved in [78] in characteristic 0 and in [100] in positive characteristic. However in these papers it is formulated under the condition that $\mathfrak{a}$ is a single ideal and $e = 1$ and we do not find the proof for this general form in any references. So we write down the proof here. It will also suggest the proof of Theorem 3.5.34.

For the first equality in the statement, it is sufficient to show the following equality:

$$\inf\left\{\frac{k_E+1}{\sum_i e_i\cdot v_E(\mathfrak{a}_i)}\,\middle|\,\begin{array}{l}E:\text{prime divisor}\\ \text{over }X\text{with center }x\end{array}\right\}=\inf_{w\in\mathbf{Z}^s_{\geq 0}}\left\{\frac{\operatorname{codim}_x(\operatorname{Cont}^w(\mathfrak{a}),X_\infty)}{\sum e_i w_i}\right\}$$

by Remark 3.5.14. First show $\geq$ of the above equality. Take a prime divisor E over X with the center at x and define $w_i := v_E(\mathfrak{a}_i)$ for every i and $w := (w_1,\ldots,w_s)$. Then, we have

$$C_X(v_E)\subset \operatorname{Cont}^w(\mathfrak{a}).$$

As the center of E is x, we obtain $x\in\overline{\pi(C_X(v_E))}$. Therefore it follows that

$$k_E+1=\operatorname{codim}(C_X(v_E),X_\infty)\geq \operatorname{codim}_x(\operatorname{Cont}^w(\mathfrak{a}),X_\infty).$$

This gives the required inequality

$$\frac{k_E+1}{\sum_i e_i\cdot v_E(\mathfrak{a}_i)}\geq\frac{\operatorname{codim}_x(\operatorname{Cont}^w(\mathfrak{a}),X_\infty)}{\sum_i e_i w_i}.$$

For the opposite inequality, take any $w=(w_1,\ldots,w_s)$ and take an irreducible component $T\subset\operatorname{Cont}^w(\mathfrak{a})$ such that $x\in\overline{\pi(T)}$ and $\operatorname{codim}(T,X_\infty)=\operatorname{codim}_x(\operatorname{Cont}^w(\mathfrak{a}),X_\infty)$. The generic point of the cylinder T gives a divisorial valuation $v_T=q\cdot v_E$ for some $q\in\mathbf{N}$ and a prime divisor E. Note that the center of E on X contains x. By the definition of the valuation, we have

$$C_X(q\cdot v_E)\supset T,\quad\text{which yields}$$

$$\operatorname{codim}(C_X(q\cdot v_E),X_\infty)\leq\operatorname{codim}_x(\operatorname{Cont}^w(\mathfrak{a}),X_\infty).$$

As $q\cdot v_E(\mathfrak{a}_i)=w_i$, we have

$$\frac{k_E+1}{\sum_i e_i\cdot v_E(\mathfrak{a}_i)}=\frac{q(k_E+1)}{\sum_i e_i w_i}\leq\frac{\operatorname{codim}_x(\operatorname{Cont}^w(\mathfrak{a}),X_\infty)}{\sum_i e_i w_i},$$

as required.

About the second equality in the statement, the inequality $\geq$ is obvious, since $\operatorname{Cont}^w(\mathfrak{a})\subset\operatorname{Cont}^{\geq w}(\mathfrak{a})$. For the opposite inequality, it is sufficient to show that for every $w=(w_1,\ldots,w_s)\in\mathbf{Z}^s_{\geq 0}$ there exists $w'=(w'_1,\ldots,w'_s)\in\mathbf{Z}^s_{\geq 0}$ such that

$$\frac{\operatorname{codim}_x(\operatorname{Cont}^{w'}(\mathfrak{a}),X_\infty)}{\sum_i e_i w'_i}\leq\frac{\operatorname{codim}_x(\operatorname{Cont}^{\geq w}(\mathfrak{a}),X_\infty)}{\sum e_i w_i}.$$

To show this, take an irreducible component $T \subset \mathrm{Cont}^{\geq w}(\mathfrak{a})$ such that $x \in \overline{\pi(T)}$ and $\mathrm{codim}(T, X_\infty) = \mathrm{codim}_x(\mathrm{Cont}^{\geq w}(\mathfrak{a}), X_\infty)$. Let v_T be the divisorial valuation defined by the generic point of T. Then $w'_i := v_T(\mathfrak{a}_i) \geq w_i$ and $T \subset \mathrm{Cont}^{w'}(\mathfrak{a})$. Hence, we obtain

$$\mathrm{codim}_x(\mathrm{Cont}^{w'}(\mathfrak{a}), X_\infty) \leq \mathrm{codim}(T, X_\infty) = \mathrm{codim}_x(\mathrm{Cont}^{\geq w}(\mathfrak{a}), X_\infty)$$

and $\sum_i e_i w'_i \geq \sum_i e_i w_i$, which yield the required inequality. □

The following shows the relation of the mld between smooth variety A and a closed subscheme X on A. It is called "Inversion of Adjunction".

Theorem 3.5.36 ([30, 32, 60]) *Let k be an algebraically closed field of arbitrary characteristic. Let A be a smooth variety over k and $X \subset A$ a closed subscheme of locally complete intersections with codimension c. Let $\widetilde{\mathfrak{a}}^e = \widetilde{\mathfrak{a}}_1^{e_1} \cdots \widetilde{\mathfrak{a}}_s^{e_s}$ be a multi-ideal on A with exponents in $\mathbf{R}_{\geq 0}$ such that $\mathfrak{a}_i := \widetilde{\mathfrak{a}}_i \mathcal{O}_X \neq 0$ for every i. Let I_X be the defining ideal of X in A. Then for a point $x \in X$ the following equality holds:*

$$\mathrm{mld}(x; X, \mathfrak{a}^e) = \mathrm{mld}(x; A, \widetilde{\mathfrak{a}}^e \cdot I_X^c).$$

For a proper closed subset $W \subset X$ the following holds:

$$\mathrm{mld}(W; X, \mathfrak{a}^e) = \mathrm{mld}(W; A, \widetilde{\mathfrak{a}}^e \cdot I_X^c).$$

Corollary 3.5.37 *Let k be an algebraically closed field of arbitrary characteristic. Let X be a normal variety of locally complete intersections with dimension d. Let $x \in X$ be a point and $W \subset X$ a proper closed subset. Then, we have the equalities:*

$$\mathrm{mld}(x; X, \mathcal{O}_X) = \inf_m\{(m+1)d - \dim \pi_m^{-1}(x)\},$$

$$\mathrm{mld}(W; X, \mathcal{O}_X) = \inf_m\{(m+1)d - \dim \pi_m^{-1}(W)\},$$

The following corollaries are proved in [30, 77] for the base field of characteristic 0 in different ways from the following proof. The proof below is based on the expression in Corollary 3.5.37 and it works for the base filed of arbitrary characteristic. In [61, Corollary 10.2.9] one can find more general statements and the proofs for them.

Corollary 3.5.38 *Let X be a normal local complete intersection variety defined over algebraically closed field k of arbitrary characteristic. Then the following hold:*

(i) X has log canonical singularities if and only if X_m is locally a complete intersection for every $m \in \mathbf{N}$,
(ii) X has canonical singularities if and only if X_m is irreducible for every $m \in \mathbf{N}$,
(iii) X has terminal singularities if and only if X_m is normal for every $m \in \mathbf{N}$.

Proof Let $d = \dim X$. As X is locally a complete intersection, X is locally defined by $c := N - d$ equations in a non-singular variety A of dimension N. Then, X_m is locally defined by $(m+1)c$ equations in a non-singular variety A_m of dimension $(m+1)N$ (cf. the construction of X_m). Therefore, we have

$$(3.5.38(i)) \qquad \dim X_m \geq (m+1)N - (m+1)c = (m+1)d,$$

where the equality holds if and only if X_m is locally a complete intersection.

First we show the equivalence in (i). We know that the restriction

$$\pi_m{}^{-1}(X_{\mathrm{reg}}) \to X_{\mathrm{reg}}$$

of π_m is a smooth morphism of relative dimension md. Therefore, by the formula in Corollary 3.5.37, X has log canonical singularities if and only if for every $m \in \mathbf{N}$, the following inequality holds:

$$(m+1)d - \dim X_m(W) \geq 0,$$

where W is the singular locus X_{sing} of X. This is equivalent to the equality in (3.5.38 (i)).

For the both implications of (ii), we may assume that X_m is locally a complete intersection of dimension $(m+1)d$ by the result (i). Actually, if we assume that X has canonical singularities, then by (i) we obtain that X_m is locally a complete intersection for every $m \in \mathbf{N}$. If we assume that X_m is irreducible, then it has dimension $(m+1)d$, because it contains an open dense subset $\pi_m^{-1}(X_{reg})$ which has dimension $(m+1)d$. As X_m is locally defined by $(m+1)(N-d)$ equations in a smooth variety A_m of dimension $(m+1)N$, the subscheme X_m is locally a complete intersection.

Now, again by the formula in Corollary 3.5.37, X has canonical singularities if and only if for every $m \in \mathbf{N}$ and the singular locus $W \subset X$, the following inequality holds:

$$(m+1)d - \dim X_m(W) \geq 1,$$

which yields $\dim X_m(W) < (m+1)d$. This is equivalent to the fact that none of the irreducible components of $X_m(W)$ can be an irreducible component of X_m, since X_m is of pure dimension $(m+1)d$. This holds if and only if X_m is irreducible for every $m \in \mathbf{N}$.

For the proof of (iii), we may assume that X_m is irreducible and locally a complete intersection of dimension $(m+1)d$ by the same reason as in the proof of (ii). We know that a local complete intersection variety is Gorenstein, in particular, it satisfies Serre's condition S_2. Thus X_m has the property S_2. Now, X has terminal singularities if and only if for every $m \in \mathbf{N}$ and the singular locus $W \subset X$, the following inequality holds:

$$(m+1)d - \dim X_m(W) \geq 2,$$

which yields $\dim X_m(W) \leq (m+1)d - 2$. Here we note that the singular locus of X_m is just $X_m(W)$. Indeed, it is obvious that the singular locus of X_m is contained in $X_m(W)$, as the compliment ${\pi_m}^{-1}(X_{\text{reg}})$ of $X_m(W)$ is non-singular. To show the opposite inclusion, denote the local Jacobian matrix of the embedding $X_m \subset A_m$ by J and the Jacobian matrix of the embedding $X \subset A$ by J_0. Then J has the following form:

$$J = \begin{pmatrix} J_0 & O \\ * & * \end{pmatrix}.$$

As we may assume that $X_m \subset A_m$ is a complete intersection, X_m is non-singular at a point p if and only if the Jacobian matrix J has full rank at p. Here, if $p \in X_m(W)$, then J_0 does not have full rank, therefore J cannot have full rank.

Hence, the inequality $\dim X_m(W) \leq (m+1)d - 2$ is equivalent to the fact that X_m is normal by the Serre's criteria for normality. □

The LSC Conjecture holds for a normal local complete intersection variety. It is proved in [31] for characteristic 0 by making use of Inversion of Adjunction (Theorem 3.5.36) and the description of mld in terms of the arc space (Theorem 3.5.34).

Theorem 3.5.39 ([31]) *Let X is be a normal, local complete intersection variety over an algebraically closed field k of arbitrary characteristic. Let $\mathfrak{a}^e$ be a multi ideal on X. Then the function $x \mapsto \operatorname{mld}(x; X, \mathfrak{a}^e)$, $x \in X$, is lower semicontinuous.*

Remark 3.5.40 By these theorems we can see the equivalence of a geometric property of X and a somehow weaker geometric property of X_m. So it is natural to ask for a condition on X_m such that it forces X to be smooth. One candidate for such a mild condition is that X_m has at worst rational singularities for every $m \in \mathbf{N} \cup \{\infty\}$ (By Proposition 3.3.16 we know that the existence of m such that X_m is smooth implies the smoothness of X, but we require a weaker condition for X_m.)

The following is a negative answer to the expectation:

Example 3.5.41 ([57]) Let k be a field of characteristic 0. Let X be a hypersurface in $\mathbf{A}_k^N$ defined by the polynomial $f = x_1^d + x_2^d + \cdots + x_N^d$. If $d > 1$, then it is clear that $(X, 0)$ is not smooth, and if $d^2 < N$, then the jet scheme X_m has at worst rational singularities for every $m \in \mathbf{N}$.

The study of singularities by making use of the space of arcs is still developing. The author hopes to write a paper including the new results in future.

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Chapter 4
Jet Schemes and Their Applications in Singularities, Toric Resolutions and Integer Partitions

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Dedicated to Monique Lejeune-Jalabert

Contents

Abstract After a brief introduction to jet schemes, this article surveys their applications in singularity theory (Nash problem, motivic integration, birational geometry, jet components graph, equisingularity, local algebras of arc spaces), in the search for toric resolution of singularities (Teissier's conjecture) and in the theory of integer partitions.

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4.1 Introduction

There exist in the literature several surveys of various aspects of jet schemes and arc spaces. This paper is meant on one hand to show the diversity of these aspects and guide the reader through it and on the other hand to survey some other aspects which do not appear in the existing surveys. First, let us say what intuitively are the jet schemes and the arc space of a variety X defined over a field $\mathbf{K}$. The arc space of X is the scheme (or the infinite dimensional variety) X_∞ which parametrizes the germs of formal curves (arcs) traced on X; *i.e.,*, a point on X_∞ corresponds to an arc traced on X. The jet schemes are finite dimensional approximations of the arc space: If we consider X embedded in an affine space, $X \subset \mathbf{A}^n$, for $m \in \mathbb{N}$, the m-th jet scheme X_m can be thought (modulo a trivial fibration) as the space of arcs in the ambient space $\mathbf{A}^n$ which have "contact" with X larger than m.

The arc space and the jet schemes of X are rather complicated compared to X : the arc space is in general infinite dimensional; the jet schemes have in general many irreducible components of different dimensions; they are in general not reduced ... but one can formulate the guiding philosophy of this article as follows: arc spaces and jet schemes can transform a difficult problem concerning a relatively simple object into a relatively simple problem concerning a difficult object.

Maybe one of the first uses of arc spaces in singularity theory goes back to Nash and is "subsequent" (1968) to the proof of existence of resolution of singularities by Hironaka. One can realize easily, that if there exists a resolution of singularities $\mu : Z \longrightarrow X$, then X admits infinitely many other resolutions, an infinite family of them is obtained by blowing up Z along regular loci. Nash wanted to codify the data which is common to all these resolutions of singularities. He suggested that this data is hidden in the arc space ; a precise form of this suggestion is what is nowadays known as the Nash problem, but also the generalized Nash problem, or the embedded Nash problem which are generalizations of the first problem. In Sect. 4.3, we will briefly discuss these problems which have made fantastic progress in the last decade.

Another momentum was the introduction of the (geometric) motivic integration (Kontsevich, Denef-Loeser) in analogy with $p-$adic integration. The arc space of a variety is the measured space in this theory; the name motivic is related to the value that takes a motivic integral, which is an element in the "Grothendieck ring" (*i.e.,* a class of a geometric object) and not a real number. This theory led to the introduction, again sometimes in analogy with $p-$adic integration but not only, of several new (motivic) invariants of singularities; this also led to another "stronger version" of the monodromy conjecture. This will be mentioned in Sect. 4.4.

The development of the geometric tools needed for motivic integration (which in particular allowed to prove the change of variables formula) led to a very effective use of jet schemes and arc spaces in the minimal model program and in "hunting" invariants of singularities of pairs (Mustaţă, Ein, Yasuda,...). This will be highlighted in Sect. 4.5.

Section 4.6 is dedicated to a weighted graph (the jet components graph) which was introduced by the author and which encodes the geometry of the jet schemes of the singularity and their truncation maps (these maps will be introduced in Sect. 4.2). We will mention some results about the structure of this graph (and hence the structure of the jet schemes) for several classes of singularities. For instance the data of this graph for irreducible plane branches, for two dimensional quasi-ordinary hypersurfaces (with H. Cobo), or for normal toric singularities is equivalent to the data of the embedded topological type of these singularities. This graph is actually determined by the irreducible components of the jet schemes, their dimensions and embedding dimensions and their behaviour with respect to the truncation maps; in particular this graph is determined by basic invariants of the jet schemes, but we can extract from it a complete invariant of the embedded topological type for these classes of singularities; the topological type is a very fine invariant of singularities. This reflects the philosophy that we mentioned above.

Section 4.7 describes how jet schemes intervene in an approach of the author to the problem of construction of embedded resolutions of singularities. This approach can be thought as a reverse Nash problem and is at the same time an approach to Teissier's conjecture on resolution of singularities with toric morphisms.

Section 4.8 concerns an equisingularity theory (Leyton-Alvarez) which is based on deformations of jet schemes, and comparisons with other equisingularity theories. This problem generalizes the study of the jet schemes of irreducible plane curves in an equisingular family which was considered by the author.

Section 4.9 describes a link (Bruschek, Mourtada, Schepers) between some aspects of classical number theory, namely the study of integer partitions and singularity theory via arc spaces. The main object of this link (the Arc Hilbert-Poincaré series) makes use of the cone structure of the arc space.

Section 4.10 concerns the structure of the localization of the algebra of arcs at two types of points (arcs): On one hand rational points (Drinfeld, Grinberg, Kazhdan) and the invariants of singularities which can be extracted of this structure (Bourqui, Sebag) and on the other hand stable points or points associated with divisorial valuations (Reguera, Mourtada-Reguera).

4.2 The Construction of Jet Schemes

Let $\mathbf{K}$ be an algebraically closed field and X an algebraic variety defined over $\mathbf{K}$. For $m \in \mathbb{N}$, the $m-$jet scheme of X is the $\mathbf{K}$-scheme X_m representing the functor

$$F_m : \text{Schemes} \longrightarrow \text{Sets}$$

which with an affine $\mathbf{K}$-scheme Spec$A$ associates the set

$$\text{Hom}_{\mathbf{K}}(\text{Spec}A[t]/(t^{m+1}), X).$$

For $m \geq p$, the natural projection $A[t]/(t^{m+1}) \longrightarrow A[t]/(t^{p+1})$ induces the truncation affine morphism $\pi_{m,p} : X_m \longrightarrow X_p$. This gives a projective system $(X_m)_{m\geq 0}$ whose limit is by definition the space of arcs

$$X_\infty := \varprojlim X_m.$$

It follows from corollary 2 in [17, 136] that X_∞ is the scheme which represents the functor F_∞ : Schemes $\longrightarrow$ Sets which with an affine $\mathbf{K}$-scheme Spec$A$ associates the set $\mathrm{Hom}_{\mathbf{K}}(\mathrm{Spec}A[[t]], X)$. In the case of an affine variety

$$X = \mathrm{Spec}\frac{\mathbf{K}[x_1, \ldots, x_n]}{(f_1, \ldots, f_r)}, \tag{4.1}$$

the jet schemes X_m and the arc space are affine varieties. Indeed, for A a $\mathbf{K}$-algebra, the data of an $A-$point of X_∞ is equivalent to the data of a $\mathbf{K}$-algebra morphism

$$\phi : \frac{\mathbf{K}[x_1, \ldots, x_n]}{(f_1, \ldots, f_r)} \longrightarrow A[[t]].$$

The morphism ϕ is completely determined by the images of $x_i, i = 1 \ldots, n$

$$x_i \longmapsto \phi(x_i) = x_i^{(0)} + x_i^{(1)} t + \cdots ; \tag{4.2}$$

these images should satisfy $f_l(\phi(x_1), \ldots, \phi(x_n)) = 0, l = 1, \ldots, r$.

If we write

$$f_l(\phi(x_1), \ldots, \phi(x_n)) = \sum_{j\geq 0} F_l^{(j)}(\mathbf{x}^{(0)}, \ldots, \mathbf{x}^{(j)})\, t^j \tag{4.3}$$

where $\mathbf{x}^{(j)} = (x_1^{(j)}, \ldots, x_n^{(j)})$. Then the data of ϕ is equivalent to giving values to $x_i^{(j)}$ in A, for $i = 1, \ldots, n; j \in \mathbb{N}$; these values should satisfy $F_l^{(j)}(\mathbf{x}^{(0)}, \ldots, \mathbf{x}^{(j)}) = 0$. This is equivalent to the data of an $A-$point in

$$X_\infty = \mathrm{Spec}\frac{\mathbf{K}[\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \ldots]}{(F_l^{(j)})_{l=1,\ldots,r}^{j\in\mathbb{N}}}.$$

Similarly, we have

$$X_m = \mathrm{Spec}\frac{\mathbf{K}[\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}]}{(F_l^{(j)})_{l=1,\ldots,r}^{j=0,\ldots,m}}.$$

Notice that by definition $X_0 = X$. We denote by π_m the truncation morphism $\pi_{m,0}$ and by Ψ_m the morphism from X_∞ to X_m induced by the fact that the arc space is the projective limit of the jet schemes. When there is an ambiguity about the variety X whose jet schemes or arc space are considered, these maps are denoted by $\pi^X_{m,p}, \pi^X_m, \Psi^X_m$.

In the case where $X = \mathbf{A}^n$ is an affine space, X_m is the affine space $X_m := \mathrm{Spec}\mathbf{K}[\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}] = \mathbf{A}^{n(m+1)}$. The truncation morphism $\pi_{m,p}$ is the map which forgets the last $n(m-p)$ coordinates $\mathbf{x}^{(p+1)}, \ldots, \mathbf{x}^{(m)}$; it is then the trivial fibration whose fiber is $\mathbf{A}^{n(m-p)}$.

The geometry of the jet schemes and the arc space of X when X is smooth is quite similar locally to the case of the affine space. This follows from the good behavior of jet schemes and arc spaces with respect to étale morphisms. One can also have a feeling of this on the level of the fibers of Ψ_0; indeed let $x \in X$ be a smooth point; let $\mathcal{O}_{X,x}$ be the local ring of X at x; a $\mathbf{K}$-arc $\gamma \in X_\infty$ centered at x, corresponds to a morphism of local rings $\gamma^* : \mathcal{O}_{X,x} \longrightarrow \mathbf{K}[[t]]$; since $\mathbf{K}[[t]]$ is complete (with respect to the $t-$adic topology), by the universal property of completeness γ^* factors through the completion $\mathcal{O}_{X,x} \longrightarrow \hat{\mathcal{O}}_{X,x}$ (with respect to the maximal ideal of $\mathcal{O}_{X,x}$). Since x is a smooth point, by Cohen structure theorem, $\hat{\mathcal{O}}_{X,x} \simeq \mathbf{K}[[x_1, \ldots, x_n]]$, n being the dimension of X at x. So the data of γ is equivalent to the data of a local morphism $\mathbf{K}[[x_1, \ldots, x_n]] \longrightarrow \mathbf{K}[[t]]$; we deduce that $(\Psi_0^X)^{-1}(x)$ is isomorphic to $(\Psi_0^{\mathbf{A}^n})^{-1}(O)$ where O is any closed point of $\mathbf{A}^n$.

The algebra of global functions on the arc space of an affine variety has a structure of a differential ring. We assume here for simplicity that the characteristic of the field $\mathbf{K}$ is zero. The algebra of global functions on $\mathbf{A}^d_\infty$ is

$$\mathcal{R}_\infty = \mathbf{K}[\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \ldots].$$

We have a derivation D on $\mathcal{R}_\infty$ defined by $D(x_i^{(j)}) = x_i^{(j+1)}$, for $i = 1, \ldots, n$; $j \in \mathbb{N}$. Assume that X is an affine variety (as in (4.1)); if we replace in the Eq. (4.2) the variables $x_i^{(j)}$ by $x_i^{(j)}/j!$ (where $j!$ is the factorial of j), we find

$$f_l(\phi(x_1), \ldots, \phi(x_n)) = \sum_{j \geq 0} \frac{\mathcal{F}_l^{(j)}(\mathbf{x}^{(0)}, \ldots, \mathbf{x}^{(j)})}{j!} t^j, \tag{4.4}$$

where $\mathcal{F}_l^{(0)} = f_l$ and $\mathcal{F}_l^{(j)}$ is recursively defined by the identity $D(\mathcal{F}_l^{(j)}) = \mathcal{F}_l^{(j+1)}$; Eq. (4.4) follows from the fact that both sides are additive and multiplicative in f_l and that this equality is obviously true for x_i. We obtain hence the desired differential structure which is induced by the derivation D on the algebra of global functions on X_∞

The differential structure is very useful to encode many geometric features of the space of arcs, for instance Kolchin's theorem which states that if X is irreducible, then X_∞ is also irreducible ; see also [110, 120] for variants of this theorem.

Building on the discussion above, we can give an explicit presentation of the arc space of the cusp singularity on the curve $X = \{x_1^2 - x_0^3 = 0\} \subset \mathbf{A}^2$. The arc space X_∞ is isomorphic to the space (scheme) whose embedding in the infinite dimensional affine space $\mathbf{A}^2_\infty = \mathrm{Spec}\mathbf{K}[\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \ldots]$ is given by the ideal

$$(x_1^{(0)^2} - x_0^{(0)^3}, 2x_1^{(0)}x_1^{(1)} - 3x_0^{(0)^2}x_0^{(1)}, \ldots).$$

4.3 The Nash Problem and Its Variants

This subject has been the subject of several surveys [43, 54, 68, 84, 115]. As mentioned in the introduction, the Nash problem seeks to detect in the arc space information common to all resolution of of singularities of a given variety. For this section, we assume for simplicity that $\mathbf{K}$ is an algebraically closed field of characteristic zero. Let X be a singular variety and let $\mu : Y \longrightarrow X$ be a divisorial resolution of singularities of X (divisorial means that the exceptional locus of μ is a divisor). Let

$$E := \mu^{-1}(\mathrm{Sing}(X)) = \cup_{i=1}^r E_i$$

be the decomposition of the exceptional locus of μ into irreducible components. Every E_i defines a divisorial valuation whose center $c_X(E_i)$ on X is included in $\mathrm{Sing}(X)$; the corresponding valuation ν_{E_i} associates with a function $h \in \mathcal{O}_{X,c_X(E_i)}$ (the local ring of X at $c_X(E_i)$) the order of annihilation of $h \circ \mu$ along E_i. Note that the center is characterized by the fact that the valuation of an element in the maximal ideal of $\mathcal{O}_{X,c_X(E_i)}$ is strictly positive and that the valuation of an element which is not in this maximal ideal is 0. Note that the center $c_Y(E_i)$ of ν_{E_i} on Y is E_i.

Definition 4.3.1 The divisor E_i is said to be an **essential** divisor if for any other resolution of singularities $\mu' : Y' \longrightarrow X$ (non-necessarily divisorial), we have that $c_{Y'}(E_i)$ is an irreducible component of the exceptional locus of μ'.

Note that ν_{E_i} has a center on every Y' as in the above definition because μ' is proper. In general, it is a difficult task to determine whether a divisor (or a divisorial valuation) is essential or not; for surface singularities, essential divisors are exactly those which are defined by the irreducible components of the exceptional locus of the minimal resolution of singularities.

The morphism μ induces a morphism $\mu_\infty : Y_\infty \longrightarrow X_\infty$; indeed seeing an arc $\gamma \in Y_\infty$ as a morphism $\gamma : \mathrm{Spec}\kappa_\gamma[[t]] \longrightarrow Y$ (where κ_γ is the residue field of X_∞ at γ), $\mu_\infty(\gamma)$ is the arc $\mu \circ \gamma : \mathrm{Spec}\kappa_\gamma[[t]] \longrightarrow X$ which belongs to X_∞. By the valuative criterion of properness we know that any arc γ on $(\Psi_0^X)^{-1}(\mathrm{Sing}(X)) \setminus \mathrm{Sing}(X)_\infty$ lifts to Y_∞, more precisely to $(\Psi_0^Y)^{-1}(E)$. Using generic smoothness of μ on the irreducible components of $\mathrm{Sing}(X)$, we can show that by restricting μ we have a dominant morphism $(\Psi_0^Y)^{-1}(E) \longrightarrow (\Psi_0^X)^{-1}(\mathrm{Sing}(X))$. We have that

$$(\Psi_0^Y)^{-1}(E) = \cup_{i=1}^r (\Psi_0^Y)^{-1}(E_i)$$

is the decomposition into irreducible components: indeed, $(\Psi_0^Y)^{-1}(E_i)$ is irreducible for every i (because Y is smooth and E_i is irreducible) and there cannot be inclusions since the E_i's are distinct, in particular the sets of constant arcs on every E_i are different. Now let $N_i = \overline{\mu_\infty((\Psi_0^Y)^{-1}(E_i))}$, where the overline indicates the Zariski closure. It follows from the discussion above that the we have the following decomposition into irreducible components

$$(\Psi_0^X)^{-1}(\mathrm{Sing}(X)) = \cup_{i \in J} N_i,$$

where $J \subset \{1, \ldots, r\}$. Moreover, if E_i is not essential, then N_i cannot be an irreducible component; this can be seen by considering another resolution $\mu' : Y \longrightarrow X$ such that the center of E_i on Y' is not an irreducible component of the exceptional locus of μ'. We conclude that $(\Psi_0^X)^{-1}(\mathrm{Sing}(X))$ has finite number of irreducible components and that we have an injection, the **Nash map**

$$\{\text{Irreducible components of } (\Psi_0^X)^{-1}(\mathrm{Sing}(X))\} \longrightarrow \{\text{Essential divisors of } X\}.$$

Nash asked (this is the Nash problem) whether the Nash map is bijective. It has been proved that this map is bijective for toric varities [73], quasi-ordinary singularities [59, 70], for surface singularities [56] and very recently for **T**-varieties of complexity one whose rational quotient is a curve of positive genus [19]; see also ([48, 87, 114] for other classes). An essential idea in attacking Nash problem is the wedge problem which was introduced by Lejeune-Jalabert [82]; an essential tool of this latter is the curve selection lemma proved by Reguera [119].

Example 4.3.2 Let $X = \{x_2^3 - x_1x_3 = 0\} \subset \mathbf{A}^3$ be the A_2 singularity. The singular locus of X is the origin $(0, 0, 0)$. We blow up the origin in $\mathbf{A}^3$ and consider the chart where the blow up morphism is given by $x_1 = uv, x_2 = v, x_3 = vw$; all the information is seen in this chart. The total transform of X is then given in this chart (which is isomorphic to $\mathbf{A}^3$ provided with the coordinates (u, v, w)) by

$$x_2^3 - x_1x_3 = v^2(v - uw) = 0.$$

The restriction of the blowup to the strict transform $\{v - uw = 0\}$ gives the minimal resolution of X; the exceptional locus of this minimal resolution has two irreducible components obtained by intersecting the exceptional divisor $\{v = 0\}$ with $\{v - uw = 0\}$: these are the curves $E_1 = \{u = v = 0\}$ and $E_2 = \{w = v = 0\}$ and both are essential (being the divisor on the minimal resolution of singularities). A direct computation gives

$$N_1 = \{x_1^{(0)} = x_1^{(1)} = x_2^{(0)} = x_3^{(0)} = 0\} \subset X_\infty$$

and

$$N_2 = \{x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = x_3^{(0)} = 0\} \subset X_\infty.$$

Since we can see from their defining equations that there are no inclusions between N_1 and N_2, we have the decomposition into irreducible components:

$$\Psi^{-1}(0) = N_1 \cup N_2;$$

hence, the Nash map for this singularity is bijective.

In general the Nash map is not bijective [42, 73, 75]. It remains a difficult problem to understand when this map is bijective or not and to understand its image in the non-surjective case: it follows for instance from [44] that terminal "divisors" belong to the image of the Nash map; Till recently, the only known Nash valuations (*i.e.,* valuations belonging to the image of the Nash map) were either minimal (minimality with respect to the order where a valuation is smaller than another if its action on every function is smaller than the action of the other valuation) or terminal and questions were made if this is always the case. Recently, counter examples to this statement were given which implies that the determination of Nash valuations is still a wide open problem [19].

Another related problem, the **generalized Nash problem** extends the above problem [71]. Given a variety X (not necessarily singular), and two irreducible divisors $E_1 \subset Y_1$ and $E_2 \subset Y_2$ where for $i = 1, 2$, we have a birational morphism $\mu_i : Y_i \longrightarrow X$, and where Y_1 and Y_2 are smooth. The problem is to to determine when do we have an inclusion $N_1 \subset N_2$? here, as above for $i = 1, 2$, $N_i := \mu_{i\infty}((\Psi_0^{Y_i})^{-1}(E_i))$. This problem is wide open even in the case where $X = \mathbf{A}^2$, [55]. An equivariant version of this problem was solved for toric varieties [69] and more recently for **T**-varieties of complexity one whose rational quotient is a curve of positive genus [19].

I should mention finally for this section, the embedded Nash problem which roughly speaking is about understanding the relation between the irreducible components of the jet schemes or contact loci and the divisors which are in some sense essential for every embedded resolution of singularities [34, 77, 99, 102].

4.4 Motivic Invariants of Singularities

Again here **K** is considered to be of characteristic zero and the varieties are defined over **K**. Motivic integration started with the proof by Kontsevich that two birationally equivalent complex projective "Calabi-Yau" varieties have the same Hodge numbers. This extends a theorem by Batyrev stating that two such varieties have the same betti numbers. The proof of Batyrev uses $p-$adic integration; by analogy with $p-$adic integration, Kontsevich introduced motivic integration (for non-singular varieties) and used an approach similar to Batyrev's proof to prove his more general result. Motivic integration was generalized to singular varieties by Denef and Loeser [24–26]. There are several excellent introductions to motivic integration [18, 26, 29, 36, 93, 135]. We mention below very little about this subject.

The function that we will be integrating are defined on constructible subset of the arc space X_∞ of a variety X (their source domain). Before saying which kind of functions we will be measuring, let us mention the Grothendieck ring $\mathscr{M}$ (actually a completion of localization of this ring) where these functions and integers will take value.

The Grothendieck ring $\mathscr{M}$ is defined by:

- The generators of $\mathscr{M}$ as a group are the classes of isomorphisms $[V]$, V being a variety over $\mathbf{K}$.
- The relations are given by: for $Y \subset Z$ a closed subvariety, $[Z \setminus Y] + [Y] = [Z]$.
- The product is defined by: for two varieties Y, Z, $[Y].[Z] = [Y \times Z]$.

The symbol $[\cdot]$ is an additive invariant, and is actually a universal additive invariant in a sense that for any other additive invariant χ (like the Euler characteristic or the Hodge polynomial) of varieties over $\mathbf{K}$ and for any two varieties Y, Z, $[Y] = [Z]$ implies $\chi(Y) = \chi(Z)$. Recall here that an invariant $\chi : Var_{\mathbb{C}} \longrightarrow A$ (where A is an abelian group and $Var_{\mathbb{C}}$ is the category of varieties over $\mathbb{C}$) is said to be additive if for X, Y varieties over $\mathbb{C}$, $X \simeq Y$ implies $\chi(X) = \chi(Y)$; and for a closed subvariety $Z \subset X$, $\chi(X) = \chi(X \setminus Z) + \chi(Z)$.

We denote $\mathbf{L}$ the class $[\mathbf{A}^1]$ of $\mathbf{A}^1$. Hence we $[\mathbf{A}^n] = \mathbf{L}^n$; by the definition of the product we know that the class of a point $[*]$ is equal to 1, the neutral element for the product.

Example 4.4.1

(i) We have that the class of the projective space of dimension 1 is

$$[\mathbf{P}^1] = [\mathbf{P}^1 \setminus *] + [*] = \mathbf{L} + 1.$$

(ii) Let $X = \{y^2 - x^3 = 0\} \subset \mathbf{A}^2$. Since he morphism $\varphi : \mathbf{A}^1 \longrightarrow X$, defined by $\varphi(t) = (t^2, t^3)$ induces an isomorphism $\mathbf{A}^1 \setminus \{0\} \longrightarrow X \setminus \{(0,0)\}$ we have

$$[X] = [X \setminus \{(0,0)\}] + [(0,0)] = \mathbf{L} - 1 + 1 = \mathbf{L}.$$

(iii) For a Zariski locally trivial fibration $P : E \longrightarrow B$ of fiber F, we have $[E] = [B][F]$.

Consider the localization $\mathscr{M}_{loc} := \mathscr{M}[\mathbf{L}^{-1}]$. The values of motivic integrals belong to a completion $\hat{\mathscr{M}}$ of $\mathscr{M}_{loc}$ where $\mathbf{L}^{-n}$ tends to zero when n tends to infinity.

A typical measurable set (*i.e.,* its measure exists, see below the definition of μ) will be a cylinder (or disjoint union of cylinders), *i.e.,* a subset $A \subset X_\infty$ such that there exist $m \in \mathbb{N}$ and a constructible subset $C_m \subset X_m$ verifying $A = \Psi^{-1}(C_m)$, The motivic measure for such A will be defined by

$$\mu(A) = \lim_{n \to \infty} [\Psi_n(A)]\mathbf{L}^{-nd},$$

d being the dimension of X. When X is smooth or when $A \cap Sing(X)_\infty = \emptyset$, the value of $[\Psi_n(A)]\mathbf{L}^{-nd}$ stabilizes for n big enough. When X is smooth, this follows from the fact that for $n \geq m$, the truncation map $\pi_{n,m}$ is a locally trivial fibration of fiber $\mathbf{A}^{d(n-m)}$; hence

$$[\Psi_n(A)]\mathbf{L}^{-nd} = [C]\mathbf{L}^{d(n-m)}\mathbf{L}^{-nd} = [C]\mathbf{L}^{-md} = [\Psi_m(A)]\mathbf{L}^{-md}.$$

The fact that the limit exists in general is a theorem of Denef and Loeser. In general, there are other types of measurable sets, of course disjoint (finite or infinite) union of cylinders as above, but also for instance for a closed subvariety $V \subset X$, we have $\mu(V_\infty) = 0$.

We now can define the motivic integral: Let A be a measurable set and let $\alpha : A \longrightarrow \mathbf{Z} \cup \{+\infty\}$ be a function whose fibers $\alpha^{-1}(n) \subset A$ are measurable for every n. We say that $\mathbf{L}^{-\alpha}$ is integrable if the series

$$\int \mathbf{L}^{-\alpha} d\mu := \sum_{n \in \mathbf{Z}} \mu(\alpha^{-1}(n))\mathbf{L}^{-n}$$

is convergent in $\hat{\mathscr{M}}$.

Example 4.4.2 Let X be a smooth variety of dimension d and let $D \subset X$ be a smooth divisor. Let us compute

$$\int_{X_\infty} \mathbf{L}^{-ord_D} d\mu$$

where $ord_D : X_\infty \longrightarrow \mathbb{Z}$ associates with an arc γ its order of contact with D. So for $n \in \mathbb{Z}_{>0}$, $ord_D^{-1}(n) = (\Psi_{n-1}^X)^{-1}(D_{n-1}) \setminus (\Psi_n^X)^{-1}(D_n)$. Since D is smooth, the truncation morphism $\pi_n^D : D_n \longrightarrow D$ is a locally trivial fibration of fiber $\mathbf{A}^{(d-1)n}$, hence $[D_n] = [D]\mathbf{L}^{(d-1)n}$. We conclude that

$$\mu(ord_D^{-1}(n)) = D_{n-1}\mathbf{L}^{-(n-1)d} - D_n\mathbf{L}^{-nd} = [D]\mathbf{L}^{-n}(\mathbf{L} - 1).$$

Notice also that $\mu(ord_D^{-1}(0)) = [X] - [D]$. Hence,

$$\int_{X_\infty} \mathbf{L}^{-ord_D} d\mu = [X] - [D] + \sum_{n \geq 1} [D]\mathbf{L}^{-n}(\mathbf{L} - 1)\mathbf{L}^{-n}$$

$$= [X] - [D] + [D]\frac{(\mathbf{L} - 1)\mathbf{L}^{-2}}{1 - \mathbf{L}^{-2}}.$$

Note that in the last line of this example, we have used the computation of a geometric series which converges in the considered completion of Grothendieck ring. Similar computations can be done for a normal crossing divisor on a smooth variety; this, with resolution of singularities and the following change of variables formula (Kontsevich, Denef-Loeser [47]) gives a very efficient way to "compute" motivic integral.

Theorem 4.4.3 *Let X be a $\mathbf{K}$-variety and let $f : Z \longrightarrow X$ be a proper birational morphism such that Z is smooth. Let $A \subset X_\infty$ be a cylinder and α be a function as above such that $\mathbf{L}^{-\alpha}$ is integrable. We have*

$$\int_A \mathbf{L}^{-\alpha} d\mu_X = \int_{f_\infty^{-1}(A)} \mathbf{L}^{-\alpha\circ f + ord_t(Jac(f))} d\mu_Z.$$

In the case where X is smooth, $Jac(f)$ is simply the ordinary jacobian determinant of f; see for the general definition. In the theorem we used the notations μ_X and μ_Z to stress the spaces where these measures are defined.

Let X be a smooth variety of dimension d (defined over the field of complex numbers) and let $f : X \longrightarrow \mathbb{C}$ be a non-constant morphism; here we consider the field of complex number because we will talk below about Milnor fibers and monodromies. Let $D = \{f = 0\}$ be the divisor defined by f on X. For $m, p \in \mathbb{N}, m \geq p$, we set

$$Cont^p(D)_m = \{\gamma \in X_m; ord_t f(\gamma) = p\}.$$

The **motivic Igusa Zeta function** [46] of f is defined by

$$Z(T) := \sum_{m\geq 0} [Cont^m(D)_m]\mathbf{L}^{-md}T^m.$$

This series was introduced by Denef and Loeser in analogy with the $p-$adic Igusa Zeta function. It is a simple exercise to see that if we define $J(T) : \sum_{m\geq 0}[D_m]T^m$, then we have the relation

$$J(T) = \frac{Z(\mathbf{L}^m T) - [X]}{\mathbf{L}^m T - 1}.$$

Using the relation

$$\int_{X_\infty} \mathbf{L}^{-ord_D} d\mu = Z(\mathbf{L}^{-1})$$

which follows from the definitions of both sides of the equality, and the change of variables formula, Denef and Loeser proved that $Z(T)$ is a "rational" function which has the following shape

$$Z(T) = (\mathbf{L} - 1) \sum_{S \subset I} [E_S^0] \prod_{s \in S} \frac{\mathbf{L}^{-\nu_s} T^{N_s}}{1 - \mathbf{L}^{-\nu_s} T^{N_s}}.$$

In this formula, the I is the index set of the irreducible components of the exceptional divisor $E = \cup_{s \in I} E_i$ of an embedded resolution of $D \subset X$; for a subset $S \subset I$, we define $E_S^0 := (\cap_{s \in S} E_s) \setminus (\cup_{i \in I \setminus S} E_i)$. The integers ν_i and N_i are also part of the data of the embedded resolution that we have considered and they are associated with the irreducible components E_i. In general, few of the $\mathbf{L}^{\nu_i/N_i}$ are actual poles of $Z(T)$ as a rational function in T.

The **Igusa motivic monodromy conjecture** [46] of Denef Loeser states that if $\mathbf{L}^{\nu_i/N_i}$ is a pole of $Z(T)$ then there exists $x \in D$ such that $e^{2\pi \mathbf{i} \nu_i/N_i}$ is an eigenvalue of the action of the local monodromy on the cohomology of the Milnor fiber of f at x. It is analogous to the "$p-$adic" Igusa's monodromy conjecture.

Another invariant which can be defined from a series similar to $Z(T)$ is the **motivic Milnor fiber** [46]: For $m \in \mathbb{N}$ and $x \in D$, define

$$\mathscr{X}_m = \{\gamma \in X_\infty \mid \gamma(0) = x \text{ and } f \circ \gamma = \gamma^*(f) = t^m + t^{m+1} h_\gamma\}.$$

Let

$$Z_{f,x}(T) = \sum_{m \geq 0} \mu_X(\mathscr{X}_m) T^m.$$

As for the motivic Igusa zeta function, $Z_{f,x}$ is rational and Denef and Loeser defined the motivic Milnor fiber at x by

$$S_{f,x} = - \lim_{T \to \infty} Z_{f,x}(T).$$

See also [118] for motivic Milnor fiber at ∞.

The last motivic invariant that we consider in this section is the "geometric" motivic series which again was introduced by Denef and Loeser in analogy with comparable series in the p-adic settings. Let Y be an algebraic variety over $\mathbf{K}$, The geometric Poincaré series [47] is defined by

$$P(T) = \sum_{m \geq 0} [\Psi_m(Y_\infty)] T^m.$$

It is a rational function which belongs to the subring of $\mathscr{M}_{loc}[[T]]$ generated by $\mathscr{M}_{loc}[T]$ and elements of the form $(1 - \mathbf{L}^a T^b)$, $a \in \mathbb{Z}$, $b \in \mathbb{N} \setminus \{0\}$. This series is difficult to compute in general [61, 86, 109].

Other invariants of "motivic" type were also considered in singularity theory, we mention here Batyrev's stringy invariants [13, 81, 135]. There has been also an important use of ideas of motivic integration in the study of real singularities [28, 33, 117].

Finally, motivic integration has been generalized to more general settings using model theory [31, 67].

4.5 Jet Schemes and Singularities of Pairs

The geometry of the arc space and the jet schemes which was needed for the proof of the change of variables formula in motivic integration allows to interpret some invariants of pairs in terms of dimensions of jet schemes. The first results in this direction are due to Mustaţă [106–108]. The first proofs of these results used motivic integration, but after [50], easier geometric proofs were found; The papers [51, 68] give a very good survey about this type of results; see also [45, 52, 53]. A key fact which intervene in many proofs of these results is the interpretation of a divisorial valuation on an variety X as the order of annihilation along arcs in an irreducible component of some contact locus. More precisely, let us consider an affine variety $X = \mathrm{Spec}(\mathcal{O}_X)$ (which is smooth for simplicity); For an ideal $I \subset \mathcal{O}_X$, consider the subvariety $Y = V(I) \subset X$; let $p \in \mathbf{N}$; the $p-$contact locus with Y is by definition

$$Cont^p(Y) := \{\gamma \in X_\infty \mid ord_t\gamma^*(I) = p\}, \tag{4.5}$$

where $\gamma^* : R \longrightarrow \mathbf{K}[[t]]$ is the $\mathbf{K}$-algebra homomorphism associated with γ and

$$ord_t\gamma^*(I) = \min_{h\in I}\left\{ord_t\gamma^*(h)\right\}.$$

Let E be a divisor over X centered at schematic point $x \in X$ and let ν_E be the associated divisorial valuation. It follows from [50] that there exists a subvariety $Y \subset X$, an integer number $p \in \mathbf{N}$ and an irreducible component $W \subset Cont^p(Y)$ such that, for every $h \in \mathcal{O}_{X,x}$

$$\nu_E(h) = \min_{\gamma\in W} ord_t(h \circ \gamma).$$

In the other (easier) direction, every fat irreducible component of $Cont^p(Y)$ (for some $p \in \mathbf{N}$ and $Y \subset X$) defines a divisorial valuation ν_E on X. Moreover, E can be constructed by a weighted blowing up performed on a log-resolution of $Y \subset X$; recall here that a log-resolution of $Y \subset X$ is an embedded resolution which factors through the blowing up of Y in X.

An important feature of the above interpretation of a divisorial valuation is that the invariants of a divisor E on X (like its discrepancy) are encoded in the geometry of W. From this, one can deduce Mustaţă's formula for the log canonical threshold of a pair $Y \subset X$ in terms of the dimensions of the jet schemes of Y which are embedded in the jet schemes of X [108]; this also leads to the characterization of rational complete intersection singularities in terms of jet schemes [106]. Note that most invariants of singularities of pairs are defined via divisors appearing on log resolutions [95]. The above results are in characteristic zero; similar results exist in positive characteristics [139]; See also [72] for recent results and questions in this direction.

4.6 The Jet-Components Graph

The study of the irreducible components of the jet schemes is significant for the search for and the understanding of the geometry of embedded resolutions of singularities. But apart from resolutions of singularities, this difficult problem has its own interest because the jet schemes contains a lot of information ([24, 25, 34, 74, 97, 106–108] etc...). But this information comes in bulk. One features of the difficulty of this problem is that while the motivic integration theory (or the geometry behind) can say something about the irreducible components of the jet schemes of maximal dimensions [108], it is much less powerful in understanding the other components which often contain the deep information about the singularities. Many questions arise in relation with these irreducible components:

What is the "structure" of the irreducible components of the jet schemes of a singular variety X?

While one can be interested in the irreducible components of the $m-$th jet scheme of X for a given $m \in \mathbb{N}$, these components come naturally in projective systems and their study becomes more exciting when we consider the variation of their geometry in these projective systems. Below we will give a meaning of the word "structure" in the question; this structure is still mysterious and very little studied, and we understand it in very few cases [32, 97, 101].

What is the relation between the geometry of the jet schemes of X and the geometry of the singular variety X ?

Finally, finding explicit relations between the local geometry of the singularities and some resolution of singularities remains a central problem in singularity theory. In this section, jet schemes stay somehow in the middle: an answer to the second question above allows to relate the geometry of the jet schemes to the geometry of singularities and the geometric approach to resolution of singularities (Sect. 4.7) links the valuations which arise from the irreducible components of the jet schemes to resolution of singularities. Apart from this approach, it is now well known that there are deep relations between resolution of singularities and jets schemes, *e.g.,* [34, 45, 71, 85, 106, 108], but these relations are far from being completely explored; the Nash problem can be thought as one of these relations. We (partially) answer the first question and "completely" the second question for quasi-ordinary and toric surface singularities. Before saying a word about these answers, let us introduce the jet components graph which will encode the structure of the irreducible components of the jet schemes.

Definition 4.6.1 ([32, 97, 101]) The jet-components graph of an algebraic variety S is the leveled weighted graph Γ obtained by

- representing every irreducible components of S_m, $m \geq 1$, by a vertex $v_{i,m}$, where the sub-index m is the level of the vertex;
- joining the vertices $v_{i_1,m+1}$ and $v_{i_0,m}$ if the morphism $\pi_{m+1,m}$ induces a morphism between the corresponding irreducible components;
- weighting each vertex by the dimension of the corresponding irreducible component.

Recall that the morphism $\pi_{m+1,m} : S_{m+1} \longrightarrow S_m$ is the truncation morphism which is induced by the algebraic morphism $\mathbf{K}[t]/(t^{m+1}) \longrightarrow \mathbf{K}[t]/(t^m)$.

This graph was introduced in [97] and was refined in [32, 101]. Sometimes, we also weight the irreducible components by their embedding dimensions; this can be necessary to recover the geometry of the singularity.

Let us present (very) briefly the singularities that we will introduce here:

Quasi-ordinary singularities of dimension d are those singularities which (locally) can be projected to an affine space $(\mathbf{A}^d, 0)$ such that the discriminant locus is a normal crossing divisor; they are particularly important in Jung's point of view on resolution of singularities and in equisingularity theory [92]. More about this type of singularities is explained in [60, 62, 90, 91, 104]. We are concerned with quasi-ordinary hypersurface singularities (over a field of characteristic 0) which are defined (locally) by a polynomial in $\mathbf{K}[[x_1, \ldots, x_d]][z]$ that we see as a polynomial in the variable z. Thanks to the Abhyankar-Jung theorem[1, 76], we know many properties of the roots of such a polynomial (in particular they can be represented as Puiseux series) and one can use these properties to introduce invariants (characteristic pairs, semigroup, Lattices) of the singularity [62, 79, 90, 91]; these are very powerful invariants that actually determine and are determined by the topological type of the singularity [57]. In [32], we determine in terms of these invariants the irreducible components of the jet schemes of a quasi-ordinary singularity of dimension 2. We determined the geometries and the dimensions of open dense subset of these irreducible components, which happen to be isomorphic to affine spaces or to trivial fibrations over some (non-normal) toric varieties which encode deeply the geometry of quasi-ordinary singularities defined by the approximate roots of our singularity; in particular they encode the geometry of the singularity itself. Note that approximate roots are roughly speaking associated with truncation of a root of a polynomial defining a quasi-ordinary singularity.

Theorem 4.6.2 ([32])
Let $(S, 0)$ be a a 2-dimensional quasi-ordinary hypersurface singularity. A canonical subgraph of the jet components graph of $(S, 0)$ determines the embedded topological type of $(S, 0)$, and the converse is true.

We show in Fig. 4.1 a part of the subgraph that appears in the theorem for a singularity whose singular locus has two irreducible components, a curve and a line. We do not put the weights here in order not to encumber the picture. Here the arrows represent a projective system of components which goes till infinity.

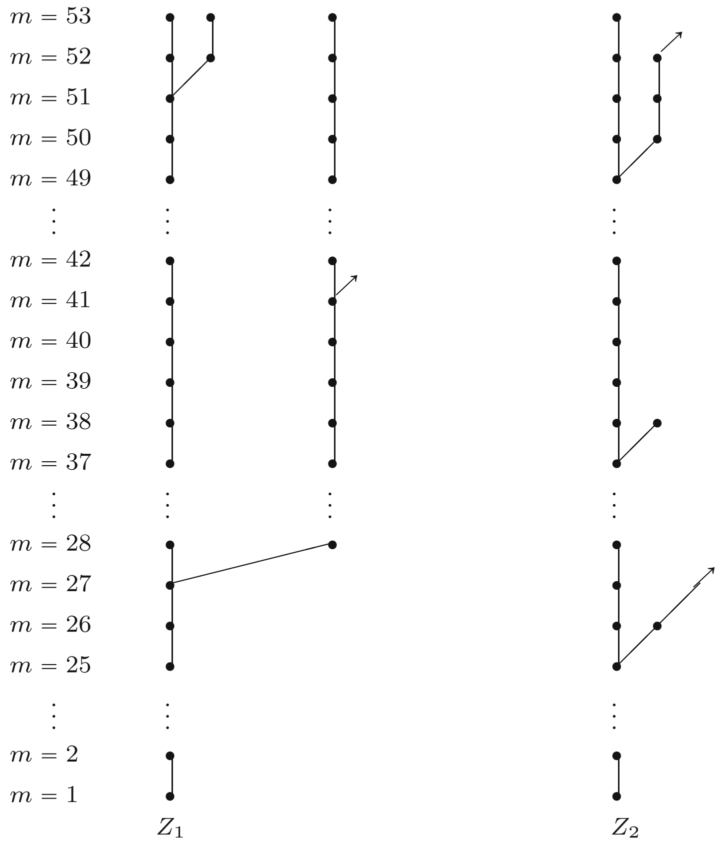

Fig. 4.1 The graph of the surface defined by $f = ((z^2 - x_1^3)^2 - x_1^7 x_2^3)^2 - x_1^{11} x_2^5 (z^2 - x_1^3)$

Here we would like to stress on the fact that only when studying how the geometry of the irreducible components varies in a projective system of irreducible components that we are able to determine the topological type of the singularity. Theorem 4.6.2 contains two very delicate results: the determination of the irreducible components and the subgraph of the jet components graph on one hand, and the fact the this subgraph determines the embedded topological type of the singularity; this is to compare with the motivic invariants which do not determine it [61].

The theorem, as we said before, partially answers the first question above and completely answer the second question. It only partially answers the first question because we don't determine all the edges in the jet components graph. This is related with and give different and new insight on the generalized Nash problem [34, 71].

We also gave in [32] examples of quasi-ordinary surface singularities embedded in $\mathbf{A}^3$ whose log canonical threshold (this is an important invariant of singularities of pairs which is computed by a divisorial valuation on a log resolution) is not obtained by a monomial valuation in any coordinates (For plane curves, the log canonical threshold is always computed by a monomial valuation, up to change of coordinates).

An important observation that one can make about the geometry of the irreducible components of the jet schemes of a quasi-ordinary surface is the following: For any such irreducible component the graded algebra of the associated valuation can be represented by the approximate roots of the singularity; this graded algebra reflects the geometry of the component and can be actually recovered from this geometry.

Normal toric surface singularities are the simplest normal toric singularities. Such a singularity is simply given by the data of two coprime numbers, its embedding dimension can be as high as one wishes and hence can be defined by a very large number of equations; moreover, apart from the case of the A_n singularities (which are hypersurfaces in $\mathbf{A}^3$) they are never locally complete intersections: this latter hypothesis is essential for many theorems about or using jet schemes [53, 106]. The structure of the jet schemes of toric singularities or even their irreducible components are not known in general [107] and determining this structure seems to be a difficult problem. We have determined the irreducible components of the jet schemes of these singularities and as for quasi-ordinary surface singularities, we determined a subgraph of the jet components graph the encodes almost completely the singularity:

Theorem 4.6.3 *([101]) The jet components graph determines the analytical type of a normal toric surface singularity in the following sense: two normal toric surface singularities are isomorphic if and only if they have the same jet components graph.*

It is worth noticing here that Motivic type invariants do not catch the analytic type ([86, 109]).

The proof of theorem 4.6.3 uses heavily the description of the defining equations of the embedding $S \subset \mathbf{A}^e$ ([121, 126]), and some syzygies of these equations that we describe and that are ad hoc to the problem. It also uses known results on the arc space of a toric variety [69, 73, 82]. The proof proceeds by induction on m (the level of the jet scheme) and on the embedding dimension e. In particular it uses a kind of approximation of the toric surface S by toric surfaces with smaller embedding dimensions. The irreducible components of the jet schemes of toric surface singularities were discovered in [98] but the complete understanding of their structure and its presentation was only completed in 2017 [101].

We close the discussion of this section by mentioning a conjectural link between the irreducible components of jet schemes and Floer theory [27].

4.7 A Geometric Approach to Resolution of Singularities via Arc Spaces

A guiding problem in singularity theory and in algebraic geometry is the problem of proving the existence of a resolution of singularities and of understanding how to determine it:

A (abstract) resolution of singularities of an algebraic variety X is a modification (a proper birational morphism: an isomorphism on a open subvariety of Y) $\mu : Y \longrightarrow X$ such that Y is non-singular.

Another more involved version of resolution of singularities is the embedded resolution of a singular variety $X \subset Z$:

An embedded resolution of singularities of an algebraic variety $X \subset Z$ is a a proper birational morphism $\mu : Y \longrightarrow Z$ such that Y is non singular and the strict transform of X by μ is non-singular and transversal to the exceptional locus of μ (the locus where μ is not an isomorphism).

Resolution of singularities has applications that range from Algebraic Geometry to Analysis, Dynamical systems, Differential Geometry, Number theory…In Algebraic Geometry or real and complex analytic geometry, it is used to transform some problems concerning singular spaces to problems concerning non singular spaces; it allows to define invariants of singularities which help in problems of classification of singularities; it also serves as a change of variables when computing integrals. An embedded resolution gives an abstract resolution by looking at its restriction to the strict transform; it contains and gives (much) more information than the information encoded in an abstract resolution. A celebrated theorem proved by Hironaka gives the existence of embedded resolution of singularities of varieties defined over a field of characteristic zero [66]. In positive characteristics, the existence of embedded resolution of singularities is proved only for varieties in dimension 2; in dimension 3, there is a proof of the existence of abstract resolution of singularities in [34, 35]. This is (with local uniformization, which is a “super” local version of resolution of singularities) a very active research subject, see *e.g.* [2, 14, 15, 35, 38, 39, 64, 78, 111, 123, 130]. See [37, 80, 125] for an introduction to resolution of singularities.

The traditional approach to resolve singularities is to iterate blowing ups at smooth centers in order to make an invariant drop. This invariant should take values in a discrete ordered set with a smallest element (which detects smoothness). It should not only detect smoothness, but also should be easy to compute so that its behavior can be followed when iterating the blowing ups. The big advantage of this approach is that it has worked in characteristic zero and that it gives an algorithm. But the construction of such a resolution is rarely linked to the deep geometry of the singularities: such a resolution is obtained as a composition of maybe 1 million blowups which are not related in general to the deep geometry of the singularities of the starting variety.

The theme of this section is a geometric approach to resolution of singularities; an approach which is based on a dialog between the following two themes:

1. The reverse Nash problem.
2. Teissier's conjecture on embedded resolution of singularities with one toric morphism.

The Reverse Nash Problem

Recall from Sect. 4.3 that the Nash problem (and its variants) searched in the arc space and jet schemes for the common data to all resolutions of singularities. What we call the reverse Nash problem is the following question:

Can we construct (or describe) a (abstract or embedded) resolution of singularities of X from its arc space and jet schemes ?

Teissier's conjecture on embedded resolution of singularities with one toric morphism

As we mentioned above, the traditional way to resolve singularities is to blowup a "permissible" center in order to make an adapted invariant drop and hence to define an algorithm which stops after finitely many steps. Such an algorithm exists in characteristic 0, thanks to the existence of a hypersurface of maximal contact (which allows an induction on the dimension of the variety) which does not exist when working in positive characteristics. Teissier asked [127, 130, 131] the following question:

Given a singular variety $X \subset \mathbf{A}^n$, does there exist an embedding $X \subset \mathbf{A}^n \hookrightarrow \mathbf{A}^N$, $N \geq n$, and a toric structure on $\mathbf{A}^N$ such that $X \subset \mathbf{A}^N$ has an embedded resolution by one toric morphism ?

We will call such an embedding torific. This question has an immediate transposition to projective varieties $X \subset \mathbf{P}^n \subset \mathbf{P}^N$. When an embedded resolution of singularities exists, a torific embedding exists for projective varieties [132]. If the reader is not familiar with the theory of toric varieties, he can think of a toric morphism as a morphism which is locally defined by monomials: a monomial morphism. In general, it is an open conjecture that the answer is yes. If true, this conjecture would imply the existence of resolution of singularities. Teissier made deep advances in the super local version of this conjecture [131]: the embedded local uniformization problem.

Let us explain in more details what we called a geometric approach to resolution of singularities: we would like to use the reverse Nash problem to construct a torific embedding; the word geometry is used since this approach is based on the geometry of the arc space and jet schemes (and sometimes of the space of valuations which does not appear here). Let us consider $X \subset \mathbf{A}^n$; we are interested in finding a torific embedding of X. We divide the problem into two questions [100]:

1. Given a divisorial valuation ν centered at $0 \in \mathbf{A}^n$, determine whether there exist an embedding $e : \mathbf{A}^n \hookrightarrow \mathbf{A}^N$, (where N depends on ν) and a toric proper birational morphism $\mu : X_\Sigma \longrightarrow \mathbf{A}^N$ such that:

$$\begin{array}{ccc} \widetilde{\mathbf{A}^n} & \longrightarrow & X_\Sigma \\ \downarrow & & \downarrow \mu \\ \mathbf{A}^n & \overset{e}{\hookrightarrow} & \mathbf{A}^N \end{array} \quad i.e.,$$

- X_Σ is a smooth toric variety (*i.e.,* Σ is a fan which is obtained by a regular subdivision of the positive quadrant $\mathbf{R}_+^N$, this quadrant is the cone defining $\mathbf{A}^N$ as a toric variety),
- the strict transform $\widetilde{\mathbf{A}^n}$ of $\mathbf{A}^n$ by μ is smooth,
- there exists a toric divisor $E' \subset X_\Sigma$ which intersects $\widetilde{\mathbf{A}^n}$ transversally along a divisor E,
- the valuation defined by the divisor E is v.

Note that a toric divisor E' centered at the origin 0 of $\mathbf{A}^N = \mathrm{Spec}\mathbf{K}[x_1, \ldots, x_N]$ corresponds to a divisorial valuation v' which is monomial, *i.e.,* there exists a vector $\alpha \in \mathbf{N}^N$ such that $v' = v_\alpha$ where

$$v_\alpha : \mathbf{K}[x_1, \ldots, x_N] \longrightarrow \mathbf{N}$$

is defined by: for $h \in \mathbf{K}[x_1, \ldots, x_N]$,

$$h = \sum_{m=(m_1,\ldots,m_N)} a_m x_1^{m_1} \cdots x_N^{m_N}, \qquad v_\alpha(h) = \min_{\{m | a_m \neq 0\}} < \alpha, m >; \tag{4.6}$$

where $< \alpha, m >$ is the usual scalar product on $\mathbf{R}^N$.

Then one can formulate the conditions above by saying that there exists an embedding $\mathbf{A}^n \hookrightarrow \mathbf{A}^N$ such that v is the trace of a monomial valuation defined on $\mathbf{A}^N$.

2. Determine a finite number of significant divisorial valuations $v_1, \ldots, v_r$ on $\mathbf{A}^n$ from the geometry of the jet schemes and the arc space of X (this step is to compare with the Nash problem that we mentioned above: very roughly speaking, as the Nash problem search for divisorial valuations that will "appear" on every resolution of singularities, here we are searching for divisorial valuations whose torifications in the sense of question (1) is essential to obtain a global torification), then embed as above $\mathbf{A}^n$ in a larger affine space $\mathbf{A}^N$ in such a way that all the valuations $v_1, \ldots, v_r$ can be seen as the traces of monomial valuations on $\mathbf{A}^N$.

If $v_1, \ldots, v_r$, are well chosen, this should guarantee that the embedding $X \subset \mathbf{A}^N$ is torific. Let us discuss this last sentence which probably for now looks a bit prophetic. Let $v = v_\alpha$ be the monomial valuation defined on $\mathbf{A}^n = \mathrm{Spec}\mathbf{K}[x_1, \ldots, x_n]$ by a vector $\alpha = (\alpha_1, \ldots, \alpha_n)$, where $\alpha_i \in \mathbb{N}, i = 1, \ldots, n$. Let $I \subset \mathbf{K}[x_1, \ldots, x_n]$ be an ideal such that the origin 0 belongs to the variety $V(I) \subset \mathbf{A}^n = \mathrm{Spec}\mathbf{K}[x_1, \ldots, x_n]$ defined by it. We will say that I or $V(I)$ is non-degenerate with respect to v at 0 if the singular locus of the variety defined by the initial ideal $in_v(I)$ of I does not intersect the torus $(\mathbf{K}^*)^n$. Note that in this context, the initial ideal of I relative to v is defined by

$$in_v(I) = \{in_v(f), f \in I\},$$

where for $f = \sum a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n} \in \mathbf{K}[x_1, \ldots, x_n]$,

$$in_v(f) = \sum_{a_{i_1,\ldots,i_n} \neq 0, i_1\alpha_1+\cdots+i_n\alpha_n = v(f)} a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n}.$$

It follows from [10, 11, 112, 130] (see also [134] for the hypersurface case) that if for every $\alpha = (\alpha_1, \ldots, \alpha_n), \alpha_i \in \mathbf{N}, i = 1, \ldots, n,$ I is Newton non-degenerate with respect to v_α at 0, then we can construct a proper toric birational morphism $Z \longrightarrow \mathbf{A}^n$ that resolves the singularities of $V(I)$ in a neighborhood of 0. Notice that I can be degenerate with respect to a valuation defined by a vector α if there exists an irreducible family of jets (having a large contact with $V(I)$) or arcs on $V(I)$ such that for a generic $\gamma = (\gamma_1(t), \ldots, \gamma_n(t))$ in this family, its **order vector** $(ord_t\gamma_1(t), \ldots, ord_t\gamma_n(t)) = \alpha$: indeed, by a Newton-Puiseux type theorem (or the fundamental theorem of tropical geometry [94]), if this is not satisfied, *i.e.,* if there is no arc, $in_{v_\alpha}(f)$ will contain monomials, hence by definition I will be non-degenerate with respect to v_α. This suggests that arcs detect Newton degeneration, and wherever there is a Newton degeneration, there is a degenerate arc passing there in the following sense: An arc defines a germ of a curve; we call an arc degenerate whenever the associated curve germ cannot be resolved with one toric morphism (this can also be detected from the properties of the arc, for instance using the notion of Nash multiplicity [83]). There are degenerate arcs that can be traced on a smooth variety: think of a (relatively) nasty plane curve like the germ of curve which is defined by $(\{(y^2 - x^3)^2 - 4x^5y - x^7 = 0\}, 0) \subset (\mathbf{A}^2, 0) = \{z = y\} \subset (\mathbf{A}^3, 0)$; it is associated with the arc $(t^4, t^6 + t^7, t^6 + t^7)$ which is traced on $\mathbf{A}^2$. The arc is degenerate but $\mathbf{A}^2 \subset \mathbf{A}^3$ is Newton non-degenerate. The moral of this part of the story is first that Newton degeneration is detected by degenerate arcs, and second that not all degenerate arcs cause Newton degeneration. Moreover, the notion of degeneration along an arc can be quantified by an invariant that one can call depth and which in the case of a plane curve is the number of Puiseux pairs minus one. Question (1) above takes care of this notion of depth and it allows by embedding in higher dimension the elimination of degeneration along a family of arcs (or jets) that defines a divisorial valuation. Question (2) concerns the determination of those families of arcs that cause Newton degeneration.

We will now give a presentation of results concerning these two questions, with some digressions in order to give applications, links between the two questions and expand a bit some problems that appear inside these questions and which are interesting for their own sakes.

Let us begin by discussing one aspect of question (1). While this question was exposed as a geometric problem, it is related to an "algebraic" problem which makes sense for any valuations: determining a generating sequence of a valuation. Let us for a moment stick to the case of a divisorial valuation centered at the origin $X = \mathbf{A}^d = \text{Spec}R$, where $R = \mathbf{K}[x_1, \ldots, x_n]$ is a polynomial ring over an algebraically closed field $\mathbf{K}$. A valuation v is then given by a mapping $v : R \longrightarrow \mathbb{N}$ which is the order of vanishing along a divisor $E \subset Z$ which satisfies $\mu(E)$ is the origin of $\mathbf{A}^n$, μ being a birational map $\mu : Z \longrightarrow \mathbf{A}^n$. Let us explain what is a generating sequence of v.

For $\alpha \in \mathbb{N}$, let

$$\mathscr{P}_\alpha = \{h \in R \mid v(h) \geq \alpha\}.$$

We define the $\mathbf{K}$-graded algebra

$$gr_v R = \bigoplus_{\alpha \in \mathbb{N}} \frac{\mathscr{P}_\alpha}{\mathscr{P}_{\alpha+1}}.$$

We call in_v the natural map

$$in_v : R \longrightarrow gr_v R, \quad h \mapsto h \mod \mathscr{P}_{v(h)+1}.$$

Definition 4.7.1 ([124]) A generating sequence of v is a set of elements of R such that their image by in_v generates $gr_v R$ as a $\mathbf{K}$-algebra.

This notion (for any valuation) is central in an earlier version of Spivakovsky's approach [124] to local uniformization and in the present approach of Teissier to the same problem [130], with the difference that Teissier restricts his analysis to minimal generating sequences for rational valuations. In general, it is very difficult to determine a generating sequence of a given valuation, apart in dimensions 1 and 2; an abstract approach follows from the valuative Cohen theorem [130]. A remarkable advance in this direction was done for (rational) valuations in [40], as discussed below. We will show below, at least on an example, the relation between this notion and question (1). We will discuss first our new approach from [100] for the study of generating sequences of divisorial valuations defined as above. For that, we will use the representation of a divisorial valuation as the order of vanishing along a family of arcs.

Let $X = \mathbf{A}^n = \text{Spec } R$, as above. We have a natural truncation morphism $X_\infty \longrightarrow X$, that we denote by Ψ_0; for a n-tuple of series, this simply gives the n-tuple of constant terms of these series. For $p \in \mathbb{N}$ and $Y = V(I) \subset X$ a subscheme defined by an ideal $I \subset R$, we consider the contact locus $Cont^p(Y)$ (see Eq. (4.5)).

With a fat irreducible component $\mathbf{W}$ of $Cont^p(Y)$, which is included in the fibre $\Psi_0^{-1}(0)$ above the origin, we associate a valuation $v_{\mathbf{W}} : R \longrightarrow \mathbb{N}$ as follows:

$$v_{\mathbf{W}}(h) = \min_{\gamma \in \mathbf{W}}\{ord_t \gamma^*(h)\},$$

for $h \in R$. It follows from [50] (see also [45, 120], prop. 3.7 (vii)), that $v_{\mathbf{W}}$ is a divisorial valuation centered at the origin $0 \in X$, and that all divisorial valuations centered at $0 \in X$, can be obtained in this way (see Sect. 4.5) for varying ideals I. We are interested in determining a generating sequence of a valuation of the form $v_{\mathbf{W}}$ with an irreducible component $\mathbf{W}$ of $Cont^p(Y)$. Recall from 4.2 or [17] the functorial definition of the arc space X_∞ : for any algebraic variety X, the arc space X_∞ represents the functor that to a $\mathbf{K}$-algebra A associates the set of $A-$valued arcs

$$X(A[[t]]) := \mathrm{Hom}_{\mathbf{K}}(\mathrm{Spec}(A[[t]]), X).$$

Hence, for a $\mathbf{K}$-algebra A we have a bijection

$$\mathrm{Hom}_{\mathbf{K}}(\mathrm{Spec}(A), X_\infty) \simeq \mathrm{Hom}_{\mathbf{K}}(\mathrm{Spec}(A[[t]]), X).$$

In particular, in our case $X = \mathbf{A}^n = \mathrm{Spec} R$, we have $X_\infty = \mathrm{Spec}(R_\infty)$, and to the identity in $\mathrm{Hom}_{\mathbf{K}}(\mathrm{Spec}(R_\infty), X_\infty)$ corresponds the universal family $\Lambda : R \longrightarrow R_\infty[[t]]$.

Let us consider the case $n = 2$, $R = \mathbf{K}[x_0, x_1]$. We have

$$R_\infty = \mathbf{K}[x_i^{(j)}; i = 0, 1; j \geq 0],$$

and Λ is given by

$$\Lambda(x_i) = x_i^{(0)} + x_i^{(1)} t + x_i^{(2)} t^2 + \cdots, \quad i = 0, 1.$$

The procedure that we give can be thought as an elimination algorithm with respect to Λ in the sense that from the equations (that we can see in R_∞) of the irreducible component of $Cont^p(Y)$ defining our valuation we will obtain elements in R that constitute the generating sequence. Let us show this on an example: Assume that the characteristic is not equal to 2. Let us consider the divisorial valuation associated with one irreducible component of $Cont^{27}(Y)$, where Y is the curve defined by the equation $(x_1^2 - x_0^3)^2 - x_0^5 x_1 = 0$. The contact locus $Cont^{27}(Y)$ has two irreducible components which are sent to the origin 0 by the truncation morphism π_{27}, the interesting one (the other one gives a monomial valuation), that we call $\mathbf{W}$ is defined in $\mathbf{A}^2_\infty$ by the ideal generated by

$$x_0^{(0)}, \ldots, x_0^{(3)}, x_1^{(0)}, \ldots, x_1^{(5)}, {x_1^{(6)}}^2 - {x_0^{(4)}}^3,$$

$$(2x_1^{(6)} x_1^{(7)} - 3{x_0^{(4)}}^2 x_0^{(5)})^2 - {x_0^{(4)}}^5 x_1^{(6)}$$

and two inequalities, the most important one of them is $x_0^{(4)} \neq 0$. Noticing that the first equation which is not that of a coordinate hyperplane being not linear, this gives us the first three elements of a generating sequence

$$x_0, x_1, x_2 = x_1^2 - x_0^3.$$

The last element was obtained by what we called an elimination process which corresponds here to dropping the indices in the parentheses from ${x_1^{(6)}}^2 - {x_0^{(4)}}^3$. Note that modulo $x_0^{(0)} = \cdots = x_0^{(3)} = x_1^{(0)} = \ldots = x_1^{(5)} = 0$, $\Lambda(x_2) = ({x_1^{(6)}}^2 - {x_0^{(4)}}^3)t^{12} + t^{13}\phi$, with $\phi \in R_\infty[[t]]$. The remaining equation, modulo the other equations, can then be rewritten

$$(2x_1^{(6)}x_1^{(7)} - 3{x_0^{(4)}}^2 x_0^{(5)})^2 - {x_0^{(4)}}^5 x_1^{(6)} = {x_2^{(13)}}^2 - {x_0^{(4)}}^5 x_1^{(6)}.$$

Again, the elimination process with respect to Λ corresponds to dropping the indices in the parentheses. The 4th and last element of the generating sequence of $\upsilon_{\mathbf{W}}$ which is then:

$$x_3 = x_2^2 - x_0^5 x_1.$$

The valuation $\upsilon_{\mathbf{W}}$ is completely determined by its generating sequence x_0, x_1, x_2, x_3 and the values $\upsilon_{\mathbf{W}}(x_0) = 4, \upsilon_{\mathbf{W}}(x_1) = 6, \upsilon_{\mathbf{W}}(x_2) = 13, \upsilon_{\mathbf{W}}(x_3) = 27$. By construction, for $i = 2, 3$ we have polynomials f_i such that

$$x_i = f_i(x_0, \ldots, x_{i-1}).$$

The functions f_i's provide an embedding $\mathbf{A}^2 \hookrightarrow \mathbf{A}^4$, which is the geometric counterpart of the following morphism

$$\mathbf{K}[x_0, x_1, x_2, x_3] \longrightarrow \frac{\mathbf{K}[x_0, x_1, x_2, x_3]}{(x_2 - f_2(x_0, x_1), x_3 - f_3(x_0, x_1, x_2))} \simeq \mathbf{K}[x_0, x_1].$$

This embedding solves question (1) for the valuation $\upsilon_{\mathbf{W}}$ and realizes this latter as the trace of the monomial valuation centered at $(\mathbf{A}^4, 0)$ and associated with the vector $\alpha = (4, 6, 13, 27)$. Here we only gave the feeling of this, but the reason why the second and the third points of question (1) are satisfied follows from the fact that if $\nu = \nu_\alpha$ then the initial ideal of $(x_2 - f_2(x_0, x_1), x_3 - f_3(x_0, x_1, x_2))$ with respect to ν is given by

$$(x_1^2 - x_0^3, x_2^2 - x_0^5 x_1),$$

which is a toric (prime) ideal and its singular locus is a point. More generally we have

Theorem 4.7.2 ([100]) *For $n = 2$, there is a constructive solution of question (1).*

It is important to mention here that determining a generating sequence is not necessary to solve question (1) for a given valuation.

We can give now an example of our geometric approach to the resolution of singularities. Let $Y \subset \mathbf{A}^2$ be again the curve defined by $(x_1^2 - x_0^3)^2 - x_0^5 x_1 = 0$. The interesting divisorial valuation is the one associated with the irreducible component of Y_{25} (or equivalently of $Cont^{26}(Y)$) which is defined by the ideal

$$\left(x_0^{(0)}, \ldots, x_0^{(3)}, x_1^{(0)}, \ldots, x_1^{(5)}, x_1^{(6)2} - x_0^{(4)3}\right).$$

We do not explain here in detail why we choose this divisor but we can say that this is the most natural choice which arises from the geometry of the jet schemes, which will be discussed below. But we can say that the space of arcs (on Y) centered at the singular point of Y has one irreducible component whose geometry is reflected by the geometry of this irreducible component of Y_{25}. Applying the procedure that we explained above, we find an embedding $\mathbf{A}^2 \hookrightarrow \mathbf{A}^3$, which is the geometric counterpart of the following morphism

$$\mathbf{K}[x_0, x_1, x_2] \longrightarrow \frac{\mathbf{K}[x_0, x_1, x_2]}{(x_2 - (x_1{}^2 - x_0{}^3))} \simeq \mathbf{K}[x_0, x_1].$$

Our curve Y seen in $\mathbf{A}^3$ is then defined by the ideal

$$I = (x_2 - (x_1{}^2 - x_0{}^3), x_2^2 - x_0^5 x_1).$$

Its (local) tropical variety (with respect to the embedding in $\mathbf{A}^3$) is the half line along the vector (4, 6, 13) (see [116] for the notion of local tropical variety). The initial ideal of I with respect to the monomial valuation associated with the vector (4, 6, 13) is given by the ideal

$$J = (x_1^2 - x_0^3, x_2^2 - x_0^5 x_1).$$

The singular locus of the variety defined by this latter ideal (which actually defines a monomial curve) is just a point so that this ideal is non-degenerate and can be resolved with one toric morphism. Hence, this embedding is torific; more generally, this gives another proof of torification for analytically irreducible plane curves [58]. Now applying our geometric approach to resolution of singularities to a reducible plane curve we were able with de Felipe and González-Pérez to prove in the following:

Theorem 4.7.3 ([41]) *For a reducible plane curve singularity, the geometric approach to resolution of singularities yields a torific embedding.*

We can actually construct a torification for curves of any embedding dimension. What makes things more complicated in higher dimensions, is that the initial ideal which is the counter part of the initial ideal that we called J above, is not toric, but it still corresponds to a $T-$ variety (*i.e.,*) a variety which is equiped with an action of a torus of smaller dimension. Some work in this direction is in the ongoing project [19].

4.8 Deformations of Jet Schemes

In this section, we work over the field of complex number **C**. Equisingularity theories were introduced by Whitney, Zariski, Teissier, Lê and others [113, 128, 129, 133, 137] to compare singularities in a family with respect to algebraic, topological, geometric or differential invariants. The theory of jet schemes allow to consider two new equisingularity conditions; let $\mathfrak{X}$ be a (flat) family of singularities defined over a base variety B. One may wonder when $\mathfrak{X} \longrightarrow B$ induces a (flat) deformation $\mathfrak{X}_m \longrightarrow B$ (or a deformation $(\mathfrak{X}_m)_{red} \longrightarrow B$ for every $m \in \mathbf{N}$. In general, this is not the case; for instance for the embedded family $\mathfrak{X} \longrightarrow (\mathbf{C}, 0)$ which is defined by

$$\mathfrak{X} = \{y^2 - ux^2 - x^3 = 0\} \subset (\mathbf{C}^3, 0),$$

where u is the parameter of the deformation, $(\mathfrak{X}_m)_{red} \longrightarrow B$ is not flat; this follows for instance from the fact that dimension of $(\mathfrak{X}_u)_5$ (where $\mathfrak{X}_u$ is the fiber over u) depends on whether $u \neq 0$ or $u = 0$: it is 6 for $u \neq 0$ and 7 if $u = 0$. For plane irreducible curves, it follows from [97] that:

Theorem 4.8.1 ([97]) *Let $\mathfrak{X} \longrightarrow B$ be a (flat) family of irreducible plane curve singularities over a smooth variety B. The induced family $(\mathfrak{X}_m)_{red} \longrightarrow B$ is flat for every $m \in \mathbf{N}$ if and only if the fibers of $\mathfrak{X}$ have the same semigroup.*

In the theorem, the notion of semigroup is attached to a plane curve and is defined via the local intersection multiplicity of the curve at the origin; this latter defines a valuation whose semigroup is by definition the semigroup of the curve [138].

Note that, all the known equisingularity theories for families of plane curves are equivalent: such a family is equisingular if every fiber has the same semigroup (or equivalently the same Puiseux pairs). It follows from [105] that this theorem is no longer true if we allow fibers which are not necessarily plane curves.

Leyton-Alvarez [87, 88] gave a sufficient condition for an embedded one parameter family of hypersurfaces $\mathfrak{X} \subset (\mathbf{C}^{d+1}, 0) \times (\mathbf{C}, 0)$ to induce a flat deformation:

Theorem 4.8.2 (Leyton-Alvarez) *Let $\mathfrak{X} \subset (\mathbf{C}^{d+1}, 0) \times (\mathbf{C}, 0)$ be a flat family of hypersurfaces. If the family $\mathfrak{X} \subset (\mathbf{C}^{d+1}, 0) \times (\mathbf{C}, 0)$ admits a simultaneous embedded resolution then it induces a flat family $(\mathfrak{X}_m)_{red} \longrightarrow (\mathbf{C}, 0)$ for every $m \in \mathbf{N}$.*

We refer to [89] for a precise definition of a simultaneous embedded resolution. The following theorem of Leyton-Alvarez, Mourtada and Spivakovsky is proved in [89].

Theorem 4.8.3 ([89]) *Let $\mathfrak{X} \subset (\mathbf{C}^{d+1}, 0) \times (\mathbf{C}, 0)$ be a flat Newton non-degenerate family of isolated hypersurface singularities. If $\mathfrak{X}$ is μ-constant then it induces a flat family $(\mathfrak{X}_m)_{red} \longrightarrow (\mathbf{C}, 0)$ for every $m \in \mathbf{N}$.*

Let us say that a family $\mathfrak{X} \longrightarrow B$ is **jet schemes equisingular** if it induces a flat family $(\mathfrak{X}_m)_{red} \longrightarrow B$ for every $m \in \mathbf{N}$. It is a very interesting line of research to compare this notion of equisingularity with the other existing notions.

4.9 Arc Spaces and Integer Partitions

This line of research again finds its origin in the study of singularities as we will show later but is now making its way into the world of combinatorics and classical number theory, so let us begin there. The following identity

$$1+\frac{e^{-2\pi}}{1+\frac{e^{-4\pi}}{1+\frac{e^{-6\pi}}{\vdots}}}=\left(\sqrt{\frac{5+\sqrt{5}}{2}}-\frac{1+\sqrt{5}}{2}\right)e^{\frac{2\pi}{5}} \tag{4.7}$$

was imagined by Ramanujan and sent to Hardy who says in the article "The Indian Mathematician Ramanujan" (Amer. Math. Monthly 44 (1937), p. 144), see also [8]:

> [These formulas] defeated me completely. I had never seen anything in the least like them before. A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true because, if they were not true, no one would have had the imagination to invent them.

Some years later, Ramanujan gave a proof of this formula by considering the following $q-$difference equation

$$F(x)=F(xq)+xqF(xq^2), \tag{4.8}$$

where $q \in \mathbf{C}^*$, and $F(x)=\sum a_n(q)x^n$ is an analytic function satisfying $F(0)=1$.

If we define $c(x,q):=\frac{F(x)}{F(xq)}$, notice that we have

$$c(x,q)=1+\frac{xq}{c(xq,q)}=1+\frac{xq}{1+\frac{xq^2}{c(xq^2,q)}}.$$

Iterating this last identity we obtain that the left member of the identity (4.7) is equal to $c(1,e^{-2\pi})$. Now if we plug $F(x)=\sum a_n(q)x^n$ in the Eq. (4.8), by comparing the coefficients of x^n we get

$$a_n(q)=\frac{q^{n^2}}{(q)_n}=\frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

The miracle arrives in the following identity

$$1+\sum_{n\geq 1}\frac{q^{n^2}}{(q)_n} \qquad = \qquad \prod_{i\equiv\ 1,4\pmod 5}\frac{1}{1-q^i}. \tag{4.9}$$

The left hand side in the identity (4.9) is $F(1)$. There is another miracle which is that $F(q)$ is also an infinite product and hence $c(1,q)$ is. And we may then deduce Ramanujan's continued fraction (4.7) by an appeal to the theory of elliptic theta functions.

The "miracles" above are called the Rogers-Ramanujan identities; they have appeared "in many different situations": in statistical mechanics, number theory, representation theory ... and we came to them first with Clemens Bruschek and Jan Schepers via Arc spaces. Before telling the story, let us state another version of the first Rogers-Ramanujan identity (4.9).

Definition 4.9.1 A partition of a positive integer n is a decreasing sequence $\Lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r)$ such that $\lambda_1+\cdots+\lambda_r = n$. The λ_i's are called the parts of this partition and r is its size.

The identity (4.9) can be stated as follows:

Theorem 4.9.2 (Rogers, Ramanujan) *The number of partitions of n with neither consecutive parts nor equal parts (of first type) is equal to the number of partitions of n whose parts are congruent to* 1 *or* 4 *modulo* 5 *(of second type).*

The generating series of the cardinality of the partitions of the first type is the left hand member in the identity (4.9) and the generating series of the cardinals of the partitions of the second type is the right hand member in (4.9), *i.e.,* the infinite product. Now we go back to algebraic geometry and to arc spaces. Let $(X,0)$ be a singularity defined over a field $\mathbf{K}$ which is assumed for simplicity to be of characteristic 0 for (0 being a closed point that after a change of coordinates may be chosen to be the origin of an affine space containing $(X,0)$). Let $X^0_\infty = \mathrm{Spec} A^0_\infty$ be the space of arcs centered at the point 0. It has a natural cone structure which induces a grading on A^0_∞ (*i.e.,* $A^0_\infty = \oplus_{h\in\mathbf{N}} A^0_{\infty,h}$) and one can consider its Hilbert-Poincaré series that we call the **Arc-Hilbert-Poincaré series** of the singularity:

$$\mathrm{AHP}_{X,0} = \sum_{h\in\mathbf{N}} \dim_{\mathbf{K}} A^0_{\infty,h}\, q^h.$$

It is not difficult to see that this is an invariant of singularities (it detects regularity) and it contains different ingredients which motivate its study from the viewpoint of singularity theory: First, if $X \subset \mathbf{A}^e$ and considering the jet schemes $X_m \subset \mathbf{A}^e_m = \mathbf{A}^{e(m+1)}$ and $X^0_m \subset (\mathbf{A}^e)^0_m = \mathbf{A}^{em}$;

(i) one notices on examples that the defining ideal of X^0_m in $\mathbf{A}^{em}$ is independent of some of the variables of the polynomial ring which is the ring of global sections of $\mathbf{A}^{em}$ and that the number of variables needed to define this ideal (modulo a linear change of variables) depends on how singular X is at 0. The more X is singular, the less variables we need for a given m; such an invariant was actually defined by Hironaka as a resolution invariant, see for instance [16] but this is another story.

(ii) The data of an $m-$jet determines its coordinates in $\mathbf{A}^{em}$ and as mentioned in the item (1), there are no constraints on some of these coordinates; but there are constraints on these "free coordinates" for the jet to be liftable to an arc and these constraints come from the equations defining X_l for $l \geq m$; the smallest l such that the equations defining X_l catch all the constraints on all the $m-$jets for them to be liftable is related to the Artin-Greenberg function which is another invariant of singularities [65, 122]: roughly speaking, Greenberg's theorem states that if a tuple $\gamma(t)$ of power series in $\mathbf{K}[[t]]$ is very close in the t-adic topology to being an arc on X (which means that γ coincides with an $r-$jet on X for some large r) then there is an actual arc γ' on X which is close to γ in the sense that γ coincides to $t-$adic order m with the $m-$jet of γ'; the Artin-Greenberg function $\beta(m)$ measures how close you need to be to an arc on X to have the same m-jet as an arc on X; again, roughly speaking, the larger the function is, the nastier the singularity $(X, 0)$ is.

The Arc Hilbert Poincaré series is related in spirit to these two types of invariants: Heuristically, the more we have free variables at the level m, the larger will be the dimension of the homogeneous components of A^0_∞ of weight less than or equal to m will be (note that the homogeneous components of weight less than or equal to m are the same as those of the ring of global sections of X^0_m) but also the larger is the Artin-Greenberg function. But this invariant is very difficult to compute, because of the complicated homological properties of A^0_∞ in general, even though sometimes for mild singularities this is possible, [26]:

Theorem 4.9.3 ([26]) *Let X be a normal hypersurface in $\mathbf{A}^n$ with a canonical singularity of multiplicity $n-1$ at the origin. Then*

$$\mathrm{AHP}_{X,0}(q) = \left(\prod_{i=1}^{n-2} \frac{1}{1-q^i}\right)^n \left(\prod_{i \geq n-1} \frac{1}{1-q^i}\right)^{n-1}.$$

This generalizes a theorem that was obtained in [99] for rational double point surface singularities. Some research is still ongoing to reveal the secrets of this invariant of singularities but let us go back now to partitions and to a beautiful link with the Arc-Hilbert-Poincaré series [25]:

Theorem 4.9.4 ([25])

$$\textit{For} \quad X = Spec \frac{\mathbf{K}[x]}{(x^2)}, \quad \mathrm{AHP}_{X,0}(q) = \prod_{i \equiv 1,4 \pmod 5} \frac{1}{1-q^i}.$$

Notice that the power series in the theorem is the right hand side of the first Rogers Ramanujan identity. The proof uses the differential structure of A^0_∞ which for $X = \mathrm{Spec}\frac{\mathbf{K}[x]}{(x^2)}$ is given by

$$A^0_\infty = \frac{\mathbf{K}[x_i, i \in \mathbf{N}_{>0}]}{[x_1^2]},$$

where $[x_1^2]$ is the differential ideal generated by x_1^2 and its iterated derivatives with respect to the derivation D which is determined by $D(x_i) = x_{i+1}$. So

$$[x_1^2] = (x_1^2, 2x_1x_2, 2x_1x_3 + 2x_2^2, \ldots) \tag{4.10}$$

The grading of A^0_∞ is induced from the weights given to the variables, x_i being of weight i. We order the monomials using an "adapted" monomial ordering, the weighted reverse lexicographical ordering; Now, it is well known that the Hilbert Poincaré series of the quotient ring by an ideal I is equal to the Hilbert Poincaré series of the quotient ring by the leading ideal (relative to a monomial ordering which respects the weight) of I. This latter is generated by the leading monomials of the elements of a Groebner basis of I. In general, it is very complicated to find a Groebner basis theoretically, even when we consider, let us say, the ideal generated by the first 5 generators of $I := [x_1^2]$, we should add many polynomials to obtain a Groebner basis [12]; the miracle is that the generators in (4.10) give a Groebner basis with respect to the weighted reverse lexicographical ordering. The proof shows actually that any S-polynomial (this is a notion used in Buchberger algorithm for computing a Groebner basis) is not relevant and it comes out, after determining its weight w, from the $(w-4)-$th derivative (by D) of the equation

$$2x_2(x_1^2) - x_1(2x_1x_2) = 0.$$

We deduce that

$$\mathrm{AHP}_{X,0}(q) = \mathrm{HP}(\frac{\mathbf{K}[x_i, i \in \mathbf{N}]}{\left(x_i^2, x_ix_{i+1}; i \in \mathbb{N}_{>0}\right)}),$$

where HP stands for the Hilbert-Poincaré series and where the ideal

$$(x_i^2, x_ix_{i+1}; i \in \mathbb{N}_{>0})$$

is the leading ideal of $[x_1^2]$. Now after a short reasoning, one sees that $\mathrm{HP}(\frac{\mathbf{K}[x_i, i \in \mathbb{N}]}{(x_i^2, x_ix_{i+1}; i \in \mathbb{N}_{>0})})$ is exactly the generating series of the number of partitions of n with neither consecutive nor equal parts. Using the first Rogers-Ramanujan identity we get the formula in the theorem.

Moreover, with very simple commutative algebra applied to

$$\mathrm{HP}\left(\frac{\mathbf{K}[x_i, i \in \mathbf{N}]}{(x_i^2, x_i x_{i+1}; i \in \mathbb{N}_{>0})}\right)$$

we find that there is a sequence of power series in the variable q which converges in the $q-$adic topology to both sides of the Rogers-Ramanujan identities giving a commutative algebra approach to these identities; this sequence was stated in an empirical way in [9].

This theorem was greatly generalized in [26]:

Theorem 4.9.5 ([26]) *For* $X = Spec\frac{\mathbf{K}[x]}{(x^n)}$,

$$AHP_{X,0}(q) = \prod_{i \not\equiv\ 0,n,n+1\ mod(2n+1)} \frac{1}{1-q^i}.$$

The proof uses similar ideas but the differential calculus is much more involved. This latter theorem is related to Gordon's identities which are partition identities generalizing the Rogers-Ramanujan identities. A commutative algebra proof of Gordon's identities was found in the PhD thesis of Pooneh Afsharijoo [4].

Now recall that in the proof of Theorem 4.9.3, we considered the Groebner basis of the ideal $[x_1^2]$ with respect to the weighted reverse lexicographical ordering; the heuristic reason of the choice of this ordering is that this allows to see first (*i.e.,* as leading monomials) the monomials which concern the larger neighborhoods from the point of view of Taylor series: for instance for the polynomial $x_2^2 + x_1x_3$, the leading term with respect to the reverse lexicographical ordering is x_2^2 which concerns an approximation of order 2 while x_1x_3 concerns an approximation of order 3. But as mentioned before, the Hilbert series of the quotient by the ideal $[x_1^2]$ is equal to the Hilbert series of the quotient by its leading monomial ideal with respect to any monomial ordering respecting the weight. With Pooneh Afsharijoo, we considered the weighted lexicographical ordering and we knew that if we catch the leading monomial ideal of $[x_1^2]$ with respect to this ordering, its Hilbert series will be equal to the generating series of the number of partitions appearing in the Rogers-Ramanujan identities, but potentially it counts partitions with different properties. The problem is that while the Groebner basis of $[x_1^2]$ with respect to the weighted reverse lexicographical ordering is differentially finite (*i.e.,* it is obtained from a finite number of polynomials -here only one polynomial- and all their derivatives), we were able to prove that with respect to the weighted lexicographical ordering, there is no Groebner basis of $[x_1^2]$ which is differentially finite [7]; A Groebner basis is then very difficult to determine; but using Groebner basis theory computations, we were able to conjecture what is the leading monomial ideal of $[x_1^2]$; this remains a conjecture but we were able to prove that the Hilbert series of the quotient by this monomial ideal is equal to the series appearing in the Rogers-Ramanujan identities. By taking a variation of the ideal $[x_1^2]$, we have been led to the following partition identities [7] where for a partition λ we denote by $s(\lambda)$ its smallest part.

Theorem 4.9.6 ([7]) *Let $n \geq k$ be positive integers. The number of partitions λ of n whose parts are larger or equal to k and whose size is less than or equal to $s(\lambda) - (k-1)$ is equal to the number of partitions of n with parts larger or equal to k and without neither consecutive nor equal parts.*

For $k = 1$, this gives another member of Rogers-Ramanujan identities: Let $n \geq 1$ be a positive integer. The number of partitions of n with size less than or equal to the smallest part is equal to the number of partitions of n without consecutive nor equal parts.

It is playful to see this last identity on the partitions of 4 but let us first call the partitions of n with size less than or equal to the smallest part, partitions of third type; partitions of first and second type were defined in Theorem 4.9.2. The partitions of 4 are

$$\begin{aligned} 4 &= 4 \\ &= 3+1 \\ &= 2+2 \\ &= 2+1+1 \\ &= 1+1+1+1 \end{aligned}$$

The partitions of 4 which are of the first type are the first and the second partitions.

The partitions of 4 which are of the second type are the first and the fifth partitions.

The partitions of 4 which are of the third type are the first and the third partitions. And as the theorem predicts, the number of these partitions, two, is the same for the three types.

Using an idea similar to the one used to guess Theorem 4.9.6, Pooneh Afsharijoo has conjectured in her thesis new identities which add new members to Gordon's identities [4, 6]; she proved this conjecture in a particular case and very recently with Pooneh Afsharijoo, Jehanne Dousse and Frédéric Jouhet, we proved these very exciting identities in general, this is the content of [5] (see also [3]).

These theorems are small steps (walking steps towards another planet) in studying what we would like to call **Ramanujan Hilbert scheme**, which parametrizes the schemes with a cone structure and whose Hilbert series is equal to $F(1)$.

There are various generalizations of these theorems or these line of thoughts. I can mention for instance [96] or a theorem on partitions of two colours in [7].

4.10 Completions of Localizations of the Algebra of Arcs

We assume for simplicity that we are working over an algebraically closed field $\mathbf{K}$ of characteristic zero. Most of the invariants that we have considered in the previous section (apart form Sect. 4.9) made use of the reduced structure of the jet schemes or the space of arcs. This latter, being often of infinite dimension, its study requires commutative algebra in infinite dimension, which till now seems to be a difficult issue. Let X be a $\mathbf{K}$-algebraic variety; the local algebras of the arc space X_∞ were studied for two types of points or arcs (for different reasons or motivations):

- The study of the local algebra of X_∞ at a $\mathbf{K}$-rational arc γ was motivated by some problems related to the Langlands program [24]. The first interesting theorem in this direction is the Drinfeld-Grinberg-Kazhdan theorem[20, 49, 63] which states that whenever $\gamma \notin (Sing(X))_\infty$, there exists a finite dimensional $\mathbf{K}$-scheme Y, a $\mathbf{K}$- point $y \in Y$ and an isomorphism

$$\hat{\mathcal{O}}_{X_\infty,\gamma} \equiv \hat{\mathcal{O}}_{Y,y} \hat{\otimes} \mathbf{K}[[T_i, i \in \mathbf{N}]]. \quad (\star)$$

  The hat denotes the completion with respect to the maximal ideal. Bourqui and Sebag considered a minimal $\hat{\mathcal{O}}_{Y,y}$ in $(\star)$, in the sense that $\hat{\mathcal{O}}_{Y,y}$ is not isomorphic to $B[[T]]$ (B being a local complete noetherian $\mathbf{K}$-algebra); this formal spectrum of $\hat{\mathcal{O}}_{Y,y}$ is then uniquely determined up to isomorphism and called the formal minimal model of X at γ. For an irreducible plane curve singularity, the formal minimal model is independent of the choice the (primitive) arc γ and thus defines an invariant of the singularity. For more about minimal formal models see [21–23, 30].
- The study of the local algebra of the arc space at a point associated with divisorial valuations (see the Sect. 4.3, see also [71, 119]); this was motivated by the Nash problem [119, 120]. It was proved by Reguera that for a divisorial valuation $\nu = \nu_E$ (E being the divisor) having a center on X, denoting by P_ν the point in the arc space associated with ν, the ring $\hat{\mathcal{O}}_{X_\infty,P_\nu}$ is noetherian. Mourtada and Reguera found a formula relating the embedding dimension of $\hat{\mathcal{O}}_{X_\infty,P_\nu}$ to the Mather discrepency $\hat{k}_E$ of E and found an upper bound of its dimension in terms of the Mather-Jacobian discrepancy [103].

Recently, there is a lot of interest on one hand (in the work of Bourqui, Sebag and others) in comparing the structure of $\hat{\mathcal{O}}_{X_\infty,P_{\nu_E}}$ with the structure of the minimal formal model at a generic $\mathbf{K}$- arc in the family of arcs associated with E (see Sect. 4.3); and on the other hand in understanding the relation between the singularities of X and the structure of $\hat{\mathcal{O}}_{X_\infty,P_\nu}$ in the continuity of [103] (see also [30]).

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Chapter 5
Indices of Vector Fields and 1-Forms

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Abstract We discuss the notions of indices of vector fields and 1-forms and their generalizations to singular varieties and varieties with actions of finite groups, as well as indices of collections of vector fields and 1-forms.

5.1 Introduction

Vector fields on a smooth manifold (a real or complex one) and their singular points play an important role in many different areas of mathematics. A classical invariant of a singular point of a vector field is its index. The notion of the index has a long history reflected in a number of classical sources. A famous result is the Poincaré–Hopf theorem which states that the sum of the indices of the (isolated) singular points of a vector field on a closed (compact, without boundary) manifold is equal to the Euler characteristic of the manifold.

There is no straightforward generalization of the notion of the index of a singular point to vector fields on singular varieties. There are several concepts of indices in this case. Some of them require special conditions on the vector fields and/or on the singular variety.

In the smooth case, there is essentially no difference between vector fields and 1-forms. In the case of singular varieties, these settings are essentially different. Traditionally the main attention was payed to indices of singular points of vector fields (cf. [22]). A suggestion to consider indices of singular points of 1-forms alongside with (or instead of) indices of vector fields was first made by V. I. Arnold in [7] (he used them for manifolds with boundaries; see also [9]). The authors started a study of indices of 1-forms on singular varieties [35]. (It should be mentioned that R. MacPherson also used particular 1-forms on singular varieties in [94] to define the local Euler obstruction.)

Indices of vector fields or of 1-forms on (compact) complex analytic manifolds are related with the Euler characteristic, that is with the top Chern number. Other Chern numbers correspond to indices of collections of vector fields or of 1-forms. Therefore it is interesting to study such indices.

There are some equivariant versions of the Euler characteristic for spaces with an action of a finite group G. Therefore it is reasonable to try to define indices of singular points of G-invariant vector fields and 1-forms on G-varieties as well.

Here we give a survey on all these concepts. We give no proofs but for every statement we give precise references to articles where one can find more details including proofs. There is a comprehensive textbook [22] by J.-P. Brasselet, J. Seade, and T. Suwa on a number of these subjects. We also wrote a survey article [41] about the developments till 2005.

Let us outline the contents of this article. In Sect. 5.2, we collect the basic notions and classical facts for the case of smooth manifolds.

In Sect. 5.3, we discuss different generalizations of the notion of the index of a singular point to vector fields and 1-forms on singular varieties.

We start (in Sect. 5.3.1) with the notion which was classically defined first and which is very general. It is the radial or Schwartz index. The idea goes back to M.-H. Schwartz who started a comprehensive study of vector fields on singular (analytic) varieties in [107]. She considered a class of vector fields important for a number of constructions related with vector fields on singular varieties: so called radial vector fields and radial extensions. For these vector fields she proved a version of the Poincaré–Hopf theorem [108–110]. Building on her work, a general notion of an index of an isolated singular point of a vector field on a singular variety was introduced in [82]. However, this preprint was not published but only circulated around (and was also presented at a number of conferences). (A revised version of it was only published almost 20 years later [83].) Because of its restricted circulation, parts of it were later (re)elaborated in publications of other authors. In the general setting, for analytic varieties, this notion of index (called Schwartz index or radial index) was defined in [34], see also [38]. (For the case of varieties with isolated singularities, it was defined and studied, in particular, in [1, 117]. The paper [1] also treated varieties with non-isolated singularities. However, in this case, not indices of singular points of a vector field were defined, but indices of a vector field for connected components of the singular locus of the variety.) It was noticed in [31] that it can also be defined for semianalytic sets. The index was used to define characteristic classes for singular varieties, see, e.g., the surveys [14, 113].

Section 5.3.2 is devoted to a notion of an index which makes only sense on varieties of special types. It is the GSV index named after X. Gómez-Mont, J. Seade, and A. Verjovsky who defined it in [64] for vector fields on hypersurfaces with isolated singularities. It was generalized to vector fields on isolated complete intersection singularities in [116]. A reinterpretation as the "virtual index" is discussed in Sect. 5.3.6. This was extended to vector fields on compact complete intersections with non-isolated singularities in [92].

In Sect. 5.3.3, we introduce the Poincaré–Hopf index for a vector field or 1-form on an isolated complex analytic singularity, a notion which is directly related to the Poincaré–Hopf theorem but is only defined for a smoothing of the singularity. It was first defined for vector fields on complex analytic surfaces in [111]. It coincides with the GSV index for isolated complete intersection singularities.

For the index of an isolated singular point of a holomorphic vector field on a complex manifold one has an algebraic formula, see Sect. 5.2.4. The search for such a formula for the index of an isolated singular point of a holomorphic vector field on a hypersurface with an isolated singularity was the motivation for Gómez-Mont to introduce the notion of a homological index in [61]. This is the subject of Sect. 5.3.4.

The radial (Schwartz) index can be defined for singular points of vector fields on arbitrary analytic (or semi-analytic) varieties. Another notion of an index with this property is the Euler obstruction which is discussed in Sect. 5.3.5. The key ingredient in its construction was defined by R. MacPherson in [94]. He defined the Euler obstruction of the differential of the squared distance function. For vector fields, it was essentially defined by Brasselet and Schwartz in [18, 23] and [22]. See a formal definition in [26].

The homological index gives rise to algebraic formulas for the GSV index of a holomorphic vector field or 1-form on certain singular varieties generalizing those of Sect. 5.2.4. These formulas are discussed in Sect. 5.3.6. We also discuss analytic and topological formulas for the index.

Some of the indices are not defined for general analytic varieties, but all the indices above are at least defined for isolated complete intersection singularities. The next more general class is the class of essentially isolated determinantal singularities introduced in [44] which recently attracted some attention. In the last subsection of Sect. 5.3 (Sect. 5.3.7), we study results on indices of 1-forms on such singularities.

In Sect. 5.4, we consider analogues of the above indices for collections of vector fields and 1-forms. An analogue of the GSV index for them was introduced in [39]. It is discussed in Sect. 5.4.1.

An analogue of the notion of the Euler obstruction for collections of 1-forms corresponding to different Chern numbers leads to the notion of Chern obstructions introduced in [42, 43]. This is considered in Sect. 5.4.2. There we also discuss relations between the Euler obstruction of a map defined in [70] and the Chern obstruction of a convenient collection of 1-forms observed by Brasselet, N. G. Grulha Jr., and M. A. S. Ruas in [15].

Finally we discuss, in Sect. 5.4.3, the generalization of the homological index to collections of 1-forms due to E. Gorsky and the second author.

The final section (Sect. 5.5) is devoted to the case that the variety carries an action of a finite group G. Through the Poincaré–Hopf theorem indices of singular points of vector fields or 1-forms are often related with the Euler characteristic of the underlying variety. The Euler characteristic (properly defined) is an additive topological invariant of spaces of some kind, say, of locally closed unions of cells in finite CW-complexes. For topological spaces with additional structures one has other additive topological invariants which can be considered as generalized Euler characteristics. One can expect appropriate notions of indices of singular points corresponding to these concepts. There are some notions of generalized Euler characteristics for spaces with an action of a finite group G. Some of them take values in the group $\mathbb{Z}$ of integers: e.g. the alternating sum of the ranks of the invariant parts of the cohomology groups with compact support, the orbifold Euler characteristic ([10, 76]), etc. Another one is the alternating sum of classes of the cohomology groups as G-modules. It takes values in the ring of representations of the group G. It was introduced in [132], see also [134]. The most general (in some sense—universal) concept of the generalized Euler characteristic for G-spaces is the equivariant Euler characteristic which takes values in the Burnside ring of the group G. It was introduced in [129]. We discuss the definition in Sect. 5.5.1.

A study of indices of singular points of G-invariant vector fields related with the Burnside ring of G was started in [93]. The indices therein had values not in the Burnside ring of G, but in a related abelian group (not a ring) depending on the underlying G-manifold. A study of indices as elements of the Burnside ring itself was initiated in [45] and continued in [46]. We first introduce these indices in the case of manifolds in Sect. 5.5.2.

The following subsections treat equivariant versions of the indices of vector fields and 1-forms on singular varieties discussed in Sect. 5.3. Namely, the equivariant radial index is treated in Sect. 5.5.3, the equivariant GSV and Poincaré–Hopf index in Sect. 5.5.4, the equivariant homological index in Sect. 5.5.5 and the equivariant Euler obstruction in Sect. 5.5.6.

The final subsection Sect. 5.5.7 discusses an attempt to generalize the Eisenbud–Levine–Khimshiashvili theorem to real quotient singularities.

5.2 The Case of Smooth Manifolds

5.2.1 The Index in the Real Case

Let M be a smooth manifold of dimension n and let X be a vector field on M. A neighbourhood of a point $p \in M$ can be identified with a neighbourhood U of the origin in $\mathbb{R}^n$. In local coordinates around a point p, X can be written as $X = \sum_{i=1}^{n} X_i(x)\frac{\partial}{\partial x_i}$, where p corresponds to the origin in $\mathbb{R}^n$. The vector field is called *continuous, smooth, analytic, etc.,* if the functions X_i are continuous, smooth, analytic, etc., respectively. A point $p \in M$ with $X(p) = 0$ is called a *zero* or a *singular point* of X. We shall define the index of a vector field at an isolated singular point.

For this purpose, we define the local degree of a mapping. Let $U \subset \mathbb{R}^n$ be an open subset and $F : U \to \mathbb{R}^n$ be a continuous mapping. Let $p \in U$ with $F(p) = 0$ and let $B_\varepsilon^n(p) = \{x \in \mathbb{R}^n \,|\, ||x - p|| \le \varepsilon\}$ be the ball of radius ε centred at p contained in U such that there are no preimages of F of the origin except p inside $B_\varepsilon^n(p)$. The *local degree* $\deg_p F$ of the mapping F at the point p is the degree of the mapping

$$\frac{F}{||F||} : S_\varepsilon^{n-1}(p) \to S_1^{n-1}$$

where $S_\varepsilon^{n-1}(p) = \partial B_\varepsilon^n(p)$ and S_1^{n-1} is the unit sphere in $\mathbb{R}^n$.

Definition 5.2.1 Let X be a continuous vector field defined on $U \subset \mathbb{R}^n$ and let $p \in U$ be an isolated zero of X. The *index* $\operatorname{ind}(X; \mathbb{R}^n, p)$ of X at the singular point $p \in U$ is the degree of the mapping $(X_1, \dots, X_n) : U \to \mathbb{R}^n$.

One can easily see that the definition of the index is independent of the choice of the local coordinates and of the sphere.

Now let the vector field X be smooth and p a non-degenerate singular point of X. This means that $J_{X,p} := \det\left(\frac{\partial X_i}{\partial x_j}(0)\right) \neq 0$. Then $\mathrm{ind}(X; \mathbb{R}^n, p) = \mathrm{sign}\, J_{X,p}$, where $\mathrm{sign}\, J_{X,p}$ denotes the sign of $J_{X,p}$, i.e.,

$$\mathrm{sign}\, J_{X,p} = \begin{cases} 1 & \text{if } J_{X,p} > 0, \\ -1 & \text{if } J_{X,p} < 0. \end{cases}$$

The index of an arbitrary isolated singular point p of a smooth vector field X is equal to the number of non-degenerate singular points $\widetilde{p}$ which split from the point p under a generic perturbation $\widetilde{X}$ of the vector field X in a neighbourhood of the point p counted with the appropriate signs $\mathrm{sign}\, J_{\widetilde{X},\widetilde{p}}$.

One of the most important properties of the index of a vector field is the Poincaré–Hopf theorem. Suppose that the manifold M is closed, i.e. compact without boundary, and that the vector field X has finitely many singular points on it. This is equivalent to say that X has only isolated singular points.

Theorem 5.2.2 (Poincaré–Hopf) *Let M be a closed (i.e. compact without boundary) manifold and X be a smooth vector field with only finitely many singular points on it. Then the sum*

$$\sum_{p \in \mathrm{Sing}\, X} \mathrm{ind}(X; M, p)$$

of indices of singular points of the vector field X is equal to the Euler characteristic $\chi(M)$ of the manifold M.

For a proof of this theorem see, e.g. [98].

Instead of a vector field on M, one can consider a 1-form ω on M. In local coordinates, ω can be written as $\omega = \sum_{i=1}^n A_i(x)dx_i$. The notions *continuous, smooth, analytic, etc.,* and *zero, singular point* are defined analogously. Indeed, using a Riemannian metric one can identify vector fields and 1-forms on a smooth (C^∞) manifold. In particular, the index $\mathrm{ind}(\omega; M, p)$ of ω at an isolated singular point p is defined to be the degree of the map $(A_1, \ldots, A_n) : U \to \mathbb{R}^n$.

5.2.2 *The Index in the Complex Case*

Now let M be a complex manifold of complex dimension n and X be a vector field on M. The index $\mathrm{ind}(X; M, p)$ of the vector field X at a singular point $p \in M$ is defined as the index of X at p on the underlying real $2n$-dimensional manifold. If the vector field is holomorphic and the singular point p is non-degenerate, then the index is equal to $+1$. The index of an isolated singular point p of a holomorphic vector field X is positive. It is equal to the number of non-degenerate singular points which split from the point p under a generic holomorphic perturbation of the vector field in the neighbourhood of p.

If M is a compact complex manifold, then its Euler characteristic $\chi(M)$ is equal to the characteristic number $\langle c_n(TM), [M]\rangle$, where $c_n(M)$ is the top Chern class of the manifold M. Therefore the Poincaré-Hopf theorem for a vector field X on a compact complex manifold M states that the sum of indices of the singular points on the vector field X is equal to $\langle c_n(TM), [M]\rangle$.

Now let ω be a complex continuous 1-form on M. There is a one-to-one correspondence between complex 1-forms and real 1-forms on the underlying real $2n$-dimensional manifold. Namely, to a complex 1-form ω one can associate the real 1-form $\eta = \operatorname{Re}\omega$. However, there is a difference in the orientation of the complex cotangent bundle T^*M and the orientation of the real cotangent bundle with its complex structure forgotten. Therefore, the index $\operatorname{ind}(\operatorname{Re}\omega; \mathbb{R}^n, p)$ does not coincide with the index of the 1-form ω at the point p, but differs from it by the sign $(-1)^n$ (see, e.g., [38, p. 235]). Therefore we define:

Definition 5.2.3 The *index* $\operatorname{ind}(\omega; \mathbb{C}^n, p)$ of the complex 1-form ω at a singular point p is $(-1)^n$ times the index of the real 1-form $\operatorname{Re}\omega$ at p:

$$\operatorname{ind}(\omega; \mathbb{C}^n, p) := (-1)^n \operatorname{ind}(\operatorname{Re}\omega; \mathbb{R}^n, p).$$

With this definition, the Poincaré-Hopf theorem for a complex 1-form ω on a compact complex manifold M states that the sum of the indices of the singular points of ω is equal to $(-1)^n\chi(M) = \langle c_n(T^*M), [M]\rangle$.

5.2.3 *Collections of Sections of a Vector Bundle*

The complex index introduced above is connected with the Euler characteristic, hence with the characteristic number $\langle c_n(TM), [M]\rangle$, where $c_n(M)$ is the top Chern class of the manifold M. In this section, we discuss indices related to other characteristic numbers.

Let M be a complex analytic manifold of dimension n and let $\{X_j\} = (X_1, \ldots, X_k)$ be a collection of k continuous vector fields on M. A *singular point* of $\{X_j\}$ is a point $p \in M$ where $(X_1(p), \ldots, X_k(p))$ are linearly dependent. A collection $\{X_j\}$ is also called a *k-field* and a non-singular one is called a *k-frame*. We recall the construction of Chern classes by obstruction theory.

For natural numbers n and k with $n \geq k$, let $M_{n,k}$ be the space of $n \times k$ matrices with complex entries and let $D_{n,k}$ be the subspace of $M_{n,k}$ consisting of matrices of rank less than k. The subset $D_{n,k}$ is a subvariety of $M_{n,k}$ of codimension $n - k + 1$. The complement $W_{n,k} = M_{n,k} \setminus D_{n,k}$ is the Stiefel manifold of k-frames (collections of k linearly independent vectors) in $\mathbb{C}^n$. It is known that $W_{n,k}$ is $(2n - 2k)$-connected and $H_{2n-2k+1}(W_{n,k}) \cong \mathbb{Z}$ (see, e.g., [78]). The latter fact also implies that the subvariety $D_{n,k}$ is irreducible. Since $W_{n,k}$ is the complement of an irreducible complex analytic subvariety of codimension $n-k+1$ in $M_{n,k}$, there is a natural choice of a generator of the homology group $H_{2n-2k+1}(W_{n,k}) \cong \mathbb{Z}$. Namely, the ("positive") generator is the boundary of a small ball in a smooth complex analytic slice in $M_{n,k} \cong \mathbb{C}^{nk}$ transversal to the irreducible subvariety $D_{n,k}$ at a non-singular point (oriented in the standard way).

Let $\{X_j\}$ be a k-field on M. Let (K) be a suitable triangulation of M and let (D) be a cell decomposition of M dual to (K). Let σ be a $2(n-k+1)$-cell of (D) which is contained in an open subset $U \subset M$ where the tangent bundle TM is trivial. Let

$$(X_1(y), \dots, X_k(y))$$

be the $n \times k$-matrix the columns of which consist of the components of the vectors $X_1(y), \dots, X_k(y)$ with respect to this trivialization. Assume that $\{X_j\}$ is a k-frame on $\partial\sigma$. Let $\psi_\sigma : \partial\sigma \cong S^{2(n-k)+1} \to W_{n,k}$ be the mapping which sends a point $y \in \partial\sigma$ to the matrix $(X_1(y), \dots, X_k(y))$.

Definition 5.2.4 The *index* $\mathrm{ind}(\{X_j\}; \sigma)$ of the k-field $\{X_j\}$ on σ is the degree of the map ψ, i.e., the obstruction to extend the k-frame $\{X_j\}$ from the boundary $\partial\sigma$ of the cell σ to its interior.

This defines a cochain $\gamma \in C^{2(n-k+1)}(M; \mathbb{Z})$ by setting $\gamma(\sigma) = \mathrm{ind}(\{X_j\}; \sigma)$ for each $2(n-k+1)$-cell and extending it linearly. This cochain is in fact a cocycle and represents the Chern class $c^{n-k+1}(M)$ of M.

Let $\pi : E \to M$ be a complex analytic vector bundle of rank m over a complex analytic manifold M of dimension n. (Special cases of interest are the tangent and the cotangent bundles of M.) We shall now generalize the construction above.

Let $\{\omega_j^{(i)}\}$ ($i = 1, \dots, s$; $j = 1, \dots, m-k_i+1$; $\sum_{i=1}^s k_i = n$) be a collection of continuous sections of the vector bundle $\pi : E \to M$. A point $p \in M$ is called non-singular for the collection $\{\omega_j^{(i)}\}$ if at least for some $i \in \{1, \dots, s\}$ the values $\omega_1^{(i)}(p), \dots, \omega_{m-k_i+1}^{(i)}(p)$ are linearly independent. This means that for this i the vectors $\omega_1^{(i)}(p), \dots, \omega_{m-k_i+1}^{(i)}(p)$ form an $(m-k_i+1)$-frame. We assume that the collection $\{\omega_j^{(i)}\}$ has only isolated singular points. We shall define an index for such a collection, cf. [39].

Let $\mathbf{k} = (k_1, \dots, k_s)$ be a sequence of positive integers with $\sum_{i=1}^s k_i = n$. Consider the space $M_{m,\mathbf{k}} = \prod_{i=1}^s M_{m,m-k_i+1}$ and the subvariety $D_{m,\mathbf{k}} = \prod_{i=1}^s D_{m,m-k_i+1}$ in it. The variety $D_{m,\mathbf{k}}$ consists of sets $\{A_i\}$ of $m \times (m-k_i+1)$ matrices such that $\mathrm{rk}\, A_i < m-k_i+1$ for each $i = 1, \dots, s$. Since $D_{m,\mathbf{k}}$ is irreducible of codimension n, its complement $W_{m,\mathbf{k}} = M_{m,\mathbf{k}} \setminus D_{m,\mathbf{k}}$ is $(2n-2)$-connected, $H_{2n-1}(W_{m,\mathbf{k}}) \cong \mathbb{Z}$, and there is a natural choice of a generator of the latter group. This choice defines a degree (an integer) of a map from an oriented manifold of dimension $2n-1$ to the manifold $W_{m,\mathbf{k}}$.

Let us choose a trivialization of the vector bundle $\pi : E \to M$ in a neighbourhood of a point p, let

$$(\omega_1^{(i)}(x), \ldots, \omega_{m-k_i+1}^{(i)}(x))$$

be the $m \times (m - k_i + 1)$-matrix the columns of which consist of the components of the sections $\omega_j^{(i)}(x)$, $j = 1, \ldots, m-k_i+1$, $x \in M$, with respect to this trivialization. Let Ψ_p be the mapping from a neighbourhood of the point p to $M_{m,\mathbf{k}}$ which sends a point x to the collection of matrices $\{(\omega_1^{(i)}(x), \ldots, \omega_{m-k_i+1}^{(i)}(x))\}$, $i = 1, \ldots, s$. Its restriction ψ_p to a small sphere $S_\varepsilon^{2n-1}(p)$ around the point p maps this sphere to the subset $W_{m,\mathbf{k}}$. The sphere $S_\varepsilon^{2n-1}(p)$ is oriented as the boundary of the corresponding ball in the complex affine space $\mathbb{C}^n$.

Definition 5.2.5 The *index* $\operatorname{ind}(\{\omega_j^{(i)}\}; M, p)$ of the collection of sections $\{\omega_j^{(i)}\}$ at the point p is the degree of the mapping $\psi_p : S_\varepsilon^{2n-1}(p) \to W_{m,\mathbf{k}}$.

One can deform the collection $\{\omega_j^{(i)}\}$ of sections or, equivalently, the map Ψ_p so that this map becomes smooth and transversal to the variety $D_{m,\mathbf{k}}$ (at smooth points of the latter one). This implies that the index $\operatorname{ind}(\{\omega_j^{(i)}\}; M, p)$ is equal to the intersection number of the germ of the image of the map Ψ_p with the variety $D_{m,\mathbf{k}}$.

Definition 5.2.6 A singular point p of the collection $\{\omega_j^{(i)}\}$ of sections is called *non-degenerate* if the map Ψ_p is smooth and transversal to the variety $D_{n,\mathbf{k}} \subset \mathscr{M}_{n,\mathbf{k}}$ at a non-singular point of it.

If p is a non-degenerate singular point of the collection $\{\omega_j^{(i)}\}$, then $\operatorname{ind}_p\{\omega_j^{(i)}\} = \pm 1$. If all the sections $\omega_j^{(i)}$ are complex analytic, then this index is equal to $+1$.

The following statement is a generalization of the well known fact that the $(2(n-k)$-dimensional) cycle Poincaré dual to the characteristic class $c_k(E)$ $(k = 1, \ldots, m)$ is represented by the set of points of the manifold M where $m-k+1$ generic sections of the vector bundle E are linearly dependent (cf., e.g., [69, p. 413]).

Theorem 5.2.7 *Let $\sum_{i=1}^s k_i = n$ and suppose that the collection $\{\omega_j^{(i)}\}$ $(i = 1, \ldots, s;\ j = 1, \ldots, m - k_i + 1)$ of sections of the vector bundle $\pi : E \to M$ over a closed complex manifold M has only isolated singular points. Then the sum of the indices of these points is equal to the characteristic number $\langle \prod_{i=1}^s c_{k_i}(E), [M] \rangle$ of the vector bundle E.*

5.2.4 Algebraic Formulas for the Indices

It turns out that, if the vector fields (or 1-forms) under consideration are analytic (and, in the real case, the singular points are algebraically isolated: see the definition below) one has algebraic formulas for the indices considered above.

The simplest algebraic formula is for the index of an isolated singular point of a holomorphic vector field on a complex manifold. In local coordinates $z = (z_1, \dots, z_n)$ centred at the singular point, a vector field can be written as $X = \sum_{i=1}^{n} X_i(z)\frac{\partial}{\partial z_i}$, where the function germs X_i are holomorphic. Let $\mathcal{O}_{\mathbb{C}^n,0}$ be the ring of germs of holomorphic functions of n variables.

The first proof of the following theorem is usually attributed to Palamodov [103]. Without a detailed proof, it was known earlier.

Theorem 5.2.8 (Palamodov) *The index* $\operatorname{ind}(X; \mathbb{C}^n, 0)$ *of the singular point of the holomorphic vector field* X *is equal to the dimension of the complex vector space* $\mathcal{O}_{\mathbb{C}^n,0}/(X_1, \dots, X_n)$*, where* $(X_1, \dots, X_n)$ *is the ideal generated by the germs* $X_1, \dots, X_n$*.*

Recall that $J_{X,0} = \det\left(\frac{\partial X_i}{\partial z_j}(0)\right)$ denotes the determinant of the Jacobian matrix of $(X_1, \dots, X_n)$. Then one has the following residue formula for the index

$$\operatorname{ind}(X; \mathbb{C}^n, 0) = \operatorname{Res}\begin{bmatrix} J_{X,0}d\mathbf{z} \\ X_1 \cdots X_n \end{bmatrix} := \frac{1}{(2\pi i)^n}\int_{\Gamma} \frac{J_{X,0}}{X_1 \cdots X_n} d\mathbf{z}$$

where $d\mathbf{z} := dz_1 \wedge \cdots \wedge dz_n$, Γ is the real n-cycle $\{\|X_k\| = \delta_k, k = 1, \dots, n\}$ for positive δ_k small enough, and Γ is oriented so that $d(\arg X_1) \wedge \cdots \wedge d(\arg X_n) \geq 0$, see also [11].

For a real analytic vector field such that its complexification has an isolated singular point (in this situation one says that the singular point is *algebraically isolated*), the index can be computed as the signature of a certain quadratic form: [50, 81].

Let $F = (f_1, \dots, f_n) : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be the germ of an analytic mapping such that $F_{\mathbb{C}}^{-1}(0) = 0$, where $F_{\mathbb{C}} : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is the complexification of F (i.e. F has an algebraically isolated preimage of the origin). Let $\mathcal{E}_{\mathbb{R}^n,0}$ be the ring of germs of analytic functions on $(\mathbb{R}^n, 0)$. By the assumption $F_{\mathbb{C}}^{-1}(0) = 0$, the factor algebra $Q_F := \mathcal{E}_{\mathbb{R}^n,0}/(f_1, \dots, f_n)$ has finite dimension. (This dimension is equal to $\dim \mathcal{O}_{\mathbb{C}^n,0}/(f_1, \dots, f_n) = \deg_0 F_{\mathbb{C}}$.) We consider on Q_F the natural residue pairing

$$\begin{aligned} B_F : Q_F \times Q_F &\longrightarrow \mathbb{R} \\ (\varphi, \psi) &\longmapsto \operatorname{Res}\begin{bmatrix} \varphi(\mathbf{x})\psi(\mathbf{x})d\mathbf{x} \\ f_1 \cdots f_n \end{bmatrix} \end{aligned}$$

and the residue is similarly defined as in the complex case, namely

$$\operatorname{Res}\begin{bmatrix} \varphi(\mathbf{x})\psi(\mathbf{x})d\mathbf{x} \\ f_1 \cdots f_n \end{bmatrix} = \frac{1}{(2\pi i)^n}\int \frac{\varphi(\mathbf{x})\psi(\mathbf{x})}{f_1 \cdots f_n} d\mathbf{x}$$

where $d\mathbf{x} := dx_1 \wedge \cdots \wedge dx_n$ and the integration is along the cycle in $\mathbb{C}^n$ given by the equations $\|f_k(\mathbf{x})\| = \delta_k$ with positive δ_k small enough.

Theorem 5.2.9 (Eisenbud–Levine–Khimshiashvili) *The degree* $\deg_0 F$ *of the map germ* F *is equal to the signature* $\operatorname{sgn} B_F$ *of the quadratic form* B_F.

For a proof of this theorem see also [8].

This can also be interpreted as a formula for the index of the singular point of the vector field $X := \sum f_i \frac{\partial}{\partial x_i}$ or of the 1-form $\omega := \sum f_i dx_i$. Moreover, the choice of a volume form permits to identify the algebra Q_F (as a vector space) with the space $\Omega_\omega = \Omega^n_{\mathbb{R}^n,0}/\omega \wedge \Omega^{n-1}_{\mathbb{R}^n,0}$.

Now let $\{\omega_j^{(i)}\}$ $(i = 1, \ldots, s;\ j = 1, \ldots, m-k_i+1;\ \sum_{i=1}^s k_i = n)$ be a collection of holomorphic sections of the (trivial) vector bundle $\pi : \mathbb{C}^m \times (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ with an isolated singular point at the origin (see Sect. 5.2.3). Let $I_{\{\omega_j^{(i)}\}}$ be the ideal in the ring $\mathcal{O}_{\mathbb{C}^n,0}$ of germs of analytic functions of n variables generated by the $(m-k_i+1) \times (m-k_i+1)$-minors of the matrices $(\omega_1^{(i)}, \ldots, \omega_{m-k_i+1}^{(i)})$ for all $i = 1, \ldots, s$. Then one has the following algebraic formula for the index (see [39, Theorem 2]).

Theorem 5.2.10 *One has*

$$\operatorname{ind}(\{\omega_j^{(i)}\}; \mathbb{C}^n, 0) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0}/I_{\{\omega_j^{(i)}\}}.$$

5.3 Vector Fields and 1-Forms on Singular Varieties

5.3.1 Radial Index

Let V be a closed (real) subanalytic variety in a smooth manifold M, where M is equipped with a (smooth) Riemannian metric. Let $V = \bigcup_{i=1}^q V_i$ be a subanalytic Whitney stratification of V (see [130] for this notion). A (continuous) *stratified vector field* on $V = \bigcup_{i=1}^q V_i$ is a vector field such that, at each point p of V, it is tangent to the stratum containing p.

Definition 5.3.1 The germ X of a vector field on the germ (V, p) is called *radial* if, for all $\varepsilon > 0$ small enough, the vector field is transversal to the boundary of the ε-neighbourhood of the point p and is directed outwards.

Let $p \in V$, let $V_{(p)} = V_i$ be the stratum containing p, $\dim V_i = k$, and let X be a stratified vector field on V in a neighbourhood of the point p. The following definition was given by Schwartz in [108].

Definition 5.3.2 The vector field X is called a *radial extension* of the vector field $X_{|V_i}$ if, for all $\varepsilon > 0$ small enough, it is transversal to the boundary of the ε-tubular neighbourhood of V_i and points outwards of the neighbourhood.

Remark 5.3.3 Note that the definitions of the radial extension in [22, Definition 2.3.2]), [34, p. 144], and [45, p. 288] use different formulations, but are somewhat inaccurate.

The existence of a radial extension of a vector field on a stratum V_i is proved in [23, Section III.7] (see also [110, Lemme 3.1.2]).

Let X be a stratified vector field on (V, p) with an isolated singular point (zero) at p. Let B_ε be a closed ε-neighbourhood in the ambient Riemannian manifold M around the point p (small enough so that the boundary ∂B_η of the η-neighbourhood of p with $0 < \eta \leq \varepsilon$ intersects all the strata V_i transversally and the vector field X has no singular points on $(V \setminus \{p\}) \cap B_\varepsilon$). One can show that there exists a (continuous) stratified vector field $\widetilde{X}$ on V such that:

1. the vector field $\widetilde{X}$ coincides with X on a neighbourhood of the intersection of V with the boundary ∂B_ε of the ε-neighbourhood around the point p;
2. the vector field has a finite number of singular points (zeros);
3. in a neighbourhood of each singular point $q \in V \cap B_\varepsilon$, $q \in V_i$, the vector field $\widetilde{X}$ is a radial extension of its restriction to the stratum V_i.

Definition 5.3.4 The *radial index* (or *Schwartz index*) $\operatorname{ind}_{\text{rad}}(X; V, p)$ of the vector field X on V at the point p is

$$\operatorname{ind}_{\text{rad}}(X; V, p) := \sum_{q \in \operatorname{Sing} \widetilde{X}} \operatorname{ind}\,(\widetilde{X}_{|V_{(q)}}; V_{(q)}, q)\,,$$

where $\operatorname{ind}\,(\widetilde{X}_{|V_{(q)}}, V_{(q)}, q)$ is the usual index of the restriction of the vector field $\widetilde{X}$ to the smooth manifold $V_{(q)}$.

The proof of the fact that the radial index is well-defined holds for this definition as well. The described notions of a radial vector field and a radial extension depend on the choice of the Riemannian metric, but the radial index does not depend on this choice, because, for instance, the space of Riemannian metrics is pathwise connected.

Now let ω be (the germ at p of) a (continuous) 1-form on (V, p), i.e. the restriction to V of a 1-form defined in a neighbourhood of the point p in the ambient manifold M. Let $V = \bigcup_{i=1}^{q} V_i$ be a subanalytic Whitney stratification of V. A point $p \in V$ is a *singular point* of ω if the restriction of ω to the stratum $V_{(p)}$ containing p vanishes at the point p.

Definition 5.3.5 The germ ω of a 1-form at the point p is called *radial* if, for all ε small enough, the 1-form is positive on the outward normals to the boundary of the ε-neighbourhood of the point p.

An example of a radial 1-form is the germ of the 1-form $d\rho^2$, where ρ is the distance function from p induced by the Riemannian metric.

Remark 5.3.6 Note that the initial definition of a radial 1-form in [38] used other words, but was somewhat inaccurate. This was noticed by an anonymous referee of the paper [73].

Let $p \in V_i = V_{(p)}$, $\dim V_{(p)} = k$, and let η be a 1-form defined in a neighbourhood of the point p. As above, let N_i be a normal slice (with respect to the

Riemannian metric) of M to the stratum V_i at the point p and h a diffeomorphism from a neighbourhood of p in M to the product $U_i(p) \times N_i$, where $U_i(p)$ is an ε-neighbourhood of p in V_i, which is the identity on $U_i(p)$.

Definition 5.3.7 A 1-form η is called a *radial extension* of the 1-form $\eta_{|V_{(p)}}$ if there exists such a diffeomorphism h which identifies η with the restriction to V of the 1-form $\pi_1^*\eta_{|V_{(p)}} + \pi_2^*\eta_{N_i}^{\mathrm{rad}}$, where π_1 and π_2 are the projections from a neighbourhood of p in M to $V_{(p)}$ and N_i respectively and $\eta_{N_i}^{\mathrm{rad}}$ is a radial 1-form on N_i.

For a 1-form ω on (V, p) with an isolated singular point at the point p there exists a 1-form $\widetilde{\omega}$ on V which possesses the obvious analogues of the properties (1)–(3) of the vector field $\widetilde{X}$ above.

Definition 5.3.8 The *radial index* $\mathrm{ind}_{\mathrm{rad}}(\omega; V, p)$ of the 1-form ω at the point p is

$$\mathrm{ind}_{\mathrm{rad}}(\omega; V, p) = \sum_{q \in \mathrm{Sing}\,\widetilde{\omega}} \mathrm{ind}(\widetilde{\omega}_{|V_{(q)}}; V_{(q)}, q)\,,$$

where $\mathrm{ind}(\widetilde{\omega}_{|V_{(q)}}; V_{(q)}, q)$ is the usual index of the restriction of the 1-form $\widetilde{\omega}$ to the stratum $V_{(q)}$.

The definition of the radial index does not depend on the stratification and on the chosen vector field $\widetilde{X}$ nor on the chosen 1-form $\widetilde{\omega}$. The sum of the indices of an appropriate deformation of the vector field or 1-form on the strata is the same for a stratification and for a refinement of it. This implies that the radial index does not depend on the stratification. (For a vector field one can consider the intersection of two stratifications. For a 1-form one can consider the minimal Whitney stratification of the variety.) Moreover, the radial index does not depend on the chosen deformation of the vector field or 1-form. This follows from the following proposition which is proved in [45, Proposition 2.1].

Proposition 5.3.9 *The number of singular points (counted with multiplicities) of the vector field $\widetilde{X}$ or of a 1-form $\widetilde{\omega}$ on a fixed stratum V_i does not depend on the choice of the vector field $\widetilde{X}$ or of the 1-form $\widetilde{\omega}$ respectively (and therefore only depends on X or ω respectively).*

Therefore the radial index is well-defined.

It follows from the definition that the radial index satisfies the *law of conservation of number*. For a vector field X this means the following: if a vector field X' with isolated singular points on V is close to the vector field X, then

$$\mathrm{ind}_{\mathrm{rad}}(X; V, p) = \sum_{q \in \mathrm{Sing}\,X'} \mathrm{ind}_{\mathrm{rad}}(X'; V, q),$$

where the sum on the right hand side runs over all singular points q of the vector field X' on V in a neighbourhood of p.

The radial index generalizes the usual index for vector fields or 1-forms on a smooth manifold. In particular, one has a generalization of the Poincaré-Hopf theorem:

Theorem 5.3.10 (Poincaré–Hopf) *For a compact real subanalytic variety V and a vector field X or a 1-form ω with isolated singular points on V, one has*

$$\sum_{Q \in \operatorname{Sing} X} \operatorname{ind}_{\mathrm{rad}}(X; V, Q) = \sum_{Q \in \operatorname{Sing} \omega} \operatorname{ind}_{\mathrm{rad}}(\omega; V, Q) = \chi(V)$$

where $\chi(V)$ denotes the Euler characteristic of the space (variety) V.

In [87], the radial index of a vector field with an isolated zero on a real closed semialgebraic set with an isolated singularity is related to an intersection index.

Now we consider the germ (V, p) of a complex analytic variety of pure dimension n embedded in the germ of a complex manifold (M, p). Then one can define analogously the notion of a radial index for a singular point of a complex vector field or complex 1-form on (V, p). In particular, for the germ of a complex 1-form ω on (V, p) the radial index $\operatorname{ind}_{\mathrm{rad}}(\omega; V, p)$ is $(-1)^n$ times the radial index $\operatorname{ind}_{\mathrm{rad}}(\operatorname{Re}\omega; V, p)$ of the real 1-form $\operatorname{Re}\omega$ on (V, p).

5.3.2 GSV Index

Let $(V, 0) \subset (\mathbb{C}^N, 0)$ be the germ of an n-dimensional complete intersection with an isolated singularity at the origin, defined by a holomorphic map germ

$$f = (f_1, \dots, f_{N-n}) : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N-n}, 0),$$

i.e. $V = f^{-1}(0)$. The germ $(V, 0)$ is called an isolated complete intersection singularity (abbreviated ICIS in the sequel). Let $z_1, \dots, z_N$ denote the coordinates of $\mathbb{C}^N$. Let $X = \sum_{i=1}^N X_i(z)\frac{\partial}{\partial z_i}$ be the germ of a (continuous) vector field on $(\mathbb{C}^N, 0)$ tangent to V, i.e. $X(z) \in T_z V$ for all $z \in V \setminus \{0\}$. Suppose that X has an isolated singular point at the origin. Then the following index is defined. It was introduced for hypersurface singularities by Gómez-Mont et al. [64] and generalized to ICIS by Seade and Suwa [117]. It is called GSV index.

Let $B_\varepsilon \subset \mathbb{C}^N$ denote the ball of radius ε centred at the origin. Let $\varepsilon > 0$ be chosen small enough so that the functions $f_1, \dots f_{N-n}$ and the vector field X are defined in a neighbourhood of B_ε, V is transversal to the sphere $S_\eta = \partial B_\eta$ for $0 < \eta \leq \varepsilon$, and the vector field X has no zeros on V inside the ball B_ε except possibly the origin. The intersection $K := V \cap S_\varepsilon$ is called the *link* of the ICIS $(V, 0)$. The link K is a $(2n-1)$-dimensional manifold and has a natural orientation as the boundary of the complex manifold $(V \cap B_\varepsilon) \setminus \{0\}$.

Define the *gradient vector field* grad f_i of a function germ f_i by

$$\operatorname{grad} f_i = \left(\overline{\frac{\partial f_i}{\partial z_1}}, \dots, \overline{\frac{\partial f_i}{\partial z_N}} \right).$$

Note that it depends on the choice of the coordinates $z_1, \dots, z_N$. The gradient vector fields grad $f_1, \dots,$ grad f_{N-n} are linearly independent everywhere on V except (possibly) at the origin. The set $\{X(z), \operatorname{grad} f_1(z), \dots, \operatorname{grad} f_{N-n}(z)\}$ is an $(N-n+1)$-frame at each point of K. This frame defines a continuous map

$$\Psi = (X, \operatorname{grad} f_1, \dots, \operatorname{grad} f_{N-n}) : K \to W_{N,N-n+1}$$

from the link K to the Stiefel manifold $W_{N,N-n+1}$ of complex $(N-n+1)$-frames in $\mathbb{C}^N$. It is known that the Stiefel manifold $W_{N,N-n+1}$ is $2(n-1)$-connected and $H_{2n-1}(W_{N,N-n+1}) \cong \mathbb{Z}$, see Sect. 5.2.3. There is a natural choice of the generator of $H_{2n-1}(W_{N,N-n+1}) \cong \mathbb{Z}$. Therefore we can make the following definition:

Definition 5.3.11 The *GSV index* $\operatorname{ind}_{\mathrm{GSV}}(X; V, 0)$ of the vector field X on the ICIS V at the origin is the degree of the map

$$\Psi : K \to W_{N,N-n+1}\,.$$

Remark 5.3.12 Note that one uses the complex conjugation for this definition. Therefore the components of the discussed map are of different tensor nature. Whereas X is a vector field, grad f_i is more similar to a covector.

One can also consider the map Ψ as a map from V to the space $M_{N,N-n+1}$ of $N \times (N-n+1)$ matrices with complex entries (defined in a neighbourhood of the ball B_ε). It maps the set $V \setminus \{0\}$ to the Stiefel manifold $W_{N,N-n+1} = M_{N,N-n+1} \setminus D_{N,N-n+1}$ (see Sect. 5.2.3). Therefore we have the following result:

Proposition 5.3.13 *The GSV index* $\operatorname{ind}_{\mathrm{GSV}}(X; V, 0)$ *of the vector field X on the ICIS V at the origin is equal to the intersection number* $(\Psi(V) \circ D_{N,N-n+1})$ *of the image $\Psi(V)$ of the ICIS V under the map Ψ and the variety $D_{N,N-n+1}$ at the origin.*

Note that, even if the vector field X is holomorphic, the image $\Psi(V)$ is in general not a complex analytic variety because we use the complex conjugation in the definition of Ψ.

Now we consider the case of a 1-form on $(V, 0)$. Let $\omega = \sum A_i(z) dz_i$ be a germ of a continuous 1-form on $(\mathbb{C}^N, 0)$ which as a 1-form on the ICIS V has (at most) an isolated singular point at the origin (thus it does not vanish on the tangent space T_pV to the variety V at all points p from a punctured neighbourhood of the origin in V). The set $\{\omega(z), df_1(z), \dots, df_{N-n}(z)\}$ is a $(N-n+1)$-frame in the space dual to $\mathbb{C}^N$ for all $z \in K$. Therefore one has a map

$$\Psi = (\omega, df_1, \dots, df_{N-n}) : K \to W_{N,N-n+1}.$$

Here $W_{N,N-n+1}$ is the Stiefel manifold of $(N-n+1)$-frames in the space dual to $\mathbb{C}^N$.

Definition 5.3.14 The *GSV index* $\mathrm{ind}_{\mathrm{GSV}}(\omega; V, 0)$ of the 1-form ω on the ICIS V at the origin is the degree of the map

$$\Psi : K \to W_{N,N-n+1}\,.$$

Just as above Ψ can be considered as a map from the ICIS V to the space $M_{N,N-n+1}$ of $N \times (N-n+1)$-matrices and the GSV index of ω is equal to the intersection number $(\Psi(V) \circ D_{N,N-n+1})$. In contrast to the case of a vector field, if the 1-form ω is holomorphic, the map Ψ and the set $\Psi(V)$ are complex analytic.

A similar construction can be considered in the real setting, see [1] for more details.

5.3.3 Poincaré–Hopf Index

Let $(V,0) \subset (\mathbb{C}^N, 0)$ be the germ of a purely n-dimensional complex analytic variety with an isolated singularity at the origin. A *smoothing* of $(V,0)$ is a 1-parameter deformation $F : (\mathscr{V}, 0) \to (\mathbb{C}, 0)$ of $(V,0)$ (that is $F^{-1}(0) = (V,0)$) such that for $t \in \mathbb{C} \setminus \{0\}$ sufficiently close to 0 the fibre $\mathscr{V}_t = F^{-1}(t)$ is smooth, cf., e.g. [68, Definition 7.3.1]. The germ $(V,0)$ is called a *smoothable singularity* if there exists a smoothing of $(V,0)$.

Let $(V,0)$ be a smoothable singularity and let $F : \mathscr{V} \to \mathbb{C}$ be a suitable representative of a smoothing of $(V,0)$. For simplicity, we assume that $\mathscr{V}$ is embedded in an open neighbourhood $U \subset \mathbb{C}^N$ of the origin. Denote by B_ε the ball of radius ε centred at the origin in $\mathbb{C}^N$ and by Δ_η the disc in $\mathbb{C}$ of radius η centred at 0. Let $\Delta_\eta^* = \Delta_\eta \setminus \{0\}$. By Lê [89, Theorem 1.1] (see also [90, Theorem 6.4.1]), for $\varepsilon \gg \eta > 0$ sufficiently small, the mapping

$$F_{|_{F^{-1}(\Delta_\eta^*)\cap B_\varepsilon}} : F^{-1}(\Delta_\eta^*) \cap B_\varepsilon \to \Delta_\eta^*$$

is the projection of a differentiable fibre bundle over Δ_η^*. Let $V_t^F := \mathscr{V}_t \cap B_\varepsilon$ be the (Milnor) fibre of this bundle over $t \in \Delta_\eta^*$.

Now let X be the germ of a continuous vector field on $(V,0)$ with an isolated singularity at 0. Then the vector field does not vanish on $V \cap S_\varepsilon$, where $S_\varepsilon = \partial B_\varepsilon$ is the boundary of the ball B_ε. Moreover, the intersection $V \cap S_\varepsilon$ is isotopic to the intersection of V_t^F with this sphere. Therefore we can assume that the vector field X is defined on the boundary ∂V_t^F of the Milnor fibre. By Brasselet et al. [22, Theorem 1.1.2], there exists an extension $\widetilde{X}$ of the vector field X to the interior of the Milnor fibre V_t^F with a finite number of singular points.

Definition 5.3.15 The *Poincaré–Hopf index* of X on $(V, 0)$ relative to the smoothing F is

$$\mathrm{ind}^F_{\mathrm{PH}}(X; V, 0) := \sum_{q \in \mathrm{Sing}\, \widetilde{X}} \mathrm{ind}(\widetilde{X}; V_t^F, q),$$

where the sum runs over the singular points of the vector field $\widetilde{X}$ on the fibre V_t^F.

The Poincaré–Hopf index depends on the choice of a smoothing, but does not depend on the choice of t and of the extension $\widetilde{X}$ [22, Proposition 3.4.1]. In particular, one has the following proposition (see also [22, Proposition 3.4.1]).

Proposition 5.3.16 *Let the vector field X be transversal to the link $K = V \cap S_\varepsilon$ of the singularity $(V, 0)$. Then*

$$\mathrm{ind}^F_{\mathrm{PH}}(X; V, 0) = \chi(V_t^F).$$

Now let $(V, 0)$ be the germ of a complete intersection with an isolated singularity at the origin. Then there is an essentially unique smoothing of $(V, 0)$, since the base space of the semi-universal deformation is smooth (see [68, Theorem 7.2.22]). Therefore we can write in this case $\mathrm{ind}_{\mathrm{PH}}(X; V, 0) := \mathrm{ind}^F_{\mathrm{PH}}(X; V, 0)$, where F is the unique smoothing of $(V, 0)$. In this case we have:

Proposition 5.3.17 *For a vector field X on an ICIS $(V, 0)$ we have*

$$\mathrm{ind}_{\mathrm{PH}}(X; V, 0) = \mathrm{ind}_{\mathrm{GSV}}(X; V, 0).$$

Therefore the Poincaré–Hopf index relative to a smoothing is called the GSV index relative to a smoothing in [22].

Seade defined in this way an index for a singular point of a vector field on a complex analytic surface with a smoothable normal Gorenstein singularity [111].

The following corollary of Propositions 5.3.16 and 5.3.17 is proved in [117, Proposition 1.4].

Proposition 5.3.18 *For a vector field X on an ICIS $(V, 0)$ we have*

$$\mathrm{ind}_{\mathrm{GSV}}(X; V, 0) = \mathrm{ind}_{\mathrm{rad}}(X; V, 0) + (-1)^n \mu,$$

where μ is the Milnor number of $(V, 0)$.

Similarly, for a 1-form one can prove [49, Proposition 2.8]:

Proposition 5.3.19 *For a 1-form ω on an ICIS $(V, 0)$ we have*

$$\mathrm{ind}_{\mathrm{GSV}}(\omega; V, 0) = \mathrm{ind}_{\mathrm{rad}}(\omega; V, 0) + \mu.$$

There is a generalisation of this index to vector fields on germs of complete intersections with non-isolated singularities [20], see also [22, Section 3.5]. Let $(V, 0)$ be the germ of a complete intersection defined by a map germ $F = (f_1, \dots, f_k) : (\mathbb{C}^N, 0) \to (\mathbb{C}^k, 0)$. We assume that a neighbourhood of the origin in $\mathbb{C}^n$ permits a Whitney stratification adapted to V and satisfying the Thom condition (a_f) [130, Definition 4.4.1]. This holds, in particular, if $(V, 0)$ is an ICIS (say, a hypersurface, i.e. $k = 1$). Let X be a stratified vector field on $(V, 0)$ with an isolated singularity at the origin. Then one can define in a similar way as above a Poincaré–Hopf or GSV index $\mathrm{ind}_{\mathrm{PH}}(X; V, 0) = \mathrm{ind}_{\mathrm{GSV}}(X; V, 0)$. For details see [22, Section 3.5].

5.3.4 Homological Index

Let $(V, 0) \subset (\mathbb{C}^N, 0)$ be a germ of a complex analytic variety of pure dimension n with an isolated singular point at the origin. Let X be a complex analytic vector field tangent to $(V, 0)$ with an isolated singular point at the origin and let ω be a holomorphic 1-form on $(V, 0)$ with an isolated singularity at the origin. We shall define an index of X and ω in a homological way.

For this purpose, we consider the module $\Omega^k_{V,0}$ of germs of holomorphic k-forms on $(V, 0)$. It is defined as follows. Let $I_{V,0} \subset \mathcal{O}_{\mathbb{C}^N,0}$ be the ideal of germs of holomorphic functions vanishing on $(V, 0)$. Consider the $\mathcal{O}_{\mathbb{C}^N,0}$-module $\Omega^k_{\mathbb{C}^N,0}$ of germs of holomorphic k-forms on $(\mathbb{C}^N, 0)$. Then

$$\Omega^k_{V,0} = \Omega^k_{\mathbb{C}^N,0} / \{f \cdot \Omega^k_{\mathbb{C}^N,0} + df \wedge \Omega^{k-1}_{\mathbb{C}^N,0} \;:\; f \in I_{V,0}\}.$$

We consider two Koszul complexes:

$$(\Omega^\bullet_{V,0}, X) : 0 \longleftarrow \mathcal{O}_{V,0} \xleftarrow{X} \Omega^1_{V,0} \xleftarrow{X} \dots \xleftarrow{X} \Omega^n_{V,0} \longleftarrow 0\,, \tag{5.1}$$

$$(\Omega^\bullet_{V,0}, \wedge\omega) : 0 \longrightarrow \mathcal{O}_{V,0} \xrightarrow{\wedge\omega} \Omega^1_{V,0} \xrightarrow{\wedge\omega} \dots \xrightarrow{\wedge\omega} \Omega^n_{V,0} \longrightarrow 0\,. \tag{5.2}$$

For the first complex $(\Omega^\bullet_{V,0}, X)$, the arrows are given by contraction with the vector field X. For the second complex $(\Omega^\bullet_{V,0}, \wedge\omega)$, the arrows are given by the exterior product with the 1-form ω. The second complex is the dual of the first one. It was used by G. M. Greuel in [67]. The sheaves $\Omega^i_{V,0}$ are coherent sheaves and the cohomology sheaves of the complexes are concentrated at the origin and hence finite dimensional.

Remark 5.3.20 If $(V, 0) = (W, 0) \times (\mathbb{C}, 0)$ and $X = \frac{\partial}{\partial t}$, where t is the coordinate on $\mathbb{C}$, the homology groups of the complex (5.1) are trivial. This implies that, if X has an isolated singular point at the origin, the homology groups $H_i(\Omega^\bullet_{V,0}, X)$ of the complex (5.1) are finite dimensional, even if V has non-isolated singularities.

Remark 5.3.21 On a germ of a complex analytic variety with an isolated singularity, holomorphic vector fields with isolated singular points always exist [12]. This is not the case for varieties with non-isolated singularities. For example, on the surface in $\mathbb{C}^3$ given by

$$xy(x+y)(x+zy)=0$$

all holomorphic vector fields vanish on the z-axis (cf. [135, Example 13.2], see also [38]).

Let us denote by $h_j(\Omega^\bullet_{V,0}, X)$ and $h_j(\Omega^\bullet_{V,0}, \wedge\omega)$ the dimension (as a $\mathbb{C}$-vector space) of the j-th homology group of the complex $(\Omega^\bullet_{V,0}, X)$ and $(\Omega^\bullet_{V,0}, \wedge\omega)$ respectively.

Definition 5.3.22

(a) The *homological index* $\mathrm{ind}_{\mathrm{hom}}(X; V, 0)$ of the vector field X on $(V, 0)$ is the Euler characteristic of the complex $(\Omega^\bullet_{V,0}, X)$:

$$\mathrm{ind}_{\mathrm{hom}}(X; V, 0) = \sum_{j=0}^{n} (-1)^j h_j(\Omega^\bullet_{V,0}, X)\,. \tag{5.3}$$

(b) The *homological index* $\mathrm{ind}_{\mathrm{hom}}(\omega; V, 0)$ of the 1-form ω on $(V, 0)$ is $(-1)^n$ times the Euler characteristic of the complex $(\Omega^\bullet_{V,0}, \wedge\omega)$:

$$\mathrm{ind}_{\mathrm{hom}}(\omega; V, 0) = \sum_{j=0}^{n} (-1)^{n-j} h_j(\Omega^\bullet_{V,0}, \wedge\omega)\,. \tag{5.4}$$

The definition of the homological index of a vector field is due to Gómez-Mont [61]. The definition was adapted to the case of a 1-form in [49]. Both indices satisfy the law of conservation of number [61, Theorem 1.2], [59].

Theorem 5.3.23 *Let $(V, 0)$ be an isolated complete intersection singularity. Then*

$$\mathrm{ind}_{\mathrm{hom}}(X; V, 0) = \mathrm{ind}_{\mathrm{GSV}}(X; V, 0),$$
$$\mathrm{ind}_{\mathrm{hom}}(\omega; V, 0) = \mathrm{ind}_{\mathrm{GSV}}\,(\omega; V, 0).$$

For a vector field X, this was proved in [61, Theorem 3.5] for a hypersurface singularity and by H.-Ch. Graf von Bothmer, Gómez-Mont and the first author in [13, Theorem 2.4] for a complete intersection singularity. For the proof for a 1-form ω see [49, Theorem 3.2 (iii)]. In [2, 3], methods of computation of the homological index are given for Cohen–Macaulay curves, graded normal surfaces, and complete intersections.

Example 5.3.24 Let $(C, 0)$ be a curve singularity and let $(\overline{C}, \overline{0})$ be its normalization. Let $\tau = \dim \mathrm{Ker}(\Omega^1_{c,0} \to \Omega^1_{\overline{C},\overline{0}})$ and $\lambda = \dim \omega_{C,0}/c(\Omega^1_{C,0})$, where $\omega_{C,0}$ is the dualizing module of Grothendieck and $c : \Omega^1_{C,0} \to \omega_{C,0}$ is the class map (see [24]). A Milnor number $\mu(f)$ of a function f on a curve singularity was introduced for curves in $\mathbb{C}^3$ in [66] and for the general case in [99]. In a similar way, one can define a Milnor number for an analytic 1-form ω with a isolated singular point on $(C, 0)$, namely

$$\mu(\omega) := \dim \omega_{C,0}/\omega \wedge \mathcal{O}_{C,0}.$$

Then one has

$$\mu(\omega) = \mathrm{ind}_{\mathrm{hom}}(\omega; C, 0) + \lambda - \tau.$$

For other formulas for $\mu(f)$ see [102].

It follows from Proposition 5.3.19 that, for a 1-form on an ICIS $(V, 0)$, the difference

$$\mathrm{ind}_{\mathrm{hom}}(\omega; V, 0) - \mathrm{ind}_{\mathrm{rad}}(\omega; V, 0)$$

between the homological index and the radial index is equal to the Milnor number of the singularity. This difference is also defined for the germ of a complex analytic space of pure dimension n with an isolated singular point at the origin. By Ebeling et al. [49, Proposition 4.1], it does not depend on the 1-form ω. Therefore one can consider this difference as a generalized Milnor number of the singularity $(V, 0)$. In [49], this invariant is computed for arbitrary curve singularities and compared with the Milnor number introduced by R. Buchweitz and G. M. Greuel [24] for these singularities. See [114] for an interesting question about this Milnor number for normal surface singularities. See [115] for a survey on relations between Milnor numbers and indices of vector fields and 1-forms on singular varieties. See also [136] for a recent extension of the Milnor number and of the homological index to a more general setting.

5.3.5 Euler Obstruction

In this section, we define the local Euler obstruction of a singular point of a vector field or a 1-form. The idea goes back to MacPherson who defined in [94] the Euler obstruction of a singular point of a complex analytic variety. As it was written above, the idea of the definition goes back to [18, 23] and [22]; an explicit definition was given in [26].

In order to introduce this notion, we need the notion of the Nash transformation of a germ of a singular variety. Let $(V, 0) \subset (\mathbb{C}^N, 0)$ be the germ of a purely

n-dimensional complex analytic variety. We assume that V is a representative of $(V, 0)$ defined in a suitable neighbourhood U of the origin in $\mathbb{C}^N$. Let $G(n, N)$ be the Grassmann manifold of n-dimensional vector subspaces of $\mathbb{C}^N$. Let V_{reg} be the non-singular part of V. There is a natural map $\sigma : V_{\text{reg}} \to U \times G(n, N)$ which is defined by $\sigma(z) = (z, T_z V_{\text{reg}})$. The *Nash transform* $\widehat{V}$ is the closure of the image $\text{Im}\,\sigma$ of the map σ in $U \times G(n, N)$. It is a usually singular analytic variety. There is the natural base point map $\nu : \widehat{V} \to V$. Let $\widehat{V}' := \widehat{V} \setminus \nu^{-1}(V \setminus V_{\text{reg}})$. Then the restriction $\nu|_{\widehat{V}'}$ maps $\widehat{V}'$ biholomorphically to V_{reg}.

The *Nash bundle* $\widehat{T}$ over $\widehat{V}$ is the pullback of the tautological bundle on the Grassmann manifold $G(n, N)$ under the natural projection map $\widehat{V} \to G(n, N)$. It is a vector bundle of rank n. There is a natural lifting of the Nash transformation to a bundle map from the Nash bundle $\widehat{T}$ to the restriction of the tangent bundle $T\mathbb{C}^N$ of $\mathbb{C}^N$ to V. This is an isomorphism of $\widehat{T}$ and $TV_{\text{reg}} \subset T\mathbb{C}^N$ over the regular part V_{reg} of V.

Let $V = \bigcup_{i=1}^q V_i$ be a subanalytic Whitney stratification of V and let X be a stratified vector field on V. Let $0 \in V$ be an isolated singular point of X on V. By [23], the vector field X has a canonical lifting to a section $\widehat{X}$ of the Nash bundle $\widehat{T}$ over the Nash transform $\widehat{V}$ without zeros outside of $\nu^{-1}(0)$. Let ε be chosen such that the vector field X is defined on $B_\varepsilon \cap V$ and does not vanish there outside of the origin where B_ε is the ball of radius ε centered at the origin.

Definition 5.3.25 The *local Euler obstruction* $\text{Eu}(X; V, 0)$ of the vector field X on V at the origin is the obstruction to extend the non-zero section $\widehat{X}$ from the preimage of a neighbourhood of the sphere $S_\varepsilon = \partial B_\varepsilon$ to the preimage of its interior. More precisely, it is the value of the obstruction (as an element of $H^{2n}(\nu^{-1}(V \cap B_\varepsilon), \nu^{-1}(V \cap S_\varepsilon)))$ on the fundamental class of the pair $(\nu^{-1}(V \cap B_\varepsilon), \nu^{-1}(V \cap S_\varepsilon))$.

Now let ω be a 1-form on U with an isolated singular point on V at the origin. Let ε be small enough such that the 1-form ω has no singular points on $V \setminus \{0\}$ inside the ball B_ε. The 1-form ω gives rise to a section $\widehat{\omega}$ of the dual Nash bundle $\widehat{T}^*$ over the Nash transform $\widehat{V}$ without zeros outside of $\nu^{-1}(0)$. The following definition was given in [38].

Definition 5.3.26 The *local Euler obstruction* $\text{Eu}(\omega; V, 0)$ of the 1-form ω on V at the origin is the obstruction to extend the non-zero section $\widehat{\omega}$ from the preimage of a neighbourhood of the sphere $S_\varepsilon = \partial B_\varepsilon$ to the preimage of its interior, more precisely, its value (as an element of the cohomology group $H^{2n}(\nu^{-1}(V \cap B_\varepsilon), \nu^{-1}(V \cap S_\varepsilon))$) on the fundamental class of the pair $(\nu^{-1}(V \cap B_\varepsilon), \nu^{-1}(V \cap S_\varepsilon))$.

We can use the definition of the local Euler obstruction of a 1-form to define the local Euler obstruction of a germ of a complex variety which was the original definition of MacPherson in [94].

Definition 5.3.27 The *local Euler obstruction* $\text{Eu}(V, 0)$ of the germ $(V, 0)$ is the local Euler obstruction of the radial 1-form $d|r|^2$ on it, where r is the distance to the origin.

The word *local* will be usually omitted.

One has the following Proportionality Theorem due to Brasselet and Schwartz [23]. (For a proof see also [22, Section 8.1.1], [19].) Let V be a complex analytic variety with a Whitney stratification $\{V_i\}$.

Theorem 5.3.28 (Proportionality Theorem for Vector Fields) *Let V_i be a stratum of the Whitney stratification and $x \in V_i$, let X_i be a vector field on V_i with an isolated singular point at x, and let X be a radial extension of X_i. Then one has*

$$\mathrm{Eu}(X; V, x) = \mathrm{Eu}(V, x) \cdot \mathrm{ind}_{\mathrm{rad}}(X; V, x).$$

Note that $\mathrm{ind}_{\mathrm{rad}}(X; V, x) = \mathrm{ind}(X_i; V_i, x)$.

There is also a Proportionality Theorem for 1-forms due to Brasselet, Seade, and Suwa [21].

Theorem 5.3.29 (Proportionality Theorem for 1-Forms) *Let V_i be a stratum of the Whitney stratification and $x \in V_i$, let ω_i be a 1-form on V_i with an isolated singular point at x, and let ω be a radial extension of ω_i. Then one has*

$$\mathrm{Eu}(\omega; V, x) = \mathrm{Eu}(V, x) \cdot \mathrm{ind}_{\mathrm{rad}}(\omega; V, x).$$

It follows from Theorem 5.3.29 that, if V_i and V_j are strata of the Whitney stratification with $V_i \subset \overline{V}_j$, then the local Euler obstruction $\mathrm{Eu}(\overline{V}_j, p)$ at any point $p \in V_i$ does not depend on p. It will be denoted by $\mathrm{Eu}(V_j, V_i)$. It is equal to the local Euler obstruction $\mathrm{Eu}(N_{ij}, p)$ of a normal slice N_{ij} of the variety $\overline{V}_j$ to the stratum V_i at the point p [17, Section 3]. If $V_i \not\subset \overline{V}_j$, we assume $\mathrm{Eu}(V_j, V_i)$ to be equal to zero.

In [18], the notion of the local Euler obstruction of a holomorphic function f with an isolated critical point on $(V, 0)$ was introduced. It is defined as follows. Let f be a holomorphic function defined in U with a isolated singular point on V at the origin. Let $\varepsilon > 0$ be small enough such that the function f has no singular points on $V \setminus \{0\}$ inside the ball B_ε. Let $\mathrm{grad}\, f$ be the gradient vector field of f as defined in Sect. 5.3.2. Since f has no singular points on $V \setminus \{0\}$ inside the ball B_ε, the angle of $\mathrm{grad}\, f(x)$ and the tangent space $T_x V_i$ to a point $x \in V_i \setminus \{0\}$ is less than $\pi/2$. Denote by $\zeta_i(x) \neq 0$ the projection of $\mathrm{grad}\, f(x)$ to the tangent space $T_x V_i$. The vector field on $V \setminus \{0\}$ which is equal to ζ_i on V_i is, in general, not continuous. It is shown in [18] that the vector fields ζ_i can be glued together to obtain a stratified vector field $\mathrm{grad}_V f$ on V such that $\mathrm{grad}_V f$ is homotopic to the restriction of $\mathrm{grad}\, f$ to V and satisfies $\mathrm{grad}_V f(x) \neq 0$ unless $x = 0$.

Definition 5.3.30 The *local Euler obstruction* $\mathrm{Eu}(f; V, 0)$ of the function f is defined to be

$$\mathrm{Eu}(f; V, 0) := \mathrm{Eu}(\mathrm{grad}_V f; V, 0).$$

The Euler obstruction of a function f is up to sign the Euler obstruction of the 1-form df. Namely, let $\omega = df$ for the germ f of a holomorphic function on $(\mathbb{C}^N, 0)$.

Then $\mathrm{Eu}(df; V, 0)$ differs from the Euler obstruction $\mathrm{Eu}(f; V, 0)$ of the function f by the sign $(-1)^n$. E.g., for the function $f(z_1, \dots, z_n) = z_1^2 + \dots + z_n^2$ on $\mathbb{C}^n$ the obstruction $\mathrm{Eu}(f; \mathbb{C}^n, 0)$ is the index of the vector field $\sum_{i=1}^n \overline{z}_i \partial/\partial z_i$ (which is equal to $(-1)^n$), but the obstruction $\mathrm{Eu}(df; \mathbb{C}^n, 0)$ is the index of the (holomorphic) 1-form $\sum_{i=1}^n z_i dz_i$ which is equal to 1.

Denote by $M_f = M_{f,t_0}$ the Milnor fibre of f, i.e. the intersection $V \cap B_\varepsilon(0) \cap f^{-1}(t_0)$ for a (regular) value t_0 of f close to 0. In [18, Theorem 3.1] the following result is proved.

Theorem 5.3.31 (Brasselet, Massey, Parameswaran, Seade) *Let* $f : (V, 0) \to (\mathbb{C}, 0)$ *have an isolated singularity at* $0 \in V$. *Then*

$$\mathrm{Eu}(V, 0) = \left(\sum_{i=1}^{q} \chi(M_f \cap V_i) \cdot \mathrm{Eu}(V, V_i) \right) + \mathrm{Eu}(f; V, 0).$$

The Euler obstruction of a vector field or 1-form can be considered as an index. In particular, it satisfies the law of conservation of number (just as the radial index). Moreover, on a smooth variety the Euler obstruction and the radial index coincide. This implies the following statement (cf. Theorem 5.3.31). We set $\overline{\chi}(Z) := \chi(Z) - 1$ and call it the *reduced* (modulo a point) Euler characteristic of the topological space Z (though, strictly speaking, this name is only correct for a non-empty space Z).

Proposition 5.3.32 *Let* $(V, 0) \subset (\mathbb{C}^N, 0)$ *have an isolated singularity at the origin and let* $\ell : \mathbb{C}^N \to \mathbb{C}$ *be a generic linear function. Then*

$$\mathrm{ind}_{\mathrm{rad}}(\omega; V, 0) - \mathrm{Eu}(\omega; V, 0) = \mathrm{ind}_{\mathrm{rad}}(d\ell; V, 0) = (-1)^{n-1}\, \overline{\chi}(M_\ell),$$

where M_ℓ *is the Milnor fibre of the linear function* ℓ *on* V. *In particular*

$$\mathrm{Eu}(df; V, 0) = (-1)^n (\chi(M_\ell) - \chi(M_f)).$$

For a stratum V_i of the Whitney stratification $\bigcup_{i=0}^q V_i$, $V_0 = \{0\}$, of V, let N_i be the normal slice in the variety V to the stratum V_i at a point of the stratum V_i and let

$$n_i = \mathrm{ind}_{\mathrm{rad}}(d\ell; N_i, 0) = (-1)^{\dim N_i - 1}\, \overline{\chi}(M_{\ell|_{N_i}})$$

be the radial index of a generic (non-vanishing) 1-form $d\ell$ on N_i. In [38, Theorem 4], the following theorem was proved.

Theorem 5.3.33 *One has*

$$\mathrm{ind}_{\mathrm{rad}}(\omega; V, 0) = \sum_{i=0}^{q} n_i \cdot \mathrm{Eu}(\omega; \overline{V}_i, 0).$$

The strata V_i of V are partially ordered: $V_i \prec V_j$ (we shall write $i \prec j$) iff $V_i \subset \overline{V_j}$ and $V_i \neq V_j$; $i \preceq j$ iff $i \prec j$ or $i = j$. In [38, Corollary 1], an "inverse" of the formula of Theorem 5.3.33 was written in the case when the variety V is irreducible and $V = \overline{V}_q$. Let n_{ij} $(i \preceq j)$ be the index of a generic 1-form $d\ell$ on the normal slice N_{ij}: $n_{ij} = (-1)^{\dim N_{ij}-1}\,\overline{\chi}(M_{\ell|_{N_{ij}}})$ (in particular $n_{ii} = 1$) and let m_{ij} be the (Möbius) inverse of the function n_{ij} on the partially ordered set of strata, i.e.

$$\sum_{i \preceq j \preceq k} n_{ij} m_{jk} = \delta_{ik}.$$

Corollary 5.3.34 *One has*

$$\mathrm{Eu}_{V,0}\,\omega = \sum_{i=0}^{q} m_{iq} \cdot \mathrm{ind}_{\mathrm{rad}}\,(\omega; \overline{V_i}, 0).$$

In [33], another proof of Corollary 5.3.34 is given and Theorem 5.3.33 and this corollary are applied to give an alternative proof of Theorem 5.3.31.

Let V be an affine variety. In [118], a *global Euler obstruction* was defined for the variety V as the obstruction to extend a radial vector field, defined outside of a sufficiently large compact subset, to a non-zero section of the Nash bundle.

In [97], the local Euler obstruction was investigated in terms of constructible sheaves and characteristic cycles.

5.3.6 Algebraic, Analytic, and Topological Formulas

In Sect. 5.2.4, we discussed algebraic formulas for the index of an analytic vector field or an analytic 1-form on a smooth manifold. It is natural to try to look for analogues of such formulas for vector fields and 1-forms on singular varieties. The homological index opens the way for such formulas.

In [61, Theorem 1], Gómez-Mont proved an algebraic formula for the homological index of a vector field with an isolated singularity in the ambient space on an isolated hypersurface singularity $(V, 0)$. O. Klehn [85] generalized this formula to the case that the vector field has an isolated singularity on the hypersurface singularity, but not necessarily in the ambient space. Graf von Bothmer, Gómez-Mont and the first author [13] gave formulas to compute the homological index in the case when $(V, 0)$ is an isolated complete intersection singularity. L. Giraldo, Gómez-Mont, and P. Mardešić [60] studied the homological index of vector fields tangent to hypersurfaces with non-isolated singularities.

Gómez-Mont and P. Mardešić derived algebraic formulas for the index of a real vector field with an algebraically isolated singular point at the origin tangent to a real analytic hypersurface with an algebraically isolated singularity at the origin as well [62, 63, 96]. The index is expressed as the signature of a certain non-degenerate quadratic form for an even-dimensional hypersurface and as the difference between the signatures of two such forms in the odd-dimensional case.

O. Klehn [86] proved that the GSV index of a holomorphic vector field X on an ICIS $(V, 0)$ coincides with the dimension of a certain explicitly constructed vector space, if X is deformable in a certain sense and V is a curve. Moreover, he gave a signature formula for the real GSV index in the corresponding real analytic case generalizing the Eisenbud–Levine–Khimshiashvili formula.

Let ω be the restriction of a holomorphic 1-form $\omega = \sum_{i=1}^{N} A_i(z)dz_i$ to an ICIS $(V, 0)$ given by a mapping $f = (f_1, \ldots, f_{N-n}) : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N-n}, 0)$. Assume that ω has an isolated singular point at the origin (on $(V, 0)$). Let I be the ideal generated by f_1, ..., f_{N-n} and the $(N-n+1) \times (N-n+1)$-minors of the matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_N} \\ \vdots & \cdots & \vdots \\ \frac{\partial f_{N-n}}{\partial z_1} & \cdots & \frac{\partial f_{N-n}}{\partial z_N} \\ A_1 & \cdots & A_N \end{pmatrix}.$$

In [35, 36], the authors proved the following formula.

Theorem 5.3.35 *One has*

$$\operatorname{ind}_{\mathrm{GSV}}(\omega; V, 0) = \dim \mathcal{O}_{\mathbb{C}^N,0}/I.$$

It generalizes the Lê–Greuel formula [67, 88] for the differential of a function. (Note that there is a minor mistake in the proof of this theorem in [36] which is corrected in [39].) In [40], the authors constructed quadratic forms on the algebra $\mathcal{O}_{\mathbb{C}^N,0}/I$ and on the space $\Omega^n_{V,0}/\omega \wedge \Omega^{n-1}_{V,0}$ generalizing the Eisenbud–Levine–Khimshiashvili quadratic form defined for smooth V.

In [51–53] A.Esterov gave formulas for the index of a 1-form on an ICIS in terms of Newton diagrams of the components under certain genericity conditions.

T. Gaffney [56] described connections between the GSV index of ω and the multiplicity of pairs of certain modules.

There are several generalizations of the residue formula of Sect. 5.2.4. P. F. Baum and R. Bott [11] considered residues of meromorphic vector fields on compact complex manifolds. An integral formula for the GSV index of a holomorphic vector field on an ICIS was given by D. Lehmann et al. in [92]. They reinterpreted the GSV-index as the *virtual index* for vector fields on complex complete intersections with arbitrary singular sets. This was done in [92] for holomorphic vector fields and in [22] in general. See [22, Chapter 5] for more details. For generalizations and related results see [79, 91, 121–125]. Klehn generalized the residue formula for the GSV index to holomorphic 1-forms on an isolated surface singularity [84]. T. Honda and Suwa [77] studied residue formulas for meromorphic functions on surfaces. The authors [37] considered indices of meromorphic 1-forms on complete intersections with isolated singularities.

In [71], a topological formula for the index of a gradient vector field grad g of an analytic function $g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ taking real values on $\mathbb{R}^n \subset \mathbb{C}^n$ is given. It expresses the index of the gradient vector field on $\mathbb{R}^n$ in terms of signatures of certain quadratic forms on the middle homology groups of specific Milnor fibres of the germ g. This formula was conjectured in [6] and also proved in [131]. In [34], a generalization of such a formula for the radial index of a gradient vector field on an algebraically isolated real analytic ICIS in $\mathbb{C}^N$ was obtained.

In [112, 116] formulas are given evaluating the GSV index of a singular point of a vector field on an isolated hypersurface or complete intersection singularity in terms of a resolution of the singularity.

5.3.7 Determinantal Singularities

The GSV index is only defined for ICIS. In the case of ICIS, the indices introduced above are best understood. In this subsection, we consider the next more general class, namely the class of determinantal singularities. An approach studying indices of 1-forms on such singularities was started in [44]. We give the basic definitions and facts following this paper.

Let $M_{m,n} \cong \mathbb{C}^{mn}$ be the space of $m \times n$-matrices with complex entries.

Definition 5.3.36 Let t be an integer with $1 \leq t \leq \min(m, n)$. The *generic determinantal variety of type* (m, n, t) is the subset

$$M^t_{m,n} := \{A \in M_{m,n} \mid \mathrm{rk}\,(A) < t\}$$

consisting of matrices of rank less than t, i.e. of matrices of which all $(t \times t)$-minors vanish.

The variety $M^t_{m,n}$ has codimension $(m-t+1)(n-t+1)$ in $M_{m,n}$. It is singular. The singular locus of $M^t_{m,n}$ coincides with $M^{t-1}_{m,n}$. The singular locus of the latter one coincides with $M^{t-2}_{m,n}$, etc. (see, e.g., [5]). The representation of the variety $M^t_{m,n}$ as the union of $M^i_{m,n} \setminus M^{i-1}_{m,n}$, $i = 1, \ldots, t$, is a Whitney stratification of $M^t_{m,n}$.

Let $U \subset \mathbb{C}^N$ be an open domain and $F : U \to M_{m,n}$ be a holomorphic map sending z to the matrix $F(z) = (f_{ij}(z))$ whose entries $f_{ij}(z)$ are complex analytic functions on U.

Definition 5.3.37 A *determinantal variety of type* (m, n, t) is the preimage $V = F^{-1}(M^t_{m,n})$ of the variety $M^t_{m,n}$ subject to the condition that $\operatorname{codim} V = \operatorname{codim} M^t_{m,n} = (m-t+1)(n-t+1)$.

The image of a generic map $F : U \to M_{m,n}$ may intersect the varieties $M^i_{m,n}$ for $i < t$. Therefore, it may not be avoided that $F^{-1}(M^t_{m,n})$ has singularities. However, a generic map F intersects the strata $M^i_{m,n} \setminus M^{i-1}_{m,n}$ of the variety $M^t_{m,n}$ transversally. This means that, at the corresponding points, the determinantal variety has "standard" singularities whose analytic type only depends on $i = \text{rk}\, F(z) + 1$. This inspired the following definitions of [44, p. 114].

Definition 5.3.38 A point $x \in X = F^{-1}(M^t_{m,n})$ is called *essentially non-singular* if, at the point x, the map F is transversal to the corresponding stratum of the variety $M^t_{m,n}$ (i.e., to $M^i_{m,n} \setminus M^{i-1}_{m,n}$ where $i = \text{rk}\, F(x) + 1$).

Definition 5.3.39 A germ $(V, 0) \subset (\mathbb{C}^N, 0)$ of a determinantal variety of type (m, n, t) has an *isolated essentially singular point* at the origin (or is an *essentially isolated determinantal singularity*: EIDS) if it has only essentially non-singular points in a punctured neighbourhood of the origin in V.

Example 5.3.40 An ICIS is an example of an EIDS: it is an EIDS of type $(1, n, 1)$.

An essentially isolated determinantal singularity $(V, 0) \subset (\mathbb{C}^N, 0)$ of type (m, n, t) (defined by a map $F : (\mathbb{C}^N, 0) \to (M_{m,n}, 0)$) has an isolated singularity at the origin if and only if $N \le (m - t + 2)(n - t + 2)$.

We shall consider deformations (in particular, smoothings) of an EIDS given by deformations of the matrix which defines the EIDS. Hence they are themselves determinantal ones.

Let $(V, 0) \subset (\mathbb{C}^N, 0)$ be an EIDS defined by a map $F : (\mathbb{C}^N, 0) \to (M_{m,n}, 0)$ ($V = F^{-1}(M^t_{m,n})$, F is transversal to $M^i_{m,n} \setminus M^{i-1}_{m,n}$ at all points x from a punctured neighbourhood of the origin in $\mathbb{C}^N$ and for all $i \le t$).

Definition 5.3.41 An *essential smoothing* $\widetilde{V}$ of the EIDS $(V, 0)$ is a subvariety of a neighbourhood U of the origin in $\mathbb{C}^N$ defined by a perturbation $\widetilde{F} : U \to M_{m,n}$ of the germ F transversal to all the strata $M^i_{m,n} \setminus M^{i-1}_{m,n}$ with $i \le t$.

A generic deformation $\widetilde{F}$ of the map F defines an essential smoothing of the EIDS $(V, 0)$ (according to Thom's Transversality Theorem). An essential smoothing is in general not smooth (for $N \ge (m - t + 2)(n - t + 2)$). Its singular locus is $\widetilde{F}^{-1}(M^{t-1}_{m,n})$, the singular locus of the latter one is $\widetilde{F}^{-1}(M^{t-2}_{m,n})$, etc. The representation of $\widetilde{V}$ as the union

$$\widetilde{V} = \bigcup_{1 \le i \le t} \widetilde{F}^{-1}(M^i_{m,n} \setminus M^{i-1}_{m,n})$$

is a Whitney stratification of it. An essential smoothing of an EIDS $(V, 0)$ of type (m, n, t) is a genuine smoothing if and only if $N < (m - t + 2)(n - t + 2)$.

There are three distinguished types of resolutions of the variety $M^t_{m,n}$.

The first one is constructed by considering $m \times n$-matrices as linear maps $\mathbb{C}^n \to \mathbb{C}^m$. Let

$$Y_1 := \{(A, W) \in M_{m,n} \times G(n - t + 1, n) \mid A(W) = 0\}.$$

The variety Y_1 is smooth and connected. Its projection to the first factor defines a resolution $\pi_1 : Y_1 \to M^t_{m,n}$ of the variety $M^t_{m,n}$.

Let us consider $m \times n$-matrices as linear maps $\mathbb{C}^m \to \mathbb{C}^n$ and let

$$Y_2 := \{(A, W) \in M_{m,n} \times G(m - t + 1, m) \mid A^T(W) = 0\}.$$

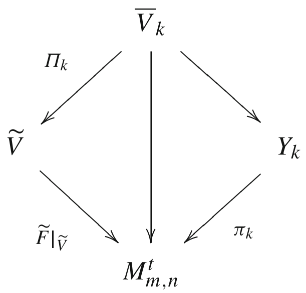
Then one gets a resolution $\pi_2 : Y_2 \to M^t_{m,n}$ of the variety $M^t_{m,n}$.

The third natural modification is given by the Nash transform $Y_3 := \widehat{M}^t_{m,n}$. One can show that $\pi_3 : Y_3 \to M^t_{m,n}$ is in fact a resolution of the variety $M^t_{m,n}$, see [44].

Let $(V, 0) = F^{-1}(M^t_{m,n}) \subset (\mathbb{C}^N, 0)$ be an EIDS and let ω be a germ of a (complex) 1-form on $(\mathbb{C}^N, 0)$ whose restriction to $(V, 0)$ has an isolated singular point (zero) at the origin. This means that the restrictions of the 1-form ω to the strata $V_i \setminus V_{i-1}$, $V_i := F^{-1}(M^i_{m,n})$, $i \leq t$, have no zeros in a punctured neighbourhood of the origin.

An essential smoothing $\widetilde{V} \subset U$ of the EIDS $(V, 0)$ (in a neighbourhood U of the origin in $\mathbb{C}^N$) is in general not smooth. To define an analogue of the PH-index one has to construct a substitute of the tangent bundle to $\widetilde{V}$. It is possible to use one of the following two natural ways.

One possibility is to use a resolution of the variety $\widetilde{V}$ connected with one of the three resolutions of the variety $M^t_{m,n}$ described above. Let $\pi_k : Y_k \to M^t_{m,n}$ be one of the described resolutions of the determinantal variety $M^t_{m,n}$ and let $\overline{V}_k = Y_k \times_{M^t_{m,n}} \widetilde{V}$, $k = 1, 2, 3$, be the fibre product of the spaces Y_k and $\widetilde{V}$ over the variety $M^t_{m,n}$:



The map $\Pi_k : \overline{V}_k \to \widetilde{V}$ is a resolution of the variety $\widetilde{V}$. For $k = 1, 2$ it is also called the *Tjurina transform* after [128], see [54]. The lifting $\omega_k := (j \circ \Pi_k)^*\omega$ (j is the inclusion map $\widetilde{V} \hookrightarrow U \subset \mathbb{C}^N$) of the 1-form ω is a 1-form on a (non-singular) complex analytic manifold $\overline{V}_k$ without zeros outside of the preimage of a small neighbourhood of the origin. In general, the 1-form ω_k has non-isolated zeros.

Definition 5.3.42 The *Poincaré–Hopf index* (*PH-index*) $\text{ind}^k_{\text{PH}}(\omega; V, 0)$, $k = 1, 2, 3$, of the 1-form ω on the EIDS $(V, 0) \subset (\mathbb{C}^N, 0)$ is the sum of the indices of the zeros of a generic perturbation $\widetilde{\omega}_k$ of the 1-form ω_k on the manifold $\overline{V}_k$ (in the preimage of a neighbourhood of the origin in $\mathbb{C}^N$).

There is also the local Euler obstruction of the 1-form ω. If one uses the Nash transform of the essential smoothing $\widetilde{V}$ of the EIDS $(V, 0)$ instead of V itself, it is called Poincaré–Hopf–Nash index.

Definition 5.3.43 The *Poincaré–Hopf–Nash index* (*PHN-index*) $\mathrm{ind}_{\mathrm{PHN}}(\omega; V, 0)$ of the 1-form ω on the EIDS $(V, 0)$ is the obstruction to extend the non-zero section $\widehat{\omega}$ of the dual Nash bundle $\widehat{T}^*$ from the preimage of the boundary $S_\varepsilon = \partial B_\varepsilon$ of the ball B_ε to the preimage of its interior, i.e. to the manifold $\overline{V}_3$, more precisely, its value (as an element of $H^{2d}(\Pi_3^{-1}(\widetilde{V} \cap B_\varepsilon), \Pi_3^{-1}(\widetilde{V} \cap S_\varepsilon)))$ on the fundamental class of the pair $(\Pi_3^{-1}(\widetilde{V} \cap B_\varepsilon), \Pi_3^{-1}(\widetilde{V} \cap S_\varepsilon))$.

In [44, Proposition 2], a formula relating the index $\mathrm{ind}^k_{\mathrm{PH}}(\omega; V, 0)$, $k = 1, 2, 3$ with the radial indices $\mathrm{ind}_{\mathrm{rad}}(\omega; V_i, 0)$, $i = 0, \dots, t$, is given. There is the following version of Theorem 5.3.33 for the PHN-index on a determinantal variety [44, Proposition 4]. Let $\ell : M_{m,n} \to \mathbb{C}$ be a generic linear form and let, for $i \leq j$,

$$n_{ij} := \mathrm{ind}_{\mathrm{rad}}(d\ell; M^{j-i+1}_{m-i+1,n-i+1}, 0).$$

By Ebeling and Gusein-Zade [44, Proposition 3], we have

$$\begin{aligned}
&\mathrm{ind}_{\mathrm{rad}}(d\ell; M^{j-i+1}_{m-i+1,n-i+1}, 0)\\
&= (-1)^{d_{ij}-1}\overline{\chi}(M^{j-i+1}_{m-i+1,n-i+1} \cap \ell^{-1}(1)) = (-1)^{(m+n)(j-i)}\binom{m-i}{m-j},
\end{aligned}$$

where d_{ij} is the dimension of $M^{j-i+1}_{m-i+1,n-i+1}$ equal to $(m-i+1)(n-i+1)-(m-j+1)(n-j+1)$.

Theorem 5.3.44 *One has*

$$\mathrm{ind}_{\mathrm{rad}}(\omega; V, 0) = \sum_{i=1}^{t} n_{it}\,\mathrm{ind}_{\mathrm{PHN}}(\omega; V_i, 0) + (-1)^{\dim V - 1}\overline{\chi}(\widetilde{V}, 0).$$

One can see that, for $i \leq j \leq t$, the integers m_{ij} from Corollary 5.3.34 are given by

$$m_{ij} = (-1)^{(m+n+1)(j-i)}\binom{m-i}{m-j}.$$

The analogue of Corollary 5.3.34, the inverse to Theorem 5.3.44, is the following statement, see [44, Proposition 5].

Corollary 5.3.45 *One has*

$$\mathrm{ind}_{\mathrm{PHN}}(\omega; V, 0) = \sum_{i=1}^{t} m_{it}\left(\mathrm{ind}_{\mathrm{rad}}(\omega; V_i, 0) + (-1)^{\dim V_i}\overline{\chi}(\widetilde{V}_i, 0)\right).$$

T. Gaffney et al. [58] proved a generalization of Theorem 5.3.44 and its inverse Corollary 5.3.45 relating it with Gaffney's multiplicities of pairs of modules.

For isolated determinantal singularities, the relations between the PH-, the PHN- and the radial indices simplify. For isolated smoothable singularities (i.e. for $N < (m-t+2)(n-t+2)$) all Poincaré–Hopf indices (including the Poincaré–Hopf–Nash index) coincide and they are equal to

$$\operatorname{ind}_{\mathrm{PH}}(\omega; V, 0) = \operatorname{ind}_{\mathrm{rad}}(\omega; V, 0) + (-1)^{\dim V}\overline{\chi}(\widetilde{V}, 0).$$

The paper [44] contains an algebraic formula for this index: Proposition 8 therein. To a regret, its proof is wrong.

N. C. Chachapoyas Siesquén [28] studies the Euler obstruction of an EIDS and gives some formulas to calculate it. In [100] (see also [101]), a formula for the Euler obstruction of a smoothable IDS is given. The papers [29] and [4] contain results on the Euler obstruction of a function on a determinantal variety.

In [104], codimension two determinantal varieties with isolated singularities are studied. The Milnor number is defined to be the middle Betti number of a generic fibre of the unique smoothing of such a singularity. For surfaces in $\mathbb{C}^4$, a Lê–Greuel formula for the Milnor number of the surface is proved. The Milnor number is also related to the Poincaré–Hopf index of the 1-form given by the differential of a generic linear projection defined on the surface. For other generalizations of the Lê–Greuel formula see [27, 32]. For other results on Milnor numbers of essentially isolated determinantal singularities see [16, 55].

5.4 Indices of Collections of Vector Fields and 1-Forms

5.4.1 GSV Index

Let $(V, 0) \subset (\mathbb{C}^N, 0)$ be an ICIS defined by a holomorphic map germ $f = (f_1, \ldots, f_{N-n}) : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N-n}, 0)$. Let $\{X_j^{(i)}\}$ be a collection of vector fields on a neighbourhood of the origin in $(\mathbb{C}^N, 0)$ ($i = 1, \ldots, s$; $j = 1, \ldots, n - k_i + 1$; $\sum k_i = n$) which are tangent to the ICIS $(V, 0) = \{f_1 = \cdots = f_{N-n} = 0\} \subset (\mathbb{C}^N, 0)$ at non-singular points of V. We say that a point $p \in V \setminus \{0\}$ is non-singular for the collection $\{X_j^{(i)}\}$ on V if at least for some i the vectors $X_1^{(i)}(p), \ldots, X_{n-k_i+1}^{(i)}(p)$ are linearly independent. Suppose that the collection $\{X_j^{(i)}\}$ has no singular points on V outside of the origin in a neighbourhood of it. Let U be a neighbourhood of the origin in $\mathbb{C}^N$ where all the functions f_r ($r = 1, \ldots, N-n$) and the vector fields $X_j^{(i)}$ are defined and such that the collection $\{X_j^{(i)}\}$ has no singular points on $(V \cap U) \setminus \{0\}$. Let $S_\delta \subset U$ be a sufficiently small sphere around the origin which intersects V transversally and denote by $K = V \cap S_\delta$ the link of the ICIS $(V, 0)$. The manifold K has a natural orientation as the boundary of a complex analytic manifold. Let Ψ_V be the mapping from $V \cap U$ to $M_{N,\mathbf{k}}$ (for the definition of $M_{N,\mathbf{k}}$ see Sect. 5.2.3) which sends a point $x \in V \cap U$ to the collection of $N \times (N - k_i + 1)$-matrices

$$\{(\text{grad } f_1(x), \dots, \text{grad } f_{N-n}(x), X_1^{(i)}(x), \dots, X_{n-k_i+1}^{(i)}(x))\}, \quad i = 1, \dots, s.$$

Here grad f_r is the gradient vector field of f_r defined in Sect. 5.3.2. Its restriction ψ_V to the link K maps K to the subset $W_{N,\mathbf{k}}$.

Definition 5.4.1 The *GSV index* $\text{ind}_{\text{GSV}}(\{X_j^{(i)}\}; V, 0)$ of the collection of vector fields $\{X_j^{(i)}\}$ on the ICIS $(V, 0)$ is the degree of the mapping $\psi_V : K \to W_{N,\mathbf{k}}$, or, equivalently, the intersection number of the germ of the image of the map Ψ_V with the variety $D_{N,\mathbf{k}}$.

For $s = 1$, $k_1 = n$, this index is the GSV index of a vector field on an ICIS defined in Sect. 5.3.2.

Let $V \subset \mathbb{CP}^N$ be an n-dimensional complete intersection with isolated singular points which is defined by homogeneous polynomials $f_1, \dots, f_{N-n}$ in $(N + 1)$ variables. Let $\{X_j^{(i)}\}$ be a collection of continuous vector fields on $\mathbb{CP}^N$ which are tangent to V. Let $\widetilde{V}$ be a smoothing of the complete intersection V, i.e. $\widetilde{V}$ is defined by $N - n$ homogeneous polynomials $\widetilde{f}_1, \dots, \widetilde{f}_{N-n}$ which are small perturbations of the functions f_i and $\widetilde{V}$ is smooth. As in Sect. 5.3.3, one can define approximations $\{\widetilde{X}_j^{(i)}\}$ of the vector fields $\{X_j^{(i)}\}$ which are tangent to $\widetilde{V}$. Then one has the following analogue of Theorem 5.2.7.

Theorem 5.4.2 *One has*

$$\sum_{p \in V} \text{ind}_{\text{GSV}}(\{X_j^{(i)}\}; V, p) = \langle \prod_{i=1}^{s} c_{k_i}(T\widetilde{V}), [\widetilde{V}] \rangle,$$

where $\widetilde{V}$ is a smoothing of the complete intersection V.

Now let $\{\omega_j^{(i)}\}$ be a collection of (continuous) 1-forms on a neighbourhood of the origin in $(\mathbb{C}^N, 0)$ with $i = 1, \dots, s$, $j = 1, \dots, n - k_i + 1$, $\sum k_i = n$. We say that a point $p \in V \setminus \{0\}$ is non-singular for the collection $\{\omega_j^{(i)}\}$ on V if at least for some i the restrictions of the 1-forms $\omega_j^{(i)}(p)$, $j = 1, \dots, n - k_i + 1$, to the tangent space $T_p V$ are linearly independent. Assume that the collection $\{\omega_j^{(i)}\}$ has no singular points on V in a punctured neighbourhood of the origin. As above, let U be a neighbourhood of the origin in $\mathbb{C}^N$ where all the functions f_r $(r = 1, \dots, N - n)$ and the 1-forms $\omega_j^{(i)}$ are defined and such that the collection $\{\omega_j^{(i)}\}$ has no singular points on $(V \cap U) \setminus \{0\}$. Let $S_\delta \subset U$ be a sufficiently small sphere around the origin. As above, let $K = V \cap S_\delta$ be the link of the ICIS $(V, 0)$. Let Ψ_V be the mapping from $V \cap U$ to $M_{n,\mathbf{k}}$ which sends a point $x \in V \cap U$ to the collection of $N \times (N - k_i + 1)$-matrices

$$\{(df_1(x), \ldots, df_{N-n}(x), \omega_1^{(i)}(x), \ldots, \omega_{n-k_i+1}^{(i)}(x))\}, \quad i = 1, \ldots, s.$$

Its restriction ψ_V to the link K maps K to the subset $W_{N,\mathbf{k}}$.

Definition 5.4.3 The *GSV index* $\mathrm{ind}_{\mathrm{GSV}}(\{\omega_j^{(i)}\}; V, 0)$ of the collection of 1-forms $\{\omega_j^{(i)}\}$ on the ICIS $(V, 0)$ is the degree of the mapping $\psi_V : K \to W_{N,\mathbf{k}}$, or, equivalently, the intersection number of the germ of the image of the mapping Ψ_V with the variety $D_{N,\mathbf{k}}$.

For $s = 1$, $k_1 = n$, this index is the GSV index of a 1-form on an ICIS defined in Sect. 5.3.2.

Let $V \subset \mathbb{CP}^N$ be an n-dimensional complete intersection with isolated singular points which is defined by homogeneous polynomials $f_1, \ldots, f_{N-n}$ in $(N + 1)$ variables. Let L be a complex line bundle on V and let $\{\omega_j^{(i)}\}$ be a collection of continuous 1-forms on V with values in L. This means that the forms $\omega_j^{(i)}$ are continuous sections of the vector bundle $T^*V \otimes L$ outside of the singular points of V. Since, in a neighbourhood of each point p, the vector bundle L is trivial, one can define the index $\mathrm{ind}_{\mathrm{GSV}}(\{\omega_j^{(i)}\}; V, p)$ of the collection of 1-forms $\{\omega_j^{(i)}\}$ at the point p as above. Let $\widetilde{V}$ be a smoothing of the complete intersection V. By using, e.g., the pull back along a projection of $\widetilde{V}$ to V, one can consider L as a line bundle on $\widetilde{V}$ as well. The collection $\{\omega_j^{(i)}\}$ of 1-forms can also be extended to a neighbourhood of V in such a way that it will define a collection of 1-forms on the smoothing $\widetilde{V}$ (also denoted by $\{\omega_j^{(i)}\}$) with isolated singular points. The sum of the indices of the collection $\{\omega_j^{(i)}\}$ on the smoothing $\widetilde{V}$ of V in a neighbourhood of the point p is equal to the index $\mathrm{ind}_{\mathrm{GSV}}(\{\omega_j^{(i)}\}; V, p)$.

One has the following analogue of Theorem 5.4.2 for 1-forms.

Theorem 5.4.4 *One has*

$$\sum_{p \in V} \mathrm{ind}_{\mathrm{GSV}}(\{\omega_j^{(i)}\}; V, p) = \langle \prod_{i=1}^{s} c_{k_i}(T^*\widetilde{V} \otimes L), [\widetilde{V}] \rangle,$$

where $\widetilde{V}$ is a smoothing of the complete intersection V.

Now let $(V, 0)$ be an ICIS defined by a holomorphic map germ $f = (f_1, \ldots, f_{N-n}) : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N-n}, 0)$ as above. Let $\{\omega_j^{(i)}\}$ ($i = 1, \ldots, s$; $j = 1, \ldots, n - k_i + 1$) be a collection of 1-forms on a neighbourhood of the origin in $\mathbb{C}^N$ without singular points on $V \setminus \{0\}$ in a neighbourhood of the origin. We now assume that all the 1-forms $\omega_j^{(i)}$ are complex analytic.

Let $I_{V,\{\omega_j^{(i)}\}}$ be the ideal in the ring $\mathcal{O}_{\mathbb{C}^N,0}$ generated by the functions $f_1, \ldots, f_{N-n}$ and by the $(N-k_i+1) \times (N-k_i+1)$ minors of all the matrices

$$(df_1(x), \ldots, df_{N-n}(x), \omega_1^{(i)}(x), \ldots, \omega_{n-k_i+1}^{(i)}(x))$$

for all $i = 1, \ldots, s$. Then we have the following algebraic formula similar to that of Theorem 5.3.35 (see [39]).

Theorem 5.4.5

$$\mathrm{ind}_{\mathrm{GSV}}(\{\omega_j^{(i)}\}; V, 0) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^N,0}/I_{V,\{\omega_j^{(i)}\}}.$$

Remark 5.4.6 In the case of collections of vector fields, the map Ψ_V is not complex analytic, whereas it is complex analytic in the case of collections of 1-forms. This is the reason that a formula similar to that of Theorem 5.4.5 does not exist for collections of vector fields. Moreover, in some cases this index can be negative (see e.g. [64, Proposition 2.2]).

5.4.2 Chern Obstruction

We now consider a generalization of the notion of the Euler obstruction to collections of 1-forms corresponding to different Chern numbers.

Let $(V, 0) \subset (\mathbb{C}^N, 0)$ be the germ of a purely n-dimensional reduced complex analytic variety at the origin. It can have a non-isolated singularity at the origin. Let $\mathbf{k} = \{k_i\}$, $i = 1, \ldots, s$, be a fixed partition of n (i.e., k_i are positive integers, $\sum_{i=1}^{s} k_i = n$). Let $\{\omega_j^{(i)}\}$ ($i = 1, \ldots, s$, $j = 1, \ldots, n-k_i+1$) be a collection of germs of 1-forms on $(\mathbb{C}^N, 0)$ (not necessarily complex analytic; it is sufficient that they are continuous). Let $\varepsilon > 0$ be small enough so that there is a representative V of the germ $(V, 0)$ and representatives $\omega_j^{(i)}$ of the germs of 1-forms inside the ball $B_\varepsilon \subset \mathbb{C}^N$.

Definition 5.4.7 A point $p \in V$ is called a *special* point of the collection $\{\omega_j^{(i)}\}$ of 1-forms on the variety V if there exists a sequence $\{p_m\}$ of points from the non-singular part V_{reg} of the variety V such that the sequence $T_{p_m}V_{\mathrm{reg}}$ of the tangent spaces at the points p_m has a limit L (in $G(n, N)$) as m tends to infinity and the restrictions of the 1-forms $\omega_1^{(i)}, \ldots, \omega_{n-k_i+1}^{(i)}$ to the subspace $L \subset T_p\mathbb{C}^N$ are linearly dependent for each $i = 1, \ldots, s$. The collection $\{\omega_j^{(i)}\}$ of 1-forms has an *isolated special point* on $(V, 0)$ if it has no special points on V in a punctured neighbourhood of the origin.

For the case $s = 1$ (and therefore $k_1 = n$), i.e. for one 1-form ω, we discussed the notion of a *singular* point of the 1-form ω on V in Sect. 5.3.1. One can easily see that a special point of the 1-form ω on V is singular, but not vice versa. (E.g. the origin is a singular point of the 1-form dx on the cone $\{x^2 + y^2 + z^2 = 0\}$, but not a special one.) On a smooth variety these two notions coincide.

Let

$$\mathscr{L}^{\mathbf{k}} = \prod_{i=1}^{s} \prod_{j=1}^{n-k_i+1} (\mathbb{C}^N_{ij})^*$$

be the space of collections of linear functions on $\mathbb{C}^N$ (i.e. of 1-forms with constant coefficients). Then one can show [42, Proposition 1.1] that there exists an open and dense set $U \subset \mathscr{L}^{\mathbf{k}}$ such that each collection $\{\ell_j^{(i)}\} \in U$ has only isolated special points on V and, moreover, all these points belong to the smooth part V_{reg} of the variety V and are non-degenerate (see Sect. 5.2.3 for the notion of a non-degenerate singular point). This implies the following proposition (see [42, Corollary 1.1]).

Proposition 5.4.8 *Let $\{\omega_j^{(i)}\}$ be a collection of 1-forms on V with an isolated special point at the origin. Then there exists a deformation $\{\widetilde{\omega}_j^{(i)}\}$ of the collection $\{\omega_j^{(i)}\}$ whose special points lie in V_{reg} and are non-degenerate. Moreover, as such a deformation one can use $\{\omega_j^{(i)} + \lambda\ell_j^{(i)}\}$ with a generic collection $\{\ell_j^{(i)}\} \in \mathscr{L}^{\mathbf{k}}$.*

Let $\{\omega_j^{(i)}\}$ be a collection of germs of 1-forms on $(V, 0)$ with an isolated special point at the origin. Let $\nu : \widehat{V} \to V$ be the Nash transformation of the variety $V \subset B_\varepsilon$ (see Sect. 5.3.5). The collection of 1-forms $\{\omega_j^{(i)}\}$ gives rise to a section $\widehat{\omega}$ of the bundle

$$\widehat{\mathbb{T}} = \bigoplus_{i=1}^{s} \bigoplus_{j=1}^{n-k_i+1} \widehat{T}^*_{i,j}$$

where $\widehat{T}^*_{i,j}$ are copies of the dual Nash bundle $\widehat{T}^*$ over the Nash transform $\widehat{V}$ numbered by indices i and j. Let $\widehat{\mathbb{D}} \subset \widehat{\mathbb{T}}$ be the set of pairs $(x, \{\alpha_j^{(i)}\})$ where $x \in \widehat{V}$ and the collection $\{\alpha_j^{(i)}\}$ of elements of $\widehat{T}_x^*$ (i.e. of linear functions on $\widehat{T}_x$) is such that $\alpha_1^{(i)}, \dots, \alpha_{n-k_i+1}^{(i)}$ are linearly dependent for each $i = 1, \dots, s$. The image of the section $\widehat{\omega}$ does not intersect $\widehat{\mathbb{D}}$ outside of the preimage $\nu^{-1}(0) \subset \widehat{V}$ of the origin. The map $\widehat{\mathbb{T}} \setminus \widehat{\mathbb{D}} \to \widehat{V}$ is a fibre bundle. The fibre $W_x = \widehat{\mathbb{T}} \setminus \widehat{\mathbb{D}}$ of it is $(2n-2)$-connected, its homology group $H_{2n-1}(W_x; \mathbb{Z})$ is isomorphic to $\mathbb{Z}$ and has a natural generator (see above). The latter fact implies that the fibre bundle $\widehat{\mathbb{T}} \setminus \widehat{\mathbb{D}} \to \widehat{V}$ is homotopically simple in dimension $2n-1$, i.e. the fundamental group $\pi_1(\widehat{V})$ of the base acts trivially on the homotopy group $\pi_{2n-1}(W_x)$ of the fibre, the last one being isomorphic to the homology group $H_{2n-1}(W_x)$: see, e.g., [120]. The following definition was made in [42] (see also [43]).

Definition 5.4.9 The *local Chern obstruction* $\mathrm{Ch}(\{\omega_j^{(i)}\}; V, 0)$ of the collections of germs of 1-forms $\{\omega_j^{(i)}\}$ on $(V, 0)$ at the origin is the (primary) obstruction to extend the section $\widehat{\omega}$ of the fibre bundle $\widehat{\mathbb{T}} \setminus \widehat{\mathbb{D}} \to \widehat{V}$ from the preimage of a neighbourhood of the sphere $S_\varepsilon = \partial B_\varepsilon$ to $\widehat{V}$. More precisely, it is the value of this obstruction (as an element of the homology group $H^{2n}(\nu^{-1}(V \cap B_\varepsilon), \nu^{-1}(V \cap S_\varepsilon); \mathbb{Z})$) on the fundamental class of the pair $(\nu^{-1}(V \cap B_\varepsilon), \nu^{-1}(V \cap S_\varepsilon))$.

The local Chern obstruction $\mathrm{Ch}(\{\omega_j^{(i)}\}; V, 0)$ can also be described as an intersection number, see [42]. Namely, let $\mathscr{D}_V^{\mathbf{k}} \subset \mathbb{C}^N \times \mathscr{L}^{\mathbf{k}}$ be the closure of the set of pairs $(x, \{\ell_j^{(i)}\})$ such that $x \in V_{\mathrm{reg}}$ and the restrictions of the linear functions $\ell_1^{(i)}, \dots, \ell_{n-k_i+1}^{(i)}$ to $T_x V_{\mathrm{reg}} \subset \mathbb{C}^N$ are linearly dependent for each $i = 1, \dots, s$. (For $s = 1$, $\mathbf{k} = \{n\}$, $\mathscr{D}_V^{\mathbf{k}}$ is the (non-projectivized) conormal space of V [127].) The collection $\{\omega_j^{(i)}\}$ of germs of 1-forms on $(\mathbb{C}^N, 0)$ defines a section $\check{\omega}$ of the trivial fibre bundle $\mathbb{C}^N \times \mathscr{L}^{\mathbf{k}} \to \mathbb{C}^N$. Then

$$\mathrm{Ch}(\{\omega_j^{(i)}\}; V, 0) = (\check{\omega}(\mathbb{C}^N) \circ \mathscr{D}_V^{\mathbf{k}})_0, \tag{5.5}$$

where $(\cdot \circ \cdot)_0$ is the intersection number at the origin in $\mathbb{C}^N \times \mathscr{L}^{\mathbf{k}}$. This description can be considered as a generalization of an expression of the local Euler obstruction as a micro-local intersection number defined in [80], see also [105, Sections 5.0.3 and 5.2.1] and [106].

Remark 5.4.10 The local Euler obstruction is defined for vector fields on singular varieties as well as for 1-forms. One can see that collections of vector fields are not well adapted to a definition of the local Chern obstructions, at least on varieties with non-isolated singularities. The reason is as follows. Vector fields on a singular variety V are required to be tangent to the smooth strata of V (of a Whitney stratification). For example, on a one-dimensional stratum, all vector fields are proportional to each other and therefore a collection cannot have an isolated special point in a natural sense. A natural definition of the Chern obstruction of a collection of vector fields makes sense for varieties with isolated singularities. (Besides that, on a singular variety (with non-isolated singularities) continuous vector fields with isolated singular points exist, whereas this may not hold for holomorphic ones, see Remark 5.3.21.)

Being a (primary) obstruction, the local Chern obstruction satisfies the law of conservation of number, i.e. if a collection of 1-forms $\{\widetilde{\omega}_j^{(i)}\}$ is a deformation of the collection $\{\omega_j^{(i)}\}$ and has isolated special points on V, then

$$\mathrm{Ch}(\{\omega_j^{(i)}\}; V, 0) = \sum \mathrm{Ch}(\{\widetilde{\omega}_j^{(i)}\}; V, p)$$

where the sum on the right hand side is over all special points p of the collection $\{\widetilde{\omega}_j^{(i)}\}$ on V in a neighbourhood of the origin. With Proposition 5.4.8 this implies the following statements. The first statement is an analogue of [119, Proposition 2.3].

Proposition 5.4.11 *The local Chern obstruction* $\mathrm{Ch}(\{\omega_j^{(i)}\}; V, 0)$ *of a collection* $\{\omega_j^{(i)}\}$ *of germs of holomorphic 1-forms is equal to the number of special points on* V *of a generic (holomorphic) deformation of the collection (lying on* V_{reg}*).*

Proposition 5.4.12 *Let* $\{\omega_j^{(i)}\}$ *be a collection of 1-forms on a compact (say, projective) variety* V *with only isolated special points. Then the sum of the local Chern obstructions of the collection* $\{\omega_j^{(i)}\}$ *at these points does not depend on the collection and therefore is an invariant of the variety.*

It is reasonable to consider this sum as ($(-1)^n$ times) the corresponding Chern characteristic number of the singular variety V. It is well known that the characteristic numbers of a compact complex manifold cannot have arbitrary values, they satisfy certain divisibility properties. A. Buryak [25] showed that, contrary to this fact, any set of integers can be the set of Chern characteristic numbers of a singular projective variety.

Let $(V, 0)$ be an ICIS. The fact that both the Chern obstruction and the GSV index of a collection $\{\omega_j^{(i)}\}$ of 1-forms satisfy the law of conservation of number and they coincide on a smooth manifold yields the following statement.

Proposition 5.4.13 *For a collection* $\{\omega_j^{(i)}\}$ *on an ICIS* $(V, 0)$*, the difference*

$$\mathrm{ind}_{\mathrm{GSV}}(\{\omega_j^{(i)}\}; V, 0) - \mathrm{Ch}(\{\omega_j^{(i)}\}; V, 0)$$

does not depend on the collection and therefore is an invariant of the ICIS.

In the framework of the definition of Schwartz–Chern classes of singular varieties [23, 107], one has to consider k-fields on n-dimensional varieties. Let us give the definition of the Euler obstruction of a k-field following [22].

Let $(V, 0) \subset (\mathbb{C}^N, 0)$ be the germ of a pure n-dimensional complex analytic variety. Let $\mathbb{C}^N = \bigcup_{i=1}^q V_i$ be a Whitney stratification of $\mathbb{C}^N$ compatible with V, i.e. $\mathbb{C}^N \setminus V$ is a stratum. Let (K) be triangulation of $\mathbb{C}^N$ subordinated to the stratification $\mathbb{C}^N = \bigcup_{i=1}^q V_i$ and (D) be a cell decomposition of $\mathbb{C}^N$ dual to (K).

Let $\{X_j\} = (X_1 \ldots, X_k)$ be a k-field, i.e. a collection of stratified vector fields, i.e. at each point $p \in V$ each vector field X_j, $j = 1, \ldots, k$, is tangent to the stratum containing p. Assume that $\{X_j\}$ has only isolated singular points in V. Let σ be a $2(N - k + 1)$-cell of (D). Note that σ is transverse to all the strata V_i, $i = 1, \ldots, q$. We assume that $\{X_j\}$ has an isolated singularity at the barycentre b_σ of σ and is a k-frame in $(\sigma \setminus \{b_\sigma\}) \cap V$, in particular it does not have any singularities on $\partial\sigma \cap V$.

Let $\nu : \widehat{V} \to V$ be the Nash transform of V and $\widehat{T}$ be the Nash bundle, (cf. Sect. 5.3.5). Each vector field X_j lifts to a section $\widehat{X}_j$ of the bundle $\widehat{T}$ over $\nu^{-1}(\partial\sigma \cap V)$, see Sect. 5.3.5. The k-frame $\{X_j\}$ lifts to k linearly independent sections $\{\widehat{X}_j\}$ of $\widehat{T}$ over $\nu^{-1}(\partial\sigma \cap V)$.

Definition 5.4.14 The *local Euler obstruction* $\mathrm{Eu}(\{X_j\}; V, \sigma)$ of the k-field $\{X_j\}$ at b_σ is the obstruction to extend $\{\widehat{X}_j\}$ to a collection of k linearly independent sections of $\widehat{T}$ over $\nu^{-1}(\sigma \cap V)$, more precisely, its value (as an element of $H^{2(N-k+1)}(\nu^{-1}(\sigma \cap V), \nu^{-1}(\partial\sigma \cap V))$) on the fundamental class of the pair $(\nu^{-1}(\sigma \cap V), \nu^{-1}(\partial\sigma \cap V))$.

Let $\{\omega_j\}$ be a collection of 1-forms on V with an isolated singularity at the barycentre b_σ of σ.

Definition 5.4.15 The *local Euler obstruction* $\mathrm{Eu}(\{\omega_j\}; V, \sigma)$ of the collection $\{\omega_j\}$ at b_σ is defined in a similar way, but now taking sections of the dual Nash bundle $\widehat{T}^*$.

There is also the definition of an Euler obstruction of a map due to N. Grulha [70].

Let $f = (f_1, \dots, f_k) : (V, 0) \to (\mathbb{C}^k, 0)$ be the germ of an analytic map. Let $\mathrm{grad}_V f_j$, $j = 1, \dots, k$, be the vector fields constructed in Sect. 5.3.5. The construction can be done in such a way that for $x \in V \setminus \{0\}$ the vector fields $(\mathrm{grad}_V f_1(x), \dots, \mathrm{grad}_V f_k(x))$ are linearly independent, see [70].

Let Σf be the singular set of f. Let us assume that there exists a cell decomposition (D) of $\mathbb{C}^N$ and a $2(N-k+1)$-cell of (D) with barycentre 0 such that $\Sigma f \cap \partial\sigma = \emptyset$.

Definition 5.4.16 The *local Euler obstruction* $\mathrm{Eu}(f; V, \sigma)$ of the map f relative to σ is defined to be

$$\mathrm{Eu}(f; V, \sigma) := \mathrm{Eu}(\{\mathrm{grad}_V f_j\}; V, \sigma).$$

It is shown in [15, Corollary 5] that the definition of $\mathrm{Eu}(f; V, \sigma)$ does not depend on a generic choice of the cell σ.

Instead of the collection of vector fields $\{\mathrm{grad}_V f_j\}$, we can consider the collection $\{df_j\}$ of 1-forms associated to f. This leads to the following definition (see [15]).

Definition 5.4.17

$$\mathrm{Eu}^*(f; V, \sigma) := \mathrm{Eu}(\{df_j\}; V, \sigma).$$

Relations between the Euler obstructions of k-fields and Chern obstructions were described in [15].

Let $(V, 0) \subset (\mathbb{C}^N, 0)$, $\{V_i\}$, (K), and (D) be as above. Assume that 0 is the barycentre of a $2k$-simplex τ of the triangulation (K) and let σ be the dual $2(N-k)$-cell. Since (K) is subordinated to the stratification, the simplex τ is contained in a stratum and the cell σ is transverse to the strata. A neighbourhood of 0 in $\mathbb{C}^N$ is homeomorphic to $\sigma \times \tau$ and one has

$$(\sigma \times \tau) \cap V = (\sigma \cap V) \times \tau \cong V \cap B_\varepsilon$$

for a ball B_ε around the origin of sufficiently small radius ε. Denote by π_1 and π_2 the projections $\pi_1 : \sigma \times \tau \to \sigma$ and $\pi_2 : \sigma \times \tau \to \tau$. Note that τ is a smooth manifold, so the index $\mathrm{ind}(\{\omega_j\}; \tau, 0)$ of a collection of 1-forms $\{\omega_j\}$ according to Sect. 5.2.3 is defined. The following theorem is proved in [15, Theorem 2.2].

Theorem 5.4.18 (Brasselet, Grulha, Ruas) *In the above setting, let* $\{\omega_j^{(1)}\}$, $j = 1, \dots, k-1$, *be a collection of germs of 1-forms on* σ *and* $\{\omega_j^{(2)}\}$, $j = 1, \dots, d-k+1$, *be a collection of germs of 1-forms on* τ. *The collection of germs of 1-forms on* $(\mathbb{C}^N, 0)$ *given by* $\{\omega_j^{(i)}\} = \{\pi_1^*(\omega_{j_1}^{(1)}), \pi_2^*(\omega_{j_2}^{(1)})\}$ *satisfies*

$$\mathrm{Ch}(\{\omega_j^{(i)}\}; V, 0) = \mathrm{Eu}(\{\omega_j^{(1)}\}; V, \sigma) \cdot \mathrm{ind}(\{\omega_j^{(2)}\}; \tau, 0).$$

The following corollary is derived from this theorem, see [15, Corollary 2.3, Corollary 2.6].

Corollary 5.4.19 (Brasselet, Grulha, Ruas) *Let* $(V, 0)$ *be as above and let* $f : (V, 0) \to (\mathbb{C}^k, 0)$ *be a map germ. Let* $\{\omega_j^{(i)}\} = \{\omega_{j_1}^{(1)}, \omega_{j_2}^{(i)}\}$ *be the collection of 1-forms defined by* $\{\omega_{j_1}^{(1)}\} = \{df_1, \dots, df_k\}$ *and* $\{\omega_{j_2}^{(2)}\} = \{\ell_1, \dots, \ell_{n-k+2}\}$ *where* $\ell_1, \dots, \ell_{n-k+1}$ *are linearly independent linear forms dual to the tangent field of* σ *and* ℓ_{n-k+2} *is a radial linear form. Then one has*

$$\mathrm{Eu}^*(f; V, \sigma) = \mathrm{Ch}(\{\omega_j^{(i)}\}; V, 0) = (\check{\omega}(\mathbb{C}^N) \circ \mathscr{D}_V^{\mathbf{k}})_0$$

(cf. Eq. (5.5)*).*

It follows from this corollary that $\mathrm{Eu}^*(f; V, \sigma)$ is independent of a generic choice of σ. Moreover, the following identity [15, Theorem 2.4] is proved using this corollary:

$$\mathrm{Eu}(f; V, \sigma) = (-1)^{n-k+1}\mathrm{Eu}^*(f; V, \sigma).$$

Therefore the Euler obstruction $\mathrm{Eu}(f; V, \sigma)$ is also independent of a generic choice of σ [15, Corollary 2.5].

In [15], also some formulas to compute the Chern obstruction are given. In particular, it is shown that the Chern obstruction is related with the polar multiplicity. Let $(V, 0) \subset (\mathbb{C}^N, 0)$ be the germ of a pure n-dimensional complex analytic variety and $f : (V, 0) \to \mathbb{C}^k$ be a generic projection. The $(n-k+1)$-polar variety $P_{n-k+1}(V)$ is the closure of the singular set $\overline{\Sigma f}$ of f. Its multiplicity at 0 is denoted by $m_{n-k+1}(V, 0)$. The following theorem is proved in [15, Theorem 3.1].

Theorem 5.4.20 (Brasselet, Grulha, Ruas) *Let* $(V, 0) \subset (\mathbb{C}^N, 0)$ *be the germ of a pure* n*-dimensional complex analytic variety and* $f : (V, 0) \to \mathbb{C}^k$ *be a generic projection. Let* $\{\omega_j^{(i)}\}$ *be a collection of germs of 1-forms on* $(\mathbb{C}^N, 0)$ *such that* $\{\omega_j^{(1)}\} = \{df_1, \dots, df_k\}$ *and* $\{\omega_j^{(i)}\}$, $i = 2, \dots, s$, $j = 1, \dots, n-k_i+1$, *are generic subcollections, where* k_i, $i = 2, \dots, s$, *are non-negative integers with* $\sum_{i=2}^{s} k_i = k - 1$. *Then one has*

$$\mathrm{Ch}(\{\omega_j^{(i)}\}; V, 0) = m_{n-k+1}(V, 0).$$

In [57, Theorem 6.1], a formula for the Chern obstruction of a collection of 1-forms on an equidimensional analytic variety is given in terms of the multiplicity of pairs of modules.

5.4.3 Homological Index

Here we consider the definition of the homological index for a collection of 1-forms due to E. Gorsky and the second author [65].

Let $(V, 0)$ be the germ of a complex algebraic variety of dimension n with an isolated singular point at the origin. Let k_i, $i = 1, \dots, s$, be positive integers such that $\sum_{i=1}^{s} k_i = n$ and let $\{\omega_j^{(i)}\}$, $i = 1, \dots, s$, $j = 1, \dots, n - k_i + 1$, be a collection of germs of holomorphic 1-forms on $(V, 0)$.

Let $W_i = \mathbb{C}^{n-k_i+1}$ be an auxiliary vector space with a basis $u_1, \dots, u_{n-k_i+1}$. We consider the complex $\mathscr{C}^{(i)} = \mathscr{C}(\omega_1^{(i)}, \dots, \omega_{n-k_i+1}^{(i)})$ of sheaves of $\mathscr{O}_{V,0}$-modules defined as follows:

$$\mathscr{C}_0^{(i)} := \Omega_{V,0}^n, \quad \mathscr{C}_t^{(i)} := \Omega^{k_i-t} \otimes S^{t-1} W_i, \quad 1 \le t \le k_i.$$

The differential $d_t : \mathscr{C}_t^{(i)} \to \mathscr{C}_{t-1}^{(i)}$ is defined by

$$d_1(\beta) := \beta \wedge \omega_1^{(i)} \wedge \dots \wedge \omega_{n-k_i+1}^{(i)},$$

$$d_t(\beta \otimes \varphi(u)) := \sum_{l=1}^{n-k_i+1} \left(\beta \wedge \omega_l^{(i)}\right) \otimes \frac{\partial \varphi}{\partial u_l}, \quad 2 \le t \le k_i.$$

The complex $(\mathscr{C}^{(i)}, d)$ is indeed a chain complex [65, Lemma 11]. It is shown in [65, Lemma 14] that the cohomology groups of $\mathscr{C}^{(i)}$ are supported on the locus of the points where the forms $\{\omega_j^{(i)}\}$ are linearly dependent. Define

$$\mathscr{C} = \bigotimes_{i=1}^{s} \mathscr{C}^{(i)},$$

where the tensor product is taken over $\mathscr{O}_{V,0}$.

Definition 5.4.21 The *homological index* $\text{ind}_{\text{hom}}(\{\omega_j^{(i)}\}; V, 0)$ of the collection of 1-forms $\{\omega_j^{(i)}\}$ is the Euler characteristic of the complex $\mathscr{C}$:

$$\text{ind}_{\text{hom}}(\{\omega_j^{(i)}\}; V, 0) := \sum_{t=0}^{n} (-1)^t \dim H^t(\mathscr{C}).$$

The homological index for a collection of 1-forms with an isolated singular point satisfies the law of conservation of number [65, Proposition 18]. The following theorem is [65, Theorem 19].

Theorem 5.4.22 (Gorsky, Gusein-Zade) *Let* $(V, 0)$ *be an ICIS and let* $\{\omega_j^{(i)}\}$ *be a collection of holomorphic 1-forms on* $(V, 0)$ *with an isolated singular point. Then*

$$\text{ind}_{\text{hom}}(\{\omega_j^{(i)}\}; V, 0) = \text{ind}_{\text{GSV}}(\{\omega_j^{(i)}\}; V, 0).$$

Since both the homological index and the Chern obstruction satisfy the law of conservation of number and coincide on a smooth manifold, one has the following statement (see [65, Proposition 40]).

Proposition 5.4.23 *Let* $(V, 0) \subset (\mathbb{C}^N, 0)$ *be the germ of a complex analytic variety of pure dimension n with an isolated singularity at the origin. The difference*

$$\text{ind}_{\text{hom}}(\{\omega_j^{(i)}\}; V, 0) - \text{Ch}(\{\omega_j^{(i)}\}; V, 0)$$

between the homological index and the Chern obstruction does not depend on the collection $\{\omega_j^{(i)}\}$ *and is an invariant of the singularity* $(V, 0)$.

5.5 Equivariant Indices

5.5.1 Equivariant Euler Characteristics

The notions of indices of vector fields and of 1-forms (on smooth manifolds and on singular varieties) are related with the Euler characteristic (through the Poincaré–Hopf theorem). Therefore it is natural to discuss equivariant versions of the Euler characteristic first.

In what follows we shall consider the additive Euler characteristic defined (for topological spaces nice enough, say, for those homeomorphic to locally compact unions of cells in finite CW-complexes) as the alternating sum of the dimensions of the cohomology groups with compact support:

$$\chi(V) = \sum_{q=0}^{\infty} (-1)^q \dim H_c^q(V; \mathbb{C}) \,. \tag{5.6}$$

This Euler characteristic coincides with the "traditional one" (defined as the alternating sum of the dimensions of the usual cohomology groups) for compact spaces (finite CW-complexes) and for complex quasi-projective varieties. The additivity of the Euler characteristic permits to use it as a sort of a (non-positive) measure for the definition of the integral with respect to the Euler characteristic: [133].

There are several generalizations of the notion of the Euler characteristic to the equivariant setting, i.e. for spaces with actions of a group (say, a finite one). The most simple (and the most straightforward) one is obtained by substituting the dimensions of the cohomology groups in (5.6) by the classes of the corresponding G-modules $H^q_c(V;\mathbb{C})$ (spaces of representations of G) in the ring $R(G)$ of representations of the group G. This analogue of the Euler characteristic is defined as an element of the ring $R(G)$. It was introduced in [132] and was used, e.g., in [134].

A finer equivariant version of the Euler characteristic of a G-space can be defined as an element of the Burnside ring $A(G)$ of the group G, i.e. the Grothendieck ring of finite G-sets. The latter is the abelian group generated by the classes $[(Z, G)]$ of finite G-sets modulo the following relations:

– if (Z_1, G) and (Z_2, G) are isomorphic, i.e., if there exists a bijective G-equivariant map $Z_1 \to Z_2$, then $[(Z_1, G)] = [(Z_2, G)]$;
– $[(Z_1 \sqcup Z_2, G)] = [(Z_1, G)] + [(Z_2, G)]$.

The multiplication in $A(G)$ is defined by the Cartesian product of sets with the natural (diagonal) G-action. The Burnside ring $A(G)$ is the free abelian group generated by the classes of irreducible G-sets which are in bijection with the classes of conjugate subgroups of the group G: the conjugacy class $[H]$ of a subgroup $H \subset G$ corresponds to the class of the G-set G/H. One has a natural ring homomorphism $A(G) \to R(G)$, sending a G-set Z to the space of functions on Z with the natural (left) action of the group G: $(g^* f)(y) = f(g^{-1}y)$. This homomorphism is, in general, neither a monomorphism nor an epimorphism. In what follows we shall mostly use the equivariant version of the Euler characteristic with values in $A(G)$ and therefore we shall refer to it as the *equivariant Euler characteristic*.

Let V be a sufficiently nice space with an action of the group G. For a point x of the space V let G_x be the isotropy subgroup $\{g \in G : gx = x\}$ of the point x. For a subgroup H of the group G let V^H be the fixed point set of the group H: $\{x \in V : G_x \supset H\}$, and let $V^{(H)}$ be the set $\{x \in V : G_x = H\}$ of points with the isotropy subgroup coinciding with H. Let $\mathrm{Conjsub}\, G$ be the set of the conjugacy classes of subgroups of G. For a conjugacy class $[H]$ of subgroups of G (which contains the subgroup H), let $V^{[H]}$ be the set of points such that each of them is fixed with respect to a subgroup conjugate to H and let $V^{([H])}$ be the set of points $x \in V$ whose isotropy subgroups G_x are conjugate to H.

Definition 5.5.1 The *equivariant Euler characteristic* of the G-space V is defined by

$$\chi^G(V) = \sum_{[H]\in \mathrm{Conjsub}\, G} \chi(V^{([H])}/G) \cdot [G/H] \in A(G)\,. \tag{5.7}$$

This notion was introduced in [129]. The equivariant Euler characteristic satisfies the additivity property: if W is a closed G-invariant subspace of a G-space V, then

$$\chi^G(V) = \chi^G(W) + \chi^G(V \setminus W)\,.$$

One can show that it is a universal invariant possessing this property (on the class of spaces homeomorphic to locally closed unions of cells in finite CW-complexes with cell actions of the group G). The equivariant analogue of the Euler characteristic with values in the ring $R(G)$ of representations of G is obtained from the equivariant Euler characteristic by the natural homomorphism $A(G) \to R(G)$ described above. Among other reductions of the equivariant Euler characteristic, one can indicate the orbifold Euler characteristic (see, e.g., [10], [76]) and its higher order analogues ([10], [126]).

5.5.2 Equivariant Indices of Vector Fields and 1-Forms on Manifolds

Let a finite group G act smoothly on (the germ of) the affine space $(\mathbb{R}^n, 0)$. Without loss of generality one can assume that the action is linear, i.e. it is defined by a representation of G on $\mathbb{R}^n$. Let X be a (continuous) vector field on $(\mathbb{R}^n, 0)$ invariant with respect to the action of G and with an isolated singular point at the origin. (One can see that, for each point p from a neighbourhood of the origin (where X is defined), the vector $X(p)$ is tangent to the subspace $(\mathbb{R}^n)^{G_p}$ of the fixed points of the isotropy subgroup G_p of the point p.)

We assume $\mathbb{R}^n$ to be endowed with a G-invariant Euclidean metric. Let $\varepsilon > 0$ be small enough so that the vector field X is defined on a neighbourhood of the closed ball B_ε of radius ε centred at the origin and has no singular points in B_ε outside the origin. It is easy to see that there exists a G-invariant vector field $\widetilde{X}$ on a neighbourhood of B_ε such that:

(1) The vector field $\widetilde{X}$ coincides with X on a neighbourhood of the sphere $S_\varepsilon = \partial B_\varepsilon$.
(2) In a neighbourhood of each singular point $p \in B_\varepsilon \setminus \{0\}$, the vector field $\widetilde{X}$ is as follows. Let $H = G_p$ be the isotropy subgroup of the point p. The germ $(\mathbb{R}^n, p)$ is in a natural way isomorphic to $((\mathbb{R}^n)^H, p) \times (((\mathbb{R}^n)^H)^\perp, 0)$, where $((\mathbb{R}^n)^H)^\perp$ is the orthogonal complement to the subspace $(\mathbb{R}^n)^H$: the direct sum of the subspaces of $\mathbb{R}^n$ corresponding to non-trivial representations

of H. In a neighbourhood of p the vector $\widetilde{X}(y_1, y_2)$ ($y_1 \in ((\mathbb{R}^n)^H, x_0)$, $y_2 \in ((\mathbb{R}^n)^H)^\perp, 0)$) is the sum $X_1(y_1) + X_2(y_2)$, where X_1 is a vector field on $((\mathbb{R}^n)^H, p)$ with an isolated singular point at p, X_2 is an H-invariant radial vector field on $((\mathbb{R}^n)^H)^\perp, 0)$. (Let us recall that, on a zero-dimensional space the only vector field (zero) is non-degenerate with the index 1 and also radial.)

Remark 5.5.2 One can assume that the vector field X_1 is smooth and has a non-degenerate singular point at x_0 (and therefore $\operatorname{ind}(X_1; (\mathbb{R}^n)^H, x_0) = \pm 1$), however, this is not necessary for the definition.

Definition 5.5.3 The *equivariant index* $\operatorname{ind}^G(X; \mathbb{R}^n, 0)$ of the vector field X at the origin is defined by the equation

$$\operatorname{ind}^G(X; \mathbb{R}^n, 0) = \sum_{\overline{p} \in (\operatorname{Sing} \widetilde{X})/G} \operatorname{ind}(\widetilde{X}_{|V_{(p)}}; V_{(p)}, p)[Gp]\,,$$

where p is a representative of the orbit $\overline{p}$. (One has $[Gp] = [G/G_p]$.)

Remark 5.5.4 One can say that the equivariant index $\operatorname{ind}^G(X; \mathbb{R}^n, 0)$ is the class $[\operatorname{Sing} \widetilde{X}] \in A(G)$ of the set $\operatorname{Sing} \widetilde{X}$ of singular points of $\widetilde{X}$ with multiplicities equal to the usual indices $\operatorname{ind}(\widetilde{X}_{|(\mathbb{R}^n)^{G_p}}; (\mathbb{R}^n)^{G_p}, p)$ of the restrictions of the vector field $\widetilde{X}$ to the corresponding fixed point sets (smooth manifolds).

One has the following equivariant version of the Poincaré-Hopf theorem.

For a subgroup $H \subset G$ there are natural maps $\mathrm{R}_H^G : A(G) \to A(H)$ and $\mathrm{I}_H^G : A(H) \to A(G)$. The *restriction map* R_H^G sends a G-set Z to the same set considered with the H-action. The *induction map* I_H^G sends an H-set Z to the product $G \times Z$ factorized by the natural equivalence: $(g_1, x_1) \sim (g_2, x_2)$ if there exists $g \in H$ such that $g_2 = g_1 g$, $x_2 = g^{-1} x_1$ with the natural (left) G-action. Both maps are group homomorphisms, however the induction map I_H^G is not a ring homomorphism.

Theorem 5.5.5 *Let M be a closed (compact, without boundary) G-manifold and let X be a G-invariant vector field on M with isolated singular points. Then one has*

$$\sum_{\overline{p} \in (\operatorname{Sing} X)/G} \mathrm{I}_{G_p}^G(\operatorname{ind}^{G_p}(X; M, p)) = \chi^G(M)\,.$$

A version of this definition of an equivariant index was given first in [93]. However, the index there takes values in an extension of the Burnside ring which depends on the G-manifold M and is not a ring. (It takes into account connected components of the fixed point sets of subgroups of G.)

Almost the same definition can be given for the equivariant index $\operatorname{ind}^G(\omega; \mathbb{R}^n, 0)$ of a G-invariant 1-form ω on $(\mathbb{R}^n, 0)$. A G-invariant Riemannian metric on a G-manifold permits to identify invariant 1-forms with invariant vector fields. In this way, the equivariant index of a 1-form is the equivariant index of the corresponding vector field.

In the complex setting one has the usual sign correction factor for a complex-valued 1-form on a complex manifold (see Sect. 5.2.2).

5.5.3 The Equivariant Radial Index on a Singular Variety

On a singular G-variety G-invariant vector fields and G-invariant 1-forms cannot be identified with each other. Therefore their equivariant indices (even being defined in similar ways) cannot be expressed through each other.

Here we shall give the definition of an equivariant version of the radial index for vector fields. The necessary changes for 1-forms are clear.

Let $(V, 0)$ be the germ of a closed real subanalytic set with an action of a finite group G. We assume $(V, 0)$ to be embedded into $(\mathbb{R}^N, 0)$ and the G-action to be induced by an (analytic) action on a neighbourhood of the origin in $(\mathbb{R}^N, 0)$. (The action on $(\mathbb{R}^N, 0)$ can be assumed to be linear.)

Let $V = \bigcup_{i=1}^{q} V_i$ be a subanalytic Whitney G-stratification of V. This means that each stratum V_i is G-invariant, the isotropy subgroups $G_p = \{g \in G : gp = p\}$ of all points p of V_i are conjugate to each other, and the quotient of the stratum V_i by the group G is connected.

Let X be a G-invariant (stratified) vector field on $(V, 0)$ with an isolated singular point at the origin. One can show that there exists a (continuous) G-invariant stratified vector field $\widetilde{X}$ on V satisfying Conditions (1)–(3) from Sect. 5.3.1.

The following definition was made in [45]. Let A be the set (a G-set) of the singular points of the vector field $\widetilde{X}$ on $V \cap B_\varepsilon$ considered with the multiplicities equal to the usual indices $\mathrm{ind}(\widetilde{X}_{|V_{(p)}}; V_{(p)}, p)$ of the restrictions of the vector field $\widetilde{X}$ to the corresponding strata (smooth manifolds).

Definition 5.5.6 The *equivariant radial index* $\mathrm{ind}^G_{\mathrm{rad}}(X; V, 0)$ of the vector field X on V at the origin is the class $[A] \in A(G)$ of the set A of singular points of $\widetilde{X}$ with multiplicities.

One can show that the equivariant radial index is well-defined: see [45].

Remark 5.5.7 As above (in the smooth case) one can write the definition as

$$\mathrm{ind}^G_{\mathrm{rad}}(X; V, 0) = \sum_{\overline{p} \in (\mathrm{Sing}\, \widetilde{X})/G} \mathrm{ind}(\widetilde{X}_{|V_{(p)}}; V_{(p)}, p)[Gp]\,,$$

where p is a representative of the orbit $\overline{p}$.

For a subgroup $H \subset G$, the vector field X is H-invariant and one has $\mathrm{ind}^H_{\mathrm{rad}}(X; V, 0) = \mathrm{R}^G_H(\mathrm{ind}^G_{\mathrm{rad}}(X; V, 0))$.

One has the following generalization of Theorem 5.5.5 (see [45, Theorem 4.6]).

Theorem 5.5.8 *Let $V = \bigcup_{i=1}^{q} V_i$ be a compact subanalytic variety and let X be a G-invariant stratified vector field on V with isolated singular points. Then one has*

$$\sum_{\overline{p}\in(\mathrm{Sing}\, X)/G} \mathrm{I}^G_{G_p}(\mathrm{ind}^{G_p}(X; V, p)) = \chi^G(V)\,.$$

Another analogue of the Euler characteristic can be defined for orbifolds: the universal Euler characteristic of orbifolds [74]. It takes values in the ring $\mathscr{R}$ generated, as a free abelian group, by isomorphism classes of finite groups. The corresponding analogue of the radial index of vector fields and 1-forms (with values in the same ring $\mathscr{R}$) was defined in [73].

5.5.4 Equivariant GSV and Poincaré–Hopf Index

Let the space $\mathbb{C}^N$ be endowed with an action of a finite group G (say, with a linear one) and let $(V, 0) = \{z \in (\mathbb{C}^N, 0) : f_1(z) = \cdots = f_{N-n}(z) = 0\}$ be an n-dimensional germ of an isolated complete intersection singularity defined by G-invariant function germs $f_i : (\mathbb{C}^N, 0) \to (\mathbb{C}, 0)$, $i = 1, \ldots, N-n$. In the usual (non-equivariant) setting (thus for the trivial group G), the GSV index of a vector field or of a 1-form on $(V, 0)$ can be defined in terms of the degree of a certain map or in terms of the intersection number of some cycles. Equivariant versions of these notions (the degree and the intersection index) are not defined (at least as elements of the Burnside ring $A(G)$). Therefore, in order to define equivariant versions of them, one has to use the fact that the GSV index agrees with the Poincaré–Hopf index (Proposition 5.3.17). Namely, let X be a G-invariant vector field on $(V, 0)$ with an isolated singularity at the origin. Let $F : (\mathscr{V}, 0) \to (\mathbb{C}, 0)$ be the essentially unique smoothing of $(V, 0)$ (see Sect. 5.3.3). The vector field $\widetilde{X}$ of Sect. 5.3.3 can be made G-invariant. (To get a G-invariant vector field, one can take an arbitrary one and take the mean over the group.) Then one can make the following definition (cf. [45, Definition 5.1]).

Definition 5.5.9 The *G-equivariant GSV index* (or *G-equivariant Poincaré–Hopf index*) is

$$\mathrm{ind}^G_{\mathrm{GSV}}(X; V, 0) = \sum_{\overline{p}\in(\mathrm{Sing}\, \widetilde{X})/G} \mathrm{ind}^G(\widetilde{X}; V_t^F, p) \in A(G), \tag{5.8}$$

where V_t^F is the Milnor fibre corresponding to the smoothing F.

It is easy to show that the right hand side of (5.8) does not depend on the extension $\widetilde{X}$ and therefore the equivariant GSV index is well-defined.

There is also a generalization to non-isolated complete intersection singularities, see [45, Definition 5.1].

The relation between the equivariant GSV index and the radial one can be described as follows. The Milnor fibre V_t^F is a manifold with a G-action and therefore its equivariant Euler characteristic $\chi^G(V_t^F) \in A(G)$ is defined. Let $\overline{\chi}^G(V_t^F) = \chi^G(V_t^F) - 1$ be the reduced equivariant Euler characteristic of V_t^F. (The element $(-1)^n\overline{\chi}^G(V_t^F) \in A(G)$ can be regarded as an equivariant version of the Milnor number of the ICIS $(V, 0)$). There is the following generalization of Proposition 5.3.18 (see [45, Proposition 5.3]).

Proposition 5.5.10 $\mathrm{ind}^G_{\mathrm{GSV}}(X; V, 0) = \mathrm{ind}^G_{\mathrm{rad}}(X; V, 0) + \overline{\chi}^G(V_t^F)$.

Let the group G act on the projective space $\mathbb{CP}^N$ by projective transformations and let $V \subset \mathbb{CP}^N$ be a G-invariant complete intersection with isolated singularities. It has a natural G-invariant smoothing $\widetilde{V} \subset \mathbb{CP}^N$. Let X be a G-invariant vector field on V with isolated singular points. Since the GSV index was defined as the Poincaré–Hopf index, it counts singular points of the vector field on the smoothing of the ICIS. Therefore one has the following version of the Poincaré–Hopf theorem (see [45, Proposition 5.2]).

Proposition 5.5.11 *Let V, $\widetilde{V}$, and X be as above. Then one has*

$$\sum_{\overline{p}\in(\mathrm{Sing}\,X)/G} \mathrm{I}^G_{G_p}(\mathrm{ind}^G_{\mathrm{GSV}}(X; V, p)) = \chi^G(\widetilde{V}) \in A(G)\,.$$

5.5.5 *Equivariant Homological Index*

Let $\mathbb{C}^N$ be endowed with an action (say, a linear one) of a finite group G and let $(V, 0) \subset (\mathbb{C}^N, 0)$ be a G-invariant germ of an analytic variety of pure dimension n. Let X be a G-invariant holomorphic vector field on $(V, 0)$ with an isolated singular point at the origin.

Remark 5.5.12 The condition that such a vector field on $(V, 0)$ exists is a rather restrictive condition on the variety. Namely, a neighbourhood in V of any point p of $V \setminus \{0\}$ has to be isomorphic to the direct product $(W_p, 0) \times (\mathbb{C}, 0)$ for a variety W_p (cf. Remark 5.3.21). This holds, in particular, if $V \setminus \{0\}$ is non-singular.

Let $\Omega^i_{V,0}$, $i = 1, 2, \ldots$, be the modules of germs of differential forms on $(V, 0)$ ($\Omega^i_{V,0} = \mathcal{O}_{V,0}$). One has natural actions of the group G on them. Consider the complex (5.1). It consists of G-modules and therefore its homology groups are (finite dimensional) G-modules as well.

Definition 5.5.13 The *equivariant homological index* of the vector field X on $(V, 0)$ is

$$\mathrm{ind}^G_{\mathrm{hom}}(X; V, 0) = \sum_{i=0}^{n} \left[H_i(\Omega^\bullet_{V,0}, X)\right] \in R_{\mathbb{C}}(G)\,,$$

where $R_{\mathbb{C}}(G)$ is the ring of (complex) representations of the group G and $[\cdot]$ is the class of a G-module in $R_{\mathbb{C}}(G)$.

Let ω be a G-invariant holomorphic 1-form on $(V, 0)$ with an isolated singular point at the origin.

Remark 5.5.14 One can see that (in contrast to the situation for vector fields) 1-forms with this property always exist.

Consider the complex (5.2). If $(V, 0)$ is smooth and $\omega(0) \neq 0$, the homology groups of the complex (5.2) are trivial. This implies that, if both V and ω have isolated singular points at the origin, the homology groups $H_i(\Omega^{\bullet}_{V,0}, \wedge\omega)$ of the complex (5.2) are finite dimensional G-modules.

Definition 5.5.15 Let $(V, 0)$ have an isolated singular point at the origin and let ω have an isolated singular point at $0 \in V$. The *equivariant homological index* of the 1-form ω on $(V, 0)$ is

$$\mathrm{ind}^G_{\mathrm{hom}}(\omega; V, 0) = \sum_{i=0}^{n} \left[H_i(\Omega^{\bullet}_{V,0}, \wedge\omega)\right] \in R_{\mathbb{C}}(G)\,.$$

Remark 5.5.16 It is not clear whether this definition makes sense for an arbitrary (G-invariant) variety $(V, 0) \subset (\mathbb{C}^N, 0)$, not necessarily with an isolated singular point at the origin, i.e. whether the homology groups $H_i(\Omega^{\bullet}_{V,0}, \wedge\omega)$ are finite dimensional in this case as well.

Assume that $(V, 0) = (\mathbb{C}^n, 0)$. It is not difficult to show that the equivariant homological index of a vector field X with an isolated singular point coincides with the reduction (under the natural homomorphism $A(G) \to R_{\mathbb{C}}(G)$) of the equivariant (radial) index of X. This follows from the fact that a G-invariant vector field on $\mathbb{C}^n$ can be deformed to one with only non-degenerate singular points and it is easy to verify the coincidence for a vector field with a non-degenerate singular point. The situation is quite different for 1-forms (for a non-trivial group G). A G-invariant 1-form cannot, in general, be deformed to one with non-degenerate singular points. It is not clear how one can describe non-removable singularities of invariant 1-forms for an arbitrary finite group G (in order to verify the coincidence of the described indices for them). This was made in [95] for the group $\mathbb{Z}_3$ of order 3. For an arbitrary finite group G the statement about the coincidence was proved in [75].

5.5.6 Equivariant Euler Obstruction

The Euler obstruction of a vector field or of a 1-form at an isolated singular point on a (quasi-projective) singular variety can be regarded as a version of an index of it. Similar to the GSV index, the usual (non-equivariant) version of the Euler obstruction is defined in terms of the (first) obstruction to extend a section of a bundle. An equivariant version of the notion of the first obstruction is not defined (at least as an element of the Burnside ring $A(G)$). Therefore, to define an equivariant version of the Euler obstruction (of a vector field or of a 1-form), one has to use another approach.

A method to define the (local) equivariant Euler obstruction of an invariant 1-form was suggested in [46]. The idea resembles the one used for the definition of the equivariant radial index. Let $(V, 0) \subset (\mathbb{C}^N, 0)$ be a germ of a complex analytic variety with an action of a finite group G and let $\{V_i\}_{i\in I}$ be a G-invariant Whitney stratification of it. Let ω be a germ of a G-invariant complex 1-form on $(V, 0)$ (that is the restriction of a G-invariant 1-form on $(\mathbb{C}^N, 0)$) with an isolated singular point at the origin. Let B_ε be a ball of a small radius ε around the origin such that representatives of V and of ω are defined in B_ε and the 1-form ω has no singular points on $V \setminus \{0\}$ inside B_ε. Let $\widetilde{\omega}$ be a G-invariant 1-form on $V \cap B_\varepsilon$ described in Sect. 5.3.5.

Definition 5.5.17 The *G-equivariant local Euler obstruction* of the 1-form ω on $(V, 0)$ is defined by

$$\mathrm{Eu}^G(\omega; V, 0) = \sum_{\overline{p}\in(\mathrm{Sing}\,\widetilde{\omega})/G} (-1)^{\dim V - \dim V_{(p)}} \mathrm{Eu}(V, V_{(p)}) \cdot \mathrm{ind}(\widetilde{\omega}_{|V_{(p)}}; V_{(p)}, p)[Gp]\,,$$

where p is a point of the orbit $\overline{p} = Gp$, $\mathrm{ind}(\cdot)$ is the usual index of a 1-form on a smooth manifold.

It is not difficult to show that the equivariant local Euler obstruction is well defined (that is, that the definition does not depend on the choice of a 1-form $\widetilde{\omega}$) and its reduction under the natural reduction homomorphism $\mathrm{R}^G_{\{e\}} : A(G) \to A(\{e\}) = \mathbb{Z}$ gives the usual Euler obstruction of the 1-form ω.

One has a global version of this notion defined either for a projective or for an affine variety, see Sect. 5.3.5. Let V be a G-invariant affine variety in $\mathbb{C}^N$ and let η be a G-invariant real 1-form on $\mathbb{C}^N$ which is radial at infinity (this means that it does not vanish on the vectors

$$\sum_i \left(x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} \right)$$

for $\|z\|$ large enough, $z = (z_1, \ldots, z_N)$, $z_j = x_j + y_j\sqrt{-1}$) and has only isolated singular points on V.

Definition 5.5.18 The *equivariant global Euler obstruction* of the affine variety V is defined by

$$\mathrm{Eu}^G(V) := \sum_{\overline{p}\in(\mathrm{Sing}\,\eta_{\mathrm{rad}})/G} \mathrm{I}^G_{G_p}(\mathrm{Eu}^{G_p}(\eta_{\mathrm{rad}}; V, p)) \in A(G).$$

The same definition makes sense for a projective (therefore compact) variety. The only difference is that one has to take an arbitrary 1-form η with only isolated singular points.

5.5.7 Real Quotient Singularities

Let a finite group G act (linearly) on the space $\mathbb{R}^n$ (and thus on its complexification $\mathbb{C}^n$). For an analytic (real) 1-form ω on $(\mathbb{R}^n, 0)$, there is defined a natural (Eisenbud–Levine–Khimshiashvili) quadratic form B on

$$\Omega_\omega := \Omega^n_{\mathbb{R}^n,0}/\omega \wedge \Omega^{n-1}_{\mathbb{R}^n,0} :$$

see Sect. 5.2.4. Its signature is equal to the index $\mathrm{ind}(\omega; \mathbb{R}^n, 0)$ of the 1-form ω. If the 1-form ω is G-invariant, its (equivariant) index $\mathrm{ind}^G(\omega; \mathbb{R}^n, 0)$ is defined as an element of the Burnside ring $A(G)$. In this case the Eisenbud–Levine–Khimshiashvili quadratic form is also G-invariant and therefore its (equivariant) signature $\mathrm{sgn}^G B$ is defined as an element of the ring $R_{\mathbb{R}}(G)$ of real representations of the group G. One can expect a relation between the equivariant signature $\mathrm{sgn}^G B$ and the equivariant index $\mathrm{ind}^G(\omega; \mathbb{R}^n, 0)$ (or rather its reduction $r(\mathrm{ind}^G(\omega; \mathbb{R}^n, 0))$ under the natural homomorphism $r : A(G) \to R_{\mathbb{R}}(G)$).

The most straightforward conjecture would be that the reduction $r(\mathrm{ind}^G(\omega; \mathbb{R}^n, 0))$ is equal to the equivariant signature $\mathrm{sgn}^G B$. In [72] and also in [30], it was explained that (for differentials of function germs) this was not the case. The reason is roughly speaking the following. For a (G-invariant) morsification of a function germ, the usual signature of the residue pairing can be expressed in terms of the real critical points of the morsification, whence an equation for the equivariant signature involves also critical points whose complex conjugates lie in the same G-orbit.

A weaker conjecture can be as follows. Let $r^{(0)} : A(G) \to \mathbb{Z}$ be the group homomorphism defined by $r^{(0)}([G/H]) = 1$. This means that $r^{(0)}\left(\sum a_H[G/H]\right) = \sum a_H$.) Let $B^G_\omega : \Omega^G_\omega \times \Omega^G_\omega \to \mathbb{R}$ be the restriction of the residue pairing to the G-invariant part Ω^G_ω of Ω_ω. It is a non-degenerate bilinear form as well. It is possible to show that the image of the index $\mathrm{ind}^G(\omega; \mathbb{C}^n, 0)$ under the map $r : A(G) \to R_{\mathbb{C}}(G)$ is equal to the class $[\Omega^{\mathbb{C}}_\omega]$ of the G-module $\Omega^{\mathbb{C}}_\omega$: [75]. Therefore, for the G-invariant part $\left(\Omega^{\mathbb{C}}_\omega\right)^G$ of $\Omega^{\mathbb{C}}_\omega$, one has

$$\dim \left(\Omega^{\mathbb{C}}_\omega\right)^G = r^{(0)}(\mathrm{ind}^G(\omega; \mathbb{C}^n, 0)) . \tag{5.9}$$

Taking into account relations between dimensions of modules in the complex case and signatures of quadratic forms in the real case in the Eisenbud–Levine–Khimshiashvili theory, one can conjecture that

$$\mathrm{sgn}\, B^G_\omega = r^{(0)}(\mathrm{ind}^G(\omega; \mathbb{R}^n, 0)) .$$

Again, in general this is not the case.

Let W be the real part of the quotient $\mathbb{C}^n/G$. Note that in general $W \neq \mathbb{R}^n/G$. A real analytic 1-form η on W defines a G-invariant analytic 1-form $\omega = \pi^*\eta$ on $\mathbb{C}^n$ ($\pi : \mathbb{C}^n \to \mathbb{C}^n/G$ is the quotient map) which is real (that is real on $\mathbb{R}^n \subset \mathbb{C}^n$) and, moreover, real on $\pi^{-1}(W)$.

One can prove the following statement ([47, 48]):

Theorem 5.5.19 *For an abelian finite group G and for a real analytic G-invariant 1-form ω one has*

$$\operatorname{sgn} B_\omega^G = r^{(0)}(\operatorname{ind}^G(\omega; \pi^{-1}(W), 0))\,. \tag{5.10}$$

Remark 5.5.20 It is very probable that the statement holds for non-abelian groups as well. However, this is not proved.

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Chapter 6
Motivic Hirzebruch Class and Related Topics

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Contents

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Abstract The motivic Hirzebruch class is a characteristic class "unifying" three distinguished characteristic homology classes of singular varieties. In this survey we discuss characteristic cohomology classes of vector bundles, characteristic homology classes of singular varieties, the motivic Hirzebruch class and related topics such as Milnor class and Fulton–MacPherson's bivariant theory etc., emphasizing "categorical aspects", namely, *natural transformations*.

6.1 Introduction

A characteristic class is usually a cohomology class of a vector bundle, i.e., an invariant of vector bundles taking values in the cohomology group of the base space. So, the reader might think that a "motivic" characteristic class means one taking values in a motivic cohomology group (e.g., [138]) of the base space (e.g., see [18, 119, 198]), but "motivic" used in this survey is adjective used in algebraic geometry, such as 'motivic' measure (e.g., [66, 114, 122]), 'motivic' invariant (e.g., see [104]) and so on. Roughly speaking, it mainly means '*additive over cutting into pieces*'.

In this survey we discuss characteristic *homology* classes from *categorical viewpoints*, namely we treat them as

"natural transformations from certain covariant functors to homology functors".

Saunders MacLane, one of the founders of category theory, is said to have remarked: *"I didn't invent categories to study functors; I invented them to study natural transformations."* (see [223, Historical notes]). Also see MacLane's book [125, §1.4, p.18]:*"As Eilenberg-MacLane first observed, 'category' has been defined in order to be able to define 'functor' and 'functor' has been defined in order to be able to define 'natural transformations' "*

Typical and important characteristic *cohomology* classes are Chern class, Stiefel–Whitney class and Pontryagin class [142]. In the case of a nonsingular variety or a differentiable manifold one can define *its characteristic cohomology class* as *that of its tangent bundle*. However, if it has singularities like algebraic varieties, one cannot define its characteristic cohomology class as above, since its tangent bundle cannot be defined because of singularities. Of course, if one could define a unique vector bundle E up to isomorphism even if the variety has singularities, one could define its characteristic cohomology class as the characteristic cohomology class of this vector bundle E. However, it seems that the existence of such a vector bundle has not been found yet.

If we consider categorical viewpoints, the characteristic cohomology class of a vector bundle can be put in as follows.[1] Let $\mathrm{Vect}(X)$ be the set of isomorphisms classes of vector bundles over a topological space X. Then we get the contravariant functor $\mathrm{Vect} : \mathscr{TOP} \to \mathscr{SET}$ from the category $\mathscr{TOP}$ of topological spaces to the category $\mathscr{SET}$ of sets. The cohomology group $H^*(X)$ is a contravariant functor $H^* : \mathscr{TOP} \to \mathscr{SET}$ if we ignore the group or ring structure of $H^*(-)$. Then, the operation $c\ell^*$ taking a characteristic cohomology class $c\ell^*(E)$ of a vector bundle E is a natural transformation $c\ell^* : \mathrm{Vect} \to H^*$. The category $\mathscr{SET}$ can be replaced by the category $\mathscr{MON}$ of monoids, since $\mathrm{Vect}(X)$ is a monoid by considering the Whitney sum $E \oplus F$. Furthermore, in the case of multiplicative characteristic classes such as Chern class, i.e., in the case when it satisfies $c\ell^*(E \oplus F) = c\ell^*(E)c\ell^*(F)$, the monoid $\mathrm{Vect}(X)$ can be replaced by the Grothendieck group (or "group completion")[2] of the monoid $\mathrm{Vect}(X)$, i.e., the K-group $K(X)$ (or $K^0(X)$) and the category $\mathscr{MON}$ can be replaced by the category $\mathscr{ABY}$ of abelian groups, and the multiplicative characteristic class can be captured as a natural transformation $c\ell^* : K(-) \to H^*(-)$ (cf. [21, 179]).

In 1960s characteristic classes or some invariants of spaces having singularities had come to be defined as homology classes or relative cohomology classes not as cohomology classes. The first one was constructed by M.-H. Schwartz. In [171] (also see [172]) Schwartz defined *a class of a singular variety X embedded into a manifold M by a class in the relative cohomology* $H^*(M, M \setminus X)$, using a stratification and its associated stratified vector bundle. Her class $Sch^p(X) \in H^{2p}(M, M \setminus X)$ is called the *p-th Schwartz class of X*. For more details, e.g., see [48, 50, 53, 57]. Another one defined as *a natural transformation* was Stiefel–Whitney homology class due to D. Sullivan [184] and its Chern class version is MacPherson's Chern class. In the study of Riemann–Roch type theorems, A. Grothendieck and P. Deligne made a conjecture about Chern class and its

[1] Here we are sloppy, ignoring whether it is a real or complex vector bundle.

[2] Note that the Grothendieck group can be in general defined for a semigroup, i.e., not requiring the existence of the unit. In the case of $\mathrm{Vect}(X)$, the class of the zero bundle $X \times \{0\}$ is the additive unit.

modified version[3] was solved by R. MacPherson [127]. For the category of complex algebraic varieties, MacPherson's Chern class is *a unique natural transformation* $c_* : F(-) \to H_*(-)$ satisfying that ("smooth condition") $c_*(\mathbb{1}_X) = c(TX) \cap [X]$ for non-singular X. Here $F(-)$ is a covariant functor such that $F(X)$ is the abelian group of constructible functions on X and $H_*(X)$ is the Borel–Moore homology group. $\mathbb{1}_X$ is the characteristic function on X and $c(TX)$ is the Chern cohomology class of the tangent bundle TX. In fact, J.-P. Brasselet and M.-H. Schwartz [50] proved that *the value* $c_*(\mathbb{1}_X)$ *is equal to the Schwartz class* $Sch^*(X)$ *under the Alexander duality isomorphism* $\mathscr{A} : H^{2p}(M, M \setminus X) \cong H_{m-2p}(X)$, where $m = \dim_{\mathbb{R}} M$ the real dimension of M; $\mathscr{A}(Sch^p(X)) = c_{m-2p}(X)$. Due to this fact, the homology class $c_*(X) := c_*(\mathbb{1}_X)$ is called the *Chern–Schwartz–MacPherson class*[4] *of X*.

After MacPherson's Chern class, Todd class and L-class of singular varieties have been constructed as natural transformations, respectively, Baum–Fulton–MacPherson's Todd class (or Baum–Fulton–MacPherson's Riemann–Roch, abbr., BFM-RR)[33] $td_* : K_0(X) \to H_*(X) \otimes \mathbb{Q}$ and Cappell–Shaneson's L-class [58] $L_* : \Omega(X) \to H_*(X) \otimes \mathbb{Q}$. Here $K_0(X)$ is the Grothendieck group of coherent sheaves on X and $\Omega(X)$ is the cobordism group of self-dual constructible complex of sheaves on X [221]. These classes should be more precisely, e.g., called MacPherson's Chern class transformations or MacPherson's Chern class natural transformations, but for the sake of simplicity just called MacPherson's Chern class. However, in order to emphasize "natural transformation", we sometimes use these expressions.

Based on such a formulation of characteristic homology classes, MacPherson wrote a survey paper [128], at the very end of which he wrote *"It remains to be seen if there exists a unified theory of characteristic classes of singular varieties."* At that time the case of L-class was not being formulated yet. Only after Intersection Homology [92](also see [93] (cf. [35]) and [47, 82, 112, 113, 129, 132]) was introduced by M. Goresky and MacPherson, an L-class of a singular variety was introduced by themselves [92]. Furthermore, Cappell and Shaneson introduced another L-class [58], which was not explicitly described as a natural transformation, but was captured as a natural transformation in [207]. For the construction of these L-classes one has to assume that X is compact.

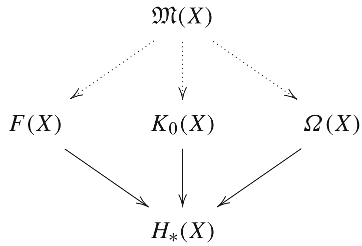
The above three characteristic homology classes c_*, td_*, L_* are formulated as Grothendieck–Riemann–Roch type theorems,[5] i.e., a natural transformation from respectively different covariant functors $F(X), K_0(X), \Omega(X)$ to the homology group. *A fundamental problem is whether one can construct a theory which "unifies" these three* c_*, td_*, L_* (cf. [209]), which is more concrete than the above

[3] In [94, II, 15.3.5, Note 87_1] (cf. [94, IV, 18.3.2, Note 164, II Cohomologie étale, 1]) Grothendieck himself explains the difference between his formulation and that of MacPherson.

[4] Thus $c_*(X)$ is sometimes denoted by $c_*^{\mathrm{SM}}(X)$.

[5] The word "Riemann–Roch" or "Grothendieck–Riemann–Roch" is not used at all in [127], as remarked by Grothendieck.

MacPherson's question. What was in the author's mind (a prototype was in [214]) is that there would be a certain covariant functor $\mathfrak{M}(X)$ as a sort of 'motif' of these three covariant functors $F(X)$, $K_0(X)$, $\Omega(X)$ and a certain natural transformation $\mathfrak{N}_{c\ell} : \mathfrak{M}(X) \to H_*(X) \otimes \Lambda_{c\ell}$ (which should be related to the characteristic cohomology class $c\ell$), from which one would obtain the above three characteristic homology classes: (1) $c_* : F(X) \to H_*(X)$ when $c\ell$ = Chern class, (2) $td_* : K_0(X) \to H_*(X) \otimes \mathbb{Q}$ when $c\ell$ = Todd class, (3) $L_* : \Omega(X) \to H_*(X) \otimes \mathbb{Q}$ when $c\ell$ = L-class. Roughly speaking, a certain covariant functor $\mathfrak{M}(X)$ was considered as in the following diagram (not commutative):



This problem was solved in [52] by considering $\mathfrak{M}(X) := K_0(\mathscr{V}/X)$ the relative Grothendieck group or "motivic group" of complex algebraic varieties and defining *the motivic Hirzebruch class* $T_{y*} : K(\mathscr{V}/-) \to H_*(-) \otimes \mathbb{Z}[y]$. Here $\mathscr{V}$ denotes the category of complex algebraic varieties.

From a historical viewpoint, clearly the *Riemann–Roch Theorem* should be mentioned. It would be safe to say that *the "origin" of the motivic Hirzebruch class is the Riemann–Roch Theorem.* Here is a very quick and rough explanation (for more details see Sect. 6.3.1). For a divisor D on a smooth complex projective curve of genus g, Riemann's inequality is $\dim_{\mathbb{C}} L(D) \geqslant \deg D + 1 - g$ and the difference between two sides of the inequality was identified by G. Roch:

$$\dim_{\mathbb{C}} L(D) - \dim_{\mathbb{C}} L(K - D) = \deg D + 1 - g,$$

which is the well-known *Riemann–Roch Theorem* (abbr., RR). This RR was extended or generalized by F. Hirzebruch to what is called *Hirzebruch–Riemann–Roch* (abbr., HRR):for a complex vector bundle E over a complex projective manifold X the following holds

$$\chi(X, E) = \int_X (ch(E)) \cup td(TX)) \cap [X].$$

In the case when X is a smooth complex projective curve, RR is "almost" equal to HRR. Namely, RR is equal to *HRR plus Serre's Duality* or *RR plus Serre's Duality* is equal to HRR. A. Grothendieck succeeded in *extending HRR to the following* *commutative diagram* for an algebraic map $f : X \to Y$ of projective complex algebraic manifolds X and Y:

$$\begin{array}{ccc} K^0(X) & \xrightarrow{ch(-)\cup td(TX)} & H^*(X)\otimes\mathbb{Q} \\ {\scriptstyle f_!}\downarrow & & \downarrow{\scriptstyle f_!} \\ K^0(Y) & \xrightarrow[ch(-)\cup td(TY)]{} & H^*(Y)\otimes\mathbb{Q}. \end{array}$$

This is called *Grothendieck–Riemann–Roch* (abbr., GRR). Indeed, *HRR is nothing but GRR for $f : X \to pt$ the map to a point*, which is the meaning of "extending HRR to GRR".

HRR was generalized by Hirzebruch to the following *generalized Hirzebruch–Riemann–Roch* (abbr., gHRR):

$$\chi_y(X, E) = \int_X \big(ch_{(1+y)}(E)) \cup T_y(TX)\big) \cap [X].$$

The three distinguished cases of gHRR for a trivial line bundle E are the following:

(i) $(y = -1)$ $\chi(X) = \int_X c(TX) \cap [X]$ (Gauss–Bonnet theorem)
(ii) $(y = 0)$ $\chi^a(X) = \int_X td(TX) \cap [X]$ (HRR for the trivial line bundle $E = 1_X$) where $\chi^a(X)$ is the arithmetic genus.
(iii) $(y = 1)$ $\sigma(X) = \int_X L(TX) \cap [X]$ (Hirzebruch's signature theorem).

These three theorems were *individually* extended to singular varieties, more strongly, as the following natural transformations for possibly singular varieties, respectively:

(i) MacPherson's Chern class: $c_* : F(-) \to H_*(-)$
(ii) Baum–Fulton–MacPherson's Todd class: $td_* : K_0(-) \to H_*(-) \otimes \mathbb{Q}$
(iii) Cappell–Shaneson's L-class: $L_* : \Omega(-) \to H_*(-) \otimes \mathbb{Q}$

These three natural transformations are unified by *the motivic Hirzebruch class* $T_{y*} : K_0(\mathcal{V}/X) \to H_*(X) \otimes \mathbb{Q}[y]$.

We also add that the above three theorems; Gauss–Bonnet theorem, HRR for the trivial line bundle, and Hirzebruch's signature theorem, were extended or generalized by M. Atiyah and I. M. Singer to *Atiyah–Singer Index Theorem*!

In this survey we discuss characteristic classes, the above motivic Hirzebruch class and related topics, such as Verdier–Riemann–Roch (abbr., VRR) type formulas, which are "contravariant-theoretical aspects" of these characteristic classes, Milnor classes, equivariant motivic Hirzebruch classes, and Fulton–MacPherson's bivariant theory, which is a theory "unifying" homology (covariant) and cohomology (contravariant) theories and was introduced with "unifying" three Riemann–Roch theorems (i.e., "SGA 6" (see (6.13) in Sect. 6.3.3), BFM-RR and VRR) as one motivation of their work.

Characteristic classes for singular varieties are discussed also in other articles in Handbook of Geometry and Topology of Singularities, III; see P. Aluffi [4], J.-P. Brasselet [48], R. Callejas-Bedregal et al. [57] and T. Suwa [187].

6.2 Characteristic Classes of Complex Vector Bundles

In this section we give a quick review of characteristic classes (e.g., see [142] and also see [48, 78]) of complex vector bundles in a way slightly different from the usual or standard one.

6.2.1 Characteristic Cohomology Classes

Let $\mathfrak{R}$ be a ring and let $H^p(X;\mathfrak{R})$ be the p-th cohomology group of a paracompact space X with the coefficient ring $\mathfrak{R}$. Let $\mathrm{Vect}_n(X)$ be the set of isomorphism classes of complex vector bundles of rank n over X. Then $\mathrm{Vect}_n : \mathscr{TOP} \to \mathscr{SET}$ is a contravariant functor. Similarly, the correspondence $H^p(-;\mathfrak{R}) : \mathscr{TOP} \to \mathscr{SET}$ taking the p-th cohomology group of a space is also a contravariant functor. *A characteristic class of degree p* (cf. [78, §3.1]) is defined to be *a natural transformation*

$$c\ell_p : \mathrm{Vect}_n(-) \to H^p(-;\mathfrak{R}).$$

Namely, for the isomorphism class of a complex vector bundle $E \to X$, $c\ell_p(E) \in H^p(X;\mathfrak{R})$, and for a continuous map $f : X \to Y$ and a complex vector bundle E over Y we have $f^*c\ell_p(E) = c\ell_p(f^*E)$, i.e., we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Vect}_n(Y) & \xrightarrow{c\ell_p} & H^p(Y;\mathfrak{R}) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathrm{Vect}_n(X) & \xrightarrow{c\ell_p} & H^p(X;\mathfrak{R}). \end{array}$$

Let Char_p be the set of all characteristic classes of degree p. Then Char_p is an abelian group with the operation $+$: for $c\ell_p, c\ell'_p \in \mathrm{Char}_p$, we define

$$c\ell_p + c\ell'_p : \mathrm{Vect}_n(-) \to H^p(-;\mathfrak{R})$$

by $c\ell_p + c\ell'_p : \mathrm{Vect}_n(X) \to H^p(X;\mathfrak{R})$ for X with $(c\ell_p + c\ell'_p)(E) = c\ell_p(E) + c\ell'_p(E)$, which certainly satisfies the above naturality. For $c\ell_p \in \mathrm{Char}_p$ and $c\ell_q \in \mathrm{Char}_q$ we define the (cup) product

$$c\ell_p \cdot c\ell_q : \mathrm{Vect}_n(-) \to H^{p+q}(-;\mathfrak{R})$$

by $c\ell_p \cdot c\ell'_p : \mathrm{Vect}_n(X) \to H^{p+q}(X;\mathfrak{R})$ for X with $(c\ell_p \cdot c\ell'_p)(E) = c\ell_p(E) \cup c\ell'_p(E)$, which certainly satisfies the above naturality.

Hence $\mathrm{Char} := \bigoplus_p \mathrm{Char}_p$ becomes a ring, which shall be called *a ring of characteristic classes of complex vector bundles of rank n* (defined on the contravariant functor). Thus a total characteristic class $c\ell = \sum_p c\ell_p$ of complex vector bundles of rank n is a natural transformation $c\ell = \sum_p c\ell_p : \mathrm{Vect}_n(-) \to H^*(-;\mathfrak{R})$.

Let $G_n(\mathbb{C}^{n+k})$ be the complex Grassmannian manifold [142] consisting of all n-dimensional sub vector spaces of the vector space $\mathbb{C}^{n+k}$. It is a compact complex manifold. The canonical inclusion $\mathbb{C}^{n+i} \subset \mathbb{C}^{n+i+i}$ defined by $(z_1, z_2, \cdots, z_{n+i}) \to (z_1, z_2, \cdots, z_{n+i}, 0)$ gives a sequence of inclusions

$$G_n(\mathbb{C}^n) \subset G_n(\mathbb{C}^{n+1}) \subset \cdots \subset G_n(\mathbb{C}^{n+k}) \subset \cdots$$

whose inductive limit $\bigcup \mathbb{C}^{n+i}$ is denoted by $G_n(\mathbb{C}^\infty)$, called *the infinite complex Grassmannian manifold* [142]. $G_n(\mathbb{C}^\infty)$ is a paracompact space and there is the tautological rank n complex vector bundle over it and denoted by $\pi : \gamma^n \to G_n(\mathbb{C}^\infty)$. $G_n(\mathbb{C}^\infty)$ is called the *classifying space* of rank n complex vector bundles and γ^n is called the *universal rank n complex vector bundle* because of the following theorem (see [142, §5 and Theorem 14.6]), which is a fundamental theorem for complex vector bundles and characteristic classes of complex vector bundles:

Theorem 6.2.1 *For a rank n complex vector bundle $E \to X$ over a paracompact space X,*

(i) *there exists a continuous map $f_E : X \to G_n(\mathbb{C}^\infty)$ such that $E \cong f_E^*\gamma^n$, and*
(ii) *the continuous map f_E is unique up to homotopy, i.e., if $g_E : X \to G_n(\mathbb{C}^\infty)$ satisfies $E \cong g_E^*\gamma^n$, then $f_E \sim g_E$.*
(iii) *If $E \cong E'$ and $E \cong f_E^*\gamma^n$ and $E' \cong g_E^*\gamma^n$, then $f_E \sim g_E$.*

In other words we have the following set isomorphism:

$$\mathrm{Vect}_n(X) \cong [X, G_n(\mathbb{C}^\infty)], \quad [E] \longleftrightarrow [f_E].$$

A characteristic class $c\ell_p$ of degree p is completely determined by the cohomology class $c\ell_p(\gamma^n) \in H^p(G_n(\mathbb{C}^\infty);\mathfrak{R})$ of the universal bundle as follows:

Let $f_E : X \to G_n(\mathbb{C}^\infty)$ be a classifying map for the given vector bundle E and consider the following commutative diagram:

$$\begin{array}{ccc}
\gamma^n \in \mathrm{Vect}_n(G_n(\mathbb{C}^\infty)) & \xrightarrow{c\ell_p} & H^p(G_n(\mathbb{C}^\infty); \mathfrak{R}) \ni c\ell_p(\gamma^n) \\
{\scriptstyle f_E^*}\downarrow & & \downarrow{\scriptstyle f_E^*} \\
[E] \in \mathrm{Vect}_n(X) & \xrightarrow[c\ell_p]{} & H^p(X; \mathfrak{R}) \ni c\ell_p(E).
\end{array} \tag{6.1}$$

Then we have

$$c\ell_p(E) = c\ell_p(f_E^*\gamma^n)) = f_E^*(c\ell_p(\gamma^n)).$$

Conversely, if we choose a cohomology class $\alpha_p \in H^p(G_n(\mathbb{C}^\infty); \mathfrak{R})$ and we define

$$c\ell_{\alpha_p} : \mathrm{Vect}_n(X) \to H^p(X; \mathfrak{R}), \quad c\ell_{\alpha_p}(E) := f_E^*\alpha_p.$$

(We could use the same symbol α_p instead of $c\ell_{\alpha_p}$ by setting $\alpha_p(E) := f_E^*\alpha_p$, but we do not do so in order to avoid some possible confusion.)

Then this is a characteristic class of degree p; $c\ell_{\alpha_p} : \mathrm{Vect}(-) \to H^p(-, \mathfrak{R})$. Indeed, for a continuous map $g : X \to Y$ and for a vector bundle $F \to Y$ we have a classifying map $f_F : Y \to G_n(\mathbb{C}^\infty)$ such that $F \cong f_F^*\gamma^n$. Hence $g^*F = g^*(f_F^*\gamma^n) = (g^*f_F^*)\gamma^n = (f_F \circ g)^*\gamma^n$. Hence $f_F \circ g : X \to G_n(\mathbb{C}^\infty)$ is a classifying map of the pullback vector bundle g^*F. Thus we have

$$c\ell_{\alpha_p}(g^*F) = (f_F \circ g)^*\alpha_p = (g^*f_F^*)\alpha_p = g^*(f_F^*\alpha_p) = g^*(c\ell_{\alpha_p}(F)),$$

which means that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Vect}_n(Y) & \xrightarrow{c\ell_{\alpha_p}} & H^p(Y; \mathfrak{R}) \\
{\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle g^*} \\
\mathrm{Vect}_n(X) & \xrightarrow[c\ell_{\alpha_p}]{} & H^p(X; \mathfrak{R}).
\end{array}$$

We also note that $c\ell_{\alpha_p}(\gamma^n) = \mathrm{id}^*_{G_n(\mathbb{C}^\infty)}\alpha_p = \alpha_p$, since a classifying map for γ^n is the identity map $\mathrm{id}_{G_n(\mathbb{C}^\infty)} : G_n(\mathbb{C}^\infty) \to G_n(\mathbb{C}^\infty)$. In other words, we have the following isomorphism (as a ring)

$$\mathrm{Char} = \bigoplus_p \mathrm{Char}_p \cong \bigoplus_p H^p\left(G_n(\mathbb{C}^\infty); \mathfrak{R}\right) = H^*\left(G_n(\mathbb{C}^\infty); \mathfrak{R}\right),$$

$$c\ell_p \leftrightarrow c\ell_p(\gamma^n) \quad \text{or} \quad c\ell_{\alpha_p} \leftrightarrow \alpha_p.$$

Hence the upshot is

Observation 6.2.2 To determine Char $= \bigoplus_p \mathrm{Char}_p$ (all the characteristic classes of complex vector bundles of rank n) is reduced to determining the cohomology ring $H^*(G_n(\mathbb{C}^\infty); \mathfrak{R})$.

6.2.2 Yoneda's Lemma

In fact, if we use the isomorphism $\mathrm{Vect}_n(X) \cong [X, G_n(\mathbb{C}^\infty)]$, then the above commutative diagram (6.1) become as follows:

$$\begin{array}{ccc} \mathrm{id}_{G_n(\mathbb{C}^\infty)} \in [G_n(\mathbb{C}^\infty), G_n(\mathbb{C}^\infty)] & \xrightarrow{c\ell_p} & H^p(G_n(\mathbb{C}^\infty); \mathfrak{R}) \ni c\ell_p(\mathrm{id}_{G_n(\mathbb{C}^\infty)}) \\ {\scriptstyle f_E^*}\downarrow & & \downarrow{\scriptstyle f_E^*} \\ [f_E] \in [X, G_n(\mathbb{C}^\infty)] & \xrightarrow[c\ell_p]{} & H^p(X; \mathfrak{R}) \ni c\ell_p(E). \end{array} \tag{6.2}$$

This is nothing but Yoneda's Lemma (e.g., see [125]):

Theorem 6.2.3 (Yoneda's Lemma, in the Contravariant Case) *Let $\mathscr{C}$ be a locally small category (i.e., $hom_{\mathscr{C}}(A, B)$ is a set) and let $F^* : \mathscr{C} \to \mathscr{SET}$ be a contravariant functor. Let $h^A : \mathscr{C} \to \mathscr{SET}$ be the (hom-set) contravariant functor $h^A := hom_{\mathscr{C}}(-, A)$. Then the set of all the natural transformations from the hom-set contravariant functor $h^A = hom_{\mathscr{C}}(-, A)$ to the contravariant functor F^* is isomorphic to the set $F^*(A)$:*

$$\mathscr{N}atural(h^A, F^*) \cong F^*(A).$$

Indeed, Yoneda's Lemma is proved by the following commutative diagram: Let $\tau : [-, A] \to F^*(-)$ be a natural transformation:

$$\begin{array}{ccc} \mathrm{id}_A \in [A, A] & \xrightarrow{\tau} & F^*(A) \ni \tau(\mathrm{id}_A) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ f \in [X, A] & \xrightarrow[\tau]{} & F^*(X) \ni \tau(f). \end{array} \tag{6.3}$$

Note that $f = f^*(\mathrm{id}_A) = f \circ \mathrm{id}_A$. Thus, $\tau(f) = \tau(f^*(\mathrm{id}_A))$, which is $f^*(\tau(\mathrm{id}_A))$ by the naturality of τ. Thus the natural transformation $\tau : [-, A] \to F^*(-)$ is completely determined by the assignment $\tau(\mathrm{id}_A) \in F^*(A)$.

Indeed, in our case, the category $\mathscr{C}$ is the homotopy category $h\mathscr{TOP}$ of topological spaces and the contravariant functor $F^* : h\mathscr{TOP} \to \mathscr{SET}$ is the cohomology theory $H^*(-;\mathfrak{R})$ forgetting the ring structure. Then, by observing the isomorphism $\mathrm{Vect}_n(-) \cong [-, G_n(\mathbb{C}^\infty)]$ we have

$$\begin{aligned} \mathrm{Char} &\cong \mathscr{N}atural\Big(\mathrm{Vect}_n(-), H^*(-;\mathfrak{R})\Big) \\ &\cong \mathscr{N}atural\Big([-, G_n(\mathbb{C}^\infty)], H^*(-;\mathfrak{R})\Big) \\ &= \mathscr{N}atural\Big(hom_{h\mathscr{TOP}}(-, G_n(\mathbb{C}^\infty)), H^*(-;\mathfrak{R})\Big) \\ &\cong H^*(G_n(\mathbb{C}^\infty);\mathfrak{R}) \quad \text{(by Yoneda's Lemma)} \end{aligned}$$

The covariant case is the following, just changing contravariant to covariant:

Theorem 6.2.4 (Yoneda's Lemma, in the Covariant Case) *Let $\mathscr{C}$ be a locally small category (i.e., $hom_{\mathscr{C}}(A, B)$ is a set) and let $F_* : \mathscr{C} \to \mathscr{SET}$ be a covariant functor. where $\mathscr{SET}$ is the category of sets. Let $h_A : \mathscr{C} \to \mathscr{SET}$ be the (hom-set) covariant functor $h_A := hom_{\mathscr{C}}(A, -)$. Then the set of all the natural transformations from the hom-set covariant functor $h_A = hom_{\mathscr{C}}(A, -)$ to the covariant functor F_* is isomorphic to the set $F_*(A)$:*

$$\mathscr{N}atural(h_A, F_*) \cong F_*(A).$$

Remark 6.2.5 It follows from the above covariant Yoneda's lemma that we have $\mathscr{N}atural\Big([G_n(\mathbb{C}^\infty), -], H_*(-;\mathfrak{R})\Big) \cong H_*(G_n(\mathbb{C}^\infty);\mathfrak{R})$. What is a geometric meaning of this?

If $\mathfrak{R} = \mathbb{Z}$, we have the following theorem [142, Theorem 14.5]:

Theorem 6.2.6 *$H^*(G_n(\mathbb{C}^\infty);\mathbb{Z}) = \mathbb{Z}[c_1, c_2, \cdots, c_n]$ where $c_i \in H^{2i}(G_n(\mathbb{C}^\infty);\mathbb{Z})$ is called the ith Chern class of the universal bundle γ^n. In particular, we have that $H^*(G_n(\mathbb{C}^\infty);\mathbb{Z})$ is a commutative ring graded in even degrees.*

Corollary 6.2.7 *A ($\mathbb{Z}$-coefficient) characteristic class $c\ell$ of complex vector bundle E of rank n is a polynomial ϕ of Chern classes $c_i(E) := f_E^* c_i$ where $f_E : X \to G_n(\mathbb{C}^\infty)$ is a classifying map for E: $c\ell(E) = \phi(c_1(E), c_2(E), \cdots, c_n(E))$. If we denote* Char$(E)$ *be the subgroup (in fact the subring) of $H^*(X;\mathbb{Z})$ consisting of all the characteristic classes of E, then we have*

$$\mathrm{Char}(E) = \mathbb{Z}[c_1(E), c_2(E), \cdots, c_n(E)].$$

Remark 6.2.8 For real vector bundles, $H^*(G_n(\mathbb{R}^\infty);\mathbb{Z}_2) = \mathbb{Z}_2[w_1, w_2, \cdots, w_n]$, where $w_i \in H^i(G_n(\mathbb{C}^\infty);\mathbb{Z}_2)$ is the i-th Stiefel–Whitney class of the universal bundle and $\mathrm{Char}(E) = \mathbb{Z}_2[w_1(E), w_2(E), \cdots, w_n(E)]$.

Lemma 6.2.9 *$H^*(\mathbb{P}^\infty;\mathbb{Z}) = \mathbb{Z}[\alpha]$ with $\alpha \in H^2(\mathbb{P}^\infty;\mathbb{Z})$.*

Here we observe that the Chern classes satisfy the following properties and in fact they are characterized by these four properties. In other words, this is an axiomatic definition of Chern class of complex vector bundles [99, Axioms of Chern class] (cf. [142, Remarks on p.38]):

Theorem 6.2.10 *For a complex vector bundle $E \to X$ with $\operatorname{rank}_{\mathbb{C}} E = n$, the total Chern class $c(E) \in H^*(X;\mathbb{Z})$ satisfies the following properties.*

(i) *(finiteness)* $c(E) = 1 + c_1(E) + c_2(E) + \cdots + c_n(E) \in H^*(X;\mathbb{Z})$ *with* $c_i(E) \in H^{2i}(X;\mathbb{Z})$ *and* $c_i(E) = 0$ *for* $i > n = \operatorname{rank}_{\mathbb{C}} E$,
(ii) *(naturality)* $f : X \to Y$ *and a vector bundle* $E \to Y$, $c(f^*E) = f^*c(E)$, *i.e., it commutes with the pullback operation,*
(iii) *(Whitney sum formula)* $c(E \oplus F) = c(E) \cdot c(F)$ *(where* $\cdot$ *is the cup product),*
(iv) *(normalization) For the tautological line bundle* γ_n^1 *of the complex projective space* $\mathbb{P}^n$, $c_1(\gamma_n^1) = [\mathbb{P}^{n-1}] \in H^2(\mathbb{P}^n;\mathbb{Z})$. *Here* $[\mathbb{P}^{n-1}]$ *is the Poincaré dual of the homology class determined by the hyperplane* $\mathbb{P}^{n-1}$.

Such a characteristic class is uniquely determined, i.e., it has to be the Chern class.

6.3 Hirzebruch–Riemann–Roch and Grothendieck–Riemann–Roch

For this section, e.g., see [99], [85, Notes and References, pp.302–304], [87] and [81, §2].

6.3.1 Riemann–Roch Theorem

In this section we quickly recall what is called the (classical[6] or original) Riemann–Roch theorem (e.g., see [84, Chapter 8], [96, III Curves, §1] and [144, §7C]) (cf. [57, §7.4.5]).

Let X be a smooth complex projective curve of genus g. Let D be a divisor on X, i.e., a formal sum $D = \sum_P n_P P$ where P is a point of X, $n_P \in \mathbb{Z}$ and $n_P = 0$ for all except for a finite number of P. We define

$$\sum_P n_P P + \sum_P m_P P := \sum_P (n_P + m_P) P.$$

$$\sum_P n_P P \succ \sum_P m_P P \Longleftrightarrow n_P \geqslant m_P.$$

$$L(D) := \{f : \text{meromorphic functions on } X \mid \operatorname{div}(f) + D \succ 0\}$$

[6] See Mumford's book [144, (7.26), p.145 and its Footnote].

where $\operatorname{div}(f) := \sum_P \operatorname{ord}_P(f)\, P$ is the divisor of a meromorphic function f. $\operatorname{div}(f) + D \succ 0$ means that $\operatorname{div}(f) \succ -D = \sum_P(-n_P)\, P$, i.e., $\operatorname{ord}_P(f) + n_P \geqslant 0$ for P.

Problem 6.3.1 Compute the complex dimension $\dim_{\mathbb{C}} L(D)$ in terms of some invariants of D and X.

In 1857 B. Riemann gave the following inequality, which is a formula for the lower bound of $\dim_{\mathbb{C}} L(D)$:

Theorem 6.3.2 (Riemann's Inequality)

$$\dim_{\mathbb{C}} L(D) \geqslant \deg D + 1 - g$$

where $\deg(D) = \sum_P n_P$.

Then, in 1865 G. Roch filled in the gap of the Riemann's inequality as the following equality, which is called the (classical or original) Riemann–Roch Theorem:

Theorem 6.3.3 (Riemann–Roch Theorem)

$$\dim_{\mathbb{C}} L(D) - \dim_{\mathbb{C}} L(K - D) = \deg D + 1 - g \tag{6.4}$$

where K is the canonical divisor of X.

In [84, Chapter 8 Riemann–Roch Theorem] Fulton uses the classical proof of Brill and Noether to prove the above (6.4).

This Riemann–Roch theorem can be interpreted in modern terms as follows: First, a divisor D determines a holomorphic line bundle $\mathscr{O}(D)$ and $L(D)$ can be described as the space of holomorphic sections of this line bundle $\mathscr{O}(D)$. Thus we have

- $\dim_{\mathbb{C}} L(D) = \dim_{\mathbb{C}} H^0(X; \mathscr{O}(D))$
- $\dim_{\mathbb{C}} L(K - D) = \dim_{\mathbb{C}} H^0(X; \mathscr{O}(K - D))$

By Serre duality [177], $H^i(X, E) \cong H^{n-i}(X, \mathscr{O}(K) \otimes E^*)^*$ for a vector bundle E over a smooth projective variety X of complex dimension n, we have

$$\dim_{\mathbb{C}} H^1(X; \mathscr{O}(D)) = \dim_{\mathbb{C}} H^0(X; \mathscr{O}(K - D))$$

since $\mathscr{O}(K) \otimes \mathscr{O}(D)^* = \mathscr{O}(K - D)$. Also we have

$$\deg D = \int_X c_1(\mathscr{O}(D)) \cap [X], \quad 1 - g = \frac{1}{2}\chi(X) = \frac{1}{2}\int_X c_1(TX) \cap [X].$$

Therefore the classical Riemann–Roch theorem (6.4) is expressed as

$$\dim_{\mathbb{C}} H^0(X; \mathscr{O}(D)) - \dim_{\mathbb{C}} H^1(X; \mathscr{O}(D)) = \int_X \left(c_1(\mathscr{O}(D)) + \frac{1}{2}c_1(TX)\right) \cap [X]. \tag{6.5}$$

6.3.2 Hirzebruch–Riemann–Roch

Let E be a holomorphic vector bundle on a compact complex manifold X. The Euler–Poincaré characteristic of E is defined as

$$\chi(X, E) := \sum_{i=0}^{\dim X} (-1)^i \dim_{\mathbb{C}} H^i(X, E).$$

Hence, the left-hand side of the above formula (6.5) is nothing but the Euler–Poincaré characteristic of $L(D)$:

$$\chi(X, \mathcal{O}(D)) = \dim_{\mathbb{C}} H^0(X, \mathcal{O}(D)) - \dim_{\mathbb{C}} H^1(X, \mathcal{O}(D)).$$

J.-P. Serre made the following conjecture (in a letter to Kodaira and Spencer at IAS of Princeton (September 29, 1953) (see [100, §1 , p.4]):

Conjecture 6.3.4 (Serre's Conjecture) There exists a polynomial $P(X, E)$ of Chern classes of the tangent bundle TX and E such that $\chi(X, E) = \int_X P(X, E) \cap [X]$.

Clearly the polynomial $c_1(\mathcal{O}(D)) + \frac{1}{2}c_1(TX)$ in the right-hand side of (6.5) is *a polynomial of Chern classes of* $\mathcal{O}(D)$ *and TX.*

Using the theory of sheaves and Thom's bordism[7] theory, F. Hirzebruch solved the above conjecture for X a complex projective manifold (in less than 3 months, circa December 10, 1953, at IAS of Princeton (see [100, §2 , p.10]) as follows [98, 99].

Before stating Hirzebruch's theorem, first we recall Chern roots. For a complex vector bundle E of rank r, we consider the following Chern polynomial $c_t(E)$ of variable t with Chern classes $c_i(E) \in H^{2i}(X, \mathbb{Z})$ as coefficients:

$$c_t(E) := 1 + c_1(E)\,t + \cdots + c_r(E)\,t^r.$$

If the Chern polynomial $c_t(E)$ is factorized as

$$c_t(E) = \prod_{i=1}^{r}(1 + \alpha_i\,t)$$

then $\alpha_1, \cdots, \alpha_r$ are called *Chern roots* of E (e.g., see [85, Remark 3.2.3] and also [99]). As one can see, these Chern roots of the Chern polynomial are *formal roots* of the polynomial. However, by "enlarging" the cohomology ring $H^*(X; \mathbb{Z})$ one can really get such roots. That is done by what is called the *splitting principle*. There exists a continuous map $f : X' \to X$ such that (1) f^*E *splits into line bundles*, i.e., $f^*E \cong L_1 \oplus \cdots \oplus L_r$ and (2) the pullback $f^* : H^*(X; \mathbb{Z}) \to H^*(X'; \mathbb{Z})$ is

[7] It is usually called "cobordism", but it is bordism. For their relationship, see Atiyah [19].

a *monomorphism*. Hence we have $f^*(c_t(E)) = c_t(f^*E) = c_t(L_1 \oplus \cdots \oplus L_r) = \prod_{i=1}^r c_t(L_i) = \prod_{i=1}^r (1 + c_1(L_i)\,t)$. This is similar to that any degree r polynomial $f(t) = 1 + a_1 t + \cdots + a_r t^r$ with real coefficients a_i can be expressed as $f(t) = \prod_{i=1}^r (1 + c_i\, t)$ with complex coefficients c_i, by enlarging the coefficient ring from the real numbers $\mathbb{R}$ to the complex numbers $\mathbb{C}$.

The important point of considering Chern roots is that

(i) *The* ith *elementary symmetric polynomial* $\sigma_i(\alpha_1, \alpha_2, \cdots, \alpha_r)$ *of these Chern roots* is the ith Chern class $c_i(E)$, i.e.,

- $c_1(E) = \sigma_1(\alpha_1, \alpha_2, \cdots, \alpha_r) = \sum_{1 \leqslant j \leqslant r} \alpha_j = \alpha_1 + \alpha_2 + \cdots + \alpha_r$,
- $c_2(E) = \sigma_2(\alpha_1, \alpha_2, \cdots, \alpha_r) = \sum_{1 \leqslant j < k \leqslant r} \alpha_j \alpha_k = \alpha_1\alpha_2 + \alpha_1\alpha_2 + \cdots + \alpha_{r-1}\alpha_r$,
- $\cdots\cdots$
- $c_r(E) = \sigma_r(\alpha_1, \alpha_2, \cdots, \alpha_r) = \alpha_1\alpha_2 \cdots \alpha_r$.

(ii) *Any symmetric polynomial* $f(\alpha_1, \alpha_2, \cdots, \alpha_r)$ of these Chern roots can be expressed uniquely as a polynomial of these elementary symmetric polynomials, i.e., *as a polynomial of Chern classes* $c_1(E), c_2(E), \cdots, c_r(E)$. (This is due to the Fundamental theorem of symmetric polynomials (e.g. see [68, Chapter 7], which also holds for symmetric power series.)

Here are two important examples of symmetric polynomials of Chern roots:

$$ch(E) := \sum_{i=1}^r e^{\alpha_i}, \qquad td(E) := \prod_{i=1}^r \frac{\alpha_i}{1 - e^{-\alpha_i}},$$

which are respectively called the *Chern character of E* and the *Todd class of E*. Clearly they are both *symmetric functions of the Chern roots*, hence they are both polynomials of Chern classes, as follows (see [85, Examples 3.2.3 and 3.2.4]):[8]

$$\begin{aligned} ch(E) =& \operatorname{rank} E + c_1 + \frac{1}{2}\left(c_1^2 - 2c_2\right) + \frac{1}{6}\left(c_1^3 - 3c_1c_2 + 3c_3\right) + \\ & \frac{1}{24}\left(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4\right) + \cdots, \\ td(E) =& 1 + \frac{1}{2}c_1 + \frac{1}{2}\left(c_1^2 + c_2\right) + \frac{1}{24}(c_1c_2) + \\ & \frac{1}{720}\left(-c_1^4 + 4c_1^2c_2 + 3c_2^2 + c_1c_3 - c_4\right) + \cdots, \end{aligned}$$

where $c_i := c_i(E)$.

Now we are ready to state the following well-known theorem due to Hirzebruch:

[8] For more higher terms, e.g., see [160, §3.1 Chern character & Chern classes, §4.4 Todd genus].

Theorem 6.3.5 (Hirzebruch–Riemann–Roch (HRR))

$$\chi(X, E) = \int_X (ch(E) \cup td(TX)) \cap [X]. \tag{6.6}$$

Here $ch(E) = \sum_{i=1}^{\mathrm{rank}\, E} e^{\alpha_i}$ *is the Chern character of the vector bundle* E *with* α_i *the Chern roots of* E *and* $td(TX) = \prod_{j=1}^{\dim X} \frac{\beta_j}{1-e^{-\beta_j}}$ *is the Todd class of the tangent bundle* TX *with* β_j *the Chern roots of* TX.

6.3.3 Grothendieck–Riemann–Roch

Grothendieck extended the above HRR to *a natural transformation* (published in [37, pp.20–71] and also published by Borel–Serre [41]):[9,10]

$$ch(-) \cup td(-) : K^0(-) \to H^*(-) \otimes \mathbb{Q}.$$

Namely, for an algebraic map $f : X \to Y$ of projective complex algebraic manifolds X and Y, the following diagram commutes (**Grothendieck–Riemann–Roch (GRR)**):

$$\begin{array}{ccc} K^0(X) & \xrightarrow{ch(-)\cup td(TX)} & H^*(X) \otimes \mathbb{Q} \\ {\scriptstyle f_!}\downarrow & & \downarrow{\scriptstyle f_!} \\ K^0(Y) & \xrightarrow[ch(-)\cup td(TY)]{} & H^*(Y) \otimes \mathbb{Q}. \end{array} \tag{6.7}$$

Here these two $f_!$ are Gysin (wrong-way) homomorphisms, as explained below and $K^0(Z)$ is the Grothendieck group of algebraic vector bundles (locally free sheaves), i.e., the quotient of the free abelian group generated by the isomorphism classes of locally free sheaves by the subgroup generated by the elements of the form $\mathscr{E} - \mathscr{E}' - \mathscr{E}''$ such that $0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$ is exact (e.g., see [85]).

Remark 6.3.6 The Grothendieck group of topological vector bundles and algebraic vector bundles are sometimes denoted by $K^0_{\mathrm{top}}(X)$ and $K^0_{\mathrm{alg}}(X)$ to avoid possible confusion. $K^0_{\mathrm{top}}(X)$ can be defined using the exact sequence as above, but in the case of topological vector bundles $0 \to E' \to E \to E'' \to 0$ is exact if and only if $E \cong E' \oplus E''$ (e.g., see [99]).

[9] "Grothendieck came along and said, 'No, the Riemann–Roch theorem is not a theorem about varieties, it's a theorem about morphisms between varieties'," said Nicholas Katz [102]

[10] Grothendieck gave 4 lectures (12 hours for 4 days) of his proof at the 1st Arbeitstagung (founded by Hirzebruch) at Bonn in 1957 (see [131, § 5.4, p.157]).

Note that $K^0(-)$ and $H^*(-)$ are contravariant functors, so $f_!$ are Gysin (wrong-way) homomorphisms defined by the following commutative diagrams:

$$\begin{array}{ccc} K^0(X) & \xrightarrow[\cong]{\mathcal{S}_X} & K_0(X) \\ f_! \downarrow & & \downarrow f_* \\ K^0(Y) & \xrightarrow[\mathcal{S}_Y]{\cong} & K_0(Y), \end{array} \qquad \begin{array}{ccc} H^*(X) & \xrightarrow[\cong]{\mathcal{P}_X} & H_*(X) \\ f_! \downarrow & & \downarrow f_* \\ H^*(Y) & \xrightarrow[\mathcal{P}_Y]{\cong} & H_*(Y) \end{array}$$

Here $\mathcal{S}_X : K^0(Z) \to K_0(Z)$ is a canonical homomorphism obtained by considering the sheaf of holomorphic local sections of a vector bundle and it is an isomorphism for smooth X (e.g., see [99]). $f_* : K_0(X) \to K_0(Y)$ is defined by $f_*\mathcal{F} := \sum_{i=0}^{\dim X}(-1)^i R^i f_*\mathcal{F}$. Thus we have $f_! = \mathcal{S}_Y^{-1} \circ f_* \circ \mathcal{S}_X$ and $f_! = \mathcal{P}_Y^{-1} \circ f_* \circ \mathcal{P}_X$, where $\mathcal{P}_Z = (-) \cap [Z] : H^*(Z) \to H_*(Z)$ denotes the Poincaré duality isomorphism for a smooth manifold Z.

Why is GRR an extension of HRR? Because we do have that

$$[\text{GRR (6.7) for } a_X : X \to pt] = \text{HRR}.$$

Indeed, [GRR (6.7) for $a_X : X \to pt$] means the following commutative diagram:

$$\begin{array}{ccccc} K^0(X) & \xrightarrow{ch(-)\cup td(TX)} & H^*(X)\otimes\mathbb{Q} & \xrightarrow[\cong]{-\cap[X]} & H_*(X)\otimes\mathbb{Q} \\ (a_X)_! \downarrow & & \downarrow (a_X)_! & & \downarrow (a_X)_* \\ K^0(pt) & \xrightarrow[ch(-)\cup td(Tpt)=ch(-)]{} & H^*(pt)\otimes\mathbb{Q}=\mathbb{Q} & \xrightarrow[\cap[pt]]{=} & H_*(pt)\otimes\mathbb{Q}=\mathbb{Q} \end{array} \tag{6.8}$$

Namely, for $E \in K^0(X)$ we have

$$ch\big((a_X)_!E\big) \cup td(T\,pt) = (a_X)_!\Big(ch(E) \cup td(TX)\Big). \tag{6.9}$$

Since $ch(V) = \dim_{\mathbb{C}} V$ for a complex vector space V (considered as a vector bundle over a point space pt) and we have

- $ch\big((a_X)_!E\big) \cup td(T\,pt) = ch\big((a_X)_!E\big) = ch\left(\sum_{i=0}^{\dim X}(-1)^i H^i(X,E)\right) = \chi(X,E)$,
- $(a_X)_!\Big(ch(E) \cup td(TX)\Big) = \displaystyle\int_X \Big(ch(E) \cup td(TX)\Big) \cap [X]$,

(6.9) means $\chi(X,E) = \int_X \Big(ch(E) \cup td(TX)\Big) \cap [X]$, i.e., HHR.

Remark 6.3.7 In fact, we see that HRR can be captured as the above commutative diagram (6.8) by simply defining $(a_X)_! : K^0(X) \to K^0(pt)$ as $(a_X)_! E = \sum_{i=0}^{\dim X} (-1)^i H^i(X, E)$, even if we do knot know GRR (6.7).

So, the above GRR (6.7) extends the left-hand side commutative diagram of the above diagram (6.8) to a map $f : X \to Y$ of non-singular varieties: The commutativity of the outer square in the following diagrams follow from that of the inner square.

$$\begin{array}{ccccccc}
K^0(X) & & & \xrightarrow{ch(-)\cup td(TX)} & & & H^*(X)\otimes\mathbb{Q} \\
 & \overset{\mathcal{S}_X}{\underset{\cong}{\searrow}} & & & & \overset{\cap[X]}{\underset{\cong}{\swarrow}} & \\
\Big\downarrow f_! & & K_0(X) & \xrightarrow{ch(\mathcal{S}_X^{-1}(-))\cup td(TX)\cap[X]} & H_*(X)\otimes\mathbb{Q} & & \Big\downarrow f_! \\
 & & \Big\downarrow f_* & & \Big\downarrow f_* & & \\
 & & K_0(Y) & \xrightarrow[ch(\mathcal{S}_Y^{-1}(-))\cup td(TY)\cap[Y]]{} & H_*(Y)\otimes\mathbb{Q} & & \\
 & \overset{\cong}{\underset{\mathcal{S}_Y}{\nearrow}} & & & & \overset{\cong}{\underset{\cap[Y]}{\nwarrow}} & \\
K^0(Y) & & & \xrightarrow[ch(-)\cup td(TY)]{} & & & H^*(Y)\otimes\mathbb{Q}.
\end{array} \tag{6.10}$$

$$\begin{array}{ccc}
K^0(X) & \xrightarrow{ch(-)\cup td(TX)} & H^*(X)\otimes\mathbb{Q} \\
f_! \Big\downarrow & & \Big\downarrow f_! \\
K^0(Y) & \xrightarrow[ch(-)\cup td(TY)]{} & H^*(Y)\otimes\mathbb{Q}
\end{array} \tag{6.11}$$

is also expressed as

$$\begin{array}{ccc}
K^0(X) & \xrightarrow{ch} & H^*(X)\otimes\mathbb{Q} \\
f_! \Big\downarrow & & \Big\downarrow f_!\left(td(T_f)\cup -\right) \\
K^0(Y) & \xrightarrow[ch]{} & H^*(Y)\otimes\mathbb{Q}.
\end{array} \tag{6.12}$$

Here $T_f := TX - f^*TY \in K^0(X)$, $td(T_f) = \dfrac{td(TX)}{f^*td(TY)} \in H^*(X)\otimes\mathbb{Q}$. Indeed, (6.11), i.e., $td(TY) \cup ch(f_! E) = \left(\mathscr{P}_Y^{-1} \circ f_* \circ \mathscr{P}_X\right)(td(TX) \cup ch(E))$ can be written as $\big(td(TY) \cup ch(f_! E)\big) \cap [Y] = f_*\Big(\big(td(TX) \cup ch(E)\big) \cap [X]\Big)$. Namely, we have $td(TY) \cap \big(ch(f_! E) \cap [Y]\big) = f_*\Big(td(TX) \cap \big(ch(E) \cap [X]\big)\Big)$. Then we have

$$
\begin{aligned}
ch(f_! E) \cap [Y] &= \frac{1}{td(TY)} \cap f_*\Big(td(TX) \cap \big(ch(E) \cap [X]\big)\Big) \\
&= f_*\left(f^*\left(\frac{1}{td(TY)}\right) \cap (td(TX) \cap (ch(E) \cap [X]))\right) \\
&= f_*\left(\frac{1}{f^* td(TY)} \cap (td(TX) \cap (ch(E) \cap [X]))\right) \\
&= f_*\left(\left(\frac{td(TX)}{f^* td(TY)} \cup ch(E)\right) \cap [X]\right) \\
&= f_*\left(\big(td(T_f) \cup ch(E)\big) \cap [X]\right).
\end{aligned}
$$

Thus $ch(f_! E) = \mathscr{P}_Y^{-1} \circ f_* \circ \mathscr{P}_X\Big(td(T_f) \cup ch(E)\Big) = f_!\Big(td(T_f) \cup ch(E)\Big)$, i.e., the commutative diagram (6.12). Furthermore, (6.12) was extended to the following

["SGA 6", [37]]: For a proper and local complete intersection (abbr., $\ell.c.i.$) morphism $f : X \to Y$

$$
\begin{array}{ccc}
K^0(X) & \xrightarrow{ch} & H^*(X) \otimes \mathbb{Q} \\
{\scriptstyle f_!}\downarrow & & \downarrow{\scriptstyle f_!\left(td(T_f)\cup -\right)} \\
K^0(Y) & \xrightarrow[ch]{} & H^*(Y) \otimes \mathbb{Q}.
\end{array}
\tag{6.13}
$$

Here $T_f \in K^0(X)$ is the virtual relative tangent bundle of f (e.g., see [85]). If $f : X \to Y$ is a map of smooth manifolds, then $T_f = TX - f^*TY \in K^0(X)$.

The inner commutative square of (6.10) was extended to singular varieties, namely Baum–Fulton–MacPherson's Riemann–Roch (or Baum–Fulton–MacPherson's Todd class) for singular varieties, which is recalled in the following section.

6.4 Three Distinguished Characteristic Classes of Complex Algebraic Varieties

As remarked in the introduction, a characteristic *cohomology* class $c\ell(M)$ of a smooth or complex manifold M is defined as *the characteristic cohomology class $c\ell(TM)$ of the tangent bundle TM of the manifold*. When it comes to complex algebraic varieties, in general a complex algebraic variety has singularities, due to which one cannot define a tangent bundle on the variety.

6.4.1 MacPherson's Chern Class c_*

In [184] D. Sullivan defined *a Whitney homology (*not cohomology*) class* $w_*(X) \in H_*(X, \mathbb{Z}_2)$ of a singular real algebraic variety X. Furthermore, Grothendieck and Deligne conjectured (cf. [94, II, 15.3.5, Note 87_1 and IV, 18.3.2, Note 164, II Cohomologie étale, 1]) and MacPherson [127] (cf. [109]) proved the existence of *Chern homology classes*. For a complex algebraic variety Z, $F(Z)$ denotes the abelian group consisting of constructible functions on Z, i.e.,

$$F(Z) = \left\{ \sum_{S \subset Z} a_S \mathbb{1}_S \;\middle|\; a_S \in \mathbb{Z}, S \text{ are subvarieties of } Z, a_S = 0 \text{ for almost all } S's \right\}.$$

Here $\mathbb{1}_S$ is the characteristic function on S, i.e., $\mathbb{1}_S(x) = 1$ for $x \in S$ and $\mathbb{1}_S(x) = 0$ for $x \notin S$.

Proposition 6.4.1 ([127]) *Let* $f : X \to Y$ *be a morphism. For a characteristic function* $\mathbb{1}_W$ *corresponding to a subvariety* W *in* X*, we define* $(f_*\mathbb{1}_W)(y) := \chi_c(f^{-1}(y) \cap W)$*, where* χ_c *is the topological Euler–Poincaré characteristic with compact support. Then the group homomorphism*

$$f_* : F(X) \to F(Y)$$

is defined by $f_*(\sum_W a_W \mathbb{1}_W) := \sum_W a_W f_*(\mathbb{1}_W)$*. Then* $F : \mathscr{V} \to \mathscr{Ab}$ *is a covariant functor from the category* $\mathscr{V}$ *of complex algebraic varieties to the category* $\mathscr{Ab}$ *of abelian groups.*

Remark 6.4.2 If $f : X \to pt$ is a map to a point pt, then we have $f_*\mathbb{1}_X = \chi_c(X)$, more precisely, $f_*\mathbb{1}_X = \chi_c(X)\mathbb{1}_{pt}$. If we identify $F(pt) = \mathbb{Z}$, we have $f_*\mathbb{1}_X = \chi_c(X)$.

Theorem 6.4.3 ([127]) *There exists a unique natural transformation* $c_* : F(-) \to H_*(-; \mathbb{Z})$ *such that ("smooth condition")* $c_*(\mathbb{1}_X) = c(TX) \cap [X]$ *for smooth* X.

As mentioned in Introduction, M.-H. Schwartz [172] defined what is called *Schwartz class* $Sch(X) \in H^*(M, M \setminus X)$ for a complex analytic variety X in a complex manifold M. For a complex algebraic variety X in a complex manifold M, J.-P. Brasselet and M.-H. Schwartz [50] proved that the Schwartz class and the value $c_*(X) := c_*(\mathbb{1}_X)$ of the characteristic function $\mathbb{1}_X$ are identical under the Alexander duality map $\mathscr{A} : H^*(M, M \setminus X) \cong H_*(X)$, i.e., $\mathscr{A}(Sch(X)) = c_*(X)$. Thus the class $c_*(X)$ is called the *Chern–Schwartz–MacPherson class*.

An idea of MacPherson's proof is the following:

(i) The uniqueness follows from (a) *resolution of singularities*[11] and (b) *the smooth condition*. Indeed, by resolution of singularities, for a constructible function $\alpha \in F(X)$ we have $\alpha = \sum_W a_W \mathbb{1}_W = \sum_V a_V (p_V)_* \mathbb{1}_V$ with each V smooth, $p_V : V \to X$ proper and $a_V \in \mathbb{Z}$.

(ii) In order to show the existence, MacPherson introduced (a) *Chern–Mather class* $c_*^{Ma}(W) \in H_*(X;\mathbb{Z})$ *for a subvariety* W, i.e., $c_*^{Ma}(W) := \nu_*\left(c(\widehat{TW}) \cap [\widehat{W}]\right)$ where $\nu : \widehat{W} \to W$ is *the Nash blow-up* and $\widehat{TW}$ is the tautological Nash tangent bundle on $\widehat{W}$, and (b) *the local Euler obstruction* Eu_W of a subvariety W (in fact, in [106] M. Kashiwara independently introduced it in the theory of D-modules), and used his *graph construction method*.

For the definition of the Chern-Mather class and the local Euler obstruction, e.g., see [46, 127] (also see [4, 48, 57]). The local Euler obstruction is *a constructible function* and is *generically the same as the characteristic function* $\mathbb{1}_W$, i.e., $\mathrm{Eu}_W \doteqdot \mathbb{1}_W$, namely $\mathrm{Eu}_W(x) = \mathbb{1}_W(x) = 1$ for $x \notin W_{sing}$ = the singular locus of W. From this we get the following key fact:

$$F(X) = \left\{ \sum_W n_W \,\mathrm{Eu}_W \,\middle|\, n_W \in \mathbb{Z}, n_W = 0 \text{ for almost all } W\text{'s} \right\}.$$

Then MacPherson defined the homomorphism $c_* : F(X) \to H_*(X;\mathbb{Z})$ by $c_*(\mathrm{Eu}_W) := c_*^{Ma}(W)$ and using the graph construction he showed that $c_* : F(-) \to H_*(-;\mathbb{Z})$ is *a natural transformation.*

Remark 6.4.4 If we define $c_* : F(X) \to H_*(X;\mathbb{Z})$ by $c_*(\mathbb{1}_W) := c_*^{Ma}(W)$ instead of $c_*(\mathrm{Eu}_W) := c_*^{Ma}(W)$, then this *does not* give us a natural transformation. For example, if X is an irreducible singular curve with only one singularity p which is a double point, then $\nu_* c_*(\mathbb{1}_{\widehat{X}}) \neq c_* \nu_*(\mathbb{1}_{\widehat{X}})$. Indeed, $\nu_*(\mathbb{1}_{\widehat{X}}) = \mathbb{1}_X + \mathbb{1}_p$, hence we have $c_*\nu_*(\mathbb{1}_{\widehat{X}}) = c_*(\mathbb{1}_X) + c_*(\mathbb{1}_p) = c_*^{Ma}(X) + [p]$. On the other hand, since $\widehat{X}$ is smooth and $T\widehat{X} = \widehat{TX}$, we have $\nu_* c_*(\mathbb{1}_{\widehat{X}}) = \nu_*(c_*^{Ma}(\widehat{X})) = \nu_*(c(T\widehat{X}) \cap [\widehat{X}]) = \nu_*(c(\widehat{TX}) \cap [\widehat{X}]) = c_*^{Ma}(X)$. In other words, for the generators of $F(X)$ *switching from the characteristic function* $\mathbb{1}_W$ *to the local Euler obstruction* Eu_W is a key trick; in other words, the local Euler obstruction Eu_W and the Chern-Mather class $c_*^{Ma}(W)$ is *a nice or good pair.*

[11] In the category of compact complex algebraic varieties, if we consider the rational homology group $H_*(X) \otimes \mathbb{Q}$ instead of the integral homology group $H_*(X)$, we can prove its uniqueness without using the resolution of singularities [110].

Remark 6.4.5 In [155] R. Piene showed the following *Todd formula* between the Chern-Mather class $c_*^{Ma}(X)$ and the polar class $P_*(X)$:

$$c_k^{Ma}(X) = \sum_{i=0}^{k} (-1)^i \binom{n+1-i}{k-i} U^{k-i} \cap P_i(X),$$

$$P_k(X) = \sum_{i=0}^{k} (-1)^i \binom{n+1-i}{k-i} U^{k-i} \cap c_i^{Ma}(X)$$

where $U = c_1(\mathcal{O}_X(1))$. In a similar manner, in [199] we defined *Segre–Mather class* $s_*^{Ma}(W) := \nu_* \left(s(\widehat{TW}) \cap [\widehat{W}] \right)$ where $s(\widehat{TW})$ is the Segre class of $\widehat{TW}$ and showed a similar Todd formula between the Segre–Mather class and another polar class $\overline{P}_*(X)$ [156]:

$$s_k^{Ma}(X) = \sum_{i=0}^{k} (-1)^i \binom{n+k}{i} U^i \cap \overline{P}_{k-i}(X),$$

$$\overline{P}_k(X) = \sum_{i=0}^{k} \binom{n+k}{i} U^i \cap s_{k-i}^{Ma}(X).$$

A similar formula is given in [4, Corollary 6.5.7].

Remark 6.4.6 The Chern–Schwartz–MacPherson class is described as follows:

$$c_*(X) = c_*^{Ma}(X) + \sum_{S \subset X_{sing}} \alpha(S) c_*^{Ma}(\overline{S}) \tag{6.14}$$

where S's are all smooth strata of the singular locus X_{sing} of a (Whitney) stratification of X and $\alpha(S)$ is a certain integer defined on the strata S. Namely, it is the Chern–Mather class of X *plus some classes coming from the singular locus*. A "Segre class version" of the above formula (6.14) was considered in [199, 201]. In [103] K. W. Johnson defined a Segre class $s_*(X) := s(d(X), X \times X)$, which is the relative Segre class of the diagonal $d(X)$ in $X \times X$ where $d : X \to X \times X$ is the diagonal map (see [85, §4.2]). Then $s_*(X) = s_*^{Ma}(X) +$ some classes supported on the singular locus X_{sing}.[12]

[12] Naive questions are (1) what is a "reasonable" Segre class $s_*(X)$ corresponding to the Chern–Schwartz–MacPherson class? and (2) can one get a formula like $s_*(X) = s_*^{Ma}(X) + \sum_{S \subset X_{sing}} \beta(S) s_*^{Ma}(S)$ with certain integers $\beta(S)$? For Segre classes, Aluffi's article [4] is recommended to read.

6.4.2 Baum–Fulton–MacPherson's Todd Class td_*

By a different approach P. Baum et al. [33] proved the following theorem, which has a description similar to that of MacPherson–Chern class transformation $c_* : F(-) \to H_*(-)$:

Theorem 6.4.7 *There exists a unique natural transformation $td_* : K_0(-) \to H_*(-;\mathbb{Q})$ such that ("smooth condition") $td_*(\mathcal{O}_X) = td(TX) \cap [X]$ for smooth X, where $\mathcal{O}_X$ is the structure sheaf of X and $td(TX)$ is the Todd class.*

This theorem is called Baum–Fulton–MacPherson's Riemann–Roch, abbr. BFM–RR, or Baum–Fulton–MacPherson's Todd class. Thus, for a proper morphism $f : X \to Y$ the following diagram commutes:

$$\begin{array}{ccc} K_0(X) & \xrightarrow{td_*} & H_*(X;\mathbb{Q}) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ K_0(Y) & \xrightarrow[td_*]{} & H_*(Y;\mathbb{Q}), \end{array} \tag{6.15}$$

Remark 6.4.8 In [33] this theorem is proved for quasi-projective varieties. Then the general case of complex algebraic varieties can be reduced to the case for quasi-projective varieties via the technique of 'Chow envelopes' as in [85, §18.3].

Remark 6.4.9 When we discuss motivation of considering the motivic Hirzebruch class, the above statement would be sufficient. However, as to the uniqueness of the natural transformation td_*, we remark the following. In [33, §0 Introduction, Riemann–Roch theorem, p. 102], their theorem (for the category of projective varieties) is stated as follows:

There exists a unique natural transformation $\tau : K_0(-) \to H_*(-;\mathbb{Q})$ such that

(i) *For any X, $\tau(\beta \otimes \alpha) = ch(\beta) \cap \tau(\alpha)$ for $\alpha \in K_0(X)$ and $\beta \in K^0(X)$, i.e., the following diagram is commutative:*

$$\begin{array}{ccc} K^0(X) \otimes K_0(X) & \xrightarrow{\otimes} & K_0(X) \\ {\scriptstyle ch\otimes\tau}\downarrow & & \downarrow{\scriptstyle \tau} \\ H^*(X) \otimes H_*(X;\mathbb{Q}) & \xrightarrow{\cap} & H_*(X;\mathbb{Q}) \end{array}$$

(ii) *If X is non-singular and $\mathcal{O}_X$ is the structure sheaf on X, then*

$$\tau(\mathcal{O}_X) = td(TX) \cap [X].$$

Thus, in the above Theorem 6.4.7 the first condition 1. is missing, but as stated in Theorem 6.4.12 below (also see [33, §0 Introduction, Uniqueness theorem. p. 103]), the uniqueness follows from only the condition that $\tau(\mathcal{O}_{\mathbb{P}_n}) = [\mathbb{P}^n] +$ classes of lower dimensions, hence the uniqueness clearly follows from a bit stronger condition that $\tau(\mathcal{O}_X) = td(TX) \cap [X]$ for smooth X, since $\tau(\mathcal{O}_{\mathbb{P}_n}) = td(T\mathbb{P}^n) \cap [\mathbb{P}^n] = [\mathbb{P}^n] +$ classes of lower dimensions. Hence we can get the uniqueness statement in the above Theorem 6.4.2 for the category of projective varieties. For the larger category of quasi-projective varieties, in [33, Chapter III, Theorem, pp.119–120] the authors proved the uniqueness of the above natural transformation td_* satisfying the following three conditions:

(i) the same as 1. above,
(ii) the same as 2. above,
(iii) If U is an open subvariety of X, then the following diagram commutes:

$$\begin{array}{ccc} K_0(X) & \xrightarrow{\tau} & H_*(X;\mathbb{Q}) \\ {\scriptstyle i^*}\downarrow & & {\scriptstyle i^*}\downarrow \\ K_0(U) & \xrightarrow[\tau]{} & H_*(U;\mathbb{Q}), \end{array}$$

where the vertical maps i^* are restrictions ($i : U \to X$ is the inclusion).

For the general category of complex algebraic varieties, see [85, §18.3].

Remark 6.4.10 Here we emphasize that in [33] the authors make no use of resolution of singularities, as written at the end of [33, (0.2), p.104]. If we use resolution of singularities, then, in the Zariski topology, for a coherent sheaf F on X we have[13] $F = \sum_V a_V (p_V)_* \mathcal{O}_V$ with V smooth, $p_V : V \to X$ proper and $a_V \in \mathbb{Z}$ (in the case when $F = \mathcal{O}_X$ such a description is also given in [33, (0.2), p.104]). In this context the uniqueness of td_* follows from the single "smooth condition".

A key ingredient of Theorem 6.4.7 is what is called *a localized Chern character* $ch^M_X(F)$ of a coherent sheaf F over X embedded into a smooth manifold M such that

- $ch^M_X(F) \in H_*(X;\mathbb{Q})$,
- $ch^X_X(F) = ch(F) \cap [X]$ if X is smooth.

[13] This is similar to the case when we deal with constructible functions, i.e., as pointed out in "An idea of MacPherson's proof", for a constructible function $\alpha \in F(X)$ we have that $\alpha = \sum_V a_V (p_V)_* 1\!\!1_V$.

Given a coherent sheaf F over a complex algebraic variety X embedded[14] into a smooth manifold M, there exists a finite resolution of the pushforward $(i_M)_*F$, i.e., there exists a complex $E_\bullet$ of vector bundles (locally free sheaves) over M such that

$$0 \to E_r \xrightarrow{d_r} E_{r-1} \to \cdots \to E_1 \xrightarrow{d_1} E_0 \to (i_M)_*F \to 0 \tag{6.16}$$

is exact. Since $((i_M)_*F)_x = 0$ for $x \in M \setminus X$, the following complex which is the above sequence (6.16) with $(i_M)_*F$ being deleted

$$0 \to E_r \xrightarrow{d_r} E_{r-1} \to \cdots \to E_1 \xrightarrow{d_1} E_0 \to 0 \tag{6.17}$$

is exact *over* $M \setminus X$, i.e., exact *off* X. Then $ch_X^M(F) := ch_X^M(E_\bullet)$ a *localized Chern character of the above complex* $E_\bullet$ *of vector bundles exact off* X *(6.17)*, which is defined in two ways; one ([33, Chap.1 Riemann–Roch by Difference-Bundles]) is by using the "difference-bundle" of Atiyah–Hirzebruch [22]:

$$ch_X^M(E_\bullet) := L\,(ch\,(d(E_\bullet))) \in H_*(X;\mathbb{Q})$$

where $d(E_\bullet) \in K^0(M, M \setminus X)$ is the "difference-bundle" of $E_\bullet$, $ch : K^0(M, M \setminus X) \to H^*(M, M \setminus X;\mathbb{Q})$ is the Chern character and $L : H^*(M, M \setminus X;\mathbb{Q}) \to H_*(X;\mathbb{Q})$ is the Lefschetz duality isomorphism. The other one [33, Chap. 1 Riemann - Roch by Grassmannian-Graph] is by using MacPherson's Grassmannian graph construction [126] (also see [85, §18.1 Graph Construction]).

Here we recall that the usual definition of the Chern character $ch(F)$ of a coherent sheaf F on a smooth complex manifold M is defined to be $ch(F) := \sum_{i=0}^r (-1)^i ch(E_i) \in H^*(M;\mathbb{Q})$, where $0 \to E_r \xrightarrow{d_r} E_{r-1} \to \cdots \to E_0 \to F \to 0$ is a finite resolution of F. Hence one could see[15] that $ch_X^M(F)$ is a "singular" version of the Poincaré dual of the Chern character of F, $ch(F) \cap [M] \in H_*(M;\mathbb{Q})$.

The localized Chern character $ch_X^M(F)$ *does not* depend on the choice of the above finite resolution (6.16) [33, Chap. 1, Proposition (4.1)], but of course *does* depend on the choice of M. However, if we consider *capping with the Todd class* $td\,((i_M)^*TM)$ of the pullback of the tangent bundle TM of the chosen ambient

[14] Here we note that a complex algebraic variety X is always embedded (as a closed subset) into $\mathbb{R}^N$ for some N, because the variety X is covered by finitely many affine varieties, which are embedded (as closed subsets) into $\mathbb{R}^n$ for some n, thus it follows from [75, §8.8 Proposition] that the variety X is itself embedded (as a closed subset) into $\mathbb{R}^N$ for some N.

[15] If we observe this, then it seems to be reasonable to think that using the above resolution (6.16) we could "simply" consider a "homology Chern character" $((i_M)^* ch((i_M)_*F)) \cap [X] \in H_*(X;\mathbb{Q})$ instead of $ch_X^M(F)$ and then, by mimicking the definition (6.18), we define $td((i_M)^*TM) \cap (((i_M)^* ch((i_M)_*F)) \cap [X]) = (i_M)^*\,(td(TM) \cup ch((i_M)_*F)) \cap [X]$. Then it remains to see whether it is independent of the resolution (6.16) and the embedding $i_M : X \to M$.

smooth manifold M, then it *does not* depend on the choice of the embedding $i_M : X \to M$ either [33, Chap.1, §6, (8)]:

$$td((i_M)^*TM) \cap ch_X^M(F) \in H_*(X;\mathbb{Q}), \tag{6.18}$$

which is nothing but Baum–Fulton–MacPherson's Todd class of a coherent sheaf F over X:

$$td_*(F) := td((i_M)^*TM) \cap ch_X^M(F) \in H_*(X;\mathbb{Q}). \tag{6.19}$$

Remark 6.4.11 In [173] M.-H. Schwartz gives anther constructions of the localized Chern character and in [116] M. Kwieciński gives a comparison between Baum–Fulton–MacPherson's construction and Schwartz's construction. In [186] T. Suwa also gives anther constructions of the localized Chern character and gives a comparison between Baum–Fulton–MacPherson's construction and his own construction.

There is the following more strengthened uniqueness theorem [33, Chap. III Uniqueness and Graded K]:

Theorem 6.4.12 (Uniqueness Theorem) $td_* : K_0(-) \to H_*(-;\mathbb{Q})$ *is the unique natural transformation satisfying either of the following conditions:*

(i) $td_*(\beta \otimes \alpha) = ch(\beta) \cap td_*(\alpha)$ *for* $\alpha \in K_0(X)$ *and* $\beta \in K^0(X)$*, and* $td_*(\mathscr{O}_{pt}) = 1$,
(ii) $td_*(\mathscr{O}_{\mathbb{P}_n}) = [\mathbb{P}^n] +$ *classes of lower dimensions*.

As remarked in [33], in the above uniqueness theorem neither condition mentions the Todd class of a bundle; the above second condition does not even mention Chern classes. The uniqueness theorem as to the second condition follows from the fact that $td_* : K_0(-)\otimes\mathbb{Q} \cong H_*(-;\mathbb{Q})$ is an isomorphism and the following theorem [83, §5 Natural Transformations,Proposition], the proof of which is due to A. Landman:

Theorem 6.4.13 *Let* $\alpha : H_*(-;\mathbb{Q}) \to H_*(-;\mathbb{Q})$ *be a natural transformation. If for each projective space* $\mathbb{P}^n$ *for* $n = 0, 1, 2, \cdots$, $\alpha([\mathbb{P}^n]) = [\mathbb{P}^n] +$ *terms of degree* $\neq n$*, then* α *is the identity.*

As mentioned in the Introduction, at the very end of [128] MacPherson remarked or posed a problem: *It remains to be seen whether there exists a unified theory of characteristic classes for singular varieties.* Motivated by the above formulation of characteristic homology classes as natural transformations, it is natural (at least for the author when he read [128]) to pose the following very naive question

Question 6.4.14 Let $c\ell$ be a multiplicative characteristic cohomology class, i.e. $c\ell(E \oplus F) = c\ell(E)c\ell(F)$. Can one define a reasonable covariant functor $\mathscr{F}_{c\ell} : \mathscr{V} \to \mathscr{A}b$ for which there exists a unique natural transformation $c\ell_* : \mathscr{F}_{c\ell}(-) \to H_*(-;\Lambda_{c\ell})$ such that

(i) when $c\ell = c$ is the Chern class, $c\ell_* : \mathscr{F}_{c\ell}(-) \to H_*(-;\Lambda_{c\ell})$ becomes MacPherson–Chern class transformation $c_* : F(-) \to H_*(-;\mathbb{Z})$ and

(ii) when $c\ell = td$ is the Todd class, $c\ell_* : \mathscr{F}_{c\ell}(-) \to H_*(-; \Lambda_{c\ell})$ becomes Baum–Fulton–MacPherson's Todd class transformation $td_* : K_0(-) \to H_*(-; \mathbb{Q})$.

This question was treated, e.g., in [200, 202–205].[16]

The uniqueness of c_* and td_* follows from the "smooth condition". If we drop this "smooth condition", we have the following results for the category of complex projective varieties.

Theorem 6.4.15 ([110]) *Any natural transformation* $\mathfrak{t} : F(-) \to H_*(-) \otimes \mathbb{Q}$ *is a linear combination* $\mathfrak{t} = \sum_{i \geqslant 0} r_i\, c_{*i} \otimes \mathbb{Q}\, (r_i \in \mathbb{Q})$ *of components* $c_{*i} \otimes \mathbb{Q} : F(-) \xrightarrow{c_*} H_*(-) \otimes \mathbb{Q} \xrightarrow{pr_{2i}} H_{2i}(-) \otimes \mathbb{Q}$ *of the rationalized MacPherson–Chern class* $c_* \otimes \mathbb{Q}$.

This follows from the following [110], which is a c_*-version of Theorem 6.4.12:

Theorem 6.4.16 $c_* \otimes \mathbb{Q} : F(-) \to H_*(-; \mathbb{Q})$ *is the unique natural transformation satisfying that* $(c_* \otimes \mathbb{Q})(\mathbb{1}_{\mathbb{P}_n}) = [\mathbb{P}^n] +$ *classes of lower dimensions*.

Similarly the following theorem follows from Theorem 6.4.12:

Theorem 6.4.17 ([204]) *Any natural transformation* $\mathfrak{t} : K_0(-) \to H_*(-) \otimes \mathbb{Q}$ *is a linear combination* $\mathfrak{t} = \sum_{i \geqslant 0} r_i\, td_{*i}\ (r_i \in \mathbb{Q})$ *of components* $td_{*i} : K_0(-) \xrightarrow{td_*} H_*(-) \otimes \mathbb{Q} \xrightarrow{pr_{2i}} H_{2i}(-) \otimes \mathbb{Q}$.

It follows from Theorems 6.4.15 and 6.4.17 that one cannot solve the above question using the covariant functors $K_0(-)$ and $F(-)$.

6.4.3 Cappell–Shaneson's L-Class L_*

After Intersection Homology [92] (cf. [47, 112, 113]) was introduced by Goresky and MacPherson, a homology L-class of a suitable compact singular variety was introduced by themselves [92] and it was also defined by P. Siegel [180] for Witt spaces, by M. Banagl [27] (also see [28, 29]) for non-Witt spaces, and by J. Cheeger [67] using L^2-forms. Later Cappell and Shaneson defined another L-class [58] using the bordism group $\Omega(Z)$ of self-dual constructible complex of sheaves on Z (for $\Omega(Z)$, e.g., see [30–32, 44, 45, 52, 58, 167, 207, 221]) and it was observed in [207] that Cappell–Shaneson's L-class is also a natural transformation, similar to c_* and td_*:

Theorem 6.4.18 ([58]) *On the category of compact complex algebraic varieties there exists a natural transformation* $L_* : \Omega(-) \to H_*(-) \otimes \mathbb{Q}$ *satisfying that* $L_*(\mathbb{Q}_X[2 \dim X]) = L^*(TX) \cap [X]$ *for a compact non-singular variety* X.

[16] At that time Cappell–Shaneson's paper [58], constructing $L_* : \Omega(-) \to H_*(-) \otimes \mathbb{Q}$, was not published yet, thus the case when $c\ell = L$ could not be considered like the other two cases.

Here $\mathbb{Q}_X[2\dim X]$ *is the shifted constant sheaf and* $L_*(TX) := \prod_{i=1}^{\dim X} \frac{\alpha_i}{\tanh \alpha_i}$ *is Hirzebruch L-class*[17] *of the tangent bundle* TX *with* α_i *the Chern roots of* TX. *So this L-class transformation is unique*[18] *on the subgroup of* $\Omega(X)$ *generated by* $f_*(\mathbb{Q}_M[2\dim M])$ *with* $f : M \to X$ *a proper morphism with* M *smooth (and pure dimensional).*

6.4.3.1 Pontryagin–Thom's Construction of *L*-Class

The construction of Goresky–MacPherson's homology L-class is an extension of Pontryagin–Thom's construction [190] (also see [157]) of the Hirzebruch L-class to the singular case, using the intersection homology. So, first we recall Pontryagin–Thom's construction (e.g., see [142, §20], [29, 82, 92, §5.7 and §6.3], [132, §3.3]). Key ingredients of Pontryagin–Thom's construction are the following:

(i) (cohomotopy set) The k-th *cohomotopy set* of a topological space X, $\pi^k(X) := [X, S^k]$ is the set of homotopy classes from X to the sphere S^k. For a manifold M of dimension n such that $n < 2k-1$, $\pi^k(M^n)$ becomes an abelian group called *Borsuk–Spanier cohomotopy group* [42, 181].

(ii) (Hurewicz map for cohomotopy and cohomology) The usual or standard Hurewicz homomorphism is $h_* : \pi_k(X) = [S^k, X] \to H_k(X)$ defined by $h_*([f]) := f_*[S^k]$, where $[S^k] \in H_k(S^k) \cong \mathbb{Z}$ is the fundamental class of S^k. Similarly we have the "dual" Hurewicz map $h^* : \pi^k(X) \to H^k(X;\mathbb{Z})$ defined by $h^*([f]) := f^*u$ where $u \in H^k(S^k) = \mathbb{Z}$ is the generator such that $< u, [S^k] >= 1$.

(iii) (Serre's theorem [176, Proposition 2', p. 289]) For a manifold[19] M of dimension n such that $n \leqslant 2k-2$, i.e., $n < 2k-1$, the above "dual" Hurewicz map $h^* : \pi^k(M^n) \to H^k(M;\mathbb{Z})$ is a $\mathscr{C}$-isomorphism,[20] i.e., both the kernel and the cokernel of h^* are finite abelian groups. In other words,

$$h^* \otimes \mathbb{Q} : \pi^k(M^n) \otimes \mathbb{Q} \cong H^k(M;\mathbb{Q}) \tag{6.20}$$

is an isomorphism.[21]

[17] See Appendix 2.

[18] In [207] it was stated that it was unique on the whole cobordism group $\Omega(X)$, thus the uniqueness is true on such a subgroup. As pointed above, the whole group $F(X)$ is generated by $f_*\mathbb{1}_M$ such that $f : M \to X$ is proper with smooth M and in the Zariski topology $K_0(X)$ is generated by $f_*\mathscr{O}_M$ such that $f : M \to X$ is proper with smooth M. Hence, in this sense the above proof of uniqueness of L_* is the same as that of c_* and td_*.

[19] In [176] Serre considers polyhedra.

[20] Let $\mathscr{C}$ denote the class of all finite abelian groups. A homomorphism $h : A \to B$ of abelian groups is called *a* $\mathscr{C}$*-isomorphism* if both $\mathrm{Ker}(h)$ and $\mathrm{Coker}(h)$ belong to $\mathscr{C}$.

[21] Since $h : A \to B$ being a $\mathscr{C}$-isomorphism implies that $\mathrm{Ker}(h \otimes \mathbb{Q}) = \mathrm{Ker}(h) \otimes \mathbb{Q} = 0$ and $\mathrm{Im}(h \otimes \mathbb{Q}) = B \otimes \mathbb{Q}$, i.e., $h \otimes \mathbb{Q} : A \otimes \mathbb{Q} \cong B \otimes \mathbb{Q}$.

(iv) (smooth approximation theorem (e.g., [43, Proposition 17.8]) Every continuous map $f : M^n \to S^k$ is homotopic to a smooth map. Hence any element $[f] \in \pi^k(M^n)$ can be represented by a smooth map $f : M^n \to S^k$.

Now, let M^n be a smooth closed oriented manifold (i.e., an oriented compact manifold without boundary). If $f : M^n \to S^{n-4i}$ is a smooth map, then it follows from the theorem of Brown and Sard that the set of regular values of f is dense in S^{n-4i}. For any regular value $p \in S^{n-4i}$, the inverse image $f^{-1}(p)$ is a smooth closed submanifold[22] (of dimension $4i$) of M and its normal bundle is trivial, since it is induced from the normal bundle at the point p in S^{n-4i}. Then consider the signature[23] $\sigma(f^{-1}(p)) \in \mathbb{Z}$. This number does not depend on the choice of a regular value, since for another regular value q, $f^{-1}(p)$ is cobordant[24] to $f^{-1}(q)$ (see [141, §7, Theorem A, p.43]), thus $\sigma(f^{-1}(p)) = \sigma(f^{-1}(q))$ since the signature is a (co)bordism invariant (Thom's theorem [189]). Furthermore the number $\sigma(f^{-1}(p))$ depends only on the homotopy class $[f]$ of f (e.g., see [29, §6.3] and [132, §3.3]). Indeed, if $f \sim g : M^n \to S^{n-4i}$, then we can have a smooth homotopy $\mathfrak{h} : M^n \times [0,1] \to S^{n-4i}$ having p as a regular value and $\mathfrak{h}^{-1}(p)$ is a smooth compact manifold with boundary, which is $f^{-1}(p) \sqcup g^{-1}(p)$, i.e., a (co)bordism from $f^{-1}(p)$ to $g^{-1}(p)$, hence $\sigma(f^{-1}(p)) = \sigma(f^{-1}(p))$. Therefore the number $\sigma(f^{-1}(p))$ depends only on the homotopy class $[f]$ of f (also see [141, §7, Theorem B, p.43]), thus it can be denoted by $\sigma([f])$. Thus we get the following map

$$\sigma : \pi^{n-4i}(M^n) \to \mathbb{Z}. \tag{6.21}$$

When $4i < \frac{n-1}{2}$, i.e., $n < 2(n-4i) - 1$, the above map (6.21) becomes a group homomorphism. Consider rationalizing (6.21):

$$\sigma \otimes \mathbb{Q} : \pi^{n-4i}(M^n) \otimes \mathbb{Q} \to \mathbb{Q}. \tag{6.22}$$

Using the above isomorphism (6.20) for $k = n-4i$, the above homomorphism (6.22) becomes

$$\sigma \otimes \mathbb{Q} : H^{n-4i}(M; \mathbb{Q}) \to \mathbb{Q}, \tag{6.23}$$

---

[22] Let $f : M^n \to S^p$ be a smooth map and let $y \in S^p$ be a regular value of f. Let $\mathfrak{b} = (v^1, \cdots, v^p)$ be a positively oriented basis of the tangent TS^p_y. Then the pair $(f^{-1}(y), f^*\mathfrak{b})$ is called the *the Pontrjagin manifold* associated with f (see [141, §7 Framed Cobordism:The Pontrjagin Construction]). Any compact framed submanifold $(N, \mathfrak{w})$ of codimension k in M is realized as a Pontrjagin manifold for some smooth map $f : M \to S^k$ [141, §7, Theorem C, p.44].

[23] Here we recall that the signature $\sigma(M)$ of a manifold M of real dimension $4r$ is *the number of positive eigenvalues minus the number of negative eigenvalues* of the intersection pairing $H_{2r}(M; \mathbb{Q}) \otimes H_{2r}(M; \mathbb{Q}) \to \mathbb{Q}$, and if $\dim_{\mathbb{R}} M$ is not divisible by 4, then $\sigma(M) := 0$.

[24] More strongly, $(f^{-1}(p), f^*\mathfrak{b})$ is *framed cobordant* to $(f^{-1}(q), f^*\mathfrak{b}')$, although we do not need framing in the above argument.

which defines a homology class $\sigma \otimes \mathbb{Q} \in H_{n-4i}(M;\mathbb{Q}) \cong Hom(H^{n-4i}(M;\mathbb{Q}),\mathbb{Q})$ by the universal coefficient theorem. Thus we get the *homology L-class* of M:

$$L_{n-4i}(M) := \sigma \otimes \mathbb{Q} \in H_{n-4i}(M;\mathbb{Q}). \tag{6.24}$$

In fact, the above condition $4i < \frac{n-1}{2}$ can be deleted by taking the product $M^n \times S^m$ with a sphere S^m of sufficiently high dimension m (e.g., see [29, §6.3] and [132, §3.3]), namely even if $4i \geqslant \frac{n-1}{2}$, by considering m such that $4i < \frac{n+m-1}{2}$ and $m - 4i > 0$ (thus $n + m - 4i > n$), we can define the above homology class $L_{n-4i}(M)$. A key fact of this "product by S^m" argument is the Künneth formula:

$$\begin{aligned} H_{n+m-4i}(M^n \times S^m) &\cong H_{n+m-4i}(M^n) \otimes H_0(S^m) \oplus H_{n-4i}(M^n) \otimes H_m(S^m) \\ &\cong H_{n-4i}(M^n). \end{aligned}$$

It turns out that this is the Poincaré dual of the (cohomology) Hirzebruch L-class $L^{4i}(M) \in H^{4i}(M;\mathbb{Q})$ (see [142], [29, §5.7], [132, §3.3]):

$$L_{n-4i}(M) = L^{4i}(M) \cap [M]. \tag{6.25}$$

In other words, the Poincaré dual of the above homology class $L_{n-4i}(M)$ *constructed "geometrically", i.e., by mapping to spheres*, is equal to the (cohomology) Hirzebruch L-class $L^{4i}(M)$. This is Pontryagin–Thom's construction of the Hirzebuch L-class of a smooth manifold.

6.4.3.2 Goresky–MacPherson's *L*-Class

Here we follow [29, §6.1 and §6.3]. Let X^n be a Whitney stratified, closed (= compact without boundary), oriented pseudomanifold [92] (cf. [47, 82, 112, 132]) which has *only strata of even codimension*:

$$X_n \supset X_{n-2} \supset X_{n-4} \supset X_{n-6} \supset \cdots$$

Compact complex algebraic varieties are such examples. Since a complex algebraic variety can be embedded into a smooth manifold , we assume that the above pseudomanifold X is also embedded in a smooth manifold M.

Definition 6.4.19 (Also See [192, Definition 4.2.11]) A continuous map $f : X \to S^k$ is called *transverse*, if

(i) f is the restriction of a smooth map $\widetilde{f} : M \to S^k$,
(ii) the north pole $N \in S^k$ is a regular value of $\widetilde{f}$, and
(iii) $\widetilde{f}^{-1}(N)$ is transverse to each stratum of X (that is, the submanifold $\widetilde{f}^{-1}(N)$ is transverse to X in the sense of Whitney stratified sets).

If $f : X \to S^{n-4i}$ is transverse, then $f^{-1}(N) = \widetilde{f}^{-1}(N) \cap X$ is Whitney stratified with strata $\widetilde{f}^{-1}(N) \cap A$ for each stratum A of X. All these $\widetilde{f}^{-1}(N) \cap A$ are of even codimension in $f^{-1}(N)$. Thus the signature $\sigma(f^{-1}(N))$ is well-defined using the middle perversity intersection homology. To be more precise, using the *lower middle perversity* $\overline{m}$,[25] we have the intersection

$$IH_{2i}^{\overline{m}}(f^{-1}(N)) \otimes IH_{2i}^{\overline{m}}(f^{-1}(N)) \to \mathbb{R}, \tag{6.26}$$

from which we get the signature $\sigma(f^{-1}(N))$. Then we have the following (see [29, Proof of Lemma 6.3.2])

Lemma 6.4.20 *The map*

$$\sigma : \pi^{n-4i}(X) \to \mathbb{Z}, \quad \sigma([f]) := \sigma(f^{-1}(N)) \tag{6.27}$$

is a well-defined homomorphism for $2(n-4i) - 1 > n$*, i.e.,* $4i > \frac{n-1}{2}$.

Then, by Serre's theorem (which holds for polyhedra, thus for pseudomanifolds as well), we have that the rationalized dual Hurewicz map $\pi^{n-4i}(X) \otimes \mathbb{Q} \cong H^{n-4i}(X) \otimes \mathbb{Q}$ is an isomorphism (cf. (6.20)). Hence, in the same way as in Sect. 6.4.3.1, we can define

$$L_{n-4i} = \sigma \otimes \mathbb{Q} \in H_{n-4i}(X; \mathbb{Q}). \tag{6.28}$$

The restriction $4i > \frac{n-1}{2}$ can be deleted by taking the product with sufficiently higher dimensional sphere, as in Sect. 6.4.3.1. This homology L-class is Goresky–MacPherson's L-class.

6.4.3.3 Cappell–Shaneson's *L*-Class

Definition 6.4.21 Let $S^\bullet$ be a bounded constructible complex of sheaves on X in the derived category $D_c^b(X)$. If the following holds, then $S^\bullet$ is called *self-dual*:

$$S^\bullet \cong \mathbf{D}(S^\bullet)[2 \dim_{\mathbb{C}} X] \tag{6.29}$$

where $\mathbf{D}$ is the Borel–Moore–Verdier dualizing functor [40, 195] (cf. [47, 132]).

For the following discussion, see [29, p.167–p.169] and also [58] . Let $X^\bullet \xrightarrow{u} S^\bullet \xrightarrow{v} Z^\bullet$ be morphisms in the derived category $D_c^b(X)$ with $v \circ u = 0$. Let $C_v^\bullet$ be a

[25] There is another middle perversity called *upper middle perversity* $\overline{n}$ such that it is complementary to the lower middle perversity $\overline{m}$, i.e., $\overline{m} + \overline{n} = \overline{t}$, where $\overline{t}$ is the top perversity, $\overline{t}(k) = k - 2$.

cone[26] of v:

$$S^\bullet \xrightarrow{v} Z^\bullet \xrightarrow{i_v} C_v^\bullet \xrightarrow[p_v]{[1]} S^\bullet[1]$$

which is a distinguished triangle. Consider its "shifted" distinguished triangle $C_v^\bullet \xrightarrow{p_v} S^\bullet[1] \xrightarrow{-v[1]} Z^\bullet[1] \xrightarrow[-i_v[1]]{[1]} C_v^\bullet[1],$ from which we get

$$C_v^\bullet[-1] \xrightarrow{-p_v[-1]} S^\bullet \xrightarrow{v} Z^\bullet \xrightarrow[i_v]{[1]} C_v^\bullet = (C_v^\bullet[-1])[1].$$

Using the homological functor $Hom(X^\bullet, -)$, we get the exact sequence

$$Hom(X^\bullet, Z^\bullet[-1]) \to Hom(X^\bullet, C_v^\bullet[-1]) \xrightarrow{p_*} Hom(X^\bullet, S^\bullet) \xrightarrow{v_*} Hom(X^\bullet, Z^\bullet).$$

Here we set $p := -p_v[-1]$. Since $v_*(u) = v \circ u = 0$, there exists $u' : X^\bullet \to C_v^\bullet[-1]$ such that $u = p_*(u') = p \circ u'$, i.e., u' is a lifting of u to $C_v^\bullet[-1]$:

$$\begin{array}{ccc} & & C_v^\bullet[-1] \\ & \overset{u'}{\nearrow} & \downarrow p \\ X^\bullet & \xrightarrow[u]{} & S^\bullet \end{array}$$

Note that this lifting u' is of course not uniquely determined, but it is the case if $Hom(X^\bullet, Z^\bullet[-1]) = 0$, *which we will assume*[27] *later*. Consider a cone $C_{u'}^\bullet$ of this lifting $u' : X^\bullet \to C_v^\bullet[-1]$:

$$X^\bullet \xrightarrow{u'} C_v^\bullet[-1] \xrightarrow{i_{u'}} C_{u'}^\bullet \xrightarrow[p_{u'}]{[1]} X^\bullet[1]. \tag{6.30}$$

---

[26] A cone is not uniquely determined, but uniquely determined up to an isomorphism. This fact is called "non-functoriality of cone construction" (cf.[89]). For example, a functor F is required to satisfy $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for an identity id_X, but it is not the case for the cone construction, i.e., the following case occurs (e.g., see [29, p.35–36]):

$$\begin{array}{ccccccc} X^\bullet & \xrightarrow{f} & Y^\bullet & \xrightarrow{i} & C_f^\bullet & \xrightarrow{[1]} & X^\bullet[1] \\ \downarrow \mathrm{id}_{X^\bullet} & & \downarrow \mathrm{id}_{Y^\bullet} & & \cong \downarrow k & & \downarrow \mathrm{id}_{X^\bullet[1]} \\ X^\bullet & \xrightarrow{f} & Y^\bullet & \xrightarrow{i'} & \widetilde{C}_f^\bullet & \xrightarrow{[1]} & X^\bullet[1] \end{array}$$

[27] In [29, 58], this vanishing condition is implicitly assumed.

This "iterated" cone $C^\bullet_{u'}$ is denoted by $C^\bullet_{u',v}$ (instead of $C^\bullet_{u,v}$, used in [29, 58]) because it *is* a mapping cone of this lifting u'. For furthermore discussion on this "iterated" cone $C^\bullet_{u',v}$ and the proof of the isomorphism (6.32) below, see Appendix 1.

Now we assume that we are given an isomorphism $\mathbf{D}X^\bullet[2\dim_{\mathbb{C}} X] \cong Z^\bullet$ such that the following diagram commutes:

$$\begin{array}{ccc} S^\bullet & \xrightarrow{\quad v \quad} & Z^\bullet \\ \cong \uparrow & & \uparrow \cong \\ \mathbf{D}S^\bullet[2\dim_{\mathbb{C}} X] & \xrightarrow[\mathbf{D}u[2\dim_{\mathbb{C}} X]]{} & \mathbf{D}X^\bullet[2\dim_{\mathbb{C}} X] \end{array} \tag{6.31}$$

Then under the commutativity of the above diagram (6.31) we can show that the above "iterated" cone $C^\bullet_{u',v}$ is also self-dual, i.e., we have:

$$C^\bullet_{u',v} \cong \mathbf{D}C^\bullet_{u',v}[2\dim_{\mathbb{C}} X]. \tag{6.32}$$

Then we have the following definition due to Cappell–Shaneson [58, §2] (also see [29, Definition 8.1.11]):

Definition 6.4.22 Let $X^\bullet \xrightarrow{u} S^\bullet \xrightarrow{v} Z^\bullet$ be morphisms in the derived category with $v \circ u = 0$ and assume $Hom(X^\bullet, Z^\bullet[-1]) = 0$. If there exists an isomorphism $Z^\bullet \cong \mathbf{D}(X^\bullet)[2\dim_{\mathbb{C}} X]$ such that the following diagram commute

$$\begin{array}{ccc} S^\bullet & \xrightarrow{\quad v \quad} & Z^\bullet \\ \cong \downarrow & & \downarrow \cong \\ \mathbf{D}(S^\bullet)[2\dim_{\mathbb{C}} X] & \xrightarrow[\mathbf{D}(u)[2\dim_{\mathbb{C}} X]]{} & \mathbf{D}(X^\bullet)[2\dim_{\mathbb{C}} X] \end{array} \tag{6.33}$$

then the "iterated" cone $S^\bullet_1 := C^\bullet_{u',v}$ is also self-dual. Then we say that $S^\bullet_1$ is obtained from $S^\bullet$ *by an elementary cobordism* or $S^\bullet_1$ is *elementarily cobordant* to $S^\bullet$.

Definition 6.4.23 $S^\bullet$ is called *cobordant* to $\widetilde{S}^\bullet$ if there exists a finite sequence $S^\bullet = S^\bullet_0, S^\bullet_1, \cdots, S^\bullet_n = \widetilde{S}^\bullet$ such that $S^\bullet_i$ is elementarily cobrodant to $S^\bullet_{i-1}$ for $1 \leqslant i \leqslant n$.

The above cobordism is an equivalence relation. The set of the cobordism classes of self-dual complexes on X is denoted by $\Omega(X)$, which becomes an abelian group by the addition operation $[S^\bullet_1] + [S^\bullet_2] := [S^\bullet_1 \oplus S^\bullet_2]$. For a morphism $f : X \to Y$ the pushforward $f_* : \Omega(X) \to \Omega(Y)$ is defined by

$$f_*([S^\bullet]) := [Rf_*S^\bullet[-\operatorname{reldim}(f)]]$$

where Rf_* is the right-derived functor and $\mathrm{reldim}(f) := \dim X - \dim Y$. With this pushforward $\Omega(-)$ becomes a covariant functor.

Cappell and Shaneson have defined a homology class $L_*(S^\bullet) \in H_*(X;\mathbb{Q})$ for a self-dual complex $S^\bullet$ on a compact singular variety X and showed the following:

(i) if $S^\bullet$ is cobordant to $\widetilde{S}^\bullet$, then $L_*(S^\bullet) = L_*(\widetilde{S}^\bullet)$.
(ii) $L_*(S_1^\bullet \oplus S_2^\bullet) = L_*(S_1^\bullet) + L_*(S_2^\bullet)$.

Thus we have a correspondence $L_* : \Omega(-) \to H_*(-;\mathbb{Q})$ defined by $L_*([S^\bullet]) := L_*(S^\bullet)$. It turns out that this is a natural transformation. This is a very rough sketch of Cappell–Shaneson's homology L-class *as a natural transformation* as stated in Theorem 6.4.18 at the beginning of Sect. 6.4.3.

We also note that Goresky–MacPherson's L-class and Cappell–Shaneson's L-class transformation L_*, denoted $L_*^{GM}(X)$ and L_*^{CS} respectively, are related. We have $L_*^{CS}([IC_{\overline{m}}^\bullet(X;\mathbb{Q})]) = L_*^{GM}(X)$ where $IC_{\overline{m}}^\bullet(X;\mathbb{Q})$ is the (lower middle perversity) intersection cohomology complex of X.

Now, how is $L_*(S^\bullet)$ defined? It is constructed by using the argument of Pontryagin - Thom construction, as in the construction of Goresky–MacPherson's L-class. For details, see [58] and also [29, §8.2.3 Construction of L-class].

The above distinguished three theories c_*, td_*, L_* are natural transformations from *different* covariant functors $F(X), K_0(X), \Omega(X)$ to the homology group. As remarked in the Introduction, a fundamental problem is whether one can construct a theory *"unifying"* these three characteristic homology classes, which is discussed in the following section.

6.5 Motivic Hirzebruch Class

6.5.1 A Generalized Hirzebruch–Riemann-Roch

As recalled above, GRR is an extension of HRR as a natural transformation. Similarly the above three distinguished characteristic classes

(i) $c_* : F(-) \to H_*(-)$
(ii) $td_* : K_0(-) \to H_*(-) \otimes \mathbb{Q}$
(iii) $L_* : \Omega(-) \to H_*(-) \otimes \mathbb{Q}$

are extensions *as natural transformations* of the following formulas (of HRR type) for X a compact smooth algebraic variety:

(i) $\chi(X) = \int_X c(TX) \cap [X]$ (Gauss–Bonnet theorem)
(ii) $\chi^a(X) = \int_X td(TX) \cap [X]$ (HRR for the trivial line bundle $E = 1_X$)
(iii) $\sigma(X) = \int_X L(TX) \cap [X]$ (Hirzebruch's signature theorem).

In [99] (cf. [101]) F. Hirzebruch introduced a characteristic class unifying the three distinguished characteristic cohomology classes $c(E), td(E), L(E)$ of a complex vector bundle E:

Definition 6.5.1 (Hirzebruch Class) Let E be a complex vector bundle over X and y be a variable. Then the following characteristic class is called the *Hirzebruch class of* E:

$$T_y(E) := \prod_{i=1}^{\operatorname{rank} E} \left(\frac{\alpha_i(1+y)}{1-e^{-\alpha_i(1+y)}} - \alpha_i y \right)$$

where α_i are the Chern roots of the vector bundle, i.e., $c(E) = \prod_{i=1}^{\operatorname{rank} E}(1+\alpha_i)$.

Indeed, the Hirzebruch class T_y specializes to

- $y = -1 : T_{-1}(E) = c(E) = \prod_{i=1}^{\operatorname{rank} E}(1+\alpha_i)$ the total Chern class,
- $y = 0 : \quad T_0(E) = td(E) = \prod_{i=1}^{\operatorname{rank} E} \frac{\alpha_i}{1-e^{-\alpha_i}}$ the total Todd class,
- $y = 1 : \quad T_1(E) = L(E) = \prod_{i=1}^{\operatorname{rank} E} \frac{\alpha_i}{\tanh \alpha_i}$ the total Hirzebruch L-class.

We note that the above specializations are due to the following:

(i) $y = -1$: $\displaystyle\lim_{y \to -1} \frac{\alpha_i(1+y)}{1-e^{-\alpha_i(1+y)}} = \lim_{y \to -1} \frac{\alpha_i}{\alpha_i e^{-\alpha_i(1+y)}} = 1$ (by l'Hôpital's rule).

(ii) $y = 0$: it is obvious.

(iii) $y = 1$: $\frac{2\alpha_i}{1-e^{-2\alpha_i}} - \alpha_i = \alpha_i \frac{1+e^{-2\alpha_i}}{1-e^{-2\alpha_i}} = \alpha_i \frac{e^{\alpha_i}+e^{-\alpha_i}}{e^{\alpha_i}-e^{-\alpha_i}} = \alpha_i \frac{\cosh \alpha_i}{\sinh \alpha_i} = \frac{\alpha_i}{\tanh \alpha_i}$.

In [142, Appendix B: Bernoulli Numbers, p.281], there is a formula $\frac{x}{\tanh x} = \frac{2x}{e^{2x}-1} + x$. By a straightforward computation it can be shown that $\frac{2x}{e^{2x}-1} + x = \frac{2x}{1-e^{-2x}} - x$.

The Hirzebruch class is used in his generalized Hirzebruch–Riemann–Roch theorem for Hirzebruch χ_y characteristic .

Definition 6.5.2 ([99]) For a compact complex algebraic manifold X the *Hirzebruch χ_y-genus* $\chi_y(X)$ of X is defined by

$$\chi_y(X) := \sum_{p \geqslant 0} \chi(X, \Lambda^p T^* X) y^p = \sum_{p \geqslant 0} \Big(\sum_{q \geqslant 0} (-1)^q \dim_{\mathbb{C}} H^q(X, \Lambda^p T^* X) \Big) y^p .$$

Or we write it as follows: $\chi_y(X) = \displaystyle\sum_{p,q \geqslant 0} (-1)^q \dim_{\mathbb{C}} H^q(X; \Omega_X^p) y^p$.

More generally we have

Definition 6.5.3 ([99]) For E a holomorphic vector bundle over X, the *Hirzebruch χ_y-characteristic of E* is defined by

$$\chi_y(X, E) := \sum_{p \geqslant 0} \chi(X, E \otimes \Lambda^p T^*X) y^p$$
$$= \sum_{p \geqslant 0} \Big(\sum_{q \geqslant 0} (-1)^q \dim_{\mathbb{C}} H^q(X, E \otimes \Lambda^p T^*X) \Big) y^p .$$

Theorem 6.5.4 (The Generalized Hirzebruch–Riemann–Roch Theorem [99])

$$\chi_y(X, E) = \int_X \big(T_y(TX) \cup ch_{(1+y)}(E) \big) \cap [X] \quad \in \mathbb{Q}[y].$$

Here $ch_{(1+y)}(E) := \sum_{j=1}^{rank\, E} e^{\beta_j(1+y)}$ is the Chern character "parameterized by $1 + y$" with β_j being Chern roots of E.

Hence $ch_{(1+y)}(E) = \operatorname{rank} E + \sum_{i \geq 1}(1 + y)^i ch_i(E)$ for $ch(E) = \operatorname{rank} E + \sum_{i \geq 1} ch_i(E)$. In particular, for the trivial line bundle E we have

$$\chi_y(X) = \int_X T_y(TX) \cap [X]. \tag{6.34}$$

We note that the three distinguished cases of (6.34) are the above three formulas, which are repeated here:

$$(y = -1) : \chi(X) = \int_X c(TX) \cap [X] \tag{6.35}$$

$$(y = 0) : \chi^a(X) = \int_X td(TX) \cap [X] \tag{6.36}$$

$$(y = 1) : \sigma(X) = \int_X L(TX) \cap [X] \tag{6.37}$$

Now that we have (6.34) for the Hirzebruch χ_y-genus $\chi_y(X)$, as GRR extends HRR, it is quite natural and reasonable to speculate that there would be a GRR-type theorem for (6.34), for which one should come up with a reasonable covariant functor which is the source of a natural transformation, and it turns out that there is one, as explained in the following sections.

Remark 6.5.5 Since we have the above three formulas (6.35), (6.36), and (6.37), we want to mention *Atiyah–Singer Index Theorem* [23–26] (for a very nice survey, e.g., see [81]). This index theorem is clearly influenced by HRR, thus it should be mentioned in Sect. 6.3.2, but we mention it in this section. The left-hand side of HRR (6.6) is an *analytic invariant* and the right-hand side of HRR (6.6) is a *topological invariant*. The Atiyah–Singer Index Theorem is also a similar formula, involving *Chern character and Todd class*. The following theorems are cited from [178, Theorem (Atiyah-Singer), p.17]) and [101, §5.2 The Atiyah–Singer index theorem]), respectively:

Theorem 6.5.6 *X be a compact smooth manifold of dimension n, let D be an elliptic operator on X and let* $\sigma(D)$ *be the symbol of D. Then*

$$\operatorname{index}(D) = (-1)^n \, (ch(\sigma(D))td(TX \otimes \mathbb{C}))\,[TX] \tag{6.38}$$

Theorem 6.5.7 *Let X be a compact, oriented, differentiable manifold of dimension* $2n$ *and* $D = (D_i : \Gamma E_i \to \Gamma E_{i+1})$ *an elliptic complex* ($i = 0, \cdots, m-1$)*, associated to the tangent bundle. Then the index of this complex is determined by the following formula (cf. [23–25]):*

$$\operatorname{ind}(D) = (-1)^n \left(\left(\frac{1}{e(T^*X)} \sum_{i=0}^{m} (1-)^i ch(E_i) \right) td(TX \otimes \mathbb{C}) \right) \cap [X]. \tag{6.39}$$

The special cases of (6.38) and (6.39) become the following:

(i) If D is the de Rham operator, then (6.38) and (6.39) become (6.35).
(ii) If D is the Dolbeault operator, then (6.38) and (6.39) become (6.36).
(iii) If D is the Hodge operator, then (6.38) and (6.39) become (6.37).

Here we do not go into details of the theorem and the above special cases, e.g., what an elliptic operator is, what the symbol is, what the de Rham operator is, etc.

Remark 6.5.8 From the viewpoint of characteristic classes of singular varieties, we just wonder what an "index theorem" in the singular case could be, namely wonder if one could pose the following questions (very vague at the moment):

(i) Could one get a reasonable formula extending the above formula (6.38) and/or (6.39) to a singular complex algebraic variety X?
(ii) If so, could one get a relative version for a map $f : X \to Y$ of possibly singular varieties, as a natural transformation? Note that, as Grothendieck generalized HRR to GRR, in [26] Atiyah and Singer extended their index theorem to a proper fiber bundle $f : X \to Y$ equipped with a family of elliptic (pseudo) differential operators along the fibers. Here X and Y are smooth.

6.5.2 Hodge–Deligne Polynomial

Hirzebruch χ_y-genus of a smooth compact variety is extended to singular varieties, using Deligne's mixed Hodge structures [70, 72] (also see [153, 154] and [183, §9.2.1]), as follows (e.g., see [69]):

Definition 6.5.9 (Hodge–Deligne Polynomial) For a mixed Hodge structure $(H_*(X), W^\bullet, F_\bullet)$ with weight filtration $W^\bullet$ and Hodge filtration $F_\bullet$, the Hodge–Deligne polynomial[28] $\chi_{u,v}(X)$ is defined by

$$\chi_{u,v}(X) := \sum_{i,p,q \geqslant 0} (-1)^i (-1)^{p+q} \dim_{\mathbb{C}}(Gr_F^p Gr_{p+q}^W H_c^i(X, \mathbb{C})) u^p v^q.$$

$\chi_{u,v}$ satisfies the following four properties:

(i) $X \cong X'$ (isomorphism) $\Longrightarrow \chi_{u,v}(X) = \chi_{u,v}(X')$,
(ii) $\chi_{u,v}(X) = \chi_{u,v}(X \setminus Y) + \chi_{u,v}(Y)$ for a closed subvariety $Y \subset X$,
(iii) $\chi_{u,v}(X \times Y) = \chi_{u,v}(X) \cdot \chi_{u,v}(Y)$,
(iv) $\chi_{u,v}(pt) = 1$.

6.5.3 Motivic Measure

The above formula $\chi_{u,v}(X) = \chi_{u,v}(X \setminus Y) + \chi_{u,v}(Y)$ is sometimes called *scissor formula* or *scissor relation* (e.g., see [95]). In other words, as mentioned at the very beginning of Introduction, the invariant $\chi_{u,v}$ is "additive over cutting into pieces". Such an invariant or measure is called *motivic invariant* or *motivic measure*. The latter term is more often used. For example, the cardinality $|-|$ counting the number of elements of a finite set, of course satisfies $|A| = |A \setminus B| + |B|$ for $B \subset A$, thus the cardinality $|-|$ is a very simple motivic measure. As we will see below, many invariants studied in geometry and topology are *motivic*. In this section we go into a bit more details of motivic measure. We will see why we use the term "motivic Hirzebruch class" with the adjective "motivic".

Let, as before, $\mathscr{V}$ denote the category of complex algebraic varieties.

Definition 6.5.10 Let A be a commutative monoid. A map $\mathfrak{a} : \mathscr{V} \to A$ is called *an A-valued additive invariant* (cf. [66, Chap.2, §1]) if it satisfies the following three conditions:

(i) $\mathfrak{a}(X) = \mathfrak{a}(X')$ if $X \cong X'$,
(ii) $\mathfrak{a}(\emptyset) = 0$,
(iii) $\mathfrak{a}(X) = \mathfrak{a}(Y) + \mathfrak{a}(X \setminus Y)$.

Remark 6.5.11 We note that if A is a group, i.e., an abelian group, then the second condition $\mathfrak{a}(\emptyset) = 0$ automatically follows from the "scissor formula" $\mathfrak{a}(X) = \mathfrak{a}(Y) + \mathfrak{a}(X \setminus Y)$ by considering $X = Y = \emptyset$.

[28] It is sometimes called the E-polynomial and denoted by $E_{u,v}(X)$.

Definition 6.5.12 Let A be a commutative ring. A map $\mathfrak{m} : \mathscr{V} \to A$ is called *an A-valued motivic measure* (cf. [66, Chap.2, §2]) or *an A-valued motivic invariant* (cf. [104]) if it satisfies the following four conditions:

(i) $\mathfrak{m}(X) = \mathfrak{m}(X')$ if $X \cong X'$,
(ii) $\mathfrak{m}(X) = \mathfrak{m}(Y) + \mathfrak{m}(X \setminus Y)$,
(iii) $\mathfrak{m}(X \times Y) = \mathfrak{m}(X)\mathfrak{m}(Y)$,
(iv) $\mathfrak{m}(pt) = 1$ for a singleton pt.

Therefore the Hodge–Deligne polynomial $\chi_{u,v}$ is a $\mathbb{Z}[u, v]$-valued motivic measure.

Remark 6.5.13 If A is a domain, then "multiplicativity" $\mathfrak{m}(X \times Y) = \mathfrak{m}(X)\mathfrak{m}(Y)$ implies $\mathfrak{m}(pt) = 0$ or $\mathfrak{m}(pt) = 1$, which follows from $\mathfrak{m}(pt) = \mathfrak{m}(pt \times pt) = \mathfrak{m}(pt)\mathfrak{m}(pt)$. If $\mathfrak{m}(pt) = 0$, then for any variety X we have $\mathfrak{m}(X) = \mathfrak{m}(pt \times X) = \mathfrak{m}(pt)\mathfrak{m}(X) = 0$, thus $\mathfrak{m}$ is a trivial invariant. So, the very natural condition $\mathfrak{m}(pt) = 1$ means in a sense that $\mathfrak{m}$ is a non-trivial one.

Remark 6.5.14 Therefore an A-valued motivic measure or invariant $\mathfrak{m}$ is *a non-trivial (condition 4.), multiplicative (condition 3.) and additive (condition 2.) invariant (condition 1.) with values in A.*

A motivic measure or invariant is a generalization of the usual counting or cardinality of a finite set. Let $\sharp(F)$ be the cardinality of a finite set F.

(i) $\sharp(A) = \sharp(A')$ if $A \cong A'$ (bijection, an isomorphism in the category of finite sets),
(ii) $\sharp(A) = \sharp(B) + \sharp(A \setminus B)$ for $B \subset A$,
(iii) $\sharp(A \times B) = \sharp(A) \cdot \sharp(B)$,
(iv) $\sharp(pt) = 1$.

Now let us consider whether there exists a $\mathbb{Z}$-valued "topological" counting $\sharp_{\mathrm{top}}$ on the category $\mathscr{TOP}$ of topological spaces, satisfying the following properties:

(i) $\sharp_{\mathrm{top}}(A) = \sharp_{\mathrm{top}}(A')$ if $A \cong A'$ (isomorphism in $\mathscr{TOP}$),
(ii) $\sharp_{\mathrm{top}}(A) = \sharp_{\mathrm{top}}(B) + \sharp_{\mathrm{top}}(A \setminus B)$ for a closed subset $B \subset A$,
(iii) $\sharp_{\mathrm{top}}(A \times B) = \sharp_{\mathrm{top}}(A) \cdot \sharp_{\mathrm{top}}(B)$,
(iv) $\sharp_{\mathrm{top}}(pt) = 1$.

Remark 6.5.15 If we consider the discrete topology on a set, then we have $\sharp_{\mathrm{top}} = \sharp$.

We can see that if there exists such a counting, then we must have that

$$\sharp_{\mathrm{top}}(\mathbb{R}) = -1,$$

which follows from a decomposition $\mathbb{R} = (-\infty, p) \sqcup \{p\} \sqcup (p, \infty)$ for any point $p \in \mathbb{R}$ and $(-\infty, p) \cong \mathbb{R} \cong (p, \infty)$. Hence we have that $\sharp_{\mathrm{top}}(\mathbb{R}^n) = (-1)^n$. The existence of such a topological counting is guaranteed by the Borel–Moore homology group $H_*^{\mathrm{BM}}(X;\mathbb{R})$ or the cohomology $H_c^*(X;\mathbb{R})$ with compact support, namely

$$\chi_c(X) := \sum_i (-1)^i \dim H_c^i(X;\mathbb{R}) = \sum_i (-1)^i \dim H_i^{\mathrm{BM}}(X;\mathbb{R})$$

satisfies the above four properties on a suitable category of topological spaces, like real semi-algebraic sets, so that in particular the dimensions above are finite dimensional and the Euler–Poincaré characteristic is well defined. In other words, for finite CW-complexes, the topological counting is nothing but the Euler–Poincaré characteristic: $\sharp_{\mathrm{top}} = \chi_c$. Motivated by this way of thinking, let us consider the following $\mathbb{Z}$ -valued "algebraic" counting:

(i) $\sharp_{\mathrm{alg}}(A) = \sharp_{\mathrm{alg}}(A')$ if $A \cong A'$ (isomorphism in $\mathscr{V}$),
(ii) $\sharp_{\mathrm{alg}}(A) = \sharp_{\mathrm{alg}}(B) + \sharp_{\mathrm{alg}}(A \setminus B)$ for a closed subvariety $B \subset A$,
(iii) $\sharp_{\mathrm{alg}}(A \times B) = \sharp_{\mathrm{alg}}(A) \cdot \sharp_{\mathrm{alg}}(B)$,
(iv) $\sharp_{\mathrm{alg}}(pt) = 1$.

In the case of complex algebraic varieties we consider the decomposition $\mathbb{P}^n = C^n \sqcup \mathbb{P}^{n-1}$ and by induction we have the following decomposition:

$$\mathbb{P}^n = \mathbb{C}^0 \sqcup \mathbb{C}^1 \sqcup \cdots \sqcup \mathbb{C}^{n-1} \sqcup \mathbb{C}^n.$$

From which we get the following formula

$$\sharp_{\mathrm{alg}}(\mathbb{P}^n) = 1 + \sharp_{\mathrm{alg}}(\mathbb{C}^1) + \sharp_{\mathrm{alg}}(\mathbb{C}^1)^2 + \cdots + \sharp_{\mathrm{alg}}(\mathbb{C}^1)^n.$$

Hence, in contrast to the case of the topological counting $\sharp_{\mathrm{top}}$, which is uniquely determined and is nothing but the Euler–Poincaré characteristic, the "algebraic" counting $\sharp_{\mathrm{alg}}$ at least depends on the value $\sharp_{\mathrm{alg}}(\mathbb{C}^1)$. Hence there exist infinitely many "algebraic" countings. As seen above, the Hodge–Deligne polynomial $\chi_{u,v}$ with any integers u, v is such an "algebraic" counting.

6.5.4 The Grothendieck Group of Complex Algebraic Varieties

Let $\mathrm{Iso}(\mathscr{V})$ be the free abelian group generated by the isomorphism classes $[X]$ of complex algebraic varieties X. Then the homomorphism $\chi_{u,v} : \mathrm{Iso}(\mathscr{V}) \to \mathbb{Z}[u,v]$ defined by $\chi_{u,v}([X]) := \chi_{u,v}(X)$ is well-defined. The subgroup of $\mathrm{Iso}(\mathscr{V})$ generated by elements of the form $[X] - [Y] - [X \setminus Y]$ with Y a closed subvariety of X shall be denoted by $\{[X] - [Y] - [X \setminus Y]\}$ and the quotient

$$K_0(\mathscr{V}) := \mathrm{Iso}(\mathscr{V}) / \{[X] - [Y] - [X \setminus Y]\}$$

is called the *Grothendieck group of complex algebraic varieties*. The equivalence class of $[X]$ in $K_0(\mathscr{V})$ is still denoted by $[X]$ for the sake of simplicity.

If we define

$$[X] \cdot [Y] := [X \times Y],$$

then $K_0(\mathscr{V})$ becomes a commutative ring with the unit $1 := [pt]$ for the singleton pt. In this case $K_0(\mathscr{V})$ is called *the Grothendieck ring of complex algebraic varieties*. Hence we have the following obvious properties:

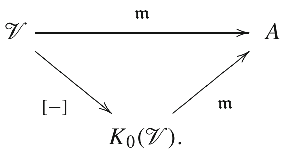
(i) $[X] = [X']$ if $X \cong X'$,
(ii) $[X] = [Y] + [X \setminus Y]$,
(iii) $[X \times Y] = [X] \cdot [Y]$,
(iv) $[pt] = 1$.

The map $[-]$ assigning the *Grothendieck class* $[X]$ to a variety X

$$[-] : \mathscr{V} \to K_0(\mathscr{V})$$

is called *the universal motivic measure* and we have the following

Proposition 6.5.16 *Let A be a commutative ring and let $\mathfrak{m} : \mathscr{V} \to A$ be a motivic measure. Then there exists a unique ring homomorphism $\mathfrak{m} : K_0(\mathscr{V}) \to A$ (using the same symbol $\mathfrak{m}$) such that the following diagram commutes:*



Thus the motivic measure $\chi_{u,v} : \mathscr{V} \to \mathbb{Z}[u, v]$ induces the ring homomorphism $\chi_{u,v} : K_0(\mathscr{V}) \to \mathbb{Z}[u, v]$.

6.5.5 The Relative Grothendieck Group of Complex Algebraic Varieties

Definition 6.5.17 ([122]) *The relative Grothendieck group*[29] $K_0(\mathscr{V}/X)$ *of* X is defined to be the free abelian group $\mathrm{Iso}(\mathscr{V}/X)$ generated by the isomorphism classes $[V \xrightarrow{h} X]$ of morphisms over X, $h : V \to X$, modulo the following additivity relation

$$[V \xrightarrow{h} X] = [Z \xrightarrow{h_{|Z}} X] + [V \setminus Z \xrightarrow{h_{|V \setminus Z}} X] \text{ for any closed subvariety } Z \subset V,$$

[29] A generalized relative Grothendieck group is considered in [169].

namely, $\mathrm{Iso}(\mathcal{V}/X)$ modulo the subgroup generated by the elements of the form $[V \xrightarrow{h} X] - [Z \xrightarrow{h_{|Z}} X] - [V \setminus Z \xrightarrow{h_{|V \setminus Z}} X]$ for any closed subvariety $Z \subset V$.

Proposition 6.5.18 *The following hold:*

(i) $K_0(\mathcal{V}/pt) = K_0(\mathcal{V})$.

(ii) $K_0(\mathcal{V}/X)$ *is a covariant functor with the pushforward* $f_* : K_0(\mathcal{V}/X) \to K_0(\mathcal{V}/Y)$ *defined by* $f_*([V \xrightarrow{h} X]) := [V \xrightarrow{f \circ h} Y]$ *for a morphism* $f : X \to Y$.

(iii) $K_0(\mathcal{V}/X)$ *is also a contravariant functor with the pullback* $f^* : K_0(\mathcal{V}/Y) \to K_0(\mathcal{V}/X)$ *defined by* $f^*([V \xrightarrow{h} Y]) := [X \times_Y V \xrightarrow{h'} X]$ *for a morphism* $f : X \to Y$. *Here we consider the fiber product*

$$\begin{array}{ccc} X \times_Y V & \xrightarrow{f'} & V \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle h} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

(iv) *The fiber product gives a ring structure to* $K_0(\mathcal{V}/X)$*:*

$$[V_1 \xrightarrow{h_1} X] \cdot [V_2 \xrightarrow{h_2} X] := [V_1 \times_X V_2 \xrightarrow{h_1 \times_X h_2} X].$$

6.5.6 Motivic Hirzebruch Class

Here is a question of wether there exists a *GRR-type theorem for* $\chi_{u,v} : K_0(\mathcal{V}) \to \mathbb{Z}[u, v]$, i.e., whether there exists a natural transformation $\tau : K_0(\mathcal{V}/-) \to H_*(-) \otimes \mathbb{Z}[u, v]$ such that for $X = pt$ a point, $\tau : K_0(\mathcal{V}/pt) \to H_*(pt) \otimes \mathbb{Z}[u, v]$ is equal to the above homomorphism $\chi_{u,v} : K_0(\mathcal{V}) \to \mathbb{Z}[u, v]$. Motivated by the fact that the three distinguished characteristic homology classes c_*, td_*, L_* are formulated *as natural transformations satisfying "smooth condition"*, as discussed in Sect. 6.4 above, we impose "smooth condition" that there exists a multiplicative characteristic cohomology class $c\ell$ such that $\tau([X \xrightarrow{\mathrm{id}_X} X]) = c\ell(TX) \cap [X]$ for smooth X. This "smooth condition" implies that $(u + 1)(v + 1) = 0$, i.e., $u = -1$ or $v = -1$. Indeed, let us consider any d-fold ($d \neq 1$) covering $\pi : \widetilde{E} \to E$ of smooth elliptic curves $E, \widetilde{E}$. Note that $T\widetilde{E} = \pi^* TE$ and $\chi_{u,v}(E) = \chi_{u,v}(\tilde{E}) = 1 + u + v + uv = (1 + u)(1 + v)$. Then we get that $(1 + u)(1 + v) = \chi_{u,v}(\widetilde{E}) = d \cdot \chi_{u,v}(E) = d(1+u)(1+v)$, i.e., $(1+u)(1+v) = d(1+u)(1+v)$. Since $d \neq 1$, $(1+u)(1+v) = 0$, i.e., $u = -1$ or $v = -1$. Therefore $\chi_{u,v}$ has to be $\chi_{-1,v}$ or $\chi_{u,-1}$. $\chi_{u,v}(X)$ is in fact symmetric with respect to (u, v), i.e., $\chi_{u,v}(X) = \chi_{v,u}(X)$, thus $\chi_{u,-1}(X) =$

$\chi_{-1,u}(X)$. We observe that $\chi_{u,-1}(X)$ is nicer than $\chi_{-1,u}(X)$ since $\chi_{u,-1}(X)$ involves only the Hodge filtration as follows: (changing u to y)

$$\chi_y(X) := \chi_{y,-1}(X) = \sum_{i,p\geqslant 0} (-1)^i \dim_{\mathbb{C}} Gr_F^p \left(H_c^i(X,\mathbb{C})\right)(-y)^p.$$

When X is non-singular and compact, it is equal to the original Hirzebruch χ_y-genus $\chi_y(X)$, therefore the above $\chi_y(X)$ is still called the Hirzebruch χ_y-genus, whether X is smooth or not. Now let us consider the following commutative diagram for $X = \mathbb{P}^n$:

$$\begin{array}{ccc} K_0(\mathscr{V}/\mathbb{P}^n) & \xrightarrow{\tau} & H_*(\mathbb{P}^n)\otimes\mathbb{Z}[y] \\ {\scriptstyle (a_{\mathbb{P}^n})_*}\downarrow & & \downarrow{\scriptstyle (a_{\mathbb{P}^n})_*} \\ K_0(\mathscr{V}) & \xrightarrow[\chi_y]{} & \mathbb{Z}[y]. \end{array}$$

Then, for $[\mathbb{P}^n \xrightarrow{\mathrm{id}_{\mathbb{P}^n}} \mathbb{P}^n] \in K_0(\mathscr{V}/\mathbb{P}^n)$ we have

$$\chi_y(a_{\mathbb{P}^n})_*([\mathbb{P}^n \xrightarrow{\mathrm{id}_{\mathbb{P}^n}} \mathbb{P}^n]) = (a_{\mathbb{P}^n})_*(\tau([\mathbb{P}^n \xrightarrow{\mathrm{id}_{\mathbb{P}^n}} \mathbb{P}^n]). \tag{6.40}$$

Then the left-hand side of (6.40) is equal to $\chi_y([\mathbb{P}^n \to pt]) = \chi_y(\mathbb{P}^n)$ and the right-hand side of (6.40) is equal to $\int_{\mathbb{P}^n} c\ell(T\mathbb{P}^n)\cap[\mathbb{P}^n]$. Thus $\chi_y(\mathbb{P}^n) = \int_{\mathbb{P}^n} c\ell(T\mathbb{P}^n)\cap[\mathbb{P}^n]$. Since $\chi_y(\mathbb{P}^n) = 1 - y + y^2 + \cdots + (-1)^n y^n$, we get

$$\int_{\mathbb{P}^n} c\ell(T\mathbb{P}^n)\cap[\mathbb{P}^n] = 1 - y + y^2 + \cdots + (-1)^n y^n.$$

In [99] Hirzebruch proved that such a characteristic class $c\ell$ *has to be* the Hirzebruch class T_y.

In our previous paper [52] (see also [135, 165, 168] and [216]), using Saito's theory of mixed Hodge modules [159], or alternatively Bittner's presentation [38] in terms of a blow-up relation as in Theorem 6.10.1 (proved via the deep 'weak factorization theorem'), we showed the following theorem:

Theorem 6.5.19 (Motivic Hirzebruch Class) *Let y be an indeterminate.*

(i) *There exists a unique natural transformation $T_{y_*}: K_0(\mathscr{V}/X) \to H_*(X)\otimes\mathbb{Q}[y]$ satisfying "smooth condition" that $T_{y_*}([X \xrightarrow{id_X} X]) = T_y(TX)\cap[X]$ for a nonsingular variety X.*
(ii) *For $X = pt$, $T_{y_*}: K_0(\mathscr{V}) \to \mathbb{Q}[y]$ is equal to $\chi_y: K_0(\mathscr{V}) \to \mathbb{Z}[y] \subset \mathbb{Q}[y]$. Namely, $T_{y_*}([V \to pt]) = \chi_y([V]) = \sum_{i,p\geqslant 0}(-1)^i \dim_{\mathbb{C}}(Gr_F^p H_c^i(V,\mathbb{C}))(-y)^p$.*

Definition 6.5.20 For any (possibly singular) variety X, $T_{y*}(X) := T_{y*}([X \xrightarrow{\mathrm{id}_X} X])$ is also called the motivic Hirzebruch class of X.

The motivic Hirzebruch class transformation $T_{y_*} : K_0(\mathscr{V}/X) \to H_*(X) \otimes \mathbb{Q}[y]$ is obtained as the composite $T_{y_*} := \widetilde{td}_{*(y)} \circ \Lambda_y^{mot}$ of the following two natural transformations:

(i) $\widetilde{td}_{*(y)} : K_0(X) \otimes \mathbb{Z}[y] \to H_*(X) \otimes \mathbb{Q}[y, (1+y)^{-1}]$, which is defined by

$$\widetilde{td}_{*(y)} := \sum_{i \geqslant 0} \frac{1}{(1+y)^i}\ td_{*i}$$

and called *a twisted Baum–Fulton–MacPherson's Todd class transformation* [209].

(ii) $\Lambda_y^{mot} : K_0(\mathscr{V}/X) \to K_0(X) \otimes \mathbb{Z}[y]$, which is the main key and denoted by mC_*, called the *motivic Chern class*, in [52]. In this paper, we use the above symbol to emphasize the following property of it:

Theorem 6.5.21 (Motivic Chern Class = "Motivic" λ_y-Class Transformation) *There exists a unique natural transformation $\Lambda_y^{mot} : K_0(\mathscr{V}/X) \to K_0(X) \otimes \mathbb{Z}[y]$ satisfying "smooth condition" that for smooth X, $\Lambda_y^{mot}([X \xrightarrow{\mathrm{id}} X]) = \sum_{p \geqslant 0}^{dim X} [\Omega_X^p] y^p = \lambda_y(T^*X) \otimes \mathscr{O}_X$. Here $\otimes \mathscr{O}_X : K^0(X) \cong K_0(X)$ is the isomorphism for smooth X, i.e., taking the sheaf of local sections.*

Note that $T_{y_*}([X \xrightarrow{\mathrm{id}} X]) = T_y(TX) \cap [X]$ for X smooth.

Remark 6.5.22 Even though the target of $\widetilde{td}_{*(y)}$ is $H_*(X) \otimes \mathbb{Q}[y, (1+y)^{-1}]$, the image of $T_{y_*} = \widetilde{td}_{*(y)} \circ \Lambda_y^{mot}$ is in $H_*(X) \otimes \mathbb{Q}[y]$.

Now, in order to define Λ_y^{mot}, first we recall the following four things:

(i) To X one can associate an abelian category of *mixed Hodge modules* $MHM(X)$, together with a functorial pullback f^* and pushforward f_* on the level of bounded derived categories $D^b(MHM(X))$ for any (not necessarily proper) map. These natural transformations are functors of triangulated categories.

(ii) Let $i : Y \to X$ be the inclusion of a closed subspace, with open complement $j : U := X \backslash Y \to X$. Then one has for $M \in D^b MHM(X)$ a distinguished triangle

$$j_! j^* M \to M \to i_! i^* M \xrightarrow{[1]}$$

Hence, by the definition of the Grothendieck group $K_0(D^b MHM(X))$ of the derived category $D^b MHM(X)$, in $K_0(D^b MHM(X))$ we have the following equality

$$[M] = [j_! j^* M] + [i_! i^* M].$$

(iii) For all $p \in \mathbb{Z}$ one has a "filtered de Rham complex" functor of triangulated categories

$$gr_p^F DR : D^b(MHM(X)) \to D^b_{coh}(X)$$

commuting with proper pushforward. Here $D^b_{coh}(X)$ is the bounded derived category of sheaves of $\mathcal{O}_X$-modules with coherent cohomology sheaves. Moreover, $gr_p^F DR(M) = 0$ for almost all p and $M \in D^b MHM(X)$ fixed.

(iv) There is a distinguished element $\mathbb{Q}^H_{pt} \in MHM(\{pt\}/k)$ such that

$$gr^F_{-p} DR(\mathbb{Q}^H_X) \simeq \Omega^p_X[-p] \in D^b_{coh}(X) \tag{6.41}$$

for X smooth and pure dimensional. Here $\mathbb{Q}^H_X := (a_X)^* \mathbb{Q}^H_{pt}$ for $a_X : X \to pt$, with $\mathbb{Q}^H_{pt}$ viewed as a complex concentrated in degree zero.

Remark 6.5.23 The above transformations are functors of triangulated categories, thus they induce functors even on the level of *Grothendieck groups of triangulated categories*, which we denote by the same symbol. We note that for these *Grothendieck groups*, by associating to a complex its alternating sum of cohomology objects, we have isomorphisms $K_0(D^b MHM(X)) \simeq K_0(MHM(X))$ and $K_0(D^b_{coh}(X)) \simeq K_0(X)$.

Definition 6.5.24 We define

(i) $mH : K_0(\mathcal{V}/X) \to K_0(MHM(X))$ by $mH([V \xrightarrow{f} X]) := [f_! \mathbb{Q}^H_V]$.

(ii) $gr^F_{-*} DR : K_0(MHM(X)) \to K_0(X) \otimes \mathbb{Z}[y, y^{-1}]$ by
$$gr^F_{-*} DR([M]) := \sum_p [gr^F_{-p} DR(M)] \cdot (-y)^p.$$

(iii) $\Lambda^{mot}_y := gr^F_{-*} DR \circ mH : K_0(\mathcal{V}/X) \xrightarrow{mH} K_0(MHM(X)) \xrightarrow{gr^F_{-*} DR} K_0(X) \otimes \mathbb{Z}[y]$.

Remark 6.5.25 By (6.41), for X smooth and pure dimensional we have that
$$gr^F_{-*} DR \circ mH([X \xrightarrow{\mathrm{id}_X} X]) = \sum_{p \geqslant 0}^{dimX} [\Omega^p_X] \cdot y^p \in K_0(X) \otimes \mathbb{Z}[y]$$

Remark 6.5.26 If $y = 0$, then $T_{0*} = \widetilde{td}_{*(0)} \circ \Lambda^{mot}_0 = td_* \circ \Lambda^{mot}_0$ and $\Lambda^{mot}_0([V \xrightarrow{f} X]) = gr^F_0 DR([f_! \mathbb{Q}^H_V])$. Λ^{mot}_0 shall be denoted by $\gamma : K_0(\mathcal{V}/X) \to K_0(X)$, which is used in Theorem 6.6.1 below.

Remark 6.5.27 For more recent works on T_{y*}, e.g., see [62–64, 133–137] (cf. [59, 60]). As to recent applications of motivic Chern class Λ^{mot}_y in Schubert calculus, e.g., see [79, 158] (cf. [5–7]).

6.5.7 A Zeta Function of Motivic Hirzebruch Class

For a finite set X let $\sharp(X)$ be the cardinality of X. Let $X^{(n)} := X^n/\mathfrak{S}_n$ be the n-th symmetric product of X. Then the formal power series

$$\zeta_{\sharp}(X)(t) := \sum_{n=0}^{\infty} \sharp(X^{(n)})t^n$$

is called *a zeta function of the cardinality* $\sharp$ *of* X. Let $\sharp(X) = m$. Then, since $\sharp(X^{(n)})$ is equal to the repeated combination ${}_mH_n = \binom{m-1+n}{m-1}$, we have

$$\zeta_{\sharp}(X)(t) = \sum_{n=0}^{\infty} \binom{m-1+n}{m-1} t^n = \frac{1}{(1-t)^m} = (1-t)^{-\sharp(X)}.$$

Remark 6.5.28 (Hasse–Weil Zeta Function and Weil Conjecture) Let X be an algebraic variety defined over the finite field $\mathbb{F}_p$. It is well-known that the Hasse–Weil zeta function $\zeta_{\sharp}(X(\mathbb{F}_p))(t) := \exp\left(\sum_{m\geqslant 1} \frac{\sharp(X(\mathbb{F}_{p^m}))}{m} t^m\right) \in \mathbb{Q}[[t]]$ is expressed as $\zeta_{\sharp}(X(\mathbb{F}_p))(t) = \sum_{n=0}^{\infty} \sharp(X^{(n)}(\mathbb{F}_p))t^n$ (e.g., see [145, Proposition 7.31]). The celebrated Weil conjecture is about rationality of this zeta function of a non- singular projective algebraic variety:

- $\zeta_{\sharp}(X(\mathbb{F}_p))(t) = \dfrac{P_1(t)P_3(t)\cdots P_{2N-1}(t)}{P_2(t)P_4(t)\cdots P_{2N}(t)}$, where $N = \dim X$ and $P_i(t)$ is an integral polynomial whose degree is equal to $\dim H^i(X)$, and
- the absolute value of the roots of $P_i(t)$ is equal to $p^{\frac{i}{2}}$.

This conjecture was solved by Deligne [71, 73].

For the Euler–Poincaré characteristic $\chi(X)$ of a topological space X

$$\zeta_{\chi}(X)(t) = \sum_{n=0}^{\infty} \chi(X^{(n)})t^n = (1-t)^{-\chi(X)} \tag{6.42}$$

was proved by I. G. Macdonald [124]. Furthermore, for the arithmetic genus $a(X)$ and the signature $\sigma(X)$ of a complex algebraic variety X

$$\zeta_a(X)(t) = (1-t)^{-a(X)}, \qquad \zeta_{\sigma}(X)(t) = (1-t)^{-\sigma(X)}.$$

were proved by B. Moonen [143] and D. Zagier [222] respectively. Macdonald's formula (6.42) was extended to the Chern–Schwartz–MacPherson class $c_*(X)$ by T. Ohmoto. To describe his formula we need some formulas:

$$\log(1-T)^{-\alpha} = -\alpha\log(1-T) = \sum_{r=1}^{\infty} \frac{T^r}{r}\alpha, \text{ i.e., } (1-T)^{-\alpha} = \exp\left(\sum_{r=1}^{\infty} \frac{T^r}{r}\alpha\right).$$

Let $\delta^k : X \to X^k$ be the diagonal map, i.e., $\delta^k(x) := \overbrace{(x, x, \cdots, x)}^{k}$ and $\pi_k : X^k \to X^{(k)}$ be the projection and let $\Delta^k := \pi_k \circ \delta^k$. Then for a homology class $\alpha \in H_*^{BM}(X)$ we define

$$(1 - t\Delta_*)^{-\alpha} = \exp\left(\sum_{r=1}^{\infty} \frac{(t\Delta_*)^r}{r}\alpha\right)$$
$$:= \exp\left(\sum_{r=1}^{\infty} \frac{t^r \Delta_*^r(\alpha)}{r}\right) \in \sum_{r=1}^{\infty} H_*^{BM}(X^{(r)}; \mathbb{Q})t^r$$

where $(t\Delta_*)^r := t^r\Delta_*^r$ and $\Delta_*^r : H_*^{BM}(X; \mathbb{Q}) \to H_*^{BM}(X^{(r)}; \mathbb{Q})$. With these definitions Ohmoto proved:

Theorem 6.5.29 ([147] (cf. [146])) $\zeta_{c_*}(X)(t) = (1 - t\Delta_*)^{-c_*(X)}$.

Furthermore Ohmoto's formula was extended to the motivic Hirzebruch class:

Theorem 6.5.30 ([63, 64, 218]) $\zeta_{T_{y*}}(X)(t) = (1 - t\Delta_*)^{-T_{y*}(X)}$.

Remark 6.5.31 In [105] M. Kapranov introduced the following motivic zeta function :

$$\zeta^{\mathrm{Kap}}(X)(t) := \sum_{n=0}^{\infty}[X^{(n)}]t^n \in K_0(\mathscr{V})[[t]].$$

He [105] showed that $\zeta^{\mathrm{Kap}}(X)(t)$ is a rational function for a nonsingular projective curve X. But M. Larsen and V. A. Lunts [117] showed that $\zeta^{\mathrm{Kap}}(X)(t)$ is not necessarily a rational function for a surface X and showed that it is a rational function if and only if the Kodaira dimension of X is negative. However, when it comes to the Grothendieck ring of Chow motives, which is finer that the Grothendieck ring $K_0(\mathscr{V})$ of algebraic varieties, Y. André [10] showed that if the Chow motive of X is Kimura-finite (see [111]) then the Chow motivic zeta function $\zeta^{\mathrm{Chow}}(X)(t) := \sum_{n=0}^{\infty}[Ch(X^{(n)}]t^n \in K_0(\mathscr{CM})[[t]]$ is a rational function. Here $\mathscr{CM}$ denotes the category of Chow motives.

6.6 A "Unification" of the Three Distinguished Characteristic Classes of Singular Varieties

The above motivic Hirzebruch characteristic class $T_{y_*} : K_0(\mathscr{V}/X) \to H_*(X) \otimes \mathbb{Q}[y]$ becomes the following three transformations for $y = -1, 0, 1$:

(i) (y=-1) $T_{-1_*} : K_0(\mathscr{V}/X) \to H_*(X) \otimes \mathbb{Q}$ is a unique natural transformation satisfying "smooth condition" that $T_{-1_*}([X \xrightarrow{\mathrm{id}_X} X]) = c(TX) \cap [X]$ for smooth X.

(ii) (y=0) $T_{0*} : K_0(\mathscr{V}/X) \to H_*(X) \otimes \mathbb{Q}$ is a unique natural transformation satisfying "smooth condition" that $T_{0*}([X \xrightarrow{\mathrm{id}_X} X]) = td(TX) \cap [X]$ for smooth X.

(iii) (y=1) $T_{1*} : K_0(\mathscr{V}/X) \to H_*(X) \otimes \mathbb{Q}$ is a unique natural transformation satisfying "smooth condition" that $T_{1*}([X \xrightarrow{\mathrm{id}_X} X]) = L(TX) \cap [X]$ for smooth X.

They are very similar to the transformations c_*, td_*, L_*. It turns out that $T_{y_*} : K_0(\mathscr{V}/X) \to H_*(X) \otimes \mathbb{Q}[y]$ "unifies" these transformations in the following sense:

Theorem 6.6.1 ([52])

$(y = -1)$: *There exists a unique natural transformation* $\epsilon : K_0(\mathscr{V}/-) \to F(-)$ *such that for X nonsingular* $\epsilon([X \xrightarrow{\mathrm{id}} X]) = \mathbb{1}_X$. *And the following diagram commutes*

$$\begin{array}{ccc} K_0(\mathscr{V}/X) & \xrightarrow{\ \epsilon\ } & F(X) \\ & {\scriptstyle T_{-1_*}} \searrow \quad \swarrow {\scriptstyle c_* \otimes \mathbb{Q}} & \\ & H_*(X) \otimes \mathbb{Q}\,. & \end{array}$$

$(y = 0)$: *There exists a unique natural transformation* $\gamma : K_0(\mathscr{V}/-) \to K_0(-)$ *such that for X nonsingular* $\gamma([X \xrightarrow{\mathrm{id}} X]) = [\mathscr{O}_X]$. *And the following diagram commutes*

$$\begin{array}{ccc} K_0(\mathscr{V}/X) & \xrightarrow{\ \gamma\ } & K_0(X) \\ & {\scriptstyle T_{0*}} \searrow \quad \swarrow {\scriptstyle td_*} & \\ & H_*(X) \otimes \mathbb{Q}\,. & \end{array}$$

$(y = 1)$: *There exists a unique natural transformation* $sd : K_0(\mathscr{V}/-) \to \Omega(-)$ *such that for X nonsingular* $sd([X \xrightarrow{\mathrm{id}} X]) = [\mathbb{Q}_X[2\dim_{\mathbb{C}} X]]$. *And the following diagram commutes for X compact:*

$$\begin{array}{ccc} K_0(\mathscr{V}/X) & \xrightarrow{\ sd\ } & \Omega(X) \\ & {\scriptstyle T_{1*}} \searrow \quad \swarrow {\scriptstyle L_*} & \\ & H_*(X) \otimes \mathbb{Q}\,. & \end{array}$$

Remark 6.6.2 This theorem can be considered as a positive "answer" to the aforementioned MacPherson's question of *whether there is a unified theory of characteristic classes of singular varieties* [128] (cf. [206, 209]).

Remark 6.6.3

(i) $(y=-1)$: $T_{-1*}(X) = c_*(X) \otimes \mathbb{Q}$.

(ii) $(y=0)$: In general, for a singular variety X we have $\Lambda_0^{mot}([X \xrightarrow{\mathrm{id}_X} X]) \neq [\mathcal{O}_X]$, thus, in general, $T_{0*}(X) \neq td_*(X)$. So, $T_{0*}(X)$ shall be called the Hodge–Todd class and denoted by $td_*^H(X)$. However, if X is a Du Bois variety, i.e., every point of X is a Du Bois singularity (note that a nonsingular point is a Du Bois singularity), we have $\Lambda_0^{mot}([X \xrightarrow{\mathrm{id}_X} X]) = [\mathcal{O}_X]$. This is due to the definition of Du Bois variety: X is called a Du Bois variety if $\mathcal{O}_X = gr_\sigma^0(DR(\mathcal{O}_X)) \cong gr_F^0(\underline{\Omega}_X^*)$ (see [182] and also [154, Definition 7.34]). Hence, for a Du Bois variety X we have $T_{0*}(X) = td_*(X)$. For example, S. Kovács [115] proved Steenbrink's conjecture that rational singularities are Du Bois, thus for the quotient X of any smooth variety acted on by a finite group we have that $T_{0*}(X) = td_*(X)$.

(iii) $(y=1)$: In general, $sd([X \xrightarrow{\mathrm{id}_X} X]) \neq \mathscr{IC}_X$, hence $T_{1*}(X) \neq L_*(X)$. So, our $T_{1*}(X)$ shall be called the Hodge–L-class and denoted by $L_*^H(X)$. A conjecture ([52, Remark 5.4]) is that $T_{1*}(X) = L_*(X)$ for a rational homology manifold. This conjecture has been affirmatively solved by J. Fernández de Bobadilla and I. Pallarés [44] and also Fernández de Bobadilla–Pallarés–Saito [45] (also see Cappell et al [61] for the hypersurface isolated singularity case, Cappell et al [62] for quotient singularity case, Maxim–Schürmann [135] for some toric variety case and Banagl [30] (also see [31]) for some threefold case.)

6.7 Verdier–Riemann–Roch and Milnor Class

6.7.1 Verdier–Riemann–Roch

The three characteristic classes c_*, td_*, Ω_* of singular varieties are natural transformations from covariant functors to the homology covariant functors. As to a *contravariant aspect* of Baum–Fulton–MacPherson's Todd classes, in [33, §3, Conjecture, p.137] (also see [88, Part II, §0.1.3]) Baum, Fulton and MacPherson conjectured that for a $\ell.c.i.$ morphism $f : X \to Y$,

$$td_*(X) = td(T_f) \cap f^!(td_*(Y)) \tag{6.43}$$

where T_f is the virtual tangent bundle of f (see [85, B.7.6]).

A morphism $f : X \to Y$ is called a $\ell.c.i.$ morphism (of codimension d) [85, §6.6] if f factors into a (closed) regular imbedding $i : X \to P$ of some (constant) codimension e, followed by a smooth morphism $p : P \to Y$ of (constant) relative dimension $e - d$. This notion is independent of such a factorization [85, Appendix B.7.6]. A morphism $f : X \to Y$ is called *smooth* (e.g., see [85, Appendix B 2.7] and also [188, Simplicity Theorem, p.584]) if f is flat of some relative dimension n and $\Omega^1_{X/Y}$ is a locally free sheaf of rank n. Note that smoothness is preserved under the base change, in particular each fiber of f is nonsingular of dimension n.

In [194] J.-L. Verdier proved the above formula (6.43) affirmatively, by showing the following more general statement, which is called Verdier–Riemann–Roch (abbr., VRR):

Theorem 6.7.1 *For a $\ell.c.i.$ morphism $f : X \to Y$, the following diagram commutes:*

$$\begin{array}{ccc} K_0(Y) & \xrightarrow{td_*} & H_*(Y) \otimes \mathbb{Q} \\ {\scriptstyle f^!}\downarrow & & \downarrow{\scriptstyle td(T_f)\cap f^!} \\ K_0(X) & \xrightarrow[td_*]{} & H_*(X) \otimes \mathbb{Q}. \end{array} \tag{6.44}$$

Here $f^! : K_0(Y) \to K_0(X)$ is the Gysin homomorphism defined by $f^!\mathscr{F} = \sum_i (-1)^i \operatorname{Tor}_i^{\mathscr{O}_Y}(\mathscr{F}, \mathscr{O}_X)$ and $f^! : H_*(Y) \otimes \mathbb{Q} \to H_*(X) \otimes \mathbb{Q}$ is the Gysin homomorphism (see [85, Example 19.2.1]).

6.7.2 Milnor Class

In [210] we showed a similar VRR-type theorem for MacPherson's Chern class transformation $c_* : F(-) \to H_*(-)$ *for a smooth morphism*:

Theorem 6.7.2 *For a smooth morphism $f : X \to Y$, the following diagram commutes:*

$$\begin{array}{ccc} F(Y) & \xrightarrow{c_*} & H_*(Y) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle c(T_f)\cap f^!} \\ F(X) & \xrightarrow[c_*]{} & H_*(X). \end{array} \tag{6.45}$$

In a general case when $f : X \to Y$ is a $\ell.c.i.$ morphism, the commutative diagram (6.45) does not necessarily hold and there is some defect for the commutativity. In the simplest case when X is a $\ell.c.i.$ variety in a smooth variety M,

$a_X : X \to pt$ is a $\ell.c.i.$ morphism. Consider the above diagram for this map $a_X : X \to pt$:

$$\begin{array}{ccc} F(pt) & \xrightarrow{c_*} & H_*(pt) \\ {\scriptstyle (a_X)^*}\downarrow & & \downarrow{\scriptstyle c(T_{a_X})\cap (a_X)^!} \\ F(X) & \xrightarrow[c_*]{} & H_*(X). \end{array} \tag{6.46}$$

For the characteristic function $\mathbb{1}_{pt}$ we have

$$(c_* \circ (a_X)^*)(\mathbb{1}_{pt}) = c_*((a_X)^*(\mathbb{1}_{pt})) = c_*(\mathbb{1}_X) = c_*(X)$$

which is Chern–Schwartz–MacPherson's class and

$$(c(T_{a_X}) \cap (a_X)^! \circ c_*)(\mathbb{1}_{pt}) = c(T_{a_X}) \cap (a_X)^!([pt]) = c(T_X^{vir}) \cap [X] \tag{6.47}$$

which is Fulton-Johnson's class $c_*^{FJ}(X)$ [86] (also see [85]). Certainly, if $a_X : X \to pt$ is smooth, i.e., X is smooth, then $T_X^{vir} = T_X$ and $c_*(X) = c(T_X) \cap [X]$, thus the above diagram commutes. Thus the defect of the commutativity of the diagram (6.46), i.e., the difference $c_* \circ (a_X)^* - c(T_{a_X}) \cap (a_X)^! \circ c_*$ evaluated on the generator $\mathbb{1}_{pt}$ of $F(pt)$ is nothing but

$$\mathscr{M}(X) := c_*(X) - c(T_X^{vir}) \cap [X] \tag{6.48}$$

which is (up to sign) what is called "Milnor class" [208, 210, 211]. Naming "Milnor class" (at that time) seems to be reasonable, considering the following preceding works:

- In [148] (also see [149–152]) A. Parusiński introduced the degree of the 0-dimensional component of $\mathscr{M}(X)$, i.e., $\int_X \mathscr{M}(X)$, as *a generalized Milnor number*.
- In the case of hypersurfaces with any singularities, in [3] (also see [1, 2]) P. Aluffi expressed the above difference $\mathscr{M}(X)$ in terms of his μ-class.
- In the case of local complete intersections with isolated singularities, in [174] J. Seade and T. Suwa expressed the degree $\int_X \mathscr{M}(X)$ as the sum of the Milnor numbers of the isolated singularities. In [185] T. Suwa expressed $\mathscr{M}(X)$ as $(-1)^{n+1} \sum_i \mu(X, p_i)[p_i]$ where $\mu(X, p_i)$ is the Milnor number of each singularity p_i. Here $n = \dim X$.

The Milnor class has been studied by many people from different motivations, e.g., see [49, 53, 55, 56, 175].

Remark 6.7.3 Here we remark that a general VRR-type theorem for MacPherson's Chern class transformation, i.e., a formula for the defect of the commutativity of the diagram (6.45) for a $\ell.c.i.$ morphism $f : X \to Y$ has been obtained by J. Schürmann [162].

Remark 6.7.4 As to Cappell–Shaneson's L-class $L_* : \Omega(-) \to H_*(-) \otimes \mathbb{Q}$, we do not know if a VRR-type formula holds for a smooth morphism. J. Schürmann pointed out that for a $\ell.c.i.$ morphism there is no well-defined Gysin map for the cobordism group $\Omega(-)$. For a smooth morphism $f : X \to Y$ the Gysin map $f^! = f^*[2\,\mathrm{reldim}\, f]$ where $\mathrm{reldim}\, f = \dim X - \dim Y$ is the dimension of the fiber of f, thus $f^!$ commutes with duality up to a shift (see [161, Corollary 3.1.5, p. 315]) and induces a map of cobordism groups. However, for a regular embedding i this is not the case. One would need some extra condition of "transversality" or "non-characteristic" (see [166], [107, Proposition 5.4.13 (ii)]), or in more geometric terms "a normally non-singular inclusion", and then i^* commutes with duality up to a shift for some adapted constructible complexes.

The motivic Hirzebruch class transformation $T_{y*} : K_0(\mathcal{V}/-) \to H_*(-) \otimes \mathbb{Q}[y]$ does satisfy a VRR-type formula for a smooth morphism $f : X \to Y$, i.e., the following diagram commutes (see [52]):

$$\begin{array}{ccc} K_0(\mathcal{V}/Y) & \xrightarrow{T_{y*}} & H_*(Y) \otimes \mathbb{Q}[y] \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle T_y(T_f)\cap f^!} \\ K_0(\mathcal{V}/X) & \xrightarrow[T_{y*}]{} & H_*(X) \otimes \mathbb{Q}[y]. \end{array} \tag{6.49}$$

Problem 6.7.5 Identify the defect of the commutativity of the above diagram (6.49) for a $\ell.c.i.$ morphism $f : X \to Y$ (possibly in a similar way as done in [162] above).

6.7.3 Generalized Motivic Milnor–Hirzebruch Classes

As in the case of MacPherson's Chern class transformation $c_* : F(-) \to H_*(-)$, if we consider the diagram (6.49) for the map $a_X : X \to pt$, then the defect of the commutativity of the diagram (6.49), i.e., $T_{y*} \circ (a_X)^* - T_y(T_{a_X}) \cap (a_X)^* \circ T_{y*}$ evaluated on the distinguished element $[pt \xrightarrow{\mathrm{id}_{pt}} pt]$ of $K_0(\mathcal{V}/pt)$ is nothing but (up to sign) $T_{y*}(X) - T_y(T_X^{vir}) \cap [X]$. Namely, this is a kind of T_{y*}-version of the Milnor class $\mathcal{M}(X)$.

The Chern–Schwartz–MacPherson class $c_*(X)$, the Baum–Fulton–MacPherson's Todd class $td_*(X)$ and Cappell–Shaneson's L-class $L_*(X)$ are all the special values of the corresponding natural transformations $c_* : F(-) \to H_*(-)$, $td_* : K_0(-) \to H_*(-) \otimes \mathbb{Q}$ and $L_* : \Omega(-) \to H_*(-) \otimes \mathbb{Q}$, respectively. Motivated by this fact, we want to capture the Milnor class, more generally the above T_{y*}-version of the Milnor class, as a special value of some natural transformation. In this section we discuss such a transformation [217].

Let S be a complex algebraic variety and fixed. Let $\mathscr{V}_S$ be the category of S-varieties, i.e., an object is a morphism $h : X \to S$ and a morphism from $h : X \to S$ to $k : Y \to S$ is a morphism $f : X \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle k} & \\ & S. & \end{array}$$

The following definition is motivated by the definition of a universal bivariant theory $\mathbb{M}^{\mathscr{C}}_{\mathscr{S}}(X \xrightarrow{f} Y)$ defined in Theorem 6.9.17.

Definition 6.7.6 Let $M^{\mathscr{P}rop}_{\ell.c.i}(\mathscr{V}/X \xrightarrow{h} S)$ be the monoid consisting of isomorphism classes $[V \xrightarrow{p} X]$ of proper morphisms $p : V \to X$ such that the composite $h \circ p : V \to S$ is a $\ell.c.i.$ morphism, with the addition $(+)$ and zero (0) defined by

- $[V \xrightarrow{h} X] + [V' \xrightarrow{h'} X] := [V \sqcup V' \xrightarrow{h+h'} X]$,
- $0 := [\phi \to X]$.

We define $K^{\mathscr{P}rop}_{\ell.c.i}(\mathscr{V}/X \xrightarrow{h} S)$ to be the Grothendieck group of the monoid $M^{\mathscr{P}rop}_{\ell.c.i}(\mathscr{V}/X \xrightarrow{h} S)$. If S is a point, $K^{\mathscr{P}rop}_{\ell.c.i}(\mathscr{V}/X \xrightarrow{h} S)$ is denoted by $K^{\mathscr{P}rop}_{\ell.c.i}(\mathscr{V}/X)$.

Lemma 6.7.7

(i) *The Grothendieck group* $K^{\mathscr{P}rop}_{\ell.c.i}(\mathscr{V}/X \xrightarrow{h} S)$ *is a covariant functor with pushforwards for proper morphisms, i.e., for a proper morphism* $f : X \to Y \in \mathscr{V}_S$

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle k} & \\ & S, & \end{array}$$

the pushforward $f_* : K^{\mathscr{P}rop}_{\ell.c.i}(\mathscr{V}/X \xrightarrow{h} S) \to K^{\mathscr{P}rop}_{\ell.c.i}(\mathscr{V}/Y \xrightarrow{k} S)$ *defined by* $f_*([V \xrightarrow{p} X]) := [V \xrightarrow{f \circ p} Y]$ *is covariantly functorial.*

(ii) *The Grothendieck group* $K^{\mathscr{P}rop}_{\ell.c.i}(\mathscr{V}/X \xrightarrow{h} S)$ *is a contravariant functor with pullbacks for smooth morphisms, i.e., for a smooth morhism* $f : X \to Y \in \mathscr{V}_S$, *the pullback* $f^* : K^{\mathscr{P}rop}_{\ell.c.i}(\mathscr{V}/Y \xrightarrow{k} S) \to K^{\mathscr{P}rop}_{\ell.c.i}(\mathscr{V}/X \xrightarrow{h} S)$ *defined by*

$f^*([W \xrightarrow{p} Y]) := [W' \xrightarrow{p'} X]$ is contravariantly functorial. Here we consider the following commutative diagrams whose top square is a fiber square:

$$\begin{array}{ccc} W' & \xrightarrow{f'} & W \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & Y \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle k} & \\ & S. & \end{array}$$

Proposition 6.7.8 *Let $c\ell : K^0(-) \to H^*(-) \otimes R$ be a characteristic class of complex vector bundles with a suitable coefficients R. Then on the category $\mathscr{V}_S$ there exists a unique natural transformation $\gamma_{c\ell *} : K^{\mathscr{P}rop}_{\ell.c.i}(\mathscr{V}/X \xrightarrow{h} S) \to H_*(X) \otimes R$ such that for a $\ell.c.i.$ morphism $h : X \to S$, $\gamma_{c\ell *}([X \xrightarrow{\mathrm{id}_X} X]) = c\ell(T_h) \cap [X]$. Namely, for a morphism $f : X \to Y$, i.e., for a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle k} & \\ & S, & \end{array}$$

the following diagram commutes:

$$\begin{array}{ccc} K^{\mathscr{P}rop}_{\ell.c.i}(\mathscr{V}/X \xrightarrow{h} S) & \xrightarrow{\gamma_{c\ell *}} & H_*(X) \otimes R \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ K^{\mathscr{P}rop}_{\ell.c.i}(\mathscr{V}/Y \xrightarrow{k} S) & \xrightarrow[\gamma_{c\ell *}]{} & H_*(Y) \otimes R. \end{array}$$

Definition 6.7.9

(i) If S is a point and $c\ell = c$ the Chern class, for a $\ell.c.i.$ variety X in a smooth manifold we have that $\gamma_{c*}([X \xrightarrow{\mathrm{id}_X} X]) = c(T_X^{\mathrm{vir}}) \cap [X]$, which is Fulton–Johnson's class $c_*^{FJ}(X)$ (see (6.47)). Thus the natural transformation $\gamma_{c\ell *} : K^{\mathscr{P}rop}_{\ell.c.i}(\mathscr{V}/X) \to H_*(X) \otimes R$ is a generalization of Fulton–Johnson's class as *a natural transformation*. It is called *a motivic Fulton-Johnson-type $c\ell$ class*, denoted by $c\ell_*^{FJ}$, since it is modeled after Fulton–Johnson's class c_*^{FJ}.

(ii) If we consider the Hirzebruch class T_y for the characteristic class $c\ell$ and we use the motivic Hirzebruch class $T_{y*} : K_0(\mathscr{V}/X) \to H_*(X) \otimes \mathbb{Q}[y]$, then the natural transformation $\gamma_{T_{y*}} : K^{\mathscr{P}rop}_{\ell.c.i}(\mathscr{V}/X \xrightarrow{h} S) \to H_*(X)\otimes\mathbb{Q}[y]$ is called *the motivic Fulton–Johnson-type Hirzebruch class* and denoted by ${T_y}_*^{FJ}$.

(iii) The Borel–Moore homology with twisted pushforward $f_\star = (-1)^{\dim Y - \dim X} f_*$ is still a covariant functor and shall be denoted by $H_\star(X)$.

Theorem 6.7.10 *We define* $\mathscr{M}T_{y_*} : K^{\mathscr{P}rop}_{\ell.c.i}(\mathscr{V}/X \xrightarrow{h} S) \to H_\star(X)\otimes\mathbb{Q}[y]$ *by*

$$\mathscr{M}T_{y_*}([V \xrightarrow{p} X]) := (-1)^{\dim V}\left({T_y}_*^{FJ} - T_{y_*}\right)([V \xrightarrow{p} X]).$$

Then we have that $\mathscr{M}T_{y_*} : K^{\mathscr{P}rop}_{\ell.c.i}(\mathscr{V}/- \xrightarrow{h} S) \to H_\star(-)\otimes\mathbb{Q}[y]$ *is a unique natural transformation such that for a* $\ell.c.i.$ *morphism* $h : X \to S$ *it satisfies that*

$$\begin{aligned}\mathscr{M}T_{y_*}([X \xrightarrow{\mathrm{id}_X} X]) &= (-1)^{\dim X}\left({T_y}_*^{FJ} - T_{y_*}\right)([X \xrightarrow{\mathrm{id}_X} X])\\ &= (-1)^{\dim X}\Big(T_y(T_X^{\mathrm{vir}})\cap[X] - T_{y*}(X)\Big).\end{aligned}$$

Definition 6.7.11

(i) Let S be a point. Then the above motivic natural transformation

$$\mathscr{M}T_{y_*} : K^{\mathscr{P}rop}_{\ell.c.i}(\mathscr{V}/X) \to H_\star(X)\otimes\mathbb{Q}[y]$$

is called *a motivic Milnor–Hirzebruch class*, even though $K^{\mathscr{P}rop}_{\ell.c.i}(\mathscr{V}/X)$ is not the motivic group $K_0(\mathscr{V}/X)$, but

- because it is a subgroup of $K_0(\mathscr{V}/X)$ and it is defined by using the motivic Hirzebruch class $T_{y_*} : K_0(\mathscr{V}/X) \to H_*(X)\otimes\mathbb{Q}[y]$ and also
- because, if we specialize $\mathscr{M}T_{y_*}$ to the case when $y = -1$ and X is a $\ell.c.i.$ variety in a smooth manifold, we have

$$\begin{aligned}\mathscr{M}T_{-1*}(X) &:= \mathscr{M}T_{-1*}([X \xrightarrow{\mathrm{id}} X])\\ &= (-1)^{\dim X}\left\{T_{-1}(T_X^{\mathrm{vir}})\cap[X] - T_{-1*}([X \xrightarrow{\mathrm{id}} X])\right\}\\ &= (-1)^{\dim X}\left(c_*^{FJ}(X)\otimes\mathbb{Q} - c_*(X)\otimes\mathbb{Q}\right)\\ &= \left((-1)^{\dim X}\left(c_*^{FJ}(X) - c_*(X)\right)\right)\otimes\mathbb{Q},\end{aligned}$$

which is the Milnor class $\mathscr{M}(X) = (-1)^{\dim X}\left(c_*^{FJ}(X) - c_*(X)\right)$ (see (6.48)) of X *modulo torsion*.

For a $\ell.c.i.$ variety X in a smooth manifold $\mathscr{M}T_{y_*}(X) := \mathscr{M}T_{y_*}([X \xrightarrow{\mathrm{id}} X])$ is called also the *motivic Milnor–Hirzebruch class of X*.

(ii) $\mathscr{M}T_{-1_*} : K^{\mathscr{P}rop}_{\ell.c.i}(\mathscr{V}/X) \to H_\star(X) \otimes \mathbb{Q}$ is called *the motivic Milnor class*.

(iii) The more general one $\mathscr{M}T_{y_*} : K^{\mathscr{P}rop}_{\ell.c.i}(\mathscr{V}/X \xrightarrow{h} S) \to H_\star(X) \otimes \mathbb{Q}[y]$ is called *a generalized motivic Milnor–Hirzebruch class*.

In [212, Theorem 2.2] we obtained a VRR-type formula of the Milnor class in a special case. The following VRR-type formula of the motivc Milnor–Hirzebruch class is a generalization of this result:

Theorem 6.7.12 *For a smooth morphism $f : X \to Y$, the twisted Gysin pullback homomorphism $f^\star : H_*(Y) \to H_*(X)$ is defined by $f^\star = (-1)^{\dim X - \dim Y} f^*$. For a smooth morphism $f : X \to Y$ in the category $\mathscr{V}_S$ as in Proposition 6.7.8, the following diagram commutes:*

$$\begin{array}{ccc} K^{\mathscr{P}rop}_{\ell.c.i}(\mathscr{V}/Y \xrightarrow{k} S) & \xrightarrow{\mathscr{M}T_{y_*}} & H_\star(Y) \otimes \mathbb{Q}[y] \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle T_y(T_f)\cap f^\star} \\ K^{\mathscr{P}rop}_{\ell.c.i}(\mathscr{V}/X \xrightarrow{h} S) & \xrightarrow[\mathscr{M}T_{y_*}]{} & H_\star(X) \otimes \mathbb{Q}[y]. \end{array}$$

6.7.4 Hirzebruch–Milnor Class via the Vanishing Cycle Functor

In [61], for a hypersurface X, globally defined as the zero-set $X = f^{-1}(0)$ of an algebraic function $f : M \to \mathbb{C}$ on a complex algebraic manifold M, the homology class $T_y(T_X^{\mathrm{vir}}) \cap [X] - T_{y*}(X)$ is denoted by $\mathscr{M}T_{y*}(X)$ and called the *Hirzebruch–Milnor class*. Hence we have that $\mathscr{M}T_{y_*}(X) = (-1)^{\dim X}\mathscr{M}T_{y*}(X)$.

In [163] J. Schürmann proves the following theorem, which is a generalization of Verdier's result [194] on the specialization of MacPherson's Chern class transformation:

Theorem 6.7.13 *Let the situation be as above and let $i : X \hookrightarrow M$ be the inclusion. Then $MHT_{y*} : K_0(MHM(-)) \to H_*(-) \otimes \mathbb{Q}[y, y^{-1}]$ commutes with the specialization, namely the following diagram commutes:*

$$\begin{array}{ccc} K_0(MHM(M)) & \xrightarrow{MHT_{y*}} & H_*(M) \otimes \mathbb{Q}[y, y^{-1}] \\ {\scriptstyle \Psi'^H_f}\downarrow & & \downarrow{\scriptstyle i^!} \\ K_0(MHM(X)) & \xrightarrow[MHT_{y*}]{} & H_*(X) \otimes \mathbb{Q}[y, y^{-1}]. \end{array}$$

Here $\Psi_f'^H = \Psi_f^H[-1] : K_0(MHM(M)) \to K_0(MHM(X))$ *is the shifted nearby cycle functor.*

Using this specialization theorem, in [61] Cappell–Maxim–Schürmann–Shaneson prove the following:

Theorem 6.7.14 *Let the situation be as above.*

$$\mathscr{M}T_{y*}(X) = MHT_{y*}(\Phi_f'^H([\mathbb{Q}_M^H])).$$

Here $\Phi_f'^H = \Phi_f^H[-1]$ *is the shifted functor of the vanishing cycle functor* $\Phi_f^H : K_0(MHM(M)) \to K_0(MHM(X))$ *defined by* $\Phi_f^H := \Psi_f^H - i^*$.

Here the following diagrams commute:

$$\begin{array}{ccccc}
K_0(\mathscr{V}/M) & \longrightarrow & K_0(\mathscr{V}/X) & \xrightarrow{T_{y*}} & H_*(X) \otimes \mathbb{Q}[y] \\
\downarrow{\scriptstyle \chi_{\mathrm{Hodge}}} & & \downarrow{\scriptstyle \chi_{\mathrm{Hodge}}} & & \downarrow \\
K_0(MHM(M)) & \xrightarrow[\Psi_f]{\Phi_f} & K_0(MHM(X)) & \xrightarrow{MHMT_{y*}} & H_*(X) \otimes \mathbb{Q}[y, y^{-1}] \\
\downarrow{\scriptstyle \mathrm{rat}} & & \downarrow{\scriptstyle \mathrm{rat}} & & \downarrow{\scriptstyle y=-1} \\
K_0(D_c^b(M)) & \xrightarrow[\Psi_f]{\Phi_f} & K_0(D_c^b(X)) & \xrightarrow{c_*\otimes\mathbb{Q}} & H_*(X) \otimes \mathbb{Q} \\
\downarrow{\scriptstyle \chi_{\mathrm{stalk}}} & & \downarrow{\scriptstyle \chi_{\mathrm{stalk}}} & \nearrow{\scriptstyle c_*\otimes\mathbb{Q}} & \\
F(M) & \xrightarrow[\Psi_f]{\Phi_f} & F(X). & &
\end{array}$$

Here we use the following:

- $\chi_{\mathrm{Hodge}} : K_0(\mathscr{V}/Z) \to K_0(MHM(Z))$ is defined by $\chi_{\mathrm{Hodge}}([V \xrightarrow{f} Z]) := f_*\mathbb{Q}_V^H$. Here $\mathbb{Q}_V^H := (a_Z)^*\mathbb{Q}_{pt}^H$ for the constant map $a_Z : Z \to pt$ with $\mathbb{Q}_{pt}^H$ being the constant pure Hodge structure $\mathbb{Q}$ of weight zero. For Z smooth $\mathbb{Q}_Z^H$ is given by $\mathbb{Q}_Z^H := (\mathscr{O}_Z, F, \mathbb{Q}_Z, W)$ with the Hodge filtration F such that $gr_F^i = 0 (i \neq 0)$ and the pure weight filtration W.
- rat denotes the forgetful functor which assigns to a complex of mixed Hodge modules the underlying rational constructible complex.
- $\chi_{\mathrm{stalk}} : K_0(D_c^b(Z)) \to F(Z)$ is defined by, for a constructible sheaf complex $\mathscr{F}_\bullet$,

$$(\chi_{\mathrm{stalk}}(\mathscr{F}_\bullet))(x) := \chi_x(\mathscr{F}_\bullet) := \sum_j (-1)^j \dim H^j(\mathscr{F}_\bullet)_x.$$

- $c_* \otimes \mathbb{Q} : K_0(D^b_c(X)) \to H_*(X) \otimes \mathbb{Q}$ is defined by the composite of $\chi_{\text{stalk}} : K_0(D^b_c(X)) \to F(X)$ and the rationalized MacPherson–Chern class homomorphism $c_* \otimes \mathbb{Q} : F(X) \to H_*(X) \otimes \mathbb{Q}$.

Remark 6.7.15 Note that the Milnor class $\mathscr{M}(X)$ is equal to $c_*((\Phi_f(\mathbb{1}_M))$. In [164] this result is extended to the case of global complete intersections.

Definition 6.7.16 ([220, §5.2]) For a hypersurface X, globally defined as the zero-set $X = f^{-1}(0)$ of an algebraic function $f : M \to \mathbb{C}$ on a complex algebraic manifold M, we define

$$\mathscr{M}T_{y*} : K_0(\mathscr{V}/X) \to H_*(X) \otimes \mathbb{Q}[y]$$

by $\mathscr{M}T_{y*}([V \xrightarrow{h} X]) := MHT_{y*}\Big(h_! h^*(\Phi'_f(\mathbb{Q}^H_M))\Big)$.

Note that $\mathscr{M}T_{y*}([X \xrightarrow{\mathrm{id}_X} X]) = MHT_{y*}(\Phi'_f(\mathbb{Q}^H_M)) = \mathscr{M}T_{y*}(X)$.

Theorem 6.7.17 ([220, §5.2]) *For a proper morphism $\rho : N \to M$ of complex manifolds and for an algebraic function $f : M \to \mathbb{C}$, we define $Y := (f \circ \rho)^{-1}(0) = \rho^{-1}(X)$ and consider the restriction map $\rho_Y : Y \to X$. Then we have the following commutative diagram:*

$$\begin{array}{ccc}
K_0(\mathscr{V}/Y) & \xrightarrow{\mathscr{M}T_{y*}} & H_*(Y) \otimes \mathbb{Q}[y] \\
{\scriptstyle (\rho|_Y)_*}\downarrow & & \downarrow{\scriptstyle (\rho|_Y)_*} \\
K_0(\mathscr{V}/X) & \xrightarrow[\mathscr{M}T_{y*}]{} & H_*(X) \otimes \mathbb{Q}[y].
\end{array}$$

6.8 Equivariant Theory

An equivariant theory (e.g., see [8, 9, 193]) is a theory to study objects equipped with actions of a group. For example, let X be a topological space with an action of a topological group G, simply called a topological G-space. Then it is quite natural to think of the quotient space X/G and/or the fixed point space X^G in order to consider invariants of this G-space. A formula describing a global invariant $\mathrm{Inv}(X)$ of a G-space X in terms of some invariants $\widetilde{\mathrm{Inv}}(X^G)$ of the fixed point space $X^G$is a "geometric" localization: $\mathrm{Inv}(X) = \widetilde{\mathrm{Inv}}(X^G)$. A typical example of "localization" in geometry is the well-known Poincaré–Hopf theorem, which says that the Euler–Poincaré characteristic of a differentiable manifold M with a vector field v on M having finitely many isolated zeros x_i's is expressed by $\chi(M) = \sum_i \mathrm{Index}_{x_i}(v)$, where $\mathrm{Index}_{x_i}(v)$ is the index of v at x_i. There is another "localization" in algebra. That is a localization of a ring or a module. Given a multiplicative closed set S, the ring $S^{-1}M := \{\frac{m}{s} | s \in S, m \in M\}$ is called the localization of M by S. A formula connecting this "algebraic" localization and "geometric" localization is a "localization" in equivariant theory. A well-known formula is Atiyah–Bott–Berline–Vergne formula [20, 36].

In this survey we mainly look at aspects of natural transformations. So, we consider *"natural transformations" between two "functors" of "geometric objects equipped with actions of groups"*.

6.8.1 Transformation Group

Definition 6.8.1 Let G be a topological group and X a topological space. A continuous map $\phi : G \times X \to X$, ($\phi(g, x)$ is simply denoted gx) is called *an action*[30] *of G on X* if it satisfies two conditions (i) for an unit $e \in G$, $ex = x$ for any $x \in X$ and (ii) for any $g, h \in G$, $g(hx) = (gh)x$. (X, G, ϕ) is called *a topological transformation group* or *a G-action on X* and X is called *a (topological) G-space.*

Remark 6.8.2 Each $g \in G$ gives rise to a continuous map $\phi_g : X \to X$ defined by $\phi_g(x) = \phi(g, x) = gx$. This continuous map is *a homeomorphism* due to the above two conditions.[31] Hence we have the *adjoint* (map to ϕ) $ad(\phi) : G \to Aut(X)$ defined by $ad(\phi)(g) := \phi_g$.

Let (X, G, ϕ) and (X', G', ϕ') be two topological spaces with actions of groups G and G' (not necessarily different). Then a morphism $(X, G, \phi) \to (X', G', \phi')$ is defined to be a continuous map $f : X \to X'$ together with a continuous group homomorphism $\psi : G \to G'$ such that $f(gx) = \psi(g)f(x)$. With this definition we get a category, called *the equivariant category of topological spaces with actions of topological groups* (e.g., see [130, §1.5]).

If we fix a topological group G and we let the above group homomorphism $\psi : G \to G$ be the identity id_G, then we get the following:

Definition 6.8.3 Let G be a topological group and X and Y be two G-spaces. Then a continuous map $f : X \to Y$ is called *G-equivariant* if $f(gx) = gf(x)$.

For a fixed topological group G, G-spaces and G-equivariant maps make a category, called *the category of G-spaces*. The category of G-spaces can be considered for many underlying categories, e.g., sets with G a group, differentiable manifolds with G a Lie group, complex manifolds with G a complex Lie group, algebraic varieties with G an algebraic group, and so on. We will use the following definitions:

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[30] This action is a left action and similarly a right action is also defined. Note that a left action can be turned into a right action by defining $xg := g^{-1}x$ and vice versa, i.e., $gx := xg^{-1}$. If we simply define $xg := gx$ instead of $xg := g^{-1}x$, then this *is not* a right action, because it does not satisfy the above second condition; indeed $(xg)h = (gx)h = h(gx) = (hg)x = x(hg)$, hence $(xg)h \neq x(gh)$. However, if we define $xg := g^{-1}x$ using the inverse g^{-1} of g, then it *does* work.

[31] Because $\phi_{g^{-1}} \circ \phi_g = \mathrm{id}_X$ (indeed, $\phi_{g^{-1}} \circ \phi_g(x) = g^{-1}(gx) = (g^{-1}g)x = ex = x$) and $\phi_g \circ \phi_{g^{-1}} = \mathrm{id}_X$.

- $G_x := \{g \in G \mid gx = x\}$ is *the isotropy group of* x or *the stabilizer of* x,
- $Gx := \{gx \mid g \in G\}$ is *the orbit of* x,
- $X^G := \{x \in X \mid G_x = G\}$ is *the fixed point set*.
- $X/G := \{Gx\}$ is *the orbit space*.
- $Gx \cong G/G_x$ (G-bijection).
- If $G_x = \{e\}$ for any $x \in X$, then the action of G is called *free*.[32]

In the category of G-sets, the following formula for a finite group G, usually called "Burnside's Lemma", is well-known:

$$|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|, \text{ namely } |X/G| = \frac{|X|}{|G|} + \frac{1}{|G|}\sum_{g\in G-\{e\}}|X^g| \tag{6.50}$$

where $|S|$ denotes the cardinality of a set S. This follows from

$$\sum_{g\in G}|X^g| = |\{(g,x) \in G \times X \mid gx = x\}| = \sum_{x\in X}|G_x|$$

and $|Gx| = |G/G_x| = \frac{|G|}{|G_x|}$. Suggested or motivated by this formula, there are similar formulas involving some geometric invariants such as the Euler–Poincaré characteristic, instead of the cardinality $|-|$, in other categories, e.g., as in Theorem 6.8.11.

6.8.2 "Simple" G-Equivariant Natural Transformations

Let $\mathscr{F}_*, \mathscr{H}_* : \mathscr{T}op \to \mathscr{A}b$ be two covariant functors and $\tau : \mathscr{F}_* \to \mathscr{H}_*$ be a natural transformation. Then we have at least the following two simple equivariant versions:

(i) ("forget") Define $\mathfrak{for} : \mathscr{T}op^G \to \mathscr{T}op$ by $\mathfrak{for}((X,G,\phi)) = X$ and $\mathfrak{for}((X,G,\phi) \xrightarrow{f} (Y,G,\psi)) := X \xrightarrow{f} Y$. Then $\mathscr{F}_*^{\mathfrak{for}} := \mathscr{F}_* \circ \mathfrak{for} : \mathscr{T}op^G \to \mathscr{A}b$ is a covariant functor and $\tau^{\mathfrak{for}} : \mathscr{F}_*^{\mathfrak{for}} \to \mathscr{H}_*^{\mathfrak{for}}$ is an equivariant natural transformation.

(ii) ("quotient") Define $\mathfrak{quot} : \mathscr{T}op^G \to \mathscr{T}op$ by $\mathfrak{quot}((X,G,\phi)) := X/G$ and $\mathfrak{quot}((X,G,\phi) \xrightarrow{f} (Y,G,\psi)) := X/G \xrightarrow{\tilde{f}} Y/G$. Then $\mathscr{F}_*^{\mathfrak{quot}} := \mathfrak{quot} \circ F_* : \mathscr{T}op^G \to \mathscr{A}b$ is a covariant functor. $\tau^{\mathfrak{quot}} : \mathscr{F}_*^{\mathfrak{quot}} \to \mathscr{H}_*^{\mathfrak{quot}}$ is an equivariant natural transformation.

[32] In this case, every fiber of the projection $\pi : X \to X/G$ at the "point" Gx is the orbit Gx, which is bijective to G. Thus, if G is a free action on X, $\pi : X \to X/G$ is something like "a principal G-bundle".

The first one is not interesting since it has nothing to do with the action of G at all. The second one is quite natural (for example, X/G appears in the above Burnside's Lemma), but whether it is interesting or not depends on the action of G. For example, consider rotating the 2-dimensional sphere S^2 in $\mathbb{R}^3$ around the z-axis, i.e., an action of S^1 on S^2, which is not free, since the isotropy groups $S^1_N = S^1_S = S^1$ for the north and south poles N and S. Since the quotient S^2/S^1 is homeomorphic to the closed interval $[-1, 1]$, thus homotopy equivalent to a point, one could not expect much from the group $\mathscr{F}_*(pt)$, for example, if $\mathscr{F}_*$ is an ordinary homology. Therefore one needs to come up with *topologically and geometrically interesting equivariant theories* $\mathscr{F}^G_*$ for a given covariant functor $\mathscr{F}_*$. The most well-know one is due to what is usually called *the Borel construction*,[33] which is recalled below.

6.8.3 Cartan Mixing Space and Cartan Mixing Diagram

We follow [193, §4.3]. Let G be a topological group, P a principal G-bundle and M a left G-space. In [65] H. Cartan constructed a fiber bundle with fiber M in the following way. Consider the right action of G on $P \times M$ by the diagonal action[34] $(p, m)g := (pg, g^{-1}m)$. Then *the Cartan mixing space*[35] $P \times_G M$ *of* P *and* M is the orbit space $P \times_G M := (P \times M)/G$ and this construction is called *the Cartan mixing construction*.[36] The following diagram, called *Cartan mixing diagram*, is commutative:

$$
\begin{array}{ccccc}
P & \xleftarrow{pr_1} & P \times M & \xrightarrow{pr_2} & M \\
{\scriptstyle \alpha}\downarrow & & {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \pi} \\
B & \xleftarrow[\tau_1]{} & P \times_G M & \xrightarrow[\tau_2]{} & M/G
\end{array}
\tag{6.51}
$$

Here $\beta : P \times M \to P \times_G M$ is the projection map $\beta(p, m) := [p, m]$, $\tau_1 : P \times_G M \to B$ is defined by $\tau_1([p, m]) := \alpha(p)$ and $\tau_2([p, m]) := \pi(m) = Gm$ is the orbit of m. Both are well-defined, since $\tau_1([pg, g^{-1}m]) = \alpha(pg) = \alpha(p) = \tau_1([p, m])$ and $\tau_2([pg, g^{-1}m]) = G(g^{-1}m) = (Gg^{-1})m = Gm = \tau_2([p, m])$.

Proposition 6.8.4 *If $\alpha : P \to B$ is a principal G-bundle and M is a left G-space, then $\tau_1 : P \times_G M \to B$ is a fiber bundle with fiber M and the projection map $\beta : P \times M \to P \times_G M$ is a principal G-bundle.*

[33] The idea originated with Henri Cartan [65]. In this sense it should be called the Cartan–Borel construction.

[34] If we simply define $(p, m)g := (pg, gm)$, then this action *does not* satisfy the second condition of action by the same reason as in the footnote of Definition 6.8.1 above.

[35] It is also called *the balanced product* of a right G-space P and a left G-space M.

[36] It is usually called *Borel mixing construction* or simply *the Borel construction*.

This proposition says that in (6.51) if $\alpha : P \to B$ is a principal G-bundle, then $\tau_1 : P \times_G M \to B$ is *a fiber bundle with fiber M*. Symmetrically, we have

$$\begin{array}{ccccc} P & \xleftarrow{pr_1} & P \times M & \xrightarrow{pr_2} & M \\ {\scriptstyle \alpha}\downarrow & & {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \pi} \\ P/G & \xleftarrow[\tau_1]{} & P \times_G M & \xrightarrow[\tau_2]{} & B \end{array} \tag{6.52}$$

where P is a G-space, M is a principal G-bundle and $\tau_2 : P \times_G M \to B$ is *a fiber bundle with fiber P*.

6.8.4 Equivariant (Co)homology by Cartan–Borel Construction

Let X and Y be path-connected spaces. If there is a continuous map $f : X \to Y$ such that $f_* : \pi_k(X) \to \pi_k(Y)$ is an isomorphism of homotopy groups for $k \geqslant 1$, then X and Y are called *weakly homotopy equivalent*.[37] If X is weakly homotopy equivalent to a point, then X is called *weakly contractible*. The following lemma follows from the above Cartan mixing diagram:

Lemma 6.8.5 *Let G be a topological group and E be a weakly contractible G-space and P be a G-space with a free action of G such that the projection $\pi : P \to P/G$ is a principal G-bundle. Then $(E \times_G P)$ and P/G are weakly homotopy equivalent.*

Indeed, consider the following Cartan diagram (using (6.52)):

$$\begin{array}{ccccc} E & \xleftarrow{pr_1} & E \times P & \xrightarrow{pr_2} & P \\ {\scriptstyle \alpha}\downarrow & & {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \pi} \\ E/G & \xleftarrow[\tau_1]{} & E \times_G P & \xrightarrow[\tau_2]{} & P/G \end{array}$$

Since $\tau_2 : E \times_G P \to P/G$ is a fiber bundle with fiber E, we get the long exact sequence: $\cdots \to \pi_k(E) \to \pi_k(E \times_G P) \to \pi_k(P/G) \to \pi_{k-1}(E) \to \cdots$. Since E is weakly homotopy contractible, i.e., $\pi_k(E) = 0$ for $k \geqslant 0$ ($\pi_0(E) = 0$ since X is path-connected), we have the isomorphism $(\tau_2)_* : \pi_k(E \times_G P) \cong \pi_k(P/G)$ for $k \geqslant 1$. Thus $(E \times_G P)$ and P/G are weakly homotopy equivalent.

---

[37] According to J.H.C. Whitehead's Theorem [197] (e.g., see also [97]), if X and Y are CW-complexes, then weakly homotopy equivalence implies homotopy equivalence.

Theorem 6.8.6 *Let G be a topological group and M a G-space. If $E \to B$ and $E' \to B'$ are principal G-bundles such that the total spaces E and E' are weakly homotopy contractible, then $E \times_G M$ and $E' \times_G M$ are weakly homotopy equivalent.*

Indeed, since $E \times M \to E \times_G M$ is a principal G-bundle, it follows from Theorem 6.8.6 that $E' \times_G (E \times M) = (E' \times (E \times G))/G$ and $E \times_G M$ are weakly homotopy equivalent. Replacing the roles of E and E', $E \times_G (E' \times M) = (E \times (E' \times G))/G$ and $E' \times_G M$ are weakly homotopy equivalent. Since $(E' \times (E \times G))/G$ and $(E \times (E' \times G))/G$ are homeomorphic, $E \times_G M$ and $E' \times_G M$ are weakly homotopy equivalent.

Now let us consider the singular homology H_* and cohomology H^* for a covariant and contravariant functor from $\mathscr{T}op$ to $\mathscr{A}b$. Since weakly homotopy equivalence (i.e., the isomorphism of homotopy groups) implies the isomorphism of singular homology and cohomology groups (e.g., see [97]), we have the following:

Corollary 6.8.7 *Let the situation be as in Theorem 6.8.6. Then $H_*(E \times_G M) \cong H_*(E' \times_G M)$ and $H^*(E \times_G M) \cong H^*(E' \times_G M)$.*

In [140] J. Milnor[38] constructed a universal principal G-bundle $\pi : EG \to BG$ such that *the total space EG is contractible*. Using this universal bundle, for a topological G-space X, we define

$$H_*^G(X) := H_*(EG \times_G X), \; H_G^*(X) := H^*(EG \times_G X)$$

which are called *the equivariant homology and cohomology* of a G-space X. $H_*^G(-)$ and $H_G^*(-)$ are respectively covariant and contravariant functors from $\mathscr{T}op^G$ to $\mathscr{A}b$, since $EG \times_G (-) : \mathscr{T}op^G \to \mathscr{T}op$ is covariant. EG is contractible, thus $EG \times X$ is homotopy equivalent to X, hence the orbit space is simply denoted by

$$X_G := EG \times_G X$$

called *the homotopy quotient of X by G*. We also see that $X_G = (EG \times X)/G \to EG/G = BG = pt_G$ is a fiber bundle[39] with fiber X. We note that $H_G^*(pt) = H^*(pt_G) = H^*(BG)$ is the cohomology group of the classifying space BG of a topological group, and for a constant map $\pi : X \to pt$ we have the above fiber bundle $\pi_G : X_G \to pt_G = BG$, which gives rise to the homomorphism $\pi_G^* : H^*(BG) \to H_G^*(X)$, thus the equivariant cohomology $H_G^*(-)$ is not only a graded algebra over $\mathbb{Z}$, but also a graded algebra over the graded algebra $H^*(BG)$ via this homomorphism.

---

[38] Milnor's construction is "functorial", i.e., any continuous homomorphism $f : G \to H$ induces a "natural" continuous map $Bf : BG \to BH$. Dold and Rashof [74] reformulated Milnor's construction for a topological monoid. R. Milgram [139] construction satisfies that there exists a "natural" homeomorphism $B(G \times H) \cong BG \times BH$.

[39] This bundle $X \hookrightarrow X_G \to BG$ is sometimes called the *Borel fibration*.

Remark 6.8.8 As in Sect. 6.2.2, the ring of characteristic classes of principal G-bundles with values in the singular cohomology $H^*(-;\mathfrak{R})$ is isomorphic to the cohomology ring $H^*(BG;\mathfrak{R})$ of the classifying space BG.

Remark 6.8.9

(i) As long as we restrict ourselves to the category of complex algebraic varieties, for example, if we consider Chow group or algebraic cycles, $\pi : EG \to BG$ cannot be used because EG and BG are infinite dimensional, thus not algebraic varieties.
(ii) In [191] B. Totaro defines a "classifying space BG", by considering an "approximation" of $EG \to BG$ by a directed system of G-bundles $E_n \to B_n$ of certain schemes E_n and B_n such that for any principal algebraic G-bundle $E \to X$ there is an affine-space bundle $g : X' \to X$ and the pullback bundle $E' = g^*E \to X'$ is obtained as the pullback of $E_n \to B_n$ by a map $X' \to B_n$. Then he defines *the Chow ring*[40] $A^*(BG)$ *of this "classifying space* BG", which is isomorphic to *the ring of characteristic classes of principal G-bundles over smooth algebraic varieties with values in the Chow ring (of the base variety)*, which is a "Chow ring version" of the topological one given in Remark 6.8.8 (cf. Sect. 6.2.2). Such a characteristic class is called an *algebraic* characteristic class.
(iii) Using Totaro's approximation, in [76] D. Edidin and W. Graham defined an equivariant Chow group $A^G_*(X)$ ($A^G_*(pt)$ is nothing but Totaro's $A^*(BG)$ above) and showed an equivariant version of Baum–Fulton–MacPherson's Riemann–Roch with Chow group. Similarly, in [54] J.-L. Brylinski and B. Zhang defined an equivariant Borel–Moore homology group and showed an equivariant version of BFM-RR with Borel–Moore homology. In [146] T. Ohmoto defined the equivariant covariant functor $F^G(-)$ of equivariant constructible functions and showed an equivariant version of MacPherson's Chern class transformation.

Remark 6.8.10 According to [8, Remark 1.8], the idea of approximating the infinite-dimensional spaces EG and BG by finite-dimensional ones can be found in the origins of equivariant cohomology [39, Remark XII.3.7].

[40] The Chow ring $A^i(X)$ is the operational Chow ring, i.e., $A^i(X) := A^i(X \xrightarrow{\mathrm{id}_X} X)$, where $A^*(X \xrightarrow{f} Y)$ is the operational bivariant theory [85, Definition 17.1] (also see Sect. 6.9.2) constructed from the covariant theory of Chow groups $A_*(X)$, i.e., the group of algebraic cylces modulo rational equivalence [85, §1.3]. Note that $A^{-p}(X \to pt) \cong A_p(X)$ ([85, Proposition 17.3.1]) and (the Poincaré duality) $A^p(X) = A^p(X \xrightarrow{\mathrm{id}_X} X) \cong A_{n-p}(X)$ for smooth X of dimension n ([85, Corollary 17.4]). In [191] the Chow ring $A^p(X)$ is defined as the Chow group $A_{n-p}(X)$ for smooth X.

6.8.5 Equivariant Motivic Hirzebruch Classes

We give a quick review of equivariant motivic Hirzebruch classes [62] of complex algebraic G-varieties for a finite group G.

Let $\mathcal{V}^G$ be the category of G-equivariant quasi-projective varieties. The relative Grothendieck group of $\mathcal{V}^G$ is a G-equivariant version of $K_0(\mathcal{V}/X)$ in Definition 6.5.17. Namely $K_0(\mathcal{V}^G/(X,G))$[41] is defined to be the free abelian group of isomorphism classes $[(Y,G) \xrightarrow{f} (X,G)]$ of G-equivariant maps, modulo the usual "scissor" relation:

$$[(Y,G) \xrightarrow{f} (X,G)] = [(Z,G) \xrightarrow{f|_Z} (X,G)] + [(Y \setminus Z, G) \xrightarrow{f|_{Y \setminus Z}} (X,G)]$$

for any G-invariant closed algebraic subspaces $Z \subset Y$. This is a covariant functor $K_0(\mathcal{V}^G/(-,G)) : \mathcal{V}^G \to \mathcal{A}b$ having the same functorial properties as the covariant functor $K_0(\mathcal{V}/-)$. For any subgroup $H < G$ we have the canonical homomorphism ("restricting" the action of G to the action of a subgroup H)

$$\mathfrak{res}^G_H : K_0(\mathcal{V}^G/(X,G)) \to K_0(\mathcal{V}^H/(X,H))$$

defined by $\mathfrak{res}^G_H([(Y,G) \xrightarrow{f} (X,G)]) := [(Y,H) \xrightarrow{f} (X,H)]$. Here we note that $\mathfrak{res}^G_H : \mathcal{V}^G \to \mathcal{V}^H$ defined by $\mathfrak{res}^G_H((X,G)) = (X,H)$ for the objects $ob(\mathcal{V}^G)$ and $\mathfrak{res}^G_H((Y,G) \xrightarrow{f} (X,G)) = (Y,H) \xrightarrow{f} (X,H)$ for the morphisms $mor(\mathcal{V}^G)$ is a covariant functor. In particular, for an element $g \in G$, consider the cyclic subgroup $< g > := \{g^k \mid k = 1, 2, \cdots n (g^n = 1)\} \subset G$. Then we have

$$\mathfrak{res}^G_{<g>} : K_0(\mathcal{V}^G/(X,G)) \to K_0(\mathcal{V}^{<g>}/(X, <g>)).$$

Using $\mathfrak{quot} : \mathcal{V}^G \to \mathcal{V}$ and the covariant functor $K_0(\mathcal{V}/-) : \mathcal{V} \to \mathcal{A}b$, we have

$$K_0^{\mathfrak{quot}}(\mathcal{V}/(-,G)) : \mathcal{V}^G \xrightarrow{\mathfrak{quot}} \mathcal{V} \xrightarrow{K_0(\mathcal{V}/-)} \mathcal{A}b$$

defined by $K_0^{\mathfrak{quot}}(\mathcal{V}^G/(X,G)) := K_0(\mathcal{V}/(X/G))$. Similarly, we have the naive "quotient" equivariant Borel–More homology:

$$H_*^{BM,\mathfrak{quot}} : \mathcal{V}^G \xrightarrow{\mathfrak{quot}} \mathcal{V} \xrightarrow{H_*^{BM}(-)} \mathcal{A}b$$

defined by $H_*^{BM,\mathfrak{quot}}((X,G) := H_*^{BM}(X/G)$. Then we have the natural transformation

$$T_{y*}^{\mathfrak{quot}} : K_0^{\mathfrak{quot}}(\mathcal{V}^G/(-,G)) \to H_*^{BM,\mathfrak{quot}}((-,G))$$

[41] In [62] it is written by $K_0^G(\mathcal{V}/X)$. In this paper we use the above notation for later presentation.

defined by $T_{y*}^{\mathfrak{quot}}([(Y,G) \xrightarrow{f} (X,G)]) := T_{y*}([Y/G \xrightarrow{\widetilde{f}} X/G])$, which is *the naive "quotient" equivariant motivic Hirzebruch class* (transformation) defined on the category $\mathcal{V}^G$.

As observed above, for an element $g \in G$ we have the homomorphism $\mathfrak{res}^G_{<g>} : K_0(\mathcal{V}^G/(X,G)) \to K_0(\mathcal{V}^{<g>}/(X,<g>))$. For the sake of simplicity, here we just write g for $<g>$ unless some confusion is possible. Then we consider the following sequence of homomorphisms (which are functorial):

$$K_0(\mathcal{V}^g/(X,g)) \xrightarrow{\chi^g_{Hodge}} K_0(MHM^g/(X,g)) \xrightarrow{MHC^g_y} K_0(Coh^g((X,g))) \otimes \mathbb{Z}[y^{\pm 1}],$$

where $K_0(Coh^g((X,g)))$ is the Grothendieck group of $<g>$-equivariant algebraic coherent sheaves on $(X,<g>)$, $\chi^g_{Hodge} = \chi^{<g>}_{Hodge}$ is the *equivariant Hodge transformation* and $MHC^g_y = MHC^{<g>}_y$ is the *equivariant motivic Chern class transformation*. Furthermore it follows from [34] that we have the *Lefschetz–Riemann–Roch transformation*

$$L_g : K_0(Coh^g((X,g))) \to K_0(X^g)$$

where $X^g = X^{<g>}$ (note that we need not consider the cyclic group $<g>$ to define $X^g = \{x \in X \mid gx = x\}$). Then, using the Baum–Fulton–MacPherson's Todd class transformation $td_* : K_0(X^g) \to H_*^{BM}(X^g) \otimes \mathbb{Q}$, combining these maps we get the homomorphism,

$$T_{y*}(g) : K_0(\mathcal{V}^G/(X,G)) \to H_*^{BM}(X^g) \otimes \mathbb{Q}[y^{\pm 1}],$$

which is called the *equivariant motivic Atiyah–Singer class* (for $g \in G$) [25].

Let $\pi_g : X^g \hookrightarrow X \to X/G$ for each element $g \in G$. Therefore we have

$$\sum_{g \in G} \pi^g_* T_{y*}(g) : K_0(\mathcal{V}^G/(X,G)) \to H_*^{BM}(X/G) \otimes \mathbb{Q}[y^{\pm}].$$

Then in [62] Cappell–Maxim–Schürmann–Shaneson obtain the following:

Theorem 6.8.11 *Let the set-up and the notations be as above. As natural transformations we have the following equality:*

$$T_{y*}^{\mathfrak{quot}} = \frac{1}{|G|} \sum_{g \in G} \pi^g_* T_{y*}(g) : K_0(\mathcal{V}^G/(-,G)) \to H_*^{BM,\mathfrak{quot}}((-,G)) \otimes \mathbb{Q}[y^{\pm 1}]$$

Remark 6.8.12 In [137] L. Maxim and J. Schürmann introduce what is called *the delocalized G-equivariant homology* as follows. The disjoint union $\bigsqcup_{g \in G} X^g$ admits an induced G-action by $h : X^g \to X^{hgh^{-1}}$ (defined by hx) so that the canonical map $i : \bigsqcup_{g\in G} X^g \to X$ defined by the inclusions becomes G-equivariant. Thus, G acts on $\bigoplus_{g\in G} H_*(X^g)$ by conjugation. Then their *delocalized G-equivariant homology* of X is defined to be the G-invariant subgroup of this conjugation action:

$$H_*^G(X) := \Big(\bigoplus_{g\in G} H_*(X^g)\Big)^G.$$

Remark 6.8.13 In [196] A. Weber defines an equivariant motivic Hirzebruch class $T^{\mathbb{T}}_{*y}(X \xrightarrow{f} M)$ for a complex torus $\mathbb{T} = (\mathbb{C}^*)^r$ and also discuss localization formulas. (Note that he defines an equivariant version of each motivic Hirzebruch class $T_{*y}(X \xrightarrow{f} M)$, but it is not defined functorially, i.e., he does not define an equivariant natural transformation $T^{\mathbb{T}}_{*y} : K_0(\mathscr{V}^{\mathbb{T}}/(-, \mathbb{T})) \to H^{\mathbb{T}}_*(-) \otimes \mathbb{Q}[y]$.)

6.8.5.1 Equivariant Motivic Chern Class mC_*^G

Let G be a complex linear algebraic group and M be a smooth quasi-projective G-variety. In Aluffi et al [7] and Fehér et al [79] (also see [80]) the authors show that there is a natural transformation

$$mC_*^G : K_0(\mathscr{V}^G/(M, G)) \to K_0((M, G)) \otimes \mathbb{Z}[y],$$

which is a G-equivariant version of the motivic Chern class (transformation) $mC_* : K_0(\mathscr{V}/-) \to K_0(-) \otimes \mathbb{Z}[y]$ in Theorem 6.5.21. Here we note that M is *smooth*,[42] thus the above natural transformation is considered for a G-equivariant proper map of *smooth* quasi-projective varieties $f : (M, G) \to (M', G)$, for which we have the following commutative diagram:

$$\begin{array}{ccc} K_0(\mathscr{V}^G/(M, G)) & \xrightarrow{mC_*^G} & K_0((M, G)) \otimes \mathbb{Z}[y] \\ \downarrow f_* & & \downarrow f_* \\ K_0(\mathscr{V}^G/(M', G)) & \xrightarrow[mC_*^G]{} & K_0((M', G)) \otimes \mathbb{Z}[y] \end{array}$$

For applications of this equivariant motivic Chern class, see [7, 79] and [80].

[42] It is not clear whether the condition of M being smooth could be dropped or not.

6.9 Bivariant Theories

Fulton and MacPherson have introduced Bivariant Theory [88] for the study of singular spaces. In particular one of their motivations[43] of introducing this theory is to unify three Riemann–Roch theorems (more accurately Grothendieck–Riemann–Roch type theorems), which are the following:

- "SGA 6" [37] (which is an extended version of the original Grothendieck–Riemann–Roch theorem [41] to the case of $\ell.c.i.$ morphism) (see (6.13) in Sect. 6.3.3) : For a proper and $\ell.c.i.$ morphism $f : X \to Y$ the following diagram commutes:

$$\begin{array}{ccc} K^0(X) & \xrightarrow{ch} & H^*(X)\otimes\mathbb{Q} \\ {\scriptstyle f_!}\downarrow & & \downarrow{\scriptstyle f_!\left(td(T_f)\cup -\right)} \\ K^0(Y) & \xrightarrow[ch]{} & H^*(Y)\otimes\mathbb{Q}. \end{array}$$

- "BFM-RR" [33] (see (6.15) in Sect. 6.4.2) : For a proper morphism $f : X \to Y$ the following diagram commutes:

$$\begin{array}{ccc} K_0(X) & \xrightarrow{td_*} & H_*(X)\otimes\mathbb{Q} \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ K_0(Y) & \xrightarrow[td_*]{} & H_*(Y)\otimes\mathbb{Q}. \end{array}$$

- "VRR" [194] (see (6.44) in Sect. 6.7.1): For a $\ell.c.i.$ morphism $f : X \to Y$ the following diagram commutes:

$$\begin{array}{ccc} K_0(Y) & \xrightarrow{td_*} & H_*(Y)\otimes\mathbb{Q} \\ {\scriptstyle f^!}\downarrow & & \downarrow{\scriptstyle td(T_f)\cap f^!} \\ K_0(X) & \xrightarrow[td_*]{} & H_*(X)\otimes\mathbb{Q}. \end{array}$$

These three theorems are unified by a Grothendieck transformation from a bivariant K-theory $\mathbb{K}_{\mathrm{alg}}$ to a bivariant homology theory $\mathbb{H}$ (see [88, Part II: Products in Riemann–Roch, §1 Statement of the theorem]):

$$\gamma : \mathbb{K}_{\mathrm{alg}}(X \to Y) \to \mathbb{H}(X \to Y)\otimes\mathbb{Q}. \tag{6.53}$$

[43] See a remark stated just before [88, 0.2 Summary of results, p. 123]: *One motivation of the present work was to unify these three Riemann–Roch theorems, and these various products and orientations.*

For the moment we just note that the bivariant homology theory $\mathbb{H}$ is a unification of the homology theory $\mathbb{H}^*(X \xrightarrow{a_X} pt) = H_{-*}(X)$ for a map $a_X : X \to pt$ to a point pt and the cohomology theory $\mathbb{H}^*(X \xrightarrow{\mathrm{id}_X} X) = H^*(X)$ for the identity map $\mathrm{id}_X : X \to X$. The bivariant product

$$\bullet : \mathbb{H}^i(X \xrightarrow{f} Y) \otimes \mathbb{H}^j(Y \xrightarrow{g} Z) \to \mathbb{H}^{i+j}(X \xrightarrow{g \circ f} Z)$$

is a generalization of the cup product $\cup$ and the cap product $\cap$:

(i) $\bullet : \mathbb{H}^i(X \xrightarrow{\mathrm{id}_X} X) \otimes \mathbb{H}^j(X \xrightarrow{\mathrm{id}_X} X) \to \mathbb{H}^{i+j}(X \xrightarrow{\mathrm{id}_X} X)$ is the cup product

$$\cup : H^i(X) \otimes H^j(X) \to H^{i+j}(X), \quad \alpha \bullet \beta = \alpha \cup \beta.$$

(ii) $\bullet : \mathbb{H}^i(X \xrightarrow{\mathrm{id}_X} X) \otimes \mathbb{H}^{-j}(X \to pt) \to \mathbb{H}^{i-j}(X \to pt)$ is the cap product

$$\cap : H^i(X) \otimes H_j(X) \to H_{i-j}(X), \quad \alpha \bullet \beta = \alpha \cap \beta.$$

Equation (6.53) is, a bit more precisely, the composite of two Grothendieck transformations:

$$\gamma = ch \circ \alpha : \mathbb{K}_{\mathrm{alg}}(X \to Y) \xrightarrow{\alpha} \mathbb{K}_{\mathrm{top}}(X \to Y) \xrightarrow{ch} \mathbb{H}(X \to Y) \otimes \mathbb{Q}, \tag{6.54}$$

where $\mathbb{K}_{\mathrm{top}}$ is the bivariant version of the topological K-theory and the Grothendieck transformation $ch : \mathbb{K}_{\mathrm{top}}(X \to Y) \to \mathbb{H}(X \to Y) \otimes \mathbb{Q}$ is constructed from the Chern character $ch : K^*(-) \to H^*(-) \otimes \mathbb{Q}$ (see [88, Part I: Bivariant Theories, §3.2 Grothendieck transformations of topological theories]). The above "SGA 6" and "VRR" follow from the following *Riemann–Roch formula*: for a $\ell.c.i.$ morphism $f : X \to Y$

$$\gamma(O_f) = td(T_f) \bullet U_f \tag{6.55}$$

(see [88, Part II:Products in Riemann–Roch, pp.133–137]) where $O_f = [\mathcal{O}_X] \in \mathbb{K}_{\mathrm{alg}}(X \xrightarrow{f} Y)$ and $U_f \in \mathbb{H}(X \xrightarrow{f} Y) \otimes \mathbb{Q}$ are canonical orientations and $td(T_f) \in \mathbb{H}(X \xrightarrow{\mathrm{id}_X} X) \otimes \mathbb{Q} = H^*(X) \otimes \mathbb{Q}$ is the total Todd class of the relative tangent bundle T_f. In this section, we give a quick survey on Fulton–MacPherson's bivariant theory and show how one gets the above three Riemann–Roch formulas from the *bivariant-theoretic* Riemann–Roch formula (6.58).

Remark 6.9.1 Speaking of "bivariant theory", there is another kind of bivariant theory introduced by G. Kasparov [108] i.e., *the bivariant K-theory*, or KK-theory, $KK(X, Y)$ (e.g., see [77]), and has been studied by many people working on C^*-algebra.

6.9.1 Fulton–MacPherson's Bivariant Theories

We make a quick review of Fulton–MacPherson's bivariant theory [88], since we refer to some axioms required on the theory in later sections.

Let $\mathscr{C}$ be a category which has a final object pt and on which the fiber product or fiber square is well-defined. Also we consider the following classes:

1. a class $\mathscr{C}$ of maps, called "confined maps" (e.g., proper maps, in algebraic geometry), which are *closed under composition and base change, and contain all the identity maps*, and
2. a class $\mathscr{I}nd$ of commutative diagrams, called "independent squares" (e.g., fiber square, "Tor-independent" square, in algebraic geometry), satisfying that

   (a) if the two inside squares in

$$\begin{array}{ccccc} X'' & \xrightarrow{h'} & X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y'' & \xrightarrow[h]{} & Y' & \xrightarrow[g]{} & Y \end{array} \qquad \text{or} \qquad \begin{array}{ccc} X' & \xrightarrow[h'']{} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[h']{} & Y \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Z' & \xrightarrow[h]{} & Z \end{array}$$

   are independent, then the outside square is also independent,

   (b) any square of the following forms are independent:

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[\mathrm{id}_X]{} & Y \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

   where $f : X \to Y$ is *any* morphism.

Remark 6.9.2 Given an independent square, its transpose is *not necessarily* independent. For example, let us consider the category of topological spaces and continuous maps. Let *any* map be confined, and we allow a fiber square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

to be *independent only if g is proper* (hence g' is also proper). Then its transpose is *not independent unless f is proper*. (Note that the pullback of a proper map by any continuous map is proper, because "proper" is equivalent to "universally closed", i.e., the pullback by any map is closed.)

Definition 6.9.3 A *bivariant theory* $\mathbb{B}$ on a category $\mathscr{C}$ with values in the category of graded abelian groups[44] is an assignment to each morphism $X \xrightarrow{f} Y$ in the category $\mathscr{C}$ a graded abelian group[45]

$$\mathbb{B}(X \xrightarrow{f} Y)$$

which is equipped with the following three basic operations. The i-th component of $\mathbb{B}(X \xrightarrow{f} Y)$, $i \in \mathbb{Z}$, is denoted by $\mathbb{B}^i(X \xrightarrow{f} Y)$.

(i) **Product**: For morphisms $f : X \to Y$ and $g : Y \to Z$, the product

$$\bullet : \mathbb{B}^i(X \xrightarrow{f} Y) \otimes \mathbb{B}^j(Y \xrightarrow{g} Z) \to \mathbb{B}^{i+j}(X \xrightarrow{gf} Z).$$

(ii) **Pushforward**: For morphisms $f : X \to Y$ and $g : Y \to Z$ with f *confined*, the pushforward $f_* : \mathbb{B}^i(X \xrightarrow{gf} Z) \to \mathbb{B}^i(Y \xrightarrow{g} Z)$.

(iii) **Pullback** : For an *independent* square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y, \end{array}$$

the pullback $g^* : \mathbb{B}^i(X \xrightarrow{f} Y) \to \mathbb{B}^i(X' \xrightarrow{f'} Y')$.

An element $\alpha \in \mathbb{B}(X \xrightarrow{f} Y)$ is sometimes expressed as follows:

$$X \xrightarrow[f]{\textcircled{\alpha}} Y$$

These three operations are required to satisfy the following seven compatibility axioms ([88, Part I, §2.2]):

---

[44] Instead of abelian groups, we consider also sets, e.g., such as the set of complex structures and the set of Spin structures (see [88, §4.3.2]), and categories, e.g., such as the derived (triangulated) category of f-perfect complexes (see [88, §7.1 Grothendieck duality]) as well.

[45] The grading is sometimes ignored.

(A_1) **Product is associative**: for

$$X \xrightarrow[f]{\textcircled{\alpha}} Y \xrightarrow[g]{\textcircled{\beta}} Z \xrightarrow[h]{\textcircled{\gamma}} Z$$

$$(\alpha \bullet \beta) \bullet \gamma = \alpha \bullet (\beta \bullet \gamma).$$

(A_2) **Pushforward is functorial**:for

$$\overset{\textcircled{\alpha}}{\overbrace{X \xrightarrow[f]{} Y \xrightarrow[g]{} Z \xrightarrow[h]{} W}}$$

with confined f, g,

$$(g \circ f)_* \alpha = g_*(f_* \alpha).$$

(A_3) **Pullback is functorial**: given independent squares

$$\begin{array}{ccccc} X'' & \xrightarrow{h'} & X' & \xrightarrow{g'} & X \\ \downarrow{f''} & & \downarrow{f'} & & \downarrow{f} \\ Y'' & \xrightarrow[h]{} & Y' & \xrightarrow[g]{} & Y \end{array}$$

$$(g \circ h)^* = h^* \circ g^*.$$

(A_{12}) **Product and pushforward commute**: $f_*(\alpha \bullet \beta) = f_* \alpha \bullet \beta$ for

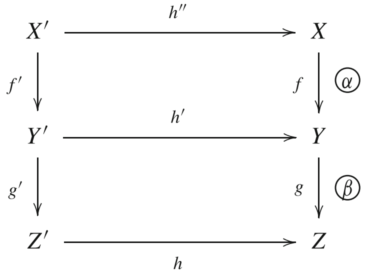
$$\overset{\textcircled{\alpha}}{\overbrace{X \xrightarrow[f]{} Y \xrightarrow[g]{} Z}} \xrightarrow[h]{\textcircled{\beta}} W$$

with confined f,

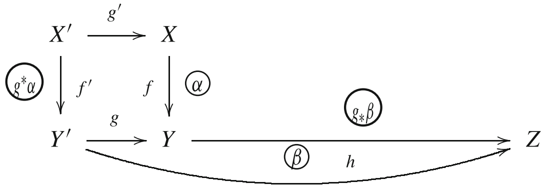
(A_{13}) **Product and pullback commute**: $h^*(\alpha \bullet \beta) = h'^* \alpha \bullet h^* \beta$ for independent squares

$$\begin{array}{ccc} X' & \xrightarrow{h''} & X \\ f' \downarrow & & f \downarrow \ \textcircled{\alpha} \\ Y' & \xrightarrow{h'} & Y \\ g' \downarrow & & g \downarrow \ \textcircled{\beta} \\ Z' & \xrightarrow[h]{} & Z \end{array}$$

(A_{23}) **Pushforward and pullback commute**: $f'_*(h^* \alpha) = h^*(f_* \alpha)$ for independent squares with f confined



(A_{123}) **Projection formula**: $g'_*(g^*\alpha \bullet \beta) = \alpha \bullet g_*\beta$ for an independent square with g confined



We also require the theory $\mathbb{B}$ to have multiplicative units:

(Units) For all $X \in \mathscr{C}$, there is an element $1_X \in \mathbb{B}^0(X \xrightarrow{\mathrm{id}_X} X)$ such that $\alpha \bullet 1_X = \alpha$ for all morphisms $W \to X$ and all $\alpha \in \mathbb{B}(W \to X)$, and such that $1_X \bullet \beta = \beta$ for all morphisms $X \to Y$ and all $\beta \in \mathbb{B}(X \to Y)$, and such that $g^* 1_X = 1_{X'}$ for all $g : X' \to X$.

A bivariant theory unifies both a covariant theory and a contravariant theory in the following sense:

Definition 6.9.4 For a bivariant theory $\mathbb{B}$, its associated covariant functors and contravariant functors are defined as follows:

(i) $\mathbb{B}_*(X) := \mathbb{B}(X \xrightarrow{a_X} pt)$ is covariant for confined morphisms and the grading is given by $\mathbb{B}_i(X) := \mathbb{B}^{-i}(X \xrightarrow{a_X} pt)$.
(ii) $\mathbb{B}^*(X) := \mathbb{B}(X \xrightarrow{\mathrm{id}_X} X)$ is contravariant for all morphisms and the grading is given by $\mathbb{B}^j(X) := \mathbb{B}^j(X \xrightarrow{\mathrm{id}_X} X)$.

A typical example of a bivariant theory is the bivariant homology theory $\mathbb{H}(X \xrightarrow{f} Y)$ constructed from the singular cohomology theory $H^*(-)$, which unifies the singular homology $H_*(X) := \mathbb{H}^{-*}(X \to pt)$ and the singular cohomology $H^*(X) := \mathbb{H}^*(X \xrightarrow{\mathrm{id}_X} X)$. Here the underlying category $\mathscr{C}$ is the category of spaces embeddable as closed subspaces of some Euclidean spaces $\mathbb{R}^n$ and continuous maps between them (see [88, §3 Topological Theories]). More generally, Fulton–MacPherson's (general) bivariant homology theory

$$h^*(X \to Y)$$

(here, using their notation) is constructed from *a multiplicative cohomology theory* $h^*(-)$ [88, §3.1] (see Definition 6.9.5 below). Here the cohomology theory h^* is either ordinary or generalized (sometimes, called extra-ordinary). A cohomology theory h^* is called *multiplicative* if for pairs (X, A), (Y, B) there is a graded pairing (exterior product)[46]

$$h^i(X, A) \times h^j(Y, B) \xrightarrow{\times} h^{i+j}(X \times Y, X \times B \sqcup A \times Y)$$

such that it is associative and graded commutative, i.e., $\alpha \times \beta = (-1)^{i+j}\beta \times \alpha$. A typical example of a multiplicative ordinary cohomology theory is the singular cohomology theory. The topological complex K-theory $K^0(-)$ and cobordism theory $\varOmega^*(-)$ are multiplicative generalized cohomology theories. We consider the category of spaces embeddable as closed subspaces in some Euclidian spaces $\mathbb{R}^N$ and continuous maps. For example, *Whitney's embedding theorem* says that any manifold of real dimension m can be embedded as a closed subspace of $\mathbb{R}^{2m}$. As remarked in a footnote in Sect. 6.4.2, we also note that a complex algebraic variety is embeddable as a closed subspace of some Euclidean space $\mathbb{R}^N$. We let a confined map be a proper map and an independent square be a fiber square.

Definition 6.9.5 For a continuous map $f : X \to Y$, choose a map $\phi : X \to \mathbb{R}^n$ such that $\varPhi = (f, \phi) : X \to Y \times \mathbb{R}^n$ defined by $\varPhi(x) := (f(x), \phi(x))$ is a closed embedding.[47] Then we define

$$h^*(X \to Y) := h^{i+n}(Y \times \mathbb{R}^n, Y \times \mathbb{R}^n \setminus \varPhi(X)). \tag{6.56}$$

Theorem 6.9.6 ([88, p.34–p.38]) *The above definition (6.56) is independent of the choice of the embedding ϕ, thus $\varPhi$, and $h^*(X \to Y)$ is a bivariant theory.*

Remark 6.9.7

(i) By the definition (6.56) we have $h^i(X \xrightarrow{\mathrm{id}_X} X) = h^i(X)$. Indeed, since $\mathrm{id}_X : X \to X$ is obviously a closed map (embedding), we can choose $\phi : X \to pt$ so that $\varPhi = (\mathrm{id}_X, \phi) : X \to X \times pt \cong X$ is a closed map. Hence by

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46 The cup product $\cup : H^i(X, A) \times H^j(X, B) \to H^{i+j}(X, A \cup B)$ is the composite $H^i(X, A) \times H^j(X, B) \xrightarrow{\times} H^{i+j}(X \times X, X \times B \sqcup A \times X) \xrightarrow{\Delta^*} H^{i+j}(X, A \cup B)$ where $\Delta : X \to X \times X$ is the diagomal map.

47 Such a map always exist, since our space is embeddable as a closed subspace of some $\mathbb{R}^N$, thus this embedding is considered as $\phi : X \to \mathbb{R}^N$, then $\varPhi = (f, \phi)$ is also a closed embedding.

the definition (6.56) we have $h^i(X \xrightarrow{\mathrm{id}_X} X) = h^i(X, (X \times pt) \setminus \Phi(X)) = h^i(X, \emptyset) = h^i(X)$.

(ii) $h^{-i}(X \xrightarrow{a_X} pt) = h^{n-i}(\mathbb{R}^n, \mathbb{R}^n \setminus \Phi(X)) =: h^{n-i}(\mathbb{R}^n, \mathbb{R}^n \setminus X)$ where $\Phi = (a_X, \phi) : X \to pt \times \mathbb{R}^n = \mathbb{R}^n$ is a closed embedding. If $h^* = H^*$ is the singular cohomology, then $h^{-i}(X \xrightarrow{a_X} pt) = h^{n-i}(\mathbb{R}^n, \mathbb{R}^n \setminus X) = H^{n-i}(\mathbb{R}^n, \mathbb{R}^n \setminus X) =: H_i^{BM}(X)$ is the Borel–More homology group (e.g., see [85], [154, B.1]).

Here is a very simple and easy example of a bivariant theory on the category of finite sets:

Example 6.9.8 (cf. [88, §6.1 The bivariant theory $\mathbb{F}$ *and §10.1.2 The Frobenius])* Let $\mathscr{F}$ be the category of finite sets, with all maps confined and all fiber squares independent. For a map $f : X \to Y$ we define $\mathbb{F}(X \xrightarrow{f} Y) := \mathbb{F}^0(X \xrightarrow{f} Y)$ to be the abelian group of $\mathbb{R}$-valued functions on the source set X. The product, pushforward and pullback are defined as follows:

(i) (product) $\bullet : \mathbb{F}(X \xrightarrow{f} Y) \otimes \mathbb{F}(Y \xrightarrow{g} Z) \to \mathbb{F}(X \xrightarrow{g \circ f} Z)$ is defined by $(\alpha \bullet \beta)(x) := \alpha(x) \times \beta(f(x))$ for $\alpha \in \mathbb{F}(X \xrightarrow{f} Y)$, $\beta \in \mathbb{F}(Y \xrightarrow{g} Z)$ and for $x \in X$.

(ii) (pushforward) For any map $f : X \to Y$ (note that any map is confined) the pushforward $f_* : \mathbb{F}(X \xrightarrow{g \circ f} Z) \to \mathbb{F}(Y \xrightarrow{g} Z)$ is defined by $f_*(\alpha)(y) := \sum_{x \in f^{-1}(y)} \alpha(x)$ for $y \in Y$.

(iii) (pullback) For a fiber square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y, \end{array}$$

the pullback $g^* : \mathbb{F}(X \xrightarrow{f} Y) \to \mathbb{F}(X' \xrightarrow{f'} Y')$ is defined by $(g^*\alpha)(x') := \alpha(g'(x'))$ (the usual functional pullback).

Definition 6.9.9 ([88, §2.7 Grothendieck transformation]) Let $\mathbb{B}, \mathbb{B}'$ be two bivariant theories on a category $\mathscr{C}$. A *Grothendieck transformation* from $\mathbb{B}$ to $\mathbb{B}'$, $\gamma : \mathbb{B} \to \mathbb{B}'$ is a collection of homomorphisms $\mathbb{B}(X \to Y) \to \mathbb{B}'(X \to Y)$ for a morphism $X \to Y$ in the category $\mathscr{C}$, which preserves the above three basic operations:

(i) $\gamma(\alpha \bullet_{\mathbb{B}} \beta) = \gamma(\alpha) \bullet_{\mathbb{B}'} \gamma(\beta)$,
(ii) $\gamma(f_*\alpha) = f_*\gamma(\alpha)$, and
(iii) $\gamma(g^*\alpha) = g^*\gamma(\alpha)$.

Remark 6.9.10 In [?, §2.7 Grothendieck transformations]a Grothendieck transformation is defined as follows. Let $\mathscr{C}$ and $\overline{\mathscr{C}}$ be categories and let $\ : \mathscr{C} \to \overline{\mathscr{C}}$ be a functor which takes confined maps in $\mathscr{C}$ to confined maps in $\overline{\mathscr{C}}$, and independent square in $\mathscr{C}$ to independent square in $\overline{\mathscr{C}}$, and the final object in $\mathscr{C}$ to the final object in $\overline{\mathscr{C}}$. We write $\overline{X}$ and $\overline{f}$ for the image in $\overline{\mathscr{C}}$ of an object X and a map f in $\mathscr{C}$. Let T be a bivariant theory on $\mathscr{C}$ and U be a bivariant theory on $\overline{\mathscr{C}}$. Then a Grothendieck transformation $t : T \to U$ is a collection of homomorphisms $t : T(X \xrightarrow{f} Y) \to U(\overline{X} \xrightarrow{\overline{f}} \overline{Y})$, one for each $f : X \to Y$ in $\mathscr{C}$, which commutes with product, pushforward and pullback. However, if we define $U(X \xrightarrow{f} Y) := U(\overline{X} \xrightarrow{\overline{f}} \overline{Y})$ then the bivariant theory U on $\overline{\mathscr{C}}$ can be considered as a bivariant theory on $\mathscr{C}$, thus a Grothendieck transformation can be defined as in Definition 6.9.9 as above.

A Grothendieck transformation $\gamma : \mathbb{B} \to \mathbb{B}'$ induces natural transformations $\gamma_* : \mathbb{B}_* \to \mathbb{B}_{*'}$ and $\gamma^* : \mathbb{B}^* \to \mathbb{B}'^*$.

Remark 6.9.11 Let $t : h^* \to \widetilde{h}^*$ be a natural transformation of two multiplicative cohomology theory. Then we get the associated Grothendieck transformation

$$t : h^*(X \xrightarrow{f} Y) \to \widetilde{h}^*(X \xrightarrow{f} Y) \tag{6.57}$$

since we have $t : h^{*+n}(Y \times \mathbb{R}^n, Y \times \mathbb{R}^n \setminus \Phi(X)) \to \widetilde{h}^*(Y \times \mathbb{R}^n, Y \times \mathbb{R}^n \setminus \Phi(X))$. For example, the Chern character $ch : K^0(-) \to H^*(-) \otimes \mathbb{Q}$ induces the Grothendieck transformation $ch : K^0(X \xrightarrow{f} Y) \to H^*(X \xrightarrow{f} Y) \otimes \mathbb{Q}$, which is $ch : \mathbb{K}_{\mathrm{top}}(X \to Y) \to \mathbb{H}(X \to Y) \otimes \mathbb{Q}$ in (6.54).

6.9.2 *Operational Bivariant Theory*

The above (general) bivariant homology theory $h^*(X \to Y)$ is constructed from a cohomology theory h^*, i.e., a contravariant theory. When it comes to a covariant theory, one can construct what is called *an operational bivariant theory* [88] (also, see [85]). Let h_* be a covariant functor (or sometimes called a homology theory). Then the *associated operational bivariant theory* $\mathbb{B}^{op}h_*(X \xrightarrow{f} Y)$ is defined as follows. For a map $f : X \to Y$, an element $c \in \mathbb{B}^{op}h_*(X \xrightarrow{f} Y)$ is defined to be a collection of homomorphisms

$$c(g) : h_*(Y') \to h_*(X')$$

for all $g : Y' \to Y$ and the fiber square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y. \end{array}$$

And these homomorphisms $c(g)$ are required to be compatible with proper pushforward, i.e., for a fiber diagram

$$\begin{array}{ccccc} X'' & \xrightarrow{k'} & X' & \xrightarrow{g'} & X \\ {\scriptstyle f''}\downarrow & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y'' & \xrightarrow[k]{} & Y' & \xrightarrow[g]{} & Y. \end{array}$$

the following diagram commutes

$$\begin{array}{ccc} h_*(Y'') & \xrightarrow{c(g\circ k)} & h_*(X'') \\ {\scriptstyle k_*}\downarrow & & \downarrow{\scriptstyle k'_*} \\ h_*(Y') & \xrightarrow[c(g)]{} & h_*(X'). \end{array}$$

If the category $\mathscr{C}$ has a final object pt and $h_*(pt)$ has a distinguished element 1, then the homomorphism $ev : \mathbb{B}^{op}h_*(X \to pt) \to h_*(X)$ defined by $ev(c) := (c(\mathrm{id}_{pt}))(1)$ is called the evaluation homomorphism. Let $\mathbb{B}$ be a bivariant theory. Then its *associated operational bivariant theory* $\mathbb{B}^{op}$ is defined to be the operational bivariant theory constructed from the covariant functor $\mathbb{B}_*(X) = \mathbb{B}(X \to pt)$. Then we have the following canonical Grothendieck transformation

$$op : \mathbb{B} \to \mathbb{B}^{op}$$

defined by, for each $\alpha \in \mathbb{B}(X \to Y)$,

$$op(\alpha) := \{(g^*\alpha)\bullet : \mathbb{B}_*(Y') \to \mathbb{B}_*(X') | g : Y' \to Y\}.$$

In this case it is not clear whether one could construct a Grothendieck transformation of the associated operational bivariant theories from a natural transformation of two covariant functors. To be more precise, if $t : h_*(-) \to \widetilde{h}_*(-)$ is a natural transformation of two covariant functors, then it is not clear whether one could construct a Grothendieck transformation $t : \mathbb{B}^{op}h_*(X \xrightarrow{f} Y) \to \mathbb{B}^{op}\widetilde{h}_*(X \xrightarrow{f} Y)$, which is an "operational bivariant theoretic analogue" of (6.57). A kind of similar problem is discussed in [88, §8.2]. Suppose that $\mathbb{B}$ is a bivariant theory, h_* is a covariant functor and there are homomorphisms $\phi(X) : \mathbb{B}_*(X) = \mathbb{B}(X \to pt) \to h_*(X)$, covariant for confined maps, and taking $1 \in \mathbb{B}^*(pt) = \mathbb{B}_*(pt)$ to $1 \in h_*(pt)$. Then a question is whether there exists a unique Grothendieck transformation $\Phi : \mathbb{B}(X \to Y) \to \mathbb{B}^{op}h_*(X \to Y)$ such that the associated map $\Phi(X) : \mathbb{B}_*(X) \to \mathbb{B}^{op}h_*(X \to Y)$ followed by the evaluation map $ev_X : \mathbb{B}^{op}h_*(X \to pt) \to h_*(X)$, i.e., $ev_X \circ \Phi(X)$, is equal to the given homomorphism $\phi(X) : \mathbb{B}_*(X) \to h_*(X)$. The answer to this question is negative, however the answer to a modified question is affirmative [213] (cf. [51]).

6.9.3 Canonical Orientation and Riemann–Roch Formula

Definition 6.9.12 ([88, Part I, §2.6.2 Definition]) Let $\mathcal{S}'$ be a class of maps in $\mathcal{C}$, which is closed under compositions[48] and contains all the identity maps. Suppose that to each $f : X \to Y$ in $\mathcal{S}'$ there is assigned an element $\theta(f) \in \mathbb{B}(X \xrightarrow{f} Y)$ satisfying

(i) $\theta(g \circ f) = \theta(f) \bullet \theta(g)$ for all $f : X \to Y, g : Y \to Z$ in $\mathcal{S}'$,
(ii) $\theta(\mathrm{id}_X) = 1_X$ for all X with $1_X \in \mathbb{B}^*(X) := \mathbb{B}^*(X \xrightarrow{\mathrm{id}_X} X)$ the unit element.

Then $\theta(f)$ is called an *canonical orientation* of f.

Example 6.9.13 A very simple example of a canonical orientation is $\theta(f) = \mathbb{1}_X$ the characteristic function on the source set X for any map $f : X \to Y$ in Example 6.9.8. Another simple example is $\theta(f) = [X \xrightarrow{\mathrm{id}_X} X] \in \mathbb{M}^{\mathcal{C}}_{\mathcal{S}}(X \xrightarrow{f} Y)$ for a "specialized" map $f : X \to Y$ in the universal bivariant theory $\mathbb{M}^{\mathcal{C}}_{\mathcal{S}}$ defined later in Theorem 6.9.17. For more non-trivial examples, see [88] (e.g., §4 Orientations in Topology).

A canonical orientation makes the covariant functor $\mathbb{B}_*(X)$ a contravariant functor for morphisms in $\mathcal{S}'$, and also makes the contravariant functor $\mathbb{B}^*$ a covariant functor for morphisms in $\mathcal{C} \cap \mathcal{S}'$, where we recall that $\mathcal{C}$ is a class of confined maps:

Proposition 6.9.14 (Gysin Homomorphisms) *Let $\mathbb{B}$ be a bivariant theory.*

(i) *As to the covariant functor $\mathbb{B}_*(X)$: For a morphism $f : X \to Y \in \mathcal{S}'$ and the canonical orientation θ on $\mathcal{S}'$, the Gysin (pullback) homomorphism*

$$f^! : \mathbb{B}_*(Y) \to \mathbb{B}_*(X) \quad \textit{defined by} \quad f^!(\alpha) := \theta(f) \bullet \alpha$$

is contravariantly functorial: $(g \circ f)^! = f^! \circ g^!$.

(ii) *As to contravariant functor $\mathbb{B}^*$: For an independent square*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_Y}, \\ X & \xrightarrow[f]{} & Y \end{array}$$

[48] In the case of confined maps, we require the stability of pullback, i.e., the pullback of a confined map is confined. For this class $\mathcal{S}'$ we do not require the stability of pullback. For example, in [88] the class of $\ell.c.i.$ morphisms is considered as such a class, and the pullback of an $\ell.c.i.$ morphism is not necessarily a $\ell.c.i.$ morphism.

where $f \in \mathscr{C} \cap \mathscr{S}'$, the Gysin (pushforward) homomorphism

$$f_! : \mathbb{B}^*(X) \to \mathbb{B}^*(Y) \quad \textit{defined by} \quad f_!(\alpha) := f_*(\alpha \bullet \theta(f))$$

is covariantly functorial: $(g \circ f)_! = g_! \circ f_!$.

We note that these Gysin maps $f^!$ and $f_!$ are denoted respectively by $f^!_\theta$ and $f^\theta_!$ in order to emphasize the canonical orientation θ.

Definition 6.9.15 (A Riemann–Roch Formula) Let $\mathbb{B}, \mathbb{B}'$ be two bivariant theories on a category $\mathscr{V}$ and let $\theta_{\mathbb{B}}, \theta_{\mathbb{B}'}$ be canonical orientations on $\mathbb{B}, \mathbb{B}'$ for a class $\mathscr{S}'$. Let $\gamma : \mathbb{B} \to \mathbb{B}'$ be a Grothendieck transformation. If there exists a bivariant element $u_f \in \mathbb{B}'(X \xrightarrow{\mathrm{id}_X} X)$ for a map $f : X \to Y \in \mathscr{S}'$ such that

$$\gamma(\theta_{\mathbb{B}}(f)) = u_f \bullet \theta_{\mathbb{B}'}(f), \qquad \begin{array}{ccc} X & \xrightarrow[\mathrm{id}_X]{\;\textcircled{u_f}\;} & X \\ \textcircled{A} \searrow & & \swarrow \textcircled{B} \\ & Y & \end{array} \tag{6.58}$$

where $A = \gamma(\theta_{\mathbb{B}}(f))$ and $B = \theta_{\mathbb{B}'}(f)$, then this formula is called *a Riemann–Roch formula* for the Grothendieck transformation $\gamma : \mathbb{B} \to \mathbb{B}'$ with respect to the orientations $\theta_{\mathbb{B}}$ and $\theta_{\mathbb{B}'}$.

In [88, Part I, §2.7 Grothendieck transformations] Fulton and MacPherson call the above formula (6.58) a Riemann–Roch formula, because a Grothendieck transformation $\gamma : \mathbb{B} \to \mathbb{B}'$ together with a formula (6.58) gives rise to the three Riemann–Roch formulas, as follows:

(i) "SGA 6"-type formula (see (6.13) in Sect. 6.3.3): for a map $f : X \to Y \in \mathscr{C} \cap S'$, the following diagram commutes:

$$\begin{array}{ccc} \mathbb{B}^*(X) & \xrightarrow{\;\gamma\;} & \mathbb{B}'^*(X) \\ {\scriptstyle f^\theta_!}\downarrow & & \downarrow{\scriptstyle f^{\theta'}_!(-\bullet u_f)} \\ \mathbb{B}^*(Y) & \xrightarrow[\gamma]{} & \mathbb{B}'^*(Y). \end{array}$$

Indeed,

$$\begin{aligned} \gamma(f^\theta_!\alpha) &= \gamma(f_*(\alpha \bullet \theta_{\mathbb{B}}(f))) = f_*\gamma(\alpha \bullet \theta_{\mathbb{B}}(f)) = f_*\Big(\gamma(\alpha) \bullet \gamma(\theta_{\mathbb{B}}(f))\Big) \\ &= f_*\Big(\gamma(\alpha) \bullet \Big(u_f \bullet \theta_{\mathbb{B}'}(f)\Big)\Big) \\ &= f_*\Big(\Big(\gamma(\alpha) \bullet u_f\Big) \bullet \theta_{\mathbb{B}'}(f)\Big) \\ &= f^{\theta'}_!(\gamma(\alpha) \bullet u_f). \end{aligned}$$

(ii) "BFM-RR" type formula (see (6.15) in Sect. 6.4.2) : for a proper map $f : X \to Y$, the following diagram commutes:

$$\begin{array}{ccc} \mathbb{B}_*(X) & \xrightarrow{\gamma} & \mathbb{B}'_*(X) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathbb{B}_*(Y) & \xrightarrow[\gamma]{} & \mathbb{B}'_*(Y), \end{array}$$

(iii) "VRR"-type formula (see (6.44) in Sect. 6.7.1): for a map $f : X \to Y \in \mathcal{S}'$, the following diagram commutes:

$$\begin{array}{ccc} \mathbb{B}_*(Y) & \xrightarrow{\gamma} & \mathbb{B}'_*(Y) \\ {\scriptstyle f^!_\theta}\downarrow & & \downarrow{\scriptstyle u_f \bullet f^!_{\theta'}} \\ \mathbb{B}_*(X) & \xrightarrow[\gamma]{} & \mathbb{B}'_*(X), \end{array}$$

$$\begin{aligned} \gamma(f^!_\theta \alpha) = \gamma(\theta_{\mathbb{B}}(f) \bullet \alpha) = \gamma(\theta_{\mathbb{B}}(f)) \bullet \gamma(\alpha) &= \Big(u_f \bullet \theta_{\mathbb{B}'}(f)\Big) \bullet \gamma(\alpha) \\ &= u_f \bullet \Big(\theta_{\mathbb{B}'}(f) \bullet \gamma(\alpha)\Big) \\ &= u_f \bullet f^!_{\theta'}(\gamma(\alpha)). \end{aligned}$$

By the same way as above, one can see that (6.58) implies (1) SGA 6, (2) BFM-RR and (3) VRR.

6.9.4 A Universal Bivariant Theory

In a category $\mathcal{C}$, like a class $\mathcal{C}$ of confined maps, we let $\mathcal{S}$ be another class of maps called "specialized maps" (e.g., smooth maps in the category of complex algebraic varieties), which is closed under composition and base change,[49] and contains all the identity maps.

Definition 6.9.16 Let $\mathbb{B}$ be a bivariant theory on a category $\mathcal{C}$ and let $\mathcal{S}$ be as above. If a canonical orientation θ is defined on $\mathbb{B}$ for the class $\mathcal{S}$ and it satisfies that for an independent square with $f \in \mathcal{S}$, hence $f' \in \mathcal{S}$

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

[49] Here we note that on the class $\mathcal{S}'$ on which a canonical orientation θ is defined in Definition 6.9.12 "closed under base change" is not required, but only "closed under composition" is required.

$\theta(f') = g^*\theta(f)$, i.e., the orientation θ preserves the pullback operation, then we call θ a *nice canonical orientation* of $\mathbb{B}$.

From now on we assume that our category $\mathscr{C}$ satisfies that any fiber square

$$\begin{array}{ccc} P' & \longrightarrow & P \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Q' & \longrightarrow & Q \end{array}$$

with f being confined, i.e., $f \in \mathscr{C}$, is an independent square. In [215] this condition is called *"$\mathscr{C}$-independence"*.

Theorem 6.9.17 (A Universal Bivariant Theory [215, Theorem 3.1]) *Let $\mathscr{C}$ be a category with a class $\mathscr{C}$ of confined morphisms, a class $\mathscr{Ind}$ of independent squares and a class $\mathscr{S}$ of specialized maps. We define $\mathbb{M}^{\mathscr{C}}_{\mathscr{S}}(X \xrightarrow{f} Y)$ to be the free abelian group generated by the set of isomorphism classes of confined morphisms $h : W \to X$ such that the composite of h and f is a specialized map: $h \in \mathscr{C}$ and $f \circ h : W \to Y$ in $\mathscr{S}$.*

(i) *The association $\mathbb{M}^{\mathscr{C}}_{\mathscr{S}}$ is a bivariant theory if the three bivariant operations are defined as follows:*

(i) **Product***: For morphisms $f : X \to Y$ and $g : Y \to Z$, the product*

$$\bullet : \mathbb{M}^{\mathscr{C}}_{\mathscr{S}}(X \xrightarrow{f} Y) \otimes \mathbb{M}^{\mathscr{C}}_{\mathscr{S}}(Y \xrightarrow{g} Z) \to \mathbb{M}^{\mathscr{C}}_{\mathscr{S}}(X \xrightarrow{gf} Z)$$

is defined by $[V \xrightarrow{h} X] \bullet [W \xrightarrow{k} Y] := [V' \xrightarrow{h \circ k''} X]$ and extended linearly, where we consider the following fiber squares

$$\begin{array}{ccccccc} V' & \xrightarrow{h'} & X' & \xrightarrow{f'} & W & & \\ {\scriptstyle k''}\downarrow & & {\scriptstyle k'}\downarrow & & {\scriptstyle k}\downarrow & & \\ V & \xrightarrow[h]{} & X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z. \end{array}$$

(ii) **Pushforward***: For morphisms $f : X \to Y$ and $g : Y \to Z$ with f confined, the pushforward*

$$f_* : \mathbb{M}^{\mathscr{C}}_{\mathscr{S}}(X \xrightarrow{gf} Z) \to \mathbb{M}^{\mathscr{C}}_{\mathscr{S}}(Y \xrightarrow{g} Z)$$

is defined by $f_*\left([V \xrightarrow{h} X]\right) := [V \xrightarrow{f \circ h} Y]$ and extended linearly.

(iii) **Pullback***: For an independent square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y, \end{array}$$

the pullback

$$g^* : \mathbb{M}^{\mathscr{C}}_{\mathscr{S}}(X \xrightarrow{f} Y) \to \mathbb{M}^{\mathscr{C}}_{\mathscr{S}}(X' \xrightarrow{f'} Y')$$

is defined by $g^*\left([V \xrightarrow{h} X]\right) := [V' \xrightarrow{h'} X']$ *and extended linearly, where we consider the following fiber squares:*

$$\begin{array}{ccc} V' & \xrightarrow{g''} & V \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle h} \\ X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y. \end{array}$$

(ii) *For a specialized morphism* $f : X \to Y$ *in* $\mathscr{S}$,

$$\theta_{\mathbb{M}^{\mathscr{C}}_{\mathscr{S}}}(f) = [X \xrightarrow{\mathrm{id}_X} X] \in \mathbb{M}^{\mathscr{C}}_{\mathscr{S}}(X \xrightarrow{f} Y)$$

is a nice canonical orientation of $\mathbb{M}^{\mathscr{C}}_{\mathscr{S}}$ *for* $\mathscr{S}$.

(iii) *(A universality of* $\mathbb{M}^{\mathscr{C}}_{\mathscr{S}}$*) Let* $\mathbb{B}$ *be a bivariant theory on the same category* $\mathscr{C}$ *with the same class* $\mathscr{C}$ *of confined morphisms, the same class* $\mathscr{I}nd$ *of independent squares and the same class* $\mathscr{S}$ *of specialized maps, and let* θ *be a nice canonical orientation of* $\mathbb{B}$ *for* $\mathscr{S}$. *Then there exists a unique Grothendieck transformation* $\gamma_{\mathbb{B}} : \mathbb{M}^{\mathscr{C}}_{\mathscr{S}} \to \mathbb{B}$ *such that for a specialized morphism* $f : X \to Y$,

$$\gamma_{\mathbb{B}} : \mathbb{M}^{\mathscr{C}}_{\mathscr{S}}(X \xrightarrow{f} Y) \to \mathbb{B}(X \xrightarrow{f} Y)$$

satisfies the normalization condition that $\gamma_{\mathbb{B}}(\theta_{\mathbb{M}^{\mathscr{C}}_{\mathscr{S}}}(f)) = \theta_{\mathbb{B}}(f)$.

Proposition 6.9.18 (Commutativity[50]) *The universal bivariant theory* $\mathbb{M}^{\mathscr{C}}_{\mathscr{S}}$ *is commutative in the sense that* $g^*(\alpha) \bullet \beta = f^*(\beta) \bullet \alpha$ *for a fiber square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & \textcircled{\scriptsize$\beta$} & \downarrow{\scriptstyle f}\ \textcircled{\scriptsize$\alpha$} \\ Z' & \xrightarrow[g]{} & Z \end{array}$$

[50] If $g^*(\alpha) \bullet \beta = (-1)^{\deg(\alpha)\deg(\beta)} f^*(\beta) \bullet \alpha$ holds, then it is called *skew-commutative* (see [88, Part I:Bivariant Theories, §2.2]).

Remark 6.9.19 In [215] (cf. [219]) the author introduced *an oriented bivariant theory* and *a universal oriented bivariant theory* for the purpose of constructing a bivariant-theoretic version $\Omega^*(X \xrightarrow{f} Y)$ of Levine–Morel's algebraic cobordism in such a way that the covariant part $\Omega^{-*}(X \to pt)$ becomes isomorphic to Levine–Morel's algebraic cobordism[51] $\Omega_*(X)$ [120] (also see [121]), thus $\Omega^*(X \xrightarrow{\mathrm{id}_X} X)$ would become a new contravariant cobordism. In [11] T. Annala has succeeded in constructing such a biavriant theory, in fact a stronger theory *entering into derived algebraic geometry*, what he calls *the bivariant derived algebraic cobordism* $\Omega^*(X \to Y)$, using the construction of Lowrey–Schürg's derived algebraic cobordism $d\Omega_*(X)$ [123] in *derived algebraic geometry* and the construction of the universal bivariant theory. Furthermore, in [16] (cf. [13]) Annala and the author constructed a bivariant version $\Omega^{*,*}(X \to Y)$ of Lee–Pandharipande's algebraic cobordism of bundles $\omega_{*,*}(X)$ [118]. For $\Omega^{*,*}(X \to Y)$, also see Annala's papers [12–15]. We remark that in [90] (cf. [91]) J. L. Gonzaléz and K. Karu have constructed *an operational bivariant algebraic cobordism*.

6.10 Bivariant Motivic Hirzebruch Characteristic Classes

6.10.1 A Bivariant Relative Grothendieck Group

The definition of a bivariant relative Grothendieck group $\mathbb{K}_0(\mathscr{V}/X \xrightarrow{f} Y)$ ([170, 218]) explained below is motivated by the construction of the above universal bivariant theory $\mathbb{M}^{\mathscr{C}}_{\mathscr{S}}(X \xrightarrow{f} Y)$.

Let $\mathscr{V}$ be the category of complex algebraic varieties and let $\mathscr{C}$ be the class of proper maps and $\mathscr{S}$ the class of smooth maps. Then we denote $\mathbb{M}^{\mathscr{C}}_{\mathscr{S}}(X \to Y)$ by $\mathbb{M}(\mathscr{V}/X \to Y)$. Also, $\mathbb{M}(\mathscr{V}/X \to pt)$ and $\mathbb{M}(\mathscr{V}/X \xrightarrow{\mathrm{id}_X} X)$ are respectively denoted by $\mathbb{M}_*(\mathscr{V}/X)$ and $\mathbb{M}^*(\mathscr{V}/X)$.

Theorem 6.10.1 (Bittner [38]) *Let $K_0(\mathscr{V}/X)$ be the relative Grothendieck group of varieties over $X \in ob(\mathscr{V})$ with $\mathscr{V} = \mathscr{V}^{(qp)}$ the category of complex algebraic (quasi-projective) varieties.*[52] *Then $K_0(\mathscr{V}/X)$ is isomorphic to $\mathbb{M}_*(\mathscr{V}/X)$ modulo the "blow-up" relation*

$$[\emptyset \to X] = 0 \quad \textit{and} \quad [Bl_Y X' \to X] - [E \to X] = [X' \to X] - [Y \to X]\,, \tag{6.59}$$

[51] Their cobordims group is a bordism group, i.e., a covariant theory.

[52] This means that $\mathscr{V}$ is the category of complex algebraic varieties and $\mathscr{V}^{qp}$ is the category of complex quasi-projective varieties. In order to define a motivic bivariant Hirzebruch class in Sect. 6.10.2 later we need the category $\mathscr{V}^{qp}$ of quasi-projective varieties.

for any cartesian diagram (called the "blow-up diagram")

$$\begin{array}{ccccc} E & \xrightarrow{i'} & Bl_Y X' & & \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} & & \\ Y & \xrightarrow{i} & X' & \xrightarrow{f} & X\,, \end{array}$$

with i a closed embedding of smooth spaces and $f : X' \to X$ proper. Here $Bl_Y X' \to X'$ is the blow-up of X' along Y with exceptional divisor E. Note that all these spaces other than X are also smooth (and quasi-projective in case $X', Y \in ob(\mathscr{V}^{qp})$).

The kernel of the quotient map $q : \mathbb{M}_*(\mathscr{V}/X) \to K_0(\mathscr{V}/X)$ is the subgroup $BL(\mathscr{V}/X)$ generated by $[Bl_Y X' \to X] - [E \to X] - [X' \to X] + [Y \to X]$ for any blow-up diagram as above. In order to define a bivariant analogue of the subgroup $BL(\mathscr{V}/X)$, we observe the following result.

Lemma 6.10.2 *Let $h : X' \to X$ be a* smooth *morphism, with $i : S \to X'$ a closed embedding such that the composite $h \circ i : S \to X$ is also* smooth *morphism. Consider the cartesian diagram*

$$\begin{array}{ccccc} E & \xrightarrow{i'} & Bl_S X' & & \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} & & \\ S & \xrightarrow[i]{} & X' & \xrightarrow[h]{} & X\,, \end{array} \tag{6.60}$$

with $q : Bl_S X' \to X'$ the blow-up of X' along S and $q' : E \to S$ the exceptional divisor map. Then:

(i) *$h \circ q : Bl_S X' \to X$ and $h \circ q \circ i' : E \to X$ are also* smooth *morphisms, with $Bl_S X'$, E quasi-projective in case $X', S \in ob(\mathscr{V}^{qp})$.*
(ii) *This blow-up diagram* commutes with any base change *in X, i.e. the corresponding fiber-square induced by pullback along a morphism $\tilde{X} \to X$ is isomorphic to the corresponding blow-up diagram of $\tilde{S} \to \tilde{X}'$.*
(iii) *The closed embeddings i, i' are* regular *embeddings, and the projection map q as well as i, i' are of* finite Tor-dimension.

Definition 6.10.3 For a morphism $f : X \to Y$ in the category $\mathscr{V} = \mathscr{V}^{(qp)}$, we consider a blow-up diagram

$$\begin{array}{ccccccc} E & \xrightarrow{i'} & Bl_S X' & & & & \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} & & & & \\ S & \xrightarrow{i} & X' & \xrightarrow{h} & X & \xrightarrow{f} & Y\,, \end{array}$$

with h proper and i a closed embedding such that $f \circ h$ as well as $f \circ h \circ i$ are smooth.

Let $\mathbb{BL}(\mathscr{V}/X \xrightarrow{f} Y)$ be the free abelian subgroup of $\mathbb{M}(\mathscr{V}/X \xrightarrow{f} Y)$ generated by

$$[Bl_S X' \xrightarrow{hq} X] - [E \xrightarrow{hiq'} X] - [X' \xrightarrow{h} X] + [S \xrightarrow{hi} X] \tag{6.61}$$

for any such diagram, and define $\mathbb{K}_0(\mathscr{V}/X \xrightarrow{f} Y) := \dfrac{\mathbb{M}(\mathscr{V}/X \xrightarrow{f} Y)}{\mathbb{BL}(\mathscr{V}/X \xrightarrow{f} Y)}$. The corresponding equivalence class of $[V \xrightarrow{p} X]$ shall be denoted by $\Big[[V \xrightarrow{p} X]\Big]$.

Note that by Lemma 6.10.2 (1) $f \circ h \circ q$ and $f \circ h \circ i \circ q'$ are smooth (with $Bl_S X'$ and E quasi-projective in the case when $\mathscr{V} = \mathscr{V}^{qp}$), so that the "relative blow-up relation" (6.61) makes sense in $\mathbb{M}(\mathscr{V}/X \xrightarrow{f} Y)$.

Theorem 6.10.4 ([170, 218]) *Let $\mathscr{V} = \mathscr{V}^{(qp)}$ be as above. $\mathbb{K}_0(\mathscr{V}/X \xrightarrow{f} Y)$ becomes a bivariant theory with the following three operations, so that the canonical projection $\mathbb{B}q : \mathbb{M}(\mathscr{V}/-) \to \mathbb{K}_0(\mathscr{V}/-)$ is a Grothendieck transformation.*

(i) **Product operation**: *For morphisms $f : X \to Y$ and $g : Y \to Z$, the product $\star : \mathbb{K}_0(\mathscr{V}/X \xrightarrow{f} Y) \otimes \mathbb{K}_0(\mathscr{V}/Y \xrightarrow{g} Z) \to \mathbb{K}_0(\mathscr{V}/X \xrightarrow{gf} Z)$ is defined by $\Big[[V \xrightarrow{h} X]\Big] \star \Big[[W \xrightarrow{k} Y]\Big] := \Big[[V \xrightarrow{h} X] \bullet [W \xrightarrow{k} Y]\Big]$ and bilinearly extended.*

(ii) **Pushforward operation**: *For morphisms $f : X \to Y$ and $g : Y \to Z$ with $f \in \mathscr{P}rop$, the pushforward $f_* : \mathbb{K}_0(\mathscr{V}/X \xrightarrow{gf} Z) \to \mathbb{K}_0(\mathscr{V}/Y \xrightarrow{g} Z)$ is defined by $f_*\Big(\Big[[V \xrightarrow{p} X]\Big]\Big) := \Big[f_*([V \xrightarrow{p} X])\Big]$ and linearly extended.*

(iii) **Pullback operation**: *For an independent square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y, \end{array}$$

the pullback $g^* : \mathbb{K}_0(\mathscr{V}/X \xrightarrow{f} Y) \to \mathbb{K}_0(\mathscr{V}/X' \xrightarrow{f'} Y')$ is defined by $g^*\Big(\Big[[V \xrightarrow{p} X]\Big]\Big) := \Big[g^*([V \xrightarrow{p} X])\Big]$ and linearly extended.

Remark 6.10.5 When Y is a point, the blow-up diagram defining $\mathbb{BL}(\mathcal{V}/X \xrightarrow{f} pt)$ is nothing but the following:

$$\begin{array}{ccccc} E & \xrightarrow{i'} & Bl_S X' & & \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} & & \\ S & \xrightarrow{i} & X' & \xrightarrow{h} & X, \end{array}$$

such that $h : X' \to X$ is proper, X' and S are nonsingular, and $q : Bl_S X' \to X'$ is the blow-up of X' along S with $q' : E \to S$ the exceptional divisor map. Hence $\mathbb{BL}(\mathcal{V}/X \xrightarrow{f} pt)$ is nothing but $BL(\mathcal{V}/X)$, i.e., we have by Bittner's theorem that $\mathbb{K}_0(\mathcal{V}/X \to pt) \simeq K_0(\mathcal{V}/X)$. Finally note that we always have a group homomorphism $\mathbb{K}_0(\mathcal{V}/X \to pt) \to K_0(\mathcal{V}/X)$ since $Bl_S X' \setminus E \simeq X' \setminus S$ in the diagram above so that $[Bl_S X' \to X] - [E \to X] = [X' \to X] - [S \to X] \in K_0(\mathcal{V}/X)$.

Before closing this section, we remark that the above $\mathbb{BL}(\mathcal{V}/X \xrightarrow{f} Y)$ is *a bivariant ideal* of $\mathbb{M}(\mathcal{V}/X \xrightarrow{f} Y)$ in the following sense [16] and its proof is implicitly in the proof of that $\mathbb{K}_0(\mathcal{V}/X \xrightarrow{f} Y)$ is a bivariant theory [170, Theorem 4.4].

Definition 6.10.6 Let $\mathbb{B}$ be a bivariant theory. A *bivariant ideal* $\mathbb{I} \subset \mathbb{B}$ consists of (graded) subgroups $\mathbb{I}(X \xrightarrow{f} Y) \subset \mathbb{B}(X \xrightarrow{f} Y)$ for each $f : X \to Y$ such that

(i) if $\alpha \in \mathbb{I}(X \xrightarrow{g \circ f} Z)$, then $f_* \alpha \in \mathbb{I}(Y \xrightarrow{g} Z)$ for $f : X \to Y$ confined;
(ii) if $\alpha \in \mathbb{I}(X \xrightarrow{f} Y)$, then $g^* \alpha \in \mathbb{I}(X' \xrightarrow{f'} Y')$ for $g : Y' \to Y$ in an independent square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y, \end{array}$$

(iii) if $\alpha \in \mathbb{I}(X \xrightarrow{f} Y)$, then $\beta \bullet \alpha \in \mathbb{I}(X' \xrightarrow{f \circ h} Y)$ for any $\beta \in \mathbb{B}(X' \xrightarrow{h} X)$ and $\alpha \bullet \gamma \in \mathbb{I}(X \xrightarrow{g \circ f} Y')$ for any $\gamma \in \mathbb{B}(Y \xrightarrow{g} Y')$.

Bivariant ideals are clearly to bivariant theories what ideals are to rings.

Proposition 6.10.7

(i) *The (object-wise) kernel of a Grothendieck transformation $\gamma : \mathbb{B} \to \mathbb{B}'$ is a bivariant ideal.*

(ii) *Given a bivariant ideal* $\mathbb{I} \subset \mathbb{B}$*, the quotient* $\mathbb{B}/\mathbb{I}$ *defined by*

$$(\mathbb{B}/\mathbb{I})(X \to Y) := \mathbb{B}(X \to Y)/\mathbb{I}(X \to Y)$$

is a bivariant theory with the induced bivariant operations defined as follows:

(a) **Product***: For morphisms* $f : X \to Y$ *and* $g : Y \to Z$*, the product operation*

$$\bullet : (\mathbb{B}/\mathbb{I})^i(X \xrightarrow{f} Y) \otimes (\mathbb{B}/\mathbb{I})^j(Y \xrightarrow{g} Z) \to (\mathbb{B}/\mathbb{I})^{i+j}(X \xrightarrow{g\circ f} Z)$$

is defined by $[\alpha] \bullet [\beta] := [\alpha \bullet \beta]$.

(b) **Pushforward***: For morphisms* $f : X \to Y$ *and* $g : Y \to Z$ *with* f confined*, the pushforward operation*

$$f_* : (\mathbb{B}/\mathbb{I})^i(X \xrightarrow{g\circ f} Z) \to (\mathbb{B}/\mathbb{I})^i(Y \xrightarrow{g} Z)$$

is defined by $f_*([\alpha]) := [f_*\alpha]$.

(c) **Pullback** *: For an* independent *square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y, \end{array}$$

the pullback operation

$$g^* : (\mathbb{B}/\mathbb{I})^i(X \xrightarrow{f} Y) \to (\mathbb{B}/\mathbb{I})^i(X' \xrightarrow{f'} Y')$$

is defined by $g^*([\alpha]) := [g^*\alpha]$.

Remark 6.10.8

(i) The definitions of the above three bivariant operations for $\mathbb{B}/\mathbb{I}$ in Proposition 6.10.7 (2) should be denoted differently to avoid some possible confusion with those on the original one $\mathbb{B}$, e.g., the product $\bullet_{\mathbb{I}}$, the pushforward $[f_*]$ and the pullback $[g^*]$, but we use the same symbols for the sake of simplicity.

(ii) These definitions $[\alpha] \bullet [\beta] = [\alpha \bullet \beta]$, $f_*([\alpha]) = [f_*(\alpha)]$ and $g^*([\alpha]) = [g^*(\alpha)]$ mean in other words that the quotient map $\Theta : \mathbb{B} \to \mathbb{B}/\mathbb{I}$ defined by $\Theta(\alpha) := [\alpha]$ is a Grothendieck transformation, i.e., $\Theta(\alpha \bullet \beta) = \Theta(\alpha) \bullet \Theta(\beta)$, $\Theta(f_*(\alpha)) = f_*\Theta(\alpha)$ and $\Theta(g^*(\alpha)) = g^*\Theta(\alpha)$.

6.10.2 A Bivariant Motivic Hirzebruch Class

Theorem 6.10.9 (A Bivariant Motivic Hirzebruch Class) *([170, Theorem 5.1], [218, Theorem 5.14]) There exists a unique Grothendieck transformation*

$$\mathbb{B}T_y : \mathbb{K}_0(\mathscr{V}^{qp}/-) \to \mathbb{H}(-) \otimes \mathbb{Q}[y]$$

satisfying "smooth condition" that for a smooth morphism $f : X \to Y$

$$\mathbb{B}T_y\Big(\Big[[X \xrightarrow{\mathrm{id}_X} X]\Big]\Big) = T_y(T_f) \bullet U_f.$$

Here $\mathbb{B}T_y$ *is the homomorphism* $\mathbb{B}T_y : \mathbb{K}_0(\mathscr{V}^{qp}/X \xrightarrow{f} Y) \to \mathbb{H}(X \xrightarrow{f} Y) \otimes \mathbb{Q}[y]$, $T_y(T_f) \in \mathbb{H}(X \xrightarrow{\mathrm{id}_X} X) \otimes \mathbb{Q}[y] = H^*(X) \otimes \mathbb{Q}[y]$ *is the Hirzebruch class of the relative tangent bundle* T_f *of* f *and* $U_f \in \mathbb{H}(X \xrightarrow{f} Y)$ *is the canonical orientation of* f.

Remark 6.10.10 In order to define $\mathbb{B}T_y : \mathbb{K}_0(\mathscr{V}^{qp}/-) \to \mathbb{H}(-) \otimes \mathbb{Q}[y]$, we appeal to Fulton–MacPherson's Grothendieck transformation $\alpha : \mathbb{K}_{\mathrm{alg}}(-) \to \mathbb{K}_{\mathrm{top}}(-)$ between the bivariant algebraic K-theory and the bivariant topological K-theory, for which Fulton and MacPherson consider[53] the category of quasi-projective varieties (see [88, Part II, §1, §2, §3]).

Remark 6.10.11 For the map to a point $a_X : X \to pt$, the above homomorphism $\mathbb{B}T_y : \mathbb{K}_0(\mathscr{V}^{qp}/X \to pt) \to \mathbb{H}(X \to pt) \otimes \mathbb{Q}[y]$ is equal to the motivic Hirzebruch class $T_{y*} : K_0(\mathscr{V}^{qp}/X) \to H_*(X) \otimes \mathbb{Q}[y]$.

Remark 6.10.12 Let $\gamma^{\mathrm{Br}} : \mathbb{F}(-) \to \mathbb{H}(-)$ be Brasselet's bivariant Chern class [46], which is a bivariant-theoretic version of MacPherson–Chern class $c_* : F(-) \to H_*(-)$. $\gamma^{\mathrm{Br}} : \mathbb{F}(-) \to \mathbb{H}(-)$ is a Grothendieck transformation satisfying "weak smooth condition" that $\gamma^{\mathrm{Br}}(\mathbb{1}_X) = c(TX) \cap [X]$ for a smooth variety X, where $\mathbb{1}_X \in F(X) = F(X \to pt)$, $c(TX) \in H^*(X) = \mathbb{H}(X \xrightarrow{\mathrm{id}_X} X)$ and $[X] = [a_X] \in \mathbb{H}(X \to pt) = H_*(X)$. A conjecture ([170, 218]) is that Brasselet's bivariant Chern class satisfies "strong smooth condition" that $\gamma^{\mathrm{Br}}(\mathbb{1}_f) = c(T_f) \bullet U_f$ for a smooth map $f : X \to Y$, where $\mathbb{1}_f = \mathbb{1}_X \in \mathbb{F}(X \xrightarrow{f} Y)$. If this conjecture is correct, then the following diagram commutes:

$$\begin{array}{ccc}
\mathbb{K}_0(\mathscr{V}^{qp}/X \to Y) & \xrightarrow{\gamma_{\mathbb{F}}} & \mathbb{F}(X \to Y) \\
& {\scriptstyle \mathbb{B}T_{-1}} \searrow \quad \swarrow {\scriptstyle \gamma^{\mathrm{Br}}} & \\
& \mathbb{H}(X \to Y). &
\end{array}$$

[53] It remains to see whether the category $\mathscr{V}^{qp}$ of complex quasi-projective varieties could be changed to the category $\mathscr{V}$ of complex algebraic varieties.

Here $\gamma_{\mathbb{F}} : \mathbb{K}_0(\mathscr{V}^{qp}/X \to Y) \to \mathbb{F}(X \to Y)$ is defined by $\gamma_{\mathbb{F}}([V \xrightarrow{h} X]) := h_* \mathbb{1}_V$.

In the case when $y = 0$, we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{K}_0(\mathscr{V}^{qp}/X \to Y) & \xrightarrow{\gamma_{\mathbb{K}_{\mathrm{alg}}}} & \mathbb{K}_{\mathrm{alg}}(X \to Y) \\ & {\scriptstyle \mathbb{B}T_0} \searrow \quad \swarrow {\scriptstyle \gamma} & \\ & \mathbb{H}(X \to Y) \otimes \mathbb{Q}. & \end{array}$$

Here we note that the Grothendieck transformation $\gamma_{\mathbb{K}_{\mathrm{alg}}} : \mathbb{K}_0(\mathscr{V}^{qp}/X \to Y) \to \mathbb{K}_{\mathrm{alg}}(X \to Y)$ is nothing but $mC_0 :: \mathbb{K}_0(\mathscr{V}^{qp}/X \to Y) \to \mathbb{K}_{\mathrm{alg}}(X \to Y)$ in [170, Corollary 5.3].

In the case when $y = 1$, the problem is to construct a bivariant version $\mathbb{B}\Omega(X \to Y)$ of the cobordism group $\Omega(X)$ and a bivariant version $\mathbb{B}L_* : \mathbb{B}\Omega(X \to Y) \to \mathbb{H}(X \to Y) \otimes \mathbb{Q}$ of Cappell–Shaneson's L-class $L_* : \Omega(-) \to H_*(-) \otimes \mathbb{Q}$ in such a way that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{K}_0(\mathscr{V}^{qp}/X \to Y) & \xrightarrow{\gamma_{\mathbb{B}\Omega}} & \mathbb{B}\Omega(X \to Y) \\ & {\scriptstyle \mathbb{B}T_1} \searrow \quad \swarrow {\scriptstyle \mathbb{B}L_*} & \\ & \mathbb{H}(X \to Y) \otimes \mathbb{Q}. & \end{array}$$

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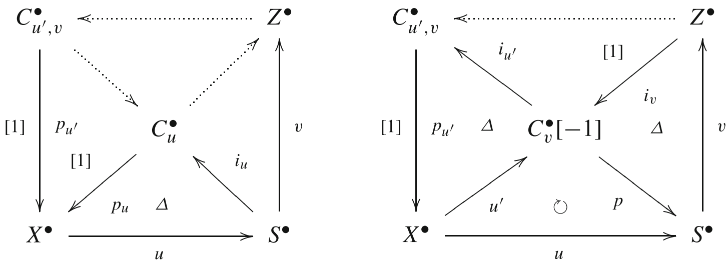
Appendices

Appendix 1

In Sect. 6.4.3.3, by (6.30) we define the "iterated" cone $C^\bullet_{u',v}$. In this appendix we discuss properties of this cone and show the self-duality (6.32). For that, furthermore, we consider a cone of $u : X^\bullet \to S^\bullet$:

$$X^\bullet \xrightarrow{u} S^\bullet \xrightarrow{i_u} C^\bullet_u \xrightarrow[p_u]{[1]} X^\bullet[1].$$

Thus, we have built up the following four triangles (*three distinguished triangles, marked Δ, and one commutative triangle, marked* $\circlearrowright$) of the octahedral diagram, where the left-hand-sided one is the upper half and the right-hand-sided one is the lower half:

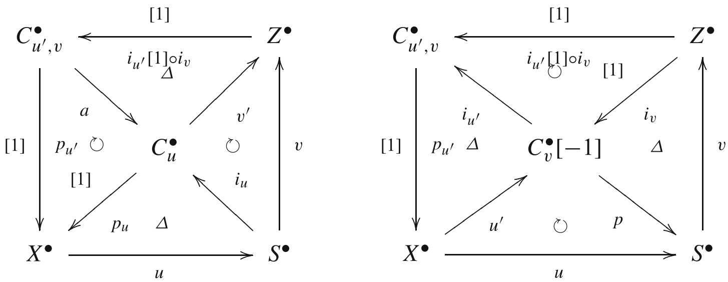


(A.1)

Then it follows from *the octahedral axiom* that there exists a distinguished triangle

$$C^\bullet_{u',v} \xrightarrow{a} C^\bullet_u \xrightarrow{v'} Z^\bullet \xrightarrow[i_{u'}[1]\circ i_v]{[1]} C^\bullet_{u',v}[1]. \tag{A.2}$$

which completes the above octahedral diagram ([A.1](#)) as follows:



(A.3)

where $a \circ i_{u'} = i_u \circ p$ and $u' \circ p_u = i_v \circ v'$.

Remark A.0.13 $v \circ u = 0$ if and only if $X^\bullet \overset{u}{\to} S^\bullet \overset{v}{\to} Z^\bullet$ is embedded into an octahedral diagram as in ([A.3](#)).

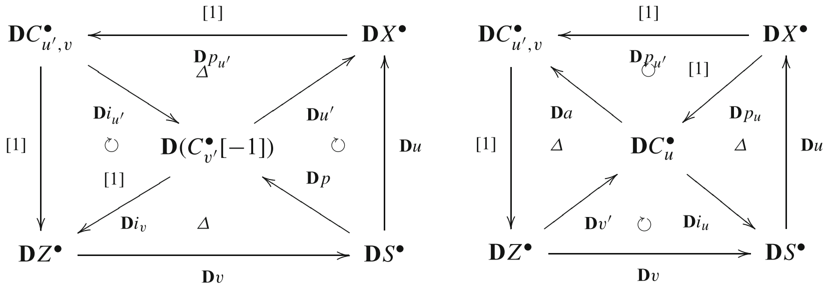
The distinguished triangle ([A.2](#)) gives rise to the distinguished triangle

$$C^\bullet_u \xrightarrow{v'} Z^\bullet \xrightarrow[i_{u'}[1]\circ i_v]{[1]} C^\bullet_{u',v}[1] \xrightarrow{-a[1]} C^\bullet_u[1], \tag{A.4}$$

which implies an isomorphism $C^\bullet_{u',v}[1] \cong C^\bullet_{v'}$, hence we get

$$C^\bullet_{u',v} \cong C^\bullet_{v'}[-1]. \tag{A.5}$$

Now, we get the following "dualizing version" of the octahedral diagram (A.3):



(A.6)

Note that the dualizing functor $\mathbf{D}$ sends a distinguished triangle to a distinguished triangle. The above octahedral diagram (A.6) is obtained by applying $\mathbf{D}$ to the octahedral diagram (A.3) with the arrows reversed, *rotating* each square by 180° around the axis connecting the upper-left and lower-right corners of the square and then exchanging the right and left squares.

From this "dualizing version" and (A.5) for the octahedral diagram (A.3), we get

$$C^\bullet_{\mathbf{D}v',\mathbf{D}u} \cong C^\bullet_{\mathbf{D}u'}[-1]. \tag{A.7}$$

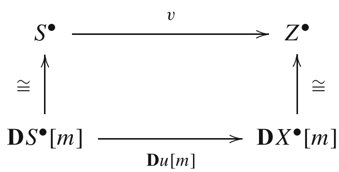
We also note that we get the following distinguished triangle from the distinguished triangle (6.30): $\mathbf{D}C^\bullet_v[1] \xrightarrow{\mathbf{D}u'} \mathbf{D}X^\bullet \longrightarrow \mathbf{D}C^\bullet_{u'}[1] \longrightarrow \mathbf{D}C^\bullet_v[2]$, from which we get the isomorphism

$$C^\bullet_{\mathbf{D}u'} \cong \mathbf{D}C^\bullet_{u'}[1], \text{ i.e., } C^\bullet_{\mathbf{D}u'}[-1] \cong \mathbf{D}C^\bullet_{u'}. \tag{A.8}$$

Since $C^\bullet_{u',v} := C^\bullet_{u'}$, it follows from (A.7) and (A.8) that we have

$$\mathbf{D}C^\bullet_{u',v} \cong C^\bullet_{\mathbf{D}v',\mathbf{D}u}. \tag{A.9}$$

Now we assume that we are given an isomorphism $\mathbf{D}X^\bullet[m] \cong Z^\bullet$ (where $m := 2\dim_{\mathbb{C}} X$) such that the following diagram commutes:



(A.10)

Here we note that this commutativity (A.10) implies the following commutativity:

$$\begin{array}{ccc} X^\bullet & \xrightarrow{\ u\ } & S^\bullet \\ \cong \uparrow & & \uparrow \cong \\ \mathbf{D}X^\bullet[m] & \xrightarrow[\mathbf{D}v[m]]{} & \mathbf{D}S^\bullet[m] \end{array} \tag{A.11}$$

Then under this assumption we claim that the above (A.9) implies the following self-duality of the above "iterated" cone $C^\bullet_{u',v}$:

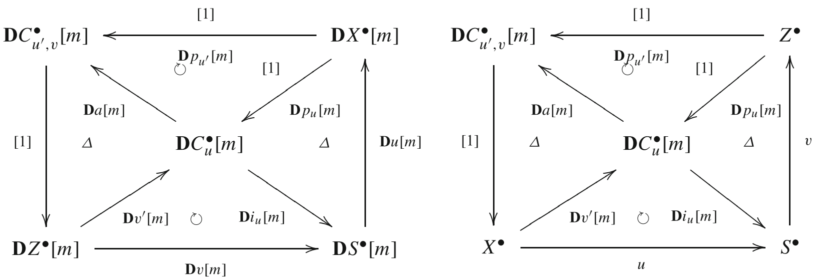
$$C^\bullet_{u',v} \cong \mathbf{D}C^\bullet_{u',v}[2\dim_{\mathbb{C}} X]. \tag{A.12}$$

In order to claim (A.12), we *assume the vanishing condition* $Hom(X^\bullet, Z^\bullet[-1]) = 0$, thus the lifting u' is unique as remarked above. We note also the uniqueness of the lifting v', because we get the following exact sequence by applying the cohomological functor $Hom(-, Z^\bullet)$ to the distinguished triangle $X^\bullet \xrightarrow{u} S^\bullet \xrightarrow{i_u} C^\bullet_u \xrightarrow{[1]} X^\bullet[1]$:

$$0 = Hom(X^\bullet[1], Z^\bullet) \to Hom(C^\bullet_u, Z^\bullet) \xrightarrow{i^*_u} Hom(S^\bullet, Z^\bullet) \xrightarrow{u^*} Hom(X^\bullet, Z^\bullet).$$

Here $Hom(X^\bullet[1], Z^\bullet) \cong Hom(X^\bullet, Z^\bullet[-1]) = 0$ and $u^* i^*_u(v') = u^*(v' \circ i_u) = u^*(v) = v \circ u = 0$.

Indeed, applying the shifting $[m]$ to the right-hand-sided square of (A.6), we get the left square below, thus $\mathbf{D}C^\bullet_{u',v}[m] \cong C^\bullet_{\mathbf{D}v',\mathbf{D}u}[m] \cong C^\bullet_{\mathbf{D}v'[m],\mathbf{D}u[m]}$:



Then, using the above commutativities (A.10) and (A.11), we get the right square above. Since the right-hand-sided triangle is distinguished, $\mathbf{D}C^\bullet_u[m] \cong C^\bullet_v[-1]$ and $\mathbf{D}i_u[m] = p$. Since the map $\mathbf{D}v'[m]$ is the unique lifting of u to $\mathbf{D}C^\bullet_u[m] \cong C^\bullet_v[-1]$, i.e., $\mathbf{D}v'[m]$ is the lifting u', thus the right square is isomorphic to the right square of (A.1). Thus we can claim that $\mathbf{D}C^\bullet_{u',v}[m] \cong C^\bullet_{u',v}$.

Then we have the following definition due to Cappell–Shaneson [58, §2] (also see [29, Definition 8.1.11]), which is Definition 6.4.22, but is repeated here for the sake of convenience:

Definition A.0.14 Let $X^\bullet \xrightarrow{u} S^\bullet \xrightarrow{v} Z^\bullet$ be morphisms in the derived category with $v \circ u = 0$ and assume $Hom(X^\bullet, Z^\bullet[-1]) = 0$. If there exists an isomorphism $Z^\bullet \cong \mathbf{D}(X^\bullet)[2\dim_{\mathbb{C}} X]$ such that the following diagram commute

$$\begin{array}{ccc} S^\bullet & \xrightarrow{\quad v \quad} & Z^\bullet \\ \cong \downarrow & & \downarrow \cong \\ \mathbf{D}(S^\bullet)[2\dim_{\mathbb{C}} X] & \xrightarrow[\mathbf{D}(u)[2\dim_{\mathbb{C}} X]]{} & \mathbf{D}(X^\bullet)[2\dim_{\mathbb{C}} X] \end{array} \tag{A.13}$$

then the "iterated" cone $S_1^\bullet := C^\bullet_{u',v}$ is also self-dual. Then we say that $S_1^\bullet$ is obtained from $S^\bullet$ *by an elementary cobordism* or $S_1^\bullet$ is *elementarily cobordant* to $S^\bullet$.

Here we emphasize that in the above definition the condition $Hom(X^\bullet, Z^\bullet[-1]) = 0$, i.e., u' being the unique lifting of u, is crucial, otherwise in the discussion above we cannot necessarily claim that the map $\mathbf{D}v'[m]$ is the lifting u', thus the self-duality (A.12) holds. Without assuming[1] this vanishing condition and also by specifying what kind of isomorphism of self-duality, the above definition of elementary cobordism is improved by defining it using *Youssin's self-dual octahedral diagrams* from the beginning so that the self-duality (A.12) holds, as done in [52, 221]. Self-dual octahedral diagrams are classified into two types; *symmetric* or *even* type and *skew-symmetric* or *odd* type . For details, see [52, 221].

Appendix 2

For this appendix, e.g., see [142]. We take a closer look at $L(E) = \prod_{i=1}^{\operatorname{rank} E} \frac{\alpha_i}{\tanh \alpha_i}$. Let E be a complex vector bundle of rank n. Then we have the following

$$L(E) = \prod_{i=1}^{n} \frac{\alpha_i}{\tanh \alpha_i} = \prod_{i=1}^{n} \left(1 + \sum_{k=1}^{\infty} B_{2k} \frac{4^k}{(2k)!} (\alpha_i^2)^k\right) \tag{A.14}$$

[1] Instead of assuming the vanishing condition $Hom(X^\bullet, Z^\bullet[-1]) = 0$ in Definition A.0.14, we could modify Definition A.0.14 just by requiring the self-duality (A.12), i.e., if (A.13) is commutative and the self-duality (A.12) holds, then we say that $S_1^\bullet$ is obtained from $S^\bullet$ by an *almost-elementary cobordism*. Since $Hom(X^\bullet, Z^\bullet[-1]) = 0$ is not assumed in Youssin's self-dual octahedral diagram, Youssin's elementary cobordism is an almost-elementary cobordism.

where B_{2k} are the Bernoulli numbers: $B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}, B_{12} = -\frac{691}{2730}, \cdots$. This is due to the following formula

$$\frac{x}{1-e^{-x}} = 1 + \frac{x}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{x^{2k}}{(2k)!}.$$

Hence $L(E)$ is of course a symmetric function (or power series) of $\alpha_1, \alpha_2, \cdots, \alpha_n$, thus it is expressed in terms of Chern classes of E. Furthermore it is a symmetric function of $\alpha_1^2, \alpha_2^2, \cdots, \alpha_n^2$, thus it is expressed in terms of the elementary symmetric polynomials of $\alpha_1^2, \alpha_2^2, \cdots, \alpha_n^2$, i.e.,

- $\sigma_1(\alpha_1^2, \alpha_2^2, \cdots, \alpha_n^2) = \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2$,
- $\sigma_2(\alpha_1^2, \alpha_2^2, \cdots, \alpha_n^2) = \sum_{1 \leqslant j < k \leqslant n} \alpha_j^2 \alpha_k^2 = \alpha_1^2\alpha_2^2 + \alpha_1^2\alpha_2^2 + \cdots + \alpha_{n-1}^2\alpha_n^2$,
- ......
- $\sigma_n(\alpha_1^2, \alpha_2^2, \cdots, \alpha_n^2) = \alpha_1^2\alpha_2^2 \cdots \alpha_n^2$.

These elementary symmetric polynomials are the terms of the expansion of

$$\prod_{i=1}^{n}(1 + \alpha_i^2).$$

This can be explained a bit more geometrically as follows. If we consider the conjugate $\overline{E}$, the the Chern polynomial of $E \oplus \overline{E}$ is

$$c_t(E \oplus \overline{E}) = c_t(E)c_t(\overline{E}) = \prod_{i=1}^{n}(1 + \alpha_i t) \prod_{i=1}^{n}(1 - \alpha_i t) = \prod_{i=1}^{n}(1 - \alpha_i^2 t^2)$$

which follows from the fact that $c_j(\overline{E}) = (-1)^j c_j(E)$. Then $c_{2j+1}(E \oplus \overline{E}) = 0$ and $c_{2j}(E \oplus \overline{E}) = (-1)^j \sigma_j(\alpha_1^2, \alpha_2^2, \cdots, \alpha_n^2)$, in other words

$$\sigma_j(\alpha_1^2, \alpha_2^2, \cdots, \alpha_n^2) = (-1)^j c_{2j}(E \oplus \overline{E}).$$

It turns out that *this is nothing but the j-th Pontryagin class* $p_j(E)$ *of the complex vector bundle E*, which is defined by

$$\begin{aligned} p_j(E) &:= p_j(E_{\mathbb{R}}) && (E_{\mathbb{R}} \text{ is the underlying real vector bundle of } E) \\ &:= (-1)^j c_{2j}(E_{\mathbb{R}} \otimes \mathbb{C}) && (\text{by the definition of Pontryagin class}) \\ &= (-1)^j c_{2j}(E \oplus \overline{E}) && (\text{since } E_{\mathbb{R}} \otimes \mathbb{C} \cong E \oplus \overline{E}) \end{aligned}$$

Therefore the above class $L(E)$ is a symmetric polynomial of Pontryagin classes $p_j(E)$, thus it is usually expressed as

$$L(E) = L\left(p_1(E), p_2(E), \cdots, p_n(E)\right). \tag{A.15}$$

Here are some lower terms:letting $p_j := p_j(E)$,

$$L(E) = 1 + \frac{1}{3}p_1 + \frac{1}{45}(7p_2 - p_1^2) + \frac{1}{945}(62p_3 - 13p_1p_2 + 2p_1^3) + \frac{1}{14175}(381p_4 - 71p_3p_1 - 19p_2^2 + 12p_2p_1^2 - 3p_1^4) + \cdots. \tag{A.16}$$

For more higher terms, e.g., see [160, §4.3 The L-genus]. The equality $p_j(E) = (-1)^j c_{2j}(E \oplus \overline{E})$, i.e., $(-1)^j p_j(E) = c_{2j}(E \oplus \overline{E})$ implies that

$$1 - p_1 + p_2 + \cdots + (-1)^n p_n = (1 + c_1 + \cdots + c_n) \times \left(1 - c_1 + c_2 + \cdots + (-1)^n c_n\right) \tag{A.17}$$

where $p_i := p_i(E)$ and $c_i := c_i(E)$. From (A.17) we get

$$p_j = c_j^2 + \sum_{k=1}^{j}(-1)^k 2c_{j-k}c_{j+k} \tag{A.18}$$

Here we note that $c_0 = 1$. Some first lower terms of (A.18) are the following:

- $p_1 = c_1^2 - 2c_2$,
- $p_2 = c_2^2 - 2c_1c_3 + 2c_4$,
- $p_3 = c_3^2 - 2c_2c_4 + 2c_1c_5 - 2c_6$,
- $p_4 = c_4^2 - 2c_3c_5 + 2c_2c_6 - 2c_1c_7 + 2c_8$,
- $p_5 = c_5^2 - 2c_4c_6 + 2c_3c_7 - 2c_2c_8 + 2c_1c_9 - 2c_{10}$.

So, by plugging (A.18) into (A.14), we can express $L(E)$ in terms of $c_1(E), \cdots, c_n(E)$. Now, due to (A.14), it is clear that the following power series

$$\prod_{i=1}^{n} \frac{\sqrt{\alpha_i}}{\tanh \sqrt{\alpha_i}} \tag{A.19}$$

is clearly a symmetric polynomial of $\alpha_1, \alpha_2, \cdots, \alpha_n$, not of $\alpha_1^2, \alpha_2^2, \cdots, \alpha_n^2$. Hence it is expressed in terms of Chern classes of E, and in (A.15) each Pontryagin class $p_j(E)$ is replaced by each Chern class $c_j(E)$, thus it is expressed as

$$L\left(c_1(E), c_2(E), \cdots, c_n(E)\right), \tag{A.20}$$

which is *usually called L-class of the complex vector bundle*. The first lower terms of this class is the same as (A.16) with p_j replaced by c_j. However, we still call $L(E) = \prod_{i=1}^{n} \frac{\alpha_i}{\tanh \alpha_i}$ the Hirzebruch L-class of E, unless some confusion with (A.19) is possible.

Remark A.0.15 This remark has nothing to do with the theme of the present survey. But just as a curiosity, we add this remark as to the term $B_{2k}\frac{4^k}{(2k)!}(\alpha_i^2)^k$ of the above formula (A.14). First we note that $B_{2k}\frac{4^k}{(2k)!}(\alpha_i^2)^k = B_{2k}\frac{(2\alpha_i)^{2k}}{(2k)!}$. If we replace α_i by π, then we have the following well-known relation (discovered by L. Euler) between the Bernoulli number B_{2k} and the Riemann zeta function $\zeta(2k)$ for any positive integer k (e.g., see [17, §5.9]):

$$\zeta(2k) = (-1)^{k+1}\frac{(2\pi)^{2k}}{2(2k)!}B_{2k} \quad \text{or} \quad B_{2k} = (-1)^{k+1}\frac{2(2k)!}{(2\pi)^{2k}}\zeta(2k).$$

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Chapter 7
Regular Vectors and Bi-Lipschitz Trivial Stratifications in O-Minimal Structures

Guillaume Valette

Contents

Abstract These notes deal with semialgebraic and subanalytic subsets of $\mathbb{R}^n$, and more generally with all the sets that are definable in a polynomially bounded o-minimal structure expanding $\mathbb{R}$, establishing existence of definably bi-Lipschitz trivial stratifications for these sets (Corollary 7.6.9). We start with basic definitions about o-minimal structures and Lipschitz geometry, and give a short survey of some historical results, such as existence of Mostowski's Lipschitz stratifications or the

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Preparation Theorem for definable functions, which gives us the opportunity to state some needed theorems as well as to describe the way the results presented in the second part fit in the landscape. Our stratification theorem (Corollary 7.6.9) is obtained as a byproduct of two foregoing results of the author that are proved in Sects. 7.5 and 7.6 respectively. The first one asserts that, given a family definable in an o-minimal structure, there is a regular vector, up to a definable family of bi-Lipschitz homeomorphisms. The second one is a bi-Lipschitz version of the famous Hardt's theorem. We give proofs of these two theorems that avoid the use of the real spectrum.

7.1 Introduction

The study of the Lipschitz geometry of singularities that arise in algebraic and analytic geometry began when T. Mostowski constructed stratifications of complex analytic sets that admit a Lipschitz version of Thom-Mather isotopy theorem [17]. This result was extended by A. Parusiński to real analytic geometry [22, 23], and his proof was then adapted to polynomially bounded o-minimal structures [18] (see also [12]). The Lipschitz geometry of singularities was investigated later independently by the author of the present paper [30–34] (see [35] for a complete expository), as well as by several other authors [1–4, 14, 15, 26] (among many others).

In [30], the author proved a bi-Lipschitz triviality theorem, which can be considered as a bi-Lipschitz version of Hardt's theorem (recalled in Theorem 7.6.3 below). These notes provide a new proof of this theorem and then derive existence of definably locally bi-Lipschitz trivial stratifications. We also include a short introduction to the theory of the Lipschitz geometry of sets that are definable in o-minimal structures, providing many necessary definitions, proofs, and references.

The existence of locally bi-Lipschitz trivial stratifications and the description of the aspect of singularities occurring in tame geometry (subanalytic, semialgebraic, or o-minimal) from the metric point of view that was achieved during the four last decades recently turned out to be valuable for applications to analysis of PDE's and geometric measure theory. In [33, 34], the author relied on it so as to compute the L^p cohomology of differential forms of bounded subanalytic manifolds, not necessarily compact. More recently, these techniques turned out to be useful to study the theory of currents [8], as well as to investigate the Sobolev spaces of these manifolds [11, 28, 29, 36], which is valuable for applications to the theory of PDE on domains with non Lipschitz boundary [37].

The main difficulty of the proof of the bi-Lipschitz version of Hardt's theorem [30] (see Theorem 7.6.3 below) is the "regular vector theorem" [30, Theorem 3.13] (see Theorem 7.3.2 and Corollary 7.3.4). This theorem asserts that, given a set $A \subset \mathbb{R}^n$ which is definable in an o-minimal structure, there is a definable bi-Lipschitz homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h(A)$ has a regular vector, which means that there is a vector such that the angle between it and all the tangent spaces to $h(A)$ (at its regular points) is bounded away from zero. We prove this result, which does not require the structure to be polynomially bounded and which is of its own interest. The regular vector theorem was actually also the key ingredient of the Lipschitz conic structure theorem in [34], itself useful later to study Sobolev spaces of definable manifolds [28, 29, 36], and was of service to M. Czapla to show existence of triangulations inducing Whitney stratifications [7].

Quite often, especially if, like in the aforementioned applications of Lipschitz geometry, Lipschitz isotopies are necessary, one needs a regular vector not only for one single set but for a definable family of sets. In [30], this is obtained by applying the regular vector theorem to the generic fiber of a family, relying on the compactness of the Stone space of the Boolean algebra of definable sets (sometimes called the real spectrum). This has nevertheless the inconvenience to involve abstract material to which specialists of PDE or geometric measure theory may be unfamiliar, and to force to work with an o-minimal structure that expands an arbitrary real closed field, that may be non archimedean and totally disconnected, which is prone to generate technical complications.

In the present article, we give a parameterized version of the regular vector theorem (Theorem 7.3.2) on o-minimal structures following the proof given in [30], but relying only on very elementary methods. We also combine it with the techniques of [34] to prove a local version, with additional properties (Theorem 7.5.14), which was used in the latter article to prove the Lipschitz conic structure of subanalytic sets. We then provide a proof of the bi-Lipschitz version of Hardt's Theorem on polynomially bounded o-minimal structures (Theorem 7.6.3), and derive existence of definably bi-Lipschitz trivial stratifications (Corollary 7.6.9).

Some Notations and Definitions Throughout this article, m, n, j, and k will stand for integers. The origin of $\mathbb{R}^n$ will be denoted $0_{\mathbb{R}^n}$. When the ambient space will be obvious from the context, we will however omit the subscript $\mathbb{R}^n$. We write $e_1, \dots, e_n$ for the canonical basis of $\mathbb{R}^n$. By $\mathscr{C}^k$ **mapping** on a set $X \subset \mathbb{R}^n$, we mean a mapping that extends to a $\mathscr{C}^k$ mapping on a neighborhood of X in $\mathbb{R}^n$.

We write $|x|$ for the euclidean norm and $d(x, y)$ for the euclidean distance (and the distance to a subset $P \subset \mathbb{R}^n$ will be denoted by $d(x, P)$). Given $x \in \mathbb{R}^n$ and $\varepsilon > 0$, we denote by $\mathbf{B}(x, \varepsilon)$ the open ball of radius ε centered at x (for the euclidean norm). The unit sphere of $\mathbb{R}^n$ centered at the origin is denoted $\mathbf{S}^{n-1}$. Given a subset A of $\mathbb{R}^n$, we respectively denote the closure and interior of A by $cl(A)$ and $int(A)$, and set $\delta A = cl(A) \setminus int(A)$.

A mapping $\xi : A \to \mathbb{R}^k$ is said to be **Lipschitz** if there is a constant L such that for all x and x' in A:

$$|\xi(x) - \xi(x')| \leq L|x - x'|.$$

We say that ξ is L**-Lipschitz** if we wish to specify the constant. The smallest nonnegative number L having this property is called the **Lipschitz constant of** ξ and is denoted L_ξ. By convention, if A is empty then ξ is Lipschitz and $L_\xi = 0$. A mapping ξ is **bi-Lipschitz** if it is a homeomorphism onto its image such that ξ and ξ^{-1} are both Lipschitz.

Given two functions ζ and ξ on a set $A \subset \mathbb{R}^n$ with $\xi \le \zeta$ we define the **closed band** $[\xi, \zeta]$ as the set:

$$[\xi, \zeta] := \{(x, y) \in A \times \mathbb{R} : \xi(x) \le y \le \zeta(x)\}.$$

The open and semi-open bands (ξ, ζ), $(\xi, \zeta]$, and $[\xi, \zeta)$, are then defined analogously.

Given a subset B of A, we write "$\xi \lesssim \zeta$ on B" when there is a positive constant C such that $\xi(x) \le C\zeta(x)$ for all $x \in B$. We write "$\xi \sim \zeta$ on B" or "$\xi(x) \sim \zeta(x)$ for x in B" whenever both $\xi \lesssim \zeta$ and $\zeta \lesssim \xi$ hold on B.

7.2 O-Minimal Structures

A **structure** (expanding the field $(\mathbb{R}, +, .)$) is a family $\mathcal{S} = (\mathcal{S}_n)_{n\in\mathbb{N}}$ such that for each n the following properties hold

(1) $\mathcal{S}_n$ is a boolean algebra of subsets of $\mathbb{R}^n$,
(2) If $A \in \mathcal{S}_n$ then $\mathbb{R} \times A$ and $A \times \mathbb{R}$ belong to $\mathcal{S}_{n+1}$,
(3) $\mathcal{S}_n$ contains $\{x \in \mathbb{R}^n : P(x) = 0\}$, for all $P \in \mathbb{R}[X_1, \dots, X_n]$,
(4) If $A \in \mathcal{S}_n$ then $\pi(A)$ belongs to $\mathcal{S}_{n-1}$, where $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the standard projection onto the first $(n-1)$ coordinates.

A structure $\mathcal{S}$ is said to be **o-minimal** if in addition:

(5) Any set $A \in \mathcal{S}_1$ is a finite union of intervals and points.

A set belonging to $\mathcal{S}_n$, for some n, is called a **definable set**, and a map whose graph is in some $\mathcal{S}_n$ is called a **definable map**.

A structure $\mathcal{S}$ is said to be **polynomially bounded** if for each definable function $f : \mathbb{R} \to \mathbb{R}$, there exists a positive number a and an $n \in \mathbb{N}$ such that $|f(x)| < x^n$ for all $x > a$.

We fix an o-minimal structure $\mathcal{S}$ for all this article. It will be assumed to be polynomially bounded in Sects. 7.2.2 and 7.6 only. For the other sections, this assumption is unnecessary.

We refer to [6, 9] for all the basic facts and definitions about o-minimal structures that we shall use all along this article, such as cell decompositions or curve selection lemma. We however recall a few definitions helpful to understand the statements of the theorems.

It is a fundamental feature of o-minimal structures that it is possible to construct a cell decomposition of $\mathbb{R}^n$ that is **compatible** with a given arbitrary finite collection of elements of $\mathcal{S}_n$, in the sense that the given sets are unions of cells of this decomposition (the word "adapted" is used in [6], instead of compatible).

The definition of cell decompositions being inductive on the dimension of the ambient space, it is obvious that if $\mathbb{C}$ is a cell decomposition of $\mathbb{R}^n$ and if $\pi : \mathbb{R}^n \to \mathbb{R}^k$ (with $k \le n$) is the canonical projection, then $\{\pi(C) : C \in \mathbb{C}\}$ is a cell decomposition. We will denote it by $\pi(\mathbb{C})$.

Definition 7.2.1 A **stratification** of a subset of $\mathbb{R}^n$ is a finite partition of it into smooth submanifolds of $\mathbb{R}^n$, called **strata**. A **stratification is compatible** with a set if this set is the union of some strata. It is **definable** if so are the strata.

Remark 7.2.2 In the above definition, we write "smooth" without specifying the degree of smoothness. It is well-known that one can construct Whitney (b) or Verdier (w) regular definable $\mathscr{C}^k$ stratifications of any given definable set [27, 35], for every given k. When the structure has $\mathscr{C}^\infty$ cell decomposition, we can construct regular stratifications that have $\mathscr{C}^\infty$ strata. Definable $\mathscr{C}^k$ manifolds admit definable $\mathscr{C}^{k-1}$ tubular neighborhoods [6].

Definition 7.2.3 We say that $(A_t)_{t\in\mathbb{R}^m}$ is a **definable family of subsets of** $\mathbb{R}^n$ if the set

$$A := \bigcup_{t\in\mathbb{R}^m} \{t\} \times A_t$$

is a definable subset of $\mathbb{R}^m \times \mathbb{R}^n$.

We will sometimes regard a definable subset $A \subset \mathbb{R}^m \times \mathbb{R}^n$ as a definable family, setting for $t \in \mathbb{R}^m$:

$$A_t := \{x \in \mathbb{R}^n \ : (t, x) \in A\}.$$

Given two definable families $A \subset \mathbb{R}^m \times \mathbb{R}^n$ and $B \subset \mathbb{R}^m \times \mathbb{R}^k$, we say that $F_t : A_t \to B_t, t \in \mathbb{R}^m$, is a **definable family of mappings** if the family of the graphs $(\Gamma_{F_t})_{t\in\mathbb{R}^m}$, is a definable family of subsets of $\mathbb{R}^{n+k}$. We will sometimes regard a function $f : A \to \mathbb{R}$, $A \in \mathcal{S}_{m+n}$, as a family of functions $f_t : A_t \to \mathbb{R}$, $t \in \mathbb{R}^m$, setting $f_t(x) := f(t, x)$.

A definable family of mappings $F_t : A_t \to B_t$, $t \in \mathbb{R}^m$ is **uniformly Lipschitz (resp. bi-Lipschitz)** if there exists a constant L such that F_t is L-Lipschitz (resp. F_t and F_t^{-1} are L-Lipschitz) for all $t \in \mathbb{R}^m$.

Given $B \in \mathcal{S}_m$ and A as above, we also define the **restriction of** A **to** B:

$$A_B := A \cap (B \times \mathbb{R}^n). \tag{7.1}$$

Define finally the m**-support** of A by

$$\mathrm{supp}_m(A) := \{t \in \mathbb{R}^m : A_t \neq \emptyset\}.$$

Definition 7.2.4 Let $A \in \mathcal{S}_{m+n}$. We will say that A is **definably topologically trivial along** $U \subset \mathbb{R}^m$ if there exist $t_0 \in U$ and a definable homeomorphism $H : U \times A_{t_0} \to A_U$, $(t, x) \mapsto (t, h_t(x))$. The mapping h is then called **a trivialization** of the set A along U.

The following theorem is sometimes called "definable Hardt's theorem", because it is the o-minimal counterpart of a theorem proved by R. Hardt about semialgebraic families of sets [13]. In this theorem, by **definable partition of a set**, we mean a *finite* partition of it into definable sets.

Theorem 7.2.5 *[[6, Theorem 5.22]] Given $A \in \mathcal{S}_{m+n}$, there exists a definable partition of $\mathbb{R}^m$ such that A is definably topologically trivial along each element of this partition.*

Remark 7.2.6 We shall make use of the following immediate consequence of this theorem: given $A \in \mathcal{S}_{m+n}$, there is a definable partition $\mathcal{P}$ of $\mathbb{R}^m$ such that, for every $B \in \mathcal{P}$, E_t is connected for every connected component E of A_B and all $t \in B$.

Definition 7.2.7 Let $A \in \mathcal{S}_{m+n}$. We will say that A is **definably bi-Lipschitz trivial along** $U \subset \mathbb{R}^m$ if it is definably topologically trivial along this set, with a trivialization $h_t : A_{t_0} \to A_t$ which is bi-Lipschitz for every $t \in U$.

In Sect. 7.6, we show (Theorem 7.6.3) that definable bi-Lipschitz triviality holds up to a definable partition of the parameter space when the o-minimal structure is polynomially bounded (which is a necessary condition), giving a Lipschitz counterpart of Theorem 7.2.5.

Remark 7.2.8 In the above definition, the Lipschitz constant of h_t (or h_t^{-1}) is a function of t. This function is not required to be bounded or locally bounded, but, since it is a definable function, it must be continuous on the elements of a definable partition of U. As a matter of fact, possibly refining the partition provided by Theorem 7.6.3, we see that we could require the Lipschitz constants of the trivialization to be locally bounded on every element of this partition (see also Remark 7.6.7 on this issue).

7.2.1 *Bi-Lipschitz Trivial Stratifications*

In his pioneer's work [17], Mostowski constructed stratifications, called today *Mostowski's Lipschitz stratifications*, that are locally bi-Lipschitz trivial along the strata (see Theorem 7.2.10 below).

Definition 7.2.9 A stratification Σ of a set X is **locally bi-Lipschitz trivial** if for every $S \in \Sigma$, there are an open neighborhood V_S of S in X and a smooth retraction $\pi_S : V_S \to S$ such that every $x_0 \in S$ has an open neighborhood W in S for which there is a bi-Lipschitz homeomorphism

$$\Lambda : \pi_S^{-1}(W) \to \pi_S^{-1}(x_0) \times W,$$

satisfying:

(i) $\pi_S(\Lambda^{-1}(x, y)) = y$, for all $(x, y) \in \pi_S^{-1}(x_0) \times W$.
(ii) $\Sigma_{x_0} := \{\pi_S^{-1}(x_0) \cap Y : Y \in \Sigma\}$ is a stratification of $\pi_S^{-1}(x_0)$, and $\Lambda(\pi_S^{-1}(W) \cap Y) = (\pi_S^{-1}(x_0) \cap Y) \times W$, for all $Y \in \Sigma$.

When everything is definable (i.e. when so are X, Λ, π_S, and Σ), we say that Σ is **definably locally bi-Lipschitz trivial**.

We say that a vector field v on a subset of a set X stratified by a stratification Σ is **tangent to the strata**, when $v(x) \in T_x S$ for all $x \in S \in \Sigma$ (at which v is defined).

One of the main achievements of [17] can then be summarized as:

Theorem 7.2.10 *Every complex analytic set $X \subset \mathbb{C}^n$ admits a stratification having the following property: for every j, each locally Lipschitz vector field tangent to the strata on X^j (denoting the union of the strata of dimension not greater than j) can be extended to a locally Lipschitz vector field on X^{j+1} that is also tangent to the strata. This stratification is locally bi-Lipschitz trivial along the strata.*

The local bi-Lipschitz trivializations of Mostowski's Lipschitz stratifications are actually provided by the flow of a Lipschitz vector field tangent to the strata. More generally, the extension property of Lipschitz tangent vector fields that enjoy Mostowski's stratifications ensures a Lipschitz version of the famous Thom-Mather's First Isotopy Lemma.

Theorem 7.2.10 was extended to the real analytic and subanalytic categories by A. Parusiński [22, 23], and then to polynomially bounded o-minimal structures [18] by N. Nguyen together with the author of the present article (see also [12]). We construct in Sect. 7.6 some stratifications that are locally *definably* bi-Lipschitz trivial. We shall make use of the so-called preparation theorem for definable functions that we recall below, with a short survey on it (suggested by the referee).

7.2.2 The Preparation Theorem

This theorem was established in the subanalytic category by A. Parusiński in [23, 24] so as to construct Lipschitz stratifications for subanalytic sets. An independent proof was then given by J.-M. Lion and J.-P. Rolin [16], who also provided a generalization to the ln-exp structure (which is non polynomially bounded). It admits a version that holds on any polynomially bounded o-minimal structure, which we present now.

We thus assume for this subsection the structure to be polynomially bounded. We denote by $\mathscr{F}$ the valuation field of the structure, which is the subfield of $\mathbb{R}$ constituted by all the real numbers α for which the function $(0, +\infty) \ni x \mapsto x^\alpha \in \mathbb{R}$ is definable.

Theorem 7.2.11 *[10, Theorem 2.1] Given some definable functions $f_1, \dots, f_l : \mathbb{R}^n \to \mathbb{R}$, there is a definable partition $\mathbb{C}$ of $\mathbb{R}^n$ such that for each set $S \in \mathbb{C}$, there are exponents $\alpha_1, \dots, \alpha_l \in \mathscr{F}$, as well as definable functions $\theta, a_1, \dots, a_l : \mathbb{R}^{n-1} \to \mathbb{R}$, satisfying $\Gamma_\theta \cap S = \emptyset$ and for $x = (\tilde{x}, x_n) \in S \subset \mathbb{R}^{n-1} \times \mathbb{R}$ and $i \in \{1, \dots, l\}$:*

$$f_i(x) \sim |x_n - \theta(\tilde{x})|^{\alpha_i} a_i(\tilde{x}). \tag{7.2}$$

A function f_i having the property (7.2) is called **reduced** (in [16, 35]), and the function $U_i(x) := \frac{f_i(x)}{|x_n-\theta(\tilde{x})|^{\alpha_i} a_i(\tilde{x})}$ is called the **unit** of its reduction. The strength of Parusiński's original version of the Preparation Theorem [23, 24] lied in the very precise description of the units U_i, which could be written $\psi \circ \Phi$, with ψ analytic function on $cl(\Phi(S))$ and $\Phi = (\Phi_1, \dots, \Phi_k)$ bounded mapping on S of the form $\Phi_j = |x_n - \theta(\tilde{x})|^{\beta_j} \cdot b_j(\tilde{x})$ (for each j), with b_j definable function and $\beta_j \in \mathscr{F}$. Theorem 7.2.11, although being much more general (it deals with an arbitrary polynomially bounded o-minimal structure), is thus a partial generalization, since it just provides units that are bounded away from zero and infinity (which is however satisfying for many purposes, see below). It was nevertheless established [18] that the units can be written as the composite of a definable Lipschitz function ψ with a mapping Φ as above. To the best knowledge of the author, it is not known whether this function ψ can always be chosen $\mathscr{C}^p$ with $p \geq 1$ (on $cl(\Phi(S))$, with Φ as above).

This precise description of the units is essential to construct Lipschitz tangent vector fields on stratified sets. Such vector fields lying in the tangent bundles of the strata, their constructions demand to have bounds for the derivative of the functions describing the strata. Nevertheless, since the present work, unlike [17, 18, 23], avoids constructing Lipschitz vector fields, we are able to spare this technical description of the units, which accounts for the fact that the far-reaching partial generalization of the Preparation Theorem given by Theorem 7.2.11 will be enough for our purpose.

The proof of Theorem 7.2.11 given in [10] relies on model theoretic principles, establishing a valuation property for definable functions which is of its own interest but which unfortunately goes beyond the scope of this survey. In the structure of semialgebraic sets [5], which is the smallest o-minimal structure (yet very valuable for applications), we however can give the following fairly short proof (taken from [30]):

Proof of Theorem 7.2.11 in the Semialgebraic Case As above, a function that satisfies (7.2) (for some semialgebraic functions a_i and θ, and $\alpha_i \in \mathbb{Q}$) will be called reduced.

We first show that the product of two reduced semialgebraic functions on a set S is reduced on every element of a semialgebraic partition of S. Take two semialgebraic functions f_1 and f_2 on a set S such that $f_1(\tilde{x}, x_n) \sim |x_n - \theta_1(\tilde{x})|^{\alpha_1} a_1(\tilde{x})$ and $f_2(\tilde{x}, x_n) \sim |x_n - \theta_2(\tilde{x})|^{\alpha_1} a_2(\tilde{x})$, for some semialgebraic functions $a_1, a_2, \theta_1, \theta_2$ and $\alpha_1, \alpha_2 \in \mathbb{Q}$. There is a semialgebraic partition $\mathscr{P}$ of S such that on each $D \in \mathscr{P}$, $(\theta_1 - \theta_2)$ has constant sign (positive, negative, or zero), and the functions $|x_n - \theta_i(\tilde{x})|$, $|\theta_i(\tilde{x}) - \theta_j(\tilde{x})|$, $i, j = 1, 2$, are comparable with each other (for $\leq$). Fix $D \in \mathscr{P}$. It is no loss of generality to assume $|x_n - \theta_2(\tilde{x})| \leq |x_n - \theta_1(\tilde{x})|$ for $(\tilde{x}, x_n) \in D$. We distinguish two cases.

Case 1: $|x_n - \theta_2| \leq |\theta_2 - \theta_1|$ on D. If $\theta_1 \equiv \theta_2$ on D then there is nothing to prove. Otherwise, either $(x_n - \theta_2)$ and $(\theta_2 - \theta_1)$ have the same sign on D or

$|\frac{x_n - \theta_2}{\theta_2 - \theta_1}| \leq \frac{1}{2}$ on this set (since we assume $|x_n - \theta_2(\tilde{x})| \leq |x_n - \theta_1(\tilde{x})|$). In either of these two possibilities:

$$|x_n - \theta_1| = |\theta_2 - \theta_1|(1 + \frac{x_n - \theta_2}{\theta_2 - \theta_1}) \sim |\theta_2 - \theta_1|,$$

which clearly yields that $f_1 \cdot f_2$ is reduced on D.

Case 2: $|\theta_2 - \theta_1| \leq |x_n - \theta_2|$ on D. Writing now

$$x_n - \theta_1 = (x_n - \theta_2)(1 + \frac{\theta_2 - \theta_1}{x_n - \theta_2}),$$

it is easy to see that a similar argument applies (see [16] or [35, lemma 1.6.7] for more details, these tricks are actually taken from [16]) to show that $f_1 \cdot f_2$ is reduced on D.

Observe that this argument has also shown that a suitable refinement of the partition allows to assume that the reductions of two given functions involve the same θ. As we can always take a common refinement of finitely many given semialgebraic partitions, we therefore just have to show the theorem in the case of one single function f.

We start by proving the theorem in the case where f is a polynomial on a semialgebraic set S. In this case, there is a semialgebraic partition $\mathscr{P}$ of S such that on each $D \in \mathscr{P}$:

$$f(\tilde{x}, x_n) = a(\tilde{x})(x_n - \xi_1(\tilde{x})) \cdots (x_n - \xi_k(\tilde{x})) \cdot (x_n^2 + \zeta_1^2(\tilde{x})) \cdots (x_n^2 + \zeta_l^2(\tilde{x})), \qquad (7.3)$$

for some semialgebraic functions $\xi_1, \ldots, \xi_k, \zeta_1, \ldots, \zeta_l$ (a being the leading coefficient). Fix $D \in \mathscr{P}$. Partitioning D if necessary, we can assume x_n^2 and $\zeta_i^2(\tilde{x})$ to be comparable (for $\leq$) on D for each i, which means that $(x_n^2 + \zeta_i^2(\tilde{x}))$ is $\sim$ on D to one of these two functions (for each i), and consequently, is reduced. By the above, since all the terms of the product displayed in the right-hand-side of (7.3) are reduced, so must be f.

We now prove the theorem for an arbitrary semialgebraic function $f : \mathbb{R}^n \to \mathbb{R}$. There is a semialgebraic partition $\mathscr{P}$ of $\mathbb{R}^n$ such that for every $S \in \mathscr{P}$, there is an $(n + 1)$-variable nonzero polynomial, say $P(x, y) = \sum_{i=0}^{d} a_i(x) y^i$, such that $P(x, f(x)) \equiv 0$ on S. Fix any $S \in \mathscr{P}$. Refining $\mathscr{P}$ if necessary, we can assume the functions $a_i(x) f(x)^i$ to be of constant sign and comparable with each other (for $\leq$) on S. Let I (resp. J) denote the set of all the integers $i \leq d$ such that $a_i(x) f(x)^i$ is nonnegative (resp. negative) on S, and let $i_0 \in I$ as well as $j_0 \in J$ be such that $a_{i_0}(x) f(x)^{i_0} = \max_{i \in I} a_i(x) f(x)^i$ and $-a_{j_0}(x) f(x)^{j_0} = \max_{j \in J} -a_j(x) f(x)^j$. We have on S:

$$a_{i_0}(x) f(x)^{i_0} \sim \sum_{i \in I} a_i(x) f(x)^i = -\sum_{j \in J} a_j(x) f(x)^j \sim -a_{j_0}(x) f(x)^{j_0},$$

and therefore (refining the partition, we can suppose that f has constant sign on S, we thus assume it to be positive)

$$f(x) \sim \left(\frac{-a_{j_0}(x)}{a_{i_0}(x)}\right)^{1/(i_0-j_0)}. \tag{7.4}$$

As $a_{i_0}(x)$ and $a_{j_0}(x)$ are polynomials, by the above, refining $\mathscr{P}$ if necessary, we can assume that these are reduced functions. Clearly, $\frac{1}{a_{i_0}(x)}$ is then also reduced, and, since we have established that the product of two reduced functions can be reduced, so will be the rational fraction $a_{j_0}(x) \cdot \frac{1}{a_{i_0}(x)}$ on every $S \in \mathscr{P}$, after an extra refinement of the partition. By (7.4), this shows the desired fact. □

7.3 The Regular Vector Theorem

We denote by $\mathbb{G}_k^n$ the Grassmannian of k-dimensional vector subspaces of $\mathbb{R}^n$, and we set $\mathbb{G}^n := \bigcup_{k=1}^n \mathbb{G}_k^n$ as well as $\mathbb{G}_*^n := \bigcup_{k=1}^{n-1} \mathbb{G}_k^n$.

Given a definable set $A \subset \mathbb{R}^n$, we denote by A_{reg} the set constituted by all the points of A at which this set is a $\mathscr{C}^1$ manifold (without boundary, of dimension $\dim A$ or smaller). Define $\tau(A)$ as the closure of the set of vector spaces that are tangent to A at a regular point, i.e.:

$$\tau(A) := cl(\{T_x A \in \mathbb{G}^n : x \in A_{reg}\}).$$

Given an element λ of $\mathbf{S}^{n-1}$ and a subset $Z \subset \mathbb{G}^n$ we set (caution, here Z is not a subset of $\mathbb{R}^n$):

$$d(\lambda, Z) := \inf\{d(\lambda, T) : T \in Z\},$$

with $d(\lambda, \emptyset) := +\infty$.

Definition 7.3.1 Let $A \in \mathcal{S}_n$. An element λ of $\mathbf{S}^{n-1}$ is said to be **regular for the set** A if there is $\alpha > 0$ such that:

$$d(\lambda, \tau(A)) \geq \alpha.$$

More generally, we say that $\lambda \in \mathbf{S}^{n-1}$ is **regular for** $A \in \mathcal{S}_{m+n}$ if there exists $\alpha > 0$ such that for any $t \in \mathbb{R}^m$:

$$d(\lambda, \tau(A_t)) \geq \alpha. \tag{7.5}$$

We then also say that λ is **regular for the family** $(A_t)_{t \in \mathbb{R}^m}$. A **subset** $C \subset \mathbf{S}^{n-1}$ **is regular** for a set $A \in \mathcal{S}_{m+n}$ if so are all the elements of $cl(C)$.

If $\lambda \in \mathbf{S}^{n-1}$ is regular for $A \in \mathcal{S}_{m+n}$, it is regular for $A_t \in \mathcal{S}_n$ for all $t \in \mathbb{R}^m$. But it is indeed even stronger since in (7.5), the angle between the vector λ and the tangent spaces to the fibers is required to be bounded below away from zero by a positive constant *independent of the parameter* t.

Regular vectors do not always exist, even if the considered set has empty interior (which is clearly a necessary condition), as it is shown by the simple example of a circle. Nevertheless, when the considered sets have empty interior, up to a definable bi-Lipschitz map, we can find such a vector:

Theorem 7.3.2 *Let $A \in \mathcal{S}_{m+n}$ such that A_t has empty interior for every $t \in \mathbb{R}^m$. There exists a uniformly bi-Lipschitz definable family of homeomorphisms $h_t : \mathbb{R}^n \to \mathbb{R}^n$, $t \in \mathbb{R}^m$, such that e_n is regular for the family $(h_t(A_t))_{t\in\mathbb{R}^m}$.*

Remark 7.3.3 In the above theorem, the family h_t is not required to be Lipschitz nor continuous with respect to the parameter $t \in \mathbb{R}^m$. It is nevertheless continuous for generic parameters (see [6, Lemma 5.17] or [35, Proposition 2.4.9]). Moreover, using Proposition 7.6.5 below, one could see that, along the elements of a suitable partition of $\mathbb{R}^m$, h_t and h_t^{-1} may be required to be Lipschitz with respect to the parameters on compact sets.

In the case $m = 0$, we have the following immediate corollary which was proved in [30]:

Corollary 7.3.4 *Let $A \in \mathcal{S}_n$ be of empty interior. There exists a definable bi-Lipschitz homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that e_n is regular for $h(A)$.*

The example of a circle that we already mentioned points out the fact that it is not possible for the homeomorphism given by this corollary to be always smooth, even if so is A.

The proof of this theorem is given in Sect. 7.5.4. In order to motivate the material that we are going to introduce for this purpose in Sects. 7.4 and 7.5 (especially Definition 7.5.2 and Theorem 7.5.4), let us now give a brief outline of the construction of this homeomorphism (we assume $m = 0$ in the outline for simplicity), with explicit references to the key results and definitions.

It is actually easy to see that given $A \in \mathcal{S}_n$, there is a covering of $\mathbb{R}^n$ by finitely many sets, say $G_1, \dots, G_l$, such that each $G_k \cap A$ has a regular vector λ_k. As a matter of fact, for each k, there is a linear automorphism h_k such that the vector e_n is regular for $h_k(G_k \cap A)$. The problem is that it is not easy to "paste" these "local embeddings" $h_1, \dots, h_l$ into a bi-Lipschitz map $h : \mathbb{R}^n \to \mathbb{R}^n$. Somehow, the idea will be to define h on $\bigcup_{i=1}^k G_i$ inductively on k, by means of the h_i's, starting with $h = h_1$.

We introduce for this purpose an "induction machinery", called *regular systems of hypersurfaces* (Definition 7.5.2). Extending h from $\bigcup_{i=1}^k G_i$ to $\bigcup_{i=1}^{k+1} G_i$ somehow requires to change coordinates, and the transition map $h_{k+1} \circ h_k^{-1}$ can be interpreted as a turn from the direction λ_k to the direction λ_{k+1}. These turns could make it difficult to extend a bi-Lipschitz mapping to a bi-Lipschitz mapping for we might come back to our starting point. Working with a regular system of hypersurfaces H makes it possible to turn "without turning back" (see (7.8) as well as (ii) of Definition 7.5.2), progressing in a zigzag but somehow always going toward the same Lipschitz upper half-space $G_b(H)$.

The main difficulty is therefore the proof of Theorem 7.5.4, which yields existence of a suitable regular system of hypersurfaces. In the proof of this theorem, the trick to avoid to "turn back" (in the sense of (ii) of Definition 7.5.2) is to choose λ_{k+1} in the same connected component as λ_k of the sets of all the regular vectors of the previous step (see Proposition 7.5.5). The key lemma on this issue is Lemma 7.5.10, which relies on the fact that the fiber $\widetilde{\pi}_e^{-1}(\lambda)$, for each e and λ in $\mathbf{S}^{n-1}$ (see (7.9)), is a connected curve of length at least 2, which leaves enough space to choose our regular vector (see Remark 7.5.6).

The proof of Theorem 7.5.4 is splitted into four steps, and a more explicit outline of it is provided before the first step (see Sect. 7.5.3). Moreover, the second and third steps are preceded by a paragraph that motivates them and gives some more details on their proof.

Remark 7.3.5 Mostowski's work [17] involved establishing results about existence of regular projections (see also [14, 20, 21, 23]). Existence of a regular vector is closely related to existence of a regular projection but not completely equivalent [19, 20]. This is however not the main difference between Corollary 7.3.4 and the theorems of [17, 20, 21, 23]: Corollary 7.3.4 provides one single vector which is regular for the whole image of the considered set by some definable bi-Lipschitz mapping whereas the theorems of [17, 20, 21, 23] provide a finite set of projections such that, at each point of the considered set itself, at least one of them is regular.

7.4 A Few Lemmas on Lipschitz Geometry

7.4.1 Regular Vectors and Lipschitz Functions

Proving Theorem 7.3.2 will require to prove parameterized versions of all the lemmas and propositions of [30]. In [15], K. Kurdyka and A. Parusiński provided a parameterized version of the "L-regular cell decomposition theorem" [14], which enabled them to generalize their proof of Thom's gradient conjecture on o-minimal structures. We start with a result of [15] that will be useful for our purpose.

Given a and b in a definable connected set A, let:

$$d_A(a, b) := \inf\{length(\gamma) : \gamma : [0, 1] \to A,\ \mathscr{C}^0 \text{ definable arc joining } a \text{ and } b\}$$

(as definable arcs are piecewise $\mathscr{C}^1$, their length is well-defined). This defines a metric on A, generally referred as **the inner metric** of A.

To avoid any confusion, we will refer to the restriction to A of the euclidean metric as the **outer metric**. When A is smooth, a $\mathscr{C}^1$ function that has bounded derivative is Lipschitz with respect to the inner metric, but is not necessary Lipschitz (w.r.t. the outer metric), these two metrics being not always equivalent. We however have the following result [15, Theorem 1.2]:

Theorem 7.4.1 *Every $A \in \mathcal{S}_{m+n}$ admits a definable partition into cells, such that for each $E \in \mathcal{P}$ and each $t \in \mathbb{R}^m$, the inner and outer metrics of E_t are equivalent. The constants of this equivalence just depend on n (and not on m and t).*

The techniques that we use in Sect. 7.4.2 to prove Proposition 7.4.13 are actually related to the main ideas of the proof of this theorem that is given in [15]. It is indeed possible to show the above theorem from the latter proposition, together with an induction on n. For more details we refer the reader to the latter article (see also [35, Chapter 3]).

Proposition 7.4.2 *Every definable Lipschitz function $\xi : A \to \mathbb{R}$, $A \in \mathcal{S}_n$, can be extended to an L_ξ-Lipschitz definable function $\tilde{\xi} : \mathbb{R}^n \to \mathbb{R}$.*

Proof Set $\widetilde{\xi}(q) := \inf\{\xi(p) + L_\xi |q - p| : p \in A\}$. By the quantifier elimination principle, it is a definable function. An easy computation shows that it is L_ξ-Lipschitz. □

Remark 7.4.3 Let $A \in \mathcal{S}_{m+n}$ and let a definable function $\xi : A \to \mathbb{R}$ be such that $\xi_t : A_t \to \mathbb{R}$ is a Lipschitz function for every $t \in \mathbb{R}^m$. The respective extensions $\tilde{\xi}_t$ of ξ_t, $t \in \mathbb{R}^m$ (with for instance $\tilde{\xi}_t \equiv 0$ if $t \notin \text{supp}_m A$), provided by the proof of the above proposition constitute a definable family of functions. We thus can extend definable families of Lipschitz functions to definable families of Lipschitz functions. This will be of service.

Lemma 7.4.4 *Let A and B in $\mathcal{S}_{n+m}$ with $B \subset A$. If $\lambda \in \mathbf{S}^{n-1}$ is regular for A, then it is regular for B.*

Proof Assume that $\lambda \in \mathbf{S}^{n-1}$ is not regular for B. It means that there is a sequence $((t_i, b_i))_{i \in \mathbb{N}}$, with $b_i \in B_{t_i,reg}$ such that $\tau := \lim T_{b_i} B_{t_i,reg}$ exists and contains λ. Choose for every i a Whitney (a) regular definable stratification of A_{t_i} (see for instance [5, 35] for the definition) compatible with B_{t_i} and $B_{t_i,reg}$ and denote by S_i the stratum containing b_i. Moving slightly b_i if necessary, we may assume that S_i is open in $B_{t_i,reg}$ (since $B_{t_i,reg}$ is open and dense in B_{t_i}), which entails that $T_{b_i} S_i = T_{b_i} B_{t_i,reg}$. As $A_{t_i,reg}$ is dense in A_{t_i}, for every $i \in \mathbb{N}$, we can find a_i in $A_{t_i,reg}$, which is close to b_i. Moreover, possibly extracting a sequence, we may assume that $\tau' := \lim T_{a_i} A_{t_i,reg}$ exists. If a_i is sufficiently close to b_i, by Whitney (a) condition, we deduce that $\tau' \supset \tau$, which contains λ. This yields that λ is not regular for A. □

Remark 7.4.5 It is worthy of notice that the proof of the above lemma has established that the corresponding number α (see (7.5)) can remain the same for B.

Given $\lambda \in \mathbf{S}^{n-1}$, we denote by $\pi_\lambda : \mathbb{R}^n \to N_\lambda$ the orthogonal projection onto the hyperplane N_λ normal to the vector λ, and by q_λ the coordinate of $q \in \mathbb{R}^n$ along λ, i.e. the number given by the euclidean inner product of q with λ.

Given $B \in \mathcal{S}_n$ and $\lambda \in \mathbf{S}^{n-1}$, with $B \subset N_\lambda$, as well as a function $\xi : B \to \mathbb{R}$, we set

$$\Gamma_\xi^\lambda := \{q \in \mathbb{R}^n : \pi_\lambda(q) \in B \quad \text{and} \quad q_\lambda = \xi(\pi_\lambda(q))\}, \tag{7.6}$$

and call this set **the graph of** ξ **for** λ.

Proposition 7.4.6 *The vector $\lambda \in \mathbf{S}^{n-1}$ is regular for the set $A \in \mathcal{S}_{m+n}$ if and only if there are finitely many uniformly Lipschitz definable families of functions $\xi_{i,t} : B_{i,t} \to \mathbb{R}$, $t \in \mathbb{R}^m$, with $B_i \subset \mathbb{R}^m \times N_\lambda$, $i = 1, \dots, p$, such that for all $t \in \mathbb{R}^m$:*

$$A_t = \bigcup_{i=1}^{p} \Gamma_{\xi_{i,t}}^\lambda.$$

Proof As the "if" part is clear, we will focus on the converse. Up to an orthogonal linear mapping we can assume that $\lambda = e_n$. Let $A \in \mathcal{S}_{m+n}$. Take a cell decomposition compatible with A and let C be a cell included in A. This cell cannot be a band since e_n is regular for A (see Lemma 7.4.4). It is thus the graph of a $\mathscr{C}^1$ function $\xi : D \to \mathbb{R}$, with $D \in \mathcal{S}_{m+n-1}$, such that ξ_t has bounded first derivative (independently of t). It therefore must be uniformly Lipschitz with respect to the inner metric. It follows from Theorem 7.4.1 that there is a definable partition $\mathscr{P}$ of D such that for each $E \in \mathscr{P}$, the inner metric of E_t is equivalent to its outer metric for all $t \in \mathbb{R}^m$, with constants that just depend on n. The family of functions ξ induces a uniformly Lipschitz family of functions on every element of $\mathscr{P}$. □

We finish this subsection with an elementary proposition that will be of service to prove Theorem 7.5.4. This proposition yields that we can replace a given collection of families of Lipschitz functions with an increasing collection of families of Lipschitz functions $\xi_{1,t} \le \cdots \le \xi_{k,t}$ in such a way that the union of the graphs is unchanged:

Proposition 7.4.7 *Let $f_{1,t}, \dots, f_{k,t}$, $t \in \mathbb{R}^m$, be definable families of functions on N_λ, $\lambda \in \mathbf{S}^{n-1}$, and let $L \in \mathbb{R}$. Assume that for all $i \le k$ and for all $t \in \mathbb{R}^m$, the function $f_{i,t}$ is L-Lipschitz. Then, there exist some definable families of functions $\xi_{1,t}, \dots, \xi_{k,t}$ on N_λ such that for all $t \in \mathbb{R}^m$*

(i) $\xi_{i,t}$ *is L-Lipschitz and all $i \le k$.*
(ii) $\bigcup_{i=1}^k \Gamma_{\xi_{i,t}}^\lambda = \bigcup_{i=1}^k \Gamma_{f_{i,t}}^\lambda$.
(iii) $\xi_{1,t} \le \cdots \le \xi_{k,t}$.

Proof Up to an orthogonal linear mapping with may assume $\lambda = e_n$. We are going to define inductively on j some definable integer valued functions $i_j : \mathbb{R}^m \times \mathbb{R}^{n-1} \to \mathbb{R}$, $j = 1, \dots, k$ such that for every $t \in \mathbb{R}^m$ and $j \le k$, the functions

$$\xi_{j,t}(x) := f_{i_j(t,x),t}(x) \tag{7.7}$$

are L-Lipschitz functions satisfying $\xi_{1,t} \leq \cdots \leq \xi_{k,t}$. Indeed, let $i_1(t,x) := \min\{i \leq k : f_{i,t}(x) = \min_{l \leq k} f_{l,t}(x)\}$. Then, assuming that $i_1, \ldots, i_{j-1}$ have been defined, let

$$i_j(t,x) := \min\{i \in I_j(t,x) : f_{i,t}(x) = \min_{l \in I_j(t,x)} f_{l,t}(x)\},$$

where $I_j(t,x)$ is the set constituted by the positive integers which are not greater than k and different from $i_1(t,x), \ldots, i_{j-1}(t,x)$. We clearly have $\xi_{1,t}(x) \leq \cdots \leq \xi_{k,t}(x)$ if $\xi_{j,t}(x)$ is defined as in (7.7).

Take a cell decomposition $\mathbb{C}$ of $\mathbb{R}^m \times \mathbb{R}^{n-1}$ such that the functions $(f_{j,t}(x) - f_{j',t}(x))$ have constant sign (positive, negative, or zero) on every cell and observe that, since the i_j's are constant on every cell, they are definable.

By construction, we have $\bigcup_{j=1}^k \Gamma_{\xi_{j,t}} = \bigcup_{j=1}^k \Gamma_{f_{j,t}}$, for all $t \in \mathbb{R}^m$, which entails that e_n is regular for the graphs of the families $\xi_{j,t}$, $t \in \mathbb{R}^m$. As a matter of fact, for each j, in order to show that $\xi_{j,t}$ is L-Lipschitz, it suffices to establish that the functions $\xi_{j|C_t}$, $C \in \mathbb{C}$, glue together into a continuous function on $\mathbb{R}^{n-1}$ for every t, which is left to the reader. □

7.4.2 *Finding Regular Directions*

Lemma 7.4.8 *Given $\nu \in \mathbb{N}$, there exists $t_\nu > 0$ such that for any $P_1, \ldots, P_\nu$ in $\mathbb{G}_*^n$ there exists a vector $\lambda \in \mathbf{S}^{n-1}$ such that for any i:*

$$d(\lambda, P_i) > t_\nu.$$

Proof Given $P_1, \ldots, P_\nu$ in $\mathbb{G}_*^n$, let $\varphi(P_1, \ldots, P_\nu) := \sup_{\lambda \in \mathbf{S}^{n-1}} \min_{i \leq \nu} d(\lambda, P_i)$. Since the P_i's have positive codimension, φ is a positive function, which, since the Grassmannian is compact, must be bounded below away from zero. □

The next lemma is a refinement of the just above lemma which says that the vector λ can be chosen among finitely many ones.

Lemma 7.4.9 *Given $\nu \in \mathbb{N}$, there exist $\lambda_1, \ldots, \lambda_N$ in $\mathbf{S}^{n-1}$ and $\alpha_\nu > 0$ such that for any $P_1, \ldots, P_\nu$ in $\mathbb{G}_*^n$ we may find $i \leq N$ such that for any $j \leq \nu$:*

$$d(\lambda_i, P_j) > \alpha_\nu.$$

Proof Let t_ν be the real number given by Lemma 7.4.8 and let $\lambda_1, \ldots, \lambda_N$ in $\mathbf{S}^{n-1}$ be such that $\bigcup_{i=1}^N \mathbf{B}(\lambda_i, \frac{t_\nu}{2}) \supset \mathbf{S}^{n-1}$. Suppose that there are $P_1, \ldots, P_\nu$ in $\mathbb{G}_*^n$ such that for any $i \in \{1, \dots N\}$ we have $d(\lambda_i, \bigcup_{j=1}^{\nu} P_j) \leq \frac{t_\nu}{2}$. This implies that any λ in $\mathbf{S}^{n-1}$ satisfies

$$d(\lambda, \bigcup_{j=1}^{\nu} P_j) < t_\nu,$$

contradicting Lemma 7.4.8. It is thus enough to set $\alpha_\nu := \frac{t_\nu}{2}$. □

The next lemma will require a definition. We estimate the angle between two vector subspaces P and Q of $\mathbb{R}^n$ in the following way:

$$\angle(P, Q) = \sup\{d(\lambda, Q) : \lambda \text{ is a unit vector of } P\}.$$

This constitutes a metric on each $\mathbb{G}_k^n$, $k \leq n$.

Definition 7.4.10 Let $\alpha > 0$ and $Z \in \mathcal{S}_{m+n}$. We say that the family $(Z_t)_{t \in \mathbb{R}^m}$ is **α-flat** if:

$$\sup\{\angle(P, Q) : P, Q \in \bigcup_{t \in \mathbb{R}^m} \tau(Z_{t,reg})\} \leq \alpha.$$

We then also say that Z is **(m, α)-flat**. When $m = 0$, we say that Z is **α-flat**.

If P and Q are two vector subspaces of $\mathbb{R}^n$ satisfying $\dim P > \dim Q$ then $\angle(P, Q) = 1$. As a matter of fact, if Z is (m, α)-flat for some $\alpha < 1$, then Z_t must be of pure dimension for all t.

Remark 7.4.11 It follows from Lemma 7.4.9 that if $Z_{1,t}, \dots, Z_{\nu,t}$, $t \in \mathbb{R}^m$, are α_ν-flat definable families (where α_ν is the constant provided by the latter lemma) of subsets of $\mathbb{R}^n$ of empty interiors then one of the λ_i's (that are also provided by the latter lemma) is regular for all these families.

Lemma 7.4.12 *Given $Z \in \mathcal{S}_{m+n}$ and $\alpha > 0$, we can find a finite partition of Z into (m, α)-flat sets.*

Proof Dividing Z into cells, we may assume that Z_t is a manifold for all $t \in \mathbb{R}^m$. We can cover the Grassmannian by finitely many balls of radius $\frac{\alpha}{2}$, which gives rise to a covering $U_1, \dots, U_k$ of Z (via the family of mappings $Z_t \ni x \mapsto T_x Z_t$) by (m, α)-flat sets. □

This leads us to the following result that originates in [30] and that will be of service in Sect. 7.6. It is closely related to the L-regular cell decompositions introduced and constructed in [14]. The difference is that we wish that the regular vector for δC can be chosen among finitely many ones. This result was then improved by W. Pawłucki [25] who has shown that we can require in addition $N = n$.

Proposition 7.4.13 *There exist* $\lambda_1, \ldots, \lambda_N$ *in* $\mathbf{S}^{n-1}$ *such that for any* $A_1, \ldots, A_p$ *in* $\mathcal{S}_{m+n}$*, there is a cell decomposition* $\mathbb{C}$ *of* $\mathbb{R}^{m+n}$ *compatible with all the* A_k*'s and such that for each cell* $C \in \mathbb{C}$ *satisfying* $\dim C_t = n$ *(for all* $t \in supp_m C$*), we may find* $\lambda_{j(C)}$*,* $1 \leq j(C) \leq N$*, regular for the family* $(\delta C_t)_{t \in \mathbb{R}^m}$*.*

Proof According to Lemma 7.4.9 (see Remark 7.4.11 and Lemma 7.4.4) it is sufficient to prove by induction on n the following assertions: given $\alpha > 0$ and $A_1, \ldots, A_p$ in $\mathcal{S}_{m+n}$, there exists a cell decomposition of $\mathbb{R}^{m+n}$ compatible with $A_1, \ldots, A_p$ and such that for every cell $C \subset \mathbb{R}^{m+n}$ of this cell decomposition satisfying $\dim C_t = n$, $(\delta C_t)_{t \in \mathbb{R}^m}$ is included in the union of no more than $2n$ definable families of empty interior that are all α-flat.

For $n = 0$ this is clear. Fix $n \in \mathbb{N}$ nonzero, $\alpha > 0$, as well as $A_1, \ldots, A_p$ in $\mathcal{S}_{m+n}$. Taking a cell decomposition if necessary, we can assume that the A_i's are cells. Apply Lemma 7.4.12 to all the A_i's, and take a cell decomposition $\mathcal{D}$ of $\mathbb{R}^{m+n}$ compatible with all the elements of the obtained coverings. Applying then the induction hypothesis to the elements of $\pi_{e_{m+n}}(\mathcal{D})$, we get a refinement $\mathcal{D}'$ of $\pi_{e_{m+n}}(\mathcal{D})$.

Given a cell D of $\mathcal{D}'$, each A_i is above D, either the graph of a definable function, say $\xi_{i,D}$, or a band, say $(\xi_{i,D}, \xi'_{i,D})$, with $\xi_{i,D} < \xi'_{i,D}$ definable functions on D (or $\pm\infty$). Let $\mathbb{C}$ be the cell decomposition given by all the graphs $\Gamma_{\xi_{i,D}}$ and $\Gamma_{\xi'_{i,D}}$, $i \leq p$, $D \in \mathcal{D}'$. To check that it has the required property, fix an open cell $C = (\xi_{i,D}, \xi'_{i,D})$, with $\xi_{i,D} < \xi'_{i,D}$ definable functions on an open cell D of $\mathcal{D}'$ (or $\pm\infty$). Since $\mathcal{D}'$ is compatible with the images under $\pi_{e_{m+n}}$ of the α-flat sets that cover the A_i's, the sets $\Gamma_{\xi_{i,D}}$ and $\Gamma_{\xi'_{i,D}}$ must be α-flat families, and since

$$\delta C_t \subset \left(\Gamma_{\xi_{i,D}}\right)_t \cup \left(\Gamma_{\xi'_{i,D}}\right)_t \cup \pi_{e_n}^{-1}(\delta D_t),$$

we see that the needed fact follows from the induction hypothesis. □

Remark 7.4.14 We have proved a stronger statement: the distance between the regular vector $\lambda_{j(C)}$ and the tangent spaces to δC_t can be bounded below away from zero by a positive number depending only on n, and not on the A_k's. This is due to the fact that we apply Lemma 7.4.9 with $\nu = 2n$.

7.5 Regular Systems of Hypersurfaces

This section is entirely devoted to the proof of Theorem 7.3.2 which requires some material. We first introduce our machinery of regular systems of hypersurfaces.

Let $Z \in \mathcal{S}_n$ and $\lambda \in \mathbf{S}^{n-1}$, with $Z \subset N_\lambda$ (see Sect. 7.4.1 for N_λ). If $A \in \mathcal{S}_n$ is the graph of a function $\xi : Z \to \mathbb{R}$ for λ, we denote by $E(A, \lambda)$ the subset constituted by the points which lie "under the graph", i.e. we set:

$$E(A, \lambda) := \{q \in \pi_\lambda^{-1}(Z) : q_\lambda \leq \xi(\pi_\lambda(q))\}. \tag{7.8}$$

Remark 7.5.1 If $A \in \mathcal{S}_{m+n}$ is such that A_t is the graph for λ of a function $\xi_t : N_\lambda \to \mathbb{R}$ for every $t \in \mathbb{R}^m$, then $E(A_t, \lambda)$, $t \in \mathbb{R}^m$, is a definable family of sets of $\mathbb{R}^m \times \mathbb{R}^n$. Indeed, regarding λ as an element of $\mathbf{S}^{n+m-1}$ (i.e., identifying λ with $(0_{\mathbb{R}^m}, \lambda)$), $E(A, \lambda)$ is also well-defined and $E(A, \lambda)_t = E(A_t, \lambda)$, for all $t \in \mathbb{R}^m$.

7.5.1 Regular Systems of Hypersurfaces

Regular systems of hypersurfaces will help us to carry out constructions inductively on the dimension of the ambient space.

Definition 7.5.2 Let $B \in \mathcal{S}_m$. A **regular system of hypersurfaces** of $B \times \mathbb{R}^n$ (parametrized by B) is a finite collection $H = (H_k, \lambda_k)_{1 \le k \le b}$ with $b \in \mathbb{N}$, of definable subsets H_k of $B \times \mathbb{R}^n$ and elements $\lambda_k \in \mathbf{S}^{n-1}$ such that the following properties hold for each $k < b$ and every $t \in B$:

(i) The sets $H_{k,t}$ and $H_{k+1,t}$ are the respective graphs for λ_k of two functions $\xi_{k,t} : N_{\lambda_k} \to \mathbb{R}$ and $\xi'_{k,t} : N_{\lambda_k} \to \mathbb{R}$ such that $\xi_{k,t} \le \xi'_{k,t}$ and which are C-Lipschitz with $C \in \mathbb{R}$ independent of t.
(ii) The following equality holds:

$$E(H_{k+1,t}, \lambda_k) = E(H_{k+1,t}, \lambda_{k+1}).$$

Let $A \in \mathcal{S}_{m+n}$. We say that H is **compatible** with A, if $A \subset \bigcup_{k=1}^{b} H_k$. An **extension** of H is a regular system of hypersurfaces (of $B \times \mathbb{R}^n$) compatible with the set $\bigcup_{k=1}^{b} H_k$.

Given a positive integer $k < b$, we set:

$$G_k(H) := E(H_{k+1}, \lambda_k) \setminus int(E(H_k, \lambda_k)).$$

We shall write $\Lambda_k(H)$ for the connected component of the set

$$\{\lambda \in \mathbf{S}^{n-1} : \lambda \text{ is regular for } H_k \cup H_{k+1}\}$$

that contains λ_k.

We will see (Proposition 7.5.5 below) that the set $G_k(H)$ may be defined using any $\lambda \in \Lambda_k(H)$ (instead of λ_k).

We will say that another regular system H' **coincides with** H **outside** $G_k(H)$ if for each j either $H'_j \subset G_k(H)$ or there exists j' such that $H'_j = H_{j'}$.

Given a regular system $H := (H_k, \lambda_k)_{k \le b}$ of $B \times \mathbb{R}^n$ and a definable set $B' \subset B$, we denote by $H_{B'}$ the regular system of hypersurfaces $(H_{k,B'}, \lambda_k)$ of $B' \times \mathbb{R}^n$, obtained by considering the sequence of the respective restrictions to B' of the H_k's (see (7.1)). We will call it the **restriction to** B' of the regular system H.

Remark 7.5.3 It is always possible to assume that the $G_k(H)_t$'s are of nonempty interior for some t. Indeed, if $int(G_k(H)_t) = \emptyset$ for all $t \in B$, then $H_k = H_{k+1}$ and in this case we may remove (H_k, λ_k) from the sequence.

Given a regular system of hypersurfaces (of $B \times \mathbb{R}^n$, $B \in \mathcal{S}_m$) H, it will be convenient to extend the notations in the following way. Set for any $t \in B$: $H_{0,t} := \{-\infty\}$ and $H_{b+1,t} := \{+\infty\}$. By convention, all the elements of $\mathbf{S}^{n-1}$ will be regular for these two sets. We will also consider that these two sets as the respective graphs of the two functions which take $-\infty$ and $+\infty$ as respective constant values. Define also $\lambda_0 := \lambda_1$, $\lambda_{b+1} := \lambda_b$, as well as $E(H_0, \lambda_0) := \emptyset$, $G_0(H) := E(H_1, \lambda_1)$, $G_b(H) := (B \times \mathbb{R}^n) \setminus int(E(H_b, \lambda_b))$, as well as $E(H_{b+1}, \lambda_{b+1}) := B \times \mathbb{R}^n$. Remark that now $B \times \mathbb{R}^n = \bigcup_{k=0}^{b} G_k(H)$.

Theorem 7.5.4 *Let $A \in \mathcal{S}_{m+n}$ be such that A_t has empty interior for all $t \in \mathbb{R}^m$. There exists a definable partition $\mathcal{P}$ of $\mathbb{R}^m$ such that for every $B \in \mathcal{P}$ there is a regular system of hypersurfaces of $B \times \mathbb{R}^n$ compatible with A_B.*

This theorem is the main ingredient of the proof of Theorem 7.3.2. The basic strategy of the proof of Theorem 7.5.4 (given in Sect. 7.5.3) relies on the following observation.

Proposition 7.5.5 *Let U be a connected subset of $\mathbf{S}^{n-1}$, $\lambda_0 \in U$, and let $\xi : N_{\lambda_0} \to \mathbb{R}$ be a continuous definable function. If U is regular for $X := \Gamma_\xi^{\lambda_0}$ then, for each $\lambda \in U$, the set X is the graph for λ of a function $\xi^\lambda : N_\lambda \to \mathbb{R}$. Moreover, $E(X, \lambda)$ is independent of $\lambda \in U$.*

Proof Let

$$C := \{\lambda \in U : \forall x \in N_\lambda, \quad card\ \pi_\lambda^{-1}(x) \cap X = 1\}.$$

We have to check that $C = U$. Let $\lambda \in C$ and set $r(\lambda) := d(\lambda, \tau(X))$.

We claim that $\mathbf{B}(\lambda, \frac{r(\lambda)}{3}) \cap U \subset C$. Pick $\lambda' \in \mathbf{B}(\lambda, \frac{r(\lambda)}{3}) \cap U$ different from λ and set $l' = \pi_\lambda(\lambda')$. We have to show that the line L generated by λ' and passing through any $x \in N_\lambda$ intersects X in exactly one point. The line L is the graph for λ of a function $\eta(x + tl') = \alpha \cdot t$ with $\alpha > \frac{2}{r(\lambda)}$ (we assume $\alpha > 0$ for simplicity).

Since $\lambda \in C$, the set X is the graph for λ of a function ξ^λ. By definition of $r(\lambda)$, $|d_x \xi^\lambda|$ (which exists almost everywhere) is bounded by $\frac{2}{r(\lambda)}$. It easily follows from the Mean Value Theorem that ξ^λ is $\frac{2}{r(\lambda)}$-Lipschitz.

This implies that for t positive large enough we will have $\eta(x+tl') \geq \xi^\lambda(x+tl')$ and $\eta(x-tl') \leq \xi^\lambda(x-tl')$ (since η is growing faster than ξ^λ). Thus, there is a point $q \in \pi_\lambda(L)$ such that $\xi^\lambda(q) = \eta(q)$, which implies that the line L cuts X. Uniqueness of the intersection point is clear from the fact that one function is growing faster than the other. This yields that $\mathbf{B}(\lambda, \frac{r(\lambda)}{3}) \cap U \subset C$.

We have shown that C is open in U. Let us now show that it is also closed in U. Consider $\lambda \in U$ and a continuous definable arc γ in C tending to λ. Since $r(\gamma(t))$ tends to $r(\lambda)$ which is nonzero, the ball $\mathbf{B}(\gamma(t), \frac{r(\gamma(t))}{3})$ contains λ for $t > 0$ small enough. The closeness of C therefore follows from the fact that $\mathbf{B}(\gamma(t), \frac{r(\gamma(t))}{3}) \cap U \subset C$. As U is connected, this yields $U = C$.

It remains to check that $E(X, \lambda)$ is independent of $\lambda \in U$. It is the closure of one of the two connected components of the complement of X. The set X is the zero locus of the function $f(q) = q_{\lambda_0} - \xi(\pi_{\lambda_0}(q))$. Locally, at a smooth point q of X it is clear that $E(X, \lambda)$ is determined by the sign of $d_q f(\lambda)$. But as U is regular for X, this function never vanishes, and consequently $E(X, \lambda)$ is independent of $\lambda \in U$.

□

Given $e \in \mathbf{S}^{n-1}$, we define a mapping $\widetilde{\pi}_e : \mathbf{S}^{n-1} \setminus \{\pm e\} \to \mathbf{S}^{n-1} \cap N_e$ by setting

$$\widetilde{\pi}_e(u) := \frac{\pi_e(u)}{|\pi_e(u)|}. \tag{7.9}$$

Remark 7.5.6 Let $e \in \mathbf{S}^{n-1}$ and suppose $\lambda_0 \in \mathbf{S}^{n-1} \cap N_e$ to be regular for a subset $A \subset N_e$. Since the elements of $\widetilde{\pi}_e^{-1}(\lambda_0)$ lie above the line generated by λ_0, for each positive real number a, the set

$$C := \widetilde{\pi}_e^{-1}(\lambda_0) \cap \{\lambda \in \mathbf{S}^{n-1} : d(\lambda, \{\pm e\}) \geq a\}$$

is regular for $\pi_e^{-1}(A)$. Moreover, by Proposition 7.5.5, if A is the graph of a Lipschitz function for λ_0 then $\pi_e^{-1}(A)$ is the graph of a Lipschitz function for each $\lambda \in C$. Furthermore, the latter proposition also entails that in this case we have for all $\lambda \in C$:

$$E(\pi_e^{-1}(A), \lambda) = \pi_e^{-1}(E(A, \lambda_0)).$$

7.5.2 Two Preliminary Lemmas

Proving Theorem 7.5.4 requires two preliminary lemmas on regular systems of hypersurfaces. The first one will make it possible for us to assume that the interiors of the $G_k(H)_t$'s are connected.

Lemma 7.5.7 *Let H be a regular system of hypersurfaces of $B \times \mathbb{R}^n$, $B \in \mathcal{S}_m$. There exists a definable partition $\mathcal{P}$ of B such that for every $B' \in \mathcal{P}$, we can find an extension $\widehat{H}$ of $H_{B'}$ such that the set $int(G_k(\widehat{H})_t)$ is connected for all $t \in B'$ and all k.*

Proof Let $1 \leq k \leq b-1$ and suppose that there is t for which $int(G_k(H)_t)$ is not connected. Applying Remark 7.2.6 to $int(G_k(H))$ provides a partition $\mathcal{P}$ of B. Let $B' \in \mathcal{P}$. Possibly replacing H with $H_{B'}$, we see that we can assume that the the property displayed in Remark 7.2.6 holds for $int(G_k(H))$.

Let $A_1, \ldots, A_\nu$ be the connected components of $int(G_k(H))$. Set $A'_i = \pi_{\lambda_k}(A_i)$ for $i \le \nu$. For $t \in B'$, every fiber $A_{i,t}$ is of the form:

$$\{q \in A'_{i,t} \oplus \mathbb{R} \cdot \lambda_k \ : \ \xi_{k,t}(\pi_{\lambda_k}(q)) < q_{\lambda_k} < \xi'_{k,t}(\pi_{\lambda_k}(q))\}.$$

Clearly $\xi_{k,t} = \xi'_{k,t}$ on the boundary of $A'_{i,t}$. We thus may define some Lipschitz functions η_i, $1 \le i \le \nu - 1$, as follows. We set over $A'_{j,t}$, $\eta_{i,t} := \xi'_{k,t}$, when $1 \le j \le i$, and $\eta_{i,t} := \xi_{k,t}$ whenever $i < j$. Extend the function $\eta_{i,t}$ by setting $\eta_{i,t} := \xi_{k,t} = \xi'_{k,t}$ on $N_{\lambda_k} \setminus \pi_{\lambda_k}(int(G_k(H)))$.

Since $\eta_{1,t} \le \cdots \le \eta_{(\nu-1),t}$, it suffices to

- let $\widehat{H}_j := H_j$ and $\widehat{\lambda}_j := \lambda_j$ if $j \le k$
- let $\widehat{H}_{j,t}$ be the graph of $\eta_{j-k,t}$ for λ_k (for every $t \in \mathbb{R}^m$) and $\widehat{\lambda}_j := \lambda_k$ for $k+1 \le j \le k+\nu-1$
- let $\widehat{H}_j := H_{j-\nu+1}$ and $\widehat{\lambda}_j := \lambda_{j-\nu+1}$ if $k+\nu \le j \le b+\nu-1$.

This is clearly a regular system of hypersurfaces. Note that the $int(G_j(\widehat{H}))$, $k \le j < k+\nu$, are the connected components of $int(G_k(H))$. □

Lemma 7.5.8 *Let $H = (H_k, \lambda_k)_{1 \le k \le b}$ be a regular system of hypersurfaces of $B \times \mathbb{R}^n$, $B \in \mathcal{S}_m$, and let $j \in \{1, \ldots, b\}$. Let X be a definable subset of $G_j(H)$ and assume that λ_j is regular for X. Then H can be extended to a regular system of hypersurfaces H' compatible with X and which coincides with H outside $G_j(H)$.*

Proof By property (i) of Definition 7.5.2, for every parameter $t \in B$, the sets $H_{j,t}$ and $H_{j+1,t}$ are the respective graphs for λ_j of two functions $\xi_{j,t}$ and $\xi'_{j,t}$. By Propositions 7.4.6 and 7.4.2, the definable family X_t, $t \in B$, may be included in a finite number of graphs for λ_j of definable families of functions on N_{λ_j}, say $\theta_{1,t}, \ldots, \theta_{\nu,t}$, C-Lipschitz for every $t \in B$ with $C \in \mathbb{R}$ independent of t. Furthermore, by Proposition 7.4.7, these families of functions can be assumed to be totally ordered (for relation $\le$), and satisfy $\xi_{j,t} \le \theta_{i,t} \le \xi'_{j,t}$, for every t. Now,

- let $H'_k := H_k$ and $\lambda'_k := \lambda_k$ whenever $1 \le k \le j$,
- let $H'_{k,t}$ be the graph of $\theta_{k-j,t}$ for λ_j and $\lambda'_k := \lambda_j$ for $j < k \le j+\nu$, $t \in \mathbb{R}$,
- let $H'_k := H_{k-\nu}$ and $\lambda'_k := \lambda_{k-\nu}$, whenever $j+1+\nu \le k \le b+\nu$.

Properties (i) and (ii) of Definition 7.5.2 clearly hold by construction. □

Remark 7.5.9 It will be of service that, in the proof of the above lemma, no extra regular vector was necessary, i.e. $\{\lambda_1, \ldots, \lambda_b\} = \{\lambda'_1, \ldots, \lambda'_{b+\nu}\}$.

7.5.3 Proof of Theorem 7.5.4

The proof is divided into four steps. The strategy is to rely on Lemma 7.5.8. The reader is invited to first glance at Step 4, which was deliberately made very short and sheds light on the reason why this lemma is helpful. The problem to get a regular system of hypersufaces compatible with a set A using this lemma is that it requires to already have a regular system $H = (H_k, \lambda_k)_{1 \le k \le b}$ such that λ_j is regular for $A \cap G_j(H)$ (for all j). This fact is not granted by the system H provided by Step 1, which satisfies a slightly weaker property. We therefore shall provide (in Step 2) another system $\widehat{H}$ (see the paragraph just before Step 2 for more details on this issue) and then construct in Step 3 what can be considered as a "common refinement" of H and $\widehat{H}$, which will be satisfying to apply Lemma 7.5.8 in Step 4.

Let $A \in \mathcal{S}_{m+n}$ be such that A_t has empty interior for every $t \in \mathbb{R}^m$. Let also $\eta \in (0, 1]$ and $\lambda \in \mathbf{S}^{n-1}$. We are actually going to prove by induction on n that, given any such A, λ, and η, there exists a definable partition $\mathcal{P}$ of $\mathbb{R}^m$ such that for every $B \in \mathcal{P}$ we can find a regular system of hypersurfaces of $B \times \mathbb{R}^n$ compatible with A_B and such that all the λ_k's (see Definition 7.5.2) can be chosen in $\mathbf{B}(\lambda, \eta) \cap \mathbf{S}^{n-1}$.

Since the result is clear for $n = 1$, we take $n \ge 2$ and assume it to be true for $(n - 1)$. Observe that it is enough to establish the claimed statement for arbitrarily small values of η. As explained just above, we split the induction step into 4 steps.

Step 1 There exists a definable partition $\mathcal{P}$ of $\mathbb{R}^m$ such that for every $B \in \mathcal{P}$, there is a regular system of hypersurfaces $H = (H_k, \lambda_k)_{1 \le k \le b}$ of $B \times \mathbb{R}^n$, with $\lambda_k \in \mathbf{S}^{n-1} \cap \mathbf{B}(\lambda, \frac{\eta}{2})$, such that for every k the set $int(G_k(H)) \cap A_B$ has a regular vector $\mu \in \mathbf{S}^{n-1} \setminus \mathbf{B}(\pm\lambda, \eta)$.

Proof of Step 1 Take $e \in \mathbf{S}^{n-1}$ such that $\pm e \notin \mathbf{B}(\lambda, \eta)$. By Lemma 7.4.12, for each $\sigma > 0$, there are finitely many $(m, \frac{\sigma}{2})$-flat sets $U_1, \dots, U_\omega$ that cover A. Consider such a partition for $\sigma = t_\nu$, where t_ν is given by Lemma 7.4.8, with ν equal to the maximal number of connected components of $\pi_e^{-1}(x) \cap A_t$, $(t, x) \in \mathbb{R}^m \times N_e$. Changing η, we may assume that $\eta \le \frac{t_\nu}{4}$.

Take a cell decomposition of $\mathbb{R}^m \times N_e$ (identify $\mathbb{R}^m \times N_e$ with $\mathbb{R}^m \times \mathbb{R}^{n-1}$) which is compatible with the $\pi_e(U_i)$, $i \le \omega$, and denote by $(W_i)_{i \in I}$ the collection of the cells of this cell decomposition that are open (in $\mathbb{R}^m \times N_e$).

Since the set A_t has empty interior for each $t \in \mathbb{R}^m$, the set $A_t \cap \pi_e^{-1}(W_{i,t})$ is (for each $i \in I$ and t), above $W_{i,t}$, the union of at most ν (possibly 0) graphs for e of continuous functions (not necessarily Lipschitz).

Choose $\eta' > 0$ such that we have in $\mathbf{S}^{n-1} \cap N_e$:

$$\mathbf{B}(\widetilde{\pi}_e(\lambda), \eta') \subset \widetilde{\pi}_e(\mathbf{B}(\lambda, \frac{\eta}{2})). \tag{7.10}$$

Apply the induction hypothesis (identify $\mathbb{R}^m \times N_e$ with $\mathbb{R}^m \times \mathbb{R}^{n-1}$) to the families $(\delta W_{i,t})_{t \in \mathbb{R}^m}$ to get a partition $\mathcal{P}$. Fix $B \in \mathcal{P}$. There is a regular system of $B \times \mathbb{R}^{n-1}$, $\overline{H} = (\overline{H}_k, \overline{\lambda}_k)_{k \le b}$, such that all the $\overline{\lambda}_k$'s belong to $\mathbf{B}(\widetilde{\pi}_e(\lambda), \eta')$.

By Lemma 7.5.7, up to a refinement of the partition, we may assume that each $int(G_k(\overline{H})_t)$ is connected for all $t \in B$. We may also assume these sets to be of nonempty interior for some t (see Remark 7.5.3). Up to an extra refinement, we may assume that it happens for all $t \in B$ (by Remark 7.2.6).

We claim that for each j and k and for every t, either $int(G_k(\overline{H})_t)$ is disjoint from $W_{j,t}$ or $int(G_k(\overline{H})_t) \subset W_{j,t}$. To see this, observe that, as $\overline{H}$ is compatible with the $\delta W_{j,t}$'s, all the sets $W_{j,t} \cap int(G_k(\overline{H})_t)$ are open and of empty (topological) boundary in $int(G_k(\overline{H})_t)$, for each t. Hence, if nonempty, these are connected components of $int(G_k(\overline{H})_t)$. But, as $int(G_k(\overline{H})_t)$ is connected, this entails that $W_{j,t} \cap int(G_k(\overline{H})_t)$ is either the empty set or $int(G_k(\overline{H})_t)$ itself, as claimed.

As the $W_{i,t}$'s are disjoint from each other, for each k there is a unique i such that $int(G_k(H)_t) \subset W_{i,t}$. After a possible refinement of the partition $\mathscr{P}$, we can assume that for each k, $int(G_k(H)_t)$ meets the same $W_{i,t}$ for all t, i.e. that for every $B \in \mathscr{P}$, i depends only on k and not on $t \in B$.

We turn to define the regular system H claimed in Step 1. For $1 \le k \le b$, let:

$$H_k := \pi_e^{-1}(\overline{H}_k).$$

Since $\overline{\lambda}_k \in \mathbf{B}(\widetilde{\pi}_e(\lambda), \eta')$, by (7.10), we have $\overline{\lambda}_k \in \widetilde{\pi}_e(\mathbf{B}(\lambda, \frac{\eta}{2}))$. Choose some $\lambda_k \in \widetilde{\pi}_e^{-1}(\overline{\lambda}_k) \cap \mathbf{B}(\lambda, \frac{\eta}{2})$.

As $\lambda_k \in \mathbf{B}(\lambda, \frac{\eta}{2})$ for all k and neither e nor $-e$ belongs to $\mathbf{B}(\lambda, \eta)$ we have:

$$d(\lambda_k, \pm e) \ge \frac{\eta}{2}, \qquad \forall\, k \le b.$$

As a matter of fact, by Remark 7.5.6, as $\overline{H}$ fulfills conditions (i) and (ii) of Definition 7.5.2, these conditions are also fulfilled by $H := (H_k, \lambda_k)_{k \le b}$.

By Lemma 7.4.8 and our choice of σ, for all k, the set $A_B \cap int(G_k(H))$ is the union of finitely many definable sets having a common regular element $\mu \in \mathbf{S}^{n-1}$ (since we have seen that each $int(G_k(\overline{H})_t)$ is included in $W_{j,t}$ for some j independent of $t \in B$). Moving slightly μ, we may assume that $d(\mu, \pm\lambda) \ge \eta$ (we have assumed $\eta \le \frac{t_\nu}{4}$). This completes the proof of the first step. □

The desired partition of $\mathbb{R}^m$ will be obtained after finitely many refinements of the partition $\mathscr{P}$. Clearly, it is enough to prove the result for all the sets A_B, $B \in \mathscr{P}$. We thus can fix B in $\mathscr{P}$ and identify A and A_B in the next steps below. For simplicity, we will not indicate either the partitions of the parameter space $\mathbb{R}^m$ resulting from the successive refinements of $\mathscr{P}$, working always with A (instead of A_B).

The flaw of the first step is that the regular vector μ that we get for $G_k(H) \cap A$ might not be in $\Lambda_k(H)$. If it belongs to this set, Proposition 7.5.5 and Lemma 7.5.8 suffice to conclude (see Step 4). Had the vector e (used in Step 1) been regular for A, we could have required $\mu \in \Lambda_k(H)$ in Step 1, using Lemma 7.5.10 below in the same way as we will do in Step 2 to construct $\widehat{H}$ by means of π_μ (see assumption (7.12)). One could therefore think that not much was achieved so far as we need a regular vector to carry out our construction and finding a regular vector is all our purpose. However, since we can focus on the sets $A \cap G_p(H)$, which all have a regular vector (provided by Step 1), by repeating in Step 2 the construction of the first step (replacing e with μ and making use of Lemma 7.5.10), we will get a system $(\widehat{H}_k, \widehat{\lambda}_k)_{k \le \widehat{b}}$ with $\widehat{\lambda}_k \in \Lambda_p(H)$ regular for $G_p(H) \cap G_k(\widehat{H}) \cap A$, for each fixed $p \le b$. It will then remain to find (in Step 3, see the paragraph before

Step 3 for more details) a common extension of H and $\widehat{H}$, obtained at Steps 1 and 2 respectively.

Step 2 Fix $p \le b$. There exists a regular system of hypersurfaces $\widehat{H} = (\widehat{H}_k, \widehat{\lambda}_k)_{k \le \widehat{b}}$ such that for every k, $\widehat{\lambda}_k \in \Lambda_p(H) \cap \mathbf{B}(\lambda, \eta)$ and is regular for $G_p(H) \cap G_k(\widehat{H}) \cap A$.

Proof of Step 2 Note that as λ_p is regular for the set $H_p \cup H_{p+1}$, there exists $r > 0$ such that $\mathbf{B}(\lambda_p, r)$ is regular for $H_p \cup H_{p+1}$. Taking r smaller if necessary, we may assume that $r \le \frac{\eta}{4}$.

Let $r' > 0$ be such that we have in $\mathbf{S}^{n-1} \cap N_\mu$:

$$\mathbf{B}(\widetilde{\pi}_\mu(\lambda_p), r') \subset \widetilde{\pi}_\mu(\mathbf{B}(\lambda_p, \frac{r}{2})). \tag{7.11}$$

To complete the proof of Step 2, we need a lemma.

Lemma 7.5.10 *Let l in $\mathbf{S}^{n-1}$, $0 < r \le 1$, and $\kappa \in \mathbb{N}$. Let C be a subset of $\mathbb{G}^n$ and $\mu \in \mathbf{S}^{n-1}$ such that*

$$d(\mu, C) > 0. \tag{7.12}$$

There exists $\alpha > 0$ such that for any $P_1, \dots, P_\kappa$ in C and any $y \in \widetilde{\pi}_\mu(\mathbf{B}(l, \frac{r}{2}))$ there exists $\widehat{\lambda} \in \mathbf{B}(l, r) \cap \widetilde{\pi}_\mu^{-1}(y)$ such that:

$$d(\widehat{\lambda}, \bigcup_{i=1}^{\kappa} P_i) \ge \alpha.$$

The proof of this lemma is postponed. We first see why it is enough to carry out the proof of Step 2. Let $\kappa \ge 1$ be the maximal number of connected components of $A \cap G_p(H) \cap \pi_\mu^{-1}(x)$, $x \in N_\mu$. Applying this lemma with this integer κ, with $C = \bigcup_{t \in B} \tau(A_t \cap G_p(H)_t)$ and $l = \lambda_p$ (μ being given by Step 1), we get a positive constant α.

By Lemma 7.4.12, we can cover $(G_p(H)_t \cap A_t)_{t \in \mathbb{R}^m}$ by $\frac{\alpha}{2}$-flat families, say $U'_{1,t}, \dots, U'_{\omega',t}$, $\omega' \in \mathbb{N}$. Take a cell decomposition $(W'_i)_{i \in I'}$ of $\mathbb{R}^m \times N_\mu$ (identify $\mathbb{R}^m \times N_\mu$ with $\mathbb{R}^m \times \mathbb{R}^{n-1}$) which is compatible with the $\pi_\mu(U'_i)$, $i \le \omega'$.

For any $i \in I'$ the family $\pi_\mu^{-1}(W'_{i,t}) \cap G_p(H)_t \cap A_t$, $t \in B$, is thus included in the union of no more than κ $\frac{\alpha}{2}$-flat families.

Lemma 7.5.10 thus implies that for any $i \in I'$ and any $y \in \widetilde{\pi}_\mu(\mathbf{B}(\lambda_p, \frac{r}{2}))$, there exists $\widehat{\lambda} \in \mathbf{B}(\lambda_p, r) \cap \widetilde{\pi}_\mu^{-1}(y)$ such that for any $t \in B$:

$$d\big(\widehat{\lambda},\ \tau(\pi_\mu^{-1}(W'_{i,t}) \cap G_p(H)_t \cap A_t)\big) \ge \frac{\alpha}{2}. \tag{7.13}$$

Apply the induction hypothesis to get a regular system of hypersurfaces $H'' = (H''_k, \lambda''_k)_{k \le b''}$ of $B \times N_\mu$ (identify N_μ with $\mathbb{R}^{n-1}$, up to a refinement of the partition

$\mathscr{P}$) compatible with the $\delta W'_{i,t}$'s. Do it in such a way that all the λ''_k are elements of $\mathbf{B}(\widetilde{\pi}_\mu(\lambda_p), r')$ (where r' is given by (7.11)).

Define now:

$$\widehat{H}_{k,t} := \pi_\mu^{-1}(H''_{k,t}). \tag{7.14}$$

The compatibility with the families $\delta W'_{i,t}$ implies that every $int(G_k(H''))_t$ is included in $W'_{i,t}$ for some i (possibly refining the partition of the parameter space), by the same argument as the one we used in Step 1 for $G_k(H)$ and the partition $(W_i)_{i\in I}$.

As a matter of fact, according to (7.13) for $y = \lambda''_k$, we know that for every integer $k \le b''$ there exists $\widehat{\lambda}_k \in \mathbf{B}(\lambda_p, r) \cap \widetilde{\pi}_\mu^{-1}(\lambda''_k)$ such that for any $t \in B$:

$$d\big(\widehat{\lambda}_k\,,\ \tau(\pi_\mu^{-1}(int(G_k(H'')_t)) \cap G_p(H)_t \cap A_t)\big) \ge \frac{\alpha}{2}. \tag{7.15}$$

Let us check that $\widehat{H} := (\widehat{H}_k, \widehat{\lambda}_k)_{k\le\widehat{b}}$ (where $\widehat{b} := b''$) is the desired regular system of hypersurfaces. For this purpose, observe that, since neither μ nor $-\mu$ belongs to $\mathbf{B}(\lambda, \eta)$, we have for each k (recall that $r \le \frac{\eta}{4}$ and $\lambda_p \in \mathbf{B}(\lambda, \frac{\eta}{2})$, as well as $\widehat{\lambda}_k \in \mathbf{B}(\lambda_p, r)$):

$$d(\widehat{\lambda}_k, \pm\mu) \ge r.$$

By Remark 7.5.6, as $\widehat{\lambda}_k \in \widetilde{\pi}_\mu^{-1}(\lambda''_k)$, this implies that the family $\widehat{H}$ fulfills the conditions of Definition 7.5.2.

Furthermore, as $\mathbf{B}(\lambda_p, r) \subset \mathbf{B}(\lambda, \eta)$ (since $r \le \frac{\eta}{4}$ and $\lambda_p \in \mathbf{B}(\lambda, \frac{\eta}{2})$), all the $\widehat{\lambda}_k$'s belong to $\mathbf{B}(\lambda, \eta)$. Note also that as $\mathbf{B}(\lambda_p, r)$ is regular for $H_p \cup H_{p+1}$, the vector $\widehat{\lambda}_k$ belongs to $\Lambda_p(H)$. This completes the proof of the second step. □

The inconvenience of Step 2 (we would like to apply Lemma 7.5.8, see Step 4) is that the provided vector is regular for $A \cap G_p(H) \cap G_k(\widehat{H})$ (instead of $A \cap G_k(\widehat{H})$). If $\widehat{H}$ were an extension of the family H constructed in Step 1, this would be no problem since in this case we would have $G_k(\widehat{H}) \subset G_p(H)$ (or $int(G_k(\widehat{H})) \cap int(G_p(H)) = \emptyset$). Thus, we will have to find a common extension $\widetilde{H}$ of H and $\widehat{H}$ given by Steps 1 and 2 respectively. This is what is carried out in the proof of Step 3.

Step 3 Fix $p \le b$. There exists an extension $\widetilde{H} = (\widetilde{H}_k, \widetilde{\lambda}_k)_{k\le\widetilde{b}}$ of H which coincides with H outside $G_p(H)$, and such that for each k, $\widetilde{\lambda}_k$ belongs to $\mathbf{B}(\lambda, \eta)$ and is regular for $A \cap G_k(\widetilde{H}) \cap G_p(H)$.

Proof of Step 3 Let $\widehat{H}$ be the regular system given by Step 2 and let $k \le \widehat{b}$ be an integer. Because $\widehat{\lambda}_k \in \Lambda_p(H)$, by Proposition 7.5.5, the sets H_p and H_{p+1} are respectively the graphs for $\widehat{\lambda}_k$ of two functions ζ_k and ζ'_k. Moreover, the set $\widehat{H}_k$ is also the graph for $\widehat{\lambda}_k$ of a function $\widehat{\xi}_k$. Define:

$$\theta_k := \min(\max(\zeta_k, \widehat{\xi}_k), \zeta'_k)$$

in order to get a function whose graph for $\widehat{\lambda}_k$ is included in $G_p(H)$. We now define the desired regular family $(\widetilde{H}_k, \widetilde{\lambda}_k)_{1 \le k \le \widetilde{b}}$ as follows:

- Let $\widetilde{H}_k := H_k$ and $\widetilde{\lambda}_k := \lambda_k$ if $k < p$.
- Let $\widetilde{H}_p := H_p$ and $\widetilde{\lambda}_p := \widehat{\lambda}_1$.
- Let $\widetilde{H}_k$ be the graph of θ_{k-p} for $\widehat{\lambda}_{k-p}$, and let $\widetilde{\lambda}_k := \widehat{\lambda}_{k-p}$, whenever $p + 1 \le k \le p + \widehat{b}$.
- And finally let $\widetilde{H}_k := H_{k-\widehat{b}}$ and $\widetilde{\lambda}_k := \lambda_{k-\widehat{b}}$ if $p + \widehat{b} + 1 \le k \le b + \widehat{b}$.

Let us check that properties (i) and (ii) of Definition 7.5.2 hold for the family $\widetilde{H}$. For $k < p - 1$, or $k \ge p + \widehat{b} + 1$, the result is clear since the family $\widetilde{H}$ is indeed the family H (after renumbering).

For $k = p - 1$, properties (i) and (ii) for $\widetilde{H}$ follow from (i) and (ii) for H and Proposition 7.5.5 since we have assumed $\widehat{\lambda}_1 \in \Lambda_p(H)$.

It remains to check (i) and (ii) for $\widetilde{H}_{k+p}$, with $0 \le k \le \widehat{b}$. We start with (i). By (i) for $\widehat{H}$, the set $\widehat{H}_{k+1}$ is the graph for $\widehat{\lambda}_k$ of a function $\widehat{\xi}'_k$ such that $\widehat{\xi}_k \le \widehat{\xi}'_k$. Define now:

$$\theta'_k = \min(\max(\zeta_k, \widehat{\xi}'_k), \zeta'_k).$$

Claim The graph of θ'_k for $\widehat{\lambda}_k$ is that of θ_{k+1} for $\widehat{\lambda}_{k+1}$.

To see this, note that the graph of θ'_k (resp. θ_{k+1}) for $\widehat{\lambda}_k$ (resp. $\widehat{\lambda}_{k+1}$) matches with $\widehat{H}_{k+1}$ over $E(H_{p+1}, \widehat{\lambda}_k) \setminus E(H_p, \widehat{\lambda}_k)$ (resp. $\widehat{\lambda}_{k+1}$). But, by Proposition 7.5.5, the sets $E(H_p, l)$ and $E(H_{p+1}, l)$ do not depend on $l \in \Lambda_p(H)$. As $\widehat{\lambda}_k$ and $\widehat{\lambda}_{k+1}$ both belong to $\Lambda_p(H)$, this already shows that the two graphs involved in the above claim match over $int(G_p(H))$.

The graph of θ'_k (resp. θ_{k+1}) for $\widehat{\lambda}_k$ (resp. $\widehat{\lambda}_{k+1}$) is also constituted by the points of $H_p \setminus int(E(\widehat{H}_{k+1}, \widehat{\lambda}_k))$ (resp. $\widehat{\lambda}_{k+1}$) on the one hand and by the points of $H_{p+1} \cap E(\widehat{H}_{k+1}, \widehat{\lambda}_k)$ (resp. $\widehat{\lambda}_{k+1}$) on the other hand. By (ii) for $\widehat{H}$, the claim ensues.

This claim proves that $\widetilde{H}_{p+k+1}$, which is by definition the graph of θ_{k+1} for $\widehat{\lambda}_{k+1}$, is indeed also the graph of θ'_k for $\widehat{\lambda}_k$. Therefore, to check (i) (for $\widetilde{H}_{k+p}$, $k \le \widehat{b}$), we just have to prove that $\theta_k \le \theta'_k$. But, as $\widehat{\xi}_k \le \widehat{\xi}'_k$, this comes down from the respective definitions of θ'_k and θ_k.

Let us check property (ii) for $\widetilde{H}_{k+p}$, for $k \le \widehat{b}$. Observe first that if $k = \widehat{b}$, it is then a consequence of Proposition 7.5.5, since we have assumed that $\widehat{\lambda}_k$ belongs to $\Lambda_p(H)$.

Let now k be such that $0 \le k \le \widehat{b} - 1$. First note that by (ii) for $\widehat{H}$ we have:

$$E(\widehat{H}_{k+1}, \widehat{\lambda}_k) = E(\widehat{H}_{k+1}, \widehat{\lambda}_{k+1}).$$

But, $E(\widetilde{H}_{k+p+1}, \widehat{\lambda}_k)$ (resp. $\widehat{\lambda}_{k+1}$) is constituted by the points of $E(H_p, \widehat{\lambda}_k)$ (resp. $\widehat{\lambda}_{k+1}$) together with the points of $E(H_{p+1}, \widehat{\lambda}_k) \cap E(\widehat{H}_{k+1}, \widehat{\lambda}_k)$ (resp. $\widehat{\lambda}_{k+1}$). As $\widehat{\lambda}_{k+1}$ and $\widehat{\lambda}_k$ both belong to $\Lambda_p(H)$, this establishes (ii) for $\widetilde{H}$.

To complete the proof of Step 3, it remains to make sure that for every $k \le \hat{b}$ the vector $\widetilde{\lambda}_{k+p}$ is regular for $G_{k+p}(\widetilde{H}) \cap G_p(H) \cap A$. By definition, we have $\widetilde{\lambda}_p = \widehat{\lambda}_1$, $\widetilde{\lambda}_{k+p} = \widehat{\lambda}_k$ for $1 \le k \le \widehat{b}$, and:

$$G_{k+p}(\widetilde{H}) \subset G_k(\widehat{H}) \cap G_p(H), \tag{7.16}$$

for each $0 \le k \le \widehat{b}$.

As for any k the vector $\widehat{\lambda}_k$ is regular for $A \cap G_k(\widehat{H}) \cap G_p(H)$ (see Step 2), this implies that for each $k \le \widehat{b}$, the vector $\widetilde{\lambda}_{k+p}$ is regular for $A \cap G_{k+p}(\widetilde{H})$. This completes the proof of the third step. □

Step 4 There is a regular system of hypersurfaces $\check{H} = (\check{H}_k, \check{\lambda}_k)_{k \le \check{b}}$ compatible with A and such that $\check{\lambda}_k \in \mathbf{B}(\lambda, \eta)$ for all k.

Proof of Step 4 By Lemma 7.5.8 (applied $(\widetilde{b}+1)$ times to $\widetilde{H}$ of Step 3), we may extend $\widetilde{H}$ to a regular system $\check{H} = (\check{H}_k, \check{\lambda}_k)_{k \le \check{b}}$ compatible with the set

$$G_p(H) \cap \bigcup_{k=0}^{\widetilde{b}} G_k(\widetilde{H}) \cap A = G_p(H) \cap A.$$

By Step 3, we have $\check{\lambda}_k \in \mathbf{B}(\lambda, \eta)$ for all k (see Remark 7.5.9). Since $\check{H}$ is an extension of H ($\widetilde{H}$ being itself an extension of H) which coincides with H outside $G_p(H)$, we may carry out this construction on all the $G_p(H)$'s successively. This provides the desired regular system. □

It remains to prove Lemma 7.5.10. The lemma below describes a property of $\widetilde{\pi}_\mu$ that we need for this purpose.

Lemma 7.5.11 *Let $\mu \in \mathbf{S}^{n-1}$, $T \in \mathbb{G}^n$, and $x \in T$. If $v \in \mathbf{S}^{n-1}$ is tangent at x to the curve $\widetilde{\pi}_\mu^{-1}(\widetilde{\pi}_\mu(x))$ then we have:*

$$d(\mu, T) \le d(v, T).$$

Proof Let w be the vector of T which realizes $d(v, T)$. Remark that the vectors x, μ, and v are in the same two dimensional vector space. Moreover (x, v) is an orthonormal basis of this plane. Write $\mu = \alpha x + \beta v$ with $\alpha^2 + \beta^2 = 1$. Then, as x and w both belong to T we have

$$d(\mu, T) \le |\mu - (\alpha x + \beta w)| = |\beta| \cdot |v - w| \le d(v, T).$$

□

Proof of Lemma 7.5.10 We will work up to a ("projective") coordinate system of $\mathbf{S}^{n-1}$ defined as follows. Let U_i^+ (resp. U_i^-) denote

$$\{x \in \mathbf{S}^{n-1} : x_i \ge \epsilon\}$$

(resp. $x_i \leq -\epsilon$) with $\epsilon > 0$. Define then $h_i : U_i^+ \to \mathbb{R}^{n-1}$ (resp. $g_i : U_i^- \to \mathbb{R}^{n-1}$) by $h_i(x_1, \dots, x_n) = (\frac{x_1}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i})$ (resp. $g_i(x_1, \dots, x_n)$, here the $\widehat{\ }$ means the term is omitted). We can assume that $\mathbf{B}(l, r)$ entirely fits in one single U_i^+ or U_i^-, say U_i^+ (up to a linear change of coordinate of $\mathbb{R}^n$).

Through such a chart, the elements $\mathbf{S}^{n-1} \cap T$, $T \in C$, will be identified with their respective images, which are affine subspaces of $\mathbb{R}^{n-1}$. The set $\mathbf{S}^{n-1} \cap N_\mu$ also becomes an affine subspace Δ, and $\widetilde{\pi}_\mu$ an orthogonal projection along a line, say L. We denote by π this projection. By Lemma 7.5.11 and hypothesis (7.12), there exists $u > 0$ such that for any $T \in C$ (the angle between affine spaces is defined as the angle between the associated vector spaces):

$$\angle(L, T) \geq u. \tag{7.17}$$

We have to find $\alpha > 0$ such that for any $P_1, \dots, P_\kappa$ in C and any $y \in \pi(h_i(\mathbf{B}(l, \frac{r}{2})))$ there exists $\widehat{\lambda} \in h_i(\mathbf{B}(l, r)) \cap \pi^{-1}(y)$ such that:

$$d(\widehat{\lambda}, \bigcup_{i=1}^{\kappa} P_i) \geq \alpha. \tag{7.18}$$

For any $y \in \pi(h_i(\mathbf{B}(l, \frac{r}{2})))$, the length of $\pi^{-1}(y) \cap h_i(\mathbf{B}(l, r))$ is bounded below away from zero by a strictly positive real number α_0 (h_i is bi-Lipschitz).

It is an easy exercise of elementary geometry to derive from (7.17) that for any $\alpha > 0$ the set $\{x \in \pi^{-1}(y) : d(x, T) \leq \alpha\}$ is a segment of length not greater than $\frac{2\alpha}{u}$, for all $T \in C$ and $y \in \Delta$.

Let $\alpha := \frac{\alpha_0 u}{4\kappa}$. By the preceding paragraph, we see that if (7.18) failed for some $y \in \pi(h_i(\mathbf{B}(l, \frac{r}{2})))$, we could cover $\pi^{-1}(y) \cap h_i(\mathbf{B}(l, r))$ by κ segments of length not greater than $\frac{\alpha_0}{2\kappa}$. This contradicts the fact that the length of $\pi^{-1}(y) \cap h_i(\mathbf{B}(l, r))$ is not less than α_0. □

7.5.4 *Proof of Theorem 7.3.2*

By Theorem 7.5.4, there is a definable partition $\mathscr{P}$ of $\mathbb{R}^m$ such that for every $B \in \mathscr{P}$ there exists a regular system of hypersurfaces compatible with A_B. Fix $B \in \mathscr{P}$ and such a regular system of hypersurfaces $H = (H_k, \lambda_k)_{1 \leq k \leq b}$.

For each $t \in B$, we shall define the desired definable mapping h_t over $E(H_{k,t}, \lambda_k)$ by induction on k, in such a way that for all $t \in B$

$$h_t(E(H_{k,t}, \lambda_k)) = E(F_{k,t}, e_n),$$

where $F_{k,t}$ is the graph for e_n of $\eta_{k,t} : \mathbb{R}^{n-1} \to \mathbb{R}$, with $(\eta_{k,t})_{t \in B}$ uniformly Lipschitz definable family of functions.

For $k = 1$, choose an orthonormal basis of N_{λ_1} and set $h_t(q) := (x_{\lambda_1}, q_{\lambda_1})$, where x_{λ_1} stands for the coordinates of $\pi_{\lambda_1}(q)$ in this basis.

Let now $k \geq 1$. By (i) of Definition 7.5.2, for any $t \in B$ the sets $H_{k,t}$ and $H_{k+1,t}$ are the respective graphs for λ_k of two functions $\xi_{k,t}$ and $\xi'_{k,t}$. For $q \in E(H_{k+1,t}, \lambda_k) \setminus E(H_{k,t}, \lambda_k)$, extend h_t by defining $h_t(q)$ as the element:

$$h_t\big(\pi_{\lambda_k}(q) + \xi_{k,t}(\pi_{\lambda_k}(q)) \cdot \lambda_k\big) + \big(q_{\lambda_k} - \xi_{k,t}(\pi_{\lambda_k}(q))\big)e_n.$$

Thanks to the property (ii) of Definition 7.5.2, we have:

$$E(H_{k+1,t}, \lambda_{k+1}) = E(H_{k+1,t}, \lambda_k),$$

and hence h_t is actually defined over $E(H_{k+1,t}, \lambda_{k+1})$. Since $\xi_{k,t}$ is C-Lipschitz with C independent of t, the h_t's constitute a uniformly bi-Lipschitz family of homeomorphisms. Note also that the image of h_t is $E(F_{k+1,t}, e_n)$, where $F_{k+1,t}$ is the graph (for e_n) of the uniformly Lipschitz family of functions:

$$\eta_{k+1,t}(x) := \eta_{k,t}(x) + (\xi'_{k,t} - \xi_{k,t}) \circ \pi_{\lambda_k} \circ h_t^{-1}(x, \eta_{k,t}(x)),$$

for $(t, x) \in B \times \mathbb{R}^{n-1}$. This completes the induction step, giving h_t over $E(H_{b,t}, \lambda_b)$. To extend h_t to the whole of $\mathbb{R}^n$ do it similarly as in the case $k = 1$.

7.5.5 Regular Vectors and Set Germs

For R positive real number and $n > 1$ we set

$$\mathscr{C}_n(R) := \{(t, x) \in [0, +\infty) \times \mathbb{R}^{n-1} : |x| \leq Rt\}.$$

We also set $\mathscr{C}_1(R) := [0, +\infty)$.

Our purpose is to show Theorem 7.5.14, which is an improvement of Corollary 7.3.4 asserting that, in the case of germs of subsets of $\mathscr{C}_n(R)$, the provided homeomorphism may be required to preserve the first coordinate in the canonical basis. This fact is an essential ingredient of the Lipschitz conic structure theorem [34, 35], which recently proved very useful to study Sobolev spaces of bounded subanalytic domains [28, 29, 33, 34, 36].

Definition 7.5.12 Let $A, B \subset \mathbb{R}^n$. A definable map $h : A \to B$ is **vertical** if it preserves the first coordinate in the canonical basis of $\mathbb{R}^n$, i.e. if for any $t \in \mathbb{R}$, $\pi(h(t, x)) = t$, for all $x \in A_t$, where $\pi : \mathbb{R}^n \to \mathbb{R}$ is the orthogonal projection onto the first coordinate.

We start with a preliminary lemma which is of its own interest. We use the notation $f(t) \ll g(t)$ to express that $f(t) \le g(t)\phi(t)$, for some function ϕ tending to zero as t goes to zero.

Lemma 7.5.13 *Let $h : (X, 0) \to (\mathscr{C}_n(R), 0)$ be a germ of vertical definable map, with $X \subset \mathscr{C}_n(R')$, for some R and R'. If $(h_t)_{t\in\mathbb{R}}$ is uniformly Lipschitz then h is a Lipschitz map germ.*

Proof Suppose that h fails to be Lipschitz. Then, by Curve Selection Lemma, we can find two definable arcs in X, say $p(t)$ and $q(t)$, tending to zero along which:

$$|p(t) - q(t)| \ll |h(p(t)) - h(q(t))|. \tag{7.19}$$

We may assume that $p(t)$ (and so $h(p(t))$) is parametrized by its first coordinate (since the first coordinate of $p(t)$ induces a homeomorphism from a right-hand-side neighborhood of zero in $\mathbb{R}$ onto a right-hand-side neighborhood of zero in $\mathbb{R}$), i.e. we may assume $p(t) = (t, p_2(t), \dots, p_n(t))$.

As $p(t)$ and $h(p(t))$ are definable arcs in $\mathbb{C}_n(R')$ and $\mathbb{C}_n(R)$ respectively, we have:

$$|h(p(t)) - h(p(t'))| \sim |t - t'| \le |p(t) - q(t)| \tag{7.20}$$

and

$$|p(t) - p(t')| \sim |t - t'| \le |p(t) - q(t)|, \tag{7.21}$$

where t' denotes the first coordinate of $q(t)$.

Therefore, we can easily derive from (7.19), (7.20), and (7.21):

$$|h(p(t)) - h(q(t))| \sim |h(p(t')) - h(q(t))| \sim |p(t') - q(t)| \lesssim |p(t) - q(t)|,$$

in contradiction with (7.19). □

Theorem 7.5.14 *Let X be the germ at 0 of a definable subset of $\mathscr{C}_n(R)$ (for some R) of empty interior. There exists a germ of vertical bi-Lipschitz definable homeomorphism (onto its image) $H : (\mathbb{C}_n(R), 0) \to (\mathbb{C}_n(R), 0)$ such that e_n is regular for $H(X)$.*

Proof We denote by $e_1, \dots, e_n$ the canonical basis of $\mathbb{R}^n$ and by $e'_1, \dots, e'_{n-1}$ the canonical basis of $\mathbb{R}^{n-1}$ (so that $e_n = (0, e'_{n-1})$). Apply Theorem 7.3.2 to X, regarded as a family of $\mathbb{R} \times \mathbb{R}^{n-1}$. This provides a uniformly bi-Lipschitz family of homeomorphisms $h_t : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$, $t \in (0, \varepsilon)$, such that e'_{n-1} is regular for the family $(h_t(X_t))_{t\in\mathbb{R}}$. Up to a family of translations, we may assume that $h(t, 0) \equiv 0$, which implies that $H : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, $(t, x) \mapsto (t, h_t(x))$, maps $\mathbb{C}_n(R)$ into $\mathbb{C}_n(R')$ for some R' (and up to a family of homothetic transformations we may assume $R = R'$). By Lemma 7.5.13, the map-germ H is bi-Lipschitz near the origin.

We now have to check that e_n is regular for the germ of the definable set $Y := H(X)$. Suppose not. Then, by Curve Selection Lemma, there exists a definable arc $\gamma : [0, \epsilon] \to Y_{reg}$ with $\gamma(0) = 0$ and $e_n \in \tau := \lim_{t\to 0} T_{\gamma(t)} Y_{reg}$. But, as e'_{n-1} is regular for the family $(Y_t)_{t\in[0,\varepsilon]}$, we have $e'_{n-1} \notin \lim_{t\to 0} T_{\tilde{\gamma}(t)} Y_{\gamma_1(t)}$, if $\gamma(t) = (\gamma_1(t), \tilde{\gamma}(t)) \in \mathbb{R} \times \mathbb{R}^{n-1}$. This implies that:

$$\tau \cap N_{e_1} \neq \lim_{t\to 0} (T_{\gamma(t)} Y_{reg} \cap N_{e_1})$$

(since the latter does not contains the vector $e_n = (0, e'_{n-1})$ while the former does). Hence, τ cannot be transverse to N_{e_1} (since otherwise the intersection with the limit would be the limit of the intersection) which means that e_1 is orthogonal to τ. This implies that the limit vector $\lim_{t\to 0} \frac{\gamma(t)}{|\gamma(t)|} = \lim_{t\to 0} \frac{\gamma'(t)}{|\gamma'(t)|} \in \tau$ is orthogonal to e_1, from which we can conclude

$$\lim_{t\to 0} \frac{\gamma_1(t)}{|\gamma(t)|} = 0,$$

in contradiction with $\gamma(t) \in \mathbb{C}_n(R)$. □

7.6 Definable Bi-Lipschitz Triviality in Polynomially Bounded O-Minimal Structures

The results of this section are valid *under the extra assumption that the structure is polynomially bounded*. It is not difficult to produce counterexamples to these results (except however to Proposition 7.6.5) as soon as this assumption fails. We are going to establish a bi-Lipschitz triviality theorem for definable families (Theorem 7.6.3), from which we will derive a stratification result (Corollary 7.6.9).

We start with a preliminary lemma. As in Sect. 7.2.2, we denote by $\mathcal{F}$ the valuation field of the structure, which is the subfield of $\mathbb{R}$ constituted by all the real numbers α for which the function $(0, +\infty) \ni x \mapsto x^\alpha \in \mathbb{R}$ is definable.

Lemma 7.6.1 *Let $\xi : \mathbb{R}^{m+n} \to \mathbb{R}$ be a definable nonnegative function. There exist some definable subsets of $\mathbb{R}^{m+n}$, say $W_1, \dots, W_k$, and a definable partition $\mathcal{P}$ of $\mathbb{R}^{m+n}$ such that for any $V \in \mathcal{P}$ there are $\alpha_1, \dots, \alpha_k$ in $\mathcal{F}$ such that for each $t \in \mathbb{R}^m$ we have on $V_t \subset \mathbb{R}^n$:*

$$\xi_t(x) \sim d(x, W_{1,t})^{\alpha_1} \cdots d(x, W_{k,t})^{\alpha_k}. \tag{7.22}$$

Proof We prove it by induction on n. For $n = 1$, the result follows from Theorem 7.2.11. Let $n \geq 2$ and assume that the proposition is true for $(n - 1)$. Let $\lambda_1, \dots, \lambda_N$ be the elements of $\mathbf{S}^{n-1}$ given by Proposition 7.4.13.

For each i, applying the Preparation Theorem (Theorem 7.2.11) to $\xi \circ \Lambda_i : \mathbb{R}^{m+n} \to \mathbb{R}$, where Λ_i is an orthogonal linear mapping of $\mathbb{R}^{m+n}$ sending $(0_{\mathbb{R}^m}, e_n)$ onto $(0_{\mathbb{R}^m}, \lambda_i)$ and preserving the m first coordinates (below, we sometimes regard Λ_i as a transformation of $\mathbb{R}^n$), we obtain a partition of $\mathbb{R}^{m+n}$. The images of all the elements of this partition under the map Λ_i provide a new partition of $\mathbb{R}^{m+n}$, denoted by $\mathscr{P}_i$. Let $(V_j)_{j\in J}$ be a common refinement of the $\mathscr{P}_i$'s. Applying Proposition 7.4.13 to the finite family constituted by all the sets of the partition $(V_j)_{j\in J}$, we get a partition Σ of $\mathbb{R}^{m+n}$ into cells.

Let $E \in \Sigma$ be an open cell. By construction and Proposition 7.4.13, there is $i \le N$ such that λ_i is regular for δE. It means that e_n is regular for the family $(\Lambda_i^{-1}(\delta E)_t)_{t\in\mathbb{R}^m}$. Hence, it follows from Proposition 7.4.6 that there is a partition $\mathscr{Q}_E$ of $\Lambda_i^{-1}(cl(E))$ into cells, such that each element C is either the graph of a uniformly Lipschitz family of functions or a set of the form

$$C = \{(z, y) \in B \times \mathbb{R} : \eta_1(z) < y < \eta_2(z)\}, \tag{7.23}$$

with $B \in \mathscr{S}_{m+n-1}$ and $\eta_1 < \eta_2$ definable functions on B such that $(\eta_{1,t})_{t\in\mathbb{R}^m}$ and $(\eta_{2,t})_{t\in\mathbb{R}^m}$ are uniformly Lipschitz.

Observe that it suffices to show the desired statement for the restriction to each cell $C \in \mathscr{Q}_E$ of the family of functions $\xi_t \circ \Lambda_i$, $t \in \mathbb{R}^m$. For the elements C of the partition $\mathscr{Q}_E$ which are graphs of some uniformly Lipschitz family of functions, one may easily deduce the result from the induction hypothesis.

Fix thus a cell $C \subset \Lambda_i^{-1}(E)$ as in (7.23). There is j such that $C \subset \Lambda_i^{-1}(V_j)$. By construction, there are $r \in \mathscr{F}$ and some functions a and θ on the basis B of C such that for $x = (\tilde{x}, x_n) \in C_t$, $t \in \mathbb{R}^m$, we have:

$$\xi_t \circ \Lambda_i(x) \sim a_t(\tilde{x})|x_n - \theta_t(\tilde{x})|^r.$$

Thanks to the induction hypothesis we thus only have to check the result for the function $|x_n - \theta_t(\tilde{x})|$.

As $\Gamma_\theta \cap C = \emptyset$, we can assume that for every $(t, \tilde{x}) \in B$, either $\theta_t(\tilde{x}) \le \eta_{1,t}(\tilde{x})$ or $\theta_t(\tilde{x}) \ge \eta_{2,t}(\tilde{x})$. Assume for instance that $\theta_t(\tilde{x}) \le \eta_{1,t}(\tilde{x})$. Writing for $t \in \mathrm{supp}_m(C)$ and $x = (\tilde{x}, x_n) \in C_t$:

$$x_n - \theta_t(\tilde{x}) = (x_n - \eta_{1,t}(\tilde{x})) + (\eta_{1,t}(\tilde{x}) - \theta_t(\tilde{x})),$$

we see that (up to a partition of C we may assume that the terms of the right-hand-side are comparable for the partial order relation $\le$) $|x_n - \theta_t(\tilde{x})|$ is $\sim$ either to $|x_n - \eta_{1,t}(\tilde{x})|$ or to $|\eta_{1,t}(\tilde{x}) - \theta_t(\tilde{x})|$.

For the latter functions, since they are $(n-1)$-variable functions, the desired result is a consequence of the induction hypothesis. Moreover, since $\eta_{1,t}$ is Lipschitz, $|x_n - \eta_{1,t}(\tilde{x})|$ is $\sim$ to the distance to the graph of $\eta_{1,t}$ for every t. This shows the result for the given cell C. □

Remark 7.6.2 The constants of the equivalence in the above lemma depend on t. However, the family of exponents $\alpha_1, \dots, \alpha_k$ just depends on $V \in \mathscr{P}$.

We recall that the structure is assumed to be polynomially bounded in this section.

Theorem 7.6.3 *Given $A \in \mathcal{S}_{m+n}$, there exists a definable partition of $\mathbb{R}^m$ such that A is definably bi-Lipschitz trivial along each element of this partition.*

Proof We prove the result by induction on n. We shall show that the trivialization H may be required to induce a trivialization of some given definable subsets of A.

Let $A \in \mathcal{S}_{m+n}$ and let $C_1, \dots, C_k$ be some definable subsets of A. Apply Theorem 7.3.2 to the set $\{(t, x) : x \in \delta A_t \cup \bigcup_{i=1}^{k} \delta C_{i,t}\}$. This provides a definable family of bi-Lipschitz maps $G_t : \mathbb{R}^n \to \mathbb{R}^n$, $t \in \mathbb{R}^m$, such that e_n is regular for the families of sets $(\delta G_t(C_{i,t}))_{t\in\mathbb{R}^m}$, $i = 1, \dots, k$, and $(\delta G_t(A_t))_{t\in\mathbb{R}^m}$.

As we can work up to a family of bi-Lipschitz maps, we will identify G_t with the identity map. By Propositions 7.4.2, 7.4.6, and 7.4.7, we can find some definable functions $\xi_1 \le \dots \le \xi_s$ on $\mathbb{R}^{m+n-1}$, with $(\xi_{i,t})_{t\in\mathbb{R}^m}$ uniformly Lipschitz for all i, and a cell decomposition $\mathscr{D}$ of $\mathbb{R}^{m+n-1}$ such that A and the C_j's are unions of some graphs $\Gamma_{\xi_{i|D}}$, $i \in \{1, \dots, s\}$, $D \in \mathscr{D}$, or bands $(\xi_{i|D}, \xi_{i+1|D})$, $i \in \{0, \dots, s\}$, $D \in \mathscr{D}$ (where $\xi_0 \equiv -\infty$ and $\xi_{s+1} \equiv +\infty$).

Refining the cell decomposition $\mathscr{D}$ if necessary (without changing notations), we can assume it to be compatible with the zero loci of the functions $(\xi_{i+1} - \xi_i)$. By Lemma 7.6.1, up to an extra refinement of the cell decomposition, we can assume that there are finitely many definable subsets $W_1, \dots, W_c$ of $\mathbb{R}^m \times \mathbb{R}^{n-1}$ such that on every cell we can find $r_1, \dots, r_c$ in $\mathscr{F}$ such that for all $i = 1, \dots, s-1$ and any $t \in \mathbb{R}^m$:

$$\xi_{i+1,t}(x) - \xi_{i,t}(x) \sim d(x, W_{1,t})^{r_1} \cdots d(x, W_{c,t})^{r_c}. \tag{7.24}$$

Refining one more time the cell decomposition $\mathscr{D}$, we may assume that the W_i's are unions of cells.

Applying now the induction hypothesis to the cells of $\mathscr{D}$ provides a partition $\mathscr{P}$. Fix $B \in \mathscr{P}$ and let $H(t, x) = (t, h_t(x))$ denote the obtained trivialization of $B \times \mathbb{R}^{n-1}$ along B. We have $h_t(C_{t_0}) = C_t$ for some $t_0 \in B$ and for all $C \in \mathscr{D}$. We are going to lift the trivialization H to an trivialization of $B \times \mathbb{R}^n$.

Given a point $(t, x) \in B \times \mathbb{R}^{n-1}$ and $1 \le i \le s-1$ let

$$\widetilde{H}(t, x, \nu\xi_{i,t_0}(x) + (1-\nu)\xi_{i+1,t_0}(x)) := (t, h_t(x), \nu\xi_{i,t}(h_t(x)) + (1-\nu)\xi_{i+1,t}(h_t(x)),$$

for all $\nu \in [0, 1]$. Set also for $\nu \in (0, \infty)$:

$$\widetilde{H}(t, x, \xi_{1,t_0}(x) - \nu) := (t, h_t(x), \xi_{1,t}(h_t(x)) - \nu),$$

as well as

$$\widetilde{H}(t, x, \xi_{s,t_0}(x) + \nu) := (t, h_t(x), \xi_{s,t}(h_t(x)) + \nu).$$

Because $\mathscr{D}$ is compatible with the zero loci of the functions $(\xi_{i+1} - \xi_i)$ and since the trivialization h was required to preserve the cells of $\mathscr{D}$, it is easily seen that $\widetilde{H}_t$ is a continuous mapping for each $t \in \mathbb{R}^m$. Observe also that, since the W_i's are unions of cells of $\mathscr{D}$, we have $h_t(W_{i,t_0}) = W_{i,t}$, for all i. Since h_t is bi-Lipschitz for every $t \in B$, we can derive from (7.24), that for each $t \in B$ we have:

$$(\xi_{i+1,t} - \xi_{i,t}) \circ h_t \sim (\xi_{i+1,t_0} - \xi_{i,t_0}).$$

This shows the bi-Lipschitzness of $\widetilde{H}_t$ on the sets $[\xi_{i,t|D_t}, \xi_{i+1,t|D_t}]$, $D \in \mathscr{D}$, $D \subset B$, $i < s$. The bi-Lipschitzness of $\widetilde{H}_t$ on the sets $(-\infty, \xi_{1,t|D_t})$ and $(\xi_{s,t|D_t}, +\infty)$ is clear since the $(\xi_{i,t})_{t\in B}$ are families of Lipschitz functions.

The continuity of H_t and H_t^{-1} with respect to t follows from a well-known fact, up to an extra refinement of the partition of the parameter space [6, Lemma 5.17 and Exercise 5.21]. □

Remark 7.6.4 We have proved a stronger statement since the trivialization is also defined on the ambient space $B \times \mathbb{R}^n$. We can also require the trivialization to preserve a finite number of given definable subfamilies of A.

In Theorem 7.6.3, the constructed trivialization h_t is Lipschitz for every t (see Definition 7.2.7). The Lipschitz condition may also be required to hold with respect to the parameter t on relatively compact sets, as it will be established by Proposition 7.6.6, which requires the following proposition.

Proposition 7.6.5 *Let $A \in \mathcal{S}_{m+n}$ and let $f_t : A_t \to \mathbb{R}$ be a definable family of functions. If f_t is Lipschitz for all $t \in \mathbb{R}^m$ then there exists a definable partition $\mathscr{P}$ of $\mathbb{R}^m$ such that for every $B \in \mathscr{P}$, $f : A \to \mathbb{R}$, $(t, x) \mapsto f_t(x)$ induces a Lipschitz function on $A \cap K$, for every compact subset K of $B \times \mathbb{R}^n$.*

Proof We prove the result by induction on m. The case $m = 0$ being vacuous, assume the result to be true for $(m - 1)$, $m \geq 1$. By Proposition 7.4.2 (see Remark 7.4.3), we may assume that $A = \mathbb{R}^{m+n}$. It is well-known that there is a definable partition $\mathscr{P}$ of the parameter space, such that f is continuous on every $B \times \mathbb{R}^n$, $B \in \mathscr{P}$ (again, see [6, Lemma 5.17 and Exercise 5.21]). Fix an element $B \in \mathscr{P}$ (we shall refine several times the partition $\mathscr{P}$).

We start with the (easier) case where $\dim B < m$. In this case, there is a partition of B such that every element of this partition has a regular vector (using for instance Remark 7.4.11), that, without loss of generality, we can assume to be $e_m \in \mathbf{S}^{m-1}$. Thanks to Proposition 7.4.6, it is therefore enough to deal with the case where B is the graph of a Lipschitz function, say $\xi : D \to \mathbb{R}$, $D \in \mathcal{S}_{m-1}$. The result in this case now follows from the induction hypothesis applied to the function $D \times \mathbb{R}^n \ni (t, x) \mapsto f(t, \xi(t), x)$.

We now address the case $\dim B = m$. The function $B \ni t \mapsto L_{f_t}$ being definable, partitioning B if necessary, we can assume this function to be continuous on this set. In particular, it is bounded on compact subsets of B. Let Z be the set of points $q \in \Gamma_f$ for which there exists a sequence $q_k \in (\Gamma_f)_{reg}$ tending to q such that

$$(0_{\mathbb{R}^m}, e_{n+1}) \in \lim T_{q_k}(\Gamma_f)_{reg},$$

where e_{n+1} is the last vector of the canonical basis of $\mathbb{R}^{n+1}$. Let $\pi : \mathbb{R}^m \times \mathbb{R}^{n+1} \to \mathbb{R}^m$ denote the projection omitting the last $(n+1)$ coordinates. We claim that $\pi(Z)$ has dimension less than m.

Assume otherwise. Take a (w)-regular definable stratification of Γ_f compatible with Z and let $S \subset Z$ be a stratum such that $\pi(S)$ has dimension m. Let S' be the set of points of S at which $\pi_{|S}$ is a submersion. Since $\pi(S)$ is of dimension m, by Sard's Theorem, the set S' cannot be empty. Moreover, by definition of S', $T_q S'$ is transverse to $\{0_{\mathbb{R}^m}\} \times \mathbb{R}^{n+1}$ at any point q of S'.

Let $q \in S' \subset Z$. By definition of Z, there is a sequence q_k tending to q such that $(0_{\mathbb{R}^m}, e_{n+1}) \in \tau_q := \lim T_{q_k}(\Gamma_f)_{reg}$. The (w) condition ensures that $\tau_q \supset T_q S'$ (S' is a manifold for it is open in S). Consequently, τ_q is transverse to $\{0_{\mathbb{R}^m}\} \times \mathbb{R}^{n+1}$ as well.

But since L_{f_t} is locally bounded (it was assumed to be continuous), the vector e_{n+1} does not belong to $\lim T_{x_k}\Gamma_{f_{t_k}}$, if $q_k = (t_k, x_k)$ in $\mathbb{R}^m \times \mathbb{R}^{n+1}$, which means that

$$(\lim T_{q_k}\Gamma_f) \cap \{0_{\mathbb{R}^m}\} \times \mathbb{R}^{n+1} \neq \lim \left(T_{q_k}\Gamma_f \cap \{0_{\mathbb{R}^m}\} \times \mathbb{R}^{n+1}\right)$$

(since the latter does not contain the vector $(0_{\mathbb{R}^m}, e_{n+1})$ while the former does), and hence, that τ_q cannot be transverse to $\{0_{\mathbb{R}^m}\} \times \mathbb{R}^{n+1}$ (since otherwise the intersection with the limit would be the limit of the intersection). A contradiction.

This establishes that $\dim \pi(Z) < m$. Since we can refine $\mathscr{P}$ into a partition which is compatible with $\pi(Z)$, we thus see that we can suppose $B \subset \mathbb{R}^m \setminus \pi(Z)$ (we are dealing with the case $\dim B = m$).

For $(t, R) \in (\mathbb{R}^m \setminus \pi(Z)) \times [0, +\infty)$ set:

$$\varphi(t, R) := \sup\{\frac{\partial f}{\partial t}(t, x) : x \in \mathbf{B}(0_{\mathbb{R}^n}, R),\ f \text{ is } \mathscr{C}^1 \text{ at } x\}$$

(which is finite, by definition of Z, since L_{f_t} is bounded). As φ is definable, up to a partition of B, this function may be assumed to be continuous (and thus bounded on compact sets) for $R \geq \zeta(t)$, with $\zeta : B \to \mathbb{R}$ definable function. The function f therefore induces a function which is Lipschitz with respect to the inner metric on every compact set of $B \times \mathbb{R}^n$. By Theorem 7.4.1, up to an extra refinement partition, we can suppose that the inner metric and the outer metric of B are equivalent, which means that so are the inner and outer metrics of $B \times \mathbb{R}^n$, establishing that f is Lipschitz on every compact set of $B \times \mathbb{R}^n$. □

As a matter of fact, the trivialization given by Theorem 7.6.3 may be required to satisfy the Lipschitz condition with respect to the parameters on compact sets:

Proposition 7.6.6 *Let $A \in \mathcal{S}_{m+n}$. Refining the partition provided by Theorem 7.6.3, we may obtain the following extra fact: for any element B of this partition, the trivialization $H : B \times A_{t_0} \to A_B$, $(t, x) \mapsto (t, h_t(x))$, induces a bi-Lipschitz mapping on $(B \times A_{t_0}) \cap K$, for every compact subset K of $B \times \mathbb{R}^n$.*

Proof This is a consequence of Theorem 7.6.3 and Proposition 7.6.5. □

Remark 7.6.7 As explained in Remark 7.2.8, refining the partition if necessary, we can assume that the Lipschitz constants of h_t and h_t^{-1} are locally bounded on each element B of the partition. It is indeed worthy of notice that, thanks to Łojasiewicz inequality, we can show that $L_{h_t} \lesssim d(t, cl(B) \setminus B)^{-k}$, with $k \in \mathbb{N}$. This is an attractive feature of Theorem 7.6.3 since bi-Lipschitz trivializations that are obtained by integration of vector fields (like on Mostowski's Lipschitz stratifications, see Sect. 7.2.1) generally have Lipschitz constants that are bounded by functions of type $\exp\left(\frac{C}{d(t,cl(B)\setminus B)}\right)$, $C > 0$, (by Gronwall's inequality).

In the above proposition, the compactness assumption is essential, as shown by the following example.

Example 7.6.8 Consider the set $A = \{(t, x, y) \in \mathbb{R}^3 : y = tx\}$. This set is bi-Lipschitz trivial along $\mathbb{R}$. However, it is easy to check that we could not require a trivialization $H(t, x, y)$ to be bi-Lipschitz with respect the parameter t, even along a compact interval (i.e., we have to require that x and y also remain in a compact set in order to ensure Lipschitzness with respect to t).

The inconvenience of bi-Lipschitz triviality theorems that are provided by integration of Lipschitz vector fields, such as the bi-Lipschitz version of Thom-Mather First Isotopy Lemma that holds on Mostowski's Lipschitz stratifications [17, 23], is that they do not provide definable trivializations. Theorem 7.6.3 enables us to construct stratifications that are locally definably bi-Lipschitz trivial, which is valuable for applications [34].

Corollary 7.6.9 *Given a definable set X, we can find a definable stratification of this set which is locally definably bi-Lipschitz trivial. This stratification may be required to be compatible with finitely many given definable subsets of X.*

Proof This follows from standard arguments of construction of definable stratifications. Theorem 7.6.3 and Proposition 7.6.5 yield that local definable bi-Lipschitz triviality holds generically, which is sometimes rephrased by saying that it is a stratifying condition (see for instance [35, Proposition 2.7.5] for more details). We can require our stratification to satisfy Whitney's (a) condition (which is also a stratifying condition [5, 27, 35]), which yields that Σ_{x_0} (in (ii) of Definition 7.2.9) exclusively consists of manifolds. □

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Chapter 8
Lipschitz Geometry of Real Semialgebraic Surfaces

Lev Birbrair and Andrei Gabrielov

Contents

Abstract We present here basic results in Lipschitz Geometry of semialgebraic surface germs. Although bi-Lipschitz classification problem of surface germs with respect to the inner metric was solved long ago, classification with respect to the outer metric remains an open problem. We review recent results related to the outer and ambient bi-Lipschitz classification of surface germs. In particular, we explain why the outer bi-Lipschitz classification is much harder than the inner classification, and why the ambient Lipschitz Geometry of surface germs is very different from their outer Lipschitz Geometry. In particular, we show that the ambient Lipschitz Geometry of surface germs includes all of the Knot Theory.

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8.1 Introduction

Lipschitz classification of semialgebraic surfaces has become in recent years one of the central questions of the Metric Geometry of Singularities. It was stimulated by the finiteness theorems of Mostowski, Parusinski and Valette (see [15, 16, 20]). They proved that there are finitely many Lipschitz equivalence classes in any semialgebraic family of semialgebraic sets. Lipschitz classification is intermediate between Smooth (too fine) and Topological (too coarse) classifications. For example, smooth classification of most singularities is not finite. It may be even infinite dimensional for non-isolated singularities.

Here we review recent developments in Lipschitz Geometry of semialgebraic surfaces (two-dimensional real semialgebraic sets). Since we are mainly interested in singularities of semialgebraic surfaces, our main object is a semialgebraic surface germ $(X, 0)$ at the origin of $\mathbb{R}^n$. Note that most results presented in this paper remain true for subanalytic sets, and for the sets definable in a polynomially bounded o-minimal structure.

A connected semialgebraic set $X \subset \mathbb{R}^n$ inherits from $\mathbb{R}^n$ two metrics: the **outer metric** $dist(x, y) = |y - x|$ and the **inner metric** $idist(x, y) =$ length of the shortest path in X connecting x and y. Note that $dist(x, y) \leq idist(x, y)$. A semialgebraic set is called Lipschitz Normally Embedded if these two metrics are equivalent (see Definition 8.2.3).

For the surface germs, there are three natural equivalence relations:

1. Inner Lipschitz equivalence: $(X, 0) \sim_i (Y, 0)$ if there is a homeomorphism $h : (X, 0) \to (Y, 0)$ bi-Lipschitz with respect to the inner metric.
2. Outer Lipschitz equivalence: $(X, 0) \sim_o (Y, 0)$ if there is a homeomorphism $h : (X, 0) \to (Y, 0)$ bi-Lipschitz with respect to the outer metric.
3. Ambient Lipschitz equivalence: $(X, 0) \sim_a (Y, 0)$ if there is an orientation preserving bi-Lipschitz homeomorphism $H : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $H(X) = Y$.

Inner Lipschitz Geometry of surface germs is relatively simple. The building block of the inner Lipschitz classification of surface germs is a β-**Hölder triangle** (see Definition 8.2.1). A surface germ $(X, 0)$ with an isolated singularity and connected link is inner Lipschitz equivalent to a β-**horn** (see Definition 8.2.2). If the singularity is not isolated, classification is made by the theory of Hölder Complexes (see [1]). A **Hölder Complex** is a triangulation (decomposition into Hölder triangles) of a surface germ. Canonical Hölder Complex (see Definition 8.2.10) is a complete invariant of the inner Lipschitz equivalence of surface germs.

Outer Lipschitz Geometry of surface germs is much more complicated. For example, the germs of all irreducible complex curves are inner Lipschitz equivalent to $(\mathbb{C}, 0)$, while the outer Lipschitz classification of the germs of complex plane curves is described by their sets of essential Puiseux pairs (see [12, 17]). Even for the union of two normally embedded Hölder triangles, the outer Lipschitz Geometry is not simple (see [3]).

A special case of a surface germ is the union of a Hölder triangle T and a graph of a Lipschitz semialgebraic function f defined on T. The outer Lipschitz equivalence of two such surface germs is equivalent to the Lipschitz contact equivalence of the two functions. This relates outer Lipschitz Geometry of surface germs with the Lipschitz Geometry of functions. A complete invariant of the contact equivalence class of a Lipschitz function f defined on a Hölder triangle T, called a "pizza," is defined in [9]. Informally, a pizza is a decomposition of T into "slices," Hölder sub-triangles $\{T_i\}$ of T, such that the order of f on each arc $\gamma \subset T_i$ depends linearly on the order of contact of γ with a boundary arc of T_i.

For the general surface germs, Lipschitz classification with respect to the outer metric is still an open problem.

The study of Lipschitz Normally Embedded, or simply Normally Embedded, sets was initiated by Kurdyka and Orro [14]. Kurdyka proved that any semialgebraic set admits a "pancake decomposition," a finite partition into normally embedded subsets. Using this partition, Kurdyka and Orro proved that any semialgebraic set admits a semialgebraic "pancake metric" equivalent to the inner metric. Normal Embedding theorem of Birbrair and Mostowski states that, for any semialgebraic set X, there is a semialgebraic and bi-Lipschitz with respect to the inner metric embedding $\Psi : X \to \mathbb{R}^m$, where $m \geq 2\dim(X) + 1$ (see [10]). Lipschitz Normal Embedding of complex analytic sets is addressed in the paper by Anne Pichon in the present volume.

The set of semialgebraic arcs in $(X, 0)$ parameterized by the distance to the origin is called the Valette link $V(X)$ of the germ $(X, 0)$ (see Definition 8.3.6). The tangency order of arcs (see Definition 8.3.9) defines a non-archimedean metric on $V(X)$.

A pancake decomposition is called minimal if it is not a refinement of another pancake decomposition. A natural question related to Lipschitz Normal Embedding of surface germs is uniqueness of a minimal pancake decomposition. The answer is negative even for a Hölder triangle. Gabrielov and Sousa in [13] gave examples of Hölder triangles having several combinatorially non-equivalent minimal pancake decompositions.

Relations between the ambient and outer equivalence of surface germs were studied in [2, 5, 6]. In the paper [2] the authors presented several outer Lipschitz and ambient topologically equivalent families of surface germs in $\mathbb{R}^3$ and $\mathbb{R}^4$, which are pairwise ambient Lipschitz non-equivalent. In [5, 6], several "Universality Theorems" were formulated. Informally, these theorems state that, even when the link of a surface germ is topologically a trivial knot, the classification problem of the ambient Lipschitz equivalence of such surface germs "contains all of the knot theory."

8.2 Inner Lipschitz Equivalence

Definition 8.2.1 For $1 \leq \beta \in \mathbb{Q}$, the standard β-Hölder triangle T_β is the germ at the origin of $\mathbb{R}^2$ of the surface $\{x \geq 0,\ 0 \leq y \leq x^\beta\}$ (see Fig. 8.1a). A *β-Hölder triangle* is a surface germ inner Lipschitz equivalent to T_β.

Definition 8.2.2 For $1 \leq \beta \in \mathbb{Q}$, the standard β-horn C_β is the germ at the origin of $\mathbb{R}^3$ of the surface $\{z \geq 0,\ x^2 + y^2 = z^{2\beta}\}$ (see Fig. 8.1b). A *β-horn* is a surface germ inner Lipschitz equivalent to C_β.

Definition 8.2.3 A semialgebraic set X is called *Lipschitz Normally Embedded* (LNE) if the inner and outer metrics on X are equivalent: $dist(x, y) \leq idist(x, y) \leq C\, dist(x, y)$ for some constant $C > 0$ and all $x, y \in X$.

For example, the germ of an algebraic curve $\{x^3 = y^2\}$ is not LNE, while the standard β-horn C_β is LNE.

Theorem 8.2.4 *Given the germ $(X, 0)$ of a semialgebraic surface with isolated singularity and connected link, there is a unique rational number $\beta \geq 1$ such that $(X, 0)$ is inner Lipschitz equivalent to the standard β-horn C_β.*

Birbrair's theory of Hölder Complexes (see [1]) is a generalization of Theorem 8.2.4 for the surface germs with non-isolated singularities.

Definition 8.2.5 A *Formal Hölder Complex* is a pair (G, β), where G is a graph and $\beta : E_G \to \mathbb{Q}_{\geq 1}$ is a function, where E_G the set of edges of G.

Definition 8.2.6 A *Geometric Hölder Complex* corresponding to a Formal Hölder Complex (G, β) is a surface germ $(X, 0)$ such that

1. For small $\varepsilon > 0$, the intersection of X with the ε-ball B_ε is homeomorphic to the cone over G, and the intersection of X with the ε-sphere S_ε is homeomorphic to G.
2. For any edge $g \in E_G$, the subgerm of $(X, 0)$ corresponding to g is a $\beta(g)$-Hölder triangle.

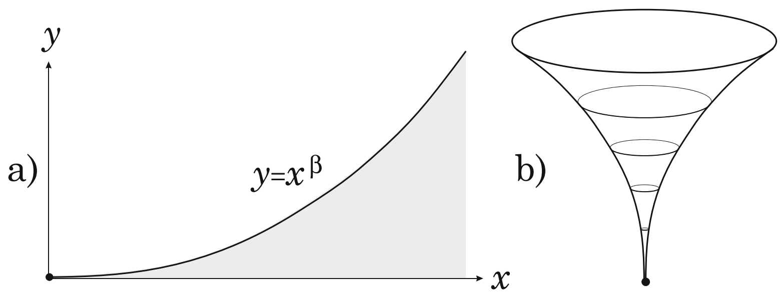

Fig. 8.1 A β-Hölder triangle and a β-horn

Theorem 8.2.7 *For any surface germ* $(X, 0) \subset \mathbb{R}^n$*, there exists a Formal Hölder Complex* (G, β) *such that* $(X, 0)$ *is a Geometric Hölder Complex corresponding to* (G, β).

Remark 8.2.8 For a given surface germ $(X, 0)$, the Formal Hölder Complex (G, β) in Theorem 8.2.7 is not unique. The simplification procedure described below reduces it to the unique Canonical Hölder Complex corresponding to $(X, 0)$.

Definition 8.2.9 We say that a vertex v_0 of the graph G is *non-critical* if it is adjacent to exactly two edges g_1 and g_2 of G, and these edges connect v_0 with two different vertices of G. A vertex v_0 of G is called a *loop vertex* if it is adjacent to exactly two different edges g_1 and g_2 of G, and these edges connect v_0 with the same vertex v_1 of G. The other vertices of G (neither non-critical nor loop vertices) are called *critical*.

Definition 8.2.10 An Abstract Hölder Complex (G, β) is *Canonical* if

1. All vertices of G are either critical or loop vertices;
2. For any loop vertex v of G adjacent to the edges g_1 and g_2, one has $\beta(g_1) = \beta(g_2)$.

Now we define a **simplification procedure**, reducing an Abstract Hölder Complex (G, β) to a Canonical one.

We start with eliminating non-critical vertices. Let v_0 be a non-critical vertex of G, connected with the vertices v_1 and v_2 of G by the edges g_1 and g_2. Then we remove the vertex v_0 from the set of vertices of G, and replace the edges g_1 and g_2 of G with the single edge g_0 connecting v_1 with v_2. Let G' be the graph obtained from G by this operation. We define an abstract Hölder complex (G', β'), setting $\beta'(g_0) = \min(\beta(g_1), \beta(g_2))$ and $\beta'(g) = \beta(g)$ for all edges g of G' other than g_0 (see Fig. 8.2a).

We repeat this operation until there are no non-critical vertices. After that, we take care of the loop vertices of G.

Let (G, β) be an Abstract Hölder Complex without non-critical vertices. If a loop vertex v_0 of G is connected by the edges g_1 and g_2 with the same vertex v_1, such that $\beta(g_1) \neq \beta(g_2)$, we define an Abstract Hölder Complex (G, β'), replacing $\beta_1 = \beta(g_1)$ and $\beta_2 = \beta(g_2)$ with $\beta'(g_1) = \beta'(g_2) = \min(\beta_1, \beta_2)$ (see Fig. 8.2b). We repeat this operation for all loop vertices of G.

The main results of the paper [1] are the following:

Theorem 8.2.11 (Inner Lipschitz Classification Theorem) *The surface germs* $(X, 0)$ *and* $(X', 0)$ *are Lipschitz equivalent with respect to the inner metric if, and only if, the corresponding Canonical Hölder Complexes are combinatorially equivalent.*

Theorem 8.2.12 (Realization Theorem) *Let* (G, β) *be an Abstract Hölder Complex. Then there exists a surface germ* $(X, 0)$ *such that* $(X, 0)$ *is a Geometric Hölder Complex corresponding to* (G, β).

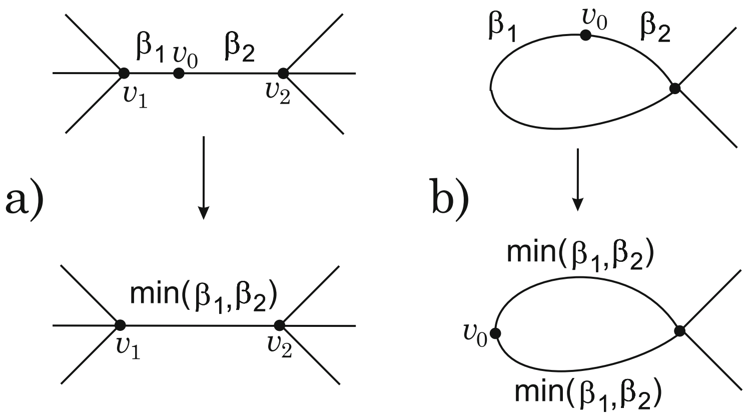

Fig. 8.2 Simplification of Hölder complexes

Remark 8.2.13 The theory of Hölder Complexes implies that a germ $(X, 0)$ of an irreducible complex curve, considered as a real surface germ, is inner Lipschitz equivalent to the germ $(\mathbb{C}, 0)$. Otherwise $(X, 0)$ is inner Lipschitz equivalent to the union of finitely many germs $(\mathbb{C}, 0)$ pinched at the origin.

8.3 Normal Embedding Theorem, Lipschitz Normally Embedded Sets

Examples of Lipschitz Normally Embedded (LNE) Surface Germs

1. The standard β-horn C_β is LNE.
2. A germ of an irreducible complex curve is LNE if, and only if, it is smooth.

There are many examples of not normally embedded surface germs. On the other hand, we have the following result:

Theorem 8.3.1 (See [7, 10]) *Let $X \subset \mathbb{R}^m$ be a connected semialgebraic set. Then there exist a normally embedded semialgebraic set $\tilde{X} \subset \mathbb{R}^q$, where $q \leq 2\dim X + 1$, and an inner bi-Lipschitz homeomorphism $p : X \to \tilde{X}$. This map is called a normal embedding of X.*

Definition 8.3.2 A subset $\tilde{X} \subset \mathbb{R}^m$ is called *Lipschitz Normally Embedded* if there exists a bi-Lipschitz homeomorphism $\Psi : \tilde{X}_{inner} \to \tilde{X}_{outer}$.

Here $\tilde{X}_{inner}$ means the space $\tilde{X}$ equipped with the inner metric, and $\tilde{X}_{outer}$ means $\tilde{X}$ equipped with the outer metric. The difference with Definition 8.2.3 is that in Definition 8.3.2 we do not a priori suppose that Ψ is the identity map.

Proposition 8.3.3 *The two definitions of Lipschitz Normally Embedded sets are equivalent.*

Pancake decomposition of Kurdyka [14] implies that there exists a decomposition of any semialgebraic set X into LNE semialgebraic subsets.

Theorem 8.3.4 *There is a decomposition of any semialgebraic set X into subsets X_i such that*

1. $\cup X_i = X$.
2. *X_i are semialgebraic LNE sets.*
3. $\dim(X_i \cap X_j) < \min(\dim X_i, \dim X_j)$ *for* $i \neq j$.

Remark 8.3.5 Using pancake decomposition, Kurdyka and Orro defined the so-called *pancake metric* (see [7, 14]). It is a semialgebraic metric equivalent to the inner metric.

Definition 8.3.6 (See [19]) An *arc* in $\mathbb{R}^n$ is (a germ at the origin of) a mapping $\gamma : [0, \epsilon) \to \mathbb{R}^n$ such that $\gamma(0) = 0$. Unless otherwise specified, arcs are parameterized by the distance to the origin, i.e., $|\gamma(t)| = t$. We usually identify an arc γ with its image in $\mathbb{R}^n$. The *Valette link* of a germ $(X, 0)$ is the set $V(X)$ of all arcs $\gamma \subset X$.

Theorem 8.3.7 (See [19]) *Let $(X, 0)$ and $(Y, 0)$ be germs of semialgebraic sets in $\mathbb{R}^n$. If these germs are semialgebraically (inner, outer or ambient) Lipschitz equivalent, then there exists a bi-Lipschitz map $h : X \to Y$ (or $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h(X) = Y$ in the case of ambient equivalence) preserving the distance to the origin, i.e., such that $h(X \cap S_\varepsilon) = Y \cap S_\varepsilon$ for small $\varepsilon > 0$.*

Definition 8.3.8 Let $f \not\equiv 0$ be (a germ at the origin of) a Lipschitz function defined on an arc γ. The *order* $ord_\gamma f$ of f on γ is the exponent $q \in \mathbb{Q}$ such that $f(\gamma(t)) = ct^q + o(t^q)$ as $t \to 0$, where $c \neq 0$. If $f \equiv 0$ on γ, we set $ord_\gamma f = \infty$.

Definition 8.3.9 The *tangency order* of arcs γ and γ' is $tord(\gamma, \gamma') = ord_\gamma|\gamma(t) - \gamma'(t)|$. The tangency order of an arc γ and a set of arcs $Z \subset V(X)$ is $tord(\gamma, Z) = \sup_{\lambda \in Z} tord(\gamma, \lambda)$. The tangency order of two subsets Z and Z' of $V(X)$ is $tord(Z, Z') = \sup_{\gamma \in Z} tord(\gamma, Z')$. Similarly, $itord(\gamma, \gamma')$, $itord(\gamma, Z)$ and $itord(Z, Z')$ are the tangency orders with respect to the inner metric.

Remark 8.3.10 If $(X, 0)$ is a germ of a semialgebraic curve, i.e., $X = \cup\gamma_i$ is a finite family of semialgebraic arcs, then the outer Lipschitz Geometry of $(X, 0)$ is completely determined by the tangency orders $tord(\gamma_i, \gamma_j)$ (see [4]).

Proposition 8.3.11 *A surface germ $(X, 0)$ is LNE if, and only if, for any two arcs γ_1 and γ_2 in X one has $tord(\gamma_1, \gamma_2) = itord(\gamma_1, \gamma_2)$.*

Proposition 8.3.12 *Let $(X, 0) \in \mathbb{R}^n$ be a surface germ inner Lipschitz equivalent to a β-horn. The Grassmannian $G(n, 2)$ can be considered as the space of all orthogonal projections $\rho : \mathbb{R}^n \to \mathbb{R}^2$. Then there exists an open dense semialgebraic subset $\widetilde{G} \subset G(n, 2)$ such that for all $\rho \in \widetilde{G}$ one has $\beta = \min_{\{\gamma_1, \gamma_2\} \subset V(X)} tord(\rho(\gamma_1), \rho(\gamma_2))$.*

The following proposition was proved first by Alexandre Fernandes [12]. A special case of this is the Arc Criterion of Normal Embedding [11].

Proposition 8.3.13 *Let* $(X, 0)$ *and* $(Y, 0)$ *be surface germs. A semialgebraic homeomorphism* $\Phi : (X, 0) \to (Y, 0)$ *preserving the distance to the origin is bi-Lipschitz if, and only if, for any two arcs* $\gamma_1, \gamma_2 \in V(X)$ *one has*

$$tord(\gamma_1, \gamma_2) = tord(\Phi(\gamma_1), \Phi(\gamma_2)). \tag{8.1}$$

A special case of Pancake Decomposition for surface germs can be stated as follows:

Theorem 8.3.14 *Let* $(X, 0)$ *be a semialgebraic surface germ. Then there exists a decomposition of* $(X, 0)$ *into the germs* $(X_i, 0)$ *such that*

1. $\cup X_i = X$.
2. *Each* X_i *is a LNE* β_i*-Hölder triangle.*
3. *For* $i \neq j$*, the intersection* $X_i \cap X_j$ *is either the origin or a common boundary arc of* X_i *and* X_j*.*

Definition 8.3.15 A pancake decomposition of a surface germ is *minimal* if the union of any two adjacent Hölder triangles X_i and X_j is not normally embedded. Two pancake decompositions are *combinatorially equivalent* if they are combinatorially equivalent as Hölder Complexes.

The answer to a natural question "Are any two minimal pancake decompositions of the same surface germ combinatorially equivalent?" is negative (see Sect. 8.5).

8.4 Pizza Decomposition of the Germ of a Semialgebraic Function

This section is related to the outer Lipschitz Geometry of a special kind of a surface germ: The union of a Lipschitz Normally Embedded (LNE) Hölder triangle and the graph of a semialgebraic Lipschitz function defined on it.

Definition 8.4.1 Given a semialgebraic Lipschitz function f defined on a β-Hölder triangle T, let

$$Q_f(T) = \bigcup_{\gamma \in V(T)} ord_\gamma f. \tag{8.2}$$

It was shown in [9] that $Q_f(T)$ is either a point or a closed interval in $\mathbb{Q} \cup \{\infty\}$.

Definition 8.4.2 A Hölder triangle T is *elementary* with respect to a Lipschitz function f if, for any $q \in Q_f(T)$ and any two arcs γ and γ' in T such that $ord_\gamma f = ord_{\gamma'} f = q$, the order of f is q on any arc in the Hölder triangle $T(\gamma, \gamma') \subseteq T$ bounded by the arcs γ and γ'.

Definition 8.4.3 Let T be a Hölder triangle and f a semialgebraic Lipschitz function defined on T. For each arc $\gamma \subset T$, the *width* $\mu_T(\gamma, f)$ of the arc γ with respect to f is the infimum of exponents of Hölder triangles $T' \subset T$ containing γ such that $Q_f(T')$ is a point. For $q \in Q_f(T)$ let $\mu_{T,f}(q)$ be the set of exponents $\mu_T(\gamma, f)$, where γ is any arc in T such that $ord_\gamma f = q$. It was shown in [9] that, for each $q \in Q_f(T)$, the set $\mu_{T,f}(q)$ is finite. This defines a multivalued *width function* $\mu_{T,f} : Q_f(T) \to \mathbb{Q} \cup \{\infty\}$. If T is elementary with respect to f, then the function $\mu_{T,f}$ is single valued. When f is fixed, we write $\mu_T(\gamma)$ and μ_T instead of $\mu_T(\gamma, f)$ and $\mu_{T,f}$.

Definition 8.4.4 Let T be a Hölder triangle and f a semialgebraic Lipschitz function defined on T. We say that T is a *pizza slice* associated with f if it is elementary with respect to f and, unless $Q_f(T)$ is a point, $\mu_{T,f}(q) = aq + b$ is an affine function on $Q_f(T)$. If T is a pizza slice such that $Q_f(T)$ is not a point, then the *supporting arc* $\tilde{\gamma}$ of T with respect to f is the boundary arc of T such that $\mu_T(\tilde{\gamma}, f) = \max_{q \in Q_f(T)} \mu_{T,f}(q)$. In that case, $\mu_T(\gamma, f) = tord(\gamma, \tilde{\gamma})$ for any arc $\gamma \subset T$ such that $tord(\gamma, \tilde{\gamma}) \leq \mu_T(\tilde{\gamma}, f)$.

Definition 8.4.5 (See [9, Definition 2.13]) Let f be a non-negative semialgebraic Lipschitz function defined on a β-Hölder triangle $T = T(\gamma_1, \gamma_2)$ oriented from γ_1 to γ_2. A *pizza* on T associated with f is a decomposition $\{T_\ell\}_{\ell=1}^p$ of T into β_ℓ-Hölder triangles $T_\ell = T(\lambda_{\ell-1}, \lambda_\ell)$ ordered according to the orientation of T, such that $\lambda_0 = \gamma_1$ and $\lambda_p = \gamma_2$ are the boundary arcs of T, $T_\ell \cap T_{\ell+1} = \lambda_\ell$ for $0 < \ell < p$, and each triangle T_ℓ is a pizza slice associated with f.

A pizza $\{T_\ell\}$ on T is *minimal* if $T_{\ell-1} \cup T_\ell$ is not a pizza slice for any $\ell > 1$.

Definition 8.4.6 (See [9, Definition 2.12]) An *abstract pizza* is a finite ordered sequence $\{q_\ell\}_{\ell=0}^p$, where $q_\ell \in \mathbb{Q}_{\geq 1} \cup \{\infty\}$, and a finite collection $\{\beta_\ell, Q_\ell, \mu_\ell\}_{\ell=1}^p$, where $\beta_\ell \in \mathbb{Q}_{\geq 1} \cup \{\infty\}$, $Q_\ell = [q_{\ell-1}, q_\ell] \subset \mathbb{Q}_{\geq 1} \cup \{\infty\}$ is either a point or a closed interval, $\mu_\ell : Q_\ell \to \mathbb{Q} \cup \{\infty\}$ is an affine function, non-constant when Q_ℓ is not a point, such that $\mu_\ell(q) \leq q$ for all $q \in Q_\ell$ and $\min_{q \in Q_\ell} \mu_\ell(q) = \beta_\ell$.

Definition 8.4.7 Two pizzas are *combinatorially equivalent* if the corresponding abstract pizzas are the same.

Theorem 8.4.8 (See [9, Theorem 4.9]) *Two non-negative semialgebraic Lipschitz functions f and g defined on a Hölder triangle T are contact Lipschitz equivalent if, and only if, minimal pizzas on T associated with f and g are combinatorially equivalent.*

Let $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma_1', \gamma_2')$ be two normally embedded β-Hölder triangles. We say that (T, T') is a *normal pair* if

$$\begin{aligned} tord(\gamma_1, T') &= tord(\gamma_1, \gamma_1') = tord(\gamma_1', T), \\ tord(\gamma_2, T') &= tord(\gamma_2, \gamma_2') = tord(\gamma_2', T). \end{aligned} \tag{8.3}$$

For example, a pair $(T, Graph(f))$ considered in this section satisfies this condition. The following question is natural: Let $T = T(\gamma_1, \gamma_2)$ and $T' = T(\gamma_1', \gamma_2')$

be a normal pair of semialgebraic β-Hölder triangles. Is it true that the union $T \cup T'$ is outer Lipschitz equivalent to the union $T \cup Graph(f)$, where f is the distance function $f(x) = dist(x, T')$ defined on T? In the paper [3] the authors give examples when this is not true. This is true, however, when T is elementary with respect to f.

8.5 Outer Lipschitz Geometry, Snakes

Definition 8.5.1 Let $(X, 0)$ be a surface germ. An arc γ of X is *Lipschitz non-singular* if there exists a normally embedded Hölder triangle $T \subset X$ such that γ is an interior arc of T and $\gamma \not\subset \overline{X \setminus T}$. Otherwise, γ is *Lipschitz singular*. It follows from the pancake decomposition that a surface germ X contains finitely many Lipschitz singular arcs. The union of all Lipschitz singular arcs in X is denoted by $Lsing(X)$. A Hölder triangle $T \subset X$ is *non-singular* if all interior arcs of T are Lipschitz non-singular.

Definition 8.5.2 If $T = T(\gamma_1, \gamma_2)$ is a non-singular β-Hölder triangle, an arc γ of T is *generic* if $itord(\gamma_1, \gamma) = itord(\gamma, \gamma_2) = \beta$. The set of generic arcs of T is denoted $G(T)$.

Definition 8.5.3 An arc γ of a Lipschitz non-singular β-Hölder triangle T is *abnormal* if there are two normally embedded Hölder triangles $T' \subset T$ and $T'' \subset T$ such that $T' \cap T'' = \gamma$ and $T \cup T'$ is not normally embedded. Otherwise γ is *normal*. The set $Abn(T)$ of abnormal arcs of T is outer Lipschitz invariant.

Definition 8.5.4 A non-singular β-Hölder triangle T is called a *β-snake* if $Abn(T) = G(T)$.

The following important property of snakes can be interpreted as "separation of scales" in outer Lipschitz Geometry.

Lemma 8.5.5 *Let T be a β-snake, and let $\{T_k\}_{k=1}^{p}$ be a minimal pancake decomposition of T. Then each T_k is a β-Hölder triangle.*

Remark 8.5.6 Minimal pancake decompositions of a snake may be combinatorially non-equivalent, as shown in Fig. 8.3. We use a planar plot to represent the link of a snake. The points in Fig. 8.3 correspond to arcs of the snake. The points with smaller Euclidean distance inside the shaded disks correspond to arcs with the tangency order higher than their inner tangency order β. Black dots indicate the boundary arcs of pancakes.

Definition 8.5.7 A β-Hölder triangle T is *weakly normally embedded* if, for any two arcs γ and γ' of T such that $tord(\gamma, \gamma') > itord(\gamma, \gamma')$, we have $itord(\gamma, \gamma') = \beta$.

Proposition 8.5.8 *Let T be a β-snake. Then T is weakly normally embedded.*

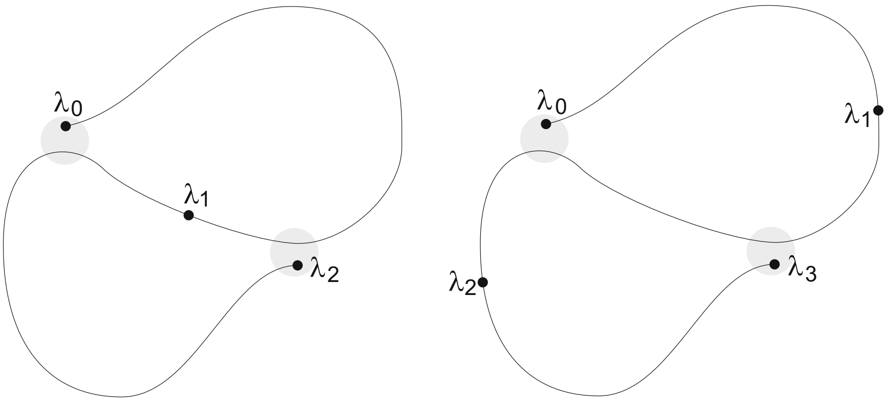

Fig. 8.3 Two combinatorially non-equivalent minimal pancake decompositions of a snake. Black dots indicate the boundary arcs of pancakes

8.6 Tangent Cones

Definition 8.6.1 The *tangent cone* C_0X of a semialgebraic set X at 0 is defined as follows:

$$C_0X = Cone\left(\lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(X \cap \{|x| = \epsilon\}\right)\right),$$

where the limit here means the Hausdorff limit.

Remark 8.6.2 There are several equivalent definitions of the tangent cone of a semialgebraic set. In particular, the tangent cone C_0X can be defined as the set of tangent vectors at the origin to all arcs in X. The tangent cone of a semialgebraic set is semialgebraic.

The tangent cone is Lipschitz invariant:

Theorem 8.6.3 (See [18]) *If two germs* $(X, 0)$ *and* $(Y, 0)$ *are outer (resp. ambient) Lipschitz equivalent, then the corresponding tangent cones* C_0X *and* C_0Y *are outer (resp. ambient) Lipschitz equivalent.*

The result is important in Theory of Metric Knots (see [2, 5, 6]) for the proof of Universality Theorem below. This result was also used to prove that, if a complex analytic set is a Lipschitz nonsingular submanifold of $\mathbb{C}^n$, then it is a smooth submanifold [8]. Moreover, the result was used in the recent study of the Zariski Multiplicity Conjecture (see the paper of Fernandes and Sampaio in the present volume).

8.7 Ambient Equivalence: Metric Knots

Definition 8.7.1 Two germs of semialgebraic sets $(X, 0)$ and $(Y, 0)$ in $\mathbb{R}^n$ are *outer Lipschitz equivalent* if there exists a homeomorphism $h : (X, 0) \to (Y, 0)$ bi-Lipschitz with respect to the outer metric. The germs are *semialgebraic outer Lipschitz equivalent* if the map h can be chosen to be semialgebraic. The germs are *ambient Lipschitz equivalent* if there exists an orientation preserving bi-Lipschitz homeomorphism $H : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, such that $H(X) = Y$. The germs are *semialgebraic ambient Lipschitz equivalent* if the map H can be chosen to be semialgebraic.

Definition 8.7.2 The *link at the origin* L_X of a germ $(X, 0)$ in $\mathbb{R}^n$ is the equivalence class of the sets $X \cap S^{n-1}_{0,\varepsilon}$ for small positive ε with respect to the ambient Lipschitz equivalence. The *tangent link* of X is the link at the origin of the tangent cone of X.

Remark 8.7.3 By the finiteness theorems of Mostowski, Parusinski and Valette (see [15, 16, 20]) the link at the origin is well defined. We write "the link at the origin" speaking of this notion of the link from Singularity Theory, reserving the word "link" for the notion of the link in Knot Theory. If $n = 4$ and X has an isolated singularity at the origin, then each connected component of L_X is a knot in S^3.

Definition 8.7.4 A *metric knot* is an ambient Lipschitz equivalence class of a surface germ $(X, 0)$ in $\mathbb{R}^4$ with an isolated singularity and connected link. In particular, the link at the origin of the germ X is an isotopy class of an ordinary topological knot in S^3.

The following result (so called **Universality Theorem** for metric knots) shows the difference between outer and ambient Lipschitz Geometry of surface germs in $\mathbb{R}^4$:

Theorem 8.7.5 (Universality Theorem) *One can associate to each knot K in S^3 a semialgebraic surface germ $(X_K, 0)$ in $\mathbb{R}^4$ so that:*

1. *The link at the origin of each germ X_K is a trivial knot;*
2. *All germs X_K are outer Lipschitz equivalent;*
3. *Two germs X_{K_1} and X_{K_2} are ambient semialgebraic Lipschitz equivalent only if the knots K_1 and K_2 are isotopic.*

The idea of a proof is illustrated in Fig. 8.4, representing the link at the origin of a surface germ X_K. A detailed explanation can be found in [6].

The following result is a more complicated version of Universality Theorem:

Theorem 8.7.6 *For any two knots K and L in S^3, one can associate a semialgebraic surface germ $\tilde{X}_{KL}$ so that:*

1. *The link at the origin of $\tilde{X}_{KL}$ is isotopic to L.*
2. *The tangent link of $\tilde{X}_{KL}$ is isotopic to K.*
3. *All surface germs $\tilde{X}_{KL}$ are outer bi-Lipschitz equivalent.*

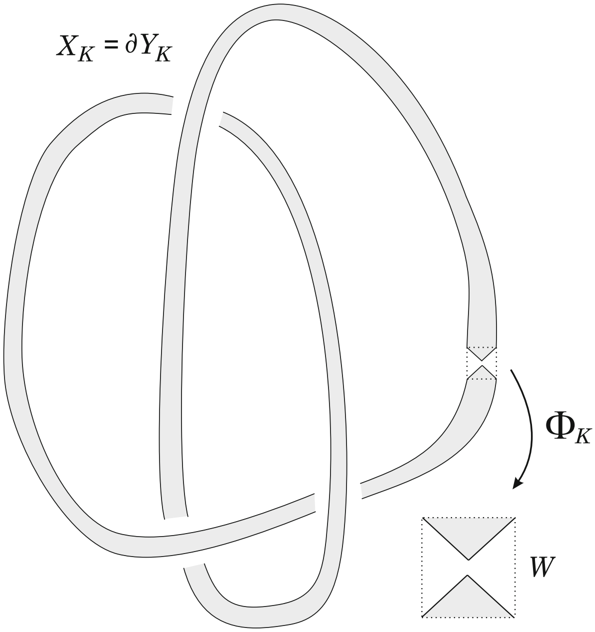

Fig. 8.4 The proof of Theorem 8.7.5

The theorem implies, for example, that for a given tangent cone one can find infinitely many outer Lipschitz equivalent, but not ambient Lipschitz equivalent surface germs.

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Chapter 9
Bi-Lipschitz Invariance of the Multiplicity

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Contents

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Abstract The multiplicity of an algebraic curve C in the complex plane at a point p on that curve is defined as the number of points that occur at the intersection of C with a general complex line that passes close to the point p. It is shown that p is a singular point of the curve C if and only if this multiplicity is greater than or equal to 2, in this sense, such an integer number can be considered as a measure of how singular can be a point of the curve C. In these notes, we address the classical concept of multiplicity of singular points of complex algebraic sets (not necessarily complex curves) and we approach the nature of the multiplicity of singular points as a geometric invariant from the perspective of the Multiplicity Conjecture (Zariski 1971). More precisely, we bring a discussion on the recent results obtained jointly with Lev Birbrair, Javier Fernández de Bobadilla, Lê Dũng Tráng and Mikhail Verbitsky on the bi-Lipschitz invariance of the multiplicity.

9.1 Introduction

Unless explicitly mentioned to the contrary, all the analytic subsets of $\mathbb{C}^n$ considered here are closed subsets of $\mathbb{C}^n$.

9.1.1 Local Analytic Structure

Let $p \in X \subset \mathbb{C}^n$ and $q \in Y \subset \mathbb{C}^m$ be analytic subsets. We say that the pair (X, p) is **analytic equivalent** to (Y, q) if there exist neighbourhoods $U \subset \mathbb{C}^n$ of p and $V \subset \mathbb{C}^m$ of q and an analytic mapping $F\colon U \cap X \to V \cap Y$; $F(p) = q$ with inverse map $G\colon V \cap Y \to U \cap X$ also analytic. This definition gives us an equivalence relation; each equivalence class is what we call a **local analytic structure**. We establish that the equivalence class of $(\mathbb{C}^n, 0)$ is the **regular local analytic structure** in dimension n. Local Analytic Geometry is the research field in Mathematics in charge of describing all local analytic structures.

Given $p \in X \subset \mathbb{C}^n$ an analytic subset, we denote by $\mathcal{O}_{X,p}$ the set of analytic functions defined in some neighbourhood of p in X equipped with natural binary operations of addition and multiplication. Defined in that way, $\mathcal{O}_{X,p}$ is a local ring with maximal ideal given by

$$\mathcal{M}_{X,p} = \{f \in \mathcal{O}_{X,p} \ : \ f(p) = 0\}.$$

Next result makes a bridge connecting Local Analytic Geometry with Commutative Algebra (see [5]).

Theorem 9.1.1 *Let $p \in X \subset \mathbb{C}^n$ and $q \in Y \subset \mathbb{C}^m$ be analytic subsets. The pair (X, p) defines the same local analytic structure as (Y, q) if, and only if, $\mathcal{O}_{X,p}$ is isomorphic to $\mathcal{O}_{Y,q}$ as local $\mathbb{C}$-algebras.*

As already mentioned here, we observe that this theorem above provides a way to study the classification of local analytic structures from the algebraic point of view. Next, from the algebraic point of view, we present one of the more important invariant of local analytic structures.

Proposition 9.1.2 *Let $X \subset \mathbb{C}^n$ be an analytic subset such that in a neighbourhood of $p \in X$ it has pure dimension d. There exists a polynomial $P(t) \in \mathbb{C}[t]$ of degree d such that $P(k)$ is equal to* $\dim \mathcal{O}_{X,p}/\mathcal{M}_{X,p}^k$ *as a $\mathbb{C}$ vector space, for k large enough. Moreover, the leading coefficient of $P(t)$ times $d!$ is equal to some positive integer e.*

Definition 9.1.3 The polynomial $P(t)$ is called **the Hilbert-Samuel Polynomial** of the pair (X, p). The integer number e provided in the above proposition is called **multiplicity of** X **at** p and denoted by $m(X, p)$.

Example 9.1.4 Let us show that $m(\mathbb{C}^n, 0) = 1$. Actually, we know that $\mathcal{O}_{\mathbb{C}^n,0}$ is isomorphic to $\mathbb{C}\{z_1, \ldots, z_n\}$, so, it has pure dimension n. Hence, $\mathcal{M}_{\mathbb{C}^n,0}^k$ is the ideal of $\mathbb{C}\{z_1, \ldots, z_n\}$ generated by $z_1^{a_1} \cdots z_n^{a_n}$ where $a_1, \ldots, a_n$ are non-negative integer numbers such that $a_1 + \cdots + a_n = k$ and, therefore,

$$\dim \mathcal{O}_{\mathbb{C}^n,0}/\mathcal{M}_{\mathbb{C}^n,0}^k = \frac{1}{n!}(n+k-1)(n+k-2)\cdots(k+1)(k)$$

$\therefore m(\mathbb{C}^n, 0) = 1$.

Example 9.1.5 Let $X \subset \mathbb{C}^2$ be the cusp defined by

$$X = \{(z_1, z_2) \in \mathbb{C}^2 \,:\, z_1^3 = z_2^2\}.$$

In this case, we see that $m(X, 0) = 2$. Indeed, $\mathcal{O}_{X,0}$ is isomorphic to $\mathbb{C}\{z_1, z_2\}/I$ where I is the ideal of $\mathbb{C}\{z_1, z_2\}$ generated by z_1, z_2 with the following relation $z_1^3 = z_2^2$. In other words, we have that $\mathcal{O}_{X,0}$ is isomorphic to

$$f(z_1) + g(z_1)z_2 \,:\, f(z_1), g(z_1) \in \mathbb{C}\{z_1\}.$$

Hence,

$$\dim \mathcal{O}_{X,0}/\mathcal{M}_{X,0}^k = 2k - 1,$$

and as $(X, 0)$ has pure dimension 1, we get $m(X, 0) = 2$.

Next result says that multiplicity of points is an invariant of the local analytic structure.

Theorem 9.1.6 *Let $p \in X \subset \mathbb{C}^n$ and $q \in Y \subset \mathbb{C}^m$ be analytic subsets. If (X, p) defines the same local analytic structure as (Y, q), then $m(X, p) = m(Y, q)$.*

Proof Let us assume that (X, p) defines the same local analytic structure as (Y, q), hence $\mathcal{O}_{X,p}$ is isomorphic to $\mathcal{O}_{Y,q}$ as local $\mathbb{C}$-algebra. Then,

$$\dim \mathcal{O}_{X,p}/\mathcal{M}_{X,p}^k = \dim \mathcal{O}_{Y,q}/\mathcal{M}_{Y,q}^k \ \forall k,$$

and (X, p) and (Y, q) have the same Hilbert-Samuel Polynomial. In particular, $m(X, p) = m(Y, q)$. □

Next we see a more geometric way to get the multiplicity of points (see [15] and [5]).

Proposition 9.1.7 *Let $X \subset \mathbb{C}^n$ be an analytic subset such that in a neighbourhood of $p \in X$ it has dimension d. If $L \colon \mathbb{C}^n \to \mathbb{C}^d$ is a generic linear projection, then its restriction to $X \cap U$ defines a finite mapping of topological degree m for each small enough neighbourhood $U \subset \mathbb{C}^n$ of the point p. Moreover, if $X \subset \mathbb{C}^n$ is an analytic subset such that in a neighbourhood of $p \in X$ it has pure dimension d then the integer number m is equal to the multiplicity $m(X, p)$.*

In the above proposition, let us make clear that a linear projection $L \colon \mathbb{C}^n \to \mathbb{C}^d$ is generic when the intersection of $Ker(L)$ with the tangent cone $C(X, p)$ (see Sect. 9.3) is only the null vector.

Example 9.1.8 Let $X \subset \mathbb{C}^n$ be a hypersurface defined as the zero set of an analytic function $f : U \subset \mathbb{C}^n \to \mathbb{C}$ in a neighbourhood U of the origin $0 \in \mathbb{C}^n$; $f(0) = 0$. Let us write

$$f(z) = f_m(z) + f_{m+1}(z) + \cdots + f_k(z) + \cdots$$

where each $f_k(z)$ is a homogeneous polynomial of degree k and $f_m \not\equiv 0$. In this case, $m(X, 0) = m$.

Definition 9.1.9 We say that $p \in X \subset \mathbb{C}^n$ is a **regular** point (of X) if $\mathcal{O}_{X,p} \cong \mathcal{O}_{\mathbb{C}^k,0}$. A point $p \in X$ is **singular** if it is not regular.

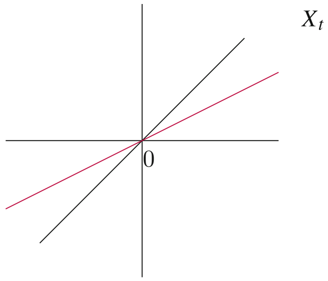
Corollary 9.1.10 *Let $X \subset \mathbb{C}^n$ be an analytic subset such that in a neighbourhood of $p \in X$ it has pure dimension d. Then, p is a regular point of X if and only if $m(X, p) = 1$.*

Although multiplicity is enough to identify the regular local analytic structures, the following example shows us that discrete invariants are not enough to describe all local analytic structures.

Example 9.1.11 (Whitney 1965) Let us consider the following family of four lines through the origin

$$X_t \colon \; x \cdot y \cdot (y - x) \cdot (y - tx) = 0.$$

For generic $s \neq t$, H. Whitney (see [35]) noted that the local analytic structures $(X_s, 0)$ and $(X_t, 0)$ are not analytic equivalent.



9.1.2 Local Topological Structure

Let $p \in X \subset \mathbb{C}^n$ and $q \in Y \subset \mathbb{C}^n$ be analytic subsets. We say that the pair (X, p) is **topological equivalent** to (Y, q) if there exist neighbourhoods $U \subset \mathbb{C}^n$ of p and $V \subset \mathbb{C}^m$ of q and a homeomorphism $F\colon U \cap X \to V \cap Y$; $F(U \cap X) = V \cap Y$ and $F(p) = q$. This definition give us an equivalence relation; each equivalence class is what we call a **local topological structure**. We establish that the equivalence class of $(\mathbb{C}^n, 0)$ is the **regular local topological structure** in dimension n.

In the 60's, results that pointed to the possibility of describing the local topology of analytical sets only with discrete invariants greatly boosted research on the subject and attracted eminent mathematicians to questions that are still open today. Actually, it was proved that there are only countably many infinite local topological structures and, in the direction of looking for nice discrete invariants of the local topological structures, in 1971 O. Zariski (see [37]) placed the following question which is still unanswered

[O. Zariski 1971] *Let $X, Y \subset \mathbb{C}^n$ be analytic subsets of codimension 1 such that (X, p) and (Y, q) are topological equivalent, is it true that $m(X, p)$ must be equal to $m(Y, q)$?*

9.1.3 Local Lipschitz Structure

Let $p \in X \subset \mathbb{C}^n$ and $q \in Y \subset \mathbb{C}^n$ be analytic subsets. We say that the pair (X, p) is **bi-Lipschitz equivalent** to (Y, q) if there exist neighbourhoods $U \subset \mathbb{C}^n$ of p and $V \subset \mathbb{C}^m$ of q and a Lipschitz map $F\colon U \cap X \to V \cap Y$; $F(p) = q$ with inverse map $G\colon V \cap Y \to U \cap X$ also Lipschitz, i.e. there exists a positive constant $\lambda \geqslant 1$ such that the bi-univocal correspondence F satisfies:

$$\frac{1}{\lambda}|x_1 - x_2| \leqslant |F(x_1) - F(x_2)| \leqslant \lambda |x_1 - x_2| \ \forall x_1, x_2 \in X \cap U.$$

This definition give us an equivalence relation; each equivalence class is what we call a **local Lipschitz structure**. We establish that the equivalence class of $(\mathbb{C}^n, 0)$

is the **regular local Lipschitz structure** in dimension n. Lipschitz Geometry of Singularities is the research field in Mathematics in charge of describing all local Lipschitz structures.

In the mid 80's (see [22]), Tadeusz Mostowski proved that there are only countably many infinite local Lipschitz structures (Parusinski proved the real version of such result in [25]); this result greatly stimulated research on the Lipschitz geometry of singularities, mainly in looking for discrete invariants for the respective classification problem. In this direction, the main objective of these notes is to bring a survey of results, some of them published with other collaborators, on the following issues:

Question AL(d): Let $X \subset \mathbb{C}^n$ and $Y \subset \mathbb{C}^m$ be two complex analytic sets with $\dim X = \dim Y = d$, $0 \in X$ and $0 \in Y$. If there exists a bi-Lipschitz homeomorphism $\varphi\colon (X, 0) \to (Y, 0)$, then is $m(X, 0) = m(Y, 0)$?

Question AL(n, d): Let $X, Y \subset \mathbb{C}^n$ be two complex analytic sets with $\dim X = \dim Y = d$ and $0 \in X \cap Y$. If there exists a bi-Lipschitz homeomorphism $\varphi\colon (\mathbb{C}^n, X, 0) \to (\mathbb{C}^n, Y, 0)$ (i.e., $\varphi\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is a bi-Lipschitz homeomorphism such that $\varphi(X) = Y$), then is $m(X, 0) = m(Y, 0)$?

We call each one of the above questions of **bi-Lipschitz invariance of the multiplicity problem**. We emphasize the importance of citing some of the main references of pioneering results in the proposed investigation of multiplicity as an invariant of local structures that are less rigid than analytic (like Lipschitz structures), namely: in the paper [14], Gau and Lipman proved the invariance of multiplicity for differentiable local structures (Ephraim, in [10], addressed the case of continuously differentiable local structures, see also the result proved by Trotman in [33]) and, in the paper [6], assuming severe restrictions on the Lipschitz constants that conjugate two pairs (X, p) and (Y, q), Comte showed that $m(X, p) = m(Y, q)$.

In recent works written in collaboration with Birbrair (UFC), Fernández de Bobadilla (BCAM), Lê (Aix-Marseille) and Verbitsky (IMPA), we proved the following results.

1. Regular local Lipschitz structure is equivalent to regular local analytic structure (see [1] and [30]);
2. In dimension 2, the multiplicity is an invariant of the local Lipschitz structure (see [3]);
3. In dimension greater than 2, the multiplicity is not an invariant of the local Lipschitz structure (see [2]).

Pham and Teissier in [26], with contributions of Fernandes in [11] and Neumann and Pichon in [24], proved that multiplicity is an invariant of the local Lipschitz structure in dimension 1 (see also [13]). Therefore, Results 2 and 3 above, together with the result of Pham and Teissier, can be summarized as follows: the multiplicity is invariant of the local Lipschitz structure only in dimensions 1 and 2.

Let us describe how these notes are organized. Section 9.2 contains preliminary results where we introduce the concept of Lipschitz normally embedded sets and we present the Pancake Decomposition Theorem. Section 9.3 is devoted to explore the notion of tangent vectors, more precisely, we introduce the concept of tangent cone to singular points and, among other results, we present a proof of the bi-Lipschitz invariance of the tangent cone of subanalytic singularities. In Section 9.4, we recover the notion of regular local Lipschitz structure and we introduce other notions of local Lipschitz regularities; we also present a proof that Lipschitz regularity of a local analytic structure implies that such structure must be analytic regular. In Section 9.5, we introduced the so-called relative multiplicities, we present a proof that such multiplicities are bi-Lipschitz invariant and we show that Question AL(d) (resp. AL(n, d)) has a positive answer if, and only if, it has a positive answer for irreducible homogeneous algebraic singularities. Finally, in Section 9.6 we address the problem of the bi-Lipschitz invariance of the multiplicity.

9.2 Lipschitz Normally Embedded Sets

In this section we define Lipschitz normally embedded sets and we present some important properties of this notion.

Let $Z \subset \mathbb{R}^n$ be a path connected subset. Given two points $q, \tilde{q} \in Z$, we define the **inner distance** on Z between q and $\tilde{q}$ by the number:

$$d_Z(q, \tilde{q}) := \inf\{\text{Length}(\gamma) \mid \gamma \text{ is an arc on } Z \text{ connecting } q \text{ to } \tilde{q}\}.$$

Definition 9.2.1 We say that Z is **Lipschitz normally embedded (LNE)**, if there is a constant $C \geqslant 1$ such that $d_Z(q, \tilde{q}) \leqslant C\|q - \tilde{q}\|$, for all $q, \tilde{q} \in Z$. We say that Z is **Lipschitz normally embedded at** p (shortly LNE at p), if there is a neighbourhood U such that $p \in U$ and $Z \cap U$ is an LNE set or, equivalently, that the germ (Z, p) is LNE. In this case, we also say that Z is C-LNE (resp. C-LNE at p).

Proposition 9.2.2 *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be non-empty subsets. Assume that there exists a bi-Lipschitz homeomorphism $\psi\colon X \to Y$. Then, X is LNE at $x_0 \in X$ if and only if Y is LNE at $\psi(x_0)$.*

Proof The proof is left as an exercise for the reader. □

Let us recall the following "Pancake decomposition" result:

Lemma 9.2.3 (Proposition 3 in [17]) *Let $X \subset \mathbb{R}^m$ be a subanalytic set and $\varepsilon > 0$. Then for each p in the closure of X, denoted by $\overline{X}$, there exist $\delta > 0$ and a finite decomposition $X \cap B_\delta(p) = \bigcup_{j=1}^{k} \Gamma_j$ such that:*

(i) each Γ_j is a subanalytic connected analytic submanifold of $\mathbb{R}^m$,
(ii) each $\overline{\Gamma}_j$ satisfies $d_{\overline{\Gamma}_j}(x, y) \leqslant (1 + \varepsilon)\|x - y\|$ for any $x, y \in \overline{\Gamma}_j$.

The above result has the following consequence:

Proposition 9.2.4 *Let $X \subset \mathbb{R}^m$ be a subanalytic set. Then d_X induces the same topology on X as the topology induced by the standard topology on $\mathbb{R}^m$.*

Proof The proof is left as an exercise for the reader. □

9.2.1 Exercises

Exercise 9.2.5 Prove that any connected compact C^1 submanifold of $\mathbb{R}^n$ is LNE.

Exercise 9.2.6 Let $X \subset \mathbb{C}^n$ be a complex analytic set. Then for any $x, y \in X$, we have that

$$d_X(x, y) = \inf_{\beta \in \Omega(x,y)} \int_0^1 \|\beta'(t)\| dt,$$

where $\Omega(x, y) = \{\beta\colon [0, 1] \to X; \beta(0) = x,\ \beta(1) = y$ and β is piecewise $C^1\}$.

Exercise 9.2.7 Prove Proposition 9.2.2.

Exercise 9.2.8 Prove Proposition 9.2.4.

Exercise 9.2.9 Let $X \subset \mathbb{C}^2$ be a complex analytic curve. Let $X_1, \ldots, X_r$ be the irreducible components of X (at 0). Then, X is LNE at 0 if and only if each X_i is smooth at 0 and for $i \neq j$, X_j and X_i meet transversally at 0.

Exercise 9.2.10 Let $X \subset \mathbb{C}^m$ be a complex algebraic set. Assume that $d_X(x, y) = \|x - y\|$ for all $x, y \in X$. Prove that X is an affine linear subspace of $\mathbb{C}^m$.

9.3 Tangent Cones

We begin this section with the definition of tangent cone of a subset $X \subset \mathbb{K}^m$ at a point, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$.

Definition 9.3.1 Let $X \subset \mathbb{K}^m$ be a set such that $x_0 \in \overline{X}$. We say that $v \in \mathbb{K}^m$ is a **tangent vector of** X **at** $x_0 \in \mathbb{K}^m$ if there are a sequence of points $\{x_i\} \subset X$ tending to x_0 and a sequence of positive real numbers $\{t_i\}$ such that

$$\lim_{i\to\infty} \frac{1}{t_i}(x_i - x_0) = v.$$

Let $C(X, x_0)$ denote the set of all tangent vectors of X at x_0. We call $C(X, x_0)$ the **tangent cone of** X **at** x_0.

Recall that a subset of $\mathbb{C}^n$ is called a **complex cone** if it is a union of one-dimensional complex linear subspaces of $\mathbb{C}^n$.

Remark 9.3.2 In the case where $X \subset \mathbb{C}^m$ is a complex analytic set such that $0 \in X$, $C(X, 0)$ is the zero locus of a set of complex homogeneous polynomials (see [36, Theorem 4D]). In particular, $C(X,0)$ is a complex algebraic subset of $\mathbb{C}^m$ and is a complex cone. More precisely, let $\mathcal{I}(X)$ be the ideal of $\mathcal{O}_{\mathbb{C}^m,0}$ given by the germs which vanishes on X. For each $f \in \mathcal{O}_{\mathbb{C}^m,0}$, let

$$f = f_k + f_{k+1} + \cdots$$

be its Taylor development where each f_j is a homogeneous polynomial of degree j and $f_k \neq 0$. So, we say that f_k is the **initial part** of f and we denote it by $\mathbf{in}(f)$. In this way, $C(X,0)$ is the affine variety of the ideal $\mathcal{I}_*(X) = \langle \mathbf{in}(f);\ f \in \mathcal{I}(X)\rangle$.

Example 9.3.3 Examples of tangent cones:

(1) If $X = \{(x,y) \in \mathbb{R}^2; x^3 = y^2\}$ then $C(X,0) = \{(x,y) \in \mathbb{R}^2; y = 0$ and $x \geqslant 0\}$;
(2) Let $f\colon (\mathbb{C}^n,0) \to (\mathbb{C},0)$ be a complex analytic function and let $f = f_k + f_{k+1} + \cdots$ be its Taylor development at the origin where each f_j is a homogeneous polynomial of degree j and $f_k \neq 0$. Then $C(V(f),0) = V(f_k)$;
(3) If $X = \{(x,y) \in \mathbb{C}^2; x^3 = y^2\}$ then $C(X,0) = \{(x,y) \in \mathbb{C}^2; y = 0\}$.

Proposition 9.3.4 *Basic properties:*

(1) If $A \subset X$ then $C(A,p) \subset C(X,p)$ for all $p \in \overline{A}$;
(2) For $X, Y \subset \mathbb{R}^m$ and $p \in \overline{X} \cap \overline{Y}$, we have

(a) $C(X \cup Y, p) = C(X,p) \cup C(Y,p)$;
(b) $C(X \cap Y, p) \subset C(X,p) \cap C(Y,p)$;

(3) For $X \subset \mathbb{R}^m$, $Y \subset \mathbb{R}^n$ and $(p,q) \in \overline{X} \times \overline{Y}$, we have $C(X \times Y,(p,q)) = C(X,p) \times C(Y,p)$;
(4) If X is a C^1-smooth submanifold of $\mathbb{R}^m$ then $C(X,p) = T_pX$ for all $p \in X$, where T_pX is the tangent space of X at p;
(5) Let $\varphi\colon (\mathbb{R}^n,p) \to (\mathbb{R}^m,\varphi(p))$ be the germ of a mapping which is differentiable at p. If $X \subset \mathbb{R}^m$ is a set such that $p \in \overline{X}$ then $D\varphi_p(C(X,p)) \subset C(\varphi(Y),\varphi(p))$, where $D\varphi_p$ is the derivative of φ at p;
(6) If $X \subset \mathbb{R}^m$ and $p \in \overline{X}$ then $C(X,p)$ is a closed subset of $\mathbb{R}^m$, $C(X,p) = C(\overline{X},p)$ and $C(X,p)$ is a real cone, i.e., for each $v \in C(X,p)$, we have that $\lambda v \in C(X,p)$ for all $\lambda > 0$.

Proof The proof is left as an exercise for the reader. □

Note that the tangent cones are not topological invariant in the sense that (X,p) can be topological equivalent to (Y,q), but $C(X,p)$ and $C(Y,q)$ are not homeomorphic . Indeed, the real cusp $X = \{(x,y) \in \mathbb{R}^2; y^3 = x^2\}$ is homeomorphic to the real line $L = \{(x,y) \in \mathbb{R}^2; y = 0\}$, but $C(X,0) = \{(0,y) \in \mathbb{R}^2; y \geqslant 0\}$ is not homeomorphic to $C(L,p) = L$, for any $p \in L$. The main result of this section is to prove that the tangent cones are bi-Lipschitz invariant.

Another way to present the tangent cone of a subset $X \subset \mathbb{R}^m$ at the origin $0 \in \mathbb{R}^m$ is via the spherical blow-up of $\mathbb{R}^m$ at the point 0 as it is going to be done in the following: let us consider the **spherical blowing-up** at the origin of $\mathbb{R}^m$

$$\begin{aligned}\rho_m\colon \mathbb{S}^{m-1}\times[0,+\infty) &\longrightarrow \mathbb{R}^m\\ (x,r) &\longmapsto rx.\end{aligned}$$

Notice that $\rho_m\colon \mathbb{S}^{m-1}\times(0,+\infty)\to\mathbb{R}^m\setminus\{0\}$ is a homeomorphism with inverse map $\rho_m^{-1}\colon \mathbb{R}^m\setminus\{0\}\to\mathbb{S}^{m-1}\times(0,+\infty)$ given by $\rho_m^{-1}(x)=(\frac{x}{\|x\|},\|x\|)$. Let us denote

$$X' := \overline{\rho_m^{-1}(X\setminus\{0\})} \text{ and } \partial X' := X'\cap(\mathbb{S}^{m-1}\times\{0\}).$$

Proposition 9.3.5 *If $X\subset\mathbb{R}^m$ is a subanalytic set and $0\in X$, then $\partial X'=\mathbb{S}_0X\times\{0\}$, where $\mathbb{S}_0X=C(X,0)\cap\mathbb{S}^{m-1}$.*

Proof The proof is left as an exercise for the reader. □

9.3.1 Characterization of Tangent Cones

In this section, we present one more way to consider tangent vectors of subanalytic sets. More precisely, the following proposition and its corollaries give us nice characterizations of tangent vector of subanalytic sets X in terms of velocity of arcs in X.

We start by recalling the following fundamental result in real algebraic geometry (see [4, Lemma 6.3]):

Lemma 9.3.6 (Curve Selection Lemma) *Let X be a subanalytic subset of $\mathbb{R}^n$ and $x\in\mathbb{R}^n$ being a non-isolated point of $\overline{X}$. Then, there exist $\delta>0$ and an analytic map $\gamma\colon(-\delta,\delta)\to\mathbb{R}^n$ such that $\gamma(0)=x$ and $\gamma((0,\delta))\subset X$.*

Proposition 9.3.7 *Let $Z\subset\mathbb{R}^m$ be a subanalytic set with $p\in\overline{Z\setminus\{p\}}$. A vector $v\in\mathbb{R}^m$ is a tangent vector of Z at p if and only if there exists a continuous subanalytic arc $\gamma\colon[0,\varepsilon)\to\overline{Z}$ such that $\gamma((0,\varepsilon))\subset Z$ and $\gamma(t)-p=tv+o(t)$, where $g(t)=o(t)$ means $\lim\limits_{t\to0^+}\frac{g(t)}{t}=0$.*

Proof Without loss of generality, we assume $p=0$. Thus, $Y=\rho_m^{-1}(X\setminus\{0\})\subset\mathbb{S}^{m-1}\times[0,+\infty)$ and $\overline{Y}$ are subanalytic sets. We are going to consider two cases:

1) Case $v\neq0$. Since v is a tangent vector of Z at 0, there are a sequence $\{s_k\}_{k\in\mathbb{N}}$ of positive real numbers and a sequence $\{z_k\}_{k\in\mathbb{N}}\subset Z$ such that $\lim\limits_{k\to+\infty}z_k=0$ and $\lim\limits_{k\to+\infty}\frac{1}{s_k}z_k=v$. In particular, $\lim\limits_{k\to\infty}\frac{z_k}{\|z_k\|}=\frac{v}{\|v\|}$ and $u=(\frac{v}{\|v\|},0)\in\overline{Y}$. Then by Curve Selection Lemma (Lemma 9.3.6), there exists an analytic arc $\beta\colon(-\delta,\delta)\to\mathbb{S}^{n-1}\times\mathbb{R}$ such that $\beta(0)=u$ and $\beta((0,\delta))\subset Y$. By writing $\beta(t)=(x(t),s(t))$, we have that $s\colon[0,\delta)\to\mathbb{R}$ is an analytic and non-constant function such that $s(0)=0$ and $s(t)>0$ if $t\in(0,\delta)$. By analyticity of s', one can suppose that s is strictly increasing in the domain $[0,\delta)$. Hence,

$s\colon [0,\delta/2] \to [0,\delta']$ is a subanalytic homeomorphism, where $\delta' = s(\frac{\delta}{2})$. We define $\gamma\colon [0,\varepsilon) \to \overline{Z}$ by

$$\gamma(t) = \rho_m \circ \beta \circ s^{-1}(t\|v\|) = \rho_m(x(s^{-1}(t\|v\|)),$$
$$s(s^{-1}(t\|v\|))) = t\|v\|x(s^{-1}(t\|v\|)),$$

where $\varepsilon = \min\{\frac{\delta'}{\|v\|}, \delta'\}$. Therefore,

$$\lim_{t\to 0^+} \frac{\gamma(t)}{t} = \lim_{t\to 0^+} \frac{t\|v\|x(s^{-1}(\|v\|t))}{t} = \lim_{t\to 0^+} \|v\|x(s^{-1}(\|v\|t))=\|v\|x(0)=v,$$

and thus $\gamma(t) = tv + o(t)$ and $\gamma((0,\varepsilon)) \subset Z$. Since γ is a composition of proper continuous subanalytic maps, it is a continuous subanalytic map as well.

2) Case $v = 0$. In this case, let $\{z_k\}_{k\in\mathbb{N}} \subset Z$ be a sequence such that $\lim\limits_{k\to+\infty} z_k = 0$. Thus, $\{\frac{x_k}{\|x_k\|}\}_{k\in\mathbb{N}}$ is, up to take subsequence, a convergent sequence. Let $v' \in \mathbb{R}^m$ be the limit of this sequence, i.e., $\lim\limits_{k\to\infty} \frac{x_k}{\|x_k\|} = v'$. Likewise as it was done in the Case 1), one can show that there exists a continuous subanalytic arc $\gamma\colon [0,\varepsilon) \to \overline{Z}$ such that $\gamma(t) = tv' + o(t)$. Let us define $\widetilde{\gamma}\colon [0,\varepsilon^{\frac{1}{2}}) \to \overline{Z}$ by $\widetilde{\gamma}(t) = \gamma(t^2)$. Thus, we have $\widetilde{\gamma}(t) = o(t) = tv + o(t)$.

Reciprocally, if there exists a continuous subanalytic arc $\gamma\colon [0,\varepsilon) \to \overline{Z}$ such that $\gamma(t) = tv + o(t)$ and $\gamma((0,\varepsilon)) \subset Z$, then for each $k \in \mathbb{N}$ we define $s_k = \frac{\varepsilon}{k+2}$ and $z_k = \gamma(s_k)$. Thus, it is clear that v is a tangent vector of Z at 0, since $\lim\limits_{k\to+\infty} z_k = 0$ and $\lim\limits_{k\to+\infty} \frac{1}{s_k} z_k = v$.

□

It is an immediate consequence of Proposition 9.3.7 the following result:

Corollary 9.3.8 *Let X be a subanalytic set in $\mathbb{R}^m$, $p \in \overline{X\setminus\{p\}}$. Then $C(X,p) = \{v \in R^m$; there exists a continuous subanalytic arc $\gamma\colon [0,\varepsilon) \to \overline{Z}$ such that $\gamma((0,\varepsilon)) \subset Z$ and $\gamma(t) - p = tv + o(t)\}$.*

Proposition 9.3.9 *Let X be a subanalytic set in $\mathbb{R}^m$, $p \in \overline{X}$. If $v \in C(X,p) \setminus \{0\}$ then we can take a continuous subanalytic arc $\gamma\colon [0,\varepsilon) \to \overline{X}$ such that $\gamma((0,\varepsilon)) \subset X$, $\gamma(t) - p = tv + o(t)$ and, moreover, satisfies anyone of the following conditions:*

(i) $\|\gamma(t) - p\| < t\|v\|$;
(ii) $\|\gamma(t) - p\| = t\|v\|$;
(iii) $\|\gamma(t) - p\| > t\|v\|$, *for all* $t \in (0,\varepsilon)$.

Proof The proof is left as an exercise for the reader. □

9.3.2 Bi-Lipschitz Invariance of Tangent Cones

The main goal of this section is to show the so-called bi-Lipschitz invariance of the tangent cones of subanalytic sets in the following sense: if (X, p) is bi-Lipschitz equivalent to (Y, q), then $C(X, p)$ is bi-Lipschitz homeomorphic to $C(Y, q)$.

Lemma 9.3.10 (McShane-Whitney-Kirszbraun's Theorem [20], [34] and [16]) *Let $h\colon X \subset \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz mapping. Then there exists a Lipschitz mapping $H\colon \mathbb{R}^n \to \mathbb{R}^m$ such that $H|_X = h$.*

Proof It is enough to consider the case that X is a closed subset and $m = 1$. Let $C > 0$ be a constant such that $|h(x) - h(y)| \leqslant C\|x - y\|$ for all $x, y \in X$. We define $H\colon \mathbb{R}^n \to \mathbb{R}$ by $H(x) = \inf\{h(y) + C\|x - y\|;\ y \in X\}$. For $x \in X$, we have that $h(x) - h(y) \leqslant \|h(x) - h(y)\| \leqslant C\|x - y\|$, and thus $H(x) = h(x)$. Given $u, v \in \mathbb{R}^n$, for each $\varepsilon > 0$, let $x_0, y_0 \in X$ such that $H(u) > h(x_0) + C\|x_0 - u\| - \varepsilon$ and $H(v) > h(x_0) + C\|x_0 - v\| - \varepsilon$. Thus,

$$H(u) - H(v) \leqslant h(y_0) + C\|y_0 - u\| - h(y_0) - C\|y_0 - v\| + \varepsilon \leqslant C\|u - v\| + \varepsilon$$

and

$$H(v) - H(u) \leqslant h(x_0) + C\|x_0 - v\| - h(x_0) - C\|x_0 - u\| + \varepsilon \leqslant C\|u - v\| + \varepsilon.$$

Finally, by taking $\varepsilon \to 0^+$, we get $|H(u) - H(v)| \leqslant C\|u - v\|$, which finishes the proof. □

Now we can state and prove the main result of this section.

Theorem 9.3.11 (Theorem of the Bi-Lipschitz Invariance of the Tangent Cones) *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be subanalytic sets. If there are constants $C_1, C_2 > 0$ and a bi-Lipschitz homeomorphism $\phi\colon (X, x_0) \to (Y, y_0)$ such that*

$$\frac{1}{C_1}\|x - y\| \leqslant \|\phi(x) - \phi(y)\| \leqslant C_2\|x - y\|, \quad \forall x, y \in X,$$

then there is a global bi-Lipschitz homeomorphism $d\phi\colon C(X, x_0) \to C(Y, y_0)$ such that $d\phi(0) = 0$ and

$$\frac{1}{C_1}\|x - y\| \leqslant \|d\phi(x) - d\phi(y)\| \leqslant C_2\|x - y\|, \quad \forall x, y \in C(X, x_0).$$

Proof This proof follows closely the proof presented in [30]. The last part of this proof follows from the ideas presented in the proof of Theorem 3.1 in [31].

Let $\phi : X \to Y$ be a bi-Lipschitz homeomorphism. By McShane-Whitney-Kirszbraun's Theorem, there exists $\widetilde{\phi} : \mathbb{R}^n \to \mathbb{R}^m$ a Lipschitz map such that $\widetilde{\phi}|_X = \phi$ and $\widetilde{\psi} : \mathbb{R}^m \to \mathbb{R}^n$ another Lipschitz map such that $\widetilde{\psi}|_Y = \phi^{-1}$. Let us define $\varphi, \psi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ as follows:

$$\varphi(x, y) = (x - \widetilde{\psi}(y + \widetilde{\phi}(x)), y + \widetilde{\phi}(x))$$

and

$$\psi(z, w) = (z + \widetilde{\psi}(w), w - \widetilde{\phi}(z + \widetilde{\psi}(w))).$$

Since φ and ψ are composition of Lipschitz maps, they are also Lipschitz maps.

Next, we show that $\psi = \varphi^{-1}$. In fact, if $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ then

$$\begin{aligned}
\psi(\varphi(x, y)) &= \psi(x - \widetilde{\psi}(y + \widetilde{\phi}(x)), y + \widetilde{\phi}(x)) \\
&= (x - \widetilde{\psi}(y + \widetilde{\phi}(x)) + \widetilde{\psi}(y + \widetilde{\phi}(x)), y + \widetilde{\phi}(x) - \\
&\qquad \widetilde{\phi}(x - \widetilde{\psi}(y + \widetilde{\phi}(x)) + \widetilde{\psi}(y + \widetilde{\phi}(x))) \\
&= (x, y + \widetilde{\phi}(x) - \widetilde{\phi}(x)) \\
&= (x, y),
\end{aligned}$$

and if $(z, w) \in \mathbb{R}^n \times \mathbb{R}^m$ then

$$\begin{aligned}
\varphi(\psi(z, w)) &= \varphi(z + \widetilde{\psi}(w), w - \widetilde{\phi}(z + \widetilde{\psi}(w))) \\
&= (z + \widetilde{\psi}(w) - \widetilde{\psi}(w - \widetilde{\phi}(z + \widetilde{\psi}(w)) + \widetilde{\phi}(z + \widetilde{\psi}(w))), w - \\
&\qquad \widetilde{\phi}(z + \widetilde{\psi}(w)) + \widetilde{\phi}(z + \widetilde{\psi}(w))) \\
&= (z + \widetilde{\psi}(w) - \widetilde{\psi}(w), w) \\
&= (z, w).
\end{aligned}$$

Therefore $\psi = \varphi^{-1}$. Finally, it is clear that $\varphi(X \times \{0\}) = \{0\} \times Y$.

Thus, by doing the identifications $X \leftrightarrow X \times \{0\}$ and $Y \leftrightarrow \{0\} \times Y$, we may assume that X and Y are subsets of same $\mathbb{R}^N$ and there is a bi-Lipschitz homeomorphism $\varphi \colon \mathbb{R}^N \to \mathbb{R}^N$ such that $\varphi|_X = \phi$.

Without loss of generality, we assume that $x_0 = y_0 = 0$. Let $K > 0$ be a constant such that

$$\frac{1}{K}\|x - y\| \leqslant \|\varphi(x) - \varphi(y)\| \leqslant K\|x - y\|, \quad \forall x, y \in \mathbb{R}^N. \tag{9.1}$$

For each $k \in \mathbb{N}$, let us define the maps $\varphi_k, \psi_k : \mathbb{R}^N \to \mathbb{R}^N$ given by $\varphi_k(v) = k\varphi(\frac{1}{k}v)$ and $\psi_k(v) = k\varphi^{-1}(\frac{1}{k}v)$. For each integer $m \geqslant 1$, let us define $\varphi_{k,m} :=$

$\varphi_k|_{\overline{B}_m} : \overline{B}_m \to \mathbb{R}^N$ and $\psi_{k,m} := \psi_k|_{\overline{B}_{mK}} : \overline{B}_{mK} \to \mathbb{R}^N$, where $\overline{B}_r$ denotes the Euclidean closed ball of radius r and with centre at the origin in $\mathbb{R}^N$. Since

$$\frac{1}{K}\|x-y\| \leqslant \|\varphi_{k,1}(x)-\varphi_{k,1}(y)\| \leqslant K\|x-y\|, \quad \forall x, y \in \overline{B}_1,\ \forall k \in \mathbb{N}$$

and

$$\frac{1}{K}\|u-v\| \leqslant \|\psi_{k,1}(u)-\psi_{k,1}(v)\| \leqslant K\|u-v\|, \quad u, v \in \overline{B}_K,\ \forall k \in \mathbb{N},$$

there exist a subsequence $\{k_{j,1}\}_{j\in\mathbb{N}} \subset \mathbb{N}$ and Lipschitz maps $d\varphi_1 : \overline{B}_1 \to \mathbb{R}^N$ and $d\psi_1 : \overline{B}_K \to \mathbb{R}^N$ such that $\varphi_{k_{j,1},1} \to d\varphi_1$ uniformly on $\overline{B}_1$ and $\psi_{k_{j,1},1} \to d\psi_1$ uniformly on $\overline{B}_K$ (notice that $\{\varphi_{k,1}\}_{k\in\mathbb{N}}$ and $\{\psi_{k,1}\}_{k\in\mathbb{N}}$ have uniform Lipschitz constants). Furthermore, it is clear that

$$\frac{1}{K}\|u-v\| \leqslant \|d\varphi_1(u)-d\varphi_1(v)\| \leqslant K\|u-v\|, \quad \forall u, v \in \overline{B}_1$$

and

$$\frac{1}{K}\|z-w\| \leqslant \|d\psi_1(z)-d\psi_1(w)\| \leqslant K\|z-w\|, \quad \forall z, w \in \overline{B}_K.$$

Likewise as above, for each $m > 1$, we have

$$\frac{1}{K}\|x-y\| \leqslant \|\varphi_{k,m}(x)-\varphi_{k,m}(y)\| \leqslant K\|x-y\|, \quad x, y \in \overline{B}_m,\ \forall k \in \mathbb{N}$$

and

$$\frac{1}{K}\|u-v\| \leqslant \|\psi_{k,m}(u)-\psi_{k,m}(v)\| \leqslant K\|u-v\|, \quad u, v \in \overline{B}_{mK},\ \forall k \in \mathbb{N}.$$

Therefore, for each $m > 1$, there exist a subsequence $\{k_{j,m}\}_{j\in\mathbb{N}} \subset \{k_{j,m-1}\}_{j\in\mathbb{N}}$ and Lipschitz maps $d\varphi_m : \overline{B}_m \to \mathbb{R}^N$ and $d\psi_m : \overline{B}_{mK} \to \mathbb{R}^N$ such that $\varphi_{k_{j,m},m} \to d\varphi_m$ uniformly on $\overline{B}_m$ and $\psi_{k_{j,m},m} \to d\psi_m$ uniformly on $\overline{B}_{mK}$ with $d\varphi_m|_{\overline{B}_{m-1}} = d\varphi_{m-1}$ and $d\psi_m|_{\overline{B}_{(m-1)K}} = d\psi_{m-1}$. Furthermore,

$$\frac{1}{K}\|u-v\| \leqslant \|d\varphi_m(u)-d\varphi_m(v)\| \leqslant K\|u-v\|, \quad \forall u, v \in \overline{B}_m \tag{9.2}$$

and

$$\frac{1}{K}\|z-w\| \leqslant \|d\psi_m(z)-d\psi_m(w)\| \leqslant K\|z-w\|, \quad \forall z, w \in \overline{B}_{mK}. \tag{9.3}$$

Let us define $d\varphi, d\psi : \mathbb{R}^N \to \mathbb{R}^N$ by $d\varphi(x) = d\varphi_m(x)$, if $x \in \overline{B}_m$ and $d\psi(x) = d\psi_m(x)$, if $x \in \overline{B}_{mK}$ and, for each $j \in \mathbb{N}$, let $n_j = k_{j,j}$ and $t_j = 1/n_j$.

Claim $\varphi_{n_j} \to d\varphi$ and $\psi_{n_j} \to d\psi$ uniformly on compact subsets of $\mathbb{R}^N$. □

Proof Let $F \subset \mathbb{R}^N$ be a compact subset. Let us take $m \in \mathbb{N}$ such that $F \subset \overline{B}_m \subset \overline{B}_{mK}$. Thus, $\{n_j\}_{j>m}$ is a subsequence of $\{k_{j,m}\}_{j\in\mathbb{N}}$ and, since $\varphi_{k_{j,m},m} \to d\varphi_m$ uniformly on $\overline{B}_m$ and $\psi_{k_{j,m},m} \to d\psi_m$ uniformly on $\overline{B}_{mK}$, it follows that $\varphi_{n_j} \to d\varphi$ and $\psi_{n_j} \to d\psi$ uniformly on F. □

Claim $d\varphi : \mathbb{R}^N \to \mathbb{R}^N$ is a bi-Lipschitz homeomorphism and $d\psi = (d\varphi)^{-1}$. □

Proof It follows from inequalities (9.2) and (9.3) that $d\varphi, d\psi : \mathbb{R}^N \to \mathbb{R}^N$ are Lipschitz mappings. Therefore, it is enough to show that $d\psi = (d\varphi)^{-1}$. In order to do that, let $v \in \mathbb{R}^N$ and $w = d\varphi(v) = \lim\limits_{j\to\infty} \frac{\varphi(t_j v)}{t_j}$. Thus,

$$\begin{aligned}
\|d\psi(w) - v\| &= \left\| \lim_{j\to\infty} \tfrac{\psi(t_j w)}{t_j} - v \right\| &&= \lim_{j\to\infty} \left\| \tfrac{\psi(t_j w)}{t_j} - \tfrac{t_j v}{t_j} \right\| \\
&= \lim_{j\to\infty} \tfrac{1}{t_j} \left\| \psi(t_j w) - t_j v \right\| &&= \lim_{j\to\infty} \tfrac{1}{t_j} \left\| \psi(t_j w) - \psi(\varphi(t_j v)) \right\| \\
&\leqslant \lim_{j\to\infty} \tfrac{K}{t_j} \left\| t_j w - \varphi(t_j v) \right\| &&= \lim_{j\to\infty} K \left\| w - \tfrac{\varphi(t_j v)}{t_j} \right\| \\
&= 0.
\end{aligned}$$

Then, $d\psi(w) = d\psi(d\varphi(v)) = v$, for all $v \in \mathbb{R}^N$, i.e., $d\psi \circ d\varphi = \mathrm{id}_{\mathbb{R}^N}$. Analogously, one can show that $d\varphi \circ d\psi = \mathrm{id}_{\mathbb{R}^N}$. □

Claim $d\varphi(C(X,0)) = C(Y,0)$. □

Proof By the previous claim, it is enough to verify that $d\varphi(C(X,0)) \subset C(Y,0)$. In order to do that, let $v \in C(X,0)$. Then, there is $\alpha\colon [0,\varepsilon) \to X$ such that $\alpha(t) = tv + o(t)$. Thus, $\varphi(\alpha(t)) = \varphi(tv) + o(t)$, since φ is a Lipschitz map. However, $\varphi(t_j v) = t_j d\varphi(v) + o(t_j)$ and then

$$d\varphi(v) = \lim_{j\to\infty} \varphi_{n_j}(v) = \lim_{j\to\infty} \frac{\varphi(t_j v)}{t_j} = \lim_{j\to\infty} \frac{\varphi(\alpha(t_j))}{t_j} \in C(Y,0).$$

□

Therefore, $d\varphi\colon C(X,0) \to C(Y,0)$ is a bi-Lipschitz homeomorphism.

Claim $\frac{1}{C_1}\|v - w\| \leqslant \|d\varphi(v) - d\varphi(w)\| \leqslant C_2\|v - w\|, \quad \forall v, w \in C(X,0)$. □

Proof Let $v \in C(X,0)$. By Proposition 9.3.7, there is a curve $\gamma\colon [0,\varepsilon) \to X$ such that $\gamma(t) = tv + o(t)$. Then, we obtain

$$\left\| \frac{\varphi(t_j v)}{t_j} - \frac{\varphi(\gamma(t_j))}{t_j} \right\| = \frac{o(t_j)}{t_j} \to 0 \text{ as } j \to +\infty.$$

Therefore,

$$\lim_{j\to+\infty}\frac{\varphi(t_j v)}{t_j}=\lim_{j\to+\infty}\frac{\varphi(\gamma(t_j))}{t_j}=d\varphi(v).$$

As $\varphi|_{X\times\{0\}}=0\times\phi$, we have

$$\lim_{j\to+\infty}\frac{\phi(\gamma(t_j))}{t_j}=d\varphi(v). \tag{9.4}$$

Therefore, if $v, w\in C(X,0)$, there are curves $\gamma,\beta\colon[0,\varepsilon)\to X$ such that $\gamma(t)=tv+o(t)$ and $\beta(t)=tw+o(t)$. Thus, by the hypothesis of the theorem, we get

$$\frac{1}{C_1}\left\|\frac{\gamma(t_j)}{t_j}-\frac{\beta(t_j)}{t_j}\right\|\leqslant\left\|\frac{\phi(\gamma(t_j))}{t_j}-\frac{\phi(\beta(t_j))}{t_j}\right\|\leqslant C_2\left\|\frac{\gamma(t_j)}{t_j}-\frac{\beta(t_j)}{t_j}\right\|.$$

Passing to the limit $j\to+\infty$ and using (9.4), we obtain

$$\frac{1}{C_1}\|v-w\|\leqslant\|d\varphi(v)-d\varphi(w)\|\leqslant C_2\|v-w\|.$$

□

This proves the theorem.

9.3.3 Exercises

Exercise 9.3.12 Prove Proposition 9.3.4.

Exercise 9.3.13 Prove Proposition 9.3.5.

Exercise 9.3.14 Prove Proposition 9.3.9.

Exercise 9.3.15 Let $X_1\subset\mathbb{R}^{m_1}$ and $X_2\subset\mathbb{R}^{m_2}$ be closed sets such that $C(X_i,p_i)=\{v\in\mathbb{R}^{m_i};$ there exists a continuous arc $\gamma\colon[0,\varepsilon)\to X_i$ such that $\gamma(t)-p_i=tv+o(t)\}$. If there is a bi-Lipschitz homeomorphism $\phi\colon(X_1,p_1)\to(X_2,p_2)$, then prove that there is a global bi-Lipschitz homeomorphism $d\phi\colon C(X_1,p_1)\to C(X_2,p_2)$ such that $d\phi(0)=0$.

9.4 Lipschitz Regularity Theorem and the Bi-Lipschitz Invariance of the Multiplicity 1

We start this section with the introduction of different notions of Lipschitz regular points (or Lipschitz regularity). The aim of this section is to show that, for any analytic subset $X \subset \mathbb{C}^n$, Lipschitz regularity of X at $p \in X$ implies that p is an analytic regular point of X (for any notion of Lipschitz regularity we work with).

9.4.1 Notions of Lipschitz Regularity

Definition 9.4.1 We say that a set $X \subset \mathbb{R}^n$ is **Lipschitz regular** (resp. a **Lipschitz submanifold**) at $p \in X$ if there exist an open neighbourhood $U \subset \mathbb{R}^n$ and a bi-Lipschitz homeomorphism $\varphi\colon U \cap X \to B^d$ (resp. $\varphi\colon U \to B^n$ such that $\varphi(U \cap X) = B^n \cap (\mathbb{R}^d \times \{0\})$), where B^k is the unit open ball of $\mathbb{R}^k$ centred at the origin.

Definition 9.4.2 We say that a set $X \subset \mathbb{R}^n$ is a **Lipschitz graph** at $p \in X$ if there exist an open neighbourhood $U \subset \mathbb{R}^n$ and a Lipschitz map $F\colon B^d \to \mathbb{R}^{n-d}$ such that $U \cap X = graph(F)$.

It is clear the following:

(1) If X is a Lipschitz graph at p then it is a Lipschitz submanifold at p;
(2) If X is a Lipschitz submanifold at p then it is Lipschitz regular at p.

However, the converses of (1) and (2) are not true in general for semialgebraic sets (see Exercises 9.4.10 and 9.4.11).

9.4.2 C^k Smoothness of Analytic Sets

In this subsection, we present other notions of regularity of sets apart from those presented in Sect. 9.4.1.

Definition 9.4.3 For $k \in \mathbb{N} \cup \{\infty, \omega\}$, we say that a set $X \subset \mathbb{R}^n$ is C^k **submanifold** or C^k **smooth** at $p \in X$ if there exist an open neighbourhood $U \subset \mathbb{R}^n$ of p and a C^k diffeomorphism (or a homeomorphism when $k = 0$) $\varphi\colon U \to B^n$ such that $\varphi(U \cap X) = B^d \times \{0\}$ and $\varphi(p) = 0$.

Since we can see $\mathbb{C}^n$ as $\mathbb{R}^{2n}$, subsets of $\mathbb{C}^n$ are subsets of $\mathbb{R}^{2n}$. Therefore the notions of regularity or smoothness introduced in Definitions 9.4.1, 9.4.2 and 9.4.3 make sense even to subsets of $\mathbb{C}^n$.

Proposition 9.4.4 *If a complex analytic subset X is C^1 smooth at $x_0 \in X$, then X is a complex analytic submanifold at x_0.*

Proof The proof is left as an exercise for the reader. □

Example 9.4.5 The set $X = \{(x, y, z) \in \mathbb{C}^3;\, y^3 = z^2 \text{ and } x = 0\}$ is C^0 smooth at any $p \in X$. Indeed, let $\phi\colon \mathbb{C} \to Y = \{(x, y) \in \mathbb{C}^2;\, y^2 = x^3\}$ be the homeomorphism given by $\phi(t) = (t^2, t^3)$. Let $\psi\colon \mathbb{C}^2 \to \mathbb{C}$ be a continuous extension of ϕ^{-1} and let $\varphi\colon \mathbb{C} \times \mathbb{C}^2 \to \mathbb{C}^2 \times \mathbb{C}$ be the map given by $\varphi(s, u) =$

$(s + \psi(u), u - \phi(s + \psi(u)))$. We have that φ is a homeomorphism such that $\varphi^{-1}(t, v) = (t - \psi(v + \phi(t)), v + \phi(t))$ and $\varphi(X) = \mathbb{C} \times \{0\}$. However, since $m(X, 0) = 2$, by Proposition 9.4.4 and Corollary 9.1.10, X is not C^1 smooth at 0.

9.4.3 Lipschitz Regularity of Analytic Sets

The following result is due to Sampaio in [30], but it was proved in a slight weaker version by Bibrair at al. in [1].

Theorem 9.4.6 (Lipschitz Regularity Theorem) *Let $X \subset \mathbb{C}^n$ be a complex analytic set. If X is Lipschitz regular at $x_0 \in X$, then X is smooth at x_0.*

Proof Let $X \subset \mathbb{C}^n$ be a d-dimensional complex analytic set. Assume that X is Lipschitz regular at $x_0 \in X$. Let $h: U \to B$ be a subanalytic bi-Lipschitz homeomorphism between an open neighbourhood U of x_0 in X and $B \subset \mathbb{R}^{2d}$, that is, an open Euclidean ball centred at the origin $0 \in \mathbb{R}^{2d}$. Let us suppose that $h(x_0) = 0$. By Theorem 9.3.11, $dh: C(X, x_0) \to T_0B$ is a bi-Lipschitz homeomorphism between the tangent cones $C(X, x_0)$ and $T_0B = \mathbb{R}^{2d}$. In particular, $C(X, x_0)$ is a topological manifold.

The next result was proved by D. Prill in [27]: □

Theorem 9.4.7 (Prill's Theorem) *Let $V \subset \mathbb{C}^n$ be a complex cone. If $0 \in V$ has a neighborhood homeomorphic to a Euclidean ball, then V is a linear subspace of $\mathbb{C}^n$.*

Now, since we consider complex analytic sets, the tangent cone at x_0 of a complex analytic set is a complex cone (see Remark 9.3.2). Then $C(X, x_0)$ is a d-dimensional linear subspace of $\mathbb{C}^n$.

Moreover, by Proposition 9.2.2, X is LNE at x_0. Thus, our main theorem is consequence of the following:

Proposition 9.4.8 *Let $X \subset \mathbb{C}^n$ be a complex analytic subset. Let $x_0 \in X$ be such that the tangent cone $C(X, x_0)$ is a linear subspace of $\mathbb{C}^n$. If there exists a neighbourhood U of x_0 in X such that U is LNE, then X is smooth at x_0.*

Proof Since $E := C(X, x_0)$ is a linear subspace of $\mathbb{C}^n$, we can consider the orthogonal projection

$$P: \mathbb{C}^n \to E.$$

We may suppose that $x_0 = 0$ and $P(x_0) = 0$. Let us choose linear coordinates (x, y) in $\mathbb{C}^n$ such that $E = \{(x, y) \in \mathbb{C}^n;\ y = 0\}$. □

Claim There exist positive constants C and ρ such that $X \cap B_\rho \subset \{(x, y); \|y\| < C\|x\|\}$. □

Proof Indeed, if this claim is not true, there exists a sequence $\{(x_k, y_k)\}_{k\in\mathbb{N}} \subset X$ such that $\lim\limits_{k\to+\infty}(x_k, y_k) = 0$ and $\|y_k\| \geqslant k\|x_k\|$. Thus, up to a subsequence, one can suppose that $\lim\limits_{k\to+\infty} \frac{y_k}{\|y_k\|} = y_0$. Since $\frac{\|x_k\|}{\|y_k\|} \leqslant \frac{1}{k}$, we obtain that $(0, y_0) \in C(X, 0)$, which is a contradiction, because $y_0 \neq 0$. Therefore, Claim 9.4.3 is true. □

Notice that the germ of the restriction of the orthogonal projection P to $X \cap B_\rho$ is a finite complex analytic map germ.

Moreover, we have the following:

Claim If $\gamma\colon [0, \varepsilon) \to X$ is a real analytic arc, such that $\gamma(0) = 0$, then the arcs γ and $P \circ \gamma$ are tangent at 0. □

Proof In order to prove this claim, let us write $\gamma(t) = (x(t), y(t))$. By the previous claim, there exists $t_0 > 0$ such that $\|y(t)\| \leqslant C\|x(t)\|$ for all $t \leqslant t_0$, since $\lim\limits_{t\to 0^+} \gamma(t) = 0$. Thus, since $\frac{x(t)}{t}$ is bounded, $\frac{y(t)}{t}$ is bounded. Let us suppose that $y(t) \neq o(t)$. Then, there exist a sequence $\{t_k\}_{k\in\mathbb{N}} \subset (0, +\infty)$ and $r > 0$ such that $t_k \to 0$ and $\frac{\|y(t_k)\|}{t_k} \geqslant r$ for all k. Since $\left\{\frac{y(t_k)}{t_k}\right\}_{k\in\mathbb{N}}$ is bounded, up to a subsequence, one can suppose that $\lim\limits_{k\to+\infty} \frac{y(t_k)}{t_k} = y_0$. Therefore, $\lim\limits_{k\to+\infty} \frac{\gamma(t_k)}{t_k} = (v', y_0) \in C(X, 0)$, where $v = (v', 0)$. However, this is a contradiction, since $\|y_0\| \geqslant r > 0$ and this implies that $y_0 \neq 0$. Then, $y(t) = o(t)$ and, therefore, $\gamma(t) = tv + o(t)$. □

In this way, the germ at 0 of $P_{|X}\colon X \to E$ is a ramified cover and the ramification locus is the germ of a codimension $\geqslant 1$ complex analytic subset Σ of the linear space E.

The multiplicity of X at 0 can be interpreted as the degree m of this germ of ramified covering map, i.e. there are open neighbourhoods U_1 of 0 in X and U_2 of 0 in E, such that m is the degree of the topological covering:

$$P_{|X}\colon X \cap U_1 \setminus P_{|X}^{-1}(\Sigma) \to E \cap U_2 \setminus \Sigma.$$

Let us suppose that the degree m is greater than 1. Since Σ is a codimension $\geqslant 1$ complex analytic subset of the space E, there exists a unit tangent vector $v_0 \in E \setminus C(\Sigma, 0)$.

Since v_0 is not tangent to Σ at 0, there exists a positive real number k such that the real cone:

$$\{v \in E \mid \|v - tv_0\| < tk,\ \forall\, 0 < t < 1\}$$

does not intersect the set Σ. Since we have assumed that the degree $m \geqslant 2$, we have at least two different liftings $\gamma_1(t)$ and $\gamma_2(t)$ of the half-line $r(t) = tv_0$, i.e. $P(\gamma_1(t)) = P(\gamma_2(t)) = tv_0$. Since P is the orthogonal projection on the tangent cone E, the vector v_0 is the unit tangent vector to the arcs γ_1 and γ_2 at 0. By construction, we have $\text{dist}(\gamma_i(t), P_{|X}^{-1}(\Sigma)) \geqslant kt$ ($i = 1, 2$), where by dist we mean the Euclidean distance.

On the other hand, any path in X connecting $\gamma_1(t)$ to $\gamma_2(t)$ is the lifting of a loop, based at the point tv_0, which is not contractible in the germ of $E \setminus \Sigma$ at 0. Thus the length of such a path must be at least $2kt$. It implies that the inner distance, $\text{d}_X(\gamma_1(t), \gamma_2(t))$, in X, between $\gamma_1(t)$ and $\gamma_2(t)$, is at least $2kt$. But, since $\gamma_1(t)$ and $\gamma_2(t)$ are tangent at 0, that is $\frac{\|\gamma_1(t) - \gamma_2(t)\|}{t} \to 0$ as $t \to 0^+$, and $k > 0$, we obtain that X is not LNE near 0. Otherwise there will be $\lambda > 0$ such that:

$$d_X(x_1, x_2) \leq \lambda \|x_1 - x_2\| \quad \text{for all} \quad x_1, x_2 \in X \quad \text{near } 0,$$

hence:

$$\begin{aligned} 2k &\leqslant \frac{\text{d}_{\text{inner}}(\gamma_1(\text{t}), \gamma_2(\text{t}))}{t} \\ &\leqslant \lambda \frac{\|\gamma_1(t) - \gamma_2(t)\|}{t} \to 0. \end{aligned}$$

which is a contradiction.

Therefore, $m = m(X, 0) = 1$, and thus by Exercise 9.4.13 X is smooth at 0.

This concludes the theorem. □

As a consequence, we obtain that the multiplicity 1 is a bi-Lipschitz invariant.

Corollary 9.4.9 *Let $X \subset \mathbb{C}^n$ and $Y \subset \mathbb{C}^m$ be complex analytic sets. If there is a bi-Lipschitz homeomorphism $\varphi\colon (X, 0) \to (Y, 0)$ then $m(X, 0) = 1$ if and only if $m(Y, 0) = 1$.*

9.4.4 Exercises

Exercise 9.4.10 Let $L = \{(x, y) \in \mathbb{C}^2; x^3 = y^2\} \cap \mathbb{S}^3$. Prove that $X = Cone(L) = \{tu; u \in L \text{ and } t \geqslant 0\}$ is Lipschitz regular at 0, but it is not a Lipschitz submanifold at 0.

Exercise 9.4.11 Give an example of a semialgebraic set $X \subset \mathbb{R}^n$ which is Lipschitz submanifold at 0 of $\mathbb{R}^n$, but it is not a Lipschitz graph at 0.

Exercise 9.4.12 Let $X = \{(x, y, z) \in \mathbb{R}^3; x^3 + y^3 + z^3 = 0\}$. Prove that:

(a) X is not C^1 smooth at 0;
(b) X is Lipschitz regular at 0.

Exercise 9.4.13 Let X be a pure dimensional complex analytic set with $0 \in X$. Prove that X is smooth at 0 if and only if $m(X, 0) = 1$.

Exercise 9.4.14 Let $X \subset \mathbb{C}^n$ and $Y \subset \mathbb{C}^m$ be complex analytic sets. If there is a bi-Lipschitz homeomorphism $\varphi\colon X \to Y$ then prove that $\varphi(Reg(X)) = Reg(Y)$ and $\varphi(Sing(X)) = Sing(Y)$, where for a complex analytic set $A \subset \mathbb{C}^k$, $Reg(A)$ denotes the subset of points $x \in A$ such that for some open neighbourhood $U \subset \mathbb{C}^k$ of x, $X \cap U$ is a complex analytic submanifold of $\mathbb{C}^k$, and $Sing(A) = A \setminus Reg(A)$.

9.5 Relative Multiplicities and Multiplicity of Homogeneous Singularities

In this section, we define the relative multiplicities of a complex analytic set at a point. Moreover, we prove that the relative multiplicities are bi-Lipschitz invariant (see Theorem 9.5.1). We also present a reduction of the Questions AL(d) and AL(n, d) for homogeneous algebraic sets (see Corollary 9.5.4).

9.5.1 Bi-Lipschitz Invariance of the Relative Multiplicities

Let $X \subset \mathbb{C}^n$ be a complex analytic set such that $0 \in X$. Let $X_1, \ldots, X_r$ be the irreducible components of $C(X, 0)$. Fix $j \in \{1, \ldots, r\}$. For a generic point $x \in (X_j \cap \mathbb{S}^{2n-1}) \times \{0\}$, the number of connected components of the germ $(\rho^{-1}(X \setminus \{0\}), x)$ is constant, and we denote this number by $k_X(X_j)$. The integer numbers $k_X(X_j)$ are called the **relative multiplities** of X at 0.

The following result, which was proved by Fernandes and Sampaio in the paper [12], shows the bi-Lipschitz invariance of the relative multiplicities.

Theorem 9.5.1 *Let $X \subset \mathbb{C}^n$ and $Y \subset \mathbb{C}^m$ be two complex analytic sets with $p = \dim X = \dim Y$, $0 \in X$ and $0 \in Y$. Let $X_1, \ldots, X_r$ and $Y_1, \ldots, Y_s$ be the irreducible components of the tangent cones $C(X, 0)$ and $C(Y, 0)$, respectively. If there exists a bi-Lipschitz homeomorphism $\varphi\colon (X, 0) \to (Y, 0)$, then $r = s$ and, up to a reordering of indices, $k_X(X_j) = k_Y(Y_j)$, $\forall\, j$.*

Proof Let $S = \{t_k\}_{k\in\mathbb{N}}$ be a sequence of positive real numbers such that

$$t_k \to 0 \quad \text{and} \quad \frac{\varphi(t_k v)}{t_k} \to d\varphi(v)$$

where $d\varphi$ is a tangent map of φ as in Theorem 9.3.11. Since $d\varphi$ is a bi-Lipschitz homeomorphism, we obtain $r = s$ and there is a permutation $\sigma\colon \{1, \ldots, r\} \to \{1, \ldots, s\}$ such that $d\varphi(X_i) = Y_{\sigma(i)}$ $\forall\, i$. This is because we can assume $d\varphi(X_i) = Y_i$ $\forall\, i$ up to a reordering of indices. Let

$$SX = \{(x, t) \in \mathbb{S}^{2n-1} \times S;\ tx \in X\}.$$

Thus, $\rho^{-1} \circ \varphi \circ \rho\colon SX \to Y'$ is an injective and continuous map that extends continuously to a map $\varphi'\colon \overline{SX} \to Y'$.

For each generic point $x \in \mathbb{S}_0 X_j \times \{0\}$, we know that $k_X(X_j)$ is the number of connected components of the set $\rho^{-1}(X \setminus \{0\}) \cap B_\delta(x)$ for $\delta > 0$ small enough. Then, $k_X(X_j)$ can be seen as the number of connected components of the set $(SX \cap \mathbb{S}^{2n-1} \times \{t_k\}) \cap B_\delta(x)$ for k large enough.

Let $\pi : \mathbb{C}^n \to \mathbb{C}^p$ be a linear projection such that

$$\pi^{-1}(0) \cap (C(X,0) \cup C(Y,0)) = \{0\}.$$

Let us denote the ramification locus of

$$\pi_{|X}\colon X \to \mathbb{C}^p \quad \text{and} \quad \pi_{|C(X,0)}\colon C(X,0) \to \mathbb{C}^p$$

by $\sigma(X)$ and $\sigma(C(X,0))$ respectively.

Given a generic point $v' \in \mathbb{C}^p \setminus (\sigma(X) \cup \sigma(C(X,0)))$ (generic here means that v' defines a direction not tangent to $\sigma(X) \cup \sigma(C(X,0)))$, let $\eta, \varepsilon > 0$ be sufficiently small such that

$$C_{\eta,\varepsilon}(v') = \{w \in \mathbb{C}^p \mid \exists t > 0;\ \|tv' - w\| < \eta t\} \cap B_\varepsilon(0) \subset \mathbb{C}^p \setminus \sigma(X) \cup \sigma(C(X,0)).$$

The number of connected components of $\pi^{-1}(C_{\eta,\varepsilon}(v')) \cap X$ is exactly $m(X)$, since $C_{\eta,\varepsilon}(v')$ is simply connected and $\pi\colon X \setminus \pi^{-1}(\sigma(X)) \to \mathbb{C}^p \setminus \sigma(X)$ is a covering map. Then, we get that $\pi|_V\colon V \to C_{\eta,\varepsilon}(v')$ is bi-Lipschitz for each connected component V of $\pi^{-1}(C_{\eta,\varepsilon}(v')) \cap X$. Therefore, for each $j = 1, \ldots, r$, there are different connected components $V_{j1}, \ldots, V_{jk_X(X_j)}$ of $\pi^{-1}(C_{\eta,\varepsilon}(v')) \cap X$ such that $C(\overline{V_{ji}}, 0) \subset X_j$, $i = 1, \ldots, k_X(X_j)$.

Let us suppose that there is $j \in \{1, \ldots, r\}$ such that $k_X(X_j) > k_Y(Y_j)$, it means that if we consider a generic point $x = (v, 0) \in \partial X' \cap X_j \times \{0\}$, there are at least two different connected components V_{ji} and V_{jl} of $\pi^{-1}(C_{\eta,\varepsilon}(\pi(v))) \cap X$ and sequences $\{(x_k, t_k)\}_{k\in\mathbb{N}} \subset \rho^{-1}(V_{ji}) \cap SX$ and $\{(y_k, t_k)\}_{k\in\mathbb{N}} \subset \rho^{-1}(V_{jl}) \cap SX$ such that $\lim(x_k, t_k) = \lim(y_k, t_k) = x$ and $\varphi'(x_k, t_k), \varphi'(y_k, t_k) \in \rho^{-1}(\widetilde{V}_{jm})$, where $\widetilde{V}_{jm}$ is a connected component of $\pi^{-1}(C_{\eta,\varepsilon}(\pi(d\varphi(v)))) \cap Y$.

Since $\varphi(t_k x_k), \varphi(t_k y_k) \in \widetilde{V}_{jm}$ $\forall\ k \in \mathbb{N}$ and $V = \widetilde{V}_{jm}$ is bi-Lipschitz homeomorphic to $C_{\eta,\varepsilon}(\pi(d\varphi(v)))$, we have

$$\|\varphi(t_k x_k) - \varphi(t_k y_k)\| = o(t_k)$$

and

$$d_Y(\varphi(t_k x_k), \varphi(t_k y_k)) \leqslant d_V(\varphi(t_k x_k), \varphi(t_k y_k)) = o(t_k).$$

Now, since X is bi-Lipschitz homeomorphic to Y, we have $d_X(t_k x_k, t_k y_k) \leqslant o(t_k)$. On the other hand, since $t_k x_k$ and $t_k y_k$ lie in different connected components of

$\pi^{-1}(C_{\eta,\varepsilon}(\pi(v)))\cap X$, there exists a constant $C > 0$ such that $d_X(t_k x_k, t_k y_k) \geqslant Ct_k$, which is a contradiction.

We have proved that $k_X(X_j) \leqslant k_Y(Y_j)$, $j = 1,\dots,r$. By similar arguments, using that φ^{-1} is a bi-Lipschitz map, we also can prove $k_Y(Y_j) \leqslant k_X(X_j)$, $j = 1,\dots,r$. □

9.5.2 Reduction to Homogeneous Algebraic Sets

Proposition 9.5.2 *Let $X \subset \mathbb{C}^n$ be a d-dimensional complex analytic set such that $0 \in X$. Let $X_1,\dots,X_r$ be the irreducible components of $C(X,0)$. Then*

$$m(X,0) = \sum_{j=1}^{r} k_X(X_j)m(X_j,0).$$

Proof The proof is left as an exercise for the reader. □

Next result follows from Theorem9.5.1 and Proposition 9.5.2.

Theorem 9.5.3 *Let $X \subset \mathbb{C}^n$ and $Y \subset \mathbb{C}^m$ be two complex analytic sets with $\dim X = \dim Y = d$, $0 \in X$ and $0 \in Y$. Let $X_1,\dots,X_r$ (resp. $Y_1,\dots,Y_s$) be the irreducible components of $C(X,0)$ (resp. $C(Y,0)$). If there exists a bi-Lipschitz homeomorphism $\varphi\colon (X,0) \to (Y,0)$, then $r = s$ and there exist a bijection $\sigma\colon \{1,\dots,r\} \to \{1,\dots,s\}$ and a bi-Lipschitz homeomorphism $d\varphi\colon C(X,0) \to C(Y,0)$ with $d\varphi(0) = 0$ and such that $d\varphi(X_i) = Y_{\sigma(i)}$ and $k_X(X_i) = k_Y(Y_{\sigma(i)})$ for all $i \in \{1,\dots,r\}$. Additionally, if $m(X_i,0) = m(Y_{\sigma(i)},0)$ for all $i \in \{1,\dots,r\}$, then $m(X,0) = m(Y,0)$.*

As a direct consequence of the above theorem, we obtain that to solve Question $AL(d)$ (resp. $AL(n,d)$), it is enough to solve it only for irreducible homogeneous algebraic sets.

Corollary 9.5.4 *Question $AL(d)$ (resp. $AL(n,d)$) has a positive answer if and only if it has a positive answer for irreducible homogeneous algebraic sets.*

9.5.3 Exercises

Exercise 9.5.5 Prove Proposition 9.5.2.

Exercise 9.5.6 Prove Theorem 9.5.3.

9.6 Bi-Lipschitz Invariance of the Multiplicity

This section is devoted to present a complete answer to the Question AL(d) on the bi-Lipschitz invariance of the multiplicity. We also present some partial results of other authors concerning the questions AL(d) and AL(n, d). We start by addressing the case of complex singularities of dimension 1.

9.6.1 Multiplicity of Curves

Theorem 9.6.1 *Let $X \subset \mathbb{C}^n$ and $Y \subset \mathbb{C}^m$ be two complex analytic curves. If $\varphi\colon (X, 0) \to (Y, 0)$ is a bi-Lipschitz homeomorphism, then $m(X, 0) = m(Y, 0)$.*

Proof Let us consider $X \subset \mathbb{C}^n$ and $Y \subset \mathbb{C}^m$ be two complex analytic curves; $0 \in X$ and $0 \in Y$. Then, we know that the tangent cones $C(X, 0)$ and $C(Y, 0)$ are union of complex lines through the origin, let us say:

$$C(X,0) = \bigcup_{i=1}^{r} L_i^X \text{ and } C(Y,0) = \bigcup_{j=1}^{s} L_j^Y.$$

Since $\varphi\colon (X, 0) \to (Y, 0)$ is a bi-Lipschitz homeomorphism, it follows, by Theorem 9.5.3, that $r = s$ and there exist a bijection $\sigma : \{1, \ldots, r\} \to \{1, \ldots, s\}$ and a bi-Lipschitz homeomorphism $d\varphi\colon C(X, 0) \to C(Y, 0)$ with $d\varphi(0) = 0$ and such that $d\varphi(L_i^X) = L_{\sigma(i)}^Y$ and $k_X(L_i^X) = k_Y(L_{\sigma(i)}^Y)$ for all $i \in \{1, \ldots, r\}$. Then, since $m(L_j^Y, 0) = 1$ and $m(L_i^X, 0) = 1$ $\forall i, j$, it follows from Theorem 9.5.3 that $m(X, 0) = m(Y, 0)$. □

9.6.2 Multiplicity of 2-Dimensional Analytic Hypersurfaces

In this subsection, we bring a positive answer to the Question AL(3,2). Notice that such a result is a consequence of the positive answer of Question AL(2) which the proof we sketch in the end of this section. Since our proof of Question AL(2) we sketch here relies on a non-trivial topological result that we do not prove here, we decide to keep this subsection with a proof of the positive answer for AL(3,2) which is quite complete and self-contained.

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a homogeneous polynomial with $\deg f = d$. We recall that the map $\phi\colon \mathbb{S}^{2n-1} \setminus f^{-1}(0) \to \mathbb{S}^1$ given by $\phi(z) = \frac{f(z)}{|f(z)|}$ is a locally trivial fibration (see [21], §4). Notice that, $\psi : \mathbb{C}^n \setminus f^{-1}(0) \to \mathbb{C} \setminus \{0\}$ defined by $\psi(z) = f(z)$ is a locally trivial fibration such that its fibres are diffeomorphic to the fibres of ϕ. Moreover, we can choose as geometric monodromy the homeomorphism h_f :

$F_f \to F_f$ given by $h_f(z) = e^{\frac{2\pi i}{d}} \cdot z$, where $F_f := f^{-1}(1)$ is the (global) Milnor fibre of f (see [21], §9).

It follows from Theorem 3.3 and Remark 3.4 in [18] the following:

Proposition 9.6.2 *Let $f, g \subset \mathbb{C}^{n+1} \to \mathbb{C}$ be two irreducible homogeneous polynomials. If $\varphi\colon (\mathbb{C}^{n+1}, V(f), 0) \to (\mathbb{C}^{n+1}, V(g), 0)$ is a homeomorphism, then the induced maps in homology of the monodromies of f and g at 0 are conjugated. In particular, the monodromies of f and g at 0 have the same Lefschetz number and $\chi(F_f) = \chi(F_g)$.*

Definition 9.6.3 Let $f\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a complex analytic function with

$$\dim SingV(f) = 1 \text{ and } SingV(f) = C_1 \cup \ldots \cup C_r.$$

Then $b_i(f)$ denotes the i-th Betti number of the Milnor fibre of f at the origin, $\mu'_j(f)$ is the Milnor number of a generic hyperplane slice of f at $x_j \in C_j \setminus \{0\}$ sufficiently close to the origin and $\mu'(f) = \sum_{i=1}^{r} \mu'_i(f)$.

Let us remind some results on Milnor number of a generic hyperplane slice.

In [19], $\mu'(f)$ is denoted by σ_f and it was proved in [19, Theorem 4.1] that it is an embedded topological invariant. We state that result here.

Proposition 9.6.4 *Let $f, g\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be complex analytic functions with 1-dimensional singular sets. If there is a homeomorphism $\varphi\colon (\mathbb{C}^{n+1}, V(f), 0) \to (\mathbb{C}^{n+1}, V(g), 0)$, then $\mu'(f) = \mu'(g)$.*

Proposition 9.6.5 (Theorem 5.11 in [28]) *Let $f \subset \mathbb{C}^{n+1} \to \mathbb{C}$ be a homogeneous polynomial with degree d and 1-dimensional singular set. Then*

$$\chi(F_f) = 1 + (-1)^n((d-1)^{n+1} - d\mu'(f)).$$

The following result was proved by Sampaio in [32].

Theorem 9.6.6 *Let $X, Y \subset \mathbb{C}^{n+1}$ be two complex analytic hypersurfaces with $0 \in X \cap Y$. Assume that each irreducible component X_i of $C(X, 0)$ satisfies $\dim SingX_i \leqslant 1$. If $\varphi\colon (\mathbb{C}^{n+1}, X, 0) \to (\mathbb{C}^{n+1}, Y, 0)$ is a bi-Lipschitz homeomorphism, then $m(X, 0) = m(Y, 0)$.*

Proof Let $\widetilde{f}, \widetilde{g}\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be two reduced complex analytic functions such that $X = V(\widetilde{f})$ and $Y = V(\widetilde{g})$. Let $f_1 \cdots f_r$ (resp. $g_1, \ldots, g_s$) be the irreducible factors of the decomposition of $\mathbf{in}(\widetilde{f})$ (resp. $\mathbf{in}(\widetilde{g})$) in irreducible polynomials. Then, $r = s$ and by reordering the indices, if necessary, there exists a bi-Lipschitz homeomorphism $\psi = d\varphi\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ such that $\psi(V(f_i)) = V(g_i)$ for all $i \in \{1, \ldots, r\}$. Fixed i, we denote $f = f_i$, $g = g_i$, $d = m(V(f), 0)$ and $e = m(V(g), 0)$. By Proposition 9.6.2, $\chi(F_f) = \chi(F_g)$. □

We have two cases to consider:

Case 1) $\chi(F_f) \neq 0$, $\chi(F_g) \neq 0$ as well.

Claim If $0 < k < d$ (respectively $0 < k < e$), then $\Lambda(h_f^k) = 0$ (respectively $\Lambda(h_g^k) = 0$), where $\Lambda(h_f^k)$ (respectively $\Lambda(h_g^k)$) denotes the Lefschetz number of h_f^k (respectively h_g^k). □

Proof We start the proof using the Topological Cylindric Structure at Infinity of Algebraic Sets (see [8], p. 26, Theorem 6.9) to justify that $F = f^{-1}(1)$ has the same homotopy type of $F_R = F \cap \{x \in \mathbb{C}^n;\ \|x\| \leqslant R\}$ for R large enough. We have that the geometric monodromy $h_f : F \to F$ given by $h_f(x) = e^{\frac{2\pi i}{d}} x$, restricted to F_R, induces a map $h = h_f|_{F_R} : F_R \to F_R$. It is clear that h^k does not have a fixed point for $0 < k < d$, hence $\Lambda(h^k) = 0$. Since h_f is homotopy equivalent to h, it follows that $\Lambda(h_f^k) = 0$ for any $0 < k < d$. □

It follows from Proposition 9.6.2 that $\Lambda(h_f^k) = \Lambda(h_g^k)$ for all $k \in \mathbb{N}$. Since f and g are homogeneous polynomials with degrees d and e respectively, $h_f^d = id : F_f \to F_f$ and $h_g^e = id : F_g \to F_g$, we get $\Lambda(h_f^d) = \chi(F_f) \neq 0$ and $\Lambda(h_g^e) = \chi(F_g) \neq 0$. Thus, it follows from previous claim that $d = e$.

Case 2) $\chi(F_f) = \chi(F_g) = 0$.

By the Lipschitz Regularity Theorem (Theorem 9.4.6), we have that $\psi(Sing(V(f))) = Sing(V(g))$ and, in particular, $\dim Sing(V(f)) = \dim Sing(V(g))$. Thus, we can suppose that $d, k > 1$. If $\dim Sing(V(f)) = 0$ then $\chi(F_f) = 1 + (-1)^n(d-1)^{n+1} = 0$ and $\chi(F_g) = 1 + (-1)^n(k-1)^{n+1} = 0$. This implies $d = k = 2$.

Thus, we can assume that $\dim Sing(V(f)) \neq 0$. In this case, we have $\dim Sing V(f) = \dim Sing(V(g)) = 1$. Since $\chi(F_f) = \chi(F_g) = 0$, by Proposition 9.6.5, we have

$$(d-1)^{n+1} - \mu'(f)(d-1) + (-1)^n - \mu'(f) = 0$$

and

$$(k-1)^{n+1} - \mu'(f)(k-1) + (-1)^n - \mu'(g) = 0.$$

Thus, we define the polynomial map $P : \mathbb{R} \to \mathbb{R}$ by

$$P(t) = t^{n+1} - \mu'(f)t + (-1)^n - \mu'(f), \quad \forall t \in \mathbb{R}.$$

Since $\mu'(f) = \mu'(g)$ (see Proposition 9.6.4), then $d - 1$ and $k - 1$ are positive zeros of P. Since $\mu'(f) = \mu'(g) \geqslant 1$, by Descartes' Rule, P has at most one positive zero. Thus, $d = k$.

As a consequence, we obtain the result proved by Fernandes and Sampaio in [12] which says that the multiplicity of surface singularities in $\mathbb{C}^3$ is invariant under bi-Lipschitz homeomorphisms.

Corollary 9.6.7 *Let* $f, g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ *be two complex analytic functions. If* $\varphi : (\mathbb{C}^3, V(f), 0) \to (\mathbb{C}^3, V(g), 0)$ *is a bi-Lipschitz homeomorphism, then* $m(V(f), 0) = m(V(g), 0)$.

9.6.3 Invariance of the Multiplicity Under Semi-Bi-Lipschitz Homeomorphisms on the Functions

In this section, we define some notions of equivalence on germs of functions and we prove some results on invariance of the multiplicity under those equivalences.

Definition 9.6.8 We say that two germs of analytic functions $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ are **semi-bi-Lipschitz equivalent**, if there are constants $C_1, C_2 > 0$ and a germ of bijection $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that

(1) $\frac{1}{C_1}\|x\| \leqslant \|\varphi(x)\| \leqslant C_1\|x\|$, for all small enough $x \in \mathbb{C}^n$;
(2) $\frac{1}{C_2}\|f(x)\| \leqslant \|g \circ \varphi(x)\| \leqslant C_2\|f(x)\|$, for all small enough $x \in \mathbb{C}^n$.

Definition 9.6.9 Let $U \subset \mathbb{C}^n$ be an open set such that $0 \in U$ and let $f : U \to \mathbb{C}$ be an analytic function. Then, for each $r > 0$ such that $B_r(0) \subset U$, we define

$$\delta_r(f) = \sup\{\delta;\ \tfrac{|f(z)|}{\|z\|^\delta} \text{ is bounded on } B_r(0) \setminus \{0\}\}.$$

Note that $\delta_r(f)$ does not depend on $r > 0$ (see Exercise 9.6.25). Thus, we define this common number by $\delta(f)$.

Proposition 9.6.10 *Let* $f, g : (\mathbb{C}^n, 0) \to \mathbb{C}$ *be a germ of an analytic function. Then,* $\mathrm{ord}_0(f) = \delta(f)$.

Proof If $\delta > m := \mathrm{ord}_0(f)$ and $f = f_m + f_{m+1} + \ldots$ with $f_m \neq 0$, then we choose $v \notin V(f_m)$. Thus, $\lim\limits_{t\to 0^+} \frac{|f(tv)|}{t^\delta} = +\infty$. Then, $\delta(f) \leqslant \mathrm{ord}_0(f)$.

If $\delta < m$, then $\lim\limits_{z\to 0} \frac{|f(z)|}{\|z\|^\delta} = 0$. Thus, there exists $r > 0$ such that $\frac{|f(z)|}{\|z\|^\delta} \leqslant 1$, for all $z \in B_r(0)$. This implies $\delta(f) \geqslant \mathrm{ord}_0(f)$.

Therefore, $\delta(f) = \mathrm{ord}_0(f)$. □

The next result is due to Comte, Milman and Trotman [7].

Theorem 9.6.11 *Let* $f, g : (\mathbb{C}^n, 0) \to \mathbb{C}$ *be two germs of analytic functions. If f and g are semi-bi-Lipschitz equivalent, then* $\mathrm{ord}_0(f) = \mathrm{ord}_0(g)$.

Proof By hypothesis, there are open neighbourhoods $U, \widetilde{U}$ of $0 \in \mathbb{C}^n$, constants $C_1, C_2 > 0$ and a bijection $\varphi : U \to \widetilde{U}$ such that

(1) $\frac{1}{C_1}\|x\| \leqslant \|\varphi(x)\| \leqslant C_1\|x\|$, for all $x \in U$;
(2) $\frac{1}{C_2}\|f(x)\| \leqslant \|g \circ \varphi(x)\| \leqslant C_2\|f(x)\|, \quad \forall x \in U$.

Let $\delta < \delta(g)$. Thus, we may take $\widetilde{r} > 0$ such that $\frac{|g(z)|}{\|z\|^{\delta}}$ is bounded on $B_{\widetilde{r}}(0) \setminus \{0\}$, $B_{\widetilde{r}}(0) \subset \widetilde{U}$ and $B_r(0) \subset U$, where $r = \frac{\widetilde{r}}{C_1}$. In particular, $\varphi(B_r(0)) \subset B_{\widetilde{r}}(0)$.

Moreover, we have

$$\begin{aligned}\frac{|f(x)|}{\|x\|^{\delta}} &= \frac{|f(x)|}{\|\varphi(x)\|^{\delta}} \frac{\|\varphi(x)\|^{\delta}}{\|x\|^{\delta}} \\ &\leqslant C_1 C_2 \frac{|g(\varphi(x))|}{\|\varphi(x)\|^{\delta}},\end{aligned}$$

for all $x \in B_r(0) \setminus \{0\}$. Since $\frac{|g(z)|}{\|z\|^{\delta}}$ is bounded on $B_{\widetilde{r}}(0) \setminus \{0\}$, then $\frac{|f(x)|}{\|x\|^{\delta}}$ is bounded on $B_r(0) \setminus \{0\}$. This implies

$$\{\rho;\ \tfrac{|g(z)|}{\|z\|^{\rho}} \text{ is bounded on } B_{\widetilde{r}}(0) \setminus \{0\}\} \subset \{s;\ \tfrac{|f(x)|}{\|x\|^{s}} \text{ is bounded on } B_r(0) \setminus \{0\}\},$$

Then, we obtain $\delta_{\widetilde{r}}(g) \leqslant \delta_r(f)$ and, since $\delta_r(f) = \delta(f)$ and $\delta_{\widetilde{r}}(g) = \delta(g)$, we have $\delta(g) \leqslant \delta(f)$. Therefore, by Proposition 9.6.10, $\mathrm{ord}_0(g) \leqslant \mathrm{ord}_0(f)$. Similarly, we obtain $\mathrm{ord}_0(f) \leqslant \mathrm{ord}_0(g)$. Thus, we have the equality $\mathrm{ord}_0(g) = \mathrm{ord}_0(f)$. □

Definition 9.6.12 We say that two germs of analytic functions $f, g\colon (\mathbb{C}^n, 0) \to \mathbb{C}$ are:

- **bi-Lipschitz right equivalent**, if there is a bi-Lipschitz homeomorphism $\varphi\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $f(x) = g \circ \varphi(x)$, for all small enough $x \in \mathbb{C}^n$;
- **bi-Lipschitz right-left equivalent**, if there are bi-Lipschitz homeomorphisms $\varphi\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ and $\phi\colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that $f(x) = \phi \circ g \circ \varphi(x)$, for all small enough $x \in \mathbb{C}^n$;
- **rugose equivalent**, if there are constants $C_1, C_2 > 0$ and a germ of bijection $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that
  (1) $\frac{1}{C_1}\|x - y\| \leqslant \|\varphi(x) - \varphi(y)\| \leqslant C_1\|x - y\|$, for all small enough $(x, y) \in \mathbb{C}^n \times f^{-1}(0)$;
  (2) $\frac{1}{C_2}\|f(x)\| \leqslant \|g \circ \varphi(x)\| \leqslant C_2\|f(x)\|$, for all small enough $x \in \mathbb{C}^n$;
- **bi-Lipschitz contact equivalent**, if there are a constant $C > 0$ and a germ of bi-Lipschitz homeomorphism $\varphi\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that

$$\frac{1}{C}\|f(x)\| \leqslant \|g \circ \varphi(x)\| \leqslant C\|f(x)\|,$$

  for all small enough $x \in \mathbb{C}^n$.

The following result is a direct consequence from the definitions.

Proposition 9.6.13 *Let* $f, g\colon (\mathbb{C}^n, 0) \to \mathbb{C}$ *be two germs of analytic functions. Let us consider the following statements:*

(1) f *and* g *are bi-Lipschitz right equivalent;*
(2) f *and* g *are bi-Lipschitz right-left equivalent;*
(3) f *and* g *are bi-Lipschitz contact equivalent;*
(4) f *and* g *are rugose equivalent;*
(5) f *and* g *are semi-bi-Lipschitz equivalent.*

Then, (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5).

Proof The proof is left as an exercise for the reader. □

We finish this subsection by stating some direct consequences of Theorem 9.6.11 and Proposition 9.6.13.

Corollary 9.6.14 (See [29]) *Let* $f, g : \mathbb{C}^n \to \mathbb{C}$ *be two germs of analytic functions. If* f *and* g *are rugose equivalent, then* $\mathrm{ord}_0(f) = \mathrm{ord}_0(g)$.

Corollary 9.6.15 *Let* $f, g\colon (\mathbb{C}^n, 0) \to \mathbb{C}$ *be two germs of analytic functions. If* f *and* g *are bi-Lipschitz contact equivalent, then* $\mathrm{ord}_0(f) = \mathrm{ord}_0(g)$.

Corollary 9.6.16 *Let* $f, g\colon (\mathbb{C}^n, 0) \to \mathbb{C}$ *be two germs of analytic functions. If* f *and* g *are bi-Lipschitz right-left equivalent, then* $\mathrm{ord}_0(f) = \mathrm{ord}_0(g)$.

Corollary 9.6.17 *Let* $f, g\colon (\mathbb{C}^n, 0) \to \mathbb{C}$ *be two germs of analytic functions. If* f *and* g *are bi-Lipschitz right equivalent, then* $\mathrm{ord}_0(f) = \mathrm{ord}_0(g)$.

9.6.4 Lipschitz Invariance of Multiplicity When the Lipschitz Constants Are Close to 1

Let us remind the result proved by Draper in [9] which says that the multiplicity of complex analytic set X at a point p is the density of X at p.

Theorem 9.6.18 (Draper [9]) *Let* $Z \subset \mathbb{C}^k$ *be a pure* d*-dimensional complex analytic subset with* $0 \in Z$. *Then*

$$m(Z, 0) = \lim_{r \to 0^+} \frac{\mathcal{H}^{2d}(Z \cap \overline{B}_r^{2k}(0))}{\mu_{2d} r^{2d}}$$

where $\mathcal{H}^{2d}(Z \cap \overline{B}_r^{2k}(0))$ *denotes the* $2d$*-dimensional Hausdorff measure of* $Z \cap \overline{B}_r^{2k}(0) = \{x \in Z; \|x\| \leqslant r\}$ *and* μ_{2d} *is the volume of* $2d$*-dimensional unit ball.*

By using the above equality, we obtain the following:

Proposition 9.6.19 *Let* $X \subset \mathbb{C}^n$ *and* $Y \subset \mathbb{C}^m$ *be two germs at* 0 *of complex analytic sets with* $\dim X = \dim Y = d$. *If there are constants* $C_1, C_2 > 0$ *and a bi-Lipschitz* *homeomorphism* $\varphi\colon (X,0) \to (Y,0)$ *such that*

$$\frac{1}{C_1}\|x-y\| \leqslant \|\varphi(x)-\varphi(y)\| \leqslant C_2\|x-y\|, \quad \forall x, y \in X$$

then

$$\frac{1}{(C_1C_2)^{2d}} m(X,0) \leqslant m(Y,0) \leqslant (C_1C_2)^{2d} m(X,0).$$

Proof The proof is left as an exercise for the reader. □

As a consequence of the above proposition, we obtain a result proved by Comte in [6, Theorem 1].

Theorem 9.6.20 *Let* $X \subset \mathbb{C}^n$ *and* $Y \subset \mathbb{C}^m$ *be two germs at* 0 *of complex analytic sets with* $\dim X = \dim Y = d$ *and* $M = \max\{m(X,0), m(Y,0)\}$. *If there are constants* $C_1, C_2 > 0$ *and a bi-Lipschitz homeomorphism* $\varphi\colon (X,0) \to (Y,0)$ *such that*

$$\frac{1}{C_1}\|x-y\| \leqslant \|\varphi(x)-\varphi(y)\| \leqslant C_2\|x-y\|, \quad \forall x, y \in X$$

and $(C_1C_2)^{2d} \leqslant 1 + \frac{1}{M}$, *then* $m(X,0) = m(Y,0)$.

In fact, we obtain a slight better result than Theorem 9.6.20.

Theorem 9.6.21 *Let* $X \subset \mathbb{C}^n$ *and* $Y \subset \mathbb{C}^m$ *be two germs at* 0 *of complex analytic sets with* $\dim X = \dim Y = d$. *Let* $X_1, \ldots, X_r$ *and* $Y_1, \ldots, Y_s$ *be the irreducible components of the tangent cones* $C(X,0)$ *and* $C(Y,0)$, *respectively and let* $M = \max\{m(X_1,0), \ldots, m(X_r,0), m(Y_1,0), \ldots, m(Y_s,0)\}$. *If there are constants* $C_1, C_2 > 0$ *and a bi-Lipschitz homeomorphism* $\varphi\colon (X,0) \to (Y,0)$ *such that*

$$\frac{1}{C_1}\|x-y\| \leqslant \|\varphi(x)-\varphi(y)\| \leqslant C_2\|x-y\|, \quad \forall x, y \in X$$

and $(C_1C_2)^{2d} \leqslant 1 + \frac{1}{M}$, *then* $m(X,0) = m(Y,0)$.

Proof By the bi-Lipschitz invariance of the tangent cones (see Theorem 9.3.11), there is a global bi-Lipschitz homeomorphism $d\varphi\colon C(X,0) \to C(Y,0)$ such that $d\varphi(0) = 0$ and $\frac{1}{C_1}\|v-w\| \leqslant \|d\varphi(v)-d\varphi(w)\| \leqslant C_2\|v-w\|, \quad \forall v, w \in C(X,0)$.

By Theorem 9.5.1, $r = s$ and, up to a re-ordering of indices, $k_X(X_j) = k_Y(Y_j)$ and $Y_j = d\varphi(X_j)$, $\forall\, j$. Moreover, by Proposition 9.5.2, we obtain

$$m(X,0) = \sum_{j=1}^{r} k_X(X_j) \cdot m(X_j,0)$$

and

$$m(Y, 0) = \sum_{j=1}^{r} k_Y(Y_j) \cdot m(Y_j, 0).$$

Since X_j and Y_j are homogeneous algebraic sets, we have $\deg(X_j) = m(X_j, 0)$ and $\deg(Y_j) = m(Y_j, 0)$. By Theorem 9.6.20, $m(X_j, 0) = m(Y_j, 0)$ for all j. Therefore, $m(X, 0) = m(Y, 0)$. □

9.6.5 Question AL(2) and Final Comments

Let us recall the Question AL(d).

Question AL(d) Let $X \subset \mathbb{C}^n$ and $Y \subset \mathbb{C}^m$ be two complex analytic sets with $\dim X = \dim Y = d$, $0 \in X$ and $0 \in Y$. If there exists a bi-Lipschitz homeomorphism $\varphi\colon (X, 0) \to (Y, 0)$, then is $m(X, 0) = m(Y, 0)$?

We finish this section bringing a complete answer to this question. Next result is a positive answer to that. Its proof was published in [3]. It is valuable to mention that Neumann and Pichon also got a positive answer to question AL(2) with the additional hypothesis that the considered surface singularities are normal (see [23]).

Theorem 9.6.22 *Let $X \subset \mathbb{C}^{N+1}$ and $Y \subset \mathbb{C}^{M+1}$ be two complex analytic surfaces. If $(X, 0)$ and $(Y, 0)$ are bi-Lipschitz homeomorphic, then $m(X, 0) = m(Y, 0)$.*

In view of what we have already proved, mainly Theorem 9.5.3 which says that it is enough to addresses Question AL(d) for irreducible homogeneous singularities, we have that the above theorem is an immediate consequence of the following result proved in [3].

Proposition 9.6.23 *Let $S \subset \mathbb{C}^n$ be a 2-dimensional homogeneous and irreducible algebraic setalgebraic sets!irreducible. Then the multiplicity $m(S, 0) = m$ is given by the following: the torsion part of $H^2(S \setminus 0, \mathbb{Z})$ is isomorphic to $\mathbb{Z}/m\mathbb{Z}$. In particular, if S, S' are 2-dimensional homogeneous and irreducible algebraic sets such that $(S, 0)$ and $(S', 0)$ are homeomorphic, then $m(S, 0) = m(S', 0)$.*

Proof In order to prove this theorem, as we saw above, by Theorem 9.5.3, it is enough to assume that $X \subset \mathbb{C}^{N+1}$ and $Y \subset \mathbb{C}^{M+1}$ are 2-dimensional homogeneous and irreducible algebraic setsalgebraic sets!homogeneous. Since a bi-Lipschitz homeomorphism between $(X, 0)$ and $(Y, 0)$ is also a homeomorphism between them, by Proposition 9.6.23, it follows that $m(X, 0) = m(Y, 0)$. □

Finally, we complete the answer of Question AL(d) by showing that, for dimension d greater then 2, we always have singularities which are bi-Lipschitz homeomorphic with different multiplicities. This result was proved in [2].

Theorem 9.6.24 *For each $n \geqslant 3$, there exists a family $\{Y_i\}_{i\in\mathbb{N}}$ of n-dimensional complex algebraic varieties $Y_i \subset \mathbb{C}^{n_i+1}$ such that:*

(a) for each pair $i \neq j$, the germs at the origin of $Y_i \subset \mathbb{C}^{n_i+1}$ and $Y_j \subset \mathbb{C}^{n_j+1}$ are bi-Lipschitz equivalent, but $(Y_i, 0)$ and $(Y_j, 0)$ have different multiplicity.
(b) for each pair $i \neq j$, there are n-dimensional complex algebraic varieties $Z_{ij}, \widetilde{Z}_{ij} \subset \mathbb{C}^{n_i+n_j+2}$ such that $(Z_{ij}, 0)$ and $(\widetilde{Z}_{ij}, 0)$ are ambient bi-Lipschitz equivalent, but $m(Z_{ij}, 0) = m(Y_i, 0)$ and $m(\widetilde{Z}_{ij}, 0) = m(Y_j, 0)$ and, in particular, they have different multiplicity.

Proof *(Sketch of the Proof)* Let $\{p_i\}_{i\in\mathbb{N}}$ be the family of odd prime numbers. For each $i \in \mathbb{N}$, let X_i be an embedding of $\mathbb{C}P^1 \times \mathbb{C}P^1$ into $\mathbb{C}P^{n_i}$ with degree $4p_i = 2\cdot 2\cdot p_i$ (we can do this by using a composition of Segre and Veronese embeddings). Let $Y_i \subset \mathbb{C}^{N+1}$ be the respective affine algebraic complex cone associated to X_i and S_i its the respective link at the origin of $\mathbb{C}^{n_i+1}$. More precisely, $S_i = Y_i \cap \mathbb{S}^{2n_i+1}$. By construction, $m(Y_i, 0) = 4p_i$ for all $i \in \mathbb{N}$. On the other hand, it was proved in [2] that S_i is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^3$ for all $i \in \mathbb{N}$ and, then $(Y_i, 0)$ is bi-Lipschitz homeomorphic to $(Y_i, 0)$ while $m(Y_i, 0) \neq m(Y_i, 0)$ for all $i \neq j$. Hence, Item (a) is proved.

Concerning to Item (b), let $f_{ij} : Y_i \to Y_j$ be a bi-Lipschitz homeomorphism such that $f_{ij}(0) = 0$ (we are considering $f_{ij}^{-1} = f_{ji}$). Let $F_{ij} : \mathbb{C}^{n_i+1} \to \mathbb{C}^{n_j+1}$ be a Lipschitz extension of f_{ij} (see Lemma 9.3.10). Let us define $\phi, \psi : \mathbb{C}^{n_i+1} \times \mathbb{C}^{n_j+1} \to \mathbb{C}^{n_i+1} \times \mathbb{C}^{n_j+1}$ as follows:

$$\phi(x, y) = (x - F_{ji}(y + F_{ij}(x)), y + F_{ij}(x))$$

and

$$\psi(z, w) = (z + F_{ji}(w), w - F_{ij}(z + F_{ji}(w))).$$

It easy to verify that ϕ and ψ are inverse maps of each other. Since ϕ and ψ are composition of Lipschitz maps, they are also Lipschitz maps. Moreover, if we denote $Z_{ij} = Y_i \times \{0\}$ and $\widetilde{Z}_{ij} = \{0\} \times Y_j$, we get $\phi(Z_{ij}) = \widetilde{Z}_{ij}$ (see [30]). Therefore, $(Z_{ij}, 0)$ and $(\widetilde{Z}_{ij}, 0)$ are ambient bi-Lipschitz equivalent, while $m(Z_{ij}, 0) = m(Y_i, 0)$ and $m(\widetilde{Z}_{ij}, 0) = m(Y_j, 0)$ are different. □

9.6.6 Exercises

Exercise 9.6.25 Prove that $\delta_r(f)$ as in Definition 9.6.9 does not depend on $r > 0$.

Exercise 9.6.26 Prove Proposition 9.6.13.

Exercise 9.6.27 Give a direct proof to Corollary 9.6.16.

Exercise 9.6.28 Prove Proposition 9.6.19.

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Chapter 10
On Lipschitz Normally Embedded Singularities

Lorenzo Fantini and Anne Pichon

This paper is dedicated to Walter Neumann, wonderful friend and outstanding mathematician.

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Abstract Any subanalytic germ $(X, 0) \subset (\mathbb{R}^n, 0)$ is equipped with two natural metrics: its *outer metric*, induced by the standard Euclidean metric of the ambient space, and its *inner metric*, which is defined by measuring the shortest length of paths on the germ $(X, 0)$. The germs for which these two metrics are equivalent up to a bilipschitz homeomorphism, which are called Lipschitz Normally Embedded, have attracted a lot of interest in the last decade. In this survey we discuss many general facts about Lipschitz Normally Embedded singularities, before moving our focus to some recent developments on criteria, examples, and properties of Lipschitz Normally Embedded complex surfaces. We conclude the manuscript with a list of open questions which we believe to be worth of interest.

10.1 Definition, First Examples, and Some General Results

10.1.1 Definition

Let (X, d_X) and (Y, d_Y) be two metric spaces. A homeomorphism $\phi\colon X \to Y$ is said to be a *bilipschitz equivalence* if there exist two positive real numbers K_1 and K_2 such that, given any two points x and x' in X, we have

$$K_1\, d_X(x, x') \leqslant d_Y\big(\phi(x), \phi(x')\big) \leqslant K_2\, d_X(x, x').$$

Two metric spaces are said to be *bilipschitz equivalent* if there exists a bilipschitz equivalence from one to the other.

A connected subanalytic subspace X of $\mathbb{R}^n$ is naturally equipped with two metrics on $(X, 0)$: its *outer metric* d_o, induced by the standard Euclidean metric of the ambient space, and its *inner metric* d_i, which is the associated arc-length metric on the germ, defined as follows:

$$d_i(x, y) = \inf\big\{\text{length}(\gamma) \,\big|\, \gamma \text{ is a rectifiable path in } X \text{ from } x \text{ to } y\big\}.$$

Note that for an arc to be *rectifiable* essentially means that its length can be computed and is finite, see [22] for details. Given any two points x and y in X, we have $d_o(x, y) \leqslant d_i(x, y)$. Moreover, the inner distance between two given points can be computed as a limit of sums of outer distances, so that two spaces which are bilipschitz equivalent for the outer metric are bilipschitz equivalent for the inner metric as well.[1] In general, the converse does not hold, but there exists a special

[1] Surprisingly, we do not know any reference for this simple fact, so here is a proof: if $f\colon X \to Y$ is a subanalytic map that is a K-Lipschitz for the outer metrics and $\gamma\colon [0, 1] \to X$ is a rectifiable path between two points x and x' of X, then $f \circ \gamma$ is a rectifiable path between $f(x)$ and $f(x')$ in Y and we have $d_i\big(f(x), f(x')\big) \leqslant \text{length}(f \circ \gamma) = \sup_n \sum_{i=1}^{n-1} d_o\big((f \circ \gamma)(i/n), (f \circ \gamma)(i + 1/n)\big) \leqslant K \sup_n \sum_{i=1}^{n-1} d_o\big(\gamma(i/n), \gamma(i + 1/n)\big) = K\,\text{length}(\gamma)$. By taking the infimum over all such paths γ, we obtain $d_i\big(f(x), f(x') \leqslant K\, d_i(x, x')$, that is f is K-Lipschitz for the inner metrics.

class of spaces, or of space germs, which have the remarkable property that their inner and outer bilipschitz classes coincide, in the following sense.

Definition 10.1.1 A connected subanalytic subspace X of $\mathbb{R}^n$ is *Lipschitz Normally Embedded* (or simply *LNE*) if there exists a subanalytic homeomorphism $f: X \to X$ which is a bilipschitz equivalence between the inner and outer metrics of X, that is such that there exists a real number $K \geqslant 1$ satisfying, for all x, y in X,

$$\frac{1}{K} d_i\big(f(x), f(y)\big) \leqslant d_o(x, y) \leqslant K d_i\big(f(x), f(y)\big).$$

If x is a point of X, the germ (X, x) is *LNE* if there is a neighborhood U of x in $\mathbb{R}^n$ such that $X \cap U$ is LNE.

Since the inner and the outer geometries of (X, x) are invariant under bilipschitz homeomorphisms (see [42, Proposition 7.2.13]), this property only depends on the subanalytic type (X, x), and not on the choice of an embedding in some smooth ambient space $(\mathbb{R}^n, 0)$.[2]

This notion was first introduced by Birbrair and Mostowski in the seminal paper [9]. Their definition is slightly different because they require the identity map, and not just any subanalytic homeomorphism, to be bilipschitz between inner and outer metrics, but in fact the two definitions are equivalent. This is a piece of folklore knowledge which is a consequence of the main result of *loc. cit*. In Sect. 10.1.3 we recall that result and include a proof of the equivalence that was kindly communicated to us by the Lev Birbrair. Note that in *loc. cit.* LNE spaces are simply called *normally embedded*; in the subsequent literature on the subject the term *Lipschitz* was added to distinguish this notion from those of projective normal embedding (in algebraic geometry) and normality (in local geometry, commutative algebra and singularity theory).

Notice that a compact space X is LNE if and only if the germs (X, x) are LNE for all points x if X. Our aim is to present a state of the art on the LNE-ness of real and complex analytic germs.

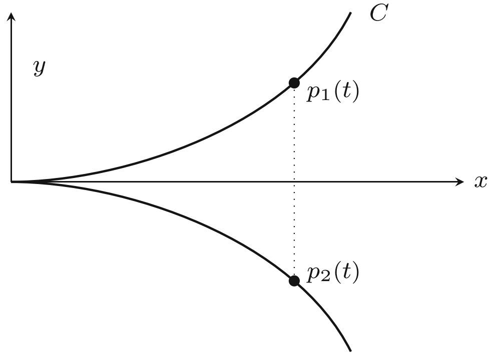
10.1.2 First Examples

Example 10.1.2 A smooth germ $(X, 0)$ is Lipschitz Normally Embedded, since it is analytically equivalent to $(\mathbb{R}^n, 0)$, where the inner metric and the outer metric coincide.

Example 10.1.3 Let $Y \subset \mathbb{R}^n$ be a subanalytic subspace of the sphere S^{n-1} of radius 1 centered at the origin of $\mathbb{R}^n$, and assume that Y is LNE. Then the cone $C(Y, 0)$

[2] Note that while the result of *loc. cit.* is only stated in the semialgebraic setting, its proof carries through for arbitrary subanalytic germs.

Fig. 10.1 The real cusp $y^2 - x^3 = 0$



over Y with apex 0, which consists of the union of the half-lines with origin 0 that intersect Y, is LNE as well.

Example 10.1.4 The germ $(C, 0)$ of the real cusp C with equation $y^2 - x^3 = 0$ in $\mathbb{R}^2$ is not LNE. Indeed, given a real number $t > 0$, consider the two points $p_1(t) = (t, t^{3/2})$ and $p_2(t) = (t, -t^{3/2})$ on C (see Fig. 10.1). Then $d_o\big(p_1(t), p_2(t)\big) = 2t^{3/2}$, so that in the germ, as t goes two zero, the outer distance between $p_1(t)$ and $p_2(t)$ has order $t^{3/2}$, which we write as $d_o\big(p_1(t), p_2(t)\big) = \Theta(t^{3/2})$.[3] On the other hand, the shortest path on C between the two points $p_1(t)$ and $p_2(t)$ is obtained by taking a path going through the origin, so that we have $d_i\big(p_1(t), p_2(t)\big) = \Theta(t)$. Therefore, taking the limit of the quotient as t tends to 0, we obtain:

$$\frac{d_o\big(p_1(t), p_2(t)\big)}{d_i\big(p_1(t), p_2(t)\big)} = \Theta(t^{1/2}) \longrightarrow 0.$$

Note that the existence of two such arcs p_1 and p_2 is due to the fact that the tangent cone T_0X of $(X, 0)$ at 0 is not reduced (it has equation $y^2 = 0$). This is an occurrence of a general result which will be stated as Theorem 10.1.29.

Example 10.1.5 A complex curve germ $(C, 0) \subset (\mathbb{C}^N, 0)$ is LNE if and only if it consists of smooth transversal curve germs. Indeed, if the latter is true then $(C, 0)$ is analytically equivalent to the germ of a union of transversal lines, which being a cone is LNE. The converse is more delicate and can be obtained by combining several results. First, if $(C, 0) \subset (\mathbb{C}^N, 0)$ is a complex curve germ, then any *generic* linear projection $\ell \colon \mathbb{C}^N \to \mathbb{C}^2$ restricts to the germ of a bilipschitz homeomorphism $\ell|_{(C,0)} \colon (C, 0) \to \big(\ell(C), 0\big)$ for the outer metric ([45, pp. 352–354]). Therefore, it suffices to prove the result for a plane curve $(C, 0) \subset (\mathbb{C}^2, 0)$. The key argument,

[3] More precisely, throughout this text, we use the *big-Theta* asymptotic notations of Bachmann–Landau in the following form: given two function germs $f, g \colon ([0, \infty), 0) \to ([0, \infty), 0)$, we say that f *is big-Theta of* g, and we write $f(t) = \Theta(g(t))$, if there exist $\eta > 0$ and $K > 0$ such that $K^{-1}g(t) \leqslant f(t) \leqslant Kg(t)$ for all t satisfying $f(t) \leqslant \eta$.

which is close to the one presented in Example 10.1.4, is that a complex curve germ $(C, 0) \subset (\mathbb{C}^2, 0)$ admitting a non essential Puiseux exponent $q > 1$ contains two arcs $p_1(t)$ and $p_2(t)$ such that $d_i\big(p_1(t), p_2(t)\big) = \Theta(t)$ and $d_o(p_1(t), p_2(t)) = \Theta(t^q)$, and therefore $(C, 0)$ cannot be LNE. For example, the complex cusp $y^2 - x^3 = 0$ in $\mathbb{C}^2$, which has has Puiseux expansion $y = x^{3/2}$, contains the two arcs $p_1(t) = (t, t^{3/2})$ and $p_2(t) = (t, -t^{3/2})$ whose inner distance is $\Theta(t)$ and whose outer distance is $\Theta(t^{3/2})$.

Notice that, more generally, the inner and outer bilipschitz types of complex curve germs are completely understood. On the one hand, the inner bilipschitz geometry of a complex curve $(C, 0)$ is trivial in the sense that for the inner metric $(C, 0)$ is bilipschitz equivalent to a straight cone over its link, that is to a union of smooth transversal curve germs (see [42, Proposition 7.2.2]). On the other hand, the outer bilipschitz type of $(C, 0)$ determines and is determined by its embedded topological type; for an algebraic proof of this result involving Lipschitz saturation of ideals, see the pioneering paper by Pham and Teissier [40] or its recent English translation [41]; for a more geometric approach, see [18] or [35].

Example 10.1.6 Starting from dimension 3 it is easy to find examples of non-LNE complex analytic germs which have non-isolated singularities, for example by taking the product of a non-LNE germ with a line. For instance, the product of a real cusp with a real line, that is the complex hypersurface in $\mathbb{C}^3$ with equation $y^2 - x^3 = 0$, is not LNE. One gets other examples by taking a homogeneous complex space with a non-LNE link; such an example is given by the hypersurface germ in $(\mathbb{C}^3, 0)$ with equation $x^2z + y^3 = 0$

Example 10.1.7 The first examples of non-LNE complex surface germs with an isolated singularity were obtained by Birbrair, Fernandes, and Neumann in 2010 [6]. It is the family of Brieskorn surfaces $x^b + y^b + z^a = 0$ where $b > a$ and a is not a divisor of b. In fact, what the authors of *loc. cit.* show is much stronger: with respect to the inner metric, those surface germ are not bi-Lipschitz equivalent to any LNE complex algebraic set.

While few examples of families of LNE singularities are known, it is still unclear whether LNE-ness is common among complex singularities with isolated singularities, even in the case of surfaces. The second part of the present paper discusses several recent advances on this front.

Example 10.1.8 The space of $n \times m$ real and complex matrices also contain remarkable families of LNE subspaces. For example, the Lie group $GL_n^+(\mathbb{R})$ consisting of $n \times n$ matrices with positive determinant is LNE, and so are the set of $n \times n$ matrices X^{n-1} with rank $n - 1$ and its closure, which is the set of matrices of determinant zero [24]. These results are generalized in [25] to the sets X_t of $m \times n$ matrices of given rank $t \leqslant \min(m, n)$ and their closures $\overline{X_t}$ by using elementary arguments of linear algebra and trigonometry, and LNE-ness is also proved in *loc. cit.* for other families such as symmetric and skew-symmetric matrices of given rank t and their closures, upper triangular matrices with determinant zero, and the intersections of those spaces with some linear subspaces.

10.1.3 The Pancake Decomposition, the Pancake Metric, and the Embedding Problem

In this section, we present three important theorems which can be considered as the first historical results around Lipschitz Normal Embeddings. We will state them in the semialgebraic setting, but they remain true in the subanalytic and polynomially bounded o-minimal categories with the obvious adaptations.

Since LNE spaces can be thought of as the simplest ones with respect to inner and outer Lipschitz geometries, it is natural to ask whether every semialgebraic subset of $\mathbb{R}^n$ admits a finite decomposition as a union of LNE sets. The answer is positive, as was established by Parusinski and Kurdyka:

Theorem 10.1.9 (Pancake Decomposition [27, 38, 39]) *Let $X \subset \mathbb{R}^n$ be a closed semialgebraic set. Then we can write*

$$X = \bigcup_{i=1}^{r} X_i$$

as a finite union of closed semialgebraic subsets of such that:

(i) *all X_i are LNE;*
(ii) *for every $i \neq j$ we have* $\dim(X_i \cap X_j) < \min(\dim X_i, \dim X_j)$.

This remarkable result has several important consequences. First, it enables to approach the following natural question: given a closed connected subset semialgebraic subset X in $\mathbb{R}^n$ is the inner metric $d_i \colon X \times X \to \mathbb{R}_{\geqslant 0}$ a semialgebraic function? Note that this is clearly the case for the outer metric on X.

The following theorem, proved by Kurdyka and Orro, states that d_i is bilipschitz equivalent to a semialgebraic metric with a bilipschitz constant as close as we want from 1. To define such a semialgebraic metric, consider a pancake decomposition $P = \{X_i\}_{i=1}^r$ of X. Given two points x, y in X let $Z_{x,y}$ be the set consisting of all the finite ordered sequences $z = (z_1, \dots, z_s)$ of points on X such that $z_1 = x$, $z_s = y$, and for every $k \in \{1, \dots, s-1\}$, there is a pancake X_{i_k} such that $X_{i_k} \cap \{z_1, \dots, z_s\} = \{z_k, z_{k+1}\}$. Finally, set

$$d_P(x, y) = \inf_{(z_1,\dots,z_s)\in Z_{x,y}} \sum_{k=1}^{s-1} d_i(z_k, z_{k+1}).$$

Theorem 10.1.10 (Pancake Metric, [28]) *The function $d_p \colon X \times X \to \mathbb{R}$ is semialgebraic and defines a metric on X (called the pancake metric) which is bilipschitz equivalent to d_i. Moreover, for all $\epsilon > 0$, there exists a pancake decomposition (obtained by refinement), such that the underlying pancake metric satisfies*

$$\forall x, y \in X,\ d_i(x, y) \leqslant d_P(x, y) \leqslant (1 + \epsilon) d_i(x, y).$$

An important application of the result above is the solution by Birbrair and Mostowski of the embedding problem, which asks whether every compact connected semialgebraic set is inner bilipschitz equivalent to a LNE semialgebraic set:

Theorem 10.1.11 ([9]) *Let X be a compact connected semialgebraic subset of $\mathbb{R}^n$. Then, for every $\epsilon > 0$, there exists a semialgebraic set $X_\epsilon \subset \mathbb{R}^m$ such that:*

(i) *X_ϵ is semialgebraically bilipschitz equivalent to X with respect to the inner metric;*
(ii) *X_ϵ is LNE;*
(iii) *the Hausdorff distance between X and X_ϵ is less than ϵ.*

Note that, when X is a complex analytic set, it is not always possible to choose X_ϵ to be complex algebraic. For instance, as already mentioned in Example 10.1.7, a surface germ defined by an equation of the form $x^b + y^b + z^a = 0$, where $b > a$ and a is not a divisor of b, does not admit a complex algebraic normal embedding, that is, it is not inner bi-Lipschitz equivalent to a LNE complex algebraic set.

We can now explain the equivalence of Definition 10.1.1 with the definition of [9], as we promised in the first section. The proof of the following corollary was communicated to us by Lev Birbrair.

Corollary 10.1.12 *A connected subanalytic subspace X of $\mathbb{R}^n$ is LNE (in the sense of Definition 10.1.1) if and only if the identity map of X is a bilipschitz equivalence between the inner and outer metrics of X.*

Proof Let $g: (X, d_i) \to (X_\epsilon, d_i)$ a bilipschitz homeomorphism between X and a LNE subanalytic subset of $\mathbb{R}^m$ as in Theorem 10.1.11 and let $f: (X, d_i) \to (X, d_o)$ be a subanalytic bilipschitz homeomorphism, which exists by Definition 10.1.1. We deduce that $g \circ f^{-1}$ is bilipschitz with respect to the outer metrics (note that here we are using the fact that the identity of X_ϵ is bilipschitz between its inner and outer metrics, which is what was intended for LNE in [9]; this proof would still be valid with the other definition, by further composing g with the appropriate homeomorphism). Therefore $g \circ f^{-1}$ is also bilipschitz with respect to the inner metrics (see Footnote 1 at p. 498). This implies that $\mathrm{Id}_X = f \circ f^{-1} = f \circ g^{-1} \circ g \circ f^{-1}$ is bilipschitz from (X, d_i) to (X, d_o). □

10.1.4 Characterization of LNE-Ness via Arcs

In this subsection we recall a necessary and sufficient condition for the LNE-ness of a semialgebraic set which was proved by Birbrair and Mendes. As in the previous section, the results stay true in the subanalytic or more generally polynomially bounded o-minimal setting (see [7, Remark 2.3]).

Definition 10.1.13 Let $(X, 0) \subset (\mathbb{R}^n, 0)$ be a semialgebraic germ. A *real arc* on $(X, 0)$ is the germ of a semialgebraic map $\delta: [0, \eta) \to X$ for some $\eta > 0$, such that $\delta(0) = 0$ and $\|\delta(t)\| = t$ (see also Remark 10.1.16).

When no risk confusion may arise, we will use the same notation for a real arc δ and for the germ $\big(\delta([0,\eta)), \delta(0)\big)$ of its parametrized image.

Definition 10.1.14 Let $(X, 0) \subset (\mathbb{R}^n, 0)$ be a semialgebraic germ and let $\delta_1 \colon [0, \eta) \to X$ and $\delta_2 \colon [0, \eta) \to X$ be two real arcs on X. The *outer contact* of δ_1 and δ_2 is defined to be infinity if $\delta_1 = \delta_2$ and is otherwise the rational number $q_o = q_o(\delta_1, \delta_2)$ defined by

$$\|\delta_1(t) - \delta_2(t)\| = \Theta(t^{q_o}).$$

The *inner contact* of δ_1 and δ_2 is the rational number $q_i = q_i(\delta_1, \delta_2)$ defined by

$$d_i\big(\delta_1(t), \delta_2(t)\big) = \Theta(t^{q_i}).$$

Remark 10.1.15 The existence and rationality of the inner contacts q_i is a consequence of the fact that the inner metric is bilipschitz equivalent to the pancake metric (Theorem 10.1.10), which is semialgebraic.

Remark 10.1.16 The inner and outer contacts $q_i(\delta_1, \delta_2)$ and $q_o(\delta_1, \delta_2)$ can also be defined taking reparametrizations by *real slices* of δ_1 and δ_2 as follows. First note that if δ_1 and δ_2 have different tangent directions then $q_i(\delta_1, \delta_2) = q_o(\delta_1, \delta_2) = 1$, so we may assume that they have the same tangent direction. We can then choose coordinates $(x_1, \dots, x_n)$ such that along the tangent half-line of δ_1 and δ_2 we have $x_1 > 0$ except at 0. For $j = 1, 2$, consider the reparametrization $\tilde{\delta}_j \colon [0, \eta) \to \mathbb{R}^n$ defined by $\tilde{\delta}_j(t) = \delta_j \cap \{x_1 = t\}$. Then we have $\|\tilde{\delta}_1(t) - \tilde{\delta}_2(t)\| = \Theta(t^{q_o})$ and $d_i\big(\tilde{\delta}_1(t), \tilde{\delta}_2(t)\big) = \Theta(t^{q_i})$. Indeed, this is an easy consequence of the following standard lemma:

Lemma 10.1.17 *Let $B \subset \mathbb{R}^n$ be any closed compact convex neighborhood of* 0 *in* $\mathbb{R}^n$ *and denote by B_1 is the unit ball of $\mathbb{R}^n$. Let $\phi \colon B \to B_1$ be the homeomorphism which maps each ray from* 0 *to ∂B linearly to the ray with the same tangent, but of length* 1. *Then the map $\phi \colon B \to B_1$ is a bilipschitz homeomorphism.*

We can now state the main result of this subsection, which is a criterion to determine if a closed semialgebraic germ is LNE using arcs and their contact orders.

Theorem 10.1.18 (Arc Criterion, [7]) *Let $(X, 0) \subset (\mathbb{R}^n, 0)$ be a closed semialgebraic germ. Then $(X, 0)$ is LNE if and only if all pairs of real arcs δ_1 and δ_2 in $(X, 0)$ satisfy $q_i(\delta_1, \delta_2) = q_o(\delta_1, \delta_2)$.*

The proof of this theorem is based on the Curve Selection Lemma. Since the latter only applies to semialgebraic metrics, the semialgebraicity of the pancake metric and Theorem 10.1.10 play again a fundamental role.

Example 10.1.19 A straightforward application of the arc criterion shows that the real surface S in $\mathbb{R}^3$ defined by the equation $x^2 + y^2 - z^3 = 0$ is LNE. See also Example 10.1.26.

The criterion given in Theorem 10.1.18 is difficult to use effectively in practice since it requires to compute the inner and outer contact orders of an immense amount of pairs of arcs. In Sect. 10.2 we state an analogous criterion for complex surface germs where the number of pairs of arcs to be tested is reduced drastically to just finitely many pairs. This makes the criterion much more efficient to prove LNE-ness and enables one to obtain several infinite families of LNE complex surface germs with isolated singularities.

10.1.5 Characterization of LNE-Ness via the Links

Recall that the *link* of a d-dimensional subanalytic germ $(X, 0) \subset \mathbb{R}^n$, which is defined by embedding $(X, 0)$ in a suitable smooth germ $(\mathbb{C}^N, 0)$ and intersecting it with a small sphere, is, up to homeomorphism, a well defined real $(2d-1)$-dimensional oriented pseudo-manifold (a smooth manifold if $(X, 0)$ has isolated singularities) which determines and is determined by the homeomorphism class of the germ $(X, 0)$. In this subsection we discuss the relation between a germ being LNE and its link being LNE. One implication is always satisfied:

Lemma 10.1.20 *Let $(X, 0)$ be a subanalytic germ in $\mathbb{R}^n$ such that $(X \setminus \{0\}, 0)$ is connected. Then, if $(X, 0)$ is LNE, so is its link.*

This is a consequence of the fact that, whenever the link of $(X, 0)$ is connected, given two real arcs δ_1 and δ_2 as in Definition 10.1.14, their inner contact can be computed as the asymptotic of the inner distances between the points $\delta_1(t)$ and $\delta_2(t)$ on the representative $X \cap \{||x|| = t\}$ of the link of $(X, 0)$. The converse implication is only true in some special cases, such as for conical subset of $\mathbb{R}^n$, as treated by Kerner, Pedersen, and Ruas:

Proposition 10.1.21 ([25, Proposition 2.8]) *Let S^{n-1} be the unit sphere centered at the origin of $\mathbb{R}^n$, let M be a compact subset of S^{n-1}, and let $X = C(M) \subset \mathbb{R}^n$ be the cone over M, that is the union of the half-lines with origin 0 and passing through points of M. Then X is LNE if and only if M is LNE (as a subset of $\mathbb{R}^n$).*

The proof is obtained by performing direct computations of inner and outer distance between points inside $C(M)$.

Remark 10.1.22 In the case where M does not intersect the meridian sphere $S^{n-2} = S^{n-1} \cap \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$, then $X = C(M)$ is also the cone $C(N)$ over the compact set $N = C(M) \cap \mathbb{R}^{n-1} \times \{\pm 1\}$, and the map $\phi \colon M \to N$ sending a point x of M to the point of the half-line through x which intersects $\mathbb{R}^{n-1} \times \{\pm 1\}$ realizes a bilipschitz homeomorphism for the outer metric. Therefore, the LNE-ness of M is equivalent to that of N in this case.

Example 10.1.23 Consider the cone $C(N)$ in $\mathbb{R}^3$ over the union of the two circles

$$N = \{(x, y, 1) \in \mathbb{R}^3 \mid ((x-1)^2 + y^2 - 1)((x+1)^2 + y^2 - 1) = 0\}.$$

Then $C(N)$ is not LNE since N is not LNE at the intersection point $q = (0, 0, 1)$ of the two circles. Indeed, the two arcs $p_1(t) = (-1 + \sqrt{1-t^2})$ and $p_2(t) = (1 - \sqrt{1-t^2})$ on (N, q) satisfy $q_o(p_1, p_2) = 3/2 \neq q_i(p_1, p_2) = 1$.

In [30], Mendes and Sampaio proved a broad generalization of Proposition 10.1.21 which provides a characterization of LNE subanalytic germs via their links. This result was further generalized by Nguyen in [36] to any definable set in a o-minimal structure (not necessarily polynomially bounded). We state here this most general version.

Theorem 10.1.24 *[30, 36] Let $(X, 0)$ be a definable germ in $(\mathbb{R}^n, 0)$ and let $\rho\colon (X, 0) \to (\mathbb{R}, 0)$ be the germ of a continuous definable function such that $\rho(x) = \Theta(\|x\|)$. Suppose that $(X \setminus \{0\}, 0)$ is connected. Then the following statements are equivalent:*

(i) $(X, 0)$ *is LNE;*
(ii) *There exist real numbers $r_0 > 0$ and $C > 0$ such that, for every $r \in (0, r_0]$, the set $X_r = \rho^{-1}(r) \cap X$ is LNE with Lipschitz constant bounded by C.*

Remark 10.1.25 Condition (ii) is stated in [30] in the case where the function ρ equals the distance to the origin. In that case, $X_r = S_r^{n-1} \cap X$ is the link of $(X, 0)$ at distance r and a germ $(X, 0)$ satisfying condition (ii) is said to be *link-LNE* (or simply *LLNE*).

The proof of Theorem 10.1.24 in the case where $(X, 0)$ and ρ are subanalytic is based on the Curve Selection Lemma, used in a similar way as in the proof of Theorem 10.1.18 given in [7], and on a result of Valette [46, Corollary 2.2] which states the existence of a bilipschitz homeomorphism $h\colon (X, 0) \to (X, 0)$ such that for all x in a neighborhood of the origin we have $\|h(x)\| = \rho(x)$.

Example 10.1.26 As an application of Theorem 10.1.24, consider again the real hypersurface S defined in $\mathbb{R}^3$ by the equation $x^2 + y^2 - z^3 = 0$ of Example 10.1.26, and fix $t > 0$. Then the intersection $S_t = S \cap \{z = t\}$ is a circle with radius $t^{3/2}$. Therefore, every S_t is LNE with Lipschitz constant $C = 2\pi$, so that S is link-LNE and hence LNE.

Example 10.1.27 (36, Proposition 3.11) Consider the semialgebraic germ $(X, 0) \subset (\mathbb{R}^3, 0)$ defined by $X = \{(t, x, z) \in \mathbb{R}^3 | 0 \leqslant x \leqslant t, z^2 = t^2x^2\}$ and the semialgebraic function $\rho\colon (X, 0) \to (\mathbb{R}^+, 0)$ define by $(t, x, z) \mapsto t$. Then $\rho(w) = \Theta(\|w\|)$ and $X_r = \rho^{-1}(r) \cap X$ is LNE but its Lipschitz constant is $\Theta(1/r)$. Therefore, condition (ii) of Theorem 10.1.24 is not satisfied, which implies that $(X, 0)$ is not LNE.

10.1.6 LNE-Ness and Moderately Discontinuous Homology

In [13], Fernández de Bobadilla, Heinze, Pe Pereira, and Sampaio defined a homology theory called *Moderately Discontinuous homology*. It produces families of groups which are invariants of the bilipschitz homeomorphism classes of subanalytic germs with respect to either the inner or the outer metric. In particular, given a subanalytic germ $(X, 0) \subset (\mathbb{R}^n, 0)$, the identity map on $(X, 0)$ induces homomorphisms between the corresponding Moderately Discontinuous homology groups of $(X, 0)$ with respect to these two metrics, and it is easy to check that if $(X, 0)$ is LNE then these homomorphisms are isomorphic. The authors asked whether the converse is true:

Question 10.1.28 Let $(X, 0) \subset (\mathbb{R}^n, 0)$ be a subanalytic germ and assume that the identity map induces isomorphisms at the level of Moderately Discontinuous homology with respect to the inner and the outer metric at every point of $(X, 0)$. Is $(X, 0)$ necessarily LNE?

In general, the answer is no: Example 10.1.27 is a counter-example, as shown by Nguyen in [36, Proposition 3.11]. Notice that that example is semialgebraic and has a non-isolated singularity. The question is still open in the case of an isolated singularity or in the complex analytic setting.

10.1.7 LNE-Ness and Tangent Cones

In this subsection, we state and discuss two necessary conditions for the LNE-ness of a subanalytic germ $(X, 0) \subset (\mathbb{R}^n, 0)$ in term of its tangent cone, proved by Fernandes and Sampaio.

Theorem 10.1.29 ([19, Corollary 3.11]) *Let* $(X, 0) \subset (\mathbb{R}^n, 0)$ *be a subanalytic germ and let* T_0X *be its tangent cone at* 0*. If* X *is LNE, then the two following conditions are satisfied:*

(i) T_0X *is LNE;*
(ii) T_0X *is reduced.*

Proof (Sketch) The proof of the first part uses the following notion of tangent cone of a subanalytic germ in $(X, 0) \subset (\mathbb{R}^n, 0)$, introduced in [19, Section 2.2], which generalizes the classical definition in the real or complex analytic setting. Let $D_0(X)$ be the set of unitary vectors v in $\mathbb{R}^n \setminus \{0\}$ such that there exist a sequence of points $(x_j)_{j\in\mathbb{N}}$ in $X \setminus \{0\}$ converging to 0 such that $\lim_{j\to+\infty} \frac{x_j}{\|x_j\|} = v$; the *tangent cone* T_0X of $(X, 0)$ at 0 is defined by

$$T_0X = \{tv \mid v \in D_0(X), t \in \mathbb{R}^+\}.$$

Assume that $(X, 0)$ is LNE. Let $0 \in U \subset X$ be a small neighborhood of 0 in X and let $\lambda > 0$ such that for all $x, y \in U, d_i(x, y) \leqslant \lambda d_o(x, y)$. The proof of the first part of the theorem presented in [19] considers two vectors v, w in T_0X and constructs an arc α in T_0X between v and w with length at most $(1 + \lambda)\|x - y\|$. The arc α is obtained by an elegant argument using the Arzelà–Ascoli Theorem, as the limit of arcs joining two sequences of points (x_j) and (y_j) in $(X, 0)$ such that $\lim \frac{x_j}{\|x_j\|} = v$ and $\lim \frac{y_j}{\|y_j\|} = w$. The second part of the theorem requires a definition of *reducedness* for the tangent cone, introduced in [4] and based on an equivalent definition of the tangent cone T_0X using spherical blowups. We refer to [19, Section 3] for details, and only remark that the definition coincides with the classical one in the case of an analytic germ. The proof consists then in the construction of a pair of arcs (p_1, p_2) which does not satisfy the arc criterion in the neighborhood of a non reduced component of T_0X, in a similar way as in Example 10.1.4. ◻

The converse of Theorem 10.1.29 is not true. It is easy to find counter-examples among semialgebraic germs with non-isolated singularities, such as $X = \{(x, y, z) \in \mathbb{R}^3 \mid (x^2+y^2-z^2)(x^2+(|y|-z-z^3)^2-z^6) = 0, z \geqslant 0\}$ [19, Example 3.12].

Note that in the example above the link of $(X, 0)$ is not LNE, hence $(X, 0)$ cannot be LNE itself thanks to Lemma 10.1.20. Therefore, it becomes natural to ask the following question: given a subanalytic germ $(X, 0) \subset (\mathbb{R}^n, 0)$ whose link is LNE and satisfying Conditions (i) and (ii) of Theorem 10.1.29, is $(X, 0)$ necessarily LNE? The answer is negative, even among complex germs with isolated singularities. A counter-example is given by Neumann and the second author in the appendix of [19]:[4]

Proposition 10.1.30 *The hypersurface germ in* $(\mathbb{C}^3, 0)$ *with equation*

$$y^4 + z^4 + x^2(y + 2z)(y + 3z)^2 + (x + y + z)^{11} = 0$$

is not LNE , it has an isolated singularity at 0*, and its tangent cone is reduced and LNE.*

To end this subsection, let us mention that Fernandes and Sampaio proved in [20] the following analogue of Theorem 10.1.29 about complex algebraic sets of any dimension which are LNE at infinity, recovering in particular the results of [16].

Theorem 10.1.31 *Let X be complex analytic set in $\mathbb{C}^n$. Assume that:*

(i) *X is Lipschitz Normally Embedded at infinity, that is there exists a compact subset K of $\mathbb{C}^n$ such that each connected component of $X \setminus K$ is Lipschitz Normally Embedded;*
(ii) *The tangent cone of X at infinity is a linear subspace of $\mathbb{C}^n$.*

Then X is an affine linear subspace of $\mathbb{C}^n$.

[4] Note that the link of a subanalytic germ with isolated singularities is smooth, and therefore LNE.

We refer to [20] for details, and in particular to [20, Section 4] for the definition of the tangent cone at infinity. This result is remarkable since it shows that an a priori mild assumption at infinity forces the rigidity of the whole X. Note that if X is Lipschitz regular at infinity, that is if outside of a large compact set in $\mathbb{R}^n$ it is bilipschitz homeomorphic to an open subset of $\mathbb{R}^k$ for some k, then X is LNE at infinity (this is [20, Corollary 3.3]).

10.1.8 LNE in Topology and Other Fields

Lipschitz Normal Embeddings have also been useful to study problems in topology. For example, Birbrair et al. [8] prove that for a large class of real analytic parametrized surfaces in $\mathbb{R}^4$ LNE-ness implies the triviality of the knot obtained as their link.

More recently, Fernandes and Sampaio [21, Theorem 3.2] showed that two LNE compact subanalytic sets which are close enough with respect to the Hausdorff distance have isomorphic fundamental groups. This leads them to give topological conditions on the link of a LNE germ that ensure that the germ is smooth, (see Theorem 4.1 in *loc. cit.*), from which they derive the following remarkable result, which is a metric version, in arbitrary dimension, of Mumford's link criterion for the smoothness of normal surface germs.

Theorem 10.1.32 ([21, Theorem 4.2]) *Let* $(X, 0)$ *be a complex analytic germ of dimension* k *with isolated singularities. Then* $(X, 0)$ *is smooth if and only if it is locally metrically conical and its link at* 0 *is* $(2k - 2)$*-connected.*

The definition for X being locally metrically conical at 0 is given in [21, page 4]. We note that an earlier result towards a metric characterization of smoothness was obtained by Birbrair et al. [5], who proved that a germ which is outer bilipschitz equivalent to a smooth germ $(\mathbb{C}^m, 0)$ is itself smooth.

We also remark that a problem related to the embedding problem discussed in Sect. 10.1.3 is studied in functional analysis. Indeed, some people working in that field are interested in studying different classes of embeddings (some of which closely resemble those of the Theorem 10.1.11 of Birbrair and Mostowski) of *discrete* metric spaces into suitable Banach spaces. We refer the interested reader to the monograph [37] and to the many references found therein.

10.2 LNE Among Complex Surface Germs

The goal of this section is to overview some recent advances on the study of LNE singularities among complex surface germs.

10.2.1 A Refinement of the Arc Criterion

As was mentioned in Sect. 10.1.4, the arc-based criterion for LNE-ness of Birbrair and Mendes given in Theorem 10.1.18 is difficult to use effectively in practice, as it requires to compute inner and outer contact orders of infinitely many pairs of arcs. Whenever $(X, 0)$ is a normal surface germ, this situation has been improved upon by Neumann, Pedersen, and the second author of this survey (see [33]), and then further in an upcoming work by Pedersen, Schober, and the two authors (see [17]), leading to a drastic reduction of the amount of pairs of real arcs whose contact orders have to be compared, down to a finite (and in fact rather small) number.

In order to state the improved criterion we need to introduce the notion of test curve. Given a sequence of point blowups $\rho\colon Y_\rho \to \mathbb{C}^2$ of $(\mathbb{C}^2, 0)$ and an irreducible component E of the exceptional divisor $\rho^{-1}(0)$ of ρ, a *test curve* at E is any plane curve germ $(\gamma, 0) \subset (\mathbb{C}^2, 0)$ whose strict transform via ρ is a smooth curve transverse to E at a smooth point of $\rho^{-1}(0)$. For the purpose of the criterion, it is sufficient to take for ρ any sequence such that the strict transform Δ^* via ρ of the discriminant curve Δ of a generic plane projection $\ell\colon (X, 0) \to (\mathbb{C}^2, 0)$ of $(X, 0)$ is a disjoint union of irreducible curves cutting the exceptional divisor $\rho^{-1}(0)$ of ρ at smooth points (such as for example any good embedded resolution of Δ), to consider a suitable subset $\{E_0, \dots, E_s\}$ of the set of irreducible components of $\rho^{-1}(0)$, and to pick one test curve γ_i at E_i for each $i = 0, \dots, s$; this gives rise to a set $\{\gamma_0, \dots, \gamma_s\}$ called a *family of test curves* for $(X, 0)$ with respect to ℓ. We can now state the criterion.

Theorem 10.2.1 ([17, 33]) *Let $(X, 0)$ be a normal surface singularity, let $\ell\colon (X, 0) \to (\mathbb{C}^2, 0)$ be a generic plane projection of $(X, 0)$, and let $\{\gamma_0, \dots, \gamma_s\}$ be a family of test curves for $(X, 0)$ with respect to ℓ. Then the following conditions are equivalent:*

(i) $(X, 0)$ *is LNE.*
(ii) *For every $j = 0, \dots, s$ and for every pair of distinct irreducible components ξ and ξ' of the principal part of $\ell^{-1}(\gamma_j)$, then ξ and ξ' have the same multiplicity as γ_j and satisfy the equality $q_i(\xi, \xi') = q_o(\xi, \xi')$.*

In the statement, the *principal part* of $\ell^{-1}(\gamma_j)$ is a curve obtained by deleting from $\ell^{-1}(\gamma_j)$ some irreducible components, namely those that do not pull back to curvettes on a suitable canonical subgraph of the minimal good resolution of $(X, 0)$. Contact orders between two complex curve germs ξ and ξ' are defined in a similar way as those between real arcs by looking at the shrinking rates as $\epsilon > 0$ goes to 0 of the inner or outer distance between the sets $\xi \cap \{||x|| = \epsilon\}$ and $\xi' \cap \{||x|| = \epsilon\}$. These are simple to compute in practice; for example this can be done by looking at the Puiseux expansions of the images of the irreducible curves ξ and ξ' through a second generic plane projection, which is easy to do with computer software such as Singular or Maple.

The version of the criterion of [17] improves upon that of [33] because the latter requires the map ρ used to construct a test family to be a good embedded resolution of the family of the projections via ℓ of the polar curves with respect to *all* generic plane projections of $(X, 0)$ (which in particular demands to determine the Nash transform of the latter) and not just to one of them, then to consider a greater number of irreducible components of $\rho^{-1}(0)$, to take *all* possible test curves at any such component, and finally to pull those curves back again with respect to *all* generic plane projections of $(X, 0)$.

10.2.2 Examples

The improvement of Birbrair and Mendes's arc criterion discussed in the previous subsection lead to the discovery of several infinite families of LNE complex surface germs with isolated singularities, no examples of which were previously known.

Theorem 10.2.2 ([34]) *Let $(X, 0)$ be a normal complex surface germ assume that it is rational. Then $(X, 0)$ is LNE if and only if it is a minimal singularity.*

This result gives the first known infinite family of non-conical LNE normal complex surface singularities, and is in fact the main reason why the criterion of [33] was developed. For a thorough discussion of rational surface singularities we refer the reader to [29, 32], here we only recall that a surface singularity $(X, 0)$ is *rational* if and only if the exceptional divisor E of its minimal good resolution consists of rational curves and its dual graph is a tree which satisfies a numerical condition (see [29, Theorem 4.2]). If moreover E is reduced, that is if the pullback of a general element of the maximal ideal of $(X, 0)$ vanishes with order one along each component of E, then $(X, 0)$ is said to be *minimal*.

The fact that a rational singularity which is LNE is minimal based on Laufer's algorithm [29, Proposition 4.1] to determine the fundamental cycle $Z_{\min}$ (see the Footnote 5 on p. 515 for the definition of $Z_{\min}$). Conversely, in order to apply the criterion of Theorem 10.2.1 and show that a minimal surface singularity is LNE, the proof of Theorem 10.2.2 relies on a detailed study of the generic polar curves of minimal surface singularities performed in [43].

More generally, an equidimensional complex germ $(X, 0)$ of multiplicity m and embedding dimension e is said to be *minimal* if it is reduced, Cohen–Macaulay, with reduced tangent cone, and Abhyankar's inequality $m \geqslant e - \dim(X, 0) + 1$ is in fact an equality. The last condition means that minimal singularities generally live in high-dimensional ambient spaces. At the other side of the spectrum, the first family of LNE normal hypersurface singularities in $\mathbb{C}^3$ was discovered later by Misev and the first author of this survey, who studied LNE-ness among *superisolated singularities*. In order to define those, consider a complex hypersurface germ $(X, 0)$ in $(\mathbb{C}^3, 0)$, defined by the equation $f(x, y, z) = 0$, and write f as a sum of polynomials $f = f_d + f_{d+1} + \ldots$, with $f_d \neq 0$ and each f_i homogeneous of degree i. Then $(X, 0)$ is said to be *superisolated* if the plane projective curve defined by $f_{d+1} = 0$ does not intersect the singular locus of the projectivized tangent cone $C_0X = \{(x : y : z) \in$

$\mathbb{P}^2_{\mathbb{C}} \mid f_d(x:y:z)=0\}$ of $(X,0)$. This implies that a single blowup of X along 0 is sufficient to resolve its singularities.

Theorem 10.2.3 ([31]) *Let $(X,0)$ be a superisolated normal complex surface germ. Then $(X,0)$ is LNE if and only if its tangent cone is reduced and LNE.*

Recall that the tangent cone of a LNE singularity has to be reduced and LNE thanks to Theorem 10.1.29.

The further improvement obtained in [17] over the criterion of [33] allows to generalize the theorem above and obtain new families of LNE normal hypersurface singularities in $\mathbb{C}^3$. In particular, the following result follows.

Theorem 10.2.4 ([17]) *Let n and k be two positive integers such that $n \geqslant k$ and let $(X,0)$ be the hypersurface in $(\mathbb{C}^3,0)$ defined by the equation*

$$\prod_{i=1}^{k}(a_i x + b_i y) - z^n = 0,$$

where the (a_i, b_i)'s are pairs of nonzero complex numbers such that the k points $(a_i : b_i)$ of $\mathbb{P}^1_{\mathbb{C}}$ are pairwise distinct. Then $(X,0)$ is LNE.

Observe that whenever $n < k$ then the tangent cone of $(X,0)$ is defined by $z^n = 0$. As the latter is non reduced, then $(X,0)$ cannot be LNE, or this would contradict Theorem 10.1.29.

10.2.3 Properties of LNE Surfaces

Lipschitz Normally Embedded complex surface singularities have many remarkable properties. For example, the authors of this survey, together with Belotto da Silva, proved the following:

Proposition 10.2.5 ([3, Proposition 2.2]) *Let $(X,0)$ be a complex LNE normal surface germ. Then the minimal resolution of $(X,0)$ factors through the blowup of X along 0, the exceptional components of this blowup are reduced, and the topological type of $(X,0)$ determines its multiplicity (which a priori is a datum of analytic nature).*

The same paper also contains the following deeper result.

Theorem 10.2.6 ([3, Theorems 1.1 and 1.2]) *Let $(X,0)$ be a complex LNE normal surface germ. Then the topological type of $(X,0)$ determines the following data:*

(i) *The dual graph of the minimal good resolution of $(X,0)$ which factors through the blowup of the maximal ideal and through the Nash transform, decorated by two families of arrows corresponding respectively to the strict transform of a* *generic hyperplane section and to the strict transform of the polar curve of a generic plane projection.*

(ii) *The (embedded) topological type of the discriminant curve of a generic projection.*

Moreover, this data can be computed explicitly from the dual graph of the minimal good resolution of $(X, 0)$.

This theorem generalizes to all LNE normal surface germs results that were previously known only for minimal surface singularities. In that special case, the first property was established by Spivakovsky [43, III, Theorem 5.4], while the second one was later proven by Bondil [11, Theorem 4.1], [12, Proposition 5.4].

This result can be thought of as a unique solution, for the class of LNE normal surface singularities, to the so-called problem of *polar exploration*, which asks to determine the generic polar variety of a singular complex surface germ. This problem was studied for a general surface germ $(X, 0)$ by the same authors together with Némethi [1], relying on the study of the *inner rates* of $(X, 0)$. Those are an infinite family of metric invariants that appeared naturally in the study of the Lipschitz geometry of $(X, 0)$ in the foundational work [10], and were then systematically studied in [2]. From this point of view, it is worth noticing that in the paper [3] referred to above it is also shown that the topological type of a LNE normal surface germ $(X, 0)$ determines its inner rates (see Proposition 5.1 of *loc. cit.*), and this combined with the main result of [2] is what allows them to determine the combinatorics of the polar curve of a generic plane projection of $(X, 0)$.

10.3 Open Questions

We conclude this survey by putting forward some open questions about LNE singularities that we find worth of interest.

10.3.1 Behavior Under Blowup and Nash Transform

It was a long-held belief by several experts in the field that the point blowup of a LNE complex surface germ $(X, 0)$ with an isolated singularity would most likely have itself only LNE singularities. The following counterexample took therefore the authors by surprise.

Example 10.3.1 The hypersurface $(X, 0)$ in $(\mathbb{C}^3, 0)$ defined by the equation

$$(x + y)(2x + y)(x + 2y) - z^5 = 0$$

is LNE (it is a special case of Theorem 10.2.4 from [17]). However, the blowup of X along 0 has a singularity whose local equation is $2x_1^3+7x_1^2y_1+7x_1y_1^2+2y_1^3-w^2=0$, whose tangent cone $w^2=0$ is non reduced.

On the other hand, the following question is still open.

Question 10.3.2 Let $(X, 0)$ be a LNE complex surface germ with an isolated singularity. Does the Nash transform of $(X, 0)$ have itself only LNE singularities?

Observe that by da Silva et al. [3, Corollary 4.7], if $(X, 0)$ is LNE then its Nash transform has only *sandwiched singularities*. Since sandwiched singularities are rational, in order to give a positive answer to Question 10.3.2 thanks to Theorem 10.2.2 one would have to show that they are minimal.

10.3.2 Topological Types of LNE Surface Singularities

It is a very natural question to study the topological properties of LNE singularities. In order to start such an investigation, it seems wise to restrict oneself to the case of normal complex surfaces. In this context, it is well-known that, by a classical result of Neumann, the topological type of a normal surface singularity $(X, 0)$ is equivalent to the datum of the weighted dual graph Γ_π of the minimal good resolution π of $(X, 0)$, where each vertex is weighted by the genus and self-intersection of the corresponding exceptional component of π.

A first observation is then that being LNE is not a topological property, as shown by the following example, kindly provided to us by Jan Stevens.

Example 10.3.3 Let X_1 be the hypersurface in $\mathbb{C}^3$ defined by the equation $x^4+y^4+z^4=0$ and let X_2 be the surface in $\mathbb{C}^4$ defined by the equations $y^2=xz$ and $w^2=x^4+z^4$. The two surface germs $(X_1, 0)$ and $(X_2, 0)$ are normal and have the same topological type, since for both of them the exceptional divisor of the minimal resolution is a single curve of genus 3 and self-intersection -4. However, $(X_1, 0)$ is LNE, since it is the cone over the smooth projective curve $x^4+y^4+z^4=0$, while $(X_2, 0)$ is not, since its tangent cone, which is defined by the equations $w^2=0$ and $y^2=xz$, is non reduced.

However, given a weighted graph Γ one can say that Γ is *LNE* if there exists a LNE normal surface singularity with resolution graph Γ. The following question is therefore very natural.

Question 10.3.4 Is there a combinatorial characterization of LNE weighted graphs?

The first results of [3] provide some obstructions for a weighted graph Γ to be LNE. Denote by $Z_{\min}$ the fundamental cycle of Γ. We then say that a vertex v of

$\Gamma(V)$ is a *numerical $\mathscr{L}$-node* of Γ if $E_v \cdot Z_{\min} < 0$.[5] It was then shown in [3, Proposition 2.2.(ii)] that whenever $(X, 0)$ is a LNE singularity whose weighted dual graph is Γ then $Z_{\min}$ coincides with the maximal ideal cycle $Z_{\max}$ of $(X, 0)$.[6] In particular, the numerical $\mathscr{L}$-nodes of Γ coincide with its usual $\mathscr{L}$-nodes, which are the vertices which correspond to the exceptional components of the blowup of X along 0. It then follows from Proposition 10.2.5 that the numerical $\mathscr{L}$-nodes of Γ are reduced, which means that whenever Γ is the dual resolution graph of a LNE surface then it satisfies the following combinatorial condition:

$$\text{writing } Z_{\min} = \sum d_v E_v, \text{ we have } d_v = 1 \text{ for every } v \text{ such that } Z_{\min} \cdot E_v < 0.$$

A weighted graph satisfying the condition above is called a *Kodaira graph*. Kodaira graphs are precisely those which can be realized as dual resolution graphs of the so-called *Kodaira singularities*, a class of surface singularities defined in terms of a suitable family of curves and introduced by Karras [23] after work of Kulikov [26]. It seems worthwhile of interest to fully investigate the relations between Lipschitz Normal Embeddings and Kodaira singularities (or the subclass of Kodaira singularities consisting of the so-called *Kulikov singularities* introduced by Stevens [44]). As a first step towards this, we mention that among rational singularities the only ones that are Kodaira are precisely the minimal singularities (see [23, Example 2.8 plus Theorem 2.9]), that is the ones that are also LNE (and Kulikov). However, not all Kulikov singularities (and therefore not all Kodaira singularities) are LNE, since their projective tangent cone is not necessarily reduced (see [44, Example 2.4] for a Kulikov singularity with reducible tangent cone; moreover its minimal resolution does not factor through the blowup of its maximal ideal).

10.3.3 Generalizations of the Arc Criterion

We have mentioned in Sect. 10.2 how improving the arc criterion for LNE-ness of Theorem 10.1.18 proved to be extremely useful in the study of LNE complex surface

[5] Let us briefly recall the precise definitions of the combinatorial notions we use here, in particular that of the fundamental cycle and how to make sense of the intersection number $E_v \cdot Z_{\min}$. A *weighed graph* is a finite connected graph Γ without loops and such that each vertex $v \in V(\Gamma)$ of Γ is weighted by two integers, its *genus* $g(v) \in \mathbb{Z}_{\geqslant 0}$ and its *self-intersection* $e(v) \in \mathbb{Z}_{\leqslant 0}$. Let $E = \bigcup_{v \in V(\Gamma)} E_v$ be a configuration of curves whose dual weighted graph is Γ, so that in particular $g(v) = g(E_v)$ and $E_v^2 = e(v)$, and let $I_\Gamma = (E_v \cdot E_{v'})$ be the *incidence matrix* of Γ. We assume that I_Γ is negative definite. A *divisor on* Γ is a formal sum $D = \sum_{v \in V(\Gamma)} m_v E_v$ over the set of the irreducible components of E with integral coefficients. The *fundamental cycle* $Z_{\min}$ of Γ is then the unique nonzero divisor on Γ which is minimal among those divisors D satisfying $D \cdot E_v < 0$ for all $v \in V(\Gamma)$. Its existence was shown by Artin, and its coefficients are all strictly positive.

[6] The cycle $Z_{\max}$ is the divisor on Γ whose coefficient at a vertex v is the order of vanishing of a generic linear form of $(X, 0)$ along the exceptional component E_v associated with v.

germs. It would therefore be very interesting to find similar improvements in a more general setting, and in particular for complex germs of higher dimensions.

Question 10.3.5 Find an improvement of the arc criterion of Birbrair and Mendes (Theorem 10.1.18) that only requires to compare the inner and outer contact orders of a finite (and in fact as small as possible) family of pairs of real arcs, for complex germs of arbitrary dimension.

Let now $(X, 0)$ be an algebraic complex germ. The *arc space* $\mathscr{L}_\infty(X, 0)$ of $(X, 0)$ is a scheme that parametrizes all complex arcs on X that are centered in 0, which are by definition the points of X with coordinates in $\mathbb{C}[[t]]$ and such that setting $t = 0$ we obtain the complex point $0 \in X$. Its geometry, and the geometry of the *jet spaces* of $(X, 0)$ (the varieties parametrizing the *jets* of $(X, 0)$, which are its complex arcs truncated at a given order), reflect interesting properties of the singularity of $(X, 0)$. Their study plays an important role in many subareas of algebraic geometry, such as in the theory of motivic integration. This leads us to formulate the following problem.

Question 10.3.6 Give a criterion for the LNE-ness of a germ $(X, 0)$ in terms of the geometries of the arc or jet spaces of $(X, 0)$.

Such a criterion should involve testing the contact orders for generic arcs (or families of arcs, which are commonly called *wedges*) of some suitable irreducible subschemes of the arc space. In dimension 2, this could be related to the irreducible components of $\mathscr{L}_\infty(X, 0)$, and hence to the *essential valuations* of $(X, 0)$, thus relating Lipschitz geometry to the notorious Nash problem solved by Fernández de Bobadilla and Pe Pereira in [14]. In arbitrary dimension, the LNE-ness of a germ $(X, 0)$ could possibly be read in terms of its *terminal valuations*, whose relation to the geometry of $\mathscr{L}_\infty(X, 0)$ was detailed by de Fernex and Docampo [15].

More generally, the relations between the geometry of arc and jet spaces and Lipschitz geometry are completely unexplored. Some recent results, such as the appearance of Mather discrepancies in [1], suggest that this may be a matter worth exploring.

10.3.4 Higher Dimensional LNE Complex Singularities

As should now be clear to the reader, very little is known about complex LNE singularities starting from the dimension 3. Since the simplest family of LNE surface germs consists of minimal surface singularities, the following question is very natural.

Question 10.3.7 Are minimal singularities in arbitrary dimension LNE?

Recall that the definition of minimal singularities in arbitrary dimensions appears after Theorem 10.2.2.

Minimal singularities form a building block in the Minimal Model Program. Therefore, more generally, can we hope to characterize LNE singularities, or at least provide new classes of higher-dimensional examples, using the invariants appearing in the Minimal Model Program?

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Chapter 11
Hilbert-Samuel Multiplicity and Finite Projections

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Abstract In this (mainly) expository notes, we study the multiplicity of a local Noetherian ring $(B, \mathfrak{m})$ at an $\mathfrak{m}$-primary ideal I, paying special attention to the geometrical aspects of this notion. To this end, we will be considering suitably defined finite extensions $S \subset B$, with S regular. We will explore some applications like the explicit description of the equimultiple locus of an equidimensional variety, or the computation of the asymptotic Samuel function.

11.1 Introduction

When it comes to measure how bad a singularity is, the case of a hypersurface in the affine space can provide some intuition. Let us assume that k is a field, let $A = k[X_1, \ldots, X_n]$ and let $f \in A$ be a non-zero polynomial defining a hypersurface $H \subset \mathbb{A}^n_k$. For a given point $\zeta \in H$, denote by $\mathfrak{p} \subset A$ the corresponding prime ideal. We can consider the *order of* f *at* $\mathfrak{p}$,

$$\nu_\zeta(f) := \max\{n : f \in \mathfrak{p}^n A_{\mathfrak{p}}\} \geqslant 1.$$

The hypersurface H is regular at ζ if $\nu_\zeta(f) = 1$, otherwise H is singular at ζ. In such case, $\nu_\zeta(f) \geqslant 2$, and we can think that the larger $\nu_\zeta(f)$ be, the more singular the point will be. It is quite natural to ask whether this measurement can be made directly at the local ring of the hypersurface at ζ. To address this question, let us first fix some notation. We will use $k(\mathfrak{p})$ to denote the residue field at $\mathfrak{p}$ and $\bar{\mathfrak{p}}^i$ to refer to $(\mathfrak{p}^i + \langle f \rangle / \langle f \rangle)$. Let $B = A/\langle f \rangle$, and set $B_{\bar{\mathfrak{p}}} = (A/\langle f \rangle)_{\bar{\mathfrak{p}}}$. Then the value $\nu_\zeta(f)$ has an impact on the dimensions of the $k(\mathfrak{p})$-vector spaces:

$$B_{\bar{\mathfrak{p}}}/\bar{\mathfrak{p}}B_{\bar{\mathfrak{p}}}, \quad B_{\bar{\mathfrak{p}}}/\bar{\mathfrak{p}}^2 B_{\bar{\mathfrak{p}}}, \quad B_{\bar{\mathfrak{p}}}/\bar{\mathfrak{p}}^3 B_{\bar{\mathfrak{p}}}, \ldots, \quad B_{\bar{\mathfrak{p}}}/\bar{\mathfrak{p}}^\ell B_{\bar{\mathfrak{p}}}, \ldots,$$

or equivalently, on the dimension of the $k(\mathfrak{p})$-vector spaces

$$B_{\bar{\mathfrak{p}}}/\bar{\mathfrak{p}}B_{\bar{\mathfrak{p}}}, \quad \bar{\mathfrak{p}}B_{\bar{\mathfrak{p}}}/\bar{\mathfrak{p}}^2 B_{\bar{\mathfrak{p}}}, \quad \bar{\mathfrak{p}}^2 B_{\bar{\mathfrak{p}}}/\bar{\mathfrak{p}}^3 B_{\bar{\mathfrak{p}}}, \ldots, \bar{\mathfrak{p}}^{\ell-1} B_{\bar{\mathfrak{p}}}/\bar{\mathfrak{p}}^\ell B_{\bar{\mathfrak{p}}}, \ldots$$

Notice that the quotients of the later sequence correspond to the j-th degree pieces of the graded ring $\mathrm{Gr}_{\bar{\mathfrak{p}}B_{\bar{\mathfrak{p}}}}(B_{\bar{\mathfrak{p}}}) = \bigoplus_i \bar{\mathfrak{p}}^i B_{\bar{\mathfrak{p}}}/\bar{\mathfrak{p}}^{i+1} B_{\bar{\mathfrak{p}}}$ of the local ring $B_{\bar{\mathfrak{p}}} = \mathcal{O}_{H,\zeta}$.

Actually, the previous approach can be applied in a more general setting. We can assume, for instance, that B is the coordinate ring of an arbitrary algebraic variety defined over a field k, or even, just any local Noetherian ring. Then, for a prime $\mathfrak{p} \subset B$, *the multiplicity* B *at* $\mathfrak{p}$ comes in naturally as an invariant when trying to measure the growth of dimension of the graded pieces of the graded ring

$$\mathrm{Gr}_{\mathfrak{p}B_{\mathfrak{p}}}(B_{\mathfrak{p}}) = \bigoplus_{i \geqslant 0} \mathfrak{p}^i B_{\mathfrak{p}}/\mathfrak{p}^{i+1} B_{\mathfrak{p}}$$

as $k(\mathfrak{p})$-vector spaces, or equivalently, the growth of the lengths of the $B_{\mathfrak{p}}$-modules, $B_{\mathfrak{p}}/\mathfrak{p}^{i+1}B_{\mathfrak{p}}$, for $i = 0, 1, \ldots$.

To be more precise, this growth is encoded asymptotically by the so called *Hilbert-Samuel polynomial* of $B_{\mathfrak{p}}$ at $\mathfrak{p}$. This is a polynomial of degree $d = \dim(B_{\mathfrak{p}})$ and the *multiplicity* at $\mathfrak{p}$, $e_{B_{\mathfrak{p}}}$, is (up to some suitable factor) the leading coefficient of that polynomial.

But, what does the multiplicity tell us about the singularity at a given point? If $\mathfrak{p}$ is a regular point in $\mathrm{Spec}(B)$ then $\mathrm{Gr}_{\mathfrak{p}B_{\mathfrak{p}}}(B_{\mathfrak{p}})$ is isomorphic to a polynomial ring in d-variables with coefficients in $k(\mathfrak{p})$. In such case the Hilbert-Samuel polynomial can be easily computed and it can be checked that the multiplicity at $\mathfrak{p}$ equals one. Moreover, under some conditions, multiplicity one implies regularity. As another example, if $B = R/\langle f\rangle$, where R is a regular ring, then the multiplicity can be computed in terms of a local writing of the equation f at each point. More precisely, if the order of f at a prime $\mathfrak{p} \in \mathrm{Spec}(R)$ is m, then the multiplicity at the corresponding prime, $\mathfrak{p}/\langle f\rangle$, in B equals m. In general, however, there is no apparent algebraic method that brings clear geometric insight on the meaning of the multiplicity.

The purpose of these notes is to focus on the geometric aspects of the multiplicity. To fix ideas, assume that B is an equidimensional ring of finite type over a field k. Now suppose that we want to present B as a finite extension of a regular ring. To do so, we could start, for instance, by considering Noether's Normalization Lemma. This tells us that if the Krull dimension of B is d, then B is a finite extension of a polynomial ring in d variables with coefficients in k, $S = k[X_1, \ldots, X_d]$. Let $K(S)$ be the fraction field of S. Then it can be shown that the multiplicity of B at any prime $\mathfrak{p}$, $e_{B_{\mathfrak{p}}}(\mathfrak{p}B_{\mathfrak{p}})$, is bounded above by the generic rank of the extension $S \hookrightarrow B$, i.e.,

$$e_{B_{\mathfrak{p}}}(\mathfrak{p}B_{\mathfrak{p}}) \leqslant [K(S) \otimes_S B : K(S)].$$

Actually, the multiplicity can be defined at any $\mathfrak{p}$-primary ideal I, $e_{B_{\mathfrak{p}}}(IB_{\mathfrak{p}})$, and it can be equally shown that, for a finite extension with S regular the same inequality holds in this more general setting, if $I \cap S$ is a prime, i.e.,

$$e_{B_{\mathfrak{p}}}(IB_{\mathfrak{p}}) \leqslant [K(S) \otimes_S B : K(S)]. \tag{11.1}$$

Here, we will be interested in the study of finite extensions $S \subset B$ with S regular for which the equality in (11.1) holds, and then we will say that $S \subset B$ is *finite-transversal with respect to I*.

Finite-transversal extensions do not always exists, see Sect. 11.3.3. However they can be constructed at the completion of the local ring $(B_{\mathfrak{p}}, \mathfrak{p}B_{\mathfrak{p}})$, maybe after enlarging the residue field, see Sect. 11.3.4.

In [30] Villamayor pointed out that finite-transversal extensions with respect to a prime $\mathfrak{p}$ can always be constructed in a local étale neighborhood of $(B_{\mathfrak{p}}, \mathfrak{p}B_{\mathfrak{p}})$, when $B_{\mathfrak{p}}$ is essentially of finite type over a perfect field k. In [7, Appendix A] such construction is described in full detail. Here we will reproduce some of the arguments in [7] with a twofold purpose. On the one hand, to show that the same procedure can be used to construct finite-transversal morphisms with respect to any $\mathfrak{p}$-primary ideal I. On the other, we will follow the different parts of the proof to illustrate distinct aspects of this construction with a list of examples.

As it turns out, finite-transversal morphisms play a role in describing the top multiplicity locus of B, via the so called *presentations of the multiplicity*. Such description, given by Villamayor in [30], was presented to show a stronger result, namely, that it is possible to resolve the singularities of an algebraic variety using the multiplicity as the main invariant (this was a question posed by Hironaka in [16]). This approach to resolution will be discussed in Sect. 11.5.

In addition, finite-transversal morphisms appear as well as a tool for the computation of the asymptotic Samuel function. In [15], Hickel gives a procedure for the computation of the asymptotic Samuel function respect to an $\mathfrak{m}$-primary ideal I in an equicharacteristic local ring $(R, \mathfrak{m})$. To this end, he considers the completion $\hat{R}$, where he constructs a finite-transversal extension with respect to $I\hat{R}$. Here we will see that his arguments are equally valid if we can find a finite transversal projection in a suitably defined étale neighborhood of $(R, \mathfrak{m})$.

This manuscript is mostly expository and some of the statements presented here are slight variations of results in [30] and also in [15]. Precise references to these articles will be given along the paper.

These notes are organized as follows. The multiplicity at an ideal of a local Noetherian ring is treated in Sect. 11.2, and some known properties are described. Finite-transversal morphisms are defined and constructed in Sect. 11.3. Finally, applications are addressed in Sects. 11.4, 11.5 and 11.6.

Notation For a quotient of a polynomial ring with coefficients in a ring A, $B = A[X_1, \ldots, X_n]/J$, we will use lowercase, x_i, to denote the class of the variable X_i in B, for $i = 1, \ldots, n$. This notation will be used in the examples along the paper.

11.2 The Multiplity Function

Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, and consider the function:

$$\begin{aligned} \mathrm{HS}_{R,\mathfrak{m}} : \mathbb{N} &\to \mathbb{N} \\ \ell &\mapsto \dim_k(R/\mathfrak{m}^{\ell+1}). \end{aligned}$$

This is referred to as the *Hilbert-Samuel function of* R *at* $\mathfrak{m}$.

Theorem 11.2.1 *Let* $(R, \mathfrak{m}, k)$ *be a Noetherian local ring. Then the Hilbert-Samuel function* $HS_{R,\mathfrak{m}}$ *is of polynomial type, i.e., there exists a polynomial* $p_{R,\mathfrak{m}}(X) \in \mathbb{Q}[X]$ *such that for* $\ell \gg 0$,

$$HS_{R,\mathfrak{m}}(\ell) = p_{R,\mathfrak{m}}(\ell).$$

In addition, the degree of $p_{R,\mathfrak{m}}(X)$ *equals* $d = \dim(R)$*, the Krull dimension of the ring* R*. Moreover,*

$$p_{R,\mathfrak{m}}(X) = e_R(\mathfrak{m})\frac{X^d}{d!} + \ldots,$$

where $e_R(\mathfrak{m}) \in \mathbb{N}$.

For a proof see for instance [29, Theorem 11.1.3], where the theorem is stated in a much more general setting. We refer to $p_{R,\mathfrak{m}}(X)$ as the *Hilbert-Samuel polynomial of* R *with respect to* $\mathfrak{m}$, and we say that $e_R(\mathfrak{m}) \in \mathbb{N}$ is the *multiplicity of the local ring* R *at* $\mathfrak{m}$ or simply *the multiplicity of* R. Sometimes we write e_R to refer to $e_R(\mathfrak{m})$.

Example 11.2.2 If $(R, \mathfrak{m}, k)$ is a d dimensional regular local ring, then

$$\dim_k R/\mathfrak{m}^{\ell+1} = \binom{d+\ell}{d},$$

and

$$p_{R,\mathfrak{m}}(X) = \frac{X^d}{d!} + \ldots.$$

Therefore, for a regular local ring, $e_R = 1$. The converse holds if we require R to be unmixed (i.e., formally equidimensional, that is, the $\mathfrak{m}$-adic completion of R is equidimensional).

Theorem 11.2.3 *[18, Theorem 40.6] A Noetherian local ring* $(R, \mathfrak{m})$ *is regular if and only if it is unmixed and* $e_R = 1$.

Remark 11.2.4 If $(R, \mathfrak{m}, k)$ is a regular local ring, $f \in R$ is a non-zero element, and $B = R/\langle f\rangle$, then

$$\nu_{\mathfrak{m}}(f) = e_B(\mathfrak{m}/\langle f\rangle),$$

see [29, Example 11.2.8]. Hence, in the hypersurface case the order of the defining equation at a given point equals the multiplicity at the point.

11.2.1 Multiplicity at $\mathfrak{m}$-Primary Ideals

Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of dimension d. As we will see in forthcoming sections, sometimes it is convenient to work with arbitrary $\mathfrak{m}$-primary ideals. If $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal, then, in the same way as we did before, the following Hilbert-Samuel function can be defined:

$$\begin{aligned} \mathrm{HS}_{R,\mathfrak{a}} : \mathbb{N} &\to \mathbb{N} \\ \ell &\mapsto \lambda_R(R/\mathfrak{a}^{\ell+1}), \end{aligned}$$

where λ_R denotes the length as R-module. An analogous of Theorem 11.2.1 holds, and there is a polynomial, *the Hilbert-Samuel polynomial of* R *at* $\mathfrak{a}$, $P_{R,\mathfrak{a}}(X)$, so that for $\ell \gg 0$,

$$\mathrm{HS}_{R,\mathfrak{a}}(\ell) = P_{\mathfrak{a}}(\ell),$$

and moreover,

$$P_{\mathfrak{a}}(X) = e_R(\mathfrak{a})\frac{X^d}{d!} + \dots,$$

with $e_R(\mathfrak{a}) \in \mathbb{N}$ (see [29, Theorem 11.1.3]). The positive integer $e_R(\mathfrak{a})$ is the *multiplicity of* R *at the* $\mathfrak{m}$*-primary ideal* $\mathfrak{a}$. From the definition it follows that

$$e_R(\mathfrak{a}) \geqslant e_R = e_R(\mathfrak{m}). \tag{11.2}$$

It is quite natural to ask under which conditions the previous inequality is an equality. First, recall that an element $r \in R$ belongs to the integral closure, $\overline{\mathfrak{a}}$, of $\mathfrak{a}$ if

$$r^{\ell} + a_1 r^{\ell-1} + \dots a_{\ell-1} r + a_{\ell} = 0$$

for some $\ell \in \mathbb{N}_{>0}$ and some $a_i \in \mathfrak{a}^i$, $i = 1, \dots, \ell$ (see [29, Section 1.1] for more details and properties). If $\mathfrak{a}$ and $\mathfrak{b}$ are two $\mathfrak{m}$-primary ideals with $\overline{\mathfrak{a}} = \overline{\mathfrak{b}}$, then $e_R(\mathfrak{a}) = e_R(\mathfrak{b})$ (see [29, Proposition 11.2.1]). What can be said if $e_R(\mathfrak{a}) = e_R(\mathfrak{b})$? The following theorem of Rees settles this question.

Theorem 11.2.5 *[24], [29, Theorem 11.3.1] Let* $(R, \mathfrak{m})$ *be a formally equidimensional Noetherian local ring and let* $\mathfrak{a} \subset \mathfrak{b}$ *be two* $\mathfrak{m}$*-primary ideals. Then* $\mathfrak{b} \subset \overline{\mathfrak{a}}$ *if and only if* $e_R(\mathfrak{a}) = e_R(\mathfrak{b})$.

In a Noetherian ring, if $\mathfrak{a} \subset \mathfrak{b}$ and $\mathfrak{b} \subset \overline{\mathfrak{a}}$ then $\mathfrak{a}$ is a *reduction of* $\mathfrak{b}$. See [29, Chapter 8] for further details.

There is a similar statement as that of Theorem 11.2.5 for ideals that are not primary to the maximal ideal of the local ring $(R, \mathfrak{m})$. But before stating that theorem we need another definition.

Definition 11.2.6 Let $(R, \mathfrak{m})$ be a local Notherian ring and let $I \subset R$ be an ideal. The *analytic spread of* I, $\ell(I)$, is defined as the Krull dimension of the ring $R[It]/\mathfrak{m}R[It]$, where t is an indeterminate. This is the same as the Krull dimension of the ring

$$\mathrm{Gr}_I(R) \otimes k(\mathfrak{m}) = \bigoplus_{i=0}^{\infty} I^i/(I^i\mathfrak{m}).$$

Observe that the analytic spread of I is the dimension of the fiber over $\mathfrak{m}$ of the blow up of R at I.

See [9] for the generalization of the notion of analytic spread to arbitrary filtrations of ideals and some advances about this invariant.

The following theorem of Böger generalizes Rees's Theorem.

Theorem 11.2.7 *[6], [29, Corollary 11.3.2] Let* $(R, \mathfrak{m})$ *be a Noetherian formally equidimensional local ring, and let* $I \subseteq J$ *be two ideals with* $\ell(I) = \mathrm{ht}(I)$*. Then* $J \subset \overline{I}$ *if and only if* $e_{R_{\mathfrak{p}}}(I_{\mathfrak{p}}) = e_{R_{\mathfrak{p}}}(J_{\mathfrak{p}})$ *for every prime ideal* $\mathfrak{p}$ *minimal over* I*.*

In the following lines we mention a couple of properties of the multiplicity: The additivity (Theorem 11.2.8) and the behavior under flat extensions Sect. 11.2.2.

Theorem 11.2.8 *[29, Theorem 11.2.4] Let* $(R, \mathfrak{m})$ *be a local Noetherian ring, let* $\mathfrak{a}$ *be an* $\mathfrak{m}$*-primary ideal, and let* $\mathscr{P}$ *be the set of minimal prime ideals* $\mathfrak{p}$ *of* R *such that* $\dim(R/\mathfrak{p}) = \dim R$*. Then*

$$e_R(\mathfrak{a}) = \sum_{\mathfrak{p}\in\mathscr{P}} e_{R/\mathfrak{p}}(\mathfrak{a}).$$

Geometrically, Theorem 11.2.8 says that the multiplicity at a point of an equidimensional algebraic variety is the sum of the multiplicities at each of the irreducible components containing the point. If the variety is not equidimensional, then the only additions to the multiplicity at a given point come from the irreducible components of maximum dimension that contain the point.

11.2.2 The Multiplicity and Flat Extensions

Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal and suppose that $(R, \mathfrak{m}, k) \to (R', \mathfrak{m}', k')$ is a flat extension of local Noetherian rings. Then it can be checked that

$$e_{R'}(\mathfrak{a}R') = e_R(\mathfrak{a}R) \cdot \lambda_{R'}(R'/\mathfrak{m}R'). \tag{11.3}$$

See [14, Chapter I, Proposition 5.1].

From there it follows that if $(R', \mathfrak{m}', k')$ is an étale extension of $(R, \mathfrak{m}, k)$ then $e_R(\mathfrak{m}) = e_{R'}(\mathfrak{m}')$. And the same equality holds if R' is the $\mathfrak{m}$-adic completion of R, i.e., $e_R(\mathfrak{m}) = e_{\hat{R}}(\hat{\mathfrak{m}})$.

11.2.3 Shortcuts for the Computation of the Multiplicity

We already saw how the multiplicity at a point of a hypersurface in a regular ring is related to the order of the ideal of definition. In general, there is no such straightforward procedure to compute this invariant. However, there are some cases in which the calculation becomes easier.

Suppose that $(R, \mathfrak{m}, k)$ is a Noetherian local ring of Krull dimension d, and let $\mathfrak{a}$ be an *ideal of definition of* R, that is, an $\mathfrak{m}$-primary ideal generated by d elements, $\mathfrak{a} = \langle a_1, \ldots, a_d \rangle$. Then:

$$e_R(\mathfrak{a}) \leqslant \lambda_R(R/\mathfrak{a}), \tag{11.4}$$

and the equality holds if and only if R is Cohen-Macaulay. This follows from considering the morphism of $R/\mathfrak{a}$-graded rings:

$$\begin{aligned} \varphi : (R/\mathfrak{a})[X_1, \ldots, X_d] &\to \mathrm{Gr}_{\mathfrak{a}}(R) = R/\mathfrak{a} \oplus \mathfrak{a}/\mathfrak{a}^2 \oplus \ldots \oplus \mathfrak{a}^n/\mathfrak{a}^{n+1} \oplus \ldots \\ X_i &\mapsto \bar{a}_i \in \mathfrak{a}/\mathfrak{a}^2. \end{aligned}$$

Since φ is surjective,

$$\lambda_R(\mathfrak{a}^j/\mathfrak{a}^{j+1}) \leqslant \lambda_R\left((R/\mathfrak{a})[X_1, \ldots, X_d]_j\right),$$

where $(R/\mathfrak{a})[X_1, \ldots, X_d]_j$ denotes the j-th degree piece of the graded $R/\mathfrak{a}$-algebra $(R/\mathfrak{a})[X_1, \ldots, X_d]$. Therefore,

$$\lambda_R(R/\mathfrak{a}^{j+1}) \leqslant \lambda_R(R/\mathfrak{a})\binom{j+d}{d}.$$

Notice that if R is Cohen-Macaulay then φ is an isomorphism and the equality in (11.4) holds. For a complete proof of this fact see [27, Theorem 19.3.11]. See also [29, Proposition 11.1.10].

The following statement gives a criterion for algorithmic computation of the multiplicity in the case of k-algebras of finite type, with k a field. For a given monomial order $>$ in a polynomial ring $k[X_1, \ldots, X_r]$ and for an ideal $I \subset k[X_1, \ldots, X_r]$, we use $L(I)$ to refer to the ideal generated by the leading monomials of the non-zero elements in I with respect to $>$.

Proposition 11.2.9 *[13, Proposition 5.5.7] Let $>$ be a local degree ordering on $k[\underline{X}] = k[X_1, \ldots, X_r]$ (that is, $deg(X^\alpha) > deg(X^\beta)$ implies $X^\alpha < X^\beta$). Let $I \subset \langle \underline{X} \rangle = \langle X_1, \ldots, X_r \rangle$ be an ideal and let $B = k[\underline{X}]_{\langle \underline{X} \rangle}/I$. Let $\mathfrak{m} = \langle \underline{X} \rangle / I$. Then*

$$HS_{B,\mathfrak{m}} = HS_{(k[\underline{X}]_{\langle \underline{X} \rangle}/L(I)), \langle \underline{X} \rangle}.$$

In particular, $k[\underline{X}]/I$ and $k[\underline{X}]_{\langle \underline{X} \rangle / L(I)}$ have the same multiplicity with respect to $\langle \underline{X} \rangle$.

11.2.4 Geometric Interpretation of the Multiplicity [14, Chapter I, pg. 15]

By examining the hypersurface case we can get some insight on the geometric meaning of the multiplicity. So let us consider a hypersurface $H \subset \mathbb{A}^n_k$ with defining ideal $\langle f \rangle \subset k[X_1, \ldots, X_n]$ and suppose that $\zeta \in H$ is the closed point with maximal ideal $\mathfrak{m} = \langle X_1, \ldots, X_n \rangle$. Assume that f is regular with respect to X_n, (i.e.,

$f(0, \ldots, 0, X_n) \neq 0$; this can always be assumed after a linear change of variables and a suitable finite extension of k). Then we can apply Weierstrass Preparation Theorem [13, Corollary 6.2.8] at the $\mathfrak{m}$-adic completion of $k[X_1, \ldots, X_n]$, and assume that, up to multiplication by a unit, f can be written as a polynomial in the variable X_n with coefficients in the ring $k[[X_1, \ldots, X_{n-1}]]$, i.e.,

$$f = X_n^r + a_1 X_n^{r-1} + \ldots + a_r \tag{11.5}$$

with $a_i \in k[[X_1, \ldots, X_{n-1}]]$. Actually, Weierstrass Preparation Theorem holds at étale neighborhood of the local ring at ζ (see [25, Theorem 6.7]), thus the expression (11.5) can also be seen as a polynomial in X_n with coefficients in some regular local ring S of dimension $n-1$, with $S[X_n]$ an étale extension of $k[X_1, \ldots, X_n]$.

Hence, after a convenient étale neighborhood of ζ is selected, we can assume that the coordinate ring of H is isomorphic to

$$B = S[X_n]/\langle X_n^r + a_1 X_n^{r-1} + \ldots + a_r \rangle.$$

Observe that B is a finite extension of S and if $K(S)$ is fraction field of S then the *generic rank* of the extension $S \subset B$ is given by $[B \otimes_S K(S) : K(S)] = r$.

Letting $\mathfrak{m}_B = \mathfrak{m}/\langle f \rangle$ we have that

$$e_B(\mathfrak{m}_B) = \nu_{\mathfrak{m}}(f) \leqslant r = [B \otimes_S K(S) : K(S)]. \tag{11.6}$$

From here it follows that, in a neighborhood of ζ, H cannot be finitely projected to a regular variety $Z = \mathrm{Spec}(S)$ with generic rank lower than $e_B(\mathfrak{m}_B)$.

Notice that, in the previous discussion, the maximal ideal $\mathfrak{m}_B \subset B$ dominates a maximal ideal $\mathfrak{m}_S \subset S$ and therefore $\mathfrak{m}_S B$ generates an $\mathfrak{m}_B$-primary ideal $I \subset B$. As we will see in the next section, the following inequality holds:

$$e_B(I) \leqslant [B \otimes_S K(S) : K(S)]. \tag{11.7}$$

In fact, we will see that the previous inequalities hold for the localization at a point of the affine coordinate ring B of an arbitrary variety over a field k. Recall that by Noether's Normalization Lemma the k-algebra B can be expressed as a finite extension of a polynomial ring over k.

However:

- it is not obvious that the generic rank of such an extension be bounded below by the maximum multiplicity of B, and
- it is not immediate either that one can find suitable finite projections to regular rings where the equality in (11.7) holds.

All these questions will be properly addressed in the next section.

11.3 Zariski's Multiplicity Formula

The starting point of this section is precisely the last discussion from the previous one. There, we were trying to understand the geometric meaning of the multiplicity of a local Noetherian ring $(R, \mathfrak{m})$ at an $\mathfrak{m}$-primary ideal $I \subset R$. To this end we want to consider finite extensions $S \subset R$ where S is a local regular ring. With this objective in mind, the goal of this section is twofold. On the one hand, we will see that under mild assumptions the inequality (11.7) holds. This will be a consequence of Zariski's multiplicity formula stated in Theorem 11.3.1 below. On the other hand, we will discuss the problem of finding a suitable finite extension $S \subset R$ so that the equality in (11.7) holds. This will lead us to the notion of *finite-transversal morphisms* which will be discussed in the second part of the section.

11.3.1 *Zariski's Multiplicity Formula for Finite Projections*

Our purpose is to study Zariski's multiplicity formula, which relates multiplicities in a finite extension of rings.

Let $A \subset B$ be a finite extension of rings. If A is local with maximal ideal $\mathfrak{M}$ then B is semi-local (see for instance [31, Th. 15, page 276]). Denote by $Q_1, \ldots, Q_r$ the maximal ideals of B. Note that the set $\{Q_1, \ldots, Q_r\}$ corresponds to the fiber over $\mathfrak{M}$ of the finite morphism,

$$\operatorname{Spec}(B) \to \operatorname{Spec}(A).$$

As we will see, Zariski's formula relates the multiplicity of A at $\mathfrak{M}$ to the multiplicities of the local rings B_{Q_i}, $i = 1, \ldots, r$, at the extension of the ideal $\mathfrak{M}B_{Q_i}$.

Theorem 11.3.1 *[31, Theorem 24, page 297 and Corollary 1, page 299] With the previous assumptions and notation, suppose furthermore that $(A, \mathfrak{M})$ is a Noetherian local domain, that B is equidimensional, and that no non-zero element of A is a zero divisor in B. Denote by $K = K(A)$ the quotient field of A, and let $L = K \otimes_A B$. Let k_0 be the residue field of A, and let k_i be the residue field of B_{Q_i}, $i = 1, \ldots, r$. Then:*

$$[L : K]e_A(\mathfrak{M}) = \sum_{i=1}^{r} [k_i : k_0] e_{B_{Q_i}}(\mathfrak{M}B_{Q_i}).$$

Note that from our assumptions $\dim(B_{Q_i}) = \dim(A) = d$ for all $i = 1, \ldots, r$, and hence all the Hilbert polynomials that are involved in the formula have degree d.

Example 11.3.2 Let $A = k[Y]_{\langle Y\rangle}$ be the localization of the polynomial ring in one variable at the maximal ideal $\langle Y\rangle$. Here $\mathfrak{M} = \langle Y\rangle A$. Let $B = A[X]/\langle f\rangle$ where $f = X^a(X^2+1)^b + Y^c$ and $c > \max\{a, b\}$.

The extension $A \subset B$ is finite, and the generic rank is $[K(A) \otimes_A B : K(A)] = a + 2b = \deg_X(f)$. Since A is regular, we have that $e_A(\mathfrak{M}) = 1$.

Assume that $k = \mathbb{R}$. Then there are two maximal ideal ideals in B, Q_1 and Q_2, corresponding to $\langle x, y\rangle$ and $\langle x^2+1, y\rangle$, respectively. The residue field of B_{Q_1} is again $\mathbb{R}$ and the residue field of B_{Q_2} is $\mathbb{C}$. Hence Zariski's multiplicity formula is expanded as follows:

$$(a+2b)\cdot 1 = [L:K]e_A(\mathfrak{M}) =$$

$$= [k_1 : k_0]e_{B_{Q_1}}(\mathfrak{M}B_{Q_1}) + [k_2 : k_0]e_{B_{Q_2}}(\mathfrak{M}B_{Q_2}) = 1\cdot a + 2\cdot b.$$

Note that $\mathfrak{M}B_{Q_i}$ is a reduction of $Q_iB_{Q_i}$, for $i = 1, 2$, therefore $e_{B_{Q_i}}(\mathfrak{M}B_{Q_i}) = e_{B_{Q_i}}(Q_iB_{Q_i})$, and by Remark 11.2.4 $e_{B_{Q_i}}(Q_iB_{Q_i}) == \nu_{k[X,Y]_{\mathfrak{q}_i}}(f)$, where $\mathfrak{q}_i/\langle f\rangle = Q_i$.

If the ground field is $k = \mathbb{C}$ then B has three maximal ideals, Q_1, Q_2' and Q_2'', all whose residue fields are isomorphic to $\mathbb{C}$. In this case Zariski's multiplicity formula splits to

$$(a+2b)\cdot 1 = [L:K]e_A(\mathfrak{M}) =$$

$$= [k_1 : k_0]e_{B_{Q_1}}(\mathfrak{M}B_{Q_1}) + [k_2' : k_0]e_{B_{Q_2'}}(\mathfrak{M}B_{Q_2'}) + [k_2'' : k_0]e_{B_{Q_2''}}(\mathfrak{M}B_{Q_2''}) =$$

$$= 1\cdot a + 1\cdot b + 1\cdot b.$$

The multiplicities involved can be computed with the same argument as above.

11.3.2 Finite-Transversal Projections

A first consequence of Zariski's multiplicity formula (Theorem 11.3.1) is that we can generalize inequality (11.6) to the non hypersurface case, and even for a wider class of rings (not only those of finite type over a field k):

Assume that S is a regular local ring and that $S \subset B$ is a finite extension under the assumptions of Theorem 11.3.1. Then, for any prime ideal $P \in \operatorname{Spec}(B)$

$$e_{B_P}(PB_P) \leqslant [L:K],$$

where $K = K(S)$ and $L = K \otimes_S B$.

Finite extensions $S \subset B$ where the equality holds will be said to be *transversal* with respect to P. In fact we state the following more general definition:

Definition 11.3.3 Let $S \subset B$ be a finite extension of excellent Noetherian rings with S regular and B equidimensional. Suppose that no non-zero element of S is a zero divisor in B. We say that the projection $\text{Spec}(B) \to \text{Spec}(S)$ (or the extension $S \subset B$) is *finite-transversal with respect to* $P \in \text{Spec}(B)$ if

$$e_{B_P}(PB_P) = [L : K].$$

Let $I \subset B$ be a P-primary ideal. We say that the projection $\text{Spec}(B) \to \text{Spec}(S)$ (or the extension $S \subset B$) is *finite-transversal with respect to the ideal* I if

$$e_{B_P}(IB_P) = [L : K],$$

and $I \cap S$ is a prime ideal of S.

Note that the conditions in Definition 11.3.3 imply that the associated primes of B are exactly the minimal primes of B, $\text{Ass}(B) = \text{Min}(B)$.

A second consequence of Theorem 11.3.1 is the following equivalence, which gives a characterization of finite-transverval projections:

Proposition 11.3.4 *[30, Corollary 4.9] Let S be regular ring and let $S \subset B$ be a finite extension. Suppose that B is Noetherian, excellent and equidimensional and that the non-zero elements of S are non-zero divisors in B.*

Let $P \in \text{Spec}(B)$ be a prime ideal of B and $I \subset B$ be a P-primary ideal. Set $\mathfrak{p} = P \cap S$. The following are equivalent:

- *(i) $e_{B_P}(IB_P) = [L : K]$ and $I \cap S = \mathfrak{p}$ is a prime ideal in S, i.e., the extension $S \subset B$ is finite-transversal w.r.t. I.*
- *(ii) The following three conditions hold:*
    - *(i) P is the only prime of B dominating $\mathfrak{p}$,*
    - *(ii) $k(P) = B_P/PB_P = S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}} = k(\mathfrak{p})$,*
    - *(iii) $\mathfrak{p}B_P$ is a reduction of the ideal IB_P.*

We will refer to (i)–(iii) as *Zariski's conditions* of the finite extension $S \subset B$ with respect to the P-primary ideal I.

Note that Proposition 11.3.4 is stated in [30, Corollary 4.9] for $I = P$ a prime ideal, but the generalization to primary ideals is straightforward.

Example 11.3.5 Let $B = k[X, Y]/\langle X^2 - Y^3\rangle$. We can consider two finite projections $\text{Spec}(B) \to \text{Spec}(S_i)$, $i = 1, 2$:

(a) $S_1 = k[Y] \subset B$, and
(b) $S_2 = k[X] \subset B$.

Let $P = \langle x, y\rangle \subset B$ and note that $e_{B_P}(PB_P) = 2$.

The finite extension in (a) is finite-transversal with respect to P, while the finite extension in (b) is not because the generic rank of $S_2 \subset B$ is 3. However, extension (b) is finite-transversal with respect to the P-primary ideal $\langle x\rangle B$.

On the other hand, note that almost any linear projection from $\text{Spec}(B)$ to a one dimensional regular linear subvariety of $\mathbb{A}^2_k$ is finite-transversal with respect to P.

11.3.3 Do Finite-Transversal Projections Exist in General?

Given a Noetherian, excellent and equidimensional ring B, a point $P \in \mathrm{Spec}(B)$ and a P-primary ideal I, we wonder if there exist a regular ring S and a finite-transversal projection w.r.t. I, $\mathrm{Spec}(B) \to \mathrm{Spec}(S)$.

The answer, in general, is negative, even for k-algebras of finite type over a field k. Note that if such projection exists then the ideal I has a reduction with $d = \dim(B)$ elements.

This last observation gives a necessary condition not always satisfied as the following example illustrates.

Example 11.3.6 Let $k = \mathbb{F}_2$ and let

$$B = k[X, Y]/\langle XY(X + Y)\rangle.$$

Then the ideal $\langle x, y\rangle \subset B$ has no reductions generated by one element, see [29, Example 8.3.2]. This means that for B and the maximal ideal $I = \langle x, y\rangle$, there does not exist a finite-transversal projection w.r.t. I.

11.3.4 Construction of Finite-Transversal Projections

If $(B, \mathfrak{m})$ is a local complete Noetherian, equidimensional ring containing an infinite coeficient field, such that $\mathrm{Ass}(B) = \mathrm{Min}(B)$, then the answer to (Sect. 11.3.3) is positive. Let $I \subset B$ be a $\mathfrak{m}$-primary ideal. Since the residue field is infinite, by Swanson and Huneke [29, Proposition 8.3.7] there exists a reduction of I generated by $d = \dim(B)$ elements, $x_1, \ldots, x_d$. Choose a coefficient field $k' \subset B$ and set $S = k'[[x_1, \ldots, x_d]] \subset B$. Note that since $x_1, \ldots, x_d$ are analytically independent, S is a ring of power series in d variables. The extension $S \subset B$ is finite by Cohen [8, Theorem 8, page 68] and we conclude that the projection $\mathrm{Spec}(B) \to \mathrm{Spec}(S)$ is finite-transversal with respect to $\mathfrak{m}$. See also [15, Proof of Theorem 1.1].

If B is a k-algebra of finite type, then Noether's normalization Lemma seems to provide a possible approach to address Sect. 11.3.3. We could find a regular ring S and a finite extension $S \subset B$, but Noether's normalization is not enough to guarantee Zariski's conditions (i)–(iii) in Proposition 11.3.4.

However, (not necessarily finite) morphisms for which conditions (ii) and (iii) hold are not hard to construct. As we will see, this will be a consequence of applying Noether's normalization to the graded ring $\mathrm{Gr}_{IB_P}(B_P)$. This motivates the following definition, which is a weaker version of the notion of finite-transversal projection.

Definition 11.3.7 Let $S \subset B$ be a (possibly non-finite) extension of Noetherian rings, with S regular and B equidimensional. Let $P \in \mathrm{Spec}(B)$ be a prime ideal and let $I \subset B$ be a PB_P-primary ideal.

We say that the projection $\mathrm{Spec}(B) \to \mathrm{Spec}(S)$ is *local-transversal with respect to the ideal I* if

(i) $\mathfrak{p} = I \cap S$ is a prime ideal and P is an isolated point in the fiber over $\mathfrak{p}$,
(ii) $k(P) = B_P/PB_P = S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}} = k(\mathfrak{p})$,
(iii) $\mathfrak{p}B_P$ is a reduction of the ideal IB_P.

Given a k-algebra B of finite type and a P-primary ideal I, we want to show that, locally for the étale topology, there exists a projection to a regular ring S which is local-transversal w.r.t. to I. This is achieved by applying the following result.

Theorem 11.3.8 *[14, Th. 10.14, page 60] Let $(B, \mathfrak{m})$ be a Noetherian local ring and let $\mathfrak{a}$ be an ideal of B. If $a_1, \dots, a_s \in \mathfrak{a}$, then the following conditions are equivalent:*

(i) $a_1, \dots, a_s \in \mathfrak{a}$ generate a reduction of $\mathfrak{a}$.
(ii) The quotient ring

$$(\mathrm{Gr}_{\mathfrak{a}}(B) \otimes_B B/\mathfrak{m}) \big/ \langle \tilde{a}_1, \dots, \tilde{a}_s \rangle$$

has Krull dimension zero, where $\tilde{a}$ is the class of a in $\mathfrak{a}/(\mathfrak{a}\mathfrak{m})$.

Proposition 11.3.9 *[7, Proposition 34.1] Let B an equidimensional k-algebra of finite type, where k is a perfect field, let $\mathfrak{m} \subset B$ be a maximal ideal, and let I be a $\mathfrak{m}$-primary ideal. Then there exists an étale extension $\lambda : B \to B'$, a maximal ideal $\mathfrak{m}' \in \mathrm{Spec}(B')$ dominating $\mathfrak{m}$, and a local-transversal projection w.r.t. $IB'_{\mathfrak{m}'} \cap B'$, $\beta : S \to B'$,*

$$\begin{array}{ccc} B & \xrightarrow{\lambda} & B' \\ & & \uparrow \beta \\ & & S. \end{array}$$

Proposition 11.3.9 can be read as saying that, after an étale extension $B \to B'$, there exist a local-transversal projection w.r.t. IB', $S \subset B'$.

Proof First, we can assume that $\mathfrak{m}$ is a rational closed point in $\mathrm{Spec}(B)$. To do so, let $k_1 = B/\mathfrak{m}$ be the residue field of B at set $B_1 = B \otimes_k k_1$. Now choose a maximal ideal $\mathfrak{m}_1 \in \mathrm{Spec}(B_1)$ mapping to $\mathfrak{m}$, and replace B and $\mathfrak{m}$ by B_1 and $\mathfrak{m}_1$.

Next, we want to construct a reduction of $IB_{\mathfrak{m}_1}$ generated by $d = \dim((B_1)_{\mathfrak{m}_1})$ elements $y_1, \dots y_d \in \mathfrak{m}_1$. Such a reduction exists, in the local ring, if the residue field is infinite, see [29, Proposition 8.3.7], but we can avoid this hypothesis enlarging k_1 if necessary. Set $R_1 = (B_1)_{\mathfrak{m}_1}$. Then the graded ring $\mathrm{Gr}_{IR_1}(R_1)$ is a k_1-algebra of finite type of dimension d (see [29, Proposition 5.1.6]). After considering a finite extension of the base field $k_1 \subset k_2$ (if needed) we can apply the graded version of Noether's normalization Lemma ([29, Theorem 4.2.3]): Set $B_2 = B_1 \otimes_{k_1} k_2$ and let $\mathfrak{m}_2 \in \mathrm{Spec}(B_2)$ be a maximal ideal mapping to $\mathfrak{m}_1$. If

$R_2 = (B_2)_{\mathfrak{m}_2}$, then there are degree one elements $\bar{y}_1, \ldots, \bar{y}_d \in \mathrm{Gr}_{IR_2}(R_2)$ such that $k_2[\bar{y}_1, \ldots, \bar{y}_d]$ is isomorphic to the polynomial ring of d variables and

$$k_2[\bar{y}_1, \ldots, \bar{y}_d] \subset \mathrm{Gr}_{IR_2}(R_2) = \mathrm{Gr}_{I(B_2)_{\mathfrak{m}_2}}\left((B_2)_{\mathfrak{m}_2}\right)$$

is finite.

Choose representatives $y_1, \ldots, y_d \in I\,(B_2)_{\mathfrak{m}_2}$ of $\bar{y}_1, \ldots, \bar{y}_d$. By Theorem 11.3.8 we conclude that $y_1, \ldots, y_d$ generate a reduction of $I\,(B_2)_{\mathfrak{m}_2}$. Select some $f \in B_2$ so that $y_1, \ldots, y_d \in (B_2)_f$. Finally $B' = (B_2)_f$, $\mathfrak{m}' = \mathfrak{m}_2$, and $S_2 = k_2[y_1, \ldots, y_d]$ give the required extension, local-transversal w.r.t. $IB'_{\mathfrak{m}'} \cap B'$. See [7, 34.3] for complete details. □

Remark 11.3.10 Note that after following the proof, the statement of Proposition 11.3.9 can reformulated as follows. There exists a finite extension $k \subset k'$, an element $f \in B \otimes_k k'$, and a maximal ideal $\mathfrak{m}' \subset B' = (B \otimes_k k')_f$, together with a smooth k'-algebra of finite type S and morphisms of finite type λ and β

$$\begin{array}{ccc} B & \xrightarrow{\lambda} & B' \\ & & \uparrow \beta \\ & & S, \end{array}$$

such that the projection β is local-transversal w.r.t. $IB'_{\mathfrak{m}'}$. Moreover S can be chosen to be a polynomial ring with d variables over k'.

Example 11.3.11 Let $B = k[X, Y]/\langle h\rangle$, where $h = X^2(X^2 + 1) + Y^5$. Let $I = \mathfrak{m} = \langle x, y\rangle \subset B$.

Note that the graded ring of $B_{\mathfrak{m}}$ at $IB_{\mathfrak{m}}$ is $\mathrm{Gr}_{IB_{\mathfrak{m}}}(B_{\mathfrak{m}}) = k[X, Y]/\langle X^2\rangle$, and the ideal $\langle y\rangle B_{\mathfrak{m}}$ is a reduction of $IB_{\mathfrak{m}}$ by Theorem 11.3.8. If $S = k[Y]$, then $S \subset B$ is local-transversal w.r.t. I.

The extension $S \subset B$ is finite but the generic rank is 4 and multiplicity of $B_{\mathfrak{m}}$ is 2. This implies that $S \subset B$ is not finite-transversal w.r.t. I.

One could consider the localization at $f = X^2 + 1$ in order to have condition (i) in Proposition 11.3.4, but then $S \subset B_f$ is not a finite extension. However, as we will see in Example 11.3.15, if we consider a convenient étale extension of S then a finite-transversal projection can be constructed.

The next lemma guarantees that the local-transversal condition is stable under étale base changes.

Lemma 11.3.12 *[7, Corollary 34.6, Example 34.8] Assume that $S \subset B$ is local-transversal w.r.t. an $\mathfrak{m}$-primary ideal I. If $S \to C$ is an étale extension and $\mathfrak{m}' \subset B \otimes_S C$ dominates $\mathfrak{m}$, then*

$$C \to B \otimes_S C$$

is again local-transversal w.r.t. $I' = I(B \otimes_S C)_{\mathfrak{m}'} \cap (B \otimes_S C)$.

Theorem 11.3.13 *[7, Proposition 31.1] Let k be a perfect field and let B be an equidimensional k-algebra of finite type with $Ass(B) = Min(B)$. Let $\mathfrak{m} \in \operatorname{Spec}(B)$ be a maximal ideal and let I be an $\mathfrak{m}$-primary ideal. Then there exist k-algebras B' and S, morphisms of finite type λ and β,*

$$\begin{array}{ccc} B & \xrightarrow{\lambda} & B' \\ & & \uparrow \beta \\ & & S', \end{array}$$

and a maximal ideal $\mathfrak{m}' \in \operatorname{Spec}(B')$, such that

(i) λ is an étale morphism, and $\mathfrak{m} = \mathfrak{m}' \cap B$,
(ii) S' is a regular ring,
(iii) β is finite-transversal w.r.t. $IB'_{\mathfrak{m}'}$.

Moreover, if $S \subset B$ is a local-transversal projection w.r.t. I, then the extension $S' \subset B'$ can be obtained by pull-back of a suitable étale map $S \to S'$ and a localization at some $f \in B \otimes_S S'$,

$$\begin{array}{ccc} B & \xrightarrow{\lambda} & B' = (B \otimes_S S')_f \\ \uparrow & & \uparrow \beta \\ S & \longrightarrow & S'. \end{array}$$

Theorem 11.3.13 is sketched in [30, 6.11] when I is a maximal ideal and full details of the proof are given in [7, Appendix A]. We reproduce here that proof to check that it also holds for an arbitrary $\mathfrak{m}$-primary ideal and to illustrate such a construction with an example, see Example 11.3.15. One of the main ingredients of the proof is Zariski's Main Theorem:

Theorem 11.3.14 *[19, Theorem 1, page 41] Let $S \subset B$ be a ring extension, and assume that B is an S-algebra of finite type. Let $A \subset B$ be the integral closure of S in B. Let $P \in \operatorname{Spec}(B)$ be a prime ideal and set $\mathfrak{n} = P \cap S$. If P is an isolated point of the fiber over $\mathfrak{n}$ then there exists $f \in A$, $f \notin P$ such that $A_f = B_f$.*

In other words, if S is essentially of finite type over a field k, Theorem 11.3.14 is saying that the (non necessarily finite) S-algebra B is, locally at P, isomorphic to a localization of an algebra which is finite over S.

Proof of Theorem 11.3.13 By Theorem 11.3.9 we can assume that there exists a local-transversal projection $S \subset B$ w.r.t. I. In fact, it comes from the proof of Theorem 11.3.13 that S can be assumed to be a polynomial ring.

First, by Zariski's Main Theorem 11.3.14, there exists $f \in A$, such that $A_f = B_f$, where A is the integral closure of S in B. Set $\mathfrak{n} = I \cap S \in \operatorname{Spec}(S)$ and let

$\mathfrak{p}_1, \ldots, \mathfrak{p}_s \in \mathrm{Spec}(A)$ be all the maximal ideals dominating $\mathfrak{n}$. We may assume that $\mathfrak{m} \cap A = \mathfrak{p}_1$.

Moreover, choosing $g \in (\mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_s) \setminus \mathfrak{p}_1$, we have that $\mathfrak{m}B_{fg}$ is the only maximal ideal in B_{fg} dominating $\mathfrak{n}$.

This means that the extension $S \subset B_{fg}$ fulfills properties (i), (ii) and (iii) in Proposition 11.3.4, but the ring extension might not be finite. We will use Lemma 11.3.12 in order to prove that the Theorem 11.3.13 holds after an étale base change extension.

Consider the henselization $\tilde{S}$ of $(S)_{\mathfrak{n}}$ (see [25, Appendix C]), where $\mathfrak{n} = I \cap S$, and the diagram:

$$\begin{array}{ccccccccc}
\mathfrak{m} & & B & \longrightarrow & B \otimes_S S_{\mathfrak{n}} & \longrightarrow & B' = B \otimes_S \tilde{S} & & \mathfrak{m}' = \mathfrak{m}'_1, \ldots, \mathfrak{m}'_{t'} \\
& & \uparrow & & \uparrow & & \uparrow & & \\
\mathfrak{p} = \mathfrak{p}_1, \ldots, \mathfrak{p}_s & & A & \longrightarrow & A \otimes_S S_{\mathfrak{n}} & \longrightarrow & A' = A \otimes_S \tilde{S} & & \mathfrak{p}' = \mathfrak{p}'_1, \ldots, \mathfrak{p}'_{s'} \\
& & \uparrow & & \uparrow & & \uparrow & & \\
\mathfrak{n} & & S & \longrightarrow & S_{\mathfrak{n}} & \longrightarrow & \tilde{S} & & \mathfrak{n}',
\end{array}$$

where $\mathfrak{n}'$ is the maximal ideal of the local ring $\tilde{S}$; note that A' (resp. B') is a semi-local ring and we are denoting by $\mathfrak{p}'_1, \ldots, \mathfrak{p}'_{s'}$ (resp. $\mathfrak{m}'_1, \ldots, \mathfrak{m}'_{t'}$) the maximal ideals dominating $\mathfrak{n}'$. Assume that $\mathfrak{p}'$ dominates $\mathfrak{p}$ and that $\mathfrak{m}'$ dominates $\mathfrak{m}$. Since $\tilde{S}$ is henselian we have that

$$A' = A'_{\mathfrak{p}'_1} \oplus \ldots \oplus A'_{\mathfrak{p}'_{s'}},$$

and each direct summand is finite over $\tilde{S}$. In particular $\tilde{S} \to A'_{\mathfrak{p}'}$ is finite.

By the choice of $g \in A$ it follows that $\mathfrak{p}$ is the only point in the fiber over $\mathfrak{n}$ of $S \to A_g$. Therefore by Bravo and Villamayor [7, Lemma 34.5] $\mathfrak{p}'$ is the only point over $\mathfrak{p}$ of $A_g \to A'_g$,

$$\begin{array}{ccccccccc}
\mathfrak{p} = \mathfrak{p}_1 & & A_g & \longrightarrow & A_g \otimes_S S_{\mathfrak{n}} & \longrightarrow & A'_g = A_g \otimes_S \tilde{S} = A'_{\mathfrak{p}'} & & \mathfrak{p}' = \mathfrak{p}'_1 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
\mathfrak{n} & & S & \longrightarrow & S_{\mathfrak{n}} & \longrightarrow & \tilde{S} & & \mathfrak{n}'.
\end{array}$$

Since fg is invertible in $A'_{\mathfrak{p}'}$ then there exists an integral equation

$$((fg)^{-1})^n + d_1((fg)^{-1})^{n-1} + \ldots + d_{n-1}(fg)^{-1} + d_n = 0,$$
$$d_i \in \tilde{S}, \quad \forall i = 1, \ldots, n. \tag{11.8}$$

Now consider a local étale neighborhood $\tilde{E}$ of $S_{\mathfrak{n}}$ containing the elements d_i, for $i = 1, \ldots, n$.

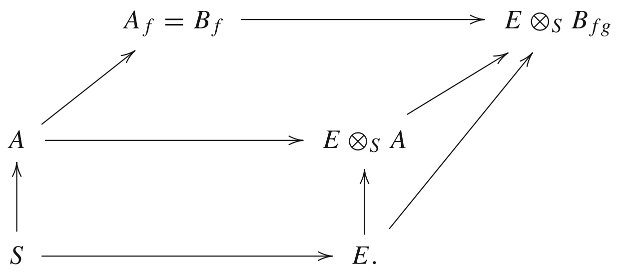
Observe that $\tilde{E} \subset \tilde{E} \otimes A_{fg}$ is a finite extension and there is a unique maximal ideal $\mathscr{P} \subset \tilde{E} \otimes A_{fg}$ dominating $\mathfrak{p}$. Therefore, $\tilde{E} \otimes A_{fg} = (\tilde{E} \otimes A)_{\mathscr{P}}$. And since $\tilde{E} \otimes A_{fg} \subset A'_{\mathfrak{p}'}$ is flat, the relation (11.8) also holds at $\tilde{E} \otimes A_{fg}$, and hence $\tilde{E} \otimes A_{fg}$ is finite over $\tilde{E}$.

Now note that,

$$\tilde{E} \otimes A_{fg} = \tilde{E} \otimes B_{fg} = \tilde{E} \otimes B_{\mathfrak{m}},$$

and therefore the extension $\tilde{E} \subset \tilde{E} \otimes B_{\mathfrak{m}}$ is finite-transversal w.r.t. $I' = I(\tilde{E} \otimes B_{\mathfrak{m}})$.

Finally observe that since $S_{\mathfrak{n}} \to \tilde{E}$ is local étale, then there exists an S-algebra of finite type E such that $S \to E$ is étale and such that $\tilde{E}$ is a localization of E (see [7, §32.4]). As consequence the finite extension is $E \to E \otimes_S B_{fg}$ is finite-transversal w.r.t. $I(E \otimes_S B_{fg})$. The following diagram summarizes the different extensions:



□

Example 11.3.15 Let us go back to Example 11.3.11 and let us assume now that the characteristic of k is different from 2. Let $\tilde{S} = k\{\{Y\}\}$ be the henselization of $k[Y]_{\langle Y \rangle}$. By Hensel's Lemma, the degree 4 polynomial $h = X^2(X^2+1)+Y^5$ factors as

$$h = h_1 \cdot h_2 = (X^2 + a_1 X + a_2)(X^2 + b_1 X + b_2) \tag{11.9}$$

where $a_1, a_2, b_1, b_2 \in \tilde{S}$ and such that $a_1, a_2, b_1, b_2 - 1 \in \langle Y \rangle$. A direct computation gives that

$$a_1 = 0, \quad b_1 = 0, \quad b_2 = 1 - a_2, \quad a_2^2 - a_2 + Y^5 = 0$$

Let $\alpha \in \tilde{S}$ be an element so that $\alpha^2 = \frac{1}{4} - Y^5$, for instance the power series

$$\alpha = \frac{1}{2}\sqrt{1 - 4Y^5} = \frac{1}{2} \sum_{i=0}^{\infty} \binom{1/2}{i} (-1)^i 4^i Y^{5i}.$$

Then $a_2 = \frac{1}{2} - \alpha$ and $b_2 = \frac{1}{2} + \alpha$ are solutions for the factorization (11.9).

Let $E = S[a_2, (2a_2 - 1)^{-1}]$ and note that $S \subset E$ is an étale extension. The extension

$$B \to B' = B \otimes_S E = E[X]/\langle X^2(X^2 + 1) + Y^5 \rangle$$

is also étale.

Let x be the class of X in B' and set $e = 1 + \frac{1}{1 - 2a_2}(x^2 + a_2)$. Note that $e^2 = e$, that the extension $E \subset B'_e$ is finite. Finally, we have that $E \subset B'_e$ is finite-transversal w.r.t. $\langle x, y \rangle B'_e$.

Another possibility is to consider an étale extension of B that contains a square root of $x^2 + 1$, see [7, §36] for further details.

11.4 Finite-Transversal Morphisms and Multiplicity

In this section we will focus on one of the main results of [30], stated below as Theorem 11.4.3. This theorem gives a procedure to describing the top multiplicity locus of a variety using finite-transversal projections. In this context, the notion of *algebraic presentations of a finite extension* plays a role (see Sect. 11.4.1). We discuss several applications and consequences of Theorem 11.4.3 in Sects. 11.4.2, 11.4.3 and 11.4.4. In addition, we present some results refining the number of generators needed for algebraic presentations in the context of finite-transversal morphisms (see Proposition 11.4.6). Finally, several examples are given in the hope that they help clarify some of the key ideas in the exposition.

We start with a generalization of a well known property of minimal polynomials for field extensions of the quotient field of an integrally closed domain. This result is essential in the exposition given in Sect. 11.4.1 which is a key step for understanding the statement of Theorem 11.4.3.

Proposition 11.4.1 *[30, Lemma 5.2] Let $S \subset B$ be a finite extension such that the non-zero elements of S are non-zero divisors in B. Assume that S is a regular ring and let $K = K(S)$ be the quotient field of S. Let $\theta \in B$ and let $f(Z) \in K[Z]$ be the monic polynomial of minimal degree such that $f(\theta) = 0$. If $S[\theta]$ denotes the S-subalgebra of B generated by θ, then*

(i) the coefficients of f are in S, $f(Z) \in S[Z]$, and
(ii) $S[\theta] \cong S[Z]/\langle f(Z) \rangle$.

Proof *(Comments on the Proof)* Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$ be the minimal primes of B. From the hypotheses we have that $\mathrm{Ass}(B) = \mathrm{Min}(B)$ and that $L = K \otimes_S B$ is the total ring of fractions of B. Then

$$L = L_1 \oplus \cdots \oplus L_m,$$

where L_i is a local Artinian ring for $i = 1, \dots, m$. Note that the minimal primes of L, $Q_1, \dots, Q_m$, are in one-to-one correspondence with $\mathfrak{q}_1, \dots, \mathfrak{q}_m$. Consider the following diagram,

$$\begin{array}{ccccccc}
S & \longrightarrow & B & & \longrightarrow & & B/\mathfrak{q}_1 \oplus \cdots \oplus B/\mathfrak{q}_m \\
\downarrow & & \downarrow{\scriptstyle\alpha} & & & & \downarrow{\scriptstyle\beta} \\
K & \longrightarrow & K \otimes_S B = L & \xrightarrow{\cong} & L_1 \oplus \cdots \oplus L_m & \longrightarrow & K(B/\mathfrak{q}_1) \oplus \cdots \oplus K(B/\mathfrak{q}_m) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
K & \longrightarrow & K[\alpha(\theta)] & \xrightarrow{\cong} & K[\alpha(\theta)_1] \oplus \cdots \oplus K[\alpha(\theta)_m] & \longrightarrow & K[\bar{\theta}_1] \oplus \cdots \oplus K[\bar{\theta}_m],
\end{array}$$

where $\bar{\theta}_i \in K(B/\mathfrak{q}_i)$ is the class of $\alpha(\theta)_i \in L_i$.

Observe that $f(Z)$ has been chosen such that $f(\alpha(\theta)) = 0 \in L$. Let $g_i(Z)$ be the minimal polynomial of $\bar{\theta}_i$ over K, for $i = 1, \dots, m$, and note that since S is normal, $g_i(Z) \in S[Z]$. Now we have that the irreducible factors of $f(Z)$ in $K[Z]$ are the $g_i(Z)$,

$$f(Z) = (g_1(Z))^{r_1} \cdots (g_m(Z))^{r_m}.$$

Hence $f(Z) \in S[Z]$.

Finally, since α is injective $f(\theta) = 0 \in B$. This gives a well defined morphism $S[Z]/\langle f(Z)\rangle \to S[\theta]$, which can be shown to be an isomorphism, see [30, page 342]. □

The following example illustrates the necessity of the hypothesis on the non-zero elements of S mapping to non-zero divisors in B.

Example 11.4.2 Let $S = k[X, Y]$ and let $B = k[X, Y, Z]/\langle (Z^2 + X^5)(Z + X^3), Y(Z^2 + X^5)\rangle$. The minimal primes of B are $\mathfrak{q}_1 = \langle z^2 + x^5\rangle$ and $\mathfrak{q}_2 = \langle y, z^2 + x^3\rangle$. We have that $K = K(S) = k(X, Y)$ and that $L = K \otimes_S B = K[Z]/\langle Z^2 + X^5\rangle$. The minimal polynomial of z over K is $f(Z) = Z^2 + X^5$. However $f(z)$ is not zero in B. In particular, $S[z] = B$ is not isomorphic to $S[Z]/\langle f(Z)\rangle$.

11.4.1 Algebraic Presentations of Finite Extensions

Let $S \subset B$ be a finite extension such that every non-zero element of S is not a zero-zivisor in B. Since the extension $S \subset B$ is finite, in particular of finite type, there are elements $\theta_1, \dots, \theta_e \in B$ such that $B = S[\theta_1, \dots, \theta_e]$. We will say that $S[\theta_1, \dots, \theta_e]$ is an *algebraic presentation* of the extension $S \subset B$.

Let $f_i(Z_i) \in K[Z_i]$ be the polynomial of minimal degree such that $f_i(\theta_i)) = 0$, $i = 1, \dots, e$. By Proposition 11.4.1, $f_i(Z_i) \in S[Z_i]$. Let $d_i = \deg(f_i(Z_i))$ be the degree of the polynomial $f_i(Z_i)$ for $i = 1, \dots, e$. We may assume that all $d_i \geqslant 2$, since otherwise $\theta_i \in S$. We have a diagram

$$\begin{array}{ccccc} & & B & & \\ & \nearrow & \uparrow & \nwarrow & \\ S[\theta_1] \cong S[Z_1]/\langle f_1(Z_1)\rangle & & \cdots & & S[\theta_e] \cong S[Z_e]/\langle f_e(Z_e)\rangle \\ & \nwarrow & \uparrow & \nearrow & \\ & & S. & & \end{array} \qquad (11.10)$$

We want to study such diagrams when $S \subset B$ is finite-transversal with respect to some prime P in B. This is the purpose of the next theorem.

Theorem 11.4.3 *[30, Proposition 5.7] Let B be an excellent and equidimensional ring, and let $S \subset B$ a finite extension such that every non-zero element of S is not a zero-zivisor in B. Fix an algebraic presentation $S[\theta_1, \ldots, \theta_e]$ of $S \subset B$. Let $\beta : \mathrm{Spec}(B) \to \mathrm{Spec}(S)$ and let $\beta_i : \mathrm{Spec}\,(S[Z_i]/\langle f_i(Z_i)\rangle) \to \mathrm{Spec}(S)$, for $i = 1, \ldots, e$. Suppose that the generic rank $n = [L : K] \geqslant 2$ and let $\mathfrak{p} \in \mathrm{Spec}(S)$. Then the following conditions are equivalent:*

- *The point $\mathfrak{p}$ is the image by β of a point of multiplicity n of $\mathrm{Spec}(B)$.*
- *For every $i = 1, \ldots, e$, the point $\mathfrak{p}$ is the image by β_i of a point of multiplicity d_i of $\mathrm{Spec}\,(S[Z]/\langle f_i(Z)\rangle)$.*

Theorem 11.4.3 has several interpretations and consequences, that we will describe in the next paragraphs.

11.4.2 Theorem 11.4.3 and Finite-Transversal Morphisms

With the notation and the hypotheses of the theorem, let $B_i = S[\theta_i] = S[Z_i]/\langle f_i(Z_i)\rangle$, for $i = 1, \ldots, e$. Then diagram (11.10) can be rewritten as

$$\begin{array}{ccccc} & & \mathrm{Spec}(B) & & \\ & \swarrow^{\alpha_1} & \downarrow^{\alpha_i} \;\; \big) & \searrow^{\alpha_e} & \\ \mathrm{Spec}(B_1) = \mathrm{Spec}\,(S[Z_1]/\langle f_1(Z_1)\rangle) & & \cdots \quad \beta & & \mathrm{Spec}(B_e) = \mathrm{Spec}\,(S[Z_e]/\langle f_e(Z_e)\rangle) \\ & \searrow_{\beta_1} & \downarrow_{\beta_i} \;\; \big) & \swarrow_{\beta_e} & \\ & & \mathrm{Spec}(S). & & \end{array} \qquad (11.11)$$

Now observe that the theorem says that the projection $\beta : \mathrm{Spec}(B) \to \mathrm{Spec}(S)$ is finite-transversal w.r.t. $P \in \mathrm{Spec}(B)$ if and only if all the projections $\beta_i : \mathrm{Spec}(B_i) \to \mathrm{Spec}(S)$ are finite-transversal w.r.t. $P_i = \alpha_i(P)$.

11.4.3 Theorem 11.4.3 and an Explicit Description of the Top Multiplicity Locus of Spec(B)

Let $F_n \subset \text{Spec}(B)$ be the set of points of multiplicity $n = [L : K]$, and assume that F_n is not empty. Then Theorem 11.4.3 gives us a procedure to describe F_n explicitly. To see this consider the following diagram

$$\begin{array}{ccccc} S[Z_1,\dots,Z_e] & \longrightarrow & C = S[Z_1,\dots,Z_e]/\langle f_1(Z_1),\dots,f_e(Z_e)\rangle & \longrightarrow & B \\ & & & & \uparrow \\ & & & & S. \end{array} \qquad S \to S[Z_1,\dots,Z_e], \quad S \to C \tag{11.12}$$

Observe that there is a closed embedding of Spec(B) in the regular scheme Spec$(S[Z_1,\dots,Z_e])$. Note that $\dim(B) = \dim(C)$ and that Spec(B) corresponds to the union of some irreducible components of Spec(C).

Let E_{d_i} be the set of points of multiplicity d_i of the hypersurface $\{f_i(Z_i) = 0\}$ in Spec$(S[Z_1,\dots,Z_e])$, for $i = 1,\dots,e$. The theorem says that

$$F_n = E_{d_1} \cap \cdots \cap E_{d_e}.$$

In other words, the top multiplicity locus of Spec(B) can be described as the top multiplicity locus of the complete intersection Spec(C). Note that the generic rank of $\beta' : \text{Spec}(C) \to \text{Spec}(S)$ is $d_1 \cdots d_e = \dim_K(C \otimes_S K)$ and that $n = \dim_K(B \otimes_S K)$ is the generic rank of $S \subset B$. Theorem 11.4.3 says that if $\mathfrak{p} \in \text{Spec}(S)$ then the following assertions are equivalent

- The point $\mathfrak{p}$ is the image by β of a point of multiplicity n of Spec(B).
- The point $\mathfrak{p}$ is the image by β' of a point of multiplicity $d_1 \cdots d_e$ of Spec(C).

If S is a k-algebra of finite type, with k a perfect field, then $S[Z_1,\dots,Z_e]$ is smooth over k, and the module of differential operators of order $\leqslant j$, $\text{Diff}_k^j(S[Z_1,\dots,Z_e])$, is free. Then we have that the closed set E_{d_i} is defined by the ideal

$$\left\langle \Delta(f_i(Z_i)) \mid \Delta \in \text{Diff}_k^{d_i-1}(S[Z_1,\dots,Z_e]) \right\rangle. \tag{11.13}$$

Actually, a similar description can also be given in a more general setting without assuming k to be perfect, see [2, Section 7] for details.

Finally, suppose we are give any equidimensional k-algebra of finite type B and let $P \in \text{Spec}(B)$ be a maximal ideal with multiplicity greater than one. Then, by Theorem 11.3.13, and after an étale extension of B, we always may assume that we have a finite-transversal projection $S \subset B$ w.r.t. P, with S smooth over k. As a consequence, we can explicitly describe the top multiplicity locus of any B as above in a conveniently chosen étale extension.

Example 11.4.4 Let $V \subset \mathbb{A}^3_k = \text{Spec}(k[X, Y, Z])$ be the monomial curve defined by

$$X = t^3, \quad Y = t^4, \quad Z = t^5.$$

Then $V = \text{Spec}(B)$, where $B = k[X, Y, Z]/J$ and $J = \langle X^3 - YZ,\ X^2Y - Z^2,\ Y^2 - XZ\rangle$. Let $S = k[X]$. Then the extension $S \subset B$ is finite since $Y^3 - X^4,\ Z^3 - X^5 \in J$. Note that these integral relations can be obtained using standard bases, see [13, Proposition 3.1.5].

The generic rank of the extension is $3 = \dim_{k(X)}(B \otimes_{k[X]} k(X))$ and we have also that $e_{\mathfrak{m}}(B) = 3$, where $\mathfrak{m} = \langle x, y, z\rangle \subset B$. In this case, $S \subset B$ is a finite-transversal extension w.r.t. $\mathfrak{m}$, and $S[y, z]$ is an algebraic presentation of B.

The minimal polynomials (in the variable W) of y and z are, respectively $W^3 - x^4$ and $W^3 - x^5$. Set $C = k[X, Y, Z]/\langle Y^3 - X^4,\ Z^3 - X^5\rangle$. Note that $\text{Spec}(C)$ has two irreducible components of dimension one. The curve $\text{Spec}(B)$ is one of the irreducible components of $\text{Spec}(C)$. By Theorem 11.4.3, the locus of points of multiplicity 3 of $\text{Spec}(B)$ (only the origin) coincides with the locus of points of multiplicity 9 of $\text{Spec}(C)$.

In Example 11.4.4 the dimension of the ring $B_{\mathfrak{m}}$ is one and the embedding dimension is three. The algebraic presentation is generated by two elements, y and z, over S and any other algebraic presentation over a regular ring is generated by two elements at least.

In general, the difference of the embedding dimension and the dimension of the local ring, the so called *excess of embedding dimension*, is a lower bound for the number of generators of an algebraic presentation. However, this lower bound can always be achieved after localization, a fact that is proved in Proposition 11.4.6 below. First we need a technical result for finite extensions of local rings.

Lemma 11.4.5 *Let* $(S, \mathfrak{n})$ *and* $(B, \mathfrak{m})$ *be Noetherian local rings and suppose that* $S \subset B$ *is a finite extension, with* S *is regular. Assume that the residue fields are equal,* $S/\mathfrak{n} \cong B/\mathfrak{m} = k$. *If* $t = \dim_{k(\mathfrak{m})}(\mathfrak{m}/\mathfrak{m}^2) - \dim(B)$ *is the excess of embedding dimension of* B, *then there are elements* $\theta_1, \dots, \theta_t \in \mathfrak{m}$ *such that*

$$B = S[\theta_1, \dots, \theta_t].$$

Proof Write $B = S[\theta_1 \dots, \theta_e]$. Since $S/\mathfrak{n} \cong B/\mathfrak{m}$ we can assume that $\theta_i \in \mathfrak{m}$ (here a translation by elements of S may be needed). We have that $\mathfrak{n}B + \langle\theta_1, \dots, \theta_e\rangle = \mathfrak{m}$, hence $e \geqslant t$. After reordering the θ_i's, we may assume that $\mathfrak{m} = \mathfrak{n}B + \langle\theta_1, \dots, \theta_t\rangle$.

Consider the S-module $N = S[\theta_1, \dots, \theta_t] \subset B$. We claim that

$$N/\mathfrak{n}N = B/\mathfrak{n}B.$$

If the claim holds then by Nakayama's Lemma we have that $N = B$ as required.

To prove the claim, let $\bar{\theta}_i$ be the class of θ_i in $\bar{B} = B/\mathfrak{n}B$, for $i = 1, \ldots, e$, and denote by $\overline{\mathfrak{m}} = \mathfrak{m}\bar{B}$ the maximal ideal of the local ring $\bar{B}$. We have that $\bar{\theta}_1, \ldots \bar{\theta}_t$ generate $\overline{\mathfrak{m}}$ in $\bar{B}$.

Note that $\bar{B}$ is an Artinian local ring, hence complete. Since $\dim_k(\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2) = t$, by Cohen's structure Theorem, $\bar{B} \cong k[[Z_1, \ldots, Z_t]]/J$ for some ideal J, where the classes of Z_i correspond to $\bar{\theta}_i$, $i = 1, \ldots, t$. Now note that every $\bar{\theta}_i$ is nilpotent, hence for some integers α_i we have $Z_i^{\alpha_i} \in J$ and then

$$\bar{B} \cong k[Z_1, \ldots, Z_t]/I$$

for some ideal $I \subset k[Z_1, \ldots, Z_t]$. □

Proposition 11.4.6 *Let $S \subset B$ be a finite-transversal extension w.r.t. an $\mathfrak{m}$-primary ideal $I \subset B$, with $\mathfrak{m}$ a maximal ideal in B. Let $t = \dim_{k(\mathfrak{m})}(\mathfrak{m}/\mathfrak{m}^2) - \dim(B_{\mathfrak{m}})$ be the excess of embedding dimension of B at $\mathfrak{m}$. Then there are elements $\theta_1, \ldots, \theta_t \in B$ and $g \in S$ such that*

$$B_g = S_g[\theta_1, \ldots, \theta_t].$$

Proof As in Lemma 11.4.5, write $B = S[\theta_1 \ldots, \theta_e]$, with $\theta_i \in \mathfrak{m}$ for $i = 1 \ldots, e$. Let $\mathfrak{n} = \mathfrak{m} \cap S$. Then the extension $S_{\mathfrak{n}} \subset B \otimes_S S_{\mathfrak{n}}$ is finite. By condition (i) in Proposition 11.3.4 $B_{\mathfrak{m}} = B \otimes_S S_{\mathfrak{n}}$. Therefore $S_{\mathfrak{n}} \subset B_{\mathfrak{m}}$ is finite, and by Lemma 11.4.5, after reordering the θ_i, we have that

$$B_{\mathfrak{m}} = S_{\mathfrak{n}}[\theta_1, \ldots, \theta_t].$$

Note that, for $j = t+1, \ldots, e$, θ_j is a polynomial in $\theta_1, \ldots, \theta_t$ with coefficients in $S_{\mathfrak{n}}$. Hence there exists some $g \in S$ such that $\theta_j \in S_g[\theta_1, \ldots, \theta_t]$, $j = t+1, \ldots, e$, and it follows that

$$B_g = S_g[\theta_1, \ldots, \theta_t].$$

□

Example 11.4.7 Let $S = k[Y]$. In the polynomial ring $S[X_1, X_2]$ consider the ideal

$$J = \langle X_1^2 - Y^5,\ X_1 - X_2 - X_1X_2(1 + X_1)\rangle,$$

and let $B = S[X_1, X_2]/J$. Note that $X_2^2(1 + Y^5 + Y^{10}) + 2Y^5X_2 - Y^5 \in J$ which gives an integral relation of X_2 with coefficients in S_f, with $f = 1 + Y^5 + Y^{10}$. The extension $S_f \subset B_f$ is finite, and $S_f[x_1, x_2]$ is an algebraic presentation of $S_f \subset B_f$. We have that $S_f \subset B_f$ is finite-transversal w.r.t. $\mathfrak{m} = \langle y, x_1, x_2\rangle B_f$.

Note that the surface with affine ring $R = S[X_1, X_2]/\langle X_1 - X_2 - X_1X_2\rangle$ is regular and $\mathrm{Spec}(B)$ is a curve in $\mathrm{Spec}(R)$. In this case the dimension of $B_{\mathfrak{m}}$ is one with embedding dimension two, since

$$\mathfrak{m}B_{\mathfrak{m}} = \langle y, x_1 \rangle B_{\mathfrak{m}} = \langle y, x_2 \rangle B_{\mathfrak{m}}.$$

Proposition 11.4.6 says that for some localization of S_f we have to be able to find an algebraic presentation of $S_f \subset B_f$ with only one generator, either x_1 or x_2.

Some Groebner basis computations show that

$$X_1(1+Y^5) - X_2(1+Y^5+Y^{10}) - Y^5 \in J,$$

an then $B_f = S_f[x_1]$. On the other hand, if $g = (1+Y^5)$ then $S_{fg} \subset B_{fg}$ is finite-transversal w.r.t. $\mathfrak{m}B_{fg}$ and $x_1 \in S_{fg}[x_2]$. Thus $B_{fg} = S_{fg}[x_2]$.

11.4.4 *Theorem 11.4.3 and Homeomorphic Copies of the Top Multiplicity Locus of* Spec(B)

Finally, there is a third main consequence of Theorem 11.4.3, whose meaning will be clarified in the Sect. 11.5, see Theorem 11.5.5 (iii) and part (iv), which is stated in Sect. 11.5.2.

Corollary 11.4.8 *[30, Corollary 5.9] Let $S \subset B$ be a finite extension of generic rank n, and suppose it is under the assumptions of Theorem 11.4.3. Assume that the set of points of multiplicity n of B, $F_n \subset \operatorname{Spec}(B)$, is not empty. Then*

(i) Zariski's conditions hold for any $P \in F_n$:

(i) β is a set theoretical bijection between F_n and $\beta(F_n)$,
(ii) if $P \in F_n$ and $\mathfrak{p} = \beta(P)$ then $k(P) = B_P/PB_P = S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}} = k(\mathfrak{p})$, and
(iii) $\mathfrak{p}B_P$ is a reduction of PB_P.

(ii) F_n is closed in Spec(B), and F_n is homeomorphic to $\beta(F_n)$.
(iii) $\beta(F_n) = Spec(S)$ if and only if $S = B_{\mathrm{red}}$.

In other words, Corollary 11.4.8 says that when $S \subset B$ is finite transversal of generic rank n then we can see a homeomorphic image of F_n in Spec(S).

11.5 Finite-Transversal Morphisms and Resolution of Singularities

In the present section we discuss the central result in [30], stated as Theorem 11.5.5 below, and its applications to resolution of singularities. The key idea here is that the description of the top multiplicity locus of a variety given in Theorem 11.4.3 is stable after blowing up a regular equimultiple center, see Theorem 11.5.5. We will make these ideas more precise along the following paragraphs.

Resolution of Singularities

Now we go back to our discussion in the Introduction. There, we mentioned the role of the order function when measuring how singular a hypersurface $H \subset \mathbb{A}^n_k$ is. When the characteristic is zero, proving the existence of a resolution of singularities is a complex task, and yet, it somehow reduces to *considering the order of ideals*. In other words and very roughly speaking, to resolve singularities Hironaka faced two main problems:

- **Problem 1.** Given a sheaf of ideals J on a smooth scheme V and a positive integer b, prove that there exists a finite sequence of blow ups so that a *suitable transform of* J has maximum order below b (see Theorem 11.5.2 below).
- **Problem 2.** Prove that *improving the singularities* of an algebraic variety X by blow ups is equivalent to solving problem 1 (see Theorem 11.5.3 below and the discussion that follows).

Our purpose is to give a few hints on these ideas. In order to do so, we start with some definitions.

Pairs and Their Role in Resolution of Singularities

Definition 11.5.1 Let V be smooth scheme of finite type over a field k, let J be a sheaf of ideals on V and let $b \in \mathbb{Z}$ a positive integer.

- We refer to (J, b) as *a pair over* V.
- The *singular locus of* (J, b) is the closed subset of V,

$$\operatorname{Sing}(J, b) := \{\zeta \in V : \nu_\zeta(J) \geqslant b\}.$$

- A *permissible center* Y for (J, b) is a closed regular subset $Y \subset V$ such that $Y \subset \operatorname{Sing}(J, b)$.
- A permissible blow up of V is the blow up of V at a permissible center Y, $V \leftarrow V_1$.
- For a permissible blow up, $V \leftarrow V_1$, with exceptional divisor E_1, the transform of the pair (J, b) is the pair, (J_1, b), where

$$J\mathscr{O}_{V_1} = \mathscr{I}(E_1)^b \cdot J_1.$$

With the previous notation, a *resolution of a pair* is a sequence of permissible blow ups,

$$\begin{array}{ccccccc} V = V_0 & \leftarrow & V_1 & \leftarrow \dots \leftarrow & V_\ell \\ (J, b) = (J_0, b) & & (J_1, b) & \dots & (J_\ell, b), \end{array}$$

so that Sing $(J_\ell, b) = \emptyset$.

To be precise, an additional condition on the permissible centers needs to be asked: they need to have normal crossings with the exceptional divisors that successively appear in the sequence.

Theorem 11.5.2 *[16] If the characteristic of k is zero, a resolution of* (J, b) *exists.*

Why Pairs?

The previous statement might lead to more questions than answers:

(i) How is the theorem proven and why the hypothesis on the characteristic?
(ii) Is it really necessary and statement about general pairs? Is it not enough to resolve pairs (J, b) with b equal to the maximum order of J at V?
(iii) While it is clear what the pair for a hypersurface could be, it is not obvious how to proceed in the general case.

Regarding to question (1): Maximal contact

Theorem 11.5.2 is proven by induction on the dimension of V: the existence of a resolution of (J, b) follows from another theorem that basically says that a resolution of (J, b) can be achieved if we know how to resolve pairs in smooth $(n-1)$-dimensional schemes. And this does not hold in general over fields of positive characteristic. For those interested in a deeper understanding on the topic we refer to the so called theory of *maximal contact* (see [12], also [4]).

Regarding to question (2): An example

Let $H : \{z^2 + (y^3 - x^5) = 0\} \subset \mathbb{A}^3_k$, where k is a field of characteristic different from 2. If we want to find a resolution of singularities of H we can start by resolving the pair $(\langle z^2 + (y^3 - x^5)\rangle, 2)$ in $\mathbb{A}^3_k$. The *theory of maximal contact* would tell us that a finding a resolution of $(\langle z^2 + (y^3 - x^5)\rangle, 2)$ is equivalent to finding a resolution of the pair $(\langle (y^3 - x^5)\rangle, 2)$ in $\mathbb{A}^2_k$. Observe that the number $b = 2$ in the second pair is **not** the maximum order of the ideal $\langle (y^3 - x^5)\rangle$ in $\mathbb{A}^2_k$. Hence, a theorem of resolution of **general** pairs as Theorem 11.5.2 is needed.

Regarding to question (3): Presentations for the Hilbert-Samuel function

If $H \subset V$ is a hypersurface of maximum order m, then it is clear that a resolution of the pair $(\mathcal{I}(H), m)$ leads to a sequence of blow ups over H so that the strict transform of H has maximum order below m. And resolution follows by induction on the order.

For arbitrary varieties, Hironaka used *presentations of the Hilbert-Samuel function*. For a variety X we will use HS_X to refer to its Hilbert-Samuel function. This function satisfies a series of properties that make it suitable as an invariant to approach resolution (see [5]):

(A) HS_X is an upper semi-continuous function on X; we will denote by $\max \mathrm{HS}_X$ its maximum value on X, and by $\underline{\mathrm{Max}}\, \mathrm{HS}_X$ the closed set of X where this maximum is achieved;
(B) HS_X is constant on X if and only if X is regular;
(C) If $Y \subset \underline{\mathrm{Max}}\, \mathrm{HS}_X$ is a closed regular center and if $X \leftarrow X_1$ is the blow up at Y, then

$$\max \mathrm{HS}_X \geqslant \max \mathrm{HS}_{X_1}.$$

The key point here is that the closed set $\underline{\mathrm{Max}}\, \mathrm{HS}_X$ can be expressed as the singular locus of a pair. But not *any* pair will work. Actually we need a *suitably defined pair* whose resolution induces a sequence of blow ups over X that forces the maximum value of HS_X to go down. This is done via the so called *standard basis* (see [3]):

Theorem 11.5.3 *(Presentations for the Hilbert-Samuel function) At an étale neighborhood of each closed point $\xi \in \underline{Max}\, HS_X$, we can assume X to be locally embedded in a smooth scheme V where we can find elements $f_1, \dots, f_r \in \mathcal{O}_{V,\xi}$ such that:*

(i) $\mathcal{I}(X)_\xi = \langle f_1, \dots, f_r \rangle$;
(ii) Denoting by m_i the maximum order of the hypersuface $H_i = \{f_i = 0\}$, we have that

$$\underline{Max}\, HS_X = \bigcap_{i=1}^{r} \underline{Max}\, HS_{H_i} = \bigcap_{i=1}^{r} \{\zeta : \nu_\zeta(f_i) = m_i\};$$

(iii) If $Y \subset \underline{Max}\, HS_X$ is a closed regular center, if $V \leftarrow V_1$ is the blow up at Y, and

$$max\, HS_X = max\, HS_{X_1},$$

then

$$\underline{Max}\, HS_{X_1} = \bigcap_{i=1}^{r} \underline{Max}\, HS_{H_{i,1}}$$

where $H_{i,1}$ is the strict transform of H_i, $i = 1, \dots, r$.

A consequence of the theorem is that ***a pair*** (J, b) ***can be naturally attached to the previous data:***

Set $M := \prod_{i=1}^{r} m_i$ and $M_i = M/m_i$. Let $J := \langle f_1^{M/m_1}, \dots, f_r^{M/m_r} \rangle$. Then resolving the pair (J, M) leads to a sequences of blow ups,

$$X \leftarrow X_1 \leftarrow \dots \leftarrow X_\ell$$

so that

$$\max \mathrm{HS}_X = \max \mathrm{HS}_{X_1} = \ldots > \max \mathrm{HS}_{X_\ell}.$$

Remark 11.5.4 It is worthwhile to make two observations to regarding Hironaka's approach to resolution:

(a) Hironaka uses the Hilbert-Samuel function, which takes values in $\mathbb{N}^{\mathbb{N}}$. The multiplicity function also satisfies properties (A), (B) and (C) from above (see [11]), takes values on $\mathbb{N}$ and has a natural geometric interpretation. Hence, it is quite natural to ask whether it can replace the role of the Hilbert-Samuel function in the resolution process. This is a question posed by Hironaka in [16] and answered affirmatively by Villamayor in [30] (this follows from Theorem 11.5.5 below).

(b) The use of the Hilbert-Samuel function requires working with an embedding of X in some smooth scheme of dimension $n \geqslant d + 1$. This leads to the definition of convenient pairs some smooth n-dimensional scheme and then induction on resolution of pairs is applied in $n - 1, n - 2, \ldots,$-dimensions. As we will see, Villamayor's approach using the multiplicity simplifies this last step. More precisely, the problem of lowering the multiplicity of a d-dimensional variety is shown to be equivalent to the resolution of a pair in some smooth scheme of the same dimension d (at least when the characteristic is zero).

11.5.1 Regarding to Remark 11.5.4 (a): Presentations for the Multiplicity Function

We dedicate the following lines to Villamayor's approach to resolution using the multiplicity function as the main invariant. We start by fixing some notation. We will use Mult_X to refer to the multiplicity function on X, $\max \mathrm{Mult}_X$ for its maximum value on X and $\underline{\mathrm{Max}}\,\mathrm{Mult}_X$ for the closed set of points of X where this value is achieved.

Let X be an equidimensional variety with $\max \mathrm{Mult}_X > 1$. A *simplification of the multiplicity of* X is a finite sequence of blow ups

$$X = X_0 \leftarrow X_1 \leftarrow \ldots \leftarrow X_n$$

so that

$$\max \mathrm{Mult}_{X_0} = \max \mathrm{Mult}_{X_1} = \ldots = \max \mathrm{Mult}_{X_{n-1}} > \max \mathrm{Mult}_{X_n}.$$

As a corollary of Theorem 11.5.5 below, we get that it is possible to resolve singularities in characteristic zero via simplifications of the multiplicity. As Hironaka's Theorem 11.5.3, Villamayor's statement is also of local nature. Hence, we will assume that X is an affine variety. Note that the parts (i) and (ii) of the next theorem have been already stated and discussed in Theorem 11.4.3 and Sect. 11.4.3.

Theorem 11.5.5 *[30, §6, Theorem 6.8] (Presentations for the Multiplicity function) Let $X = Spec(B)$ be an affine equidimensional algebraic variety of dimension d defined over a perfect field k, and let $\xi \in \underline{Max}\, Mult_X$ be a closed point. Then, there is an étale neighborhood B' of B, mapping $\xi' \in \mathrm{Spec}(B')$ to ξ, a smooth k-algebra S together with a finite-transversal morphism at ξ', $\beta : \mathrm{Spec}(B') \to \mathrm{Spec}(S)$ so that if $B' = S[\theta_1, \ldots, \theta_e]$ and $f_i(Z_i) \in K(S)[Z_i]$ denote the minimum polynomial of θ_i over $K(S)$ for $i = 1, \ldots, e$, then $f_i(Z_i) \in S[Z_i]$ and there is a diagram:*

$$\begin{array}{ccccc} S[Z_1, \ldots, Z_e] & \longrightarrow & S[Z_1, \ldots, Z_e]/\langle f_1(Z_1), \ldots, f_e(Z_e)\rangle & \longrightarrow & B' \\ & & \uparrow & \nearrow & \\ & & S & & \end{array}$$

for which the following hold:

(i) Let $V = \mathrm{Spec}(S[Z_1, \ldots, Z_e])$, and let $\mathscr{I}(X)$ be the defining ideal of X at V. Then

$$\langle f_1, \ldots, f_e\rangle \subset \mathscr{I}(X);$$

(ii) Denoting by m_i the maximum order of the hypersuface $H_i = \{f_i = 0\} \subset V$, we have that

$$\underline{Max}\, Mult_X = \bigcap_{i=1}^{e} \underline{Max}\, Mult_{H_i} = \bigcap_{i=1}^{e} \{\zeta : \nu_\zeta(f_i) = m_i\};$$

(iii) Let $Y \subset \underline{Max}\, Mult_X$ be a closed regular center. Then $\beta(Y)$ is regular in $\mathrm{Spec}(S)$. Now, if $X \leftarrow X_1$ is the blow up at Y, and $\mathrm{Spec}(S) \leftarrow T_1$ is the blow up at $\beta(Y)$ then there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(B) & \longleftarrow & X_1 \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta_1} \\ \mathrm{Spec}(S) & \longleftarrow & T_1 \end{array} \tag{11.14}$$

where the horizontal maps are blow ups and the vertical are finite morphisms. Moreover, if

$$max\, Mult_X = max\, Mult_{X_1},$$

then $\beta_1 : X_1 \to T_1$ *is finite-transversal w.r.t. any point in* $\underline{Max}\,Mult_{X_1}$ *and if* $V \leftarrow V_1$ *is the blow up at* Y*, then*

$$\underline{Max}\,Mult_{X_1} = \bigcap_{i=1}^{e} \underline{Max}\,Mult_{H_{i,1}}$$

where $H_{i,1}$ *is the strict transform of* H_i *in* V_1*, for* $i = 1, \ldots, e$.

A consequence of the theorem is that ***a pair*** (J, b) ***can be naturally attached to the previous data:***

Set $M := \prod_{i=1}^{e} m_i$ and $M_i = M/m_i$. Let $J := \langle f_1^{M/m_1}, \ldots, f_e^{M/m_e} \rangle$. Then resolving the pair (J, M) leads to a sequences of blow ups,

$$X \leftarrow X_1 \leftarrow \ldots \leftarrow X_\ell$$

so that

$$\max \mathrm{Mult}_X = \max \mathrm{Mult}_{X_1} = \ldots > \max \mathrm{Mult}_{X_\ell},$$

i.e., to a simplification of the multiplicity of X.

11.5.2 Regarding to Remark 11.5.4 (b): Resolution of Pairs in Dimension $d = \dim X$

Notice that Villamayor's Presentations of the Multiplicity come equipped with a finite projection to dome d-dimensional smooth scheme. This finite-transversal projection has one additional property:

(iv) If $T \subset \beta(\underline{\mathrm{Max}}Mult_X) \subset \mathrm{Spec}(S)$ is a regular closed subscheme, then $\beta^{-1}(T)_{\mathrm{red}}$ is also regular and the simultaneous blow ups at T and $\beta^{-1}(T)_{\mathrm{red}}$ lead to a commutative diagram as (11.14) with the same properties as in Theorem 11.5.5 (iii).

When the characteristic of the base field is zero, there is a pair on $\mathrm{Spec}(S)$ (which is a d-dimensional scheme) whose resolution induces a resolution of the pair (J, M). Thus, a simplification of the multiplicity of X is directly achieved via the resolution of a d-dimensional pair, where $d = \dim X$, see [1, Chapter 7] for more details.

11.6 Finite-Transversal Morphisms and the Asymptotic Samuel Function

The *asymptotic Samuel function* was introduced by Samuel in [26] and afterwards studied by D. Rees [20–23]. Here we review the definition and some properties. For further details we refer the reader to [17] and [29]. Our purpose here is to prove

Theorem 11.6.8 which is a slightly modified version of a theorem of Hickel on the computation of the asymptotic Samuel function (see Theorem 11.6.7 for Hickel's statement).

Suppose A is a commutative ring with unit 1, and let $I \subset A$ be a proper ideal. For each $f \in A$ consider the value $\nu_I(f) = \sup\{\ell \in \mathbb{N} \cup \{\infty\} \mid f \in I^\ell\}$. Observe that for $f, g \in A$ we have $\nu_I(f+g) \geqslant \min\{\nu_I(f), \nu_I(g)\}$ and $\nu_I(f \cdot g) \geqslant \nu_I(f)+\nu_I(g)$. In particular, for $m \in \mathbb{N}$, $\nu_I(f^m) \geqslant m\nu_I(f)$ and the inequality could be strict. The asymptotic Samuel function is a normalized version of the ordinary function, which, as we will see, has a nicer behavior.

Definition 11.6.1 The *asymptotic Samuel function at* I, $\bar{\nu}_I : A \to \mathbb{R} \cup \{\infty\}$, is defined as:

$$\bar{\nu}_I(f) = \lim_{n\to\infty} \frac{\nu_I(f^n)}{n}, \qquad f \in A. \tag{11.15}$$

The limit (11.15) exists in $\mathbb{R}_{\geqslant 0} \cup \{\infty\}$ (see [17, Lemma 0.2.1]. When $(A, \mathfrak{m})$ is a local regular ring, $\overline{\nu}_{\mathfrak{m}} = \nu_{\mathfrak{m}}$. The next proposition summarizes some of the main properties of the asymptotic Samuel function.

Proposition 11.6.2 *[17, Corollary 0.2.6, Proposition 0.2.9] The function $\bar{\nu}_I$ is an order function, i.e., it satisfies the following properties:*

(i) $\bar{\nu}_I(f+g) \geqslant \min\{\bar{\nu}_I(f) + \bar{\nu}_I(g)\}$*, for all* $f, g \in A$*,*
(ii) $\bar{\nu}_I(f \cdot g) \geqslant \bar{\nu}_I(f) + \bar{\nu}_I(g)$*, for all* $f, g \in A$*,*
(iii) $\bar{\nu}_I(0) = \infty$ *and* $\bar{\nu}_I(1) = 0$.

Furthermore, for each $f \in A$ and each $r \in \mathbb{N}$:

(iv) $\bar{\nu}_I(f^r) = r\bar{\nu}_I(f)$*;*
(v) $\bar{\nu}_{I^r}(f) = \frac{1}{r}\bar{\nu}_I(f)$.

Note that if $f \in A$ is nilpotent then $\bar{\nu}_I(f) = \infty$.

Example 11.6.3 If $A = k[X, Y]/\langle X^2 + Y^3\rangle$ and if $\mathfrak{m} = \langle x, y\rangle \subset A$, then it can be checked that $\overline{\nu}_{\mathfrak{m}}(y) = 1$, while $\overline{\nu}_{\mathfrak{m}}(x) = 3/2$. However, if $A = k[X, Y, Z]/\langle X^2 + Y^2 + Z^3\rangle$, $\mathfrak{m} = \langle x, y, z\rangle$ and the characteristic is different from 2, then $\overline{\nu}_{\mathfrak{m}}(x) = \overline{\nu}_{\mathfrak{m}}(y) = \overline{\nu}_{\mathfrak{m}}(z) = 1$.

The Asymptotic Samuel Function on Noetherian Rings

When A is Noetherian, the number $\overline{\nu}_I(f)$ measures how deep the element f lies in the integral closure of powers of I:

Proposition 11.6.4 *[29, Corollary 6.9.1] Suppose A is Noetherian. Then for a proper ideal $I \subset A$ and every $a \in \mathbb{N}$,*

$$\overline{I^a} = \{f \in R \mid \bar{\nu}_I(f) \geqslant a\}.$$

Corollary 11.6.5 *Let A be a Noetherian ring and $I \subset A$ a proper ideal. If $f \in A$ then*

$$\bar{\nu}_I(f) \geqslant \frac{a}{b} \Longleftrightarrow f^b \in \overline{I^a}.$$

See also [10] for a generalization of the asymptotic Samuel function to arbitrary filtrations of ideals and properties.

The previous characterization of $\overline{\nu}_I$ leads to the following result that gives a valuative version of the function.

Theorem 11.6.6 *Let A be a Noetherian ring, and let $I \subset A$ be a proper ideal not contained in a minimal prime of A. Let $v_1, \ldots, v_s$ be a set of Rees valuations of the ideal I. If $f \in A$ then*

$$\bar{\nu}_I(f) = \min\left\{\frac{v_i(f)}{v_i(I)} \mid i = 1, \ldots, s\right\}.$$

Proof See [29, Lemma 10.1.5, Theorem 10.2.2] and [28, Proposition 2.2]. □

In particular, it follows from here that when A is a Noetherian ring, $\overline{\nu}_I(f)$ always takes values in $\mathbb{Q} \cup \{\infty\}$.

On a Explicit Formula for the Computation of the Asymptotic Samuel Function

In [15] M. Hickel presented a series of nice results regarding the asymptotic Samuel function. In particular, he proved the following theorem with an explicit method for its calculation:

Theorem 11.6.7 *[15, Theorem 2.1] Let $(R, \mathfrak{m}, k)$ be a complete local Noetherian domain of equal characteristic and Krull dimension d. Let $I \subset R$ be an $\mathfrak{m}$-primary ideal and suppose that I has a reduction generated by d elements, $J = \langle x_1, \ldots, x_d\rangle \subset R$. Let $r \in R$. Let $A := k[[x_1, \ldots, x_d]] \cong k[[X_1, \ldots, X_d]]$, let $\mathfrak{m}_A \subset A$ be the maximal ideal, and let*

$$p(Z) = Z^\ell + \sum_{i=1}^{\ell} a_i Z^{\ell-i}$$

be the minimal polynomial of r over $K(A)$. Then:

$$\overline{\nu}_I(r) = \min_{1 \leqslant i \leqslant \ell} \frac{\nu_{\mathfrak{m}_A}(a_i)}{i}.$$

The theorem gives a method for the explicit computation of the asymptotic Samuel function for a local Noetherian ring $(R, \mathfrak{m}, k)$ of equal characteristic, by passing to the completion and then reducing to the domain case. The existence of a reduction of the ideal I generated by d elements can be achieved by extending the residue field.

Following the arguments in Sect. 11.3, the previous result can be shown to hold in a suitable étale neighborhood of R, when R is equidimensional and an algebra of finite type over a perfect field k. Thus, under these additional hypotheses, the completion is not needed and the reduction to the domain case is avoided.

Theorem 11.6.8 *Let $(B, \mathfrak{m})$ be a Noetherian equicharacteristic, equidimensional local ring of Krull dimension d. Let $I \subset \mathfrak{m}$ be an $\mathfrak{m}$-primary ideal. Assume that there exists a local étale neighborhood $(B', \mathfrak{m}')$ of $(B, \mathfrak{m})$ with a finite-transversal morphism w.r.t. I, $S \subset B'$. Let $b \in B$. If*

$$p(Z) = Z^{\ell} + a_1 Z^{\ell-1} + \ldots + a_{\ell}$$

is the minimal polynomial of $b \in B'$ over the fraction field of S, $K(S)$, then $p(Z) \in S[Z]$ and

$$\overline{\nu}_I(b) = \min \left\{ \frac{\nu_{\mathfrak{m}_S}(a_i)}{i} : i = 1, \ldots, \ell \right\}, \tag{11.16}$$

where $\mathfrak{m}_S = \mathfrak{m}' \cap S$.

Proof We follow the ideas of Hickel in the proof of [15, Theorem 2.1] to check that the result also holds in this different setting. To ease the notation let us assume that $B = B'$, and let $\mathfrak{m}_S = \mathfrak{m} \cap S$. Since the extension $S \subset B$ is assumed to be finite-transversal w.r.t. I we have that $\mathfrak{m}_S B$ is a reduction of I generated by d-elements. Let $b \in B$. Then $L = B \otimes_S K(S)$ is a finite extension of $K(S)$, although not necessarily a domain. Let

$$p(Z) = Z^{\ell} + a_1 Z^{\ell-1} + \ldots + a_{\ell} \in K(S)[Z]$$

be the minimal polynomial of b over $K(S)$. Observe that $p(Z)$ might not be irreducible over $K(S)$. By Proposition 11.4.1 $p(Z) \in S[Z]$ and the subring of $S[b] \subset B$ is isomorphic to $S[Z]/\langle p(Z) \rangle$. Thus $S[b]$ is free of rank ℓ over S. Taking the $\mathfrak{m}_S$-adic completion of S we have a commutative diagram,

$$\begin{array}{ccc}
B & \longrightarrow & B \otimes_S \hat{S} = R \\
\uparrow & & \uparrow \\
S[b] & \longrightarrow & S[b] \otimes_S \hat{S} = T \\
\uparrow & & \uparrow \\
S & \longrightarrow & \hat{S} \cong \tilde{k}[[X_1, \ldots, X_d]]
\end{array}$$

where the vertical maps are finite while the horizontal maps are faithfully flat and $\tilde{k}$ is the residue field of S (which is also the residue field of B). Since the rank of

$S[b] \otimes_S \hat{S}$ over $\widehat{S}$ is ℓ, $p(X)$ is also the minimal polynomial of $b \in S[b] \otimes_S \hat{S}$ over $\widehat{S}$. By Cohen [8, Theorem 8], $B \otimes_S \hat{S}$ is complete, and then the claim follows from [15, Theorem 2.1] if $B \otimes_S \hat{S}$ is a domain.

Otherwise, we will see that the same argument given in [15, Theorem 2.1] can be used in this case to prove a similar result. Set $J = \langle X_1, \ldots, X_d \rangle \subset \hat{S}$, let $J_1 = \langle X_1, \ldots, X_d \rangle T$ and let $J_2 = \langle X_1, \ldots, X_d \rangle R$. Then J_2 is a reduction of IR. And we have that that

$$\overline{\nu}_I(b) = \overline{\nu}_{IR}(b) = \overline{\nu}_{J_2}(b) = \overline{\nu}_{J_1}(b),$$

where the last equality comes from the fact that $T \subset R$ is a finite extension and using Corollary 11.6.5. On the one hand, if $h(Z) = Z^m + c_1 Z^{m-1} + \ldots + c_m \in \hat{S}[Z]$ is any monic polynomial with $q(b) = 0$ then

$$\overline{\nu}_{J_1}(b) \geqslant \min \left\{ \frac{\nu_{\mathfrak{m}_{\hat{S}}}(c_i)}{i} : i = 1, \ldots, s \right\},$$

(see [15, pg. 1374]). To verify the equality (11.16), we follow the argument in the proof of [15, Theorem 2.1] to check that the hypothesis on T being a domain is not needed.

If b is a unit or nilpotent, then we are done. Suppose otherwise that $\overline{\nu}_I(b) = r/s > 0$, where $r, s \in \mathbb{N}_{>0}$ and consider the diagram:

$$\begin{array}{ccc} S'[b^s] & \longrightarrow & S[b] \otimes_S \hat{S} = \hat{S}[Z]/\langle p(Z) \rangle = T \\ \uparrow & & \uparrow \\ S' = k[[X_1^r, \ldots, X_n^r]] & \longrightarrow & \hat{S} \cong k[[X_1, \ldots, X_n]] \end{array}$$

Notice that all the maps are finite, and that the first vertical is under the assumptions of Proposition 11.4.1. Hence if $q(Z)$ is the minimal polynomial of b^s over $K(S')$, we have that $q(Z) \in S'[Z]$ and moreover, $S'[b^s] \cong S[Z]/\langle q(Z) \rangle$.

In addition, the conditions in Proposition 11.3.4 hold for the first vertical finite map:

(i) There is a unique maximal ideal $\mathfrak{m}'$ in $S'[b^s]$ dominating the maximal ideal $\mathfrak{m}_{S'}$ of S';
(ii) The residue fields at $\mathfrak{m}'$ and at $\mathfrak{m}_{S'}$ are the same, hence $\mathfrak{m}' = \mathfrak{m}_{S'} + \langle b^s \rangle$;
(iii) The expansion of the maximal ideal of S', $\mathfrak{m}_{S'}$ in $S'[b^s]$, generates a reduction of $\mathfrak{m}'$.

Hence, by Zariski's multiplicity formula for finite projections, the multiplicity of $S[b^s]$ at $\mathfrak{m}'$ is the same as the generic rank of the finite extension $S' \subset S'[b^s]$. In other words, $S' \subset S'[b^s]$ is finite-transversal w.r.t. $\mathfrak{m}'$. Therefore, the polynomial $q(Z) \in S'[Z]$ determines a hypersurface in $\mathrm{Spec}(S'[Z])$ whose multiplicity at

$\langle X_1^r, \ldots, X_d^r, Z\rangle$ is the same generic rank of the finite extension. Thus, if

$$q(Z) = Z^m + c_1 Z^{m-1} + \ldots + c_m$$

necessarily $c_i \in \langle X_1^r, \ldots, X_n^r\rangle^i S'$.

Next, since $q(Z^s)$ is a multiple of $p(Z)$, following word by word the proof of Hickel in [15, pgs. 1374-5], we get that

$$\overline{\nu}_J(b) \leqslant \min_i \left\{ \frac{\nu_{\mathfrak{m}_{\hat{S}}}(a_i)}{i} : i = 1, \ldots, \ell \right\}.$$

□

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Chapter 12
Logarithmic Comparison Theorems

Francisco J. Castro-Jiménez, David Mond, and Luis Narváez-Macarro

We dedicate this chapter to the memory of our colleagues Jim Damon and José Luis Vicente Córdoba

Contents

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Abstract In this chapter we study the comparison between the logarithmic and the meromorphic de Rham complexes along a divisor in a complex manifold. We focus on the case of free divisors, starting with the case of locally quasihomogeneous divisors, and we explain how D-module theory can be used for this comparison.

12.1 Introduction

We survey a number of results known as *comparison theorems*, along the lines of the comparison theorem of Grothendieck. Grothendieck's Comparison Theorem states that, for any hypersurface D in a complex manifold X, the cohomology of the meromorphic de Rham complex with respect to $D \subset X$ (i.e. $h^q(\Omega^\bullet_X(\star D))$ for $0 \leqslant q \leqslant \dim X$) coincides with $\mathbb{R}^q j_* \mathbb{C}_U$ where $j : U := X \setminus D \to X$ is the inclusion and $\mathbb{C}_U$ is the constant sheaf $\mathbb{C}$ on U. In effect, the hypercohomology of this complex is the (topological) cohomology $H^\bullet(U; \mathbb{C})$. If D is a normal crossing divisor (NCD), this comparison result was proved by Atiyah and Hodge, and Grothendieck's proof reduces the general case to the case of a NCD by using Hironaka's resolution of singularities. When D is an arrangement of hyperplanes in $\mathbb{C}^n$, the Brieskorn complex $B^\bullet$ computes the singular cohomology of the complement $U = \mathbb{C}^n \setminus D$, $H^\bullet(U; \mathbb{C})$; and this is also a comparison result. There is a class of divisors, the so-called *free divisors*, introduced by Kyoji Saito, for which a certain number of comparison results can be proven. We survey these results and show how $\mathscr{D}$-module theory provides a way to deal with them.

The content of the paper is as follows. In Sect. 12.2.1 we first recall Grothendieck's comparison theorem (see also Sect. 12.3.6), that there is a canonical isomorphism

$$h^q(\Omega^\bullet_X(\star D)) \to \mathbb{R}^q j_* \mathbb{C}_U$$

for $0 \leqslant q \leqslant \dim X$. If X is Stein, then the global morphisms analogous to the previous ones

$$h^q(\Gamma(X, \Omega^\bullet_X(\star D))) \to H^q(U; \mathbb{C})$$

are isomorphisms. We then recall Brieskorn's Theorem proving that if D is a finite union of hyperplanes in $X = \mathbb{C}^n$, with equations $h_j = 0$, then the cohomology of the complement $H^q(X \setminus D; \mathbb{C})$ can be computed as the cohomology of the so called Brieskorn complex $B^\bullet$, which is the $\mathbb{C}$-subalgebra of the exterior algebra $\Gamma(X, \Omega^\bullet_X(\star D))$ generated by the forms $\frac{dh_j}{h_j}$.

Kyoji Saito (see Sect. 12.2.3) introduced the notion of *logarithmic meromorphic form* with respect to a divisor D in a complex manifold X: a meromorphic form ω has a *logarithmic pole* along the divisor D if both $h\omega$ and $hd\omega$ are holomorphic, where h is a local equation for D. Note in particular that all of Brieskorn's forms have logarithmic poles along an arrangement of hyperplanes $D \subset \mathbb{C}^n$. Logarithmic meromorphic forms form an $\mathscr{O}_X$-subcomplex of the meromorphic de Rham complex

$\Omega_X^\bullet(\star D)$. A natural question asks for the class of divisors $D \subset X$ for which the inclusion

$$\Omega_X^\bullet(\log D) \hookrightarrow \Omega_X^\bullet(\star D)$$

is a quasi-isomorphism (i.e. when the two complexes have the same cohomology). By analogy with Grothendieck's comparison theorem, if the morphism $\Omega_X^\bullet(\log D) \hookrightarrow \Omega_X^\bullet(\star D)$ is a quasi-isomorphism we say that the divisor D satisfies the *logarithmic comparison theorem* (LCT), or that LCT holds for D. Theorem 12.2.5 states that any *locally quasihomogeneous free divisor* $D \subset X$ satisfies LCT. In 1977 H. Terao conjectured (see Sect. 12.2.4) that LCT always holds for hyperplane arrangements (which are, of course, locally quasihomogeneous), and this conjecture was finally proved by Daniel Bath in [2].

A free divisor is *linear* if the module $\mathscr{D}er(-\log D)$ of logarithmic derivations with respect to D (as defined by K. Saito in [58], see Sect. 12.2.3) has a basis of vector fields whose coefficients are linear forms. In Sect. 12.2.5 we recall a number of results on linear free divisors and in particular the theorem of [28], which gives a completely independent proof that a global version of LCT holds for "reductive" linear free divisors.

In Sect. 12.3 we introduce the background in $\mathscr{D}$-module theory that is used in Sect. 12.4. This last section is devoted to state a $\mathscr{D}$-module criterion for the Logarithmic Comparison Theorem for free divisors. First, and following the paper of F.J. Calderón-Moreno [10], we consider the sheaf $\mathscr{V}_X^D$ of logarithmic differential operators on X, with respect to a free divisor $D \subset X$ and the logarithmic Spencer complex $\mathrm{Sp}^\bullet(\log D)$ which is a locally free resolution of the $\mathscr{V}_X^D$-module $\mathscr{O}_X$. By using [13] one generalizes the construction of this last complex to an arbitrary left $\mathscr{V}_X^D$-module. This constructions are used to prove the main theorem in this context. This theorem, proved by Calderón-Moreno and Narváez-Macarro ([13, Cor. 4.2] and [53, Corollary 1.7.2]), states that a free divisor $D \subset X$ satisfies the Logarithmic Comparison Theorem if and only if the natural morphism

$$\varrho : \mathscr{D}_X \overset{\mathbb{L}}{\otimes}_{\mathscr{V}_X^D} \mathscr{O}_X(D) \longrightarrow \mathscr{O}_X(\star D)$$

is an isomorphism in the derived category of $\mathscr{D}_X$-modules (see Theorem 12.4.7). Fixing a point $p \in D$, a reduced local equation f of the germ (D, p) and a basis $\{\delta_1, \ldots, \delta_n\}$ of $\mathscr{D}er(-\log D)_p$, the morphism ϱ_p is an isomorphism if and only if the complex $\mathscr{D}_{X,p} \overset{\mathbb{L}}{\otimes}_{\mathscr{V}_{X,p}^D} \mathscr{O}_{X,p}(D)$ is concentrated in cohomological degree 0, $\mathscr{D}_{X,p} f^{-1} = \mathscr{O}_{X,p}(\star D)$ and the $\mathscr{D}_{X,p}$-annihilator of $f^{-1} \in \mathscr{O}_{X,p}(\star D)$ equals the left ideal $\mathscr{D}_{X,p}(\delta_1 + \alpha_1, \ldots, \delta_n + \alpha_n)$, where $\delta_i(f) = \alpha_i f$ for $i = 1, \ldots, n$, see Sect. 12.4.3. That gives a $\mathscr{D}$-module criterion to test if LCT holds for a given free divisor $D \subset X$. Finally we treat some examples to show how this criterion can be applied in practice.

12.2 Comparison Theorems

12.2.1 Grothendieck's Comparison Theorem

To calculate the cohomology of a space with constant coefficients $\mathbb{C}$, one can use a resolution of the constant sheaf $\mathbb{C}_X$ and then calculate the hypercohomology of the resolution. For example, on a complex manifold the holomorphic Poincaré lemma (see e.g. [26, Ch. 2, Par. 2, Ex. 2.5.1]—the proof for C^∞ forms is easily adapted to the holomorphic case) shows that the complex of holomorphic differential forms

$$0 \longrightarrow \mathbb{C}_X \longrightarrow \mathcal{O}_X = \Omega^0_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \cdots \tag{12.1}$$

is exact, so that $\Omega^\bullet_X$ is a resolution of $\mathbb{C}_X$ by locally free $\mathcal{O}_X$-modules. If X is in addition a Stein space—for example, the complement of a divisor D in $\mathbb{C}^n$, or a convex open set in $\mathbb{C}^n$—then all of the sheaves in the complex are *acyclic*: $H^j(X, \Omega^k_X) = 0$ for $j > 0$ and $k \geqslant 0$ (in effect, this is the definition of 'Stein space'). In this case the double complex with which one calculates the hypercohomology is reduced to the complex of global sections

$$0 \to \Gamma(X, \mathcal{O}_X) \to \Gamma(X, \Omega^1_X) \to \cdots$$

and so one has the *analytic de Rham theorem*:

Theorem 12.2.1 *If X is a Stein manifold then*

$$h^q(\Gamma(X, \Omega^\bullet_X)) \longrightarrow H^q(X; \mathbb{C}) \tag{12.2}$$

is an isomorphism.

Here if we view $H^j(X; \mathbb{C})$ as the singular cohomology group, then the arrow is given by the integration of differential forms along singular chains, which we will refer to as the *de Rham morphism*. Note that this theorem already implies the non-trivial fact that the cohomology of a Stein manifold vanishes above middle real dimension.

If X is the complement of a divisor (a hypersurface) $D \subset \mathbb{C}^n$, then Ω^k_X (or more precisely $j_*\Omega^k_X$, where $j : X \to \mathbb{C}^n$ is inclusion) is equal to the sheaf of germs of holomorphic forms with arbitrary singularities along D. This is strictly larger than the sheaf $\Omega^k(\star D)$ of meromorphic forms with poles along D, since it places no restriction on the behaviour of the extension to D, and in particular allows also forms with essential singularities along D. In [32, Th. 2],[1] Alexander Grothendieck

[1] It is noted in [32] that the contents of this paper formed part of a letter of the author to M. F. Atiyah, dated October 14, 1963, except for some remarks dated November 1963 and July 1965.

showed that despite this difference, there are canonical isomorphism of sheaves of complex vector spaces

$$h^q\left(\Omega^\bullet_{\mathbb{C}^n}(\star D)\right) \simeq \mathbb{R}^q j_* \mathbb{C}_{\mathbb{C}^n \smallsetminus D}, \quad q = 0, \ldots, n, \tag{12.3}$$

where $\mathbb{R}^q j_*$ is the q-th right derived functor of the left-exact functor j_*, see e.g. [35, I.7]. The maps in (12.3) are given by the composition of the adjunction

$$\Omega^\bullet_{\mathbb{C}^n}(\star D) \to j_* j^{-1} \Omega^\bullet_{\mathbb{C}^n}(\star D) = j_* \Omega^\bullet_{\mathbb{C}^n \smallsetminus D}$$

with the inverse of Poincaré quasi-isomorphism $\mathbb{C}_{\mathbb{C}^n \smallsetminus D} \xrightarrow{\sim} \Omega^\bullet_{\mathbb{C}^n \smallsetminus D}$.

Taking into account that $\mathbb{C}^n$ is Stein, by Grothendieck [32, Corollary, page 97], one has $H^j(\mathbb{C}^n, \Omega^k_{\mathbb{C}^n}(\star D)) = 0$ for $j > 0$ and any k, and there are canonical isomorphisms of complex vector spaces

$$h^q(\Gamma(\mathbb{C}^n, \Omega^\bullet_{\mathbb{C}^n}(\star D))) \longrightarrow H^q(\mathbb{C}^n \smallsetminus D; \mathbb{C}) \tag{12.4}$$

for $q \geqslant 0$.

These maps are formal in the sense that they come from adjunction and Poincaré lemma. A well known (but not obvious) fact, see Remark 12.2.2, is that these maps coincide with the de Rham map given by integration of meromorphic forms along singular chains composed with the isomorphism between singular cohomology with coefficients in $\mathbb{C}$ and sheaf cohomology with coefficients in the constant sheaf $\mathbb{C}_{\mathbb{C}^n \smallsetminus D}$ (see [56, Ch. 4; Th. 4.14]).

The isomorphism in (12.3) can be stated as an isomorphism in the derived category, which is of course valid on any complex manifold X and any divisor $D \subset X$, namely

$$\Omega^\bullet_X(\star D) \simeq \mathbb{R} j_* \mathbb{C}_{X \smallsetminus D}. \tag{12.5}$$

This is known as *Grothendieck's Comparison Theorem* (see Theorem 12.3.11). The core assertion is that one can ignore essential singularities along D.

To end this section, let us emphasize that Grothendieck's proof of (12.5) used in an essential way Hironaka's resolution of singularities (in the complex algebraic and the complex analytic settings), and became a model to follow for the cohomological study of algebraic varieties over an arbitrary field. This fact brought conferred upon the resolution of singularities in positive characteristic a privileged place in the construction of cohomological theories, to the extent that it appeared as a matter of substance in this context. On the other hand, Grothendieck's comparison theorem became a completely new and unexpected way to understand regular singular points (at infinity) of integrable connections on algebraic fiber bundles (see footnote (13) in [32], dated July 1965, and the crucial work [21]), and more generally, a conceptual way to incorporate *regularity* as a central notion in $\mathscr{D}$-module theory (see [45]). Finally, $\mathscr{D}$-module theory itself has provided new tools to understand

Grothendieck's comparison theorem and to avoid resolution of singularities in its treatment [47, 48] (see also [49]).

In this paper we would like: (1) to explain what the logarithmic comparison means, drawing some analogies with Grothendieck's comparison theorem; (2) to survey the original proof of the Logarithmic Comparison Theorem for locally quasihomogeneous free divisors (see Theorem 12.2.5); and (3) to explain how $\mathscr{D}$-module theory provides a characterization of those free divisors which satisfy the Logarithmic Comparison Theorem (see Sect. 12.4.3). We emphasise that in all of them resolution of singularities does not play any rôle.

Remark 12.2.2 For any topological space X, one denotes by $H_q(X;\mathbb{C})$ the singular homology group of X with values in $\mathbb{C}$, for $q \geqslant 0$. Recall that the singular homology group $H_q(X;\mathbb{C})$ is by definition the homology group $h_q(C_\bullet(X), \partial)$ of the complex $(C_\bullet(X), \partial)$ of singular chains of X with coefficients in $\mathbb{C}$, where ∂ denotes the boundary map.

Recall also that the complex of singular cochains $(C^\bullet(X), \delta)$ is defined as

$$C^\bullet(X) := \mathrm{Hom}_{\mathbb{C}}(C_\bullet(X), \mathbb{C}),$$

where δ is the coboundary map associated with the boundary map ∂. The singular cohomology group $H^q(X;\mathbb{C})$, of X with values in $\mathbb{C}$ for $q \geqslant 0$, is by definition the cohomology group $h^q(C^\bullet(X), \delta)$.

There exists a natural morphism

$$H^q(X;\mathbb{C}) \xrightarrow{\psi_q} H_q(X;\mathbb{C})^* := \mathrm{Hom}_{\mathbb{C}}(H_q(X;\mathbb{C}), \mathbb{C}).$$

For any $[\eta] \in H^q(X;\mathbb{C})$ and any $[\gamma] \in H_q(X;\mathbb{C})$, one has

$$\psi_q([\eta])([\gamma]) = \eta(\gamma)$$

where [ ] means equivalence class in the corresponding (co)homology group. The morphism ψ_q is an isomorphism, for $q \geqslant 0$, since $\mathbb{C}$ is a field.

If X is a complex manifold with $\dim X = n \geqslant 1$, the de Rham morphism

$$dR_q : h^q(\Gamma(X, \Omega_X^\bullet)) \longrightarrow H^q(X;\mathbb{C})$$

is defined (or induced) by integration of differential q-forms over singular q-chains:

$$dR_q([\omega])([\gamma]) := \int_\gamma \omega \in \mathbb{C}$$

where ω is a q-form, γ is a singular q-chain and [ ] means equivalence class in the corresponding (co)homology group. Here, to be precise, one needs to consider C^∞ chains, and the point is that the inclusion of the complex of C^∞ singular chains of X with coefficients in $\mathbb{C}$ in $C_\bullet(X)$ is a quasi-isomorphism (see [44, Ap. A, Th. 2.1]).

The de Rham theorem states that if X is a Stein manifold then dR_q is an isomorphism for $q = 0, \dots, n$; see e.g. [44, App. A, Th. 3.1]. In the C^∞ category, the de Rham theorem holds in much greater generality, for all paracompact differentiable manifolds, (see *loc. cit.*).

Recall that by the Poincaré Lemma the complex (12.1), i.e. the complex

$$0 \to \mathbb{C}_X \to \Omega_X^\bullet,$$

is exact. Since the category $\mathrm{Mod}(\mathbb{C}_X)$, of sheaves of $\mathbb{C}$-vector spaces, has enough injectives (see e.g. [65, Ex. 2.3.12]), there exists an injective resolution

$$0 \to \mathbb{C}_X \to I_X^\bullet$$

and a natural morphism

$$\Omega_X^\bullet \to I_X^\bullet,$$

unique up to homotopy equivalence, making the obvious diagram commute (see [31, Rq. 3, Th. 2.4.1] and e.g. [65, 2.5]). So for any q, there is a natural morphism

$$\alpha_q : h^q(\Gamma(X, \Omega_X^\bullet)) \to h^q(\Gamma(X, I_X^\bullet)) = H^q(X; \mathbb{C}_X)$$

where the last equality is just the definition of the sheaf cohomology of the constant sheaf $\mathbb{C}_X$.

Moreover, as mentioned in the introduction, if X Stein, then each Ω_X^q is acyclic for the "global sections functor" $\Gamma(X; \ )$, and hence the natural morphism α_q is an isomorphism (see [31, Rq. 3, Th. 2.4.1] and e.g. [65, 2.5]).

Finally, for any complex manifold X (in fact, for any locally contractible topological space) let us consider, see e.g. [56, Ch. 4, Th. 4.14], the complex

$$0 \to \mathbb{C}_X \to \widetilde{S}_X^\bullet \tag{12.6}$$

where, for $q \geqslant 0$, one denotes by $\widetilde{S}_X^q$ the sheaf associated to the presheaf S_X^q of singular q-cochains on X. Local contractibility means that each point has a fundamental system of neighbourhoods which are contractible. In each of these, the complex of singular cochains is exact. It follows that the complex of sheaves (12.6) is exact. As before for the complex $\Omega_X^\bullet$ (but now with the complex $\widetilde{S}_X^\bullet$ in its place), and again using the injective resolution $I_X^\bullet$ of $\mathbb{C}_X$, for each q there is a natural morphism

$$\beta_q : H^q(X; \mathbb{C}) = h^q(\Gamma(X, \widetilde{S}_X^\bullet)) \to h^q(\Gamma(X, I_X^\bullet)) = H^q(X; \mathbb{C}_X)$$

where the first equality is just the definition of the singular cohomology $H^q(X; \mathbb{C})$ (see [56, Ch. 4, Prop. 4.12]). Since the sheaves $\widetilde{S}_X^q$ are all flabby they are also acyclic for the functor $\Gamma(X;\ )$ (see [56, Ch. 4, ex. 1.10, 2 and Prop. 3.3]). Hence the morphism β_q is an isomorphism for $q \geqslant 0$.

So, for any q we have the following diagram of isomorphisms

$$\begin{array}{ccc} h^q(\Omega^\bullet(X)) & \xrightarrow{dR_q} & H^q(X;\mathbb{C}) \\ \downarrow{\scriptstyle \alpha_q} & \swarrow{\scriptstyle \beta_q} & \\ H^q(X;\mathbb{C}_X). & & \end{array} \tag{12.7}$$

It is "well-known" that the diagram commutes, up to a sign, although a concrete reference for this result seems to be elusive.

12.2.2 The Brieskorn Complex

For a special class of divisors, namely hyperplane arrangements in affine space $X = \mathbb{C}^n$, Brieskorn, in [7] generalising Arnol'd in [1], had already given a way of calculating the cohomology of the complement. If D is the union of hyperplanes H_j with equations $h_j = 0$, $j = 1, \ldots, N$, then the collection of meromorphic 1-forms $\frac{dh_j}{h_j}$ generates a $\mathbb{C}$-subalgebra $B^\bullet$ of the exterior algebra $\Gamma(X, \Omega^\bullet_X(\star D))$. Since $d\left(\frac{dh_j}{h_j}\right) = 0$, all exterior derivatives on $B^\bullet$ are zero, and $B^\bullet$ is a subcomplex of the complex $\Gamma(X, \Omega^\bullet_X(\star D))$ with derivative zero, known as the *Brieskorn complex*. Brieskorn showed

Theorem 12.2.3 *The de Rham morphism $B^p \to H^p(U;\mathbb{C})$ is an isomorphism.* □

Note that each of Brieskorn's forms

$$\omega_{j_1,\ldots,j_k} := \frac{dh_{j_1}}{h_{j_1}} \wedge \cdots \wedge \frac{dh_{j_k}}{h_{j_k}}$$

has at most a first order pole along D, so in this special case his result is stronger than Grothendieck's.

An earlier version of Brieskorn's result, that it holds for a normal crossing divisor (NCD)([33]) played an important role in Deligne's mixed Hodge theory: every quasiprojective variety U has a projective compactification $U \hookrightarrow X$ in which $D := X \smallsetminus U$ is a NCD. Similarly, every singular variety has a resolution whose exceptional divisor is once again an NCD. Deligne used the poles to define the weight filtration on the cohomology of U.

If $D = \bigcup_{j=1}^N \{h_j = 0\}$ is a union of more general irreducible divisors $\{h_j = 0\}$, the analogous Brieskorn complex $B^\bullet$ generated by the closed forms $\frac{dh_j}{h_j}$ does *not* in general calculate the cohomology of the complement: for example, if D is irreducible, this complex reduces to $0 \to B^0 \to B^1 \to 0$, whereas $X \smallsetminus D$

may have cohomology in higher dimensions. Deligne and Dimca explored the relation between pole order and the Hodge filtration in the cohomology of complex projective varieties in [22].

12.2.3 The Logarithmic Comparison Theorem

Kyoji Saito, in [58], introduced a fruitful generalisation of the Brieskorn complex. A meromorphic form ω has a *logarithmic pole* along the divisor D if both $h\omega$ and $hd\omega$ are regular, where h is a local equation for D. Note in particular that all of Brieskorn's forms have logarithmic poles along D.

Denote the sheaf of germs of meromorphic k-forms with logarithmic poles by $\Omega^k_X(\log D)$. Then $\Omega^\bullet_X(\log D)$ is again a subcomplex of $\Omega^\bullet_X(\star D)$. In the same article Saito also described the dual of $\Omega^1_X(\log D)$, namely the sheaf of "logarithmic derivations", $\mathscr{D}er(-\log D)$, whose stalk at $x \in D$ consists of germs of vector fields on X which are tangent to D at its smooth points. The duality arises from the contraction pairing

$$\Omega^1_X(\log D) \times \mathscr{D}er(-\log D) \to \mathscr{O}_X, \quad (\omega, \xi) \mapsto \iota_\xi(\omega) = \omega(\xi).$$

Saito showed that this is a perfect pairing, so that $\Omega^1_X(\log D)$ is also the dual of $\mathscr{D}er(-\log D)$. Saito's interest was especially in the case where D is a *free divisor*, that is, where $\mathscr{D}er(-\log D)$ (or, equivalently, $\Omega^1_X(\log D)$) is a locally free sheaf of $\mathscr{O}_X$-modules. Because the dual of any $\mathscr{O}_X$-module has depth at least 2 when depth $\mathscr{O}_X \geqslant 2$ (see e.g. [60], or [38, Lemma 9.2]), it follows from this that every plane curve is a free divisor. More interesting is the fact, proved by K. Saito, that the discriminant in the base of a versal deformation of an isolated hypersurface singularity is a free divisor. Looijenga generalised this by showing in [38, Corollary 6.13] that it holds also for the discriminant in the base of a versal deformation of an isolated complete intersection singularity (ICIS), and later authors ([8, 63]) have extended this to the discriminants of a range of non-ICIS singularities, to discriminants in quiver representation spaces [9] and more generally in prehomogeneous vector spaces [29].

Since for any divisor $D \subset X$ one has a natural morphism

$$\Omega^\bullet_X(\log D) \hookrightarrow \Omega^\bullet_X(\star D)$$

one can ask for the class of divisors $D \subset X$ such that the previous natural morphism is a quasi-isomorphism (or an isomorphism in the derived category of sheaves of $\mathbb{C}_X$-vector spaces).

Definition 12.2.4 If the morphism $\Omega^\bullet_X(\log D) \hookrightarrow \Omega^\bullet_X(\star D)$ is a quasi-isomorphism we say that the divisor D satisfies the *logarithmic comparison theorem* (LCT), or that *the LCT holds for D*.

By Grothendieck's comparison theorem, the LCT holds for a divisor D if and only if the morphism $h^q(\Omega^\bullet_X(\log D)) \to \mathbb{R}^q j_* \mathbb{C}_U$ is an isomorphism for $0 \leqslant q \leqslant \dim X$. Any NCD satisfies the LCT, after [21, Cap. II, Lemme 6.9].

Theorem 12.2.5 (Logarithmic Comparison Theorem, [15]) *If $D \subset X$ is a locally quasihomogeneous free divisor with complement U, then the de Rham morphism*

$$\Omega^\bullet_X(\log D) \to \mathbb{R} j_* \mathbb{C}_U \tag{12.8}$$

is a quasi-isomorphism.

Here *locally quasihomogeneous* means that at each point $p \in D$ there are local coordinates on X in which D has a weighted homogeneous local equation with all weights strictly positive—in other words, locally there is a good $\mathbb{C}^*$-action centred at p. Divisors with this property are also known as *strongly quasihomogeneous*, and as *positive*. Evidently hyperplane arrangements have this property, and it also holds for the discriminants of stable maps in Mather's "nice dimensions", because of the remarkable (and unexplained) fact that in the nice dimensions, all stable germs are quasihomogeneous in suitable coordinates (see [51, Section 7.4] for a pedestrian proof of this).

Local quasihomogeneity is used twice in the proof of Theorem 12.2.5. First, it allows an inductive argument on the dimension of the divisor.

Lemma 12.2.6 *Let X be a complex manifold of dimension n, let D be a strongly quasihomogeneous divisor in X, and let $p \in D$. Then there is an open neighbourhood U of p such that for each $q \in U \cap D$, with $q \neq p$, the germ at q of the pair (X, D) is isomorphic to the germ at 0 of a product $(\mathbb{C}^{n-1} \times \mathbb{C}, D_0 \times \mathbb{C})$ where D_0 is a strongly quasihomogeneous divisor.*

To prove that (12.8) is an isomorphism at $p \in D$, one uses induction on the dimension of the divisor. It is easy to see that if (12.8) holds for a a divisor $D_0 \subset \mathbb{C}^{n-1}$ then it holds for $D_0 \times \mathbb{C} \subset \mathbb{C}^{n-1} \times \mathbb{C}$. Lemma 12.2.6 therefore says that we may assume by induction that (12.8) holds at all points of $D \cap U \setminus \{p\}$. Note that the induction begins with the divisor $0 \subset \mathbb{C}$; it is well known that $H^1(\mathbb{C} \setminus \{0\}; \mathbb{C}) \simeq \mathbb{C}$, generated by the logarithmic form $\frac{dz}{z}$. Thus in what follows we assume $n \geqslant 2$.

The main body of the proof of Theorem 12.2.5 then consists in showing that (12.8) also holds at p (which for convenience we suppose is the point $(0, \ldots, 0)$), by showing that for any sufficiently small polycylinder V centred at p, the de Rham morphism

$$h^p(\Gamma(V, \Omega^\bullet_X(\log D)) \to H^p(V \setminus D; \mathbb{C})$$

is an isomorphism. The argument involves two Čech-de Rham double complexes associated with Stein covers $\{V_i\}$ of $V \setminus \{0\}$, with $V_i = V \setminus \{x_i = 0\}$, and $\{V'_i = V_i \setminus D\}$ of $V \setminus D$, and the two standard spectral sequences associated to each one.

The complexes are

$$K^{p,q} = \bigoplus_{i_0<i_1<\cdots<i_q} \Gamma\left(\bigcap_{j=0}^{q} V_{i_j}, \Omega_X^p(\log D)\right) \tag{12.9}$$

and

$$\tilde{K}^{p,q} = \bigoplus_{i_0<i_1<\cdots<i_q} \Gamma\left(\bigcap_{j=0}^{q} V'_{i_j}, \Omega_X^p\right) \tag{12.10}$$

with differentials d, the exterior derivative, and $\check{d}$, the Cech differential. The inclusion $V'_i \hookrightarrow V_i$ determines a morphism $\rho_0 : K^{p,q} \to \tilde{K}^{p,q}$ which commutes with d and $\check{d}$ and thus gives rise to morphisms of the associated spectral sequences, which we denote by ${}'\rho_\ell$ and ${}''\rho_\ell$, where the subindex ℓ refers to the page of the spectral sequence.

First Spectral Sequences
Applying d to (12.9) and (12.10), we get the first page of the first spectral sequence associated to each double complex:

$${}'E_1^{p,q} = h^p(K^{\bullet,q}) = \bigoplus_{i_0<i_1<\cdots<i_q} h^p\left(\Gamma\left(\bigcap_{j=0}^{q} V_{i_j}, \Omega_X^\bullet(\log D)\right)\right)$$

and

$$\begin{aligned} {}'\tilde{E}_1^{p,q} = h^p(\tilde{K}^{\bullet,q}) &= \bigoplus_{i_0<i_1<\cdots<i_q} h^p\left(\Gamma\left(\bigcap_{j=0}^{q} V'_{i_j}, \Omega_X^\bullet\right)\right) \\ &= \bigoplus_{i_0<i_1<\cdots<i_q} H^p(\bigcap_{j=0}^{q} V'_{i_j}; \mathbb{C}). \end{aligned}$$

Since 0 is excluded from all of the open sets in the covers, the induction hypothesis implies that ${}'\rho_1^{p,q} : {}'E_1^{p,q} \to {}'\tilde{E}_1^{p,q}$ is an isomorphism for all p, q, and it follows that ${}'\rho_\infty^{p,q}$ is also an isomorphism. Thus, ρ_0 induces an isomorphism of the cohomology of the total complexes of $K^{\bullet,\bullet}$ and $\tilde{K}^{\bullet,\bullet}$.

Second Spectral Sequences

Because $\{V_i\}$ and $\{V_i'\}$ are Stein covers, applying $\check{d}$ gives, as first pages of the second spectral sequences,

$$''E_1^{p,q} = H^q(V \smallsetminus \{0\}, \Omega_X^p(\log D)) \quad \text{and} \quad ''\tilde{E}_1^{p,q} = H^q(V \smallsetminus D, \Omega_X^p).$$

Because D is a free divisor, all of the $\Omega_X^p(\log D)$, as well as the Ω_X^p, are free $\mathcal{O}_X$-modules, and thus since $H^q(V \smallsetminus \{0\}, \mathcal{O}_X)$ is zero except for $q = 0$ and $q = n-1$ (see e.g. [38, (8.14)]), $H^q(V \smallsetminus \{0\}, \Omega_X^p(\log D))$ also vanishes except in these dimensions. In other words

$$''\tilde{E}_1^{p,q} = 0 \;\text{ except for } q = 0 \text{ and } q = n-1.$$

Now because $V \smallsetminus D$ is a Stein space and Ω_X^p is coherent, $''\tilde{E}_1^{p,q} = H^q(V \smallsetminus D, \Omega_X^p)$ is equal to 0 for all $q > 0$, and the spectral sequence $''\tilde{E}$ collapses at E_2, with

$$''\tilde{E}_\infty^{p,0} = H^p(V \smallsetminus D; \mathbb{C}), \quad ''\tilde{E}_\infty^{p,q} = 0 \;\text{ if } q > 0. \tag{12.11}$$

We claim that $''E$ also collapses at E_2. In view of the vanishing already remarked upon, it is enough to show that the complex $(''E_1^{\bullet,q}, d)$, i.e.

$$\begin{aligned} 0 \to H^{n-1}(V \smallsetminus \{0\}, \Omega_X^0(\log D)) &\to H^{n-1}(V \smallsetminus \{0\}, \Omega_X^1(\log D)) \to \cdots \\ \to H^{n-1}(V \smallsetminus \{0\}, \Omega_X^t(\log D)) &\to 0 \end{aligned} \tag{12.12}$$

is exact. This is the only point at which the argument departs from "general nonsense".

Each of the groups in (12.12) is generated by classes of the form $c_\alpha \omega_\alpha$ where $c_\alpha \in H^{n-1}(V \smallsetminus \{0\}, \mathcal{O}_X)$ and ω_α is a basis element of $\Omega_X^p(\log D)$ in V.

Recall that $(D, 0)$ is assumed weighted homogeneous in V, with respect to positive weights $w_1, \ldots, w_t$ for the coordinate functions. We have

$$H^{n-1}(V \smallsetminus \{0\}, \mathcal{O}_X) = \frac{\Gamma(V \smallsetminus \bigcup_i \{x_i = 0\}, \mathcal{O}_X)}{\sum_j \Gamma(V \smallsetminus \bigcup_{i \neq j} \{x_i = 0\}, \mathcal{O}_X)}.$$

Each term in the numerator can be represented by a Laurent series in which all exponents are negative; series in the denominator have all exponents negative, bar one. An easy lemma shows that $\Omega_X^p(\log D)_0$ has a basis consisting of forms of weighted degree strictly less than $\sum_j w_j$. It follows that *the complex* (12.12) *has zero weight-zero part*. But since the exterior derivative d preserves weighted degrees, it follows that using the Lie derivative with respect to the Euler form, one can construct a contracting homotopy from the complex to its weight zero part. Thus, it is exact.

We are left with

$$''E_\infty^{p,0} = h^p(\Gamma(V \setminus \{0\}, \Omega_X^\bullet(\log D)) = h^p(\Gamma(V, \Omega_X^\bullet(\log D))),$$
$$''E_\infty^{p,q} = 0 \ \text{ if } q > 0 \tag{12.13}$$

(since $tn > 1$ and $\Omega^p(\log D)$ is locally free, 0 is a removable singularity). The theorem now follows from (12.11), (12.13) and the isomorphism of the cohomology of the total complexes of $K^{\bullet,\bullet}$ and $\tilde{K}^{\bullet,\bullet}$.

The importance of local quasihomogeneity for logarithmic comparison theorem remains unclear. It was shown to be necessary for plane curves in [14], and for free surfaces in 3-space in [30], but in higher dimensions, local weak quasihomogeneity is sufficient, see [19], [53, Remark 1.7.4] and Theorem 12.4.14. In [34] the authors characterize the family of quasi-homogeneous divisors with isolated singularity satisfying LCT. In [3, Cor. 1] it is proven that if D is a divisor with isolated singularities and D satisfies LCT then it is locally quasi-homogeneous.

Definition 12.2.7 A divisor D is locally weakly quasihomogeneous [19, Def. 2.1] if at each point $p \in D$ there are local coordinates on X in which D has a weak weighted homogeneous local reduced equation $f = 0$, that is: all the weights are non negative and not all of them are 0, with f of strictly positive weight.

12.2.4 Conjecture of Terao

H. Terao conjectured in [61, Conjecture 3.1] that LCT always holds for hyperplane arrangements. After partial results of Wiens and Yuzvinsky in [66], the conjecture was finally proved by Daniel Bath in [2]. Bath's proof uses induction on the codimension of the flats of the arrangement, and a spectral sequence argument similar to the argument of [15] given above. The argument of [2] is not obviously extendable outside the case of hyperplane arrangements, due to the particular strategy employed for studying the Castelnuovo-Mumford regularity of logarithmic forms for arrangements.

12.2.5 Linear Free Divisors and Logarithmic Comparison Revisited

A free divisor $D \subset \mathbb{C}^n$ is *linear* if $\mathrm{Der}(-\log D) = \Gamma(\mathbb{C}^n, \mathscr{D}er(-\log D))$ has a basis of vector fields whose coefficients are linear forms. Normal crossing divisors are of course the simplest examples, and are in fact the only linear free divisors among hyperplane arrangements. In [9] R.-O. Buchweitz and the second author showed that the discriminant in the representation space of a Dynkin quiver, with a root of the underlying diagram as dimension vector, is a linear free divisor. All irreducible linear free divisors are classified, though not under that name, in [59], where the operation of *castling* derives them from four basic examples.

If $D \subset \mathbb{C}^n$ is a linear free divisor then the set of weight zero vector fields in $\mathrm{Der}(-\log D)$ is a Lie algebra of dimension n; it is isomorphic to the Lie algebra of the Lie subgroup of G_D consisting of linear automorphisms of the pair $(\mathbb{C}^n, D)$. For linear free divisors, [28] gives an argument for the logarithmic comparison theorem which is quite different from the proof of [15]. When D is a linear free divisor, then it turns out that $\mathbb{C}^n \smallsetminus D$ is a single orbit of the identity component G_D^0 of the group G_D, so $H^*(\mathbb{C}^n \smallsetminus D; \mathbb{C})$ is the cohomology of G_D^0/S_p, where S_p is the isotropy subgroup of a point p. Since G_D^0 is path connected, the action of S_p on $H^*(G_D^0; \mathbb{C})$ is trivial, so $H^*(G_D^0/S_p; D) = H^*(G_D^0; \mathbb{C})$. Second, in this case the weight zero subcomplex of $\Gamma(\mathbb{C}^n, \Omega^\bullet_{\mathbb{C}^n}(\log D))$ coincides with the complex of Lie algebra cohomology, with complex coefficients, of the Lie algebra $\mathfrak{g}_D$ of G_D^0. By an argument using weighted homogeneity, $\Gamma(\mathbb{C}^n, \Omega^\bullet_{\mathbb{C}^n}(\log D))$ is chain homotopic to its weight zero subcomplex. Thus, $\Gamma(\mathbb{C}^n, \Omega^\bullet_{\mathbb{C}^n}(\log D))$ coincides with the Lie algebra cohomology of $\mathfrak{g}_D$ with complex coefficients. For compact connected Lie groups G, a well-known argument shows that the Lie algebra cohomology coincides with the topological cohomology of the group. For linear free divisors the group G_D^0 is never compact, but the isomorphism also holds good for the larger class of reductive groups, and for a significant class of linear free divisors, including all those mentioned above, G_D^0 is indeed reductive.

12.2.6 A Pairing $H^p(X \smallsetminus D) \times H^q(D) \to H^{p+q}(D)$

The paper [50] introduces a variant $\check{\Omega}^k_D$ of the module of Kähler forms Ω^k_D, defined by

$$\check{\Omega}^k_D := \frac{\Omega^k_X}{h\Omega^k_X(\log D)},$$

where h is a reduced equation for D. Note that $\check{\Omega}^0_D = \Omega^0_D = \mathcal{O}_D$. Since $\frac{dh}{h}$ has a logarithmic pole and $\Omega^k_X(\log D) \supset \Omega^k_X$, it follows that $h\Omega^k_X(\log D) \supseteq dh \wedge \Omega^{k-1}_X + h\Omega^k_X$, so $\check{\Omega}^k_D$ is a quotient of Ω^k_D. If D is a free divisor then each $\check{\Omega}^k_D$ is a maximal Cohen-Macaulay $\mathcal{O}_D$-module, and coincides with Ω^k_D at smooth points of D. The $\check{\Omega}^k_D$ form a complex with respect to the usual exterior derivative: if $\omega \in \Omega^k_X(\log D)_x$ then $\frac{dh}{h} \wedge \omega \in \Omega^{k+1}_X(\log D)$, from which it follows that $d(h\omega) \in h\Omega^{k+1}_X(\log D)$. We denote the exterior derivative on this complex by $\check{d}$. If D is quasihomogeneous then $(\check{\Omega}^\bullet_D, \check{d})$ is a resolution of $\mathbb{C}_D$ ([50, Lemma 3.3]).

Straightforward calculations show

(i) there is a well-defined pairing

$$\Omega^k_X(\log D) \times \check{\Omega}^\ell_D \to \check{\Omega}^{k+\ell}_D, \quad \text{defined by } (\omega_k, \omega_\ell) \mapsto \omega_k \wedge \omega_\ell. \qquad (12.14)$$

Note that there is no comparable wedge pairing

$$\Omega^k_X(\log D) \times \Omega^\ell_D \to \Omega^{k+\ell}_D.$$

For such a pairing to be well defined, the wedge of $\omega_k \in \Omega^k_X(\log D)$ with $\omega_\ell \in (dh \wedge \Omega^{\ell-1}_X + h\Omega^\ell_X)$ would have to lie in $dh \wedge \Omega^{k+\ell-1}_X + h\Omega^{k+\ell}_X$, and in general this does not hold. For example, if $h = h_1 h_2$, and we take $\omega_1 = \frac{dh_1}{h_1} \in \Omega^1_X(\log D)$ and $\omega_1' = dh \in dh \wedge \Omega^0_X$, then

$$\omega_1 \wedge \omega_1' = \frac{dh_1}{h_1} \wedge \big(h_1 dh_2 + h_2 dh_1\big) = dh_1 \wedge dh_2 \notin \big(dh \wedge \Omega^1_X + h\Omega^2_X\big).$$

(ii) The pairing (12.14) descends to a pairing on the homology of the two complexes,

$$h^k(\Gamma(X, \Omega^\bullet_X(\log D)) \times h^\ell(\Gamma(X, \check{\Omega}^\bullet_D)) \to h^{k+\ell}(\Gamma(X, \check{\Omega}^\bullet_D)).$$

(iii) When D is locally quasihomogeneous then in view of the LCT and the exactness of $\check{\Omega}^\bullet_D$, this gives a pairing

$$H^k(X \setminus D; \mathbb{C}) \times H^\ell(D; \mathbb{C}) \to H^{k+\ell}(D; \mathbb{C}).$$

The properties of this pairing remain to be explored.

12.3 $\mathscr{D}_X$-modules

In this section we recall some of the basics of $\mathscr{D}$-module theory. We mainly follow [37, 46] and [27]. The section contains the necessary terminology and results that are used in the subsequent sections.

12.3.1 Basic Objects

Let X be a complex manifold of dimension $n \geqslant 1$ and $\mathscr{O}_X$ (or simply $\mathscr{O}$) be the sheaf of germs of holomorphic functions on X. The sheaf of rings (or more precisely of $\mathbb{C}$-algebras) $\mathscr{O}_X$ is coherent: this is Oka's theorem [41, 55].

We denote by $\mathscr{D}_X$ (or simply by $\mathscr{D}$) the sheaf of linear differential operators on X with holomorphic coefficients [37, I, §1], [27, Def. 3]. Sometimes we say *operators (or differential operators) on* X instead of linear differential operators on X with holomorphic coefficients. For any open set $U \subseteq X$, $\mathscr{D}(U)$ is a $\mathscr{O}(U)$-algebra and so $\mathscr{D}$ is a sheaf of $\mathscr{O}$-algebras.

If $(U; x_1, \dots, x_n)$ is a chart in X and $U \subset X$ is connected, any operator $P \in \mathscr{D}(U)$ can be written in a unique way as a finite sum

$$P = \sum_{\alpha\in\mathbb{N}^n} c_\alpha \partial^\alpha = \sum_{\alpha\in\mathbb{N}^n} c_\alpha \partial_1^{\alpha_1} \cdots \partial_n^{\alpha^n}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, $c_\alpha \in \mathscr{O}(U)$ and ∂_i is the partial derivative $\frac{\partial}{\partial x_i}$. For any $x \in X$, the stalk $\mathscr{D}_{X,x}$ is a non-commutative $\mathscr{O}_{X,x}$-algebra, since $\partial_i x_i - x_i \partial_i = 1$ for $1 \leqslant i \leqslant n$.

If $P \in \mathscr{D}(U)$ is non zero, the order of P is the non negative integer

$$\operatorname{ord}(P) = \max\{|\alpha|\,;\ c_\alpha \neq 0\}$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We also write $\operatorname{ord}(0) = -\infty$.

For $k \in \mathbb{N}$, we denote by $F^k(\mathscr{D})$ the subsheaf of $\mathscr{D}$ whose sections can be written locally as operators of order less than or equal to k. Each $F^k(\mathscr{D})$ is a sheaf of coherent $\mathscr{O}$-modules and the family $F^\bullet := (F^k(\mathscr{D}))_k$ is a discrete increasing exhaustive filtration of $\mathscr{D}$. One has $F^0(\mathscr{D}) = \mathscr{O}$ and $F^k(\mathscr{D})F^\ell(\mathscr{D}) = F^{k+\ell}(\mathscr{D})$ for $k, \ell \geqslant 0$. The associated sheaf of graded rings (and more precisely, of graded $\mathbb{C}$-algebras) is

$$\operatorname{gr}_{F^\bullet}(\mathscr{D}) := \bigoplus_{k\in\mathbb{N}} \frac{F^k(\mathscr{D})}{F^{k-1}(\mathscr{D})}$$

where we write $F^{-1}(\mathscr{D}) = \{0\}$.

If P, Q are local sections in $\mathscr{D}$, the difference $[P, Q] := PQ - QP$ is called the commutator of (P, Q). For $x \in X$, the stalk $\operatorname{gr}_{F^\bullet}(\mathscr{D}_x) \simeq \operatorname{gr}_{F^\bullet}(\mathscr{D})_x$ is a commutative ring since $\operatorname{ord}([P, Q]) \leqslant \operatorname{ord}(P) + \operatorname{ord}(Q) - 1$. Thus, $\operatorname{gr}_{F^\bullet}(\mathscr{D})$ is a sheaf of commutative $\mathbb{C}$-algebras.

The sheaves of rings $\mathscr{D}_X$ and $\operatorname{gr}_{F^\bullet}(\mathscr{D}_X)$ are coherent (see [37, I, Th. 3.2], [27, Prop. 9]).

We denote

$$\sigma_k : F^k(\mathscr{D}) \to \frac{F^k(\mathscr{D})}{F^{k-1}(\mathscr{D})}$$

the natural projection map (which is a morphism of $\mathscr{O}$-modules) and we call it the kth *symbol map*.

If P is an operator of order k we simply write

$$\sigma(P) = \sigma_k(P)$$

and we call this element *the principal symbol* of P.

If $(x_1, \dots, x_n)$ is a system of local coordinates around a point $x \in X$, then $\partial_i = \frac{\partial}{\partial x_i} \in F^1(\mathscr{D}_x)$ and we write $\xi_i = \sigma(\partial_i)$ for $1 \leqslant i \leqslant n$.

There is an isomorphism of $\mathscr{O}_x$-algebras

$$\mathrm{gr}_{F^\bullet}(\mathscr{D}_x) \longrightarrow \mathscr{O}_x[\xi_1, \dots, \xi_n] \tag{12.15}$$

see e.g. [37, I, §1], [27, Prop. 4]; the image of an element $P + F^{k-1}(\mathscr{D}_x)$, with $P \in F^k(\mathscr{D}_x)$, being simply $\sigma_k(P)$. Compare this morphism with (12.3.3). Since $\mathrm{gr}_{F^\bullet}(\mathscr{D}_x)$ is noetherian, a standard argument using induction on the order of the operators, proves that $\mathscr{D}_x$ is a left and right noetherian ring.

12.3.2 Filtrations

Let $\mathscr{M}$ be a left $\mathscr{D}$-module (i.e. a sheaf of left $\mathscr{D}$-modules). A filtration of $\mathscr{M}$ is a collection, $(\mathscr{M}_k)_{k\in\mathbb{N}}$, of $\mathscr{O}$-submodules of $\mathscr{M}$ such that:

(i) For any $k, \ell \in \mathbb{N}$: $\mathscr{M}_k \subset \mathscr{M}_{k+1}$ and $F^\ell(\mathscr{D})\mathscr{M}_k \subset \mathscr{M}_{k+\ell}$.
(ii) $\mathscr{M} = \cup_{k\in\mathbb{N}}\mathscr{M}_k$.

To any filtration $\Gamma := (\mathscr{M}_k)_{k\in\mathbb{N}}$ of $\mathscr{M}$ we associate its graded $\mathscr{O}$-module

$$\mathrm{gr}_\Gamma(\mathscr{M}) := \bigoplus_{k\in\mathbb{N}} \frac{\mathscr{M}_k}{\mathscr{M}_{k-1}}$$

with $\mathscr{M}_{-1} := \{0\}$. More generally, $\mathrm{gr}_\Gamma(\mathscr{M})$ is a $\mathrm{gr}_{F^\bullet}(\mathscr{D})$-module. A filtration $(\mathscr{M}_k)_k$ of $\mathscr{M}$ is said to be *good* is the following two conditions hold

(i) For any $k \in \mathbb{N}$, $\mathscr{M}_k$ is a coherent $\mathscr{O}_X$-module.
(ii) There exist $k_0 \in \mathbb{N}$ such that for any $\ell \in \mathbb{N}$, $F^\ell(\mathscr{D})\mathscr{M}_{k_0} = \mathscr{M}_{k_0+\ell}$.

Any coherent $\mathscr{D}$-module $\mathscr{M}$ admits locally (i.e. after restriction to sufficiently small open subsets in X) good filtrations (see e.g. [37, I, p. 7], [27, Prop. 10]).

Let $\Gamma := (\mathscr{M}_k)_k$ be a filtration of $\mathscr{M}$. Then Γ is locally good (i.e. after restriction to sufficiently small open subsets in X, Γ is a good filtration) if and only if $\mathrm{gr}_\Gamma(\mathscr{M})$ is a coherent $\mathrm{gr}_{F^\bullet}(\mathscr{D})$-module (see [37, I, Prop. 4.1], [27, Th. 1]).

Let $\mathscr{M}$ be a coherent left $\mathscr{D}_X$-module. Then there exists a coherent sheaf of ideals $\mathscr{J}(\mathscr{M})$ of $\mathrm{gr}_{F^\bullet}(\mathscr{D}_X)$ with the following property: for any open subset $U \subset X$ such that the restriction $\mathscr{M}_U$ admits a good filtration, one has $\mathscr{J}(\mathscr{M})_U \simeq \sqrt{\mathrm{ann}_{\mathrm{gr}(\mathscr{D}_U)}(\mathrm{gr}(\mathscr{M}_U))}$, where $\mathrm{gr}(\mathscr{D}_U) = \mathrm{gr}_{F^\bullet}(\mathscr{D}_U)$ and $\mathrm{ann}_{\mathrm{gr}(\mathscr{D}_U)}(\mathrm{gr}(\mathscr{M}_U))$ is the sheaf of annihilating ideals of the coherent $\mathrm{gr}(\mathscr{D}_U)$-module $\mathrm{gr}(\mathscr{M}_U)$ (see [27, Prop.17; Rmk. 6]).

Definition 12.3.1 Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. The closed analytic subset of the cotangent bundle T^*X defined by the coherent sheaf of ideals $\mathscr{J}(\mathscr{M})$ is called *the characteristic variety* of $\mathscr{M}$ and is denoted by $\mathrm{Char}(\mathscr{M})$.

Example 12.3.2

1) The characteristic variety of the $\mathscr{D}_X$-module $\{0\}$ is $\emptyset$, and $\mathrm{Char}(\mathscr{D}_X) = T^*X$ and then $\dim \mathrm{Char}(\mathscr{D}_X) = 2n = 2\dim X$.
2) If $\mathscr{M} = \frac{\mathscr{D}_X}{\mathscr{I}}$ where $\mathscr{I} \subset \mathscr{D}_X$ is a coherent sheaf of left ideals in $\mathscr{D}_X$, the characteristic variety $\mathrm{Char}(\mathscr{M})$ is the closed analytic subset of T^*X defined by the coherent sheaf of ideals $\mathrm{gr}_{F^\bullet}(\mathscr{I}) \subset \mathrm{gr}_{F^\bullet}(\mathscr{D}_X)$, since the filtration on the quotient $\mathscr{M}$, induced by the $F^\bullet$ filtration on $\mathscr{D}_X$, is a locally good filtration.
3) The $\mathscr{D}_X$-module $\mathscr{O}_X$ can be presented locally as a quotient $\mathscr{D}_{X,x}/\mathscr{D}_{X,x}(\partial_1, \cdots, \partial_n)$ where $\{\partial_1, \ldots, \partial_n\}$ is a basis of $\mathscr{D}er(\mathscr{O}_{X,x})$. The graded ideal $\mathrm{gr}(\mathscr{D}_{X,x}(\partial_1, \cdots, \partial_n))$ equals $\mathrm{gr}(\mathscr{D})(\xi_1, \ldots, \xi_n)$ (recall that ξ_i is the principal symbol of ∂_i (12.3.1)). Then $\mathrm{Char}(\mathscr{O}_X) = T^*_X X$, that is, the zero section of the cotangent bundle T^*X, and $\dim \mathrm{Char}(\mathscr{O}_X) = n$.

The characteristic variety $\mathrm{Char}(\mathscr{M}) \subset T^*X$ of a coherent $\mathscr{D}_X$-module is involutive (e.g. [25, 42], [27, App. B]). Hence, if $\mathscr{M}$ is a non zero module, one has $2n \geqslant \dim(\mathrm{Char}(\mathscr{M})) \geqslant n$. This is called Bernstein's inequality. It is proved in [4] for modules over the Weyl algebra.

Definition 12.3.3 A coherent $\mathscr{D}_X$-module is said to be *holonomic* if either $\mathscr{M} = \{0\}$ or $\dim(\mathrm{Char}(\mathscr{M})) = \dim X$.

The $\mathscr{D}_X$-module $\mathscr{O}_X$ is holonomic while $\mathscr{D}_X$ is not holonomic as $\mathscr{D}_X$-module. If $\mathscr{M} = \frac{\mathscr{D}_X}{\mathscr{I}}$ where $\mathscr{I}$ is a sheaf of non zero locally principal ideals (i.e. locally generated by a single differential operator) then $\mathscr{M}$ is holonomic if and only if $\dim X = 1$, since $\dim \mathrm{Char}(\frac{\mathscr{D}_X}{\mathscr{I}}) = 2n - 1$. A central result in $\mathscr{D}_X$-module theory is

Theorem 12.3.4 *[36] If D is a hypersurface in a complex manifold X then the $\mathscr{D}_X$-module $\mathscr{O}_X(\star D)$ of meromorphic functions in X with poles on D is holonomic.*

We finish this subsection with a result that we use later on (see Sect. 12.3.5):

Theorem 12.3.5 *A non zero coherent $\mathscr{D}_X$-module $\mathscr{M}$ is holonomic if and only if $\mathscr{E}xt^i_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X) = 0$ for $i \neq n$. In this case, $\mathscr{E}xt^n_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X)$ is a coherent holonomic right $\mathscr{D}_X$-module.*

The proof follows from [37, IV; Ths. 4.2.5, 4.2.6], see also [27, Ths. 7, 8].

12.3.3 The Natural Isomorphism $\mathbf{Sym}_{\mathscr{O}_X}(\mathscr{D}er(\mathscr{O}_X)) \xrightarrow{\sim} \mathbf{gr}_{F^\bullet}(\mathscr{D}_X)$

There is a natural injective morphism of $\mathscr{O}_X$-modules

$$\mathscr{D}er(\mathscr{O}_X) \xrightarrow{\iota} \mathrm{gr}^{(1)}_{F^\bullet}(\mathscr{D}) \subset \mathrm{gr}_{F^\bullet}(\mathscr{D})$$

mapping a local section $\delta \in \mathscr{D}er(\mathscr{O}_X)$ to $\sigma(\delta) \in \mathrm{gr}^{(1)}_{F^\bullet}(\mathscr{D})$. Then, by the universal property of the symmetric algebra (see [6, §6, Prop. 2]) there exists a unique morphism of $\mathscr{O}_X$-algebras

$$\kappa : \mathrm{Sym}_{\mathscr{O}_X}(\mathscr{D}er(\mathscr{O}_X)) \to \mathrm{gr}_{F^\bullet}(\mathscr{D}_X), \tag{12.16}$$

extending the morphism $\iota : \mathscr{D}er(\mathscr{O}_X) \to \mathrm{gr}_{F^\bullet}(\mathscr{D})$. Since $\mathrm{gr}_{F^\bullet}(\mathscr{D}_X)$ is a graded $\mathscr{O}_X$-algebra and $\iota(\mathscr{D}er(\mathscr{O}_X)) \subset \mathrm{gr}^{(1)}_{F^\bullet}(\mathscr{D})$, by [Remark 1, p. 498, loc.cit.] the morphism κ is graded so that we have, for any $k \geqslant 0$ a morphism of $\mathscr{O}_X$-modules

$$\kappa_k : \mathrm{Sym}^{(k)}_{\mathscr{O}_X}(\mathscr{D}er(\mathscr{O}_X)) \to \mathrm{gr}^{(k)}_{F^\bullet}(\mathscr{D}_X).$$

By Bourbaki [6, §6, Th. 1] and since $\mathscr{D}er(\mathscr{O}_X)$ is a locally free $\mathscr{O}_X$-module of rank n, for any $x \in X$, $\mathrm{Sym}_{\mathscr{O}_{X,x}}(\mathscr{D}er(\mathscr{O}_{X,x}))$ is canonically isomorphic to the polynomial algebra $\mathscr{O}_{X,x}[T_1, \ldots, T_n]$, the canonical isomorphism being obtained by mapping ∂_i to T_i for $1 \leqslant i \leqslant n$. Locally, for $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $|\alpha| = \sum_i \alpha_i = k$, one has $\kappa_k(\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}) = \sigma(\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n})$ where $\sigma(\ )$ means the principal symbol (see (12.3.1)). Then κ_k is an isomorphism for any k and κ is a graded isomorphism. The isomorphism κ is the intrinsic version of the isomorphism (12.15).

12.3.4 The Spencer Complex

Let X be a complex manifold of dimension $n \geqslant 1$.

Definition 12.3.6 ([46, Chap I, (2.1)]) The Spencer complex in X is the following complex of left $\mathscr{D}_X$-modules, denoted $\mathrm{Sp}^\bullet_X$ (or simply $\mathrm{Sp}^\bullet$):

$$0 \to \mathscr{D} \otimes_{\mathscr{O}_X} \bigwedge^n \mathscr{D}er(\mathscr{O}_X) \xrightarrow{\epsilon_{-n}} \cdots \xrightarrow{\epsilon_{-2}} \mathscr{D} \otimes_{\mathscr{O}_X} \mathscr{D}er(\mathscr{O}_X) \xrightarrow{\epsilon_{-1}} \mathscr{D}$$

where the differential $\epsilon_\bullet$ is defined by

$$\epsilon_{-1}(P \otimes \delta) = P\delta,$$

$$\epsilon_{-p}(P \otimes (\delta_1 \wedge \cdots \wedge \delta_p)) = \sum_{i=1}^{p} (-1)^{i-1} P\delta_i \otimes (\delta_1 \wedge \cdots \widehat{\delta_i} \cdots \wedge \delta_p)) +$$

$$\sum_{1 \leqslant i<j \leqslant p} (-1)^{i+j} P \otimes ([\delta_i, \delta_j] \wedge \delta_1 \wedge \cdots \widehat{\delta_i} \cdots \widehat{\delta_j} \cdots \wedge \delta_p) \ \ (2 \leqslant p \leqslant n).$$

We denote by $\widetilde{\mathrm{Sp}^\bullet}_X$ (or simply $\widetilde{\mathrm{Sp}^\bullet}$) the augmented complex

$$\mathrm{Sp}^\bullet_X \xrightarrow{\epsilon_0} \mathscr{O}_X \to 0$$

where $\epsilon_0(P) = P(1)$.

Remark 12.3.7 The complex $\mathrm{Sp}^\bullet_X$ is a locally free resolution of the left $\mathscr{D}_X$-module $\mathscr{O}_X$. Since $\mathscr{D}er(\mathscr{O}_X)$ is a locally free $\mathscr{O}_X$-module, each left $\mathscr{D}_X$-module

$$\mathrm{Sp}^p_X := \mathscr{D}_X \otimes_{\mathscr{O}_X} \bigwedge^p \mathscr{D}er(\mathscr{O}_X)$$

is locally free (of rank $\binom{n}{p}$).

To prove that $\mathrm{Sp}^\bullet_X$ is a resolution of $\mathscr{O}_X$, or equivalently that $\widetilde{\mathrm{Sp}^\bullet}_X$ is acyclic, we can proceed locally at each point $x \in X$. If we choose local coordinates $(x_1, \ldots, x_n)$ on X around x, the partial derivatives $(\partial_1, \ldots, \partial_n)$—where $\partial_i = \frac{\partial}{\partial x_i}$—form a basis of the free $\mathscr{O}_X$-module $\mathscr{D}er(\mathscr{O}_X)_x$. The differential $\epsilon_{-p,x}$ can be written as

$$\epsilon_{-p,x}(P \otimes (\partial_{i_1} \wedge \cdots \wedge \partial_{i_p})) = \sum_{j=1}^{p} (-1)^{j-1} P\partial_{i_j} \otimes (\partial_{i_1} \wedge \cdots \widehat{\partial_{i_j}} \cdots \wedge \partial_{i_p}).$$

So, the complex $\widetilde{\mathrm{Sp}^\bullet}_{X,x}$ is a Koszul complex (which is denoted by $K(\partial_1, \ldots, \partial_n; \mathscr{D}_X)$ in [43, I.2]. This complex is exact (see e.g. [24]). We give here a proof of the acyclicity of $\widetilde{\mathrm{Sp}^\bullet}_{X,x}$ based on [10, Th. 3.1.2; Prop. 4.1.3] (which proves a more general result and follows a suggestion of B. Malgrange [43, I.2]).

We consider a discrete increasing filtration $G^\bullet := G^\bullet\left(\widetilde{\mathrm{Sp}^\bullet}_X\right)$ on the complex $\widetilde{\mathrm{Sp}^\bullet}_X$ (or more precisely on the complex $\widetilde{\mathrm{Sp}^\bullet}_{X,x}$). The discrete filtration $G^\bullet$ is compatible with the differentials and the associated graded complex is exact. This implies that the complex $\widetilde{\mathrm{Sp}^\bullet}_X$ is exact since the filtration $G^\bullet$ is discrete (i.e. $G^k = 0$ if $k < 0$).

The definition of $G^\bullet$ is as follows: for $0 \leqslant p \leqslant n$ and $k \in \mathbb{N}$, write

$$G^{k,-p} := G^k\left(\mathscr{D} \otimes_{\mathscr{O}_X} \bigwedge^p \mathscr{D}er(\mathscr{O}_X)\right) = F^{k-p}(\mathscr{D}) \otimes_{\mathscr{O}_X} \bigwedge^p \mathscr{D}er(\mathscr{O}_X)$$

where $F^\bullet(\mathscr{D})$ is the order filtration in $\mathscr{D}$. We also write $G^{k,1} := G^k(\mathscr{O}_X) = \mathscr{O}_X$ for all $k \in \mathbb{N}$.

For each k, p one has $\epsilon_{-p}(G^{k,-p}) \subset G^{k,-p+1}$ since for any $P \in F^{k-p}(\mathscr{D})$ one has $P\partial_i \in F^{k-p+1}(\mathscr{D})$. Thus, $G^{k,\bullet}$ is a complex of $\mathscr{O}_X$-modules for any k and $(G^{k,\bullet})_k$ is a discrete increasing exhaustive filtration (simply denoted by $G^\bullet$) of the complex $\widetilde{\mathrm{Sp}^\bullet}_X$.

For each $0 \leqslant p \leqslant n$, the family $(G^{k,-p})_k$ is a discrete increasing exhaustive filtration of the $\mathscr{D}$-module $\mathscr{D} \otimes \bigwedge^p \mathscr{D}er(\mathscr{O}_X)$ whose associated graded $\mathscr{O}_X$-module

$$\mathrm{gr}_{G^{\bullet,-p}} \left(\mathscr{D} \otimes \bigwedge^{p} \mathscr{D}er(\mathscr{O}_X) \right)$$

is naturally isomorphic to

$$\mathrm{gr}_{F^\bullet}(\mathscr{D})[-p] \otimes \bigwedge^{p} \mathscr{D}er(\mathscr{O}_X).$$

The last isomorphism follows from the fact that $\bigwedge^p \mathscr{D}er(\mathscr{O}_X)$ is a locally free $\mathscr{O}_X$-module and therefore $\mathscr{O}_X$-flat.

Then the associated graded complex $\mathrm{gr}_{G^\bullet}(\widetilde{\mathrm{Sp}^\bullet})$ is

$$0 \to \mathrm{gr}_{F^\bullet}(\mathscr{D})[-n] \otimes \bigwedge^{n} \mathscr{D}er(\mathscr{O}_X) \xrightarrow{\tau_{-n}} \cdots \xrightarrow{\tau_{-1}} \mathrm{gr}_{F^\bullet}(\mathscr{D}) \xrightarrow{\tau_0} \mathscr{O}_X \to 0$$

where the differential $\tau_\bullet = \mathrm{gr}(\epsilon_\bullet)$ is acting by

$$\tau_0(Q) = Q_0,$$
$$\tau_{-1}(Q \otimes \partial_i) = Q\sigma(\partial_i),$$
$$\tau_{-p}(Q \otimes (\partial_{i_1} \wedge \cdots \wedge \partial_{i_p})) = \sum_{j=1}^{p} (-1)^{j-1} Q\sigma(\partial_{i_j}) \otimes (\partial_{i_1} \wedge \cdots \widehat{\partial_{i_j}} \cdots \wedge \partial_{i_p}))$$
$$(2 \leqslant p \leqslant n),$$

where the tensor product is taken over $\mathscr{O}_X$, Q_0 is the $0-th$ homogenous component of $Q \in \mathrm{gr}_{F^\bullet}(\mathscr{D})$ and $\sigma(\,)$ is the principal symbol of the corresponding differential operator (see (12.3.1)). So, the last complex is nothing but the augmented Koszul complex with respect to the regular sequence $\{\sigma(\partial_1), \ldots, \sigma(\partial_n)\}$ in the commutative ring $\mathrm{gr}_{F^\bullet}(\mathscr{D})$. Hence this complex is acyclic.

12.3.5 The de Rham Complex of a $\mathscr{D}_X$-Module

We follow here [46, Chap. I (2.6)]. Let $\mathscr{M}$ be a left $\mathscr{D}_X$-module. The left exact functor

$$\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X, *) : \mathrm{Mod}(\mathscr{D}_X) \longrightarrow \mathrm{Mod}(\mathbb{C}_X)$$

can be derived to give a functor

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, *) : D^b(\mathcal{D}_X) \longrightarrow D^b(\mathbb{C}_X).$$

Here $\mathrm{Mod}(A_X)$ stands for the category of (left) A_X-modules and $D^b(A_X)$ for the derived category of complexes of A_X-modules with bounded cohomology.

Definition 12.3.8 Let $\mathcal{M}$ be a left $\mathcal{D}_X$-module. The complex $\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M})$ in $D^b(\mathbb{C}_X)$ is called the de Rham complex of $\mathcal{M}$. It is denoted by $\mathrm{DR}(\mathcal{M})$.

Let $\mathcal{M}$ be a left $\mathcal{D}_X$-module. Since $\mathcal{M}$ carries an integrable connection, there is a natural morphism of sheaves of $\mathbb{C}_X$-vector spaces

$$\nabla : \mathcal{M} \longrightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{M}$$

given locally by

$$\nabla(m) = \sum_{i=1}^{n} dx_i \otimes \partial_i(m).$$

Proposition 12.3.9 ([46, Lemme (2.6.3)]) *For any left $\mathcal{D}_X$-module $\mathcal{M}$, the complex* $\mathrm{DR}(\mathcal{M})$ *can be represented by the complex*

$$\Omega_X^\bullet(\mathcal{M}) := 0 \to \mathcal{M} \xrightarrow{\nabla} \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{M} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega_X^n \otimes_{\mathcal{O}_X} \mathcal{M} \to 0$$

concentrated in degrees $[0, n]$*, where* $\nabla(\omega \otimes m) = \omega \wedge \nabla(m) + (-1)^{\deg(\omega)} d\omega \otimes m$.

Proof By Remark 12.3.7 the complex $\widetilde{\mathrm{Sp}}^\bullet$ is a locally free resolution of the left $\mathcal{D}_X$-module $\mathcal{O}_X$. Thus $\mathrm{DR}(\mathcal{M})$ is represented by $\mathcal{H}om_{\mathcal{D}_X}(\mathrm{Sp}^\bullet, \mathcal{M})$. Moreover, for $p = 0, \ldots, n$, the $\mathcal{O}_X$-modules $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^p \mathcal{D}er(\mathcal{O}_X), \mathcal{M})$ and $\Omega_X^p \otimes \mathcal{M}$ are isomorphic. And there exists a natural quasi-isomorphism, in $D^b(\mathbb{C}_X)$, from $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^\bullet \mathcal{D}er(\mathcal{O}_X), \mathcal{M})$ to $\Omega_X^\bullet \otimes \mathcal{M}$. □

Following [46, Ch. 1, (4.1)], if $\mathcal{M}$ is a complex in the derived category $D_c^b(\mathcal{D}_X)$ (complexes of left $\mathcal{D}_X$-modules, with bounded and coherent cohomology), the complex

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)$$

of right $\mathcal{D}_X$-modules has bounded and coherent cohomology.

Definition 12.3.10 Let $\mathcal{M}$ be a complex in $D_c^b(\mathcal{D}_X)$. The dual of $\mathcal{M}$ is the complex $\mathcal{M}^*$ in $D_c^b(\mathcal{D}_X)$ defined by

$$\mathcal{M}^* = \mathcal{H}om_{\mathcal{O}_X}(\Omega_X^n, \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X))[n]$$

where $n = \dim X$.

The dual $\mathscr{M}^*$ is also denoted $\mathbb{D}(\mathscr{M})$. The dual $\mathscr{M}^*$ of a holonomic $\mathscr{D}_X$-module $\mathscr{M}$ is also holonomic (see [46, Ch. I, (4.1)] and Theorem 12.3.5). The dual $\mathscr{O}_X^*$ is naturally isomorphic to $\mathscr{O}_X$ and there is a natural isomorphism in the derived category $D^b(\mathbb{C}_X)$

$$\mathrm{DR}(\mathscr{M}) \xrightarrow{\sim} \mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}^*, \mathscr{O}_X).$$

The last complex is called the (holomorphic) solution complex of $\mathscr{M}^*$.

12.3.6 Grothendieck's Comparison Theorem Revisited

The statement of (a version of) Grothendieck's Comparison Theorem is as follows (see Sect. 12.2.1):

Theorem 12.3.11 ([32, Th. 2]) *If D is a divisor in the complex manifold X, and $j : U := X \smallsetminus D \hookrightarrow X$ is the inclusion, then the de Rham morphism*

$$\Omega_X^\bullet(\star D) \longrightarrow \mathbb{R}j_*\mathbb{C}_U \tag{12.17}$$

is a quasi-isomorphism.

Remark 12.3.12 If D is a normal crossing divisor, the result is due to Atiyah-Hodge [33]. The proof of Grothendieck uses Hironaka's resolution of singularities to reduce the general case to the normal crossing divisor one.

Theorem 2 in [32] is more general. It holds for a reduced complex analytic space X, an analytic closed subset $D \subset X$, assuming $U := X \smallsetminus D$ is non singular and dense in X, and that U can be defined locally by one equation.

Mebkhout [46, Chap. 2, §2][2] interprets Grothendieck's comparison theorem as the *regularity* of the $\mathscr{D}_X$-module $\mathscr{O}_X$. This regularity is equivalent to the fact that, in the derived category $D^b(\mathbb{C}_X)$, the natural morphism

$$\mathrm{DR}(\mathscr{O}_X(\star D)) \to \mathrm{DR}(\mathbb{R}j_*j^{-1}\mathscr{O}_X)$$

is a quasi-isomorphism for any divisor D. This last morphism is the de Rham morphism (12.17) since $\mathrm{DR}(\mathscr{O}_X(\star D))$ equals $\Omega_X^\bullet(\star D)$ and $\mathrm{DR}(\mathbb{R}j_*j^{-1}\mathscr{O}_X)$ is quasi-isomorphic to $\mathbb{R}j_*j^{-1}\,\mathrm{DR}(\mathscr{O}_X) = \mathbb{R}j_*j^{-1}\Omega_X^\bullet \simeq \mathbb{R}j_*\mathbb{C}_U$, by Poincaré Lemma.

[2] Following Mebkhout's Ph.D. Thesis.

12.4 Free Divisors and Logarithmic D-modules

12.4.1 The Sheaf of Logarithmic Differential Operators with Respect to a Free Divisor

Let X be a complex analytic manifold of dimension n and $D \subset X$ a hypersurface. Let us denote $U = X \setminus D$ and $j : U \hookrightarrow X$ the corresponding open immersion.

When D is a free divisor, there is a nice sheaf of subrings of the sheaf $\mathscr{D}_X$ of linear differential operators in X, namely the sheaf of *logarithmic differential operators* with respect to D, denoted by $\mathscr{V}_X^D$. It is the sheaf of subrings of $\mathscr{D}_X$ with stalks

$$\mathscr{V}_{X,x}^D = \{P \in \mathscr{D}_{X,x} \mid P(\mathscr{J}_x^j) \subset \mathscr{J}_x^j \ \forall j \geqslant 0\}.$$

Moreover, $\mathscr{V}_X$ is a sheaf of filtered rings, with the induced filtration by the order filtration in $\mathscr{D}_X$, whose graded ring is commutative and the canonical map

$$\mathrm{Sym}_{\mathscr{O}_X} \mathscr{D}er(-\log D) \longrightarrow \mathrm{gr}\mathscr{V}_X^D \tag{12.18}$$

is an isomorphism of commutative graded $\mathscr{O}_X$-algebras (see [10, Cor. 2.1.6]; compare with Sect. 12.3.3). As a consequence, $\mathscr{V}_X^D$ is generated by $\mathscr{O}_X$ and $\mathscr{D}er(-\log D)$, and $\mathscr{V}_X^D$ is the enveloping algebra of $\mathscr{D}er(-\log D)$ considered as a Lie algebroid (cf. [20, §(2.1)]). From there we deduce that $\mathscr{V}_X^D$ is a left and right coherent sheaf of rings (one can proceed as in the case of $\mathscr{D}_X$, Sect. 12.3.1, or as in [5, Th. 1.2.5]) and its stalk at each point of X is a left and right Noetherian ring of finite global homological dimension $\leqslant 2n$ (cf. [5, App. IV, Prop. 4.14 and Th. 5.1]).

12.4.2 The Logarithmic Spencer Complex

From now on, we will assume that $D \subset X$ is a free divisor in a complex manifold X of dimension $n \geqslant 1$.

Definition 12.4.1 ([10, Def. 3.1.1]) The logarithmic Spencer complex associated with $D \subset X$ is the following complex of left $\mathscr{V}_X^D$-modules, denoted $\mathrm{Sp}^\bullet(\log D)$:

$$0 \to \mathscr{V}_X^D \otimes_{\mathscr{O}_X} \bigwedge^n \mathscr{D}er(-\log D) \xrightarrow{\epsilon_{-n}} \cdots \xrightarrow{\epsilon_{-2}} \mathscr{V}_X^D \otimes_{\mathscr{O}_X} \mathscr{D}er(-\log D) \xrightarrow{\epsilon_{-1}} \mathscr{V}_X^D$$

where the differential $\epsilon_\bullet$ is defined by

$$\epsilon_{-1}(P \otimes \delta) = P\delta,$$

$$\epsilon_{-p}(P \otimes (\delta_1 \wedge \cdots \wedge \delta_p)) = \sum_{i=1}^{p} (-1)^{i-1} P\delta_i \otimes (\delta_1 \wedge \cdots \widehat{\delta_i} \cdots \wedge \delta_p)) +$$

$$\sum_{1 \leqslant i < j \leqslant p} (-1)^{i+j} P \otimes ([\delta_i, \delta_j] \wedge \delta_1 \wedge \cdots \widehat{\delta_i} \cdots \widehat{\delta_j} \cdots \wedge \delta_p) \quad (2 \leqslant p \leqslant n).$$

We denote by $\widetilde{\mathrm{Sp}}^\bullet(\log D)$ the augmented complex

$$\mathrm{Sp}^\bullet(\log D) \xrightarrow{\epsilon_0} \mathscr{O}_X$$

where $\epsilon_0(P) = P(1)$.

For a free divisor D the complex $\mathrm{Sp}^\bullet(\log D)$ is a locally free resolution of the left $\mathscr{V}_X^D$-module $\mathscr{O}_X$ [10, Th. 3.1.2] (this is a particular case of Proposition 12.4.6).

Remark 12.4.2 The logarithmic Spencer complex should be compared with the Spencer complex (see Definition 12.3.6).

The logarithmic Spencer complex generalizes the one given in [23, App. A (A.4)] for a normal crossing divisor $D \subset X$. Actually, this definition is a sheaf version of the Rinehart complex of a Lie-Rinehart algebra (see [57]).

A *logarithmic connection* (with respect to D) on an $\mathscr{O}_X$-module $\mathscr{E}$ is a $\mathbb{C}$-linear map

$$\nabla : \mathscr{E} \to \Omega_X^1(\log D) \otimes_{\mathscr{O}_X} \mathscr{E}$$

satisfying the Leibniz rule, i.e. $\nabla(ae) = da \otimes e + a\nabla(e)$ for any holomorphic function a and any local section e of $\mathscr{E}$.

As in the classical case (see Sect. 12.3.5), such a ∇ may be extended to a family of $\mathbb{C}$-linear maps

$$\nabla^p : \Omega_X^p(\log D) \otimes_{\mathscr{O}_X} \mathscr{E} \to \Omega_X^{p+1}(\log D) \otimes_{\mathscr{O}_X} \mathscr{E},$$

with $\nabla^p(\alpha \otimes e) = d\alpha \otimes e + (-1)^p \alpha \wedge \nabla(e)$ for any logarithmic p-form α and any local section e of $\mathscr{E}$.

We say that the logarithmic connection ∇ is *integrable* if $\nabla^{p+1} \circ \nabla^p = 0$ for all $p \geqslant 0$. In such a case, the *logarithmic de Rham complex* of $(\mathscr{E}, \nabla)$ is by definition the complex (of sheaves of complex vector spaces)

$$\Omega_X^\bullet(\log D)(\mathscr{E}) := \mathscr{E} \to \Omega_X^1(\log D) \otimes_{\mathscr{O}_X} \mathscr{E} \to \Omega_X^2(\log D) \otimes_{\mathscr{O}_X} \mathscr{E} \to \cdots$$
$$\to \Omega_X^n(\log D) \otimes_{\mathscr{O}_X} \mathscr{E},$$

where $\mathscr{E}$ is placed in degree 0.

Any logarithmic connection ∇ on an $\mathscr{O}_X$-module $\mathscr{E}$ gives rise to an action of logarithmic vector fields $\mathscr{D}er(-\log D)$ on $\mathscr{E}$

$$(\delta, e) \longmapsto \nabla_\delta(e) = \langle \delta, \nabla(e) \rangle$$

for any logarithmic vector field δ and any local section e of $\mathscr{E}$, where $\langle \delta, \nabla(e) \rangle$ is induced by the contraction of logarithmic 1-forms by logarithmic vector fields.

Obviously, the exterior derivative $d : \mathscr{O}_X \to \Omega_X^1(\log D)$ is a logarithmic connection which is integrable, and $\Omega_X^\bullet(\log D)(\mathscr{O}_X) = \Omega_X^\bullet(\log D)$.

When D is free, the following result holds, by essentially the same proof as in the classical case.

Proposition 12.4.3 *Assume that D is a free divisor and let ∇ be a logarithmic connection on an $\mathscr{O}_X$-module $\mathscr{E}$. The following properties are equivalent:*

(i) ∇ *is integrable.*
(ii) $\nabla_{[\delta,\delta']} = [\nabla_\delta, \nabla_{\delta'}]$ *for all logarithmic vector fields* δ, δ'.

Since the sheaf of logarithmic differential operators $\mathscr{V}_X^D$ is the enveloping algebra of the Lie algebroid $\mathscr{D}er(-\log D)$ provided that D is a free divisor, we obtain the following corollary.

Corollary 12.4.4 *Assume that D is a free divisor and let $\mathscr{E}$ be an $\mathscr{O}_X$-module. The following data are equivalent:*

(a) An integrable logarithmic connection ∇ on $\mathscr{E}$.
(b) A structure of left $\mathscr{V}_X^D$-module on $\mathscr{E}$ extending its $\mathscr{O}_X$-module structure.

Moreover, the action of a logarithmic derivation δ on $\mathscr{E}$ in (b) is given by $\delta \cdot e = \nabla_\delta(e)$ for each local section e of $\mathscr{E}$.

From now on, we assume that D is a free divisor.

Any locally free $\mathscr{O}_X$-module of finite rank endowed with an integrable logarithmic connection will be called an ILC, for short. Examples of ILCs are the invertible $\mathscr{O}_X$-modules $\mathscr{O}_X(kD)$, $k \in \mathbb{Z}$, which carry an obvious structure of left $\mathscr{V}_X^D$-module.

We define the logarithmic Spencer complex of an arbitrary left $\mathscr{V}_X^D$-module in the following way.

Definition 12.4.5 ([13]) Let $\mathscr{E}$ be a left $\mathscr{V}_X^D$-module. The logarithmic Spencer complex of $\mathscr{E}$ (with respect to $D \subset X$) is the following complex of left $\mathscr{V}_X^D$-modules, denoted $\mathrm{Sp}^\bullet(\log D)(\mathscr{E})$:

$$0 \to \mathscr{V}_X^D \otimes_{\mathscr{O}_X} \bigwedge^n \mathscr{D}er(-\log D) \otimes_{\mathscr{O}_X} \mathscr{E} \xrightarrow{\epsilon_{-n}} \cdots \xrightarrow{\epsilon_{-2}}$$
$$\mathscr{V}_X^D \otimes_{\mathscr{O}_X} \mathscr{D}er(-\log D) \otimes_{\mathscr{O}_X} \mathscr{E} \xrightarrow{\epsilon_{-1}} \mathscr{V}_X^D \otimes_{\mathscr{O}_X} \mathscr{E}$$

where the differential $\epsilon_\bullet$ is defined by

$$\epsilon_{-1}(P \otimes \delta \otimes e) = P\delta \otimes e - P \otimes \delta e,$$
$$\epsilon_{-p}(P \otimes (\delta_1 \wedge \cdots \wedge \delta_p)) \otimes e$$
$$= \sum_{i=1}^{p} (-1)^{i-1} P\delta_i \otimes (\delta_1 \wedge \cdots \widehat{\delta_i} \cdots \wedge \delta_p)) \otimes e$$
$$- \sum_{i=1}^{p} (-1)^{i-1} P \otimes (\delta_1 \wedge \cdots \widehat{\delta_i} \cdots \wedge \delta_p)) \otimes \delta_i e$$
$$+ \sum_{1 \leqslant i < j \leqslant p} (-1)^{i+j} P \otimes ([\delta_i, \delta_j] \wedge \delta_1 \wedge \cdots \widehat{\delta_i} \cdots \widehat{\delta_j} \cdots \wedge \delta_p) \otimes e \ (2 \leqslant p \leqslant n).$$

We denote by $\widetilde{\mathrm{Sp}}^\bullet(\log D)(\mathscr{E})$ the augmented complex

$$\mathrm{Sp}^\bullet(\log D)(\mathscr{E}) \xrightarrow{\epsilon_0} \mathscr{E}$$

where $\epsilon_0(P \otimes e) = Pe$.

Note that for $\mathscr{E} = \mathscr{O}_X$ we have $\mathrm{Sp}^\bullet(\log D)(\mathscr{O}_X) = \mathrm{Sp}^\bullet(\log D)$.

Proposition 12.4.6 *For any ILC $\mathscr{E}$, the logarithmic Spencer complex of $\mathscr{E}$ (with respect to a free divisor $D \subset X$) $\mathrm{Sp}^\bullet(\log D)(\mathscr{E})$ is a locally free resolution of the $\mathscr{V}_X^D$–module $\mathscr{E}$.*

The proof of this proposition is similar to the proof in Remark 12.3.7. Namely, we consider the discrete increasing filtration $G^\bullet := G^\bullet\big(\widetilde{\mathrm{Sp}}^\bullet(\log D)(\mathscr{E})\big)$ on the complex $\widetilde{\mathrm{Sp}}^\bullet(\log D)(\mathscr{E})$ given, for $0 \leqslant p \leqslant n$ and $k \in \mathbb{N}$, by

$$G^{k,-p} := G^k \left(\mathscr{V}_X^D \otimes_{\mathscr{O}_X} \bigwedge^p \mathscr{D}er(-\log D) \otimes_{\mathscr{O}_X} \mathscr{E} \right)$$
$$= F^{k-p}(\mathscr{V}_X^D) \otimes_{\mathscr{O}_X} \bigwedge^p \mathscr{D}er(-\log D) \otimes_{\mathscr{O}_X} \mathscr{E}$$

where $F^\bullet(\mathscr{V}_X^D)$ is the filtration induced by the order filtration in $\mathscr{D}$. We also write $G^{k,1} := G^k(\mathscr{E}) = \mathscr{E}$ for all $k \in \mathbb{N}$. The associated graded complex turns out to be (locally) the tensor product over $\mathscr{O}_X$ of $\mathscr{E}$ and the augmented Koszul complex with respect to $\{\sigma(\delta_1), \dots, \sigma(\delta_n)\} \subset \operatorname{gr}\mathscr{V}_X^D$, $\{\delta_1, \dots, \delta_n\}$ being a local $\mathscr{O}_X$-basis of $\mathscr{D}er(-\log D)$, which is exact since $\mathscr{E}$ is locally free over $\mathscr{O}_X$ and $\operatorname{gr}\mathscr{V}_X^D$ is (locally) a polynomial ring over $\mathscr{O}_X$ in the variables $\{\sigma(\delta_1), \dots, \sigma(\delta_n)\}$ (see the isomorphism (12.18)).

By using the logarithmic Spencer resolution Proposition 12.4.6, as in the case of $\mathscr{D}_X$ (see Sect. 12.3.4), we obtain a canonical isomorphism of complexes of sheaves of complex vector spaces

$$\mathscr{H}om_{\mathscr{V}_X^D}(\mathrm{Sp}^\bullet(\log D), \mathscr{E}) \simeq \Omega_X^\bullet(\log D)(\mathscr{E}),$$

for any left $\mathscr{V}_X^D$-module $\mathscr{E}$, and so an isomorphism in the derived category

$$\mathbb{R}\mathscr{H}om_{\mathscr{V}_X^D}(\mathscr{O}_X, \mathscr{E}) \simeq \Omega_X^\bullet(\log D)(\mathscr{E}).$$

This is completely similar to Proposition 12.3.9.

12.4.3 A $\mathscr{D}$-Module Criterion for LCT

Grothendieck's comparison theorem (see (12.2.1) and (12.3.6)) tells us that the natural map $\Omega_X^\bullet(\star D) \to \mathbb{R}j_*\mathbb{C}_{X\setminus D}$ obtained by composition of the adjunction map $\Omega_X^\bullet(\star D) \to \mathbb{R}j_*j^{-1}\Omega_X^\bullet(\star D) = \mathbb{R}j_*\Omega_{X\setminus D}^\bullet = j_*\Omega_{X\setminus D}^\bullet$ with the inverse of the induced map by the Poincaré quasi-isomorphism $\mathbb{C}_{X\setminus D} \to \Omega_{X\setminus D}^\bullet$ is an isomorphism in the derived category of sheaves of complex vector spaces. Definition 12.2.4 tells us that LCT holds for D when the natural morphism

$$\Omega_X^\bullet(\log D) \hookrightarrow \Omega_X^\bullet(\star D) \tag{12.19}$$

is a quasi-isomorhism. Let us explain how this map can be interpreted in terms of $\mathscr{D}$-module theory in the case of free divisors.

On one hand, we have canonical isomorphisms (in the derived category)

$$\Omega_X^\bullet(\log D) \simeq \mathbb{R}\mathscr{H}om_{\mathscr{V}_X^D}(\mathscr{O}_X, \mathscr{O}_X) \simeq \mathbb{R}\mathscr{H}om_{\mathscr{D}_X}\left(\mathscr{D}_X \overset{\mathbb{L}}{\otimes}_{\mathscr{V}_X^D} \mathscr{O}_X, \mathscr{O}_X\right).$$

By taking $\mathscr{D}_X$-duals $\mathbb{D}(-)$ (see Definition 12.3.10) we obtain

$$\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}\left(\mathscr{D}_X \overset{\mathbb{L}}{\otimes}_{\mathscr{V}_X^D} \mathscr{O}_X, \mathscr{O}_X\right) \simeq \mathbb{R}\mathscr{H}om_{\mathscr{D}_X}\left(\mathbb{D}\,\mathscr{O}_X, \mathbb{D}\left(\mathscr{D}_X \overset{\mathbb{L}}{\otimes}_{\mathscr{V}_X^D} \mathscr{O}_X\right)\right) \simeq$$
$$\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}\left(\mathscr{O}_X, \mathbb{D}\left(\mathscr{D}_X \overset{\mathbb{L}}{\otimes}_{\mathscr{V}_X^D} \mathscr{O}_X\right)\right) = \mathrm{DR}\left(\mathbb{D}\left(\mathscr{D}_X \overset{\mathbb{L}}{\otimes}_{\mathscr{V}_X^D} \mathscr{O}_X\right)\right),$$

and by using the duality formula in [13, Cor. 3.1.2] (see also [54, Th. (A.32)]), we obtain

$$\mathbb{D}\left(\mathscr{D}_X \overset{\mathbb{L}}{\otimes}_{\mathscr{V}_X^D} \mathscr{O}_X\right) \simeq \mathscr{D}_X \overset{\mathbb{L}}{\otimes}_{\mathscr{V}_X^D} \mathscr{O}_X(D),$$

and so

$$\Omega_X^\bullet(\log D) \simeq \cdots \simeq \mathrm{DR}\left(\mathscr{D}_X \overset{\mathbb{L}}{\otimes}_{\mathscr{V}_X^D} \mathscr{O}_X(D)\right). \tag{12.20}$$

On the other hand we have another canonical isomorphism (see Proposition 12.3.9)

$$\Omega_X^\bullet(\star D) = \Omega_X^\bullet(\mathscr{O}_X(\star D)) \simeq \mathbb{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{O}_X(\star D)) = \mathrm{DR}(\mathscr{O}_X(\star D)). \tag{12.21}$$

The point is that the inclusion $\mathscr{O}_X(D) \subset \mathscr{O}_X(\star D)$ induces a canonical left $\mathscr{D}_X$-linear map

$$\varrho : \mathscr{D}_X \overset{\mathbb{L}}{\otimes}_{\mathscr{V}_X^D} \mathscr{O}_X(D) \longrightarrow \mathscr{O}_X(\star D), \tag{12.22}$$

and we have the following theorem ([13, Cor. 4.2] and [53, Cor. 1.7.2]):

Theorem 12.4.7 *If $D \subset X$ is a free divisor, the following properties hold:*

(1) The logarithmic comparison map (12.19) corresponds to the map (12.22) under the DR *functor and the canonical isomorphisms (12.20) and (12.21).*

(2) The following properties are equivalent:

(2–1) The logarithmic comparison theorem holds for D, i.e. the map (12.19) is a quasi-isomorphism.

(2–2) The map (12.22) is an isomorphism (in the derived category of $\mathscr{D}_X$-modules).

(2–3) The canonical map

$$j_!\mathbb{C}_{X\smallsetminus D} \longrightarrow \Omega_X^\bullet(\log D)(\mathscr{O}_X(-D))$$

is a quasi-isomorphism.

Remark 12.4.8

(a) Let us notice that property (2–3) above comes from taking Grothendieck-Verdier duals. Namely, $\left(\mathbb{R}j_*\mathbb{C}_{X\smallsetminus D}\right)^\vee \simeq j_!\mathbb{C}_{X\smallsetminus D}$ and

$$\left(\Omega^\bullet_X(\log D)\right)^\vee \simeq \left(\mathrm{DR}\left(\mathscr{D}_X \overset{\mathbb{L}}{\otimes}_{\mathscr{V}^D_X} \mathscr{O}_X(D)\right)\right)^\vee$$

$$\overset{(\star)}{\simeq} \mathbb{R}\mathscr{H}om_{\mathscr{D}_X}\left(\mathscr{D}_X \overset{\mathbb{L}}{\otimes}_{\mathscr{V}^D_X} \mathscr{O}_X(D), \mathscr{O}_X\right) \simeq$$

$$\mathbb{R}\mathscr{H}om_{\mathscr{V}^D_X}\left(\mathscr{O}_X(D), \mathscr{O}_X\right) \simeq \mathbb{R}\mathscr{H}om_{\mathscr{V}^D_X}\left(\mathscr{O}^*_X, \mathscr{O}_X(D)^*\right) \simeq$$

$$\mathbb{R}\mathscr{H}om_{\mathscr{V}^D_X}\left(\mathscr{O}_X, \mathscr{O}_X(-D)\right) \simeq \Omega^\bullet_X(\log D)(\mathscr{O}_X(-D)),$$

where we have used Mebkhout local duality formula in $(\star)$ (see [46, Chap. I, (4.3)]; see also [52]).

(b) Let us also notice that the complex $\Omega^\bullet_X(\log D)(\mathscr{O}_X(-D))$ is a subcomplex of $\Omega^\bullet_X$, since locally $\Omega^\bullet_X(\log D)(\mathscr{O}_X(-D)) = f\,\Omega^\bullet_X(\log D)$ with $f = 0$ a reduced local equation of D, and $f\,\Omega^p_X(\log D) \subset \Omega^p_X$ for $p = 0, \ldots, n$. Moreover,

$$\Omega^\bullet_X(\log D)(\mathscr{O}_X(-D)) \simeq \cdots$$

$$\simeq \mathbb{R}\mathscr{H}om_{\mathscr{D}_X}\left(\mathscr{D}_X \overset{\mathbb{L}}{\otimes}_{\mathscr{V}^D_X} \mathscr{O}_X(D), \mathscr{O}_X\right) \simeq$$

$$\mathbb{R}\mathscr{H}om_{\mathscr{D}_X}\left(\mathbb{D}\,\mathscr{O}_X, \mathbb{D}\left(\mathscr{D}_X \overset{\mathbb{L}}{\otimes}_{\mathscr{V}^D_X} \mathscr{O}_X(D)\right)\right)$$

$$\simeq \mathbb{R}\mathscr{H}om_{\mathscr{D}_X}\left(\mathscr{O}_X, \mathscr{D}_X \overset{\mathbb{L}}{\otimes}_{\mathscr{V}^D_X} \mathscr{O}_X\right) = \mathrm{DR}\left(\mathscr{D}_X \overset{\mathbb{L}}{\otimes}_{\mathscr{V}^D_X} \mathscr{O}_X\right)$$

and we can prove that the inclusion $\Omega^\bullet_X(\log D)(\mathscr{O}_X(-D)) \hookrightarrow \Omega^\bullet_X$ comes from the map of $\mathscr{D}_X$-modules

$$\mathscr{D}_X \overset{\mathbb{L}}{\otimes}_{\mathscr{V}^D_X} \mathscr{O}_X \longrightarrow \mathscr{O}_X, \quad P \otimes a \longmapsto P(a), \tag{12.23}$$

by applying the $\mathrm{DR}(-)$ functor.

Remark 12.4.9 Let us understand in concrete terms the significance of the map (12.22) being an isomorphism. First of all, (12.22) is an isomorphism if and only if it is so stalkwise, and obviously ϱ_p is an isomorphism for each $p \in X \smallsetminus D$. Let us take a point $p \in D$, a reduced local equation $f \in \mathscr{O}_{X,p}$ of D and a basis $\delta_1, \ldots, \delta_n \in \mathscr{D}er(-\log D)_p$ with $\delta_i(f) = \alpha_i f$, $\alpha_i \in \mathscr{O}_{X,p}$, $i = 1, \ldots, n$. On the other hand, $\mathscr{O}_{X,p}(D)$ is generated as $\mathscr{V}^D_{X,p}$-module by f^{-1} and the kernel of

$$P \in \mathscr{V}^D_{X,p} \longmapsto P(f^{-1}) \in \mathscr{O}_{X,p}(D)$$

is the left ideal generated by $\delta_1 + \alpha_1, \ldots, \delta_n + \alpha_n$, i.e.

$$\mathscr{O}_{X,p}(D) \simeq \mathscr{V}^D_{X,p}/\mathscr{V}^D_{X,p}\langle \delta_1 + \alpha_1, \ldots, \delta_n + \alpha_n \rangle.$$

Consequently the map $\varrho_p : \mathscr{D}_{X,p} \overset{\mathbb{L}}{\otimes}_{\mathscr{V}^D_{X,p}} \mathscr{O}_{X,p}(D) \longrightarrow \mathscr{O}_{X,p}(\star D)$ is an isomorphism (in the derived category of $\mathscr{D}_{X,p}$-modules) if and only if the following properties hold:

(i) The complex $\mathscr{D}_{X,p} \overset{\mathbb{L}}{\otimes}_{\mathscr{V}^D_{X,p}} \mathscr{O}_{X,p}(D)$ is exact in cohomological degrees $\neq 0$, and
(ii) $\mathscr{O}_{X,p}(\star D) \simeq \mathscr{D}_{X,p}/\mathscr{D}_{X,p}\langle \delta_1 + \alpha_1, \ldots, \delta_n + \alpha_n \rangle$.

The isomorphism in (ii) comes from the map

$$P \in \mathscr{D}_{X,p} \longmapsto P(f^{-1}) \in \mathscr{O}_{X,p}(\star D),$$

and so property (ii) is equivalent to

(ii-1) $\mathscr{O}_{X,p}(\star D)$ is generated as $\mathscr{D}_{X,p}$-module by f^{-1}, and
(ii-2) The $\mathscr{D}_{X,p}$-annihilator of $f^{-1} \in \mathscr{O}_{X,p}(\star D)$ is the left ideal generated by $\delta_1 + \alpha_1, \ldots, \delta_n + \alpha_n$.

Notice that property (ii-2) is equivalent to the fact that the $\mathscr{D}_{X,p}$-annihilator of f^{-1} is generated by order 1 operators. From [62] we know that the last property implies that the b-function of the germ f has no integer roots strictly less than -1, and so property (ii-2) implies property (ii-1) (by the Bernstein functional equation). We conclude that the map ϱ_p is an isomorphism if and only if the following properties hold:

(i) The complex $\mathscr{D}_{X,p} \overset{\mathbb{L}}{\otimes}_{\mathscr{V}^D_{X,p}} \mathscr{O}_{X,p}(D)$ is exact in cohomological degrees $\neq 0$, and
(ii-2) The $\mathscr{D}_{X,p}$-annihilator of $f^{-1} \in \mathscr{O}_{X,p}(\star D)$ is the left ideal generated by $\delta_1 + \alpha_1, \ldots, \delta_n + \alpha_n$.

Let us recall the following definition [16].

Definition 12.4.10 A free divisor $D \subset X$ is called *Spencer* if the complex $\mathscr{D}_X \overset{\mathbb{L}}{\otimes}_{\mathscr{V}^D_X} \mathscr{O}_X$ is concentrated in cohomological degree 0 and $\mathscr{D}_X \otimes_{\mathscr{V}^D_X} \mathscr{O}_X$ is holonomic.

The free divisor $D = ((xz + y)(x^4 + y^5 + xy^4) = 0) \subset \mathscr{C}^3$ is not a Spencer divisor since the complex in Definition 12.4.10 is neither concentrated in degree 0 nor holonomic in degree 0 (see [13] example 5.1).

If the the LCT holds for a free divisor D, then $\mathbb{D}(\mathscr{O}_X(\star D)) \simeq \mathbb{D}\left(\mathscr{D}_X \overset{\mathbb{L}}{\otimes}_{\mathscr{V}_X^D} \mathscr{O}_X(D)\right) \simeq \mathscr{D}_X \overset{\mathbb{L}}{\otimes}_{\mathscr{V}_X^D} \mathscr{O}_X$, and so D is Spencer and $\mathbb{D}(\mathscr{O}_X(\star D)) \simeq \mathscr{D}_X \otimes_{\mathscr{V}_X^D} \mathscr{O}_X$ (since both $\mathscr{O}_X(\star D)$ and its dual are holonomic).

To check whether the complex $\mathscr{D}_X \overset{\mathbb{L}}{\otimes}_{\mathscr{V}_X^D} \mathscr{O}_X$ is concentrated in cohomological degree 0, we can use the logarithmic Spencer complex $\mathrm{Sp}^\bullet(\log D)$, which is a locally free resolution of $\mathscr{O}_X$ as a left $\mathscr{V}_X^D$-module. So we have to check whether the complex $\mathscr{D}_X \otimes_{\mathscr{V}_X^D} \mathrm{Sp}^\bullet(\log D)$ is concentrated in cohomological degree 0 or not.

For this, we can consider again the discrete increasing filtration $G^\bullet := G^\bullet\left(\mathscr{D}_X \otimes_{\mathscr{V}_X^D} \mathrm{Sp}^\bullet(\log D)\right)$ on the complex $\mathscr{D}_X \otimes_{\mathscr{V}_X^D} \mathrm{Sp}^\bullet(\log D)$ given by, for $0 \leqslant p \leqslant n$ and $k \in \mathbb{N}$,

$$G^{k,-p} := G^k\left(\mathscr{D}_X \otimes_{\mathscr{O}_X} \bigwedge^p \mathscr{D}er(-\log D)\right) = F^{k-p}(\mathscr{D}_X) \otimes_{\mathscr{O}_X} \bigwedge^p \mathscr{D}er(-\log D)$$

where $F^\bullet(\mathscr{D}_X)$ is the filtration by the order filtration in $\mathscr{D}$. The associated graded complex turns out to be (locally) the Koszul complex with respect to $\{\sigma(\delta_1), \ldots, \sigma(\delta_n)\} \subset \mathrm{gr}\mathscr{D}_X$, $\{\delta_1, \ldots, \delta_n\}$ being a local $\mathscr{O}_X$-basis of $\mathscr{D}er(-\log D)$, but in general, this is not a regular sequence[3] and so $\mathrm{gr}_{G^\bullet}\left(\mathscr{D}_X \otimes_{\mathscr{V}_X^D} \mathrm{Sp}^\bullet(\log D)\right)$ is not concentrated in cohomological degree 0. This fact motivates the following definition [10, Def. 4.1.1].

Definition 12.4.11 We say that a free divisor $D \subset X$ is *Koszul* at a point $p \in D$ if for some (and hence for any) local basis $\{\delta_1, \ldots, \delta_n\}$ of $\mathscr{D}er(-\log D)_p$, the symbols $\{\sigma(\delta_1), \ldots, \sigma(\delta_n)\}$ form a regular sequence in $\mathrm{gr}\mathscr{D}_{X,p}$; and we say that D is *Koszul* if it so at any point $p \in D$.

Let us notice that the Koszul property for a free divisor $D \subset X$ is equivalent to saying that the complex $\mathrm{gr}\mathscr{D}_X \overset{\mathbb{L}}{\otimes}_{\mathrm{gr}\mathscr{V}_X^D} \mathscr{O}_X$ is concentrated in cohomological degree 0. In that case, the module $\mathrm{gr}\mathscr{D}_X \otimes_{\mathrm{gr}\mathscr{V}_X^D} \mathscr{O}_X$ has automatically dimension $n = \dim X$.

Any Koszul free divisor D is Spencer since, from the very definition, the complex $\mathscr{D}_X \otimes_{\mathscr{V}_X^D} \mathrm{Sp}^\bullet(\log D)$ is concentrated in cohomological degree 0 (its graded complex with respect to $G^\bullet$ is concentrated in cohomological degree 0), and $\mathscr{D}_X \otimes_{\mathscr{V}_X^D} \mathscr{O}_X$ is holonomic since it is locally presented as $\mathscr{D}_{X,p}/\mathscr{D}_{X,p}\langle\delta_1, \ldots, \delta_n\rangle$, where $\{\delta_1, \ldots, \delta_n\}$ is a basis of $\mathscr{D}er(-\log D)_p$, and the quotient of $\mathrm{gr}\mathscr{D}_{X,p}$ by the ideal $\langle\sigma(\delta_1), \ldots, \sigma(\delta_n)\rangle$ has dimension n. Moreover, in this case $\mathrm{gr}\left(\mathscr{D}_X \otimes_{\mathscr{V}_X^D} \mathscr{O}_X\right) \simeq \mathrm{gr}\mathscr{D}_X \otimes_{\mathrm{gr}\mathscr{V}_X^D} \mathscr{O}_X$, $\{\delta_1, \ldots, \delta_n\}$ is an involutive basis of the ideal $\mathscr{D}_{X,p}\langle\delta_1, \ldots, \delta_n\rangle$

[3] Remember that $\{\sigma(\delta_1), \ldots, \sigma(\delta_n)\}$ is a regular sequence in $\mathrm{gr}\mathscr{V}_X^D$ since this ring is (locally) a polynomial ring in the variables $\{\sigma(\delta_1), \ldots, \sigma(\delta_n)\}$ with coefficients in $\mathscr{O}_X$.

with respect to the order filtration and the characteristic variety of $\mathscr{D}_X \otimes_{\mathscr{V}^D_X} \mathscr{O}_X$ is (locally) given by $\sigma(\delta_1) = \cdots = \sigma(\delta_n) = 0$.

Actually, the condition for a free divisor to be Koszul is equivalent to the fact that the logarithmic stratification of D [58, §3] is locally finite, or equivalently, that any logarithmic stratum of D is holonomic in the sense of *loc. cit.* (see [28, Theorem (7.4)]). In particular, any plane curve and any free hyperplane arrangement is a Koszul free divisor.

On the other hand, we know that any locally quasihomogeneous free divisor is Koszul [11].[4] Furthermore, the roots of the b-function of a reduced local equation $f = 0$ of any locally quasihomogeneous free divisor are symmetric with respect to -1 (see [12, Theorem 5.6], [54, Corollary (4.2)]) and the $\mathscr{D}[s]$-annhilator of f^s is generated by order one operators [12, Corollary 5.8]. From there we deduce a purely algebraic proof of Theorem 12.2.5, based on Theorem 12.4.7 (see [54, Corollary (4.5) and Remark (4.6)]).

Remark 12.4.12 Notice also that, for any hyperplane arrangement, free or not, with equation $f = 0$, reduced or not, the $\mathscr{D}_{X,p}$-annihilator of $f^{-1} \in \mathscr{O}_{X,p}(*D)$ if generated by operators of order 1 [64, Th. 5.3]. As mentioned before, see Sect. 12.2.4, LCT holds for any hyperplane arrangement [2].

If a free divisor $D \subset \mathbb{C}^n$ is Spencer with a polynomial defining equation $f = 0$, then by Castro-Jiménez and Ucha-Enríquez [17, Crit. 3.1 and 4.1], D satisfies LCT if and only if the annihilating ideal in $\mathscr{D}$ of $1/f$ is generated by operators of order 1; that is, if condition (ii.2) in Remark 12.4.9 holds. In [17, Rk. 5.8] examples are given of three free divisors in $\mathbb{C}^3$, defined by quasi-homogeneous polynomials (with strictly positive weights), that do not satisfy LCT and hence, by Theorem 12.2.5, they are not locally quasi-homogeneous. That gives a negative answer to a question proposed in [12, Prob. 6.5].

Remark 12.4.13 The sheaf $\check{\Omega}^\bullet_D$ introduced in [50] (see Sect. 12.2.6) is the quotient of $\Omega^\bullet_X$ by the subcomplex $\Omega^\bullet_X(\log D)(\mathscr{O}_X(-D))$, and so there is a commutative diagram of complexes of sheaves of $\mathbb{C}$-vector spaces

$$\begin{array}{ccccccccc} 0 & \longrightarrow & j_!\mathbb{C}_{X\setminus D} & \xrightarrow{\text{adj.}} & \mathbb{C}_X & \longrightarrow & \mathbb{C}_D & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \text{q-i}} & & \downarrow{\scriptstyle \overline{\lambda}} & & \\ 0 & \longrightarrow & \Omega^\bullet_X(\log D)(\mathscr{O}_X(-D)) & \xrightarrow{\text{incl.}} & \Omega^\bullet_X & \longrightarrow & \check{\Omega}^\bullet_D & \longrightarrow & 0 \end{array} \tag{12.24}$$

with exact rows, where the vertical arrows are the natural ones. The bottom row comes from the triangle

$$\mathscr{D}_X \overset{\mathbb{L}}{\otimes}_{\mathscr{V}^D_X} \mathscr{O}_X \longrightarrow \mathscr{O}_X \longrightarrow K^\bullet \xrightarrow{+1}$$

[4] Actually, it is easy to see that for any locally quasihomogeneous divisor, free or not, the logarithmic stratification of D is locally finite.

by applying the DR(−) functor.

The middle vertical arrow in (12.24) is a quasi-isomorphism by Poincaré lemma, and λ is a quasi-isomorphism if and only if $\overline{\lambda}$ is a quasi-isomorphism. So, we can add another equivalent property to (2) in Theorem 12.4.7:

(2–4) The canonical map $\mathbb{C}_D \longrightarrow \check{\Omega}^{\bullet}_D$ is a quasi-isomorphism.

We deduce a new proof of Theorem 12.2.5 by using [50, Lemma 3.3], where it is proven that the map $\overline{\lambda}$ is a quasi-isomorphism for any locally quasihomogeneous divisor (not necessarily free).

In fact, a similar argument shows that LCT holds for any locally weakly quasihomogeneous free divisor and *a posteriori* any such free divisor is Spencer (see Definition 12.2.7) (see [19, Remark 3.11] and [53, Remark 1.7.4], although the statement in [19, Remark 3.11] that any logarithmic differential form has a non negative weight is wrong). Namely, we have the following.

Theorem 12.4.14 *If $D \subset X$ is a locally weakly quasihomogeneous free divisor, then D satisfies the logarithmic comparison theorem.*

Proof By Theorem 12.4.7 (2–3), we need to prove that the complex $\Omega^{\bullet}_X(\log D)(\mathscr{O}_X(-D))_p$ is exact for any $p \in D$. Since D is locally weakly quasihomogeneous, there are local coordinates $x_1, \ldots, x_r, x_{r+1}, \ldots, x_n$ centered at p, $r > 0$, weights $(w_1, \ldots, w_r, 0, \ldots, 0)$, $w_i > 0$ for all $i = 1, \ldots, r$, and a reduced local equation f of D at p which is quasihomogeneous in these coordinates with $\mathrm{wt}(f) > 0$.

By the standard argument (see Lemma 3.3 in [50]), the complex

$$\Omega^{\bullet}_X(\log D)(\mathscr{O}_X(-D))_p = f\,\Omega^{\bullet}_X(\log D)_p$$

is homotopic to its weight zero subcomplex, and the theorem is a consequence of the following lemma. □

Lemma 12.4.15 *Let f be a quasihomogeneous polynomial in $\mathbb{C}\{x_{r+1}, \ldots, x_n\}[x_1, \ldots, x_r]$, of strictly positive weight with respect to weights $(w_1, \ldots, w_r, 0, \ldots, 0)$ with $w_1, \ldots, w_r$ all strictly positive. Then, for any $p > 0$ and any non-zero quasihomogeneous logarithmic p-form ω, we have $\mathrm{wt}\,\omega > -\mathrm{wt}\,f$.*

Proof Since $f\omega$ must be holomorphic, we know that $\mathrm{wt}\,\omega \geqslant -\mathrm{wt}\,f$. We may assume $\partial f/\partial x_1 \neq 0$ (i.e. f effectively depends on x_1).

Suppose that ω is a non-zero logarithmic p-form, and that $\mathrm{wt}\,\omega = -\mathrm{wt}\,f$. Then, $\alpha = f\omega$ is a non-zero holomorphic p-form with $\mathrm{wt}\,\alpha = 0$. That means that α only depends on the variables $x_{r+1}, \ldots, x_n$ of weight 0:

$$\alpha = \sum_{r+1 \leqslant i_1 < \ldots < i_p \leqslant n} \alpha_{i_1,\ldots,i_p}(x_{r+1}, \ldots, x_n) dx_{i_1} \wedge \ldots \wedge dx_{i_p}.$$

Reordering the variables, we may assume $\alpha_{r+1,\ldots,r+p} \neq 0$.

Since ω is logarithmic, the $p+1$-form $df \wedge \omega$ is holomorphic (this follows from the fact that $f\omega$ and $f d\omega$ are holomorphic). The coefficient of $dx_1 \wedge dx_{r+1} \wedge \ldots \wedge dx_{r+p}$ in $df \wedge \omega$, which must be holomorphic, is

$$\frac{\alpha_{r+1,\ldots,r+p}}{f} \frac{\partial f}{\partial x_1}.$$

So f divides $\alpha_{r+1,\ldots,r+p} \frac{\partial f}{\partial x_1}$ in the ring of power series $\mathbb{C}\{x_1, \ldots, x_r, x_{r+1}, \ldots, x_n\}$. However, f and $\partial f/\partial x_1$ are quasihomogeneous polynomials in $\mathbb{C}\{x_{r+1}, \ldots, x_n\}[x_1, \ldots, x_r]$. By equating homogeneous parts, we deduce that f also divides $\alpha_{r+1,\ldots,r+p} \partial f/\partial x_1$ in $\mathbb{C}\{x_{r+1}, \ldots, x_n\}[x_1, \ldots, x_r]$. Since $\mathrm{wt}(\partial f/\partial x_1) = \mathrm{wt}(f) - \mathrm{wt}(x_1)$, this is impossible. □

A natural question is to characterize free divisors for which LCT holds. This has been done in [14] for plane curves, and more generally in [62, Cor. 1.8] and [54, Th. (4.7)] for Koszul free divisors. For general free divisors the following conjecture remains open [14, Conjecture 1.4]:

Conjecture 12.4.16 Let $D \subset X$ a free divisor. If LCT holds for D, then D is *strongly Euler homogeneous*, i.e. for each $p \in D$ there is a local reduced equation of D at p and a germ of vector field χ at p, singular at p, such that $\chi(f) = f$.

In [18, §3, 4] the authors provide an infinite family of free divisors for which LCT does not hold. Any divisor of this family is defined by a polynomial

$$f_{p,q} = (x_1^p - x_2^q) \prod_{i=3}^{n} (x_1 x_i + x_2)$$

for $n \geqslant 3$ and $3 \leqslant p < q$.

12.4.4 The Logarithmic Spencer Complex Revisited

Recall that, see (12.4.2), for a free divisor D in a complex manifold X, the logarithmic Spencer complex $\mathrm{Sp}^\bullet(\log D)$ is a locally free resolution of the left $\mathscr{V}_X^D$-module $\mathscr{O}_X$ [10, Th. 3.1.2]. We are going to describe the complex $\mathscr{D}_X \otimes_{\mathscr{V}_X^D} \mathrm{Sp}^\bullet(\log D)$ in terms of a (local) basis $\{\delta_1, \ldots, \delta_n\}$ of the free $\mathscr{O}_X$-module $\mathscr{D}er(-\log D)$. We can write

$$[\delta_i, \delta_j] = \sum_{k=1}^{n} \alpha_k^{ij} \delta_k$$

for $1 \leqslant i < j \leqslant n$ where $\alpha_k^{ij} \in \mathcal{O}_X$ for $k = 1, \ldots, n$. We also have

$$\mathcal{D}_X \otimes_{\mathcal{V}_X^D} \mathrm{Sp}^p(\log D) = \mathcal{D}_X \otimes_{\mathcal{V}_X^D} \mathcal{V}_X^D \otimes_{\mathcal{O}_X} \bigwedge^p \mathcal{D}er(-\log D)$$
$$\simeq \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^p \mathcal{D}er(-\log D)$$

for $p = 0, \ldots, n$, and we also denote ϵ_{-p} the differential of this complex of left $\mathcal{D}_X$-modules. We denote this complex by $\mathrm{Sp}^{\bullet}_{\mathcal{D}_X}(\log D)$.

For $p \in \mathbb{N}$ we denote $\Lambda_p := \{(i_1, \ldots, i_p) \in \mathbb{N}^p \mid 1 \leqslant i_1 < i_2 < \ldots < i_p \leqslant n\}$. The free left $\mathcal{D}_X$-module $\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^p \mathcal{D}er(-\log D) \simeq \bigoplus_{\underline{\mathbf{i}} \in \Lambda_p} \mathcal{D}_X \, e_{\underline{\mathbf{i}}}$ has rank $\binom{n}{p}$ with basis $\{e_{\underline{\mathbf{i}}} \,|\, \underline{\mathbf{i}} \in \Lambda_p\}$, for $p = 0, \ldots, n$. An isomorphism from $\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^p \mathcal{D}er(-\log D)$ onto $\bigoplus_{\underline{\mathbf{i}} \in \Lambda_p} \mathcal{D}_X \, e_{\underline{\mathbf{i}}}$ maps

$$1 \otimes \delta_{i_1} \wedge \cdots \wedge \delta_{i_p} \mapsto e_{\underline{\mathbf{i}}}.$$

For $p \in \mathbb{N}$ and $\underline{\mathbf{i}} \in \Lambda_p$ we fix the following notations:

- The support of $\underline{\mathbf{i}}$ is $\mathrm{supp}(\underline{\mathbf{i}}) = \{i_1, \ldots, i_p\}$.
- $\overline{\mathrm{supp}(\underline{\mathbf{i}})} = \{1, \ldots, n\} \setminus \mathrm{supp}(\underline{\mathbf{i}})$.
- $\underline{\mathbf{i}}(\widehat{k}) := (i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_p) \in \Lambda_{p-1}$, for $1 \leqslant k \leqslant p$.
- $\underline{\mathbf{i}}(\widehat{k, \ell}) := \underline{\mathbf{i}}(\widehat{k})(\widehat{\ell - 1}) \in \Lambda_{p-2}$, $1 \leqslant k < \ell \leqslant p$.
- $\sigma(q; \underline{\mathbf{i}}) := \max\{j \in \{1, \ldots, p\} \mid i_j < q\}$ and $\sigma(q; \underline{\mathbf{i}}) = 0$ if $q < i_1$.
- $\underline{\mathbf{i}}(\check{q}) = (i_1, \ldots, i_{k-1}, q, i_{k+1}, \ldots, i_p) \in \Lambda_{p+1}$ for $k = \sigma(q; \underline{\mathbf{i}})$ and $q \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_p\}$.

The complex $(\mathrm{Sp}^{\bullet}_{\mathcal{D}_X}(\log D), \epsilon_{-\bullet})$ can be written as the complex $(\bigoplus_{\underline{\mathbf{i}} \in \Lambda_\bullet} \mathcal{D}_X \, e_{\underline{\mathbf{i}}}, \widetilde{\epsilon_{-\bullet}})$ where

$$\widetilde{\epsilon_{-p}} : \bigoplus_{\underline{\mathbf{i}} \in \Lambda_p} \mathcal{D}_X \, e_{\underline{\mathbf{i}}} \longrightarrow \bigoplus_{\underline{\mathbf{j}} \in \Lambda_{p-1}} \mathcal{D}_X \, e_{\underline{\mathbf{j}}}$$

is the morphism of left $\mathcal{D}_X$-modules defined by

$$\widetilde{\epsilon_{-p}}(e_{\underline{\mathbf{i}}}) = \sum_{k=1}^{p} (-1)^{k-1} \delta_{i_k} e_{\underline{\mathbf{i}}(\widehat{k})} + \sum_{1 \leqslant \ell < m \leqslant p} (-1)^{\ell+m}$$
$$\sum_{q \in \overline{\mathrm{supp}(\underline{\mathbf{i}}(\widehat{\ell, m}))}} (-1)^{\sigma(q; \underline{\mathbf{i}}(\widehat{\ell, m}))} \alpha_q^{i_\ell i_m} e_{\underline{\mathbf{i}}(\widehat{\ell, m})(\check{q})}.$$

Notice that the complex $(\bigoplus_{\underline{\mathbf{i}} \in \Lambda_\bullet} \mathcal{D}_X \, e_{\underline{\mathbf{i}}}, \widetilde{\epsilon_{-\bullet}})$ can be viewed as a non commutative version of a Koszul complex.

Let us write down an example for $n = 3$.
One has:

$$\epsilon_{-1}(e_i) = \delta_i \text{ for } i = 1, 2, 3,$$

$$\epsilon_{-2}(e_{ij}) = \delta_i e_j - \delta_j e_i - \sum_{q=1}^{3} \alpha_q^{ij} e_q \text{ for } 1 \leqslant i < j \leqslant 3,$$

$$\begin{aligned}\epsilon_{-3}(e_{123}) &= \delta_1 e_{23} - \delta_2 e_{13} + \delta_3 e_{12} + (\alpha_1^{13} + \alpha_2^{23}) e_{12} \\ &\quad + (-\alpha_1^{12} + \alpha_3^{23}) e_{13} + (-\alpha_2^{12} - \alpha_3^{13}) e_{23} \\ &= (\delta_3 + \alpha_1^{13} + \alpha_2^{23}) e_{12} + (-\delta_2 - \alpha_1^{12} + \alpha_3^{23}) e_{13} + (\delta_1 - \alpha_2^{12} - \alpha_3^{13}) e_{23}.\end{aligned}$$

12.4.5 The Annihilator Ideal of $1/f$ for a reduced equation f of a normal crossing divisor in $\mathbb{C}^n$

Write $\mathscr{D} = \mathscr{D}_{\mathbb{C}^n,0}$ and $f = x_1 \cdots x_r$ for $1 \leqslant r \leqslant n$. One has the following equality

$$Ann_{\mathscr{D}}(1/f) = \mathscr{D}(\partial_{r+1}, \ldots, \partial_n, x_1\partial_1 + 1, \ldots, x_r\partial_r + 1).$$

Since the inclusion $A_n := A_n(\mathbb{C}) \subset \mathscr{D}$ is flat, it is enough to prove a similar equality for the Weyl algebra A_n instead of $\mathscr{D}$. The inclusion

$$A_n(\partial_{r+1}, \ldots, \partial_n, x_1\partial_1 + 1, \ldots, x_r\partial_r + 1) \subset \operatorname{Ann}_{A_n}\left(\frac{1}{f}\right)$$

is obvious since each generator of the first ideal annihilates $1/f$. To prove the opposite inclusion, let us assume first $1 = r = n$ and write $x = x_1$ and $\partial = \partial_x$. First of all, any $P = P(x, \partial) \in A_1$ can be written as

$$P = Q(x\partial + 1) + R + S$$

for unique $Q \in A_1$, $R := R(x) \in \mathbb{C}[x]$ and $S := S(\partial) \in \mathbb{C}[\partial](\partial)$. If $P(\frac{1}{x}) = 0$ and $S(\partial)$ is not zero with order $s \geqslant 1$, then $\frac{R}{x} + S(\partial)(\frac{1}{x}) = 0$ which is a contradiction since the order of the pole of the rational function $\frac{R}{x} + S(\partial)(\frac{1}{x})$ is $s+1$. Then $S = 0$ and then $R = 0$ and $P \in A_1(x\partial + 1)$. The general case $1 \leqslant r \leqslant n$ can be reduced to $r = n$ by taking, for any operator P annihilating $\frac{1}{x_1 \cdots x_r}$, the remainder of the division of P by $\partial_{r+1}, \ldots, \partial_n$. Finally, for the case $r = n$ we proceed by division of an operator P annihilating $\frac{1}{x_1 \cdots x_n}$, with respect to $\partial_1 x_1, \ldots, \partial_n x_n$.

The $\mathscr{D}$-module criterion for LCT gives another proof that LCT holds for normal crossing divisors. Indeed, any such divisor $D \equiv (x_1 \cdots x_r = 0) \subset \mathbb{C}^n$ is Koszul free and then the complex $\mathscr{D} \otimes_{\mathscr{V}} \operatorname{Sp}^\bullet(\log D)$ is concentrated in cohomological degree 0 and $\mathscr{D}\otimes_{\mathscr{V}}\mathscr{O}$ is locally presented as the quotient $\mathscr{D}/\mathscr{D}(\partial_{r+1},\ldots,\partial_n,x_1\partial_1+1,\ldots,x_r\partial_r+1)$, since $\{x_1\partial_1,\ldots,x_r\partial_r,\partial_{r+1},\ldots,\partial_n\}$ is a basis of $\mathrm{Der}(-\log D)$. Finally, one has

$$\frac{\mathscr{D}}{\mathscr{D}(\partial_{r+1},\ldots,\partial_n,x_1\partial_1+1,\ldots,x_r\partial_r+1)}\simeq \mathscr{D}f^{-1}=\mathscr{O}(\star D).$$

The last equality follows from the well-know fact that the (global) Bernstein-Sato polynomial of $f=x_1\cdots x_r$ is $(s+1)^r$.

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