

Chapter 2

Instructional Supports for Mathematical Problem Solving and Learning: Visual Representations and Teacher Gesture



Martha W. Alibali, Anna N. Bartel, and Amelia Yeo

Abstract Teachers' efforts to guide students' attention are potentially important for students' learning. In this chapter, we consider two types of external supports that teachers frequently use to guide students' attention: diagrams and gestures. We argue that teachers use diagrams and gestures to schematize specific features of mathematical problems or tasks, such as important elements and structural relations. In turn, teachers' schematizing increases the likelihood that students encode those features. If the schematized features are relevant to the problem or task at hand, students' appropriate encoding of those features will support their performance and learning. We present a selective review of research (including our own) on the roles of diagrams and teacher gestures in helping students encode key features and discern structure in instructional material.

Keywords Mathematics learning · Gesture · Diagrams · Encoding · Problem solving · Instruction · Teachers

2.1 Introduction

At any moment during instruction, there are many possible targets for students' attention. Given the abundant possibilities, and given the importance of attention for learning, it seems likely that teachers' efforts to guide students' attention are important for students' learning. In this chapter, we consider some of the techniques that teachers routinely use to guide students' attention in classroom settings. We focus specifically on two types of external supports for attention: diagrams and gestures.

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Imagine a middle-school teacher giving a lesson about a mathematical story problem. The teacher would like the students to attend to the mathematical relationships expressed in the story, rather than focusing on the details of the people and objects in the story. The teacher could draw a diagram to depict the mathematical relationships, in an effort to highlight those relationships and draw students' attention to them.

Now imagine another middle-school teacher presenting a lesson about linear equations. At the outset of the lesson, this teacher would like the students to attend, in a general way, to a specific symbolic equation and the associated graph, which are written on the board. At a later point in the lesson, the teacher would like the students to attend to specific elements of these representations, namely, the y -intercept in the equation and the y -intercept on the graph. The teacher could use pointing gestures, first to guide students' attention to the graph and the equation in a general way, and then to zero in on the y -intercept in each inscription.

The teachers' actions in these examples are quite ordinary, and perhaps even mundane. But in our view, these actions merit deeper consideration in terms of their function in the ongoing instruction. In each case, the instructional material is complex, and as each lesson unfolds, the teacher wishes for students to attend to certain aspects of the material at hand and not others. To do so, the teachers use external supports—in one case, a diagram, and in the other, a series of pointing gestures—to help students attend to important aspects of the material. With these external supports, teachers highlight the specific aspects of the material to which they would like students to attend in the moment.

It stands to reason that students will learn more if they pay attention to the “right” things at the “right” times. Of course, some forms of learning occur in the absence of focused attention (Conway, 2020). However, most forms of learning require that learners attend to the to-be-learned material. Students need to attend to instructionally relevant information in order to encode and operate on that information (Fisher et al., 2014). When students sustain attention to relevant lesson material, they are more likely to learn and retain that content.

What kinds of practices do teachers use in their efforts to manage students' attention to instructional material? In considering this issue, it is important to bear in mind the distinction between “paying attention” in general, and attending to specific, relevant features of the context at hand. Practices that encourage students to stay alert—such as varying instruction and implementing exercise or stretching breaks—may help students to pay attention and to learn, in general (Drollette et al., 2012; Hill et al., 2010). However, other sorts of practices may help students to focus on specific, relevant features of the instructional material or the ongoing instruction as it unfolds.

In this chapter, we consider how teachers manage students' attention to specific, instructionally relevant information, with a focus on two approaches that teachers regularly use: diagrams and hand gestures. We argue that these supports help students to attend to and discern key elements and structural features of mathematical problems and inscriptions.

Many past studies on diagrams and gestures have yielded evidence that these supports are beneficial for performance and learning. For example, diagrams have been shown to support performance in equation solving (Chu et al., 2017) and story problem solving (Cooper et al., 2018; Múñez et al., 2013). Gestures have been shown to support learning about missing-value equations (Cook et al., 2013; Koumoutsakis et al., 2016), linear equations (Alibali et al., 2013), conservation of quantity (Church et al., 2004), and bilateral symmetry (Valenzeno et al., 2003). A few studies have revealed null or negative effects of these supports in some settings or for some subgroups of learners. For example, Yeo et al. (2017b) found that middle-school students learned less about links between equations and graphs after a lesson that included gestures to the equations than after a comparable lesson that did not include such gestures. Booth and Koedinger (2012) found that low-ability sixth-grade students performed less well on story problems with diagrams than on comparable, text-only story problems—however, low-ability eighth graders performed better with diagrams. On the whole, the bulk of past research suggests that diagrams and teacher gestures are beneficial for students' performance and learning, but the size of the benefit varies depending on characteristics of the setting and the learner, as well as on the specifics of the diagrams or the gestures. Thus, diagrams and teachers' gestures can support students' performance and learning, but there are variations in the size and consistency of the benefits.

Why are diagrams and teacher gestures generally beneficial for problem solving and learning? We argue that both diagrams and gestures can schematize relevant information, and this makes students more likely to encode and use that information. We define schematizing as the process of highlighting or preserving some elements or relations and neglecting others (see Kita et al., 2017, for discussion). When teachers use diagrams or gestures to schematize specific aspects of problems or inscriptions (and consequently, to guide attention away from others), this increases the likelihood that students encode the highlighted elements and relations. If the schematized elements or relations are relevant to the problem or task at hand, appropriate encoding will in turn support students in performing or learning those tasks. From this perspective, instructional practices that schematize relevant aspects of the structure of instructional material should support students' encoding, and therefore their performance and learning.

Thus, we suggest that teachers use diagrams and gestures to guide students' attention in ways that help students "see" key features or aspects of structure in the task or problem at hand. It is sometimes difficult for students to identify important elements within complex mathematical inscriptions, so teachers may use gestures or diagrams that highlight or depict those elements. Likewise, it is sometimes difficult for students to discern relevant structure in mathematical problems, so teachers may use diagrams or gestures that highlight or depict that structure. By helping them focus their attention, these external supports, in turn, can support students in encoding features and in discerning structure.

It is worth noting that, in addition to highlighting key elements and structures in other representations, diagrams and gestures can also represent mathematical information directly. For example, a diagram of a story problem is itself a representation

of the story problem, as well as a means to highlight key elements in the story problem text. A gesture that traces a right angle is itself a representation of that specific angle, as well as a means to highlight structure in the geometric figure to which it refers. Thus, in using these supports, teachers both highlight key elements and structures in other representations and provide students with additional representations of targeted mathematical information.

In this chapter, we present a selective review of research (including our own) on the roles of diagrams and teachers' gestures in supporting students' attention during instruction. We focus on the role of these supports in helping students encode key features and discern structure in instructional material. We begin by considering teachers' gestures as an external support for students' encoding of lesson-relevant information.

2.2 Teacher Gesture as an External Support for Attending to Instructionally Relevant Information

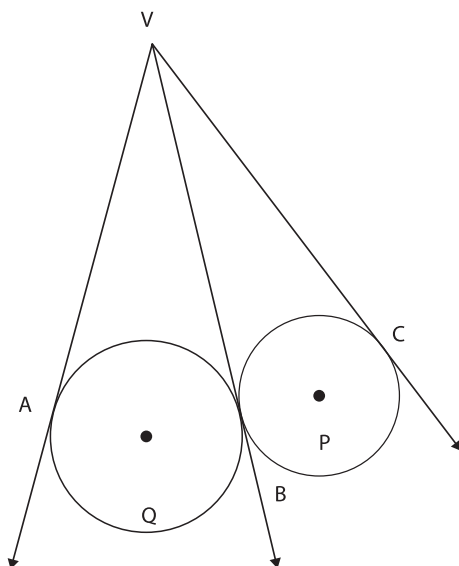
Teachers regularly produce gestures in classroom instruction. A large body of literature has considered the functions of gesture in communication (see, e.g., Church et al., 2017). Along with several other functions, this body of literature highlights the role of gesture in guiding attention to relevant information in settings that involve communication, including in instruction (Alibali et al., 2011; Church et al., [in press](#)).

One type of gesture that teachers commonly use to guide students' attention is deictic gestures, which include pointing and tracing gestures that are directed toward specific objects, inscriptions, or locations (see Cooperrider & Mesh, 2022, and Kita, 2003, for further information on this class of gestures). Such gestures highlight the referents of those gestures—that is, the objects, inscriptions, or locations to which those gestures refer—as the current focus of the discourse. In some cases, teachers may also use blocking or covering-up gestures in an effort to direct their listeners' attention away from specific objects, inscriptions, or locations.

A brief example from a high school geometry lesson illustrates how one teacher used gesture, both to highlight information that she wanted the students to focus on, and to “anti-highlight” or downplay other information that she wanted students not to focus on. The geometric figure that is the focus of the example is shown in Fig. 2.1, and an excerpt of the teacher's speech and gestures during the lesson are presented in Table 2.1. In the excerpt, the teacher is focusing on a problem that the students had been asked to solve, namely, proving that the three line segments depicted in the diagram in Fig. 2.1 (i.e., the segments from the vertex to points A, B, and C) are congruent.

At the outset of the excerpt, the teacher wishes for students to focus on the three line segments. She refers to the line segments in her speech (saying, “all three of these segments”; Unit 1 in Table 2.1) and simultaneously uses gestures to guide

Fig. 2.1 Inscription that is the focus of the lesson excerpted in Table 2.1



students' attention sequentially to each one of them. She does so by indicating the endpoints of each segment with her right hand at the vertex (labeled V for ease of reference in Fig. 2.1, although it was not labeled on the board) and her left hand indicating, in turn, the point of tangency for each of the segments (points A, B, and C) (Unit 1 in Table 2.1; see Fig. 2.2, Panel A [top left]). Moments later, when she wishes to emphasize segments VA and VB, she traces along the length of each segment in gesture (Unit 3). Thus, she uses two different forms of deictic gestures—pointing and tracing—to guide students' attention to the portions of the inscription that are most relevant in the current moment.

When she wishes students to focus solely on the two line segments that are tangent to circle Q, the teacher refers to the other segment, VC, saying “Let’s just forget about this one” (Unit 2 in Table 2.1). She uses her right arm to physically cover segment VC, putting it out of students' sight (and presumably, out of students' minds) for the moment (Fig. 2.2, Panel B, [top right]). Along with her speech, her “blocking” gesture guides students' attention away from a (momentarily) irrelevant part of the inscription. In this way, the teacher seeks to ensure that the students are attending to the information that is critical in the moment, rather than other irrelevant information, which could potentially be confusing or could overload students' memories.

The teacher also wants the students to connect the current problem to a specific theorem, the two-tangent theorem, that they had proved in class the day before. (The two-tangent theorem holds that any two tangent segments that are drawn to a circle from the same external point are congruent.) The teacher refers to that previous exercise, both in speech (“That’s what we did in the lab yesterday, right?”; Unit 4 in Table 2.1) and in gesture. She takes a step forward so that she can point to the relevant inscription from the day before, which was still present on the board (Unit 5 in

Table 2.1 Excerpt from geometry lesson

Unit	Modality	Transcript
1	Speech	Really, they're saying all three of {[these segments] [are the same] []}
	Gesture	1. RH index-finger point to V (vertex where the three segments come together). <i>Note: She holds this gesture in place while producing each of the next three gestures</i> 2. LH index-finger point to A (point of tangency for segment VA on circle Q) 3. Left hand index-finger point to B (point of tangency for segment VB on circles Q and P) 4. LH index-finger point to C (point of tangency for segment VC on circle P)
2	Speech	Let's just {forget about this one right here, for a second...
	Gesture	1. Places right arm with hand extended over segment VC, fully covering it
3	Speech	Are [these two segments gonna be the] [same?]
	Gesture	<i>Note: Her right arm is covering segment VC through this entire utterance</i> 1. LH holding marker traces from V to A 2. LH holding marker traces from V to B, holds at B
4	Speech	That's what we did in the lab yesterday, right?}}
	Gesture	<i>Note: Her gestures in this utterance are held from the preceding utterance: right arm covering segment C, LH holding marker at B</i>
5	Speech	That's what we just proved [right here], right...
	Gesture	1. RH index finger point to circle in a different diagram (located to the left of the focal diagram) on the board
6	Speech	...is that [this one's] gonna be congruent to [this one.]
	Gesture	1. Draws tick mark on segment VA 2. Draws tick mark on segment VB

Within each unit, the speech transcript is in the top row and the gesture transcript is in the bottom row. The words that accompany each gesture are indicated in brackets. Curly brackets indicate segments in which the teacher holds a gesture with one hand while producing additional gestures with the other hand (indicated in square brackets and in the text description)

Table 2.1; see Fig. 2.2, Panel C [bottom]), and then she steps back to return to the example at hand (Unit 6). This segment of the discourse is what Alibali and colleagues (2014) have called a “linking episode”, in which a teacher seeks to connect ideas in some way. Here, the teacher seeks to connect a general theorem—the focus of the previous lesson—to the current problem, which draws on that theorem. By pointing to the inscription used when proving the theorem in the previous class period, the teacher helps students reactivate the concept that they learned with that inscription, so that they can apply it to the example at hand. Pointing to the inscription provides students with an additional cue for retrieving the relevant information—a cue that may be more effective at reactivating those concepts than her words, which are quite general and even vague (“what we did in the lab”, “what we just proved”).

In summary, the teacher first guided students' attention to the line segments, and she then asked students to “forget about” one of them, while she zeroed in on the other two. Thus, in this brief excerpt, we see that the teacher uses gestures to help students focus on relevant elements of this highly complex inscription. She also

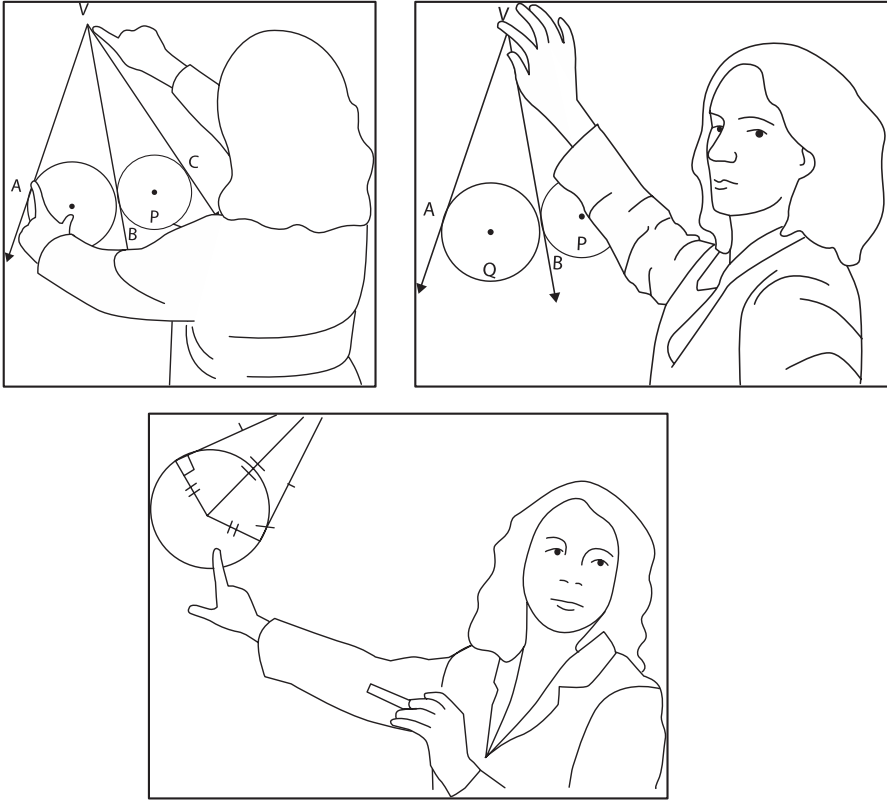


Fig. 2.2 Images from the excerpt presented in Table 2.1. Panel A (top left): Teacher using pointing gestures to indicate line segment; Panel B (top right): teacher using “covering up” gesture to encourage students to “forget about” a line segment for a moment; Panel C (bottom): teacher pointing to related inscription to link it to the current example

uses gestures to connect the example at hand to a previous inscription that the class had used earlier in proving a related theorem.

2.3 Do Teachers’ Gestures Help Students Encode Instructionally Relevant Information?

The example above focuses on how a teacher uses gesture to highlight certain elements of the inscription at hand and to “anti-highlight” or downplay other elements. Naturalistic data of this sort are compelling, but such data cannot address a key question: do teachers’ gesture actually influence how students encode instructionally relevant information? To address this question, one approach is to use an

experimental design that compares students' encoding of problem features when the teacher gestures in different ways.

Numerous studies have investigated whether students learn more when teachers produce gestures than when they do not (e.g., Church et al., 2004; Cook et al., 2013; Koumoutsakis et al., 2016; Valenzeno et al., 2003). These studies align with other research showing that gestures have a beneficial effect on comprehension of speech in other, non-instructional settings. Indeed, two comprehensive meta-analyses of the effects of speakers' gestures on listeners' comprehension (Dargue et al., 2019; Hostetter, 2011) have demonstrated beneficial effects of gesture. Both revealed a medium effect size (.61 for Hostetter, 2011, and .54 for Dargue et al., 2019 [for gesture observation]). These values indicate that across studies, roughly 70% of the participants who saw speakers' gestures scored above the mean score for participants who did not see speakers' gestures. If gestures had no effect on comprehension, one would expect this value to be 50%.

Studies investigating the effects of gesture on comprehension have been conducted in a range of settings, and many have focused on conversational settings rather than instructional settings (e.g., Kelly et al., 1999). Studies that focus on learning have used diverse types of gestures in the experimental stimuli and have examined diverse outcome measures. Although many studies have included pointing and/or tracing gestures, the dependent variables have generally not focused on students' encoding of the referents of the gestures, but rather on other, "downstream" outcomes, such as whether students learned from the lessons.

Two recent studies have investigated student attention to features of the instructional material using eye-tracking methodology. One of these studies focused on elementary-school students' learning to solve missing-addend mathematical equations of the format $3 + 4 + 5 = ___ + 5$ from a brief instructional video, and it compared students' eye movement patterns in two conditions: one in which the instructor produced gestures while providing the verbal instruction, and one in which the instructor provided instruction in speech alone (Wakefield et al., 2018). Students who viewed the video that included instructor gestures showed different patterns of attention to the instructional material than students who viewed the speech-alone lesson. They looked more at the problem that the instructor pointed to, and they looked less at the instructor. They were also more likely to align their visual attention with the content of the instructor's speech. And, not surprisingly, participants who viewed the speech-and-gesture lesson solved more of the posttest problems correctly than did participants who viewed the speech-alone lesson.

The second study used a similar eye-tracking approach to examine young children's visual attention in a lesson about analogical reasoning (Guarino et al., 2021). In the experimental task used in this study, children were asked to identify an item in a target scene that corresponded relationally to a specific item in a source scene, in the context of a distractor item that matched the target item in features but not relations. For example, given a source scene that showed a dog chasing a cat, the participant might be asked to identify the item that corresponded to the dog in a target scene that showed a boy chasing a girl, and that also included another dog. The lesson video in this study focused on teaching children to make relational

comparisons; one version of the lesson was presented in speech alone, and the other included gestures that highlighted the relational comparisons in the source and target scenes, and in so doing, directed attention away from the distractors that matched in features but not relations. Guarino and colleagues found that children who received instruction that included gestures attended less to the distractor items, and they were more likely to align their visual attention with the content of the instructor's speech. Thus, the instructor's gesture helped the children to attend to relevant visual information at the appropriate times. However, in this study, gesture did not benefit learning; there were similar levels of learning about analogical reasoning in both conditions.

Importantly, neither of these studies included measures of whether participants actually encoded the information that the instructor highlighted in gestures in the lessons that included gesture. To examine whether people encode information more effectively when speakers highlight that information in gestures, our research team has conducted three experiments examining whether students' encoding differs, depending on the teacher's gestures. All three studies investigated this question in the context of graphs of linear equations.

Our primary research question was whether students encoded the intercepts and slopes of the lines in the graphs, and whether the teacher's gestures to those features of the line would influence students' encoding. To assess encoding, we asked students to reconstruct the lines they had seen by drawing them on provided, blank graph frames; this measure is based on the assumption that students who had encoded the y -intercept and slope of a line on a graph when the teacher presented it would be able to reconstruct that line moments later, if requested to do so.

Our studies used a software-based teacher avatar (see Fig. 2.3), developed in prior work (Anasingaraju et al., 2016; Vest et al., 2020), that can gaze, speak, gesture, and write. We used the avatar so that we could perfectly control the teacher's gestures and speech. The experimental stimuli were presented in brief video excerpts of the teacher avatar presenting linear graphs. On all trials, the teacher said, "Take a look at this line." On some trials, the teacher pointed to the y -intercept of the line while uttering this statement, and on other trials, she traced the slope increase near the center of the graph by tracing a right angle under the line, starting from the line, tracing over one unit and then up to meet the line. In one experiment, we also included trials in which the teacher simply gazed at the line while speaking and did not produce any gestures. The teacher's speech was identical across conditions; the only way in which the stimuli differed was in the nature of the teacher's gestures.

In two experiments with undergraduate participants, we found that the avatar instructor's gesture influenced participants' encoding of slope in the given graphs (Yeo et al., 2017a). One experiment showed that participants were significantly more likely to correctly encode the slopes of the lines when the teacher used over-and-up tracing gestures to indicate the lines' slopes than when she used no gestures. The second experiment showed that participants were significantly more likely to correctly encode the slopes of the lines when the teacher used over-and-up tracing gestures to indicate the lines' slopes than when she pointed to the y -intercepts of the lines. Neither study revealed a beneficial effect of the teacher's pointing to the

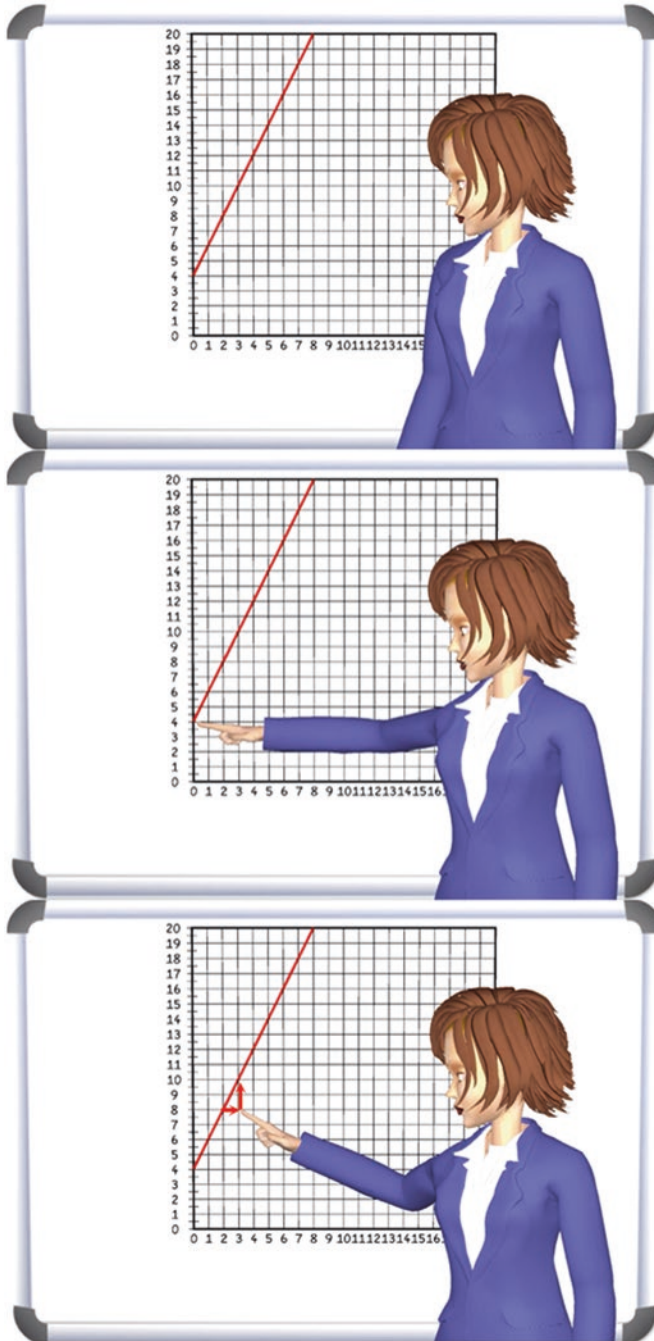


Fig. 2.3 The teacher avatar presenting a line without gesture (top), with a pointing gesture to the y-intercept (middle), and with a gesture tracing the unit increase in slope (indicated with red arrows that were not present on the graph)

y-intercepts on participants' encoding of the y-intercepts of the lines, perhaps because the participants were very successful at encoding the y-intercepts, even without teacher gestures to the y-intercepts. These findings align with the hypothesis that teachers' gestures influence participants' encoding of instructionally relevant information. In both experiments, teachers' gestures to visual representations of linear functions supported undergraduate students in successfully encoding a key feature—the slopes of the lines.

Our third study focused on middle-school students, and it compared students' encoding when the avatar teacher used over-and-up tracing gestures and when she pointed to the lines' y-intercepts (Yeo et al., [in preparation](#)). Like the undergraduate participants, the middle-school students were significantly more likely to correctly encode the slopes when the teacher used over-and-up tracing gestures than when she pointed to the y-intercepts. Also similar to the undergraduates, the middle-school students' encoding of the y-intercepts of the lines did not vary with the teachers' gestures, because the students were highly successful at encoding the y-intercepts, regardless of whether the teacher pointed to the y-intercepts or not. Thus, the instructor's gesture supported students' encoding of the slopes of the lines.

The effect sizes in each of these experiments were small, but for both undergraduates and middle-school students, students were more likely to correctly encode slope when the avatar teacher produced a gesture that highlighted slope than when she produced a gesture that highlighted intercept. It is worth noting that on each trial, the teacher uttered only a single sentence (“Take a look at this line”) and produced only a single gesture (a point to the intercept vs. an over-and-up tracing gesture to highlight slope). In our view, the fact that this very small manipulation yielded reliable effects is noteworthy. When one considers the number of gestures that a teacher produces over the course of single lesson, the potential, cumulative impact of such gestures on students' encoding of the instructional material is potentially large.

The work reviewed here suggests that teachers' gestures do indeed support students' encoding of instructionally relevant information. By guiding students where to look, teachers' gestures enhance the likelihood that students attend to and encode such information. Such gestures guide students' attention to specific features of inscriptions; for example, gesture to the slopes of the lines yielded benefits for encoding of slope, but not for encoding of intercepts. In this way, the teacher's gestures helped students accurately encode relevant aspects of the inscription at hand and ignore irrelevant features—thus, helping students to schematize key information.

2.4 Diagrams as External Supports for Discerning Structure

Another type of external support for learning that teachers commonly use in classroom settings is diagrams. Diagrams are two-dimensional visual representations that are schematic, in the sense that they selectively depict some aspects of the represented entity or situation and omit others (Bryant & Tversky, [1999](#); Tversky,

2011). Because of their schematic nature, different diagrams of the same entity or situation may depict different features of that entity or situation. Diagrams make particular features salient, and in so doing, they influence students' attention to those features. Just as gestures that highlight different features of mathematical representations lead to differences in learners' encoding of those representations, diagrams that make salient different aspects of mathematical entities or situations may lead to differences in how learners attend to those entities or situations. From this perspective, then, diagrams that schematize different information may have different effects on how people interpret and conceptualize the entities or situations depicted in the diagrams.

One study in the domain of scientific reasoning has addressed the possibility that diagrams that schematize different features lead students to conceptualize situations differently. In this study, Lee (2010) presented 9th-grade students with diagrams depicting the earth's orbit around the sun, and he examined whether different diagrams were associated with different incorrect conceptualizations of the cause of the seasons. Lee found that certain combinations of diagram features were associated with specific misconceptions; for example, students were more likely to offer side-based explanations (which incorrectly attribute the seasons to one side of the earth facing the sun and the other side facing away) when diagrams included shading of half of the earth and depicted an elongated orbital path.

Research on data visualization has also examined how people interpret graphical depictions of data, and whether different graphical features lead people to make different inferences about the underlying data distributions (Shah et al., 1999; Shah & Freedman, 2011; Zacks & Tversky, 1999). This work has shown that people who view line graphs tend to interpret the underlying data in terms of continuous trends, and they tend to describe the data using continuous, trend-related language (e.g., "Height increases with age"; Zacks & Tversky, 1999). In contrast, people who view bar graphs of the same data tend to interpret the data in terms of individual data points, and they tend to describe the data using discrete comparisons (e.g., "12-year-olds are taller than 10-year-olds"; Zacks & Tversky, 1999). Thus, different visual representations of the same data lead people to focus on different aspects of the distributions when interpreting and describing the data.

Building on this related work, we sought to examine whether different diagrams would differentially support learners in understanding algebraic story problems and symbolizing them in equations. We selected story problems as our task domain, in light of past work showing that people find such problems challenging (Koedinger & Nathan, 2004; Mayer, 1982; Nathan et al., 1992; Reed, 1999). Story problems often include information that is irrelevant to the symbolization and solution process (such as specific details of the cover stories) and learners often have difficulty identifying the critical features of the story situations.

More complex story problems present greater challenges than simpler ones. For example, Heffernan and Koedinger (1997) reported that people find it more difficult to correctly symbolize two-operator story problems than to correctly symbolize pairs of corresponding one-operator problems. Their findings suggest that integrating multiple operations into a single structure is challenging and error prone. An

illustrative example is provided in Table 2.2. Based on Heffernan and Koedinger’s (1997) results, students should be less likely to successfully symbolize the two-operator problem (first row of Table 2.2) than to successfully symbolize the pair of corresponding one-operator equations (second row of Table 2.2).

In light of this prior work, we investigated the role of two different types of diagrams in supporting learners’ symbolization of two-operator story problems (Bartel & Alibali, 2021). Both diagrams were in the form of “tape diagrams”, which are diagrams that represent relevant quantities in horizontal strips that resemble pieces of tape (Chu et al., 2017; Murata, 2008). One of the diagrams—which we call the integrated diagram—directly represented the integration of the two operations. The other—which we call the discrete diagram—represented the two operations separately. A sample problem and the two corresponding diagrams are presented in Fig. 2.4. We hypothesized that the integrated diagrams would help students to grasp the structure of the story problems and to symbolize the story problems in integrated, two-operator equations.

We tested these predictions in two experiments with undergraduate participants. We examined whether participants generated accurate representations of the story problem structure, either in one-operator equations (e.g., for the problem in Fig. 2.4, $22 - 7 = x$, $x * 5 = n$) or in a single integrated two-operator equation (e.g., for the problem in Fig. 2.4, $(n/5) + 7 = 22$). We analyzed the data from the two experiments both separately and in combination. The analysis of the combined dataset showed that participants in the diagram conditions were more likely to accurately symbolize the problem structure than participants who did not receive diagrams. However, a close look at the data revealed that this beneficial effect of diagrams was driven by participants who had lower visuospatial abilities. This subgroup of participants represented the problems more accurately with the support of diagrams that highlighted the operations and/or their integration, whereas participants with strong visuospatial abilities tended to accurately represent the problems, whether diagrams were present or not.

Table 2.2 Sample two-operator problem and corresponding pair of one-operator problems

Problem type	Example	Equations
Two-operator	Neil bought a package of 40 sunflower seeds. He emptied the bag and planted an equal number of seeds in each of four flowerpots. The next day, Neil decided he wanted to save some seeds, so he took two seeds out of each flowerpot. Write an expression for how many sunflower seeds were in each flowerpot.	$(40/4) - 2 = x$
Two one-operator	(1) Neil bought a package of 40 sunflower seeds. He emptied the bag and planted an equal number of seeds in each of four flowerpots. Write an expression for the number of sunflower seeds Neil planted in each flowerpot.	$40/4 = x$
	(2) The next day, Neil decided he wanted to save some seeds, so he took two seeds out of each flowerpot. Write an expression for how many sunflower seeds were in each flowerpot.	$x - 2 = y$

Modeled after Heffernan and Koedinger (1997)

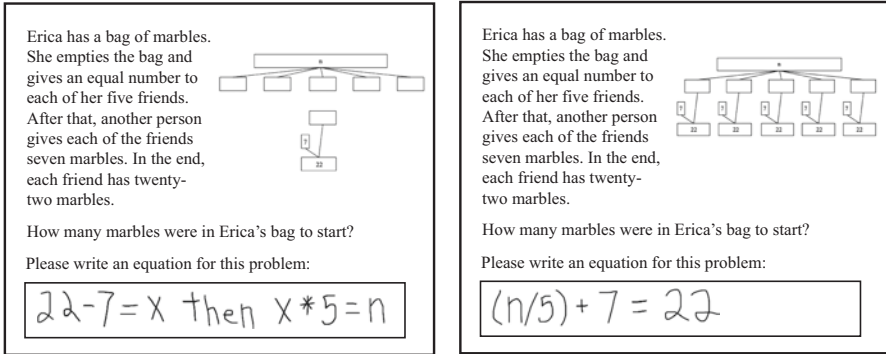


Fig. 2.4 Two-operator story problem with the discrete diagram (left panel), which depicts each of the operations, and the integrated diagram (right panel), which depicts the operations and their relationships

We also considered whether the discrete and integrated diagrams were differentially beneficial. We had predicted that the integrated diagrams, which depicted the operations and their relationships, might be more helpful than the discrete diagrams, which depicted the operations but not their relationships. Because the relationship between operations was a key aspect of the problems' structure, we predicted that diagrams that schematized this relationship might be more beneficial. Indeed, this was the case for a specific subgroup of participants, namely, participants who had more negative attitudes towards mathematics. These participants were more likely to accurately symbolize the problems' structures when the problems were accompanied by integrated diagrams than when they were accompanied by discrete diagrams.

We were also interested in whether diagrams supported undergraduate students in generating integrated, two-operator equations, in light of Heffernan and Koedinger's past work highlighting the challenges of combining operations into an integrated structure. To address this question, we compared the likelihood that participants generated integrated equations (e.g., $(n/5) + 7 = 22$) in the two diagram conditions. In one of the experiments, participants were indeed more likely to generate integrated equations in the integrated diagram condition, as we had predicted—but in the other experiment, participants were similarly likely to generate integrated equations in both diagram conditions. Given that each experiment represents an independent test of this effect, these mixed results suggest that the predicted effect may be small, or it may depend on other factors that were not measured in our experiment.

In summary, both types of diagrams were beneficial for students with less strong visuospatial skills, and the integrated diagram was especially beneficial for participants with negative attitudes towards mathematics. The findings suggest that the diagrams were beneficial for participants who, because of their pattern of skills or attitudes, were unlikely to effortfully engage with the story problems, either because they found that visualizing the relationships expressed in the stories was challenging, or because they had negative attitudes toward mathematics. Further, in one

experiment, the integrated diagrams supported students in generating integrated equations, providing some support for the idea that the schematic nature of the diagram guided students' conceptualizations in a particular way. It is worth noting that both of these studies were conducted with undergraduate students. It seems likely that the findings would generalize to younger students; however, further research is needed to be certain. It is also worth emphasizing that the studies involved symbolizing algebraic story problems—a task that is common in middle and high school.

We suggest that diagrams are beneficial largely because they are schematic. Diagrams distill and depict the most important elements and relations in a mathematical situation, so they can help learners to discern what is important and what is not. This view is supported by other research suggesting that visual representations that incorporate many rich, perceptual details may be less beneficial for learning and transfer than diagrams that are more bland and schematic (e.g., Cooper et al., 2018; Kaminiski & Sloutsky, 2013; Kaminski et al., 2008; Menendez et al., 2020). Visual representations that include “seductive details” can be visually appealing, but they can also limit generalization and transfer. Extraneous features may provide additional targets for visual attention, making it more challenging for learners to attend to the critical features of the task at hand.

Our broader point is that diagrams that appropriately schematize key aspects of problem structure can support students' performance on tasks, such as symbolization, that require discerning and attending to structural features of the problems. Diagrams are not equally beneficial for all subgroups of students, but many students can profit from the support for discerning structure that diagrams provide.

2.5 Implications for Educational Practice

Teachers often wish to support their students in successfully attending to and encoding instructional material in mathematics lessons, and they have many tools at their disposal for doing so. In this chapter, we have sought to highlight two tools that teachers commonly use to guide students' attention and to help students discern and encode structure: gestures and diagrams. Both gestures and diagrams can schematize information, and they can therefore support students' encoding.

Teachers may wish to reflect on what specific features or relations they would like to highlight, given the goals of the current lesson, and to consider what means of guiding attention to and schematizing those features or relations may be most effective. Gestures are always readily available, and they can be generated “on the spot” to address challenges or “trouble spots” in instructional communication. Diagrams may require more advanced planning and preparation—but diagrams can also be spontaneously created, depending on the tools and media that are available, and they can be readily used in both in-person and virtual instruction.

Gestures and diagrams have different affordances, and these affordances may make one or the other form of support better suited for a particular lesson or a particular instructional goal. Gestures are fleeting, and once produced, their “moment”

has passed. If a student happens to be inattentive at the moment when a gesture is produced, that student might miss out on the potential support that the gesture could provide. Diagrams are longer-lasting, and they are generally still present if one looks away and then looks back. However, diagrams are static, and as such, they may be easy to ignore. Gestures are dynamic, and they involve movement and force—so they may attract attention in ways that diagrams do not. Gestures can also be produced over or on top of other representations (e.g., tracing a line on a graph)—so that they are spatially contiguous with other, related representations (Mayer, 2009)—though they can also be produced in “neutral space”, away from the representations to which they refer (or over imaginary representations). In contrast, diagrams are generally placed alongside other representations, and as they are not spatially contiguous, they often require learners to engage in a mapping process, which can be quite challenging. These distinct affordances may influence how students use gestures and diagrams as supports—but at present, there is limited scientific understanding of these affordances and their implications for student learning.

We began this chapter with two examples of ordinary classroom moments—a teacher drawing a diagram to illustrate a story problem, and a teacher pointing to elements of a graph and an equation. Although the teachers’ actions in each case may seem quite unremarkable, we have argued that they are critically important for guiding students’ attention and supporting students’ encoding of problem features and their discerning of mathematical structure. There is more to be learned about precisely how and for whom diagrams and gestures are beneficial; however, at a minimum, it is clear that both diagrams and gestures play a role in effective pedagogy. As such, we encourage scholars of teaching and learning to more deeply consider the roles of gestures and diagrams in fostering students’ understanding, and we encourage teachers to be playful about how they use gestures and diagrams in instruction.

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