Katherine M. Robinson
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Adam K. Dubé *Editors*

Mathematical Teaching and Learning

Perspectives on Mathematical Minds in the Elementary and Middle School Years

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Contents

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Abbreviations

CAST	Center for Applied Special Technology
CCM	Cognitive-Communicative Model
CCSS	Common Core State Standards
CCSS-LA+	Common Core State Standards for English Language Arts & Literacy in History/Social Studies, Science, and Technical Subjects
CCSS-M	Common Core State Standards for Mathematics
DBCCC	Developmental Bio-Cultural Co-Constructivism
DHNP	Digital Home Numeracy Practices
EDA	Equilibrated Development Approach
ELL	English Language Learners
EQAO	Education Quality and Accountability Office
ERPs	Event-Related Potentials
FH2T	From Here to There!
FH2T:E	From Here to There!: Elementary
GM	Graspable Math
HLE	Home Learning Environment
HME	Home Numeracy Environment
HNP	Home Numeracy Practices
LIFG	Left Inferior Frontal Gyrus
MAS	Math Anxiety Survey
MAT	Math Achievement Test
MLD	Mathematical Learning Difficulties
MPES	Math Print Exposure Survey
MRHS	Math Reading Habits Survey
NRC	National Research Council
NTC	Number Talk Club
OECD	Organization for Economic Co-operation and Development
O-P Framework	Opportunity-Propensity Framework
PD	Professional Development
S.D.	Standard Deviation

SCAMPER	Substitute, Combine, Adjust, Modify, Put, Eliminate, Reverse
SDG4	Sustainable Development Goal 4 of the 2030 Agenda for Sustainable Development
SDT-SMaLL	Symbol Decision Task for SMaLL
SES	Socio-Economic Status
SMALL	Symbolic Mathematics Language Literacy
SPST	Student Program Support Teacher
TMSJ	Teaching Math for Social Justice
TWI	Two-Way Immersion
UDL	Universal Design for Learning
UN	United Nations
UNESCO	United Nations Educational, Scientific and Cultural Organization
VHQMI	Vision of High-Quality Mathematics Instruction

Chapter 1
An Introduction to Mathematics Teaching and Learning in the Elementary and Middle School Years

Donna Kotsopoulos, Adam K. Dubé, and Katherine M. Robinson

Abstract This edited collection focuses on interdisciplinary approaches to understanding teaching and learning during the elementary and middle school years. The elementary and middle school years are a formative period in children's mathematical learning making effective teaching critical for mathematical success during this period and later. The collection encompasses international perspectives from the fields of mathematics education, mathematical development, and cognition utilizing relevant disciplinary theoretical orientations and methodologies. In this chapter we introduce Part I which focusses on innovative pedagogical approaches to teaching and Part II which focusses on important methods and factors relating to mathematical learning. The collection highlights both key developments in research but also on how research can be best put into practice.

Keywords Elementary · Interdisciplinary · Mathematics · Middle school · Mathematics teaching · Mathematics learning · Mathematics education

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1.1 An Introduction to Mathematics Teaching and Learning in the Elementary and Middle School Years

There is an enduring need in research to undertake interdisciplinary theories and methodologies to understand mathematical teaching and learning. Practical and effective recommendations for teachers, parents, schools, and policy makers are always needed. As an example, consider a teacher supporting a student in the fourth grade who is struggling with multiplication. The teacher has used a variety of pedagogical approaches to support teaching and learning based on their understanding of how to teach multiplication, but to no avail. The student is still not grasping the concepts. This approach of trial and error pedagogy is not uncommon in classrooms and in homes as teachers, caregivers, and parents alike often find themselves struggling to support learners of all ages. What might the research say about how to support mathematical teaching and learning given a student's challenges? Cognitive science may offer a perspective related to cognitive load. Educational neuroscience might illustrate why certain approaches are not productive because of the way neural pathways are activated. Educational psychology may offer perspectives related to motivation or learning theories. Developmental psychologists may offer a perspective related to working memory. This simple example illustrates the immense complexities of understanding teaching and learning in mathematics – especially when there are challenges. The complexities underscore the importance of interdisciplinary approaches for understanding teaching and learning of mathematics.

This edited collection, focuses on interdisciplinary approaches to understanding teaching and learning during the elementary and middle school years. It is an extension of our edited collection, *Mathematical learning and cognition in early childhood: Integrated interdisciplinary research* (Robinson et al., 2019). The aim of this earlier book was to advance the sometimes disparate perspectives of education, developmental psychology, educational psychology, mathematics, cognitive science, and neuroscience. The focus was on early childhood and covered the preschool and early school years. The success of this book was undoubtedly linked to the dearth of interdisciplinary perspectives and the recognition of the importance of mathematical learning in the early years. These were also motives for us to continue advancing these perspectives. Like our first edited collection, this book will be relevant to scholars/educators in the field of mathematics education and also those in mathematical development and cognition. Each chapter also includes practical recommendations and implications for teachers, parents, caregivers, school, and policy makers. This edited collection brings together interdisciplinary and international perspectives and includes contributions from esteemed scholars from various fields of knowledge, theoretical orientations, and methodologies. The book is organized in two sections. Part I focuses on the pedagogical approaches to teaching. Part II focuses on mathematical learning.

A focus on the elementary and middle school years, in addition to being a natural extension to the first edited collection focused on the early years, is a developmentally significant period for teaching and learning of mathematics. Children who fall behind during these years are at significant risk for later difficulties in more advanced mathematics including algebra and calculus (Lee & Mao, 2020; McEachin et al., 2020). Falling behind in mathematics during the elementary and middle school years leaves children at risk of limitations for their educational attainment and then vocational options (Dougherty et al., 2017). This collection approaches this issue by taking diverse disciplinary perspectives on how children are taught mathematics and how they learn mathematics. Contributing researchers across multiple disciplines are increasingly turning their attention to elementary and middle school mathematics and investigating typically developing children as well as children with or at-risk of learning disabilities. This edited collection makes a significant and needed contribution to current knowledge for both researchers and educators in the field.

Our edited collection begins in Part I with a focus on teaching in Chap. 2. Alibali, Bartel, and Yeo begin their chapter with what they describe as an "ordinary moment" in a classroom where a teacher presents a diagram and uses gestures during instruction. Their chapter reviews literature on the role and use of diagrams and gestures by teachers to guide students' attention. These scholars from the US and Singapore show that teachers use both to schematize-specific features of mathematical problems or tasks, such as important elements and structural relations. More importantly, the use of these supports increases the likelihood that students encode those features and supports student performance and learning.

In Chap. 3, Polotskaia, Savard, Fellus, and Freiman focus on teaching strategies that inform student's approaches to solving word problems. In the chapter, they outline a rationale for the Equilibrated Development Approach (EDA) to word problem solving and provide its principles and epistemological stance. The authors share examples of original teaching-learning activities fostering students' mathematical thinking and sense making in solving word problems, as well as classroom observations. Their discussion highlights the intersection point of mathematics education, educational psychology, learning theories, and studies in neuro-education to demonstrate how the EDA approach coalesces insights garnered from these diverse study areas to constitute an innovative way of teaching word-problem solving in elementary school.

In Chap. 4, LeSage and Mamolo describe research that focuses on the tensions of teaching mathematics for social justice. The authors describe the experiences of a Canadian educated middle school teacher's attempts to introduce socially relevant project-based teaching while teaching in a South American international school. The authors explore and analyse the tensions the teacher experienced as she navigated the competing perspectives and expectations of the school community. This chapter offers insights into the uncertain journey of curricular change.

The theme of social justice in teaching and learning continues in Chap. 5. Italian scholars, Morselli and Robotti, engage in design-based research to design and implement inclusive activities for the teaching and learning of algebraic proof. These scholars draw from a combination of interdisciplinary theoretical tools and references from neuroscience, cognitive science, education, and mathematics education. With a focus on Universal Design for Learning principles, they detail how to design inclusive educational activities to improve and optimize teaching and learning for all students.

Part I concludes with Chap. 6 from Dickson, Kotsopoulos, and Musio who explore a board-wide professional development (PD) initiative aimed at improving mathematical achievement. Where many studies focus on teacher learning through PD, their research emphasize student learning from PD. The importance of sustained PD as opposed to "one-off" efforts were emphasized.

In Part II, our attention shifts to mathematical learning and the learner. Alam and Dubé in Chap. 7 turn our attention to the home environment. These researchers describe the Digital Home Numeracy Practice (DHNP) Model and detail its components. They explore the affect of digital factors on middle school children's mathematics practice. The model addresses how different aspects of family, such as parental factors (e.g., socio-economic situation, mathematics attitude and beliefs), children's factors (e.g., cognition, motivation, and self-regulation in general, and mathematics attitude in specific) and parent-child relationship may contribute to children's digital mathematics learning.

Woods in Chap. 8 describes "number talks" where teachers encourage students to mentally solve mathematics problems and then come together as a class to share their mathematical reasoning through whole class and small group discussion. The chapter highlights how a teacher leverages number talks to support students to (a) develop agency, (b) distribute authority, and (c) share mathematical reasoning. Mental computations were found to play an important role.

In Chap. 9, Anchan and Soylu present evidence-based recommendations for teaching mathematics to bilingual and multilingual middle school children. Classroom recommendations were developed drawing on psychological and neural mechanisms of bilingual mathematical learning and cognition, as well as sociocultural issues and implications for classroom practice. Key recommendations for practice include techniques such as code switching. Additionally, there is a discussion about the need for changes in teacher training and educational policy-making in order to increase awareness about bilingual children's needs.

Gontijo and Fonseca focus on mathematical creativity in 5th grade students in Chap. 10. The authors describe primary classroom research focused on a series of workshops that explicitly introduced creativity techniques to instruction. Pre- and post-tests of creativity, motivation, and mathematical performance showed improvement in students over time. The potential for creativity in mathematics instruction and in learner responses is highlighted. An important contribution of this chapter is the description of the creativity techniques.

Chapter 11 by Headley, Plano Clark, Stitzlein, Brown, and Swoboda focuses on the critical relationship between literacy and mathematics. This chapter introduces Symbolic Mathematics Language Literacy (SMaLL) as a framework to conceptualize reading and writing using the symbols and syntax of mathematics. It presents a mixed methods study that highlights how variations in SMaLL can be experienced among adolescents. A key contribution is the proposition that SMaLL exposes implicit literacy-for-mathematics demands in learning standards and offers researchers a useful framework for investigating them.

Part II concludes with Chap. 12 by Chan, Closser, Smith, Lee, Drzewiecki, and Ottmar. These authors present Graspable Math (GM) which is an online dynamic algebra notation system designed based on the research of cognitive, perceptual, and affective processes to support student learning. The chapter describes how log data recorded in GM offer a window into students' mathematical cognition, perceptual processes, and problem-solving strategies that can inform both research and instructional practice. The chapter informs classroom instruction and future research by providing teachers and researchers with in-depth feedback on students' use of mathematical strategies and understanding from log data.

We conclude with a prelude to our upcoming third volume that will extend this work: *Mathematical cognition and understanding: Perspectives on mathematical minds in the elementary and middle school years* (Robinson et al., In press). This third volume will shift the focus to the cognitive underpinnings involved in elementary and middle school students' mathematical knowledge and the diverse, interdisciplinary approaches to investigating and increasing the conceptual underpinnings of mathematical understanding. Our gratitude to the incredible scholars who contributed to this edited collection, Springer for their enduring confidence in our collective work, and our readers who support these collections.

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Part I
Pedogical Approaches to Teaching

Chapter 2
Instructional Supports for Mathematical Problem Solving and Learning: Visual Representations and Teacher Gesture

Martha W. Alibali, Anna N. Bartel, and Amelia Yeo

Abstract Teachers' efforts to guide students' attention are potentially important for students' learning. In this chapter, we consider two types of external supports that teachers frequently use to guide students' attention: diagrams and gestures. We argue that teachers use diagrams and gestures to schematize specific features of mathematical problems or tasks, such as important elements and structural relations. In turn, teachers' schematizing increases the likelihood that students encode those features. If the schematized features are relevant to the problem or task at hand, students' appropriate encoding of those features will support their performance and learning. We present a selective review of research (including our own) on the roles of diagrams and teacher gestures in helping students encode key features and discern structure in instructional material.

Keywords Mathematics learning · Gesture · Diagrams · Encoding · Problem solving · Instruction · Teachers

2.1 Introduction

At any moment during instruction, there are many possible targets for students' attention. Given the abundant possibilities, and given the importance of attention for learning, it seems likely that teachers' efforts to guide students' attention are important for students' learning. In this chapter, we consider some of the techniques that teachers routinely use to guide students' attention in classroom settings. We focus specifically on two types of external supports for attention: diagrams and gestures.

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Imagine a middle-school teacher giving a lesson about a mathematical story problem. The teacher would like the students to attend to the mathematical relationships expressed in the story, rather than focusing on the details of the people and objects in the story. The teacher could draw a diagram to depict the mathematical relationships, in an effort to highlight those relationships and draw students' attention to them.

Now imagine another middle-school teacher presenting a lesson about linear equations. At the outset of the lesson, this teacher would like the students to attend, in a general way, to a specific symbolic equation and the associated graph, which are written on the board. At a later point in the lesson, the teacher would like the students to attend to specific elements of these representations, namely, the y-intercept in the equation and the y-intercept on the graph. The teacher could use pointing gestures, first to guide students' attention to the graph and the equation in a general way, and then to zero in on the y-intercept in each inscription.

The teachers' actions in these examples are quite ordinary, and perhaps even mundane. But in our view, these actions merit deeper consideration in terms of their function in the ongoing instruction. In each case, the instructional material is complex, and as each lesson unfolds, the teacher wishes for students to attend to certain aspects of the material at hand and not others. To do so, the teachers use external supports—in one case, a diagram, and in the other, a series of pointing gestures—to help students attend to important aspects of the material. With these external supports, teachers highlight the specific aspects of the material to which they would like students to attend in the moment.

It stands to reason that students will learn more if they pay attention to the "right" things at the "right" times. Of course, some forms of learning occur in the absence of focused attention (Conway, 2020). However, most forms of learning require that learners attend to the to-be-learned material. Students need to attend to instructionally relevant information in order to encode and operate on that information (Fisher et al., 2014). When students sustain attention to relevant lesson material, they are more likely to learn and retain that content.

What kinds of practices do teachers use in their efforts to manage students' attention to instructional material? In considering this issue, it is important to bear in mind the distinction between "paying attention" in general, and attending to specific, relevant features of the context at hand. Practices that encourage students to stay alert—such as varying instruction and implementing exercise or stretching breaks—may help students to pay attention and to learn, in general (Drollette et al., 2012; Hill et al., 2010). However, other sorts of practices may help students to focus on specific, relevant features of the instructional material or the ongoing instruction as it unfolds.

In this chapter, we consider how teachers manage students' attention to specific, instructionally relevant information, with a focus on two approaches that teachers regularly use: diagrams and hand gestures. We argue that these supports help students to attend to and discern key elements and structural features of mathematical problems and inscriptions.

Many past studies on diagrams and gestures have yielded evidence that these supports are beneficial for performance and learning. For example, diagrams have been shown to support performance in equation solving (Chu et al., 2017) and story problem solving (Cooper et al., 2018; Múñez et al., 2013). Gestures have been shown to support learning about missing-value equations (Cook et al., 2013; Koumoutsakis et al., 2016), linear equations (Alibali et al., 2013), conservation of quantity (Church et al., 2004), and bilateral symmetry (Valenzeno et al., 2003). A few studies have revealed null or negative effects of these supports in some settings or for some subgroups of learners. For example, Yeo et al. (2017b) found that middle-school students learned less about links between equations and graphs after a lesson that included gestures to the equations than after a comparable lesson that did not include such gestures. Booth and Koedinger (2012) found that low-ability sixth-grade students performed less well on story problems with diagrams than on comparable, text-only story problems—however, low-ability eighth graders performed better with diagrams. On the whole, the bulk of past research suggests that diagrams and teacher gestures are beneficial for students' performance and learning, but the size of the benefit varies depending on characteristics of the setting and the learner, as well as on the specifics of the diagrams or the gestures. Thus, diagrams and teachers' gestures can support students' performance and learning, but there are variations in the size and consistency of the benefits.

Why are diagrams and teacher gestures generally beneficial for problem solving and learning? We argue that both diagrams and gestures can schematize relevant information, and this makes students more likely to encode and use that information. We define schematizing as the process of highlighting or preserving some elements or relations and neglecting others (see Kita et al., 2017, for discussion). When teachers use diagrams or gestures to schematize specific aspects of problems or inscriptions (and consequently, to guide attention away from others), this increases the likelihood that students encode the highlighted elements and relations. If the schematized elements or relations are relevant to the problem or task at hand, appropriate encoding will in turn support students in performing or learning those tasks. From this perspective, instructional practices that schematize relevant aspects of the structure of instructional material should support students' encoding, and therefore their performance and learning.

Thus, we suggest that teachers use diagrams and gestures to guide students' attention in ways that help students "see" key features or aspects of structure in the task or problem at hand. It is sometimes difficult for students to identify important elements within complex mathematical inscriptions, so teachers may use gestures or diagrams that highlight or depict those elements. Likewise, it is sometimes difficult for students to discern relevant structure in mathematical problems, so teachers may use diagrams or gestures that highlight or depict that structure. By helping them focus their attention, these external supports, in turn, can support students in encoding features and in discerning structure.

It is worth noting that, in addition to highlighting key elements and structures in other representations, diagrams and gestures can also represent mathematical information directly. For example, a diagram of a story problem is itself a representation of the story problem, as well as a means to highlight key elements in the story problem text. A gesture that traces a right angle is itself a representation of that specific angle, as well as a means to highlight structure in the geometric figure to which it refers. Thus, in using these supports, teachers both highlight key elements and structures in other representations and provide students with additional representations of targeted mathematical information.

In this chapter, we present a selective review of research (including our own) on the roles of diagrams and teachers' gestures in supporting students' attention during instruction. We focus on the role of these supports in helping students encode key features and discern structure in instructional material. We begin by considering teachers' gestures as an external support for students' encoding of lesson-relevant information.

2.2 Teacher Gesture as an External Support for Attending to Instructionally Relevant Information

Teachers regularly produce gestures in classroom instruction. A large body of literature has considered the functions of gesture in communication (see, e.g., Church et al., 2017). Along with several other functions, this body of literature highlights the role of gesture in guiding attention to relevant information in settings that involve communication, including in instruction (Alibali et al., 2011; Church et al., in press).

One type of gesture that teachers commonly use to guide students' attention is deictic gestures, which include pointing and tracing gestures that are directed toward specific objects, inscriptions, or locations (see Cooperrider & Mesh, 2022, and Kita, 2003, for further information on this class of gestures). Such gestures highlight the referents of those gestures—that is, the objects, inscriptions, or locations to which those gestures refer—as the current focus of the discourse. In some cases, teachers may also use blocking or covering-up gestures in an effort to direct their listeners' attention away from specific objects, inscriptions, or locations.

A brief example from a high school geometry lesson illustrates how one teacher used gesture, both to highlight information that she wanted the students to focus on, and to "anti-highlight" or downplay other information that she wanted students not to focus on. The geometric figure that is the focus of the example is shown in Fig. 2.1, and an excerpt of the teacher's speech and gestures during the lesson are presented in Table 2.1. In the excerpt, the teacher is focusing on a problem that the students had been asked to solve, namely, proving that the three line segments depicted in the diagram in Fig. 2.1 (i.e., the segments from the vertex to points A, B, and C) are congruent.

At the outset of the excerpt, the teacher wishes for students to focus on the three line segments. She refers to the line segments in her speech (saying, "all three of these segments"; Unit 1 in Table 2.1) and simultaneously uses gestures to guide

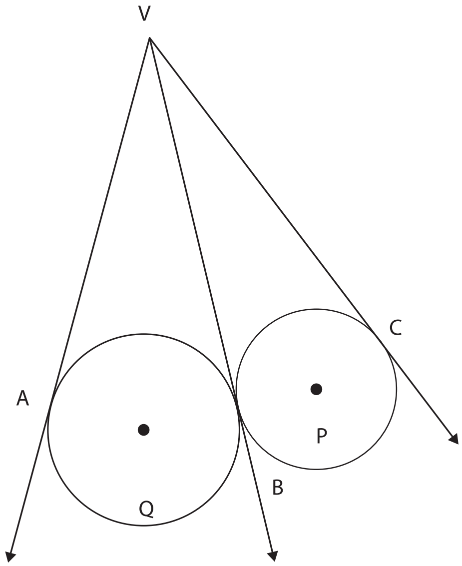

Fig. 2.1 Inscription that is the focus of the lesson excerpted in Table 2.1

students' attention sequentially to each one of them. She does so by indicating the endpoints of each segment with her right hand at the vertex (labeled V for ease of reference in Fig. 2.1, although it was not labeled on the board) and her left hand indicating, in turn, the point of tangency for each of the segments (points A, B, and C) (Unit 1 in Table 2.1; see Fig. 2.2, Panel A [top left]). Moments later, when she wishes to emphasize segments VA and VB, she traces along the length of each segment in gesture (Unit 3). Thus, she uses two different forms of deictic gestures—pointing and tracing—to guide students' attention to the portions of the inscription that are most relevant in the current moment.

When she wishes students to focus solely on the two line segments that are tangent to circle Q, the teacher refers to the other segment, VC, saying "Let's just forget about this one" (Unit 2 in Table 2.1). She uses her right arm to physically cover segment VC, putting it out of students' sight (and presumably, out of students' minds) for the moment (Fig. 2.2, Panel B, [top right]). Along with her speech, her "blocking" gesture guides students' attention away from a (momentarily) irrelevant part of the inscription. In this way, the teacher seeks to ensure that the students are attending to the information that is critical in the moment, rather than other irrelevant information, which could potentially be confusing or could overload students' memories.

The teacher also wants the students to connect the current problem to a specific theorem, the two-tangent theorem, that they had proved in class the day before. (The two-tangent theorem holds that any two tangent segments that are drawn to a circle from the same external point are congruent.) The teacher refers to that previous exercise, both in speech ("That's what we did in the lab yesterday, right?"; Unit 4 in Table 2.1) and in gesture. She takes a step forward so that she can point to the relevant inscription from the day before, which was still present on the board (Unit 5 in

Table 2.1 Excerpt from geometry lesson

Unit	Modality	Transcript
1	Speech	Really, they're saying all three of {[these segments] [are the same] [ ]}
	Gesture	1. RH index-finger point to V (vertex where the three segments come together). *Note: She holds this gesture in place while producing each of the next three gestures* 2. LH index-finger point to A (point of tangency for segment VA on circle Q) 3. Left hand index-finger point to B (point of tangency for segment VB on circles Q and P) 4. LH index-finger point to C (point of tangency for segment VC on circle P)
2	Speech	Let's just {forget about this one right here, for a second…
	Gesture	1. Places right arm with hand extended over segment VC, fully covering it
3	Speech	Are [these two segments gonna be the] [same?
	Gesture	*Note: Her right arm is covering segment VC through this entire utterance* 1. LH holding marker traces from V to A 2. LH holding marker traces from V to B, holds at B
4	Speech	That's what we did in the lab yesterday, right?]}
	Gesture	*Note: Her gestures in this utterance are held from the preceding utterance: right arm covering segment C, LH holding marker at B*
5	Speech	That's what we just proved [right here], right…
	Gesture	1. RH index finger point to circle in a different diagram (located to the left of the focal diagram) on the board
6	Speech	…is that [this one's] gonna be congruent to [this one.]
	Gesture	1. Draws tick mark on segment VA 2. Draws tick mark on segment VB

Within each unit, the speech transcript is in the top row and the gesture transcript is in the bottom row. The words that accompany each gesture are indicated in brackets. Curly brackets indicate segments in which the teacher holds a gesture with one hand while producing additional gestures with the other hand (indicated in square brackets and in the text description)

Table 2.1; see Fig. 2.2, Panel C [bottom]), and then she steps back to return to the example at hand (Unit 6). This segment of the discourse is what Alibali and colleagues (2014) have called a "linking episode", in which a teacher seeks to connect ideas in some way. Here, the teacher seeks to connect a general theorem—the focus of the previous lesson—to the current problem, which draws on that theorem. By pointing to the inscription used when proving the theorem in the previous class period, the teacher helps students reactivate the concept that they learned with that inscription, so that they can apply it to the example at hand. Pointing to the inscription provides students with an additional cue for retrieving the relevant information—a cue that may be more effective at reactivating those concepts than her words, which are quite general and even vague ("what we did in the lab", "what we just proved").

In summary, the teacher first guided students' attention to the line segments, and she then asked students to "forget about" one of them, while she zeroed in on the other two. Thus, in this brief excerpt, we see that the teacher uses gestures to help students focus on relevant elements of this highly complex inscription. She also

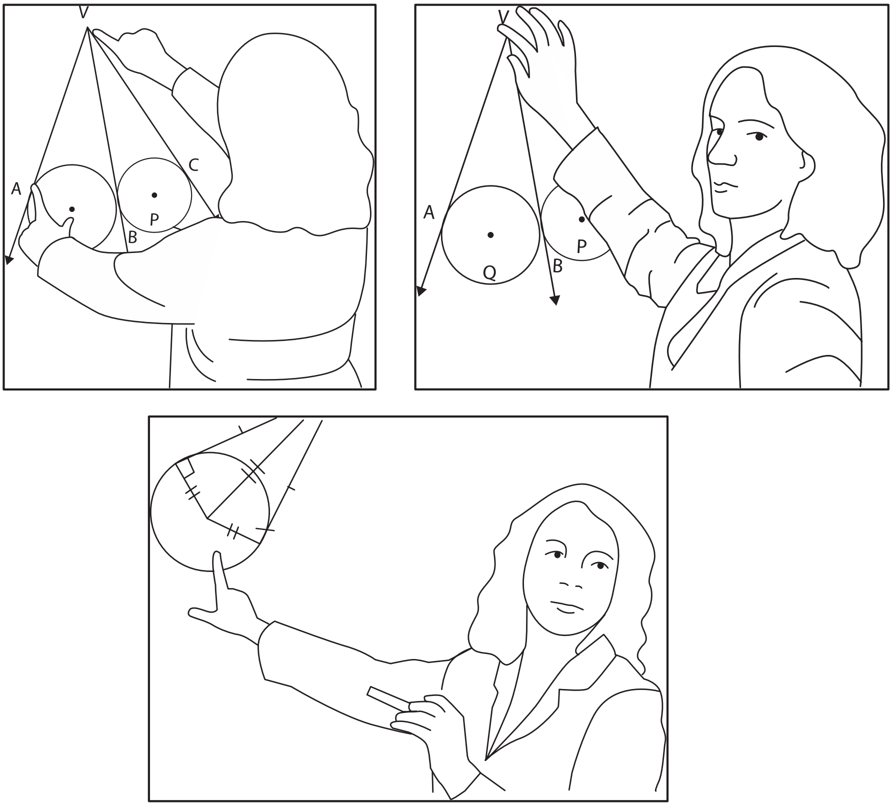

Fig. 2.2 Images from the excerpt presented in Table 2.1. Panel A (top left): Teacher using pointing gestures to indicate line segment; Panel B (top right): teacher using "covering up" gesture to encourage students to "forget about" a line segment for a moment; Panel C (bottom): teacher pointing to related inscription to link it to the current example

uses gestures to connect the example at hand to a previous inscription that the class had used earlier in proving a related theorem.

2.3 Do Teachers' Gestures Help Students Encode Instructionally Relevant Information?

The example above focuses on how a teacher uses gesture to highlight certain elements of the inscription at hand and to "anti-highlight" or downplay other elements. Naturalistic data of this sort are compelling, but such data cannot address a key question: do teachers' gesture actually influence how students encode instructionally relevant information? To address this question, one approach is to use an experimental design that compares students' encoding of problem features when the teacher gestures in different ways.

Numerous studies have investigated whether students learn more when teachers produce gestures than when they do not (e.g., Church et al., 2004; Cook et al., 2013; Koumoutsakis et al., 2016; Valenzeno et al., 2003). These studies align with other research showing that gestures have a beneficial effect on comprehension of speech in other, non-instructional settings. Indeed, two comprehensive meta-analyses of the effects of speakers' gestures on listeners' comprehension (Dargue et al., 2019; Hostetter, 2011) have demonstrated beneficial effects of gesture. Both revealed a medium effect size (.61 for Hostetter, 2011, and .54 for Dargue et al., 2019 [for gesture observation]). These values indicate that across studies, roughly 70% of the participants who saw speakers' gestures scored above the mean score for participants who did not see speakers' gestures. If gestures had no effect on comprehension, one would expect this value to be 50%.

Studies investigating the effects of gesture on comprehension have been conducted in a range of settings, and many have focused on conversational settings rather than instructional settings (e.g., Kelly et al., 1999). Studies that focus on learning have used diverse types of gestures in the experimental stimuli and have examined diverse outcome measures. Although many studies have included pointing and/or tracing gestures, the dependent variables have generally not focused on students' encoding of the referents of the gestures, but rather on other, "downstream" outcomes, such as whether students learned from the lessons.

Two recent studies have investigated student' attention to features of the instructional material using eye-tracking methodology. One of these studies focused on elementary-school students' learning to solve missing-addend mathematical equations of the format 3 + 4 + 5 = ___ + 5 from a brief instructional video, and it compared students' eye movement patterns in two conditions: one in which the instructor produced gestures while providing the verbal instruction, and one in which the instructor provided instruction in speech alone (Wakefield et al., 2018). Students who viewed the video that included instructor gestures showed different patterns of attention to the instructional material than students who viewed the speech-alone lesson. They looked more at the problem that the instructor pointed to, and they looked less at the instructor. They were also more likely to align their visual attention with the content of the instructor's speech. And, not surprisingly, participants who viewed the speech-and-gesture lesson solved more of the posttest problems correctly than did participants who viewed the speech-alone lesson.

The second study used a similar eye-tracking approach to examine young children's visual attention in a lesson about analogical reasoning (Guarino et al., 2021). In the experimental task used in this study, children were asked to identify an item in a target scene that corresponded relationally to a specific item in a source scene, in the context of a distractor item that matched the target item in features but not relations. For example, given a source scene that showed a dog chasing a cat, the participant might be asked to identify the item that corresponded to the dog in a target scene that showed a boy chasing a girl, and that also included another dog. The lesson video in this study focused on teaching children to make relational comparisons; one version of the lesson was presented in speech alone, and the other included gestures that highlighted the relational comparisons in the source and target scenes, and in so doing, directed attention away from the distractors that matched in features but not relations. Guarino and colleagues found that children who received instruction that included gestures attended less to the distractor items, and they were more likely to align their visual attention with the content of the instructor's speech. Thus, the instructor's gesture helped the children to attend to relevant visual information at the appropriate times. However, in this study, gesture did not benefit learning; there were similar levels of learning about analogical reasoning in both conditions.

Importantly, neither of these studies included measures of whether participants actually encoded the information that the instructor highlighted in gestures in the lessons that included gesture. To examine whether people encode information more effectively when speakers highlight that information in gestures, our research team has conducted three experiments examining whether students' encoding differs, depending on the teacher's gestures. All three studies investigated this question in the context of graphs of linear equations.

Our primary research question was whether students encoded the intercepts and slopes of the lines in the graphs, and whether the teacher's gestures to those features of the line would influence students' encoding. To assess encoding, we asked students to reconstruct the lines they had seen by drawing them on provided, blank graph frames; this measure is based on the assumption that students who had encoded the y-intercept and slope of a line on a graph when the teacher presented it would be able to reconstruct that line moments later, if requested to do so.

Our studies used a software-based teacher avatar (see Fig. 2.3), developed in prior work (Anasingaraju et al., 2016; Vest et al., 2020), that can gaze, speak, gesture, and write. We used the avatar so that we could perfectly control the teacher's gestures and speech. The experimental stimuli were presented in brief video excerpts of the teacher avatar presenting linear graphs. On all trials, the teacher said, "Take a look at this line." On some trials, the teacher pointed to the y-intercept of the line while uttering this statement, and on other trials, she traced the slope increase near the center of the graph by tracing a right angle under the line, starting from the line, tracing over one unit and then up to meet the line. In one experiment, we also included trials in which the teacher simply gazed at the line while speaking and did not produce any gestures. The teacher's speech was identical across conditions; the only way in which the stimuli differed was in the nature of the teacher's gestures.

In two experiments with undergraduate participants, we found that the avatar instructor's gesture influenced participants' encoding of slope in the given graphs (Yeo et al., 2017a). One experiment showed that participants were significantly more likely to correctly encode the slopes of the lines when the teacher used over-and-up tracing gestures to indicate the lines' slopes than when she used no gestures. The second experiment showed that participants were significantly more likely to correctly encode the slopes of the lines when the teacher used over-and-up tracing gestures to indicate the lines' slopes than when she pointed to the y-intercepts of the lines. Neither study revealed a beneficial effect of the teacher's pointing to the

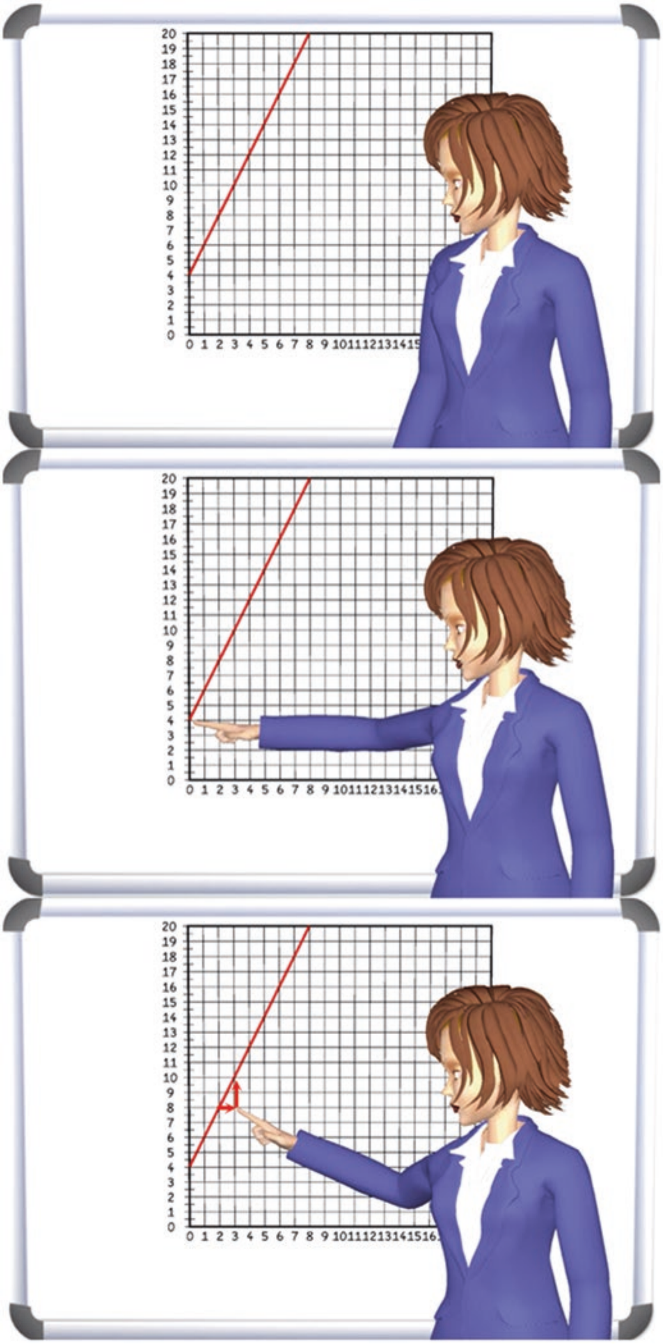

Fig. 2.3 The teacher avatar presenting a line without gesture (top), with a pointing gesture to the y-intercept (middle), and with a gesture tracing the unit increase in slope (indicated with red arrows that were not present on the graph)

y-intercepts on participants' encoding of the y-intercepts of the lines, perhaps because the participants were very successful at encoding the y-intercepts, even without teacher gestures to the y-intercepts. These findings align with the hypothesis that teachers' gestures influence participants' encoding of instructionally relevant information. In both experiments, teachers' gestures to visual representations of linear functions supported undergraduate students in successfully encoding a key feature—the slopes of the lines.

Our third study focused on middle-school students, and it compared students' encoding when the avatar teacher used over-and-up tracing gestures and when she pointed to the lines' y-intercepts (Yeo et al., in preparation). Like the undergraduate participants, the middle-school students were significantly more likely to correctly encode the slopes when the teacher used over-and-up tracing gestures than when she pointed to the y-intercepts. Also similar to the undergraduates, the middle-school students' encoding of the y-intercepts of the lines did not vary with the teachers' gestures, because the students were highly successful at encoding the y-intercepts, regardless of whether the teacher pointed to the y-intercepts or not. Thus, the instructor's gesture supported students' encoding of the slopes of the lines.

The effect sizes in each of these experiments were small, but for both undergraduates and middle-school students, students were more likely to correctly encode slope when the avatar teacher produced a gesture that highlighted slope than when she produced a gesture that highlighted intercept. It is worth noting that on each trial, the teacher uttered only a single sentence ("Take a look at this line") and produced only a single gesture (a point to the intercept vs. an over-and-up tracing gesture to highlight slope). In our view, the fact that this very small manipulation yielded reliable effects is noteworthy. When one considers the number of gestures that a teacher produces over the course of single lesson, the potential, cumulative impact of such gestures on students' encoding of the instructional material is potentially large.

The work reviewed here suggests that teachers' gestures do indeed support students' encoding of instructionally relevant information. By guiding students where to look, teachers' gestures enhance the likelihood that students attend to and encode such information. Such gestures guide students' attention to specific features of inscriptions; for example, gesture to the slopes of the lines yielded benefits for encoding of slope, but not for encoding of intercepts. In this way, the teacher's gestures helped students accurately encode relevant aspects of the inscription at hand and ignore irrelevant features—thus, helping students to schematize key information.

2.4 Diagrams as External Supports for Discerning Structure

Another type of external support for learning that teachers commonly use in classroom settings is diagrams. Diagrams are two-dimensional visual representations that are schematic, in the sense that they selectively depict some aspects of the represented entity or situation and omit others (Bryant & Tversky, 1999; Tversky, 2011). Because of their schematic nature, different diagrams of the same entity or situation may depict different features of that entity or situation. Diagrams make particular features salient, and in so doing, they influence students' attention to those features. Just as gestures that highlight different features of mathematical representations lead to differences in learners' encoding of those representations, diagrams that make salient different aspects of mathematical entities or situations may lead to differences in how learners attend to those entities or situations. From this perspective, then, diagrams that schematize different information may have different effects on how people interpret and conceptualize the entities or situations depicted in the diagrams.

One study in the domain of scientific reasoning has addressed the possibility that diagrams that schematize different features lead students to conceptualize situations differently. In this study, Lee (2010) presented 9th-grade students with diagrams depicting the earth's orbit around the sun, and he examined whether different diagrams were associated with different incorrect conceptualizations of the cause of the seasons. Lee found that certain combinations of diagram features were associated with specific misconceptions; for example, students were more likely to offer side-based explanations (which incorrectly attribute the seasons to one side of the earth facing the sun and the other side facing away) when diagrams included shading of half of the earth and depicted an elongated orbital path.

Research on data visualization has also examined how people interpret graphical depictions of data, and whether different graphical features lead people to make different inferences about the underlying data distributions (Shah et al., 1999; Shah & Freedman, 2011; Zacks & Tversky, 1999). This work has shown that people who view line graphs tend to interpret the underlying data in terms of continuous trends, and they tend to describe the data using continuous, trend-related language (e.g., "Height increases with age"; Zacks & Tversky, 1999). In contrast, people who view bar graphs of the same data tend to interpret the data in terms of individual data points, and they tend to describe the data using discrete comparisons (e.g., "12-year-olds are taller than 10-year-olds"; Zacks & Tversky, 1999). Thus, different visual representations of the same data lead people to focus on different aspects of the distributions when interpreting and describing the data.

Building on this related work, we sought to examine whether different diagrams would differentially support learners in understanding algebraic story problems and symbolizing them in equations. We selected story problems as our task domain, in light of past work showing that people find such problems challenging (Koedinger & Nathan, 2004; Mayer, 1982; Nathan et al., 1992; Reed, 1999). Story problems often include information that is irrelevant to the symbolization and solution process (such as specific details of the cover stories) and learners often have difficulty identifying the critical features of the story situations.

More complex story problems present greater challenges than simpler ones. For example, Heffernan and Koedinger (1997) reported that people find it more difficult to correctly symbolize two-operator story problems than to correctly symbolize pairs of corresponding one-operator problems. Their findings suggest that integrating multiple operations into a single structure is challenging and error prone. An illustrative example is provided in Table 2.2. Based on Heffernan and Koedinger's (1997) results, students should be less likely to successfully symbolize the two-operator problem (first row of Table 2.2) than to successfully symbolize the pair of corresponding one-operator equations (second row of Table 2.2).

In light of this prior work, we investigated the role of two different types of diagrams in supporting learners' symbolization of two-operator story problems (Bartel & Alibali, 2021). Both diagrams were in the form of "tape diagrams", which are diagrams that represent relevant quantities in horizontal strips that resemble pieces of tape (Chu et al., 2017; Murata, 2008). One of the diagrams—which we call the integrated diagram—directly represented the integration of the two operations. The other—which we call the discrete diagram—represented the two operations separately. A sample problem and the two corresponding diagrams are presented in Fig. 2.4. We hypothesized that the integrated diagrams would help students to grasp the structure of the story problems and to symbolize the story problems in integrated, two-operator equations.

We tested these predictions in two experiments with undergraduate participants. We examined whether participants generated accurate representations of the story problem structure, either in one-operator equations (e.g., for the problem in Fig. 2.4, $22 - 7 = x$, $x * 5 = n$) or in a single integrated two-operator equation (e.g., for the problem in Fig. 2.4, $(n/5) + 7 = 22$). We analyzed the data from the two experiments both separately and in combination. The analysis of the combined dataset showed that participants in the diagram conditions were more likely to accurately symbolize the problem structure than participants who did not receive diagrams. However, a close look at the data revealed that this beneficial effect of diagrams was driven by participants who had lower visuospatial abilities. This subgroup of participants represented the problems more accurately with the support of diagrams that highlighted the operations and/or their integration, whereas participants with strong visuospatial abilities tended to accurately represent the problems, whether diagrams were present or not.

Table 2.2 Sample two-operator problem and corresponding pair of one-operator problems

Problem type	Example	Equations
Two-operator	Neil bought a package of 40 sunflower seeds. He emptied the bag and planted an equal number of seeds in each of four flowerpots. The next day, Neil decided he wanted to save some seeds, so he took two seeds out of each flowerpot. Write an expression for how many sunflower seeds were in each flowerpot.	$(40/4) - 2 = x$
Two one-operator	(1) Neil bought a package of 40 sunflower seeds. He emptied the bag and planted an equal number of seeds in each of four flowerpots. Write an expression for the number of sunflower seeds Neil planted in each flowerpot.	$40/4 = x$
	(2) The next day, Neil decided he wanted to save some seeds, so he took two seeds out of each flowerpot. Write an expression for how many sunflower seeds were in each flowerpot.	$x - 2 = y$

Modeled after Heffernan and Koedinger (1997)

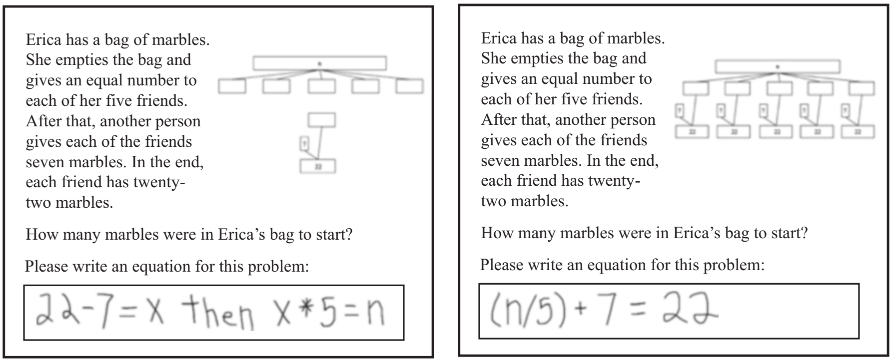

Fig. 2.4 Two-operator story problem with the discrete diagram (left panel), which depicts each of the operations, and the integrated diagram (right panel), which depicts the operations and their relationships

We also considered whether the discrete and integrated diagrams were differentially beneficial. We had predicted that the integrated diagrams, which depicted the operations and their relationships, might be more helpful than the discrete diagrams, which depicted the operations but not their relationships. Because the relationship between operations was a key aspect of the problems' structure, we predicted that diagrams that schematized this relationship might be more beneficial. Indeed, this was the case for a specific subgroup of participants, namely, participants who had more negative attitudes towards mathematics. These participants were more likely to accurately symbolize the problems' structures when the problems were accompanied by integrated diagrams than when they were accompanied by discrete diagrams.

We were also interested in whether diagrams supported undergraduate students in generating integrated, two-operator equations, in light of Heffernan and Koedinger's past work highlighting the challenges of combining operations into an integrated structure. To address this question, we compared the likelihood that participants generated integrated equations (e.g., $(n/5) + 7 = 22$) in the two diagram conditions. In one of the experiments, participants were indeed more likely to generate integrated equations in the integrated diagram condition, as we had predicted—but in the other experiment, participants were similarly likely to generate integrated equations in both diagram conditions. Given that each experiment represents an independent test of this effect, these mixed results suggest that the predicted effect may be small, or it may depend on other factors that were not measured in our experiment.

In summary, both types of diagrams were beneficial for students with less strong visuospatial skills, and the integrated diagram was especially beneficial for participants with negative attitudes towards mathematics. The findings suggest that the diagrams were beneficial for participants who, because of their pattern of skills or attitudes, were unlikely to effortfully engage with the story problems, either because they found that visualizing the relationships expressed in the stories was challenging, or because they had negative attitudes toward mathematics. Further, in one experiment, the integrated diagrams supported students in generating integrated equations, providing some support for the idea that the schematic nature of the diagram guided students' conceptualizations in a particular way. It is worth noting that both of these studies were conducted with undergraduate students. It seems likely that the findings would generalize to younger students; however, further research is needed to be certain. It is also worth emphasizing that the studies involved symbolizing algebraic story problems—a task that is common in middle and high school.

We suggest that diagrams are beneficial largely because they are schematic. Diagrams distill and depict the most important elements and relations in a mathematical situation, so they can help learners to discern what it important and what is not. This view is supported by other research suggesting that visual representations that incorporate many rich, perceptual details may be less beneficial for learning and transfer than diagrams that are more bland and schematic (e.g., Cooper et al., 2018; Kaminiski & Sloutsky, 2013; Kaminski et al., 2008; Menendez et al., 2020). Visual representations that include "seductive details" can be visually appealing, but they can also limit generalization and transfer. Extraneous features may provide additional targets for visual attention, making it more challenging for learners to attend to the critical features of the task at hand.

Our broader point is that diagrams that appropriately schematize key aspects of problem structure can support students' performance on tasks, such as symbolization, that require discerning and attending to structural features of the problems. Diagrams are not equally beneficial for all subgroups of students, but many students can profit from the support for discerning structure that diagrams provide.

2.5 Implications for Educational Practice

Teachers often wish to support their students in successfully attending to and encoding instructional material in mathematics lessons, and they have many tools at their disposal for doing so. In this chapter, we have sought to highlight two tools that teachers commonly use to guide students' attention and to help students discern and encode structure: gestures and diagrams. Both gestures and diagrams can schematize information, and they can therefore support students' encoding.

Teachers may wish to reflect on what specific features or relations they would like to highlight, given the goals of the current lesson, and to consider what means of guiding attention to and schematizing those features or relations may be most effective. Gestures are always readily available, and they can be generated "on the spot" to address challenges or "trouble spots" in instructional communication. Diagrams may require more advanced planning and preparation—but diagrams can also be spontaneously created, depending on the tools and media that are available, and they can be readily used in both in-person and virtual instruction.

Gestures and diagrams have different affordances, and these affordances may make one or the other form of support better suited for a particular lesson or a particular instructional goal. Gestures are fleeting, and once produced, their "moment" has passed. If a student happens to be inattentive at the moment when a gesture is produced, that student might miss out on the potential support that the gesture could provide. Diagrams are longer-lasting, and they are generally still present if one looks away and then looks back. However, diagrams are static, and as such, they may be easy to ignore. Gestures are dynamic, and they involve movement and force—so they may attract attention in ways that diagrams do not. Gestures can also be produced over or on top of other representations (e.g., tracing a line on a graph)—so that they are spatially contiguous with other, related representations (Mayer, 2009)—though they can also be produced in "neutral space", away from the representations to which they refer (or over imaginary representations). In contrast, diagrams are generally placed alongside other representations, and as they are not spatially contiguous, they often require learners to engage in a mapping process, which can be quite challenging. These distinct affordances may influence how students use gestures and diagrams as supports—but at present, there is limited scientific understanding of these affordances and their implications for student learning.

We began this chapter with two examples of ordinary classroom moments—a teacher drawing a diagram to illustrate a story problem, and a teacher pointing to elements of a graph and an equation. Although the teachers' actions in each case may seem quite unremarkable, we have argued that they are critically important for guiding students' attention and supporting students' encoding of problem features and their discerning of mathematical structure. There is more to be learned about precisely how and for whom diagrams and gestures are beneficial; however, at a minimum, it is clear that both diagrams and gestures play a role in effective pedagogy. As such, we encourage scholars of teaching and learning to more deeply consider the roles of gestures and diagrams in fostering students' understanding, and we encourage teachers to be planful about how they use gestures and diagrams in instruction.

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Chapter 3
Equilibrated Development Approach to Word Problem Solving in Elementary Grades: Fostering Relational Thinking

Elena Polotskaia, Annie Savard, Olga Fellus, and Viktor Freiman

Abstract The issue of how teaching strategies inform students' approaches to solving word problems has been, and still remains, the focus of attention in mathematics education. In this book chapter, we outline a rationale for the Equilibrated Development Approach (EDA) to word problem solving and provide its principles and epistemological stance. The EDA is informed by and critically engages with the tenets of sociocultural theory, the Vygotskian paradigm for learning, and Davydov's and Galperin's theoretical and empirical work (Vygotsky LS, Educational psychology. St. Lucie Press, Boca Raton, 1997; Davydov VV, Problems of developmental instruction: a theoretical and experimental psychological study. Nova Science Publishers, Hauppauge, 2008; Galperin P, Georgiev L, The formation of elementary mathematical notions. Soviet Stud Psychol Learn Teach Math 1:189–216, 1969). For a decade now, we have used the EDA to construct, refine, and gradually implement new ways of teaching problem solving. We share examples of original teaching-learning activities fostering students' mathematical thinking and sense making in solving word problems. We also share examples from our classroom observations to suggest alternative teaching strategies in elementary mathematics. The purpose of our work is to surface processes of understanding quantitative relationships and to shed light on the role they play in one's capacity-building in solving word problems. In this chapter, we discuss our work within the intersection point of mathe-

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matics education, educational psychology, learning theories, and studies in neuro-education to demonstrate how the EDA approach coalesces insights garnered from these diverse study areas to constitute an innovative way of teaching word-problem solving in elementary school.

Keywords Word problem solving · Relational thinking · Teaching mathematics · Elementary school

3.1 Introduction

Teaching mathematics through problems is integral to K-12 school curricula. At elementary school level, students are expected to first tackle problems that involve additive structures (addition and subtraction) and then work on problems that present multiplicative structures (multiplication and division). In our work, we observed that a traditional way of teaching word problems in class constitutes a top-down approach, in which students are expected to first master the arithmetic operations and only then apply these operations in word problems (Cavalcante et al., 2019). This top-down strategy generates multiple, varied, and potentially counterproductive challenges for the learners, because the tendency to focus on prescribed calculations leaves little, if any, space to focusing on mathematical structure and, in turn, to making sense of the problem at hand. To address this tension, we bring perspectives from different fields of research to demonstrate the relevance, and necessity, of studying the mathematical structures in word problem instead of focusing on performing operations on numerical values.

3.2 Theoretical Background

3.2.1 Operational Paradigm

A copious body of research approaches word problem solving from the point of view of the mathematical structure problems represent. To identify this structure, the meaning of a particular action-related cue or relationship given in the text is usually used (e.g., Greer, 1992; Schwartz, 1996). Many researchers are in consensus that the entry point in word problems is understanding the wording as a reflection of an arithmetic operation (Carpenter et al., 1999; Nesher et al., 1982; Riley et al., 1984). Savard et al. (2018) identified this approach as the Operational Paradigm because it emphasizes arithmetic operation as a means to understanding a problem. Within the Operational Paradigm, the arithmetic operations are considered the most important mathematical knowledge to be developed. In classrooms that are oriented toward the operational paradigm, students first learn about elementary arithmetic operations in situations where the wording corresponds directly to an operation. For example, the problem "*Marta has 3 apples. Mark gives her 2 more apples.*

How many apples does Marta have now?" corresponds to the addition operation 3 + 2. Students see the problem as a story, reproduce the story using objects in a sequential way, and construct the answer as the final state of this story. This matching of word constructions to specific operations has been identified as the essential entry point to the development of problem-solving skills within the Operational Paradigm (e.g., Nesher et al., 1982; Riley et al., 1984).

Evidently, not all problems can be solved using wording that corresponds directly to an operation. For example, the problem "*Jim had 5 cookies. He received some more cookies. Now he has 8 cookies. How many cookies has he received?*" requires a non-intuitive transformation of the semantic meaning of the story (*receive more*) into a subtraction operation (8-5 =?), and thus, is more difficult for students to carry out. At some point of this developmental path, students are expected to reason about the problem in a holistic flexible way, and see the problem as a structure that can be transformed into different operations (Lesh & Zawojewski, 2007). To that end, a student "is able to read the word 'more,' and yet perform a subtraction operation" (Nesher et al., 1982, p. 392). Thus, the Operational Paradigm implies that at some point, the learner should jump from a sequential intuitive thinking to a non-intuitive transformation of the semantic structure of a problem.

The idea of different levels of difficulty in word problems was supported by studies conducted by Pape (2003, 2004). Pape's notions of *consistent* (e.g., The Marta problem) and *inconsistent language* (e.g., The Jim problem) can be used to explain students' difficulties in word problem solving. To solve a consistent problem, no structure transformation is required. A structure transformation that is required to solve *inconsistent* problems represents a substantial difficulty for students. Within the Operational Paradigm, there is no clear theoretical solution to this gap.

3.2.2 Insights from Neuro-education and the Developmental Aspects of Learning

Research in neuro-education can shed light on the way mathematical thinking is developed and organized. Stavy and Babai (2010) used brain imaging to analyze the process of solving certain geometric problems. In some problems, a congruent condition—cases where figures with smaller perimeters have smaller areas—was used, so the solution could be intuitive and straightforward. In other cases, incongruent conditions, where the relationship between the change in the perimeters and the change in the areas was opposite, necessitated some logical analysis to solve the task correctly. Stavy and Babai (2010) showed that the parts of the brain that were most active during the problem-solving process differed when it came to solving direct intuitive (congruent condition) versus solving reflexive inverse (incongruent condition) problems. This study is relevant to our work as it suggests that the difference in brain activation for intuitive versus reflexive thinking can also be applied to arithmetic word problem solving because it helps explain the difficulty in transitioning to types of word problems that require a transformation of the semantic structure of the problem.

Piaget's perception of learning grounds the learning of new skills on already constructed knowledge (Piaget, 1964). If the sequential thinking of arithmetic operations (i.e., direct sequential thinking) is what students develop first, then it is the only available skill and is automatically retrieved when solving word problems. Stavy and Babai's (2010) study helps us understand that a different way of thinking activates a different area in the brain thus necessitating learning a different set of skills. Problems that do not align with the sequential thinking of arithmetic operations may require reflection and transformation because their semantic structure does not linearly correspond to a particular arithmetic operation and this, in turn, requires activation of other brain areas that were not previously used. If we consider intuitive thinking as something that heavily relies on knowledge of operations and if we acknowledge that transformations of semantic structure of problems are treated differently in the brain, we can better understand why many students struggle with the shift from sequential intuitive thinking to holistic and flexible thinking. It is possible that the practice in intuitive thinking does not so much contribute to the development of structural analysis and transformation skills.

Another aspect in neuroscience research that can inform the process of word problem solving is the attention the solver pays to available information. The neuroscientist Robertson (2017) explains that word-problem solvers might sometimes ignore the essence of a problem by paying more attention to its superficial aspects. He explains that this is quite natural at the beginning of the learning process. Robertson (2017) also points to the important role mental schema plays in the process of analyzing and solving a problem. He argues that mental schema directly influences the way the solver sees the situation from the beginning of the process of solving the problem. We suggest that initial mental schema can guide students in representing a problem. Using available knowledge, students create representations of what they pay attention to. Thus, if thinking about objects or about the exact number of objects is what students pay attention to in the problem's text, then they think about the numerical value and not necessarily about the relationship between quantities.

To draw students' attention to the wording of arithmetic problems, teachers often highlight keywords (e.g., won, removed, shared, times more) and the known data—numbers in the text. Yet, this approach can distract students' attention from the mathematical structure of the problem. Not paying attention to structure is more often than not the de-facto instruction. This naïve ignorance of inconsistent word problems can develop into a firm solving strategy, that, in turn, may be difficult or even impossible to unlearn or reform. The long-term effect of not attending to inconsistent structures manifests itself at post-secondary level as researchers observed the "keywords translate to operation" strategy that is instilled in elementary school mathematics in some university students (Hegarty et al., 1995). Furthermore, Bednarz (2009) builds on the work of others and explains that the first learned strategy persists and generates obstacles years after its formation potentially as a result of years of practice that solidify this way of thinking.

These studies in neuroeducation and mathematics education help us see why the developmental trajectory based on the operational paradigm fails so many learners. The introduction of the operations followed by a disproportionate use of word problems that can be directly interpreted into arithmetic operations can produce the "intuitive thinking" effect. Excessive attention paid to numbers and keywords together with the visual representations of discrete objects and numerical symbols can hinder the goal of structural understanding of a problem. Later, the initially constructed strategy of the direct translation of keywords into operations can become an obstacle to the holistic and flexible understanding of the mathematical structure of more complex problems. We suggest that in the long term, the operational approach can potentially exacerbate difficulties of some students in mathematics.

A growing number of studies over the last decades brings new perspectives to highlight the relationship between mathematical structures and word problem solving (e.g., Cai et al., 2005; Ng & Lee, 2009; Xin et al., 2011). They emphasize the idea that understanding a situation mathematically requires students to recognize relationships between given quantities. Thus, many researchers suggest that to ensure success in word problem solving, it is relational thinking that students should develop more.

3.2.3 Relational Paradigm

In the second half of the twentieth century, Russian psychologists Galperin and Georgiev (1969) and then Davydov (1982) promoted the idea that teaching numbers and operations should be **preceded by an intentional teaching of quantitative relationships** in elementary school. In addition to demonstrating that the idea of equivalence and related additive principles arise from relational thinking, Davydov also argued that the concept of number emerges from the multiplicative comparison of two magnitudes, one functioning as a unit of measure and the other as the quantity that is being measured. For example, measuring a string can give **5.5 inches**, **14 centimeters and 4 lengths of a paperclip**, thus the number depends on the unit of measurement in relation to the measured quantity. However, many traditional teaching contexts do not build on the idea of measurement and its relational nature at the beginning of the learning trajectory. Davydov (1982) went as far as claiming that starting with counting and operations precludes relational thinking as well as the understanding of underlying mathematical principles. He argued:

> Although children of ages 6 or 7 can readily solve in an abstract (e.g., 3 + 2 = ? or 8 - 5 = ?) or concrete (e.g., add two apples to three apples) form, they cannot explain what numbers are, how they arose, or why in using numbers it is necessary to add or subtract (p. 225).

At the very beginning of a new learning trajectory that was proposed by Davydov and his colleagues, students were asked to manipulate real objects comparing quantities of water, areas of surfaces, and lengths of ropes with the objective of developing holistic and generalized understanding of the underlying quantitative relationships. In this learning trajectory, teachers discussed with students the schematic and symbolic representations to model the underlying relationships in problems. Only when some basic relationships are mustered by learners do they start to use this knowledge to develop their formal understanding of natural numbers. Operations appear as tools to find unknown values in a relationship, which preserve its equilibrium. For example, the value 3 (=8–5) preserves the equilibrium of the relationship described in the Jim problem (see p. 2).

Davydov's relational approach to the development of number sense has been recognized as an important contribution to elementary mathematics teaching theory (Iannece et al., 2009; Lins & Kaput, 2004; Sophian, 2007) thus opening a possibility of a new developmental trajectory grounded within the relational paradigm (Polotskaia & Savard, 2018). The development of Davydov's ideas yielded the reconsidering of problem solving as a learning activity. In our work, we questioned the types of thinking that should support a successful process of solving a word problem and favored the quantitative relationships over operations. Yet, we acknowledged research (e.g., Nesher et al., 1982; Okamoto, 1996; Riley et al., 1984) that has confirmed the important role that number knowledge and operations play in supporting successful problem solving. Keeping in mind these two arguments, we considered both numerical and relational thinking as two necessary but not sequenced stages of problem-solving knowledge development. We organize teaching so that students can develop holistic relational thinking and numerical thinking in harmony and coherence with each other (Polotskaia, 2015).

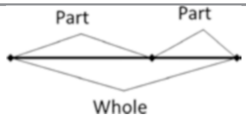
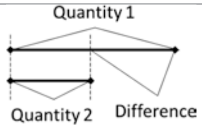
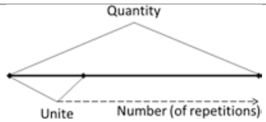
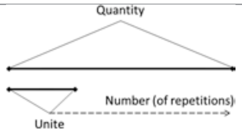
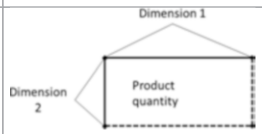
3.3 Equilibrated Development Approach

Further developing Davydov's ideas, our study proposes a balanced approach. We consider the learning trajectory not as a straight line, but as a network of learning paths that can be efficiently interconnected. Thus, we argue for an Equilibrated Development Approach (EDA) to teaching elementary mathematics. In a recent study, we develop the EDA by identifying and integrating into the traditional curriculum the notions of two additive and three multiplicative relationships (Polotskaia & Savard, 2018; Polotskaia & Savard, 2021) (see Appendix 1). These quantitative relationships function as mental tools giving access to the mathematical meaning of a problem, one that is sometimes replaced in the student's mind by a simplistic image of an operation with numbers. The understanding of a problem as network of relationships contributes to equilibrate the complex process of the developing mathematical knowledge. Furthermore, constructing visual representations of relationships allows students to express their relational understanding making relationships explicit through the use of tangible tools that can be used for a mathematical discussion. Thus, the ability to construct sound visual representations of relationships is an important element of the developmental path towards relational thinking.

To summarize, the Equilibrated Development Approach puts forward:

- Paying attention to studying five basic relationships (see Table 3.1);
- Using visual representations of relationships by constructing and analyzing diagrams; and

Table 3.1 Five simple multiplicative relationships

Relationship	Description	Model	Examples
Additive composition	This relationship can be used if a quantity is composed of two (or more) parts		1. There are red apples and green apples 2. Ann has marbles. She wins more marbles
Additive comparison	This relationship can be used when two quantities are compared to highlight the difference		1. Ann has fewer buttons than Olga.
Multiplicative composition	This relationship can be used if one quantity is composed of a number of equal parts (the number can be real).		1. Max has many boxes with the same number of marbles in each. 2. A car moving with a constant speed made a certain distance in a certain time.
Multiplicative comparison	This relationship can be used if one quantity is compared in a multiplicative way to the other quantity. The latter is physically distinct from the former, and the comparison yields a number whether it is known or unknown.		1. Max has three times as many marbles as Maya. 2. Max's shoe is twice as long as Maya's shoe. 3. How many times is Maya younger than max?
Cartesian product	This relationship can be used if all three elements have different physical origins and none of them can be seen as a pure number or as a unit of measurement.		1. One uses a number of skirts and a number of blouses to create costumes. 2. One evaluates a rectangular area in relation to its length and width.

- Conducting conceptually rich mathematical discussions about relationships by conjecturing, justifying, proving, making inferences, verbalizing, and making sense.

The developmental impact of the EDA on learners is fundamentally different from the one often promoted within the operational paradigm. A simple (1 operation) arithmetic problem can be analyzed and understood as a simple relationship. Relational paradigm suggests that all word problems presenting one quantitative relationship require two steps: (1) transforming the semantic meaning of the text into a visually represented relationship, then (2) using the properties of the relationship at hand to deduce an operation, which allows, in turn, to find the unknown element. The transformations of meaning among the text, the visual representation, and the operation require more effort from the learner to solve the problem than a simple keyword translation. At the same time, the learner will develop relational thinking instead of automatizing the direct translation of keywords.

A traditional task of solving a word problem requires a calculation of a numerical answer. This explicit expectation by itself, can distract students' attention from the relationships in favor of numbers and operations. Thus, the goal of studying relationships requires new forms of learning tasks and activities.

3.3.1 Activities to Promote Relational Thinking and Modeling

Our teaching experiment and design projects (Freiman et al., 2017; Polotskaia, 2014; Polotskaia & Savard, 2018) helped us create several new forms of activities for students in grades 2 to 6. Such activities direct students' attention to the quantitative relationships and explicitly require students to model those relationships visually. A visual representation of a relationship serves at least two objectives. To represent a quantity in a relationship, the learner needs to mentally transform the idea of a number into a line segment (Davydov & Mikulina, 1988). The mental exercise of connecting a discrete and a continuous representation of a number contributes to the development of number sense. A visual expression of the learner's relational thinking makes it easy for the teacher and other learners to access the student's thinking and react in more adequate ways. Below, we describe some types of activities putting forward the visual modeling of relationships.

When the time comes to solve word problems, the relational analysis and calculation are carefully integrated into a cyclic process described by the ethnomathematical model (Mukhopadhyay & Greer, 2001; Polotskaia, 2014; Savard, 2008). The model is represented graphically in Fig. 3.1; an example of a solving process will follow in the section titled *How it Works in Class*.

According to the ethno-mathematical model, to solve a problem, the solver should become familiar with the sociocultural context of the problem through reading the text (story). Using her mental representation of the problem, the solver should express the situation within the mathematical context by creating a model producing a holistic view of the quantitative relationships involved. At this moment, the numbers given in the problem are not included in the analysis (see "~~97~~" on Fig. 3.1). From the model, the arithmetic operation to apply can be derived, thus transforming the holistic view into a calculation plan. At this point, the numbers are used to conduct the calculation (see "97" on Fig. 3.1). The learner should then make sense of the numerical results in terms of the sociocultural context evaluating it in relation to the wording of the problem. Thus, the problem-solving process is organized in a cycle, potentially supporting the development of both relational and numerical thinking in learners. This process of solving requires multiple transformations of meaning thus fostering learners' mathematical thinking development.

The teacher's explicit request to construct a representation transforms the step "understand the problem" into an analysis and a modelling process. Instead of

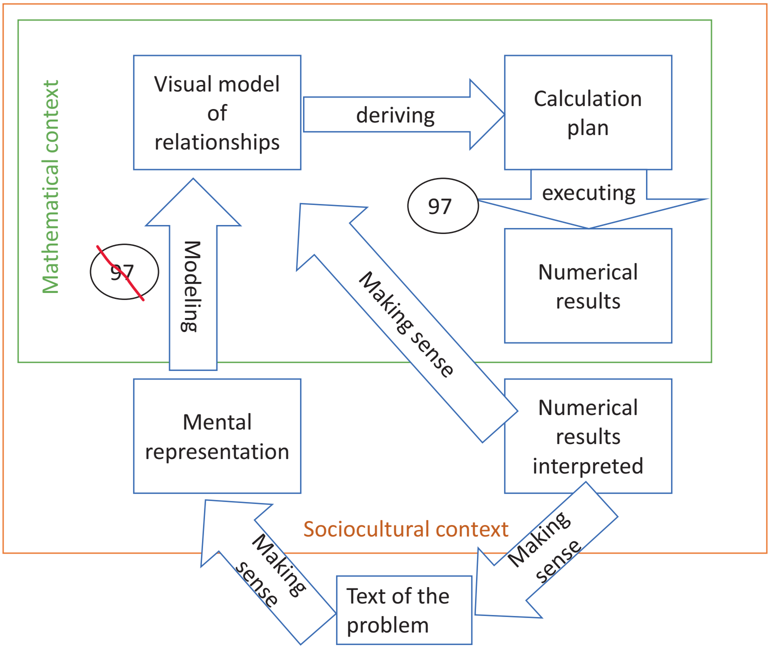


Fig. 3.1 Ethno-mathematical model of the problem-solving process

focusing on known data and keywords in the text of the problem, the teacher invites students to think about the type of situation at hand (comparison or not) and to represent all important elements as **related together**. In the next section, we provide a partial verbatim of a lesson in one of the experimental classes to illustrate this type of analysis.

The ethno-mathematical model for problem solving offers multiple entry points depending on the intention of the activity. In turn, it occasions different opportunities for the learner to enhance their relational thinking. See below different examples of activities, which illustrate diverse pathways through the ethno-mathematical model.

3.3.1.1 Activity 1. Communicating the Mathematical Structure of a Problem: The Captain's Game

This activity is a game that can be applied to any traditional word problem (Ducharme & Polotskaia, 2010). The game consists of communicating the mathematical structure of a problem between a team and its captain. A team of 3–4 students chooses its captain and the captain leaves the room. The rest of the team analyzes a word problem and tries to represent it visually while respecting the following rules:

Rule	Didactic goal
They may not write words or letters.	Use a different semiotic system.
They may not use mathematical or other symbols for operations.	Use an analog visual way to represent a relationship.
Only the numbers appearing in the problem can be used.	Represent a quantity as a continuous object rather than a set of discrete objects and identify those quantities using numbers from the problem.
They can use the symbols “=” and “?“as well as any drawings.	Signal an equality by using “=”. Mark with “?” the missing quantity.

On our classroom shelf there are 45 books and 8 boxes of games. Each box contains the same number of games. In total there are 32 games. How many games are stored in each box?

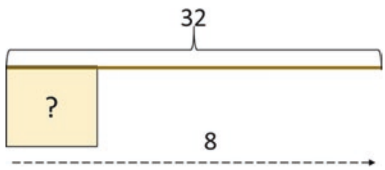


Fig. 3.2 Word problem and its possible representation for the Captain’s game. (Polotskaia et al., 2023)

This representation should then serve the captain as a message describing the problem. The captain, who did not read the word problem, uses the message to propose arithmetic operations with appropriate numbers to calculate the numerical answer to the initial problem. The team that wins the game (against other teams) is the one whose message clearly represents the problem in a way that the captain is able to obtain a correct numerical answer. This is only possible if the quantitative relationships are well represented. Figure 3.2 presents a word problem and its representation according to the rules of the game.

3.3.1.2 Activity 2. Mathematically Impossible Situations (MIS)

This activity is described as the “Who’s wrong?” activity in Savard and Polotskaia (2017). Starting with a traditional word problem that usually includes a story and a question, the teacher replaces the question (e.g., “How many in each box?”) with a statement (e.g., “In each box, there are 6 apples”) in such a way that the story becomes mathematically impossible.

There are 24 apples in boxes.
There are 12 boxes.
In each box, there are 6 apples.

The teacher invites students to see whether the situation is mathematically sound and to make suggestions about what may be mathematically impossible. It could be any of the three facts. The teacher then covers the numbers using post-it notes and constructs together with the students a visual representation without numbers to make sense of the quantitative relationship presented in the story. Together, they evaluate three hypotheses about which number is not correct and for each hypothesis, they calculate the number corresponding to the two other values in the relationship.

3.3.1.3 Activity 3. Working in a Computer Environment

To allow students to work with word problems while ignoring numbers, we designed a computer environment (Freiman et al., 2017). Here, the text of a problem on the screen shows boxes with letters where numbers are usually present (Fig. 3.3). At any time, a student can click on any letter to obtain a numerical value that is 'hidden' behind the letter.

To solve a problem, students need to analyze it in a general way, understand, and represent relations visually (e.g., using paper and a pencil). Using the representation, students try to figure out the operation to use and construct the solution in a general form (using letters). For example, in the Lucie c) problem (Fig. 3.3), the intended expression is N − D. The computer environment can evaluate this general solution and provide an immediate feedback to the student.

Working in the computer environment was organized as a cycle. At the first step, students are expected to find one or two problems they were not successful with. At

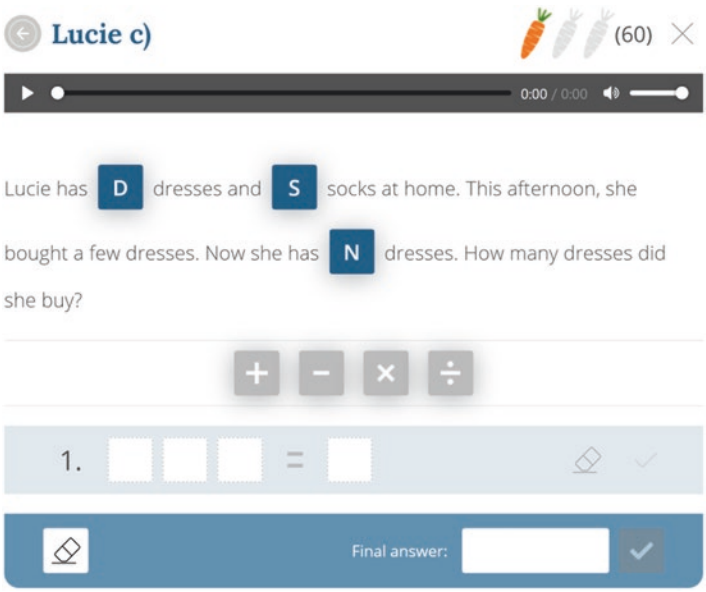
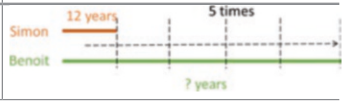


Fig. 3.3 Working in a computer environment. (www.elenapolotskaia.com)

Table 3.2 Additive versus multiplicative comparison

Wording	*Simon is 12 years old. He is 5 years younger than Benoit. How old is Benoit?*	*Simon is 12 years old. He is 5 times younger than Benoit. How old is Benoit?*
Convenient representation		

the second step, students decide collectively which "non-successful" problem(s) they wish to discuss together with the teacher. During the discussion, students interact with the teacher and with each other to formulate and justify their thinking about relationships, thus solidifying their shared understanding of the problem, its graphical representation, and its solution. In our approach, the whole-class discussion, and not the computer, is the main pedagogical tool helping learners to construct deeper understanding. They then return to individual work in the computer environment to test their new knowledge, thus engaging in a new cycle of knowledge construction.

3.3.1.4 Activity 4. Differentiation Between Additive and Multiplicative Relationships

The goal of this activity is to direct students' attention to the differences in expressions of additive versus multiplicative comparison (see Table 3.2). The teacher proposes simultaneously two different word problems with similar wording, one presenting an additive relationship, and the other a multiplicative relationship. The teacher then invites students to analyze and visually represent and compare the two situations.

While each of these activities emphasizes different aspects of the ethnomathematical model, the main goal for all of them is to help students grasp quantitative relationships, be able to represent them visually, and derive mathematical conclusions based on the relational understanding of a situation. This quest for understanding is supported by mathematically rich discussions, recognized by many researchers as an important tool for the development of learners' mathematical thinking (e.g., Biron et al., 2016; Stein et al., 2008). In what follows, we showcase how such discussion is organized in the classroom.

3.4 How It Works in Class

The following excerpt presents a part of a lesson in grade 4. Previously, in grade 3, these students have been using diagrams to represent and solve additive word problems. In grade 4, they participated in two activities to discuss multiplicative composition relationships. In this new lesson, the students discuss, for the first time, a situation where quantities are compared in a multiplicative way. The teacher is assisted by a consultant because the approach and the activity are both new for her.

T- teacher.
C- Consultant.

At the beginning of the activity, the teacher briefly reminds the students that they have already discussed some relationships and proposes to analyze and represent the following situation.

The 2nd cycle students will participate in different sporting activities. They have a choice between hockey, climbing, or swimming.
We know that:

- *Two times more students chose hockey than climbing.*
- *Three times more students chose swimming than climbing.*
- *Some students chose climbing.*

The situation described on the whiteboard contains only comparison statements, but not numerical values for the mentioned quantities. There is no question either. The formulation of the task makes the calculation impossible. To begin with, the teacher explicitly invites students to read and reformulate comparison statements. Students speak to their partner to express their thinking orally and negotiate the meaning of the problem together.

T: (after 2 minutes) What do you think about the first statement? How can you reformulate it?
Claudie: I would say, there are more students participating in hockey than in climbing.
T: First of all, there are more students participating in hockey than in climbing, how do you know this?
Claudie: They say two times more students chose hockey than climbing. … Two times fewer to climbing.
T: They say two times more students chose hockey than climbing and you deduce that …?
Claudie: two times fewer…
T: Where?
Claudie: to climbing.
T: So, two times fewer to climbing.
Claudie: (put the proposed expression on the blackboard).
T: Antoine, do you want to say something? Is it exactly the same? Please go ahead.
Antoine: There are two times more to hockey, so for climbing there are two times fewer.
Tania: You need to take twice the number of students who will do hockey.
T: We take twice the number of students participating in hockey… why?
Tania: To find the number for hockey…
T: To find the number of students in hockey, we need to take twice the number of students to hockey …? Is it so? There is something here. Think more.
C: Anybody can complete the statement? We need to take twice the number of students … where?

Claudie: Take twice the number of persons to climbing and to hockey to know how many in total.

C: To find the total. OK. But, if I take twice the number of people to climbing, what will I find?

Nik: The number of persons to hockey.

T: Anything else? OK.

Together the group formulates two new versions of the relational expression describing the same relationship between the number of participants to climbing and to hockey. It is difficult for individual students to formulate a complete relational expression. But together they arrive at an expression using an opposite adjective (fewer instead of more) and to another one using an action (take twice). These important logical connections between the original expression, inverse expression and an action (operation) can potentially support the understanding of the situation in a relational way as a network of ideas rather than as a calculation recipe.

The teacher discusses the second statement in the same way. They arrive to the following formulations:

- There are three times fewer students to climbing than to swimming.
- You need to take three times the number for climbing to know the number for swimming.
- If we know the number for climbing, we need to multiply it by three to find the number for swimming.

The teacher distributes the task (the wording of the situation and an annotated place for a representation) to students. She asks students to represent the situation by using segments. Each segment represents the number of students to each sport. The shortest segment represents the sport with fewest students, and the longest segment represents the sport with the largest number of participants. The place for the representation is pre-organized so the students need only to decide on the length of each segment.

Students work in teams of two. The teacher observes the students' work and decides to discuss the representation with the whole group. She invites all students to sit near the whiteboard.

T: We will start with the first statement, and you will tell me what to do to represent it.

T: There are two times more students to hockey than to climbing. Can I draw segments using this information?

Students: You draw a long segment for hockey and a short one for climbing.

T: (Draws a very long segment for hockey and a very short one for climbing) Like this? Does it work? You say, I should draw a long segment for hockey and a short one for climbing. We said two times more for hockey. (Points to the long segment.) Should this line be like this? (See Fig. 3.4.)

The teacher uses the student's idea suggesting one line to be longer than the other, but she intentionally exaggerates the length in her drawing to attract students'

Fig. 3.4 The teacher's first proposal of a representation "more hockey than climbing"

Fig. 3.5 The teacher shows "two times the value of climbing for hockey"

attention to the quantitative part of the comparison expression without directly indicating the problem with her drawing.

Nik: It doesn't work.
T: How should it be then?
Nik: It should be smaller.
T: It should be smaller because there is another one bigger? Shall we ask for help?
Matis: It should be two times longer than climbing.
T: A-a! I know that in hockey, there are two times more students than in climbing, so my line should be twice as long as for climbing. If for climbing it is like this (shows the length with her fingers), so for hockey it should be two times this (shows two times the same distance with her fingers on the line for hockey). (See Fig. 3.5)

The teacher makes specific gestures to mimic the "two times" expression on the diagram. In future activities, students will use this meaningful gesture (Wagner et al., 2008) to deepen their understanding of multiplication, division, and multiplicative relationships.

C: Or you can start with the hockey. Your line for hockey is beautiful!
T: A-a!
Mike: You divide it in two.
C: Why should we divide it in two?
Antoine and Mike together: Because one half is the line for climbing.
T: Why can we take the half for the climbing?
Antoine: Because it's two times. You divide in two.
T: (Draws a segment for climbing and shows that it compares to a half of hockey). Like this? Makes sense? (See Fig. 3.6)

In the episode above, students are able to formulate an inverse operation: based on the "more" statement, they propose a division to find the climbing representation.

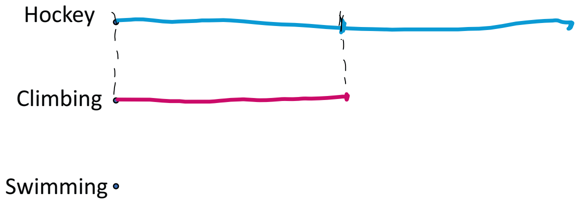


Fig. 3.6 The teacher constructs a half

T: Rosalie, what did I do to find the line for climbing?
Rosalie: You took a half of hockey.
T: Why?
Rosalie: Because … (Seems to be blocked.)
T: Claudie?
Claudie: Because at first, we had a big line. And for climbing… We know that in hockey there are two times more than for climbing. For hockey, we need to take the climbing two times.
T: Ok, for hockey, we need to take the line of climbing two times. But we found the line for climbing from the line for hockey. How did we do this?
Antoine: We divided it in two parts.
T: (does a gesture of cutting the line of hockey) We divided it in two. We separated it in two parts. There are two times fewer for the climbing.

The teacher continued to discuss the division idea even though it was already formulated and carried out. Not all students can see the inversion or can explain it (see Rosalie above). More work is needed to allow for each student to recognize the equivalence of "times more" comparison and "times fewer" comparison. The discussion is always about the multiplicative comparison relationship at hand. Using the visual representation on the whiteboard makes the relationship visible and lends itself available for students to describe it in different words, and to operate on it.

In what follows, the teacher discusses the second statement in the same way. She suggests that the students are ready to construct a representation on the whiteboard themselves.

T: Rosalie, let's do it, you can do it (gives the pen to Rosalie). Three times more students chose swimming than climbing.
Rosalie: (Draws a segment equal to climbing, stops and observes, continues the line, draws the second part, observes, and draws the third part.)
T: How did you find the length of the line for swimming?
Rosalie: The line for swimming is three times longer than the line for climbing. (Shows that the edge of the first part corresponds to the end of the segment for the climbing.) I divided it in three.
T: Did you divide it in three? (Points to the climbing.)
T: (Shows with her two hands the segment for climbing.) Did you divide **this** in three?
Roslaie: No.

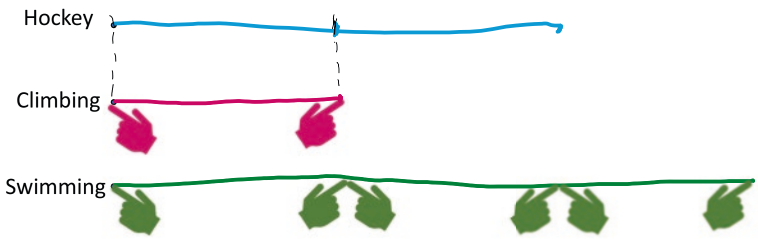


Fig. 3.7 Construction of the line of swimming

T: What did you do?
Roslaie: This line (shows the line for swimming), I divided it in three.
T: How did you construct this line?
Rosalie: (Shows the segment for climbing) I took **this** three times.
T: A-a! You took the line for climbing three times.
T: (Shows the segment for climbing with her two hands and repeats the gesture three times on the line for swimming.) You took climbing and you repeated it three times. (See Fig. 3.7.)

The teacher pays careful attention to the words the student uses to explain her work. The student uses the word "divide" to reflect the fact that the line she drew is divided in three parts. However, this verb does not correspond to the action needed to construct the line for swimming (which is taking three times).

Noah: But…
T: Noah?
Noah: It is not long enough.
T: How is it not?
Noah: If hockey is two times …
T: Come please and show how we should arrange this.
Noah: (compares the segment for hockey with the segment for swimming to show that two parts of swimming do not correspond to two parts of hockey.)
T: Please, arrange it. Use a cursive line to compare.
Noah: (puts a vertical cursive line and makes the third part of swimming line equal to the first part, to obtain three equal parts).
Ron: Exactly!
T: Is it better now?
Everybody: Yes!

The above episode shows the advantage of the visual representation of relationships. The students visually analyze the correspondence of all three segments to the two comparative statements. Thus, they can see that the visual representation of the swimming segment does not respect the proportions of the hockey segment.

The teacher invites students to adjust their representations before continuing the discussion about different ways to formulate the two relationships. By analyzing the constructed visual representation, the group formulates more statements about the situation.

- To obtain the line for climbing, you need to take a half of the line for hockey.
- Two times climbing gives hockey.
- Three times the number of students in climbing gives the number of students in swimming.
- Three times the line of climbing gives the line of swimming.
- You take the line for swimming, and you cut it in tree pieces to obtain the line of climbing.
- The climbing represents one third of the swimming.

While observing their work students try to discover more relationships without any prompt.

Nic: Can we compare the line of hockey with the line of swimming? I can take the line of hockey plus one time the line of climbing and it gives the line of swimming.

In this class, many students wanted to continue the discussion and compare swimming and hockey, but it implies a multiplicative comparison, which seemed too difficult to figure out (2/3) and the lesson was over. Only in the next lesson the students would know the numbers and the question of the problem. Then they would use their representations and the mathematical understanding of the story to propose a calculation to find the numerical answer.

The lesson we described shows the careful work of the teacher, through a mathematically rich discussion, to construct students' deeper understanding of multiplicative comparison and the relationship between multiplication and division. The calculation part of the solving process is intentionally delayed to allow the discussion about relationships. Students' attention is constantly directed toward the relationships between quantities. The teacher together with the students constructs visual representations of the relationships. Notably, the process of modelling and the constructed visual model helps students to see each relation from multiple perspectives and to find multiple logical connections between different formulations. All this work is about the coordination between the sense provided in the textual expressions and the sense coming from the visual representation of relationships.

The word problem activity discussed above is qualitatively different from typical classroom work that is observed in a more traditional approach and that follows an operational paradigm. The traditional approach draws students' attention to key words and known numbers for the purpose of finding a numerical solution. Within the relational approach, however, that we showcase above, students learn to appreciate the system of relationships in a word problem and consequently they can construct a network of logical connections through a process of sensemaking. In this process, students use gestures meaningfully to build up a more profound understanding of mathematical relationships. Only later, will they get to numerical calculations. Taken together, the relational approach exemplifies capacity building in and through negotiation and coordination of mathematical meaning.

3.5 Conclusion

The EDA is an innovative approach to teaching problem-solving. The focus on quantitative relationships and their visual modeling allows students to develop relational thinking which, as many researchers increasingly believe, is the foundation of algebraic thinking (Kieran, 2018). Relational analysis of problems requires important intellectual work from students thus contributing to a sustainable mathematical thinking development. We argue that including relational analysis in a form of the tasks and activities illustrated in this chapter in every day classroom work would be strongly beneficial for enhancing mathematical thinking in all students.

Previous work (e.g., Freiman et al., 2017; Polotskaia & Savard, 2018; Savard & Polotskaia, 2017) demonstrates the increased interest students pay to activities guided by the ethno-mathematical model and mathematical discussions organized by the teacher. Specifically, Polotskaia and Savard (2018) reported a qualitative change in how students approach additive word problems, which seems to contribute to the observed significant improvement in students' capacity in solving them. Moreover, these findings suggest that for students who learned to think relationally, the traditionally difficult (inconsistent) problems are much less difficult than for students who learned within the operational paradigm.[1] In addition, from our collective theoretical work and practical experience, we conjecture that learning about relationships is accessible for learners at any stage of the learning trajectory. While it is certainly better to start with relationships at the very beginning, our practice shows that students in grades 2, 3 and 4, as well as teachers, can successfully integrate this powerful practice to their previous knowledge of numbers and operations.

It is important to highlight that the positive outcome for students we observed in our research is not due to the introduction and use of diagrams per se. The careful work on the relationships, the constant sense making and negotiation through the mathematically rich oral discussions allow the students to make visible their relational thinking and use it to solve word problems. Therefore, the role of the teacher in the process is crucial.

The EDA approach is new and very different from what mathematics teaching usually looks like in regular school settings. As shown by other researchers (e.g., Blanton et al., 2015; Gjære & Blank, 2019; Malara & Navarra, 2018), an essentially different approach requires from teachers a thorough rethinking and rebuilding of their teaching practices. This process of reconstruction usually takes a lot of time and requires special training and support. The teachers in all our studies reported an important and positive shift in students' thinking and thus appreciated the approach saying that they substantially changed the way they conceive their teaching. However, all of the teachers we worked with highlighted the need for ongoing training and support to be able to fully implement the EDA in their classes. We suggest that EDA is incorporated not only in teacher education programs but also in

[1] See Polotskaia and Savard (2018) for quantitative and qualitative analyses of evidence from the experimental and control groups.

professional development continual support to introduce a much-needed change in how early mathematics is taught.Funding InformationThe research projects discussed in this chapter were funded by the Quebec ministry of éducation (“Chantier 7” funding opportunity 2012 and 2015).

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Chapter 4
Experiences of Tension in Teaching Mathematics for Social Justice

Ann LeSage and Ami Mamolo

Abstract Teachers who commit themselves to change embark upon a difficult and sometimes lonely journey of self-doubt. In this chapter, we meet Nora, a Canadian educated middle school teacher who was hired as part of a school-wide initiative to introduce curricular change in an elite South American international school. Nora's attempts to introduce socially relevant project-based teaching to her mathematics class were met with resistance, even amongst a backdrop of reform. We explore and analyse the tensions that emerged for Nora as she navigated the competing perspectives and expectations of her supervisors, colleagues, students and parents. This chapter contributes new insight into the experiences, supports, and shifts needed to help teachers persist through the uncertain journey of curricular change.

Keywords Teacher tensions · Teaching mathematics for social justice · Social justice context problems · Collegial tensions · Student tensions

4.1 Introduction

This chapter explores the tensions surrounding and experienced by Nora, a Canadian educated teacher who was hired by an elite South American school (colegio) to support curricular change in their middle school mathematics program. Nora brought with her a practice that was characterized by student-centered, socially responsible interdisciplinary approaches that emphasized conceptual understanding, reasoning and communication, and realistic worldly applications. Teachers who consider adopting such approaches that deviate from traditional expectations for classroom mathematics instruction can find themselves stuck in a web of competing perspectives with little professional support (administrative, pedagogical, mathematical, or emotional) as they navigate the tensions that invariably emerge. In this chapter we

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explore the tensions that emerged for Nora as she navigated the competing perspectives and expectations of her supervisors, colleagues, students and parents.

Nora's tensions are not unique. In general, teachers are required to respond to novel challenges that emerge in their classrooms. Lampert and Ball (1999) recognized that teachers must "be prepared for the unpredictable … [and]…figure out what is right practice in the situation" (p. 39). Rowland and Zazkis (2013) agree, and state further that teaching "involves attending to students' questions, anticipating some difficulties and dealing with unexpected ones, taking advantage of opportunities, making connections, and extending students' horizons beyond the immediate tasks" (p. 138). Through this narrative inquiry research, we shed light on some of the anticipated and unexpected difficulties that emerged as tensions for Nora as she introduced reform-oriented practices in an environment that embraced traditional approaches to teaching and learning mathematics. Nora faced resistance from her administration, colleagues, students and their parents, and despite her experience and competence as a mathematics teacher, she felt stuck in a web of competing forces. We analyze Nora's stories with an eye toward how teacher education programs might better support pre-service teachers to be prepared for the unpredictable.

4.2 Background and Context

4.2.1 The Colegio Context

The colegio in which Nora taught is a private, coeducational, non-denominational international school, which follows American curricula and boasts having earned a reputation as a progressive educational leader in Latin America. It is a bi-language school (English and Spanish) that is internationally competitive, on par with leading K–12 institutions across the globe. The school is divided into four divisions: Pre-Primary, Primary, Middle School and High School. Each division has a Principal, who reports to the School Director. The Director is the most senior administrator at the school. He, along with the school's Board of Governors are the decision makers for all matters concerning school operations.

The colegio has been a pillar of the community for generations, and students are often from families where multiple generations of alumni l. There is a strong sense of pride, history and ownership within the tightly-knit school community and many alumni stay active with this community. There is a culture of high expectations on students to *do well* and become the future leaders of their country.

The colegio imports approximately 25% of their teachers, head-hunted specifically from the United States and Canada. Nora was one of the imported teachers and was recruited by the school's Director while she was completing her Master of Education in Canada. The Director who hired Nora had been at the school for over two decades and had a strong vision for how they intended to maintain and advance the colegio's reputation as a leading progressive educational institution. They had initiated reform-oriented programs across the colegio's departments and programs, and was on a head-hunting mission to recruit like-minded international instructors when he met and recruited Nora. The Director made clear to Nora at the time of her recruitment that they expected her to introduce major changes to the mathematics department, in line with their vision for the colegio.

Nora was brought to the colegio to better align their mathematics programming with the rest of the colegio's progressive programming, which was seen by the Director as lagging behind in reform efforts. It became clear to Nora early on that the Director's vision was not shared amongst the majority of her mathematics colleagues. The resistance to the Director's vision and Nora's approaches was strong early on, and Nora soon realized that she was not in a position to implement or advocate for the large-scale changes for which she was recruited. When she expressed this to the Director, they encouraged her to keep true to her pedagogical approach and make changes where she could.

4.2.2 Introducing Nora

Nora is a 35-year-old female English-speaking Canadian. At the time of this research, Nora had been teaching for six years, including at international schools in the United Arab Emirates, and had just completed her Master in Education degree. Nora brought a varied educational background and work experience to her teaching practice. Nora's professional life has been laden with circumstances that required her to assume life-changing risks (i.e., pursuing a non-traditional first career; leaving her first career without a complete vision for her next career; moving overseas to teach at international schools; moving to countries where English was not the primary language). As such, she has developed an openness to accepting ambiguity and emotionally risky situations. She has learned that, generally, positive outcomes result from what might be perceived as more risky life choices. She enjoys being immersed in situations that require her to think creatively or being in spaces of uncertainty where she is provided with new opportunities to learn and discover. We describe Nora's orientation to teaching in the following section.

4.3 Methodology and Data

We adopted a person-centred research approach (Waite et al., 2010) to understand Nora's experiences as she struggled to introduce reform-oriented mathematics pedagogies at an international school in South America. We explore Nora's stories and discuss the internal and external tensions she experienced through the lens of narrative inquiry space (Connelly & Clandinin, 1985, 1994, 2000). Connelly and Clandinin's narrative research provides the framework for understanding the influence of teachers' stories, experiences and reflection on the development of their teaching knowledge, practice, and beliefs. Connelly and Clandinin assert that teacher narratives are human constructs which are both personal and social. Personal, "reflecting a person's life history, and social, reflecting the professional contexts in which the teacher lives" (p. 318). They emphasize that narrative inquiry space is three-dimensional; encompassing the elements of temporality (past, present, and future), personal individuality, and place or context. Thus, the stories teachers narrate are situated and developed within a time and place, and are interpreted and understood through the experiences of the teacher. Nora's stories provide insight into her beliefs, teaching practice, and the tensions she experienced during a time of change, in the specific context of one international school in South America.

To understand the tension stories Nora experienced, we collected data from Nora's written reflective accounts and through semi-structured interviews. Nora's written and oral stories helped us understand her lived experiences and the tensions she encountered in her teaching and professional life. Among the tensions experienced by Nora were ones she witnessed her students going through; that is, her perception of students' experiences of tension resulted in tensions of her own.

4.3.1 Nora's Orientation to Mathematics Teaching and Learning

Nora possesses a dynamic view of mathematics, coupled with a philosophy of teaching that is best described as a problem-solving, non-traditional (Raymond, 1997) or social constructivist (Ernest, 1989, 1994; Vygotsky, 1978). She believes that learning mathematics requires teaching in a way that engages students. She endeavours to expose her students to a broad range of mathematical concepts and encourages them to solve complex, open-ended problems which are embedded in real-life contexts. Nora strives to create a classroom environment that nurtures students' and teachers' development of mathematics competence and confidence. As a constructivist teacher, Nora advocates the creation of mathematical communities that encourage both students and teachers to actively construct mathematical assumptions through investigations and explorations.

An example of how Nora enacted these practices in her teaching were through multi-week projects that incorporated a variety of math concepts and allowed students to explore other subjects through a mathematical lens. Given the magnitude of these large-scale projects, they were introduced only once or twice per year. The projects required students to work collaboratively in groups to solve real-life open-ended, complex problems. Nora developed two such projects for her students: The *Design Your House Project* and the *Environmental Project*.

The *Design Your House Project* required students to design the interior and exterior, including landscaping, of a home. Students were given budget constraints for materials and labour costs. The budgets varied from group to group to add an element of realism to the project and to emphasize issues of fairness. The students created architectural floor plans, then designed the building ensuring to stay within their limited budget constraints. After designing the 2-dimensional plan, selecting construction materials and establishing their construction costs, the students built three-dimensional scale models of their homes including landscaping features (all of which were built to scale). Upon completion of their homes, each group created a brochure introducing their property to their school peers.

The *Environmental Project* required students to investigate local environmental issues (e.g., at the school, city or country level), determine solutions to address the issue, develop a plan to implement the solution then present their plan to the appropriate governmental body. As an example, a group of students decided to tackle the problem of excessive waste on the school campus. With the assistance of the school custodial staff, students collected and organized the garbage on the school property over a 1-week period. They determined how to categorize the waste and explored how the school could begin to reduce the amount of waste put into the landfill. Through exploration of the data, they determined that the colegio and students could take action to address the organic and recyclable materials. They proposed a school-wide recycling program and worked with the custodial staff and gardeners to create recycling and composting sites throughout the school campus. The students presented their environmental plan to the school's Board of Governors, who passed the plan, and took the necessary steps to put their plan into action the following school year.

4.4 Teaching Math for Social Justice

The *House Project* and *Environment Project* align with what Mamolo et al. (2018) describe as "social justice context problems," (p. 378) which are in-depth and extended mathematical explorations that foster deep mathematical thinking, procedural fluency, and a broader understanding of the relevance of mathematics for understanding and addressing various societal issues. The problems involve creating artefacts that address a social issue via a mathematically informed perspective. The problems and artefact creation require extended experiences grappling with and applying mathematical concepts; such experiences are considered crucial in fostering conceptual understanding and deep learning (e.g., Flewelling & Higginson, 2001; Watson, 2008). Such projects aim to foster critical awareness of social injustices in the lived experiences of others around them while emphasizing the importance of exploring, questioning, negotiating and discussing ideas, approaches, and solutions. Such projects are modeled after Gutstein's (2006) framework for Teaching Math for Social Justice (TMSJ). In his framework, Gutstein (2006) posits two dimensions of Teaching Mathematics for Social Justice, which are summarized in Table 4.1.

The TMSJ framework was developed by Gutstein through his work with underprivileged middle school students. He describes the importance of using mathematics to change the world, as it can foster in students a sense that they are "capable of making change" and may help them develop "a sense of social agency" (p. 27). In

Table 4.1 TMSJ pedagogical goals

Social justice pedagogical goals		Mathematics pedagogical goals	
Reading the world with mathematics	Using mathematics to understand world issues, inequities, and opportunities	Reading the mathematical word	Developing mathematical power, understanding, and literacy
Writing the world with mathematics	Using mathematics to take action and initiate changes in your community and beyond	Succeeding academically in a traditional sense	Succeeding in school, standardized tests, and post-secondary school.
Developing positive social and cultural identities	Being able to see yourself as able, confident, competent, and valuable	Changing one's orientation to mathematics	Seeing mathematics as relevant, connected, and powerful for understanding real issues.

Gutstein (2006)

line with these perspectives, Bartell (2013) suggests, "The purpose of education is not to integrate those who are marginalized into the existing society but rather to change society so that all are included" (p. 3). In Nora's case, the school and students were not the marginalized individuals but rather came from families that were among the wealthiest and most powerful in the country. Students were accustomed to travelling internationally and had accrued worldly experiences that aligned with the expectations that they would be future leaders, destined for important things. This reality sharpened Nora's resolve to help her students empathize and understand the lived realities that existed beyond their communities. She explained:

> Budgets were always a part of the lessons because [in their reality] the kids had more money than they could spend. It was eye-opening for a lot of them, for example, when they were designing their houses and couldn't afford some of the luxuries they were used to at home. [They would say,] "What do you mean we can't have an indoor pool and theater?" Um, yea… most of the world doesn't live this way!

For Nora, the disconnect between her students' lived experiences and those of "most of the world" became an important factor in the social justice pedagogical goals of her projects. For instance, in the housing project, students were assigned realistic budgets based on a scale factor of the number of letters in their name as a way to help read the world with mathematics. For example, Daily Labor Rates were calculated as follows: each vowel of a student's surname was valued at $16.25 plus each letter "L" in a student's full name (first+middle+surname) was valued at $15. The number of days, Total Labor Days, was determined by the number of letters in a student's first name. The project required students to research actual costs of labor, materials, and so forth, and even students with very long names were required to revisit their original plans, reassess, redesign, and confront how this constraint impacted the ability to realize their dream. Constraints in the environment project impacted the issue students could tackle as well as solutions they could propose, and helped connect students' understanding of reading the world with mathematics to their understanding of writing the world with mathematics. The waste reduction program proposed by students to the school Board of Governors is an example of taking action to initiate change, which resulted in new environmental policy and procedures in their school. Through their work on the projects, students developed a broader and more empathetic awareness of how mathematics is relevant for solving real world problems, as well as of the social inequities around them and what it might mean to make valuable contributions to their communities. Mathematically, the projects were quite rich, yet in line with other research (Bartell, 2013; Garii & Rule, 2009; Gustein, 2006; Nolan, 2009; Xenofontos et al., 2020) balancing mathematical and social justice pedagogical goals was challenging. As Nora described:

> These were Pre-Algebra students, about grade 7 in the American system. But we went well beyond, to Algebra 2 content. The projects took a long time, 6-8 weeks, and love them or hate them, the students were embedded in them. The first project [the housing project] was tough at first, and I'm not sure we explored the social side of it as much as I might have liked. My pedagogy was really different from what they were used to, and there was a lot of resistance. By the time we did the environment project, the students were more on board and we could dive deeper to where students were actually presenting plans to make changes in their communities and some of these changes were actually put into action. It was awesome.

4.4.1 Tensions in Teaching Math for Social Justice

Tensions necessarily emerge in teaching as teachers negotiate unanticipated situations which elicit an inner turmoil about what course of action to take next (Berry, 2007; Floden & Buchmann, 1993; Katz & Raths, 1992; Lampert, 1985). Tensions in teaching are seen as ubiquitous and inescapable (Mason, 1988) though they can ebb and flow over time (Berry, 2007). Tensions can be unsolvable and recurring (Cuban, 1992; Lampert, 1985), and while it is possible to discuss tension in isolation of other tensions, they are not independent of them (e.g, Katz & Raths, 1992; Lampert, 1985; Mason, 1988). Tensions are "related in complex ways that reflect the complexity of teaching situations" (Sparrow & Frid, 2001, p. 453).

Balancing pedagogical goals aimed at fostering both mathematics learning and social understanding can elicit specific tensions around whether the social issues will take precedence over the mathematics learning (e.g., Wager & Stinson, 2012), or whether the social issue is inadequately explored (e.g., Bartell, 2013), or whether to embrace real and messy data versus made-up data to smooth out the mathematics (Mamolo, 2018). Mamolo and Pinto (2015) discuss tensions in the form of perceived and actual risks involved in enacting pedagogies that jar with the belief that education ought to be value neutral and avoid controversial topics. Mamolo and Pinto (2015, p. 90) write:

> Mistaking value neutrality as a characteristic of education undermines the goal of social justice. Items that appear in curriculum documents privilege certain knowledge, skills, and attitudes… in particular, within mathematics, values underpin the conventions, approaches, and nature of what are viewed as acceptable ways of engaging in the discipline (Ernst, 1989)

Preservice teachers who engaged with social justice context problems as part of their mathematics professional development also experienced tensions that led to an articulation of pedagogical goals for teaching math for social justice in teacher education (Mamolo, 2018). These goals and their associated tensions include (Mamolo, 2018, p. 39):

1. *Reading the mathematical world*: this goes beyond understanding of content (the mathematical word), to include knowledge of mathematical sensibilities, values, and ways of being. Tensions emerged for participants who believed that debate and ambiguity do not belong in mathematics;
2. *Fostering success in non-traditional ways*: using complexities of social justice contexts with local, national, or global connections to promote and reveal mathematical understanding. Tensions emerged regarding participants' struggles to recognize and articulate the mathematics in their work; and
3. *Changing one's orientation toward mathematics class*: this goal speaks to the perceived purposes of school mathematics and the role of a teacher. Tensions emerged for participants who believed that the class content was irrelevant to most pupils yet necessary to master.

As with the participants from Mamolo (2018), Nora's experiences teaching with social justice context problems surfaced tensions within and across these three goals. In what follows, we analyze the tensions experienced by Nora as she faced resistance from her students and their parents, her colleagues, and her supervisor. Tensions surfaced with respect to *reading the mathematical world*, *fostering success in non-traditional ways*, and *changing one's orientation toward mathematics class*. Each tension was experienced simultaneously with other tensions and elicited new ones. Nora was like a fly caught in a spider's web, suspended in tension, where a disruption of one strand causes reverberations across the others. For the sake of the paper, we disaggregate individual tensions first and discuss how they manifested for Nora. Such a disaggregation allows us to unweave the complexity of the spider's web and zoom in on how Nora's experiences in tension had unanticipated consequences that created new tensions.

4.5 Nora's Tensions

Prior to beginning her teaching at the colegio, Nora anticipated there would be tensions with the culture and language differences between her and her students, as well as with her reform-based pedagogy and the traditional approaches to teaching mathematics of the school. Indeed, it was precisely these tensions that brought Nora to the colegio. Her adventurous spirit had drawn her to international teaching in the past. Nora recounted being head-hunted by the Director of the colegio, who was a fixture in the community.

> The Director was a brilliant man, and also charismatic, as he somehow convinced me to accept a teaching position at the school. I had no intentions of going to South America (I was hoping to secure a position in Madagascar). However, he convinced me that I would be a perfect fit and that I was exactly what the Math Department needed at the time… someone that didn't teach traditionally; someone that focused on manipulatives, problem solving and

> conceptual understanding. He informed me that, although he strongly supported reform ideals, that the students, parents and the other math teachers may not embrace the same philosophy. I knew that it was not going to be easy and that there was going to be resistance to this change. However, I knew that he had my back. And at the end of the day, he would fight for me and help me stand up for what I believed in.

4.5.1 Student Tensions: Fostering Success and Changing Orientation

Nora anticipated that her pedagogical approaches which included *fostering success in non-traditional ways* would elicit resistance from her students and their parents, and had experienced such tensions in the past. Nora wrote:

> As with many Middle School students, they were verbal about their distaste for the level of detail required in the writing (justifying answers, explaining thinking, explaining *why* an equation works, etc.). Some complained to their parents and the Middle School Principal. Some parents voiced their concerns to the Principal, as their child's grades were lower in the first Quarter of the year (compared to previous years). Grades were very important in this school, as such, any slipping of the grades was problematic for the students and their parents.

> I recall during the first Parent Interviews in the Fall term, having to explain to parents that once their child became more proficient with explaining their thinking, that their grades would rebound and that they would learn more than if we simply used the traditional textbook approach. I recall inviting parents and the Principal to my classroom to see what we were doing. None of the parents accepted my invitation, but the Principal did come to my class (a few times) and interacted with the students.

The heavy writing component of the social justice projects was especially frustrating for students. In general, teachers face resistance when introducing reform methods (Remillard, 1999, 2000; Ross, 1999; Ross et al., 2003; Roulet, 1998) such as ones that require students to communicate and explain their thinking in writing – after all, there is a perception that math is about numbers, and numbers are universal. Nora's students, who were learning the material in their non-native language, were resistant to this change. Students, and their parents, were accustomed to the school's textbook tradition, which focused on computational or procedural competence, rather than conceptual understanding. Such approaches fall short of realistic mathematics used in everyday or academic contexts (e.g., Moschkovich, 2002) and do little to help students read the word with mathematics or develop connections between their identities and their school learning (e.g., Gustein, 2006).

Previous research on the effects of textbooks on teaching and learning suggests that textbooks can be a substantial impediment to nontraditional teaching. There is a persistent theme running through these studies that textbooks reinforce a traditional approach to mathematics teaching, one dominated by a reliance on recall of procedures in place of fostering student thinking (Eisenhart et al., 1993; Frykholm, 1996; Grant et al., 1996; Spillane, 2000). Further, as Mamolo and Pinto (2015) note, traditional or conventional textbooks tend to under-emphasize social justice contexts or issues in their address of subject matter, and as such, reliance on them works against social justice aims.

The tension experienced by Nora as she confronted resistance to her nontraditional approaches, relates to the social justice pedagogical goal for teacher education: *fostering success in nontraditional ways*. Nora felt tensions regarding students' (and their parents') uncertainty about whether her nontraditional approaches could foster success. The writing requirement emerged as a recurring challenge for students and, as such, emerged as a recurring tension for Nora. For instance, Nora recalled that:

> Pretty much all of my students equated math teaching with textbook teaching. They were used to working from a textbook and were open with me about how much easier that was for them. Using a textbook was less work. But they also described textbooks as boring and lacking challenge. Students told me that while they might have learned less if I taught from the textbook, their grades would have been better. I knew their parents were worried about grades. And I knew that my students were feeling the pressure from their parents too.

Nora had planned for, and had experience with, students' resistance to her approaches. She knew that students' grades would rebound once they adapted to the sociomathematical norms of her classroom. However, a new tension emerged with respect to the parental resistance she experienced.

> The school was well established in the community. My students' parents went to this school, and so did their grandparents. Generations of the same families went there, and families were very much a part of the school culture. Parents had been part of the school for their whole lives pretty much, and they really felt that they "knew more about the school" than any new teacher would. The parents were very well educated, Ivy League graduates, very well off, very intimidating… They applied intense pressure on my Principal, and it was because of their request that the Principal visited my classes to appraise me and give feedback.
>
> After the Principal's class visits, we would debrief the lesson and she would provide me with feedback. Most of her feedback focused on how I interacted with the students and helping me understand the cultural differences (i.e., softening my approach, more positive reinforcement, more comforting/more physical contact, etc.).

Part of the tensions Nora experienced was from a pedagogical disconnect between her and her students. Nora knew that students were not accustomed to explaining their thinking, and that it took time for them to appreciate this as a teaching strategy. This was a recurring tension that Nora perceived in her students and it led to a new tension for Nora as she adapted her practice to better connect with her students. Nora decided to allow her students to work and discuss their ideas in their native language (Spanish), rather than in the language of their instruction (English). This created a new tension for Nora who did not speak Spanish and could not understand what her students were saying in her class. Nora had to trust her students to be on task and as such release some of her control over the class. This tension relates to dynamics of control (the teacher's) and autonomy (the students') and the social justice pedagogical goal for teacher education: *changing one's orientation to* *mathematics class*. By allowing students to speak in a language with which she was unfamiliar, Nora took a risk and tensions emerged as she negotiated a changing orientation to her ideas of classroom activity, student autonomy and trust. As Mamolo and Pinto (2015) noted:

> Conventional and enduring "cultural myths" (Nuthall, 2004) place the teacher in control (at the front of the class with quiet students), the curriculum and textbook as the authorities of what and how students should be taught, and undebatable truths as the requisite knowledge to be acquired. There are risks involved on all three counts – social risks involved in managing a class that seems "out of control", subject-matter risks in diverting from prescribed approaches, and an intersection of social and subject-matter risks in negotiating disparate interpretations of mathematical "truths." (p. 92)

Tensions for Nora extended beyond the classroom community to the broader school community, in which students and their parents were integral members. Nora identified the school community as an external impediment (tension). Tensions emerged with other members of the school community as well, including with her principal and colleagues. Fullan (2001) cited a positive community of practice as a quintessential contributor to successful school and board-wide reform efforts. Nora felt that she did not have a positive community of practice supporting her change efforts. She cited the following elements within the school community as negatively influencing her efforts to implement non-traditional mathematics pedagogy: the traditional beliefs and experiences of parents; the traditional beliefs and practices of colleagues; and the lack of collegial support or collaboration within the school.

4.5.2 Collegial Tensions: Changing Orientation and Mathematical Being

Nora's feelings of being unsupported in her reform efforts are reflected in the research literature, where studies have cited a multitude of factors that impede movement toward mathematics reform teaching practices. Such explanations include the social teaching norms of the school (including school politics and parental involvement) and the nature of the immediate classroom situation (LeSage, 1999; Raymond, 1997; Remillard, 1999, 2000; Roulet, 1998); and feelings of isolation from colleagues (Roulet, 1998). Nora explained the effect that the Principal's visit had on her:

> The Principal was a good teacher. She had similar ideas about fostering student thinking and getting them to explain their ideas. I remember early in my first year teaching at the colegio, my students complained to the Principal that I was always asking them to explain things. I would say things like, "oh, what are you doing there? Explain it to me, I don't understand." It was a way for me to get them talking without feeling like I was testing them. But the students were worried that I really didn't understand! I still think it's hilarious that they thought this. They went to the Principal and complained, and she had my back. She assured them I knew what I was doing and reinforced me. That changed when she started getting feedback from parents.

> I think the parents kind of panicked when they saw the first Quarter grades. There were some students who were used to getting A+'s in their typical rote-based courses, and were now getting B+'s in my conceptual-based course. So, I guess I'm not surprised that parents were worried. The resistance did not surprise me. Change is difficult for everyone. However, when faced with resistance to my teaching, it forced me to question my abilities and myself. It would have been easier to teach from the textbook. It would have been less work for me; and my students would have faced fewer frustrations. Knowing in your mind that you will have to face resistance is one thing, but having to experience it is a totally different thing. Because the resistance comes from so many different places, it can be hard to see through the fog of resistance to that light at the end, where you know eventually the students will benefit, the parents will be on board, and the Principal will be on board. But you really need the students to go with you and be on board. It can be exhausting because you are continually fighting them, their beliefs about what you should be doing, what you should know… even if I wasn't doubtful about my abilities before, I become doubtful because there was so much resistance.

The resistance Nora felt from her Principal could be articulated as a tension experienced by the Principal regarding Nora's reform teaching and the perceived purposes of school mathematics and the role of the teacher. This tension aligns with the social justice pedagogical goal for teacher education of *changing one's orientation toward mathematics class*, and highlights the Principal's dilemma of how to support Nora and her students while responding to the pressures and demands of their parents. The Principal's tension translated into tension for Nora as she had to work to earn the support of her Principal, despite the fact that she had been headhunted by the colegio's Director because of her approaches and successes in teaching mathematics. One of the consequences felt by the resistance and actions of the Principal was that Nora began to doubt her expertise and this doubt led to tensions in Nora's sense of her mathematical abilities. Thus, for Nora, tensions around changing orientation toward mathematics class elicited a new tension in her reading the mathematical world. Reading the mathematical world includes knowledge of the multifaceted ways that individuals engage with mathematics, for what different purposes, and to what different ends (Mamolo, 2018). It speaks to a holistic understanding of the *mathematical world* and ways of navigating within it. Mathematical sensibilities such as explaining the why, deductive reasoning, asking what if, negotiating realistic constraints, and debating possible solutions were parts of the mathematical world that Nora could navigate fluently and which she brought into her classes. The entrenched school culture of textbook teaching and the tensions that emerged because of her different view of the mathematical world elicited in Nora what she described as a "strong fear of imposter syndrome", which was contributed to by her interactions with colleagues at the colegio. Nora wrote:

> The greatest resistance from colleagues were from the Secondary Math teachers; in particular, one senior teacher. He had a PhD in Mathematics, taught a number of the Secondary math courses as well as mathematics (part time) at a local university. He was very traditional in his teaching methods and beliefs about how mathematics should be taught. His depth of mathematical knowledge and his status in the school community intimidated me. He had taught at the school for about 15 years. At the time, I had my M.Ed. and did not have a degree in Mathematics. What did I know? It took me until my second year at the school before I could begin to defend my teaching with any level of confidence or articulation.

In line with observations from Mamolo (2018), Nora struggled to articulate the mathematical power of learning activities that required students to explicate their thinking and that contextualized mathematics in issues of social justice. Nora made efforts to meet regularly for lunchtime chats with her most resistant colleague.

> We would meet over lunch and chat. At first it was difficult, but our relationship warmed up over time. This particular teacher did not understand why I required my students to write and did not see the value in such activities… His perspective was that numbers / math are universal and that requiring students to write and explain their ideas as they tackled these complex problems was not appropriate for math class. There was truth to his argument that having students whose English was not strong write in math was challenging for them, and I considered allowing students to write in Spanish, but could not figure out how to assess their work if I allowed them to do so.

Nora's approach was open-minded, and she accepted feedback and advice, and worked to adapt her practice to best support her students. She met regularly with the Principal, her colleagues, and the Director.

The continued support Nora received from her Director helped her address some of the hurdles she faced: "He pushed me forward and always encouraged me to continue to fight for change." Nora was also buoyed by the support of a friend who taught English at the school:

> She was a fabulous support and wonderful cheerleader for change. I'm not sure my second year would have been as positive if she were not a significant person in my life. As an English teacher, she was an advocate for writing across the curriculum, and always encouraged me to continue to help the students write about their thinking and about their math experiences.

Tensions around her ability to read the mathematical world, and the legitimacy of how she did so, recurred for Nora throughout her teaching at the colegio. Nora described this tension as "not a constant pressure, but intermittent". She reflected:

> It doesn't make any sense on the surface. My fear is that people are going to figure out that I don't know how to teach and I don't know math. But my belief that my students are more important than my fear, that's the tension… imposter syndrome and me.… my fear is outweighed by the benefit to my students… so the scale tips and you face those fears because you're supposed to, because it's your job. Because it doesn't make any other sense, that I pushed through to make this change… it's exhausting, but we do it anyway because at the end of day we want systemic change to happen.

Nora's resilience in the face of a multitude of tensions and pressure coming from every direction seemed to come from an unshakable belief in the importance of what she was doing. Like the fly caught in a web, the tensions experienced by Nora, from her colleagues, principal, students, and parents, were interconnected. Experiences of tension in one area elicited new tensions in other areas, creating a complex space in which Nora was suspended in fear. Nora's belief in student-centered benefits of her teaching approaches, as well as her personality and openness to risk-taking, contributed to her openness to feedback, criticism, and even surveillance. She reflected:

> It certainly was not easy. But, in the end, I knew my students could do it. If they were willing, we could explore some exciting topics in more interesting ways than any textbook could offer. … and we did! Over my two years at the colegio, I did see changes. Changes in students' perspectives of mathematics, more support from my high school teaching colleagues as they saw changes in their incoming students' understanding of mathematics and their mindset, and less resistance from incoming parents (it was a tight community, so lots of talk/gossip about this 'new' math).

> I remember a significant moment in my relationship with my Math PhD colleague, when he acknowledged my teaching practice was influencing student understanding of math. I believe he learned about what my students were doing from some of his own students who had siblings in my class. I recall my students telling me that they helped older siblings with their math homework … and this information was relayed to the High School teacher. I remember this as a turning point. Afterwards, we would have lunch hour conversations where I would show him what my students were working on, and sometimes he helped me see the depth of their mathematical thinking that I sometimes did not see. At one point, he gave me test questions from his Grade 10 class; and my students were able to answer them … some more proficient than his own students. I remember laughing about this with him over a lunchtime chat and teasing him that perhaps my ideas weren't so outrageous.

4.6 Implications for Teaching

The findings from this study hold particular implications for the mathematics education community as teachers who commit themselves to changing their teaching practice embark upon a difficult and sometimes lonely journey of self-doubt. This journey is frequently complicated by community and/or student resistance to change (LeSage, 1999, 2005; Manouchehri, 2003; Raymond, 1997; Ross, 1999; Ross et al., 2002; Roulet, 1998). For Nora, the journey also exposed her to tensions that are sometimes hidden – the tensions that can arise amongst administrators with divergent positions and pressures. Whereas the Director who hired Nora was accountable to the Board of Directors and was leading the teaching changes, the Principal who oversaw Nora was accountable to students' parents and was leading the teachers. While Nora described her Director as a support, she described her Principal as an appraiser, and we suggest that understanding the tensions experienced at the administrative level and their impact on fostering curricular changes may be an important part of understanding teachers' journey toward reform. In Nora's case, the resistance she felt from students, parents, administrators and colleagues manifested as tensions felt about her approaches to, and mathematics pedagogical knowledge for, teaching mathematics for social justice. Such approaches necessarily come with risk and tension for which teachers may be ill-supported (e.g., Mamolo & Pinto, 2015). This chapter contributes new and refined understanding about the interplay of tensions impacting teachers who attempt to introduce these new practices in their classes.

4.6.1 Elicited and Eliciting Tensions

Nora's story is significant because she is not the typical middle school teacher. Her orientation toward mathematics teaching already aligned strongly with the aforementioned pedagogical goals for teacher education, which is not the norm. Moreover, Nora is an individual who embraces risk and had accrued several non-conventional experiences before teaching at the colegio – she had relevant industry experience as an environmental engineer, had previously taught in international schools, she held an advanced degree in education, and had strong convictions about the effectiveness of her approaches. Nevertheless, she was deeply shaken by the tensions that emerged.

Tensions Nora experienced with respect to one pedagogical goal elicited tensions in another; these tensions were recurring, complex, and layered. For Nora, *fostering student success in non-traditional ways* elicited tensions in her *orientation toward mathematics class*. The interdisciplinary social projects elicited resistance from students regarding writing and explaining their thinking in English, which led to a change in Nora's practice that allowed for Spanish language discussions in her class. This elicited tension in Nora's *orientation toward mathematics class*, as she had to release part of her teacher control over the class discussions and allow for an unprecedented amount of student autonomy. Further, Nora's *orientation toward mathematics class* elicited tensions in her *reading the world with mathematics*. The resistance expressed by Nora's colleagues elicited recurring feelings of self-doubt and self-efficacy as she struggled to articulate the mathematical relevance of her activities. Nora's experiences are in-line with teachers' tensions when learning to teach math for social justice, including tensions from balancing pedagogical goals and from external factors such as parents and colleagues (e.g., Xenofontos et al., 2020). For instance, Mamolo (2018) noted that:

> The ability to articulate relevant mathematics pedagogical goals, as encouraged by the structures in the [social justice projects], may also help address some of the concerns and pressures experienced by teachers who may be reluctant to stray from more "stereotypical" approaches to mathematics teaching (p. 51).

4.6.2 Preparing for Sticky Situations

Nora's story highlights the complexities and interconnection of tensions that are elicited and experienced when introducing curricular change in mathematics. Even in a setting that was recognized as progressive, traditional expectations for mathematics were prevalent and strongly held. While Nora anticipated some resistance to her approaches, she was unprepared for the emotional toll she experienced while feeling tensions amongst what she was hired to do, what she was pressured to do, and what she believed she needed to do for her students.

These tensions, coupled with feelings of self-doubt, can derail teachers from attempting to introduce a new way of engaging with mathematics in their classrooms. Teacher education programs that prepare individuals to teach mathematics for social justice need to help build resilience to these anticipated tensions and pressures (as well as to the unanticipated ones). We suggest that part of the answer lies in mathematics and the pedagogical goals of *reading the mathematical world, fostering success in non-traditional ways,* and *changing one's orientation toward mathematics class* (Mamolo, 2018). Extended experiences with mathematical tasks such as the social justice projects can help preservice teachers overcome their own resistance to such approaches, particularly when time is spent explicitly articulating the mathematical nature, relevance, and importance of those experiences.

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Chapter 5
Designing Inclusive Educational Activities in Mathematics: The Case of Algebraic Proof

Francesca Morselli and Elisabetta Robotti

Abstract In the chapter we address the issue of designing and implementing inclusive activities for the teaching and learning of mathematics and, in particular, for algebraic proof. To this aim, we present design-based research that benefits from a combination of theoretical tools and references from neuroscience, cognitive science, education and mathematics education. We rely on the Universal Design for Learning principles to design inclusive educational activities to improve and optimize teaching and learning for all students, and we promote the activation of formative assessment strategies, so as to create an educational path each student is led to become responsible of his/her learning. In the chapter, we detail the design process, showing how the theoretical tools contribute to the creation and implementation of inclusive activities for the teaching and learning of algebraic proof, and we provide evidence of the effectiveness of the approach in terms of proof understanding and inclusion.

Keywords Inclusive mathematics education · Universal design for learning · Multimodality · Formative assessment · Algebra · Proof and proving

5.1 Introduction

UNESCO defines inclusive education as an ongoing process aimed at offering quality education for all, while respecting diversity and the different needs and abilities, characteristics, and learning expectations of students and communities, eliminating all forms of discrimination. UNESCO's Incheon Declaration highlights the importance of developing high quality educational research, taking in account the specificities of national education systems, to ensure inclusive and equitable quality education opportunities for all: "no education target should be considered met

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unless met by all" (Education 2030, Incheon Declaration and Framework for Action , 2015, p. 7).

Inclusion and how it is approached should be an urgent consideration for mathematics education. Indeed, data from the latest OECD assessment (OECD, 2019) show that more than half of the world's adolescent population does not meet the minimum standards in mathematics and reading (in Italy 1 student out of 4 at the age of 15 is in a state of mathematical illiteracy, and only 2% of these students reach the highest level). Therefore, the UN set the following goals for the 2030 agenda: "By 2030, ensure that all girls and boys complete a program free, fair and quality primary and secondary education"and also "Build and improve educational facilities that are child, disability and gender sensitive and provide safe, non-violent, inclusive and effective learning environments for all" (Education 2030, Incheon Declaration and Framework for Action, 2015, p. 20).

In this perspective, we address the issue of designing and implementing inclusive activities for the teaching and learning of mathematics and, in particular, for algebraic proof. Therefore, we present a design-based research that benefits from a combination of theoretical tools and references from neuroscience, cognitive science, education and mathematics education.

In the last decade there was a growing interest for networking theories in mathematics education, seen as a research practice that may improve the understanding of didactical phenomena (Prediger et al., 2008). There are different networking strategies, for instance we refer to the combining strategy, where a conceptual framework is built by juxtaposing elements from different theories. We point out here that Prediger and colleagues refer to all theories in mathematics education. In the same vein, in this contribution we explore the viability of an interdisciplinary approach (combining theoretical tools coming from cognitive psychology, education and mathematics education) to design and implement inclusive educational activities for a first encounter with algebraic proof.

In order to design and implement efficient educational activities with a focus on inclusion, it is important to take into account the student dimension. Both cognitive and educational science researchers refer to learning profiles (Armstrong et al., 2012), although there is not a shared definition of them. This is also evidenced by Karahiannalis and Nöel (2020) who establish common ground, at a cognitive level, attempting to transpose relevant aspects of the cognitive psychology literature into the field of mathematics education (Karagiannakis et al., 2017; Karagiannakis & Baccaglini-Frank, 2014). Exploiting these studies, some researchers have been actively trying to elaborate theoretical grounding, both for research on students with low achievement in mathematics and students with Mathematical Learning Difficulties (MLD), when teaching and learning include physical and digital artifacts (e.g., Baccaglini-Frank et al., 2014; Baccaglini-Frank & Robotti, 2013; Robotti et al., 2015). For instance, Baccaglini-Frank and Robotti (2013) referred to a "radical" approach in the development of technological tools for MLD. Within this approach, software is designed to propose fundamental mathematical content (e.g., the notion of "variable" or "function") in ways that take advantage of particular hardware and software affordances. The interactions with software designed according to this approach are frequently less constrained: tasks within the environment need to be designed by an educator (as they might not be part of the software), input and feedback may be given in various ways, and the role of the teacher becomes fundamental in mediating the meanings developed by the students within the environment. Thus, inclusion is realized by designing educational activities that consider the educational context and the learning profiles of the students (also students with MLD), combining theoretical tools and exploiting software that was purposefully created.

We argue that inclusive educational activities should be planned so that they address a plurality of students, with a plurality of learning profiles. According to researchers in cognitive science and neuroscience, (Núñez & Lakoff, 2005) and in mathematics education as well (Radford et al., 2009, Arzarello & Robutti, 2010, Nemirowsky, 2003), this can be done exploiting a multimodal approach. Indeed, mathematical thinking develops through a multimodal approach, where many modalities may be activated simultaneously. Multimodal activities seem especially effective for students with low achievement in mathematics (or even a diagnosis of Mathematical Learning difficulties). In the chapter, we detail the design process, showing how the theoretical tools contribute to the creation and implementation of inclusive activities for the teaching and learning of algebraic proof, and we provide evidence of the effectiveness of the approach.

5.2 Theoretical Framework

5.2.1 Multimodal Approach

The notion of multimodality arose within the paradigm of embodied cognition, a cognitive science theory that recognizes the central role of the body in shaping thinking. According to Loncke and colleagues (Loncke et al., 2006), multimodality is the use of two or more forms of communication from the two main modalities, namely auditory and visual, and is deeply intertwined with perceptuo-motor activities. Multimodality concerns cognitive science and also neuroscience, because it details how the body is involved in thinking and learning. Indeed, it emphasizes sensory and motor functions, and their importance for successful interaction with the environment. In the perspective of neuroscience, the sensory-motor system of the brain is multimodal rather than modular: "an action like grasping… (1) is neurally enacted using neural substrates used for both action and perception, and (2) the modalities of action and perception are integrated at the level of the sensory-motor system itself and not via higher association areas." (Gallese & Lakoff, 2005, p. 459).

We are particularly interested in the effect that the multimodality perspective may have on mathematics education. Arzarello (2006), quoting Nemirovsky, points to how research in math education suggests that the paradigm of multimodality implies that "the understanding of a mathematical concept rather than having a definitional essence, spans diverse perceptuomotor activities, which become more or less active depending o f the context" (Nemirovsky, 2003, p. 108). Also, Radford (2006) highlights that the understanding of relationships between bodily actions carried out through artifacts (objects, technological tools, etc.) and linguistic and symbolic activity is essential to understand human cognition, and mathematical thinking in particular. In other words, the multimodality perspective suggests that human activity is multimodal and cognitive processes should be analyzed taking into account all the involved modalities (Radford et al., 2009). During mathematical activities involving media, students produce a variety of signs such as words, gestures, and actions on the tools, interactions, and written or oral signs of whatever nature" (Arzarello & Robutti, 2010, p.718). In this perspective, we consider signs produced by students in math activities with media as expression of multimodal approach devoted to the development of algebraic proof. Our reference to the Universal design for learning (UDL), that will be discussed in the next paragraph, is coherent with the multimodality perspective.

5.2.2 Universal Design for Learning

Universal Design for Learning (UDL) is a framework to improve and optimize teaching and learning for all people (Rose & Meyer, 2006), based on scientific insights into how humans learn. Indeed, neuroscience research has identified three primary neurological networks that impact learning (CAST, 2018) Cherrier et al., 2020):

- The recognition network deals with perception, language and symbols (incoming stimuli) and affects "what" students learn
- The strategic network mediates "how" students process incoming information (physical action, expression and communication) based on past experience or background knowledge
- The affective network regulates students'attitudes about incoming information as well as their motivation to engage in specific activities — the "why" students want to learn and "why" they are engaged.

Successful teaching and learning involve all the three networks simultaneously and it is designed on UDL's three principles:

- Multiple means of engagement - tap into learners' interests, offer appropriate challenges, and increase motivation
- Multiple means of representation - give learners various ways of acquiring information and knowledge
- Multiple means of expression - provide learners alternatives for demonstrating what they know.

CAST (Center for Applied Special Technology), a nonprofit education research and development organization, created the Universal Design for Learning framework and the UDL Guidelines (https://udlguidelines.cast.org/more/research-evidence). In Fig. 5.1 the three principle of UDL are presented in more detail.

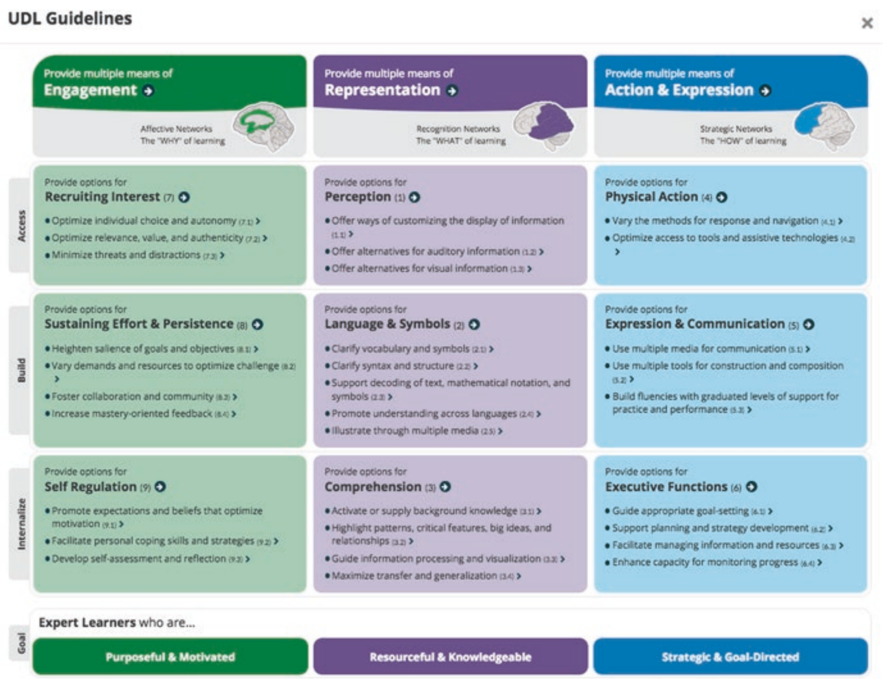


Fig. 5.1 UDL Guidelines. (https://udlguidelines.cast.org/action-expression/expression-communication/construction-composition)

The first UDL principle focuses on providing multiple means of engagement: indeed, besides recognizing the necessity of recruiting students' interest, one must know that not all the learners will find the same activities or information equally relevant or valuable. For our focus, we rely specifically to the following principles: vary demands and resources to optimize challenge (UDL 8.2), foster collaboration and community (UDL 8.3), develop self-assessment and reflection (UDL 9.3),

The second UDL principle focuses on providing multiple means of representation. As far as representation of math objects is concerned, this principle suggests to provide options for perception in terms of alternative means (i.e., registers) of representation (algebraic language, for instance) through which the math object can be represented (UDL 1.2, UDL 1.3). This principle suggests to lead students to improve comprehension about information at disposal, activating background knowledge (UDL 3.1), highlighting big ideas and new ideas to answer tasks (UDL 3.2) and to guide information processing (UDL 3.3) and design strategies of solution (UDL 2.3, UDL 2.4). It also suggests to move students towards generalization (UDL3.4), so that they will be able to transfer their learning to new contexts (UDL3.4). In the teaching and learning sequence we will discuss, the proposed tasks will allow active participation, exploration and experimentation.

Providing multiple means of action and expression is the third UDL principle. In the teaching and learning sequence we will discuss, physical action on representations of math objects (UDL 5.2) supports mathematical thinking. Similarly, actions, like gestures and moving objects (cf., just writing), support communication (UDL 5.1). Managing information, resources, ideas, … in order to develop math thinking (UDL 6.3), supports the executive functions which are essential in guiding appropriate goal setting (UDL. 6.1) and monitoring progress (UDL 6.4). Such a focus on goal setting and progress monitoring suggests a link with formative assessment, that will be treated in the subsequent section.

5.2.3 Formative Assessment

Formative assessment is defined as a method of teaching in which “evidence about student achievement is elicited, interpreted, and used by teachers, learners, or their peers, to make decisions about the next steps in instruction that are likely to be better, or better founded, than the decisions they would have taken in the absence of the evidence that was elicited” (Black & Wiliam, 2009, p. 7). A key element of formative assessment is feedback, that is any “information provided by an agent (e.g., teacher, peer, book, parent, self, experience) regarding aspects of one’s performance or understanding” (Hattie & Timperley, 2007, p.81). Wiliam and Thompson (2007) provide a description of five main strategies to perform formative assessment in class: (FA1) clarifying and sharing learning intentions and criteria for success; (FA2) engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding; (FA3) providing feedback that moves learners forward; (FA4) activating students as instructional resources for one another; (FA5) activating students as the owners of their own learning. Such strategies may be activated by the teacher, but also by the peers and by the student himself. Indeed, formative assessment should in principle lead to auto-regulation.

5.2.4 Algebraic Proof

As already mentioned, we are interested in the design and implementation of inclusive activities for the teaching and learning of mathematics. In the present contribution, we address the topic of proof. There is a wide amount of research on the teaching and learning of proof, see the work of Stylianides et al. (2016) for a recent overview. Here we focus on students’ first encounter with proof, an important learning experience encompassing both understanding what a proof is and in learning how to prove (Balacheff, 1982). As De Villiers (1990) points out, it is crucial to make students aware of the different functions that proof has in mathematical activity: verification/conviction, explanation, systematization, discovery, communication.

Lin et al. (2012) present a series of principles for task design aimed at promoting conjecturing, proving, and the transition between conjecture and proof. In relation to conjecturing, it is important to provide students with an opportunity to engage in: (C1) observing specific cases and generalizing; (C2) constructing new knowledge based on prior knowledge; (C3) transforming prior knowledge into a new statement; (C4) reflecting on the conjecturing process and on the produced conjectures. Concerning the transition from conjecture to proof, the teacher should propose tasks that raise students' need to prove. Moreover, the teacher should establish "social norms that guide the acceptance or rejection of participants' mathematical arguments" (p. 317), emphasizing that the acceptance /rejection is based on the logical structure of the argument and not on the authority of the instructor. In relation to proving, it is important to guide students: P1) to express in different modes of argument representation (verbal arguments, symbolic notations, etc.); P2) to understand that "different modes of argumentation are appropriate for different types of statements" (p. 318); P3) to create and share their own proofs and to evaluate proofs produced by the teacher, thus "changing roles"; and P4) to become aware of the problem of sufficient and necessary proof.

Focusing on algebraic proof, we refer to Boero (2001) who describes the fundamental cycle of *formalization*, *transformation* and *interpretation*. Performing a proof by algebraic language encompasses the following crucial issues: the choice of the formalization, that must be correct but also goal-oriented; the validity and usefulness of the transformations; the correct and purposeful interpretation of algebraic expressions in a given context of use.

We draw from the aforementioned theoretical tools, coming from different domains, to sketch a theoretical framework for the design and implementation of inclusive activities in mathematics concerning algebraic proof. In Sects. 5.3, 5.4 and 5.5 we discuss a teaching and learning sequence that is conceived within this theoretical frame.

5.3 Method: A Design-Based Approach

From a methodological point of view, the sequence is the result of cycles of design, enactment, analysis and redesign, according to the design-based approach (DBCR, 2003). Following this approach, we take as a starting point specific theoretical claims concerning the teaching and learning process (UDL principles, guidelines for the first approach to proof, formative assessment strategies) and we aim to understand "the relationships among theory, designed artifacts, and practice" (DBCR, 2003, p. 6), also considering the design of the teaching and learning sequence as an outcome of the research in itself. Moreover, the research is characterized by a strong interaction and collaboration between researchers and teachers, who take part in the design, implementation and a posteriori analysis.

The teaching and learning sequence we present is the result of cycles of design, enactment, analysis and redesign that started in 2012 and involved four teachers of a lower secondary school in the North of Italy. At present, five cycles were performed in grade 7 (pupils' age: 12–13), involving 8 classes (about 160 pupils). For our analysis, we rely on teacher's notes, observer's notes, video recordings of the class discussions and written productions of the students.

In Sect. 5.4 we discuss the design of the teaching and learning sequence on algebraic proof; in Sect. 5.5 we give a general overview of the implementation and in Sect. 5.6 we carry out a qualitative analysis of relevant excerpts from the implementation.

Our analysis addresses the following research questions:

1. Is the combined framework efficient in helping to promote inclusive educational activities?
2. And, in that case, are they effective for the teaching and learning of algebraic proof?

5.4 From Theoretical Tools to Design: An Educational Sequence on Isoperimetric Rectangles

The teaching and learning sequence concerns isoperimetric rectangles. At the core of the sequence is the conjecture and explanation of the fact that, among all the rectangles with fixed perimeter, the square has the maximum area. In the designed sequence, the students start from the empirical evidence to grasp the idea of variation of the area according to the length of the sides. Then, students are asked to conjecture about the maximum area. Afterwards, they are guided by the teacher to feel the need for a general explanation and to appreciate the power of algebra in leading to a proof. The task sequence is organized in the following steps.

Step 1: Individually, students complete the explorative paper and pencil task in paper and pencil: "*Draw four rectangles with a perimeter of 20 cm*". Afterwards, students work in small groups on the following task: "*Compare the methods you used to draw the rectangles and synthesize*".

Step 2: Students work in groups and cut cardboard to create a set of isoperimetric rectangles.

Step 3: Students work in groups, aided by paper and pencil, on the following questions: "*Do you think all the rectangles have the same area? If not, what is the rectangle with the biggest area?*"

After each of the steps 1, 2, 3 the teacher promotes a mathematical discussion.

Step 4: Once established that the square is the rectangle with the biggest area, the teacher guides the students to prove the property. The proof, carried out in algebraic language, is presented at the blackboard, with the teacher involving the students via open questions. When presenting the proof, the teacher refers to the previous steps. For instance, the teacher underlines that using algebra allows one to generalize from a specific rectangle to the generic rectangle, that is a rectangle with the same perimeter as the square.

Step 5: Each student receives a sheet containing the written proof of the property, and is asked to fill some open sentences concerning the proving process (for instance: "I put the rectangle over the square in order to…"; "I use letters because….").

After step 5, the teacher collects written answers from the students and promotes a discussion amongst them.

Step 6: Each student is asked to answer to the following open question: "*Looking back at the previous steps, you may note that we worked on the problem of iso-perimetric rectangles by means of different approaches: we used paper and pencil, cardboard, we drew a rectangle over a square on the blackboard, we used letters. What does each approach tell you?? Do they make you understand the same thing? Were they equally easy to follow and understand?*"

This teaching and learning sequence is conceived under the perspective of multimodality. Indeed, the construction of meanings is strictly bound throughout all the activities; on one hand, to the use of tools (e.g., paper and pencil, cardboards); on the other hand, to the interactions between people working together (small groups, class discussions). This way of working is typical of perceptuo-motor activities described in literature (e.g. Nemirovsky, 2003), where students are involved in solving mathematical problems individually or in groups.

Further, the sequencing is conceived according to the principles by Lin et al. (2012): students observe specific cases and generalize, so as to formulate a conjecture (principle C1). When conjecturing, they construct new knowledge based on prior knowledge (C2), so as to transform prior knowledge into a new statement (C3). Concerning the transition to proof, the teacher proposes different modes of argument representation (principle P1), paying special attention to the link between geometric and algebraic representations. During the guided proof and the subsequent individual reconstruction, students are led to reflect on the cycle of algebra, from the formalization (using letters to express relations) to the transformation and interpretation of algebraic expressions.

The sequence is conceived according to the UDL principles to ensure that all students can access and participate in meaningful, challenging learning activities about proof. There are different ways to take in and categorize information, to make sense of letters, symbols, colors and shapes, to connect new learning to prior knowledge. This is done so that each student may have a privileged channel of access and processing of information: visual-verbal, visual non-verbal, auditory and kinesthetic (UDL7). The steps involve different registers of representation, hence many channels of access to information (UDL1). Throughout the sequence, each student may address the problem on the basis of the privileged channel. For instance, the student may do some conjecture on the biggest area on the basis of the drawing, or on the basis of the manipulation of cardboards (UDL4). During group work and discussion the students compare individual strategies with different means of expression and communication (UDL5). In line with the UDL principles, modes of representation alternate so as to scaffold the solving process of any student. Working on different modes of representation (UDL2) is a support for reasoning, and also promotes motivation (UDL8). Guiding students to tackle the problem in multiple ways promotes sustaining effort and persistence and provides multiple means of engagement (UDL8). During steps 5 and 6 students are led to internalize the proving process by means of a guided reflection (UDL6.1). Such activity supports self-regulation and metacognition (UDL9).

The cycles of design, implementation and redesign were performed taking into account the UDL perspective. For instance, the use of cardboard was suggested by a student. In the first cycle of design, the students just worked on paper and pencil, produced the conjecture and were guided in the algebraic proof. In the algebraic proof, the idea of posing the square over the rectangle, and after comparing the non-coinciding areas, is crucial. One student, Bianca, interacting with the teacher during the guided proof, made the gesture of superimposing two concrete figures. This gave the idea of making students construct figures on cardboard (UDL1.2, UDL5.2). In the second cycle of implementation, each group was asked to construct a set of isoperimetric rectangles with a freely chosen perimeter, and after the set was given to another group, with no information on the measures, so as to promote an exploration without reference to the specific rectangles, that could pave the way to generalization (UDL3.4). Anyway, some groups still measured the sides, because they preferred to ground their exploration also on the numerical aspects (UDL3). From the third cycle on, each group could keep the set of rectangles they constructed and ground their reasoning on the drawing, the cardboards or the measures, according to their own privileged way of approaching the problem (UDL2.3, UDL2.4).

During the designed sequence many formative assessment strategies are activated. In steps 1 and 3, the student is asked to explain respectively the procedure for drawing isoperimetric rectangles and the conjecture. Explaining makes the student *responsible for his/her learning* (strategy FA5). During group work and all the class discussions, students act as instructional resources for the classmates (strategy FA4). During the guided proof of the statement (step 4), the teacher explicates the learning objectives of the activity (strategy FA1). During steps 5 and 6, students are encouraged by the task itself to become responsible for their own learning (strategy FA5); the teacher may gather information on the learning process, and use it to provide individual feedback (FA3).

Table 5.1 presents a synthesis of the steps, outlining which principles were put into action in each step. The presence of principles in the same line suggests the coherency of the combined theoretical framework, while the absence of principles in some rows (for instance, in steps 5 and 6 there is no reference to principles referring to conjecture and proof) shows the necessity of combining theoretical references.

Table 5.1 Synthesis of the steps of the designed educational sequence and of the principles put into action in each step

Step	Algebraic proof	UDL	Formative assessment
1. "*Draw four rectangles with a perimeter of 20 cm*". "*Compare the methods you used to draw the rectangles and synthesize*".	C1	UDL2 UDL7.1 UDL7.2	FA5 FA4
2. Construct a set of isoperimetric rectangles in cardboard	C1 C2	UDL1.2 UDL2.1 UDL2.4 UDL2.5 UDL4 UDL5.1 UDL5.2	FA5 FA4
3. "*Do you think all the rectangles have the same area? If not, what is the rectangle with the biggest area?*"	C1 C2 C3	UDL3.3 UDL3.4 UDL2.3 UDL 2.4	FA5
4.Guided proof	P1		FA1
5. Individual reconstruction of the proof		UDL1.4 UDL8.1 UDL6.4 UDL5.2	FA5 FA3
6. Individual "looking back"		UDL1.4 UDL 6.1 UDL9.1	FA5 FA3

Note. (C1) observing specific cases and generalizing; (C2) constructing new knowledge based on prior knowledge; (C3) transforming prior knowledge into a new statement; (C4) reflecting on the conjecturing process and on the produced conjectures; (P1) to express in different modes of argument representation (verbal arguments, symbolic notations, etc.); (FA1) clarifying and sharing learning intentions and criteria for success; (FA2) engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding; (FA3) providing feedback that moves learners forward; (FA4) activating students as instructional resources for one another; (FA5) activating students as the owners of their own learning. For UDLs, see Fig. 5.1

5.5 The Teaching and Learning Sequence: An Overview

In this section we present a general overview of the implementation of the teaching and learning sequence through the theoretical tools previously introduced. We refer to data collected in a class of 22 students that was involved in the third cycle of experimentation.

Steps 1 and 2 paved the way to the conjecture concerning the biggest area (step 3). This excerpt from a groupwork is representative of student answers: students go back and forth from the geometric representation of the problem to the arithmetic one.

Students often did not insert the square as a special case of rectangle, although the sides of the square represent a suitable solution for the arithmetic problem (finding two numbers with a fixed sum). The square was refused because it is not in their concept image of rectangle. A further analysis of the provided explanations may be found in (Levenson & Morselli, 2014).

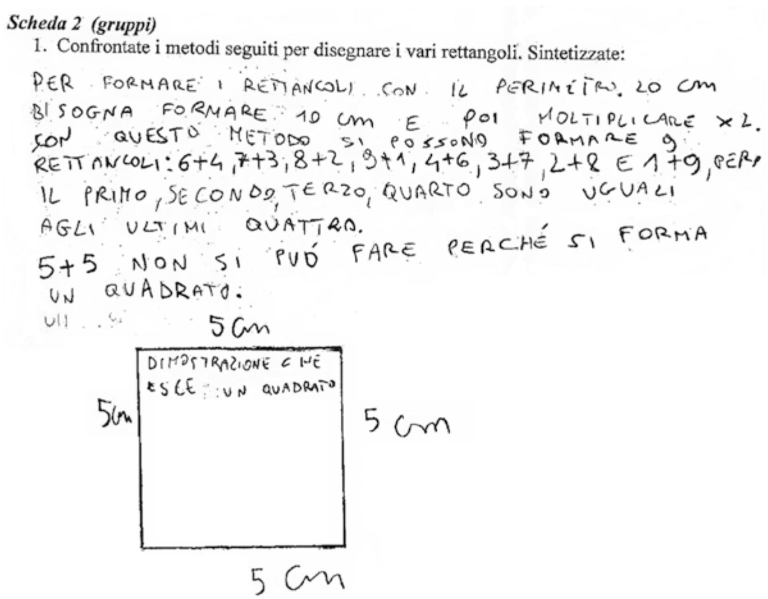
Ready access to sets of cardboard rectangles strongly supported students' reasoning. By moving, overlapping and cutting cardboard, students realized that the area varies (when they overlap figures and cut the exceeding part, they cannot cover all the figure), realized that area increases when the rectangle has the two consecutive sides with similar measures, and conjectured about the square as the rectangle with the biggest area. Other students relied on the measures, thus shifting back to the numerical mode of representation of the problem (see Fig. 5.2).

Figure 5.3 shows the "ladder" of rectangles in cardboard and the idea of "cutting and overlapping" with cardboards and represented in paper and pencil.

This part of the sequence is efficiently narrated in the following student excerpt:

> … after cutting them we overlapped them, finding out that there was always a part exceeding. This made me understand that the exceeding part may be cut out into the figure, and the part that was not exceeding could be put "in common" with the other figures. In my group we also out [the rectangles] in a sequence, from the smallest to the biggest. And we found that if we went on [the rectangle] became small and thin, or vice versa with a bigger area. And we found that we got a square and got the conclusion that the square is a special kind of rectangle.

The last sentence of the student ("the square is a special kind of rectangle") is linked to the previously mentioned issue of steps 1 and 2: not all the students, at first,


Scheda 2 (gruppi)
1. Confrontate i metodi seguiti per disegnare i vari rettangoli. Sintetizzate:

PER FORMARE I RETTANGOLI CON IL PERIMETRO 20 CM BISOGNA FORMARE 10 CM E POI MOLTIPLICARE ×2. CON QUESTO METODO SI POSSONO FORMARE 9 RETTANGOLI: 6+4, 7+3, 8+2, 9+1, 4+6, 3+7, 2+8 E 1+9, PERÒ IL PRIMO, SECONDO, TERZO, QUARTO SONO UGUALI AGLI ULTIMI QUATTRO.
5+5 NON SI PUÓ FARE PERCHÉ SI FORMA UN QUADRATO.

Fig. 5.2 Group work on step 1. (Translation: "In order to make rectangles with a perimeter of 20 cm, one must make 10 cm and then multiply by 2. With this method one can make 9 rectangles: 6+4, 7+3, 8+2, 9+1, 4+6, 3+7, 2+8 and 1+9, but the first, second, third and fourth one are equal to the last four. 5+5 cannot be done because a square is made")

		
Fig. 3.1 – The «ladder»	Fig. 3.2 - "Cutting and overlapping" with cardboards	Fig. 3.3 - "Cutting and overlapping" with paper and pencil

Fig. 5.3 From the students'productions

inserted the square into the set of rectangles, because the square did not correspond to their concept image of rectangle. Interestingly, students were keener to accept the square as a special case of rectangle at the end of step 3, when they found out that the square has the biggest area. In other words, they extended their concept image of a rectangle so as to include the square in order to have the square as the rectangle with the biggest area. Here an intuition supported by visualization and manipulation contributes to the refinement of the concept image. This shows how the multimodal approach, according to the UDL principles, contributes to mathematical thinking.

Step 3 was followed by a class discussion, where the issue of generality emerged. This paved the way to step 4, where the teacher guided the students to explain in general terms why the square is the rectangle with the biggest area. The proof relies on the previous experience with the cardboards (overlapping the rectangle on the square and comparing the areas of the non-coinciding parts), and algebraic languages adds generality (letters represent measures, so that the rectangle is the generic rectangle with the same perimeter as the square). During the proof there was a back and forth between visual and algebraic modes of representation, which we can relate to the UDL principles concerning modes of representation and means of expression and communication. During such activity, all the students were involved in the class discussion (under the mediation of the teacher) because they could rely on the different modes of representation to support their reasoning and also to communicate it. Thus, the class discussion was a crucial moment not only for promoting a first approach to proof, but also for inclusion: all the students had at their disposal means to act and communicate—allowing them to take part in the discussion and then contribute to the construction of meaning. Moreover, the teacher used the guided proof to clarify the learning intentions of the activity (formative assessment strategy FA1).

During step 5 (individual reflection on the algebraic proof) students were led to reflect on each step of the proving process, giving meaning to all the process (e.g., the formalization, the transformation of the inequality in an equivalent one, the final interpretation of the inequality) and reflecting again on the employed modes of representation (figure, letters, …). Each step of the proving process was justified in natural language, thus promoting a link between algebraic and natural language modes of representation.

Finally, in step 6, students were asked to recall and reflect on the whole teaching and learning sequence, giving meaning to all the steps. Students' narratives and reflections will be analyzed in the subsequent paragraph, with the aim of understanding whether the designed sequence constitutes an inclusive activity on algebraic proof.

5.6 Analysis

We focus on Step 6, analyzing student excerpts[1] that provide evidence of the efficacy of the sequence in terms of inclusion and teaching and learning of algebraic proof. We choose to focus on step 6 because it is the last one of the sequence, and because it requires students to adopt a reflective stance on all the sequences. Due to space constraints it is not possible to report on all the students' productions, so we will focus on those excerpts in which the reflection on relevant themes (conjecturing and proving processes, use of multiple means of representations, progress in understanding, encountered difficulties) was particularly rich, clear and complete.

> **Camilla** (translated): In the first approach I understood well what is the meaning of isoperimetric rectangle. In the second approach I understood well which was the rectangle with the biggest area because overlapping the cardboards one group created a square which is a special rectangle. We understood that [the square] has the biggest area. In the third approach we specified better why the square has the biggest area.

Camilla describes the journey from discovery to explanation. Moreover, Camilla ascribes to each approach a specific role in terms of construction of meaning: the first one (figural) makes the students understand the problem and the relations at issue, the second approach (kinesthetic) leads to the conjecture and its perceptive verification, the third approach (verbal, non-visual, symbolic) allows to generalize and reach an explanation. The evolution is accompanied and mediated by the different registers of representation, action and communication, thus supporting the efficacy of the UDL principles as a guide for the design of the sequence.

> **Erika** (translated): The first approach was not very useful to me because, since we used a particular measure, I did not know whether what I understood could be applied to any rectangle. Moreover, it was not very useful because just drawing you could not see anything special and if you noticed something, you could hardly see it. The second method was very useful because we all had the idea of overlapping them to see which was the one with the biggest area and we understood it was the square. And also thanks to a sort of "ladder" with the square as a starting point and each step was a rectangle with longer basis and shorter height in comparison with the side of the square. But in this way you don't understand why the square is the rectangle with the biggest area. The third method was the most important because it gave motivation to the fact that the square is the rectangle with the biggest area.

[1] All the excerpts were transcribed verbatim and translated by the authors.

Erika proposes meta-level reflections on the function of each approach towards meaning construction: for instance, she points out that the drawing is too specific and static and doesn't allow seeing invariants. On the contrary, the dynamic actions on concrete figures in cardboards allowed her to see invariants and relations. Erika is aware of the fact that the cardboards give a perceptual evidence for the conjecture ("*also thanks to a sort of "ladder" with the square as a starting point*") but do not provide a general explanation ("*in this way you don't understand why the square is the rectangle with the biggest area*"). Erika also recognizes the value of the algebraic language as a generalizing tool ("*The third method was the most important because it gave an explanation*"). Erika also explicates a link between action on figures in cardboard and algebraic transformation. We point out that Erika judges the approaches in terms of "usefulness", thus expressing her personal preference and functionality in relation to the objective of conjecturing and proving. Erika seems to be fully aware of the learning intentions of the activity.

> **Gaia** (translated): I had more difficulties in understanding the last activity because with the cardboards and without letters it is easier… you can move figures, you cut the pieces that are left, you add to what is missing… but the concept is not as accurate as the one with letters.

Gaia points out that concrete representation on cardboards allows action (thus promoting conjecture) and communication to the classmates. At the same time, the concrete representation does not hinder the necessity of moving to another representation (algebra) in order to generalize.

> **Beatrice** (translated): we did many approaches, but the easiest was the one where we had to draw four rectangles with the same perimeter but drawn in different ways; the approach of cutting cardboards was not difficult, the only problem was to draw rectangles that were equal to those on the cardboard; the method on the blackboard seemed to me more difficult to understand. On the blackboard there were a square and a rectangle overlapped, they had the same perimeter and, by means of calculation, we had to explain why the area of the square is bigger than the area of the rectangle. They all say the same thing, but in different ways, for example they want to make understand that rectangles that are isoperimetric to the square are infinite, but with drawing and the cutting you understand less because you cannot draw infinite ones, whilst with the mind and numbers you can go on to infinity. For me, a student in difficulty should try the first two methods, but a student not in difficulty should try the third one. With the third method I had difficulty because I could not understand well, while with the first two method I understood the concept of area, but I could not immediately grasp the idea of infinity but it was not possible to do it. The first two approaches are also more amusing because you can compare your ideas with the ones of your mates and if you don't understand the mates can help him, while if you are alone you have to understand by your own, which is more difficult. […] Not everybody understands letters and figures and calculations, but with the drawing and easier explanations you understand more.

Beatrice recognizes that the sequence was organized in terms of evolution of generality. She is also aware at metacognitive level of the fact that the algebraic approach, although valuable in terms of generality, is more demanding in terms of cognitive load. She adds considerations in terms of engagement and points out the different methods are suitable for different students.

> **Ivan** (translated): They are all useful and each of us can use his own method but they all take the same result, so there is not a wrong method among them. They all take to the result and each of us may use his own.

Ivan, in a very synthetic way, seems to confirm that the designed sequence achieved its main goals: provide all students with multiple modes of representation, occasions of action and motivations to address the problem, and construct the meaning of algebraic proof.

5.7 Discussion

In this contribution we presented a combined theoretical framework, consisting of theoretical tools from mathematics education, education, cognitive science, and neuroscience. In a design-based research perspective, we used them in combined theoretical framework to design, implement and analyze a teaching and learning sequence to realize an inclusive activity on algebraic proof.

The students, starting from the empirical evidence, grasped the idea of variation of the area according to the length of the sides. Afterwards, they were guided by the teacher to feel the need for a general explanation and to appreciate the power of algebra in leading to a general explanation. During the guided proof and the subsequent individual reconstruction, students were led to reflect on the cycle of algebra, from the formalization (using letters to express relations) to the transformation and interpretation of algebraic expressions. We point out that students at the same time grasped the necessity of a new mode of representation (algebraic language) and learn how to use it, with the semantic level (reference to the meaning of the expressions) supporting the syntactic one (transformation). As we discussed in the previous section, students' answers in step 6 are promising, because they show that students highly appreciated having at disposal multiple means of representation to ground their reasoning. At the same time, students recognized the explanatory and generalizing power of the algebraic language.

Students' reflections in step 6, conceived as a formative assessment occasion, suggest to us that the designed sequence promoted an inclusive approach to conjecturing and algebraic proving. From the point of view of algebraic proof, students were able to report the evolution from exploration, to conjecture, to proof. Moreover, they were generally aware of the need for a general explanation and appreciated the proving power of algebraic language. From the point of view of inclusion, we saw the effectiveness of the three principles of UDL "in action". A crucial issue is that in the teaching and learning sequence each register has its own status (and students show to be aware of this), but for each student a register may play a specific function with respect to the learning objective (UDL 2.4, UDL2.5). All students are using muliple sources of information to make sense of the other, with their preference reflected in their excerpts. Some students are completely aware of the generalizing power of algebra, other students appreciate the necessity of algebra after having dealt with the dynamism of cardboards. In general, figures in cardboards are efficient in activating the reasoning that leads to the conjecture, because the dynamic work on the figures allows one to identify geometric invariants (UDL 4.1). Interestingly, students themselves are aware of the two dimensions of the registers (status in reference to the designed teaching sequence, function in relation to the personal learning experience throughout the sequence). This suggests that providing multiple registers of representation enriches and makes the teaching and learning sequence really inclusive, without losing the content-related objectives of the activity (approach to proof). Moreover, we observed that students link their appreciation of one register to the fact that working in that register fosters understanding (UDL2.4). Having at disposal more than one register of representation, students could not only formulate their conjectures, but also communicate their conjecturing process to the teacher and the peers (UDL5.1, UDL 2.5).

5.8 Implications and Conclusions

On the basis of these findings, we argue that the combined framework was efficient for designing inclusive educational activities. It was particularly effective for the teaching and learning of algebraic proof. Such results suggest that integrating theoretical tools coming from neuropsychology, cognitive science and education may help with designing inclusive teaching and learning sequences in mathematics education. More specifically, we suggest to integrate UDL principles into the design principles for promoting conjecturing, proving, and the transition between conjecture and proof (Lin et al., 2012). To this aim, the observation of specific cases to gain their generalization (C1) should be performed with multiple means of representation, action and communication. Using multiple means of representation (UDL2) (for instance, concrete materials and symbolic representations) is effective in the conjecturing phase; it is important that the work on concrete materials is followed by the verbal and symbolic formulation of the conjectures, exploiting multiple means of representation and communication. Multiple means of action, representation and communication promote generalization, and effectively support students' engagement in the task (UDL7) and students' sharing of ideas (UDL5). Moreover, special attention should be payed to the communication of the conjecturing process and of the produced conjecture. Once the conjecture is proved, it is important to look back at all the process (C4) and give meaning to the different phases as well as to the different means of representation that were used in each of them (C4, P1). It is important to give meaning to the algebraic proof by connecting it to the different means of action and representation. This helps to make the acquired knowledge more stable. The "looking back" phase is also crucial in sustaining effort (UDL8), because it makes students focus on the objectives of the activity and on all the conjecturing and proving process. Moreover, the "looking back" phase supports a control system on the processes. Therefore, this phase also supports self-regulation (UDL9).

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Chapter 6
A Sustained Board Level Approach to Elementary School Teacher Mathematics Professional Development

Brandon Allan Dickson, Donna Kotsopoulos, and Carolyn Mussio

Abstract In this chapter, we explored a school board's response to improving mathematics achievement. While mathematics achievement is an area of interest in many jurisdictions, competing interests often limit professional development (PD) in mathematics to 'one-off' sessions. This research was completed in the context of a study analyzing student mathematical achievement. The school board in which this study took place had mathematics scores below the provincial average on provincial standardized tests. Multiple provincial level mathematics curriculum and strategy changes also occurred immediately prior to the year of PD programming. In response, the school board implemented yearlong, twice weekly virtual interactions led by school-based facilitators, focused specifically on grades 1–3 teachers. The emphasis of the PD was on the development of school-based professional learning communities made up of elementary school teachers, the principal, and Student Program Support Teachers. Where many studies focus on teacher learning through PD, we emphasize student learning from PD. Results from one school within the board with lower mathematics scores, but high teacher participation in the PD suggest the success of this sustained PD. This research contributes to the literature related to professional development and teachers' mathematical content knowledge. Implications for school boards and directions for further research are discussed.

Keywords Professional development · Mathematics education · Teacher educators · Systemic change · Instructional leadership

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6.1 Introduction

Preservice and beginning teachers increasingly state that they feel uncomfortable developing and implementing effective mathematics programming at the elementary school level (Brady & Bowd, 2006; Wessels, 2014). One of the main ways that teachers enhance their practice is through professional development (PD; Miller et al., 2015) and it is often focused primarily on teacher's learning (Carney et al., 2016). The impact on student learning is often understated or overlooked (Sztajn et al., 2007).

In this chapter, we report research that emphasizes student mathematical learning as a result of PD. We also provide a framework for PD which targets student learning. We focus on the experience of one school board which implemented a district wide, virtual approach to mathematics PD. In response to declining mathematics scores on provincial standardized tests across the district and a province wide focus on improving mathematics achievement, a sustained PD program was organized for elementary school teachers and implemented over the course of one full school year. Beyond regional concerns about declining achievement, mathematics is recognized as a skill which is particularly important to develop in young children. This is because, the biggest predictor of mathematics success in later grades, is mathematics success in the early years (Duncan et al., 2007). Teachers' mathematical knowledge has a huge impact on student learning. In fact, it is one of the greatest predictors of early mathematics success (Tchoshanov, 2011); hence, the importance of this effort to link PD and student-level outcomes.

We focus on the results of this initiative in one school at which all three elementary school teachers and a Student Program Support Teacher (SPST) participated in the PD. Our study combines qualitative observation with quantitative data from student test scores to explore the outcomes of this year-long, multi-pronged PD initiative. Implications for curriculum implementation and directions for areas for further research will be shared.

6.2 Professional Development (PD) Programming

There is abundant literature related to effective professional development in mathematics education. Research shows that teachers who participate in PD tend to have already successfully implemented new learning received from past PD in their own practice (Guskey, 2002). As a result, effective PD must be connected to classroom practice to keep teachers engaged (Grimmett, 2014). Unsurprisingly, students have shown improvement in their learning in classrooms where teachers engage in PD (Fishman et al., 2013). Effective PD still requires the learning be linked to opportunities for implementation in the classroom and collaboration with other teachers (Desimone & Garet, 2015). Particularly impactful are opportunities for in-school coaching with professionals who can help to advance teachers' practice in real time

(Koellner & Jacobs, 2015). This implementation however requires a great deal of investment of time and resources from leadership at the school and board level (Darling-Hammond et al., 2005).

PD programming is often run from the school board level; however, school board approaches vary drastically. Koellner and Jacobs (2015) describe PD along a continuum from highly adaptable to highly specific. One example on continuum they mention would be the range from one-on-one coaching sessions, which are highly adaptable, to large group workshops which are more rigid in their structure. They also mention that school board level PD programs tend to be focused on individual subjects and do not often provide individualized support to specific educators, but rather focus on broad approaches which are applicable to everyone (Desimone et al., 2006; Koellner & Jacobs, 2015; Landry et al., 2009). Traditional PD is often measured in hours or 'seat-time' in a specific location and is often not evaluated based on the relevance to the practice of individual teachers (Polly et al., 2018; Polly & Hannafin, 2010). The literature generally agrees that PD focused on the experience of the individual teachers as learners is far more effective than approaches which focus on broad concepts not relevant to everyone (Hill et al., 2013; Polly et al., 2018).

Despite the fact that large, board level PD may not be the most effective method, conducting smaller PD sessions is a challenge given the limited resources available for PD. These limitations include both physical items such as technology to deliver such programming, and the limited time of facilitators and teachers for such programming (Shernoff et al., 2017). There is a general recognition that school board level approaches to PD are necessary and may ~~even~~ be effective if implemented correctly. In general, it is recognized that school board level approaches to PD are more successful when they focus on interactions between PD organizers and teachers as well as interactions between teachers. This can come in a variety of forms, and therefore, it is not the form of PD which matters so much as the ability to make contact (Desimone & Garet, 2015; Hochberg & Desimone, 2010; Koellner & Jacobs, 2015).

Even when PD is content focused, the quality of interactions in the PD programming is a key factor in the success of PD, rather than strictly the content which is taught (Carney et al., 2019). There is less consensus however on how to make PD focused on teachers' learning. Some claim that smaller PD sizes and an increased number of facilitators are needed to make traditional PD more meaningful and allow for these interactions to take place (Hochberg & Desimone, 2010). Others argue that online videoconferencing and frequent follow-ups through email can provide opportunities to supplement traditional PD programming, even in larger groups (Carney et al., 2019).

Regardless of format, student-centred approaches to PD need to be sustained over long periods of time rather than being one-offs. They also ought to focus on specific content, rather than broad instructional strategies, and should use the time to support depth of learning in the content area being taught (Polly et al., 2018). This kind of learning can be incentivized by school board funded resources such as technology or lesson plans which promote engagement and encourage teachers to attend PD programming (Wayne et al., 2008). Further, regardless of format, student-centred approaches can be built through hands-on learning opportunities and can be promoted through opportunities for in-class evaluation and support to provide reflection in real time (Polly & Hannafin, 2010). The success of any PD relies on teachers seeing the success of their own learning in student achievement, which will ultimately promote continued engagement. Therefore, promoting student centred PD programming is essential (Petrides & Nodine, 2005). School board support, in the form of planning support and release time, for small school-based communities of teachers to work together as a form of PD has shown to increase teacher accountability (White & Lim, 2008) as teachers learn from peers and increase their own learning (Doig & Groves, 2011).

The study of how to best facilitate this kind of teacher success and learning in mathematics PD is well documented. Typically, mathematics PD operates as single session programming in large group sessions with the potential for small group breakouts (Polly et al., 2018). Often, PD is intended to provide teachers with strategies for development of their teaching rather than helping with the development or planning of whole lessons or units. Often PD programs do not address specific mathematical content areas or deal directly with curriculum expectations (Simpson & Linder, 2014). If collaboration occurs, it is usually between teachers within a school, rather than knowledge dispersal within a whole school board. It also frequently happens because of the decisions of individual teachers' initiative, rather than because of the design of PD sessions. (Remoe-Gillen, 2017). Of the few studies that exist with a focus on student learning, there is consensus that when consistent PD is delivered and sessions are layered to promote stronger curricular understanding, with intersystem collaboration, both teacher learning and student learning benefit (Polly et al., 2016; Polly et al., 2017; Remoe-Gillen, 2017).

6.3 Assessing Mathematics Professional Development

The success of mathematics PD programs has been measured in different ways. Traditionally, teacher learning, and teacher self-efficacy have been used as the evaluation criteria of PD programs as teachers are the recipients of PD (Carney et al., 2016). The danger with this is that it prioritizes educators' learning with the assumption it will result in student learning.

Miller et al. (2015) have developed a more complex framework analyzing the success of PD programs through studying interconnected systems that contribute to PD success: "the learning activity system (PD activities, their coherence, opportunities for reflection and time for supervised application of new learning), the teacher learning system (teacher's beliefs, values and perceptions) and the school/school board systemic context (school practice, routine, and policies)" (p. 320). The study of PD programs through this lens, while important to ensure that the system supports the success of the program, still does not evaluate student learning as a result of the PD, which, even recognizing that student success is a result of numerous factors, can be an indicator of the overall success and suitability of the PD programming. This can create a cycle whereby the impact on student learning is a secondary success criterion to the impact on teacher learning. We base our study on Miller et al.'s (2015) framework to understand the potential for impact of the PD on the various systems. However, we build on this understanding of successful PD using the simple premise that for PD to be a success the impact on students must be central (Grossman et al., 2001; Sztajn et al., 2007). Therefore, while our qualitative findings will seek to understand success of the various systems of PD, our quantitative analysis evaluates the impact of this PD on student mathematics achievement.

6.4 Provincial Context

This research was completed in the wider context of a study analyzing PD and student mathematical achievement. Our specific study takes place in a publicly funded school board in Southwestern Ontario. In the province of Ontario, mathematics and literacy scores are measured on the provincial standardized test which is administered by the Education Quality and Accountability Office (EQAO, 2019). Tests at the elementary level are administered in Grade 3 and Grade 6 for both mathematics and English. The tests are graded on a scale of Level 1–4 (corresponding with letter grades D-A). The criteria to meet the provincial expectations on any test is Level 3 or 70%. While these tests are not a progression requirement for students from grade-to-grade, and do not impact students' school-level report cards, the EQAO reports are used by both school board and school level administrators as one gauge of the successes of the programming and the challenges being experienced by their schools. It is also used by the provincial government to adjust provincial education objectives, develop curriculum and shift funding.

In the Canadian province of Ontario, scores on the provincially administered standardized mathematics test at the Grade 3 level had declined from 63% of students at or above the provincial standard in mathematics in 2015–2016, to 61% in the 2017–2018 school year. In response to provincially declining mathematics, in 2016 a *Renewed Math Strategy* (Ontario Ministry of Education, 2016b) was introduced. This was expanded on in 2018 by a province mandated 'back to the basics' approach to mathematics called, *Focusing on the Fundamentals of Math: A Teacher's Guide* (*Fundamentals*) (Ontario Ministry of Education, 2018). These new documents were incorporated across the province. Recognizing that there are inherent equity issues with standardized tests and that these tests are often inaccurate in their description of student learning, the emphasis in Ontario on the results of the EQAO tests makes the declining scores on this test relevant to students and board level responses. While this is not the entirety of our analysis, it provides important context for the PD sessions that were implemented.

6.5 The Professional Development Sessions

Our research focuses on the experience of three elementary school classes in Bayfield school (pseudonym used throughout) a school within this school board in the Canadian province of Ontario. Two of the three elementary school teachers at this school were early career teachers with fewer than three years of teaching experience. The third teacher had over 10 years of teaching experience and was a mathematics lead at a former school. In total, 32 students participated in this research. There were 19 girls and 13 boys in either grade 1 or grade 2.

In the school board in which our study took place, the PD format described was also inspired by a lack of supply teachers, either certified occasional teachers, or uncertified support personnel to supervise classrooms and the subsequent inability to procure release time. The school board in which our study took place had experienced a decline in standardized test scores similar to that experienced at the provincial level. In the 2017–2018 school year the board had 55% of Grade 3 students at or above the provincial standard in math, 6% below the provincial average (EQAO, 2019). At this same school, the test at the grade 6 level showed only 41% of students were at or above the provincial standard in mathematics in 2017–2018. These tests were relatively consistent across years, with student scores on standardized tests declining substantially more than the provincial average between their grade 3 and grade 6 tests.

In response to this decline, a team of four facilitators were tasked with designing and implementing mathematics PD to improve student mathematics achievement across the entire school board. Facilitators are generalist elementary teachers (i.e., no curricular specialization) who are seconded from classroom instruction to work at the school board level to assist with professional learning. They are typically assigned to work with 5–6 elementary schools to support teachers and provide individualized programming approaches. Facilitators regular work involves them being in the classroom with teachers for co-planning, co-teaching and co-assessing. These facilitators had existing relationships with classroom teachers, as a result of this school-based work, and were able to build on these relationships to run virtual mathematics PD. As such facilitators are aware not just of system level priorities, but the needs of individual schools and classrooms.

In the 2018–2019 school year, the role of the facilitators shifted to a focus on supporting implementation of the government's new initiative, *Fundamentals* (Ontario, 2018). The PD program implemented by the facilitators was to be sustained across the whole year with multiple points of contact, with a focus on numeracy in the elementary school grades. The PD was designed to directly link curricular expectations to the government's new *Fundamentals* initiative and support teachers in understanding how they might space practice of these fundamental skills throughout the school year.

Facilitators began this programming based on feedback from elementary school teachers that it was difficult to envision anything other than a 'unit-by-unit' approach to mathematics instruction. As a result, they wanted to develop a program that would support educators in developing a common understanding of what mathematics learning and teaching could ‘look like and sound like’ in the elementary school years.

The facilitators adopted and modified a Ministry recommended scope and sequence approach (Ontario, 2016a) which focused on layered learning through repetition. A series of unit layouts and lesson plan bundles that laid out week-to-week learning goals for the whole school year were put in a web-based drive that was shared with all elementary teachers. These were intended to supplement, rather than replace individual teachers’ lesson plans. Full, ready-to-use lesson plan resources were distributed as well. Professional reading such as government policy documents and academic journals were also provided in the drive.

The focus of the PD was not simply giving access to these resources, but to use them to structure professional learning throughout the year. A multi-step PD program was developed that involved weekly live video conference training, weekly emails with highlights from the video conference learning and related tasks, in addition to traditional face-to-face PD sessions and one-on-one in class support at some schools. In total, from the end of September to the beginning of June, approximately 22 weekly video sessions occurred where facilitators would combine content learning with answering questions and modelling use of tasks as laid out in the Scope and Sequence documents. Some of the key features for successful PD that were included in this study are outlined in Table 6.1.

Although our current study focussed on one specific school with three elementary school teachers, all elementary school teachers across the school board were invited to attend these weekly sessions, and recordings of the sessions were posted

Table 6.1 Key features of successful PD Programming

Key PD features	
Key feature	Definition
Sustained regular live contact	Providing multiple opportunities for live PD on a related topic over the course of a series of weeks or months, rather than single sessions, to provide opportunities for learning and questions
Regular follow-up	Opportunities for engagement between scheduled sessions, such as emails or pre-recorded videos
One-to-one support	Giving teachers an opportunity for facilitators to come into the classroom and work with the teacher to implement the PD learning into their instruction
Focus on classroom implementation and student learning	Live sessions, emails, and lesson plans should be focused on classroom implementation to demonstrate the potential benefits of the PD to teachers. This can include connection to curriculum
Model lesson plans	Lesson plans created by facilitators or co-created with teachers as a model for new lessons, or as examples of how to implement the PD learning in their own classroom
Release time	Opportunities for supply teachers to take over the class so that the teacher can participate in the PD during time
Collaboration between teachers	Opportunities as part of the PD for teachers to work in groups within their schools to share ideas and co-plan

online for any teacher unable to attend. Additionally, 29 weekly emails were sent out to principals, SPSTs and all elementary school teachers in the board's 45 elementary schools with additional resources, and information on assessment or learning for that week. Two board-wide, full day PD sessions were also planned on designated PD days for all elementary teachers providing expert speakers and collaborative learning for teachers. Two additional half days of release time were provided by the board so that teachers could meet in small groups to engage in content learning and co-teach in another elementary school classroom within their own school. Figure 6.1 below is an example of one of the emails sent to teachers. The email refers to an image which teachers were familiar with which noted the three pillars of successful mathematics classrooms: three-part lessons, building community and mini lessons to practice.

The email from Fig. 6.1 followed an online session based on assessment in mathematics. The weekly live PD was focused on the process of scaffolding mathematics teaching and understanding where students are, and how to move them to the next step of their learning. The emails were intended to summarize key learnings from the live PD sessions and to support teachers in their classroom practice by providing tasks they could try with their students. Figure 6.2 below shows an example of how emails included specific resources for teachers to use.

By connecting curriculum expectations and providing links to specific resources, this PD provided ways for teachers to incorporate the weekly learning into their own practice. Teachers at all levels of knowledge were able to apply the new learning directly into their classrooms.

By the end of the school year, teachers had incorporated much of this mathematics PD into their own practice. One classroom teacher noted that many of the mathematics lessons they used were either directly from the PD programming provided or had resulted out of co-planning sessions with other teachers.

For our next section, we analyze the resultant data from this PD programming. Data collected in this project includes scores from provincial standardized

***please note that most of the tasks we have sent so far in the Friday emails fall into the third pillar of mini-lessons and practice. This is ONE piece of an effective numeracy learning experience.**

When you look at the pillars, what pieces do you feel really good about? What pieces do you need to focus more on, or learn more about?

Something else we are reflecting on this week is assessment for learning.

What have you learned about your students' mathematical thinking in the past six weeks? What are their strengths? What are some areas that you will continue to focus on and provide opportunity for guided practice? What evidence do you have?
In what ways are you tracking students' progress towards Term One learning goals?

Put simply, how are your students doing, how do you know, and where are you going next?

Fig. 6.1 Example email on assessment

Our second, weekly Primary Numeracy Message focuses on Patterning and Algebra. We dug into some patterning content during our Wednesday Primary Skype, and we'd like to keep this conversation going.

One of our goals this year is to help educators make connections between the strands. Specifically, we are hoping to highlight where there are connections to sense of number, or the FUNdamentals. Take a look at the **end of year** grade level expectations from the Curriculum Document (below). **Where do you see connections between the strands of Patterning & Algebra and Number Sense & Numeration?**

Grade 1	Grade 2	Grade 3
Overall Expectation Identify, describe, extend, and create repeating patterns.	**Overall Expectation** Identify, describe, extend, and create repeating patterns, growing patterns, and shrinking patterns	**Overall Expectation** Describe, extend, and create a variety of numeric patterns and geometric patterns
Specific Expectations • identify, describe, and extend, through investigation, geometric repeating patterns involving one attribute • identify and extend, through investigation, numeric repeating patterns • describe numeric repeating patterns in a hundreds chart • identify a rule for a repeating pattern • create a repeating pattern involving one attribute • representing a given repeating pattern in a variety of ways	**Specific Expectations** • identify and describe, through investigation, growing patterns and shrinking patterns generated by the repeated addition or subtraction of 1s, 2s, 5s, 10s, and 25s on a number line and on a hundreds chart. • identify, describe, and create, through investigation, growing patterns and shrinking patterns involving addition and subtraction, with and without the use of calculators. • identify repeating, growing, and shrinking patterns found in real-life contexts • represent a given growing or shrinking pattern in a variety of ways • create growing or shrinking patterns • create a repeating pattern by combining two attributes • demonstrate, through investigation, an understanding that a pattern results from repeating an operation or making a repeated change to an attribute	**Specific Expectations** • identify, extend, and create a repeating pattern involving two attributes, using a variety of tools • identify and describe, through investigation, number patterns involving addition, subtraction, and multiplication, represented on a number line, on a calendar, and on a hundreds chart • extend repeating, growing, and shrinking number patterns • create a number pattern involving addition or subtraction, given a pattern represented on a number line or a pattern rule expressed in words • represent simple geometric patterns using a number sequence, a number line, or a bar graph • demonstrate, through investigation, an understanding that a pattern results from repeating an action, repeating an operation, using a transformation, or making some other repeated change to an attribute.

Seeing these connections helps us view Patterning as more than one 'unit' of study. Students will be working on these fundamental numeracy concepts **throughout the year**. **We are moving from "doing patterns" to "focusing on mathematical pattern and structure"** (from Ch 12 in Learning and Teaching Early Math).

"Students must see all math as a search for patterns, structure, relationships, as a process of making and testing ideas, and in general, making sense of quantitative and spatial situations (Schoenfeld, 2008). Only if they do so throughout their work with math will they be well prepared for later math, including algebra." *(p. 220 Learning and Teaching Early Math by Clements and Sarama)*

What does it mean to focus on mathematical pattern and structure? We have two examples of tasks below, in the context of repeating and growing patterns.

Repeating Patterns (Earlier Primary)

Fig. 6.2 Example email with expectations

mathematics tests, Prime Number test results (Nelson, 2005) as a pre and post-test, and quotes from teachers at the end of the year recorded by the first author. Prime tests are mathematics tests developed by Nelson Education to align with the Ontario and Canadian mathematics curriculums to allow teachers to evaluate students' level of understanding and develop programming to support them. These tests evaluate each of the strands of the curriculum including: number and operations, patterns and algebra, geometry, data management and probability and measurement.

6.6 Results

At Bayfield school, all three elementary school teachers saw great value in the PD programming as was evidenced by their attendance and use of the various aspects of the PD. In total, 20 of 22 video conference sessions were attended as a group including all of the teachers and the SPST. Email follow-ups were read together and incorporated into planning. Day long PD sessions were also attended as a school unit. A facilitator came into the school five times to work with teachers on mathematics lesson planning. Mathematics lessons were often co-planned by all three teachers using the Scope and Sequence documents as the framework while incorporating session information into their planning. Teachers reported that between their three classes, they tried almost every activity they were introduced to through the PD at least once. The existence of key PD features in this programming such as sustained regular live contact, regular follow-up, one-to-one support and collaboration between teachers clearly promoted high uptake and meaningful interaction.

Below we detail two tasks as examples of the results of the PD programming and how it manifested itself in Bayfield School. These tasks were not direct lessons from the PD programming, but rather were identified by teachers after attending the PD sessions (Fig. 6.3).

In this activity students drew two numbers out of a bag which went on the left side of the sheet, for example, numbers 7 and 3. They then placed that number of blocks into the scale. Students then drew a third number from the bag. This number had to be smaller than the sum of the first two. This number was the place in the first box on the right and they place this many blocks in the box. Students then had to determine how many more blocks would need go into the other side of the scale. If the scale balanced, students knew they were right and repeated the activity.

In subtraction smash (Fig. 6.4), students used a paper clip as the first spinner to determine the minuend. They then placed pieces of sticky tack on the ten frames. They then used the second spinner to determine the subtrahend and took away that number of pieces of tack. They wrote these numbers below and determined the difference.

These lessons illustrate opportunities for students to develop basic numeracy skills through games identified as a result of teacher learning from the PD. Neither of these lessons was one of the pre-made lessons given as part of the PD. These lessons, however were identified by the teachers as a result of their learning from the PD programming. For example, one teacher noted that the balance the scale activity lesson idea resulted from PD programming on mathematical fluency and recognizing that there can be multiple ways in which to add to the same number.

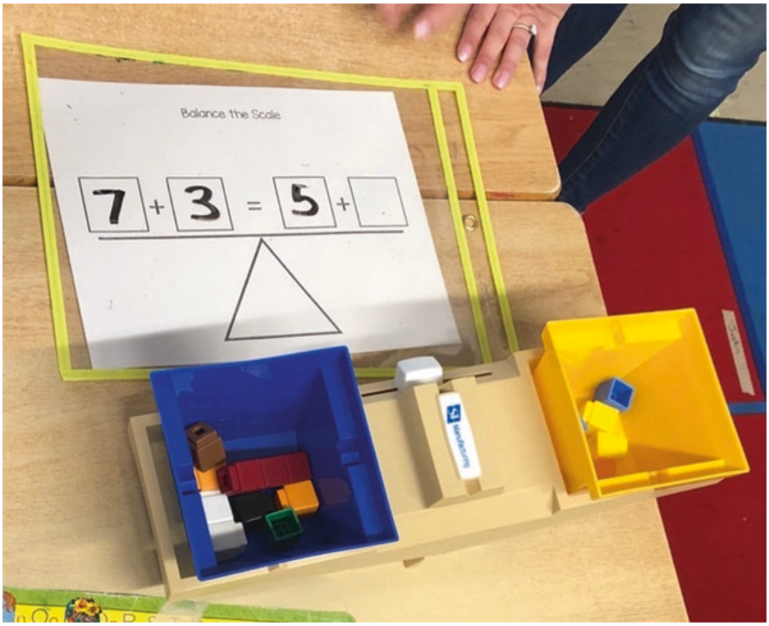

Fig. 6.3 Balance the scale activity

Fig. 6.4 Subtraction Smash activity

These tasks were coupled with teaching instructions and support from facilitators to implement the PD in their classrooms. This is evidence of the impact of a the PD on classroom implementation. Teachers at various levels of mathematics knowledge were able to introduce these fundamental numeracy skills in ways that students found engaging.

6.6.1 Changes in Mathematical Achievement

Prime assessments (Nelson, 2005) were administered by teachers before the PD programming began and at the end of the school year. Release time was provided for supply coverage so that teachers could administer tests one-on-one to students to reduce the potential impact on students tests of having an unknown researcher conducting their tests. Prime tests are utilized by some school boards in the province of Ontario to measure elementary students' basic mathematical proficiency. Students are given a score out of 20. On these tests, a score under 13 is considered pre-phase 1 (beginner), a score between 13–17 is considered phase 1 (concrete) and over 17 is considered phase 2 (whole number comfort). In total 32 students across the three classes consented to participating in the study.

On Prime tests at the start of the year, students' mean score was 9.2 out of 20. On post-tests, students' mean score was 14 out of 20. The mean score saw students move from beginner to whole number comfort. A Wilcoxon Signed-Ranks indicated that the median post-test ranks were statistically significantly higher than the median pre-test ranks, $Z = 4.19$, $p < .000$. Due to the limited number of participants and logistical challenges, our study did not have a control group to compare the increase in student learning against. However, our findings are corroborated by provincial standardized mathematics test scores, which indicated that Bayfield School saw an 8% increase in elementary school students at or above the provincial standard in mathematics from 2017–2018 before the PD program was implemented to 2018–2019 when this PD began. The school board in which this study took place saw a 2% increase over this span. The province as a whole saw a 3% decrease in Grade 3 student achievement in the same year. Provincial test results from Bayfield school showed that in 2017–2018, before the PD intervention began, 50% of students had been at or above the provincial standard in mathematics. This was 5% below the school board average.

6.6.2 Teacher Perceptions

One teacher who was new to the profession noted the particular challenges of teaching mathematics saying, "It is hard to come up with new ways to teach math concepts, so you teach the way you remember being taught. This PD allowed me to learn from the other teachers and see what they did and how they used the lessons the team gave us. I think I learned a lot about how to teach math." This teacher noted that they did not have a background in mathematics and therefore had felt somewhat uncomfortable teaching mathematics before this PD programming. They mentioned that their mathematics anxiety would likely go away with time, but that this PD programming increased teachers' comfort with teaching mathematics. The teachers are noting the importance of collaboration between teachers in their school and their increased comfort with teaching.

The most experienced teacher, who had been a mathematics lead previously, noted that they were not used to this kind of sustained PD and wished this level of support had been available earlier in their career. The teacher noted specifically that this PD included curricular links and a focus on classroom implementation, therefore they did not have to struggle to connect the lessons from the PD to the curriculum. The teacher also noted that this PD felt different because it was focused on classroom instruction and student success criteria, rather than on their own learning as teachers. In specific reference to their own instruction, they noted:

> I used to count by 2s and by 5s with the students at the start of everyday. They knew it off by heart. But when I did the tests, I was disappointed how few were able to complete that part (of the test). They knew the numbers but not the meaning of them. I didn't know how else to teach it though. In doing the PD, I got lesson plan ideas and was able to use those. The students seemed to do much better on that part of the test the second time.

Both of these quotes illustrate not just that teachers were satisfied with the programming and felt that they learned something, but also that their practice changed as a result of the programming. Student success as a result of these changes is detailed below.

6.7 Discussion

6.7.1 Success of the Professional Development Programming

The focus of this study was on a school board level approach to professional development programming for elementary school teachers. The study evolved from looking simply at the experience of teachers' perceptions of the PD programming to evaluating the impact of this kind of programming on students' mathematics performance. Our results indicated that this kind of PD which included several key features such as sustained programming, regular follow-up, model lesson plans and focus on classroom implementation also led to significant increases in student achievement in mathematics. While the provincial standardized mathematics scores continued to decline, this PD may have led to an increase in school board level mathematics scores. The fact that such an intensive, targeted approach to increasing numeracy in the elementary school grades was not a surprise but rather the hope and goal of this kind of PD focused primarily on the fundamentals of specific content areas such as numeracy (Polly et al., 2018). This increase in student achievement aligns well with research showing the potential positive impacts of PD on student success (Fishman et al., 2013).

A large part of the increase in students' mathematics understanding may be attributed to increased teacher comfort with content and their subsequent ability to teach content. Rather than just telling teachers how to teach content, this PD programming gave teachers the resources such as model lesson plans to apply in their classroom and opportunities to practice with one-to-one support. This approach is consistent with the literature on the need to connect PD to classroom practice to keep teachers engaged (Grimmett, 2014). Further, by seeing student achievement resulting from the PD programming, the teachers in this school were willing to continue to incorporate learning from the PD into their practice (Petrides & Nodine, 2005). In the case of Bayfield School, this resulted in a high uptake from the three teachers participating in a high percentage of the sessions and meaningful collaboration between teachers within the school. The opportunity for facilitators' support in classes to help teachers in implementing their learning in the classroom and release time also gave teachers an opportunity for coaching on their own practice (Koellner & Jacobs, 2015). Where teachers noted that they did not know in the past how to teach new content, being given not just the means to do so but the skills to do it effectively through student-centred PD programming and collaboration allowed teachers to make meaningful shifts in their own practice (Carney et al., 2019).

The format of collaborative video-conference sessions and addressing questions publicly in weekly emails allowed for a level of engagement and inter-school collaboration that had not previously occurred. This resulted in teachers learning from the experience of others, corroborating findings that collaboration between teachers in different schools promotes more successful mathematics PD (Remoe-Gillen, 2017). This research also supports Simpson and Linder's (2014) discussion of the importance of curricular ties in PD to make lesson planning relevant to educators, as teachers noted that they tried most of the activities they were offered because they could see the relevance. In the present study, the focus on the key features of successful PD such as student-centered approach to PD with activities intended for classroom use to meet student success criteria resulted in increased scores by students on Prime tests and an increased school score on standardized tests. This suggests that sustained, student centered, collaborative PD should be leveraged for a variety of academic subjects to enrich the learning of students.

All of this is not to say that the teachers at Bayfield school or teachers in the school board studied are lesser mathematics teachers and therefore required this kind of PD to effectively teach content. Nor is this an indictment of new teachers or schools with many beginning teachers. Like teachers at many schools, teachers at Bayfield school were doing remarkable things in mathematics teaching prior to this intervention, one teacher had even been a mathematics lead previously. Rather this is to note that, given the vast differences in teachers comfort levels with and the ability to teach mathematics, this kind of programming can positively impact teachers at all levels of experience and all levels of comfort.

6.7.2 Limitations and Next Steps

One potential limitation of this research was the short time frame. A more longitudinal approach to the evaluation of PD programming would also provide insight into how further teacher development would continue to support students, and at what point this sort of programming becomes redundant and could be replaced with additional PD efforts. This research would benefit from a longitudinal analysis to understand if teacher engagement remains high in subsequent years and if decreased engagement corresponds to decreasing student achievement. Further, research using a control class to compare the results of the PD programming would give better perspective on the impact of this PD programming specifically.

Additionally, as one teacher noted, by having teachers conduct the Prime testing they were aware of their students' level of understanding and what they needed to improve on. While we recognize the potential problems of teachers 'teaching to the test' after they learned where their students could improve, there is also a need to recognize that through this PD programming, teachers were given training on using Assessment for Learning skills (Ontario Ministry of Education, 2020), which evaluates where students are at and allows teaching to adjust accordingly. Through this PD and research, in conducting pre-tests, teachers were given the opportunity to recognize what they needed to learn more about and adjusted their instruction accordingly to support meaningful student learning.

One problem that we considered was the potential scalability of this kind of programming While it was possible to give release time to a certain number of teachers as an incentive to participate in the PD, school boards may not be able to support and fund such release time for intra-school collaboration if this PD were scaled to include more grades and resulted in higher teacher participation school board wide. Additionally, this kind of programming in a virtual setting is still heavily reliant on teachers independently engaging with the scheduled PD programming. To make this effective at a larger scale would certainly require more resources and support from administration to ensure this collaboration is sustained. Further, the teachers noted that this kind of virtual programming did not allow for consistent regular interaction with teachers from other schools, and so a focused effort on maintaining inter-school relationships is key to ensuring this PD continues to produce innovative practices and does not become routine.

6.8 Implications for School Boards

These preliminary results suggest that the PD program implemented in this school board produced positive early results on student learning, which occurred despite the fact that provincial elementary school mathematics scores decreased the year that this PD and the new *Fundamentals* approach was introduced. The implementation of the key features of successful PD promoted teacher success in this school board, which resulted in student learning. The reality is that students' mathematics achievement will vary with the quality of the learning environment and knowledge of the teacher. In providing this kind of PD programming however, school districts can begin to level the playing field and mitigate the variance between student success, by providing all teachers with a base of knowledge and materials in a subject that is so often fraught with anxiety.

Additionally, as this programming was not mandatory for teachers across the board, our results provide important implications for teachers, beyond just boards trying to create PD which is meaningful for teachers and encourages participation. As teachers are looking for potential PD to participate in, our findings provide important context on the potential implications for students. Where teachers are on the lookout for quality PD programming to build on their skills, focusing on the presence of the key features of PD we identified in potential PD programs will help teachers access programming with the best chance of learning for both them and their students. Further, the success of this programming in developing meaningful student learning ought to serve as a call to action for educators to rally around and seek out meaningful PD opportunities.

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Part II
Mathematical Learning

Chapter 7
A Digital Home Numeracy Practice (DHNP) Model to Understand the Digital Factors Affecting Elementary and Middle School Children's Mathematics Practice

Sabrina Shajeen Alam and Adam K. Dubé

Abstract The home is an important environment for individualized mathematics instruction, one that must be strongly considered given that children spend more time at home than in schools. As a result, researchers argue that we must understand how exposure to numeracy activities at home can provide a foundation for children's mathematics education. In this chapter, we outline how digital home numeracy practices (DHNPs) could serve as a primary means of home mathematics learning. We also propose a DHNP model and detail its components. The model addresses how different aspects of family, such as parental factors (e.g., socioeconomic situation, mathematics attitude and beliefs), children's factors (e.g., cognition, motivation, and self-regulation in general, and mathematics attitude in specific) and parent-child relationship may contribute to children's digital mathematics learning. Further, it differentiates between indirect and direct practices of home numeracy activities using technology. Finally, we discuss the potential avenues for future research on and practical implications for DHNP during the elementary and middle school years.

Keywords Mathematics education · Home numeracy · DHNP model

7.1 Introduction

Educational technology has been shown to have a positive impact on children's educational attainment (e.g., mathematics achievement; Cheung & Slavin, 2013). Using technological tools has become a dominant learning culture altering how educators and parents are teaching and how children are learning (Kukulska-Hulme, 2010). One could say that the learning culture of today has shifted from paper-pen

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to finger-screen (i.e., digital devices like tablets and smartphones). This shift is particularly noticeable in mathematics education and has become commonplace for middle school aged children.

As soon as computers became affordable and prevalent, there was a shift from learning to operate computers for mathematics activities (i.e., computing) to using computers to support mathematics learning (Durmus & Karakirik, 2006). Children are now extending their mathematics knowledge with the support of a range of digital tools (e.g., e-textbooks, videos, games). Previous studies have established the overall effectiveness of technology-based mathematics learning and instruction in classrooms; for example, Fabian et al. (2016) conducted a meta-analysis of over 60 studies on the efficacy of mobile devices as a tool for classroom-based mathematics learning and found positive gains across most studies.

Over the past decade, majority of children in North American homes now have access to digital devices (Funk et al., 2017). We are now poised to see the same shift seen in middle school classrooms repeated in the home, with digital devices having the potential to become a ubiquitous learning tool for home numeracy. However, research has largely focused on digital mathematics practices in formal context (i.e., school) and there is a dearth of research on digital mathematics practice at home.

Home numeracy is an established and expanding research field, led by mathematical cognition researchers, that primarily focuses on identifying analog (non-digital) home numeracy practices (HNP), their frequency, and their link to later mathematics achievement (Blevins-Knabe & Austin, 2016). Previous research shows that home environment during middle childhood predict mathematics achievement in late adolescence (Tang & Davis-Kean, 2015). Given the proliferation of digital devices in the home, it is conceivable that digital home numeracy practices (DHNP) should be incorporated into the study of home numeracy. Considering this, in this chapter we propose a model of home numeracy that centers DHNP and can be used as a guide for understanding the digital factors impacting middle school children's mathematics practice.

7.2 Digital Home Numeracy Practice (DHNP)

We define DHNP as the use of digital devices (such as cellphones, tablets) to foster children's mathematics understanding and learning at home. In USA, majority of homes (84%) have at least one mobile device available (Funk et al., 2017) and educational apps on mobile devices are affordably priced (*m* = $15 CAD, Dubé et al., 2020). Consequently, there are increasing opportunities for children from diverse socio-economic backgrounds to access them (Callaghan & Reich, 2018). More than 94% of North American parents let their children access digital devices (Wood et al., 2016), Given this widespread use and availability, it is fair to surmise that DHNP may already be part of the home learning environment and parents might have a positive attitude towards DHNP. This may be particularly true for middle schoolers, considering that the top commercially developed mathematics apps target mathematics topics for this age group (Dubé et al., 2020). In the next section, we describe how digital tools help children learn mathematics from a theoretical standpoint using theories and research that help us understand human-machine interactions.

7.2.1 Theoretical Underpinnings for DHNP

A child's central experience of learning with a digital device is their interaction with the object itself. The theoretical model has three components:

7.2.1.1 Cognitive-Communicative Model (CCM)

McEwen and Dubé (2017) propose that a mutual understanding between a user and a digital tool is co-constructed in an active and dynamic way during child-tablet computer interaction. When using a digital device, an interactive form of communication arises where user and device both serve as senders and receivers of information. Thus, the bi-directional interactive relationship between a child and a device is central to conceptualizing how digital devices shape mathematics learning. When considering how children learn from mathematics apps, we must think of both units in the interaction (child and app) as active agents that are affecting the other. Within the CCM, the affordances of the device are critical to successful child-device interactions.

7.2.1.2 Affordance Theory

Originally coined by Gibson (1977), affordances are the product of an interaction between an organism and an object (e.g., human + hammer = hammering) that depend on the physical attributes of the object and a person's knowledge of their use. In digital technology use, affordances are a product between a user and a digital device reflecting possible actions the device can perform. Children come to understand affordances of a mathematics app through exploring and using it (Dubé & McEwen, 2017). Research suggests that learning with digital devices is more effective when the affordance is more concrete and aligns with the learning content. For example, practicing number line estimation on a touch-screen device is more effective if the user is allowed to drag across the number line rather than tap it (i.e., a continuous gesture for a continuous concept, Dubé & McEwen, 2015). The reason that concrete and congruent affordances may matter is because these physical interactions between children and their educational app screens are supported by embodied cognition.

7.2.1.3 Embodied Cognition

Embodied cognition states that our physical movement (or motor behavior) influences cognition (Schneegans & Schöner, 2008); such that, there is a close link between motor (generate action) and sensory surfaces (give sensory signal to the surroundings), and our motor and sensory systems shape and influence how we think and learn by contributing additional information that is incorporated into cognition. Thus, the level of embodiment that occurs during touch-screen interaction (c.f., a game controller, watching a screen) increases the overall sensory experience and contributes information to the interaction that may influence learning (Tran et al., 2017). Critically, many embodiments are determined by the software design and not the physical device itself. For example, allowing a child to use a drag gesture for a number line activity is determined by the software designer and not the device manufacturer. So educational media running on devices must be designed with the correct features.

7.2.2 Design Features of Effective Educational Media

Research has identified specific design features of educational media that can improve children's learning with digital devices (Hillmayr et al., 2020), such as virtual manipulatives, digital feedback, and digital scaffolding.

7.2.2.1 Virtual Manipulatives

Children experience better sensory involvement by directly touching and physically moving touch-screen devices. On touch screen devices, interactive and dynamic objects (e.g., interactive 3-D pictures, Moyer-Packenham et al., 2002) are called virtual manipulatives. Virtual manipulatives can be thought of as cognitive technological tools (Zbiek et al., 2007). For mathematics specifically, they can be defined as interactive, technology-enabled visual representations of dynamic physical mathematics objects (e.g., pattern blocks, cubes; Moyer-Packenham et al., 2002). Virtual manipulatives are helpful because they create an interactive environment for learners to pose and answer their own questions and to connect mathematics concepts with operations (Durmus & Karakirik, 2006). Virtual manipulatives have two fundamentally different affordances than physical ones. First, children can easily be allowed to alter a virtual object whereas a real object cannot be altered (e.g., size, shape, and color of objects). Second, apps with virtual manipulatives often give users hints and digital feedback that guide the experience (Anderson-Pence, 2014).

7.2.2.2 Digital Feedback

Digital feedback (Bokhove & Drijvers, 2010) refers to information provided by digital tools (such as mathematics apps) to users on their performance on a given task that is intended to shape their behavior, interactions within the task/app, and overall understanding (e.g., Plass et al., 2011). Common sensory feedback that occurs during digital mathematics activities are audio (a sound contingent on a behavior/response; Blair, 2013) and visual feedback (written text or imagery for correct/wrong answers; see an example in Alam & Dubé, 2022, 2023). There are several forms of digital feedback (Cayton-Hodges et al., 2015; Johnson & Priest, 2014). Status feedback provides assistance during problem solving (e.g., suggesting the next step). Corrective feedback informs learners of a mistake and guides them to correct mistakes. Conceptual feedback asks learners questions to reconsider their perceptions of the tasks. Finally, explanatory feedback helps learners reduce the amount of extraneous processing that occurs when selecting information to complete a task (e.g., look here, this is important) and entails providing learners with a principle-based evaluation of the accuracy of their answers (e.g., this is wrong because). The quality of mathematics learning likely depends on the type of feedback provided by digital tools. For example, research indicates that explanatory feedback can be more beneficial for learning than corrective feedback (Hattie & Timperley, 2007), as it often uses adaptive features (i.e., intelligent tutoring system; Hillmayr et al., 2020).

7.2.2.3 Digital Scaffolding

Digital scaffolding delivers support to learners so they can meaningfully participate in a learning task and subsequently gain skills they would not be able to acquire unaided (i.e., computer-based support). Overall, the impact of digital scaffolding on learning outcome shows robust positive effects; $g = 0.46$, $p < .01$ (see the meta-analysis by Belland et al., 2017). Cayton-Hodges et al. (2015) categorized various forms of scaffolds, including on-demand hints (i.e., offered at the learner's requests), on-error hints (i.e., offered when mistakes happen), guiding questions, and reflections (i.e., prompted reflections through logical questions). Thus, mathematics apps can function as cognitive tools by assisting learners in their completion of cognitive tasks via digital scaffolds (Lajoie, 2005). Research is needed to determine which kinds of digital scaffolding optimizes learning (Lajoie, 2014).

7.3 The DHNP Model

The DHNP model consists of two structural components: an outer and inner model (see Fig. 7.1 for the conceptual model). In the following sections, we describe and delineate these components as well as discuss the underlying theoretical assumptions of DHNP and how they may affect mathematics learning during elementary and middle school years.

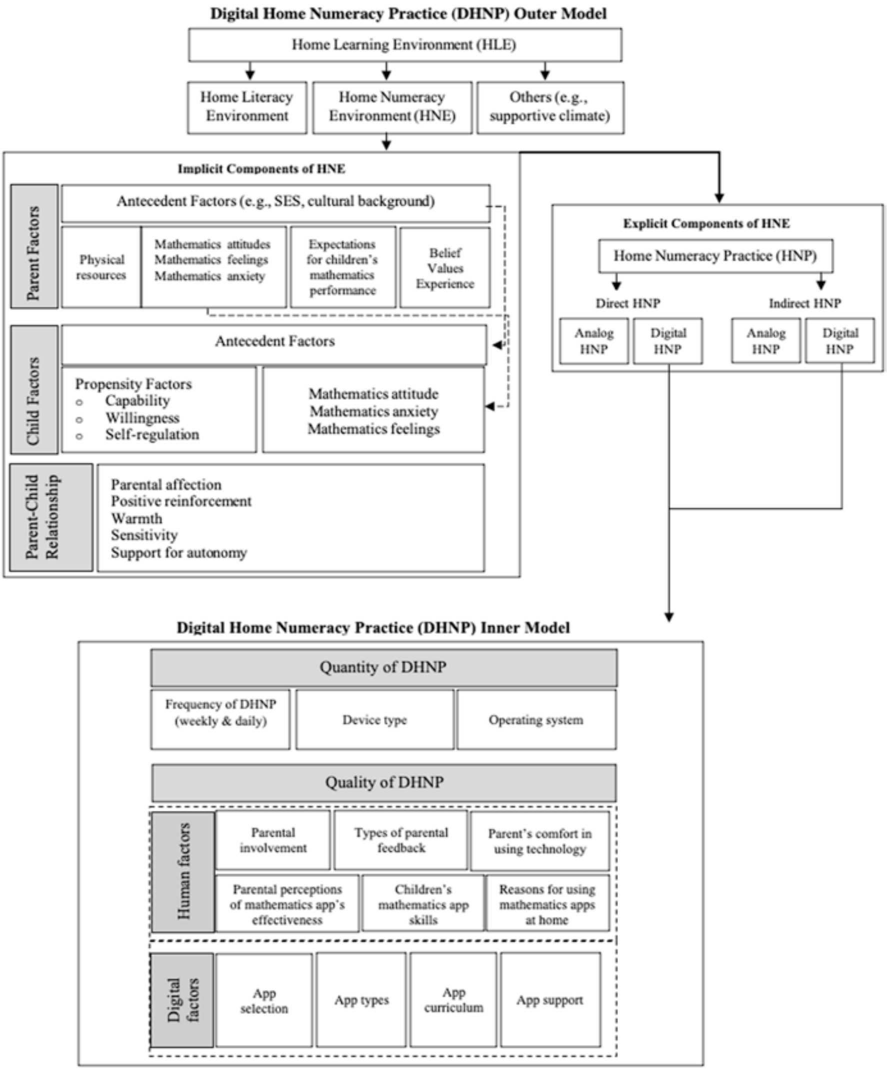

Fig. 7.1 Conceptual Model of DHNP

7.3.1 DHNP Outer Model

7.3.1.1 Home Learning Environment (HLE)

The family is a child's foremost teacher and advocate (Lehrl et al., 2020). When accounting for weekends, summer, and holidays, children spend only an estimated 1000 hours of a year in school in the USA (Hull & Newport, 2011), whereas they have approximately 5000 hours outside of school (Cross & Cross, 2017). Thus, children spend most of their time in home environments with parents and caretakers. The HLE is, therefore, a critical source of support for children's overall academic achievement (Niklas & Schneider, 2017). Three key features of HLE accelerate children's academic development: their involvement in learning exercises, parent-child interaction quality, and the educational resources available (Bradley & Corwyn, 2002). More broadly, the HLE has several components such as home literacy, home numeracy, and supportive climate (Kluczniok, 2017); each contributes to successful developmental outcomes (Lehrl et al., 2020). Thus, the HLE significantly contributes to children's educational and social development (e.g., Tamis-LeMonda et al., 2019), due to its central role in children's daily lives (Lehrl et al., 2020).

7.3.1.2 Home Numeracy Environment (HNE)

The HNE, which falls under the HLE, consists of the numerical activities (Hart et al., 2016) and supports (Zippert & Rittle-Johnson, 2020) parents provide to develop children's early mathematics skills. Previous research has used a variety of HNE measures to identify activities (e.g., playing card games) that support children's mathematics learning and predict their numerical knowledge (Mutaf et al., 2018). Importantly, it is not a simple matter of the more activities the better. Instead, some HNE activities do not correlate to children's math performance (e.g., quantity and counting skills; Missall et al., 2015). A possible explanation for some of these inconsistent findings might be that parents sometimes do not recognize age-appropriate activities for their children (Fluck et al., 2005; Skwarchuk, 2009). Thus, other aspects of the HNE need to be considered to understand which activities are beneficial and why. We have categorized the HNE into two components: explicit (factors directly affecting numeracy activities) and implicit (greater context in which numeracy activities occur).

7.3.1.3 Implicit Components of the HNE: Parental Factors, Child Factors and Parent-Child Relationships

Parental Factors Parental involvement is beneficial for children's academic achievement (Levine et al., 2010); they are supposed to act as active advocates for quality mathematics education. As argued, parents are a major influence on the

HNE (Blevins-Knabe, 2016). However, there is more to parental factors than just their numeracy knowledge. Critically, implicit parent factors affect the HNE indirectly by creating a context for direct numeracy practices.

Antecedent Factors Demographic variables can be considered antecedent factors that parents bring to the HNE, as they affect but are not affected by other factors. They include indices of socio-economic status (SES, i.e., income, education level, family composition, and housing) as well as cultural backgrounds that affect to what extent parents get involved in home numeracy activities with their children (Del Rio et al., 2017). The following SES factors affect children's mathematics performance.

- *Parents' education.* Anders et al. (2012) conducted a study where they found that mothers in Germany who hold a graduate degree engaged in more mathematics learning activities (such as mathematics games) more than mothers who only completed high school education.
- *Economic condition.* The U.S department of Education (2007) reveals that economically disadvantaged children show two times more mathematics deficiencies by fifth grade than well-off children (National Center of Education Statistics, 2007).
- *Socio-cultural background* influences parental values and academic expectations for their children. For example, European American parents in the USA had lower academic expectations than minority parents, even though their ninth graders had higher scores on exams than those of minority parents (Hossler & Stage, 1992). Regrettably, researchers' understanding of the HNE of low-SES families is still limited (Cheung et al., 2020); specifically, fewer studies have considered the impact of SES on mathematics performance during middle childhood.

Physical Resources A meta-analysis of family influences on children's overall academic performance reveals that home mathematics learning resources help children to do better in mathematics (Sirin, 2005). Home numeracy resources (such as the number of mathematics books; Zhu & Chiu, 2019) are a key component of the home numeracy experience; however, little attention has been given to identifying the physical resources at home (Anders et al., 2012); even less attention has been given to the digital home mathematics resources.

Mathematics Attitudes, Feelings, and Anxiety Parental mathematics attitudes have a large impact on children's behavior, attitude, and beliefs about their own mathematics abilities and academic achievement overall (e.g., Eccles, 1983; Eccles et al., 1982). These attitudes are conveyed to children through parental behavior and activities (Gunderson et al., 2012). Greater parental mathematics anxiety reduces parent-child verbal (i.e., conversation) and physical involvement (i.e., numeracy activities), which has been linked to children's poor mathematics performance at school (Berkowitz et al., 2015). During middle childhood, parent's mathematics anxiety may become particularly salient due to the transition from elementary to high school in which parents have trouble helping children with more advanced mathematics homework (Soni & Kumari, 2015). Specifically, parent's mathematics anxiety can result in more controlling support during mathematics homework, and this is negatively related to children mathematics achievement (Retanal et al., 2021). Given that 93% and 17% of adults report either some or high levels of mathematics anxiety, respectively (USA sample, Ashcraft & Moore, 2009; Blazer, 2011), it seems likely that parents' mathematics anxiety must be considered when understanding the HNE.

Expectations for Children's Mathematics Performance Parents' expectations for their children's academic performance may affect the HNE. Alexander et al. (1994) defined parental expectation as their reliance on children's academic performance and judgements of their child's future success. Academic expectations are based on children's grades and the resources available to them (i.e., more resources more expectations; see Yamamoto & Holloway, 2010). Several determinants of parental expectations (e.g., ethnic background) influence the HNE and have both direct and indirect effects on children's subsequent mathematics performance (Yamamoto & Holloway, 2010). However, evidence for the role of parental expectations is still mounting and more work is needed (Del Rio et al., 2017).

Beliefs, Values, and Experience The HNE includes parents' beliefs, values, and experience that altogether have an impact on children's mathematics success or failure (e.g., Eccles et al., 1982, Yee & Eccles, 1988).

- *Parental beliefs.* Positive parental beliefs regarding the importance of the HNE are linked to greater HNP; however, belief and practice are not perfectly aligned (Napoli et al., 2021).
- *Parents' mathematics value* affects children via two pathways: how much parents value mathematics overall and how much they value their way to teach mathematics (cf., school math, Muir, 2012). There is a disconnect between how parents were taught mathematics and how it is learned today (Marshall & Swan, 2010) and this contributes to valuing mathematics less when it comes to helping children with the subject. How much parents value mathematics overall influences how much importance they place on the HNE and is shown to affect children's future mathematics motivation (Jacobs et al., 2005).
- *Parents' mathematics and technology experience* may influence the opportunities of technology exposure for mathematics learning offered to children, with research showing that parents who use technology less in their own lives providing fewer technology interactions to their children (McPake et al., 2013).

Child Factors Many child factors affecting the HNE stem from parental factors and this directionality (parent to child) can be best understood with the opportunity-propensity framework.

Opportunity-Propensity (O-P) Framework Byrnes and Miller's (2007), opportunity-propensity (o-p) framework proposes that there are three factors (antecedent, propensity, and opportunity) that independently but collectively determine children's academic achievement.

Antecedent factors are latent aspects of the HNE, such as parent's SES (education, income) and parental expectations (as previously discussed). These factors set the stage for opportunity and propensity. Propensity factors support or influence children's mathematics learning when children are given opportunities (Zhu & Chiu, 2019), such as being exposed to mathematics related activities. According to Byrnes and Miller (2007), propensity has three primary aspects: capability or cognition (e.g., intelligence), willingness or motivation (e.g., interest, expectations), and self-regulation (e.g., strategic thoughts and behaviors guiding learning goal). The likelihood of HNP being effective depends on how children's propensity enables them to take advantage of the provided opportunities (i.e., numeracy practice) at home (see Byrnes & Miller, 2007). Children who have more willingness to do mathematics show better mathematics performance (Fisher et al., 2012).

Mathematics Attitude, Anxiety, and Feelings The Expectancy value model of Eccles et al. (1982) argues that parental beliefs and behaviors have a strong relation with children's attitudes, beliefs, and cognitions. Following this theory, parental mathematics factors may act as precursors to their children's mathematics attitude, anxiety and feelings, and this conjecture has been supported by previous research (e.g., Maloney et al., 2015; Soni & Kumari, 2015). Several measures have been used to understand children's mathematics attitudes; however, more research is needed to understand the extent to which children's mathematics attitudes influence the HNE. Within the O-P framework, attitudes are propensities that moderate how children take advantage of opportunities in the HNE.

Parent-Child Relationships As children grow older, continuous parental affection, positive reinforcement, warmth, sensitivity, and support for autonomy (Edwards & Knoche, 2010) all contribute to children's long-term academic success (National Scientific Council on the Developing Child, 2004). For mathematics, positive and warm parental support of children's home mathematics learning is associated with their further academic success (Jacobs et al., 2005), as it helps children to develop positive attitudes about mathematics and their ability to learn new mathematics skills (Chavkin, 1994). Thus, how much home numeracy practice improves children's mathematics outcomes depends on not just the frequency of opportunities but also the quality of those opportunities.

7.3.1.4 Explicit Components of the HNE

The implicit components of the HNE reviewed previously are factors that indirectly affect home numeracy practices (HNP), which are the explicit factors of the HNE. The explicit factors can be defined as the frequency of numeracy related activities at home (e.g., writing numbers), the extent of numerical exposure, and the frequency and level of numeracy conversations between parents and children (LeFevre, Sowinski, et al., 2009b; Levine et al., 2010). These HNP discussions (verbal) and activities (physical) are associated with children's mathematics knowledge and performance (e.g., Mutaf et al., 2018). These are often categorized as either formal or informal (i.e., direct or indirect; LeFevre, Skwarchuk, et al., 2009a). Despite these seemingly clear terms, HNPs are not always easily delineated (LeFevre et al., 2010).

Direct HNP are opportunities in which parents directly instruct their children about mathematics with the goal of improving specific numeracy skills (Kleemans et al., 2012; LeFevre et al., 2002) or overall mathematics knowledge (Skwarchuk et al., 2014). The implicit factors of the HNE can mediate the frequency of direct HNP. For example, parents who have positive mathematics attitudes and high academic expectations are more involved in direct HNP (Susperreguy et al., 2020). Indirect HNP are opportunities (e.g., playing games involving a numbered board) in which mathematics learning may occur incidentally even though teaching about mathematics is not the purpose of the activities (Skwarchuk et al., 2014). Children's formal mathematics knowledge is primarily developed during indirect HNP through parent-child discussion (Ginsburg et al., 2008; LeFevre, Skwarchuk, et al., 2009a) about mathematics concepts, numbers, and quantity that arise during everyday activities (e.g., cooking and talking about quantity; Bjorklund et al., 2004; Vandermaas-Peeler et al., 2012).

Analog and Digital Forms of HNP Each aspect of HNP (i.e., direct, and indirect) can be further broken down into analog and digital. *Analog HNP (AHNP)* are mathematics activities with non-digital objects, such as counting apples or shopping (indirect AHNP) and solving mathematics problems on paper (direct AHNP). AHNP has been the focus of research interest for several decades. Despite this focus, there are relatively few studies that have investigated the full range of AHNP to gain a comprehensive understanding of how non-digital and/or conventional forms of home mathematics practice affect mathematics outcomes. *DHNP* is a relatively recent practice that has arisen due to the availability and affordance of digital devices in the home. Indirect DHNP are digital mathematics applications or apps that are designed as play-like experiences (e.g., game-based learning) that teach mathematics via the intrinsic integration of mathematics into play activities (e.g., DragonBox Algebra) whereas direct DHNP are e-textbooks and online open-learning systems that more closely mirror formal mathematics instruction (e.g., khan academy).

7.3.2 DHNP Inner Model

Our proposed model of DHNP consists of constructs that detail both the quantity and quality of DHNP. This is a newly developed model that differentiates between analog and digital that can guide future HNE research by providing a detailed breakdown of the factors that may contribute to the effect of DHNP on children's mathematics outcomes.

7.3.2.1 DHNP Components

Having provided a brief coverage of theories used to study and conceptualize how children interact with and learn from digital tools, we now provide a model that identifies the key components researchers should consider when studying how technology use in the home affects children's mathematics understanding.

Quantity of DHNP This component of the model is foremost concerned with the number of opportunities for DHNP experiences as well as detailing the nature of those opportunities (device type, operating system).

- *Frequency of DHNP* is the rate of weekly or even daily children's home digital tools use (e.g., mathematics apps) is a central measure for studying DHNP. Such a measure can be obtained via standard self-report data but can also be captured via operating-system-level software, available on most modern devices, that tracks the frequency and length of software use on devices (e.g., iOS's Screen Time feature). Other aspects of this component provide insights into the nature of these uniquely digital opportunities. Understanding the frequency of DHNP is important because learning from devices is the result of ongoing interaction/communication with the device (CCM theory).
- *Operating system* refers software that runs devices contributes to different affordances (e.g., accessibility features provided by iOS are different from Android) and should be considered along with device type.
- *Device type*. Educational apps on touch-screen devices (such as a smartphone) provide very different interaction affordances than traditional PCs or even laptops (Segal, 2011), such as more embodied, gesture-based interactions (Dubé & McEwen, 2017). As new mathematics learning technologies are created, each with unique affordances (e.g., augmented reality mathematics apps like Photomath), it is important to account for the type of device providing the learning opportunity and not just how frequently it occurs.

Quality of DHNP As argued by the CCM, DHNP consists of an interaction between the child and the digital mathematics tool. Thus, the factors determining the quality of DHNP must consider both human and device factors.

Human Factors There are several factors related to parents and children which directly influence the quality of DNP. These are more than just the setting for the HNE (implicit components), but rather factors that are proposed to have a direct effect on the quality of DNP and may account for improvements in children's mathematics ability.

- *Parental involvement.* In the USA, 66% of families report engaging in weekly parent-child digital play sessions (Wartella, 2019) and these are opportunities for parental involvement in DHNP. The level of involvement will likely range from accompanying children's mathematics app use by playing together to simply monitoring their child's independent use. Parents' mathematics attitudes and anxiety (as already discussed in the DHNP outer model) may mediate parental involvement in DHNP and must also be considered.
- *Types of parental feedback*. Parental feedback (i.e., discussions about mathematics arising in the game) during or following app use may also shape children's digital home numeracy experience just as it does with AHNP (Vandermaas-Peeler et al., 2012).
- *Parents' comfort with technology.* Parents' role in children's digital practices is likely influenced by their comfort with technology (Hatzigianni & Margetts, 2014). If parents are comfortable with technology, they may provide more opportunities for DHNP and may be more likely to engage in them (i.e., provide feedback, play together).
- *Parental perceptions of mathematics app effectiveness.* Their involvement may also extend to their perceptions of children's digital device use, with some parents having a negative view of children's digital device use more generally (e.g., screen time, Delen et al., 2015). In contrast, many parents think that mathematics games are beneficial for their children's mathematics learning (Wartella, 2019); perhaps, because mathematics games are perceived as fun and effective (Cheng & Su, 2012). Parental perceptions of digital tools may also be influenced by their SES, technology knowledge, and previous exposure to digital devices. So, understanding DHNP must account for the role of parents.
- *Reasons for using mathematics apps at home*. In addition to parental factors, researchers should investigate the reasons for using mathematics apps at home. Possible reasons could include teachers assigning apps as homework, parents giving mathematics apps to children as a form of distraction or education, and/or children seeking out DHNP themselves for practice or entertainment (i.e., playing games; Oliemat et al., 2018). Who is providing the opportunity and why could account for varying frequencies of direct and indirect DHNP and impact children's propensity?
- *Children's mathematics app skills.* Children's' capacity to independently engage in and take advantage of DHNP may also depend on their level of digital expertise or children's mathematics app skills. As such, children's digital device capacity or skill should be considered as a primary propensity (see Byrnes & Miller, 2007) that mediates their DHNP exposure. Children who understand the affordances (e.g., what actions are possible) of a specific game may engage in more effective communication with the digital devices, such as selecting the appropriate gesture (e.g., Wang et al., 2021), and completing mathematics apps efficiently with fewer interaction mistakes (Dubé & McEwen, 2017; Moyer-Packenham et al., 2016). They do not need their parents to initiate the DHNP experiences or not even require adult guidance.

Digital Factors Besides human factors, app factors are also important— the digital aspects of DHNP. There are several digital factors that may affect the quality of DHNP. They include app selection, app type, app curriculum, and app support.

- *App selection.* From the 200,000+ educational apps currently available (Apple, 2019), parents, teachers, and/or children must choose which mathematics apps to download. Selecting mathematics apps with learning benefits, that is quality apps, is obviously important; however, the task of identifying and selecting quality apps can be difficult (Yusop & Razak, 2013), and little is known about how people select apps. When searching for mathematics apps in an App Store, the app selection process is informed not only by their children's learning needs but also how App Stores are designed and the information they provide (Dubé et al., 2020; Vaala et al., 2015). For each app, App Stores provide textual information on price, consumer reviews and ratings, the popularity of the app via download rates, and written description of the app; as well as visuals depicting the app including pictures and videos. App selection could be influenced by any combination of these factors (e.g., selecting visually appealing top-rated apps without reading written description). Eye tracking research shows that educators look at written descriptions of apps more than visuals and other text information (Pearson et al., 2021), but nothing is known about parents' app selection process.
- *App types.* There are a variety of mathematics apps available, with the most common being videos, e-textbook, tutoring, and games. Mathematics *videos* provide non-interactive, multi-sensory learning experiences that explain or demonstrate mathematics concepts or procedures to learners (Carr, 2012; Outhwaite et al., 2019; Pavio, 1986). *E-textbooks* are multimedia digital books (i.e., video, audio) that can include limited interactivity (e.g., interactive quizzes, 3D images, graphs, Van Horne et al., 2017; Rockinson-Szapkiw et al., 2013). *Tutoring* apps are interactive drill-and-practice or quiz activities (Kaur et al., 2017) that often provide immediate feedback and virtual reward systems (e.g., points). *Games* are interactive, goal-oriented activities (Dubé & Keenan, 2016) in which learning content is transformed into a play behavior and used to complete the goal of the game (e.g., beat the boss by solving a fraction puzzle, Bedwell et al., 2012; Kafai, 1996). Games provide a unique way to teach mathematics by intrinsically integrating the learning contents (i.e., curriculum) into the game activities (Kafai, 1996), a special feature that makes games unique from other types. The nature and level of interaction differs both quantitively and qualitatively between each type of Mathematics app; therefore, the study of DHNP must consider app type when describing the HNE.
- *App curriculum.* An app's curriculum refers to the mathematics topics it contains. Generally, a mathematics app's curriculum tends to focus on a specific mathematics topic (e.g., apps just for arithmetic, measurement, geometry, spatial reasoning, Dubé et al., 2020). However, it is possible for a single mathematics app to contain more than one mathematics topic and build multiple mathematics skills. It is more common for videos, e-textbooks, and tutors to cover multiple mathematics topics than games. This occurs because different mathematics top-

ics lend themselves to different game types that can be hard to combine into a single app. Also, mathematics game curriculums tend to focus on elementary to middle school mathematics topics while there are more video, e-textbooks, and tutor apps for mathematics topics typically taught to adolescents. Thus, mathematics apps with topics appropriate for children in middle childhood are more likely to be games than other app types and this means that the DHNP of middle childhood may be comprised of more indirect than direct HNP. Not by a choice the parents make, but by virtue of the apps available.

- *App supports.* Design features from effective educational media (i.e., manipulatives, feedback, scaffolds, Hillmayr et al., 2020) should be incorporated into mathematics apps so that they properly support children's learning. For example, some mathematics apps provide hands-on or physical experiences (such as tapping or dragging blocks, dice, 3-D shapes, graphs, cards) by including virtual manipulatives to help children understand math. One such app is virtual manipulative by abcya.com. In the app, interactive visual representations of percentages, fractions, and decimals are provided to help children 8+ to visually understand the relationships among these mathematics concepts through finger tapping, pressing, and dragging. Digital feedback and scaffolds can also be included in mathematics apps by providing hints and guidance based on their mathematics performance. Researchers have identified a broad range of feedback (Cayton-Hodges et al., 2015) and scaffold types that digital learning tools can provide (Lajoie, 2014). Research suggests that the majority of the top 90 mathematics apps in the Apple App store do not contain these features (Dubé et al., 2020). Future DHNP studies must determine whether parents are selecting apps containing these features for use in their homes.

7.4 How Does the Proposed DHNP Model Contribute to Middle School Mathematics Education?

The DHNP model combines various components of HNE, and it also integrates several digital components, which is a unique contribution in the field of home numeracy. Furthermore, the proposed model is based on the principle that digital home numeracy is an individualized learning tool that is directly influencing children's learning (Hillmayr et al., 2020). This proposition calls for an avenue of new research to investigate how digital technologies are already being used by children as a mathematics learning tool in the home, how effective this form of numeracy practice really is, and which factors impact children's m mathematics learning the most. Now, why should we answer these questions? Generally, learning progress becomes optimized when there is a strong educational alliance between school, home, and community (Groves et al., 2006; Nokali et al., 2010; Vincent et al., 2005). If children are learning at home, they can eventually transfer their acquired knowledge from informal (i.e., home) to formal context (i.e., school) and thus obtain better mathematics performance in the future; however, empirical evidence on DHNP is required.

7.5 Potential Avenues for Practical Implications on DHNP

What is the importance of DHNP for researchers, parents, and teachers? Researchers need to identify how DHNP helps children's mathematics learning, whereas parents and teachers need research-backed guidelines to make DHNP more efficient for their children and students. Currently, there are no formal guidelines for parents and educators on how to properly implement digital mathematics apps in the classroom or home (Dubé & Dubé, 2020). The components of the DHNP model can serve as the basis for such guidelines. Based on the previous discussion the following theory-driven guidelines are proposed.

1. DHNP encompasses both indirect/'fun' mathematics apps like games and more direct/'traditional' forms of digital mathematics practice.
2. The quality of DHNP must be considered alongside quantity.
3. Quality of DHNP is affected by more than just the apps themselves, parental and child factors are also important.
4. Parents should be involved in their child's DHNP, give children feedback on their app use, and be comfortable with fundamentals of digital app use.
5. Children vary in their mathematics apps skill; some children may need more support than others.
6. Not all mathematics apps are equal:
   (a) There are different types of mathematics apps, ranging from interactive games to passive educational videos, and selecting appropriate ones takes effort.
   (b) Some apps are more aligned with the school curriculum than others.
   (c) Select apps that contain a range of in-app supports, such as feedback, virtual manipulatives, and scaffolds.

Empirical research is necessary to evaluate the efficacy of these guidelines. Preliminary data from Canadian parents of middle school children suggests that parental involvement in DHNP is related to children's mathematics ability (Alam & Dubé, 2022, 2023). Further research is needed to understand how the various DHNP components impact children's mathematics ability and to test if these theory-driven guidelines are empirically supported.

7.6 Summary

This chapter proposed a DHNP model containing outer and inner components. Existing research is available on the outer components of the DHNP model (i.e., HLE, HNE). In contrast, the DHNP inner model is a novel framework in need of empirical support to grow and/or refine its components. We hold that the DHNP model can be used to study how digital mathematics experiences shape children's mathematics knowledge and the underlying factors which influence those experiences. Going forward, we would argue the DHNP model should be investigated as a holistic approach to understanding the home learning environment. Exploring the DHNP model during elementary and middle school years is crucial as research on the impact of digital technology during this developmental phase is less prevalent (Blumberg et al., 2019). The next step is for researchers to test and refine this DHNP model or propose contrasting models that can guide researchers, educators, and parents' understanding of how an increasingly common home numeracy activity is impacting children's mathematics learning in the elementary and middle school years.

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Chapter 8
How Number Talks Assist Students in Becoming Doers of Mathematics

Dawn M. Woods

Abstract Number talks are discussions where teachers encourage their students to mentally solve mathematics problems and then come together as a class to share their mathematical reasoning. As students share, listen, and discuss their solution strategies, they begin to make connections between how procedures are the same, different, and/or more efficient. In this chapter, I explore how a teacher leverages number talks to support students in becoming doers of mathematics. Findings from this study reveal how the teacher supported students to (a) develop agency, (b) distribute authority, and (c) share mathematical reasoning. Further, it was found that mental computation played an important role since it supported students to discover ingenious, effective, and efficient ways of solving mathematical problems.

Keywords Number talks · Doers of mathematics · Agency · Authority · Mathematical reasoning · Sociomathematical norms · Mental computation

8.1 Introduction

Number talks are five-to-fifteen-minute discussions where a teacher encourages their students to mentally solve computation problems and then come together as a class to share and discuss their mathematical reasoning (Parrish, 2010/2014). Typically, this whole class discussion progresses through five phases where the teacher: (a) sets the stage by supporting students to enter into the discussion in a way that makes sense to them, (b) launches the discussion by sharing a mathematical representation (i.e., mathematical expressions, equations, models) and provides students with enough time to mentally solve it, (c) gathers student responses, (d) facilitates a whole class discussion where students' share mathematical reasoning,

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and (e) summarizes key ideas and conclusions (e.g., Humphreys & Parker, 2015; Parrish, 2010/2014).

During number talks, students productively struggle with number relationships, apply their new understandings to computation strategies, and then discuss and analyze their reasoning. For this chapter, mathematical reasoning is defined as an evolving process where students conjecture, generalize, and investigate mathematical ideas (Lannin et al., 2011; Russell, 1999). As students engage in mathematical reasoning during number talks, they have opportunity to become doers of mathematics, characterized by seeing sense in mathematics, perceiving it as useful, and believing that steady effort in learning mathematics pays off (Aguirre et al., 2013; Jackson, 2009; Martin, 2009; National Research Council [NRC], 2001).

Because of the explicit focus on number relationships and number theory, practitioner focused articles have considered number talks to be an instructional practice that improves students' number sense, mental mathematics, and mathematical reasoning abilities (Gerstenschlager & Strayer, 2019; Humphreys & Parker, 2015; Parker & Humphreys, 2018; Parrish, 2010/2014, 2011; Parrish & Dominick, 2016; Sun et al., 2018). While these practitioner focused articles are helpful in how understanding how number talks may support students in becoming doers of mathematics, little empirical research explores how the classroom community is developed to support number talks. For these reasons, in this chapter, I share the story of how Ms. Jones (pseudonym), a third-grade teacher, uses number talks to assist students in becoming doers of mathematics.

8.2 Conceptual Framework

I begin by describing how the situative perspective is a lens to view how Ms. Jones and her students participated in learning centered on number talks. Then, I discuss the role mental computation plays in number talks and conclude by highlighting the importance of developing sociomathematical norms to support students in becoming doers of mathematics.

8.2.1 A Situative Perspective on Knowing and Learning

A situative perspective is a lens to understand how students learn to become doers of mathematics as they actively participate within a learning community centered on number talks (Greeno, 1998, 2006; Lave, 1988; Lave & Wenger, 1991). As students in Ms. Jones' class engage in number talks, they not only learn how to build relationships that enable them to think with each other (Wenger, 1998), but also have opportunity to develop productive dispositions that support them to become doers of mathematics (Aguirre et al., 2013; Jackson, 2009; Martin, 2009; NRC, 2001). In this space, Ms. Jones builds habits where students are encouraged to try new strategies and receive social support for their efforts within their learning community. Because Mrs. Jones takes time to explicitly build these habits, the distribution of authority–who's in charge of making mathematical contributions–shifts (Cobb et al., 2009; Engle & Greeno, 2003). As students engage in higher order thinking with others, they realize that they have conceptual agency or the ability, permission, and obligation to exercise control over their learning process and view themselves as an effective learner and doer of mathematics (Cobb et al., 2009; NRC, 1987).

8.2.2 The Role of Mental Computation During Number Talks

Number talks begin with students mentally solving a computation problem (Parrish 2010/2014). In this space, students are encouraged to work on a problem in their heads (e.g., solving a mathematical problem by mirroring traditional strategies or using a nontraditional procedure) in order produce an exact answer (Reys, 1984; Sowder, 1988; Trafton, 1978). In Ms. Jones' class, students select a strategy that is most efficient and effective for them (Blöte et al., 2000; Torbeyns & Verschaffel, 2016) since the computation problem is problematized to support sensemaking (Hiebert et al., 1996). Because students are mentally solving a problem, prior research confirms that this process provides students with an opportunity to (a) develop conceptual understanding into why algorithms work, (b) advance creative and independent thinking, (c) improve problem solving skills, and (d) advance computational estimation skills (Reys, 1984; Reys et al., 1995).

Once students mentally solve the computation problem, Ms. Jones leads a whole class discussion supporting students to share what they know and understand about the many ways to solve the problem. This opportunity to share and listen to each other's solution strategies has the potential to build mathematical capacity. As students share and listen to the different ways of solving the problem, Ms. Jones builds mathematical vocabulary, highlights mathematical strategies, and supports students in connecting how nontraditional and traditional procedures are the same, different, and/or more efficient when solving problems mentally. Here, students glean an understanding of the structure of number and their properties because of the varied procedures invented by students. As students listen and make connections, they are emboldened to create innovative ways of manipulating numbers (Cobb & Merkel, 1989; Sowder, 1992). Hence, the role of mental computation during number talks may support students in becoming doers of mathematics since they are actively solving, sharing, listening, and making mathematical connections.

8.2.3 *Developing Sociomathematical Norms for Doing Mathematics*

For number talks to assist students in becoming doers of mathematics, the teacher takes on the role of a learning coach supporting students to explain and make sense of their own and their peers' mathematical thinking. As part of this role, the teacher attends to the development of sociomathematical norms so that generative learning occurs (Yackel & Cobb, 1996). Research characterizes four sociomathematical norms that are particularly salient for supporting conceptual thinking. These sociomathematical norms are: (a) explanation is not a procedural description, but consists of a mathematical argument, (b) understanding the relationships between multiple strategies supports mathematical thinking, (c) mistakes provide opportunities to reconceptualize a problem by exploring alternative strategies, and (d) collaborative discussion involves individual accountability and using mathematical argumentation to reach consensus (Kazemi & Stipek, 2001).

Consequently, as teachers develop these sociomathematical norms in their classrooms, number talks shift from a focus on correct answers and sharing procedural descriptions to drawing mathematical connections between different solution methods (Ball, 1993; Stein et al., 2008; Wood & Turner-Vorbeck, 2001). In this space, discussions also include deliberation on which strategies might be most accurate, efficient, and/or flexible (Nathan & Knuth, 2003). Since students' mathematical thinking is made visible, taken-as-shared understandings are developed and students gain autonomy as they fully participate within the community of practice (Lave & Wenger, 1991; Wood et al., 2006; Yackel & Cobb, 1996).

8.3 Study Context

Ms. Jones was a third-grade teacher who had 11 years of experience and taught in a large suburban district in the southwestern United States. During this study, Ms. Jones engaged in number talks to leverage student-to-student talk in ways that supported them to become doers of mathematics. Over the course of a 15-week period, her students (n = 12) took ownership of their learning as Ms. Jones built habits that supported her students to listen, try new strategies, share their thinking, and learn how to accept support from others. Hence, the focus of this study was to understand how Ms. Jones leveraged number talks to build a community that assisted her students in becoming doers of mathematics. A secondary focus was to explore how mental mathematics may have played a supporting role in assisting students in reasoning about mathematics.

A variety of data were collected to document how number talks assisted student in becoming doers of mathematics. Over 15 weeks, number talk observations ranging from eight to 28 min were digitally recorded and transcribed. A second data source was participation in a weekly Number Talk Club (NTC) where five third-grade teachers collaborated on leveraging number talks to develop learning communities supporting students in doing mathematics. These NTC sessions ranged in length from 25 to 52 min, were digitally recorded, and transcribed. A third data source was a modified version of the Vision of High-Quality Mathematics Instruction (VHQMI) interview protocol (Munter, 2014). This interview protocol was conducted at the beginning and the end of the study to track changes in the discourses teachers employed to describe ideal classroom practice (e.g., Hammerness, 2001). These interviews were digitally recorded and transcribed. Alternating between number talk observations, NTC sessions, and interviews, deductive (i.e., theory, hypothesis, observation, and confirmation)/inductive reasoning (i.e., observation, pattern, tentative hypothesis) was used in a circular approach to emphasize observations and to understand how number talks (and mental computation) supported students to become doers of mathematics (Dubbels, 2011; Hutchins, 1995).

8.4 How Did Number Talks Assist Ms. Jones' Students in Becoming Doers of Mathematics?

From the beginning of the study, Ms. Jones demonstrated that she was an experienced and reflective practitioner with a vision of mathematics instruction that supported developing mathematical practices such as problem solving, reasoning and proof, communication, representation, and mathematical connections in her students (Common Core State Standards for Mathematics, 2010). During the initial interview, she revealed that mathematics instruction begins by *hooking* students so that they are interested, as well as offering hands-on explorations (with tools and/or models) in ways that problematized mathematics as relevant and meaningful for students. She believed that the role of a teacher was to be a facilitator, using formative assessment (i.e., pre- and post-testing) to drive instruction, while purposefully employing questioning strategies to check students' understanding of the mathematical content. Yet, missing from this vision of mathematics instruction was how students developed productive dispositions that support them to see and make sense of mathematics.

Because of this, three dilemmas surfaced during NTCs where Ms. Jones, supported by her colleagues, replayed, rehearsed, and re-visioned (c.f. Horn, 2010) how to use number talks. These dilemmas were: (a) "I'm hearing just the same stuff", (b) how can all students "feel a little bit safer" to share their mathematical thinking, and (c) "some of them still struggle with explaining their reasoning." Therefore, the following sections make visible how Ms. Jones re-visioned these dilemmas to provide her students with opportunities to become doers of mathematics.

8.4.1 Building Agency by Establishing Sociomathematical Norms

As Ms. Jones and colleagues talked about number talks during week three of the study, Ms. Jones expressed, "I'm hearing just the same stuff" regarding the strategies elicited from students during number talks. She explained how number talks were a "show and tell of traditional strategies" and wondered what approach she could use to support students to share different ways of solving the computation problems. In addition, she wondered about how to provide opportunities for students to develop an understanding of what numbers mean and an ability use numbers flexibly to make comparisons and perform operations (Berch, 2005; Gersten & Chard, 1999). As Ms. Jones and colleagues analyzed these concerns, they concluded that number talks should be a space where mental computation encouraged students to move away from using traditional methods as they selected strategies (i.e., decomposition, place value, standard algorithm, compensation) that were most efficient and effective for them. They also determined that when students listened to each other and then tried out a "new to them strategy" or when mathematical connections were made between the strategies that number sense was being developed.

Equipped with these new understandings, Ms. Jones' established the sociomathematical norm of exploring new strategies. Evidence of this was heard during a number talk later in week three. She stated to the students,

> When you have a solution, go ahead and give me a thumbs up and start preparing in your head your way. Now remember mathematicians, we talked earlier this week about pushing yourself to try out a new strategy. Maybe one that is a little less familiar, but you've heard another friend use.

In this moment, Ms. Jones invited students to extend their mathematical thinking by trying out a new strategy. Next, she wrote the mathematical expression 62–16 on the board and students followed the established number talk routine by solving the problem mentally. After students shared their first-time answers, ranging from 46 to 54, Ms. Jones asked the students to turn and talk with a partner and to listen in way where they would be able to share "the method that your partner came up with." Not only was this a new feature of the established number talk routine, but it also provided students with the opportunity to listen for understanding in a less public space as they practiced and clarified their mathematical contribution (Chapin et al., 2013).

After a few moments of partner talk about solving 62–16, Ms. Jones began the whole class discussion phase of the number talk. In the following vignette, notice how Ms. Jones supported students (all names are pseudonyms), in trying out a strategy that was new for them.

1. Ms. Jones: Well, let's go ahead and let Charlie build on what you started. You started how he solved it and then why don't you talk to us about why or how you started this way?
2. Charlie: Um, I did it because Sai did it and I wanted to try it.

3. Ms. Jones: Okay. You never tried it before, and you tried it like Sai had done before. Okay. So, tell me, what did you do next?
4. Charlie: I did six minus one which equals five.
5. Ms. Jones: Hold on really quick. So, you did six minus one. What is the six? What does that mean?
6. Andy: Sixty.
7. Ms. Jones: [To Andy:] Thank you for adding on.
8. Charlie: So, 60 minus 10 and then that equals 50.
9. Ms. Jones: Okay.
10. Charlie: And then I added back the four to 50.
11. Ms. Jones: You added back the four to 50, like this? [Modeling it on the whiteboard.]
12. Charlie: Uh-huh, and that equals 54.
13. Ms. Jones: Yes. I like, I like that. You tried it like another student in the room. We talk about when we write, we can use the strategies of other authors. You just used the strategy of another mathematician! How neat. So, let's continue with this and let's see what other solutions we came up with or what other ways.

Although Charlie's answer was incorrect, this vignette illustrated how the sociomathematical norm of exploring new strategies was taken up by the class. Here, Ms. Jones did not hear "just the same stuff", as students took risks to share their peers thinking, as well as to try out a strategy that was new to them. For example, in line 2, Charlie revealed that his strategy was like one that Sai shared the other day, thereby providing evidence that Charlie was stretching himself to try out a new and maybe a more flexible way to subtract two numbers.

Also evident in this vignette is that Ms. Jones made mathematics visible as she focused the conversation to build taken-as-shared meanings that could further support her students to stretch themselves to try out a new strategy. For example, in line 5 she paused Charlie's explanation to make mathematics visible by stating, "Hold on really quick. So, you did six minus one. What is the six? What does that mean?" Here, she requested for Charlie to acknowledge the connection between his explanation and the (missing) place value of the digits so all students could understand his thinking. The vignette also revealed that at least one other student, Andy, was following Charlie's explanation since he interjected, "Sixty" in line 6.

Notably, the error that was shared by Charlie was not ignored, but instead became an opportunity for students to explore differences within solutions. An illustration of how errors became opportunities began as Tucker, who was also trying out Sai's strategy, realized that he made a computational error as his partner Juliana was explaining his solution strategy. He interjected, "I did two minus six, which I probably got that one wrong. It is actually negative four, but I accidentally made it negative six." Then, Tucker continued by explaining his revised strategy, "I did 60 minus 10 which equals 50. And 50 minus four would equal, would equal to 46." In this moment, not only did Tucker revise his thinking, but this exchange also provided Charlie with the insight into his error about how the negative four should be subtracted and not added to the fifty. As the remainder of the number talk unfolded, there was evidence that at least three other students pushed themselves to try out Sai's strategy resulting in answers ranging from 44 to 46.

Providing an opportunity for all students to make mathematical connections during the number talk, as well as to try out a new strategy, Ms. Jones asked for students to share strategies that were different from Sai's. Here, Ms. Jones called on Bhavna to share a strategy that could help students to make mathematical connections.

1. Bhavna: I got 46. I decomposed the 62 to a 60 and a two.
2. Ms. Jones: Okay.
3. Bhavna: I did 60 minus 10. That equals 50.
4. Ms. Jones: How did you get the 10?
5. Bhavna: From the 16.
6. Ms. Jones: What did you do?
7. Bhavna: I decomposed the 16 to a ten and six.
8. Ms. Jones: Okay.
9. Bhavna: I did 60 minus 10 and that equals 50. Umm… [pause]
10. Ms. Jones: Does anybody want....
11. Bhavna: Oh, yeah...
12. Ms. Jones: You got it. Okay.
13. Bhavna: I did 50 minus six and that equals 44. [pause] I did 44 plus two which equals to 46.
14. Ms. Jones: Okay. Do I have anybody that wants to agree or disagree with this answer? And tell me why? I now have two solutions for 46. So, do you agree or disagree with this and why? Elijah?
15. Elijah: I agree because she decomposed that and then she added back the two from, because she never subtracted it, so she added back the two from the 60.
16. Ms. Jones: She added back the two from the 60. All right, Sofia, do you agree or disagree?
17. Sofia: I agree because... Umm [pause].... I just agree.
18. Ms. Jones: You just agree, you just think she was correct. Okay. Did anybody have an additional way before we start unpacking this just a little bit more? Okay. Because we've got a lot of answers. We tried a lot of new ways, which was super, super awesome.

During this exchange, Bhavna shared a different, flexible solution strategy resulting in the answers of 46. Next, in line 14, Ms. Jones invited students to reason about the answer of 46 since this was the answer that was emerging as correct. In this moment, Elijah, in line 15 offered up a claim and evidence to why 46 was the answer. First, he claimed that he agreed with Bhavna. Then he stated evidence, "she decomposed that and then she added back the two from because she never subtracted it, so she added back the two from the 6". Although Elijah did not connect the evidence back to the claim with a warrant, he was beginning to engage in mathematical argumentation to convince other members of his learning community that 46 was the true answer.

Before the number talk concluded, two other students shared different strategies based on place value confirming that the answer was in fact, 46. Then Ms. Jones summarized the number talk process as she reflected out loud with her students attentively listening,

> I really pushed you on your thinking on that one. I can tell, because you got a little uncertain, but you know what? That happens when we're trying a new strategy. It's not always going to work for us the exact first time we try it. But what you guys did was you communicated, you got creative with your solutions and some of you tried some things you hadn't done before.

In this moment, Ms. Jones realized a shift as she shared what she noticed during the number talk. Here, she not only reflected on how students creatively pushed themselves to explore new strategies, but how they also explored differences in strategies, claimed errors, and used math talk as they made important mathematical connections.

A salient aspect of the dilemma, "I'm hearing just the same stuff", was for Ms. Jones to navigate how to support her students' shifting roles as they became doers of mathematics. In this, and future, number talks Ms. Jones decentered herself so that her students could try out a new strategy and listen to each other share their ingenious ways of handling numbers. Because of this, authority (i.e., who's in charge of making mathematical contributions) became jointly distributed between students and the teacher (Cobb et al., 2009). This shift in authority lead to students exercising conceptual agency or control over their learning process; a key factor in mathematical learning (Cobb et al., 2009; Boaler & Greeno, 2000). Further, it could be argued that mental computation played an important role since it supported students to generate creative ways of handling numbers (Cobb & Merkel, 1989; Reys, 1984; Sowder, 1992) by using methods that were most efficient and effective for them (e.g., Blöte et al., 2000; Torbeyns & Verschaffel, 2016).

8.4.2 Shifting Authority Through Small Group Number Talks

As the NTC in week 7 began, Ms. Jones revealed that she wanted to modify her number talk routine because she noticed that some students may not have the space to share their mathematical thinking. Thinking aloud, she re-visioned how she could continue to shift authority so that all students felt comfortable talking during number talks. She concluded, "that way I can kind of sit with some students, and other groups could just do their number talk. That way they [those I sit with] feel a little bit safer." Other members of the NTC added on as they reflected on the dilemma of who was in charge of making mathematical contributions.

Equipped with this new idea, Ms. Jones tried small group number talks to support students in developing confidence while continuing to distribute authority among students so that they felt safe to make mathematical contributions. During week 8, the students mentally solved 70–59 = 70–60 + 1 to determine if it was true or false. Then, students discussed their solution strategies in small groups of three or four. Here, they asked each other questions about their answers and defended their strategies as Ms. Jones guided from the periphery while supporting quiet students to interject their thinking. After small groups had time to share and defend their answers, Ms. Jones transitioned from small groups to a whole class number talk. In this space, small group leaders summarized by sharing a strategy that was new to them, consensus on true or false, and mistakes that were claimed. For example, one group shared,

1. Small Group Leader: He did 59 and he rounded it to 60 and then he did 70 minus 60 equals 10. And then he did 10 plus one, which equals 11. And then he did 70 minus 60 again and got 10, and then 10 plus one equals 11 and he got true.
2. Ms. Jones: Okay. Did the rest of your group members agree?
3. Small Group Leader: Yes. Everybody but Melissa. She got false and said she thought she messed up.
4. Ms. Jones: Okay, what got her confused?
5. Small Group Leader: She did 59 plus one, which equals 60. And then she did 70 minus 60 equals 10. And then she did 70 minus 60 again, which equals 10 plus one, which equals 11.

This moment revealed that as students shared their thinking during small group number talks, they had opportunities to tell their mathematical ideas, as well as agree and disagree about them. In fact, these conversations shifted into a process where students defended an answer and investigated mathematical ideas together. Further, this process supported students to enact the sociomathematical norm of claiming mistakes as they engaged in mathematical argumentation to reach consensus.

During the next NTC in week eight, Ms. Jones replayed what the students said about the small group experience: “I said, ‘What’d y’all think?’ They’re like, ‘This is awesome. Everybody got to talk in the smaller group.’ My whole goal was to get all students talking. So, in a smaller group, it did!” In this replay, Ms. Jones shared that her students not only took-up but were excited about the opportunity to share their mathematical reasoning during small group number talks.

Taken together, these moments revealed that small group number talks felt “a little bit safer”. As Ms. Jones guided small group number talks from the periphery, she provided an extra layer of encouragement and support so that students felt comfortable in making mathematical contributions. Further, small group number talks provided opportunities for students to share in progress thinking and claim mistakes, while revising their mathematical thinking as they discussed the mental computation problem.

8.4.3 Learning How to Share Mathematical Reasoning During Whole Group Number Talks

As the teachers reflected on what they noticed about their students' mathematical thinking during the NTC in week 13, Ms. Jones revealed, "They're starting to say, 'I would like to add on to that", or "I think she's saying…" without prompting. Yet, she noted "they're getting better at sharing it [their strategy, but] some of them still struggle with explaining their reasoning." Since conjecturing, generalizing, and investigating mathematical ideas is the essence of mathematical activity (Lannin et al., 2011; Russell, 1999), Ms. Jones wanted to support students to engage in reasoning, specifically argumentation. In a mathematics classroom, argumentation is typically defined as a line of reasoning where students defend why their answers are true (Sriraman & Umland, 2014) for the purpose of concept development (Stapes & Newton, 2016). To this end, Ms. Jones wondered if sentence stems such as, "I agree with _____ because _____" or "I disagree with _____ because _____" would be an entry point into argumentation during a number talk.

In week 15, Ms. Jones introduced the sentence stems designed to promote reasoning and then displayed the number talk 68 + 36 = 104, as represented by a strip diagram on the board. Since she was continuing to shift agency by supporting students to feel a bit safer in sharing their mathematical thinking, the number talk on this day followed the routine of individual students solving the problem mentally and then providing evidence of their claims in small groups of three before the whole group discussion.

As the whole group discussion phase of the number talk began, Ms. Jones asked, "Would anybody like to defend an answer?" Hazel volunteered to defend her answer of 68 by providing evidence that she, "did 104 minus 36 and then I got 68. So, I checked my answer. I did 68 plus 36 and I got 104." Next, Elizabeth offered her argument.

1. Elizabeth: I would like to agree with Hazel and um ... I would like to agree with the answer 68 because I did my math thinking, I decomposed 104 and 36 and then doing all the math…
2. Ms. Jones: Could you come up here and show me that, really quick? How [did] you decompose 104 and 36? I hear some kids are impressed. Let's see how you did this. And what was the reason you did this?
3. Elizabeth: Because I couldn't solve the problem just by it being the big numbers. I couldn't solve it. So, I decomposed it to where it would help me.
4. Ms. Jones: Okay. So then talk just about what you did.
5. Elizabeth: And then I did 4 minus 6, which equals negative 2. And then I did zero minus 30, which equals negative 30. I did negative 30 plus negative 2 which equals negative 32. And then I did 32 minus 104.

6. Ms. Jones: 32 minus 104? Or ...
7. Elizabeth: Wait, let me think.
8. Ms. Jones: I see some shared thinking (referring to the shared thinking hand signal that many students were doing in support of Elizabeth's thinking). I see some friends that are thinking that you did something that they did.
9. Elizabeth: I mean, and then I did 100 minus 36.
10. Ms. Jones: 36?
11. Elizabeth: 32.
12. Ms. Jones: Okay. And you got ...
13. Elizabeth: And I got 68.
14. Mrs. Jones: Okay, so you're just showing us a different strategy for how you solved for that?
15. Elizabeth: Yes. I got the same answer in a different way.

During this exchange, Elizabeth established a mathematical argument, supported by a claim, evidence, and warrant. In line 1 she made a claim that she agreed with Hazel. In line 3, she shared that her evidence was different than Hazel's and then provided her thinking to support her claim. Even though the evidence provided was messy (an error in subtraction in line 5), it was an attempt at reasoning about a mathematical idea. Then in line 15, Elizabeth concluded with a warrant which connected her evidence back to the claim that she agreed with Hazel.

When Ms. Jones realized that "some of them still struggle with explaining their reasoning," this provided the opportunity to revise what explaining mathematical thinking could sound like during number talks. Students, like Hazel and Elizabeth, were actively working on establishing a mathematical argument by making claims and providing evidence to justify the claim. Although the warrants in these data typically did not take the form of generalizations or connections to rules that made the evidence true, students were sharing their mathematical thinking in ways that built engagement in mathematics.

Recall from Ms. Jones' initial interview that the development of students' productive dispositions that support them to see and make sense of mathematics was missing in her vision of mathematics instruction. Yet, these findings suggest that Ms. Jones took steps to develop students' agency and shift authority, to support students in sharing their mathematical reasoning during number talks.

The final interview confirmed these shifts as Ms. Jones shared how students should take ownership of their learning as they "facilitate themselves", meaning that students did the work of mathematicians as they mentally solved, clarified, and reasoned about solution strategies together. She stated,

> If kids are not able to communicate their math thinking, then you really don't know where they are at until you can get them to really explain it and talk through the process. As a teacher [a number talk] lets you see that they really grasp it.

For Ms. Jones and the students, number talks became a shared learning space where students made mistakes, pushed themselves to explore new strategies, and shared their mathematical reasoning. Therefore, it could be argued that because of these rich opportunities to learn together during number talks, students in Ms. Jones' classroom became doers of mathematics.

8.5 Discussion

Throughout the 15-week study, Ms. Jones noticed three main dilemmas hindering the way students participated in number talks: (a) "I'm hearing just the same stuff", (b) how can all students "feel a little bit safer" to share their mathematical thinking, and (c) "some of them still struggle with explaining their reasoning". With the support of colleagues during the NTC, she replayed, rehearsed, and re-visioned how number talks could become a generative learning space to support her students in sharing their reasoning during these mathematical discussions (Woods, 2018).

Ms. Jones realized different structures were needed to support students in building habits leading to productive dispositions about mathematics (NRC, 2001). In terms of "hearing just the same stuff", Ms. Jones deliberately promoted the sociomathematical norm of trying out a "new to them" strategy during number talks (e.g., Yackel & Cobb, 1996). As students took-up this norm and stretched their thinking to explore new and different strategies, Ms. Jones encouraged them to practice sharing, as well as actively listening, with a partner during turn and talks (Chapin et al., 2013). In this less public space students not only revised their thinking, but also heard "new to them" strategies and asked clarifying questions to aid understanding. As students shared, discussed, and refined strategies they discovered they had ability, permission, and obligation to exercise agency over their learning process (Cobb et al., 2009; NRC, 1987). Evidence of this was found in the variety of different strategies that students used over the course of the study that were different from the "same stuff" that they typically shared during number talks.

As students exercised agency, Ms. Jones decentered herself as students explored ingenious ways of handling numbers and shared their thinking. Because of this, authority or who's in charge of making mathematical contributions became jointly distributed between students and the teacher (Cobb et al., 2009). Yet, Ms. Jones realized that not all students had the opportunity to make contributions to number talks. Therefore, she developed a new support–small group number talks–so that all students had the space to share their mathematical thinking.

The goal of these small group number talks was to provide support to students who needed extra encouragement to share their thinking. In this space, students felt "a little bit safer" and shared their mathematical ideas, agreed and disagreed about them, and revised their thinking. Evidence of this was found in Ms. Jones' replay during the NTC in week eight. Here, she reported how her students were excited about the structure because "everybody got to talk in the smaller group", thereby shifting authority to make mathematical contributions among more students.

As more students participated during number talks, Ms. Jones realized that "some of them still struggle with explaining their reasoning." Because of this, Ms. Jones realized the need to explicitly focus on mathematical reasoning since it is the essence of mathematical activity (Lannin et al., 2011; Russell, 1999), as well as provided connections to content learning (Staples & Newton, 2016). In fact, some of Ms. Jones' students engaged in argumentation as early as week three when they stated a claim, defended why their answers were true, or agreed/disagreed with a solution strategy (Sriraman & Umland, 2014). Yet, Ms. Jones wanted to provide all students access to this mathematical activity. For this reason, she illuminated a path to argumentation using sentence stems. As students used statements such as "I agree with _____ because _____" during a number talk, they entered a space where they engaged in mathematical reasoning (Chapin et al., 2013) as they stated a claim, provided evidence, and a warrant.

In these data, rich moments of mathematical argumentation were just beginning. Often, arguments were missing warrants and if there were warrants, generalizations or connections to rules that made the evidence true was often missing. Yet, students were engaging in number talks differently by the end of the study. This could be because Ms. Jones noticed and wondered about how she could leverage number talks to support students in doing mathematics. First, Ms. Jones enacted sociomathematical norms such as trying out a "new to them" strategy (e.g., Yackel & Cobb, 1996). Second, she decentered herself so that authority could be jointly distributed so that students could exercise conceptual agency (Cobb et al., 2009). Third, she supported students to feel safer in making contributions, thereby further distributing authority among all students. I argue that these three moves created the space for all students to engage in mathematical reasoning since students discovered that they had an obligation to exercise agency and control over their learning processes (Cobb et al., 2009; NRC, 1987).

It can also be argued that mental computation played an invaluable role in assisting students to become doers of mathematics during number talks. Prior research revealed how mental computation provided students with moments to not only develop conceptual understanding into why algorithms work, but also advance creative and independent thinking (Reys, 1984; Reys et al., 1995). Further, mental math gave students permission to find ways to creatively handle numbers (Cobb & Merkel, 1989; Reys, 1984; Sowder, 1992). Throughout the study, Ms. Jones not only encouraged students to use mental math, but also to find methods that were most efficient and effective for them (e.g., Blöte et al., 2000; Torbeyns & Verschaffel, 2016), as they tried out different strategies.

8.6 Implications for Teaching and Learning

Although more and different types of data are needed to confirm that students developed productive dispositions that support them to become doers of mathematics, evidence provided in this chapter illustrates that students were generating their own understanding as they talked about different solution strategies during number talks. Noticeably, number talks became a shared learning space where students made mistakes, pushed themselves to explore new strategies, and shared their mathematical reasoning.

So, what are the implications from this study for teachers, like Ms. Jones, who want to empower their students to do mathematics? First, this generative mathematics learning space built upon the features of community already established within the classroom. For example, Ms. Jones had routines established, such as morning meeting, to address students daily social-emotional needs. Because of this, there was already a culture of care and safety established within the classroom on which to build a rich, mathematics-focused learning community.

Second, as Ms. Jones took an inquiry stance into her practice by noticing what the students were doing during number talks and then wondering how to leverage the routine to support deeper learning, the established community began to shift into a community of practice. In this space, students were encouraged to enact sociomathematical norms while receiving social support for their efforts within their learning community.

Third, as Ms. Jones shared authority with her students, there was a notable shift in agency – or students' ownership of their learning – as habits were built that supported students to listen, try new strategies, share their thinking and mistakes, and accept support from others. As students took ownership of their learning, Ms. Jones decentered herself (as small group number talks were established) so that students could engage in mathematical reasoning. This shift in agency and authority supported students in Ms. Jones' classroom to be doers of mathematics.

Because of these three implications, the daily 10-to-15-minute number talks became a shared learning space where students made mistakes, pushed themselves to explore new strategies, and shared their mathematical reasoning. Notable was how Ms. Jones took the time to develop a classroom community that ultimately shifted authority in ways where students could exercise conceptual agency. Because of these shifts, and the role mental mathematics played, it could be argued that as students reasoned about mathematics together during number talks, they were doers of mathematics.

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Chapter 9
Language Matters: Mathematical Learning and Cognition in Bilingual Children

Mona Anchan and Firat Soylu

Abstract Although many bilingual children receive formal training in their non-dominant language in the USA and other multicultural societies, educational programs tailored for the needs of bilingual children are scarce. Like in other areas of instruction, bilingual children face additional challenges when learning math, given the language divide between home numeracy and formal school environments. This chapter presents evidence-based recommendations for teaching math to bilingual and multilingual children in elementary and middle schools. To ground these recommendations in research findings, psychological and neural mechanisms of bilingual mathematical learning and cognition are discussed, as well as sociocultural issues and implications for classroom practice. To support bilingual children's math learning in their non-dominant language, we recommend allowing code-switching and other off-loading strategies, strengthening fact retrieval in both languages, incorporating the child's home and cultural contexts, instructing in their home language or finding online alternatives, providing culturally-relevant math instruction and feedback, and making connections between mathematics and children's everyday lives. We also discuss the need for changes in teacher training and educational policy-making in order to increase awareness about bilingual children's needs and to transition bilingualism from being a disadvantage in formal education to being a quality that can enrich and enhance children's educational experiences.

Keywords Bilingual math · ELL · Math education · Math cognition · Bilingual children · Bilingualism · Home language · Teaching recommendations

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9.1 Introduction

According to the 2019 national report about mathematical performance in USA, 16% of English-speaking children in fourth grade did not achieve basic proficiency in math; this, however, is not as alarming as the 41% of fourth grade children categorized as bilinguals who lack basic mathematical proficiency (National Assessment of Educational Progress, 2019). With 23% of the U.S. school population being bilingual and growing (U. S. Census Bureau, 2021), understanding how bilingual individuals process mathematics is especially relevant for effectively teaching mathematics and improving their mathematical learning outcomes. This is especially important to ensure mathematics instruction is accessible and equitable for a subset of those bilingual children classified as English Language Learners (ELL) who constitute 10% of the U.S. school population (U. S. Department of Education, 2020).

If mathematics test outcomes were to serve as a proxy for children's understanding of basic mathematics (as they often do in policy circles), then the tradition of teaching mathematics in English or another dominant cultural language often yields difficulties for bilingual children. When bilingual children do not have access to educational opportunities or resources to learn basic mathematics in their home language early on, lack of connections between English and their home language may lead to academic difficulties in the short-term and potential long-term setbacks such as fewer career prospects and a lower quality of life (Dowker, 2019; Shin, 2013). However, these contextual matters are rarely considered in neuroscientific studies designed to understand how the brain processes different mathematical tasks. This unintentional oversight thereby continues to exacerbate existing inequities. For example, if most mathematical cognition studies that utilize neuroimaging proceed with the unspoken assumption that language proficiency does not play a primary role in mathematical processing by not classifying their participants' language status (see a list of all neuroimaging studies in numerical cognition in the review by Peters & De Smedt, 2018), then resulting recommendations to improve academic outcomes would incorrectly presume a person's language status as inconsequential for mathematics teaching and learning. Even considering the few mathematical cognition studies that do examine bilingualism, bilingual participants are treated categorically without qualifying their exact proficiency levels in their two languages.

Therefore, Whitford and Luk (2019) suggest treating bilingualism as the dynamic experience that it is. Most bilinguals, regardless of their proficiency, often have to perform mental gymnastics (Kroll et al., 2015) to cognitively manage the language system in which they are operating by activating the language they are currently using while suppressing competing representations in their other language. However, the quality and efficiency of bilinguals' mental gymnastics depend on their past and current interactions. For instance, a child who learns English as a second language in a natural/unstructured environment with high exposure to that language (e.g., home, relatives, neighbors, media, etc.) will have a different proficiency level than someone who learns English as a second language in a more structured environment like school (Bedore et al., 2016; Ruiz-Felter et al., 2016).

Since a child's language proficiency and bilingualism experience varies based on the interaction between their two languages and various environmental factors, Whitford and Luk (2019) suggest considering how bilingualism impacts cognition across ages and different sociocultural factors by examining the interaction between language factors (exposure, background, proficiency) and a variety of cognitive (e.g., verbal and non-verbal IQ) and demographic (e.g., education, socioeconomic status) variables. Understanding how bilingualism affects experiences (of mathematical learning or otherwise) is especially important now, given the economic and social shifts towards operating in multiple languages in this increasingly globalized world (Surrain & Luk, 2019). Outside of research that directly studies bilingualism, there is also a call to report and treat a child's language exposure, proficiency, and demographic variables in all developmental studies due to its potential to be a hidden moderator (Byers-Heinlein et al., 2019). Like the research sector, it is important to provide similar evidence-based recommendations for teaching and learning purposes as well. To provide such recommendations to mathematics educators who work with bilingual children and adults, findings from various fields (i.e., psycholinguistics, mathematics education, cognitive neuroscience, educational policy) were consolidated with insights from mathematical cognition about bilingual mathematics learning in international contexts.

While the cognitive demands of bilingualism are ever present in a bilingual child's development (Whitford & Luk, 2019), these cognitive demands seem to unequally affect students' mathematics performance (Anchan & Soylu, 2021d), which could have downstream effects on their future numerical development. Understanding numerical development requires connecting theories and findings across different levels, and studying how environmental and cultural factors (e.g., language, home numeracy environment, socio-economic level), as well as biological factors (e.g., neural, genetic) contribute to the development of numerical skills.

In this chapter, we utilize the interdisciplinary lenses of educational neuroscience to present how bilingual children in the elementary and middle school years develop their mathematical thinking and learn mathematics (Han et al., 2019; Knox, 2016). The pragmatic epistemology of this framework allows an educational concern (i.e., poor mathematical performance in bilingual children) to define a course of action by drawing on interdisciplinary insights about children's language proficiency, their cognitive processes, and sociocultural factors related to their school and home environments. But extrapolating insights from multiple disciplines has its own challenges. For example, it is not always possible to keep terminology consistent while crossing disciplinary boundaries. Therefore, at the risk of compromising precision for a pragmatic cause, we have used the term 'bilingual' broadly in this chapter, only qualifying bilingual proficiency when reported by the original study. It is our hope that this lack of precision will advance research in the areas of bilingual mathematical cognition and mathematics education by prompting further conversations and discussions not just across academic disciplines but also among practitioners, administrators, and parents.

9.2 Biological and Cultural Evolution of Mathematical Skills

Given that human mathematical skills are to a large extent an outcome of recent cultural evolution, the human brain does not have dedicated systems that originally evolved to support mathematical cognition. Instead, similar to many other cognitive skills, mathematical cognition makes use of neural systems that evolved to support other functions (Anderson, 2010, 2014). Early studies focused on understanding the role of language, symbolic, visuospatial, and sensorimotor systems for fundamental mathematical skills, like numerosity estimation, subitizing, counting, and arithmetic (e.g., Dehaene, 1992). Across nearly 30 years of research, we learned that we share some fundamental mathematical skills with other animals, enabling estimation of physical and numerical magnitudes, but beyond that mathematical development strongly relies on body-based, visuospatial, symbolic, and verbal representations, which are embedded in sociocultural contexts where development takes place (O'Shaughnessy et al., 2021). But with the world becoming increasingly connected and globalized, evidence about bilingual brains is calling a fundamental assumption in these studies into question (Whitford & Luk, 2019). Can mathematical cognition findings about neural processing be extended to all populations when many of these studies did not account for individuals' language status (monolingual, bilingual, multilingual) or include it as a variable? And how does this affect how bilingual children are taught math? With this goal in mind, this chapter outlines evidence about brain development and mathematical processing in bilinguals, followed by some recommendations to integrate these findings in the mathematics classroom (or home) while teaching bilingual children.

9.3 Bilingual Brains Process Information Differently

While monolingual and bilingual brains both process languages and cognitive tasks efficiently, the brain networks carrying out similar tasks and the associated outcomes may differ (Anderson et al., 2018). The fundamental architecture and language processing mechanisms involved in the bilingual brain may not always be accessible or examinable in monolinguals (Kroll et al., 2015). This insight is reiterated in neuroimaging and behavioral studies that show monolingual and bilingual children and adults performing differently on similar tasks or using different brain networks (Bialystok, 1999; Bialystok & Martin, 2004; Anderson et al., 2018). For example, in an fMRI study, Anderson et al. (2018) compared monolinguals and English-French bilingual adults on verbal and nonverbal task-switching experiments. While monolinguals used 2 different networks to process the verbal and non-verbal tasks, bilinguals used a common network for both tasks. Other studies suggest that bilinguals have enhanced executive functioning skills such as attentional control (Bialystok & Majumder, 1998; Bialystok, 1999), mental flexibility

(Mielecki et al., 2017) and inhibitory control (Bialystok & Martin, 2004; Kroll et al., 2008; van Heuven & Dijkstra, 2010) as a result of juggling two languages.

9.4 Bilingual Mathematical Development

The implicit assumption in many elementary and middle school classrooms is that mathematics is learned in a language-independent way (Anchan, 2019). A growing body of neurocognitive evidence suggests otherwise. When examining mathematical problem solving in Turkish-German bilingual elementary school students, for example, Kempert et al. (2011) found language proficiency in the language of instruction/testing to be predictive of their mathematical performance. Bilinguals activate mathematical representations in both languages at all times. Therefore, they would either actively inhibit representations in one language while performing operations in the other (Kroll et al., 2008) or they would use some cognitive subprocess to choose one language over the other (Dijkstra & Van Heuven, 2002). As a result of these background processes, bilinguals may react significantly slower than monolinguals on some tasks. Juggling additional subprocesses (Kroll, 2008) may also lead bilinguals to make more errors on mathematical tasks, leading to some disadvantages in bilinguals. Venkatraman et al. (2006) fMRI-scanned English-Chinese bilinguals as they performed two arithmetic tasks—base-7 addition and percentage estimation—to study exact and approximate number processing. They performed the tasks in both English and Chinese where they were trained in one language and untrained in the other. Language switching effects were found in both types of number processing – approximate number processing (left inferior frontal gyrus [LIFG], left inferior parietal lobule, angular gyrus), and exact number processing (bilateral posterior intraparietal sulcus, LIFG) – suggesting that mathematical calculation (which depends on retrieval of mathematical facts) relies on verbal and language-related networks. Therefore, mathematical processing is not independent of language.

More specifically, mathematical retrieval, calculation, and performance seem to depend upon the primary language of mathematics instruction. When bilingual high schoolers were trained on multiplication and subtraction problems in one language (German or French) and tested in both languages, Saalbach et al. (2013) found cognitive costs related to language switching when language of arithmetic instruction differed from the students' frequently used language. Similarly, in a sample of 193 German-French bilinguals between the ages of 12–23, Van Rinsveld et al. (2015, 2016) found bilingual participants' language proficiency to be crucial for solving simple and complex addition problems. While extended amounts of practice in both languages helped bilingual participants to perform equally well on simple single-digit addition problems, this was not true for more complex addition problems that involved double-digit or larger numbers. The number words used to describe the numbers also made a difference in how bilinguals processed numbers. For example, a bilingual whose primary language is English may read 24 as “twenty-four” but a bilingual who primarily speaks German may read is as “four-and-twenty.” These small but significant differences seem to compound over time leading bilinguals to process mathematics problems faster and more easily in their first language. In the case of simple multiplication, Salillas and Wicha (2012) similarly showed that the memory networks established in a bilingual individual’s childhood does not affect their retrieval process in adulthood even if the other language is dominant.

While many of the previously mentioned studies support the notion that arithmetic facts are encoded in verbal memory in the language of mathematics instruction (Dehaene & Cohen, 1995), there is also evidence to suggest that bilinguals represent mathematical facts for each language separately (Campbell & Xue, 2001). Martinez-Lincoln et al. (2015) compared bilingual teachers’ arithmetic performance in their primary or secondary language of instruction. They found that when the teachers performed arithmetic in their primary language of instruction, they maintained a primary language advantage. When their performance in their teaching and non-teaching languages were compared using Event-Related potentials (ERPs, i.e., brain signals), teachers showed more efficient access to their language they taught in, regardless of whether it was their first or second language. This suggests that access to terms even in the secondary language of instruction improved with use and practice. Although this study was done in adults, it could have implications for mathematical learning in children.

Cerda et al. (2019) recorded ERPs in bilingual children as they verified the correctness of multiplication problems that were presented as spoken number words in Spanish and English blocks. Even though participants showed a language bias, they elicited comparable N400 amplitudes (i.e., brain response to encountering something unexpected that usually happens when the number presented does not match the expected answer) for both languages, which suggests similar cognitive processes in both registers at the semantic level. According to these adult and child studies, if a bilingual child’s development in both languages is almost balanced, disadvantages arising from mathematics instruction in their second language could be mitigated by increased functional use of their second language. This, however, may not be the case for bilingual children with partial or limited use of their second language such as English Language Learners. For such bilinguals, Van Rinsveld et al. (2016) found that providing contextual cues in their home language during instruction helped bilingual participants perform better in mathematics even in their second language. This means that despite their level of proficiency in both languages, bilingual individuals must learn mathematical facts in both languages to retrieve them at the same rate; otherwise, they will face cognitive costs when performing in their second language, because they would most likely retrieve facts in the language of instruction and translate them into the other language (Schwartz & Sprouse, 1996).

9.5 Insights from Bilingual Mathematical Education in USA and Other Countries

Research about bilingual mathematical education corroborates much of the cognitive findings about bilingual mathematical development presented earlier. For example, while examining the math learning success of Filipino-English bilingual children in the fifth grade, Bernardo and Calleja (2005) found that they were more likely to understand and solve word problems in their first language. This first language advantage was observed even in bilinguals whose first language was English (Bernardo, 2002). A parallel observation was documented outside the USA, where Clarkson and Galbraith (1992) found Papua New Guinean sixth grade students who were proficient in both their languages (English, and a second native language, e.g., Tok Pisin or Hiri Motu) scoring considerably higher on two different mathematical tests compared to their less proficient bilingual peers. Some bilingual students even performed better than their monolingual peers, despite the latter belonging to schools with more resources. Further examination led Clarkson (1992) to conclude that this may be due to Papua New Guinea's national policy promoting the use of students' original languages in school, allowing them to easily understand difficult mathematical concepts in classrooms. Planas and Civil (2013) compared students from Mexico in Tucson, USA, and students from Latin America in Barcelona, Spain. In both cases, the primary language of instruction were English and Catalan respectively. They showed that students' level of participation in the mathematics classroom depended on the language of instruction; low levels of participation were associated with instructing in students' non-home language. These studies from other countries support the recommendation to incorporate bilinguals' home language in mathematics instruction in U.S. classrooms as well.

9.5.1 Frequency of Language Use

Mathematical cognition models in sync with insights about word frequency (Ashcraft, 1992) and information processing (Anderson, 1983) state that when a language in which mathematics is taught and learned is used frequently, those mathematical facts will be stored and retrieved in that language most efficiently. Campbell and Clark's (1988) encoding-complex model of mathematical cognition also posits that each language has its own representation of arithmetic facts, and the rate of retrieving those facts is dependent on experience (Campbell & Epp, 2004). Since 'reaction time' or the time to access mathematical facts and solve problems is a primary measure of mathematical performance in schools, understanding how efficiently children retrieve their learned mathematical facts (which depends on experience and practice) is important for teaching mathematics to bilingual children. The frequency of language use seems to be important to children (Thordardottir, 2019) not only when they are learning mathematics but also while maintaining learned facts through retrieval-based procedures such as practice and problem-solving. In the next section, based on a review of bilingual and mathematical education research in the USA as well as other countries, we make evidence-based recommendations for math teachers and parents to reduce elementary and middle school bilingual children's cognitive load, supplement their math learning processes, and improve their math learning outcomes.

9.6 Evidence-Based Recommendations

9.6.1 Allowing Code-Switching

An overwhelming body of research seems to suggest that 'code-switching' or switching between their two languages should not be discouraged or penalized among bilingual children, thereby encouraging, and even normalizing bilingual instruction for all children. Parvanehnezhad and Clarkson (2008) studied language switching in Iranian bilingual children as they solved mathematical problems and explained their reasoning in an interview setting. Students reported switching between their home language (Farsi) and the language of instruction (English) when they found the problem to be difficult, when they were more familiar with the Farsi version of the numbers or words being used, and when they were in a Persian school environment. It is equally important to consider the cognitive processes a bilingual student may be employing while learning mathematics in either language so instruction and communication of mathematical concepts can be tailored to their 'zone of proximal development' (Zaretskii, 2009). In addition to promoting students' understanding of mathematical concepts, teaching mathematics in students' first language also seems to nurture socioemotional aspects of their learning. In a comparison of five multicultural schools in Sweden, bilingual students between the ages of 9 and 16 reported higher levels of confidence, engagement and learning when bilingual mathematics teachers instructed their students and engaged them in mathematical activities using both languages. Students also felt secure using both languages while doing and understanding mathematics problems (Norén, 2008).

9.6.2 Allowing Other 'Off-loading' Strategies

Similar to code-switching, implementing other sensorimotor strategies (e.g., finger counting, sketching, diagramming, visual aids, etc.) can also help bilingual children to offload some of their persistent cognitive load, thereby allowing for better mathematical processing and performance. Children who use their fingers to count and do arithmetic in the early school years (K to second grades) were found to perform better in the later school years (Baroody & Wilkins, 1999; Crollen & Noël, 2015;

Long et al., 2016). Before children switch to mental number representations entirely, fingers help children as a cognitive offload or embodied processing mechanism, later to be replaced by fact retrieval, which is more efficient and makes cognitive resources available for learning of more advanced arithmetic and algebra. Neuroimaging studies both with children (Berteletti & Booth, 2015) and adults (Soylu & Newman, 2016) show an association of the finger sensorimotor system with number processes. In addition, multiple studies showed that children's finger gnosis (the ability to individuate fingers) scores correlate with or predict their mathematical skills (Fayol et al., 1998; Noël, 2005), even though there are also some studies not showing such an association (Long et al., 2016). There are also studies showing that fine motor ability correlates with (Fischer et al., 2018) or predicts (Luo et al., 2007) mathematical skills in young children. Similarly, visual mathematical representations (VMRs) and computer-based Mathematical Cognitive Tools (CMTs) help educators to scaffold their instruction and help children to cognitively offload while learning (Sedig & Liang, 2006).

9.6.3 Strengthen Retrieval of Mathematical Facts in Both Languages

Since basic mathematical facts are the building blocks of higher-level math, strengthening bilingual children's retrieval of basic mathematical facts in both their languages is crucial for their mathematical development. Neuroimaging studies show that with higher arithmetical skills, both children (Rosenberg-Lee et al., 2011) and adults (Grabner et al., 2007; Prado et al., 2011) show less activation in the intraparietal sulcus (i.e., associated with calculation) and more in angular gyrus (associate with retrieval) during arithmetic tasks, particularly for addition and multiplication, given the higher reliance on retrieval of arithmetic facts for these operations. Further, in a study conducted with adults, parietal activation during a complex multiplication task shifted from intraparietal sulcus (i.e., calculation) to angular gyrus (i.e., retrieval), as these adults were trained with the multiplication facts included in the task (Grabner et al., 2009). Automatizing calculation processes by switching to retrieval via practice could be another way to help students reduce their cognitive load.

9.6.4 Incorporating Home and Cultural Contexts

Another recommendation to improve bilingual children's learning outcomes in mathematics is to draw on the children's home and cultural contexts. Whether monolingual, bilingual, or multilingual, Secada and De La Cruz (1996) suggested that students need to make sense of mathematics instruction in order to perform satisfactorily, and that is often achieved by connecting children's problem-solving strategies to mathematics instruction in school. Building on this suggestion, they recommend using children's home and cultural backgrounds to promote mathematical understanding among students from varied cultural and linguistic backgrounds. According to Secada and De La Cruz (1996), teachers should consider adopting these four principles in their teaching practice: (1) assessing students' understanding constantly; (2) allowing students to choose from a variety of mathematical content and levels that is interesting, open-ended, and accessible; (3) building on students'prior knowledge and home experiences; (4) developing mathematical language in their cultural and linguistic context. For bilingual students, the last two suggestions are all the more crucial from a standpoint of equity; it would allow them to build their own understanding of mathematical concepts from a similar starting point as their monolingual peers.

9.6.5 Mathematics Instruction in the Home Language

While it is recommended that bilingual children's home language is utilized for mathematics instruction, the context, setting, and manner of this instruction should also be considered. For example, the Redwood City study for Mexican American bilingual children (Cohen, 1976) found that separating bilingual children from their monolingual peers to instruct them in Spanish did not necessarily have the desired effect in academic achievement. In a study of third to fourth grade children who were instructed bilingually for 6 years, bilingual children outperformed their public-school peers (instructed in English) in Spanish reading, vocabulary, and storytelling. However, their public-school peers, whether monolingual or bilingual, performed better in English storytelling, which means that separating bilingual students could result in them having lesser opportunities to practice syntactic improvisation in English. The comparisons between the bilingual-schooled and public-schooled children yielded mixed results in mathematics and English vocabulary performance.

A meta-analysis of bilingual education programs conducted by Willig (2012) found similar mixed results with small to moderate differences favoring bilingual education in reading, language skills, mathematics, and total achievement when the tests were in English, and in reading, language, mathematics, writing, social studies, listening comprehension, and attitudes toward school or self when tests were in other languages. The mixed performance in mathematics and other subjects seen in these studies further suggest that separate instruction may not be the most effective strategy in mathematics instruction for bilinguals.

Since most classrooms have bilingual children who speak more than one non-English language which mathematics teachers may not speak themselves, it is not practically feasible to instruct all bilingual children in their home language. One recent innovation that some mathematics teachers have implemented to deal with this challenge is to connect with teachers in other countries teaching the same material and provide students some 1-on-1 online instruction or recordings of the instruction in the student's home language (WestEd, 2020). Teachers also reported organizing their classes in rotating stations where students with various home languages were grouped together and each station included instruction in a different home language; students would then take turns learning from the instruction at each station, wherein the student familiar with the language of instruction would explain what they learned to their station peers who were not familiar with the language, thereby 'flipping the script' (WestEd, 2020). Despite such creative solutions, there is a shortage of mathematics instructional practices that is inclusive for all bilingual students.

9.6.6 Immersive Bilingual Programs or Structured Immersive Sessions

In 1996, Rossel and Baker reported only 25% of the 300 evaluated immersive bilingual programs to be methodologically acceptable. Surprisingly, Willing (2012) reported similar results 16 years later. Willig (2012) called for quality research in the field of bilingual education to remedy the high prevalence of methodological shortfalls seen in this domain. This suggests that there has been a need for effective bilingual programs in USA for at least 25 years. According to Rossell and Baker (1996), only 9% of the methodologically acceptable programs showed bilingual education to be more effective than regular classroom instruction for mathematical performance.

A few studies found bilingual education more effective than regular classroom instruction but even then, structured immersion was still considered as the ideal format for bilinguals with limited English proficiency. Structured immersion programs help bilingual children acquire language skills in their second language so they can succeed in a classroom where mathematics and other instruction occurs primarily in English. A meta-analysis conducted by Greene (1997) similarly recommended using their native language (versus English-only instruction) when instructing bilingual children with limited English proficiency for moderate learning benefits.

From a teaching point of view, therefore, it seems that in addition to some educational innovation, providing mathematics instruction in a bilingual child's home language in early elementary grades without separating them from their monolingual or balanced bilingual/multilingual peers might prove most beneficial. This could be done in the form of supplementary structured immersion sessions where English is taught explicitly to low-proficiency bilinguals who are grouped and instructed according to their English proficiency. Another way to do this might be to start or maintain quality two-way immersion (TWI) programs that have succeeded in preparing bilingual students for better mathematical understanding and performance in both their languages. Lindholm-Leary and Borsato (2005) examined general school-related attitudes, mathematics coursework, and mathematical achievement in three groups of high school students—Hispanics who used to be ELLs, native English-speaking Hispanics, and Caucasian English-speaking monolinguals —enrolled in a TWI program throughout all the elementary grades. They found all three groups had positive attitudes toward mathematics and school, were enrolled in college preparation mathematics courses, and were performing at average or above average levels in math. Marian et al. (2013) confirmed this finding in their comparison of test scores across different elementary school programs. They found that Bilingual TWI programs positively affected students' mathematics and reading performance regardless of their language status. Bilingual students in TWI programs outperformed their peers in transitional programs of instruction. Similarly, English-speaking students in TWI programs outperformed their peers in regular classrooms.

9.6.7 Feedback and Culturally Relevant Mathematics Instruction

But any program or type of instruction is only as effective as the sum of its components, and educators play the most vital role in this equation. This is very much in line with Clarkson and Gabraith (1992) who cautioned against treating bilingualism as a unidimensional factor, and instead advocated designing research and programs by accounting for the myriad of factors that play a role in educating bilingual children. Teacher-driven learning supplements, such as feedback and culturally relevant mathematics instruction, is one such factor that has been shown to be effective in instructing bilingual children. Cardelle-Elawar (1990) trained four pre-service mathematics teachers to provide oral feedback to their low-performing sixth graders who were bilingual. The oral feedback was modeled on Mayer's model of metacognition and 4-step-problem solving: (1) translation, (2) integration, (3) planning and monitoring, (4) solution execution. Effective implementation of this model showed that just 6 h of feedback led to higher mathematics performance in low-performing bilingual students.

Cahnmann and Remillard (2002) qualitatively examined teachers' role in making mathematics instruction accessible to students from diverse backgrounds. Individual cases found two effective strategies for instructing bilingual students in math, given the implicit assumption that they are teacher-generated: (1) drawing connections between the student's culture and mathematical concepts; (2) pursuing the complexities of mathematics and making it meaningful to the student. Cahnmann and Remillard also called on administrators to provide generous scaffolding and support to their teachers so they, in turn, can provide similar levels of support to their bilingual students.

9.6.8 Discussions About Mathematics and Culture

Supporting teachers as they do this complex non-formulaic work of teaching and supporting bilingual students is crucial due to the dearth of existing structural supports in the educational system. Bose and Remillard (2011) examined national policy reports detailing U.S. mathematics instruction to identify ways to render mathematics education more equitable. Due to its definition and focus being restricted to procedural and factual knowledge, resulting recommendations for mathematics instruction focused on supporting teacher content knowledge over other forms of knowledge. This is unfortunate since evidence suggests that teachers must wield various types of knowledge and skills to teach students effectively and further facilitate student learning.

For example, Bernardo and Calleja (2005) showed that bilingual students usually neglected to consider real-life constraints and connections while solving word problems. This can be easily rectified if teachers are supported and encouraged to teach mathematics in the context of a student's everyday experiences (including language) instead of the typical procedural manner devoid of linguistic markers. Moschkovich (2007a, b) suggested that having a mathematical discussion with bilingual students would draw on existing sociolinguistic resources allowing them to make meaning of mathematical concepts and integrate it into their lives more willingly. Dominguez (2011) similarly advocates capitalizing on students' experiences as bilinguals as cognitive resources to teach math. In their study, pairs of students who solved problems showed differences in their communication and thought patterns, depending on the context and language.

9.6.9 Making Connections Between Mathematics and Aspects of Children's Lives

There are additional strategies used by elementary and middle school teachers in regular classrooms that, not so surprisingly, have been found to be effective in instructing bilinguals. Gutiérrez (2002) highlighted three high school mathematics teachers who successfully instructed many Hispanic students. The strategies they used to do so included building on students' previous knowledge, using supplementary textbook materials, and promoting teamwork which allowed students to work in their primary language alongside peers. Musanti et al. (2009) conducted a case study of a first-grade bilingual teacher learning and teaching Cognitively Guided Instruction, a framework used to understand student's understanding of contextualized word-problems. They found that ongoing reflections, collegial conversations, and constantly analyzing students' work enriched a teacher's understanding of how students learned math; this, in turn, allowed them to provide more opportunities for students to explain their solutions and thinking, thereby creating an effectual feedback loop between instruction and performance. Therefore, instructing bilinguals in their home language or a mixture of both languages while connecting the content to their own life and culture in the form of discussions, reflections, and teamwork might foster understanding of mathematical concepts and boost mathematical performance in bilinguals.

9.6.10 Confirmations from Non-USA Contexts

The recommendations suggested in previous sections have also been replicated in countries outside North America. Gale et al. (1981) tested elementary school children's academic performance in both English-only and bilingual program classes in Milingimbi, an Aboriginal community in Australia. Although not immediately apparent, by their seventh year in the program, the children enrolled in bilingual classes outperformed their English-only peers in seven out of ten tests, mathematics being one of them. Based on a study of bilingual students in Norway, Özerk (1996) similarly suggested that a case could be made for the adoption of bilingual education grounded in pedagogical evidence. Özerk compared mathematics teaching and learning between two groups of linguistic-minority Norwegian students; one group was instructed in the students' second language (typical monolingual setting like most U.S. classrooms) and the other in a bilingual classroom. The linguistic-minority students instructed in a bilingual classroom performed at the same level or better in mathematics than their peers from the monolingual classroom. Based on high attendance and promotion rates coupled with low dropout rates for Guatemalan bilingual schools, Patrinos and Velez (2009) proposed switching from regular to bilingual education programs for students belonging to disadvantaged populations, estimating national cost savings of $five million. According to them, students enrolled in bilingual schools performed above average on all subject matters.

These cross-sectional findings were further substantiated by a 4-year longitudinal study in The Netherlands investigating the effects of using English as the language of instruction during the first 4 years of secondary education (Admiraal et al., 2006). Academic performance of these bilingual students, who were equally proficient in English and Dutch, was compared to students instructed in Dutch. They found that the students enrolled in the bilingual program outperformed their peers in regular classrooms, and also showed higher English language proficiency. The same was found to be true in Cambodia (Lee et al., 2015) and Mozambique (Benson, 2000). Lee et al. (2015) recommended using a student's first language for mathematics instruction to foster understanding of mathematical concepts, much like researchers' recommendations in the U.S. context. Similar to the U.S. context, Benson (2000) found teaching in two different Bantu languages in transitional bilingual programs more promising for educating bilinguals than instructing them in their non-native language. While transitional programs are a step in the right direction when considering equitable bilingual mathematics education, international data lends support to U.S. findings about two-way immersion programs suggesting that they still might be the most effective for instructing and educating bilingual as well as monolingual children.

Similar support for TWI programs was found in a Canadian study. Math, English, and French performance of elementary-school students (ages 6–12) in Montreal's four different public-school programs—French-as-a-second-language, delayed and early French immersion, and full French-medium schooling (i.e., teaching all subjects in French)—were compared in a longitudinal study (Lambert et al., 1993). The control group for this study consisted of students enrolled in an all-English and all-French school. Except for French oral skills, they found students in French-medium and French immersion programs to be indistinguishable from students in all-French schools on written aspects of French, English, and Math. This also supports the view that students' oral proficiency in a language is determined by the opportunities for social interaction available to them.

9.6.11 Need for More Innovation and Research

Although American and international research corroborates the benefits of bilingual education and the importance of teaching mathematics in a child's home language, its effectiveness is contingent upon thoughtful implementation of bilingual instruction as well as consideration of students' needs. Examining the implementation of mathematics instruction in Malaysian bilingual classrooms, Lim and Chew (2007) pointed out that approaching mathematics instruction in a procedural manner when instructing in a non-dominant language led to poor understanding of mathematical concepts among students and therefore, poor mathematical performance. In the same vein, Tsung and Cruickshank (2009) showed the detrimental effects on mathematical performance when students do not receive adequate instruction in their mother tongue. They conducted case studies of two schools in China, a rural minority elementary school that instructed in a minority language (not Mandarin or Cantonese) and an urban mixed minority elementary school where Mandarin was the primary language of instruction. Minority ethnic children performed poorly in all three—their mother tongue, Mandarin, and English—compared to their peers instructed in Mandarin. These studies suggest that teaching bilingual children mathematics in their home language in an immersive environment is necessary but not sufficient. To address the need for more research and innovation, educators and parents are encouraged to partner with interdisciplinary researchers (e.g., educational psychologists/neuroscientists) so their practical knowledge and insights can become an integral part of the efforts to improve math teaching and learning for bilingual and multilingual children (Anchan, 2022).

9.7 Conclusion

Using an interdisciplinary lens, this chapter outlined the similarities and differences in how bilingual children learn and process mathematics compared to the established monolingual norm. To avoid conceptualizing bilinguals as two monolinguals in one (Grosjean, 1989), recommendations were made for teaching mathematics to bilingual children in the elementary and middle school years based on studies from the fields of mathematics education, educational policy, and educational psychology which were further supported by evidence from cognitive neuroscience, mathematical cognition, psycholinguistics, and cognitive science. Proper and suitable execution of mathematics instruction in a bilingual child's home language as well as the dominant school language is only part of the whole picture. Student-level factors such as their pre-existing knowledge, cultural background, interests, and motivations must be given equal consideration to tailor effective mathematics instruction for bilingual students. Additionally, offloading bilingual students' persistent cognitive load by recruiting their other sensory modalities while creating an immersive learning environment can also help. Mixed methods research (Anchan & Soylu, 2021a, b, c, d, e) is currently underway to precisely target various aspects of this topic and address this educational concern pragmatically and cohesively.

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Chapter 10
Mathematical Creativity of Learning in 5th Grade Students

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Abstract In this chapter, we present a differentiated approach to fifth grade students' mathematical learning, including the development of creative thinking in mathematics. Interdisciplinary workshops based on creativity techniques were carried out with 51 Brazilian students weekly for 3 months. Pre- and post-tests of mathematical creativity, mathematical motivation scale, and tests of school mathematical performance were completed. Our findings demonstrate that students developed creative thinking in mathematics, as well as several unusual solutions that show us how a child can be potentially creative in mathematics when encouraged to do so.

Keywords Mathematical creativity · Mathematical learning

10.1 Introduction

Creativity is essential in all areas of knowledge and is recognized as a specific production resulting from specific processes in an area and evaluated by their specialists (Chamberlin & Moon, 2005). This leads to the need to seek ways to characterize it in different fields of knowledge. As Sak et al. (2017) point out, a creative novel belongs to literature, a creative painting belongs to the arts, the invention of a new energy source belongs to science, and a new theory of numbers belongs to mathematics. Some inventions or discoveries require specific work from various domains, such as mathematical physics, genetics, or business. In this chapter, we will discuss mathematical creativity.

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Creativity can be complex because it involves many factors generating changes in the artistic, scientific, technological, and other fields. It "occurs in the interaction between skills, processes and the environment, an interaction through which something is produced that is defined as new and useful in a given social context" (Morais & Fleith, 2017, p. 22). This interaction can be seen in classrooms when teachers pose a problem to students and use creativity techniques to generate ideas that lead to the solution. For example, brainstorming in a whole group setting is a creativity technique that aims to generate many ideas without judgments a priori. All students can contribute by presenting the ideas. Then, everyone gets involved in judging the ideas presented, looking for the one that can best lead to the solution of the problem.

Some research has shown that encouraging creativity can contribute to improving student performance in mathematics. For example, Fonseca (2019) analyzed the effects of an after-school creativity techniques-based course on high school students' performance in mathematics. Students were divided into control and experimental groups. Instruction for the control group was conventional and similar to school learning. For the experimental group, the classes were based on techniques aimed to stimulate creative thinking in mathematics. Statistically significant results were only found in the experimental group, which obtained the greatest increase in all variables. The inclusion of creativity techniques in mathematics classes favors both the capacity for creative thinking and motivation and, consequently, a better performance in mathematics (Fonseca, 2019).

In this chapter, we define mathematical creativity, review important aspects of mathematical creativity, and showcase how techniques for creative mathematics problem solving were applied in a classroom setting. We report on the results of interdisciplinary workshops based on creativity techniques were carried out with fifth grade Brazilian students. The following research question was undertaken: *How can creativity thinking be developed in mathematics in elementary school students?*

10.2 Mathematical Creativity

Studies of mathematical creativity are enduring. There are indications that Poincaré was the first mathematician to investigate creativity in the field of mathematics based on observation of his own practice (Gontijo et al., 2019; Hadamard, 1954; Muir, 1988; Sriraman, 2004). Poincaré said:

> What is, in fact, mathematical creation? It does not consist in making new combinations with already known mathematical entities. Anyone could do this, but the combinations that could be obtained in this way would be limited in number and, for the most part, totally devoid of interest. Creating consists, precisely, not in building useless combinations, but those that are useful and that are in a tiny minority. To create is to discern, to choose. (Poincaré, 1911/1995, p. 8)

Building on Poincaré's view, creating in mathematics consists of interweaving associations that collaborate to reach a goal – which can be finding the result of an expression/equation or the solving of different problems.

Over the years, other authors have proposed refinements and new concepts about mathematical creativity, such as Hadarmard (1954), Krutetskii (1976), Livne et al. (1999) and Gontijo (2007). It is worth noting that although their definitions of creativity may vary slightly, they essentially converge on a capacity to produce different ideas to solve mathematical problems.

For this chapter we used the definition about mathematical creativity proposed by Gontijo (2007), which considers

> mathematical creativity as the ability to present countless possibilities of solutions appropriate to a problem-situation, so that they focus on different aspects of the problem and/or different ways of solving it, especially unusual ways (originality), both in situations that require the resolution and elaboration of problems and in situations that require the classification or organization of objects and/or mathematical elements according to their properties and attributes, whether textually, numerically, graphically or in the form of a sequence of actions. (p. 49)

First, this definition emphasizes that creativity is the ability to present many answers to the same problem as a fluency of thought. The ability to present answers that can be classified into different categories, as they focus on different aspects of the problem or different ways of solving them, is what we call flexibility of thought. Originality corresponds to the ability to present infrequent or unusual responses. The characterization of an answer as original depends on the group where that answer was produced, observing a set of variables (Fonseca & Gontijo, 2020; Fonseca, 2019; Gontijo, 2007, 2018).

Second, this definition is useful for the study of mathematical creativity, as it highlights the desire that the individual can leave a static mental framework towards the construction of multiple innovative solutions for the different problems that are present every day, being able to use different resources for this – which is also in part called divergent thinking (Lee et al., 2003; Lev-Zamir & Leikin, 2013). This way, we consider that the way in which mathematics is worked in classrooms needs to be changed.

We emphasize that the activities with the greatest potential to favor the development of mathematical creativity or of creative thinking in mathematics are those that contemplate open problems, that is, the problems that enable the student to generate innumerable ways to solve them (Gontijo, 2020). Closed questions that only demand previously studied algorithmic applications are not enough. It is necessary to include questions that allow the formulation of conjectures, hypotheses, multiple answers and/or adoption of multiple solution paths.

10.3 Creativity Techniques

There are a variety of creativity techniques, each with its own purpose. According to Gontijo (2015, p. 17),

Creativity techniques aim to encourage students to solve problems favoring the creation of original solutions; rules, principles and generalizations; new algorithms;

new questions and problems and new mathematical models. Some techniques also enable a deep understanding of mathematical conceptions while students investigate a problem. [...] Furthermore, the use of creativity techniques can be a very effective way for students to develop a passion for learning Mathematics.

For our research, we have adapted creativity techniques in mathematics, such as (1) Brainstorming; (2) Braindrawing; (3) Brainwriting; (4) List of attributes; (5) Checklist; (6) Reworking of tasks; (7) Scamper; (8) What if . . .; (9) Forced relationships; (10) Alternative uses; and (11) Dramatization. Regardless of the modality technique, it is important to build an environment free from punishment and criticism, so that participants feel free to create (Wechsler et al., 2018) and observe four basic rules: (a) no criticism, in order to provide a climate pleasant and free for the participants; (b) chain generation, which refers to the fact that listed ideas can and should be used to generate new ideas; and (c) mutation and combination, a way of combining the ideas generated or even improving them (Bianchi, 2008; Conklin & Dacey, 2004; Wechsler et al., 2018).

The Brainstorming (i.e., oral), Braindrawing (i.e., drawing) and Brainwriting (i.e., written) techniques are similar in their conception, differing only in the modality of the way the generated ideas are registered. One activity that can be carried out using these techniques is to record on the blackboard the following numbers: 3902 — 51062 — 7250. The teacher then asks students to list all the common characteristics of the numbers presented (orally or in writing). Then he/she asks them to write as many numbers as they can that have characteristics common to the numbers presented.

Another creativity technique is the attribute list. This technique involves identifying optimal characteristics or attributes. It involves a process of partitioning the product or process under analysis into smaller units, which helps to 'see' it, understand it, reinvented it, or used it more effectively. Three major steps characterize this technique: (a) list the identified attributes; (b) modify attributes in different ways; (c) transfer the modified attributes to other situations. For example, an activity that can be developed from the list of attributes refers to identifying the properties of quadrilaterals. Students may be asked to record the characteristics of squares, rectangles, diamonds, trapezoids and parallelograms. Based on these characteristics, they analyze what they have in common and what makes them different, consolidating their knowledge of these geometric shapes. The checklist provides the list of attributes, rather than generating the list, and invites participants to reflect on and compare among the listed attribute to the analyzed object.

The "re-elaboration of tasks" and the "what if..." are part of a technique called SCAMPER, where each letter represents an action in the creative process: S – Substitute, C – Combine, A – Adjust, M – Modify, P – Put (to other uses), E – Eliminate and R – Reverse. For a fuller description of each, refer to Wechsler et al. (2018). To see how this technique might be applied to developing mathematical creativity, consider this common problem: "Marcos has a daughter and needs to buy her school supply. The initial amount of R$ 250.00 can be divided into two equal installments: one in the act and one for 30 days. Another option is to pay in cash with a 10% discount. How much Marcos will save if he chooses a lump sum cash payment instead of installments?"

In this case, we can encourage students to make "variations" in some of problem element(s) using the Scamper technique. The following questions may contribute to this:

- Which other people, places, values could be put into this problem? (**S**ubstituting some elements of the problem);
- Which kind of concepts could be explored in this problem? An example would be indicate a fine and interest fees that would be applied in the delay of the installment (**C**ombining elements);
- Is it possible to adapt this problem, creating other questions with the same data presented? (**A**djusting the problem);
- Can we reframe or redefine the problem? (**M**odifying the problem);
- What other types of purposes could be attributed to the reasoning used to solve this problem? (**P**ut – to other uses);
- Could any information of the problem be suppressed? If yes, which one(s)? What would be the implication(s) of this suppression? (**E**liminating elements of the problem);
- After working with this activity, can you imagine a mathematical problem that involves this context, but with a totally different question? (**R**eversing the situation. Arranging the situation differently and unusually).

The technique of forced relationships, or object-focus as mentioned by Bianchi (2008), refers to proposing associations that involve attributes and qualities between two non-commonly related objects. In general, you select an initial object and this must be contrasted with a second "strange" object.

Thinking about alternative uses for the same object is another technique to stimulate creativity. According to Conklin and Dacey (2004), this is a technique linked to the development of originality. It is associated with exploring different possibilities from the same process or product, in the case of this chapter from the same problem. The authors offer as an example the function $y = 5x + 2$ and how it can be understood through a context or graphically. Thus, offering different meanings for this function would be a way of alternative uses to interpret it, for example, to represent the price quotation of some type of transport or the amount to be charged by a service provider, among others.

And, finally, the technique of dramatization, which can also include actions such as mime, dance and singing with the purpose of providing students with moments for an improvisation creation that uses a corporal expression. From an "immersion in a given character", they can see something from a different perspective than the traditional one. The goal is freedom of thought to increase concentration, vocabulary, sense of humor, etc. Furthermore, it can help reduce shyness, when this is the case (Fonseca & Gontijo, 2021).

10.4 Workshop Model to Stimulate Creative Thinking in Mathematics

Looking for a way to stimulate mathematical creativity, Gontijo (2020) systematized a model of workshops. This model consists of six follow described steps:

- Warm up: motivational moment – classroom climate;
- Approach to the task: bridge between warm-up and the investigative problem;
- Investigative problem: problem that encourages student participation by generating, evaluating and selecting ideas
- Formalization of concepts and definitions: moment of systematization of concepts and definitions used during the workshop
- Retrospective: reflection on what was produced and systematized;
- Future projections: reflection on how the was experienced in the workshop can be used in other contexts

During the execution of the warm-up steps, approach to the task and the investigative problem, creativity techniques are used to stimulate the generation of ideas, which is intended to enrich the moment with debate and the collective construction of knowledge, as well as the development of creativity thinking in mathematics. Below, we present an example of a series of workshops planned for pedagogical work with students in the fifth year of elementary school, based on the workshop model shown in Image 1. The workshops aimed to explore problems involving the operations of addition, subtraction, multiplication, and division, in a playful context mediated by games and games, through which students were asked about the strategies used to “win” the games and how they built these strategies to achieve their goals.

10.5 Application of Mathematical Creativity

Our study was conducted with two classes of fifth Grade students (n = 51) from a public school from Federal District, Brasilia/Brazil. In each class, eight workshops were held, each supported by creativity techniques Data collection occurred over six additional sessions. The eight workshops were planned and executed once a week, with an average duration of 3 h each and are outlined in Table 10.1.

Table 10.1 Workshops

Workshop	Objective	Creativity techniques
1	Mathematical (magical) divinations: Students conjecture, establish hypotheses and identify patterns, based on guesswork or on critical thinking. In this opportunity, the contents of addition, multiplication and numerical expressions were explored	Brainstorming What if …

(continued)

Table 10.1 (continued)

Workshop	Objective	Creativity techniques
2	Estimate: Students investigate how many pencils there were at school, producing reasonable arguments that could justify the answers presented. The second activity was related to estimating the mass of school kits. These activities, in addition to exploring the ability to make estimates and work with measurements, also sought to develop skills to solve and elaborate problems with natural numbers involving the different meanings of operations	Brainstorming Forced relationships Scamper
3	Investigating movements: Students made balloon-powered cars with the aim of investigating the factors that could contribute to the cars reaching the greatest distance in the shortest time. Thus, students could come to test ways to increase the distance covered by the carts and thus, stimulating scientific thinking, as well as participating in a collaborative and playful experience, as a way to foster creativity in the school space	Brainwriting Checklist Scamper Reworking of tasks
4	Calendar secrets: Students identify regularities and patterns in calendars, which is a great instrument to carry out investigations in everyday contexts, as well as to stimulate logical reasoning. Students were instructed to look for patterns of numbers and operations, based on problems presented by the instructors	Brainstorming Reworking of tasks What if …
5	Healthy and conscious consumption: Students carried out activities in which they had to solve and elaborate problems involving purchase and sale situations, payment methods, used terms such as change and discount, performed mental calculations, estimates, reading of graphs and elaboration of tables, construction and interpretation of bar graphs. All these actions were developed in a context of discussion about sustainability, ethical, conscious and responsible consumption, from a contextualized approach	Brainstorming Checklist Dramatization Scamper
6	Geometric Constructions: Students established links between the geometric shapes that compose it and the real world, in order to build geometric concepts (area, perimeter, side, angles, etc.) from the identification of flat figures, their characteristics, description and classification	Braindrawing List of attributes Alternative uses
7	Numerical Problems: Students explored mathematical "guessing" and one "game". In relation to the game, among other objectives, it was sought to exercise calculation of simple divisions and times tables; explore the concept of di-visors of a number and analyze the possible values for the remainders of the divisions of the numbers on the board by the numbers of the dice. During the game, a set of investigative questions was presented to students to encourage the use of creative thinking in the activity	Brainstorming Forced relationships Reworking of tasks
8	Playing with numbers: Students explores problems involving the operations of addition, subtraction, multiplication and division, in a playful context mediated by games, in which students were asked about the strategies used to "win" the games and how they built these strategies	Brainstorming Reworking of tasks

10.6 Data Sources

Two versions of the Mathematical Creativity Test (Carvalho, 2019) were administered, one before and one after the workshops. Both versions are composed of three items that involve the following activities: the first item involves performing arithmetic operations, the second item involves geometry, requiring the division of rectangles into a certain number of parts with the same size and third item asks for the elaboration of mathematical problems from information presented in a graph. For each item, students received an answer sheet and were instructed to produce as many solutions as possible in the time given for each item. Thus, the responses were analyzed considering fluency (number of valid responses produced), flexibility (number of categories resulting from the grouping of responses according to the similarity among the resolution strategies used) and originality (rare responses among study participants) (Carvalho, 2019; Fonseca, 2015; Gontijo, 2007). The scores were calculated using the model developed by Leikin (2009), called Multiple Solution Tasks, which resulted in a general score encompassing elements related to fluency, flexibility and originality of the answers presented.

The second instrument applied was the 21-item Classroom Climate Likert Scale for Mathematical creativity (Gontijo, 2018). Responses include Never, Few times, Often, and Always. Some example items are: (a) I was happy with my performance in the mathematics workshops; (b) I found myself creative in the mathematics workshops; (c) I became interested in the contents taught in the mathematics workshops.

10.7 Results and Discussion

The table below shows the results of the mean, median and standard deviation obtained with the application of the Mathematics Creativity Test, Version A (in the pre-test) and Version B (post-test), by class and by the set of students (Table 10.2).

The analysis of these data shows us that the workshops increased students' mathematical creativity scores, evidencing that these developed fluency, flexibility, and originality of thinking measured by the tests. We note that while in the pre-test, the results were mostly concentrated between scores from 1.02 to 1.54; the results of the post-test are mostly concentrated between the scores of 1.47 and 1.89.

Significant differences were found from the comparison between the means with the student t-test between the pre- and post-tests applied (Total of students – $t(50) = -18.38$, $p < .05$). It is worth mentioning that significant differences were

Table 10.2 Mean, Median and Standard Deviation of the scores on the Mathematical Creativity Test

	General N = 51			Grade A N = 25			Grade B N = 26		
	Average	Median	S.D.	Average	Median	S.D.	Average	Median	S.D.
Pre-test	1.35	1.36	0.27	1.31	1.33	0.25	1.37	1.36	0.29
Post-test	1.71	1.76	0.22	1.67	1.73	0.22	1.76	1.79	0.22

also found when each group was compared separately, which suggests that the interventions achieved the expected effect (Class A – $t(24) = -12.07$, $p < .05$; Class B – $t(25) = -13.87$, $p < .05$).

The increase in the average scores between the pre-test and the post-test results from the stimuli that the students had during the workshops, as they had contact with different types of activities that required fluency, flexibility, and originality of thought in solving mathematics problems. As an example, we highlight workshops 7 and 8, which aimed to work with arithmetic operations and problems involving numbers. The creativity techniques Brainstorming, Forced relationships and Reworking of tasks enabled students to develop creative thinking strategies to respond to the post-test satisfactorily, manipulating numbers creatively to compose and solve numerical expressions with mathematical operations. This type of activity was explored in the first item of the mathematical creativity test.

The results obtained in the post-test can also be understood from the skills developed explicitly in workshop 6, which had geometry as its main focus. The work with the identification and construction of geometric shapes, exploring their characteristics and classification, as well as perimeter and area calculations, brought the students closer to activities such as the second test item, which required the division of a rectangle into a certain number of equal parts. The Braindrawing and List of attributes creativity techniques may have contributed to developing the skills required in the second item of the test.

Another example related to the increase in scores refers to the activities developed in workshop 5, in which the students had to solve and elaborate mathematical problems. The creativity techniques used in this workshop, especially Scamper (Wechsler et al., 2018), provide students with skills to transform a given problem-situation into other problem-situations, and this type of activity was present in the third item of the mathematical creativity test.

These results are compatible with the results found in the research by Carvalho (2019) and Fonseca (2019). The two research involved the application of pre-test and post-test with an intervention between them, based on mathematics classes/workshops using creativity techniques. The compatibility found with the results of these research allow us to infer that the workshops presented here created an atmosphere in the classroom that contributed to so that students feel confident to manifest fluency, flexibility, and originality of thought. That is, the classes contributed favorably to the development of creativity in mathematics.

It is worth mentioning that the creativity techniques that was used by the teacher during the workshop are pedagogical strategies. The techniques are alternatives that can help the teacher stimulate creativity, nurturing a favorable climate and contributing to the generation of ideas (Conklim & Dacey, 2004; Gontijo, 2015). We defend that is not necessary a direct instruction on the techniques with the students, but the using it naturally, as brought by Fonseca and Gontijo (2021).

The analysis of the data obtained by Classroom Climate Likert Scale for Mathematical Creativity shows that students have positive perceptions about themselves in relation to mathematics based on the activities developed in the workshops. To illustrate these results, we present answers given to the item "I was happy with my performance in the mathematics workshops". Most students, 52%, said they were always satisfied with their performance. Another 36% said they were often satisfied and only 12% said they were rarely satisfied. It is noteworthy that none of the students said they were ever satisfied with their own performance. Overall, 88% of students chose alternatives that express a positive perception of their performance. We consider this fact relevant, as it was relevant that no student did not express dissatisfaction with its result. The workshop model also contributes to the development of motivation in mathematics. It is worth noting that the literature highlights that one of the characteristics of a creative individual is their motivation with the field, after all, the individual needs to be motivated in what one intends to be creative (Gontijo, 2007, 2020; Grégoire, 2016).

The students'positive perceptions may be related to the structure of the workshops, based on the six steps that comprise them. The initial activities – Warm up, were designed to motivate students and contribute to the development of positive attitudes towards mathematics. The study by Borges (2019) was developed from the inclusion of warm-up activities in math classes. The results showed positive effects when comparing groups that received these activities with groups that did not. In addition to having more positive attitudes towards mathematics, they also performed better in school assessments. The present research found similar elements with regard to student motivation and task involvement.

We highlight as the most relevant point, the development of workshops enabled students to generate, evaluate and select ideas considered appropriate to solve problem situations. These types of problems require more than mastering facts and procedures. They require the search for answers based on mathematical concepts and procedures and their relationships (Gontijo, 2020; Schoenfeld, 2013). In addition, in the case of the workshops held, problems were sought that could foster the interest and motivation of students, favoring engagement in tasks.

10.8 Considerations and Implications

In the field of mathematics, discussions about teaching strategies that can favor the development of creative thinking are still scarce. This is despite this area of knowledge playing an important role in the learning. Possibly, the way the teaching of mathematics has historically been conducted in schools has not encouraged teachers and students to think of other ways to organize the pedagogical work with this subject.

In this chapter, we sought to address some theoretical bases on creative thinking in mathematics, presenting some ways to assess this type of thinking and how to stimulate it through pedagogical workshops, based on the presentation of empirical evidence. We also presented a practical example of a workshop designed for students in the fifth year of elementary school. It is worth nothing that the adopted model can be used in different school years, given its validation with elementary, secondary and higher education students.

The workshop model and the mathematical creativity techniques mentioned can contribute to the engagement of students in mathematical tasks, especially when the activities are guided by real-world problems, as they stimulate the ability to transform ideas into actions, taking risks, planning and managing study projects to achieve objectives. Benefits may extend to students' progression in their education, but also in their daily lives at home and in society, preparing them for the civic experience and for professional practice. By stimulating creative thinking in mathematics, we provide students with deeper engagement and understanding, as we encourage them to "switch on" their brains and actively engage in math learning through inquiry, problem solving. As our results demonstrate, this contributes to a positive self-concept in relation to mathematics.

We emphasize that the use of the workshop model presented in this chapter must be accompanied by a series of strategies to favor the development of other characteristics in students, which are important for creativity, among them: strengthening personality traits such as self-confidence, curiosity, persistence, independent thinking and courage; explore new situations and deal with the unknown; helping students overcome emotional blocks, such as fear of failure, fear of being criticized, and feelings of inferiority and insecurity; and implementation of activities that present challenges and opportunities for creative activity (Alencar et al., 2018).

In addition to adopting the strategies mentioned above, mathematics teachers must prioritize the use of problem situations in the planning of mathematics lessons, offering challenging activities, based both on the students'life context and on abstract situations that require the use of formal language. and specific procedures characteristic of mathematics. Thus, activities involving the formulation and resolution of problems, and involving the redefinition of mathematical elements, can become a valuable instructional resource for learning mathematics and promoting creativity in this area.

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Chapter 11
Symbolic Mathematics Language Literacy: A Framework and Evidence from a Mixed Methods Analysis

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Abstract The relationship between literacy and mathematics is poorly understood. Existing theoretical frameworks, such as disciplinary literacy and mathematical discourse, do not carefully account for the uniqueness and ubiquity of symbolic mathematics in curriculum. Symbolic mathematics, the writing system of mathematics, is so unique that its symbols and conventions are not addressed in language arts classrooms. It is so ubiquitous in mathematics classrooms that discourse revolves around it. This chapter introduces Symbolic Mathematics Language Literacy (SMaLL) as a framework to conceptualize reading and writing using the symbols and syntax of mathematics. Also, it presents a mixed methods study that highlights how variations in SMaLL can be experienced among adolescents. The study used mixed methods spectrum analysis to explore the range of cognitive and metacognitive variations in students' reading of symbolic mathematics (i.e, $\sqrt{x}$) in isolation and within expository text. Cognitive reactions to reading symbolic mathematics were measured quantitatively using an orthographic error-detection task. Metacognitive strategies for reading symbolic mathematics were elicited using qualitative interviews. The theoretical discussion of SMaLL exposes implicit literacy-for-mathematics demands in learning standards and offers researchers a useful framework for investigating them. The empirical findings provide educators pedagogical knowledge about the spectrum of SMaLL among adolescents.

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Did you read the problem carefully? Students may hear this question in mathematics classes, from elementary school to university. Some mathematics teachers may ask variations of this question multiple times during a single class. It may be used to prompt persistence in independent problem-solving. Alternatively, it may be posed to direct students' attention to mathematical text. Carefully questioning students about when and how they read mathematical text can offer teachers insights into how reading symbolic mathematics differs among students. This chapter is inspired by former students who had the courage to say *I don't know how to read math* and the willingness to develop reading skills in mathematics class.

Research substantiates a reliable relationship between literacy and mathematics skills (Singer & Strasser, 2017; Vukovic & Lesaux, 2013). In cognitive psychology, researchers include measures of reading skills in studies of mathematical development to tease apart literacy- and mathematics-specific skills. In education, researchers design studies aimed at understanding how literacy and mathematics skills work together to support mathematical learning (Casa et al., 2020; Drageset, 2015). Despite synergistic agendas across fields, researchers face formidable challenges to identifying component literacy skills that support mathematical development and understanding how literacy skills operate in mathematics classrooms.

Multiple frameworks attempt to explain relationships between literacy and mathematics. Content area literacy, for example, suggests skills learned in language arts classrooms should transfer to mathematics (Fang & Schleppegrell., 2008). Disciplinary literacy suggests students should develop the literacy skills that mathematicians use in their work (Johnson et al., 2011). Finally, mathematical discourse suggests specialized conversation skills should support learning mathematics (Kysh et al., 2007). Each framework reveals important insights. However, none fully account for the centrality of symbolic mathematics in curriculum and instruction.

Symbolic mathematics refers to the formal writing system of mathematics education and mathematicians. Classroom discourse often revolves around symbolic mathematics written on papers or boards. For example, a teacher might gesture to text-based communication such as $\boxed{A = \pi r^2}$ and say, *Let's discuss what you already know and what you want to learn about this* (Nessel & Baltas, 2007). Symbolic mathematics is so commonplace during instruction that students must read it to engage in everyday lessons. Also, they must be fluent in it to demonstrate achievement on assessments.

The purpose of this chapter is to define and illustrate *Symbolic Mathematics Language Literacy* (SMaLL; Headley, 2016). SMaLL is a framework for inspecting the reading and writing skills necessary to use the symbols and syntax of academic mathematics. These skills are undertheorized and understudied despite the widespread use of symbolic mathematics in instruction. SMaLL offers researchers a useful theoretical foundation for investigating the role of literacy in mathematical development and offers educators a new way of thinking about students' cognitive and metacognitive interactions with mathematical text.

The chapter is organized in four parts. First, we share an analysis of symbolic mathematics in curriculum to justify the need for a framework like SMaLL. Second, we present SMaLL as a theoretically grounded literacy-for-mathematics framework. Next, we present a mixed methods study of SMaLL among adolescents to highlight strategies for generating empirical evidence of variations in SMaLL. In conclusion, the potential of SMaLL as a tool for advancing interdisciplinary research and generating pedagogical knowledge about differences in mathematical development is discussed.

11.1 Symbolic Mathematics in Curriculum

The inception of SMaLL began with an analysis of elementary, middle, and high school mathematics standards through the lens of curriculum theory. More specifically, the Common Core State Standards (CCSS; National Governors Association [NGA], 2016) was examined to explore the degree to which literacy and mathematics are described as separate areas of study and the extent to which literacy skills are implicit in the mathematics standards. In this section, we discuss that analysis to explain how two issues – framing content and unacknowledged curricular demands (Au, 2012) – underscore the significance of developing a SMaLL framework.

The CCSS, written in two documents, isolates mathematics standards (CCSS-M) from literacy standards (CCSS-LA+) for the language arts (in the language of instruction) and other content areas. One document, the *Common Core State Standards for English Language Arts & Literacy in History/Social Studies, Science, and Technical Subjects* (CCSS-LA+, NGA, 2010a), presents an "integrated model of literacy" (p. 4). It describes literacy as a cohesive collection of reading, writing, listening, and speaking skills that are used with "increasing fullness and regularity" (p. 7) as literacy emerges. The CCSS-LA+ distinguishes between literary text (e.g., poems, stories, novels) and informational text in content areas (e.g., historical documents, scientific explanations, technical instructions). The CCSS-LA+ explicitly identifies mathematics in a statement of "what is not covered" and indicates that literacy standards in mathematics "modeled on [CCSS-LA+] are strongly encouraged" (p. 6).

The CCSS-LA+ does not give a concise definition of reading or writing. Reading is described in terms of development over time as

> a steadily growing ability to discern more from and make fuller use of text, including making an increasing number of connections among ideas and between texts, considering a wider range of textual evidence, and becoming more sensitive to inconsistencies, ambiguities, and poor reasoning in texts (p. 8).

This implies a common definition of an instance of reading as "the process of extracting and constructing meaning from text" (Faust & Kandelshine-Waldman, 2011, p. 546). Writing is portrayed as reading's counterpart in text-based communication: the process of generating text from which meaning can be extracted.

The other document, the *Common Core State Standards for Mathematics* (CCSS-M; NGA, 2010b) describes what students should know and be able to do as a result of mathematics instruction. Specific standards are organized by grade and domain (e.g., fractions, algebra, functions, statistics). Notably, the CCSS-M specifies eight overarching standards of mathematical practice (see Table 11.1) that apply to the study of mathematics across all grade-levels and domains. There is no explicit description of the role of literacy in mathematics in the CCSS-M. The introduction, however, suggests reading is, minimally, an inherent element of mathematics education. For example, it indicates that screen reader or speech-to-text technology are appropriate supports for some students. The statement is an acknowledgement that text is an expected mode of communication. Furthermore, it suggests that students who cannot read mathematical text for themselves are dependent on devices (or others) to read aloud to them.

Table 11.1 SMaLL skills in CCSS-M mathematical practices

Mathematical practice	Exemplar of relevant SMaLL skill(s)
1. Make sense of problems and persevere in solving them.	Students can **read** expressions and equations and explore their relationships to other representations.
2. Reason abstractly and quantitatively.	Students can determine when it is possible and useful to **write** variable expressions or algebraic equations to represent a quantity or relationship.
3. Construct viable arguments and critique the reasoning of others.	Students can **read** expressions and equations and examine their validity. Also, students can **write** expressions and equations in an organized progression to clarify their strategies and justify their claims.
4. Model with mathematics.	Given a real-world problem, students can identify constants, variable quantities, and relationships. Also, they can **write** expressions and equations useful for generating reasonable solutions.
5. Use appropriate tools strategically.	Students can **read** symbolic mathematics as needed to employ learning tools (e.g., Desmos). Also, students can **write** expressions and equations in formats required for tools such as Excel and graphing calculators.
6. Attend to precision.	Students can **read** $A = \pi r^2$ and understand its relationship to the statement, "Area equals pi times the radius squared." Also, students can **write** unambiguous expressions and equations using conventional symbolic mathematics to express their meaning (e.g., write x^2 but not $2x$ or $x2$ to convey the statement, "X times itself").
7. Look for and make use of structure.	Students can **read** the distributive property, $a(b + c) = ab + bc$, and make use of the pattern to simplify or factor expressions with similar structures.
8. Look for and express regularity in repeated reasoning.	Given a table with the sum of the interior angles of a triangle, quadrilateral, and pentagon, students can **write** $(n - 2)180$ to express the sum of the interior angles of a polygon with n sides.

Note: The words *read* and *write* are presented in bold to highlight the examples of literacy demand

Although the CCSS largely treats mathematics as distinct from literacy, a close read of the CCSS-M reveals underlying reading and writing requirements throughout. Table 11.1 itemizes each mathematical practice and provides an exemplar to make implicit literacy demands explicit. Literacy skills are also unacknowledged in some grade- and domain-specific standards. For example, an eighth-grade standard is "Apply the Pythagorean Theorem" (p. 52). The description suggests that, presented with $\boxed{a^2 + b^2 = c^2}$, students should be able to render a reading such as "A squared plus B squared equals C squared" or "the sum of the squares of the short sides of a right triangle is equal to the square of its hypotenuse." There are, however, some grade- and domain-specific standards that describe literacy goals precisely. For example, a fifth-grade standard is, "Read, write, and compare decimals to thousandths" (p. 35). The example is a multidigit decimal number written in expanded form. A clarification indicates comparing decimals entails reading mathematical symbols (<, >, =).

The CCSS is not necessarily representative of global standards for mathematics. Standards and curricula outside the US may differ in three ways: language of instruction; framing of literacy and mathematics as similar/distinct content areas; or degree to which the literacy demands of mathematics are implied or explicit. Despite these differences, reading and writing symbolic mathematics are universal requirements for mathematics achievement. Assessments used to make international comparisons substantiate this. For example, The International Mathematics and Science Study (TIMSS) requires eighth graders to read and write algebraic expressions to demonstrate their mathematics knowledge and skill (see item M032761; International Association for the Evaluation of Educational Achievement [IEA], 2013; 2017).

In summary, symbolic mathematics plays an important role in mathematics education. It is an historically significant writing system that mathematicians developed to transcend complications of everyday languages, to recognize patterns, and to solve important problems (Devlin, 2000). It remains the writing system of mathematics education and the medium of mathematics assessment. Conceptualizing mathematics as distinct from language arts can obscure the domain-specific literacy skills underlying mathematical development. It is, however, clear that learning to read and write symbolic mathematics is essential to mathematics achievement.

11.2 Symbolic Mathematics Language Literacy (SMaLL)

SMaLL is a reading-focused framework aligned with theories of human development and reading acquisition. It addresses limitations of extant literacy-for-mathematics frameworks such as disciplinary literacy (Brozo & Crain, 2018; Fang & Chapman, 2020; Johnson et al., 2011; Wilson, 2011) and mathematical discourse (Bennett, 2014; Bertolone-Smith & Gillette-Koyen, 2019; Herbel-Eisenmann et al., 2013; Moschkovich, 2007; Sfard, 2007). Unlike disciplinary literacy, SMaLL creates a platform for investigating students' development of domain-specific literacy in relationship to grade-level expectations. Unlike mathematical discourse, SMaLL prioritizes reading as a literacy practice during the mathematical learning process.

SMaLL is grounded in the developmental bio-cultural co-constructivism (DBCCC) theory of human development. DBCCC holds that human development is a multilevel process of "continuous, interdependent, co-productive" (Baltes et al., 2006, p. 3) transaction between culture, behavior, and neurobiology. In the context of DBCCC, SMaLL is conceptualized as one aspect of human development. SMaLL posits that classroom discourse experiences related to reading, intentional use of metacognitive strategies for reading, and automated cognitive reading processes contribute to individual differences in SMaLL acquisition (see Fig. 11.1). DBCCC guards against deterministic views of SMaLL development by acknowledging students' behavioral agency as well as cultural and neurobiological conditions beyond their control.

SMaLL is also informed by research produced during the reading wars between whole language and phonics (Pearson, 2004). The debates fueled research aimed at revealing necessary, sufficient, and ideal conditions for reading acquisition. Research related to whole language explored top-down environmental (e.g.,

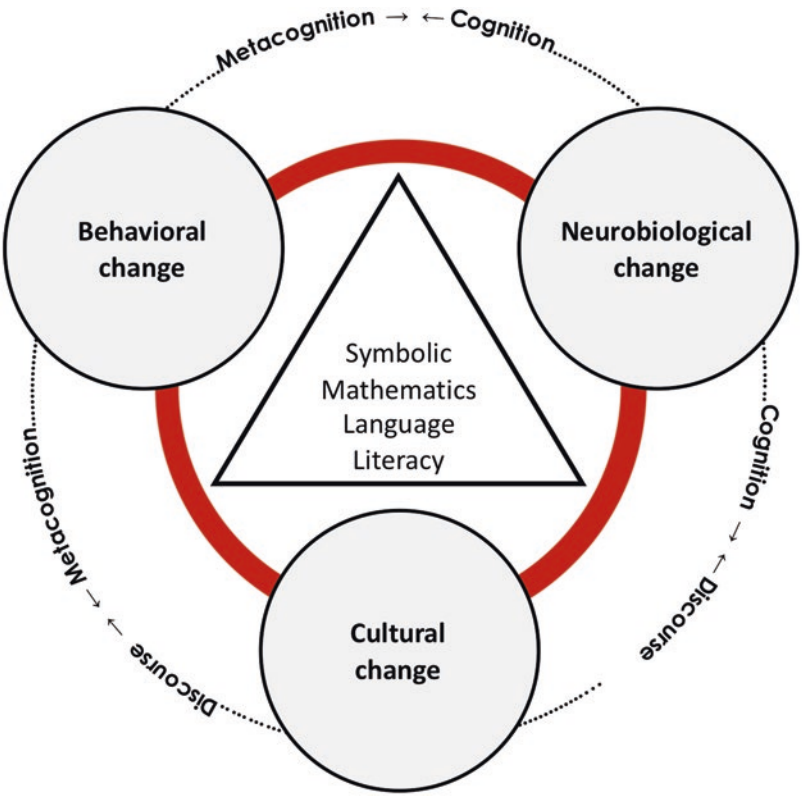

Fig. 11.1 Symbolic Mathematics Language Literacy (SMaLL). (Adapted from Baltes et al. 2006)

at-school, at-home) processes of reading acquisition. Advocates of whole language argue that reading skills develop somewhat naturally in the context of a "literate environment" (Hempenstall, 1997, p. 400). Research related to phonics explored bottom-up cognitive processes correlated to reading acquisition and gave rise to multicomponent models of readings (Coltheart et al., 2001; Frost, 2012; Wolf, 2007). Advocates of phonics argue that phonemic, morphologic, orthographic, syntactic, and semantic skills are necessary skills for learning to read text in any language (National Institute of Child Health and Human Development [NICHHD], 2000; Brady et al., 2011; Wolf et al., 2009). The SMaLL framework incorporates elements from both sides of the debate, asserting that reading symbolic mathematics is a complex multilevel phenomenon that is influenced by both cultural and cognitive processes.

The inclusion of cognition in the SMaLL framework sets it apart from other literacy-for-mathematics frameworks. In particular, SMaLL is built on interdisciplinary findings from reading and mathematics cognition research. The triple-code model (Dehaene, 2011; Dehaene et al., 2003) posits that mathematics cognition involves three brain regions – quantity, verbal, and visual circuits – in adults (e.g., Schmithorst & Brown, 2004) and children (Ansari & Dhital, 2006; LeFevre et al., 2010). Similar cognitive processes are implicated in both reading and mathematics performance, particularly among struggling students (e.g., Dirks et al., 2008; Rubinsten, 2009; Simmons & Singleton, 2008). Meta-analytic evidence suggests differences in orthography impact correlations found between measures of reading and mathematics (Singer & Strasser, 2017). Maruyama et al. (2012) argues that, during development, reading symbolic mathematics puts strong demands on language processing regions of the brain until simple expressions become "compiled" (p. 1457) in visual regions. For a summary of evidence of cognitive relationships between reading and mathematics see Brown (2018).

The scope of SMaLL is intentionally focused in important ways. SMaLL does not account for literacy demands of the language of instruction on mathematical development. It also does not attempt to account for mathematical literacy (i.e., conceptual understanding of mathematical ideas or the ability to do mathematics). For example, given $\boxed{\sqrt{x^2}}$, SMaLL is concerned with how a student reads the text (e.g., "the square root of X squared"); and it does not attempt to account for how students understand this as a concise representation of $\boxed{\sqrt{x \cdot x}}$ or that it can be simplified to $\boxed{|x|}$. For these reasons, SMaLL should be regarded as a complementary theory that can be compared, contrasted, or combined with other literacy-for-mathematics frameworks and comprehensive theories about mathematical development.

The SMaLL framework holds open the possibility that literacy-for-mathematics skills include separable components of mathematical development that are worthy of exploration in their own right. Thinking of SMaLL as a domain-specific reading skill influenced by neurobiological processes, individual behavior, and classroom culture allows researchers and educators to think about how reading relates to mathematical development through a new lens. They can consider what SMaLL skills students develop under various conditions at various time points and explore how SMaLL relates to mathematical development. For example, in the absence of formal instruction, a preschool child should have few SMaLL skills (e.g., naming single digit Arabic numerals) and limited mathematical development (e.g., counting small sets). Later, assuming access to high-quality CCSS-M instruction through first grade, the child should be able to read, for example, $\boxed{8 > 5}$ as "eight is greater than five" (and, vice versa, write the spoken phrase). SMaLL compels consideration of other possibilities as well. For example, a child may be able to do one of these things but not the other. Also, a child may be able to do both in the absence of comprehension of the relative magnitude of the values.

The SMaLL framework facilitates interdisciplinary investigation of literacy for mathematics and its component skills assuming that symbolic mathematics are "graphemic strings of symbolic characters governed by a grammar" (Ranney, 1987, p. 29). Initial hypotheses about top-down cultural components include the supposition that students need access to mathematical texts and encouragement to read them. Initial hypotheses about bottom-up cognitive components include the supposition that reading symbolic mathematics requires the same skills necessary for reading other writing systems: phonological skills (abilities related to mapping symbolic mathematics to sounds or oral/inner-speech), orthographic skills (abilities related to recognizing symbols and conventionally combined units of symbols), semantic skills (abilities related to assigning meaning to symbolic mathematics), and syntactic skills (abilities related to de/composing symbols into meaningful units). Another important hypothesis, which follows from the DBCCC foundation, is that students' behavioral agency and deployment of metacognitive strategies shape their SMaLL development.

11.3 Empirical Exploration of SMaLL Variations

To explore whether empirical evidence aligned with the literature-driven framework, we conducted a study of SMaLL among adolescents learning mathematics under the CCSS-M (Headley, 2016). The study used an innovative multilevel mixed methods research design (Headley & Plano Clark, 2020) and addressed these research questions:

> ***Cognitive level.*** Do adolescents vary in their cognitive reactions to reading symbolic mathematics?
>
> ***Metacognitive level.*** In what ways do adolescents vary in their metacognitive strategies for reading symbolic mathematics?
>
> ***Multilevel.*** In what ways does the integrated evidence support a theory of SMaLL as a multilevel aspect of human development?

We employed a quantitative strand of inquiry using operationalized variables and statistics alongside a qualitative strand of inquiry using interviews and grounded theory principles (Charmaz, 2015). The two strands were designed with crosswalks (Yin, 2006) to facilitate across-level and across-strand integration and theory-building (Greene, 2007; Rossman & Wilson, 1985). The primary integration technique was spectrum analysis, which yielded evidence-based refinements to the SMaLL framework.

The research site and participants were recruited to generate a sample of adolescents who had a shared school culture and a range of differences in terms of classroom culture and academic achievement. The site was a middle school serving all seventh- and eighth-grade students in a large district in a midwestern state in the US.

Each participant was enrolled in a grade-appropriate core mathematics course (aligned with the CCSS-M) in addition to one of two electives: Math Extension or Math Intervention. Table 11.2, which describes the quantitative and qualitative samples, suggests the sampling strategies effectively captured similar variation in grade-level, gender, and elective enrollment for both strands of inquiry. Participants were told the study was about reading in mathematics class and assured they would not be asked to make calculations or solve mathematics problems.

11.3.1 Quantitative Strand: Cognitive Evidence of SMaLL

Table 11.3 provides a summary of the quantitative measurement tools and variables. The Symbol Decision Task, the centerpiece of the quantitative strand, was a measure of reading symbolic mathematics at a cognitive level. The remaining data collection tools generated data that served two methodological purposes. First, the variables allowed for exploration of relationships predicted by the SMaLL framework and relevant literature. Second, the data supported the across-strand and across-level integration necessary to address the mixed methods research question. Only Mathematics Assessment Test (MAT) data were secondary, collected from school records. The remaining data were primary, collected in person by the authors.

The Symbol Decision Task for SMaLL (SDT-SMaLL) was an innovative cognitive task that measures orthographic awareness. It was a modified version of a timed lexical decision task (Lepore & Brown, 2002; Naples et al., 2012). A lexical decision task presents correctly spelled words and incorrectly spelled word-like

Table 11.2 Participant demographics by strand

Quantitative ($N = 158$)	*n*	%	Qualitative[a] ($N = 18$)	*n*	%
Grade			*Grade*		
7th	88	56	7th	10	56
8th	70	44	8th	8	44
Gender			*Gender*		
Female	65	41	Female	10	56
Male	93	59	Male	8	44
Elective enrollment			*Elective enrollment*		
Math intervention	134	85	Math intervention	15	83
Math extension	24	15	Math extension	3	17

[a] The qualitative sample is a subset of students from the quantitative sample

Table 11.3 Quantitative measurement tool and variable guide

Measurement tool	Variable	Directional interpretation
Symbol decision task	SDT-SMaLL	Higher sum, more orthographic awareness
Math print exposure survey	MPES	Higher sum, more exposure
Math Reading habits survey	MRHS	Higher sum, more productive behaviors
Math anxiety survey	MAS	Lower sum, less anxiety
Math achievement test	MAT	Higher score, higher achievement

foils such as *brain* and *brane,* respectively. To complete a lexical decision task, respondents make a judgment about whether the text is written correctly or not. Likewise, the SDT-SMaLL presented conventionally composed chunks of symbolic mathematics and visually similar unconventionally composed foils such as $\boxed{x^2}$ and $\boxed{x2}$, respectively. In both tasks, orthographic awareness is demonstrated by accurately distinguishing between correctly written text and incorrectly written text. The SDT-SMaLL was a reading-only task and did not require the ability to identify mathematically correct solutions or equivalent expressions (as in Kroeger, 2012).

Before completing the SDT-SMaLL, participants read along with oral instructions telling them to read chunks of text and respond *yes* or *no* to the question: *Is the text readable*? After reviewing a description of the display-response pattern, participants completed a practice round to acclimate themselves to the 1500 ms limit on viewing time. The display-response timing forced a cognitive reaction (i.e., a response made before metacognitive activity can be completed; Staresina & Wimber, 2019). During the data collection round, each participant viewed and responded to 60 items in random order. Half of the items were conventionally composed (relative to near grade-level CCSS-M standards). The remaining 30 items were visually similar unconventionally composed foils.

The Math Print Exposure Survey (MPES) and Math Reading Habits Survey (MRHS) were novel data collection tools analogous to surveys used in literacy research (Cunningham et al., 2001; Cunningham & Stanovich, 1990). The former assessed access to mathematical text. Each of the 15 items is a formula (e.g., $\boxed{A = \pi r^2}$) associated with near grade-level mathematics curriculum. To complete the survey, participants selected *Never*, *Only one or two times*, *A few times*, or *Many times* to indicate how often they had seen the formula in any context. The latter measured the frequency of literacy-for-mathematics behaviors. Participants selected *Never*, *Rarely*, *Sometimes*, *Frequently*, or *Always* in response to 10 statements such as *When I do a math problem, I read the problem more than once* and *When I see math symbols, I get someone else to read them to me.*

The remaining measurement tools are well-described in the literature. The Math Anxiety Survey (MAS) was a minimally modified – adapted to reduce reading level – version of a survey of positive and negative affect towards mathematics (Bai et al., 2009). The Math Achievement Test (MAT) was a grade-level assessment used for statewide school accountability (Ohio Department of Education, 2021).

The quantitative results were, for the most part, consistent with the literature-driven theory of SMaLL. Participants varied in their cognitive reactions to reading symbolic mathematics. Correlations (see Table 11.4) suggest that more orthographic awareness was associated with more math print exposure, lower mathematics anxiety, and higher mathematics achievement. However, the correlation between orthographic awareness and math reading habits was not statistically significant as expected. Regression analyses suggest that orthographic awareness may be useful in modeling grade-level mathematics achievement (see Table 11.5). Among eighth graders, for instance, 50% of the variation in mathematics achievement scores was explained by orthographic awareness and mathematics anxiety while controlling for measures of math print exposure and math reading habits. In summary, the results suggests that orthographic awareness may be an essential cognitive component of SMaLL.

11.3.2 Qualitative Strand: Metacognitive Evidence of SMaLL

Participants in the qualitative strand of inquiry completed a semi-structured interview (30–50 minutes). Ten expository text selections related to near grade-level mathematics topics were the centerpiece of the qualitative strand. The text selections, adapted from open-access instructional support websites, were used to induce

Table 11.4 Orthographic awareness: Correlational analysis

7th graders	SDT-SMaLL	MPES	MRHS	MAS
MPES	0.34 (0.00)			
MRHS	0.18 (0.12)	0.43 (0.00)		
MAS	−0.43 (0.00)	−0.43 (0.00)	−0.37 (0.00)	
MAT6	0.47 (0.00)	0.44 (0.00)	0.20 (0.07)	−0.49 (0.00)
8th graders	SDT-SMaLL	MPES	MRHS	MAS
MPES	0.56 (0.00)			
MRHS	0.14 (0.25)	0.41 (0.00)		
MAS	−0.42 (0.00)	−0.31 (0.01)	−0.58 (0.00)	
MAT7	0.56 (0.00)	0.28 (0.04)	0.33 (0.01)	−0.66 (0.00)

Note: The p-values are displayed in parentheses below the Pearson's r value. SDT-SMaLL = Symbol Decision Task, Symbolic Mathematics Language Literacy version; MPES = Math Print Exposure Survey; MRHS = Math Reading Habits Survey; MAS = Math Anxiety Survey; MAT6 = Math Achievement (sixth grade content; MAT7 = Math Achievement (seventh grade content)

Table 11.5 Mathematics achievement: Regression model summaries

	Variable	*B*	*SE*(*B*)	*t*	Sig. (*p*)
7th grade					
	(intercept)	403.83	44.34	9.109	< 0.01
	MAS	−1.47	0.56	−2.61	0.01
	SDT-SMaLL	2.67	1.22	2.19	0.03
	MPES	0.95	0.46	2.06	0.04
$F(3, 68) = 12.88$, $p < 0.01$ Adj R^2=.33					
8th grade					
	(intercept)	410.27	43.32	9.47	< 0.01
	MAS	−2.08	.43	−4.87	< 0.01
	SDT-SMaLL	3.78	1.37	2.75	< 0.01
$F(2, 53) = 28.79$, $p < 0.01$ Adj R^2=.50					

Note: *B* = unstandardized regression coefficient; *SE*(*B*) = standard error of the coefficient

metacognitive behavior for reading symbolic mathematics. Each text selection had symbolic mathematics features in common with SDT-SMaLL or MPES items to create crosswalks. For example, one reading selection discussed inequalities and included $\boxed{>}$ so that, during the mixed methods integration, it could be analyzed in conjunction with SDT-SMaLL items that included visually similar symbols. The text selections varied in multiple ways: length, topic, complexity of symbolic mathematics, and ratio of English to symbolic mathematics text. Each selection was expository only (i.e., none presented a problem to be solved). In preparation for the interview, the text selections were formatted using a similar font, pasted into text-boxes of the same width, printed, and cut to length.

During the interview, the text selections were displayed in three rounds. In the first round, the participant viewed all 10 selections and (quickly) selected the most readable. After participants read the text aloud, they answered follow-up questions about why that text was selected and how choices were made during reading. The second round was like the first. However, for the third round, the participant (quickly) identified the text selection that was most unreadable. After attempting to read the text aloud, they answered follow-up questions about their strategies for reading when it is effortful. The purpose of this strategy was three-fold. First, it simulated reading instances (ranging in difficulty) that might occur in the classroom. Second, it allowed participants to refer to particular features of text to explain their experience of reading mathematical text. Importantly, it increased the likelihood of activating a variety of crosswalks (e.g., producing metacognitive descriptions of reading inequalities that could be compared to the cognitive reactions of SDT-SMaLL items with $\boxed{>}$).

The qualitative analysis began with coding text selections as readable or unreadable based on manifest content (Cho & Lee, 2014). That was followed by an iterative process of generating tentative codes and returning to the data to test themes (Creswell & Miller, 2000). The iterative analysis was guided by grounded-theory principles in the sense that the goal was to reveal "the *possible* range of empirical meanings, actions, and process" (Charmaz, 2015, p. 1616, emphasis in original). The analysis resulted in concept maps for *readability criteria* and *reading process* (see Headley, 2016).

Readability criteria concepts included four themes related to identifying mathematical text as readable or unreadable. The first, and most dominant, criterion was *print features*. Participants noticed two distinct writing systems: symbolic mathematics and English. They also noticed how the text was structured and the relative length of text in each system. They differed in opinions about what made mathematical text more readable. For example, Jackson said, "[It] has a long paragraph section . . . I know I could read that." However, Nicky said, "When I'm reading math, I don't think that I'd have to read words first, just numbers and variables and stuff."

The remaining three criteria were *familiarity*, *translation*, and *math self-efficacy*. Text selections related to familiar (i.e., recent or repeated classroom topics) and unfamiliar topics were generally regarded as readable and unreadable, respectively. However, familiarity was not sufficient for readability. For example, Delaney had no trouble describing the topic of the text selection she identified as unreadable: "Well, [this] is about slope and lines and point-slope . . ." Some text selections were identified as unreadable because the participant could not generate a coherent translation. For example, Bobby explained, "I just didn't really know how to say it." Self-efficacy for doing mathematics also played a role in readability. Elena, referring to "all the little squared and cubes and stuff" in an unreadable text selection, explained: "I've never really done super well with exponents."

Two reading process themes emerged: *perception* and *production*. Perception involved receiving the text as visual stimuli. Production involved generating a translation in the form of oral/inner speech. Three perception processes – automatic, intentional, assisted – were related to mathematical text that were easy, effortful, and difficult to read, respectively. Amelia described her automatic perception as "natural" saying "[the symbols] look weird, but in your brain, they kind of trigger something." Gil explained his intentional strategy this way: "You just kind of have to stop and re-read it . . ." Robin explained her strategy for getting assistance: "I would [ask] a person . . . if I still didn't understand it, I'd use videos on YouTube."

Production processes entailed making decisions about whether to read text in a compressed or expanded fashion (e.g., read $\boxed{A}$ as "A" or "area"). Participants' rationales for compressing or expanding depended on ideas about the purpose for reading (e.g., self-teaching, writing during problem solving, discourse with others). For example, Owen noted,

> I sometimes use the words… When I am more like learning the equations and the variables, that's when I'd be more likely to use the words…But once I know what it is and I can just memorize; I can do it easily. I just do it like the shortened version.

A compressed reading did not always indicate efficiency or comprehension. In some cases, compressing text indicated reading difficulty. Martin, for instance, gave a halting compressed reading of $\boxed{d = rt}$. Asked about his hesitation, he explained, "I was sort of deciding if I should use the word[s]… or just the letter[s]. I used the letter[s]…. I didn't know what they stand for."

In summary, the qualitative results suggest SMaLL involves a variety of metacognitive skills. Students use different criteria to judge text with symbolic mathematics. In addition, students apply different processes to reading text judged as readable and unreadable. Notably, reading symbolic mathematics using a compression process may indicate either a lack of comprehension or a shift towards efficiency.

11.3.3 Mixed Methods Integration: Multilevel Evidence of SMaLL

Integration (Bryman, 2006; Tashakkori & Teddlie, 2008) was conducted using what we called *spectrum analysis*. Spectrum analysis involves the purposeful use of both graphical and narrative strategies for integration (i.e., joint displays and weaving narrative, respectively; Fetters et al., 2013; Plano Clark & Sanders, 2015). Spectrum analysis was designed to embrace the theory-building nature of the inquiry and honor the inclusive mission of mathematics education. Its name is an acknowledgement of its aim to offer a summary of the majority or most-likely experiences in the context of the possible range of unique experiences that may exist. Conducting spectrum analysis involves mixing the quantitative tradition of focusing on central tendency with the qualitative tradition of recognizing unique voices. It is accomplished by iteratively juxtaposing quantitative and qualitative data/findings (e.g., tables, profiles, expository composition) and testing assertions (Cronenberg & Headley, 2019).

Spectrum analysis began with the construction of an across-participant joint display of quantitative data illustrating the relative location of qualitative participants for each quantitative measure (see Fig. 11.2). Locating the qualitative participants on quantitative scales offered an initial glimpse of who might experience SMaLL in typical or unusual ways. Next, within-participant profiles of each qualitative participant were developed (see Headley, 2016). Finally, the profiles were reviewed, grouped and regrouped, and ordered and reordered in a variety of ways to test assertions about the center and extremes of the SMaLL spectrum.

The spectrum analysis revealed that students like Owen, who appeared to be somewhat typical in terms of quantitative measures relevant to SMaLL, were not homogenous in their qualitative experiences of SMaLL. To illustrate how the qualitative data complemented the quantitative data at the extremes of the SMaLL spectrum, we provide excerpts of narratives for Loren and Peter. Loren and Peter were opposites in both quantitative and qualitative ways.

Loren, a seventh grader enrolled in Math Intervention, appears to the left on the spectra in Fig. 11.2. Her mathematics achievement score fell just short of the state

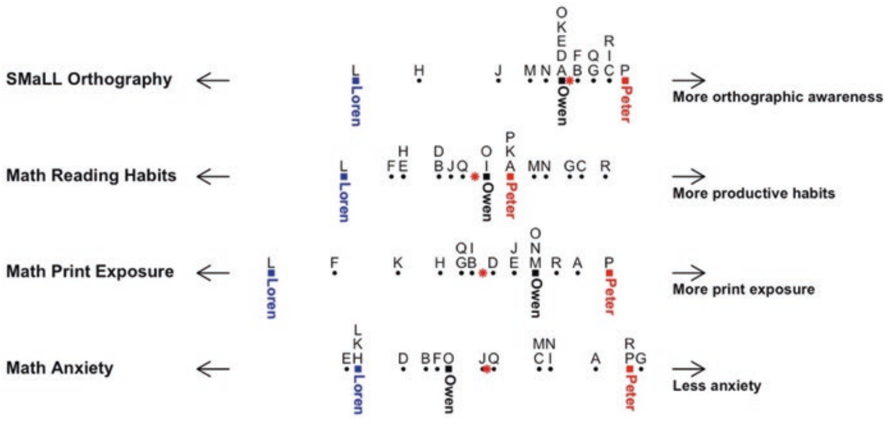

Fig. 11.2 Across-participant Joint Display: Quantitative Spectra. Letters correspond to qualitative participants' pseudonyms. Stacked letters indicate participants at the same location on the spectrum. Names below the spectra highlight participants whose locations were similar (left to right) relative to one another for each of the spectra. The asterisk indicates the grand mean among adolescents in the quantitative sample

standard for proficiency. In selecting readable text selections, Loren looked for brevity in terms of both symbolic mathematics and English. She identified a text selection describing four rules for solving inequalities as unreadable for her because "It had a lot more words and numbers… and then the bolded stuff." She attempted to read one rule, "The inequality X minus two is *something to* five [has the] same solutions as the inequality X *something to* seven" (emphasis added to highlight translation corresponding to $\boxed{>}$). Then she explained, "I couldn't tell if [the symbols] were greater than or less than." Asked what she might do if she were reading aloud in class and came to symbolic mathematics she could not read, she explained without hesitation, "I would just stop, and eventually the teacher would say the answer."

Peter, an eighth grader enrolled in a Math Extension elective, appears to the right on the spectra in Fig. 11.2. His mathematics achievement score was well above the state standard for proficiency. Peter selected the two longest text selections as readable. Referring to the same text selection Loren identified as unreadable, he explained that it was readable for him because it had more words and more "structure" (referring to numbered rules, bold text, and indented examples). He read $\boxed{d = rt}$ as "distance equals rate times time" and explained, "The reason I prefer to read it [that] way is because it sort of reminds me of what all the variables mean." Peter identified a text selection with subscripts as the most unreadable because "sometimes if I'm reading subscripts… I've switched X one and X two around before." Describing how he approached difficult-to-read mathematical text, he identified multiple strategies including decoding symbols and searching for English-language context clues.

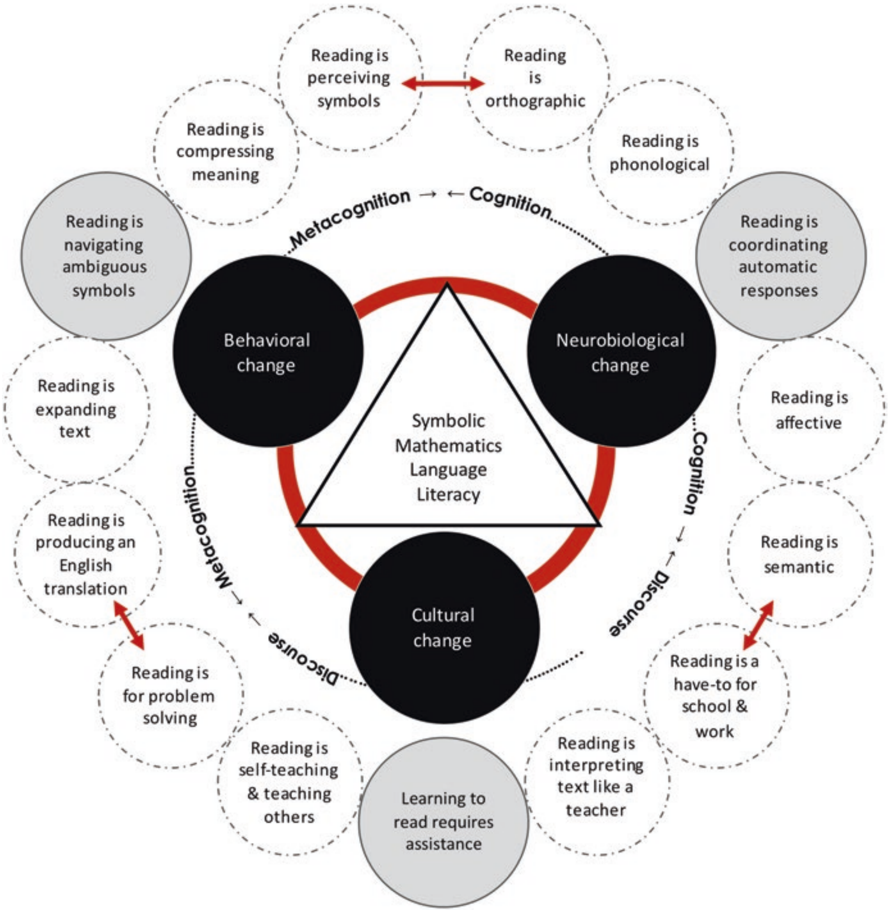

Fig. 11.3 Empirical Model of SMaLL Among Adolescents in Middle School. The shaded circles represent the general conclusions, one for each level. The dashed circles represent themes related to nearby general conclusions for this sample. The dashes and double-ended arrows are a reminder that, according to the framework, the across-level processes should be understood as interactive, transactional, and reciprocal

The spectrum analysis yielded rich insights into qualitative variations in SMaLL among participants who appeared similar from a quantitative perspective. It also revealed that students at the extremes of the SMaLL spectrum may have different kinds of metacognitive strategies for reading symbolic mathematics. Figure 11.3 shows the updated model of SMaLL summarizing the findings supported by the mixed methods analysis. It illustrates the general conclusions (one for each level), across-level interactions and transactions, and variables and themes that warrant further exploration.

11.4 Discussion and Conclusions

SMaLL is an innovative framework for conceptualizing literacy for mathematics. SMaLL names symbolic mathematics as the writing system for the language of mathematics and identifies reading symbolic mathematics as a process worthy of inspection in the endeavor to understand mathematical development. It is fundamentally different from extant literacy-for-mathematics frameworks because it is grounded in theories of human development and aligned with reading acquisition research findings. The framework invites investigations of mathematics-specific reading skills that might be shaped by cultural shifts in curriculum and instruction, behavioral efforts to develop metacognitive strategies, and neurobiological changes that automate cognition.

In this chapter, we highlighted variations in SMaLL. The quantitative results indicates that orthographic awareness of symbolic mathematics can vary and is related to mathematics achievement, mathematics anxiety, and exposure to common mathematical formulas. The qualitative results suggest adolescents vary in terms of the criteria they use to judge the readability of mathematical text as well as the strategies they use to generate oral/inner speech translations of mathematical text. Finally, the mixed methods analysis supports the theory of SMaLL as a multilevel aspect of development with similarities – at the cognitive, metacognitive, and cultural levels – to reading acquisition. Evidence from this first empirical exploration of SMaLL among middle school students supports and extends the literature-driven theory. The portraits can reasonably contribute to pedagogical knowledge about possible differences in SMaLL among middle school students. However, more research is necessary to determine what SMaLL skills are probable among students and develop descriptions of how transactions between cognition, metacognition, and discourse typically impact SMaLL development.

SMaLL is a departure from frameworks like content area literacy that suggest it is specialized language-of-instruction skills (e.g., English) that promote mathematical development. It raises questions about whether reading skills specific to symbolic mathematics are requisite for the mathematical development imagined in curricula. It opens the possibility that differences in curricula (e.g., elementary school vs. middle school, English-language vs. Chinese-language) might require more (or less) or different kinds of SMaLL skills. The strong theoretical foundation of SMaLL offers solid grounding for empirical studies of these issues.

For now, the greatest value of the SMaLL framework lies in its potential to reinvigorate interdisciplinary literacy-for-mathematics research and inspire teachers to address the pedagogical demands of teaching students whose SMaLL skills differ in number and kind. In cognitive science, researchers might advance SMaLL theory by interrogating the SDT-SMaLL (Douglas et al., 2020), developing measures of orthographic awareness for other grade levels (Xu et al., 2021), or developing measurement tools to explore other plausible component skills. In learning sciences, researchers might advance SMaLL theory by questioning the relationship between the reading culture in mathematics classrooms and students' experiences of SMaLL.

Research aimed at refining the theory of SMaLL and revealing relationships between SMaLL skills and mathematics achievement is critical to improving mathematics education. If orthographic awareness is as important to learning mathematics as phonological awareness is to reading acquisition, opportunities to improve mathematics education are being missed. More broadly, if SMaLL is essential to mathematical development, literacy for mathematics should be reframed in curricula. Reframing will require care because curriculum is a "tool that structures the accessibility of knowledge" (Au, 2012, p. 44). On the one hand, allowing SMaLL demands to remain unacknowledged limits all students' access to instruction that supports SMaLL. On the other hand, standards that outline a single SMaLL trajectory may limit particular students' access to rich mathematical experiences (Keefe & Copeland, 2011). Viable strategies for making SMaLL explicit in curricular documents depend on further study and conscientious integration of research into practice.

Being able to read the text they see in mathematics class matters to students. However, for some, reading symbolic mathematics is daunting. Elena, for instance, made this plea for help:

> We've learned how to do this [math], and we've learned what it's for and where to use it. But we've never really been taught how to read it… It just seems a little backwards… [teachers] should teach us more about that.

It is possible that SMaLL operates as a secret gateway – or gatekeeper – to mathematics achievement and STEM careers (Vilorio, 2014). We are eager to see the SMaLL framework employed to advance literacy-for-mathematics research with the hope that it will inspire progress towards explicit curriculum and instructional support.

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Chapter 12
Grasping Patterns of Algebraic Understanding: Dynamic Technology Facilitates Learning, Research, and Teaching in Mathematics Education

Jenny Yun-Chen Chan, Avery Harrison Closser, Hannah Smith, Ji-Eun Lee, Kathryn C. Drzewiecki, and Erin Ottmar

Abstract Prior work has established that cognitive and perceptual processes influence students' attention to notational structures in mathematical expressions, which in turn affects their problem-solving approaches and performance. Advances in educational technology provide opportunities to further investigate these processes, improve student learning, and inform classroom instruction. Chapter 12 presents Graspable Math (GM), an online dynamic algebra notation system designed based on the research of cognitive, perceptual, and affective processes to support student learning. The log data recorded in GM offer a window into students' mathematical cognition, perceptual processes, and problem-solving strategies that can inform both research and instructional practice. First, we review the evidence of using GM to support algebra learning with elementary and middle school students. Next, we describe how log data from GM provide opportunities to research students' problem-solving processes and their uses of mathematical strategies. To conclude, we discuss how this work can inform classroom instruction and future research by providing teachers and researchers with in-depth feedback on students' use of mathematical strategies and understanding.

Keywords Educational technology · Algebra learning · Dynamic notations · Perceptual learning · Graspable Math

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As students transition from arithmetic to algebra, learning how to attend to notational structures is an important component of developing algebraic thinking that can later impact students' performance (Kieran, 1989). In addition to cognitive and affective factors, perceptual features, such as the color and spacing of symbols, influence students' attention to notational structures in mathematical expressions, consequently impacting students' problem-solving approaches and performance (Alibali et al., 2018; Kirshner & Awtry, 2004; Landy & Goldstone, 2010; Marghetis et al., 2016). For instance, students may be inclined to solve problems from left to right, which sometimes violates mathematical rules, such as the order of operations (e.g., $3 + 4 \times 5$). Perceptual features, such as spatial proximity between symbols, can direct students' attention to important elements of the notation that guide their problem-solving approach (e.g., $3\ +\ 4{\times}5$; the spacing of symbols provides grouping that is congruent with the order of operations). To that end, technology-based learning tools that leverage visual, auditory, and/or sensory features of instructional materials to direct students' attention towards key patterns in the notational structure of mathematical expressions and equations may positively impact students' development of mathematical thinking.

Tapping into perceptual motor systems during algebra practice may provide unique opportunities for students to explore the structures of algebra both physically and visually. The key to designing successful perceptual practice for algebra relies on tools that conceptually embody mathematical rules. Over the past several years, members of our team have developed a digital learning platform called Graspable Math (GM; activities.graspablemath.com). GM is a dynamic algebra notation system in which numbers and mathematical symbols can be physically moved and rearranged through specified gesture-actions (i.e., mouse or touch screen actions, such as dragging, shaking, and tapping symbols) that result in fluid, real-time transformations on the screen. These gesture-actions were developed as analogies to the dynamics of algebraic problem solving, providing students with "gestural congruency" between the gesture-actions in GM and the mathematical meaning of the notation (Lindgren & Johnson-Glenberg, 2013; Segal, 2011) to explore algebraic structures (Ottmar et al., 2015).

Beyond supporting student learning and engagement (Landy & Goldstone, 2007, 2010; Ottmar et al., 2015; Ottmar & Landy, 2017; Weitnauer et al., 2016), the log data recorded within GM (e.g., mouse clicks and actions, the timestamp for each action) enable a granular examination of learning by providing a window into students' cognitive processes and the underlying mechanisms of learning. The data can also inform instruction by providing teachers with detailed information about their students' problem-solving processes and behaviors during practice activities.

In this chapter, we present GM as a technology-based pedagogical tool for elementary and middle school students, and as a research and teaching tool that provides rich information on students' mathematical cognition, perceptual strategies, and problem-solving processes. We synthesize theoretical and empirical work on GM, describe how researchers can use this tool to unpack underlying mechanisms of mathematics learning, and discuss ways for practitioners to incorporate GM in classroom instruction.

12.1 Theories of Perceptual Learning and Embodied Cognition

Perceptual learning theory suggests that reasoning and learning about mathematics are inherently perceptual; the way that students perceive visual, auditory, and sensory information guides the way that they process materials and learn (Goldstone et al., 2017; Jacob & Hochstein, 2008; Kellman et al., 2010; Kirshner & Awtry, 2004; Patsenko & Altmann, 2010). Perceptual learning allows students to make extensive use of perceptual-motor routines and motion-based metaphors when solving problems and equations (Goldstone et al., 2017). For example, the spatial proximity between terms can help learners consistently follow the order of operations when solving equations. Students are more likely to solve equations correctly when the spacing between symbols strategically highlights the grouping of the symbols and the operation that should be completed first (e.g., $6 + 2{\times}9$; Landy & Goldstone, 2010). Regardless of a student's level of mathematical knowledge, the tendency to use perceptual features and groupings in mathematics notation is somewhat automatic (Harrison et al., 2020; Marghetis et al., 2016), and has implications for the ways in which individuals interpret, compute, and produce mathematics notation.

Researchers have found that the visual features of abstract mathematical structures influence students' ability to learn appropriate rules. For instance, when the rules are visually salient in notation (e.g., $2(x - y) = 2x - 2y$), middle school students are better able to remember and recognize these rules compared to when the rules are not visually salient (e.g., $x^2 - y^2 = (x - y)(x + y)$; Kirshner & Awtry, 2004). Further, ample evidence suggests that the visual presentation of notation impacts how students reason, process, understand, and learn mathematics (e.g., Braithwaite et al., 2016; Harrison et al., 2020; Landy & Goldstone, 2010). As an example, using perceptual features, such as color, can direct students' attention to relevant information (e.g., highlighting the equal sign in red within an equation, $4 + 7 = 13 - \_\_$, to support reasoning of equivalence) and help adapt their perceptual experiences to support high-level cognition (Alibali et al., 2018; Gibson, 1969; Goldstone et al., 2017). With this understanding, researchers, developers, and teachers can leverage perceptual features in instructional materials to support student learning by directing their attention to important visual cues in notation during problem solving.

Beyond shaping students' thinking processes, perceptual features also impact students' actions which reflect and further influence their learning. Embodied cognition theories contend that students' physical experiences in the world impact their cognitive processes, including thinking and reasoning in mathematics (Abrahamson et al., 2020; Foglia & Wilson, 2013; Nathan et al., 2014; Shapiro, 2010; Wilson, 2002). Specifically, Alibali and Nathan (2012) posit that mathematical cognition is "based in perception and action, and it is grounded in the physical environment" (p. 247). In other words, students' learning environments influence the way they perceive instructional materials which, in turn, informs their cognitive processes, learning, and problem solving.

12.2 Leveraging Perceptual and Embodied Learning Within Graspable Math

GM uses perceptual features to direct students' attention to notational structures; it also embeds embodied features to allow dynamic manipulation of symbols. Among other design choices, GM supports perceptual and embodied learning through three distinct features: (1) the visual presentation of spacing between terms and operands in mathematical notation, (2) students' ability to manipulate notation on the screen, and (3) the fluid transformations that provide immediate feedback on students' actions.

12.2.1 Perceptual Features Guide Students' Attention to Notational Structures

Gestalt principles of grouping posit that we tend to perceive groups of objects as a whole rather than individual objects (Hartmann, 1935). For instance, when viewing "×", we perceive one symbol rather than four intersecting lines. Following the Gestalt principle of grouping by spatial proximity, GM intentionally displays notations in a way that encourages students to group terms in the order of operations—terms surrounding higher-precedence operations (e.g., ×, ÷) are closer together than terms surrounding lower-precedence operations (e.g., +, −). Spatial proximity has been shown to impact mathematical reasoning during problem solving (e.g., Landy & Goldstone, 2010). For example, people tend to perform operations that are physically spaced closer together, even if those operations conflict with the order of operations (e.g., incorrectly simplifying $6+2 \times 9$ to 72 by performing addition before multiplication; Harrison et al., 2020). This phenomenon supports the notion that people rely on perceptual systems to process symbolic notations and are influenced by spatial properties of mathematical notation (Goldstone et al., 2017; Wagemans et al., 2012). Such research suggests that perception plays a key role in mathematical thinking. Further, strategically leveraging spatial proximity when presenting mathematical notations in digital learning tools may be an effective approach to support student learning. By systematically varying the spatial proximity between terms following the order of operations, GM aims to help direct students' attention to the correct groupings of terms and operands, which in turn may support their development of perceptual-motor routines for transforming algebraic notation.

12.2.2 Transforming Abstract Symbols into Objects Makes Algebra Concrete for Learners

GM is a dynamic algebra notation system where all symbols are individual objects that can be manipulated by dragging and dropping each object with a mouse or on a touch screen. By treating symbols like objects on a screen, students can work through problems by moving symbols to transform expressions and equations, resulting in a tangible learning experience. This design reflects embodied cognition theories which posit that thinking does not occur internally; instead, it is a process grounded in our physical experiences with tactile or imaginary objects (Abrahamson et al., 2020; Nathan et al., 2014). For instance, aligned with Melcer and Isbister's (2016) Embodied Learning Games and Simulations Framework, GM utilizes object-centered embodiment by having students manipulate numbers and symbols as objects on the screen. Students can tap, drag, and manipulate symbols in a physical-to-digital format of embodiment where students' physical gesture-actions (i.e., mouse movement or touchscreen activity) result in changes to digital objects on the screen, allowing students to connect their actions with mathematical transformations. Given that individuals tend to treat abstract symbols as physical objects distributed in space (De Lima & Tall, 2008; Dörfler, 2003; Landy & Goldstone, 2009, 2010), GM provides a digital playground for students to explore mathematical symbols as manipulatable objects. Through dynamic manipulations that result in fluid visualizations of algebraic principles, students can learn which actions are appropriate and valid in particular mathematical contexts.

12.2.3 Immediate Visual Feedback Informs Students' Problem Solving

GM provides a fluid visualization that allows students to see the transformation process of algebraic expressions and equations. When students complete a valid gesture-action, GM responds with a fluid visualization of an expression or equation transformation in real-time, so students receive immediate visual feedback on their actions. For example, students can tap the "+" in "7 + 3" and see the two numbers combined into "10" (Fig. 12.1a); they can also drag and drop the "3" from right to left in "2 + 3" to change the expression to "3 + 2" (Fig. 12.1b). In GM, students are able to learn patterns of problem-solving behavior and algebraic principles through interacting with symbols and viewing the fluid visualizations that provide automatic feedback. Compared to solving equations using paper and pencil, the fluid visualizations may help direct students' attention towards structural patterns in algebraic notation by offloading the cognitive demands of calculations onto the system and shifting students' focus to the problem-solving process as a whole. By developing students' perceptual-motor routines of algebraic equation solving, students can encounter, discover, and practice mathematical principles in action (Nathan et al., 2016, 2017).

These features of GM have been intentionally designed to support students' perception and action that influence mathematical thinking and learning. By developing a system based on theories of perceptual and embodied learning, GM is uniquely situated to advance learning theories by using log data to address research questions on how students learn algebra, how students' problem-solving strategies and behavior develop over time, and how systems like GM can support and inform classroom

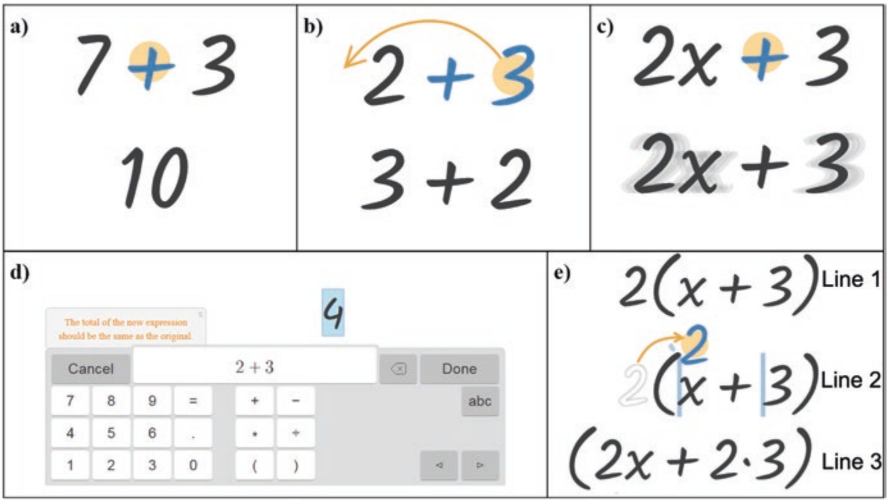

Fig. 12.1 Mathematical expressions are digital objects in Graspable Math. Students can use gesture-actions in GM to (**a**) add numbers together, and (**b**) commute terms. GM also provides visual feedback on invalid gesture-actions through (**c**) shaking and (**d**) an error message. (**e**) GM records a history of students' actions within the system

instruction. Next, we describe GM and the ongoing research efforts to understand the development of mathematical cognition and student learning in GM.

12.3 Graspable Math: A Tool to Advance Theory, Research, and Practice

Developed based on the tenets of perceptual learning and embodied cognition, GM is an interactive algebra notation system that allows students to pick up and transform mathematical expressions and equations (Weitnauer et al., 2016). As students transform expressions, GM provides immediate feedback and fluid visualizations of their mathematically valid gesture-actions (Fig. 12.1a, b). It is important to note that GM only enacts valid mathematical actions. When students attempt mathematically invalid actions, GM provides visual feedback to students. For example, if a student attempts to combine "$2x$" and "3" by tapping the addition sign, the expression shakes and remains as "$2x + 3$", indicating that the action is invalid (Fig. 12.1c). If a student tries to substitute a number (e.g., 4) with a non-equivalent expression (e.g., 2 + 3), the system does not enact the incorrect substitution. Instead, a message appears on the screen informing the student that "the total of the new expression should be the same as the original" (Fig. 12.1d). By allowing students to manipulate and transform notation with immediate visual feedback, they can explore mathematical properties and concepts, such as commutativity, associativity, distributivity, and equivalence (Chan et al., 2022a; Knuth et al., 2006; Prather & Alibali, 2009), and experience the consequences of valid and invalid transformations through perception and action.

GM also has an extensive data logging system that records all of students' actions as they interact with the system. In GM, each mouse click, movement, error, and moment-by-moment problem-solving process is recorded and time-stamped. For instance, as a student picks up and drags "2″ into "($x + 3$)", the system records the initial (i.e., $2(x + 3)$) and end (i.e., $(2x + 2 \cdot 3)$) states of the expression and time-stamps the actions of dragging and dropping the "2″ (Fig. 12.1e). As such, the data in GM reveal students' steps, errors, and the timing of these actions. These detailed logs of students' actions allow researchers and teachers to study or monitor the microstructure of students' problem-solving processes *during* mathematical tasks. Further, by analyzing these actions across problems, we can gain insights into students' behavioral patterns, and their approaches to problem solving. We can also utilize these data to identify common misconceptions or behavioral strategies shared by multiple students within a class.

GM aims to promote students' intuitive, efficient, and mathematically valid perceptual-motor routines while they engage with and explore algebraic concepts in a dynamic environment. Specifically, by leveraging the dynamic capabilities of technology tools, designing activities that target common gaps in student knowledge (NGA Center & CCSSO, 2010), and providing feedback on students' problem-solving processes, tools like GM may help students develop appropriate "structural intuition" (Kellman et al., 2010, p. 299) and perceptual-motor routines (Goldstone et al., 2017) of algebraic notations. Building upon the GM approach and cognitive theories, our team has been designing practical tools, available on activities.graspablemath.com, for classroom uses. These tools include discovery puzzle-based games (e.g., *From Here to There!*), an interactive whiteboard for in-class demonstrations (*Graspable Math Canvas*), and a series of activities that leverage the dynamic notation system for algebra learning (*Graspable Math Activities*). We are also designing dashboards that allow teachers to identify students' potential misconceptions, tailor classroom instruction to students' needs, and respond in real time when students are struggling by seeing automatic updates on students' progress within the activities.

Further, we have begun to explore how digital tools can be used for students to explore the interconnections between multiple representations of algebraic equations, and for teachers to demonstrate these connections. Research has shown that students struggle to make connections between representations (e.g., Bernardo & Okagaki, 1994; Clement et al., 1981; Landy et al., 2014; Martin & Bassok, 2005), hindering student learning and understanding of algebraic symbols (Koedinger & Nathan, 2004). By integrating Geogebra (i.e., a dynamic geometry tool) within GM, teachers and students can link the algebraic equations with coordinate graphs (Fig. 12.2). As teachers or students apply gesture-actions on one representation, such as dragging a line up or down to change the slope in the graph, they can see the corresponding changes to the slope value in the equation. The synchronous changes between an equation and its corresponding graph can help demonstrate the relations

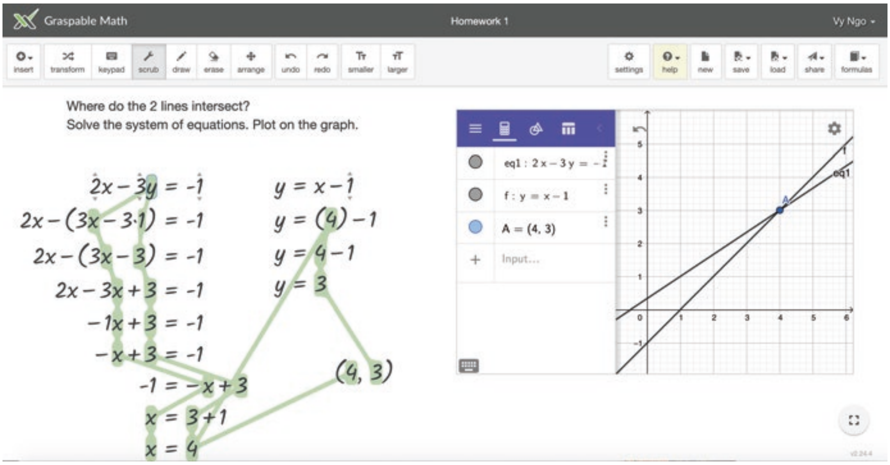

Fig. 12.2 A sample task in Graspable Math with Geogebra integration where students solve and graph a system of equations

between these two representations, as well as the connections between each element of the representations. Further, the green paths connecting the system of equations trace *x* and *y* through the history derivation, allowing students to follow the transformation process for each term.

In summary, tools like GM allow students to experience fluid visualizations of expressions and the interconnections of algebraic representations, as well as allowing teachers to model and discuss mathematical principles using gesture-actions on expressions. The log data collected through these tools also provide researchers a window into students' thinking processes. In these ways, GM acts as an instructional tool to support teachers and students and as a research platform to advance our understanding of how perceptual features and embodied actions impact students' behavior and learning.

12.4 Research on Mathematical Cognition and Student Learning

12.4.1 Evidence of Student Learning in Graspable Math and From Here to There!

The effectiveness of GM has been examined across several studies over the past decade. Early studies demonstrated the usability of GM as well as student benefits of using the tool, such as being able to work through algebra problems more efficiently and with fewer errors than using paper and pencil (Ottmar et al., 2012; Weitnauer et al., 2016). Since then, several classroom-based studies have demonstrated the impact of GM on student learning through playing a gamified version of GM, *From Here to There*!. These studies test GM's effects on learning compared to other educational technologies (Chan et al., 2022a; Decker-Woodrow et al., in press), the potential predictors of these effects (Hulse et al., 2019), the underlying mechanisms (Chan et al., 2023; Ottmar et al., 2015), and ways in which GM can be effectively incorporated in classroom instruction (Ottmar & Landy, 2017).

From Here To There! (FH2T) is an interactive mathematics puzzle game that leverages the GM technology and has been developed through iterative design and testing cycles (Ottmar et al., 2015). The game presents problems that challenge students to transform starting expressions and equations in the center of the screen into a specified goal state (located in the white box; Fig. 12.3)—an expression or equation that is mathematically equivalent to, but visually different from, the starting expression or equation. While the starting expression and the goal state are not connected by an equal sign, the transformation process demonstrates and provides students practice with mathematical equivalence, grounding abstract concepts in physical movements (Abrahamson et al., 2020). Different from other instantiations of GM, problems in FH2T are presented with gamified elements, such as challenges and rewards. For instance, students can retry problems and receive up to three clovers when they solve the problem in the most efficient way. The clovers act as points that help students monitor their performance and serve as an extrinsic motivator for efficient problem solving (Liu et al., 2022; von Ahn, 2013). In FH2T, students can also request hints as needed to receive support so the problems can be challenging without eliciting excessive frustration (Aleven & Koedinger, 2002). Through this design, FH2T integrates GM technology with engaging features to create a playful learning environment for students to practice algebraic skills.

To measure the effectiveness of FH2T compared to other learning tools, we conducted a randomized controlled trial with 475 middle school students (Chan et al., 2022a). Students were randomly assigned to play FH2T or complete online problem sets adapted from open-source curricula. Students completed four 30-minute

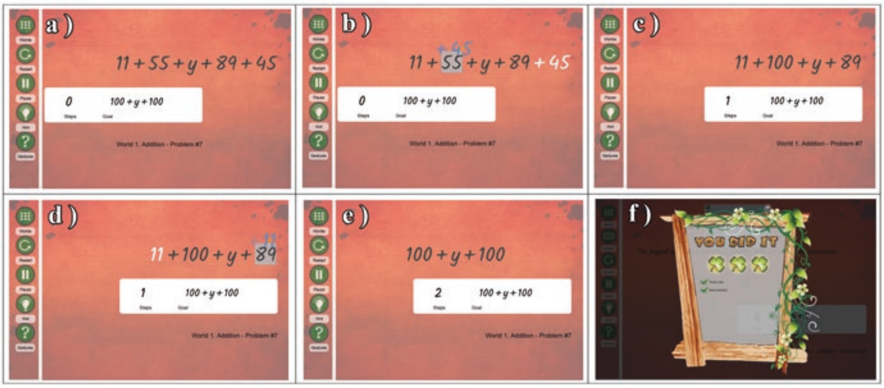

Fig. 12.3 (**a**) A sample problem in From Here to There! and (**b**, **c**, **d**) a potential transformation process involving two steps to (**e**) reach the goal state and (**f**) gain rewards

sessions (a total of 2 hours) of mathematics problem solving using their assigned technology. It is important to note that the online problem sets provided hints and correctness feedback during problem solving, and these supports were previoiusly shown to improve middle school students' mathematical learning compared to traditional paper-and-pencil homework (Mendicino et al., 2009). Results indicated that students, regardless of condition, improved their understanding of mathematical equivalence from pretest ($M = 63.33\%$ [percentage of correct answers], 3.80 out of 6 points) to posttest ($M = 69.00\%$, 4.14 points). Further, students in the FH2T condition ($M = 71.67\%$, 4.30 points) scored 5% (0.30 points) higher on the posttest compared to their counterparts in the online problem set condition ($M = 66.67\%$, 4.00 points), Hedge's $g = 0.16$ and improvement index = 6.4. While the effect size might seem small, practically speaking, the benefit of FH2T did emerge after only a 2-hour intervention in comparison to an established and effective educational technology. Further, the What Works Clearinghouse (2020) improvement index of 6.4 suggests that an average student at the 50th percentile may improve to 56.4th percentile after a two-hour intervention of FH2T compared to completing online problem sets. These findings suggest that FH2T may be effective at improving middle school students' understanding of mathematical equivalence above and beyond traditional online problem sets.

The Elementary version of FH2T (FH2T:E) has also been shown to be effective for classroom use. In FH2T:E, the problems are designed to promote understanding of early mathematical concepts among elementary students. With 185 second graders, we found that completing more problems within FH2T:E was associated with higher posttest scores and this effect was significant above and beyond students' prior knowledge (Hulse et al., 2019). Specifically, for every one standard deviation increase in the number of completed problems within FH2T:E, students scored an average of 3.07% (0.46 out of 15 points) higher on the posttest ($M = 74.20\%$, 11.13 points) when controlling for pretest ($M = 65.93\%$, 9.89 points). In summary, the two studies (Chan et al., 2022a; Hulse et al., 2019) demonstrate that the FH2T games can improve mathematics performance in both elementary and middle school students, showing promise as a digital tool to promote mathematical learning across grade levels.

We have further investigated potential mechanisms of learning behind FH2T and how playing the game leads to gains in students' notation fluency by comparing two versions of FH2T—fluid visualization and retrieval practice (Ottmar et al., 2015). Students in the *fluid visualization* condition used gesture-actions to dynamically manipulate terms and saw the expression automatically transformed on the screen. Students in the *retrieval practice* condition also used gesture-actions, but entered the resulting expression instead of viewing the automatic transformation (e.g., tapping the addition sign in 7 + 3 then typing in 10). Results showed that, after four 30-minute intervention sessions, students in the fluid visualization condition (pretest: $M = 33.07\%$, 9.92 out of 30 points; posttest: $M = 36.27\%$, 10.88 points) showed a 3.20% (0.96 points) increase in their equation-solving performance whereas the students in the retrieval practice condition (pretest: $M = 36.67\%$, 11.00

points; posttest: $M = 34.87\%$, 10.46 points) did not (average gain: −1.80%, −0.54 points). One interpretation of the findings is that rather than requiring students to focus on computations and typing in answers at each step, the fluid visualization liberates students to focus on the overall transformation process of the algebraic expressions. Thus, fluid visualization may help alleviate the cognitive demands of computations, providing opportunities for students to practice and improve their fluency in the perceptual-motor routines of algebra (Goldstone et al., 2010, 2017; Landy & Goldstone, 2007). Further work is needed to understand how these findings relate to prior work on the benefit of practicing arithmetic fact retrievals (Ashcraft & Christy, 1995; McNamara, 1995).

Given that the fluid visualization within FH2T improves learning, we conducted a study to examine *when*, in the instructional sequence, dynamic manipulation combined with fluid visualization is effective for learning (Ottmar & Landy, 2017). In that study, seventh graders who had little knowledge of algebraic equation solving (pretest: $M = 12.61\%$, 2.27 out of 18 points) used GM to transform and solve equations for one hour either *before* or *after* a one-hour lesson of equation solving using paper and pencil. Students who practiced equation solving using GM first scored higher on the immediate algebra posttest ($M = 87.28\%$, 15.71 points) and the retention test one month later ($M = 84.78\%$, 15.26 points) compared to the students who received the traditional instruction with paper and pencil first (posttest: $M = 74.61\%$, 13.43; retention: $M = 77.89\%$, 14.02 points). These findings suggest that using GM early in algebra lessons may support and prepare students for future learning.

In summary, this body of work has demonstrated that GM and FH2T can be powerful tools to support student learning in algebra. Specifically, it provides evidence for the positive impacts of FH2T compared to traditional online problem sets and the influence of progress on this positive impact. It also suggests that coupling dynamic manipulations of mathematical symbols with fluid visualizations of expression transformations may potentially be more beneficial for learning than having students practice mental calculations. Further, providing students the opportunity to dynamically interact with abstract symbols prior to, instead of after, explicit instruction may better prepare students for learning. Beyond their effects on student learning, GM and FH2T also collect rich data that allow researchers to investigate the cognitive processes underlying students' problem solving.

12.4.2 Analyzing Students' Problem-Solving Processes in Graspable Math

Beyond an instructional tool, GM is a research tool that can provide insights into students' problem-solving processes for researchers and teachers. It logs all student actions and mouse-movements, allowing researchers to examine, analyze, and visualize students' problem-solving processes as well as their mathematical errors at scale. By doing so, teachers and researchers can go beyond the correctness of student responses to investigate *how* students solve problems and *what* mathematical misconceptions students may hold. For instance, the log data may show that a student repeatedly tries to add an integer with a variable (e.g., $2 + x$), suggesting that the student may have misconceptions about operations with unlike terms. In short, GM makes student thinking visible for both researchers and teachers.

By leveraging the log data within GM and applying methods and approaches from different fields, we have conducted a number of studies that expand the literature on perceptual learning, student engagement, and problem solving. For example, we have examined students' behavioral engagements (Lee et al., 2022a), problem-solving errors (Bye et al., 2022), and steps in the problem-solving processes (Chan et al., 2022c). Here, we review the findings on the variability and productivity of problem-solving steps as well as predictors of strategy efficiency in middle school students. Efficient and flexible problem solving is a primary goal in mathematics education (NGA Center & CCSSO, 2010), and by understanding *how* students solve problems and *what* influences their problem-solving behaviors, we can inform researchers and educators to design instruction that better support students' problem solving and mathematical learning. By reviewing these findings, we aim to provide examples of the ways in which analyzing the rich log data can offer critical insights into students' problem-solving processes and different learning theories. Further, by situating this work within the larger context of related work, we aim to demonstrate the unique affordances of GM for research and practice.

Observing individual and aggregate visualizations of all student actions in GM and FH2T has revealed notable variation in students' problem-solving approaches and factors that impact their approaches. To visualize students' problem-solving processes across the entire sample, we created Sankey diagrams of paths in students' problem-solving processes to see the variability and frequencies of how students move from step to step (See Fig. 12.4a; Lee et al., 2022c). For example, in the problem of transforming $9 \cdot 4$ into $3 \cdot 6 \cdot 2$, we found remarkable variations in the number of steps that students took to reach the goal state, the sequence of transformations, and the mathematical strategies and properties they used. In this particular example, 64% of the students (the two blue paths at the upper left of Fig. 12.4a) made a productive first step that brought them closer to the goal state, and these students tended to solve the problem using an efficient strategy that involved the fewest number of steps. In contrast, the remaining 36% of the students who made a non-productive first step (the remaining red paths at the lower left of Fig. 12.4a) tended to solve the problem in suboptimal ways that involved more steps than necessary. These findings demonstrate that the log data within GM can provide valuable insights into students' thinking process and decisions as they solve problems. Specifically, students vary in their problem-solving approaches and their first steps on a problem may have important implications on their strategy efficiency. Additionally, these visualizations can help teachers identify patterns of problem-solving behavior to discuss during instruction.

To investigate the factors that impact students' first steps, we have examined how different features of problems influence the productivity of students' solution strategies. Prior work has suggested that students use proximity as a perceptual cue to

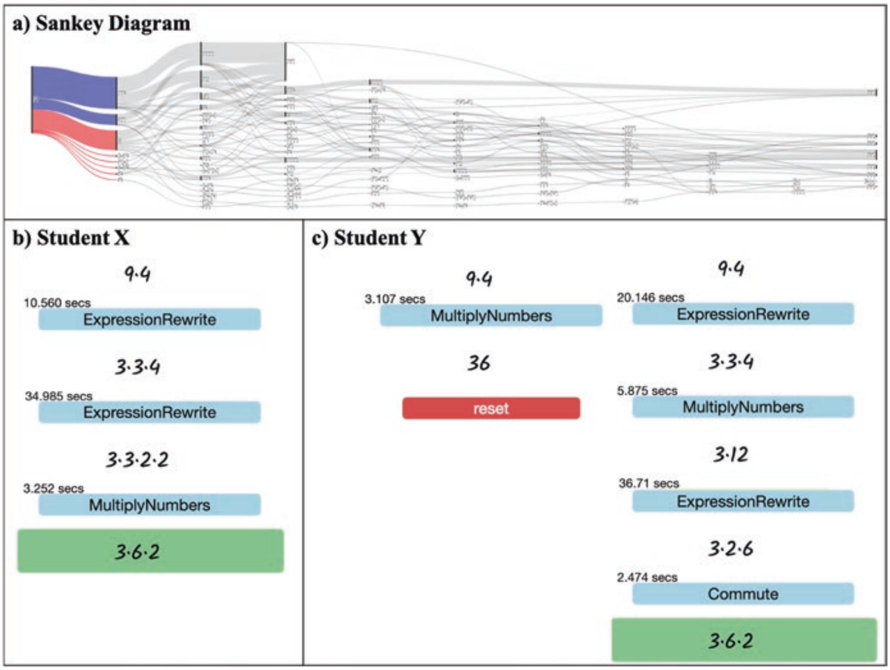

Fig. 12.4 Visualizations of students' problem-solving process. (**a**) A Sankey Diagram showing variations of solution strategies among 343 students on one problem (adapted from Lee, Stalin, et al., 2022). Individual student examples show (**b**) efficient and (**c**) inefficient solution strategies on problem 9 · 4
Note: For readability and interpretability, Fig. 12.4a was truncated to the first 10 steps

group symbols aligning with the order of operations (Landy & Goldstone, 2010), and that mathematics standards tend to focus on base 10 numbers, such as 10 or 100 (NGA Center & CCSSO, 2010). To examine how these factors would interact to influence students' solution strategies, we designed problems that varied in whether the numbers to be combined were adjacent (e.g., 47 + 53 + *b* → 100 + *b*) or non-adjacent to each other (e.g., 47 + *b* + 53 → 100 + *b*) and whether the problem involved 100 or non-100 numbers (e.g., 47 + 52 + *b* → 99 + *b*; Lee et al., 2022b). Using the log data within GM, we coded whether students' first steps were productive or non-productive, and found that students' first steps were more likely to be productive when the numbers to be combined were adjacent versus non-adjacent to each other and when the goal was to make 100 versus non-100 numbers. These findings extend prior work demonstrating the effects of problem features on problem solving. Through the log data, we see that the structure and presentation of problems impact students' first step on a problem, and consequently students' problem-solving process and performance.

In addition to providing information on *what* steps students take to solve problems, the log data also reveal *when* actions are taken. For example, to transform 9 · 4 into 3 · 6 · 2, Student X first paused for 10.56 seconds, made a productive first step by factoring 9, then reached the goal state in three steps—the fewest steps possible to complete this problem (Fig. 12.4b). Making a non-productive first step, Student Y first multiplied 9 and 4 to make 36, reset the problem, then reached the goal state after another four steps, taking a total of 68 seconds (Fig. 12.4c). As shown in these examples, these data visualizations demonstrate that students vary in the amount of time they take between each step while problem solving, and they take different series of mathematically allowable steps to link two states of an expression.

To further explore the microstructure of students' problem-solving processes, we have examined the role of students' pause time (i.e., time paused before first action / total problem-solving time) on their strategy efficiency (i.e., the total number of steps taken to solve problems) in FH2T (Chan et al., 2022b). We focused on pause time because previous studies have shown a positive relation between pausing and mathematical performance, and it has been used as a behavioral indicator of thinking and planning (e.g., Gobert et al., 2015; Paquette et al., 2014). Analyses of the log data in GM have revealed that students with longer pause time use more efficient strategies involving fewer steps, and that pause time remains a strong and significant predictor even when accounting for students' algebraic knowledge, mathematics anxiety, and mathematics self-efficacy. The results extend previous findings in the algebraic problem-solving literature (e.g., Ramirez et al., 2016; Star & Rittle-Johnson, 2008) by suggesting that pause time is a unique predictor of strategy efficiency above and beyond prior knowledge and affective factors. Further, they provide evidence for the importance of examining students' problem-solving processes in digital learning platforms. In particular, the log data offer unique opportunities to examine the relations between students' behavioral patterns and their solution strategies.

In summary, analyzing log data collected in educational technologies like GM allows researchers to efficiently examine students' problem-solving processes at a fine-grained level and effectively visualize the variability of solution strategies in problem-solving contexts across a large group of students. Using a number of analytics techniques, we have created visualizations that reveal students' problem-solving processes, identified ways in which problem features impact students' solution strategies, and found behavioral indicators that predict students' strategy efficiency. This work extends prior literature on mathematics problem solving, provides implications for instruction, and shows promise in using log data to examine potential mechanisms through which educational technologies may improve learning.

12.5 Implications for Research and Education

GM and the larger theoretical framework of this research have several implications. First, the findings show that subtle design choices in the presentation of instructional materials may have consequential impacts on students' thinking and reasoning of mathematics. For students, GM is a tool to actively explore algebraic concepts and develop fluency in mathematical reasoning through dynamic interactions with mathematical symbols and notation. GM allows students to discover relational properties of arithmetic and algebra through exploration and play. For educators, GM is an instructional tool that grounds mathematics teaching and learning in perception and embodiment. GM can facilitate instruction by serving as a formative assessment tool to gain insights into students' understanding and misconceptions. The variety of GM tools, such as the Canvas and Activities, can also supplement classroom instruction and provide teachers with a dynamic platform to design activities for their students. For researchers, GM is a platform to collect fine-grained data about student behavior and performance during problem solving that can inform classroom practice. Further, the iterative design cycle we undertake can serve as a guidepost for researchers to develop theory-driven educational technologies. Looking ahead, future work on GM and similar educational technologies should leverage the log data within the systems to further understand students' thinking process, develop tools that efficiently identify students' misconceptions, and design instructional resources that effectively improves students' mathematical learning.

12.6 Conclusion

Developed based on theories of perceptual learning and embodied cognition, GM is an interactive algebra notation system designed to leverage cognitive, perceptual, and affective processes during learning and instruction. To date, multiple studies have revealed the benefits of using GM on mathematical learning in elementary and middle school students. Further, researchers can utilize the log data in GM to explore students' mathematical cognition, perceptual processes, and problem-solving strategies. This work has advanced our understanding of the mechanisms underlying mathematical learning and informed both research and instructional practice. In conclusion, GM and its extensions allow teachers and students to experience algebraic concepts through dynamic manipulation of symbols. Further, the research on GM can inform classroom instruction and future research by providing teachers and researchers with in-depth, actionable feedback on students' knowledge and use of mathematical strategies.

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Index

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