The Evolution of Research on Mathematics Teachers' Competencies, Knowledge and Skills

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1 Introduction

What characterizes competent mathematics teachers, what types of knowledge do they need in order to be able to teach successfully, and what skills do they draw upon for successful teaching? These questions have long concerned mathematics education research, teacher education and educational policy. The NCTM standards (2000), for example, refer specifically to teacher knowledge as a ground to start from, stating that "[t]eachers must know and understand deeply the mathematics they are teaching and be able to draw on that knowledge with flexibility in their teaching tasks" (p. 17). Teachers need not only sufficient disciplinary mathematical knowledge and knowledge of the school subject (Bromme, 1994). As Shulman (1986, 1987) argues, teachers need a specialized knowledge base for teaching that is different from pure mathematical knowledge and that differs from other professions, thus coining the term of pedagogical content knowledge. However, to determine what teachers should know, what other aspects constitute teacher competencies, and to specify how teachers acquire these in teacher education, as well as how teachers use their skills and act competently in practical situations in teaching is not an easy task. Teacher competencies are also related to underlying beliefs about the role of teachers

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A. Manizade et al. (eds.), *The Evolution of Research on Teaching Mathematics*, Mathematics Education in the Digital Era 22,
https://doi.org/10.1007/978-3-031-31193-2_3

and the teaching profession, which are culturally shaped. All these entities are subject to change over time and are continually evolving. For example, further challenges arise over time as new demands emerge in the context of teachers' professional practice—such as the increasing integration of technology and digitization, which also require thinking about additional necessary teacher competencies.

In the framework of presage-process–product research underlying this volume (see Chap. 1), Medley (1987) names the facet of teacher competence (Type E) as a central variable within the research on teaching, which he understands as the "knowledges [note: plural!], skills, and values which a teacher possesses" (p. 105) and which he considers being "the tools of teaching" (ibid.). Teacher competence has thus an impact on student learning (the outcome of teaching), as it enables teachers to teach successfully and competently in classroom situations. However, it becomes clear that in order to be able to assess this effect, additional mediating variables should be taken into account as good as possible (see Chap. 2). For example, pre- and post-active teacher activities (Type D), such as planning, assessment, reflection and out-of-class activities of mathematics teaching (see Chap. 3) and interactive mathematics teacher activities (Type C), that take place when in the presence of the students (see Chap. 4). Yet, teacher competencies play a central role in the quality of instruction.[1] Characterized by a cognitivist and individualist perspective, what most research on teacher competence today seem to agree on is that teachers' professional knowledge is central within teacher competence and is considered an essential component of the job-specific prerequisites for successful classroom action. It represents an important cognitive resource for interpreting classroom situations and generating informed decisions for actions needed for successful and competent teaching (Baumert & Kunter, 2006; Gitomer & Zisk, 2015; Guerriero, 2017).

Since the beginning of presage-process–product research, and based on theoretical reflections on a subject-specific characterization of teacher cognitions in teaching, which were initiated in the U.S. in the late 1980s, the question of the theoretical conceptualization and empirically examination of teachers' professional knowledge has become increasingly important (e.g. Carpenter & Fennema, 1992; Carpenter et al., 1988, 1989; Fennema et al., 1996; Neubrand, 2018; Petrou & Goulding, 2011; Rowland, 2014). The research initially sought to identify and isolate more general variables of successful teaching, but has since taken somewhat different forms. Presage-process–product research meanwhile is transitioned into the content-dependent, more situation-specific study of teachers' professional knowledge and its implications for the quality of mathematics instruction. In recent years, a new branch of research on the theoretical description and empirical measurement of professional knowledge of mathematics teachers has become firmly established in the international mathematics education research discipline (a.o. Ball et al., 2008; Baumert et al., 2010; Buchholtz et al., 2014; Carrillo-Yañez et al., 2018; Davis & Simmt, 2006; Even & Ball, 2009; Hill et al., 2004, 2008a, 2008b; Kaiser et al., 2014, 2017;

[1] For further student-related variables as well as external and internal context variables that play a role in the relation of teacher competence and student outcome see Chapter 1 and the other Chapters in this volume.

Krauss et al., 2008; Kunter et al., 2013; Lindmeier, 2011; Manizade & Martinovic, 2016; Manizade & Mason, 2011; Rowland & Ruthven, 2011; Scheiner et al., 2019). However, this work already builds on research approaches that have developed over the past 30 years, as we will show in this chapter.

On the theoretical level, following the seminal work of Shulman (1986, 1987), different dimensions of knowledge are often distinguished in the study of teachers' professional knowledge, depending on assumed aspects of content, referring to the so-called domain specificity. This classification of teachers' professional knowledge has also been used in large-scale international comparative studies of the effectiveness of teacher education programs, such as TEDS-M 2008 (Blömeke et al., 2014a, 2014b; Tatto et al., 2012) and its predecessor study MT21 (Schmidt et al., 2007, 2011). Despite the abundance of studies in this area, however, there is still no agreement on a unified theoretical conceptualization because different conceptualizations are based on different domains attributed to teachers' professional knowledge, differ in their theoretical assumptions, and also have different grain-size of the knowledge elements considered (Even, et al., 2017; Neubrand, 2018).

However, the complexity of the construct of professional knowledge in contemporary research on Type E has not only increased as a result of different theoretical conceptualizations, but also because of the question of the extent to which it is situationally available in school practice as a cognitive prerequisite 'in the head of a teacher' in the form of requirements-related knowledge and skills. When such knowledge is operationalized and measured context-independently for empirical studies (for example in psychometric scalable knowledge tests), research to date showed mixed results as to whether or not it is possible to separate different knowledge domains empirically (Bednarz & Proulx, 2009; Buchholtz et al., 2014; Charalambous et al., 2019; Depaepe et al., 2013). Current discourses within research on teaching, however, put up for discussing the extent to which a context-independent investigation of teachers' professional knowledge seems to be useful at all. Thus, on various occasions, the importance of approaches that allow for a more situation-specific measurement of teachers' cognitive processes in teaching has been pointed out to strengthen the contextual study of teacher competence (e.g., Kaiser et al., 2015; Shavelson, 2010). Since then, scholarly advancements in the last decade have consisted in the differentiation of the current conceptualizations for teaching mathematics according to the theoretically-sound and empirically-based integration of action-oriented knowledge facets (Blömeke et al., 2015; Kaiser et al., 2017; Neubrand, 2018). Among other things, this has led to current mathematics education research approaches to the study of teacher competencies, such as the Knowledge Quartet (Rowland, 2008a; b), Lindmeier's action-based competence approach (Lindmeier, 2011; Lindmeier et al., 2020), and the German TEDS research program (Kaiser & König, 2019). These research approaches focus more on the situational manifestation of professional knowledge and its relation to perceived instructional quality (Even et al., 2017). At the same time, however, the call for a situation-embedded study of knowledge is countered by the fact that the more contextually knowledge is analyzed in studies, the even more difficult it becomes to empirically distinguish knowledge from other factors such as teacher personality or affective variables (Even et al., 2017).

Newer challenges in the description and study of teachers' professional knowledge are also posed by the ever-changing demands of professional practice, which have increased significantly since the late 1980s so there is a constant need to rethink what specialized teacher competencies are needed for successful teaching. Medley (1987) identifies this as a distinct branch of research in teacher competence (Type E and Type D research, p. 111), which is normatively oriented and includes both preactive teacher behaviors like planning or evaluating as well as situational aspects of competence. For example, current topics in research on teaching include the study of teachers' diagnostic skills. These are becoming increasingly important because of the need to deal with an ever-increasing linguistic and cultural diversity of students in the classroom due to transnationalization processes and multiple cultural attributions. Furthermore, novel challenges concerning competencies in the use of technology and digital media in mathematics teaching and dealing with the challenges of digitization play a role (e.g., Mishra & Koehler, 2006) as well as skills and attitudes for achieving equity and educational justice in mathematics classrooms (Schoenfeld et al., 2019).

This chapter provides an overview of the most important developments in the field of describing professional competencies of mathematics teachers, especially taking up the perspective of the development of research over time since Medley's (1987) reflections. However, this overview chapter does not follow the criteria of a systematic review; rather, we provide a narrative review (Snyder, 2019) to give as good and comprehensive as possible an overview on the progress of the research in the field. As a result, however, the perspective is inevitably subjective, and not all work is included. First, we will discuss the development of research on teacher competencies, knowledge and skills over time, before discussing various facets of teacher competencies and teacher professional knowledge separately in Sect. 2. Section 3 deals with the different conceptualization and operationalization of teacher competencies in key studies and research programs and the further development of research towards the consideration of situational aspects. We conclude the chapter with an outlook on the further development of Type E research and a summary reflection.

2 Evolution of Research on Teacher Competencies, Knowledge and Skills

Research on teachers' competencies, knowledge and skills has been influenced by different research directions over time. To be able to chronologically situate the developments and to describe the further developments in terms of thematic content, it is necessary to reflect on the underlying paradigms of research on teaching. Research on the teaching profession has undergone several paradigm shifts since the 1960s, changing the underlying theories and the research approaches used. In the process, existing paradigms were critically examined for weaknesses and further developed so that today's research on teacher competencies, knowledge and skills is based on different paradigmatic approaches which have complementary strengths and set different accents.

The so-called personality paradigm or traits paradigm, which prevailed until about the 1960s, attempted to attribute the pedagogical effectiveness of teachers' actions to measured personality traits (e.g., patience or emotional stability). However, the paradigm had its weaknesses in that it was unable to explain how these characteristics impact different classroom situations (Bromme, 2001). Since its research has produced few or only trivial results on the relationship between teacher action and learning success, the paradigm is not considered very fruitful today.

Originating in teacher effectiveness research, Medley's reflections on directly detectable relationships between different variables in the chain of effects (Medley, 1987) on the outcomes of teaching can be assigned to the presage-process–product paradigm, which took over from the 1960s when research on teaching became more systematic and empirical. This research paradigm questions what effects certain characteristics of teachers have on the desired learning outcomes of their students, assuming stable behavior (Floden, 2001).

In the past, researchers following this paradigm deliberately did not examine teachers' cognitions, but rather behavioral features that are easy to control and observe, e.g., the number and level of questions asked, the waiting time after questions, or the frequency of feedback on students' responses (e.g., Gage & Needels, 1989). An assumption of many studies was that effective teaching practices were domain-general, and researchers could look across teaching in different domains and make generalizations about what teaching expertise looked like overall (Russ et al., 2011). The assumption of the paradigm, that a teacher's behavior exerted a direct influence on student's learning experienced significant criticism in later years, in part because the focus in observing teachers in some studies tended to be only on isolated surface characteristics and did not look at more complex structures of instructional quality (e.g., deep structures rather than surface structures) or the combination of multiple variables, including those of Types E and F[2] (Bromme, 2001). It further became clear that the impact of specific teacher actions depended on the context and the learner much more than assumed and findings on teacher behavior were not as transferable to the realities of different classrooms as one had hoped for (Weinert et al., 1989). However, back in the 1980's, researchers only had access to different (less advanced) research tools (e.g., compared to today's multilevel structure equation modelling), and considered different evidence in their work. One outcome of the criticism was the programmatic remodelling of the paradigm, basically in the expert paradigm (Ornstein, 1995). But presage-process–product research nevertheless continued to evolve. An important aspect of presage-process–product research that continues to shape research on the teaching profession today is the holistic approach that seeks to make connections between teacher behavior and student

[2] This criticism does not undermine the overall framework developed by Medley in general, as it is open to the conceptualization of the variables studied and also takes into account corresponding contextual variables.

learning in particular ways. It thus identifies relevant variables for successful teaching as shown in recent meta-analyses on the effectiveness of teaching (Hattie, 2009; Seidel & Shavelson, 2007). The presage-process–product paradigm thus continues to influence research on instructional quality today and has established its standards.

Based on findings from cognitive psychology research, since the mid-1980s the individual cognitions of the teacher had become the focus of interest in research on the teaching profession. This approach was initially promising in that it was hoped that an understanding of the teacher's thinking would provide insight into why teachers behaved in certain ways in the classroom. Again, however, the focus was in the beginning on cross-domain, rather than initially subject-specific, approaches (Russ et al., 2011). This changed mainly due to the growing influence of the research program "Knowledge Growth in Teaching" by Lee Shulman (Shulman, 1986, 1987) and the work of his research group at Stanford University. Shulman pointed out the importance of subject matter in the study of teacher knowledge. In his famous Presidential Address at the 1985 annual meeting of the American Educational Research Association and the article published in 1986, Shulman cautioned against teacher effectiveness evaluations at the time that focused purely on generic teacher behaviors (such as orientation to simple rules like appropriate waiting times on student responses). He proposed a classification of teachers' professional knowledge that accounted for subject-specific viewpoints and, he saw subject matter knowledge as central to the pedagogical preparation and accessibility of subject content in the classroom. The most important consideration for the research on teacher cognitions at that time was his postulation of pedagogical content knowledge (PCK), which differs from the knowledge required by other professions, such as mathematicians. PCK is specifically oriented towards teaching and includes knowledge about different student cognitions and teaching approaches. "Within the category of pedagogical content knowledge I include, for the most regularly taught topics in one's subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the ways of representing and formulating the subject that make it comprehensible to others" (Shulman, 1986, p. 9). Shulman himself did not aim for the development of a catalog of corresponding knowledge content but specified his idea of PCK in his article published the following year, "Knowledge and Teaching: Foundations of the New Reform" (Shulman, 1987) as a "specific amalgam" of knowledge about subject content and pedagogy, which focuses on subject representations and concepts of understanding as well as misconceptions. Shulman (1987, p. 8) distinguishes various forms of knowledge in his typology of professional knowledge:

- Content knowledge,
- General pedagogical knowledge (strategies of classroom management and organization),
- Curricular knowledge (including materials that serve as "tools of the trade" for teachers),

- Pedagogical content knowledge, a special "amalgam" of subject content and pedagogy that is found exclusively among teachers and forms the basis of their professional understanding,
- Knowledge of learners and their characteristics,
- Knowledge of educational contexts (e.g., about working of groups, administration and funding of school districts, or the character of communities and cultures),
- Knowledge of educational goals and values and their philosophical and historical grounds.

Later on, pedagogical knowledge, content knowledge, and pedagogical content knowledge had a great impact in terms of the theoretical design of research studies on teachers' professional knowledge.

According to Shulman, teachers must transform subject content into pedagogical forms such as examples, illustrations, and classroom tasks that make the content accessible to learners. This transformation of subject matter into pedagogically effective forms of learning is understood as the central intellectual task of the teacher and has become the defining characteristic of pedagogical content knowledge (Deng, 2007a, 2007b). Thus, for Shulman, PCK means the integration of subject matter knowledge and pedagogical knowledge that enables teachers to translate subject matter knowledge into pedagogically effective forms of presentation that match learners' abilities and interests. Shulman's work, however, did not go uncriticized and the criticism led to further developments in research on teacher knowledge. Among other things, it was noted that Shulman had a static understanding of knowledge as something that could be acquired and applied regardless of the complexity of the instructional context, and that the idea of "transforming" or "translating" subject matter into pedagogical forms amounted to a routine, mechanistic transmission of a fixed canon of knowledge. Shulman's critics objected that mathematical knowledge itself could also be assumed to be multidimensional and dynamic in nature, from which it follows that teachers' knowledge is characterized by its "interactive and dynamic nature" (Fennema & Franke, 1992, p. 162). Other scholars adopted this dynamic view of knowledge, essentially viewing it as physically and socially situated in the act of teaching in a particular context (Bednarz & Proulx, 2009; Döhrmann et al., 2018; Meredith, 1995).

This situatedness of teachers' cognitions was taken up by the so-called expert paradigm. The presage-process–product research at that time looked more for the general abilities and skills of teachers and was less concerned with the question of whether these individual bundles of behaviors could actually be found in a person in reality. The expert paradigm focused on the successful teacher "as a whole" (Bromme, 2001; Schön, 1983), and the focus henceforth was on teachers' knowledge and skills. Central to this development was the work of Berliner (2001), for example, in which he calls the teacher an 'expert teacher' and speaks of 'teaching expertise'. According to the expert paradigm, teachers are called experts because they can successfully manage a very specialized, complex task such as school teaching. Expertise is manifested, for example, in the immediacy of action expected of a responsible teacher in his or her teaching and the resulting time pressure of acting as well as in acting under information deficit concerning the current situation, the complexity and dynamics of which are continuously changing due to the students' behavior. In this context, teachers draw on specific knowledge and skills, which can be technically described within the research approach through detailed analyses of requirements—such as those derived from psychology (e.g., Bromme, 1992, 2008).

A recent further development of the expert paradigm has been the approach of professional competence of teachers for about twenty years (Kunter et al., 2013). In this approach, teachers' knowledge and skills are not only identified using requirement analyses in terms of the expertise paradigm but are furthermore complemented by the examination of personality traits such as motivation and self-regulation. The concept of competence was introduced into the discussion by Franz Emmanuel Weinert (1999, 2001) about twenty years ago as part of an influential review of different definitions of competence in a report prepared for the OECD. In describing professional action competence, Weinert states:

"The theoretical construct of action competence comprehensively combines those intellectual abilities, content-specific knowledge, cognitive skills, domain-specific strategies, routines and subroutines, motivational tendencies, volitional control systems, personal value orientations, and social behaviors into a complex system. Together, this system specifies the prerequisites required to fulfill the demands of a particular professional position, social role, or personal project" (Weinert, 1999, p. 10). In summary, competence can thus be defined as "the ability to successfully meet complex demands on a particular context through the mobilization of psychosocial prerequisites (including both cognitive and noncognitive aspects)" (Rychen & Salganik, 2003, p. 43).

A feature of this definition of competence is that it is first understood as context-based. Second, in addition to purely cognitive components, it includes affective components such as volitional, motivational and social readiness to apply the competence in situations. It should also be noted that there is a distinction between competence as a general overarching concept, and the distinction between individual competencies if individual content-related competence facets are meant. According to this understanding, the professional competence of mathematics teachers consists of subject-related and subject-overlapping cognitive dispositions—teachers' professional knowledge (cf. also Baumert & Kunter, 2006)—as well as additional affective personality traits like beliefs, motivation or values (Hannula et al., 2019) specifically for the subject mathematics. These form the basis for mastery of specific situations that arise in professional demands.

Today's research on teacher competencies, knowledge, and skills invokes the different approaches of these paradigms. These are perceived as complementary so that the boundaries between the different paradigms often fade. For example, the current approach to professional competence combines the systematic analysis of teachers' characteristics and abilities of the presage-process–product paradigm with the approach of researching teacher cognitions and the approach of looking at certain characteristics of teachers' personality, such as motivation and values. Consequently, Medley's variables of Type E are still valid as the main units of analyses in research studies, even with today's advances in research on mathematics teaching and teacher education.

3 Components of Teachers' Professional Competencies

Taking into account Shulman's (1986, 1987) reflections on the professional requirements of teachers, which we will discuss in more detail in the next section, the professional competencies of mathematics teachers and its components.

3.1 Content Knowledge

Teachers need knowledge of relevant facts, concepts, and their relations oriented to the subject body of knowledge, as well as subject-specific procedures for generating knowledge and justifying it. This means that teachers of mathematics must be proficient in mathematics, which can be expressed, for example, by the "five strands" of mathematical proficiency by Kilpatrick et al. (2001), which are: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. The deeper understanding of reasoning also implies that argumentation and proving is part of the professional knowledge of teachers so that they are able "to explain why a particular proposition is deemed warranted, why it is worth knowing, and how it relates to other propositions" (Shulman, 1986, p. 9). Neubrand et al. (2009) address the connections of teachers' content knowledge to more general mathematical skills such as explaining, communicating, and even modeling, and include insights into the history and epistemology of mathematics among the content knowledge of mathematics teachers. Somewhat later than Shulman, Bromme (1994)—a representative of the expert paradigm—also formulated on this basis the central insight that when describing teachers' content knowledge, a distinction should be made between the knowledge of the discipline and that of the school subject, since the school subject has a "life of its own" (p. 74), with its own body of knowledge and epistemologies. In mathematics education research, this distinction by Bromme contributed to the identification of professional knowledge of school mathematics (Deng, 2007a, 2007b), or elementary mathematics from a higher standpoint in relevant studies (Buchholtz et al., 2013) going back to approaches by Felix Klein (1908/2016). Dreher et al. (2018), for example, conceptualized this type of knowledge as so-called school related content knowledge (SRCK).

3.2 Pedagogical Content Knowledge

Although Shulman identified two components that are central to PCK, namely knowledge of instructional strategies and representations, and knowledge of students' (mis)conceptions, he did not specify PCK for mathematics. To describe the subject-specific PCK for mathematics, it is not sufficient to focus only on mathematical content, which would neglect cognitive and social preconditions of the learning processes of students. In terms of content, mathematical pedagogical content knowledge presupposes an understanding of subject knowledge, but central to this is knowledge of the potential of school subject matter for learning processes (curricula and syllabi, learning goals and principles), knowledge of subject-related student cognitions (student ideas and errors, learning prerequisites), and knowledge of subject-specific instructional strategies (representations, subject-related diagnostics, performance measurement, and subject-related explanatory and mediation strategies). Subsequently, Shulman's model has been refined more and more, also in response to criticism (for an overview, see the systematic review on PCK by Depaepe et al., 2013). Grossman (1990) for example distinguished four components that are central to teachers' PCK: (1) knowledge of students' understanding, (2) knowledge of curriculum, (3) knowledge of instructional strategies, and (4) knowledge of purposes for teaching. Depaepe et al. (2013) even distinguish a total of eight different facets based on their systematic review: (1) knowledge of students' (mis)conceptions and difficulties, (2) knowledge of instructional strategies, (3) knowledge of mathematical tasks and cognitive demands, (4) knowledge of educational ends, (5) knowledge of curriculum and media, (6) context knowledge, (7) content knowledge, and (8) pedagogical knowledge. A relevant extension of Shulman's understanding of PCK was undertaken in the U.S. in the late 2000s in the Learning Mathematics for Teaching (LMT) project, amongst others, through the formulation of the construct mathematical knowledge for teaching (MKT) or content knowledge for teaching mathematics (CKTM) by Ball and colleagues (e.g., Ball et al., 2008; Hill et al., 2004, 2005, 2008a, 2008b), which we will discuss in more detail below.

If one takes a closer look at the relationship between mathematics and pedagogy within the construct of PCK, however, some aspects can be identified that are more strongly influenced by the subject, while other aspects are more clearly related to pedagogy (see also Chick et al., 2006). With respect to a normative description of the content of the PCK, the perspectives of referring to the scientific discipline of mathematics education (i.e., mathematics, psychology, educational science, general didactics), which have been discussed since the 1970s and which continue to shape the mathematics education discourse today, provide orientation (Buchholtz et al., 2014). By more subject-related pedagogical content knowledge we can therefore understand primarily mathematical aspects of teaching and learning mathematics. This includes, for example, knowledge about subject-specific approaches to teaching, basic ideas, and mental representations of mathematical content, e.g., fractions, percentages, or the concept of derivation, and being able to identify critical mathematical components within concepts that are fundamental for understanding; knowledge about the interconnectedness and interdependence of mathematical concepts (to establish connections between the different subject areas of mathematics education and their mathematical backgrounds, connections to other subjects in the sense of interdisciplinary learning, and connections between mathematical concepts and the real world (Freudenthal, 1991)); knowledge about fundamental mathematical ideas and mathematical activities (e.g., abstraction or algorithmic thinking); knowledge of students' subject-specific preconcepts and barriers to understanding, as well as levels of conceptual rigour and formalization (important in analysing and interpreting student solutions and student questions); knowledge of the role of everyday language and mathematical language in concept formation; knowledge of subject-motivated approaches to mathematical content (e.g., different approaches to the concept of probability; justifications for number range extensions); knowledge about subject-matter-based diagnostics of student solutions and errors (e.g., student misconceptions; appropriateness of student solutions); as well as knowledge about different types of tasks (important for using tasks as a starting point for learning processes).

Under more teaching-related pedagogical content knowledge in mathematics, we can locate perspectives beyond mathematical subject knowledge, which focus more on educational-psychological areas, but which are constitutive for mathematics education. These include knowledge about concepts of mathematical education (e.g., theoretical concepts of mathematical thinking and general competencies such as modeling, problem-solving, and reasoning); knowledge about dealing with different forms of heterogeneity in mathematics education (e.g., the use of different teaching goals in mathematics education, differentiation, and individualization); knowledge about dyscalculia, giftedness, and special education support (important for developing support plans for dyscalculic and gifted learners or inclusive learning groups, taking into account specific learning requirements); knowledge about forms and concepts for teaching and learning mathematics in schools (e.g., genetic learning, discovery learning, dialogical learning, extracurricular learning); knowledge about educational standards, curricula, and textbooks for the subject of mathematics; and knowledge about aims and forms of assessment in mathematics education (formative and summative).

The different requirements for PCK make clear that this knowledge is closely connected to content knowledge because the teacher consciously must choose between all the possible representations the subject provides for teaching (Neubrand et al., 2009). This may be one of the reasons for which there are still mixed findings of the empirical separation of these different knowledge facets (Charalambous et al., 2019; Depaepe et al., 2013), depending on respective measures. However, it is also clear from these lists that there are overlaps with general pedagogical knowledge—which we describe in the next section, for example in the area of assessment and in the area of dealing with heterogeneity, and that subject-specific curricular aspects also play a role (Grossman, 1990), which Shulman (1987) had rather assigned to general curricular knowledge.

3.3 General Pedagogical Knowledge

What kind of general pedagogical knowledge a mathematics teacher should possess is not an easy question. As König et al. (2011) indicate, the shape of general pedagogy is strongly influenced by cultural perspectives on the objectives of schooling and on the role of teachers (Hopmann & Riquarts, 1995). However, König et al. (2011), identify, based on a literature review, two core tasks: instruction and classroom management. "Less agreement exists as to what extent and what kind of knowledge about counseling and nurturing students' social and moral development or knowledge about school management should also be included in the area of general pedagogy" (König et al., 2011, p. 189). When it comes to knowledge about effective instruction, theories of learning, an understanding of the various educational philosophies, and general knowledge about learners (Grossmann & Richert, 1988) should be added to teachers' GPK along with knowledge about effective classroom management. By combining research on the quality of instruction and general didactics based on task analyses, König and colleagues were able to develop a framework for mathematics teachers' GPK consisting of four different dimensions of pedagogical knowledge. Thus GPK in the model of König et al. (2011) comprises knowledge about structures (structuring of learning objectives, lesson planning and structuring the lesson process, lesson evaluation), knowledge about motivation, and classroom management (achievement motivation; strategies to motivate single students or the whole group, strategies to prevent and counteract interferences, effective use of allocated time and routines). Furthermore knowledge about adaptivity (strategies of differentiation, use of a wide range of teaching methods) and knowledge about assessment (assessment types and functions, evaluation criteria, teacher expectation effects).

3.4 Beliefs

Research on teacher action assumes that the application of professional knowledge in action situations presupposes corresponding subjective beliefs (Felbrich et al., 2014; Schmotz et al., 2010). This relation makes the connection between Medleys Type E and Type F (Chapter 1.1 on pre-existing mathematics teacher characteristics) clear since pre-service teachers already have initial beliefs about teaching and learning and about mathematics at the beginning of their studies, which also influences the acquisition of professional knowledge (Blömeke et al., 2014a, 2014b; Buchholtz, 2017). Beliefs are thought to serve an orienting and action-guiding function for applying learned knowledge (Schmotz et al., 2010; Schoenfeld, 1998; Thompson, 1992). However, despite intensive research on teachers' beliefs, especially in the context of pedagogical-psychological oriented approaches, no precise and selective definition of the concept of beliefs can be discerned so far (Leder, 2019; Törner, 2002). Philipp (2007) defines beliefs as "the lenses through which one looks when interpreting the world" (p. 258). Richardson (1996) proposes a domain-unspecific definition of beliefs that are based on a broader understanding. She understands beliefs to be "psychologically held understandings, premises, or propositions about the world that are felt to be true" (Richardson, 1996, p. 103). This refers to a person's epistemological stands towards an object, which includes affective attitudes and the readiness to act (cf. Grigutsch et al., 1998) and which, in contrast to knowledge, are dependent on the degree of individual agreement (Beswick, 2005, 2007). Still beliefs are seen by many researchers as largely cognitive in nature (Beswick, 2018). So far, however, it has not been sufficiently clarified to what extend beliefs contain cognitive components, and which components can be identified. With regard to long-term development of beliefs, however, it can be assumed according to the current state of research that they are relatively stable with respect to restructuring, and to a certain extent can act as psychological "filters" and/or "barriers" (Reusser et al., 2011). On the other hand, however, beliefs can change in teachers' professional development (Eichler & Erens, 2015; Swars et al., 2009). For mathematics teachers, despite the vagueness of the term, there is a broad consensus on the differentiation of profession-related beliefs (Ernest, 1989). Among others, it is assumed that beliefs can be domain-specific (Eichler & Erens, 2015; Törner, 2002) or even situation-specific (Kuntze, 2011; Schoenfeld, 2010). With respect to epistemological beliefs about the structure of mathematics, according to Grigutsch et al. (1998), the emphasis on the formal aspect of mathematics (formalism aspect) or an orientation towards procedures and calculation schemes (schema orientation) can be brought to the fore with respect to static views. With respect to dynamic views, the application aspect and the processual character of mathematics are mostly emphasized (cf. Grigutsch et al., 1998). In addition, beliefs about the acquisition of mathematical knowledge or the teaching and learning of mathematics (Handal, 2003; Kuntze, 2011; Staub & Stern, 2002) represent another significant dimension of epistemological beliefs. Here, transmission-oriented beliefs, in which students are viewed as passive recipients of knowledge, are often distinguished from constructivist-influenced beliefs that endorse the principles of constructive learning (Staub & Stern, 2002). Although the question of how teacher beliefs influence student achievement is far from conclusive, it is likely that dynamic beliefs about mathematics and constructivist teaching–learning beliefs are more strongly related to an emphasis on processual, iterative mathematics in instructional settings (Reusser et al., 2011).

3.5 Motivation and Self-regulation Skills

Motivational research in psychology counts motivation as a personal trait which refers to the individually varying personal characteristics that constitute the reasons for and the persistence of human behaviour (Kunter, 2013; Pintrich, 2003; Rheinberg, 2006). It serves as an important predictor of how successful people can handle situational demands that occur in teaching. Thus, motivation and self-regulation are vital for teachers to succeed in their profession in the long term (Alexander, 2008; Kunter et al., 2013; Woolfolk Hoy, 2008). The beginnings of research on teacher motivation in the 1970s were still in the study of why people decide to become teachers (Lortie, 1975). Within presage-process–product research, the motivational orientation would likely be described as a characteristic of beginning teacher candidates (Type F, Chapter 1.1) and connections would be sought between career choice motivation at career entry and teachers' learning outcomes (with the goal of selective admission to the teaching career). Medley (1987) describes this as Type FE research (research in teacher selection, p. 111). However, because affective personality traits have (re)entered the professional competence research, contemporary research on professional teacher competencies examines differences in motivation and self-efficacy between practicing teachers, such as in the form of intrinsic motivation and enthusiasm for the subject of mathematics and for teaching, and further, what influence these forms of enthusiasm have on teaching quality and, if applicable, student achievement (Kunter, 2013). By this, the research goes far beyond Type F and Type E research. The description of the manageable psychological construct of self-efficacy by Bandura (1997) in the late 1990s also contributed significantly to this development. Self-regulatory skills are now also part of many studies of professional teacher competence, as the teaching profession is believed to have implications for teacher health and well-being due to its high demands. In order to meet the demanding challenges over extended periods of time, teachers need to develop self-regulation skills in order to maintain their occupational commitment over time and to preclude unfavorable motivational and emotional outcomes (Kunter et al., 2013).

4 Different Conceptualizations of Teacher Knowledge

As knowledge is considered a major component of teacher competencies, we will focus on recent conceptualizations of mathematics teacher knowledge in the following. Worldwide, many conceptualizations of professional knowledge are based on Shulman's fundamental description, such as in the U.S. the Learning Mathematics for Teaching project by the research group around Deborah Ball (LMT; cf. Hill et al., 2008a, 2008b), the study on Mathematics Knowledge in Teaching (Rowland & Ruthven, 2011) in the U.K., as well as different frameworks in Australia (Beswick & Chick, 2020; Chick et al., 2006). In Germany, the COACTIV study builds on this work (Kunter et al., 2013) but also frameworks developed by other researchers (Buchholtz et al., 2013; Dreher et al., 2016, 2018). International comparative studies such as MT21 (Schmidt et al., 2007, 2011) or the Teacher Education and Development Study in Mathematics (TEDS-M; Blömeke et al., 2014a, 2014b; Tatto, et al., 2012), also built on this work and investigated teachers' professional knowledge at the end of their education with a framework based on Shulman. A more systematic overview of the description of professional knowledge by teachers can be found, for example, in the ICMI study by Even and Ball (2009), in the Handbook by Wood et al. (2008), or in various different publications such as by Cochran-Smith and Zeichner (2005), Rowland (2014), Neubrand (2018) or Manizade and Orrill (2020). In the following, we describe some of these key frameworks that have been more widely received internationally.

4.1 Mathematical Knowledge for Teaching (MKT)

A model that has been widely acknowledged and applied internationally is the Michigan group's Mathematical Knowledge for Teaching. This approach to describing and measuring teachers' professional knowledge consists of developing a practice-based theory of the mathematical resources entailed by the work of teaching on the basis of the knowledge facets identified by Shulman. To this end, extensive observational categories were derived from mathematics tasks and observation of primary teachers' practical work with students. Thus, rather than normatively specifying Shulman's classification in technical terms, the project took, as its starting point, a requirements analysis that first identified three key responsibilities of teachers. The requirements were "(1) [t]o provide effective opportunities to learn substantial mathematics and treat the mathematics with intellectual integrity (Bruner, 1960); (2) to be able to hear student thinking, take it seriously, and make it an integral part of the instruction; and (3) to be committed to the learning of every student, and further to the learning of the class as an intellectual community" (Ball & Bass, 2009, p. 26). The goal of the project was initially to empirically study instruction to characterize the mathematical knowledge necessary "to carry out the work of teaching mathematics" (Hill et al., 2005, p. 373; Ball & Bass, 2003). In the process, knowledge facets were also specified in more detail (Ball et al., 2005, 2008), resulting in the development of a model of professional knowledge (the MKT model). MKT covers three categories that relate to teachers' subject matter knowledge: (1) common content knowledge (CCK, i.e., mathematical knowledge and skills used in settings other than teaching), which describes knowledge held in common with professionals in other mathematically intensive fields; (2) specialized content knowledge (SCK, i.e., mathematical knowledge and skills that are unique to the teaching of mathematics); and (3) horizon content knowledge (HCK, i.e., an awareness of how distinct mathematical topics are related to each other), which Bass and Ball (2009) described as an "elementary perspective on advanced knowledge that equips teachers with a broader and also more particular vision and orientation for their work" (Bass & Ball, 2009, p. 34). In contrast, there are three categories that can be considered constituent of teachers' PCK: (4) knowledge of content and students (KCS, i.e., knowledge about students' mathematical thinking or typical student errors); (5) knowledge of content and teaching (KCT, i.e., knowledge to introduce a new concept or method); and (6) knowledge of content and curriculum (i.e., knowledge on educational goals, standards, and grade levels where particular topics are typically taught) (Ball et al., 2008). Later on, the project developed measures of MKT (Hill et al., 2004) and used teachers' scores as a predictor of students' mathematics achievement. They found that "teachers' mathematical knowledge was significantly related to student achievement gains in both first and third grades […]" (Hill et al., 2005, p. 371).

Internationally, the model gained much recognition and was transferred or applied in many other countries including Ireland, Norway and Indonesia (Blömeke & Delanay, 2012; Delanay et al., 2008; Fauskanger, 2015; Ng et al., 2012). However, although widely used the model has also been criticized as the empirical differentiation of the dimensions has not been shown sufficiently and it is not clear whether the model can be transferred to the secondary level (Speer et al., 2015). Furthermore, its operationalization for the empirical measurement of teachers' knowledge and the use of multiple-choice operationalization items in a respective instrument have been criticized because this operationalization might underestimate the complexity of some of the knowledge facets (especially those involving students learning and thinking) (Manizade & Mason, 2011).

4.2 The Knowledge Quartet

Tim Rowland and his colleagues in the United Kingdom took a perspective away from the empirical testing of teachers' knowledge that is present in the Michigan project and other projects. They analyzed videotaped data from classroom observations and proposed a framework for describing the knowledge the teacher enacts in the classroom. The aim of their project, which became known as "Knowledge Quartet", was to make visible and describe the professional knowledge and beliefs acquired during training in classroom teaching situations in which this knowledge becomes visible (Rowland, 2008a, 2008b). Their theoretical framework for the observation, analysis and development of mathematics teaching has been developed in the context of primary education, although approaches to transfer to the secondary level exist (Rowland et al., 2011). The approach of the study followed methods similar to grounded theory research. The identified theoretical model consisted of four categories: (1) foundation, which describes the teachers' knowledge base; (2) transformation, which includes situations in which knowledge about chosen representations, examples, analogies, explanations, etc. is revealed—a category that takes up the ideas of PCK; (3) connection, which describes situations in which students' misconceptions are revealed, and the teacher knows about what is 'hard' or 'easy' to grasp for the students; and finally, (4) contingency, which refers to unexpected, unplanned moments, i.e. students' unexpected responses and questions (Rowland et al., 2005). The framework is now used in several countries by collaborating colleagues (including Norway, U.K., the U.S., Ireland, Turkey, Italy, Cyprus and Australia). However, qualitative reconstructive studies with a rather smaller sample size dominate the study of teacher knowledge here (e.g., Maher et al., 2022; Petrou, 2009).

4.3 Modelling Teachers' Knowledge in Relation to Teaching Practice

Researcher groups from Australia, Canada and the U.S. developed frameworks for empirical research on teachers' knowledge which especially account for the blurriness of Shulmans knowledge domains when it comes to teaching practice. The work of the Michigan group was criticized for that "the precise way in which they conceive of knowledge and how aspects of such a conception beyond 'facts that are known' is incorporated in their model is not clear" (Beswick et al., 2012, p.133). Furthermore, teachers "do not always employ the same sort of knowledge in apparently equivalent situations, and they draw upon a range of types of knowledge concerning many of their everyday tasks, moving among them seamlessly and flexibly" (ibid., p.154). In the work of the Australian researchers Beswick and colleagues, therefore, the conception of knowledge also includes teachers' beliefs and confidence as central components in corresponding frameworks, thus also taking into account affective competence characteristics in particular, which were thought to be more intertwined with knowledge facets here than in other frameworks because they have such a major impact on teachers' actions in practice (Beswick & Chick, 2020; Beswick et al., 2012). To investigate the professional knowledge of Tasmanian middle school teachers in mathematics, a profile framework was developed with eight different facets. Specifically, the framework refers to teachers' knowledge and readiness: (1) to nominate how they would improve middle school students' mathematical understandings and how mathematics might be used to enhance students' learning more broadly; (2) to outline a plan for teaching a mathematics concept that they considered important; and (3) to rate their confidence about developing their students' understanding of a range of middle school mathematics topics, and their ability to make connections between mathematics and other curriculum areas. Furthermore (4) to use of mathematics in everyday life; (5) their beliefs on mathematics teaching and learning; (6) and to anticipate appropriate and inappropriate responses that their students might give to mathematics problems and to describe how they could use each of the items in their classroom. The framework furthermore contains teachers' background variables and their perceived professional learning needs (Beswick et al., 2012). The model developed and operationalized for an empirical study thus acts as counter to highly analytic models such as MKT. In order to provide evidence-based insights into how Australian teacher education prepares mathematics teachers for their professional requirements, empirical studies examined the teacher knowledge of primary and secondary mathematics teacher education students in MCK and PCK using Rasch-scaled knowledge tests (Beswick & Goos, 2012; Goos, 2013). In particular, the studies found close empirical relationships between the two knowledge facets. Chick and her colleagues on the other hand developed a framework for analysing primary teachers' PCK for teaching decimals (Chick et al., 2006). Their framework shows especially the blurriness between content knowledge and pedagogical knowledge. It entails three categories with a large number of sub-categories in which pedagogy and content are thought intertwined and set in a mutual context.

Their PCK framework contains the knowledge of teaching strategies, knowledge of students' thinking, knowledge of representations, knowledge of the cognitive demand of tasks, knowledge of explanations, as well as resources and the curriculum. Furthermore, a category "content knowledge in a pedagogical context," covers a profound understanding of fundamental content, knowledge to deconstruct content to its key components, an awareness of mathematical structure and connections, as well as procedural knowledge when for example solving problems or using an algorithm. The third category of the framework is "pedagogical knowledge in a content context." It contains sub-categories of knowledge of the goals of learning, assessment practices, and classroom techniques that are needed for example when students need to work in groups. (Beswick & Chick, 2020).

The Canadian framework "Mathematics-for-Teaching" (Davis & Simmt, 2006) considers the complex structure of professional knowledge dynamically and distinguishes in knowledge acquisition "between the relatively stable aspects of mathematical knowledge itself and the somewhat more volatile qualities" (Davis & Simmt, 2006, p. 297). The model distinguishes relatively stable aspects of knowledge e.g. about curriculum structures or mathematics and dynamic aspects of "knowing", e.g. classroom collectivity or a subjective understanding to attend to both explicit and tacit aspects of teachers' mathematical knowledge. Other researchers describe the professional knowledge of mathematics teachers as situated within a specific mathematical content.

Important in this context are the works of Manizade and Martinovic on professional-situated knowledge in geometry (Manizade & Martinovic, 2016, 2018; Manizade & Mason, 2011) in the U.S. and Canada, respectively, which are characterized by the fact that Shulman's CK and PCK are situated and considered and scrutinized for very specific mathematical topics commonly taught in secondary mathematics, such as the area of trapezoids (see also e.g., rational numbers, Depaepe et al., 2015). The researchers highlight the importance of the development of measures of professionally-situated knowledge. They focus on developing valid and reliable measurements of mathematics teachers' situational manifestation of PCK and CK within specific geometry contexts. In their work, Manizade and Martinovic (2016) describe the following five dimensions of such knowledge, including: (1) geometry knowledge; (2) knowledge of student challenges and conceptions; (3) ability to ask diagnostic questions; (4) knowledge of applicable instructional strategies and tools; and (5) ability to provide geometric extensions with respect to a specific topic in geometry. Martinovic and Manizade (2017, 2018) describe the development of instruments—which they referred to as probes—for assessing teachers' knowledge for teaching geometry. Unlike assessing mathematics teacher competence on a more generic level, they argue the benefits of developing assessment instruments within a well-defined and narrow topic in mathematics, and of combining different measures to ensure the validity of the assessed construct.

4.4 Teachers' Knowledge About the Integration of Technology in the Classroom

With the increase of the integration of technologies and digital tools in the teaching of mathematics, necessary new developments emerged for conceptualizations of teacher knowledge. Based on the premise that technology integration efforts should be creatively designed or structured for particular subject matter ideas in specific classroom contexts, Mishra and Koehler (2006) developed the TPACK framework based on Shulman's description of PCK to describe the teacher knowledge needed when integrating technology in teaching. The TPACK framework was also later revised and adapted (Koehler & Mishra, 2008, 2009). The framework includes seven categories of knowledge: Technological knowledge (TK) includes the technical knowledge of using emerging media, including digital media, such as programs, devices, or hardware. It also includes pedagogical knowledge (PK), content knowledge (CK), and four other categories defined by the intersections of these knowledge categories. These facets embrace the technological content knowledge (TCK), which is the knowledge of how technology and subject knowledge affect each other. From the perspective of mathematics education, this includes knowledge about technical possibilities for representing mathematics, for example, through dynamic geometry programs, pedagogical content knowledge (PCK), and technological pedagogical knowledge (TPK), which is knowledge about how the use of technologies affects general teaching and learning processes. In the intersection of all knowledge areas lies the so-called technological pedagogical content knowledge (TPACK), which describes a combination of subject-specific PCK with knowledge about the use of technology for learning. TPACK also takes into account the relationship between teachers' decisions and the contextual factors of teaching, such as class size, environment, resources, and culture (Koehler & Mishra, 2009). The TPACK framework was specifically designed to enable research on the knowledge teachers need to effectively integrate technology into their teaching in a particular content area. Mathematics educational research has increasingly adopted the rather generic framework in recent years to describe mathematics-specific requirements of each knowledge facet and to explore how these develop, for example, for the area of curriculum development or in terms of describing instructional practices (Niess et al., 2009). Furthermore, the framework has been applied to observe mathematics teachers' practices in using technology in teaching and to describe them at the level of the knowledge facets involved (Muir et al., 2016; Patahuddin et al., 2016).

4.5 COACTIV

Also based on the approaches of the Michigan group and the work of Shulman, a study with representative samples of German secondary school teachers developed in the mid-2000s to investigate teachers' professional knowledge and its empirical relation to student achievement. The key factor was the facilitation of a national extension of the 2003 PISA sample, in which individual and grade-level aggregated student performance from the PISA study could be extended longitudinally and related to teacher characteristics of about 300 teachers teaching in these grades. The COACTIV research program (Baumert et al., 2010; Kunter et al., 2013) aimed to investigate the professional competencies of practicing mathematics teachers, including making statements about the relationship to student achievement. Standardized achievement tests of teacher professional knowledge were used (Krauss et al., 2008). The framework for teacher knowledge developed by COACTIV is based on content knowledge, but identifies three different facets of subject-specific knowledge: first, knowledge of student conceptions and prior knowledge (e.g., knowledge about typical student errors or the difficulty of mathematical tasks); and secondly, knowledge of subject-specific instructional strategies (for example, knowledge about representations and making content "accessible"). An innovative feature of the COACTIV theoretical framework was that subject-specific knowledge was operationalized in part through knowledge about task quality and the cognitive potential of the tasks used in the classroom. In this context, a corresponding classification of tasks used placed particular emphasis on the content-specific cognitive activation of mathematical tasks (Neubrand et al., 2013). This classification allowed "the recognition, for example, of how conceptual thinking is incorporated in a lesson, how teachers select the tasks, and if that selection influences the learning progress of the students" (Neubrand, 2018, p. 606). The research program investigated the competence of practicing German mathematics teachers differentiated in the areas of content knowledge and pedagogical content knowledge. Among other things, COACTIV found that systematic differences in performance existed between teachers for higher track secondary level in content knowledge, some of which could be attributed to differences in teacher education characteristics. A central finding of the study was also that the content knowledge of teachers was a necessary prerequisite for the acquisition of pedagogical content knowledge, but that ultimately the pedagogical content knowledge of a teacher had a greater explanatory power for predicting student performance than their content knowledge (cf. Kunter et al., 2013)—which did not mean, however, that content knowledge was less important in teacher education.

4.6 TEDS-M

The studies from the TEDS-M research program focus on different aspects of professional competencies, each with a different emphasis. While earlier international comparative studies such as TEDS-M 2008 (Blömeke et al., 2014a, 2014b; Tatto et al., 2012) or its predecessor study MT21 (Schmidt et al., 2007, 2011) focused mainly on knowledge-related (dispositional) aspects and knowledge at the end of teacher education, subsequent studies of the TEDS-M research program in Germany included in addition situational aspects of professional competence and thus also focus to a greater extent on the competencies of practicing teachers. Particular attention in the following first is given to the results of the TEDS-M 2008 study, which was commissioned by the International Association for the Evaluation of Educational Achievement (IEA) and examined the teacher professional knowledge of prospective primary and secondary mathematics teachers in 16 participating countries. With regard to the underlying framework, the TEDS-M 2008 study and its predecessor study MT21 refer to the different knowledge facets of Shulman (1986, 1987) and differentiate PCK two-dimensionally, namely along with different requirements for teachers (Döhrmann et al., 2012). Within the theoretical framework between teaching-related demands like "Mathematics Curricular Knowledge" and "Knowledge of Planning for Mathematics Teaching," as well as learning process-related demands like "Enacting Mathematics for Teaching and Learning" are distinguished (Tatto et al., 2012, p. 131). Curricular and instructional planning requirements include the selection of subject-specific teaching content for students, as well as its justification, simplification, and preparation using various representations. This therefore includes knowledge of mathematics curricula, assessment methods, and teaching methods. Interaction-related requirements, which reflect the teacher's activities during the lesson, intend to include the classification of student answers against the background of cognitive levels, possible errors, and error patterns. These are therefore analytical and diagnostic skills that prospective teachers should possess. An overview of international research findings is provided by Tatto et al. (2012). Furthermore, Blömeke and Delanay (2012) describe the current state of research from TEDS-M 2008 in a review article from the perspective of similarities and differences between TEDS-M 2008 and the Learning Mathematics for Teaching study (LMT; Hill et al., 2008a, 2008b). Meanwhile, several complementary and in-depth national analyses have emerged from TEDS-M 2008 and MT21, looking in detail at specific issues in participating countries (2014a, 2014b; Blömeke et al., 2009a, 2009b). Furthermore, within the TEDS-M research program TEDS-LT followed as a new study, expanding the concepts of TEDS-M 2008 for a German sample to both a longitudinal design and more subjects, as German and English were included besides mathematics (Blömeke et al., 2011, 2013).

5 Recent Extensions in the Concept of Mathematics Teacher Competence

Despite the blurry lines between CK and PCK, like Kaiser and König (2019) note, several studies to date have provided evidence that the knowledge facets as proposed by Shulman (1987) can be theoretically and empirically differentiated and separated (e.g., Blömeke et al., 2016; Krauss et al., 2008), provided that appropriate instruments, topics and sampling are used. Fundamental to this were scientific studies that examined the structure of professional knowledge in particular. Regarding the correlations between the specific facets, it turned out that, "as Shulman (1987) with his "amalgam" hypothesis on the nature of PCK suggested PCK is related to both CK and GPK, whereas CK and GPK are more distant to each other" (Kaiser & König,

2019. p. 603f.). For example, in the COACTIV study, a strong correlation between CK and PCK was found (0.61) (Baumert et al., 2010). Important scientific developments about the professional competence of teachers can be located especially in the last five to ten years. Since teachers access different forms of their professional knowledge in different instructional contexts—so the assumption—it seems reasonable to focus not only on the structure but especially on its application in different teaching situations when examining professional knowledge (Even et al., 2017; Kaiser et al., 2015; Rowland, 2008b). Thus, as a new guiding question in research on mathematics teacher competencies, knowledge and skills, if we follow up on Medley's Type E, it was added how content knowledge, pedagogical content knowledge, and general pedagogical knowledge can be surveyed in connection with teaching practice using suitable instruments, which led in particular to the investigation of situation -specific skills, in other studies referred to as professional noticing (Sherin et al., 2011; Van Es & Sherin, 2008).

5.1 Situational Aspects of Mathematics Teachers' Professional Competencies

When situational aspects of teachers' professional competencies are addressed in the context of empirical studies, the main aim is to survey competencies as closely as possible to real situations from everyday teaching. With their conceptualization of competence as a continuum, Blömeke et al. (2015) aimed to overcome an opposition that had increasingly emerged between different approaches to understanding competence. On one hand, there existed the analytical approach of dispositional aspects of competence, which formed essentially the basis of cognitively oriented empirical studies from educational research mainly using paper-and-pencil tests. According to this approach, one starts from analytically separable areas of competence (e.g., the knowledge facets) which can then be measured and considered in terms of their structural relationships. The goal here is to promote specific competencies as a resource for behavior in specific situations. As we described, competence here includes both cognitive and affective-motivational domains. The analytical approach was now opposed by a holistic approach in the research tradition from organizational psychology, which focused on the observation of behavior and performance in an appropriate real-life context. Competence then influences this behavior, whereby competence is still understood as a collection of diverse cognitive and affective-motivational components that constantly change—depending on the situation and requirements. The idea of Blömeke and her colleagues was to combine both approaches in a common continuous model. Specifically, they assume that the behavior of, for example, a teacher in concrete situations is influenced by his or her competence (in the sense of the holistic approach). However, competence is then not understood as a constantly changing collection of different components, but as a fixed sum of clearly describable individual components (in the sense of the analytical approach).

The starting point of the new model of competence as a continuum is the disposition of a teacher, which is characterized by cognitive (CK, PCK, GPK) and affective-motivational areas (a.o. beliefs). These cognitive and affective-motivational dispositions are complemented by situation-specific skills, which are also referred to as professional noticing (here, the teachers' noticing discussion plays a role, in particular, see Sherin et al., 2011). That is, in a specific situation, a teacher first perceives the situation, interprets what is perceived, and makes appropriate decisions. The teacher does this influenced by the situation at hand, but of course also by their basic disposition. Based on the teacher's perception, interpretation, and decision, their actual actions in the situation then emerge. It is therefore said that professional noticing consisting of the areas of perception, interpretation, and decision-making plays a mediating or transforming role between disposition and actual action which is an observable performance. While pure surveys with tests represent a proven possibility for the investigation of competence in the sense of the analytical approach (for example with instruments of MKT or TEDS-M), it is immediately clear that situational aspects are difficult to assess in this way, because the reality of teaching can only be represented in test items to a limited extent. An alternative way of assessing competence in a situation-related manner is the use of video vignettes or dynamic geometry software as a stimulus for answering test items. Subsequently, many recent studies investigating situational teacher competence built on the use of video vignettes (e.g., Bruckmaier et al., 2016; Kaiser & König, 2019; Kersting, 2008; Kersting et al., 2010; Knievel et al., 2015; Seidel & Stürmer, 2014). Martinovic and Manizade (2020) for instance used interactive dynamic instruments (that incorporate dynamic software such as GeoGebra) to mimic the classroom simulations and a variety of student responses to a given mathematics problem question. This way, they evaluated teachers' professionally situated knowledge (PCK and CK) based on teachers' responses to the questions that follow up a dynamic simulation.

5.2 *Further Developments of the Studies of the TEDS-M Research Program*

The further developments of the TEDS-M research program in Germany, which aim at investigating the competence development of mathematics teachers in the first years of their professional activity, are also based on this approach. Central to this is the outlined understanding of competence as a continuum. In addition, expertise research (Berliner, 2001) with its basic distinction between experts and novices forms a central pillar of the conceptual framework for further developments. Specifically, the different areas of teacher knowledge from TEDS-M were conceptually supplemented by the situation-specific skills of professional noticing, which were surveyed with video vignettes aimed at eliciting different aspects of expertise. The TEDS-M

Follow up study (TEDS-FU) for example measured perception, interpretation, and decision-making as facets of professional noticing of in-service mathematics teachers (Kaiser et al., 2015); The relation of knowledge and noticing concerning GPK was evaluated by König et al. (2014), differentiating mathematics teachers' pedagogical competence into knowledge and noticing facets. Kaiser and König (2019, p. 605) also report structural connections in this context, with a connection between dispositional and situational facets of professional competence being particularly evident in interpreting classroom perception: "Whereas teacher knowledge and interpretation skills are moderately related to each other (0.37), perception is only loosely related to interpretation (0.17) and knowledge (0.13)."

5.3 Relationships of Teacher Competencies to Instructional Quality and Student Achievement

The results presented so far give us clues about the relationship between teachers' knowledge and their skills. What needs to be questioned, however, is why appropriate skills were considered valuable components of teacher competence in the first place. One obvious answer is that skills in the area of professional noticing help with the design of instruction and are linked to this assumption that ultimately student achievement can also be improved by good instruction. Specifically, some studies in recent years have surveyed the direct relationship between teacher skills and instructional quality (Hill et al., 2008a, 2008b; Santagata & Lee, 2021). In the TEDS-Instruct study and the TEDS-Validate study, for example, two observers each assessed lower secondary mathematics teaching on different criteria using a comprehensive rating manual that focused on four facets of teaching quality, namely efficient classroom management, constructive support, the potential for cognitive activation, and content-related structuring (for details Schlesinger et al., 2018). At the same time, results from the subject-related competence facets were available for the participating teachers, which were collected using TEDS-M and TEDS-FU instruments (Blömeke et al., 2020). Thereby, efficient classroom management did not correlate significantly with the subject-related competence facets. The remaining three quality dimensions correlate significantly positively with teachers' professional noticing of mathematics teaching, but not consistently with subject-related knowledge facets (Jentsch et al., 2021). As TEDS-Validate and TEDS-Instruct furthermore had access to the results of students' achievement tests, the studies especially offer the opportunity to fully observe the linkage between teachers' competences, instructional quality, and students' achievements. Results revealed that with regard to the dimensions of instructional quality cognitive activation was found as a predictor for students' progress in achievement. In addition, general pedagogical knowledge and situation-specific classroom management expertise (CME) serve as predictors for instructional quality (GPK for all three dimensions, CME only for cognitive activation). Furthermore, there is a direct effect of teachers' professional competence on students' achievement but without mediation by the instructional quality (König et al., 2021). Also, other studies investigated the relationships between professional competence, teaching quality, and student achievement (cf. Kaiser & König, 2019, p. 606). In the COACTIV study, a strong positive effect of PCK on student learning progress was found to be mediated by the quality of instruction. In particular, the dimensions of cognitive activation and individual learning support played a crucial role. For CK, however, the mediation model applied only to a very limited extent. Despite the high correlation with PCK, teachers' CK had lower predictive power for students' learning progress (Baumert et al., 2010). Similarly, Hill et al. (2005) and Hill and Chin (2018) furthermore showed that teachers' knowledge and their instructional quality were significantly related to students' outcomes.

6 Concluding Remarks

In the present chapter, we provided an overview of important lines of development and the evolution of mathematics education research on professional teacher competencies, knowledge and skills. Research has evolved from the process–product paradigm and has been developed especially in the period of 30 years after Medley's (1987) reflections. The starting point in this process were basic theoretical reflections on teachers' professional knowledge, which were strongly influenced by cognitivism. Subsequently, an independent branch of research in mathematics education developed, which dealt with the professional competence of teachers, thus broadening the focus by not only taking single cognitive aspects into account. As in Medley's time, the starting point to this shift in the research was the intention to measure and describe what makes a good teacher and how to improve student achievement in mathematics. From the critique of the studies in the following years, research evolved further towards the inclusion of more situation-specific teacher competencies, examining connections and effects between the different variables within the chain of effects, namely teachers' competence, instructional quality, and student achievements.

What have these developments in common? The developments represent decisive improvements with regard to the systematic inclusion of personality characteristics of teachers as well as the contextual conditions in which teacher competencies come into play. It is clear that different conceptualizations of teacher competencies still take into account, to varying degrees, the same variables that Medley (1987) had already considered, although in the meantime a stronger emphasis on the subject-specific characteristics of mathematics has also been taken into account.

However, new conceptual challenges arose as a result of further developments. Thus, after many years, as we describe, currently a large variety of frameworks on teacher competencies, knowledge and skills is available internationally, each describing teacher competence differently and thus setting different emphases. Conceptualizations are based on different domains attributed to teachers' professional knowledge, differ in their theoretical assumptions, and also have different grain-size of the knowledge elements considered (Even et al., 2017; Neubrand, 2018). On the one hand, the boundaries of what is understood by teacher competencies in certain domains are pragmatically determined from theoretical considerations or in the context of empirical studies, but on the other hand, Delphi methods, or the Grounded Theory Approach, for example, could also be used to develop content-valid conceptualizations (Manizade & Mason, 2011; Martinovic & Manizade, 2017). Either way, however, the conceptualizations of teacher competencies, knowledge, and skills for research purposes remain normative—and thus dependent on cultural traditions, epistemologies, and values. We expect the field to evolve further with great progress in the next years.

While we often assume that mathematics education is culture-neutral, research indicates that the way in which we express ourselves and view mathematics is in fact highly cultural (Leung et al., 2006). Although many of these different frameworks are used in several countries to assess teacher competencies, the cultural dependency of the frameworks should not be overlooked (Blömeke & Delanay, 2012), so that a transfer to other educational systems is by no means trivial and should require validation studies (e.g. Yang et al., 2018). In the future, therefore, it can be assumed that the cultural sensitivity of research on teacher competencies will be more critically scrutinized. International research on teacher competencies can nevertheless benefit from this polyphony, although it suffers from it at the same time. The multiplicity and diversity of frameworks need not be seen as confusion but can be seen as richness—if one takes a comparative perspective, however, it seems profitable when frameworks and conceptualizations are synthesized and compared based on their similarities and differences.

What is clear from our overview, however, is that after more than three decades of developing research on teacher competencies, knowledge, and skills, there are still methodological challenges to empirical measurement. Certainly, current tools of measuring allow us to capture teacher competencies more accurately than in the past. Methodological advances such as multilevel structural equation modelling (Teo & Khine, 2009) allow for the consideration of numerous relevant (background) variables and differences between individuals, classes, and schools when examining relationships between teacher competencies and student outcomes. These analyses can be used to identify interactions between teachers' characteristics, personal and affective traits, and various other factors, all those that are related to teacher competencies. Nevertheless, even today we do not have the means to realize Medley's vision of taking into account the interrelationships of all variables in studies, and often only proxies can be used for variables to be measured so that even in the future the validity of measurement instruments, in particular, will have to be critically analyzed. However, these methodological advances should still not lead research on teacher competencies to neglect mediating variables in the chain of connections between teacher competencies and student outcomes. Teaching activities in instructional quality as a mediating variable, and thus the situatedness of teacher competencies has played and probably will play an increasingly central role as a site for observing and measuring competencies, especially in recent years.

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