The Evolution of Research on Teaching Mathematics: International Perspectives in the Digital Era: Introduction

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1 Introduction

Mathematics teaching is subject to cultural and temporal conditions. Not only do school and societal conditions shift, and with them the composition of the student body, but also curricular regulations and new mathematical and pedagogical insights determine the content to be taught and the approach to learning used in mathematics classes. To reflect on mathematics teaching in a changing world, there is a need for continuous scientific research into this process of teaching mathematics. Results of this research also have a retrospective impact on mathematics teacher education insofar as the conditions of education need to be continuously adapted to the professional requirements of teachers in practice. Research on teaching mathematics thus bears a great responsibility and is a constantly evolving field of research for scholars around the globe.

This book comes at the time when the world is facing an ongoing global pandemic and experiencing violence and unrest due to active war. This publication symbolizes

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A. Manizade et al. (eds.), *The Evolution of Research on Teaching Mathematics*, Mathematics Education in the Digital Era 22,
https://doi.org/10.1007/978-3-031-31193-2_1

a professional commitment and international collaboration par excellence apropos teaching mathematics. The editors from three different continents and researchers who represent sixteen institutions and eight countries worked constructively and collaboratively with utmost respect for each other, with intentions to reflect on existing research knowledge and to create new knowledge that can be shared and used by other educators and researchers across the world.

In preparation for this book, our international group of researchers shared current issues related to the evolution of research on teaching mathematics. We examined the present state of research on mathematics teaching and discussed the theoretical and methodological challenges associated with it, including issues related to conceptualization, instrumentation, and design. Additionally, we explored the likely direction of future research developments. In our literature review and discussions on this project, it became evident that studies on teaching frequently establish direct relationships between units of analysis that, at first glance, cannot be assumed to be directly related in a chain of effects. There are examples of studies presented in this book that directly relate teacher competencies to student achievements using empirical measurement models in a causal or relational way. Without criticizing these studies across the board, however, it seems reasonable to consider moderating or intermediate variables in this chain of effects (Baron & Kenny, 1986), such as the initiated student learning activities observable by teachers in the classroom, aspects of instructional quality (e.g., classroom management or cognitive activation), or corresponding student variables such as attention and cooperation in class or students' prior knowledge (e.g., Fig. 1).

Although there are researchers who do indeed study mediating variables (e.g., Blömeke et al., 2022), it became clear to us that there is a lack of a systematic scientific overview of the complete chain of effects between teacher characteristics, activities, and students' learning processes. Overviews of precisely these aspects of research on teaching and respective studies are scarce, which inspired this book.

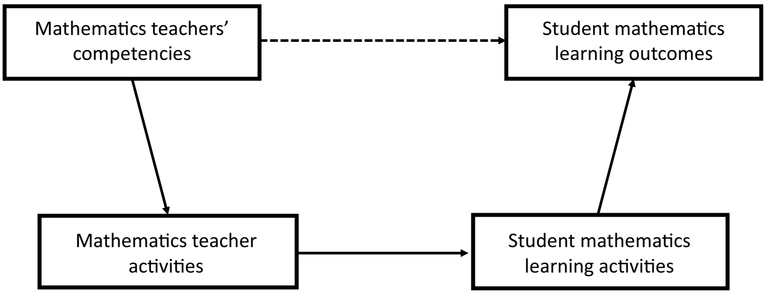

Fig. 1 Example of a chain of effect in teaching

2 The Purpose of the Book

Research that aims to relate teachers' observable actions with students' gains in achievement is referred to as *process–product research*. The term was first used by Donald M. Medley and Harold E. Mitzel (Hunt et al., 2010; Medley & Mitzel, 1963). *Presage-process–product research* then also considered other important variables, namely all the preceding and mediating variables that influence the actions of teachers in the classroom, such as teachers' professional training, knowledge, competencies, skills, personality traits, and teachers' abilities to plan a lesson or assess students. The framework for this book was based on a 1987 seminal work called "Evolution of research on teaching" by Medley (1987), in which he discussed literature on the development of research on teaching for thirty years prior to that publication vis-à-vis the presage-process–product standpoint. In it, he described a set of essential variables of research on teaching as given in Fig. 2, which he labelled online variables - "ones which lie along a direct line of influence of the teacher on pupil learning" (p. 105) and offline variables, "ones which affect pupil learning but are not under the direct control of the teacher." (ibid.).

Updating this framework is timely and, since it has not been described for mathematics teaching in particular, the framework was adapted and applied in the context of mathematics teaching and mathematics teacher education, as presented in Fig. 3 (Manizade et al., 2019). In the past twenty to thirty years, research on teaching has evolved further, and researchers have used a wide range of conceptual and theoretical frameworks in an effort to advance knowledge in presage-process–product research in mathematics education (e.g., Blömeke et al., 2016; Buchholtz, 2017; Liljedahl, 2016; Manizade & Martinovic, 2018). For this reason, the terms of the variables used by Medley (1987) have been adapted to the current research discourse. Although the

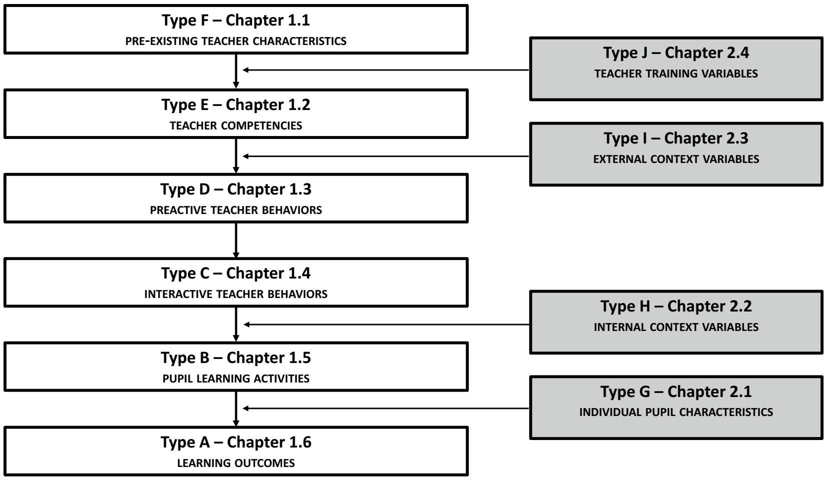

Fig. 2 Representation of Medley's 1987 framework mapped to the book's chapters

field of research on teaching mathematics has considerably advanced during the past twenty to thirty years, we find that the main units of analysis in the current research studies have remained the same: thus, Medley's framework is still valuable as it gives an orientation to all possible variables that become apparent qua the chain of effects from teacher behavior to student achievements. Moreover, the abiding challenges associated with the conceptualization, instrumentation, operationalization, and research design that Medley described are still complex, despite recent advances in technology and research methodology in the digital era.

One of the aims of the book is to update and situate Medley's framework within mathematics education research of the last three decades. Societal and educational realities have changed significantly since Medley wrote his seminal paper. Therefore, based on current research, additional variables must be considered in the chain of effects. Another goal is to provide researchers, who are scientifically concerned with more than one main unit of analysis—as described in Fig. 3—with current knowledge and methods apropos of the respective variables in the overview chapters. Each chapter of the book is based on reviews of research conducted over the past twenty to thirty years and written by leading experts in the respective fields. The chapters therefore also address cultural and technological aspects of the research on the respective variables.

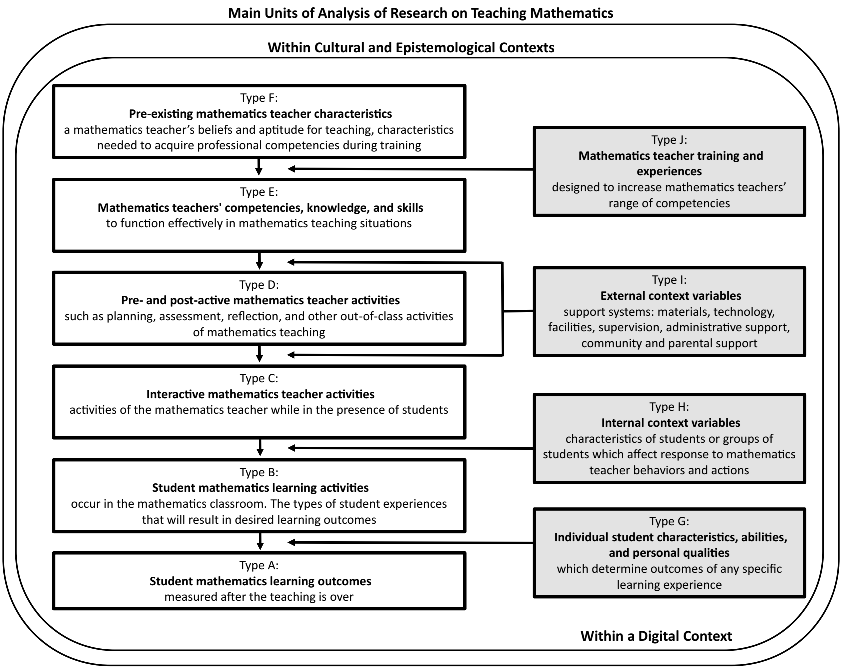

Fig. 3 Updated framework of research on teaching mathematics

Additionally, in his original work, Medley focused on discussion surrounding *good teaching* and the complexity of defining such a term in research (Medley, 1987). In the past twenty to thirty years, myriad of new theoretical perspectives on teaching mathematics have emerged in the field. These perspectives assume that a wide range of mathematics learning goals based on theoretical frameworks are enacted by teachers in the classroom (Manizade et al., this volume). Depending on these goals, the definition of *good teaching* and what is valued in the mathematics classroom can have an array of meanings (Manizade et al., this volume). These include reproducing the perfect sequence of steps when solving a mathematical problem, engaging students in productive struggle and productive failure, developing mathematical constructs through collaborative discourse, and addressing students' lived cultural experiences as mathematical experiences, to name a few. The updated framework, therefore, considers the epistemological contexts of research on teaching mathematics with respect to main units of analysis, in addition to considering the cultural and digital contexts that also affect all units of analyses of research presented in the framework (Fig. 3).

3 Book Structure

The book is comprised of two parts. In part one, we examine research in mathematics education with focus on units of analysis that Medley called *online variables* (Medley, 1987). In contrast to current use, the term *online* has a distinct and different meaning in Medley's work. *Online variables* are units of analysis of research that can be under the control of mathematics teachers. They included research on mathematics teaching and teacher education that examined: pre-existing mathematics teacher characteristics (Type F); mathematics teacher competencies, knowledge, and skills (Type E); pre-post-active mathematics teacher activities (Type D); interactive mathematics teacher activities (Type C); student mathematics learning activities (Type B); and student mathematics learning outcomes (Type A) (Fig. 3).

In part two, we examine mathematics education research with main units of analysis that are not under the direct control of teachers. These include *offline research variables* (Medley, 1987) such as individual student characteristics, abilities, and personal qualities (Type G); internal context variables (Type H); external context variables (Type I); and mathematics teacher training and experiences (Type J). A detailed discussion of both parts of the book is presented later in this chapter. Because the offline (Types J, I H, and G) research foci that are not under the direct control of mathematics teachers are so broad, our authors selected a subset of research variables within each type to discuss in their respective chapters included in part two of the book. We understand the importance of each research focus and unit of analysis and acknowledge that a larger publication would be needed to include all their components.

In the following section, we give an overview of the individual units of analysis of research on teaching mathematics, as well as the chapters of the book.

4 Part 1: Online Variables

4.1 Pre-Existing Mathematics Teacher Characteristics

Pre-existing teacher characteristics include abilities, knowledge, and attitudes that a candidate for admission to a teacher preparation program possesses on entry, as well as a candidate's aptitude for teaching. In order for teachers to learn the necessary competencies for teaching in teacher education processes, they must possess appropriate entry-level prerequisites that sustain competency development.

Mathematics teacher competencies include, for example, cognitive abilities such as prior mathematical and pedagogical knowledge at the point of study entry, attitudes toward mathematics as a subject or toward the learning and teaching mathematics, as well as motivational and volitional variables such as enthusiasm for the subject of mathematics and personality traits and identity aspects such as one's own understanding of one's role, self-regulation and self-concept, and ability to reflect and collaborate with students and with colleagues. More recent research also counts emotional aspects such as personal well-being or stress resilience among personal factors that play a role in competence acquisition at entry level. It should be noted here that all the influencing variables themselves also change in the context of teacher education. That is, in line with Medley, the changeability of personality structures is assumed.

In Chap. 2, Olive Chapman compiles findings on these main research units of analyses based on extensive literature reviews spanning over more than twenty years. With respect to the prior mathematical knowledge of pre-service teachers, Chapman focuses on studies in the content area of fractions, whole number operations, geometry and algebraic thinking and problem-posing. Many of the current studies demonstrate, in part, large gaps in knowledge related to conceptual understanding of elementary mathematical concepts and operations, which pose an ongoing challenge to teacher education. For the area of prior mathematics-related pedagogical knowledge, Chapman focuses on studies examining skills in observing instruction and noticing and analyzing student work and thinking and evaluating tasks. Here, too, the systematic review revealed weaknesses among beginning pre-service teachers who, for example, can generate few pedagogical decisions from observations of instruction or fail to recognize the potential of mathematics tasks. In the area of attitudes, Chapman adds to existing findings with those related to attitudes toward technology use and mathematical processes or specific mathematics areas such as algebra.

Overall, Chapman notes a shift in studies over the past twenty years away from focusing on single "hard" categories, such as high school graduation or mathematics grades, to examining content aspects of prior knowledge and learning conditions including those influenced by culture and technology at the beginning of the teacher education program. Finally, Chap. 2 also addresses methodological challenges and future directions for Type F research, including different survey formats, designs, and methods of research analysis.

4.2 Mathematics Teachers' Competencies, Knowledge and Skills

Medley described Type E teacher competencies as knowledges, skills, and values that a teacher possesses. Without going into detail about what exactly is meant by competencies, knowledges, or skills, he describes these as the "tools" of teaching in an instrumental, functional sense. They are the prerequisites for successful and competent teacher action in various situations. This assumes that the prerequisites for teaching can be precisely specified for a given situation - as is done in later research, for example, through requirements analysis by observing teachers. Interestingly, Medley also included values in these prerequisites and thus included affective characteristics of teachers among the competencies. A conceptual understanding of competency can be discerned here, the scope of which was recognized in the early 2000s in the educational psychology discussion on the conceptual understanding of competencies and was more widely received. In contrast to Type F, however, Medley saw this online variable less as the personality characteristics of teachers. He understood teacher competencies as a measurable outcome of teacher education and experiences - in contrast to Type F, pre-existing mathematics teacher characteristics. Teacher competencies thus always remain a potential trait in the exclusive research of Type E, since the (measurable) performance of these competencies only takes place in the actual preparation and implementation of teaching (Type D and C).

In Chap. 3, Nils Buchholtz, Björn Schwarz, and Gabriele Kaiser describe the development of mathematics education research on teacher competencies in the last 30 years, especially the research on teacher knowledge and affective variables such as beliefs or self-regulatory skills. For the subject of mathematics, normative requirements have always been formulated for teachers in terms of their content knowledge. However, the researchers see the starting point of research on Type E in psychological cognition research, which has strongly influenced research on mathematics teaching and teacher education. At its starting point, research on Type E was thus still closely aligned with Medley's description. However, Buchholtz, Schwarz and Kaiser describe how Lee Shulman's work in particular inspired, developed, and advanced the research. A broad research field of qualitative and quantitative studies on teacher cognitions developed, resulting in a plurality of different conceptualizations of teacher knowledge that refer to different knowledge bases (mainly: content knowledge, pedagogical content knowledge, pedagogical knowledge). Teacher competencies are thus conceived in research as a multidimensional construct, the complexity of which poses major challenges to research in terms of its measurability. Different ways of measurement (especially through knowledge tests) have been used in research. Overall, the plurality in a research field is perceived as a strength, especially since it is broadly based internationally. In recent years, research on teacher competencies has started to focus more on the situational performance of competencies, which has already extended the focus from Type E to Types

D and C. The reason for this development has been on the one hand methodological developments through video-based competence measurement, and on the other hand the increasing conviction that teacher competencies can only be examined to a limited extent outside of the situational context of practical teaching. That is, an isolated consideration of Type E is less insightful. To this end, the chapter provides an overview of current research on situational-based mathematics teacher competency measurement and the relationships among teacher competencies, instructional quality, and student outcomes.

4.3 Pre- and Post-Active Mathematics Teacher Activities

In his original work, Medley referred to the online variable, Type D, as *preactive teacher behaviors.* These included such activities as "planning, evaluation, and other out-of-class activities of teaching, the things a teacher does to promote pupil learning while no pupils are present". These are practices that demonstrate how teachers' professional competencies knowledge and skills (Type E) affect the quality of their classroom interactions with students (Type C), and therefore, indicate how successfully the teacher can meet their goals for teaching.

In their Chap. 4, Agida Manizade, Alex Moore, and Kim Beswick named this variable *pre- and post- active* because several of the Type D activities (e.g., lesson, and unit planning) are performed prior to teaching, while others (e.g., reflection, and assessment) are conducted after lessons have been taught. Manizade, Moore, and Beswick focused on lesson planning, assessment, and reflection as the key actions that teachers perform when students are not present in the classroom. These "pre- and post-" actions are the most direct ways through which teachers shape observable teaching work, as mediated by their goals for teaching. These goals are representations of teachers' epistemological commitments apropos teaching mathematics, whether those commitments be consciously espoused or unconsciously reproduced due to constraints within which they work. The researchers surveyed the literature on lesson planning, assessment, and reflection according to eight epistemological paradigms that are known in the field of mathematics teaching, namely Situated Learning Theory, Behaviorism, Cognitive Learning Theory, Social Constructivism, Structuralism, Problem Solving, Culturally Relevant Pedagogy, and Project- and Problem-Based Learning. They place other perspectives on learning theory, which are derivatives of these prevailing paradigms, within this overarching frame. They detail each perspective, providing a definition, goals for teaching, pros and cons of each theoretical perspective, and examples from the literature on teaching mathematics. The chapter revealed that some of the theoretical perspectives are well-reported in the literature whilst others have not received the same amount of attention from researchers. The researchers recognized that the chapter focused on the western cultural context and more research is needed in a variety of cultural settings, considering each of the settings affects every unit of analysis in research on mathematics teaching and teacher education (Fig. 3). The researchers posited that, amidst cultural contexts and the technological advent of the digital era of mathematics education, researchers must engage more explicitly with the theoretical perspectives identified as underserved and must themselves reckon with their own epistemological commitments more intentionally when engaging and reporting on studies regarding Type D.

4.4 Interactive Mathematics Teacher Activities

Medley (1987) described interactive teacher behaviors as "the behaviors of the teacher while in the presence of students" (p. 105). He explained that these behaviors are typically what are referred to as teaching and are the means through which teachers influence students. They are directly observable actions through which teachers translate their pre-post-active behaviors (i.e., planning and other out-of-class activities, Type D) into learning experiences for students. They are the bridge between teachers' plans to promote student learning (Type D) and the things that students do that result in their learning (Type B).

In Chap. 5 Kim Beswick, Felicity Rawlings-Sanaei, and Laura Tuohilampi discuss the research literature related to the activities that mathematics teachers engage in when they are with students. Importantly in the digital era teachers can be with students without being physically with them. Teachers' interactive behaviors in online or virtual contexts remain under-researched but have attracted increased attention in recent years in which the pandemic forced the closure of schools for periods of weeks or months in many countries, necessitating a move to online interaction.

The authors structure their chapter in two main parts. The first surveys what we know about normative teaching practices; the things that typically happen in mathematics classrooms whether physical or virtual. They rely primarily on large scale studies, principally the *Trends in International Mathematics and Science Study* (TIMSS) and the *Programme for International Student Assessment* surveys (PISA). These studies rely on teacher self-reports as well as student reports of the activity that occurs in their mathematics classrooms. TIMSS video studies provided more direct access to teacher behaviors but have still relied on teachers to indicate the extent to which the video-recorded lessons were typical of their practice. The second part of Chap. 5 deals with teachers' interactive behaviors documented by researchers interested in promoting or supporting teachers to implement particular behaviors or to adopt in some way an approach to mathematics teaching that the researchers believe will be beneficial. Beswick et al. describe the interactive behaviors reported in these studies as atypical because they represent approximations of changed behaviors that align with the researchers' perspective.

4.5 Student Mathematics Learning Activities

The variable Type B was described by Medley as student learning activities. By this, he meant all types of student experiences within the classroom that result in the learning outcomes desired by the teacher. These student activities and behaviors always take place under teaching objectives, in that they are directed or oriented by the teacher and therefore a direct result of an interactive teacher's behavior (Type C). For a direct influence of teaching activities on student learning to be assumed, Medley presupposed that all learning is based on learners' activity. That is, student activity can be used as an indicator of learning processes. Most particularly, therefore, it is important that any activity is perceived as purposeful.

Maria Timmerman addresses this purpose of students' learning activities from a mathematical perspective in Chap. 6, presenting different ways in which students' learning activities could be understood and seen as productive or purposeful for learning mathematics. She illustrates that effective and equitable experiences of students are related to how mathematics learning has been defined over recent decades, in different countries internationally and also under different educational premises, whereby respective curricula can provide an orienting framework.

Timmerman notes a shift in mathematics education research towards student thinking over the last 30 years, where the focus is no longer exclusively on student behavior. This has been driven by developments of new epistemological perspectives on mathematics teaching, and the development of new curricular objectives, including but not limited to problem-solving, or project-based learning activities. Additionally, process-oriented goals, in contrast to the teaching of pure factual knowledge as well as the competence orientation, have fundamentally changed student learning activities by broadening the horizon of what over the years is considered as a learning activity in mathematics.

Regarding the development of the theoretical perspective on student learning activities, Timmerman describes different conceptualizations of student learning in mathematics, including the theory of progressive coordination of actions and the development of cognitive schemata, the research model of learning through activity, and research on student engagement, which plays a particularly important role in problem solving processes. Timmerman also focuses on how in the context of such activities, the affective learning conditions of the students, such as productive dispositions or student perseverance, which can positively influence student learning activities (e.g., when students are struggling or failing and can use this for learning processes). This also brings the individual prerequisites of students (Type H) more into focus when examining the effectiveness of learning activities.

4.6 Student Mathematics Learning Outcomes

Medley identified student learning outcomes as the first online variable (Type A), which he associated with each type of "changes in pupils" (p. 105) that can be measured after teaching has been completed. He referred to the outcome of a completed learning process, which at that time was primarily measured in the form of achievement gains on standardized tests. In this sense, he called it a "production of learning outcomes" (p.105) as a result of teaching with the attention given to progress towards teaching goals that could be detected through close observation. Learning outcomes are seen as the ultimate goal and the measurability criterion of teaching effectiveness. There are, however, challenges associated with the measurability of this criterion, that specifically relate to different theoretical frameworks and approaches used for teaching mathematics. These challenges, therefore, continue to be a part of the mathematics education research discourse.

In Chap. 7, Jelena Radišić presents an overview of the challenge of describing mathematical understanding and knowledge as a measurable learning outcome, addressing different conceptualizations of mathematical competence, literacy, or proficiency. Making something as vague as mathematical understanding measurable based of certain criteria remains a challenge of mathematics education research to this day. Various mathematical activities, such as problem-solving, modelling, reasoning, and proving have continuously found their way into mathematics education curricula internationally over the last 30 years and still elude measurability of mastery. For this reason, teaching effectiveness that is measured according to students' acquisition of these skills, is challenging. Jelena Radišić's research perspective is based on international large-scale assessment studies (ILSAs), which have been developed internationally since the late 1980s for comparative educational monitoring and which still today systematically collect and compare learning outcomes on the basis of high scientific standards. Since the studies are almost exclusively methodologically quantitative and use big data by collecting a large number of variables on many cases, they now allow the simultaneous statistical correlation of multiple variables and consideration of different contextual conditions in the tradition of presage-process–product research. Whereas Medley's assessment of "good teaching" with respect to Type A tended to be general in its maximization of learning outcomes, today's Type A research takes a more nuanced view in measuring effectiveness of learning for students with individual learning needs.

The fact that specific methodological problems arise with the measurement of student outcomes is addressed in the chapter, as is the growing influence that technology has on learning and therefore on our understanding of learning outcomes. Finally, Radišić takes a new perspective on research on Type A by describing affective variables such as student motivation and self-belief as learning outcomes in their own right. Affective variables remain underrepresented in research on teaching.

5 Part 2: Offline Variables

5.1 Individual Student Characteristics, Abilities and Personal Qualities

In Medley's model, individual student characteristics (Type G), that is abilities and other personal qualities of students, mediate between student learning activities (Type B) and student learning outcomes (Type A). This mediating offline variable is explained by the observation that students do not show the same outcome even under identical learning conditions. Learning processes in the classroom depend to a large extent on individual students' cognitive and affective preconditions, which can be shaped by family, social, cultural identity-forming experiences, and physical conditions.

Education is increasingly characterized by high levels of student diversity in many countries due to migration movements and cultural and transnational multiple attributions. Individual student characteristics can, therefore, include variables such as race, gender, or socio-economic background. The language requirements of students today are diversified to a greater extent than in Medley's time. In many countries, students with special educational needs are included in mainstream education, so that learning processes are also influenced by students' physical or social-emotional development and how they can overcome learning difficulties or learning disabilities. Mathematics education research also takes up emotional and physical characteristics such as resilience, mathematics anxiety, or students well-being as psychological variables influencing the individual learning process.

In Chap. 8, Rhonda Faragher describes central aspects of Type G in an overview and focuses on the subset of Type G, namely learners with intellectual disabilities, learning difficulties, and learned difficulties. She starts by describing two significant developments in the last decades: the recognition of streaming (tracking) as harmful; and the recognition of inclusive education as beneficial. These have changed the nature of mathematics classrooms substantially. Faragher first describes different approaches of mathematics education, neuro-psychological research, and general pedagogical research on special needs education to understand learning difficulties and learning disabilities of students and to make them accessible for research. She then presents different approaches that have developed in recent years to address the impact of these learning difficulties and learning disabilities on student achievement in the classroom and to provide equal opportunities for all students. The researcher claims that in doing so, teachers can adapt instruction in ways such as by the use of Universal Design for Learning (UDL), using digital tools that make instructional content more accessible to students, or adapting curriculum and learning activities to students' achievement levels and prior knowledge. Faragher uses case studies of achieving equity for students with Down syndrome to illustrate the latter throughout the chapter. Faragher argues that with the increasing acceptance and implementation of inclusive learning in the classroom, in research the Type G offline variable is ultimately not only a mediator between Type B and Type A, but as the direction of future research, this offline variable must also play a role in other research variables, for example when teachers' lesson-planning is analyzed or appropriate support structures are created in schools.

5.2 Internal Context Variables

Internal context variables (Type H) affect individual or group student responses to any teacher actions in the classroom. They mediate between the interactive teacher behaviors (Type C) and the learning activities (Type B), thus influencing the way students respond to the teacher in social interaction and behave during initiated or mediated learning activities. By its nature, the Type H variable is close in content to the Type G variable, as psychosocial factors of student diversity are both evident at the individual level of learning processes and express their collective expression in the responses of students or groups of students to the teacher's teaching activities. This may include, for example, students' work behavior, motivation, self-efficacy, or self-regulation. Recent mathematics education research has also focused on the social and emotional experience of students and their well-being in the classroom. The offline variable, Type G, addresses intrapersonal cognitive preconditions and processing, as well as affective attitudes of the students, and thus primarily focuses on individual appropriation processes of the students against the background of diversity, the variable Type H. Additionally, this main unit of research analysis focuses on social and interpersonal factors of the students' diversity, which become particularly important in the interaction between student and teacher and leads to different observable actions of the students in the classroom.

Megan Che and Even Baker, in Chap. 9, follow this broader perspective on context variables by focusing on identity-creating aspects of individual student personality in their description of the Type H variable. The central thesis of their chapter is that the identity of students is not only based on individual elements, but also on collective elements and the learning context, i.e., the mathematical experiences of the students as doers of mathematics, which consequently requires a situated consideration of identity-forming aspects and internal context variables both in research and in teaching within external contexts. In their description of the future direction of research on student internal context, Megan Che and Evan Baker call for further consideration of research approaches based on critical theory and postmodern perspectives on educational contexts. The researchers claim that these perspectives can provide additional insight into "understandings of students' mathematical identities and internal social contexts in a variety of technological mathematical learning environments, including gaming environments, online mathematics classrooms, and social media environments" (Che & Baker, this volume) without dismissing the importance of students' access to the technology. Additionally, they discuss another future research focus, "online communities and the potential to inhabit yet another identity as a virtual being in virtual worlds." (Che & Baker, this volume).

5.3 External Context Variables

External context variables stand for the support system within which teachers act and thus exploit and develop the potential of their competencies for professional practice. Medley understood this as the material, the facilities, the supervision, and administrative support provided by the school or the community of practitioners. Since these offline variables are mediating factors between teacher competencies and pre-post-active teacher activities, external context variables mainly influence how teachers carry out activities such as lesson planning, evaluation, and reflection depending on contingently given formal and material structures in the global educational system or the local school. Medley illustrated this dependency by highlighting that teachers with the same, or even assumedly identical competency profiles would act differently in differently supported instructional settings.

What does the support mean within the school context in the sense of mathematics educational research on Type I? If we look at research on textbooks and curricula, for example, culturally shaped task and examination cultures and national educational standards come into view, and form the normative guidelines for teachers' work in formulating learning goals and planning lessons. For the practical implementation of these guidelines, lack of free access to teaching materials and books is too often an obstacle. The collegial support of mathematics teachers at school can also be counted as part of this support system. The opportunities for further training through involvement in informal or national teacher associations, access to professional development (PD) and local feedback structures at school, for example through the principal, parents, or peers, are part of the support system described.

In Chap. 10, Birgit Pepin and Ghislaine Gueudet consider an offline variable of the technological support of teaching. This new variable, which Medley could not yet include among the external context variables at the end of the 1980s, has continuously shaped the schoolwork of teachers within the last 30 years. In their chapter, Pepin and Gueudet shed light on the educational policy preconditions and anchors for the use of digital resources and educational technologies, as well as research on the willingness and preconditions for teachers to use or not use technology and digital resources in the classroom, or on the reasons why they do not. Overall, they note, the role of the teacher is changing toward supporting the learning process as students become more self-regulated learners in their engagement with digital learning tools. The integration of programming into mathematics instruction, which has been increasingly promoted over many years, also requires new knowledge on the part of the teacher. Research on the quality criteria of digital resources is also receiving attention, for example, on the development of electronic curriculum materials, electronic textbooks, and dynamic mathematics tasks that, in terms of student learning of mathematics, require teachers not only to integrate these materials into the classroom, but also to design their instruction around them.

5.4 *Mathematics Teacher Training and Experiences*

The duration and quality of teacher training can differ qualitatively and quantitatively across teachers, as Medley described in the Type J offline variable. Different teacher training factors are the influential variables that mediate teachers' personal characteristics (Type F) and learned competencies (Type E). This means, for example, the extent to which teachers can develop their personal potentials in the context of training processes and translate them into learned competencies and skills is influenced by aspects of their training. Medley (1987) understood this as the experiences during teacher training designed to increase the "teacher's repertoire of competencies" (p. 106). Thus, indirectly, the abilities and mediation approaches of teacher educators, coaches and trainers come into view, as well as engagement in teacher PD.

In the field of mathematics education research, there have long been many approaches to assessing the quality of teacher education and training and to evaluating the influence of corresponding variables on the development of teacher competencies by means of empirical studies. International studies have considered, for example, the duration of teacher training, the quality of the courses offered, and the number of courses attended during training. The form of teacher training (e.g., how courses are structured or which seminars and courses are effective in teacher training to acquire mathematical knowledge for teaching) can also be analyzed and assessed from the perspective of cultural and national educational policy influences or normative values of "good" teaching. The importance of continuous professional development for teachers has increased over recent decades. As a result, respective corresponding variables are considered, such as engagement and participation in teacher PD. Recent mathematics education research also focuses on incorporating variables such as duration, structure, and quality of PD as well as effectiveness of PD assessment measures.

In Chap. 11, Joyce Peters-Dasdemir, Lars Holzäpfel, Bärbel Barzel and Timo Leuders, describe a special unit of analysis assigned to Type J—the qualification of teacher educators or adult educators providing PD. This unit of analysis refers to the qualification of facilitators of PD in mathematics, which is an area that has been insufficiently researched and that Medley did not consider. The teaching profession is characterized by experiential and lifelong learning and continuous professional development has gained traction in educational studies. This development has led to scientific research on the quality of PD. The chapter's central idea here in terms of advancing research on teaching and Medley's framework is to extend the chain of effects upward to include the corresponding effectiveness of those engaged in teacher education. To this end, Peters-Dasdemir et al. developed a competency framework model that can be used to describe the necessary professional profile of facilitators. Based on the results of overview studies on the criteria of effective teacher training, development, and based on systematic findings in adult education, the model includes aspects of the role of trainers as facilitators, their content and field-specific knowledge, professional values, and beliefs. In addition, their role identity, professional self-monitoring skills, and social competencies. The PD facilitators need to have fundamental professional knowledge and skills of the school subject that go beyond the knowledge of teachers (e.g., regarding curricular standards or current relevant empirical research findings).

5.5 Research Methods, Techniques, and Tools for Research on Teaching in the Digital Era

Following the description of the ten online and offline variables, Medley (1987) pointed out methodological issues to be considered in research on teaching. These methodological issues can refer to all stages of the research process in relation to the variables, their conceptualization, their instrumentation in empirical studies, the design of studies to investigate them, and the quality of the analysis of the data collected in studies. In relation to the conceptualization of the variables in research, Medley noted that the critical definition of effectiveness, that is, of "good teaching," varies intersubjectively, so all variables can potentially be affected by researcher bias. Challenges are also posed by the instrumentation of studies, that is, how the variables under study are operationalized in studies. Here, the evolution of research on teaching has led to increasingly better refinement of methods, which is taken up by all the authors in this volume. Medley further identified challenges of a more methodological nature in how studies examining the different variables must be specifically designed and what forms of data collection must take place. Finally, statistical data analyses and interpretation of results also pose challenges to researchers, but Medley recognized an ongoing elaboration of statistical analysis procedures. With increased sophistication of technological tools access to powerful statistical procedures has improved. Due to the fact that in the 1980s, the primary research methods accepted in the education community were first and foremost quantitative, Medley's work focused on quantitative methods of analysis. However, his concerns related to conceptualization, instrumentation, and design in research on teaching are still valid and relevant today, even with new technological and methodological developments and a wide range of modern qualitative and mixed methods used in mathematics education research.

Chandra Orrill, Zarina Gearty and Kun Wang in Chap. 12, provide information about methodological developments in mathematics education research and how it is positioned in the twenty-first century. They note that in addition to the quantitative research that Medley had in mind, qualitative research methods continued to be developed steadily in the 1980s and have led to profound insights in the research on teaching. Since overcoming of what has been characterized as trench warfare between quantitative and qualitative methodologies, a growing number of mixed-methods studies have also been observed with respect to the main units of analysis of research described by Medley. Looking specifically at quantitative research, Orrill et al. consider the item response theory (IRT) as an influential psychometric model which has significantly contributed to the further development of methodology in mathematics education research on teaching – especially, when it comes to the measurement of effectiveness. However, the researchers also present methodological advancements related to study design. For example, they describe teaching experiments, design-based-research, and cultural historical activity theory as new developments of design frameworks that meet the specific demands and needs of mathematics education research. Orrill et al. also separately address technological developments in research (e.g., eye-tracking, DGS and 360° video capture), and how these have led to both new insights and further development of methods in research.

6 Conclusion

Through the process of writing this book, we updated the original framework considering current research on teaching mathematics (Fig. 3). In addition to presenting new connections between main units of analyses of research, we acknowledge that each research variable must be considered within its cultural context and changes from one culture to another. The book focused on a western cultural perspective. Additionally, epistemological contexts are major factors in considering every unit of analysis of research on teaching mathematics. Depending on researchers' conceptual framework, the ideas surrounding Medley's "good teaching" change as the goals of teaching are directly tied to epistemological stances. Ultimately, new developments in technology change the way we can define (e.g., students' digital identities), evaluate (e.g., new instruments/measures of teachers' knowledge), and connect (e.g., modern research tools, methods, and techniques) main units of analysis described in framework presented in Fig. 3.

Finally, in Medley's original work, he warned against using variables that were far removed from one another within one study. New research methods and techniques described in Chap. 12 show that there are ways to consider multiple units of analyses, as well as the ones that are not adjacent to each other within the framework (Fig. 3). However, even with new technologies and advances, we found through writing this book that units of analyses (Types A though E) further removed from each other have less predictive value in contrast to those variables within the framework that are closer to each other. Although researchers considered and studied mediating variables between those that they intended to measure and report, it became clear to us was that there is a lack of a systematic scientific overview of the complete chain between the units of analysis described in Medley's original framework. Our intention was to provide such an overview and to offer scholars potential directions for research related to each unit of analysis as presented in the chapters of this book. This was the inspiration for our project, and we hope the chapters broaden the readers' horizons just as our views were expanded through collaboration with this international team of scholars.

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