The Supergeometric Algebra: The Square Root of the Geometric Algebra

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Abstract. This paper gives a pedagogical account of the Supergeometric Algebra (SGA), the square root of the Geometric Algebra (GA). The fact that a spinor can be treated as a bitcode is emphasized.

Keywords: Supergeometric Algebra · Geometric Algebra · spinors

1 Introduction

It is remarkable that the foundations of the Clifford algebra, or Geometic Algebra (GA), were established a century and a half ago by Grassmann [1,2] and Clifford [3], but it took a David Hestenes [4–7] to berate the physicists that the GA is something they really ought to pay attention to.

I think the Supergeometric Algebra (SGA), the extension of the GA to include spinors, deserves similar close attention by a wider audience. The name follows the common practice of physicists to prepend the word "super" to spinorial extensions of theories. From a strictly mathematical perspective, there's nothing new in this paper. Cartan, who introduced spinors to mathematics in 1913 [8], was thoroughly familiar with everything to do with geometric algebras and spinor algebras [9].

The present paper aims to give a pedagogical introduction to the SGA. A more formal exposition can be found in [10]. The most important single concept I hope to convey is that

$$\boxed{\text{A spinor is a bitcode.}} \tag{1}$$

The second important concept is that, as proved by Brauer & Weyl (1935) [11],

$$\boxed{\text{The Geometric Algebra is the Supergeometric Algebra squared.}} \tag{2}$$

If you are a computer scientist, you should be intrigued by the notion that a spinor is a bitcode. If you are interested in the GA, you should be aware of the fact that there is a natural way to represent objects in the GA with a bitcode.

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A prominent application of the SGA is to the fermions and forces of physics. A companion [12] to the present paper shows how the Dirac algebra of spacetime symmetries (the GA of Spin(3, 1)) and the geometric algebra of the group Spin(10), well known as a possible grand unified group, combine as commuting subalgebras of the Spin(11, 1) geometric algebra in 11+1 spacetime dimensions, unifying the four forces of Nature. The paper [12] is based on [13].

For simplicity, the treatment in the present paper takes all dimensions to be spatial. All results generalize to arbitrary dimensions of space and time. For the most part, time dimensions can be treated mathematically as if they were imaginary (with respect to the imaginary i) spatial dimensions.

2 Spinors as a Bitcode

Background. When a gymnast or ballet dancer rotates by one full turn, they return to where they started. Human experience might suggest that this is a law of Nature, that anything rotated by one full turn would necessarily return to its original state. Cartan first showed in 1913 [8] that mathematically there are more fundamental objects, which he called spinors (French spineurs), that require two full turns to return them to their original state. Cartan showed moreover that, within the context of rotations, there is nothing more fundamental than spinors. These properties stem from the topological properties of the rotation group: the usual rotation group (the special orthogonal group SO(N), in N spatial dimensions) is not simply-connected, but it has a double cover (Spin(N)) that is simply-connected. Figure 1 illustrates Dirac's belt trick [14], a well-known demonstration of the non-trivial topological properties of the rotation group.

Dirac in 1928 [15] rediscovered spinors from a physics perspective when he discovered his eponymous equation, a relativistic version of Schrödinger's [16] non-relativistic equation of quantum mechanics. Dirac's equation provided an experimentally successful description of the behavior of the electron, and predicted that an electron should have an antiparticle, a positron, which was discovered in 1932 [17].

With the establishment of the standard model of physics in the 1960s s and 1970s s (see [12] for more), it has become apparent that all known matter (fermions and quarks) is made of spinors, and that all known forces (the three forces of the standard model, plus gravity) emerge from symmetries of spinors. If indeed spinors are so fundamentally plumbed into the laws of Nature, then we humans would do well to pay attention.

Spinors, Vectors, and Rotors. In physics, a spinor is an object of spin $\frac{1}{2}$, whereas a vector is an object of spin 1. Whereas a vector rotates to itself under a rotation by 360°, a spinor requires two turns, 720°, to rotate it back to itself. Spinors and vectors exist in arbitrary dimensions of space, and more generally of spacetime.

If you are familiar with the GA, you may perhaps have heard the idea that a spinor is a rotor. That is not the right way to think about a spinor. A rotor is an element of the group Spin(N) of rotations in N space(time) dimensions. As a

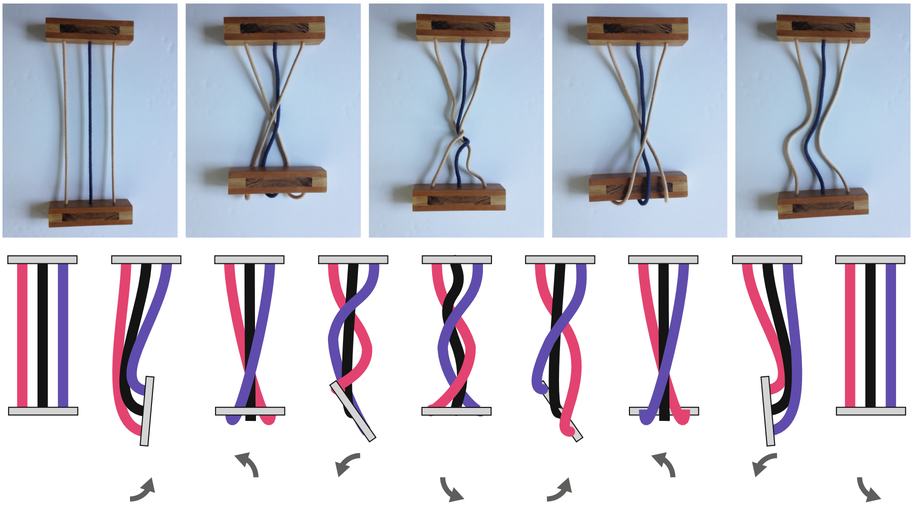

Fig. 1. A version of Dirac's belt trick [14], which illustrates the non-trivial topological properties of the rotation group. The trick demonstrates how an object tethered by ropes to another object gets tangled up when rotated by one full turn, but can be returned to its original state by rotating it a second full turn. The upper row of images are photographs of blocks and hawsers crafted by Tomas Herrera. The graphic in the lower row is by Liberty S. Hamilton.

multivector, a rotor is an element of the even geometric algebra; more specifically, a rotor is an element of the group Spin(N) obtained by exponentiating the bivectors of the GA. Any multivector $\boldsymbol{a}$ in the GA transforms under a rotor R as

$$R : \ \boldsymbol{a} \to R\boldsymbol{a}\overline{R} \ , \tag{3}$$

where $\overline{R}$ is the reverse (inverse) of R, and since a rotor is a multivector, that is how a rotor transforms. By contrast, a spinor ψ transforms under a rotor R as

$$R : \ \psi \to R\psi \ . \tag{4}$$

It is true that the transformation law (4) makes it legitimate to conceptualize that a spinor encodes a rotation (a Dirac spinor in 3+1 spacetime dimensions is indeed a "Lorentz gyroscope"), but a spinor is mathematically different from a rotor.

In mathematics, a dimension-n representation of a group is a set of $n \times n$ matrices multiplication of which reproduces the action of the group, along with a set of n-dimensional column vectors upon which the matrices act, rotating the vectors among each other.

A Cartesian vector is an element of the fundamental representation of the group SO(N) of orthogonal rotations in N dimensions. The dimension of the vector representation is N. The index i of a vector x_i runs over $i = 1, ..., N$. I was thrilled to learn this secret in high school, that a vector could be represented as an algebraic object x_i with a Cartesian index i. It meant that geometry problems could be solved by translating them into algebra. I could throw away my geometry textbook.

A spinor is an element of the fundamental representation of the group Spin(N), the covering group (double cover) of the orthogonal group SO(N). The dimension of the spinor representation is $2^{[N/2]}$. The index of a spinor can be expressed as a bitcode with $[N/2]$ bits, each of which can be either up $\uparrow$ or down $\downarrow$.

Examples. The simplest spinor is a Pauli spinor, which lives in $N = 2$ or 3 dimensions. A Pauli spinor has $[N/2] = 1$ bit, and $2^{[N/2]} = 2$ complex components, a total of 4°C of freedom. The one bit can be either up $\uparrow$ or down $\downarrow$. In Dirac's bra-ket notation, basis Pauli spinors are sometimes denoted $|\uparrow\rangle$ and $|\downarrow\rangle$. The distinction between even and odd dimensions N is addressed in Section 5.

The next simplest example is a Dirac spinor, which lives in $N = 3{+}1$ spacetime dimensions. A Dirac spinor has $[N/2] = 2$ bits, and $2^{[N/2]} = 4$ complex components, 8°C of freedom altogether. The Dirac bits comprise a boost bit, which can be either up $\Uparrow$ or down $\Downarrow$, and a spin bit, which can likewise be either up $\uparrow$ or down $\downarrow$. The spinor is said to be right-handed if the boost and spin bits align, $\Uparrow\uparrow$ or $\Downarrow\downarrow$, left-handed if they anti-align, $\Uparrow\downarrow$ or $\Downarrow\uparrow$. The chiral components of a Dirac spinor, right- or left-handed, are called its Weyl components. Only massless spinors can be purely chiral: a massive spinor, such as an electron, is necessarily a (complex) linear combination of right- and left-handed spinors. Chirality plays a central role in the standard model of physics, in that only the left-handed chiral components of Dirac spinors couple to the weak force: the right-handed components do not feel the weak force.

It has been known since the mid 1970s s [18,19] that each generation of fermions of the standard model organizes elegantly as spinors of the group Spin(10) in $N = 10$ dimensions. The companion paper [12] shows how the standard model and the Dirac algebra can be combined as commuting subalgebras of the Spin(11, 1) geometric algebra in $N = 11{+}1$ spacetime dimensions. Spinors of Spin(11, 1) have $[N/2] = 6$ bits, and $2^{[N/2]} = 64$ components.

Formalities. How should spinors be thought of geometrically? Start with N-dimensional Euclidean space $\mathbb{R}^N$. Partition the orthonormal basis vectors of Euclidean space into $[N/2]$ pairs, and call them $\boldsymbol{\gamma}_i^+$ and $\boldsymbol{\gamma}_i^-$, $i = 1, ..., [N/2]$. If the number N of dimensions is odd, one vector, $\boldsymbol{\gamma}_N$, remains unpaired (see §5 for more on the GA in odd dimensions). Unlike the GA, the SGA requires a complex structure from the outset, involving a commuting imaginary i which can be identified naturally as the quantum mechanical imaginary (do not confuse the index i with the imaginary i). The grouping of vectors into pairs $\boldsymbol{\gamma}_i^+$ and $\boldsymbol{\gamma}_i^-$ stems from this inevitable intrinsic complex structure. This is a good thing, because quantum mechanics requires a complex structure, which must come from somewhere. Chiral combinations $\boldsymbol{\gamma}_i$ and $\boldsymbol{\gamma}_{\bar{\imath}}$ (with a barred index $\bar{\imath}$) of the orthonormal basis vectors are defined by

$$\boldsymbol{\gamma}_i \equiv \frac{\boldsymbol{\gamma}_i^+ + i\boldsymbol{\gamma}_i^-}{\sqrt{2}} , \quad \boldsymbol{\gamma}_{\bar{\imath}} \equiv \frac{\boldsymbol{\gamma}_i^+ - i\boldsymbol{\gamma}_i^-}{\sqrt{2}} , \tag{5}$$

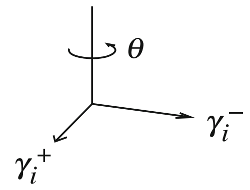

Fig. 2. Right-handed rotation by angle θ in the $\boldsymbol{\gamma}_i^+\boldsymbol{\gamma}_i^-$ plane. The $[N/2]$ conserved charges of Spin(N) are eigenvalues of quantities under rotations in $[N/2]$ planes $\boldsymbol{\gamma}_i^+\boldsymbol{\gamma}_i^-$, $i = 1, ..., [N/2]$. A spinor is a bitcode with $[N/2]$ bits, each of which specifies the corresponding charge of the spinor, either $+\frac{1}{2}$ ($\uparrow$), or $-\frac{1}{2}$ ($\downarrow$).

which are normalized so that $\boldsymbol{\gamma}_i \cdot \boldsymbol{\gamma}_{\bar{\imath}} = 1$. The vectors $\boldsymbol{\gamma}_i^+$ and $i\boldsymbol{\gamma}_i^-$ can be thought of as, modulo a normalization, the real and imaginary parts of a complex vector $\boldsymbol{\gamma}_i$ whose complex conjugate is $\boldsymbol{\gamma}_{\bar{\imath}}$.

A pillar of modern physics is Noether's (1918) theorem [20], which states that with each symmetry of a system is associated a conserved charge. Spin(N) has $[N/2]$ conserved charges, which are the eigenvalues of quantities under rotations in each of the $\boldsymbol{\gamma}_i^+\boldsymbol{\gamma}_i^-$ planes, Fig. 2. The $[N/2]$ bits of a spinor specify its $[N/2]$ charges, each of which can be either $+\frac{1}{2}$ (signified up $\uparrow$) or $-\frac{1}{2}$ (signified down $\downarrow$).

For "ordinary" spatial rotations, the conserved charge is the projection of the angular momentum, or spin, in the $\boldsymbol{\gamma}_i^+\boldsymbol{\gamma}_i^-$ plane (in fundamental units, $\hbar = 1$). However, in other applications of the SGA, the word charge may refer to other conserved charges, such as the conserved charges of the standard model of physics [12].

Under a right-handed rotation by angle θ in the $\boldsymbol{\gamma}_i^+\boldsymbol{\gamma}_i^-$ plane, Fig. 2, the chiral basis vectors $\boldsymbol{\gamma}_i$ and $\boldsymbol{\gamma}_{\bar{\imath}}$ transform by a phase, Fig. 3,

$$\boldsymbol{\gamma}_i \to e^{-i\theta}\,\boldsymbol{\gamma}_i \ , \quad \boldsymbol{\gamma}_{\bar{\imath}} \to e^{i\theta}\,\boldsymbol{\gamma}_{\bar{\imath}} \ . \tag{6}$$

The signs follow the physics convention that a right-handed rotation by angle θ rotates a phase by $e^{-i\theta}$. (If one of the two orthonormal dimensions, say $\boldsymbol{\gamma}_i^-$, is a time dimension, then the rotation in the $\boldsymbol{\gamma}_i^+\boldsymbol{\gamma}_i^-$ plane becomes a Lorentz boost, and the transformation (6) becomes

$$\boldsymbol{\gamma}_i \to e^{\theta}\,\boldsymbol{\gamma}_i \ , \quad \boldsymbol{\gamma}_{\bar{\imath}} \to e^{-\theta}\,\boldsymbol{\gamma}_{\bar{\imath}} \ .) \tag{7}$$

The transformation (6) identifies the chiral basis vectors $\boldsymbol{\gamma}_i$ and $\boldsymbol{\gamma}_{\bar{\imath}}$ as having i-charge equal to $+1$ and -1. All other chiral basis vectors, $\boldsymbol{\gamma}_j$ and $\boldsymbol{\gamma}_{\bar{\jmath}}$ with $j \neq i$, along with the unpaired basis vector $\boldsymbol{\gamma}_N$ if N is odd, remain unchanged under a rotation in the $\boldsymbol{\gamma}_i^+\boldsymbol{\gamma}_i^-$ plane, so have zero i-charge. The i-charge of a multivector (or tensor of multivectors) can be read off from its covariant chiral indices:

$$i\text{-charge} = \text{number of } i \text{ minus } \bar{\imath} \text{ covariant chiral indices} \ . \tag{8}$$

A spinor ψ,

$$\psi = \psi^a \boldsymbol{\epsilon}_a \ , \tag{9}$$

Fig. 3. The spiral lines track the phase angle $\mp\theta$ of right- and left-handed chiral basis vectors $\boldsymbol{\gamma}_i$ and $\boldsymbol{\gamma}_{\bar{\imath}}$ (left two images), Eqs. (6), and $\mp\theta/2$ of basis spinors $\boldsymbol{\epsilon}_{...i...}$ and $\boldsymbol{\epsilon}_{...\bar{\imath}...}$ with i'th bit up and down (right two images), Eqs. (11), under a rotation by angle θ in the $\boldsymbol{\gamma}_i^+\boldsymbol{\gamma}_i^-$ plane. It takes one full turn, $\theta = 2\pi$, to rotate vectors $\boldsymbol{\gamma}_i$ and $\boldsymbol{\gamma}_{\bar{\imath}}$ to themselves, but two full turns, $\theta = 4\pi$, to rotate the spinors $\boldsymbol{\epsilon}_{...i...}$ and $\boldsymbol{\epsilon}_{...\bar{\imath}...}$ to themselves.

is a complex (with respect to the imaginary i) linear combination of $2^{[N/2]}$ basis spinors $\boldsymbol{\epsilon}_a$. Chiral basis spinors comprise $2^{[N/2]}$ basis spinors $\boldsymbol{\epsilon}_a$,

$$\boldsymbol{\epsilon}_a \equiv \boldsymbol{\epsilon}_{a_1...a_{[N/2]}} \tag{10}$$

where $a = a_1...a_{[N/2]}$ denotes a bitcode of length $[N/2]$. Each bit a_i is either up $\uparrow$ or down $\downarrow$. For example, one of the basis spinors is the all-bit-up basis spinor $\boldsymbol{\epsilon}_{\uparrow\uparrow...\uparrow}$.

Under a right-handed rotation by angle θ in the $\boldsymbol{\gamma}_i^+\boldsymbol{\gamma}_i^-$ plane, basis spinors $\boldsymbol{\epsilon}_{...i...}$ and $\boldsymbol{\epsilon}_{...\bar{\imath}...}$ with i-bit respectively up and down transform as, Fig. 3,

$$\boldsymbol{\epsilon}_{...i...} \rightarrow e^{-i\theta/2}\,\boldsymbol{\epsilon}_{...i...} \ , \quad \boldsymbol{\epsilon}_{...\bar{\imath}...} \rightarrow e^{i\theta/2}\,\boldsymbol{\epsilon}_{...\bar{\imath}...} \ . \tag{11}$$

The transformation (11) shows that basis spinors $\boldsymbol{\epsilon}_{...i...}$ and $\boldsymbol{\epsilon}_{...\bar{\imath}...}$ have i-charge respectively $+\frac{1}{2}$ and $-\frac{1}{2}$ in each of its $[N/2]$ bits. The i-charge of a spinor (or tensor of spinors) can be read off from its covariant chiral indices:

$$i\text{-charge} = \tfrac{1}{2}(\text{number of } i \text{ minus } \bar{\imath} \text{ covariant chiral indices}) \ . \tag{12}$$

whereas an orthonormal Cartesian basis vector $\boldsymbol{\gamma}_i^+$ or $\boldsymbol{\gamma}_i^-$ sticks out in one dimension at a time, a basis spinor $\boldsymbol{\epsilon}_a$ sticks out in all dimensions at once. This sticking-out-in-all-dimensions-at-once, like a hedgehog, is perhaps one of the things that makes it hard to visualize a spinor. The reason a Cartesian vector can stick out in a single dimension i is that it can be constructed from a tensor product of spinor pairs in which the i'th bit of each of the two spinors points in the same direction, while all bits other than i point in opposite directions, canceling each other out.

3 Spinor Metric

The existence of a metric is fundamental to the GA, and the existence of a spinor metric is similarly fundamental to the SGA. The Euclidean metric δ_{ij} (or Minkowski metric η_{mn}) is that vectorial tensor of rank 2 that remains invariant under SO(N) (or SO(K,M) in $K{+}M$ spacetime dimensions). Similarly, the spinor metric ε_{ba} is that spinor tensor of rank 2 that remains invariant under Spin(N) (or Spin(K, M) in $K+M$ spacetime dimensions).

The scalar product of two spinors χ and ψ can be denoted with a dot,

$$\chi \cdot \psi \ . \tag{13}$$

The fact that the scalar product must be a scalar, therefore carry zero charge, implies that the spinor metric $\varepsilon_{ba} \equiv \boldsymbol{\epsilon}_b \cdot \boldsymbol{\epsilon}_a$ can be non-zero only between basis spinors whose indices are bit-flips of each other, $b = \bar{a}$. Each non-zero component $\varepsilon_{\bar{a}a}$ of the spinor metric equals ± 1, with the sign depending on the component a and the number N of dimensions. See [10] for details.

Whereas the Euclidean (or Minkowski) metric must be symmetric, the spinor metric can be either symmetric or antisymmetric. There prove to be two possible choices for the spinor metric, differing from each other by a factor of the pseudoscalar, denoted ε and ε_{alt}. In Nature, it is Nature that makes the choice. The following chart shows the symmetry of the spinor metric ε or ε_{alt} in N spacetime dimensions:

N(mod 8) :	1	2	3	4	5	6	7	8
ε^2	+	+	−	−	−	−	+	+
$\varepsilon^2_{\text{alt}}$	+	−	−	−	−	+	+	+

(14)

The chart exhibits the well known period-8 Cartan-Bott periodicity [21,22] of geometric algebras.

The chart (14) shows that in 3 or 4 spacetime dimensions, the spinor metric is necessarily antisymmetric. Thus the Pauli metric in $N = 3$ dimensions, and the Dirac metric in $N = 3+1$ spacetime dimensions, are necessarily antisymmetric.

4 Column Spinors and Row Spinors

It is not only mathematically correct (in the context of representation theory), but also conceptually helpful, to think of a spinor ψ as a column vector (of dimension $2^{[N/2]}$), and a rotor R as a matrix that acts on the column spinor ψ. More generally, any multivector $\boldsymbol{a}$ can be represented as a $2^{[N/2]} \times 2^{[N/2]}$ matrix that acts by matrix multiplication on a column spinor ψ, yielding $\boldsymbol{a}\psi$.

Associated with any column spinor ψ is a row spinor $\psi\cdot$, equal to the transpose of the column spinor ψ multiplied by the spinor metric tensor ε,

$$\psi\cdot \equiv \psi^\top \varepsilon \ . \tag{15}$$

A scalar product $\chi \cdot \psi$ of spinors can be thought of as the matrix product of a row spinor $\chi\cdot$ with a column spinor ψ,

$$\chi \cdot \psi = \chi^\top \varepsilon \, \psi \ . \tag{16}$$

The notation $\psi\cdot$ for a row spinor, with a trailing dot symbolizing the spinor metric, is extremely convenient. The dot immediately distinguishes a row spinor from a column spinor; and the dot makes transparent the application of the associative rule to a sequence of products of spinors, Eq. (19). A row spinor $\psi\cdot$ transforms under a rotor R as

$$R:\ \psi\cdot \to \psi\cdot\overline{R}\ , \tag{17}$$

as follows from the fact that a spinor transforms as (4), and a scalar product of a row and column spinor must be a scalar.

In opposite order, the product of a column spinor ψ and a row spinor $\chi\cdot$ defines their outer product $\psi\chi\cdot$. The outer product transforms under a rotation in the same way (3) as a multivector,

$$R:\ \psi\chi\cdot \equiv \psi\chi^{\top}\varepsilon \to (R\psi)(R\chi)^{\top}\varepsilon = R(\psi\chi\cdot)\overline{R}\ . \tag{18}$$

Multiplication of outer products satisfies the associative rule

$$(\psi\chi\cdot)(\varphi\xi\cdot) = \psi(\chi\cdot\varphi)\xi\cdot\ , \tag{19}$$

which since $\chi\cdot\varphi$ is a scalar is proportional to the outer product $\psi\xi\cdot$. The associative rule (19) makes it straightforward to simplify long sequences of products of column and row spinors, a process known in quantum field theory as Fierz rearrangement.

A core property of spinors in physics is that they satisfy an exclusion principle. The exclusion principle underlies much of the richness of the behavior of matter at low energy. According to the usual rules of matrix multiplication, a row matrix can multiply a column matrix, yielding a scalar, and a column matrix can multiply a row matrix, yielding a matrix, but a row matrix cannot multiply a row matrix, and a column matrix cannot multiply a column matrix:

$$\begin{pmatrix} \quad & \quad \end{pmatrix}\begin{pmatrix} \\ \\ \end{pmatrix} = \begin{pmatrix} \ \end{pmatrix} \qquad \text{inner product} = \text{scalar}\ , \tag{20a}$$

$$\begin{pmatrix} \\ \\ \end{pmatrix}\begin{pmatrix} \quad & \quad \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} \qquad \text{outer product} = \text{multivector}\ , \tag{20b}$$

$$\begin{pmatrix} \quad & \quad \end{pmatrix}\begin{pmatrix} \quad & \quad \end{pmatrix} = \varnothing \qquad \text{forbidden}\ , \tag{20c}$$

$$\begin{pmatrix} \\ \\ \end{pmatrix}\begin{pmatrix} \\ \\ \end{pmatrix} = \varnothing \qquad \text{forbidden}\ . \tag{20d}$$

These rules resemble the rules for fermionic creation and destruction operators in quantum field theory: creation following destruction is allowed, and destruction following creation is allowed, but creation following creation is forbidden, and destruction following destruction is forbidden. It can be shown that the multiplication rules for row and column spinors indeed reproduce those of fermion creation (row) and destruction (column) operators in quantum field theory.

It would seem that the distinction between column and row spinors, as realized in Nature, is profound.

5 The GA Is the Square of the SGA

Brauer & Weyl (1935) [11] first proved the theorem that the algebra of outer products of spinors is isomorphic to the GA, in any number of even N spacetime dimensions. They used a language familiar to physicists, that of tensors, and representations of groups. [10] gives a proof of the theorem in the language of the GA.

In the notation of the present paper, the Brauer-Weyl isomorphism says that there is an invertible linear mapping between outer products of column and row basis spinors $\boldsymbol{\epsilon}_a$ and $\boldsymbol{\epsilon}_b \cdot$ and basis multivectors $\boldsymbol{\gamma}_A$ of all grades,

$$\boldsymbol{\epsilon}_a \boldsymbol{\epsilon}_b \cdot = c_{ab}^{A} \boldsymbol{\gamma}_A \ , \quad \boldsymbol{\gamma}_A = c_A^{ab} \boldsymbol{\epsilon}_a \boldsymbol{\epsilon}_b \cdot \ , \tag{21}$$

that respects the algebraic structure, that is, it respects addition and multiplication of spinors and multivectors. The outer product is neither symmetric nor antisymmetric in ab. The full set of $2^{N/2} \times 2^{N/2}$ outer products of basis spinors yields the entire 2^N-dimensional geometric algebra.

The simplest example of the Brauer-Weyl isomorphism is the Pauli SGA in $N = 2$ dimensions, where the outer products of the two spinors $\uparrow$ and $\downarrow$ map to the basis multivectors of the GA in 2 dimensions, consisting of one scalar, 2 vectors, and one pseudoscalar, a total of $1+2+1 = 4 = 2^2$ multivectors,

$$\underset{\text{1 scalar}}{1 = (\downarrow\uparrow - \uparrow\downarrow)\cdot \ ,} \quad \underset{\text{2 vectors}}{\boldsymbol{\gamma}_1 = \sqrt{2} \uparrow\uparrow\cdot \ , \quad \boldsymbol{\gamma}_{\bar{1}} = -\sqrt{2} \downarrow\downarrow\cdot \ ,} \quad \underset{\text{1 pseudoscalar}}{I_2 = -i(\downarrow\uparrow + \uparrow\downarrow)\cdot \ .} \tag{22}$$

The spinor metric adopted in the algebra (22) is the antisymmetric choice (right column) in the chart (14), which ensures that the algebra is the same as that of the Pauli algebra in $N = 3$ dimensions, Eq. (25).

The natural complex structure of spinors means that spinors live naturally in even spacetime dimensions N. The group Spin(N) on the other hand exists in either even or odd dimensions, and likewise the GA lives in both even and odd dimensions. There are two ways to extend the SGA to odd N dimensions.

The first is to project the odd N-dimensional GA into one lower dimension, by identifying the pseudoscalar I_N of the odd-dimensional GA with the unit multivector (times a phase), whereupon the pseudoscalar I_{N-1} of the one-dimension-lower even-dimensional algebra is promoted to a vector in the N-dimensional algebra.

An example is the Pauli algebra in $N = 3$ dimensions. The orthonormal basis vectors of the Pauli algebra, here denoted $\boldsymbol{\gamma}_1^+$, $\boldsymbol{\gamma}_1^-$, and $\boldsymbol{\gamma}_3$, are commonly denoted σ_i, $i = 1, 2, 3$. The pseudoscalar I_3 of the Pauli algebra, the product of the three vectors, is identified with the imaginary i times the unit scalar 1,

$$I_3 \equiv \boldsymbol{\gamma}_1^+ \boldsymbol{\gamma}_1^- \boldsymbol{\gamma}_3 \, (= \sigma_1 \sigma_2 \sigma_3) = i \, 1 \ . \tag{23}$$

As a result of the identification (23), the pseudoscalar I_2 of the 2-dimensional algebra is promoted to a vector of the 3-dimensional algebra,

$$I_2 \equiv \boldsymbol{\gamma}_1^+ \boldsymbol{\gamma}_1^- \, (= \sigma_1 \sigma_2) = i \boldsymbol{\gamma}_3 \, (= i \sigma_3) \ . \tag{24}$$

The 3-dimensional geometric algebra differs from the 2-dimensional geometric algebra in that the former possesses a higher level of symmetry: whereas in 2 dimensions there is just one rotation, generated by the bivector $\sigma_1\sigma_2$, in 3 dimensions there are 2 more rotations, generated by the bivectors $\sigma_1\sigma_3$ and $\sigma_2\sigma_3$.

The Pauli SGA in $N = 3$ dimensions (with the standard choice ε of spinor metric, the center column in the chart (14)) is essentially identical to the Pauli SGA (22) in $N = 2$ dimensions, except that the 2D pseudoscalar I_2 is promoted to a vector $\boldsymbol{\gamma}_3$, Eq. (24),

$$\underbrace{1 = (\downarrow\uparrow - \uparrow\downarrow)\cdot\ ,}_{\text{1 scalar}} \quad \underbrace{\boldsymbol{\gamma}_1 = \sqrt{2}\uparrow\uparrow\cdot\ , \quad \boldsymbol{\gamma}_3 = -(\downarrow\uparrow + \uparrow\downarrow)\cdot\ , \quad \boldsymbol{\gamma}_{\bar{1}} = -\sqrt{2}\downarrow\downarrow\cdot\ .}_{\text{3 vectors}} \tag{25}$$

The rest of the Pauli GA, comprising the 1 pseudoscalar and 3 bivectors, are just i times the 1 scalar and 3 vectors, since the pseudoscalar I_3 has been identified with the imaginary, Eq. (23).

The other way to extend the SGA to odd dimensions is to embed the odd N-dimensional algebra into one higher dimension $N+1$, and to treat the extra vector $\boldsymbol{\gamma}_{N+1}$ as a scalar. The extra scalar vector $\boldsymbol{\gamma}_{N+1}$ serves the role of a parity operator (or a time-reversal operator, if one of the dimensions is a time dimension), by virtue of anticommuting with all the original N orthonormal vectors.

6 Conjugation

Spinors have an intrinsic complex structure, and there is a discrete operation, complex conjugation, that converts spinors (and multivectors) into their complex conjugates. The basis spinors $\boldsymbol{\epsilon}_a$ are treated as real, so the complex conjugate of a spinor $\psi \equiv \psi^a \boldsymbol{\epsilon}_a$ is the spinor with complex-conjugated coefficients,

$$\psi^* \equiv (\psi^a)^* \boldsymbol{\epsilon}_a \ . \tag{26}$$

In quantum field theory, complex conjugation turns a spinor into an anti-spinor.

The operation (26) of complex conjugation is not however Lorentz-covariant; under a rotor R, the complex conjugate spinor ψ^* transforms as

$$R: \ \psi^* \to (R\psi)^* = R^* \psi^* \ . \tag{27}$$

The conjugation operator C is introduced to restore Lorentz covariance. The conjugate spinor $\bar{\psi}$ is defined to be the product of the conjugation operator C and the complex conjugate spinor ψ^*,

$$\bar{\psi} \equiv C\psi^* \ . \tag{28}$$

(Do not confuse conjugation with reversion in the GA; the conjugation overbar $\bar{}$ is shorter and thinner than the reversion overbar $\overline{\ }$.) The conjugation operator C is defined as a Lorentz-invariant operator with the property that commuting it through any rotor R converts the rotor to its complex conjugate, $CR^* = RC$. With the conjugation operator so defined, the conjugate spinor $\bar{\psi}$ transforms under a rotor in the same way as any other spinor,

$$R: \ \bar{\psi} \to C(R\psi)^* = RC\psi^* = R\bar{\psi} \ . \tag{29}$$

The conjugate spinor $\bar{\psi}$ is the antiparticle of the spinor ψ, expressed in a Lorentz-covariant fashion.

In physics, the operation of conjugation is often conflated with the operation of converting a column spinor to a row spinor, so that the conjugate of a spinor ψ is taken to be the conjugate row (or row conjugate) spinor $\bar{\psi}\cdot$. The reason for the conflation is that in quantum field theory a field is a linear combination of creation and destruction operators, and the partner of a fermion destruction operator (column spinor) is the anti-fermion creation operator (conjugate row spinor). However, the two operations are distinct, and it is wise to keep them so. All four operations — fermion creation and destruction, and anti-fermion creation and destruction — occur in quantum field theory.

If all spatial N dimensions are spatial (no time dimensions), then the conjugation operator C coincides with the spinor metric tensor ε. If there is a time dimension, then in the chiral representation the conjugation operator is, up to a phase, the product of the spinor metric and the time vector (or a product of all the time vectors, if there is more than one time dimension). Some texts refer to the spinor metric tensor as the conjugation operator, which I find egregiously confusing.

7 Supersymmetry

The Supergeometric Algebra is *not* the same as the algebra of supersymmetry. The supersymmetry algebra is the extension of the Poincaré algebra to include symmetries generated by spinors. The Poincaré algebra is the algebra of global translations and Lorentz transformations of flat (Minkowski) space. The Poincaré algebra is not the same as the Dirac algebra (the GA in 3+1 spacetime dimensions). The Poincaré and Dirac algebras share the property of having both vector and bivector generators, but the vectors of the Poincaré algebra, the momentum generators P_m, commute, whereas the vectors γ_m of the Dirac algebra anticommute.

In the SGA in 4 spacetime dimensions, the 4 symmetrized outer products of the 2 right-handed with the 2 left-handed spinors yield the 4 chiral basis vectors. The coefficients of the mapping coincide with those of the supersymmetry algebra, which is unsurprising since the mapping must respect the properties of spinors and vectors under Lorentz transformations.

My own idiosyncratic view is that supersymmetry may not be Nature's way. The algebra of outer products of spinors yields the entire geometric algebra, including multivectors of all grades, not just vectors. String theory is apparently a theory not merely of strings (whose worldtubes are generated by bivectors), but of branes of all dimensions (whose worldtubes are generated by multivectors of all grades). Hopefully Nature's way will in due course become apparent to enterprising observers and experimentalists, as has happened in the past.

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