On Semimonotone Z-Matrices

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Abstract. In 1981, Stone conjectured that a fully semimonotone Q_0 matrix is contained in P_0. In 1995, Murthy proved that for $n = 5$, if $A \in \mathbb{R}^{n\times n} \cap E_0^f \cap Q_0$ and $a_{ii} > 0$, then $A \in P_0$. Here, we show that for matrices with some specific sign patterns this conjecture is true. Murthy showed that fully semimonotone Z-matrices are P_0, that is $E_0^f \cap Z \subseteq P_0$. Here, we show that semimonotone Z-matrices are contained in P_0, that is, we exempt the condition of fully semimonotone with semimonotone. Further, we show the equivalency of E_0-matrices and E_0^f-matrices for Z-matrices. Precisely, we are characterizing the matrices in $P_0 \cap Q_0$. These classes have been found to be interesting in view of the fact that these are processable by Lemke's algorithm.

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1 Introduction

Given a matrix $A \in \mathbb{R}^{n\times n}$. Let $q \in \mathbb{R}^n$ be a vector. The linear complementarity problem, $LCP(q, A)$ can be described as follows:

We want to find $x \in \mathbb{R}^n$ such that

$$x \geq 0, \tag{1}$$

$$Ax + q = w \geq 0, \tag{2}$$

$$x^t w = 0. \tag{3}$$

If such a vector exists, we call this x as a solution to the $LCP(q, A)$. For a given $q \in \mathbb{R}^n$, if some vector $x \in \mathbb{R}^n$ satisfies (1) and (2), we call it as a feasible solution for $LCP(q, A)$. We call a matrix Q_0 if for all q, whenever $LCP(q, A)$ has a feasible solution, it also has a solution satisfying (1), (2), and (3). We call A to be a Q-matrix if $LCP(q, A)$ has a solution for every q. We call A to be an R_0-matrix, if $LCP(0, A)$ has a unique solution. If all the principal minors of a matrix A are positive (nonnegative), then we call A to be a P (P_0)-matrix. We call A to be a Z-matrix, if $a_{ij} \leq 0$ for $i \neq j$. We denote the classes of matrices in each of the above cases by Q, Q_0, R_0, P, P_0, and Z respectively. For further references, the reader may refer to [3,5,10] and the results therein.

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Characterizing the number of solutions to an LCP has been found to be interesting among the researchers. LCP associated to a P-matrix has a unique solution for every vector q. In 1983, Cottle and Stone introduced a new class U known as the class of U-matrices. We say a matrix U if $LCP(q, A)$ has a unique solution whenever q is in the interior of the union of complementary cones. Further, the class is expanded for the unique solution of $LCP(q, A)$ for q in the interior of any non-degenerate complementary cone. This class is known as the class of fully semimonotone matrices, denoted by E_0^f. The class of E_0^f was introduced by Stone. For further results refer to [9].

In [4], Cottle and Stone proved $P \subseteq U \subseteq E_0^f$. In [13], Stone proved that $U \cap Q_0 \subseteq P_0$. Further, he raised the conjecture $E_0^f \cap Q_0 \subseteq P_0$. For matrices with some specific sign patterns, we show that this conjecture is true. In particular, we show that semimonotone Z-matrices are contained in P_0. In addition, we also prove that fully semimonotone matrices having specific sign patterns are P_0.

In proving some of our results we use the concept from the Completely mixed matrix games. A two person zero-sum game may be described as following:

Let Player 1 and player 2 choose integers $i \in \bar{m}$ and $j \in \bar{n}$ respectively. Then Player 2 receives an amount a_{ij} from Player 1. This amount a_{ij} may be negative, positive, or zero. A mixed strategy for Player 1 and 2 are the probability vectors $x = (x_1, x_2, ..., x_m)^t$, and $y = (y_1, y_2, ..., y_n)^t$, respectively, where $x_i \geq 0$ for all i and $\sum_{i=1}^{m} x_i = 1$ and $y_j \geq 0$ for all j and $\sum_{i=1}^{n} y_j = 1$. We call (x^*, y^*) to be the optimal strategies for Player 1 and Player 2 respectively, if the following conditions hold

$$\sum_i x_i^* a_{ij} \leq v \qquad \text{for } j = 1, 2, .., n, \tag{4}$$

$$\sum_j y_j^* a_{ij} \geq v \qquad \text{for } i = 1, 2, .., m. \tag{5}$$

It is known that such a v exists and is unique. We denote $v = val(A)$ and call it the value of the matrix game $A = (a_{ij})$. In describing (4) and (5), we assumed Player 1 to be the minimizer where Player 2 to be the maximizer. If each entry of the vector $x = (x_1, x_2, ..., x_m)^t$ is positive, then we call such a vector x as a completely mixed strategy for player 1. Similarly, if each entry of the vector $y = (y_1, y_2, ..., y_n)^t$ is positive, then we call such a vector y as a completely mixed strategy for player 2. If each of the optimal pair (x^*, y^*) is completely mixed for a game associated with A, then we call it a completely mixed game.

Kaplansky [7] has characterized a completely mixed (c.m.) matrix game. He showed the following:

Consider a game associated with matrix $A \in \mathbb{R}^{m \times n}$ and suppose $val(A) = 0$, then the game associated with matrix A is c.m. if and only if $m = n$, $r(A) = n-1$, and each of the cofactor of A is nonzero and have same sign.

The organisation of this manuscript is as following: In Sect. 2, we present a few basic results that are used in further sections. In Sect. 3, we provide our main theorem regarding the Z-matrices. Section 4 contains some more results related to E_0^f. In Sect. 5, we provide some open problem for future work and conclude the paper.

2 Preliminaries

Notation: In this manuscript, we used signs at many places instead of a fix value. The meaning for these signs is as following: $+$ implies positive, $\ominus$ means nonpositive, $-$ denotes negative, $\oplus$ means nonnegative, and $*$ denotes any real value.

$A \in \mathbb{R}^{m\times n} = (a_{ij}); i = 1, 2, ..., m$ and $j = 1, 2, ..., n$ denotes a matrix. $\bar{n}$ denotes the set $\{1, 2, ..., n\}$. Let $\alpha, \beta \subseteq \bar{n}$ and the complements are $\bar{\alpha} = \bar{n}\backslash\alpha$ and $\bar{\beta} = \bar{n}\backslash\beta$. If we delete rows of A corresponding to $\bar{\alpha}$ and columns of A corresponding to $\bar{\beta}$, then the resulting matrix is a submatrix of A, denoted by $A_{\alpha\beta}$. We call $A_{\alpha\beta}$ to be a principal submatrix of A if $\alpha = \beta$. $|\alpha|$ denotes the cardinality of the set α. The determinant of the matrix A is denoted as $|A| = det(A)$. We say a vector $x \geq 0$ $(x > 0)$, if every coordinate of x is nonnegative (positive). Similarly, we say a matrix $A \geq 0$ $(A > 0)$, if each entry of A is nonnegative (positive).$r(A)$ denotes the rank of matrix A.

This section contains some basic definitions and results from the literature. These results are used in the next sections.

A.W. Tucker introduced the concept of PPTs. PPTs (principal pivot transforms) play a crucial role in the consideration of LCP. A detailed treatment of PPT was given by Tsatsomeros [14].

Let $A \in \mathbb{R}^{n\times n}$ and $\alpha \subseteq \bar{n}$.

$$A = \begin{pmatrix} A_{\alpha\alpha} & A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha} & A_{\bar{\alpha}\bar{\alpha}} \end{pmatrix}$$

If $A_{\alpha\alpha}^{-1}$ exists, then for such an α, the PPT is defined. We denote such a PPT of A as ppt(A, α).

$$\text{ppt}(A, \alpha) = \begin{pmatrix} A_{\alpha\alpha}^{-1} & -A_{\alpha\alpha}^{-1}A_{\alpha\bar{\alpha}} \\ A_{\bar{\alpha}\alpha}A_{\alpha\alpha}^{-1} & A/A_{\alpha\alpha} \end{pmatrix},$$

where $A/A_{\alpha\alpha} = A_{\bar{\alpha}\bar{\alpha}} - A_{\bar{\alpha}\alpha}A_{\alpha\alpha}^{-1}A_{\alpha\bar{\alpha}}$ is known as the Schur complement. For any α, if $A_{\alpha\alpha}$ is invertible, then PPT exists for the corresponding α. We call all those PPTs as legitimate PPTs.

Semimonotone matrices were first initiated by Eaves [5], and initially these are denoted by L_1. Later, in [3], this class was denoted by E_0. The name "semimonotone" was initiated by Karamardian. It is known that $P_0 \subseteq E_0$.

Definition 1. We call $A \in \mathbb{R}^{n\times n}$ to be a semimonotone matrix if, for any nonzero nonnegative vector x, there is some k in such a manner that x_k is positive and $(Ax)_k$ is nonnegative. We call A as a fully semimonotone matrix, denoted by E_0^f, if A and all its legitimate principal pivot transforms are in E_0.

Some of the useful properties of semimonotone matrices from [3,10] are stated below.

Theorem 1. *If $A \in \mathbb{R}^{n\times n} \cap E_0$, then we can conclude the following:*

1. *$A_{\alpha\alpha} \in E_0$ for all $\alpha \subseteq \bar{n}$.*
2. *$A^t \in E_0$.*
3. *a_{ii} are nonnegative for all $i = 1, 2, ..., n$.*
4. *For any vector $q > 0$, there is a unique solution for $LCP(q, A)$, that is $x = 0$.*
5. *$val(A) \geq 0$.*

Tsatsomeros and Wendler [15] provided the following result:

Theorem 2. *Let $A \in \mathbb{R}^{2\times 2}$ and $a_{ii} > 0$. Then $A \in E_0$ if and only if either $A \geq 0$ or $det(A) \geq 0$.*

Definition 2. We call $A \in \mathbb{R}^{n\times n}$ to be a copositive matrix if for any nonnegative vector x, $x^tAx \geq 0$. We denote the class of such matrices by C_0. A matrix A is called fully copositive matrix, denoted by C_0^f, if A and each of its legitimate PPTs is in C_0.

The class E_0^f includes the class C_0^f. In [8], Murthy and Parthasarathy provide a result for C_0^f.

Theorem 3. *Let $A \in C_0^f \cap Q_0 \cap \mathbb{R}^{n\times n}$. Then $A \in P_0$.*

Next we state a few known results for fully semimonotone Q_0 matrices from [11].

Theorem 4. *Let $A \in E_0^f \cap Q_0 \cap \mathbb{R}^{n\times n}$. Further, assume $det(A_{\alpha\alpha}) \geq 0$ for all $|\alpha| = n - 1$. Then $A \in P_0$.*

Theorem 5. *Let $A \in \mathbb{R}^{n\times n} \cap E_0^f \cap R_0$. Then A is a P_0-matrix.*

The following corollary is proved already. Here we provide another proof.

Corollary 1. *Let $A \in \mathbb{R}^{n\times n} \cap E_0^f \cap Q_0$. Further, suppose that $A_{\alpha\alpha} \in P$ for $|\alpha| \leq (n - 2)$. Then $A \in P_0$.*

Proof. Let $B = A_{\alpha\alpha}$ where $|\alpha| = n - 1$. We claim that $det(B) \geq 0$.

Suppose $det(B) < 0$. Since $A_{\alpha\alpha} \in P$ for $|\alpha| \leq (n - 2)$, that is each proper principal submatrix of B is also a P-matrix. Hence, the diagonal entries of B^{-1} are negative.

Since $A \in E_0^f$, observe that $B \in E_0$ and $B^{-1} \in E_0$. It is not possible for E_0-matrix to have negative diagonal entry. Hence, $det(B) \geq 0$. Therefore, using Theorem 4, $A \in P_0$. □

Remark 1. We have used the fact that if $A \in E_0^f$, then any proper principal submatrix is E_0. If some proper principal submatrix (say B) is non-singular, then $B^{-1} \in E_0$.

The result given below is the Theorem 4.1.2 in [3]. This result is useful in proving next theorem.

Theorem 6. *[3] Let $M = ppt(A, \alpha)$. Then for any submatrix $M_{\beta\beta}$ of M*

$$det(M_{\beta\beta}) = \frac{det(A_{\gamma\gamma})}{det(A_{\alpha\alpha})}$$

where $\gamma = \alpha \triangle \beta$.

We need the following result.

Theorem 7. *Let A be a P_0-matrix. Then any PPT of A is also a P_0-matrix.*

Proof. Let $A \in \mathbb{R}^{n\times n} \cap P_0$ and $\alpha \subseteq \bar{n}$ such that $det(A_{\alpha\alpha}) \neq 0$. Therefore, $ppt(A, \alpha)$ exists. Let us call it M. Now for any $\beta \subseteq \{1, 2, ..., n\}$, using Theorem 6, we have

$$det(M_{\beta\beta}) = \frac{det(A_{\gamma\gamma})}{det(A_{\alpha\alpha})} \tag{6}$$

where $\gamma = \alpha \triangle \beta$. Observe that $A_{\alpha\alpha}$ and $A_{\gamma\gamma}$ are principal submatrices of A. Since $A \in P_0$. Hence, $det(A_{\alpha\alpha}) > 0$, $det(A_{\gamma\gamma}) \geq 0$. Therefore, on putting these in 6, we have $det(M_{\beta\beta}) \geq 0$. Since β was arbitrary, hence $M \in P_0$. Therefore, ppt of a P_0-matrix is also a P_0-matrix. □

Remark 2 [3]. It is known that $P \subseteq P_0 \subseteq E_0^f \subseteq E_0$. In the next section we show that for the Z-matrices, P_0, E_0^f and E_0 are equal.

The following two results of game theory are used in proving our results.

Theorem 8 [2]. *Let $A \in \mathbb{R}^{n\times n} \cap Z$. Consider a game is associated with matrix A. Suppose $val(A) > 0$. Then A has to be a P-matrix.*

Theorem 9 [7]. *Consider a game associated with matrix $A \in \mathbb{R}^{m\times n}$ and $val(A) = 0$. Then A is c.m. if and only if $m = n$, $r(A) = n - 1$, and each of the cofactor of A is nonzero and have same sign.*

The next two results are known for Q and R_0 matrices.

Theorem 10 [1]. *Let $A \in \mathbb{R}^{n\times n} \cap P_0$. Then $A \in Q$ if and only if $A \in R_0$.*

Theorem 11 [6]. *Let $A \in R_0 \cap \mathbb{R}^{n\times n}$ and $LCP(q, A)$ has a unique solution, for some $q > 0$. Then $A \in Q$.*

3 Main Results

Murthy and Parthasarathy [9] showed that $E_0^f \cap Z \subseteq P_0$. In this section, we show that the condition of fully semimonotone is not necessary. In particular, we show that the semimonotone, Z-matrices are P_0.

Theorem 12. *Let $A \in \mathbb{R}^{n\times n} \cap E_0 \cap Z$, then $A \in P_0$.*

Proof. This proof is done using the mathematical induction. For $n = 1$, it is obvious.

For $n = 2$, the sign pattern would be

$$\begin{pmatrix} \oplus & \ominus \\ \ominus & \oplus \end{pmatrix}.$$

Since the diagonal entries are nonnegative, both proper principal minors are nonnegative. If any of the off-diagonal entry is zero, then the determinant would be nonnegative. Hence, $A \in P_0$. If both the off-diagonal entries are negative, then using Theorem 2, the determinant of the matrix A is nonnegative. Hence, $A \in P_0$.

Now for $n = 3$. Since $A \in E_0$, using Theorem 1, each of its principal submatrix is E_0 and the diagonal entries are nonnegative. From Theorem 2, either $A_{\alpha\alpha} \geq 0$ for $|\alpha| = 2$ or $det(A) \geq 0$. That means $A_{\alpha\alpha} \in P_0$ for all $|\alpha| = 2$. Consider $x \geq 0$ be an optimal for A, so $Ax \geq 0$,

$$\begin{pmatrix} \oplus & \ominus & \ominus \\ \ominus & \oplus & \ominus \\ \ominus & \ominus & \oplus \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

a) If exactly one coordinate of x is non-zero (say x_1), then $A \in E_0$ implies that A must be of the type

$$A = \begin{pmatrix} \oplus & \ominus & \ominus \\ 0 & \oplus & \ominus \\ 0 & \ominus & \oplus \end{pmatrix}.$$

Here, $det(A) \geq 0$. Hence, $A \in P_0$.

b) If exactly two coordinates of x are non-zero (say x_1, x_2), then $A \in E_0$ implies that A must be of the type

$$A = \begin{pmatrix} \oplus & \ominus & \ominus \\ \ominus & \oplus & \ominus \\ 0 & 0 & \oplus \end{pmatrix}.$$

Since, $A \in E_0$, using 1, the submatrix B of A, on omitting last row and last column, is also E_0. Using the Theorem 2, either $B \geq 0$ or $det(B) \geq 0$. Hence, $det(A) \geq 0$. Therefore, $A \in P_0$.

c) Let $x > 0$. Then the game associated with A is c.m. We have the hypothesis that A is a semimonotone matrix, hence using Theorem 1, $val(A) \geq 0$. Now, we will check for both the cases when the value is zero or the value is positive. If $val(A) > 0$, then by Theorem 8, $A \in P$. Therefore, $A \in P_0$.
Now for $val(A) = 0$, from Theorem 9, $rank(A) = n - 1$. That means one of the row is linear combination of others. Hence $det(A) = 0$ and therefore $A \in P_0$.

Hence, it is true for $n = 3$. Now for the induction hypothesis, let it is true up to any $n - 1$ order. Now we will show that it is true for n.

a) Let all the coordinates of x be positive, that is, the game is completely mixed. Since $A \in E_0$, hence $val(A) \geq 0$. If $val(A) > 0$, then by Theorem 8, $A \in P$. Therefore, $A \in P_0$. If $val(A) = 0$, then using Theorem 9, $rank(A) = n - 1$. That means one of the row is linear combination of others. Hence $det(A) = 0$ and therefore $A \in P_0$.

b) Let x has k non-zero coordinates such that $k \leq n - 1$. We can partition our given matrix as

$$A = \begin{pmatrix} C & B \\ 0 & D \end{pmatrix}$$

where $C \in \mathbb{R}^{k\times k}, B \in \mathbb{R}^{k\times n-k}, D \in \mathbb{R}^{n-k\times n-k}$ and 0 is null matrix of order $n - k \times k$. From partitioned matrix properties, we know that $det(A) = det(C).det(D)$. Since from induction we know that for any $k \leq n - 1$, $det(C) \geq 0, det(D) \geq 0$. Hence, $det(A) \geq 0$. Therefore, $A \in P_0$.

Hence, for any n, $A \in \mathbb{R}^{n\times n} \cap E_0 \cap Z$ implies $A \in P_0$. □

In the above theorem, both the conditions of A being E_0 and Z are necessary. We can see this by the following two examples.

Example 1. Let

$$A = \begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix}.$$

It can be seen that $A \in Z$ and A is not an E_0-matrix (since $Ax < 0$ for some vector $x = (1, 1)^t$). Notice that $det(A) = -2$. Therefore, it is not a P_0-matrix. □

Example 2. Let

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

Since A is a nonnegative matrix, it can be easily verified that $A \in E_0$. Since off-diagonal entries are positive, $A \notin Z$. But the determinant of A is negative, hence A is not a P_0-matrix. □

Remark 3. It is known that $P_0 \subseteq E_0$. From Theorem 12, we can conclude that within Z-matrices, P_0 is equivalent to E_0. From Example 2, it can be seen that the Z-property is necessary for the equivalence to hold. From this result, we can conclude the next two theorems.

In [12], Parthasarathy, Ravindran and Sunil showed that within the class of E_0, R_0-matrices and Q-matrices are equivalent for matrices up to order 3. They provided counter examples of matrices which are $E_0 \cap Q$ but not R_0 for order 4 and above. But here we prove the equivalence with the additional assumption of Z for any order of matrices.

Theorem 13. *Let $A \in \mathbb{R}^{n\times n} \cap E_0 \cap Z$. Then $A \in Q$ iff $A \in R_0$.*

Proof. Let $A \in \mathbb{R}^{n\times n} \cap E_0 \cap Z$. Using Theorem 12, $A \in P_0$. Then Theorem 10 states that within P_0, Q is equivalent to R_0. Hence, $A \in Q$ if and only if $A \in R_0$. Therefore, within the class of E_0 and Z-matrices, R_0 is equivalent to Q □

Theorem 14. *Let $A \in \mathbb{R}^{n\times n} \cap Z$. Then $A \in E_0$ if and only if $A \in E_0^f$.*

Proof. Whenever $A \in E_0^f$, it is obvious A is an E_0-matrix. Now for the converse part let us assume $A \in E_0$. Since $A \in Z$, by Theorem 12, $A \in P_0$. Using remark 2, it is known that $P_0 \subseteq E_0^f$. Therefore, $A \in E_0^f$.

Hence, within the class of Z-matrices, E_0 is equivalent to E_0^f. □

In general, E_0-matrix are not equivalent to E_0^f. Hence, in the above theorem, the condition of matrix being a Z-matrix is necessary. It can be seen by the example below.

Example 3. Let

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Since A is a nonnegative matrix, it can be easily verified that $A \in E_0$. Let $\alpha = \{1, 2\}$. Consider $ppt(A, \alpha)$.

$$ppt(A, \alpha) = \begin{pmatrix} -1 & 2 & -2 \\ 1 & -1 & 1 \\ -1 & 2 & -2 \end{pmatrix}.$$

Observe that the diagonal entries of the ppt corresponding to the above α are negative. Using Theorem 1, ppt(A, α) is not an E_0-matrix. Therefore, $A \notin E_0^f$. □

4 Results for Matrices with Specific Sign Patterns

In this section, we consider matrices with some specific sign patterns and show some properties of such matrices.

Theorem 15. *Let $A \in \mathbb{R}^{n\times n}$ and $a_{ii} > 0$. Further suppose that all the entries below the diagonal are nonnegative. Then $A \in Q$.*

Proof. First we show that $A \in R_0$ for the given sign pattern of A. On the contrary, suppose $A \notin R_0$. That is, there is a non-zero vector $x \geq 0$ such that $Ax = w \geq 0$ and $x^t w = 0$.

$$Ax = \begin{pmatrix} + & * & * & \dots & * \\ \oplus & + & * & \dots & * \\ \oplus & \oplus & + & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \oplus & \oplus & \oplus & \dots & + \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_k \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} w_1 \\ \vdots \\ w_k \\ \vdots \\ w_n \end{pmatrix} = w$$

WLOG, let $x_k > 0$ for any $1 \leq k \leq n$, and $x_m = 0$ for $m > k$. Then it is easy to observe that w_k is positive. Hence, $x_k w_k \neq 0$. Hence, x_k cannot be positive for any k. Therefore, $A \in R_0$.

Let $d \in \mathbb{R}^n$ be any positive vector. Similarly, we can show that$x = 0$ is the only solution for $LCP(d, A)$. Hence, using Theorem 11, it can be concluded that $A \in Q$. □

Theorem 16. *Let $A \in \mathbb{R}^{n\times n}$ and $a_{ii} > 0$. Further suppose that all the entries below the diagonal are nonnegative and all the entries above the diagonal are non-positive. Then $A \in P_0$.*

Proof. Since A has all its diagonal entries positive. Hence, for $n = 1$, $A \in P_0$.

Now for $n = 2$,

$$A = \begin{pmatrix} + & \ominus \\ \oplus & + \end{pmatrix}$$

Here, $det(A) > 0$, hence $A \in P$ as well as $A \in P_0$.

For $n = 3$, every proper principal submatrix have the same sign pattern. Since we have seen such a matrix is P_0 up to $n = 2$, that is, proper principal minor are nonnegative. Hence, from Theorem 4, $A \in P_0$.

Assume it is true for the matrices up to order $n - 1$, that is, every matrix of the given sign pattern up to order $n - 1$ is P_0. For n, every proper principal minor is nonnegative. Therefore, using Theorem 4, such a matrix A is always a P_0-matrix. □

Remark 4. Since we know that $P_0 \subseteq E_0^f$, the matrices with given sign pattern in the above theorem are E_0^f. Therefore, Stone's conjecture holds for the matrices with all its diagonal entries positive, all the entries below the diagonal are nonnegative, and all the entries above the diagonal are non-positive.

Next, we show another sign pattern such that the conjecture holds for that pattern too. This pattern is almost similar to the above. But for the sake of completeness we are also giving an another way of proving it.

Theorem 17. *Let $A \in \mathbb{R}^{n\times n}$ and $a_{ii} > 0$. Further suppose that all the entries above the diagonal are nonnegative. Then $A \in R_0$.*

Proof. For given A, consider $LCP(0, A)$.

$$Ax = \begin{pmatrix} + & \oplus & \oplus & \ldots & \oplus \\ * & + & \oplus & \ldots & \oplus \\ * & * & + & \ldots & \oplus \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \ldots & + \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ x_k \\ \vdots \\ x_n \end{pmatrix} = w$$

where $*$ is any real number and $w \in \mathbb{R}^n$. Here for least value of k such that $x_k > 0$, $(Ax)_k = w_k$ is also positive. It contradicts the condition of complementarity. Hence, no x_k is positive. Therefore, $A \in R_0$. □

Remark 5. In the above theorem, if we further assume that A is fully semimonotone, then $A \in P_0$.

Theorem 18. *Let $A \in \mathbb{R}^{n\times n} \cap E_0^f$ and $a_{ii} > 0$. Further suppose that all the entries above the diagonal are nonnegative. Then $A \in P_0$.*

Proof. Let $A \in \mathbb{R}^{n\times n}$ and $a_{ii} > 0$. All the entries above the diagonal are nonnegative. Theorem 17 implies that $A \in R_0$. Since $A \in E_0^f$, Theorem 5 implies that $A \in P_0$. □

5 Conclusions

We have proved that a fully semimonotone and Q_0-matrix with specific sign patterns is a P_0-matrix. We also have proved that a matrix that is semimonotone and Z is contained in the class of P_0-matrices. Further, we have shown that for Z-matrices, the semimonotone matrices are the fully semimonotone matrices. Observe that these classes are subsets of $P_0 \cap Q_0$, and hence these classes are processable by Lemke's algorithm, that is, for each q, either Lemke's algorithm gives a solution or terminates in a ray.

Open Problem: Now, we state an open problem. The following conjecture is due to R.E. Stone.

Let $A \in \mathbb{R}^{n\times n}$. Further assume that A is a fully semimonotone Q_0-matrix. Can we say A is a P_0-matrix?

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