The Mexican Hat Wavelet Transform on Generalized Quotients and Its Applications

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Abstract. The theory of wavelet analysis is used to characterize functions and distribution spaces intrinsically. It is a field that is constantly evolving and is a mathematical approach widely used for many applications. Recently, the theory of Mexican hat wavelet transform (MHWT) on distributions and its properties are derived by Pathak et al. [10]. Further, Singh et al. [18] constructed Representation theorems for the same transform with some applications.

In this chapter, we study the Mexican hat wavelet transform (MHWT) to the space of generalized quotients with its operational properties and applications. We extend MHWT as a continuous linear map between the spaces of generalized quotients. An inversion and a characterization theorem for the MHWT of generalized quotients are also discussed. Further, Mexican hat wavelet transformation is defined on the space of tempered generalized quotients by employing the structure of exchange property. We study the exchange property for the Mexican hat wavelet transform by applying the theory of the Mexican hat wavelet transform of distributions. Furthermore, different properties of Mexican hat wavelet transform are discussed on the space of tempered generalized quotients with applications.

Keywords: Fourier transform · Wavelet transform · Schwartz distributions · quotient spaces

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1 Introduction

The field of wavelet has lately drawn a substantial amount of attention from mathematical scientists from domains of different subject areas. That is forming a generic bond among physicists, mathematicians, and electrical engineers.

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The topic of wavelets has always been a prevalent cause of discussion in numerous engineering and scientific gatherings at present. Few considered wavelets as a unique basis for representing functions, others view it as a method for the analysis of time-frequency, and rest believe that it is an advanced topic of mathematics. Indeed, all of these theories are correct, given the fact that "wavelets" are flexible mechanisms that are extremely rich in mathematical scope and have a significant number of applications.

Wavelets are the latest area in the frontiers of mathematics, signal processing, image processing, and scientific computing. It is a versatile tool in every aspect of mathematical context and possesses great potential for applications, as wavelets can be viewed as a unique basis for representing functions for time-frequency analysis. The theory of Fourier analysis is well established and popular subject at the core of pure and applied mathematical analysis. The basic building blocks of the Fourier transform (complex exponentials: $e^{i2\pi tu}$) oscillate over all of the time ($-\infty < t < \infty$). As a result, it is difficult for the Fourier transform to represent signals that are localized in time. Thus, it fails to accumulate information that varies with time. As it does not provide the time at which frequency exists hence, it is only ideal for stationary signals. Hence, Fourier methods are not very effective in recapturing the non-smooth signal. In these cases, wavelet analysis is often very efficient, as it presents a simple approach for dealing with the local aspects of a signal. For the last two decades, the advancement of wavelet transform in the field of signal analysis is expanding making it an important mathematical tool. The main reason is wavelet transform can represent a function of the time domain in a time-frequency plane. Therefore, it works as a frequency and time localization operator. Also, wavelets can change according to time intervals to obtain high and low-frequency components. Hence, enhancing the study of signal analysis with localized impulses and oscillations. In particular, wavelet analysis is efficient in extracting noise from signals that complement the classical methods of Fourier analysis. Wavelet analysis has been one of the major research directions in both pure and applied mathematics and is still undergoing rapid growth.

The wavelets were developed mostly during the last three decades and are associated with the classical theories of different disciplines, including pure and applied mathematics and engineering. The theory of wavelets can be seen as syntheses of different ideas that started from various areas, including physics (coherent states formalism in quantum mechanics), mathematics (Calderòn Zygmund operators and Littlewood - Paley theory), and engineering (in signal and image processing). The mathematical interpretation of the wavelet transform started in the year 1985 when Y. Meyer discovered the results given by Morlet and the Marseille group. He noticed a link of Morlet's algorithm to the resolution of identity in the harmonic analysis due to A. Calderón in 1964. Therefore, Meyer built the mathematical foundation of wavelet analysis and hence may be regarded as the founder of it. He still actively promotes the field of wavelet analysis as an interdisciplinary area of research. Recently, applications of wavelet analysis have been extended across various fields of mathematics, physics, computer science, and engineering. The term wavelet refers to a short wave. This indicates that every wavelet is localized and has to have at least some oscillations. Wavelets were introduced to represent functions more efficiently than the Fourier series. Further, wavelets comprise a family which contains functions indexed by two parameters, one for scaling and the other for positioning. They are developed from one single function called the mother wavelet. A function ψ is called wavelet if it satisfies

$$\int_{-\infty}^{\infty} \psi(t)dt = 0. \tag{1.1}$$

This condition indicates that ψ switches sign in $(-\infty, \infty)$, and it fades at $\pm\infty$. By applying position and scaling parameters on the basic function $\psi \in L^2(\mathbb{R})$, the wavelet $\psi_{b,a}(t)$ is defined by

$$\psi_{b,a}(t) = (\sqrt{a})^{-1}\psi\left(\frac{t-b}{a}\right), \quad t \in \mathbb{R}, \tag{1.2}$$

where the normalizing factor $(\sqrt{|a|})^{-1}$ ensures that $||\psi_{b,a}(x)||$ is independent of the position parameter b and scaling parameter a. Also, $\psi_{b,a}(t)$ is called the mother wavelet and it satisfies the admissibility condition given by [9] as follows:

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(u)|^2}{|u|} du < \infty. \tag{1.3}$$

The wavelet is called admissible if $C_\psi < \infty$. Therefore,

$$\Psi(0) = \int_{-\infty}^{\infty} \psi(t)dt = 0. \tag{1.4}$$

The wavelet $\psi(t)$ acts as a impulse response of a band-pass filter that decays as fast as $|t|^{1-\epsilon}$. Practically, the wavelet $\psi(t)$ should decay much faster to provide good time-localization. The mother wavelet emerges as a local oscillation such that the energy of each oscillation in the physical space is discovered in the limited province. Then by the uncertainty principle, the positioning of the function in the physical space restricts the positioning in the frequency domain. The dilation or scaling parameter 'a'controls the width and the frequency of $\psi_{b,a}(t)$. The position or translation parameter 'b'relocates the wavelet across the whole domain and is beneficial for identifying the location of the wavelets.

The wavelet transform of $\phi \in L^2(\mathbb{R})$, with respect to (1.2), is defined by [9]

$$(W\phi)(b,a) = \int_{\mathbb{R}} \phi(t)\overline{\psi_{b,a}}(t)dt, \quad t, b \in \mathbb{R}, \quad a > 0. \tag{1.5}$$

and the inversion for (1.5) is given by

$$\phi(x) = \frac{2}{C_\psi}\int_0^\infty \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{a}}(W\phi)(b,a)\psi\left(\frac{x-b}{a}\right)db\right]\frac{da}{a^2}, \quad x \in \mathbb{R}, \tag{1.6}$$

where,

$$\frac{C_\psi}{2} = \int_0^\infty \frac{|\hat{\psi}(v)|^2}{|v|} dv = \int_0^\infty \frac{|\hat{\psi}(-v)|^2}{|v|} dv < \infty \qquad [3,\ \text{p. } 64]. \tag{1.7}$$

If (1.5) exists, then $(W\phi)(b,a)$ maps each square integrable function ϕ on $\mathbb{R}$ to wavelet transform function W on $\mathbb{R} \times \mathbb{R}_+$. Therefore, from (1.5),

$$(W\phi)(b,a) = (\phi * h_{a,0})(b), \tag{1.8}$$

where $h(t) = \bar{\psi}(-t)$.

If $\phi \in L^p(\mathbb{R})$ and $\psi \in L^q(\mathbb{R})$, then

$$\phi * h_{a,0}(b) \in L^r(\mathbb{R}), \qquad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1. \tag{1.9}$$

Now, applying Fourier transform to (1.8), we get

$$\begin{aligned}(W\phi)(b,a) &= \frac{|a|^{1/2}}{(2\pi)} \int_{\mathbb{R}} e^{i\omega b} \overline{\hat{\psi}(a\omega)} \hat{\phi}(\omega) d\omega \\ &= \frac{|a|^{1/2}}{(2\pi)} \mathcal{F}^{-1} \left[\hat{\phi}(\omega) \overline{\hat{\psi}(a\omega)} \right] (b,a).\end{aligned} \tag{1.10}$$

Hence,

$$\mathcal{F}\left[(W\phi)(b,a)\right](\omega) = |a|^{1/2} \hat{\phi}(\omega) \overline{\hat{\psi}(a\omega)}. \tag{1.11}$$

This relation holds in general, for $\phi \in L^p(\mathbb{R})$ and $\psi \in L^q(\mathbb{R})$, where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1;\ 1 \le p, q, r \le 2$.

The Mexican hat wavelet is constructed by taking the negative normalized second derivative of a Gaussian function which, up to scale and normalization, is the second Hermite function. It is a special case of the family of continuous wavelets known as Hermitian wavelets and is defined by [10,29]

$$\psi(t) = e^{-(\frac{t^2}{2})}(1 - t^2) = -\frac{d^2}{dt^2} e^{-(\frac{t^2}{2})} \tag{1.12}$$

such that

$$\psi_{b,a}(t) = -a^{\frac{3}{2}} D_t^2 e^{-\frac{(b-t)^2}{2a^2}}, \qquad \left(D_t = \frac{d}{dt}\right). \tag{1.13}$$

Thus, the wavelet transform (1.5) can be written as:

$$(W\phi)(b,a) = -a^{\frac{3}{2}} \int_{\mathbb{R}} \phi(t)\ D_t^2 e^{-\frac{(b-t)^2}{2a^2}} dt, \qquad a > 0 \tag{1.14}$$

which then, under certain conditions on ϕ is

$$(W\phi)(b,a) = -a^{\frac{3}{2}} \int_{\mathbb{R}} \phi^{(2)}(t)\ \ e^{-\frac{(b-t)^2}{2a^2}} dt, \qquad a > 0. \tag{1.15}$$

From the above two equations, the MHWT can also be considered as the Weierstrass transform of $D_t^2\phi(t) = \phi^{(2)}(t)$. Hence, we may infer various properties of MHWT from the known theory of the Weierstrass transform. The distributional Weierstrass transform has been studied in [8]. For a suitable space of generalized functions $(\mathscr{W}_{\alpha,\beta}^{\gamma})^{'}$ the generalized MHWT is given by [10]

$$(W\phi)(b,a) = -a^{\frac{3}{2}}\left\langle \phi(t),\ D_t^2 exp\left(-\frac{(b-t)^2}{2a^2}\right)\right\rangle, \qquad \frac{\alpha}{\gamma} < \text{Re}\, b < \frac{\beta}{\gamma}. \quad (1.16)$$

A function $k(b,a)$ is defined by

$$k(b,a) = \frac{1}{\sqrt{2\pi a}} e^{\left(\frac{-b^2}{2a}\right)}, \quad (1.17)$$

where $a \in (0,\infty)$ and $b = \sigma + i\omega$. Then

$$D_t^2 k(b-t,a^2) = \frac{1}{\sqrt{2\pi}a} D_t^2\left(e^{\frac{-(b-t)^2}{2a^2}}\right). \quad (1.18)$$

Therefore, by (1.13)

$$\psi_{b,a}(t) = -(2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} D_t^2 k(b-t,a^2)$$

and $$(W\phi)(b,a) = (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \int_{\mathbb{R}} \phi(t)\ D_t^2\ k(b-t,\ a^2)\ dt. \quad (1.19)$$

The most general theory of the MHWT is investigated on the generalized function space $(\mathscr{W}_{\alpha,\beta}^{\gamma})^{'}$ developed by Pathak *et al.* [10]. It is proved that the MHWT $(W\phi)(b,a)$ of $\phi \in (\mathscr{W}_{\alpha,\beta}^{\gamma})^{'}$, is given by $\langle \phi^{(2)}(t), k_{a^2}(b-t)\rangle$ is an analytic function in the strip $\frac{\alpha}{\gamma} < \text{Re}\, b < \frac{\beta}{\gamma}$ for some $\alpha,\beta,\gamma \in \mathbb{R}$. Therefore, it follows that the MHWT of $\phi \in (\mathscr{W}_{\alpha,\beta}^{\gamma})^{'}$ is an entire function and hence its restriction to the real axis is in $C^\infty(\mathbb{R})$ and further this restriction uniquely determines the analytic function $(W\phi)(b,a)$. For our purposes the MHWT of ϕ denotes this restriction only.

1.1 Generalized Quotients

In recent years the theory of distributions or generalized functions is at its peak bringing a great revolution in mathematical analysis. In 1935, Sergei L. Sobolev derived the theory of generalized functions while working on the second-order hyperbolic partial differential equations. But in the 1950s, L. Schwartz introduced the concept of distributions that opened a new area of mathematical research [28]. This concept supported the development of several mathematical disciplines, such as transformation theory, operational calculus, ordinary and partial differential equations, and functional analysis. Another approach for this theory was given by S. Bochner around 1930s, to generalize the Fourier transformation for functions $f(t)$ that grow as t approaches infinity. The concept of distribution gives a better mechanism for analysing various entities, such as the delta function, which arise naturally in several mathematical sciences and which can be corrected using distributions. The idea behind distribution is assigning a function not by its values but by its behaviour as a functional on some space of testing functions. Here the space of testing functions is represented by $\mathcal{D}$ which contains all complex-valued functions that are infinitely smooth and have compact support. A continuous linear functional on the space $\mathcal{D}$ is called a distribution and space of all distributions is dual of the space $\mathcal{D}$, denoted by $\mathcal{D}'$.

In the theory of distributional analysis, differentiation is a continuous operation as every distribution has derivatives of all orders. Consequently, distributional differentiation commutes with different limiting processes such as integration and infinite summation. This is the contrast to classical analysis wherein either such operations cannot be interchanged or the inversion of the order must be justified by an additional argument. Though not very recently, yet during the last five decades the theory of generalized functions and integral transforms has been combined, which gave rise to fruitful results in the theory of integral transforms associated with distributions, known as distributional transform analysis. Recently, there were many applications of wavelet and other transforms in distribution spaces [11–15, 17]. Further, the investigation of the wavelet transform of distributions, tempered distributions, and ultra-distributions has extended the applications of the wavelet transform [7, 19–21].

One of the recent generalizations of L. Schwartz's theory of distributions is the Mikusiński's algebraic approach or the sequential approach, used to define generalized quotient spaces (Boehmians). The theory of generalized quotients in 1973 by T. K. Boehme, brought a new change in the theory of applicable functional analysis [2]. The motivation for the development of the theory of generalized quotients lies in the core of regular operators, proposed by J. Mikusiński and P. Mikusiński in [4–6], which form a subalgebra of the field of Mikusiński operators.

The generalized quotients are defined by an abstract algebraic construction which is the same as the construction of the field of quotients. Instead of the normal quotients, here we use quotients of sequences where the numerator is a sequence of some set G and the denominator is a delta sequence. This space of generalized quotients includes all regular operators, all distributions, and some objects which are neither operators nor distributions. Also, it is possible to construct generalized quotients even if there are zero divisors, such as the space of all continuous functions, say C. Application of this construction to function spaces with the convolution product provides different spaces of generalized functions. Therefore, different integral transforms have been defined for various spaces of generalized quotients and their properties are investigated in [1, 16, 22–25].

In the next section, we discuss some of the basic results required for the investigation of MHWT on the generalized quotient space. Also we show the MHWT becomes a continuous linear map from one space of generalized quotient into another. The operational properties of MHWT and an inversion formula in the context of generalized quotients is also discussed in this section. In the last section, we deals with Mexican hat wavelet transformation on the space of tempered generalized quotients by employing the structure of exchange property. Furthermore, different properties of Mexican hat wavelet transform are discussed on the space of tempered generalized quotients with applications.

2 The Mexican Hat Wavelet Transform (MHWT) on the Space $\mathcal{H}$

Now, we take suitable generalized quotient spaces on which the MHWT can be derived. The construction of the space $\mathscr{B}(C^\infty, \Delta)$ is given by Pathak [8] and the construction of $\mathscr{B}(\mathcal{H}, \Delta)$ is as follows:

Let the space $\mathcal{H}$ consists of functions $\phi \in C^\infty$ such that

$$\sup_{x\in\mathbb{R}} e^{\frac{-x^2}{2}} \rho_{p,q}^{-1}(x)\ |\phi(x)| \ \leq\ M(p,q) \text{ for all } -\infty < p < 0 < q < \infty,$$

where $$\rho_{p,q}(x) = \begin{cases} e^{\frac{-px}{2}}, & x < 0 \\ e^{\frac{-qx}{2}}, & x \geq 0 \end{cases}$$

and $M(p,q)$ is a constant which depends on p and q. The n^{th} semi norm for $N = 0, 1, 2, ...$ on $\mathcal{H}$ is defined as,

$$\|\phi\|_N = \sup_{x\in\mathbb{R}} |e^{\frac{-x^2}{2}} \rho_{-N,N}^{-1}(x)\phi(x)|, \tag{2.1}$$

where $\mathcal{H}$ becomes a Fréchet space under the above mentioned family of semi norms. A sub semi group of $\mathcal{H}$, denoted by S is taken as a testing function space i.e., $S = \mathcal{D}$ and let Δ be a class of sequences (δ_n) from $\mathcal{D}$ which satisfies the following conditions:

(i) $\int_{\mathbb{R}} \delta_n = 1,$
(ii) $\int_{\mathbb{R}} |\delta_n| \leq M,$
(iii) supp $\delta_n \to 0\ as\ n\ \to \infty.$

The set of all continuous linear functionals defined on $\mathcal{D}$ is denoted by $\mathcal{D}'$.

Now we consider the Mexican hat wavelet transform of the function $\phi(t)$ as the convolution of $\phi^{(2)}(t)$ with the function $k_{a^2}(b)$. Hence, the classical inverse wavelet transform will produce the second derivative of the function $\phi(t)$. If $\phi(t), \varphi(t) \in \mathcal{H}$, then the convolution product $\phi * \varphi$ is given by

$$(\phi * \varphi)(x) = \int_{\mathbb{R}} \phi(u)\varphi(x-u)du. \tag{2.2}$$

The MHWT of $\phi \in \mathcal{H}$, is given by,

$$\begin{aligned}(W\phi)(b,a) &= (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} (\phi^{(2)} * k_{a^2})(b) \\ &= (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \int_{\mathbb{R}} \phi^{(2)}(t) k_{a^2}(b) dt, \quad b \in \mathbb{C},\ a \in \mathbb{R}^+,\end{aligned} \tag{2.3}$$

where $k_{a^2}(b) = k(b-t, a^2) = \frac{1}{\sqrt{2\pi}a} e^{\frac{-(b-t)^2}{2a^2}}$.

Theorem 1. *For a function $\phi \in \mathcal{H}$ and $t \in \mathbb{R}$,*

$$(W\phi)(b,a) = (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} (\phi^{(2)} * k_{a^2})(b) = (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \lim_{n\to\infty} ((\phi^{(2)} * k_{a^2})\ e^{-\frac{t^2}{2n}})(b).$$

Proof. Consider,

$$\begin{aligned}(2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \lim_{n\to\infty} ((\phi^{(2)} * k_{a^2})\ e^{-\frac{t^2}{2n}})(b) &= (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \lim_{n\to\infty} \int_{\mathbb{R}} \phi^{(2)}(t)\ k(b-t,a^2)\ e^{-\frac{t^2}{2n}} dt \\ &= a^{\frac{3}{2}} \lim_{n\to\infty} \int_{\mathbb{R}} \phi^{(2)}(t)\ e^{-\frac{(b-t)^2}{2a^2}}\ e^{-\frac{t^2}{2n}} dt \\ &= a^{\frac{3}{2}} \int_{\mathbb{R}} \phi^{(2)}(t) e^{-\frac{(b-t)^2}{2a^2}} dt,\end{aligned}$$

(by Lebesgue dominated convergence theorem).

Therefore,

$$(W\phi)(b,a) = (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \lim_{n\to\infty} ((\phi^{(2)} * k_{a^2})\ e^{-\frac{t^2}{2n}})(b).$$

□

Theorem 2. *For $\phi \in \mathcal{W}'(-\infty,\infty)$ and $\varphi \in \mathcal{D}$, we have*

$$(W(\phi * \varphi))(b,a) = (W\phi)(b,a) * \varphi.$$

Theorem 3. *Let $\phi_n^{(2)} \to \phi^{(2)}$ uniformly as $n \to \infty$ in $\mathcal{H}$, then $(W\phi_n)(b,1) \to (W\phi)(b,1)$ as $n \to \infty$, for $b = \sigma + i\omega$.*

Lemma 1. *Let $\phi, g \in \mathcal{H}$ such that $(W\phi)(b,a) = (Wg)(b,a)$, then $\phi^{(2)} = g^{(2)}$ in $\mathcal{H}$.*

Proof. The proof is similar to [[27], Lemma 4.4.4], in the case of Weierstrass transform. □

Definition 1. Let $X = \left[\frac{\phi_n}{\varphi_n}\right] \in \mathscr{B}(\mathcal{H},\Delta)$, then the MHWT of X as a generalized quotient is defined by,

$$Y = (WX)(b,a) = \left[\frac{(W\phi_n)(b,a)}{\varphi_n}\right].$$

It is well defined since, if $X = \left[\frac{\phi_n}{\varphi_n}\right] = Y = \left[\frac{g_n}{\psi_n}\right]$ in $\mathscr{B}(\mathcal{H},\Delta)$, then

$$\begin{aligned}\phi_m * \psi_n &= g_n * \varphi_m \quad \forall m,n \in \mathbb{N} \\ (W(\phi_m * \psi_n))(b,a) &= (W(g_n * \varphi_m))(b,a) \\ (W\phi_m)(b,a) * \psi_n &= (Wg_n)(b,a) * \varphi_m \quad \text{(by Theorem 2)} \\ \left[\frac{(W\phi_n)(b,a)}{\varphi_n}\right] &= \left[\frac{(Wg_n)(b,a)}{\psi_n}\right].\end{aligned}$$

Theorem 4. *For $\phi \in \mathcal{W}'(-\infty,\infty)$, Definition 3 is consistent with the classical definition.*

Proof. By considering the map $\phi \rightarrow \left[\frac{\phi*\delta_n}{\delta_n}\right]$, any $\phi \in \mathcal{W}'(-\infty,\infty)$ can be considered as an element of $\mathscr{B}(\mathcal{H},\Delta)$ by [27, Theorem 4.3.9], i.e., if $X = \left[\frac{\phi*\delta_n}{\delta_n}\right]$, then

$$(WX)(b,a) = \left[\frac{W(\phi*\delta_n)(b,a)}{\delta_n}\right] = \left[\frac{(W\phi)(b,a)*\delta_n}{\delta_n}\right] = (W\phi)(b,a).$$

□

3 Operational Properties

This section introduces the operational properties of the MHWT on generalized quotient space. Further, through inversion it is shown that the generalized quotient in $\mathscr{B}(\mathcal{H},\Delta)$ approximates to a function in C^∞ in a distributional sense.

Theorem 5. *(i) For $X, Y \in \mathscr{B}(\mathcal{H},\Delta)$, $X+Y \in \mathscr{B}(\mathcal{H},\Delta)$ and $(W(X+Y))(b,a) = (WX)(b,a) + (WY)(b,a)$.*
(ii) For $X \in \mathscr{B}(\mathcal{H},\Delta)$ and $\alpha(\neq 0) \in \mathbb{C}$, $\alpha X \in \mathscr{B}(\mathcal{H},\Delta)$ and $(W(\alpha X))(b,a) = \alpha(WX)(b,a)$.

Theorem 6. *For $X \in \mathscr{B}(\mathcal{H},\Delta)$ and $\psi \in \mathcal{D}$, $(W(X*\psi))(b,a) = (WX)(b,a)*\psi$.*

Proof. Let $X = \left[\frac{\phi_n}{\varphi_n}\right] \in \mathscr{B}(\mathcal{H},\Delta)$ and $\psi \in \mathcal{D}$, then

$$X*\psi = \left[\frac{\phi_n*\psi}{\varphi_n}\right] \in \mathscr{B}(\mathcal{H},\Delta). \tag{3.1}$$

Thus,

$$\begin{aligned}(W(X*\psi))(b,a) &= \left[\frac{(W(\phi_n*\psi))(b,a)}{\varphi_n}\right]\\ &= \left[\frac{(W\phi_n)(b,a)*\psi}{\varphi_n}\right]\\ &= \left[\frac{(W\phi_n)(b,a)}{\varphi_n}\right]*\psi\\ &= (WX)(b,a)*\psi.\end{aligned}$$

□

Now, we show that the MHWT on $\mathscr{B}(\mathcal{H},\Delta)$ is continuous in the sense that it carries δ-convergent sequences onto δ-convergent sequences.

Theorem 7. *Let* (X_n) *be a sequence of generalized quotients such that* $X_n \xrightarrow{\delta} X$ *in* $\mathscr{B}(\mathcal{H}, \Delta)$, *then* $(WX_n)(b, 1) \xrightarrow{\delta} (WX)(b, 1)$ *in* $\mathscr{B}(C^\infty, \Delta)$.

In the next theorem we show the inversion of the MHWT of generalized quotients belonging to the space $\mathscr{B}(\mathcal{H}, \Delta)$.

Theorem 8. *Let* $Y = \left[\frac{g_n}{\varphi_n}\right] \in \mathscr{B}(C^\infty, \Delta)$ *be such that* $Y = (WX)(b, a)$ *for some* $X \in \mathscr{B}(\mathcal{H}, \Delta)$. *Then* $X = \left[\frac{\phi_n}{\varphi_n * \varphi_n}\right]$ *where* ϕ_n*'s are defined as follows:*

$$\phi_{n,k}(t) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-k}^{k} g_n(iy, a) k(y + it, a)\, dy$$

and
$$F_n^{(2)} = lim_{k\to\infty} \phi_{n,k},$$

where the limit is taken in $\mathcal{D}'$, *then* $\phi_n = F_n * \varphi_n$.

The next theorem indicates the characterization of MHWT for generalized quotients on compact subsets of $\mathbb{R}$.

Theorem 9. *A generalized quotient* $Y = \left[\frac{g_n}{\psi_n}\right]$ *in* $\mathscr{B}(C^\infty, \Delta)$ *is the MHWT of a generalized quotient* $X = \left[\frac{\phi_n}{\varphi_n}\right]$ *in* $\mathscr{B}(\mathcal{H}, \Delta)$ *if and only if for each* n, g_n *can be extended as an entire function satisfying* $|g_n(b, a)| \leq C_n e^{\frac{\omega^2}{2}} P_n(|\omega|)$, *where* $b = \sigma + i\omega$ *as* σ *varies on compact subsets of* $\mathbb{R}$ *and* $P_n(|\omega|)$ *is a polynomial in* $|\omega|$ *depending on both* n *and on the compact set in which* σ *varies.*

Proof. Let $Y = (WX)(b, a)$, then by applying Mexican hat wavelet transform on Theorem 1.3.15 of [27],

$$|g_n(b, a)| \leq C_n e^{\frac{\omega^2}{2}} P_n(|\omega|) \text{ for every } n \in \mathbb{N}. \tag{3.2}$$

Conversely, let g_n can be extended as an entire function which satisfies (3.2) for every $n \in \mathbb{N}$, and by the same Theorem there exists, $h_n \in \mathcal{W}'(-\infty, \infty)$ such that $(Wh_n)(b, a) = g_n$. Since $h_n \in \mathcal{W}'(-\infty, \infty)$ and $\psi_n \in \Delta$, therefore, by Lemma 4.3.8 of [27], $h_n * \psi_n \in G$.

Now, put $\phi_n = h_n * \psi_n$ and $\varphi_n = \psi_n * \psi_n$ so that $\phi_n \in \mathcal{H}$ and $(\varphi_n) \in \Delta$. Clearly, $\frac{\phi_n}{\varphi_n}$ is a quotient in $\mathscr{B}(\mathcal{H}, \Delta)$, as $\frac{g_n}{\psi_n}$ is a quotient in $\mathscr{B}(\mathcal{H}, \Delta)$.

Also, $g_n = (Wh_n)(b,a)$, i.e., $X = \left[\frac{\phi_n}{\varphi_n}\right] \in \mathscr{B}(\mathcal{H}, \Delta)$ and

$$\begin{aligned}(WX)(b,a) &= \left[\frac{(W\phi_n)(b,a)}{\varphi_n}\right] \\ &= \left[\frac{W(h_n * \psi_n)(b,a)}{\psi_n * \psi_n}\right] \\ &= \left[\frac{(Wh_n)(b,a) * \psi_n}{\psi_n * \psi_n}\right] \\ &= \left[\frac{g_n * \psi_n}{\psi_n * \psi_n}\right] \\ &= \left[\frac{g_n}{\psi_n}\right] \\ &= Y.\end{aligned}$$

□

4 The Exchange Property

In this section, the space of tempered generalized quotients is constructed by applying the exchange property. This construction for generalized quotients indicates that the role of convergence is not necessary.

Let the space of rapidly decreasing smooth functions on $\mathbb{R}^n$ and $\mathbb{R}^n \times \mathbb{R}_+$ be denoted by $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}(\mathbb{R}^n \times \mathbb{R}_+)$. The dual of $\mathscr{S}$ is the space of tempered distributions, represented by $\mathscr{S}'$. The spaces $\mathscr{S}$ and $\mathscr{S}'$ have been introduced and developed in [1]. The class $\mathscr{S}'$ of t empered distributions is contained in $(\mathscr{W}^{\gamma}_{\alpha,\beta})'$. Therefore the Mexican hat wavelet transform theory can be made applicable to $\mathscr{S}'$. Further, the Mexican hat wavelet transform can be expanded to the space of tempered generalized quotient, as the space is a natural expansion of tempered distributions. In this paper, we extend the Mexican hat wavelet transformation to a class of generalized quotient space that have quotients of sequences in the form of ϕ_n/φ_n, where the numerator contains terms of the sequence from some set $\mathscr{S}'$ and the denominator is a delta sequence such that it satisfies the following condition

$$\phi_n * \varphi_m = \phi_m * \varphi_m, \quad \forall m, n \in \mathbb{N}. \tag{4.1}$$

Further, the delta sequences are defined as sequences of functions $\{\varphi_n\} \in \mathscr{S}$ such that

1. $\int_{\mathbb{R}^n} \varphi_n(x)dx = 1$ for all $n = 1, 2, 3, \cdots$.
2. There exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^n} |\varphi_n(x)|\, dx \le C \text{ for all } n = 1, 2, 3, \cdots.$$

3. $\lim_{n\to\infty} \int_{\|x\|\ge\epsilon} \|x\|^k \,|(\varphi_j(x))|\, dx = 0$ for every $k \in \mathbb{N}$ and $\epsilon > 0$.

In particular, we extend the transformation to generalized quotient space by defining an exchange property for the Mexican hat wavelet transform. We discuss some of the basic results required for the investigation of MHWT on the generalized quotient space. Further, we describes some algebraic properties of MHWT in the context of tempered generalized quotients.

Theorem 10. *For a function $\phi \in \mathscr{S}'$ and $t \in \mathbb{R}$,*

$$(W\phi)(b,a) = (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} (\phi^{(2)} * k_{a^2})(b) = (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \lim_{n\to\infty} ((\phi^{(2)} * k_{a^2}) e^{-\frac{t^2}{2n}})(b).$$

Proof. Consider,

$$\begin{aligned}(2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \lim_{n\to\infty} ((\phi^{(2)} * k_{a^2}) e^{-\frac{t^2}{2n}})(b) &= (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \lim_{n\to\infty} \int_{\mathbb{R}} \phi^{(2)}(t) k_{a^2}(b) e^{-\frac{t^2}{2n}} dt \\ &= a^{\frac{3}{2}} \lim_{n\to\infty} \int_{\mathbb{R}} \phi^{(2)}(t) e^{-\frac{(b-t)^2}{2a^2}} \ e^{-\frac{t^2}{2n}} dt \\ &= a^{\frac{3}{2}} \int_{\mathbb{R}} \phi^{(2)}(t) e^{-\frac{(b-t)^2}{2a^2}} dt.\end{aligned}$$

Therefore,

$$(W\phi)(b,a) = (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \lim_{n\to\infty} ((\phi^{(2)} * k_{a^2}) e^{-\frac{t^2}{2n}})(b).$$

□

Theorem 11. *For $\phi \in \mathscr{S}'$ and $\varphi \in \mathscr{S}$, we have*

$$(W(\phi * \varphi))(b,a) = (Wf)(b,a) * \varphi.$$

Proof. By using [[27], Lemma 4.3.8], $(\phi * \varphi) \in \mathscr{S}'$ and hence $(W(\phi * \varphi))(b,a)$ is defined. Also, by Theorem 10

$$(W(\phi * \varphi))(b,a) = (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \lim_{n\to\infty} (((\phi^{(2)} * \varphi) * k_{a^2}) e^{-\frac{t^2}{2n}})(b).$$

Consider,

$$\begin{aligned}(2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} (((\phi^{(2)} * \varphi) * k_{a^2}) e^{-\frac{t^2}{2n}})(b) &= (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \int_{\mathbb{R}} (\phi^{(2)} * \varphi)(t) k(b-t, a^2) e^{-\frac{t^2}{2n}} \ dt \\ &= a^{\frac{3}{2}} \int_{\mathbb{R}} (\phi^{(2)} * \varphi)(t) e^{-\frac{(b-t)^2}{2a^2}} e^{-\frac{t^2}{2n}} \ dt \\ &= a^{\frac{3}{2}} \int_{\mathbb{R}} \langle \phi^{(2)}(s), \varphi(t-s) \rangle e^{-\frac{(b-t)^2}{2a^2}} e^{-\frac{t^2}{2n}} \ dt \\ &= a^{\frac{3}{2}} \int_{\mathbb{R}} \langle \phi^{(2)}(s), \varphi(t-s) \rangle \psi_n(t) dt, \qquad (4.2)\end{aligned}$$

where $\psi_n(t) = e^{-\frac{(b-t)^2}{2a^2}} e^{-\frac{t^2}{2n}}$.

By [10, Lemma 4.3], we have

$$a^{\frac{3}{2}} \int_{-m}^{m} \langle \phi^{(2)}(s), \varphi(t-s) \rangle \psi_n(t) dt = a^{\frac{3}{2}} \left\langle \phi^{(2)}(s), \int_{-m}^{m} \varphi(t-s) \psi_n(t) dt \right\rangle, \ \forall m > 0,$$

which converges to

$$a^{\frac{3}{2}} \left\langle \phi^{(2)}(s), \int_{-m}^{m} \varphi(t-s)\psi_n(t)dt \right\rangle \text{ as } m \to \infty,$$

Therefore,

$$\begin{aligned} \int_{-\infty}^{\infty} \langle \phi^{(2)}(s), \varphi(t-s) \rangle e^{-\frac{(b-t)^2}{2a^2}} e^{-\frac{t^2}{2n}} \, dt &= \left\langle \phi^{(2)}(s), \int_{-\infty}^{\infty} \varphi(t-s)\psi_n(t) \, dt \right\rangle \\ &= \langle \phi^{(2)}(s), (\varphi * \psi_n)(s) \rangle. \end{aligned} \tag{4.3}$$

Let us now consider,

$$\begin{aligned} (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} ((\phi^{(2)} * k_{a^2}) * \varphi)(b) &= (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \int_{\mathbb{R}} (\phi^{(2)} * k_{a^2})(b-t)\varphi(t) \, dt \\ &= (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \int_{-M}^{M} \langle \phi^{(2)}(s), k_{a^2}(b-t-s) \rangle \varphi(t) \, dt, \end{aligned}$$

where supp $\varphi \subseteq [-P, P]$. Now by [10, Lemma 4.3],

$$\begin{aligned} (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} ((\phi^{(2)} * k_{a^2}) * \varphi)(b) &= (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \int_{-M}^{M} \langle \phi^{(2)}(s), k_{a^2}(b-t-s) \rangle \varphi(t) \, dt \\ &= (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \left\langle \phi^{(2)}(s), \int_{-\infty}^{\infty} k_{a^2}(b-t-s)\varphi(t) \, dt \right\rangle \\ &= (2\pi)^{\frac{1}{2}} a^{\frac{5}{2}} \left\langle \phi^{(2)}(s), \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} a} \psi(t-s)\varphi(t) \, dt \right\rangle \\ &= a^{\frac{3}{2}} \left\langle \phi^{(2)}(s), \int_{-\infty}^{\infty} \psi(t-s)\varphi(t) \, dt \right\rangle \\ &= a^{\frac{3}{2}} \langle \phi^{(2)}(s), (\varphi * \psi)(s) \rangle. \end{aligned} \tag{4.4}$$

From (4.3) and (4.4), we obtain

$$(W(\phi * \varphi))(b, a) = (W\phi)(b, a) * \varphi.$$

□

Definition 2. For a family $\{\varphi_j\}_{j \in J}$, where $\varphi_j \in \mathcal{S}$, we define

$$M\Big(\{\varphi_j\}_J \Big) = \{x \in \mathbb{R}^n : \varphi_j(x) = 0, \;\; \forall j \in J\}. \tag{4.5}$$

A family of pairs $\{(\phi_j, \varphi_j)\}_J$, where $\phi_j \in \mathcal{S}'$ and $\varphi_j \in \mathcal{S}$, have the exchange property if

$$\phi_j * \varphi_k = \phi_k * \varphi_j, \forall j, k \in J. \tag{4.6}$$

Let set $\mathcal{A}$ denotes the collection of $\{(\phi_j, \varphi_j)\}_J$, where $\phi_j \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi_j \in \mathcal{S}(\mathbb{R}^n), \; \forall j \in J$, with exchange property such that $M\Big(\{\varphi_j\}_J \Big) = \emptyset$.

Lemma 2. *If* $M\left(\{\varphi_j\}_J\right)=\emptyset$ *and* $M\left(\{\lambda_k\}_K\right)=\emptyset$*, then* $M\left(\{\varphi_j*\lambda_k\}_{J\times K}\right)=\emptyset$.

Theorem 12. *If* $\{(\phi_j,\varphi_j)\}_J\in\mathcal{A}$*, then there exists a unique* $F\in\mathcal{S}'(\mathbb{R}^n\times\mathbb{R}_+)$ *such that* F *is the Mexican hat wavelet transform of the family of functions* $\{(\phi_j,\varphi_j)\}_J$*, i.e.,* $F=(W\{(\phi_j,\varphi_j)\}_J)$.

Proof. Let us consider family of sequences $\{(\phi_j,\varphi_j)\}_J\in\mathcal{A}$, where $\phi_j\in\mathscr{S}'(\mathbb{R}^n)$ and $\varphi\in\mathscr{S}$, $\forall j\in J$, with exchange property such that for some $\epsilon>0$, we have $|\varphi(x)|>\epsilon$, $\forall x\in M\left(\{\varphi_j\}_J\right)^c$. Then, in some open neighborhood of x, we define

$$F=\frac{(W\phi_j)}{\varphi_j}. \tag{4.7}$$

Case 1: We show that for some open neighborhood of x we have a quotient F that is unique in that neighborhood, i.e., F does not depend on $j\in J$. Let U and V be some open neighborhood of x such that $|\varphi_j(x)|>\epsilon$, $\forall x\in U$ and $|\varphi_k(x)|>\epsilon$, $\forall x\in V$. Then since $\{(\phi_j,\varphi_j)\}\in\mathcal{A}$, hence it satisfy the exchange property and therefore,

$$\phi_j*\varphi_k=\phi_k*\varphi_j,\ \forall j,k\in J. \tag{4.8}$$

Applying Mexican hat wavelet transform to (4.8), we get

$$\begin{aligned}(W(\phi_j*\varphi_k))&=(W(\phi_k*\varphi_j))\\(W\phi_j)*\varphi_k&=(W\phi_k)*\varphi_j\quad\text{(by Theorem 11)}\\\frac{(W\phi_j)}{\varphi_j}&=\frac{(W\phi_k)}{\varphi_k}.\end{aligned} \tag{4.9}$$

Hence, we get a unique quotient $F=\dfrac{(W\phi_j)}{\varphi_j}$ on $U\cap V$.

Case 2: We need to show that there exists a unique quotient $F\in\mathscr{S}'(\mathbb{R}^n\times\mathbb{R}_+)$. From (4.7) and (4.9), for any $j,k\in J$, we have

$$(W\phi_k)=F\varphi_k,\ \forall k\in J \tag{4.10}$$

such that there exists a unique $F\in\mathscr{S}'(\mathbb{R}^n\times\mathbb{R}_+)$ which implies exchange property.

Clearly, for a total sequence, say $\{\varphi_j\}_\mathbb{N}$, where $\varphi_j\in\mathcal{S}(\mathbb{R}^n)$ for all $j\in\mathbb{N}$, there is an $\phi_j\in\mathcal{S}'(\mathbb{R}^n)$ such that $(W\phi_j)=\varphi_jF$. Hence, $\{(\phi_j,\varphi_j)\}_\mathbb{N}\in\mathcal{A}$ and $F=(W(\{(\phi_j,\varphi_j)\}_\mathbb{N}))$. □

Lemma 3. *For the family of pairs of sequences* $\{(\phi_j,\varphi_j)\}_J$, $\{(g_k,\lambda_k)\}_K\in\mathcal{A}$ *has an Equivalence Relation, i.e.,* $\{(\phi_j,\varphi_j)\}_J\ \ \{(g_k,\phi_k)\}_K$ *if*

$$\phi_j*\lambda_k=g_k*\varphi_j,\quad\forall j\in J,\ k\in K. \tag{4.11}$$

Theorem 13. *Let* $\{(\phi_j, \varphi_j)\}_J, \{(g_k, \lambda_k)\}_K \in \mathcal{A}$. *Then* $\{(\phi_j, \varphi_j)\}_J \sim \{(g_k, \lambda_k)\}_K$ *iff* $(W(\{(\phi_j, \varphi_j)\}_J)) = (W(\{(g_k, \lambda_k)\}_K))$.

Proof. Let $\{(\phi_j, \varphi_j)\}_J \sim \{(g_k, \lambda_k)\}_K$, hence, they satisfy the exchange property, defined as

$$\phi_j * \lambda_k = g_k * \varphi_k, \ \forall j \in J, k \in K.$$

Let F and G denotes the Mexican hat wavelet transform of some family of sequences such that $F = (W(\{(\phi_j, \varphi_j)\}_J))$ and $G = (W(\{(g_k, \lambda_k)\}_K))$. Now, consider,

$$\begin{aligned} \varphi_j F * \lambda_k &= (W\phi_j) * \lambda_k \\ &= (W(\phi_j * \lambda_k)) \\ &= (W(g_k * \varphi_j)) \\ &= (W g_k) * \varphi_j \\ &= \lambda_k G * \varphi_j. \end{aligned}$$

Now, by applying Lemma 2, we get $F = G$.

Conversely, we need to show that the family of sequences $\{(\phi_j, \varphi_j)\}_J$ and $\{(g_k, \lambda_k)\}_K$ are equivalent. Let us consider

$$\begin{aligned} & F = G \\ \implies & (W\phi_j) * \lambda_k = (W g_k) * \varphi_j \\ \implies & (W(\phi_j * \lambda_k)) = (W(g_k * \varphi_j)) \\ \implies & \phi_j * \lambda_k = g_k * \varphi_j. \end{aligned} \tag{4.12}$$

Hence, $\{(\phi_j, \varphi_j)\}_J \sim \{(g_k, \lambda_k)\}_K$. □

From the above theorem it is shown that there is an equivalence relation on $\mathcal{A}$ and hence splits $\mathcal{A}$ into equivalence classes. The equivalence class contains the generalized quotient $\frac{\phi_n}{\varphi_n}$ and is denoted by $\left[\frac{\phi_n}{\varphi_n}\right]$. These equivalence classes are called generalized quotients or Boehmians and the space of all such generalized quotients is denoted by $\mathcal{B}$.

Definition 3. Let $X = \left[\frac{\phi_n}{\varphi_n}\right] \in \mathscr{B}$, then the MHWT of X as a generalized quotient is defined by,

$$Y = (WX)(b, a) = \left[\frac{(W\phi_n)(b, a)}{\varphi_n}\right].$$

It is well defined since, if $X = \left[\frac{\phi_n}{\varphi_n}\right] = Y = \left[\frac{g_n}{\psi_n}\right]$ in $\mathscr{B}$, then

$$\begin{aligned}
\phi_m * \psi_n &= g_n * \varphi_m \quad \forall m, n \in \mathbb{N} \\
(W(\phi_m * \psi_n))(b,a) &= (W(g_n * \varphi_m))(b,a) \\
(W\phi_m)(b,a) * \psi_n &= (Wg_n)(b,a) * \varphi_m \quad \text{(by Theorem 11)} \\
\left[\frac{(W\phi_n)(b,a)}{\varphi_n}\right] &= \left[\frac{(Wg_n)(b,a)}{\psi_n}\right].
\end{aligned}$$

Further, by considering the map $\phi \rightarrow \left[\frac{\phi * \delta_n}{\delta_n}\right]$, any $\phi \in \mathcal{W}'(-\infty, \infty)$ can be considered as an element of $\mathscr{B}$ by [27, Theorem 4.3.9], i.e., if $X = \left[\frac{f * \delta_n}{\delta_n}\right]$, then

$$(WX)(b,a) = \left[\frac{W(\phi * \delta_n)(b,a)}{\delta_n}\right] = \left[\frac{(W\phi)(b,a) * \delta_n}{\delta_n}\right] = (W\phi)(b,a).$$

This definition extends the theory of MHWT to more general spaces than $(\mathscr{W}^{\gamma}_{\alpha,\beta})'$.

From Theorem 12 and Theorem 13, it is clear that the Mexican hat wavelet transform is a bijection from the space of generalized quotients to the space of distributions.

Theorem 14. *For every $\mathcal{X} \in \mathcal{B}_{\mathscr{S}'(\mathbb{R}^n)}$ there exists a delta sequence (φ_n) such that $\mathcal{X} = [\{(\phi_n, \varphi_n)\}_{\mathbb{N}}]$ for some $\phi_n \in \mathscr{S}'(\mathbb{R}^n)$.*

Proof. Let $(\phi_n) \in \mathscr{S}(\mathbb{R}^n)$, be a delta sequence and $X \in \mathcal{B}_{\mathscr{S}'(\mathbb{R}^n)}$. Then, $(WX) * \phi_n \in \mathscr{S}'$, since $(WX) \in \mathscr{S}'$. Consequently, $(WX) * \phi_n = (Wg_n)$ for some $g_n \in \mathscr{S}'$. Therefore, we have

$$X = \left[\frac{g_n * \phi_n}{\phi_n * \phi_n}\right]. \tag{4.13}$$

Hence, $\phi_n = (g_n * \phi_n) \in \mathscr{S}'$ and by using the property of delta sequences $\phi_n * \phi_n \in \mathscr{S}$ is a delta sequence. This completes the proof. □

5 Conclusions

Wavelet analysis is a field that is constantly evolving and is a mathematical approach widely used for many applications. The Mexican hat wavelet transform (MHWT) is considered to have one of the most appropriate wavelet basis constructed by using Gaussian function. Therefore, it is symmetrical and satisfies the Gaussian decays in both space and frequency, which helps to extract data in the space-frequency window. The space of generalized quotients includes regular operators, distributions, ultra-distributions and also objects which are neither regular operators nor distributions. It may be concluded here that the space of tempered generalized quotient is constructed in a simple way by using the exchange property.

In this chapter, the MHWT has been investigated explicitly on generalized quotient space and its operational properties are obtained with its inverse. The characterization of the MHWT for generalized quotients is also achieved. Further, the Mexican hat wavelet has one of the most appropriate wavelet basis functions which is localized in both space and frequency, hence it can give strong applications for the analysis of space-frequency and other digital modulation. This generalized quotient space can be used to examine Mexican hat wavelet transformation on various manifolds. Moreover, the results can be applied to solving ordinary and partial differential equations, Cauchy problem, mixed boundary value problems, approximation theory, mathematical modeling and computation. Moreover, the aforesaid analysis can be used to obtain approximation theory, mixed boundary value problems, and Paley-Wiener-Schwartz theorem for the MHWT.

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