Numerical Solution of Eighth Order Boundary Value Problems by Using Vieta-Lucas Polynomials

Rakesh Kumar[1], Shivani Aeri[1(✉)], and Poonam Sharma[2]

[1] Srinivasa Ramanujan Department of Mathematics, Central University of Himachal Pradesh, Shahpur Campus, Shahpur 176206, H.P., India
shivvani.aeri@gmail.com

[2] Department of Mathematics, PGDC Hamirpur, Hamirpur, H.P., India

Abstract. In this paper, the eighth order boundary value problems (BVPs) are solved by utilizing the Vieta-Lucas polynomials based scheme. The operational matrix of derivative of shifted Vieta-Lucas polynomials is used. The corresponding algebraic equations are handled by taking the roots of Vieta-Lucas polynomials as collocation points. The illustrative examples provide the favorable comparison with other existing methods that demonstrates the efficiency and accuracy of the scheme.

Keywords: Vieta-Lucas polynomials · collocation method · Eighth order BVPs

1 Introduction

Higher-order BVPs have a variety of usage in engineering and sciences [1]. These kind of equations can be found in fluid dynamics, hydrodynamics, astrophysics, beam theory, astronomy, induction motors, and other fields [2]. The physics of various hydrodynamic stability issues are governed by eighth-order differential equations [3]. In this paper, we offer a strategy based on Vieta-Lucas polynomials for solving eighth order boundary value problems. Numerous scholars have worked on eighth order BVPs using diverse approaches. Using finite difference methods Boutayeb and Twizell [4] solved these kind of problems, Wazwaz [5] proposed a numerical technique that employed the Adomian decomposition method as well as a modified Adomian decomposition approach. Siddiqi and Twizell [6] introduced differential quadrature and generalised differential quadrature rules, Nonic spline and nonpolynomial nonic spline methods were utilised by Siddiqi and Akram [7], variational iteration decomposition was suggested by Noor and Mohyud-Din [8], and homotopy perturbation was employed by Golbabai and Javidi [9]. Costabile and Napoli [10] employed collocation techniques and particular classes of polynomials to solve ninth order BVPs, whereas Akram and Rehman [11] used the reproducing kernel space approach. Xu et al. [12] introduced a collocation approach based on second kind Chebyshev wavelets.

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Elahi et al. [13] employed the Legendre Galerkin approach to solve eighth order boundary value problems, whereas Islam et al. [14] used the Galerkin method. Agarwal [1] investigated the existence and uniqueness of these equations.

Different kinds of differential equations are handled analytically [15–19] however it is not always possible to find the analytical solutions, thus the researchers are interested in the development of new numerical schemes that provide better approximations such as the operational matrix approach [20–24] has been widely used for the approximation purposes. Vieta-Lucas polynomials (VLPs) and their shifted forms have recently become popular for numerically solving several types of differential equations [25,26]. In this paper, we solved eighth order boundary value problems using a Vieta-Lucas polynomials based scheme.

This work is organised as follows: In Sect. 2, we discuss the necessary background and terminologies. Section 3 describes the mathematical model and the proposed method. Section 4 gives the estimates for convergence and error. Section 5 includes various illustrated examples to demonstrate the proposed approach's simplicity and applicability. In Sect. 6, the obtained results are compared to the approximate solutions of other known techniques. A reliable excellent degree of accuracy is achieved in all of the circumstances tested. The final remarks are found in Sect. 7.

2 Preliminaries

In this part, we will go through some of the fundamental definitions and properties of Vieta-Lucas polynomials, which are used in this study.

Definition 2.1. The Vieta-Lucas polynomials $\mathrm{VL_n}(\zeta)$ of degree n $(\mathrm{n} \in \mathbb{N} \cup \{0\})$ can be defined as [27]:

$$\mathrm{VL_n}(\zeta) = 2\cos(\mathrm{n}\delta), \tag{1}$$

where $\delta = \arccos\left(\frac{\zeta}{2}\right)$ and $\mid \zeta \mid \in [-2, 2]$, $\delta \in [0, \pi]$.

The recurrence relation for Vieta-Lucas polynomials $\mathrm{VL_n}(\zeta)$ is given by [27]:

$$\mathrm{VL_n}(\zeta) = \zeta \mathrm{VL_{n-1}}(\zeta) - \mathrm{VL_{n-2}}(\zeta), \quad \mathrm{m} \geq 2, \tag{2}$$

with $\mathrm{VL_0}(\zeta) = 2$ and $\mathrm{VL_1}(\zeta) = \zeta$.

The first few Vieta-Lucas polynomials are given as:

$$\begin{aligned}
\mathrm{VL_0}(\zeta) &= 2,\\
\mathrm{VL_1}(\zeta) &= \zeta,\\
\mathrm{VL_2}(\zeta) &= \zeta^2 - 2,\\
\mathrm{VL_3}(\zeta) &= \zeta^3 - 3\zeta,\\
\mathrm{VL_4}(\zeta) &= \zeta^4 - 4\zeta^2 + 2,\\
\mathrm{VL_5}(\zeta) &= \zeta^5 - 5\zeta^3 + 5\zeta,\\
\mathrm{VL_6}(\zeta) &= \zeta^6 - 6\zeta^4 + 9\zeta^2 - 2.
\end{aligned}$$

In terms of power series expansion, the Vieta-Lucas polynomials are expressed as [27]:

$$\mathrm{VL_n}(\zeta) = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^{j} \frac{n(n-j-1)!}{j!(n-2j)!} \zeta^{n-2j}, \quad n \geq 1. \tag{3}$$

The Vieta-Lucas polynomials $\mathrm{VL_n}(\zeta)$ and $\mathrm{VL_m}(\zeta)$ are orthogonal over $[-2, 2]$ with respect to weight function $w(\zeta) = \frac{1}{\sqrt{4-\zeta^2}}$ and satisfy the following condition [25]:

$$\langle \mathrm{VL_n}(\zeta), \mathrm{VL_m}(\zeta) \rangle_{w(\zeta)} = \int_{-2}^{2} \mathrm{VL_n}(\zeta)\mathrm{VL_m}(\zeta) w(\zeta)\, d\zeta = \begin{cases} 4\pi, & n = m = 0, \\ 2\pi, & n = m \neq 0, \\ 0, & n \neq m. \end{cases} \tag{4}$$

Proposition 2.1. The basic properties of Vieta-Lucas polynomials are given as:

(i) $\mathrm{VL_n}(\zeta)(\mathrm{VL_m}(\zeta)) = \mathrm{VL_{nm}}(\zeta)$.
(ii) $\mathrm{VL_n}(\zeta)\, \mathrm{VL_m}(\zeta) = \mathrm{VL_{n+m}}(\zeta) + \mathrm{VL_{|n-m|}}(\zeta)$.
(iii) $\zeta \mathrm{VL_n}(\zeta) = \mathrm{VL_{n+1}}(\zeta) + \mathrm{VL_{n-1}}(\zeta)$.
(iv) $(4 - \zeta^2)\mathrm{VL_n}(\zeta) = -\mathrm{VL_{n+2}}(\zeta) + 2\mathrm{VL_n}(\zeta) - \mathrm{VL_{n-2}}(\zeta)$.

Proof. Omitted □

2.1 Shifted Vieta-Lucas Polynomials and Its Operational Matrix of Differentiation

Definition 2.2. The shifted VLPs $\mathrm{VL}^*_n(\zeta)$ over $[0, 1]$ with degree $n \in \mathbb{N} \cup \{0\}$ can be defined as [25]:

$$\mathrm{VL}^*_n(\zeta) = \mathrm{VL_n}(4\zeta - 2). \tag{5}$$

The recurrence relation of shifted VLPs is [25]:

$$\mathrm{VL}^*_n(\zeta) = (4\zeta - 2)\mathrm{VL}^*_{n-1}(\zeta) - \mathrm{VL}^*_{n-2}(\zeta), \tag{6}$$

provided $\mathrm{VL}^*_0(\zeta) = 2$ and $\mathrm{VL}^*_1(\zeta) = 4\zeta - 2$.

The power series expansion of shifted VLPs are [25]:

$$\mathrm{VL}^*_n(\zeta) = 2n \sum_{j=0}^{n} (-1)^{j} \frac{4^{n-j}(2n-j-1)!}{j!(2n-2j)!} \zeta^{n-j}, \quad n \geq 1. \tag{7}$$

The shifted VLPs satisfy the following orthogonality property [25]:

$$\langle \mathrm{VL}^*_n(\zeta), \mathrm{VL}^*_m(\zeta) \rangle_{w^*(\zeta)} = \int_{0}^{1} \mathrm{VL}^*_n(\zeta)\mathrm{VL}^*_m(\zeta) w^*(\zeta)\, d\zeta = \begin{cases} 4\pi, & n = m = 0, \\ 2\pi, & n = m \neq 0, \\ 0, & n \neq m, \end{cases} \tag{8}$$

where $w^*(\zeta) = \frac{1}{\sqrt{\zeta-\zeta^2}}$ is the weight function of shifted Vieta-Lucas polynomials. Assume $y(\zeta)$ defined on the interval [0,1] be a Lebesgue square integrable function. So it can be written in terms of shifted VLPs as

$$y(\zeta) = \sum_{j=0}^{\infty} c_j \mathrm{VL}_{\mathrm{j}}^*(\zeta), \tag{9}$$

where c_j are unknown coefficients and can be obtained by following expressions

$$c_j = \frac{1}{\alpha_j \pi} \int_0^1 \frac{y(\zeta)\mathrm{VL}_{\mathrm{j}}^*(\zeta)}{\sqrt{\zeta - \zeta^2}} d\zeta, \tag{10}$$

where

$$\alpha_j = \begin{cases} 4, & j = 0, \\ 2, & j \geq 1. \end{cases}$$

Now, the truncated series can be written as

$$y_N(\zeta) = \sum_{j=0}^{N} c_j \mathrm{VL}_{\mathrm{j}}^*(\zeta) = \mathrm{C}^{\mathrm{T}} \Phi(\zeta),$$

where

$$\mathrm{C}^{\mathrm{T}} = [\mathrm{c}_0, \mathrm{c}_1, \mathrm{c}_2, \ldots, \mathrm{c}_{\mathrm{N}}], \qquad \Phi(\zeta) = [\mathrm{VL}_0^*(\zeta), \mathrm{VL}_1^*(\zeta), \mathrm{VL}_2^*(\zeta), \ldots, \mathrm{VL}_{\mathrm{N}}^*(\zeta)].$$

The shifted VLPs operation matrix is defined as

$$\frac{dy_N}{d\zeta} = C^T D^{(1)} \Phi(\zeta), \tag{11}$$

where $D^{(1)}$ is the operation matrix of differentiation of shifted VLPs of order $(N+1) \times (N+1)$ are given as:

$$D^{(1)} = d_{ij} = \begin{cases} \frac{4i}{\alpha_j}, & j = i - h \begin{cases} h = 1, 3, \cdots, N & \text{if } N \text{ even}, \\ h = 1, 3, \cdots, N-1 & \text{if } N \text{ odd}, \end{cases} \\ 0, & \text{otherwise}. \end{cases} \tag{12}$$

where $\alpha_0 = 2$ and $\alpha_k = 1 (k \geq 1)$.

For any $n \in \mathbb{N}$, it can be generalized as:

$$\frac{d^n \Phi(\zeta)}{d\zeta^n} = (D^{(1)})^n \Phi(\zeta) = D^{(n)} \Phi(\zeta), \text{ where } n \in \mathbb{N}. \tag{13}$$

For example: for N = 6, we get

$$D^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 \\ 6 & 0 & 12 & 0 & 0 & 0 \\ 0 & 16 & 0 & 16 & 0 & 0 \\ 10 & 0 & 20 & 0 & 20 & 0 \end{pmatrix}.$$

3 Mathematical Model and Numerical Scheme

This section includes the mathematical description of the model followed by the numerical scheme that describes the utility of differentiation matrix of shifted Vieta-Lucas polynomial to solve the eighth order BVPs. The eight order differential equation is formulated as

$$\frac{d^8y}{d\zeta^8}+\sum_{j=0}^{7}a_j\frac{d^jy}{d\zeta^j}=f(\zeta),\qquad \zeta\in[0,1], \tag{14}$$

where $f(\zeta)$ and a_j are the continuous functions on the interval $[0,1]$. Subject to supplementary conditions

$$\frac{d^iy}{d\zeta^i}\mid_{\zeta=0}=u_i,\frac{d^iy}{d\zeta^i}\mid_{\zeta=1}=v_i,\ i=0,1,2,3. \tag{15}$$

Let $y_N(\zeta)$ be the shifted Vieta-Lucas polynomials approximation given as

$$y_N(\zeta)=\sum_{j=0}^{N}c_j\mathrm{VL}^*_\mathrm{j}(\zeta)=\mathrm{C}^\mathrm{T}\Phi(\zeta), \tag{16}$$

where the unknowns are $C=[c_0,c_1,c_2,\dots,c_N]^T$.

Using shifted Vieta-Lucas polynomial operational matrix of derivative, Eq. (14) can be expressed as

$$C^TD^{(8)}\Phi(\zeta)+\sum_{j=0}^{7}a_jC^TD^{(j)}\Phi(\zeta)=f(\zeta). \tag{17}$$

Thus, the residual term can be written as

$$R_N(\zeta)=C^TD^{(8)}\Phi(\zeta)+\sum_{j=0}^{7}a_jC^TD^{(j)}\Phi(\zeta)-f(\zeta). \tag{18}$$

Now, by using collocation method, we get

$$R_N(\zeta_i)=0,\qquad i=0,1,2,\dots,N-8. \tag{19}$$

where collocation points are taken as

$$\zeta_i=\frac{1+\cos\left(\frac{(2i+1)\pi}{2(N-8)}\right)}{2},\qquad i=0,1,\dots,N-8. \tag{20}$$

The corresponding boundary conditions gives

$$\frac{d^iy}{d\zeta^i}\mid_{\zeta=0}=C^T\Phi(0)=u_i,\frac{d^iy}{d\zeta^i}\mid_{\zeta=1}=C^T\Phi(1)=v_i,\ i=0,1,2,3. \tag{21}$$

This yields N nonlinear equations. This nonlinear system can be solved to determine the values of coefficients of vector C. By substituting the value of C, we obtain the numerical solution $y_N(\zeta)$.

4 Convergence and Error Analysis

Theorem 4.1 [25]. *Let* $y(\zeta) \in L^2_\omega[0,1]$ *and* $\frac{d^2y}{d\zeta^2} \leq H$*, where* H *is arbitrary constant. Then* $y(\zeta)$ *can be expressed as*

$$y(\zeta) = \sum_{j=0}^{\infty} c_j \mathrm{VL}^*_\mathrm{j}(\zeta), \tag{22}$$

and $y_N(\zeta)$ *is defined in* (16)*. Furthermore, this numerical solution uniformly converges to* $y(\zeta)(y_N(\zeta) \to y(\zeta)$ *as* $N \to \infty)$*. Also, the coefficients* c_i *are bounded, i.e.,*

$$|c_i| \leq \frac{H}{4i(i^2-1)}. \tag{23}$$

Lemma 4.2 [28]. *Let* $f(\zeta)$ *be a function such that* $f(k) = c_k$ *and assume the following:*

1. $f(\zeta)$ *is a continuous, decreasing, positive function for* $\zeta \geq N$.
2. $\sum c_N$ *is convergent, and* $R_N = \sum_{k=N+1}^{\infty} c_k$.

Then

$$R_N \leq \int_N^\infty f(\zeta)d\zeta. \tag{24}$$

Theorem 4.3 [25]. *If Theorem* (4.1) *is satisfied by the function* $y(\zeta)$*, and* $y_N(\zeta) = \sum_{i=0}^{n} c_i VL^*_i(\zeta)$*, then the estimated error(in* $\mathbb{L}^2[0,1]$ *norm) can be given as:*

$$||y(\zeta) - y_N(\zeta)|| < \frac{H}{12N^{\frac{3}{2}}}. \tag{25}$$

5 Numerical Examples

We provide the following test examples in this section to validate the accuracy and efficiency of the proposed method.

Example 5.1. Let us consider the eighth order differential equation as

$$\frac{d^8y}{d\zeta^8} + \zeta y = -(48 + 15\zeta + \zeta^3)e^\zeta, \qquad \zeta \in [0,1]. \tag{26}$$

with

$$y\mid_{\zeta=0} = 0, \quad \frac{dy}{d\zeta}\mid_{\zeta=0} = 1, \quad \frac{d^2y}{d\zeta^2}\mid_{\zeta=0} = 0, \quad \frac{d^3y}{d\zeta^3}\mid_{\zeta=0} = -3,$$
$$y\mid_{\zeta=1} = 0, \quad \frac{dy}{d\zeta}\mid_{\zeta=1} = -e, \quad \frac{d^2y}{d\zeta^2}\mid_{\zeta=1} = -4e, \quad \frac{d^3y}{d\zeta^3}\mid_{\zeta=1} = -9e.$$

Apply our proposed method as follows

$$y_N(\zeta) = \sum_{j=0}^{9} c_j \mathrm{VL}_{\mathrm{j}}^*(\zeta) = \mathrm{C}^{\mathrm{T}}\Phi(\zeta). \tag{27}$$

Now using operational matrix of derivative approach

$$\frac{d^8 y_N}{d\zeta^8} = C^T D^{(8)} \Phi(\zeta), \tag{28}$$

where

$$D^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 16 & 0 & 16 & 0 & 0 & 0 & 0 & 0 \\ 10 & 0 & 20 & 0 & 20 & 0 & 0 & 0 & 0 \\ 0 & 24 & 0 & 24 & 0 & 24 & 0 & 0 & 0 \\ 14 & 0 & 28 & 0 & 28 & 0 & 28 & 0 & 0 \\ 0 & 32 & 0 & 32 & 0 & 32 & 0 & 32 & 0 \end{pmatrix},$$

$$D^{(8)} = \begin{pmatrix} 0 & 0\,0\,0\,0\,0\,0\,0\,0 \\ 0 & 0\,0\,0\,0\,0\,0\,0\,0 \\ 0 & 0\,0\,0\,0\,0\,0\,0\,0 \\ 0 & 0\,0\,0\,0\,0\,0\,0\,0 \\ 0 & 0\,0\,0\,0\,0\,0\,0\,0 \\ 0 & 0\,0\,0\,0\,0\,0\,0\,0 \\ 0 & 0\,0\,0\,0\,0\,0\,0\,0 \\ 0 & 0\,0\,0\,0\,0\,0\,0\,0 \\ 1321205760 & 0\,0\,0\,0\,0\,0\,0\,0 \end{pmatrix},$$

and

$$\Phi(\zeta) = \begin{pmatrix} 2 \\ 4\zeta - 2 \\ 2 - 16\zeta + 16\zeta^2 \\ -2 + 36\zeta - 96\zeta^2 + 64\zeta^3 \\ 2 - 64\zeta + 320\zeta^2 - 512\zeta^3 + 256\zeta^4 \\ -2 + 100\zeta - 800\zeta^2 + 2240\zeta^3 - 2560\zeta^4 + 1024\zeta^5 \\ 2 - 144\zeta + 1680\zeta^2 - 7168\zeta^3 + 13824\zeta^4 - 12288\zeta^5 + 4096\zeta^6 \\ -2 + 196\zeta - 3136\zeta^2 + 18816\zeta^3 - 53760\zeta^4 + 78848\zeta^5 - 57344\zeta^6 + 16384\zeta^7 \\ 2 - 256\zeta + 5376\zeta^2 - 43008\zeta^3 + 168960\zeta^4 - 360448\zeta^5 + 425984\zeta^6 - 262144\zeta^7 + 65536\zeta^8 \end{pmatrix}.$$

Substituting these values in Eq. (26), we get residual function as:

$$R_N(\zeta) = C^T D^{(8)} \Phi(\zeta) + \zeta(C^T \Phi(\zeta)) + (48 + 15\zeta + \zeta^3)e^{\zeta}. \tag{29}$$

Now using the collocation method, we get

$$c_0 - c_2 + c_4 - c_6 + 2642411521 c_8 + \frac{445\sqrt{e}}{8} = 0. \tag{30}$$

and from the boundary conditions, we have

$$C^T\Phi(0) = 0, \quad C^TD^{(1)}\Phi(0) = 1, \quad C^TD^{(2)}\Phi(0) = 0, \quad C^TD^{(3)}\Phi(0) = -3, \tag{31}$$

$$C^T\Phi(1) = 0, \quad C^TD^{(1)}\Phi(1) = -e, \quad C^TD^{(2)}\Phi(1) = -4e, \quad C^TD^{(3)}\Phi(1) = -9e. \tag{32}$$

On solving Eqs. (30) together with (31) and (32), we get the values of unknown coefficients and which leads to the required solution as

$$y_N(\zeta) = 4.20 \times 10^{-17} + \zeta + 8.32 \times 10^{-17}\zeta^2 - 0.49\zeta^3 + \cdots - 0.002\zeta^8. \tag{33}$$

Example 5.2. Consider the following eighth order differential equation

$$\frac{d^8y}{d\zeta^8} + \frac{d^7y}{d\zeta^7} + 2\frac{d^6y}{d\zeta^6} + 2\frac{d^5y}{d\zeta^5} + 2\frac{d^4y}{d\zeta^4} + 2\zeta\frac{d^3y}{d\zeta^3} + 2\frac{d^2y}{d\zeta^2} + \zeta^2\frac{dy}{d\zeta} + \zeta y(\zeta)$$
$$= -(\zeta^4 - 2\zeta^3 + 14\zeta - 27)\cos\zeta - (3\zeta^3 - 13\zeta^2 + 11\zeta + 17)\sin\zeta, \qquad \zeta \in [0, 1].$$

with

$$y\mid_{\zeta=0} = 0, \quad \frac{dy}{d\zeta}\mid_{\zeta=0} = -1, \quad \frac{d^2y}{d\zeta^2}\mid_{\zeta=0} = 0, \quad \frac{d^3y}{d\zeta^3}\mid_{\zeta=0} = 7,$$
$$y\mid_{\zeta=1} = 0, \quad \frac{dy}{d\zeta}\mid_{\zeta=1} = 2\sin 1, \quad \frac{d^2y}{d\zeta^2}\mid_{\zeta=1} = 4\cos 1 + 2\sin 1, \quad \frac{d^3y}{d\zeta^3}\mid_{\zeta=1} = 6\cos 1 - 6\sin 1.$$

Similarly, using the approximation

$$y_N(\zeta) = \sum_{j=0}^{9} c_j \mathrm{VL}_{\mathrm{j}}^*(\zeta) = \mathrm{C}^{\mathrm{T}}\Phi(\zeta). \tag{34}$$

which gives the required solution as

$$y_N(\zeta) = 3.95 \times 10^{-18} - \zeta - 5.50 \times 10^{-17}\zeta^2 + 1.16\zeta^3 + \cdots - 0.0006\zeta^8. \tag{35}$$

Example 5.3. The eighth order differential equation is considered as

$$\frac{d^8y}{d\zeta^8} + \frac{d^7y}{d\zeta^7} + 2\frac{d^6y}{d\zeta^6} + 2\frac{d^5y}{d\zeta^5} + 2\frac{d^4y}{d\zeta^4} + 2\zeta\frac{d^3y}{d\zeta^3} + 2\frac{d^2y}{d\zeta^2} + \frac{dy}{d\zeta} + y(\zeta)$$
$$= 14\cos\zeta - 16\sin\zeta - 4\zeta\sin\zeta, \qquad \zeta \in [0, 1].$$

with conditions

$$y\mid_{\zeta=0} = 0, \quad \frac{dy}{d\zeta}\mid_{\zeta=0} = -1, \quad \frac{d^2y}{d\zeta^2}\mid_{\zeta=0} = 0, \quad \frac{d^3y}{d\zeta^3}\mid_{\zeta=0} = 7,$$
$$y\mid_{\zeta=1} = 0, \quad \frac{dy}{d\zeta}\mid_{\zeta=1} = 2\sin 1, \quad \frac{d^2y}{d\zeta^2}\mid_{\zeta=1} = 4\cos 1 + 2\sin 1, \quad \frac{d^3y}{d\zeta^3}\mid_{\zeta=1} = 6\cos 1 - 6\sin 1.$$

For

$$y_N(\zeta) = \sum_{j=0}^{9} c_j \mathrm{VL}_{\mathrm{j}}^*(\zeta) = \mathrm{C}^{\mathrm{T}}\Phi(\zeta). \tag{36}$$

Hence, the required solution are obtained as

$$y_N(\zeta) = 1.73 \times 10^{-18} - \zeta - 1.94 \times 10^{-16}\zeta^2 + 1.16\zeta^3 + \cdots - 0.00082\zeta^8. \tag{37}$$

Example 5.4. Assume the nonlinear eighth order differential equation as

$$\frac{d^8y}{d\zeta^8} = e^{-\zeta}y^2(\zeta), \qquad \zeta \in [0,1].$$

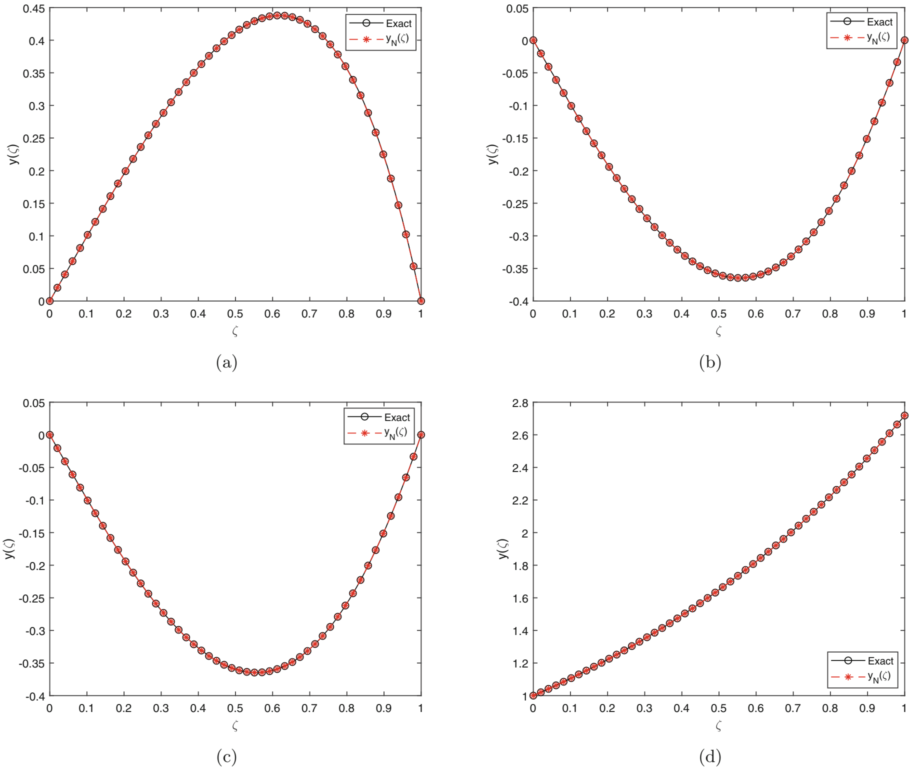

Fig. 1. Solution curves for (a) Example 5.1, (b) Example 5.2, (c) Example 5.3 and (d) Example 5.4.

with supplementary conditions

$$y\,|_{\zeta=0}= 1, \quad \frac{dy}{d\zeta}\,|_{\zeta=0}= 1, \quad \frac{d^2y}{d\zeta^2}\,|_{\zeta=0}= 1, \quad \frac{d^3y}{d\zeta^3}\,|_{\zeta=0}= 1,$$
$$y\,|_{\zeta=1}= e, \quad \frac{dy}{d\zeta}\,|_{\zeta=1}= e, \quad \frac{d^2y}{d\zeta^2}\,|_{\zeta=1}= e, \quad \frac{d^3y}{d\zeta^3}\,|_{\zeta=1}= e.$$

Let

$$y_N(\zeta) = \sum_{j=0}^{11} c_j \mathrm{VL}_{\mathrm{j}}^*(\zeta) = \mathrm{C}^{\mathrm{T}}\Phi(\zeta). \tag{38}$$

Which leads to the desired solution as

$$y_N(\zeta) = 5.58\times10^{-17} - 0.99\zeta - 1.14\times10^{-16}\zeta^2 - 0.49\zeta^3 + \cdots - 0.000041\zeta^8. \tag{39}$$

6 Results and Discussions

Figure 1a, 1b, 1c and 1d demonstrates the solution plots of the exact solution and approximate solution ($y_N(\zeta)$) obtained from the proposed numerical scheme for Example 5.1, 5.2, 5.3 and 5.4 respectively. It is observed from the figure that the approximate solution is in good agreement with the exact solution. Which signifies that the proposed numerical scheme is capable to solve the problem effectively.

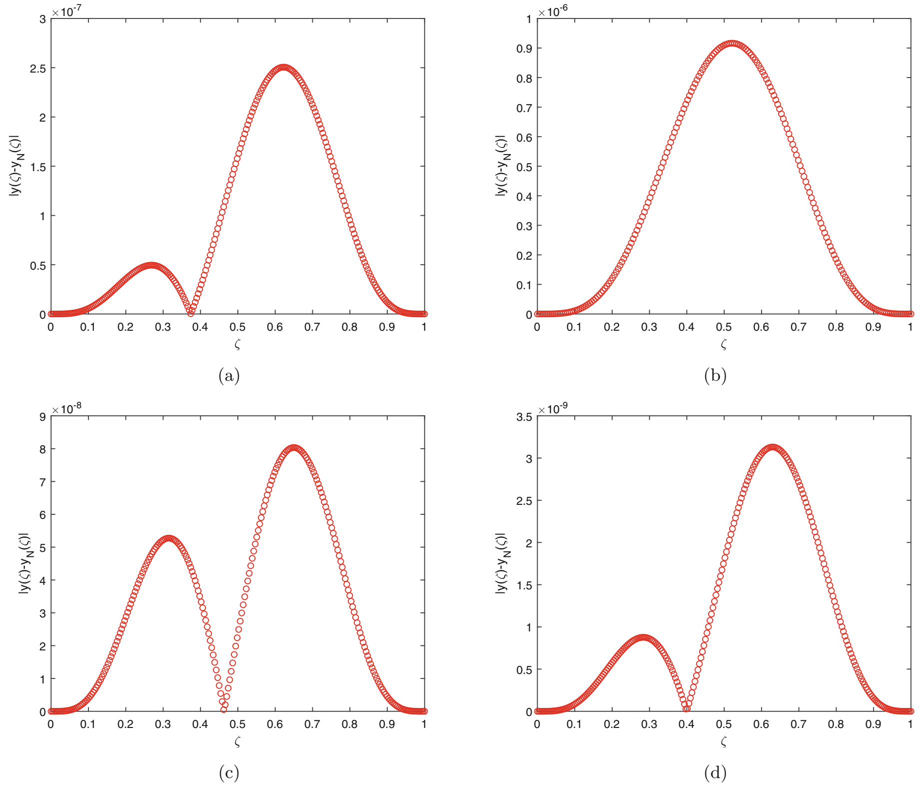

Fig. 2. Absolute error plots for (a) Example 5.1, (b) Example 5.2, (c) Example 5.3 and (d) Example 5.4.

Figure 2a, 2b, 2c and 2d depicts the absolute error plots on the interval $[0, 1]$ for Example 5.1, 5.2, 5.3 and 5.4 respectively. Which shows that the order of error is less and the error is bounded in the interval $[0, 1]$ that clearly represents the reliability of the proposed numerical scheme.

Table 1 compares the absolute errors obtained by the proposed method and from the other existing methods. From Table 1, it is observed that the proposed

Table 1. Absolute error comparisons for Example 5.1, Example 5.2, Example 5.3 and Example 5.4.

ζ	Exact Solution	Viswanadham [29]	Elahi et al. [13]	$\mid y(\zeta) - y_N(\zeta) \mid$
		Example 5.1		
0.1	0.09946	5.21×10^{-8}	1.44×10^{-7}	5.55×10^{-9}
0.2	0.19542	2.22×10^{-6}	1.45×10^{-6}	3.57×10^{-8}
0.3	0.28347	7.00×10^{-6}	4.38×10^{-6}	4.55×10^{-8}
0.4	0.35803	1.11×10^{-5}	7.59×10^{-6}	2.79×10^{-8}
0.5	0.41218	1.22×10^{-5}	9.06×10^{-6}	1.60×10^{-7}
0.6	0.43730	8.88×10^{-6}	7.81×10^{-6}	2.47×10^{-7}
0.7	0.42288	2.53×10^{-6}	4.64×10^{-6}	2.11×10^{-7}
0.8	0.35608	1.81×10^{-6}	1.58×10^{-6}	9.44×10^{-8}
0.9	0.22136	2.04×10^{-6}	1.61×10^{-7}	1.18×10^{-8}
		Example 5.2		
0.1	-0.09883	4.23×10^{-6}	5.03×10^{-8}	1.08×10^{-8}
0.2	-0.19072	9.98×10^{-6}	5.14×10^{-7}	1.20×10^{-7}
0.3	-0.26892	5.09×10^{-6}	1.55×10^{-6}	3.88×10^{-7}
0.4	-0.32711	7.62×10^{-6}	2.71×10^{-6}	7.18×10^{-7}
0.5	-0.35956	1.49×10^{-5}	3.26×10^{-6}	9.09×10^{-7}
0.6	-0.36137	2.28×10^{-5}	2.82×10^{-6}	8.25×10^{-7}
0.7	-0.32855	2.27×10^{-5}	1.68×10^{-6}	5.14×10^{-7}
0.8	-0.25824	1.94×10^{-5}	5.77×10^{-7}	1.83×10^{-7}
0.9	-0.14883	1.32×10^{-5}	5.88×10^{-8}	1.93×10^{-8}
		Example 5.3		
0.1	-0.09883	3.79×10^{-7}	5.03×10^{-8}	3.98×10^{-9}
0.2	-0.19072	2.14×10^{-6}	5.14×10^{-7}	2.86×10^{-8}
0.3	-0.26892	5.63×10^{-6}	1.55×10^{-6}	5.22×10^{-8}
0.4	-0.32711	9.74×10^{-6}	2.71×10^{-6}	3.40×10^{-8}
0.5	-0.35956	1.13×10^{-5}	3.26×10^{-6}	2.36×10^{-8}
0.6	-0.36137	1.01×10^{-5}	2.82×10^{-6}	7.32×10^{-8}
0.7	-0.32855	7.27×10^{-6}	1.68×10^{-6}	7.35×10^{-8}
0.8	-0.25824	3.87×10^{-6}	5.77×10^{-7}	3.48×10^{-8}
0.9	-0.14883	1.43×10^{-6}	5.88×10^{-8}	4.48×10^{-9}
		Example 5.4		
ζ	Exact Solution	Bernstein poly. [14]	Legendre poly. [14]	$\mid y(\zeta) - y_N(\zeta) \mid$
0.1	1.10517	5.43×10^{-7}	8.54×10^{-6}	8.57×10^{-11}
0.2	1.22140	7.34×10^{-7}	1.73×10^{-6}	5.75×10^{-10}
0.3	1.34986	9.54×10^{-7}	1.33×10^{-6}	8.60×10^{-10}
0.4	1.49182	1.73×10^{-7}	2.97×10^{-6}	8.45×10^{-12}
0.5	1.64872	4.99×10^{-8}	9.49×10^{-7}	1.78×10^{-9}
0.6	1.82212	2.40×10^{-7}	1.24×10^{-6}	3.05×10^{-9}
0.7	2.01375	4.30×10^{-8}	9.54×10^{-6}	2.70×10^{-9}
0.8	2.22554	7.75×10^{-7}	7.75×10^{-7}	1.22×10^{-9}
0.9	2.45960	3.20×10^{-7}	2.32×10^{-6}	1.55×10^{-10}

numerical method provides less error in comparison to the other existing methods. Thus, it clearly demonstrates the accuracy and efficiency of the proposed numerical scheme.

7 Conclusion

In this work, we presented a reliable strategy for solving eighth order boundary value problems numerically. Based on a class of shifted VLPs, this approach is developed. The operational matrix of derivative of shifted VLPs are used to formulate the numerical scheme. From the illustrative examples, it is observed that the method is efficient for solving linear/nonlinear eighth order BVPs effectively. The resulting findings are also compared to the previous results, which show good agreement. Which demonstrates the efficiency and reliability of the proposed approach.

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