On Weighted Fractional Operators with Applications to Mathematical Models Arising in Physics

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Abstract. In recent study, we develop the weighted generalized Hilfer-Prabhakar fractional derivative operator and explore its key properties. It unifies many existing fractional derivatives like Hilfer-Prabhakar and Riemann-Liouville. The weighted Laplace transform of the newly defined derivative is obtained. By involving the new fractional derivative, we modeled the free-electron laser equation and kinetic equation and then found the solutions of these fractional equations by applying the weighted Laplace transform.

Keywords: weighted Hilfer-Prabhakar fractional derivative · weighted Laplace transform · free-electron laser equation · fractional kinetic equation

1 Introduction

Fractional calculus is a natural development of classical calculus with a long mathematical history. The fractional calculus concept has been applied in numerous models. In a variety of domains, fractional models can be utilized to capture and comprehend complex processes (see related literature [1,2]). Fractional calculus has progressed significantly as a result of its applications in practical mathematics such as chemistry, mechanics, physics, engineering, and biology [3–7]. In literature, several fractional operators exist with wide applications such as the Riemann-Liouville [8], the ψ-Hilfer fractional derivative and its properties [9], generalized Hilfer-Prabhakar fractional derivative with arising physical models [31], the fractional calculus iteration procedure on conformable derivatives [10], the Hadamard fractional calculus and Hadamard-type fractional differential equations [11], kernel Hilbert space method for nonlinear partial differential equations [12].

The generalization of fractional integral and derivative operators have achieved remarkable attention in recent decades [13–20]. Various special functions arise in the kernels of fractional operators, including the Wright function,

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the Gauss hypergeometric function, Mittag-Leffler function, Fox H-function and Meijer G-function. The Hilfer-Prabhakar fractional derivative, which is an extension of the Caputo and the Riemann-Liouville fractional derivatives was presented by R. Hilfer in [13]. The fractional Prabhakar integral and derivative operators are established by involving the generalized Mittag-Leffler function in the kernel of the Riemann-Liouville fractional operators [14].

This research is inspired by the widespread use of fractional differential equations in engineering, economics, physics, and a variety of other fields of science [21–27]. The purpose of this work is to enhance the existing literature on fractional calculus and to provide the strong applicability in sciences.

We start this study by recalling some relevant definitions and notions.

Definition 1. *[28] The $\jmath$-gamma function is defined as:*

$$\Gamma_\jmath(\vartheta) = \int\limits_0^\infty x^{\vartheta-1} e^{\frac{-x^\jmath}{\jmath}} dx, \, \Re(\vartheta) > 0, \, \jmath > 0.$$

Note that $\Gamma(\vartheta) = \lim_{\jmath \to 1} \Gamma_\jmath(\vartheta)$ *and* $\Gamma_\jmath(\vartheta) = \jmath^{\frac{\vartheta}{\jmath}-1}\Gamma(\frac{\vartheta}{\jmath})$.

Definition 2. *[28] For* $\Re(\vartheta) > 0$, $\jmath > 0$ *and* $\Re(\varsigma) > 0$, *the $\jmath$-beta function is defined as*

$$B_\jmath(\vartheta, \varsigma) = \frac{1}{\jmath}\int_0^1 \tau^{\frac{\vartheta}{\jmath}-1}(1-\tau)^{\frac{\varsigma}{\jmath}-1} d\tau.$$

$\Gamma_\jmath$ *and* $B_\jmath$ *functions are related as* $B_\jmath(\vartheta, \varsigma) = \frac{\Gamma_\jmath(\vartheta)\Gamma_\jmath(\varsigma)}{\Gamma_\jmath(\vartheta+\varsigma)}$.

Definition 3. *For* $F \in C^n[\hat{a}, \ell]$ *and* $F'(\varsigma) > 0$ *on* $[\hat{a}, \ell]$. *Then*

$$AC^n_F[\hat{a}, \ell] = \Big\{\Psi : [\hat{a}, \ell] \to \mathbb{C} \;\; \Psi^{[n-1]} \in AC[\hat{a}, \ell]\Big\},$$

where $\Psi^{[n-1]} = \left(\frac{1}{g'(\varsigma)}\frac{d}{d\varsigma}\right)^{n-1}\Psi$.

Definition 4. *[29] Let* $n \in N$, $\jmath \in \mathbb{R}^+$, $\alpha, \varrho, \epsilon \in \mathbb{C}$, $\Re(\varrho) > 0$, $\Re(\alpha) > 0$, *then $\jmath$-Mittag-Leffler function is defined by*

$$E^\epsilon_{\jmath,\varrho,\alpha}(\vartheta) = \sum_{n=0}^\infty \frac{(\epsilon)_{n,\jmath}\vartheta^n}{\Gamma_\jmath(\varrho n + \alpha)n!}.$$

Weighted generalized fractional integral operator involving $\jmath$-Mittag-Leffler function introduced in [30] is described in the following definition.

Definition 5. *For* $s \in \mathbb{R}\backslash\{-1\}$, $\jmath \in \mathbb{R}^+$, $\alpha, \varrho, \omega, \epsilon \in \mathbb{C}$, $\Re(\varrho) > 0$, $\Re(\epsilon) > 0$, $\Re(\alpha) > 0$. *Let Φ be a positive increasing function on* $(\delta, \ell]$, $\delta > 0$ *having continuous derivative* Φ' *on* $(0, \ell)$, *and* $\Psi \in L_1[\delta, \ell]$, *then*

$$\begin{aligned}({}_\Phi{}^s_\jmath\mathfrak{I}^{\omega,w,\epsilon}_{\delta^+;\varrho,\alpha}\Psi)(\vartheta) = \tfrac{(s+1)^{1-\frac{\alpha}{\jmath}}}{\jmath} w^{-1}(\vartheta) \textstyle\int_\delta^\vartheta (\Phi^{s+1}(\vartheta) - \Phi^{s+1}(\varsigma))^{\frac{\alpha}{\jmath}-1}\Phi^s(\varsigma) \\ \times E^\epsilon_{\jmath,\varrho,\alpha}(\omega(\Phi^{s+1}(\vartheta) - \Phi^{s+1}(\varsigma))^{\frac{\varrho}{\jmath}})\Phi'(\varsigma)w(\varsigma)\Psi(\varsigma)d\varsigma. \qquad (1)\end{aligned}$$

Definition 6. *[31] Let $\Psi \in C^1[\delta,\ell]$, $\delta > 0$, $0 < \vartheta < \ell < \infty$, $s \in \mathbb{R}\backslash\{-1\}$, $\jmath, \varrho > 0$, $\omega, \epsilon \in \mathbb{R}$, $\alpha \in (0,1)$, $\Bbbk \in [0,1]$ and $\big(\Psi * {}^s_\jmath\mathfrak{J}^{\omega,-\epsilon(1-\Bbbk)}_{0^+;\varrho,(1-\Bbbk)(\jmath-\alpha)}\big)(\vartheta) \in AC^1[\delta,\ell]$, then generalized Hilfer-Prabhakar derivative is defined as*

$$ {}^s_\jmath\mathfrak{D}^{\epsilon,\alpha,\varpi}_{\delta^+;\varrho,\omega}\Psi(\vartheta) = \jmath\Big({}^s_\jmath\mathfrak{J}^{\omega,-\epsilon\varpi}_{\delta^+;\varrho,\varpi(\jmath-\alpha)}\Big(\frac{1}{\vartheta^s}\frac{d}{d\vartheta}\Big){}^s_\jmath\mathfrak{J}^{\omega,-\epsilon(1-\varpi)}_{\delta^+;\varrho,(1-\varpi)(\jmath-\alpha)}\Psi\Big)(\vartheta). $$

Weighted generalized Laplace transform introduced in [32] is defined as follows:

Definition 7. *For the real valued functions Ψ, $w(x) \neq 0$ and Φ is such that $\Phi'(\xi) > 0$ on $[a,\infty)$, the weighted generalized Laplace transform of Ψ is given by*

$$ \mathfrak{L}^w_\Phi\{\Psi(t)\}(u) = \int_a^\infty e^{-u(\Phi(t)-\Phi(a))}w(t)\Psi(t)\Phi'(t)dt, \qquad (2) $$

for all values of u.

Definition 8. *[33] For some $\alpha \in \mathbb{R}$, the Caputo derivative of non integer order α with $\Psi(x) \in AC^n([a,b])$ is given by*

$$ ({}_{\mathfrak{C}}\mathfrak{D}^\alpha_{a^+}\Psi)(\vartheta) = \tfrac{1}{\Gamma(n-\alpha)}\int_a^\vartheta(\vartheta-x)^{n-\alpha-1}\Psi^{(n)}(x)dx,\ n \in \mathbb{N} \qquad (3) $$

where $n = [\alpha]+1$.

Definition 9. *[20] Let $\Psi, \psi \in C^1[\delta,\ell]$, $\delta > 0$, $0 < \vartheta < \ell < \infty$, $\varrho > 0$, $\epsilon \in \mathbb{R}$, $\alpha \in (0,1)$, $\varpi \in [0,1]$ and $\big(\Psi * \mathfrak{J}^{-\epsilon(1-\varpi)}_{\delta^+;\varrho,(1-\varpi)(1-\alpha)}\big)(\vartheta) \in AC^1[\delta,\ell]$, then weighted generalized Hilfer-Prabhakar fractional derivative is defined as*

$$ \mathfrak{D}^{\epsilon,\alpha,\varpi}_{\delta^+;\varrho,\omega}\Psi(\vartheta) = \Big(\mathfrak{J}^{\omega,-\epsilon\varpi}_{\delta^+;\varrho,\varpi(1-\alpha)}\Big(\frac{d}{d\vartheta}\Big)\mathfrak{J}^{\omega,-\epsilon(1-\varpi)}_{\delta^+;\varrho,(1-\varpi)(1-\alpha)}\Psi\Big)(\vartheta), $$

Samraiz *et al.* [34] proposed the modified weighted $(\jmath, s)$-Riemann-Liouville fractional integral of order ρ, which is stated as follows:

Definition 10. *Suppose that Ψ be a continuous function on $[a,b]$ and Φ is strictly increasing differentiable function. Then modified weighted $(\jmath, s)$-Riemann-Liouville fractional integral of order ρ is given by*

$$ ({}_\Phi{}^s_\jmath\mathfrak{J}^\rho_{a^+,w}\Psi)(\vartheta) $$
$$ = \tfrac{(s+1)^{1-\frac{\rho}{\jmath}}w^{-1}(\vartheta)}{\jmath\Gamma_\jmath(\rho)}\int_a^\vartheta(\Phi^{s+1}(\vartheta)-\Phi^{s+1}(t))^{\frac{\rho}{\jmath}-1}\Phi^s(t)\Phi'(t)w(t)\Psi(t)dt,\ \vartheta \in [a,b], $$

where $\rho, \jmath > 0$, $\omega(\vartheta) \neq 0$ and $s \in \mathbb{R}\backslash\{-1\}$.

Theorem 1. *[32] Let $\mathfrak{D}^\jmath_w\Psi$, $\jmath = 0,1,2,...,m-1$ be weighted Φ-exponential order with $\Psi \in AC^{m-1}_w[a,\xi)$. Furthermore, if the function $\mathfrak{D}^n_w\Psi$ is piecewise continuous on an $[a,T]$, then the weighted Laplace transform of $\mathfrak{D}^n_w\Psi$ is defined by*

$$ \mathfrak{L}^w_\Phi\{\mathfrak{D}^n_w\Psi\}(u) = u^n\mathfrak{L}^w_\Phi\{\Psi(\xi)\}(u) - \sum_{\jmath=0}^{n-1}u^{n-\jmath-1}\Psi_\jmath(a). $$

Definition 11. *[32] The convolution of Ψ and Φ is defined by*

$$(\Psi *_{\Phi}^{w} h)(\xi) = \quad w^{-1}(\xi)\int_a^{\xi} w(\Phi^{-1}(\Phi(\xi)+\Phi(a)-\Phi(t))) \\ \times\Psi(\Phi^{-1}(\Phi(\xi)+\Phi(a)-\Phi(t)))w(t)h(t)\Phi^{'}(t)dt.$$

Theorem 2. *[30] Let Ψ be a piecewise continuous w-weighted Φ-exponential order function on interval $[a,\vartheta]$. Then*

$$\mathfrak{L}_{\Phi}^{w}\{{}_{\Psi,\jmath}^{\ s}\mathfrak{I}^{w,\omega,\epsilon}_{\delta+;\varrho,\alpha}\Psi(\vartheta)\}(s) \\ = (s+1)^{1-\frac{\alpha}{\jmath}}(\jmath s)^{-\frac{\alpha}{k}}\Big(1-k\omega(ks)^{-\frac{\varrho}{k}}\Big)^{-\frac{\epsilon}{k}}\mathfrak{L}_{\Phi}^{w}\{\Psi(\vartheta)\}(s),$$

with $|\jmath\omega(\jmath s)^{-\frac{\varrho}{\jmath}}| < 1$.

The present work is one in a sequence of studies starting by Garra *et al.* [20] in 2014, then modified by Samraiz *et al.* [31] in 2020.

2 Weighted Generalized Hilfer-Prabhakar Fractional Derivative and Weighted Laplace Transform

In the current section, we describe the weighted generalized Hilfer-Prabhakar fractional derivative. The weighted Laplace transform of the novel operator is also evaluated.

Definition 12. *Let $\Psi,\psi \in C^1[\delta,\ell]$, $\delta > 0$, $0 < \vartheta < \ell < \infty$, $s \in \mathbb{R}\backslash\{-1\}$, $\eta,\varrho > 0$, $\omega,\epsilon \in \mathbb{R}$, $\alpha \in (0,1)$, $\varpi \in [0,1]$ and $\big(\Psi * {}_{\psi,\eta}^{\ s}\mathfrak{I}^{\omega,w,-\epsilon(1-\varpi)}_{\delta+;\varrho,(1-\varpi)(\eta-\alpha)}\big)(\vartheta) \in AC^1[\delta,\ell]$, then weighted generalized Hilfer-Prabhakar fractional derivative is defined as*

$${}_{\psi,\eta}^{\ s}\mathfrak{D}^{\epsilon,w,\alpha,\varpi}_{\delta+;\varrho,\omega}\Psi(\vartheta) = \eta\Big({}_{\psi,\eta}^{\ s}\mathfrak{I}^{\omega,w,-\epsilon\varpi}_{\delta+;\varrho,\varpi(\eta-\alpha)}\Big(\frac{w^{-1}(\vartheta)}{\psi^s(\vartheta)\psi'(\vartheta)}\frac{d}{d\vartheta}\Big)w(\vartheta)\,{}_{\psi,\eta}^{\ s}\mathfrak{I}^{\omega,w,-\epsilon(1-\varpi)}_{\delta+;\varrho,(1-\varpi)(\eta-\alpha)}\Psi\Big)(\vartheta), \quad (4)$$

where $\psi^s(\vartheta) = (\psi(\vartheta))^s$, $\psi^s(\vartheta) \neq 0$ and $w(\vartheta) \neq 0$.

We observe that, the generalized Hilfer-Prabhakar fractional derivative operator given in [31] can be achieved with choice of parameters $\psi(\vartheta) = \vartheta$ and $w(\vartheta) = 1$ in (4). The choice of the parameters $\psi(\vartheta) = \vartheta, w(\vartheta) = 1, \eta = 1, s = 0$, gives Hilfer-Prabhakar fractional derivative introduced by Garra et al. in [20] presented in Definition 9. If we set $\psi(\vartheta) = \vartheta$, $w(\vartheta) = 1$, $\varpi = 0$ and $\epsilon = 0$ in (4), we get (η,s)-Riemann-Liouville fractional derivative operator given in [35]. Corresponding to the choice of the parameters $\psi(\vartheta) = \vartheta$, $w(\vartheta) = 1$ and $\varpi = 0$ in (4), we obtain (η,s)-Prabhakar fractional derivative given in [36]. If we substitute $\psi(\vartheta) = \vartheta$, $w(\vartheta) = 1$, $\varpi = 0$, and $s = 0$ in (4), we get η-Prabhakar fractional derivative operator given in [8], the choice of the parameters $\psi(\vartheta) = \vartheta$, $w(\vartheta) = 1$, $\varpi = 0$, $s = 0$ and $\eta = 1$ in (4) gives Prabhakar fractional derivative operator presented in [14].

Proposition 1. *Let* $s \in \mathbb{R}\backslash\{-1\}$, $\eta \in \mathbb{R}^+$, $\alpha, \varrho, \omega, \epsilon \in \mathbb{C}$, $\Re(\varrho) > 0$, $\Re(\alpha) > 0$ *and* $\ell > 0$ *then integral operator* ${}_{\psi,\eta}^{\;s}\mathfrak{I}_{\delta^+;\varrho,\alpha}^{\omega,w,\epsilon}$ *is bounded on* $C[\delta,\ell]$, $\delta \geq 0$ *i.e.,*

$$|({}_{\psi,\eta}^{\;s}\mathfrak{I}_{\delta^+;\varrho,\alpha}^{\omega,w,\epsilon}\Psi)(\vartheta)| \leq G\|w\Psi\|_{C[\delta,\ell]},$$

where

$$\|w\Psi\|_{C[\delta,\ell]} = \max\{|w\Psi| : 0 < x < \ell\}$$

and

$$G = \frac{(s+1)^{-\frac{\alpha}{\eta}}(\psi^{s+1}(\ell)-\psi^{s+1}(\delta))^{\frac{\alpha}{\eta}}}{\eta}\sum_{m=0}^{\infty}\frac{|(\epsilon)_{m,\eta}\omega^m|}{|\Gamma_\eta(\varrho m+\alpha)|m!}$$
$$\times\frac{(\psi^{s+1}(\ell)-\psi^{s+1}(\delta))^{\frac{\varrho}{\eta}m}}{\left[\frac{m}{\eta}(\varrho+\alpha)\right]}. \quad (5)$$

Proof. We first prove that the series in the Eq. (5) is convergent. Let b_m denotes the m^{th} term of the series, then we have

$$\frac{b_{m+1}}{b_m} = \frac{|(\epsilon)_{m+1,\eta}\omega^{m+1}(\psi^{s+1}(\ell)-\psi^{s+1}(\delta))^{\frac{\varrho(m+1)}{\eta}}|}{|(\epsilon)_{m,\eta}\omega^{m}(\psi^{s+1}(\ell)-\psi^{s+1}(\delta))^{\frac{\varrho(m)}{\eta}}|}$$
$$\times\frac{\left[\frac{m\varrho+\alpha}{\eta}\right]|\Gamma_\eta(\varrho m+\alpha)|m!}{\left[\frac{(m+1)\varrho+\alpha}{\eta}\right]|\Gamma_\eta(\varrho(m+1)+\alpha)|(m+1)!}$$
$$= \eta^{\frac{-\varrho}{\eta}}\frac{|m+\frac{\epsilon}{\eta}|}{m+1}\frac{\left|\Gamma\left(\frac{\varrho m}{\eta}+\frac{\alpha}{\eta}\right)\right|}{\left|\Gamma\left(\frac{\varrho m}{\eta}+\frac{\varrho}{\eta}+\frac{\alpha}{\eta}\right)\right|}$$
$$\times\frac{\left[\frac{m\varrho+\alpha}{\eta}\right]}{|\left[\frac{(m+1)\varrho+\alpha}{\eta}\right]|}\omega|(\psi^{s+1}(\ell)-\psi^{s+1}(\delta))^{\frac{\varrho}{\eta}}|$$
$$\sim\frac{\omega|(\psi^{s+1}(\ell)-\psi^{s+1}(\delta))^{\frac{\varrho}{\eta}}|}{(\left|\frac{\varrho}{\eta}m\right|)^{\frac{\varrho}{\eta}}} \to 0(m\to\infty).$$

This implies that the series on the right side of (5) is convergent and hence G is finite.

Now, Consider

$$|({}_{\psi,\eta}^{\;s}\mathfrak{I}_{\delta^+;\varrho,\alpha}^{\omega,w,\epsilon}\Psi)(\vartheta)| \leq \frac{(s+1)^{1-\frac{\alpha}{\eta}}}{\eta}w^{-1}(\vartheta)\int_\delta^\vartheta(\psi^{s+1}(\vartheta)-\psi^{s+1}(\varsigma))^{\frac{\alpha}{\eta}-1}\psi^s(\varsigma)$$
$$\times\left|E_{\eta,\varrho,\alpha}^{\epsilon}(\omega(\psi^{s+1}(\vartheta)-\psi^{s+1}(\varsigma))^{\frac{\varrho}{\eta}})w(\varsigma)\Psi(\varsigma)\right|\psi'(\varsigma)d\varsigma. \quad (6)$$

Substitute $u = \psi^{s+1}(\vartheta) - \psi^{s+1}(\varsigma)$ on the right side of (6), we get

$$\begin{aligned} |({}_{\psi,\eta}^{\ s}\mathfrak{J}_{\delta^+;\varrho,\alpha}^{\omega,w,\epsilon}\Psi)(\vartheta)| &\leq \frac{(s+1)^{-\frac{\alpha}{\eta}}}{\eta} w^{-1}(\vartheta) \int_0^{\psi^{s+1}(\vartheta)-\psi^{s+1}(\delta)} u^{\frac{\alpha}{\eta}-1} \\ &\quad \times |E_{\eta,\varrho,\alpha}^{\epsilon}(\omega(u)^{\frac{\varrho}{\eta}})| \|w\Psi\|_{C[\delta,\ell]} du \\ &\leq \|w\Psi\|_{C[\delta,\ell]} \frac{(s+1)^{-\frac{\alpha}{\eta}}}{\eta} w^{-1}(\ell) \int_0^{\psi^{s+1}(\ell)-\psi^{s+1}(\delta)} u^{\frac{\alpha}{\eta}-1} \\ &\quad \times |E_{\eta,\varrho,\alpha}^{\epsilon}(\omega(u)^{\frac{\varrho}{\eta}})| du \\ &\leq \|w\Psi\|_{C[\delta,\ell]} \frac{(s+1)^{-\frac{\alpha}{\eta}}}{\eta} w^{-1}(\ell) \sum_{m=0}^{\infty} \frac{|(\epsilon)_{m,\eta}\omega^m|}{|\Gamma_\eta(\varrho m+\alpha)|m!} \\ &\quad \times \int_0^{\psi^{s+1}(\ell)-\psi^{s+1}(\delta)} (u)^{(\frac{\varrho}{k\eta})m+(\frac{\alpha}{\eta})-1} du \\ &\leq \|w\Psi\|_{C[\delta,\ell]} \frac{(s+1)^{-\frac{\alpha}{\eta}} (\psi^{s+1}(\ell)-\psi^{s+1}(\delta))^{(\frac{\alpha}{\eta})}}{\eta} \sum_{m=0}^{\infty} \frac{|(\epsilon)_{m,\eta}\omega^m|}{|\Gamma_\eta(\varrho m+\alpha)|m!} \\ &\quad \times \frac{(\psi^{s+1}(\ell)-\psi^{s+1}(\delta))^{(\frac{\varrho}{\eta})m}}{\left[\frac{m}{\eta}(\varrho+\alpha)\right]}, \end{aligned}$$

which gives

$$|({}_{\psi,\eta}^{\ s}\mathfrak{J}_{\delta^+;\varrho,\alpha}^{\omega,w,\epsilon}\Psi)(\vartheta)| \leq G\|w\Psi\|_{C[\delta,\ell]}.$$

This completes the proof of Proposition 1.

Theorem 3. *Let s be any real number except -1, $\eta, \varrho > 0$, $\omega, \epsilon \in \mathbb{R}$, $\alpha \in (0,1)$, $\varpi \in [0,1]$. If $\Psi \in L_1[\delta,\ell]$, then the weighted generalized Hilfer-Prabhakar fractional derivative ${}_{\psi,\eta}^{\ s}\mathfrak{D}_{\delta^+;\varrho,\omega}^{\epsilon,w,\alpha,\varpi}$ is bounded on $C[\delta,\ell]$*

$$\|{}_{\psi,\eta}^{\ s}\mathfrak{D}_{\delta^+;\varrho,\omega}^{\epsilon,\alpha,\varpi}\Psi(\vartheta)\| \leq A_1 A_2 \|w\Psi\|_{[\delta,\ell]},$$

where

$$\begin{aligned} A_1 &= \frac{(s+1)^{-\frac{\varpi(\eta-\alpha)}{\eta}} (\psi^{s+1}(\ell)-\psi^{s+1}(\delta))^{\frac{\varpi(\eta-\alpha)}{\eta}}}{\eta} \\ &\quad \times \sum_{n=0}^{\infty} \frac{|(-\epsilon\varpi)_{n,\eta}\omega^n|}{|\Gamma_\eta(\varrho n+\varpi(\eta-\alpha))|n!} \frac{(\psi^{s+1}(\ell)-\psi^{s+1}(\delta))^{\frac{\varrho n}{\eta}}}{\left[\frac{\varrho n}{\eta}+\frac{\varpi(\eta-\alpha}{\eta})\right]}, \end{aligned} \tag{7}$$

and

$$\begin{aligned} A_2 &= \frac{(s+1)^{-\frac{(1-\varpi)(\eta-\alpha)-\eta}{\eta}} (\psi^{s+1}(\ell)-\psi^{s+1}(\delta))^{\frac{(1-\varpi)(\eta-\alpha)-\eta}{\eta}}}{\eta} \\ &\quad \times \sum_{m=0}^{\infty} \frac{|(\epsilon(\varpi-1))_{m,\eta}\omega^m|}{|\Gamma_\eta(\varrho m+(1-\varpi)(\eta-\alpha))|m!} \frac{(\psi^{s+1}(\ell)-\psi^{s+1}(\delta))^{\frac{\varrho m}{\eta}}}{\left[\frac{\varrho m}{\eta}+\frac{(1-\varpi)(\eta-\alpha}{\eta})\right]}. \end{aligned} \tag{8}$$

Proof. Using Proposition 1, we have

$$\begin{aligned}
&\|{}_{\psi,\eta}^{\;s}\mathfrak{D}^{\epsilon,w,\alpha,\varpi}_{\delta^+;\varrho,\omega}\Psi(\vartheta)\| \\
&= \left\|\eta\Big({}_{\psi,\eta}^{\;s}\mathfrak{J}^{\omega,w,-\epsilon\varpi}_{\delta^+;\varrho,\varpi(\eta-\alpha)}\Big(\tfrac{w^{-1}(\vartheta)}{\psi^s(\vartheta)\psi'(\vartheta)}\tfrac{d}{d\vartheta}\Big)w(\vartheta)\Big({}_{\psi,\eta}^{\;s}\mathfrak{J}^{\omega,w,-\epsilon(1-\varpi)}_{\delta^+;\varrho,(1-\varpi)(\eta-\alpha)}\Psi\Big)(\vartheta)\right\| \\
&\leq \eta A_1\left\|\Big(\tfrac{w^{-1}(\vartheta)}{\psi^s(\vartheta)\psi'(\vartheta)}\tfrac{d}{d\vartheta}\Big)w(\vartheta)\Big({}_{\psi,\eta}^{\;s}\mathfrak{J}^{\omega,w,-\epsilon(1-\varpi)}_{\delta^+;\varrho,(1-\varpi)(\eta-\alpha)}\Psi\Big)(\vartheta)\right\| \\
&= A_1\left\|\Big({}_{\psi,\eta}^{\;s}\mathfrak{J}^{\omega,w,-\epsilon(1-\varpi)}_{\delta^+;\varrho,(1-\varpi)(\eta-\alpha)-\eta}\Psi\Big)(\vartheta)\right\| \\
&\leq A_1A_2\|\Psi\|_{[\delta,\ell]},
\end{aligned}$$

where both A_1 and A_2 are given by (7) and (8).

Proposition 2. *Let* $s\in\mathbb{R}\setminus -1$, $\eta,\varrho,\rho>0$, $\omega,\epsilon,\sigma\in\mathbb{R}$, $\alpha\in(0,1)$, $\varpi\in[0,1]$, $\rho>\alpha+\varpi\eta-\alpha\varpi$ *and* $\Psi\in L_1[\delta,\ell]$, *then*

$$\Big({}_{\psi,\eta}^{\;s}\mathfrak{D}^{\epsilon,w,\alpha,\varpi}_{\delta^+;\varrho,\omega}\big({}_{\psi,\eta}^{\;s}\mathfrak{J}^{\omega,w,\sigma}_{\delta^+;\varrho,\rho}\Psi\big)\Big)(\vartheta)=\big({}_{\psi,\eta}^{\;s}\mathfrak{J}^{\omega,w,\sigma-\epsilon}_{\delta^+;\varrho,\rho-\alpha}\Psi\big)(\vartheta).$$

In particular

$$\Big({}_{\psi,\eta}^{\;s}\mathfrak{D}^{\epsilon,w,\alpha,\varpi}_{\delta^+;\varrho,\omega}\big({}_{\psi,\eta}^{\;s}\mathfrak{J}^{\omega,w,\epsilon}_{\delta^+;\varrho,\alpha}\Psi\big)\Big)(\vartheta)=\Psi(\vartheta).$$

Proof. Using the Definition 12 and semi group property of (5) given in [30], we have

$$\begin{aligned}
&\Big({}_{\psi,\eta}^{\;s}\mathfrak{D}^{\epsilon,w,\alpha,\varpi}_{\delta^+;\varrho,\omega}\big({}_{\psi,\eta}^{\;s}\mathfrak{J}^{\omega,w,\sigma}_{\delta^+;\varrho,\rho}\Psi\big)\Big)(\vartheta) \\
&=\eta\Big({}_{\psi,\eta}^{\;s}\mathfrak{J}^{\omega,w,-\epsilon\varpi}_{\delta^+;\varrho,\varpi(\eta-\alpha)}\Big(\tfrac{w^{-1}(\vartheta)}{\psi^s(\vartheta)\psi'(\vartheta)}\tfrac{d}{d\vartheta}w(\vartheta)\Big){}_{\psi,\eta}^{\;s}\mathfrak{J}^{\omega,w,-\epsilon(1-\varpi)}_{\delta^+;\varrho,(1-\varpi)(\eta-\alpha)}\big({}_{\psi,\eta}^{\;s}\mathfrak{J}^{\omega,w,\sigma}_{\delta^+;\varrho,\rho}\Psi\big)\Big)(\vartheta) \\
&=\eta\Big({}_{\psi,\eta}^{\;s}\mathfrak{J}^{\omega,w,-\epsilon\varpi}_{\delta^+;\varrho,\varpi(\eta-\alpha)}\Big(\tfrac{w^{-1}(\vartheta)}{\psi^s(\vartheta)\psi'(\vartheta)}\tfrac{d}{d\vartheta}w(\vartheta)\Big)\Big({}_{\psi,\eta}^{\;s}\mathfrak{J}^{\omega,w,-\epsilon(1-\varpi)+\sigma}_{\delta^+;\varrho,(1-\varpi)(\eta-\alpha)+\rho}\Psi\Big)\Big)(\vartheta) \\
&=\Big({}_{\psi,\eta}^{\;s}\mathfrak{J}^{\omega,w,-\epsilon\varpi}_{\delta^+;\varrho,\varpi(\eta-\alpha)}\Big({}_{\psi,\eta}^{\;s}\mathfrak{J}^{\omega,-\epsilon(1-\varpi)+\sigma}_{\delta^+;\varrho,(1-\varpi)(\eta-\alpha)+\rho-\eta}\Psi\Big)\Big)(\vartheta) \\
&=\Big({}_{\psi,\eta}^{\;s}\mathfrak{J}^{\omega,\sigma-\epsilon}_{\delta^+;\varrho,\rho-\alpha}\Psi\Big)\Big)(\vartheta).
\end{aligned}$$

The proof of the Proposition 2 is completed.

Theorem 4. *For* $s\in\mathbb{R}\backslash\{-1\}$, $\eta,\varrho,\rho>0$, $\omega,\epsilon\in\mathbb{R}$, $\alpha\in(0,1)$, $\varpi\in[0,1]$, $\rho>\alpha+\varpi\eta-\alpha\varpi$ *and* $\Psi\in L_1[\delta,\ell]$ *then*

$$\Big({}_{\psi,\eta}^{\;s}\mathfrak{J}^{\rho}_{\delta^+,w}\big({}_{\psi,\eta}^{\;s}\mathfrak{D}^{\epsilon,w,\alpha,\varpi}_{\delta^+;\varrho,\omega}\Psi\big)\Big)(\vartheta)=\big({}_{\psi,\eta}^{\;s}\mathfrak{J}^{\omega,w,-\epsilon}_{\delta^+;\varrho,\rho-\alpha}\Psi\big)(\vartheta).$$

Proof. Using the Definition 12 and Theorem 2.5 given in [36], we get

$$\begin{aligned}
&\Big({}_{\psi,\eta}^{\ \ s}\mathfrak{I}^{\rho}_{\delta+,w}\big({}_{\psi,\eta}^{\ \ s}\mathfrak{D}^{\epsilon,w,\alpha,\varpi}_{\delta+;\varrho,\omega}\big)\varPsi\big)\Big)(\vartheta)\\
&=\eta\Big({}_{\psi,\eta}^{\ \ s}\mathfrak{I}^{\rho}_{\delta+,w}\,{}_{\psi,\eta}^{\ \ s}\mathfrak{I}^{\omega,w,-\epsilon\varpi}_{\delta+;\varrho,\varpi(\eta-\alpha)}\Big(\frac{w^{-1}(\vartheta)}{\psi^{s}(\vartheta)\psi'(\vartheta)}\frac{d}{d\vartheta}w(\vartheta)\Big)\,{}_{\psi,\eta}^{\ \ s}\mathfrak{I}^{\omega,w,-\epsilon(1-\varpi)}_{\delta+;\varrho,(1-\varpi)(\eta-\alpha)}\varPsi\Big)(\vartheta)\\
&=\eta\Big({}_{\psi,\eta}^{\ \ s}\mathfrak{I}^{\omega,w,-\epsilon\varpi}_{\delta+;\varrho,\varpi(\eta-\alpha)+\rho}\Big(\frac{w^{-1}(\vartheta)}{\psi^{s}(\vartheta)\psi'(\vartheta)}\frac{d}{d\vartheta}w(\vartheta)\Big)\,{}_{\psi,\eta}^{\ \ s}\mathfrak{I}^{\omega,w,-\epsilon(1-\varpi)}_{\delta+;\varrho,(1-\varpi)(\eta-\alpha)}\varPsi\Big)(\vartheta)\\
&=\Big({}_{\psi,\eta}^{\ \ s}\mathfrak{I}^{\omega,w,-\epsilon\varpi}_{\delta+;\varrho,\varpi(\eta-\alpha)+\rho}\big({}_{\psi,\eta}^{\ \ s}\mathfrak{I}^{\omega,-\epsilon(1-\varpi)}_{\delta+;\varrho,(1-\varpi)(\eta-\alpha)-\eta}\varPsi\big)\Big)(\vartheta)\\
&=\Big({}_{\psi,\eta}^{\ \ s}\mathfrak{I}^{\omega,-\epsilon}_{\delta+;\varrho,\rho-\alpha}\varPsi\Big)(\vartheta).
\end{aligned}$$

This completes the proof.

Theorem 5. *The weighted Laplace transform of generalized Hilfer-Prabhakar fractional derivative is given by*

$$\begin{aligned}
\mathfrak{L}^{w}_{\psi}\{{}_{\psi,\eta}^{\ \ s}\mathfrak{D}^{\epsilon,w,\alpha,\varpi}_{\delta+;\varrho,\omega}\varPsi(\vartheta)\}(u)&=(s+1)^{\frac{\alpha-\eta}{\eta}}(\eta u)^{\frac{\alpha}{\eta}}\Big(1-\eta\omega(\eta u)^{-\frac{\varrho}{\eta}}\Big)^{\frac{\epsilon}{\eta}}\\
&\times\mathfrak{L}^{w}_{\psi}\{\varPsi(\vartheta)\}(u)-\eta(s+1)^{-\frac{\varpi(\eta-\alpha)}{\eta}}(\eta u)^{\frac{-\varpi(\eta-\alpha)}{\eta}}\\
&\times\Big(1-\eta\omega(\eta u)^{-\frac{\varrho}{\eta}}\Big)^{\frac{\epsilon\varpi}{\eta}}\,{}_{\psi,\eta}^{\ \ s}\mathfrak{I}^{\omega,w,-\epsilon(1-\varpi)}_{\delta+;\varrho,(1-\varpi)(\eta-\alpha)}\varPsi(\delta^{+}).
\end{aligned}$$

Proof. Using the Definition 12 and Theorem 1, we have

$$\begin{aligned}
&\mathfrak{L}^{w}_{\psi}\{{}_{\psi,\eta}^{\ \ s}\mathfrak{D}^{\epsilon,w,\alpha,\varpi}_{\delta+;\varrho,\omega}\varPsi(\vartheta)\}(u)\\
&=\eta(s+1)^{-\frac{\varpi(\eta-\alpha)}{\eta}}(\eta u)^{\frac{-\varpi(\eta-\alpha)}{\eta}}\Big(1-\eta\omega(\eta u)^{-\frac{\varrho}{\eta}}\Big)^{\frac{\epsilon\varpi}{\eta}}\\
&\times\mathfrak{L}^{w}_{\psi}\Big\{{}_{\psi,\eta}^{\ \ s}\mathfrak{I}^{\omega,w,-\epsilon(1-\varpi)}_{\delta+;\varrho,(1-\varpi)(\eta-\alpha)}{}^{[1]}\varPsi(\vartheta)\Big\}(u)\\
&=\eta(s+1)^{-\frac{\varpi(\eta-\alpha)}{\eta}}(\eta u)^{\frac{-\varpi(\eta-\alpha)}{\eta}}\Big(1-\eta\omega(\eta u)^{-\frac{\varrho}{\eta}}\Big)^{\frac{\epsilon\varpi}{\eta}}\\
&\times\Big[u\mathfrak{L}^{w}_{\psi}\{{}_{\psi,\eta}^{\ \ s}\mathfrak{I}^{\omega,w,-\epsilon(1-\varpi)}_{\delta+;\varrho,(1-\varpi)(\eta-\alpha)}\varPsi(\vartheta)\}(u)-{}_{\psi,\eta}^{\ \ s}\mathfrak{I}^{\omega,w,-\epsilon(1-\varpi)}_{\delta+;\varrho,(1-\varpi)(\eta-\alpha)}\varPsi(\delta^{+})\Big]\\
&=(\eta u)(s+1)^{-\frac{\varpi(\eta-\alpha)}{\eta}}(\eta u)^{\frac{-\varpi(\eta-\alpha)}{\eta}}\Big(1-\eta\omega(\eta u)^{-\frac{\varrho}{\eta}}\Big)^{\frac{\epsilon\varpi}{\eta}}\\
&\times\mathfrak{L}^{w}_{\psi}\{{}_{\psi,\eta}^{\ \ s}\mathfrak{I}^{\omega,w,-\epsilon(1-\varpi)}_{\delta+;\varrho,(1-\varpi)(\eta-\alpha)}\varPsi(\vartheta)\}(u)-\eta(s+1)^{-\frac{\varpi(\eta-\alpha)}{\eta}}(\eta u)^{\frac{-\varpi(\eta-\alpha)}{\eta}}\\
&\times\Big(1-\eta\omega(\eta u)^{-\frac{\varrho}{\eta}}\Big)^{\frac{\epsilon\varpi}{\eta}}\,{}_{\psi,\eta}^{\ \ s}\mathfrak{I}^{\omega,w,-\epsilon(1-\varpi)}_{\delta+;\varrho,(1-\varpi)(\eta-\alpha)}\varPsi(\delta^{+})\\
&=(\eta u)(s+1)^{-\frac{\varpi(\eta-\alpha)}{\eta}}(\eta u)^{\frac{-\varpi(\eta-\alpha)}{\eta}}\Big(1-\eta\omega(\eta u)^{-\frac{\varrho}{\eta}}\Big)^{\frac{\epsilon\varpi}{\eta}}\\
&\times\Big[(s+1)^{-\frac{(1-\varpi)(\eta-\alpha)}{\eta}}(\eta u)^{\frac{-(1-\varpi)(\eta-\alpha)}{\eta}}\Big(1-\eta\omega(\eta u)^{-\frac{\varrho}{\eta}}\Big)^{\frac{\epsilon(1-\varpi)}{\eta}}
\end{aligned}$$

$$\times\mathfrak{L}_{\psi}^{w}\{\Psi(\vartheta)\}(u)\Big] - \eta(s+1)^{-\frac{\varpi(\eta-\alpha)}{\eta}}(\eta u)^{\frac{-\varpi(\eta-\alpha)}{\eta}}\left(1-\eta\omega(\eta u)^{-\frac{\varrho}{\eta}}\right)^{\frac{\epsilon\varpi}{\eta}}$$
$$\times{}_{\psi,\eta}^{\;\;s}\mathfrak{I}_{\delta^{+};\varrho,(1-\varpi)(\eta-\alpha)}^{\omega,w,-\epsilon(1-\varpi)}\Psi(\delta^{+})$$
$$= (s+1)^{\frac{\alpha-\eta}{\eta}}(\eta u)^{\frac{\alpha}{\eta}}\left(1-\eta\omega(\eta u)^{-\frac{\varrho}{\eta}}\right)^{\frac{\epsilon}{\eta}}\mathfrak{L}_{\psi}^{w}\{\Psi(\vartheta)\}(u) - \eta(s+1)^{-\frac{\varpi(\eta-\alpha)}{\eta}}$$
$$\times(\eta u)^{\frac{-\varpi(\eta-\alpha)}{\eta}}\left(1-\eta\omega(\eta u)^{\frac{-\varrho}{\eta}}\right)^{\frac{\epsilon\varpi}{\eta}}{}_{\psi,\eta}^{\;\;s}\mathfrak{I}_{\delta^{+};\varrho,(1-\varpi)(\eta-\alpha)}^{\omega,w,-\epsilon(1-\varpi)}\Psi(\delta^{+}),$$

this proves the proof of the Theorem 5.

3 Free-Electron Laser Equation Involving Weighted Generalized Fractional Operators

In recent decades, several methods for tackling the generalized fractional integro-differential free-electron laser problem have been proposed. In the present section, we offer a more generalized model of the free-electron laser problem involving the newly defined operator.

Theorem 6. *The solution of free-electron laser problem*

$${}_{\psi,\eta}^{\;\;s}\mathfrak{D}_{\delta^{+};\varrho,\omega,w}^{\epsilon,\alpha,\varpi}\Psi(\vartheta) = \rho\,{}_{\psi,\eta}^{\;\;s}\mathfrak{I}_{\delta^{+};\varrho,\alpha}^{\sigma,\omega,w}\Psi(\vartheta) + f(\vartheta), \tag{9}$$

$${}_{\psi,\eta}^{\;\;s}\mathfrak{I}_{\delta^{+};\varrho,(1-\varpi)(\eta-\alpha),w}^{\omega,-\epsilon(1-\varpi)}\Psi(\delta^{+}) = D, \quad D \geq 0, \tag{10}$$

where $\vartheta \in (0,\infty)$, $f \in L_1[0,\infty)$, $\alpha \in (0,1)$, $\varpi \in [0,1]$, ω, $\rho \in \mathbb{R}$, $\delta > 0$, $\varrho > 0$, ϵ, $\sigma \geq 0$ *is given by*

$$\Psi(\vartheta) = D\sum_{m=0}^{\infty}\rho^{m}(s+1)^{-\frac{\varpi(\eta-\alpha)+\alpha-\eta}{\eta}}\left(\psi^{s+1}(\vartheta)-\psi^{s+1}(\delta)\right)^{\frac{\varpi(\eta-\alpha)+\alpha(1+2m)}{\eta}-1}$$
$$\times E_{\eta,\varrho,\varpi(\eta-\alpha)+\alpha(1+2m)}^{(\epsilon+\sigma)m-\epsilon(\varpi-1)}\left(\omega\left(\psi^{s+1}(\vartheta)-\psi^{s+1}(\delta)\right)^{\frac{\varrho}{\eta}}\right)$$
$$+\sum_{m=0}^{\infty}\rho^{m}(s+1)^{2m}\left({}_{\psi,\eta}^{\;\;s}\mathfrak{I}_{\eta,\varrho,\alpha(1+2m)}^{\omega,w,(\epsilon+\sigma)m+\epsilon}f\right)(\vartheta).$$

Proof. Applying weighted Laplace transform on both sides of (9) and using Theorems 2 and 5, we get

$$\mathfrak{L}_{\psi}^{w}\{{}_{\psi,\eta}^{\;\;s}\mathfrak{D}_{\delta^{+};\varrho,\omega,w}^{\epsilon,\alpha,\varpi}\Psi(\vartheta)\}(u) = \rho\mathfrak{L}_{\psi}^{w}\{{}_{\psi,\eta}^{\;\;s}\mathfrak{I}_{\delta^{+};\varrho,\alpha}^{\sigma,\omega,w}\Psi(\vartheta)\}(u) + \mathfrak{L}_{\psi}^{w}\{f(\vartheta)\}(u). \tag{11}$$

The Eq. (11) can be presented as

$$\mathfrak{L}_{\psi}^{w}\{\Psi(\vartheta)\}(u) = \frac{D\eta(s+1)^{-\frac{\varpi(\eta-\alpha)}{\eta}}(\eta u)^{\frac{-\varpi(\eta-\alpha)}{\eta}}\left(1-\eta\omega(\eta u)^{\frac{\varrho}{\eta}}\right)^{\frac{\epsilon\varpi}{\eta}}}{(s+1)^{\frac{\alpha-\eta}{\eta}}(\eta u)^{\frac{\alpha}{\eta}}\left(1-\eta\omega(\eta u)^{\frac{-\varrho}{\eta}}\right)^{\frac{\epsilon}{\eta}}}$$
$$+\frac{\mathfrak{L}_{\psi}^{w}\{f(\vartheta)\}(u)}{(s+1)^{\frac{\alpha-\eta}{\eta}}(\eta u)^{\frac{\alpha}{\eta}}\left(1-\eta\omega(\eta u)^{\frac{-\varrho}{\eta}}\right)^{\frac{\epsilon}{\eta}}}$$
$$\times\left(1-\rho(\eta u)^{-\frac{2\alpha}{\eta}}\left(1-\eta\omega(\eta u)^{\frac{-\varrho}{\eta}}\right)^{-\frac{\epsilon+\sigma}{\eta}}\right)^{-1}.$$

By using the binomial expansion, we get

$$\mathfrak{L}_{\psi}^{w}\{\Psi(\vartheta)\}(u)$$

$$= \frac{D\eta(s+1)^{-\frac{\varpi(\eta-\alpha)}{\eta}}(\eta u)^{\frac{-\varpi(\eta-\alpha)}{\eta}}\left(1-\eta\omega(\eta u)^{\frac{\varrho}{\eta}}\right)^{\frac{\epsilon\varpi}{\eta}}}{(s+1)^{\frac{\alpha-\eta}{\eta}}(\eta u)^{\frac{\alpha}{\eta}}\left(1-\eta\omega(\eta u)^{\frac{-\varrho}{\eta}}\right)^{\frac{\epsilon}{\eta}}}$$

$$+\frac{\mathfrak{L}_{\psi}^{w}\{f(\vartheta)\}(u)}{(s+1)^{\frac{\alpha-\eta}{\eta}}(\eta u)^{\frac{\alpha}{\eta}}\left(1-\eta\omega(\eta u)^{\frac{-\varrho}{\eta}}\right)^{\frac{\epsilon}{\eta}}}$$

$$\times\sum_{m=0}^{\infty}\rho^{m}(\eta u)^{-2\frac{\alpha m}{\eta}}\left(1-\eta\omega(\eta u)^{\frac{-\varrho}{\eta}}\right)^{-\frac{(\epsilon+\sigma)}{\eta}m},$$

$$= D\eta\sum_{m=0}^{\infty}\rho^{m}(s+1)^{-\frac{\varpi(\eta-\alpha)+\alpha-\eta}{\eta}}$$

$$\times(\eta u)^{-\frac{\varpi(\eta-\alpha)+\alpha(1+2m)}{\eta}}\left(1-\eta\omega(\eta u)^{\frac{-\varrho}{\eta}}\right)^{-\frac{(\epsilon+\sigma)m-\epsilon(\varpi-1)}{\eta}}$$

$$+\sum_{m=0}^{\infty}\rho^{m}(s+1)^{-\frac{\alpha-\eta}{\eta}}(\eta u)^{-\frac{\alpha(1+2m)}{\eta}}$$

$$\times\left(1-\eta\omega(\eta u)^{\frac{-\varrho}{\eta}}\right)^{-\frac{\epsilon+m(\epsilon+\sigma)}{\eta}}\mathfrak{L}_{\psi}^{w}\{f(\vartheta)\}(u).$$

Applying inverse Laplace transform, we obtain

$$\Psi(\vartheta) = D\sum_{m=0}^{\infty}\rho^{m}(s+1)^{-\frac{\varpi(\eta-\alpha)+\alpha-\eta}{\eta}}\left(\psi^{s+1}(\vartheta)-\psi^{s+1}(\delta)\right)^{\frac{\varpi(\eta-\alpha)+\alpha(1+2m)}{\eta}-1}$$

$$\times E_{\eta,\varrho,\varpi(\eta-\alpha)+\alpha(1+2m)}^{(\epsilon+\sigma)m-\epsilon(\varpi-1)}\left(\omega(\psi^{s+1}(\vartheta)-\psi^{s+1}(\delta))^{\frac{\varrho}{\eta}}\right)$$

$$+\sum_{m=0}^{\infty}\rho^{m}(s+1)^{2m}\left({}_{\psi,\eta}^{s}\mathfrak{J}_{\eta,\varrho,\alpha(1+2m)}^{\omega,w,(\epsilon+\sigma)m+\epsilon}f\right)(\vartheta).$$

Hence the proof is completed.

Remark 1. Let $s=0$, $\psi(\vartheta)=1$, $w(\vartheta)=1$, $\eta=1$, $\epsilon=\varpi=0$, $\sigma=\varrho=1$, $\alpha\to 1$, $f(\vartheta)=0$, $\omega=ir$, $\rho=-i\Pi p$ with $r,p\in\mathbb{R}$, then the Theorem 6 convert to the following free-electron laser equation

$$\frac{d}{d\vartheta}\Psi(\vartheta) = -ip\Pi\int_{0}^{\vartheta}(\vartheta-t)e^{ir(\vartheta-t)}\Psi(t)dt,\ \Psi(0)=1.$$

Corollary 1. *Let $\psi(\vartheta)=1$, $w(\vartheta)=1$, $s=0$ and $\eta=1$, then we have the problem given in [20] is defined as follows:*

$$\mathfrak{D}_{\delta^+;\varrho,\omega}^{\epsilon,\alpha,\varpi}\Psi(\vartheta) = \rho\mathfrak{J}_{\delta^+;\varrho,\alpha}^{\sigma,\omega}\Psi(\vartheta)+f(\vartheta), \tag{12}$$

$$\mathfrak{J}_{\delta^+;\varrho,(1-\varpi)(\eta-\alpha)}^{\omega,-\epsilon(1-\varpi)}\Psi(\delta^+) = A,\quad A\geq 0,$$

where $\vartheta\in(0,\infty)$, $f\in L_1[0,\infty)$; $\alpha\in(0,1)$, $\varpi\in[0,1]$, $\omega,\rho\in\mathbb{R}$, $\varrho>0$, ϵ, $\sigma\geq 0$. The solution to the fractional equation is

$$\Psi(\vartheta) = C\sum_{m=0}^{\infty}\rho^{m}(\vartheta)^{\varpi(1-\alpha)+\alpha(1+2m)-1}$$

$$\times E_{\varrho,\varpi(1-\alpha)+\alpha(1+2m)}^{(\epsilon+\sigma)m-\epsilon(\varpi-1)}\left(\omega(\vartheta)^{\varrho}\right)$$

$$+\sum_{m=0}^{\infty}\rho^{m}\left(\mathfrak{J}_{\delta^+,\varrho,\alpha(1+2m)}^{\omega,(\epsilon+\sigma)m+\epsilon}f\right)(\vartheta).$$

4 Fractional Kinetic Equation Involving Weighted Generalized Fractional Operators

Physics, control systems, dynamic systems, and engineering have all become more interested in developing mathematical models of various physical phenomena due to their importance in the field of applied research. The fundamental equations of mathematical physics and the natural sciences known as the kinetic equations, define the continuation of a substance's motion. In the present segment, we establish a generalization of the kinetic equation. The reader is referred to related literature [37–40].

Theorem 7. *Then solution to the Cauchy fractional problem*

$$c_{\psi},{}^{s}_{\eta}\mathfrak{D}^{\epsilon,\alpha,\varpi}_{\delta^{+};\varrho,\omega,w}M(t) - M_0 f(t) = b_{\psi},{}^{s}_{\eta}\mathfrak{I}^{\omega,w,\sigma}_{\delta^{+};\varrho,q}M(t),\ f \in L_1[0,\infty); \tag{13}$$

subject to

$$_{\psi},{}^{s}_{\eta}\mathfrak{I}^{\omega,w,-\epsilon(1-\varpi)}_{\delta^{+};\varrho,(1-\varpi)(\eta-\alpha)}M(\delta) = d, \quad d \geq 0,$$

with $s \in [0,\infty)$, $\varpi \in [0,1]$ $\omega \in \mathbb{C}$, $c, b \in \mathbb{R}(c \neq 0)$, $\alpha, \varrho, q, \eta > 0$, $\delta, \epsilon, \sigma \geq 0$ *is given by*

$$\begin{aligned}M(t) = d\sum_{n=0}^{\infty}\left(\tfrac{b}{c}\right)^{n}(s+1)^{-\frac{\varpi(\eta-\alpha)+(\alpha-\eta)(n+1)+qn}{\eta}}(\psi^{s+1}(t)-\psi^{s+1}(\delta))^{\frac{\varpi(\eta-\alpha)+\alpha+(q+\alpha)n}{\eta}-1}\\ \times E^{(\epsilon+\sigma)n+\epsilon(1-\varpi)}_{\eta,\varrho,\varpi(\eta-\alpha)+\alpha+(q+\alpha)n}(\omega(\psi^{s+1}(t)-\psi^{s+1}(\delta))^{\frac{\varrho}{\eta}})\\ +\tfrac{M_0}{c}\sum_{n=0}^{\infty}\left(\tfrac{b}{c}\right)^{n}(s+1)^{n+1}{}_{\psi},{}^{s}_{\eta}\mathfrak{I}^{\omega,(\epsilon+\sigma)n+\epsilon}_{\delta^{+};\varrho,(q+\alpha)n+\alpha}f(t).\end{aligned}$$

Proof. Applying weighted generalized Laplace transform on both sides of (14), we have

$$c\mathfrak{L}^{w}_{\psi}\{_{\psi},{}^{s}_{\eta}\mathfrak{D}^{\epsilon,\alpha,\varpi}_{\delta^{+};\varrho,\omega}M(t)\}(u) - M_0\mathfrak{L}^{w}_{\psi}\{f(t)\}(u) = b\mathfrak{L}^{w}_{\psi}\{_{\psi},{}^{s}_{\eta}\mathfrak{I}^{\omega,\sigma}_{\delta^{+};\varrho,q}M(t)\}(u).$$

By considering the hypothesis of Theorems 2 and 5, we get

$$\begin{aligned}c\Big[(s+1)^{\frac{\alpha-\eta}{\eta}}(\eta u)^{\frac{\alpha}{\eta}}\left(1-\eta\omega(\eta u)^{-\frac{\varrho}{\eta}}\right)^{\frac{\epsilon}{\eta}}\mathfrak{L}^{w}_{\psi}\{M(t)\}(u) - \eta(s+1)^{-\frac{\varpi(\eta-\alpha)}{\eta}}(\eta u)^{\frac{-\varpi(\eta-\alpha)}{\eta}}\\ \times\left(1-\eta\omega(\eta u)^{-\frac{\varrho}{\eta}}\right)^{\frac{\epsilon\varpi}{\eta}}{}_{\psi},{}^{s}_{\eta}\mathfrak{I}^{\omega,-\epsilon(1-\varpi)}_{\delta^{+};\varrho,(1-\varpi)(\eta-\alpha)}M(\delta^{+})\Big] - M_0\mathfrak{L}^{w}_{\psi}\{f(t)\}(u) = b(s+1)^{-\frac{\alpha}{\eta}}(\eta u)^{\frac{-\alpha}{\eta}}\\ \times\left(1-\eta\omega(\eta u)^{\frac{-\varrho}{\eta}}\right)^{\frac{-\sigma}{\eta}}\mathfrak{L}^{w}_{\psi}\{M(t)\}.\end{aligned}$$

We can rewrite the above equations as

$$\left[\frac{c-b(s+1)^{-\frac{\alpha-\eta+q}{\eta}}(\eta u)^{-\frac{\alpha+q}{\eta}}\left(1-\eta\omega(\eta u)^{\frac{-\varrho}{\eta}}\right)^{-\frac{\epsilon+\sigma}{\eta}}}{(s+1)^{-\frac{\alpha-\eta}{\eta}}(\eta u)^{-\frac{\alpha}{\eta}}\left(1-\eta\omega(\eta u)^{\frac{-\varrho}{\eta}}\right)^{-\frac{\epsilon}{\eta}}}\right]\mathfrak{L}_{\psi}^{w}\{M(t)\}(u)$$
$$= c\eta d(s+1)^{-\frac{\varpi(\eta-\alpha)}{\eta}}(\eta u)^{-\frac{\varpi(\eta-\alpha)}{\eta}}\left(1-\eta\omega(\eta u)^{-\frac{\varrho}{\eta}}\right)^{\frac{\epsilon\varpi}{\eta}} + M_0\mathfrak{L}_{\psi}^{w}\{f(t)\}(u),$$
$$\mathfrak{L}_{\psi}^{w}\{M(t)\}(u) = c\eta d\left[\frac{(s+1)^{-\frac{\varpi(\eta-\alpha)+(\alpha-\eta)}{\eta}}(\eta u)^{-\frac{\varpi(\eta-\alpha)+\alpha}{\eta}}\left(1-\eta\omega(\eta u)^{-\frac{\varrho}{\eta}}\right)^{\frac{\epsilon(\varpi-1)}{\eta}}}{c-b(s+1)^{-\frac{\alpha-\eta+q}{\eta}}(\eta u)^{-\frac{\alpha+q}{\eta}}\left(1-\eta\omega(\eta u)^{\frac{-\varrho}{\eta}}\right)^{-\frac{\epsilon+\sigma}{\eta}}}\right]$$
$$+\left[\frac{(s+1)^{-\frac{\alpha-\eta}{\eta}}(\eta u)^{-\frac{\alpha}{\eta}}\left(1-\eta\omega(\eta u)^{\frac{-\varrho}{\eta}}\right)^{-\frac{\epsilon}{\eta}}}{c-b(s+1)^{-\frac{\alpha-\eta+q}{\eta}}(\eta u)^{-\frac{\alpha+q}{\eta}}\left(1-\eta\omega(\eta u)^{\frac{-\varrho}{\eta}}\right)^{-\frac{\epsilon+\sigma}{\eta}}}\right]M_0\mathfrak{L}_{\psi}^{w}\{f(t)\}(u).$$

Taking $\left|\frac{b}{c}(s+1)^{-\frac{\alpha-\eta+q}{\eta}}(\eta u)^{-\frac{\alpha+q}{\eta}}\left(1-\eta\omega(\eta u)^{\frac{-\varrho}{\eta}}\right)^{-\frac{\epsilon+\sigma}{\eta}}\right| < 1$, we get

$$\mathfrak{L}_{\psi}^{w}\{M(t)\}(u) = \Big[\eta d(s+1)^{-\frac{\varpi(\eta-\alpha)+(\alpha-\eta)}{\eta}}(\eta u)^{-\frac{\varpi(\eta-\alpha)+\alpha}{\eta}}\left(1-\eta\omega(\eta u)^{-\frac{\varrho}{\eta}}\right)^{\frac{\epsilon(\varpi-1)}{\eta}}$$
$$+(s+1)^{-\frac{\alpha-\eta}{\eta}}(\eta u)^{-\frac{\alpha}{\eta}}\left(1-\eta\omega(\eta u)^{\frac{-\varrho}{\eta}}\right)^{-\frac{\epsilon}{\eta}}c^{-1}M_0\mathfrak{L}_{\psi}^{w}\{f(t)\}(u)\Big]$$

$$\times\sum_{n=0}^{\infty}\left(\frac{b}{c}\right)^{n}(s+1)^{-\frac{(\alpha-\eta+q)n}{\eta}}(\eta u)^{-\frac{(\alpha+q)n}{\eta}}\left(1-\eta\omega(\eta u)^{\frac{-\varrho}{\eta}}\right)^{-\frac{(\epsilon+\sigma)n}{\eta}}$$
$$= d\eta\sum_{n=0}^{\infty}\left(\frac{b}{c}\right)^{n}(s+1)^{-\frac{\varpi(\eta-\alpha)+(\alpha-\eta)(n+1)+qn}{\eta}}(\eta u)^{-\frac{\varpi(\eta-\alpha)+\alpha+(\alpha+q)n}{\eta}}$$
$$\times\left(1-\eta\omega(\eta u)^{\frac{-\varrho}{\eta}}\right)^{-\frac{(\epsilon+\sigma)n+\epsilon(1-\varpi)}{\eta}} + \frac{M_0}{c}\sum_{n=0}^{\infty}\left(\frac{b}{c}\right)^{n}(s+1)^{-\frac{(\alpha-\eta)(n+1)+qn}{\eta}}(\eta u)^{-\frac{\alpha+(\alpha+q)n}{\eta}}$$
$$\times\left(1-\eta\omega(\eta u)^{\frac{-\varrho}{\eta}}\right)^{-\frac{(\epsilon+\sigma)n+\epsilon}{\eta}}.$$

Applying inverse Laplace transform, we get

$$M(t) = d\sum_{n=0}^{\infty}\left(\frac{b}{c}\right)^{n}(s+1)^{-\frac{\varpi(\eta-\alpha)+(\alpha-\eta)(n+1)+qn}{\eta}}(\psi^{s+1}(t)-\psi^{s+1}(\delta))^{\frac{\varpi(\eta-\alpha)+\alpha+(q+\alpha)n}{\eta}-1}$$
$$\times E_{\eta,\varrho,\varpi(\eta-\alpha)+\alpha+(q+\alpha)n}^{(\epsilon+\sigma)n+\epsilon(1-\varpi)}(\omega(\psi^{s+1}(t)-\psi^{s+1}(\delta))^{\frac{\varrho}{\eta}})$$
$$+\frac{M_0}{c}\sum_{n=0}^{\infty}\left(\frac{b}{c}\right)^{n}(s+1)^{n+1}{}_{\psi,\eta}^{\;\;s}\mathfrak{I}_{\delta^{+};\varrho,(q+\alpha)n+\alpha}^{\omega,(\epsilon+\sigma)n+\epsilon}f(t).$$

Hence the proof is done.

In next example, we establish the corresponding growth model and shows the behaviour by sketching its graph.

Example 1. The solution to the growth model

$$ {}_{\psi,\eta}^{\;\;s}\mathfrak{D}_{\delta^{+};\varrho,\omega,w}^{\epsilon,\alpha,\varpi}M(t) - M(t) = 0 \tag{14}$$

subject to

$$ {}_{\psi,\eta}^{\ s}\mathfrak{J}^{\omega,w,-\epsilon(1-\varpi)}_{\delta+;\varrho,(1-\varpi)(\eta-\alpha)}M(0)=d_{\circ}, $$

is

$$ M(t)=\sum_{n=0}^{\infty}\frac{t^{2n}}{\Gamma 2(n+0.5)},\ 0\le t\le 1. \tag{15} $$

Solution 1. By setting $c=b=1$, $s=0$, $\psi(t)=t$, $\varpi=0$, $\eta=1$, $\sigma=0$, $\epsilon=0$, $\delta=0$, $\alpha=1$, $q=1$, $d=d_0=1$ $M_0=0$, we obtained Eq. (15). The graph of this equation is

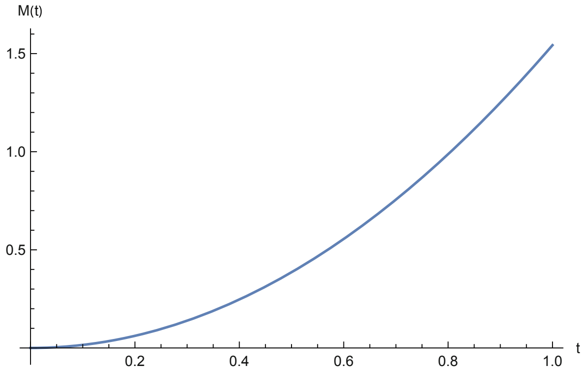

Fig. 1. For $\psi(t)=t$ the graph in Fig. 1 shows the increasing behaviour with $0^{+}\le t\le 1$.

Remark 2. [37] If we take $s=0$, $\psi(t)=1$, $w(t)=1$, $\eta=1$, $\sigma=\epsilon=\varpi=0$, $\alpha\to 0$, and $b=-c^{p}$, then we have

$$ M(t)-M_0 f(t)=-c^{p}D^{p}_{\delta+}M(t),\ \ M(0)=d,\ d\ge 0, $$

where $D^{p}_{\delta+}$ is the Riemann-Liouville fractional derivative operator.

Corollary 2. *Let $s=0$ $\psi(t)=1$, $w(t)=1$ and $\varpi=0$, then we obtain the Cauchy problem given in [40] and is defined by*

$$ c_{\eta}\mathfrak{D}^{\epsilon,\alpha}_{\delta+;\varrho,\omega}(t)-M_0 f(t)=b_{\eta}\mathfrak{J}^{\omega,\sigma}_{\delta+;\varrho,q}M(t),\ f\in L_1[0,\infty); $$
$$ {}_{\eta}\mathfrak{J}^{\omega,-\epsilon}_{\delta+;\varrho,\eta-\alpha}M(0)=d,\ \ d\ge 0, $$

with $\omega\in\mathbb{C}$, $c,b\in\mathbb{R}(c\ne 0)$, $\alpha,\varrho,q,\eta>0$, $\epsilon,\sigma\ge 0$. The resolution to the equation is given by

$$ M(t)=d\textstyle\sum_{n=0}^{\infty}\left(\frac{b}{c}\right)^{n}t^{\frac{\alpha+(q+\alpha)n}{\eta}-1}E^{(\epsilon+\sigma)n+\epsilon}_{\eta,\varrho,\alpha+(q+\alpha)n}(\omega(t)^{\frac{\varrho}{\eta}}) $$
$$ +\tfrac{M_0}{c}\textstyle\sum_{n=0}^{\infty}\left(\frac{b}{c}\right)^{n}{}_{\eta}\mathfrak{J}^{\omega,(\epsilon+\sigma)n+\epsilon}_{\delta+;\varrho,(q+\alpha)n+\alpha}f(t). $$

5 Conclusions

In this article, we introduced a new weighted generalized Hilfer-Prabhakar fractional derivative operator. This operator generalized many existing fractional derivatives. The novel operator was applied to the kinetic differintegral equation and the free-electron laser equation to create their fractional models as applications. The classical Laplace fails to find the solutions to these models, so we utilized a weighted Laplace transform. By using the specific values of the parameters the fractional growth model is presented that is strongly applicable in the field of science. The graph of the explored model is sketched that has increasing behaviour. We conclude, that the results presented in this article are more general and this idea may use to explore new weighted version of Furthermore, such fractional operators will be helpful to developed physical models. Then these new operators can be utilized to modeled physical problems like free electron laser equation and kinetic equation. The solutions to these models can be found by using Laplace transform of some other significant ways.

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