A Computational Study of Local Fractional Helmholtz and Coupled Helmholtz Equations in Fractal Media

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Abstract. In this manuscript, an approximate analytical solution of the Helmholtz and coupled Helmholtz equations of fractional order is obtained using local fractional Sumudu decomposition method (LFSDM). The Helmholtz equations play an important role in the study of various physical problems such as seismology, tsunamis, optics, acoustics, medical imaging, electrostatics and quantum mechanics. To validate the efficiency and reliability of the employed scheme, the Helmholtz and coupled Helmholtz equations are considered. The results obtained with this scheme are in a good agreement with previous works. Moreover, the graphical presentations for obtained solutions are also illustrated for distinct values of order of a partial derivative.

Keywords: Helmholtz equations · Local fractional derivative · Fractal media · Sumudu transform

1 Introduction

In last decades, fractional calculus has been applied very frequently in the field of applied sciences and technology. Actually, the area of fractional calculus is concerned with integral and derivatives of real order and it significantly handles scientific and engineering problems by formulating them in the form of fractional differential equations such as the diffusion equations [1], the gas dynamic equation [2], telegraph equation [3], wave equation [4–7], Fokker-Planck equation [8, 9], Laplace equation [10], Klein-Gordon equations [11], Helmholtz equation [12], and Burger's equations [13].

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Various local fractional schemes have been used to solve the local fractional PDEs (LFPDEs) such as the local fractional decomposition method [13–15], local fractional variational iteration method [16–22], local fractional differential transform method [23, 24], local fractional series expansion method [25, 26], local fractional Sumudu decomposition method [27], local fractional reduce differential transform method [28], local fractional Laplace variational iteration method [29], local fractional Laplace decomposition method [30], and local fractional Laplace homotopy perturbation method [31, 32]. This paper presents the copulation of LFST and LFADM, which is called as LFSDM, to solve the local fractional Helmholtz and coupled Helmholtz equations.

The paper is arranged in the following way: The basic definitions for calculus and fractional integration are presented in Sect. 2, the method used are analyzed in Sect. 3, illustrative examples are given that explain the effectiveness of the method proposed in Sect. 4, the numerical results and discussion are described in the Sect. 5 and finally, the conclusion is provided in Sect. 6.

2 Mathematical Fundamentals

Definition 2.1. The LF derivative of $\varphi(\mu)$ of order ε at μ_0 is [14–16]:

$$\varphi^{(\varepsilon)}(\mu_0) = \lim_{\mu \to \mu_0} \frac{\Gamma(1+\varepsilon)[\varphi(\mu) - \varphi(\mu_0)]}{(\mu - \mu_0)^{\varepsilon}}, \quad 0 < \varepsilon \leq 1 \tag{1}$$

Definition 2.2. The Mittage-Leffler function is defined by [14]:

$$E_{\varepsilon}\left(\mu^{\varepsilon}\right) = \sum_{k=0}^{\infty} \frac{\mu^{k\varepsilon}}{\Gamma(1+k\varepsilon)}, \quad \mu \in R, \quad 0 < \varepsilon \leq 1 \tag{2}$$

Definition 2.3. The LFST of $\varphi(\mu)$ given by [26]

$$ST_{\varepsilon}\{\varphi(\mu)\} = \frac{1}{\Gamma(1+\varepsilon)} \int_{0}^{\infty} E_{\varepsilon}\left(-w^{\varepsilon}\mu^{\varepsilon}\right) \frac{\varphi(\mu)}{w^{\varepsilon}} (d\mu)^{\varepsilon}. \tag{3}$$

Following (4), its inverse formula is defined by

$$ST_{\varepsilon}^{-1}(ST_{\varepsilon}\{\varphi(\mu)\}) = \varphi(\mu), \quad 0 < \varepsilon \leq 1. \tag{4}$$

The properties for LFST are:

1. $ST_{\varepsilon}\left\{\frac{\mu^{\varepsilon}}{\Gamma(1+\varepsilon)}\right\} = w^{\varepsilon}$.
2. $ST_{\varepsilon}\left\{\frac{\partial^{m\varepsilon}\varphi(\mu,\tau)}{\partial\mu^{m\varepsilon}}\right\} = \frac{1}{w^{m\varepsilon}}\left[ST_{\varepsilon}\{\varphi(\mu,\tau)\} - \sum_{k=0}^{m-1} w^{k\varepsilon} \frac{\partial^{k\varepsilon}\varphi(0,\tau)}{\partial\tau^{k\varepsilon}}\right]$.

3 Analysis of LFSDM

Let us consider the PDE with LFDOs:

$$L_\varepsilon\varphi(\mu,\tau)+R_\varepsilon\varphi(\mu,\tau)=g(\mu,\tau),\qquad 0<\varepsilon\le 1 \tag{5}$$

where $L_\varepsilon\varphi(\mu,\tau)=\frac{\partial^{m\varepsilon}}{\partial\mu^{m\varepsilon}}\varphi(\mu,\tau)$, R_ε denotes linear LFDO, and $g(\mu,\tau)$ is the non-differentiable source term.

Applying the LFST on Eq. (5), and using the property of the LFST, we get

$$ST_\varepsilon\{\varphi(\mu,\tau)\}=\sum_{k=0}^{m-1}w^{k\varepsilon}\frac{\partial^{k\varepsilon}\varphi(0,\tau)}{\partial\tau^{k\varepsilon}}+w^{m\varepsilon}ST_\varepsilon\{g(\mu,\tau)\}-w^{m\varepsilon}ST_\varepsilon\{R_\varepsilon\varphi(\mu,\tau)\}. \tag{6}$$

Taking the inverse of LFST on Eq. (6), we have

$$\begin{aligned}\varphi(\mu,\tau)=&\sum_{k=0}^{m-1}w^{k\varepsilon}\frac{\partial^{k\varepsilon}\varphi(0,\tau)}{\partial\tau^{k\varepsilon}}\frac{\mu^{k\varepsilon}}{\Gamma(1+k\varepsilon)}+ST_\varepsilon^{-1}\big[w^{m\varepsilon}ST_\varepsilon\{g(\mu,\tau)\}\big]\\&-ST_\varepsilon^{-1}\big[w^{m\varepsilon}ST_\varepsilon\{R_\varepsilon\varphi(\mu,\tau)\}\big].\end{aligned} \tag{7}$$

Now, procedure of ADM suggests the decomposition of the unknown function $\varphi(\mu,\tau)$ as an infinite series in the following way

$$\varphi(\mu,\tau)=\sum_{n=0}^{\infty}\varphi_n(\mu,\tau). \tag{8}$$

By making use of the Eq. (8) in Eq. (7), it yields the following result:

$$\begin{aligned}\sum_{n=0}^{\infty}\varphi_n(\mu,\tau)=&\sum_{k=0}^{m-1}w^{k\varepsilon}\frac{\partial^{k\varepsilon}\varphi(0,\tau)}{\partial\tau^{k\varepsilon}}\frac{\mu^{k\varepsilon}}{\Gamma(1+k\varepsilon)}+ST_\varepsilon^{-1}\big[w^{m\varepsilon}ST_\varepsilon\{g(\mu,\tau)\}\big]\\&-ST_\varepsilon^{-1}\left[w^{m\varepsilon}ST_\varepsilon\left\{R_\varepsilon\sum_{n=0}^{\infty}\varphi_n(\mu,\tau)\right\}\right].\end{aligned} \tag{9}$$

Matching both sides of (9) provides

$$\begin{aligned}\varphi_0(\mu,\tau)&=\sum_{k=0}^{m-1}w^{k\varepsilon}\frac{\partial^{k\varepsilon}\varphi(0,\tau)}{\partial\tau^{k\varepsilon}}\frac{\mu^{k\varepsilon}}{\Gamma(1+k\varepsilon)}+ST_\varepsilon^{-1}\big[w^{m\varepsilon}ST_\varepsilon\{g(\mu,\tau)\}\big]\\\varphi_1(\mu,\tau)&=-ST_\varepsilon^{-1}\big[w^{m\varepsilon}ST_\varepsilon\{R_\varepsilon[\varphi_0(\mu,\tau)]\}\big],\\\varphi_2(\mu,\tau)&=-ST_\varepsilon^{-1}\big[w^{m\varepsilon}ST_\varepsilon\{R_\varepsilon[\varphi_1(\mu,\tau)]\}\big],\\\varphi_3(\mu,\tau)&=-ST_\varepsilon^{-1}\big[w^{m\varepsilon}ST_\varepsilon\{R_\varepsilon[\varphi_2(\mu,\tau)]\}\big],\\&\ \vdots\end{aligned} \tag{10}$$

The general form of above obtained local fractional recursive relations is

$$\begin{aligned}\varphi_0(\mu,\tau)&=\sum_{k=0}^{m-1}w^{k\varepsilon}\tfrac{\partial^{k\varepsilon}\varphi(0,\tau)}{\partial\tau^{k\varepsilon}}\tfrac{\mu^{k\varepsilon}}{\Gamma(1+k\varepsilon)}+ST_\varepsilon^{-1}\big[w^{m\varepsilon}ST_\varepsilon\{g(\mu,\tau)\}\big],\\\varphi_n(\mu,\tau)&=-ST_\varepsilon^{-1}\big[w^{m\varepsilon}ST_\varepsilon\big\{R_\varepsilon\big[\varphi_{n-1}(\mu,\tau)\big]\big\}\big],\quad n\ge 1,0<\varepsilon\le 1\end{aligned} \tag{11}$$

4 Application of LFSDM

Example 4.1. Consider the Helmholtz equation with LFDO:

$$\frac{\partial^{2\varepsilon}\varphi(\mu,\tau)}{\partial\mu^{2\varepsilon}}+\frac{\partial^{2\varepsilon}\varphi(\mu,\tau)}{\partial\tau^{2\varepsilon}}+\varphi(\mu,\tau)=\frac{\mu^{\varepsilon}}{\Gamma(1+\varepsilon)}\frac{\tau^{\varepsilon}}{\Gamma(1+\varepsilon)}, \tag{12}$$

with

$$\varphi(0,\tau)=0, \quad \varphi^{(\varepsilon)}(0,\tau)=\frac{\tau^{\varepsilon}}{\Gamma(1+\varepsilon)}. \tag{13}$$

Taking LFST of (12), we get

$$\begin{aligned}ST_{\varepsilon}\{\varphi(\mu,\tau)\}&=\sum_{k=0}^{1}w^{k\varepsilon}\frac{\partial^{k\varepsilon}\varphi(0,\tau)}{\partial\mu^{k\varepsilon}}+w^{2\varepsilon}ST_{\varepsilon}\left\{\frac{\mu^{\varepsilon}}{\Gamma(1+\varepsilon)}\frac{\tau^{\varepsilon}}{\Gamma(1+\varepsilon)}\right\}\\&\quad-w^{2\varepsilon}ST_{\varepsilon}\left\{\frac{\partial^{2\varepsilon}\varphi(\mu,\tau)}{\partial\tau^{2\varepsilon}}+\varphi(\mu,\tau)\right\}\\&=w^{\varepsilon}\frac{\tau^{\varepsilon}}{\Gamma(1+\varepsilon)}+w^{3\varepsilon}\frac{\tau^{\varepsilon}}{\Gamma(1+\varepsilon)}-w^{2\varepsilon}ST_{\varepsilon}\left\{\frac{\partial^{2\varepsilon}\varphi(\mu,\tau)}{\partial\tau^{2\varepsilon}}+\varphi(\mu,\tau)\right\}.\end{aligned}$$

The inversion of LFST implies that

$$\begin{aligned}\varphi(\mu,\tau)&=\frac{\mu^{\varepsilon}}{\Gamma(1+\varepsilon)}\frac{\tau^{\varepsilon}}{\Gamma(1+\varepsilon)}+\frac{\mu^{3\varepsilon}}{\Gamma(1+3\varepsilon)}\frac{\tau^{\varepsilon}}{\Gamma(1+\varepsilon)}\\&\quad-ST_{\varepsilon}^{-1}\left[w^{2\varepsilon}ST_{\varepsilon}\left\{\frac{\partial^{2\varepsilon}\varphi(\mu,\tau)}{\partial\tau^{2\varepsilon}}+\varphi(\mu,\tau)\right\}\right].\end{aligned} \tag{14}$$

Now, procedure of ADM suggests the decomposition of the unknown function $\varphi(\mu,\tau)$ as an infinite series in the following way

$$\varphi(\mu,\tau)=\sum_{n=0}^{\infty}\varphi_n(\mu,\tau). \tag{15}$$

Substituting (15) in (14), it yields the following result:

$$\begin{aligned}\sum_{n=0}^{\infty}\varphi_n(\mu,\tau)&=\frac{\mu^{\varepsilon}}{\Gamma(1+\varepsilon)}\frac{\tau^{\varepsilon}}{\Gamma(1+\varepsilon)}+\frac{\mu^{3\varepsilon}}{\Gamma(1+3\varepsilon)}\frac{\tau^{\varepsilon}}{\Gamma(1+\varepsilon)}\\&\quad-ST_{\varepsilon}^{-1}\left[w^{2\varepsilon}ST_{\varepsilon}\left\{\frac{\partial^{2\varepsilon}}{\partial\tau^{2\varepsilon}}\sum_{n=0}^{\infty}\varphi_n(\mu,\tau)+\sum_{n=0}^{\infty}\varphi_n(\mu,\tau)\right\}\right].\end{aligned} \tag{16}$$

On comparing both sides of (16), we have:

$$\varphi_0(\mu,\tau) = \frac{\mu^{\varepsilon}}{\Gamma(1+\varepsilon\ )}\frac{\tau^{\varepsilon}}{\Gamma(\ 1+\varepsilon)} + \frac{\mu^{3\varepsilon}}{\Gamma(1+3\varepsilon\ )}\frac{\tau^{\varepsilon}}{\Gamma(1+\varepsilon\ )},$$

$$\varphi_1(\mu,\tau) = -ST_{\varepsilon}^{-1}\left[w^{2\varepsilon}\ ST_{\varepsilon}\left\{\frac{\partial^{2\varepsilon}\varphi_0(\mu,\tau)}{\partial\tau^{2\varepsilon}} + \varphi_0(\mu,\tau)\right\}\right]$$

$$= -\frac{\mu^{3\varepsilon}}{\Gamma(1+3\varepsilon\ )}\frac{\tau^{\varepsilon}}{\Gamma(\ 1+\varepsilon\ )} - \frac{\mu^{5\varepsilon}}{\Gamma(1+5\varepsilon)}\frac{\tau^{\varepsilon}}{\Gamma(\ 1+\varepsilon)}$$

$$\varphi_2(\mu,\tau) = -ST_{\varepsilon}^{-1}\left[w^{2\varepsilon}\ ST_{\varepsilon}\left\{\frac{\partial^{2\varepsilon}\varphi_1(\mu,\tau)}{\partial\tau^{2\varepsilon}} + \varphi_1(\mu,\tau)\right\}\right]$$

$$= \frac{\mu^{5\varepsilon}}{\Gamma(1+5\varepsilon\ )}\frac{\tau^{\varepsilon}}{\Gamma(1+\varepsilon\ )} + \frac{\mu^{7\varepsilon}}{\Gamma(1+7\varepsilon\ )}\frac{\tau^{\varepsilon}}{\Gamma(1+\varepsilon\ )}$$

$$\varphi_3(\mu,\tau) = -ST_{\varepsilon}^{-1}\left[w^{2\varepsilon}\ ST_{\varepsilon}\left\{\frac{\partial^{2\varepsilon}\varphi_2(\mu,\tau)}{\partial\tau^{2\varepsilon}} + \varphi_2(\mu,\tau)\right\}\right]$$

$$= -\frac{\mu^{7\varepsilon}}{\Gamma(1+7\varepsilon\ )}\frac{\tau^{\varepsilon}}{\Gamma(1+\varepsilon\ )} - \frac{\mu^{9\varepsilon}}{\Gamma(1+9\varepsilon\ )}\frac{\tau^{\varepsilon}}{\Gamma(1+\varepsilon\ )}.$$

$$\vdots$$

Therefore, the approximate solution $\varphi(\mu, \tau)$ of Eq. (12) is expressed by

$$\varphi(\mu,\tau) = \sum_{n=0}^{\infty}\varphi_n(\mu,\tau) = \frac{\mu^{\varepsilon}}{\Gamma(1+\varepsilon)}\frac{\tau^{\varepsilon}}{\Gamma(1+\varepsilon)}. \tag{17}$$

The result is the same as the one which is obtained by the LFLADM [12] and LFLHPM [31].

Example 4.2. Now we examine the coupled Helmholtz equations with LFDOs:

$$\begin{aligned}\frac{\partial^{2\varepsilon}\varphi(\mu,\tau)}{\partial\mu^{2\varepsilon}} + \frac{\partial^{2\varepsilon}\psi(\mu,\tau)}{\partial\tau^{2\varepsilon}} - \varphi(\mu,\tau) = 0,\\ \frac{\partial^{2\delta}\psi(\mu,\tau)}{\partial\mu^{2\delta}} + \frac{\partial^{2\delta}\varphi(\mu,\tau)}{\partial\tau^{2\delta}} - \psi(\mu,\tau) = 0,\end{aligned} \tag{18}$$

with

$$\begin{aligned}&\varphi(0,\tau) = 0,\ \varphi^{(\varepsilon)}(0,\tau) = E_{\varepsilon}\left(\tau^{\varepsilon}\right),\\ &\psi(0,\tau) = 0, i\psi^{(\varepsilon)}(0,\tau) = E_{\varepsilon}\left(\tau^{\varepsilon}\right).\end{aligned} \tag{19}$$

Taking LFLT of (18), we obtain

$$ST_\varepsilon\{\varphi(\mu,\tau)\} = w^\varepsilon E_\varepsilon\left(\tau^\varepsilon\right) + w^{2\varepsilon} ST_\varepsilon\left\{\varphi(\mu,\tau) - \frac{\partial^{2\varepsilon}\psi(\mu,\tau)}{\partial\tau^{2\varepsilon}}\right\},$$
$$ST_\varepsilon\{\Psi(\mu,\tau)\} = -w^\varepsilon E_\varepsilon\left(\tau^\varepsilon\right) + w^{2\varepsilon} ST_\varepsilon\left\{\psi(\mu,\tau) - \frac{\partial^{2\varepsilon}\varphi(\mu,\tau)}{\partial\tau^{2\varepsilon}}\right\}, \tag{20}$$

The inversion of LFST implies that

$$\varphi(\mu,\tau) = \frac{\mu^\varepsilon}{\Gamma(1+\varepsilon)} E_\varepsilon\left(\tau^\varepsilon\right) + ST_\varepsilon^{-1}\left[w^{2\varepsilon} ST_\varepsilon\left\{\varphi(\mu,\tau) - \frac{\partial^{2\varepsilon}\psi(\mu,\tau)}{\partial\tau^{2\varepsilon}}\right\}\right],$$
$$\psi(\mu,\tau) = -\frac{\mu^\varepsilon}{\Gamma(1+\varepsilon)} E_\varepsilon\left(\tau^\varepsilon\right) + ST_\varepsilon^{-1}\left[w^{2\varepsilon} ST_\varepsilon\left\{\psi(\mu,\tau) - \frac{\partial^{2\varepsilon}\varphi(\mu,\tau)}{\partial\tau^{2\varepsilon}}\right\}\right]. \tag{21}$$

Now, we compose the unknown functions $\varphi(\mu,\tau)$ and $\psi(\mu,\tau)$ in the form of infinite series as

$$\varphi(\mu,\tau) = \sum_{n=0}^{\infty} \varphi_n(\mu,\tau),$$
$$\psi(\mu,\tau) = \sum_{n=0}^{\infty} \psi_n(\mu,\tau). \tag{22}$$

On making use of (22) in (21), it yields the following result:

$$\sum_{n=0}^{\infty} \varphi_n(\mu,\tau) = \frac{\mu^\varepsilon}{\Gamma(1+\varepsilon)} E_\varepsilon\left(\tau^\varepsilon\right)$$
$$+ ST_\varepsilon^{-1}\left[w^{2\delta} ST_\varepsilon\left\{\sum_{n=0}^{\infty} \varphi_n(\mu,\tau) - \frac{\partial^{2\varepsilon}}{\partial\tau^{2\varepsilon}}\left(\sum_{n=0}^{\infty} \psi_n(\mu,\tau)\right)\right\}\right],$$
$$\sum_{n=0}^{\infty} \psi_n(\mu,\tau) = -\frac{\mu^\varepsilon}{\Gamma(1+\varepsilon)} E_\varepsilon\left(\tau^\varepsilon\right)$$
$$+ ST_\varepsilon^{-1}\left[w^{2\varepsilon} LT_\varepsilon\left\{\sum_{n=0}^{\infty} \psi_n(\mu,\tau) - \frac{\partial^{2\varepsilon}}{\partial\tau^{2\varepsilon}}\left(\sum_{n=0}^{\infty} \varphi_n(\mu,\tau)\right)\right\}\right]. \tag{23}$$

Now, comparison of both sides of (23) yields

$$\varphi_0(\mu,\tau) = \frac{\mu^{\varepsilon}}{\Gamma(1+\varepsilon)} E_{\varepsilon}(\tau^{\varepsilon}),$$
$$\psi_0(\mu,\tau) = -\frac{\mu^{\varepsilon}}{\Gamma(1+\varepsilon)} E_{\varepsilon}(\tau^{\varepsilon}),$$

$$\varphi_1(\mu,\tau) = ST_{\varepsilon}^{-1}\left[w^{2\varepsilon} ST_{\varepsilon}\left\{\varphi_0(\mu,\tau) - \frac{\partial^{2\varepsilon}}{\partial\tau^{2\varepsilon}}\psi_0(\mu,\tau)\right\}\right]$$
$$\psi_1(\mu,\tau) = ST_{\varepsilon}^{-1}\left[w^{2\delta} ST_{\varepsilon}\left\{\psi_0(\mu,\tau) - \frac{\partial^{2\varepsilon}}{\partial\tau^{2\varepsilon}}\varphi_0(\mu,\tau)\right\}\right]$$

$$= \frac{2\mu^{3\varepsilon}}{\Gamma(1+3\varepsilon)} E_{\varepsilon}(\tau^{\varepsilon}),$$
$$= -\frac{2\mu^{3\varepsilon}}{\Gamma(1+3\varepsilon)} E_{\varepsilon}(\tau^{\varepsilon}),$$

$$\varphi_2(\mu,\tau) = ST_{\varepsilon}^{-1}\left[w^{2\varepsilon} ST_{\varepsilon}\left\{\varphi_1(\mu,\tau) - \frac{\partial^{2\varepsilon}}{\partial\tau^{2\varepsilon}}\psi_1(\mu,\tau)\right\}\right]$$
$$\psi_2(\mu,\tau) = ST_{\varepsilon}^{-1}\left[w^{2\varepsilon} ST_{\varepsilon}\left\{\psi_1(\mu,\tau) - \frac{\partial^{2\varepsilon}}{\partial\tau^{2\varepsilon}}\varphi_1(\mu,\tau)\right\}\right]$$

$$= \frac{4\mu^{5\varepsilon}}{\Gamma(1+5\varepsilon)} E_{\varepsilon}(\tau^{\varepsilon}),$$
$$= -\frac{4\mu^{5\varepsilon}}{\Gamma(1+5\varepsilon)} E_{\varepsilon}(\tau^{\varepsilon}),$$

$$\varphi_3(\mu,\tau) = ST_{\varepsilon}^{-1}\left[w^{2\varepsilon} ST_{\varepsilon}\left\{\varphi_2(\mu,\tau) - \frac{\partial^{2\varepsilon}}{\partial\tau^{2\varepsilon}}\psi_2(\mu,\tau)\right\}\right]$$
$$\psi_3(\mu,\tau) = ST_{\varepsilon}^{-1}\left[w^{2\varepsilon} ST_{\varepsilon}\left\{\psi_2(\mu,\tau) - \frac{\partial^{2\varepsilon}}{\partial\tau^{2\varepsilon}}\varphi_2(\mu,\tau)\right\}\right]$$

$$= \frac{8\mu^{7\varepsilon}}{\Gamma(1+7\varepsilon)} E_{\varepsilon}(\tau^{\varepsilon}),$$
$$= -\frac{8\mu^{7\varepsilon}}{\Gamma(1+7\varepsilon)} E_{\varepsilon}(\tau^{\varepsilon}),$$

$$\vdots$$

Hence, the solutions are expressed as

$$\varphi(\mu,\tau)=\sum_{n=0}^{\infty}\varphi_n(\mu,\tau)=E_\varepsilon\left(\tau^\varepsilon\right)\frac{sinh_\varepsilon\left(\sqrt{2}\mu^\varepsilon\right)}{\sqrt{2}}.$$

$$\psi(\mu,\tau)=\sum_{n=0}^{\infty}\psi_n(\mu,\tau)=-E_\varepsilon\left(\tau^\varepsilon\right)\frac{sinh_\varepsilon\left(\sqrt{2}\mu^\varepsilon\right)}{\sqrt{2}}. \tag{24}$$

The result (24) is the same as the one which is obtained by the LFADM [12] and LFHPM [31].

5 Numerical Results and Discussion

In this segment, the numerical simulations for solution of Helmholtz and coupled Helmholtz equations obtained via LFSDM are presented. The numerical investigation of Helmholtz and coupled Helmholtz equations considers different values of $\varepsilon = 1,\ \frac{\log 2}{\log 3}$. Here, the Matlab software is utilized to draw all the 3D plots. Figures 1 & 2 show the 3D surface plot for solution $\varphi(\mu,\tau)$ for Example 1 for $\varepsilon = 1.0$ and $\varepsilon = \frac{\log 2}{\log 3}$, respectively. Figure 2 represents the variation of $\varphi(\mu,\tau)$ in fractal dimension. Similarly, the 3D surface plots for solution $\varphi(\mu,\tau)$ for Example 2 are depicted in Figs. 3 and 4 for $\varepsilon = 1.0$ and $\varepsilon = \frac{\log 2}{\log 3}$, respectively. Figures 5 & 6 represent the 3D variation for solution

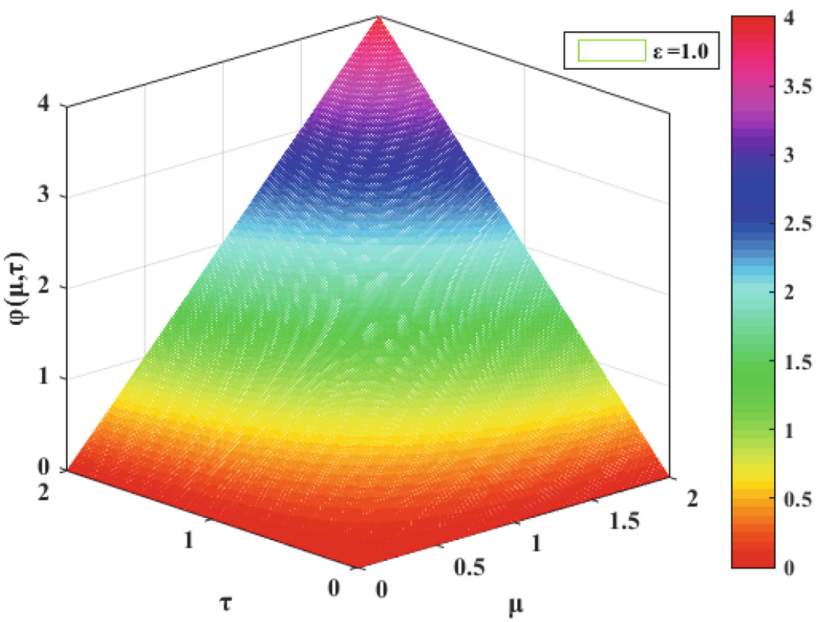

Fig. 1. 3D behaviour of $\varphi(\mu,\tau)$ for Example 1 w.r.t. μ and τ for $\varepsilon = 1.0$

$\psi(\mu,\ \tau)$ for Example 2 for $\varepsilon = 1,\ \frac{\log 2}{\log 3}$, respectively. Figure 6 represents the variation of $\varphi(\mu, \tau)$ on Cantor set.

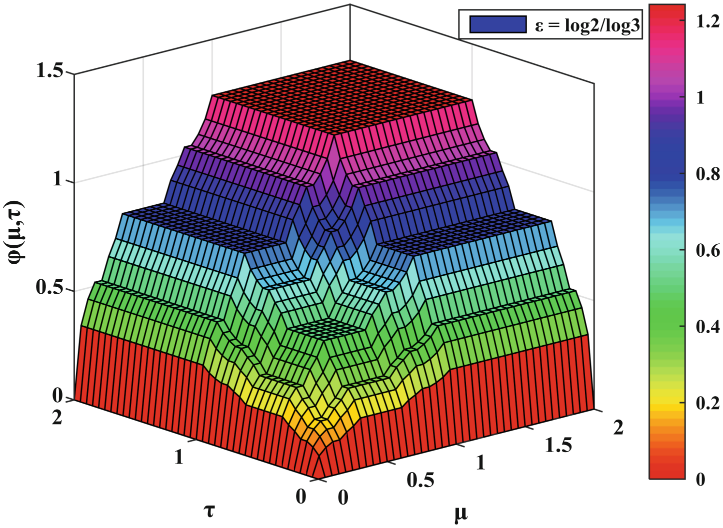

Fig. 2. 3D behaviour of $\varphi(\mu, \tau)$ for Example 1 w.r.t. μ and τ for $\varepsilon = \frac{\log 2}{\log 3}$

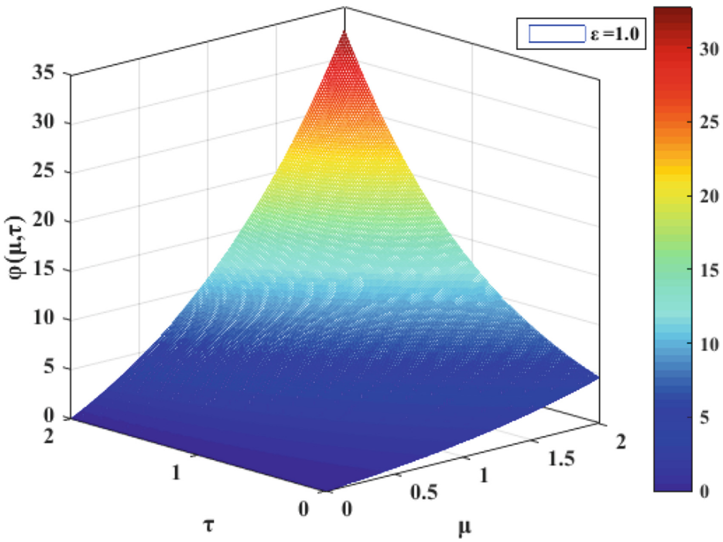

Fig. 3. 3D variation of $\varphi(\mu, \tau)$ for Example 2 w.r.t. μ and τ for $\varepsilon = 1.0$

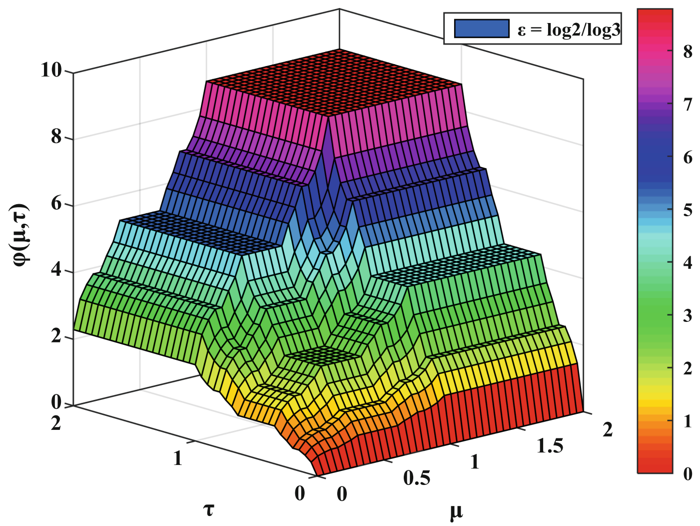

Fig. 4. 3D variation of $\varphi(\mu, \tau)$ for Example 2 w.r.t. μ and τ for $\varepsilon = \frac{\log 2}{\log 3}$

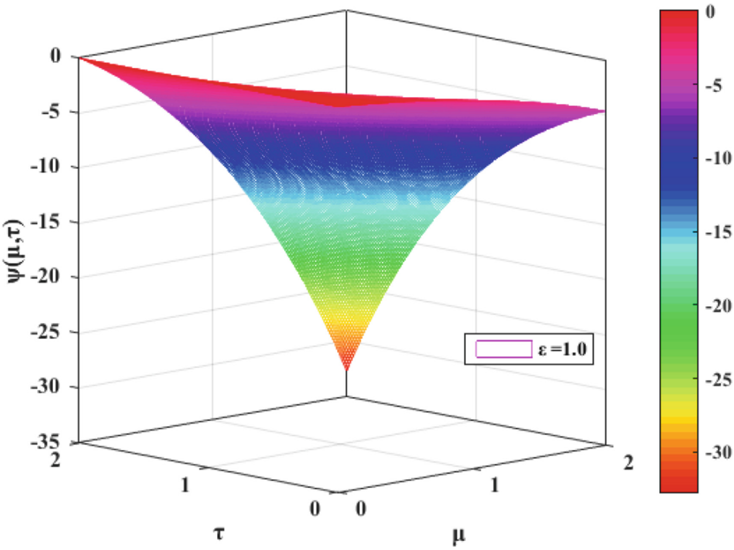

Fig. 5. 3D behaviour of $\psi(\mu, \tau)$ for Example 2 w.r.t. μ and τ for $\varepsilon = 1.0$

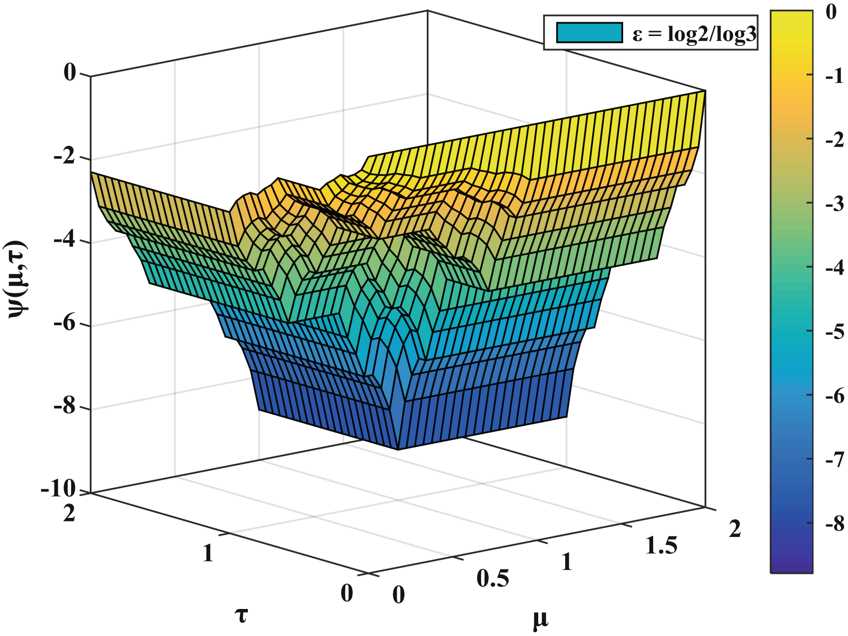

Fig. 6. 3D behaviour of $\psi(\mu,\ \tau)$ for Example 2 w.r.t. μ and τ for $\varepsilon = \frac{\log 2}{\log 3}$

6 Conclusions

In this work, the LFSDM is conveniently employed to obtain the approximate solution of Helmholtz and coupled Helmholtz equations within LFDOs. The proposed algorithm provides a solution in a series form that converges rapidly to an exact solution if it exists. From the obtained results, it is clear that the FSDM yields very accurate solutions using only a few iterates. The method is very powerful and efficient in finding semi-analytical solutions for wide classes of LFPDEs.

References

1. Jafari, H., Jassim, H.K., Vahidi, J.: Reduced differential transform and variational iteration methods for 3D diffusion model in fractal heat transfer within local fractional operators. Therm. Sci. **22**, S301–S307 (2018)
2. Baleanu, D., Jassim, H.K.: A modification fractional variational iteration method for solving nonlinear gas dynamic and coupled KdV equations involving local fractional operators. Therm. Sci. **22**, S165–S175 (2018)
3. Jafari, H., Jassim, H.K.: Application of the local fractional adomian decomposition and series expansion methods for solving telegraph equation on cantor sets involving local fractional derivative operators. J. ZankoySulaimani-Part A **17**, 15–22 (2015)
4. Jassim, H.K., Ünlü, C., Moshokoa, S.P., KhaliqueC.M.: Local fractional Laplace variational iteration method for solving diffusion and wave equations on cantor sets within local fractional operators. Math. Probl. Eng. **2015**, 1–9 (2015)

5. Baleanu, D., Jassim, H.K.: Approximate solutions of the damped wave equation and dissipative wave equation in fractal strings. Fractal Fract. **3**(26), 1–12 (2019)
6. Jassim, H.K.: The approximate solutions of three-dimensional diffusion and wave equations within local fractional derivative operator. Abstr. Appl. Anal. **2016**, 1–5 (2016)
7. Wang, S.Q., Yang, Y.J., Jassim, H.K.: Local fractional function decomposition method for solving inhomogeneous wave equations with local fractional derivative. Abstr. Appl. Anal. **2014**, 1–7 (2014)
8. Yang, X.J., et al.: Local fractional variational iteration method for Fokker-Planck equation on a Cantor set. Acta Universitaria **23**, 3–8 (2013)
9. Jassim, H.K.: New approaches for solving fokker planck equation on cantor sets within local fractional operators. J. Math. **2015**, 1–8 (2015)
10. Yan, S.P., Jafari, H., Jassim, H.K.: Local fractional Adomian decomposition and function decomposition methods for solving Laplace equation within local fractional operators. Adv. Math. Phys. **2014**, 1–7 (2014)
11. Yang, A.M., et al.: Application of local fractional series expansion method to solve Klein-Gordon equations on Cantor sets. Abstr. Appl. Anal. **2014**, 1–6 (2014)
12. Jassim, H.K.: The approximate solutions of Helmholtz and coupled Helmholtz equations on cantor sets within local fractional operator. J. Zankoy Sulaimani-Part A **17**, 19–25 (2015)
13. Jassim, H.K.: A new approach to find approximate solutions of Burger's and coupled Burger's equations of fractional order. TWMS J. Appl. Eng. Math. **11**(2), 415–423 (2021)
14. Baleanu, D., Jassim, H.K.: Approximate analytical solutions of goursat problem within local fractional operators. J. Nonlinear Sci. Appl. **9**, 4829–4837 (2016)
15. Jafari, H., et al.: Local fractional adomian decomposition method for solving two dimensional heat conduction equations within local fractional operators. J. Adv. Math. **9**, 2574–2582 (2014)
16. Jassim, H.K., Khafif, S.A.: SVIM for solving Burger's and coupled Burger's equations of fractional order. Prog. Fract. Differ. Appl. **7**(1), 1–6 (2021)
17. Jafari, H., et al.: Local Fractional variational iteration method for nonlinear partial differential equations within local fractional operators. Appl. Appl. Math. **10**, 1055–1065 (2015)
18. Xu, S., Ling, X., Zhao, Y., Jassim, H.K.: A novel schedule for solving the two-dimensional diffusion in fractal heat transfer. Therm. Sci. **19**, S99–S103 (2015)
19. Yang, X.J.: Advanced Local Fractional Calculus and Its Applications. World Science Publisher, New York (2012)
20. Yang, X.J.: Local Fractional Functional Analysis and Its Applications. Asian Academic Publisher Limited, Hong Kong (2011)
21. Jafari, H, Jassim, H.K., Vahidi V.: Reduced differential transform and variational iteration methods for 3D diffusion model in fractal heat transfer within local fractional operators. Therm. Sci. **22**, S301–S307 (2018)
22. Jassim, H.K., Shahab, W.A.: Fractional variational iteration method to solve one dimensional second order hyperbolic telegraph equations. J. Phys. Conf. Ser. **1032**(1), 1–9 (2018)
23. Yang, X.J., Machad, J.A., Srivastava, H.M.: A new numerical technique for solving the local fractional diffusion equation: two-dimensional extended differential transform approach. Appl. Math. Comput. **274**, 143–151 (2016)
24. Jafari, H., Jassim, H.K., Tchier, F., Baleanu, D.: On the approximate solutions of local fractional differential equations with local fractional operator. Entropy **18**, 1–12 (2016)
25. Jassim, H.K., Baleanu, D.: A novel approach for Korteweg-de Vries equation of fractional order. J. Appl. Comput. Mech. **5**(2), 192–198 (2019)
26. Singh, J., Jassim, H.K., Kumar, D.: An efficient computational technique for local fractional Fokker-Planck equation. Phys. A **555**(124525), 1–8 (2020)
27. Baleanu, D., Jassim, H.K.: Exact solution of two-dimensional fractional partial differential equations. Fractal Fract. **4**(21), 1–9 (2020)

28. Jafari, H., Jassim, H.K., Moshokoa, S.P., Ariyan, V.M., Tchier, F.: Reduced differential transform method for partial differential equations within local fractional derivative operators. Adv. Mech. Eng. **8**, 1–6 (2016)
29. Baleanu, D., Jassim, H.K., Al Qurashi, M.: Solving Helmholtz equation with local fractional derivative operators. Fractal Fract. **3**(43), 1–13 (2019)
30. Jassim, H.K.: Analytical approximate solutions for local fractional wave equations. Math. Methods Appl. Sci. **43**(2), 939–947 (2020)
31. Baleanu, D., Jassim, H.K.: A modification fractional homotopy perturbation method for solving Helmholtz and coupled Helmholtz equations on cantor sets. Fractal Fract. **3**(30), 1–8 (2019)
32. Singh, J., et al.: On the local fractional wave equation in fractal strings. Math. Method Appl. Sci. **2019**, 1–8 (2019)