On a Class of New *q*-Hypergeometric Expansions as Discrete Analogues of the Erdélyi Type *q*-Integrals

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Abstract. Recently, Vyas et al. have developed an alternative way of proof for the Gasper's discrete analogue of an Erdélyi integral and inspired from this new type of derivation they resolved the problem of finding the discrete extensions of all the Erdélyi type integrals in the form of several new hypergeometric expansions for certain ${}_{q+1}F_q$. Motivated from the above-mentioned work, here in this paper, our objective is to resolve the problem of finding the discrete extensions of the Erdélyi type q-integrals in the form of several new q-hypergeometric expansions for certain ${}_{r+1}\Phi_r$. The motivation behind this work is the fact that the q-series and basic q-polynomials, specifically the q-gamma and basic q- hypergeometric functions and basic q-hypergeometric polynomials, are applicable particularly in several diverse areas of science and engineering, viz. Statistics, number theory, combinatorial analysis, nonlinear electric circuit theory, combinatorial generating functions, quantum mechanics, mechanical engineering, lie theory, theory of heat conduction, particle physics and cosmology.

Keywords: Erdélyi integrals · Erdélyi type q-integrals · Basic q-hypergeometric functions · classical summation theorem · Basic q- hypergeometric expansions

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1 Introduction

Heine [7, 8] first developed the idea of basic analogue or q–analogue of the Gauss hypergeometric function ${}_2F_1$ as an infinite series.

The q–shifted factorials are described in the literature for arbitrary (real or complex) q, a and $|q| < 1$ as:

$$(\alpha; q)_n = \begin{cases} 1, & n = 0 \\ (1-\alpha)(1-\alpha q)(1-\alpha q^2)\ldots(1-\alpha q^{n-1}), & n \in \mathbb{N} \end{cases}. \tag{1.1}$$

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Using this notation, we can write the Heine's series analogous to the notation for Gauss' series as:

$$ {}_2\Phi_1(\alpha,\ \beta\ ;\ \gamma\ ;\ q, z)\ \equiv\ {}_2\Phi_1\left[\begin{matrix}\alpha,\ \beta; \\ \gamma;\end{matrix}\ q,\ z\right] = \sum_{k=0}^{\infty} \frac{(\alpha; q)_k(\beta; q)_k z^k}{(\gamma; q)_k(q; q)_k}. \tag{1.2} $$

A generalized q-hypergeometric series [24, p. 347], see also [2] and [5] with v numerator parameters $\alpha_1, \alpha_2, \cdots, \alpha_v$ and u denominator parameters $\beta_1, \beta_2, \cdots, \beta_u$ is defined by

$$ {}_v\Phi_u(\alpha_1, \cdots, \alpha_v; \beta_1, \cdots, \beta_u; q, z) \equiv $$

$$ {}_v\Phi_u\left[\begin{matrix}\alpha_1, \cdots, \alpha_v; \\ \beta_1, \cdots, \beta_u;\end{matrix}\ q,\ z\right] = \sum_{k=0}^{\infty} \frac{(\alpha_1; q)_k \cdots (\alpha_v; q)_k z^k}{(\beta_1; q)_k \cdots (\beta_u; q)_k (q; q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+u-v} \tag{1.3} $$

where $\binom{k}{2} = k(k-1)/2\,.$

For the convergence conditions of the above hypergeometric series, please see [5]. A complete list of important properties and formulas for q-shifted factorial, to be used frequently while deriving the q-hypergeometric expansion of Sect. 3, can be found in [5, Appendix 1, pp. 351–352].

The Euler's integral representation of Gauss hypergeometric function is given in [19, p. 47, Theorem 16] and its Thomae's q-analogue is mentioned in [5].

In 1939, Erdélyi [1] used fractional calculus method to develop three integrals [4, Eqs. (1.3)–(1.5)], known as "Erdélyi integrals" in the literature, which extend Euler's integral for ${}_2F_1(z)$ [19, p. 47, Theorem 16] and Bateman's integral [4, Eq. (1.2)]. Gasper [3] derived the discrete extension of one of the Erdélyi integrals [4, Eq. (1.6)] as stated below [6, Eq. (26)]:

$$ {}_3F_2\left(\begin{matrix}\alpha,\ \beta,\ -n; \\ \gamma,\ \delta;\end{matrix}\ 1\right) = \sum_{k=0}^{n} \binom{n}{k} \frac{(\mu)_k(\lambda+\delta-\alpha-\beta)_k(\gamma-\mu)_{n-k}}{(\gamma)_n(\delta)_k} $$
$$ \times\ {}_3F_2\left(\begin{matrix}\lambda-\alpha,\ \lambda-\beta,\ -k; \\ \mu,\ \lambda+\delta-\alpha-\beta;\end{matrix}\ 1\right) {}_3F_2\left(\begin{matrix}\alpha+\beta-\lambda,\ \lambda-\mu,\ k-n; \\ \gamma-\mu,\ \delta+k;\end{matrix}\ 1\right). \tag{1.4} $$

Gasper [3] proved (1.4) by following the steps analogous to Erdélyi's fractional calculus proof of [4, Eq. (1.3)].

Later, Gasper [4] motivated from the proof of the above-mentioned discrete analogue, developed three expansions identities for the terminating balanced ${}_4\Phi_3$ series and obtained the q-analogues of Erdélyi's integrals [4, Eqs. (1.8), (1.9), (1.13) and (1.14)] and corresponding discrete analogues and discrete q-extensions, see [3, Eqs. (1.6), (1.7), (2.9), (3.4), (1.11) and (1.12)].

The discrete extensions of the q-analogue of Erdélyi's integrals [4, Eq. (1.8)] or q-analogue of (1.4), developed by [4] is as given below:

$$
{}_3\Phi_2\left(\begin{matrix}\alpha,\ \beta,\ q^{-n};\\ \gamma,\ \delta;\end{matrix}\ q,\ q\right) = \sum_{k=0}^{n} \binom{n}{k}_q \frac{(\mu;q)_k \left(\frac{\lambda\delta}{\alpha\beta};q\right)_k \left(\frac{\gamma}{\mu};q\right)_{n-k} \mu^n \left(\frac{\alpha\beta}{\lambda\mu}\right)^k}{(\gamma;q)_n(\delta;q)_k}
$$
$$
\times\ {}_3\Phi_2\left(\begin{matrix}\frac{\lambda}{\alpha},\ \frac{\lambda}{\beta},\ q^{-k};\\ \mu,\ \frac{\lambda\delta}{\alpha\beta};\end{matrix}\ q,\ q\right) {}_3\Phi_2\left(\begin{matrix}\frac{\alpha\beta}{\lambda},\ \frac{\lambda}{\mu},\ q^{k-n};\\ \frac{\gamma}{\mu},\ \delta q^k;\end{matrix}\ q,\ q\right). \quad (1.5)
$$

The two expansions [4, Eqs. (2.8) and (2.9)] were further applied to obtain expansion formulas for the orthogonal polynomials like Racah polynomials, Askey-Wilson polynomials and their q-analogues. The application of q-Erdélyi integral [4, Eq. (1.12)] in driving the q-analogue of a Kampé de Fériet summation; conjectured by Joris Ven der Jeugt in his work on the evaluation of the $9-j$ recoupling coefficients appearing in the quantum theory of angular momentum, are also discussed in [4].

The recent research papers [22, 23] and many others, cited therein, are examples of ongoing trend and interest in the field of q-analysis and q-calculus. Srivastava [22] presents an excellent set of discussion and comments on the study of post-quantum or (p, q)-version of the classical q-analysis.. In a review article by Srivastava [23], the overview and recent developments in the theory of several extensively studied higher transcendental functions along with their applications in widely investigated areas of various sciences have been nicely presented. For some recent developments in the field of special functions, we refer to the following research paper [14, 15, 20, 21, 27] and [28]. Further, the inspiration to work on q-hypergeometric functions and basic q- hypergeometric polynomials, is because of their vast applicability in several diverse areas of science and engineering (see, for details, [29, p. 235]). The above-mentioned analysis and observations motivate us to study q-discrete expansions of Erdélyi's type integrals investigated by [26].

2 Motivation and Objective

Several researchers, for example, [6, 17, 18, 26] and [30] have studied and investigated the expansions which involve integrals and represent the hypergeometric functions because of the several applications of such integrals (see, for example, [4, 5]). In this context, Joshi and Vyas [9] gave an alternative way to prove Erdélyi's integrals by utilizing the classical series rearrangement techniques [19, 25] and some classical hypergeometric summation theorems. This kind of proof motivated them to establish seven Erdélyi type integrals including a generalization and unification of Erdélyi integrals [9, Eqs. (3.1) to (3.7) and Eq. (4.1)] for certain ${}_{q+1}F_q(z)$. Taking this work forward, Joshi and Vyas [11] investigated two different classes of the q-integrals in the form of basic q-extensions of all Erdélyi type integrals due to [9], along with various special cases and applications. Further, following [4], Joshi and Vyas [12] obtained two q-hypergeometric expansions for ${}_{12}\Phi_{11}(q)$ and ${}_r\Phi_s(q)$. As applications, these expansion formulas were set to give some ${}_{10}\Phi_9(q)$ expansions applicable to the top class ${}_{10}\Phi_9(q)$ biorthogonal rational functions which on specialization lead us to the gasper's ${}_4\Phi_3(q)$ expansion formulas.

Recently, Vyas et al. [26] investigated the discrete analogues of the Erdélyi type integrals due to [9] and Luo and Raina [16], along the lines of a triple series manipulation-based derivation of the Erdélyi's integrals due to [9]. A careful investigation of the papers [10–13, 16] and [26] depicts that classical series rearrangement technique is a versatile technique which helps in deriving the higher order hypergeometric identities and q-hypergeometric expansions which are not available in the literature.

Motivated from the above-mentioned work and an analysis of the method of proof discussed in [26], here, in this research paper we establish new q-hypergeometric expansions as q-discrete analogues of the Erdélyi type integrals. Inline to [11], we obtain two types of q-analogues of Erdélyi type of integrals. It may be noted from [11] that all Erdélyi type of integrals possesses first type of q-analogues (having t^{-1} in the numerator), while some possess the second type of q-analogue (do not have t^{-1} in the numerator). To derive these q-expansion formulas, we express the right side of each of the q-expansion formulas as a triple series and then apply the double series manipulation lemma [21, p. 57, Lemma 10] or [24, p. 100, Eq. (2)]:

$$\sum_{n_2=0}^{\infty}\sum_{n_1=0}^{n_2}\Omega(n_1,n_2)=\sum_{n_2=0}^{\infty}\sum_{n_1=0}^{\infty}\Omega(n_1,n_2+n_1), \tag{2.1}$$

and ${}_3\Phi_2$ transformation formula [5, p. 212, Eq. (III.12)]

$$ {}_3\Phi_2\left[\begin{matrix}-n,\ b,\ c\\ d,\ e\end{matrix};\ q,q\right]=\frac{\left(\frac{e}{c},q\right)_n}{(e,q)_n}c^n{}_3\Phi_2\left[\begin{matrix}q^{-n},\ c,\ \frac{d}{b}\\ d,\ \frac{c}{e}q^{1-n}\end{matrix};\ q,\ \frac{qb}{e}\right]. \tag{2.2}$$

At the end, the application of triple series manipulation lemma [9, p. 128]:

$$\sum_{n_3=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_1=0}^{\infty}\Omega(n_3,n_2,n_1)=\sum_{n_3=0}^{\infty}\sum_{n_2=0}^{n_3}\sum_{n_1=0}^{n_3-n_2}\Omega(n_3-n_2-n_1,n_2,n_1), \tag{2.3}$$

and use of two appropriate q-classical summation theorems to solve the involved inner series, lead us to the desired discrete q-extensions.

In Sect. 3, we state all of the investigated new q-hypergeometric expansions. All the q-hypergeometric expansions stated in Sect. 3 include terminating series only, hence the question of convergence doesn't arise because terminating series are always convergent. Further, to convert the results of Sect. 3 into their corresponding Erdélyi type q-integrals, the procedure mentioned in [26, p. 2 and p. 5 (Remark 2)] can be applied in a straight forward manner. In Sects. 4 and 5, we give the brief outline of the derivations of the new q-hypergeometric expansions, to illustrate the difference between the derivations of the discrete extensions or the new q-hypergeometric expansions corresponding to the Erdélyi type q-integrals with and without the presence t^{-1} as one of the numerator parameters in one of the involved hypergeometric functions.

3 Discrete Extensions of Erdélyi Type *q*-Integrals or New *q*-Hypergeometric Expansions

The results from (3.1) to (3.11) provide the basic (or q-) expansion formulas or the discrete analogue of the Erdélyi type q-integrals given [11, Eqs. (1.2) to (1.12)], respectively. The usual condition for the convergence of q-hypergeometric series that is the denominator parameters are non negative integers. This condition is applied to all the following q-hypergeometric expansions:

$$
{}_4\Phi_3\left[\begin{matrix}-N,\ \nu,\ \xi,\ \lambda;\\ \varepsilon,\ \gamma,\ \delta;\end{matrix}\ q;\ q\right].=\frac{(q^{\gamma-\mu};q)_N}{(q^{\gamma};q)_N}\sum_{k=0}^{\infty}\frac{(q^{-N};q)_k\,(q^{\mu};q)_k(q^{\varepsilon-\nu-\xi+\delta};q)_k}{(q^{\varepsilon};q)_k(q^{1+\mu-\gamma-N};q)_k(q;q)_k}(q^{1+\nu+\xi-\delta-\gamma})^k q^{\mu N}
$$

$$
{}_3\Phi_2\left[\begin{matrix}\frac{\nu\xi}{\delta},\ \frac{\lambda}{\mu},q^{-N+k};\\ \frac{\gamma}{\mu},\ q^{\varepsilon+k};\end{matrix}\ q;\ q\right]{}_4\Phi_3\left[\begin{matrix}\frac{\delta}{\xi},\ \frac{\delta}{\nu},\lambda,q^{-k};\\ \frac{\varepsilon\delta}{\nu\xi},\mu,\delta;\end{matrix}\ q;\ q\right] \tag{3.1}
$$

$$
{}_4\Phi_3\left[\begin{matrix}-N,\ \nu,\ \xi,\ \lambda;\\ \varepsilon,\ \gamma,\ \delta;\end{matrix}\ q;\ \frac{q\mu}{\lambda}\right]=\frac{(q^{\gamma-\mu};q)_N}{(q^{\gamma};q)_N}\sum_{k=0}^{\infty}\frac{(q^{-N};q)_k\,(q^{\mu};q)_k(q^{\varepsilon-\nu-\xi+\delta};q)_k q^{\mu N}}{(q^{\varepsilon};q)_k(q^{1+\mu-\gamma-N};q)_k(q;q)_k}(q^{1+\nu+\xi-\delta-\gamma})^k
$$

$$
{}_3\Phi_2\left[\begin{matrix}\frac{\lambda}{\mu},\ \frac{\nu\xi}{\delta},q^{-N+k};\\ \frac{\gamma}{\mu},q^{\varepsilon+k};\end{matrix}\ q;\ \frac{q}{\lambda}\right]{}_4\Phi_3\left[\begin{matrix}\frac{\delta}{\xi},\ \frac{\delta}{\nu},\lambda,q^{-k};\\ \frac{\varepsilon\delta}{\nu\xi},\ \mu,\delta;\end{matrix}\ q;\ \frac{q\mu}{\lambda}\right] \tag{3.2}
$$

$$
{}_6\Phi_5\left[\begin{matrix}-N,\ q^{\alpha},\ \sqrt{\frac{\gamma}{\beta}},\ -\sqrt{\frac{\gamma}{\beta}},\ \sqrt{\frac{q\gamma}{\beta}},\ -\sqrt{\frac{q\gamma}{\beta}};\\ \varepsilon,\sqrt{\gamma}\ -\sqrt{\gamma},\ \sqrt{q\gamma},\ -\sqrt{q\gamma};\end{matrix}\ q,\ \frac{q\beta}{\alpha}\right]=\frac{\left(q^{\alpha+\beta};\,q\right)_N}{(q^{\gamma};\,q)_N}
$$

$$
\sum_{k=0}^{\infty}\frac{(q^{-N};q)_k(q^{\gamma-\alpha-\beta};q)_k(q^{\varepsilon-\beta};q)_k}{(q^{\varepsilon};q)_k(q^{1-\alpha-\beta-N};q)_k(q;q)_k}(q^{1+\beta-\gamma})^k{}_3\Phi_2\left[\begin{matrix}\frac{\alpha}{\beta},\ \frac{\gamma}{\beta},\ q^{-k};\\ \frac{\gamma}{\alpha\beta},\ \frac{\varepsilon}{\beta};\end{matrix}\ q,\ \frac{q}{\alpha}\right]
$$

$$
{}_8\Phi_7\left[\begin{matrix}\alpha,\sqrt{\beta},-\sqrt{\beta},\sqrt{\beta q},-\sqrt{\beta q},\ \frac{q\beta}{\varepsilon m},\ \frac{\alpha q^{-N+k}}{\beta},q^{-N+k}\\ \sqrt{\alpha\beta},-\sqrt{\alpha\beta},\sqrt{\alpha\beta q},-\sqrt{\alpha\beta q},\ \frac{q\beta}{\varepsilon k},\gamma q^{N},q^{\varepsilon+k}\end{matrix};\ q,\ q\right] \tag{3.3}
$$

$$
{}_4\Phi_3\left[\begin{matrix}-N,\ \alpha\ ,\ \beta,\ \frac{q\gamma}{\lambda\mu};\\ \varepsilon\ ,\ \frac{q\gamma}{\lambda},\ \frac{q\gamma}{\mu};\end{matrix}\ q,\ \frac{q\mu}{\alpha}\right]=\frac{(q^{\,1+\gamma-\beta};q)_N}{(q^{\,1+\gamma};q)_N}\sum_{k=0}^{\infty}\frac{(q^{\,-N};q)_k(q^{\beta};q)_k(q^{\,\varepsilon-\mu};q)_k q^{\beta N}}{(q^{\varepsilon};q)_k(q^{\beta-\gamma-N};q)_k(q;q)_k}(q^{\mu-\gamma})^k
$$

$$
{}_4\Phi_3\left[\begin{matrix}\frac{\alpha}{\mu},\beta q^{k},\ \frac{\varepsilon q^{k}}{\mu},q^{-N+k};\\ \varepsilon q^{k},\ \frac{\varepsilon}{\mu},\ \frac{\beta q^{-N+k}}{\gamma};\end{matrix}\ q,\ \frac{q\mu}{\alpha\gamma}\right]
$$

$$
{}_8\Phi_8\left[\begin{matrix}\gamma,q\sqrt{\gamma},-q\sqrt{\gamma},\mu,\lambda,\alpha,\ \frac{q\gamma q^{N}}{\beta q^{k}},q^{-k};\\ \sqrt{\gamma},-\sqrt{\gamma},\ \frac{q\gamma}{\beta},\ \frac{q\gamma}{\lambda},\ \frac{q\gamma}{\mu},\ \frac{\mu q^{1-k}}{\varepsilon},\gamma q^{1-k},0;\end{matrix}\ q,\ \frac{q\lambda}{\varepsilon\alpha\gamma}\right] \tag{3.4}
$$

$$
{}_4\Phi_3\left[\begin{matrix}-N,\ \alpha,\ \beta,\ \frac{q\gamma}{\lambda\mu};\\ \varepsilon,\ \frac{q\gamma}{\lambda},\ \frac{q\gamma}{\mu};\end{matrix}\ q,q\right]=\frac{(q^{1+\gamma-\beta};q)_N}{(q^{1+\gamma};q)_N}\sum_{k=0}^{\infty}\frac{(q^{-N};q)_k(q^{\beta};q)_k(q^{\varepsilon-\mu};q)_k q^{\beta N}}{(q^{\varepsilon};q)_k(q^{\beta-\gamma-N};q)_k(q;q)_k}(q^{\mu-\gamma})^k
$$

$$
{}_4\Phi_3\left[\begin{matrix}\frac{\alpha}{\mu},q^{-N+k},\beta q^{k},\ \frac{\varepsilon q^{k}}{\mu};\\ \frac{\varepsilon}{\mu},q^{\varepsilon+k},\ \frac{\beta q^{k}}{\gamma q^{N}};\end{matrix}\ q,\ \frac{q}{\gamma}\right]
$$

$$
{}_8\Phi_8\left[\begin{matrix}\gamma,q\sqrt{\gamma},-q\sqrt{\gamma},\mu,\lambda,\alpha,\ \frac{\gamma q^{1-k+N}}{\beta},q^{-k};\\ \sqrt{\gamma},-\sqrt{\gamma},\ \frac{q\gamma}{\mu},\ \frac{q\gamma}{\lambda},\ \frac{q\gamma}{\beta},\gamma q^{1-N},\ \frac{\mu q^{1-k}}{\varepsilon},0;\end{matrix}\ q,\ \frac{q\lambda}{\varepsilon\mu}\right] \tag{3.5}
$$

$$
{}_4\Phi_3\left[\begin{matrix} -N,\ \alpha,\ \beta,\ \frac{q\gamma}{\lambda\mu}; \\ \varepsilon,\ \frac{q\gamma}{\lambda},\ \frac{q\gamma}{\mu}; \end{matrix}\ q, q\right] = \frac{(q^{1+\gamma-\beta};q)_N}{(q^{1+\gamma};q)_N} \sum_{k=0}^{\infty} \frac{(q^{-N};q)_k (q^{\beta};q)_k (q^{\varepsilon-\mu};q)_k}{(q^{\varepsilon};q)_k (q^{\beta-\gamma-N};q)_k (q;q)_k} (q^{\mu-\gamma})^k q^{\beta N}
$$

$$
{}_3\Phi_2\left[\begin{matrix} \frac{\alpha}{\mu},\ \frac{\varepsilon q^k}{\mu},\ q^{-N+k}; \\ \frac{\varepsilon}{\mu},\ \varepsilon q^k; \end{matrix}\ q, q\right] {}_9\Phi_7\left[\begin{matrix} \alpha, \mu, \gamma, \lambda, q^{-k}, \beta q^k, q\sqrt{\gamma}, -q\sqrt{\gamma}, 0; \\ \frac{q\gamma}{\mu}, \frac{q\gamma}{\lambda}, \frac{q\gamma}{\beta}, \frac{q\gamma}{N}, q^{\varepsilon+k}, \sqrt{\gamma}, -\sqrt{\gamma}; \end{matrix}\ q, \frac{q^{1+N}\gamma}{\lambda\mu\beta}\right] \tag{3.6}
$$

$$
{}_7\Phi_6\left[\begin{matrix} -N,\ \alpha,\ \beta,\ \sqrt{\frac{\gamma}{\mu}}, -\sqrt{\frac{\gamma}{\mu}}, \sqrt{\frac{q\gamma}{\mu}}, -\sqrt{\frac{q\gamma}{\mu}}; \\ \varepsilon,\ \frac{\gamma}{\mu},\ \sqrt{\gamma}, -\sqrt{\gamma}, \sqrt{q\gamma}, -\sqrt{q\gamma}; \end{matrix}\ q, q\right] = \frac{(q^{\gamma-\beta};q)_N}{(q^{\gamma};q)_N}
$$

$$
\sum_{k=0}^{\infty} \frac{(q^{-N};q)_k (q^{\beta};q)_k (q^{\varepsilon-\mu};q)_k q^{\beta N}}{(q^{\varepsilon};q)_k (q^{1+\beta-\gamma-N};q)_k (q;q)_k} (q^{1+\mu-\gamma})^k
$$

$$
{}_4\Phi_3\left[\begin{matrix} \alpha,\ \mu,\ \beta q^k,\ q^{-N+k}; \\ \frac{\gamma}{\beta}, \gamma N,\ q^{\varepsilon+k}; \end{matrix}\ q, \frac{\gamma q^{1+N}}{\alpha\beta}\right] {}_2\Phi_1\left[\begin{matrix} \frac{\alpha}{\mu}, q^{-k}; \\ \frac{\varepsilon}{\mu}; \end{matrix}\ q, \frac{q}{\alpha}\right] \tag{3.7}
$$

$$
{}_5\Phi_4\left[\begin{matrix} -N,\ \alpha,\ \beta,\ \gamma,\ \frac{\alpha q\mu}{\beta\gamma}; \\ \varepsilon,\ \frac{\alpha q}{\beta},\ \frac{\alpha q}{\gamma},\ \frac{\beta\gamma}{\mu}; \end{matrix}\ q, \frac{q\lambda}{\alpha}\right] = \frac{(q^{1+\alpha-\mu-\lambda};q)_N}{(q^{1+\alpha-\mu};q)_N} \sum_{k=0}^{\infty} \frac{(q^{-N};q)_k (q^{\lambda};q)_k (q^{\varepsilon-\mu};q)_k}{(q^{\varepsilon};q)_k (q^{\mu-\alpha+\lambda-N};q)_k (q;q)_k} (q^{2\mu-\alpha})^k
$$

$$
{}_4\Phi_3\left[\begin{matrix} \mu,\ \frac{\alpha}{\lambda},\ \frac{\alpha q^{1-k+N}}{\mu\lambda},\ q^{-k}; \\ \frac{q\alpha}{\lambda\mu},\ \frac{q\mu}{\varepsilon},\ \frac{q^{1-k}}{\lambda} \end{matrix}\ q, \frac{q^{2-N}}{\alpha\varepsilon}\right]
$$

$$
{}_{12}\Phi_{11}\left[\begin{matrix} \frac{\alpha}{\mu}, q\sqrt{\frac{\alpha}{\mu}}, -q\sqrt{\frac{\alpha}{\mu}}, \frac{\beta}{\mu}, \frac{\gamma}{\mu}, \frac{q\alpha}{\beta\gamma}, \sqrt{\alpha}, -\sqrt{\alpha}, \\ \sqrt{\frac{\alpha}{\mu}}, -\sqrt{\frac{\alpha}{\mu}}, \frac{q\alpha}{\beta}, \frac{\alpha\beta}{\gamma}, \frac{\beta\gamma}{\mu}, \sqrt{\gamma}, -\sqrt{\gamma}, \end{matrix}\right.
$$

$$
\left.\begin{matrix} q\sqrt{\alpha}, -q\sqrt{\alpha}, \frac{\varepsilon q^k}{\mu}, q^{-N+k}; \\ q\sqrt{\lambda}, -q\sqrt{\lambda}, \frac{q^{1+N}\alpha}{\mu}, q^{\varepsilon+k}; \end{matrix}\ q, \frac{q^{1+N}\gamma\mu}{\alpha}\right] \tag{3.8}
$$

$$
{}_5\Phi_4\left[\begin{matrix} -N,\ \alpha,\ \beta,\ \gamma,\ \frac{\alpha q}{\beta\gamma}; \\ \varepsilon,\ \frac{\alpha q}{\beta},\ \frac{\alpha q}{\gamma},\ \frac{\beta\gamma}{\mu}; \end{matrix}\ q, q\right] = \frac{(q^{1+\alpha-\mu-\lambda};q)_N}{(q^{1+\alpha-\lambda};q)_N} \sum_{k=0}^{\infty} \frac{(q^{-N};q)_k (q^{\lambda};q)_k (q^{\varepsilon-\mu};q)_k}{(q^{\varepsilon};q)_k (q^{\mu-\alpha+\lambda-N};q)_k (q;q)_k} (q^{2\mu-\alpha})^k
$$

$$
{}_4\Phi_3\left[\begin{matrix} \mu,\ \frac{\alpha}{\lambda},\ \frac{\alpha q^{1-k+N}}{\mu\lambda},\ q^{-k}; \\ \frac{q\alpha}{\lambda\mu},\ \frac{q^{1-k}}{\lambda},\ \frac{\mu q^{1-k}}{\varepsilon}; \end{matrix}\ q, \frac{q^{2-N}}{\varepsilon}\right]
$$

$$
{}_{13}\Phi_{12}\left[\begin{matrix} \frac{\alpha}{\beta}, q\sqrt{\frac{\alpha}{\mu}}, -q\sqrt{\frac{\alpha}{\mu}}, \frac{\beta}{\mu}, \frac{\gamma}{\mu}, \frac{q\alpha}{\beta\gamma}, \sqrt{\alpha}, -\sqrt{\alpha}, \\ \sqrt{\frac{\alpha}{\mu}}, -\sqrt{\frac{\alpha}{\mu}}, \frac{q\alpha}{\beta}, \frac{q\alpha}{\gamma}, \frac{\beta\gamma}{\mu}, \sqrt{\lambda}, -\sqrt{\lambda}, \end{matrix}\right.
$$

$$
\left.\begin{matrix} q\sqrt{\alpha}, -q\sqrt{\alpha}, \frac{\varepsilon q^k}{\mu}, \lambda q^k, q^{-N+k}; \\ q\sqrt{\lambda}, -q\sqrt{\lambda}, \frac{\mu\lambda q^k}{\alpha}, \frac{q^{1+N}\alpha}{\mu}, q^{\varepsilon+k}; \end{matrix}\ q, \frac{q^{1+N}\mu}{\alpha}\right] \tag{3.9}
$$

$$
{}_8\Phi_7\left[\begin{matrix} -N,\ \beta,\ \gamma,\ \frac{\alpha q}{\mu\lambda}, q\sqrt{\mu}, -q\sqrt{\mu}, \sqrt{q\alpha}, -\sqrt{q\alpha}; \\ \varepsilon,\ \frac{\alpha q}{\lambda},\ \frac{\beta\gamma}{\mu}, \sqrt{\beta\gamma}, -\sqrt{\beta\gamma}, \sqrt{q\beta\gamma}, -\sqrt{q\beta\gamma}; \end{matrix}\ q, q\right] = \frac{(q^{2\gamma+\beta-\alpha-1};q)_N}{(q^{\gamma+\beta};q)_N}
$$

$$\sum_{k=0}^{\infty} \frac{(q^{-N};q)_k (q^{1+\alpha-\gamma};q)_k (q^{\varepsilon-\gamma};q)_k}{(q^{\varepsilon};q)_k (q^{2+\alpha-2\gamma-N-\beta};q)_k (q;q)_k} (q^{1-\beta})^k$$

$$ {}_4\Phi_3\left[\begin{matrix} \frac{2\gamma\beta q^{1+n+N}}{\alpha}, \frac{\beta\gamma}{\alpha q}, \gamma, -k; \\ \frac{\lambda q^{-m}}{\alpha}, \frac{\mu\gamma q^{1+n}}{\varepsilon}, \frac{2\gamma\beta q^{2n-1}}{\alpha}; \end{matrix}\; q, \frac{q^{1-N}}{\varepsilon}\right]$$

$$ {}_9\Phi_8\left[\begin{matrix} \alpha, q\sqrt{\mu}, -q\sqrt{\mu}, \beta, \mu, \lambda, \frac{2\gamma\beta q^{N-1}}{\alpha}, \gamma q^{2m}, q^{-N+k}; \\ \sqrt{\alpha}, -\sqrt{\alpha}, \frac{q\alpha}{\mu}, \frac{q\alpha}{\lambda}, \frac{2\gamma\beta}{\alpha q}, q^{\varepsilon+k}; \end{matrix}\; q, \frac{q\varepsilon}{\alpha\mu}\right] \tag{3.10}$$

$$ {}_9\Phi_8\left[\begin{matrix} -N, \varepsilon, \beta, \gamma, \frac{q\alpha^2}{\beta\gamma\mu}, q\sqrt{\mu}, -q\sqrt{\mu}, \sqrt{q\mu}, -\sqrt{q\mu}; \\ \varepsilon, \frac{q\alpha}{\beta}, \frac{q\alpha}{\gamma}, \frac{\beta\gamma\mu}{\alpha}\sqrt{\lambda}, -\sqrt{\lambda}, \sqrt{q\lambda}, -\sqrt{q\lambda}; \end{matrix}\; ; q, q\right] = \frac{(q^{\lambda-\alpha};q)_N}{(q^{\lambda};q)_N}$$

$$\sum_{k=0}^{\infty} \frac{(q^{-N};q)_k (q^{\alpha};q)_k (q^{\varepsilon-\alpha+\mu};q)_k}{(q^{\varepsilon};q)_k (q^{1+\alpha-\lambda-N};q)_k (q;q)_k} (q^{1+\alpha-\lambda-\mu})^k$$

$$ {}_4\Phi_3\left[\begin{matrix} \frac{\alpha}{\mu}, \frac{\lambda}{q\mu}, \alpha q^k, q^{-N+k}; \\ \frac{\lambda}{\alpha}, \lambda q^N, q^{\varepsilon+k}; \end{matrix}\; q, \frac{q^2\mu}{\alpha}\right] \tag{3.11}$$

4 Proof of q-Hypergeometric Expansion (3.1) Corresponding to Erdélyi Type Integrals with t^{-1} as One of the Numerator Parameters

Denoting the right hand side of (3.1) by Δ and replacing the hypergeometric series by their series form and then using the double series manipulation (2.1), we get

$$\Delta = \frac{\left(\frac{\gamma}{\mu};q\right)_N q^{\mu N}}{(\gamma;q)_N} \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{k} \frac{(q^{-N};q)_{k+m+n}(\mu;q)_{k+m}\left(\frac{\varepsilon\delta}{\nu\xi};q\right)_{k+m}\left(\frac{q\nu\xi}{\delta\gamma}\right)^{k+m}}{(\varepsilon:q)_{k+m+n}\left(\frac{\mu q^{1-N}}{\gamma};q\right)_{k+m}(q;q)_k}$$
$$\times \frac{\left(\frac{\nu\xi}{\delta};q\right)_n\left(\frac{\lambda}{\mu};q\right)_n\left(\frac{\delta}{\xi};q\right)_m\left(\frac{\delta}{\nu};q\right)_m(\lambda;q)_m(-q)^m q^n q^{\binom{m}{2}-m^2-mk}}{\left(\frac{\gamma}{\mu};q\right)_n\left(\frac{\varepsilon\delta}{\gamma\xi};q\right)_m(\mu;q)_m(\delta;q)_m(q;q)_m(q;q)_n}. \tag{4.1}$$

Taking inner series in above equation, we obtain the following equation:

$$\Delta = \frac{\left(\frac{\gamma}{\mu};q\right)_N q^{\mu N}}{(\gamma;q)_N} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(q^{-N};q)_{m+n}(\mu;q)_m\left(\frac{q\nu\xi}{\delta\gamma}\right)^m}{(\varepsilon:q)_{m+n}\left(\frac{\mu q^{1-N}}{\gamma};q\right)_m}$$
$$\times \frac{\left(\frac{\nu\xi}{\delta};q\right)_n\left(\frac{\lambda}{\mu};q\right)_n\left(\frac{\delta}{\xi};q\right)_m\left(\frac{\delta}{\nu};q\right)_m(\lambda;q)_m(-q)^m q^n q^{\binom{m}{2}-m^2}}{\left(\frac{\gamma}{\mu};q\right)_n(\mu;q)_m(\delta;q)_m(q;q)_m(q;q)_n}$$

$$\times\ {}_3\Phi_2\left[\begin{matrix} q^{-(N-m-n)},\ \mu q^m,\ \frac{\varepsilon\delta}{\nu\xi}q^m; & \dfrac{q\nu\xi q^{-m}}{\delta\gamma} \\ \varepsilon q^{m+n},\ \frac{\mu q^{1-N+m}}{\gamma}: & \end{matrix}\right]. \tag{4.2}$$

Now, applying the formula (2.2) (reversed) on the ${}_3\Phi_2$ of above equation, we can write

$$\Delta=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(q^{-N};q)_{k+m+n}(\mu;q)_{k+m}\left(\frac{\nu\xi}{\delta};q\right)_{n+k}\left(\frac{\nu\xi}{\delta\gamma}\right)^m}{(\varepsilon:q)_{k+m+n}(\gamma:q)_{k+m+n}}$$
$$\times\ \frac{\left(\frac{\lambda}{\mu};q\right)_n\left(\frac{\delta}{\xi};q\right)_m\left(\frac{\delta}{\nu};q\right)_m(\lambda;q)_m q^{k+m+n}\mu^n q^{mn}\gamma^m}{(\mu;q)_m(\delta;q)_m(q;q)_k(q;q)_m(q;q)_n}. \tag{4.3}$$

Next, applying the triple series manipulation (2.3) on the above equation and then taking an inner series in n gives:

$$\Delta=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(q^{-N};q)_k(\mu;q)_k\left(\frac{\nu\xi}{\delta};q\right)_k}{(\varepsilon:q)_k(\gamma:q)_k(q;q)_k}$$
$$\times\ \frac{\left(\frac{\delta}{\xi};q\right)_m\left(\frac{\delta}{\nu};q\right)_m(\lambda;q)_m\left(q^{-k};q\right)_m q^{m+k}}{\left(\frac{\delta q^{1-k}}{\nu\xi};q\right)_m(\mu;q)_m(\delta;q)_m(q;q)_m}\ {}_2\Phi_1\left[\begin{matrix} q^{m-k},\ \frac{\lambda}{\mu}; & q \\ \frac{q^{1-k}}{\mu}; & \end{matrix}\right]. \tag{4.4}$$

Now, applying Chu-Vandermonde summation theorem [5, p. 354, Eq. (II.6)] and then taking an inner series in m leads us to the following equation:

$$\Delta=\sum_{k=0}^{\infty}\frac{(q^{-N};q)_k\left(\frac{\nu\xi}{\delta};q\right)_k q^k}{(\varepsilon:q)_k(\gamma:q)_k(q:q)_k}\ {}_3\Phi_2\left[\begin{matrix} \frac{\delta}{\xi},\ \frac{\delta}{\nu},\ q^{-k}; & q \\ \delta,\ \frac{\delta q^{1-k}}{\nu\xi}; & \end{matrix}\right]. \tag{4.5}$$

The application of q-Pfaff-Saalschütz summation theorem [5, p. 355, Eq. (II.12)] leads to the left side of (3.1).

5 Proof of the New q-hypergeometric Expansion (3.2) Corresponding to Erdélyi Type Integrals Without t^{-1} as One of the Numerator Parameters

Denoting the right hand side of (3.2) by Δ and replacing the hypergeometric series by their series form and then using the double series manipulation (2.1), we get

$$\Delta=\frac{\left(\frac{\gamma}{\mu};q\right)_N q^{\mu N}}{(\gamma;q)_N}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{k}\frac{(q^{-N};q)_{k+m+n}(\mu;q)_{k+m}\left(\frac{\varepsilon\delta}{\nu\xi};q\right)_{k+m}\left(\frac{q\nu\xi}{\delta\gamma}\right)^{k+m}}{(\varepsilon:q)_{k+m+n}\left(\frac{\mu q^{1-N}}{\gamma};q\right)_{k+m}(q;q)_k}$$

$$\times \frac{\left(\frac{\nu\xi}{\delta};q\right)_n\left(\frac{\lambda}{\mu};q\right)_n\left(\frac{\delta}{\xi};q\right)_m\left(\frac{\delta}{\nu};q\right)_m(\lambda;q)_m\left(-\frac{\mu}{\lambda}q^{1-n}\right)^m\left(\frac{q}{\lambda}\right)^n q^{\binom{m}{2}-m^2-mk}}{\left(\frac{\gamma}{\mu};q\right)_n\left(\frac{\varepsilon\delta}{\gamma\xi};q\right)_m(\mu;q)_m(\delta;q)_m(q;q)_m(q;q)_n}. \tag{5.1}$$

Taking inner series in above equation, we obtain the following equation:

$$\Delta = \frac{\left(\frac{\gamma}{\mu};q\right)_N q^{\mu N}}{(\gamma;q)_N}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\left(q^{-N};q\right)_{m+n}(\mu;q)_m\left(\frac{q\nu\xi}{\delta\gamma}\right)^m}{(\varepsilon:q)_{m+n}\left(\frac{\mu q^{1-N}}{\gamma};q\right)_m}$$
$$\times \frac{\left(\frac{\nu\xi}{\delta};q\right)_n\left(\frac{\lambda}{\mu};q\right)_n\left(\frac{\delta}{\xi};q\right)_m\left(\frac{\delta}{\nu};q\right)_m(\lambda;q)_m\left(-\frac{\mu}{\lambda}q^{1-n}\right)^m\left(\frac{q}{\lambda}\right)^n q^{\binom{m}{2}-m^2}}{\left(\frac{\gamma}{\mu};q\right)_n(\mu;q)_m(\delta;q)_m(q;q)_m(q;q)_n}$$
$$\times {}_3\Phi_2\left[\begin{matrix} q^{-(N-m-n)},\ \mu q^m,\ \frac{\varepsilon\delta}{\nu\xi}q^m; & \frac{q\nu\xi q^{-m}}{\delta\gamma} \\ \varepsilon q^{m+n},\ \frac{\mu q^{1-N+m}}{\gamma}\ : & \end{matrix}\right]. \tag{5.2}$$

Now, applying the formula (2.2) (reversed) on the ${}_3\Phi_2$ of above equation, we can obtain

$$\Delta = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\left(q^{-N};q\right)_{k+m+n}(\mu;q)_{k+m}\left(\frac{\nu\xi}{\delta};q\right)_{n+k}\left(\frac{\nu\xi}{\delta\lambda}\right)^m}{(\varepsilon:q)_{k+m+n}(\gamma:q)_{k+m+n}}$$
$$\times \frac{\left(\frac{\lambda}{\mu};q\right)_n\left(\frac{\delta}{\xi};q\right)_m\left(\frac{\delta}{\nu};q\right)_m(\lambda;q)_m q^{k+m+n}\mu^{m+n}}{(\mu;q)_m(\delta;q)_m(q;q)_k(q;q)_m(q;q)_n}. \tag{5.3}$$

Next, applying the triple series manipulation (2.3) on Eq. (5.3) and then taking an inner series in n gives:

$$\Delta = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{\left(q^{-N};q\right)_k(\mu;q)_k\left(\frac{\nu\xi}{\delta};q\right)_k}{(\varepsilon:q)_k(\gamma:q)_k(q;q)_k}$$
$$\times \frac{\left(\frac{\delta}{\xi};q\right)_m\left(\frac{\delta}{\nu};q\right)_m(\lambda;q)_m\left(q^{-k};q\right)_m\mu^m q^{m+k}}{\left(\frac{\delta q^{1-k}}{\nu\xi};q\right)_m(\mu;q)_m(\delta;q)_m(q;q)_m}\,{}_2\Phi_1\left[\begin{matrix} q^{m-k},\ \frac{\lambda}{\mu}; & q^{1-m} \\ \frac{q^{1-k}}{\mu}; & \lambda \end{matrix}\right]. \tag{5.4}$$

Now, applying Chu-Vandermonde summation theorem [5, p. 354, Eq. (II.7)] and then taking an inner series in m, we can write

$$\Delta = \sum_{k=0}^{\infty}\frac{\left(q^{-N};q\right)_k\left(\frac{\nu\xi}{\delta};q\right)_k(\lambda;q)_k q^k\left(\frac{\mu}{\lambda}\right)^k}{(\varepsilon:q)_k(\gamma:q)_k(q:q)_k}\,{}_3\Phi_2\left[\begin{matrix} \frac{\delta}{\xi},\ \frac{\delta}{\nu},\ q^{-k}; & q \\ \delta,\ \frac{\delta q^{1-k}}{\nu\xi}; & \end{matrix}\right]. \tag{5.5}$$

The application of q-Pfaff-Saalschütz summation theorem [5, p. 355, Eq. (II.12)] leads to the left side of (3.1).

6 Conclusion

In conclusion, this paper shows the superiority of the series manipulation method discussed by Vyas et al. [26], in deriving the discrete analogues of the Erdélyi type q-integrals in the form of the new q-hypergeometric expansions. More significantly, the main results, Eqs. (3.1) to (3.11), provide the generalizations and a set of different q-analogues (in some cases) and thus lead to many of the ordinary hypergeometric expansions derived in [26], on setting $q \to 1$. The Eqs. (3.1) and (3.2) provide two q-analogues of the result [26, Theorem 2, p.5], the Eq. (3.3) is a q-analogue of [26, Theorem 5, p. 6], the Eqs. (3.4) to (3.6) are q-analogues of [26, Theorem 6, p. 6], the Eq. (3.7) is a q-analogue of [26, Theorem 7, p. 6], the Eqs. (3.8) to (3.9) are q-analogues of [26, Theorem 8, p. 7], the Eq. (3.10) is a q-analogue of [26, Theorem 9, p. 7] and the Eq. (3.11) is a q-analogue of [26, Theorem 10, p. 8]. However, the q-analogues of [26, Theorems 1, 3 and 4, p. 5] can't be developed until the required q-analogue of the extended Saalschütz theorem (see [26, Eq. (17), p. 3]) is determined, and hence it remains an open problem. Some future directions for further research for the q-hypergeometric expansions obtained in this paper may be to discover the further generalizations of the expansions along the line of [4, 12] and in addition, these results may also be specialized along the line of [11] to produce known and new q-hypergeometric transformations, which will form the subject matter of our subsequent paper in the foreseeable future. Moreover, in [22], the $(p; q)$-calculus was exposed to be a rather trivial and inconsequential variation of the classical q-calculus, the additional parameter p being redundant. This observation by Srivastava [22] will indeed apply also to any future attempt to produce the rather straightforward $(p; q)$-variants of the results of this paper.

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