Approximate Numerical Solution of the Nonlinear Klein-Gordon Equation with Caputo-Fabrizio Fractional Operator

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Abstract. This work analyze the linear and nonlinear fractional Klein-Gordon equation using fractional homotopy perturbation transform method (FHPTM) via Caputo-Fabrizio derivative. The proposed technique is used to solve fractional model without any restrictive assumptions. The acquired results ratify that the proposed method is acceptable and credible for approximate analytic treatment of the extensive types of nonlinear physical processes.

Keywords: FHPTM · Fractional Klein-Gordon equation (FKGE) · Caputo-Fabrizio (CF) derivative · Laplace transform (LT) · Approximate solution

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1 Introduction

Nonlinear fractional differential equation (NFDEs), which is one of the emerging area that can be classified as an applied model, for example, physics, computational fluid dynamic, chemical science, natural science, optics, plasma physics etc. The difficulty to search exact solution for the NFDEs has led researchers to explore the approximate and numerical methods to find the solution of these systems [1–7]. There are various numerical methods like projective Riccati equation method [8], HPTM is used to solve the NFDEs [9], Collocation method [10], reduced differential transform method [11], Laplace transform method [12], q-homotopy analysis Sumudu transform technique [13–16], Sine-Gordon expansion method [17, 18], Caputo-Fabrizio fractional derivative [19, 20, 25], Homotopy perturbation technique [21–23], Atangana-Baleanu fractional derivative [24], FVIM [26], and many others [27–54].

The aim of this paper is to analyze the nonlinear fractional KGE with CF derivative by FHPTM. Let us assume the fractional KGE [27, 30] as:

$$h_{tt}(x,t) - h_{xx}(x,t) + ah(x,t) = r(x,t), \tag{1}$$

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having initial conditions

$$h(x,0) = f(x), \text{ and } h_t(x,0) = g(x). \tag{2}$$

where a is real constant, $h(x,t)$ is a complex valued function, t is a time variable and x is a space variable. The fractional KGE appear in physics. The KGE is solved by modified Adomian decomposition method by Yokchoo et al. (2020) [27] and HATM by Kumar et al. (2014) [30].

The present manuscript is organized as follows: Some definitions are given in Sect. 2, in Sect. 3 general description of FHPTM using CF derivative is discussed, in Sect. 4 test example is presented and in Sect. 5, we conclude our work.

2 Basic Tools

In this section, we present some fundamental notion of fractional calculus and Laplace transform, which are essential in the present framework.

Definition 2.1. The Caputo fractional derivative of order $\alpha \geq 0$ and $n \in N \cup \{0\}$ is define as:

$${}_0^{CF}D_t^{\alpha} h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{(n-\alpha-1)} \frac{d^n}{dt^n} h(\xi) d\xi, \tag{3}$$

where $n-1 \leq \alpha < n$.

Definition 2.2. Let $h \in K^1(a,b)$, $b > a$, then the Caputo-Fabrizio fractional differential operator is defined as:

$${}_0^{CF}D_t^{\alpha} h(t) = \frac{M(\alpha)}{(1-\alpha)} \int_a^t \exp\left[-\frac{\alpha(1-\xi)}{1-\alpha}\right] h'(\xi) d\xi, t \geq 0, 0 < \alpha < 1, \tag{4}$$

where $M(\alpha)$ is a normalisation function which satisfies $M(0) = M(1) = 1$.
${}_0^{CF}D_t^{\alpha} h(t) = 0$, if h is a constant function.

Definition 2.3. The CF integral of order $0 < \alpha < 1$ is given by

$${}_0^{CF}I_t^{\alpha} h(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} u(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t h(\xi) d\xi, t \geq 0, \tag{5}$$

Definition 2.4. The Laplace transform (LT) for the CF fractional operator of order $0 < \alpha \leq 1$ for $m \in N$ is given as:

$$\begin{aligned} L\left[{}_0^{CF}D_t^{(m+\alpha)} h(t)\right](s) &= \frac{1}{1-\alpha} L\left[h^{(m+1)}(t)\right] L\left[\exp\left(\frac{-\alpha}{(1-\alpha)} t\right)\right] \\ &= \frac{s^{(m+1)} L[h(t)] - s^m h(0) - s^{(m-1)} h'(0) \ldots\ldots.. - h^{(m)}(0)}{s + \alpha(1-s)}. \end{aligned} \tag{6}$$

In particular, we have

$$L\left({}_0^{CF}D_t^{\alpha} h(t)\right)(s) = \frac{sL(h(t)) - h(0)}{s + \alpha(1-s)}, \quad m = 0,$$

$$L\left({}_0^{CF}D_t^{\alpha+1} h(t)\right)(s) = \frac{s^2 h(f(t)) - sh(0) - h'(0)}{s + \alpha(1-s)}, \quad m = 1.$$

3 General Description of FHPTM Using CF Operator

We have considered the following nonlinear partial differential equation in the CF sense as:

$$ {}_{0}^{CF}D_{t}^{(m+\alpha)}h(x,t)+\rho h(x,t)+\sigma h(x,t)=k(x,t), n-1<\alpha+m\leq n, \tag{7}$$

with initial conditions:

$$\frac{\partial^{j}h(x.0)}{\partial t^{j}}=f_{j}(k), j=0,1,2\ldots\ldots.., m-1. \tag{8}$$

Applying the LT on both Eq. (7) and Eq. (8), we get

$$L[h(x,t)]=\eta(x,s)-\left(\frac{s+\alpha(1-s)}{s^{(m+1)}}\right)L[\rho h(x,t)+\sigma h(x,t)], \tag{9}$$

where

$$\eta(x,s)=\frac{1}{s^{(m+1)}}\left[s^{m}f_{0}(x)+s^{m-1}f_{1}(x)+\ldots\cdots+f_{m}(x)\right]+\frac{s+\alpha(1-s)}{s^{(m+1)}}\tilde{t}h(x,s). \tag{10}$$

Taking the inverse LT the Eq. (9) yields

$$h(x,t)=\eta(x,s)-L^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{(m+1)}}\right)L[\rho h(x,t)+\sigma h(x,t)]\right]. \tag{11}$$

where $\eta(x,s)$ arises from the source term.

Now, we apply the FHPTM to obtain the solution of Eq. (11) starting by the hypothesis that $h(x,t)$ expressed below is a solution of this equation.

$$h(x,t)=\sum\nolimits_{n=0}^{\infty}z^{m}h_{m}(x,t), \tag{12}$$

where, $h_m(x,t)$ are known functions, the nonlinear term can be decomposed as:

$$\sigma h(x,t)=\sum\nolimits_{n=0}^{\infty}z^{m}H_{m}(x,t). \tag{13}$$

The polynomials $H_m(x,t)$ [28] are

$$H_{n}(h_{0},h_{1},h_{2},\ldots\ldots,h_{m})=\frac{1}{m!}\frac{\partial^{m}}{\partial p^{m}}\left[\sigma\left(\sum\nolimits_{i=0}^{\infty}z^{i}h_{i}\right)\right]_{z=0}, m=0,1,2,\ldots. \tag{14}$$

Substituting Eq. (12) and Eq. (13) into (11), we get

$$\sum\nolimits_{n=0}^{\infty}h_{m}(x,t)=\eta(x,s)-zL^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{(m+1)}}\right)L\left[\rho\sum\nolimits_{m=0}^{\infty}z^{m}h_{m}(x,t),+\sigma\sum\nolimits_{m=0}^{\infty}z^{m}H_{m}\right]\right], \tag{15}$$

comparing the coefficients of z^0, z^1, z^2, and z^3, we get

$$\begin{aligned}
&z^0 : h_0(x,t) = \eta(x,s),\\
&z^1 : h_1(x,t) = -L^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{(m+1)}}\right)L[\rho h_0(x,t) + H_0(u)]\right],\\
&z^2 : h_2(x,t) = -L^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{(m+1)}}\right)L[\rho h_1(x,t) + H_1(u)]\right],\\
&z^3 : h_3(x,t) = -L^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{(m+1)}}\right)L[\rho h_2(x,t) + H_2(u)]\right],\\
&z^{m+1} : h_{m+1}(x,t) = -L^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{(m+1)}}\right)L\big[\rho h_{m+1}(x,t) + H_{m+1}(u)\big]\right].
\end{aligned} \tag{16}$$

4 Applications

In this segment, four different examples are solved by FHPTM.

Example 4.1. Let us assume the linear fractional KGE as [27, 30]:

$$D_{tt}^{\alpha}h(x,t) - h_{xx}(x,t) + h(x,t) = 0, \tag{17}$$

having initial conditions

$$h(x,0) = f(x), \quad h_t(x,0) = g(x). \tag{18}$$

Taking the LT on Eq. (17) both sides and from Eq. (18), we get

$$L[h(x,t)] = \frac{1}{s^2}x + \left(\frac{s+\alpha(1-s)}{s^2}\right)L[h_{xx} + h]. \tag{19}$$

Applying the inverse of the LT to Eq. (19), we get

$$h(x,t) = xt + L^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^2}\right)L[h_{xx} + h]\right]. \tag{20}$$

Now, we apply the HPTM, we have

$$\sum_{m=0}^{\infty} h_n(x,t) = xt + zL^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^2}\right)L\left[\sum_{m=0}^{\infty} z^m h_m(x,t) - \sum_{m=0}^{\infty} z^m h_m(x,t)\right]\right], \tag{21}$$

$$\begin{aligned}
z^0 &: h_0(x,\ t) = xt,\\
z^1 &: h_1(x,\ t) = xt\left[t(1-\ \alpha) + \frac{t^2\,\alpha}{2}\right],\\
z^2 &: h_2(x,\ t) = xt\left[\frac{t^2}{2}\left(1-2\,\alpha+\alpha^2\right) - \frac{t^3}{3}\left(-\alpha+\ \alpha^2\right) + \frac{t^4\,\alpha}{24}\right].
\end{aligned} \tag{22}$$

Example 4.2. Let us assume the linear fractional KGE as [27, 30]:

$$D_{tt}^{\alpha} h(x,t) - h_{xx}(x,t) + h(x,t) = 2\sin x, \tag{23}$$

having initial conditions

$$h(x,0) = \sin x, \quad \text{and} \quad h_t(x,0) = 1. \tag{24}$$

Taking the LT on Eq. (22) both sides and from Eq. (23), we have

$$L[h(x,t)] = \frac{1}{s^2}\sin x + \frac{1}{s^2} + \frac{s+\alpha(1-s)}{s^2}2\sin x + \left(\frac{s+\alpha(1-s)}{s^2}\right)L[h_{xx} - h]. \tag{25}$$

Applying inverse LT to Eq. (24), we get

$$h(x,t) = 2t\sin x - 2t\alpha\sin x + t^2\alpha\sin x + t + L^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^2}\right)L[h_{xx} - h]\right]. \tag{26}$$

Now, we apply the HPTM, we have

$$\begin{aligned}
&\sum_{m=0}^{\infty} h_m(x,t) = 2t\sin x - 2t\alpha\sin x + t^2\alpha\sin x + t\\
&+zL^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^2}\right)L\left[\sum_{n=0}^{\infty} z^n h_m(x,t) - \sum_{n=0}^{\infty} z^n h_n(x,t)\right]\right],
\end{aligned} \tag{27}$$

$$\begin{aligned}
&z^0 : h_0(x,t) = 2t\sin x - 2t\alpha\sin x + t^2\alpha\sin x + t,\\
&z^1 : h_1(x,t) = \frac{t^2}{2}\left(-1+\alpha-4\sin x + 8\alpha\sin x - 4\alpha^2\sin x\right)\\
&+\frac{t^3}{2}\left(-\alpha - 8\alpha\sin x + 8\alpha^2\sin x\right) - \frac{t^2\alpha^2}{6}\sin x,\\
&z^2 : h_2(x,t) = \frac{t^3}{6}\left(-1+2\alpha+\alpha^2+8\sin x - 24\alpha\sin x + 24\alpha^2\sin x - 8\alpha^3\sin x\right)\\
&+\frac{t^4}{12}\left(\alpha-\alpha^2+12\alpha\sin x + 8\alpha^2\sin x\right)\\
&+\frac{t^5}{120}\left(\alpha^2+24\alpha^2\sin x - 24\alpha^3\sin x\right) - \frac{t^6\alpha^3}{90}\sin x.
\end{aligned} \tag{28}$$

Example 4.3. Let us assume the linear fractional KGE as [27, 30]:

$$D_{tt}^{\alpha}h(x,t) - h_{xx}(x,t) + h^2(x,t) = x^2t^2, \tag{29}$$

having initial conditions

$$h(x,0) = 0, \text{ and } h_t(x,0) = x. \tag{30}$$

Taking the LT on Eq. (29) both sides and from Eq. (30), we have

$$L[h(x,t)] = \frac{1}{s^2}x + +\frac{s+\alpha(1-s)}{s^2}\left(\frac{2x^2}{s^3}\right) + \left(\frac{s+\alpha(1-s)}{s^2}\right)L\Big[h_{xx} - h^2\Big]. \tag{31}$$

Applying the inverse of the LT to Eq. (31), we have

$$h(x,t) = xt + 2x^2\left[\frac{1}{6}t^3(1-\alpha) + \frac{t^4\alpha}{24}\right] + L^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^2}\right)L\Big[h_{xx} - h^2\Big]\right]. \tag{32}$$

Now, we apply the HPTM, we have

$$\begin{aligned}&\sum\nolimits_{m=0}^{\infty} h_m(x,t) = xt + 2x^2[\frac{1}{6}t^3(1-\alpha) + \frac{t^4\alpha}{24}]\\ &+zL^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^2}\right)L\Big[\sum\nolimits_{m=0}^{\infty} z^m h_m(x,t) - \sum\nolimits_{m=0}^{\infty} z^m H_m(x,t)\Big]\right].\end{aligned} \tag{33}$$

where $H_m(h)$ is the He's polynomial used to decomposed the nonlinear term is defined as:

$$\begin{aligned}&H_0(h) = h_0^2,\\ &H_1(h) = \frac{\partial}{\partial z}\Big[(h_0 + zh_1)^2\Big]_{z=0} = 2h_0h_1.\end{aligned} \tag{34}$$

Comparing the coefficient of z in Eq. (33), we get

$$\begin{aligned}&z^0 : h_0(x,t) = xt + \frac{x^2t^2}{3}(1-\alpha) + \frac{t^4\alpha x^2}{12},\\ &z^1 : h_1(x,t) = \frac{t^3}{3}\Big(-x^2 + x^4\alpha\Big) - \frac{t^4}{12}\Big(-2 + 4\alpha^2x^2 - 2\alpha^2\Big)\\ &-\frac{t^5}{15}\Big(2x^3 - \alpha - 4x^3\alpha + \alpha^2 + 2x^3\alpha^2\Big) + \frac{t^6}{180}\Big(-9x^3\alpha + \alpha^2 + 9x^3\alpha^2\Big)\\ &+\frac{t^7}{252}\Big(-4x^2 + 12x^4\alpha - x^3\alpha^2 - 12x^4\alpha^2 + 4x^2\alpha^3\Big)\\ &-\frac{t^8}{112}\Big(x^4\alpha - 2x^4\alpha^2 + x^4\alpha^3\Big) + \frac{t^9}{648}\Big(-x^4\alpha^2 + x^4\alpha^3\Big) - \frac{x^4\alpha^3t^{10}}{12960},\\ &z^2 : h_2(x,t) = \frac{t^4}{16}\Big(-1 + 2\alpha + \alpha^2\Big) + \frac{t^5}{15}\Big(2x^3 - \alpha - 4x^3\alpha + \alpha^2 + 2x^3\alpha^2\Big)\\ &-\frac{t^6}{180}\Big(34x - 102x\alpha - 9x^3\alpha + \alpha^2 + 102x\alpha^2 + 9x^3\alpha^2 - 34x\alpha^3\Big)\\ &-\frac{t^7}{1260}\Big(-88x^4 + 112x\alpha + 264x^4\alpha - 224x\alpha^2 - 5x^3\alpha^2 - 264x^4\alpha^2 + 112x\alpha^3 + 88x^4\alpha^3\Big) + \dots.\end{aligned} \tag{35}$$

Example 4.4. Let us assume the linear fractional KGE as [27, 30]:

$$D_{tt}^{\alpha} h(x,t) - h_{xx}(x,t) + h^2(x,t) = 2x^2 - 2t^2 + x^4 t^4, \tag{36}$$

having initial conditions

$$h(x,0) = 0, \text{ and } h_t(x,0) = 0. \tag{37}$$

Taking the LT on Eq. (36) both sides and from Eq. (37), we have

$$L[h(x,t)] = \frac{s+\alpha(1-s)}{s^2}\left(\frac{-4}{s^3} + \frac{2x^2}{s} + \frac{2x^4}{s^3}\right) + \left(\frac{s+\alpha(1-s)}{s^2}\right) L\left[h_{xx} - h^2\right]. \tag{38}$$

Applying the inverse of the LT to Eq. (38), we have

$$h(x,t) = t^2x^2\alpha - 2t\left(-x^2 + x^2\alpha\right) + \frac{t^4}{12}\left(-2\alpha + x^4\alpha\right) - \frac{t^3}{3}\left(2 - x^4 - 2\alpha + x^4\alpha\right) + L^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^2}\right) L\left[h_{xx} - h^2\right]\right]. \tag{39}$$

Now, we apply the HPTM, we have

$$\sum_{m=0}^{\infty} h_m(x,t) = t^2x^2\alpha - 2t\left(-x^2 + x^2\alpha\right) + \frac{t^4}{12}\left(-2\alpha + x^4\alpha\right) - \frac{t^3}{3}\left(2 - x^4 - 2\alpha + x^4\alpha\right) + zL^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^2}\right) L\left[\sum_{m=0}^{\infty} z^m h_m(x,t) - \sum_{m=0}^{\infty} z^m H_m(x,t)\right]\right]. \tag{40}$$

where $H_m(h)$ is the He's polynomial used to decomposed the nonlinear term is defined as:

$$\begin{aligned} H_0(h) &= h_0^2, \\ H_1(h) &= \frac{\partial}{\partial z}\left[(h_0 + zh_1)^2\right]_{z=0} = 2h_0h_1. \end{aligned} \tag{41}$$

Comparing the coefficient of z in Eq. (40), we get

$$
\begin{aligned}
&z^0 : h_0(x,t) = t^2x^2\alpha - 2t\left(-x^2 + x^2\alpha\right) + \frac{t^4}{12}\left(-2\alpha + x^4\alpha\right) \\
&-\frac{t^3}{3}\left(2 - x^4 - 2\alpha + x^4\alpha\right), \\
&z^1 : h_1(x,t) = 2t^2\left(1 - 2\alpha + \alpha^2\right) + \frac{4t^3}{3}\left(-x^4 + \alpha + 3x^4\alpha - \alpha^2 - 3x^4\alpha^2 + x^4\alpha^3\right) \\
&-\frac{2t^5}{15}(4x^2 - 2x^6 - 9x^2\alpha + 6x^6\alpha + 9x^2\alpha^2 - 3x^4\alpha^2 - 6x^6\alpha^2 - 4x^2\alpha^3 \\
&+3x^4\alpha^3 + 2x^6\alpha^3) \\
&-\frac{t^6}{90}\left(-38x^2\alpha + 19x^6\alpha + 73x^2\alpha^2 - 38x^6\alpha^2 + 3x^4\alpha^3 + 19x^6\alpha^3\right) \\
&+\frac{t^7}{63}\left(-4 + 4x^4 - x^8 + 12\alpha - 12x^4\alpha + 3x^8\alpha - 12\alpha^2 + 6x^2\alpha^2 + 12x^4\alpha^2\right. \\
&\left.-3x^6\alpha^2 - 3x^8\alpha^2 + 4\alpha^3 - \ldots\right) \\
&-\frac{t^8}{648}(12\alpha - 12x^4\alpha + 3x^8\alpha - 24\alpha^2 + 24x^4\alpha^2 - 6x^8\alpha^2 + 12\alpha^3 - 2x^2\alpha^3 \\
&-12x^4\alpha^3 + x^6\alpha^3 + 3x^8\alpha^3) \\
&+\frac{t^9}{648}\left(-4\alpha^2 + 4x^4\alpha^2 - x^8\alpha^2 + 4\alpha^3 - 4x^4\alpha^3 + x^8\alpha^3\right) \\
&-\frac{t^{10}}{12960}\left(4\alpha^3 - 4x^4\alpha^3 + x^8\alpha^3\right), \\
&z^2 : h_2(x,t) = -6t^4\left(x^4 - 4x^2\alpha + 6x^2\alpha^2 - 4x^2\alpha^3 + x^2\alpha^4\right) - \frac{2t^5}{15}(-3 - 8x^6 + 9\alpha \\
&+47x^2\alpha + 40x^6\alpha - 9\alpha^2 - 141x^2\alpha^2 - 80x^6\alpha^2 + 3\alpha^2 + 141x^2\alpha^3 + 80x^6\alpha^3 - \ldots.) \\
&+\frac{t^6}{45}\left(28 - 100x^4 - 103\alpha + 400x^4\alpha + 68x^6\alpha + 150\alpha^2 - 99x^2\alpha^2 - 600x^4\alpha^2 - 272x^6\alpha^2 - 103\alpha^3 + \ldots\right) \\
&-\frac{t^7}{315}(176x^4 - 88x^8 - 176\alpha - 2778x^4\alpha + 440x^8\alpha + 519\alpha^2 \\
&-46x^4\alpha^2 - 252x^6\alpha^2 - 880x^8\alpha^2 - \ldots) + \ldots.
\end{aligned}
\tag{42}
$$

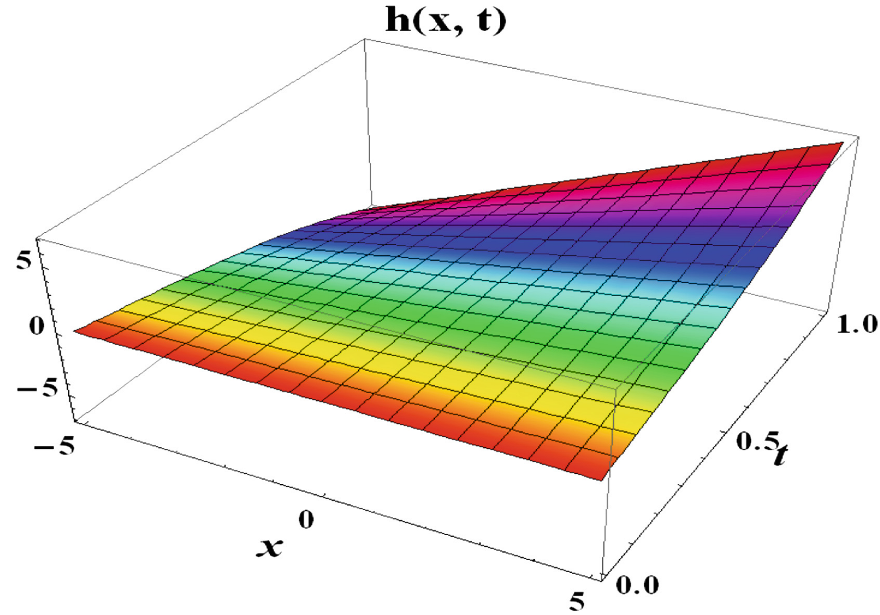

Fig. 1. FHPTM solution $h(x,t)$ at $\alpha = 2$ for Ex.4.1.

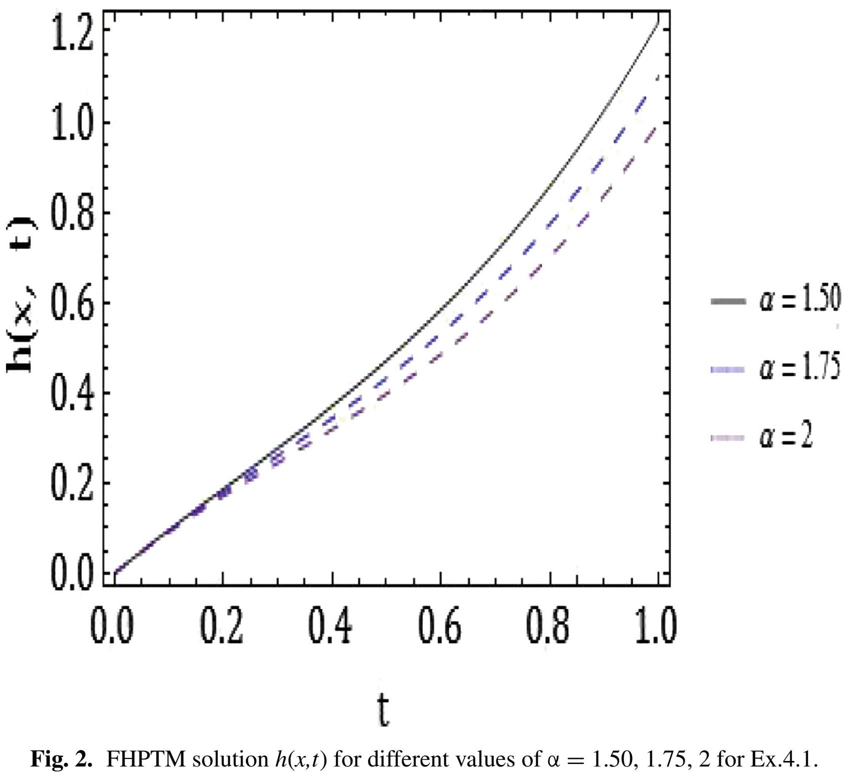

Fig. 2. FHPTM solution $h(x,t)$ for different values of $\alpha = 1.50, 1.75, 2$ for Ex.4.1.

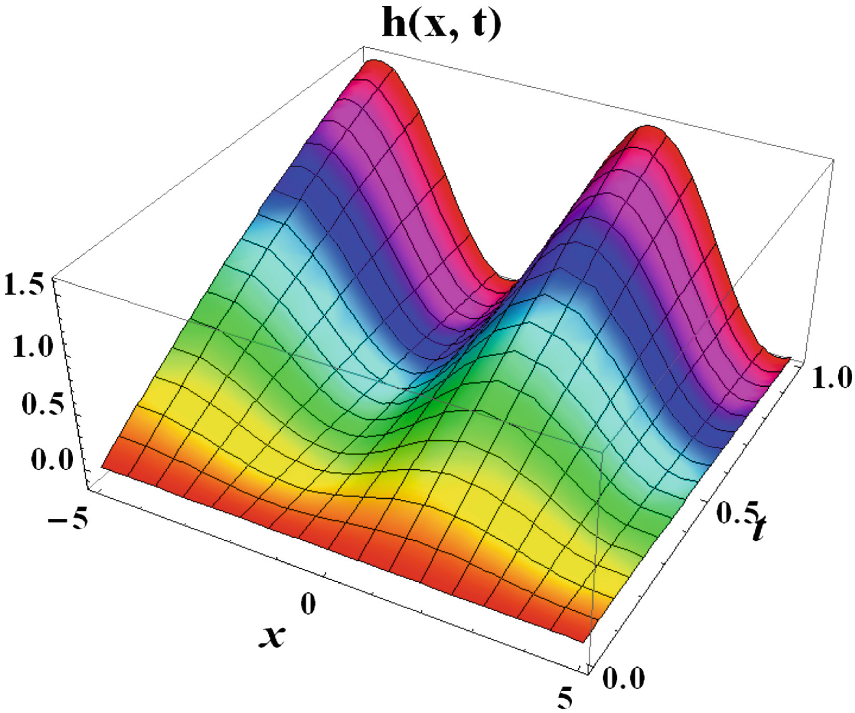

Fig. 3. FHPTM solution $h(x,t)$ at $\alpha = 2$ for Ex.4.2.

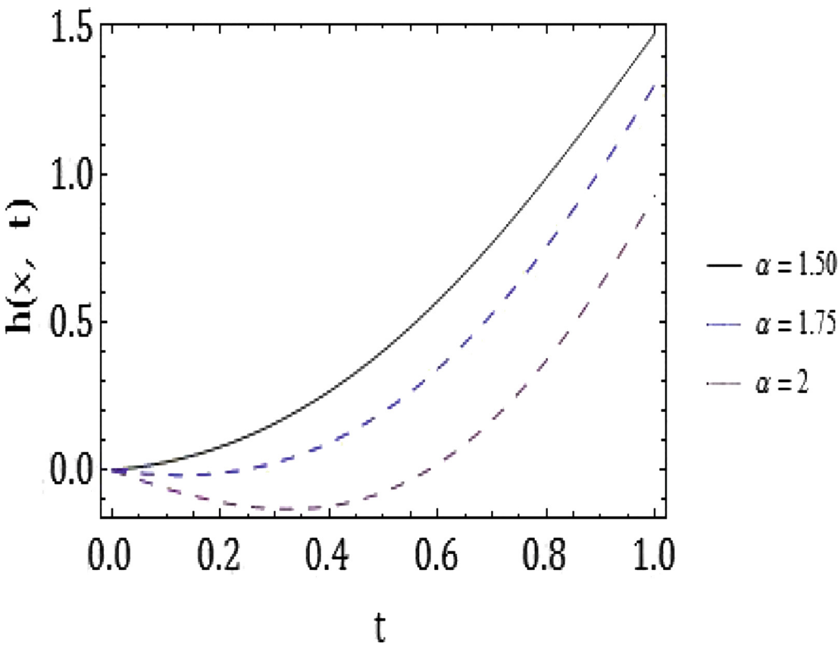

Fig. 4. FHPTM solution $h(x,t)$ for distinct values of $\alpha = 1.50, 1.75, 2$ for Ex.4.2.

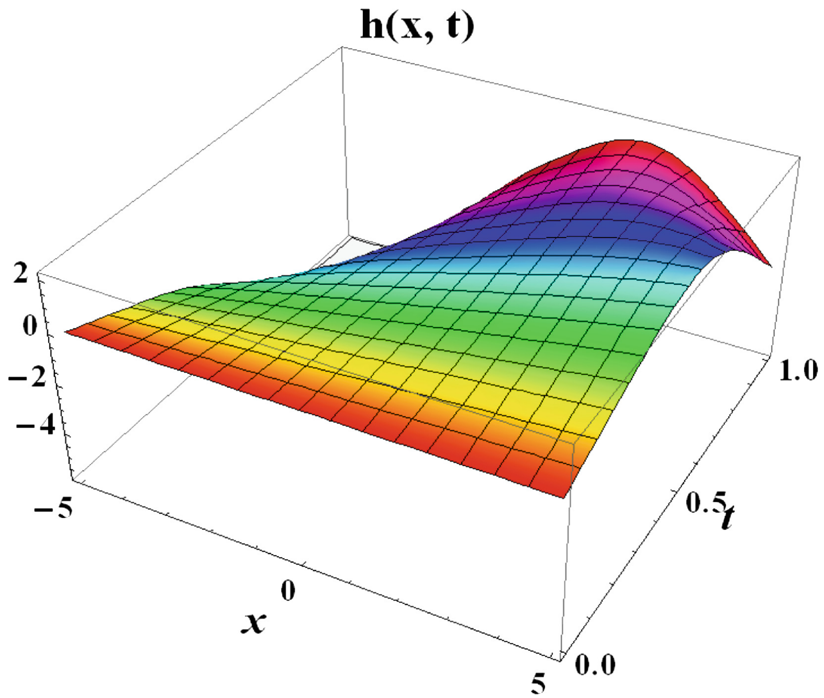

Fig. 5. FHPTM solution $h(x,t)$ at $\alpha = 2$ for Ex.4.3.

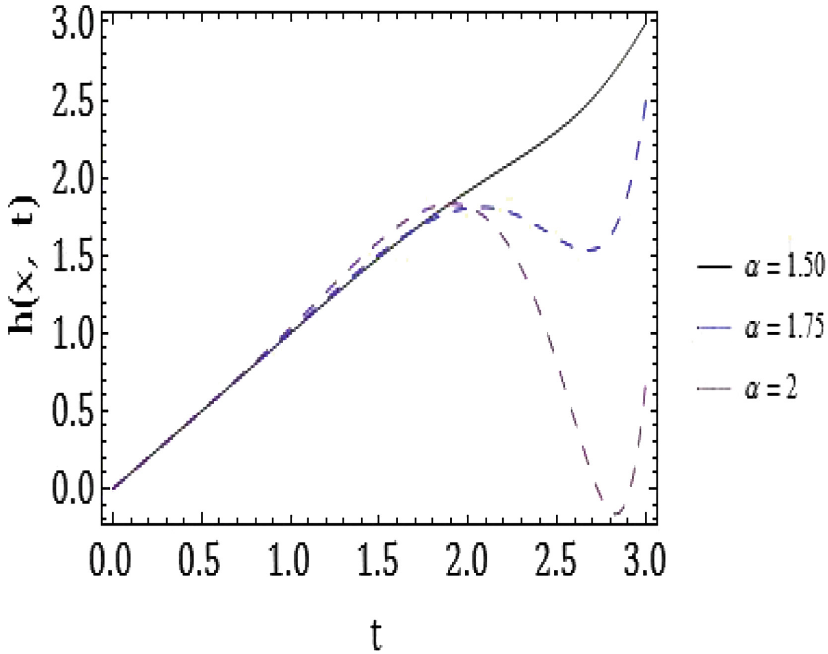

Fig. 6. FHPTM solution $h(x,t)$ for distinct values of $\alpha = 1.50, 1.75, 2$ for Ex.4.3.

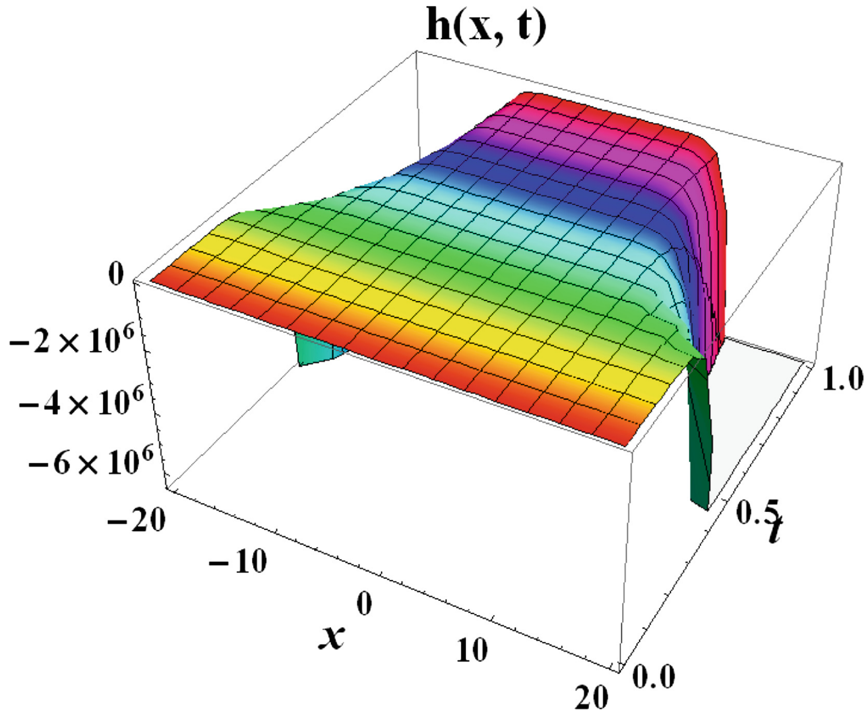

Fig. 7. FHPTM solution $h(x,t)$ at $\alpha = 2$ for Ex.4.4.

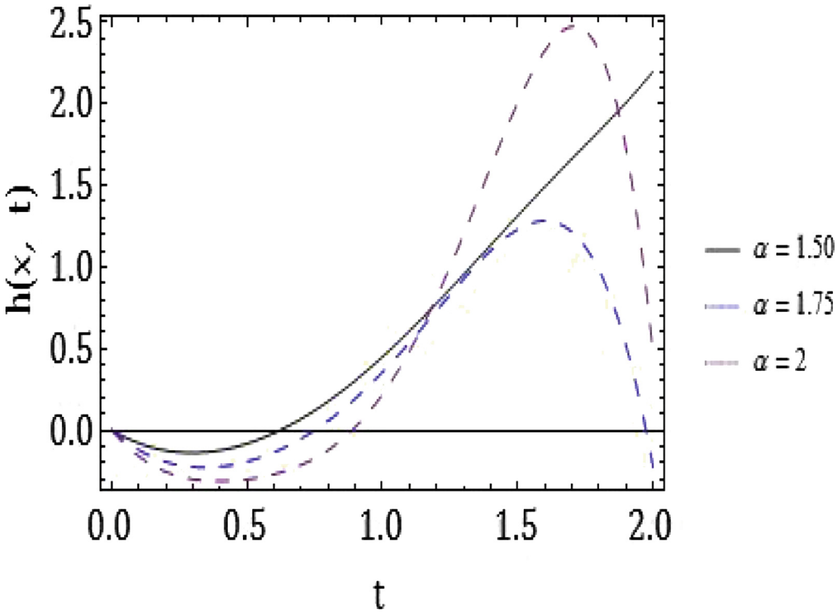

Fig. 8. FHPTM solution $h(x,t)$ for distinct values of $\alpha = 1.50, 1.75, 2$ for Ex.4.4.

5 Conclusion

In this present work, the FHPTM is applied on the fractional linear and nonlinear KGE via CF derivative (Figs. 1, 2, 3, 4, 5, 6, 7 and 8). We note that the approximate series solutions acquired for the first three terms is very suitable and converges very strongly with solutions to real physical problems. The above method is reliable, simple and dominant in seeking approximate solution to different nonlinear Klein-Gordon fractional order equation. Finally, we can draw the conclusion that the proposed FHPTM is highly expressive and can be used to analyze broad class of the fractional order linear and nonlinear models to perceive the behavior of the phenomena that emerged in linked sciences and engineering areas.

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