Multivariate Fuzzy-Random and Stochastic Arctangent, Algebraic, Gudermannian and Generalized Symmetric Activation Functions Induced Neural Network Approximations

George A. Anastassiou(✉)

Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA
ganastss@memphis.edu

Abstract. In this article we study the degree of approximation of multivariate pointwise and uniform convergences in the q-mean to the Fuzzy-Random unit operator of multivariate Fuzzy-Random Quasi-Interpolation arctangent, algebraic, Gudermannian and generalized symmetric activation functions based neural network operators. These multivariate Fuzzy-Random operators arise in a natural way among multivariate Fuzzy-Random neural networks. The rates are given through multivariate Probabilistic-Jackson type inequalities involving the multivariate Fuzzy-Random modulus of continuity of the engaged multivariate Fuzzy-Random function. The plain stochastic extreme analog of this theory is also met in detail for the stochastic analogs of the operators: the stochastic full quasi-interpolation operators, the stochastic Kantorovich type operators and the stochastic quadrature type operators.

Keywords: Fuzzy-Random analysis · Fuzzy-Random neural networks and operators · Fuzzy-Random modulus of continuity · Fuzzy-Random functions · Stochastic processes · Jackson type fuzzy and probabilistic inequalities

1 Fuzzy-Random Functions and Stochastic Processes Background

See also [18], Ch. 22, pp. 497–501.

We start with

Definition 1 *(see [35]). Let $\mu : \mathbb{R} \to [0, 1]$ with the following properties:*

(i) is normal, i.e., $\exists\, x_0 \in \mathbb{R} : \mu(x_0) = 1$.

(ii) $\mu(\lambda x + (1 - \lambda) y) \geq \min\{\mu(x), \mu(y)\}$, $\forall\, x, y \in \mathbb{R}$, $\forall\, \lambda \in [0, 1]$ *(μ is called a convex fuzzy subset).*

J. Singh et al. (Eds.): ICMMAAC 2022, LNNS 666, pp. 1–32, 2023.
https://doi.org/10.1007/978-3-031-29959-9_1

(iii) μ *is upper semicontinuous on* $\mathbb{R}$*, i.e.,* $\forall\ x_0 \in \mathbb{R}$ *and* $\forall\ \varepsilon > 0$*,* $\exists$ *neighborhood* $V(x_0) : \mu(x) \leq \mu(x_0) + \varepsilon,\ \forall\ x \in V(x_0)$.

(iv) the set $\overline{supp(\mu)}$ *is compact in* $\mathbb{R}$ *(where* $supp(\mu) := \{x \in \mathbb{R}; \mu(x) > 0\}$*).*

We call μ *a fuzzy real number. Denote the set of all* μ *with* $\mathbb{R}_{\mathcal{F}}$.

E.g., $\chi_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$*, for any* $x_0 \in \mathbb{R}$*, where* $\chi_{\{x_0\}}$ *is the characteristic function at* x_0.

For $0 < r \leq 1$ *and* $\mu \in \mathbb{R}_{\mathcal{F}}$ *define* $[\mu]^r := \{x \in \mathbb{R} : \mu(x) \geq r\}$ *and* $[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) > 0\}}$.

Then it is well known that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded interval of $\mathbb{R}$. For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \ \ [\lambda \odot u]^r = \lambda [u]^r, \ \forall\ r \in [0,1],$$

where $[u]^r + [v]^r$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $\lambda [u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g., [35]). Notice $1 \odot u = u$ and it holds $u \oplus v = v \oplus u$, $\lambda \odot u = u \odot \lambda$. If $0 \leq r_1 \leq r_2 \leq 1$ then $[u]^{r_2} \subseteq [u]^{r_1}$. Actually $[u]^r = \left[u_-^{(r)}, u_+^{(r)}\right]$, where $u_-^{(r)} < u_+^{(r)}$, $u_-^{(r)}, u_+^{(r)} \in \mathbb{R}$, $\forall\ r \in [0,1]$.

Define

$$D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+ \cup \{0\}$$

by

$$D(u,v) := \sup_{r \in [0,1]} \max \left\{ \left| u_-^{(r)} - v_-^{(r)} \right|, \left| u_+^{(r)} - v_+^{(r)} \right| \right\},$$

where $[v]^r = \left[v_-^{(r)}, v_+^{(r)}\right]$; $u, v \in \mathbb{R}_{\mathcal{F}}$. We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$. Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [35], with the properties

$$\begin{gathered} D(u \oplus w, v \oplus w) = D(u,v), \ \forall\ u, v, w \in \mathbb{R}_{\mathcal{F}}, \\ D(k \odot u, k \odot v) = |k|\, D(u,v), \ \forall\ u, v \in \mathbb{R}_{\mathcal{F}}, \ \forall\ k \in \mathbb{R}, \\ D(u \oplus v, w \oplus e) \leq D(u,w) + D(v,e), \ \forall\ u, v, w, e \in \mathbb{R}_{\mathcal{F}}. \end{gathered} \tag{1}$$

Let (M,d) metric space and $f, g : M \to \mathbb{R}_{\mathcal{F}}$ be fuzzy real number valued functions. The distance between f, g is defined by

$$D^*(f,g) := \sup_{x \in M} D(f(x), g(x)).$$

On $\mathbb{R}_{\mathcal{F}}$ we define a partial order by "$\leq$": $u, v \in \mathbb{R}_{\mathcal{F}}$, $u \leq v$ iff $u_-^{(r)} \leq v_-^{(r)}$ and $u_+^{(r)} \leq v_+^{(r)}$, $\forall\ r \in [0,1]$.

$\overset{*}{\sum}$ denotes the fuzzy summation, $\widetilde{o} := \chi_{\{0\}} \in \mathbb{R}_{\mathcal{F}}$ the neutral element with respect to $\oplus$. For more see also [36,37].

We need

Definition 2 *(see also [30], Definition 13.16, p. 654).* *Let $(X, \mathcal{B}, P)$ be a probability space. A fuzzy-random variable is a $\mathcal{B}$-measurable mapping $g : X \to \mathbb{R}_{\mathcal{F}}$ (i.e., for any open set $U \subseteq \mathbb{R}_{\mathcal{F}}$, in the topology of $\mathbb{R}_{\mathcal{F}}$ generated by the metric D, we have*

$$g^{-1}(U) = \{s \in X; g(s) \in U\} \in \mathcal{B}). \tag{2}$$

The set of all fuzzy-random variables is denoted by $\mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$. Let $g_n, g \in \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $n \in \mathbb{N}$ and $0 < q < +\infty$. We say $g_n(s) \underset{n\to+\infty}{\overset{\text{"q-mean"}}{\to}} g(s)$ if

$$\lim_{n\to+\infty} \int_X D(g_n(s), g(s))^q P(ds) = 0. \tag{3}$$

Remark 1 (see [30], p. 654). If $f, g \in \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, let us denote $F : X \to \mathbb{R}_+ \cup \{0\}$ by $F(s) = D(f(s), g(s))$, $s \in X$. Here, F is $\mathcal{B}$-measurable, because $F = G \circ H$, where $G(u, v) = D(u, v)$ is continuous on $\mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}}$, and $H : X \to \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}}$, $H(s) = (f(s), g(s))$, $s \in X$, is $\mathcal{B}$-measurable. This shows that the above convergence in q -mean makes sense.

Definition 3 *(see [30], p. 654, Definition 13.17).* *Let $(T, \mathcal{T})$ be a topological space. A mapping $f : T \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$ will be called fuzzy-random function (or fuzzy-stochastic process) on T. We denote $f(t)(s) = f(t, s)$, $t \in T$, $s \in X$.*

Remark 2 (see [30], p. 655). Any usual fuzzy real function $f : T \to \mathbb{R}_{\mathcal{F}}$ can be identified with the degenerate fuzzy-random function $f(t, s) = f(t)$, $\forall\ t \in T$, $s \in X$.

Remark 3 (see [30], p. 655). Fuzzy-random functions that coincide with probability one for each $t \in T$ will be consider equivalent.

Remark 4 (see [30], p. 655). Let $f, g : T \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$. Then $f \oplus g$ and $k \odot f$ are defined pointwise, i.e.,

$$\begin{aligned}
&(f \oplus g)(t, s) = f(t, s) \oplus g(t, s), \\
&(k \odot f)(t, s) = k \odot f(t, s)\ \ , t \in T, s \in X,\ k \in \mathbb{R}.
\end{aligned}$$

Definition 4 *(see also Definition 13.18, pp. 655–656, [30]).* *For a fuzzy-random function $f : W \subseteq \mathbb{R}^N \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $N \in \mathbb{N}$, we define the (first) fuzzy-random modulus of continuity*

$$\Omega_1^{(\mathcal{F})}(f, \delta)_{L^q} =$$

$$\sup\left\{\left(\int_X D^q(f(x, s), f(y, s)) P(ds)\right)^{\frac{1}{q}} : x, y \in W,\ \ \|x - y\|_\infty \le \delta\right\},$$

$0 < \delta$, $1 \le q < \infty$.

Definition 5 [16]. *Here $1 \le q < +\infty$. Let $f : W \subseteq \mathbb{R}^N \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $N \in \mathbb{N}$, be a fuzzy random function. We call f a (q-mean) uniformly continuous fuzzy* *random function over* W, *iff* $\forall\, \varepsilon > 0\ \exists\, \delta > 0$:*whenever* $\|x-y\|_\infty \le \delta$, $x, y \in W$, *implies that*

$$\int_X \left(D\left(f\left(x,s\right), f\left(y,s\right)\right)\right)^q P\left(ds\right) \le \varepsilon.$$

We denote it as $f \in C_{FR}^{U_q}\left(W\right)$.

Proposition 1 [16]**.** *Let* $f \in C_{FR}^{U_q}\left(W\right)$, *where* $W \subseteq \mathbb{R}^N$ *is convex.*
Then $\Omega_1^{(\mathcal{F})}\left(f,\delta\right)_{L^q} < \infty$, *any* $\delta > 0$.

Proposition 2 [16]**.** *Let* $f, g : W \subseteq \mathbb{R}^N \to \mathcal{L}_{\mathcal{F}}\left(X, \mathcal{B}, P\right)$, $N \in \mathbb{N}$, *be fuzzy random functions. It holds*

(i) $\Omega_1^{(\mathcal{F})}\left(f,\delta\right)_{L^q}$ *is nonnegative and nondecreasing in* $\delta > 0$.
(ii) $\lim_{\delta \downarrow 0} \Omega_1^{(\mathcal{F})}\left(f,\delta\right)_{L^q} = \Omega_1^{(\mathcal{F})}\left(f,0\right)_{L^q} = 0$, *iff* $f \in C_{FR}^{U_q}\left(W\right)$.

We mention

Definition 6 *(see also* [6]*).* *Let* $f\left(t,s\right)$ *be a random function (stochastic process) from* $W \times \left(X, \mathcal{B}, P\right)$, $W \subseteq \mathbb{R}^N$, *into* $\mathbb{R}$, *where* $\left(X, \mathcal{B}, P\right)$ *is a probability space. We define the q-mean multivariate first modulus of continuity of* f *by*

$$\Omega_1\left(f,\delta\right)_{L^q} :=$$

$$\sup\left\{\left(\int_X \left|f\left(x,s\right) - f\left(y,s\right)\right|^q P\left(ds\right)\right)^{\frac{1}{q}} : x, y \in W,\ \ \|x-y\|_\infty \le \delta\right\}, \quad (4)$$

$\delta > 0$, $1 \le q < \infty$.

The concept of f being (q-mean) uniformly continuous random function is defined the same way as in Definition 5, just replace D by $|\cdot|$, etc. We denote it as $f \in C_{\mathbb{R}}^{U_q}\left(W\right)$.

Similar properties as in Propositions 1, 2 are valid for $\Omega_1\left(f,\delta\right)_{L^q}$.

Also we have

Proposition 3 [3]**.** *Let* $Y\left(t,\omega\right)$ *be a real valued stochastic process such that* Y *is continuous in* $t \in [a,b]$. *Then* Y *is jointly measurable in* $\left(t,\omega\right)$.

According to [28], p. 94 we have the following

Definition 7. *Let* $\left(Y, \mathcal{T}\right)$ *be a topological space, with its* σ*-algebra of Borel sets* $\mathcal{B} := \mathcal{B}\left(Y, \mathcal{T}\right)$ *generated by* $\mathcal{T}$. *If* $\left(X, \mathcal{S}\right)$ *is a measurable space, a function* $f : X \to Y$ *is called measurable iff* $f^{-1}\left(B\right) \in \mathcal{S}$ *for all* $B \in \mathcal{B}$.

By Theorem 4.1.6 of [28], p. 89 f as above is measurable iff

$$f^{-1}\left(C\right) \in \mathcal{S} \text{ for all } C \in \mathcal{T}.$$

We mention

Theorem 1 *(see [28], p. 95)*. *Let $(X, \mathcal{S})$ be a measurable space and (Y, d) be a metric space. Let f_n be measurable functions from X into Y such that for all $x \in X$, $f_n(x) \to f(x)$ in Y. Then f is measurable. I.e., $\lim_{n\to\infty} f_n = f$ is measurable.*

We need also

Proposition 4 [16]. *Let f, g be fuzzy random variables from $\mathcal{S}$ into $\mathbb{R}_{\mathcal{F}}$. Then*

(i) Let $c \in \mathbb{R}$, then $c \odot f$ is a fuzzy random variable.
(ii) $f \oplus g$ is a fuzzy random variable.

Proposition 5. *Let $Y\left(\overrightarrow{t}, \omega\right)$ be a real valued multivariate random function (stochastic process) such that Y is continuous in $\overrightarrow{t} \in \prod_{i=1}^{N} [a_i, b_i]$. Then Y is jointly measurable in $\left(\overrightarrow{t}, \omega\right)$ and $\int_{\prod_{i=1}^{N}[a_i,b_i]} Y\left(\overrightarrow{t}, \omega\right) d\overrightarrow{t}$ is a real valued random variable.*

Proof. Similar to Proposition 18.14, p. 353 of [7].

2 About Neural Networks Background

2.1 About the Arctangent Activation Function

We consider the

$$\arctan x = \int_0^x \frac{dz}{1+z^2}, \ x \in \mathbb{R}. \tag{5}$$

We will be using

$$h(x) := \frac{2}{\pi} \arctan\left(\frac{\pi}{2} x\right) = \frac{2}{\pi} \int_0^{\frac{\pi x}{2}} \frac{dz}{1+z^2}, \ x \in \mathbb{R}, \tag{6}$$

which is a sigmoid type function and it is strictly increasing. We have that

$$h(0) = 0, \ h(-x) = -h(x), \ h(+\infty) = 1, \ h(-\infty) = -1,$$

and

$$h'(x) = \frac{4}{4+\pi^2 x^2} > 0, \text{ all } x \in \mathbb{R}. \tag{7}$$

We consider the activation function

$$\psi_1(x) := \frac{1}{4}\left(h(x+1) - h(x-1)\right), \ x \in \mathbb{R}, \tag{8}$$

and we notice that

$$\psi_1(-x) = \psi_1(x), \tag{9}$$

it is an even function.

Since $x+1>x-1$, then $h(x+1)>h(x-1)$, and $\psi_1(x)>0$, all $x\in\mathbb{R}$.

We see that

$$\psi_1(0)=\frac{1}{\pi}\arctan\frac{\pi}{2}\cong 0.319. \tag{10}$$

Let $x>0$, we have that

$$\psi_1'(x)=\frac{1}{4}\left(h'(x+1)-h'(x-1)\right)=$$
$$\frac{-4\pi^2 x}{\left(4+\pi^2(x+1)^2\right)\left(4+\pi^2(x-1)^2\right)}<0. \tag{11}$$

That is

$$\psi_1'(x)<0, \text{ for } x>0. \tag{12}$$

That is ψ_1 is strictly decreasing on $[0,\infty)$ and clearly is strictly increasing on $(-\infty,0]$, and $\psi_1'(0)=0$.

Observe that

$$\begin{aligned}&\lim_{x\to+\infty}\psi_1(x)=\tfrac{1}{4}\left(h(+\infty)-h(+\infty)\right)=0,\\&\text{and}\\&\lim_{x\to-\infty}\psi_1(x)=\tfrac{1}{4}\left(h(-\infty)-h(-\infty)\right)=0.\end{aligned} \tag{13}$$

That is the x-axis is the horizontal asymptote on ψ_1.

All in all, ψ_1 is a bell symmetric function with maximum $\psi_1(0)\cong 18.31$.

We need

Theorem 2 *([19], p. 286).* *We have that*

$$\sum_{i=-\infty}^{\infty}\psi_1(x-i)=1,\ \forall\ x\in\mathbb{R}. \tag{14}$$

Theorem 3 *([19], p. 287).* *It holds*

$$\int_{-\infty}^{\infty}\psi_1(x)\,dx=1. \tag{15}$$

So that $\psi_1(x)$ is a density function on $\mathbb{R}$.

We mention

Theorem 4 *([19], p. 288).* *Let $0<\alpha<1$, and $n\in\mathbb{N}$ with $n^{1-\alpha}>2$. It holds*

$$\sum_{\substack{k=-\infty\\ :|nx-k|\geq n^{1-\alpha}}}^{\infty}\psi_1(nx-k)<\frac{2}{\pi^2\left(n^{1-\alpha}-2\right)}=:c_1(\alpha,n). \tag{16}$$

Denote by $\lfloor\cdot\rfloor$ the integral part of the number and by $\lceil\cdot\rceil$ the ceiling of the number.

We need

Theorem 5 *([19], p. 289). Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_1(nx-k)} < \frac{1}{\psi_1(1)} \cong \mathbf{4.9737} =: \alpha_1, \ \forall\ \mathbf{x} \in [a,b]. \tag{17}$$

Note 1 ([19], pp. 290–291).

i) We have that

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_1(nx-k) \neq 1, \tag{18}$$

for at least some $x \in [a,b]$.

ii) For large enough $n \in \mathbb{N}$ we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$.

In general, by Theorem 2, it holds

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_1(nx-k) \leq 1. \tag{19}$$

We introduce (see [24])

$$Z_1(x_1,...,x_N) := Z_1(x) := \prod_{i=1}^{N} \psi_1(x_i), \ x = (x_1,...,x_N) \in \mathbb{R}^N, \ N \in \mathbb{N}. \tag{20}$$

Denote by $a = (a_1,...,a_N)$ and $b = (b_1,...,b_N)$.

It has the properties:

(i) $Z_1(x) > 0, \ \forall\ x \in \mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty} Z_1(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} Z_1(x_1-k_1,...,x_N-k_N) = 1, \tag{21}$$

where $k := (k_1,...,k_n) \in \mathbb{Z}^N$, $\forall\ x \in \mathbb{R}^N$,
hence

(iii)

$$\sum_{k=-\infty}^{\infty} Z_1(nx-k) = 1, \tag{22}$$

$\forall\ x \in \mathbb{R}^N$; $n \in \mathbb{N}$,
and

(iv)
$$\int_{\mathbb{R}^N} Z_1(x)\,dx = 1, \tag{23}$$

that is Z_1 is a multivariate density function.

(v) It is clear that
$$\sum_{\begin{cases} k = -\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\infty} Z_1(nx-k) < \frac{2}{\pi^2 (n^{1-\beta} - 2)} = c_1(\beta, n), \tag{24}$$

$0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \mathbb{R}^N$.

(vi) By Theorem 5 we get that
$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_1(nx-k)} < \frac{1}{(\psi_1(1))^N} \cong (4.9737)^N =: \gamma_1(N), \tag{25}$$

$\forall\ x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$, $n \in \mathbb{N}$.

Furthermore it holds
$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_1(nx-k) \neq 1, \tag{26}$$

for at least some $x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$.

Above it is $\|x\|_\infty := \max\{|x_1|, ..., |x_N|\}$, $x \in \mathbb{R}^N$, also set $\infty := (\infty, ..., \infty)$, $-\infty = (-\infty, ... - \infty)$ upon the multivariate context.

2.2 About the Algebraic Activation Function

Here see also [20].

We consider the generator algebraic function
$$\varphi(x) = \frac{x}{\sqrt[2m]{1 + x^{2m}}},\ m \in \mathbb{N},\ x \in \mathbb{R}, \tag{27}$$

which is a sigmoidal type of function and is a strictly increasing function.

We see that $\varphi(-x) = -\varphi(x)$ with $\varphi(0) = 0$. We get that
$$\varphi'(x) = \frac{1}{(1 + x^{2m})^{\frac{2m+1}{2m}}} > 0,\ \forall\ x \in \mathbb{R}, \tag{28}$$

proving φ as strictly increasing over $\mathbb{R}, \varphi'(x) = \varphi'(-x)$. We easily find that $\lim_{x \to +\infty} \varphi(x) = 1$, $\varphi(+\infty) = 1$, and $\lim_{x \to -\infty} \varphi(x) = -1$, $\varphi(-\infty) = -1$.

We consider the activation function

$$\psi_2(x) = \frac{1}{4}\left[\varphi(x+1) - \varphi(x-1)\right]. \tag{29}$$

Clearly it is $\psi_2(x) = \psi_2(-x)$, $\forall\ x \in \mathbb{R}$, so that ψ_2 is an even function and symmetric with respect to the y-axis. Clearly $\psi_2(x) > 0$, $\forall\ x \in \mathbb{R}$.

Also it is

$$\psi_2(0) = \frac{1}{2\sqrt[2m]{2}}. \tag{30}$$

By [20], we have that $\psi_2'(x) < 0$ for $x > 0$. That is ψ_2 is strictly decreasing over $(0, +\infty)$.

Clearly, ψ_2 is strictly increasing over $(-\infty, 0)$ and $\psi_2'(0) = 0$.

Furthermore we obtain that

$$\lim_{x\to+\infty} \psi_2(x) = \frac{1}{4}\left[\varphi(+\infty) - \varphi(+\infty)\right] = 0, \tag{31}$$

and

$$\lim_{x\to-\infty} \psi_2(x) = \frac{1}{4}\left[\varphi(-\infty) - \varphi(-\infty)\right] = 0. \tag{32}$$

That is the x-axis is the horizontal asymptote of ψ_2.

Conclusion, ψ_2 is a bell shape symmetric function with maximum

$$\psi_2(0) = \frac{1}{2\sqrt[2m]{2}},\ \ m \in \mathbb{N}. \tag{33}$$

We need

Theorem 6 [20]**.** *We have that*

$$\sum_{i=-\infty}^{\infty} \psi_2(x-i) = 1,\ \forall\ x \in \mathbb{R}. \tag{34}$$

Theorem 7 [20]**.** *It holds*

$$\int_{-\infty}^{\infty} \psi_2(x)\, dx = 1. \tag{35}$$

Theorem 8 [20]**.** *Let* $0 < \alpha < 1$, *and* $n \in \mathbb{N}$ *with* $n^{1-\alpha} > 2$. *It holds*

$$\sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}}^{\infty} \psi_2(nx-k) < \frac{1}{4m\left(n^{1-\alpha}-2\right)^{2m}} =: c_2(\alpha, n),\ \ m \in \mathbb{N}. \tag{36}$$

We need

Theorem 9 [20]**.** *Let* $[a,b] \subset \mathbb{R}$ *and* $n \in \mathbb{N}$ *so that* $\lceil na \rceil \leq \lfloor nb \rfloor$*. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_2 (nx-k)} < 2 \left(\sqrt[2m]{1+4^m} \right) =: \alpha_2, \tag{37}$$

$\forall\ x \in [a,b]$, $m \in \mathbb{N}$.

Note 2. 1) By [20] we have that

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_2 (nx-k) \neq 1, \tag{38}$$

for at least some $x \in [a,b]$.

2) Let $[a,b] \subset \mathbb{R}$. For large $n \in \mathbb{N}$ we always have $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$.
In general it holds that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_2 (nx-k) \leq 1. \tag{39}$$

We introduce (see also [25])

$$Z_2 (x_1, ..., x_N) := Z_2 (x) := \prod_{i=1}^{N} \psi_2 (x_i), \ x = (x_1, ..., x_N) \in \mathbb{R}^N, \ N \in \mathbb{N}. \tag{40}$$

It has the properties:

(i) $Z_2 (x) > 0, \ \forall\ x \in \mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty} Z_2 (x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} Z_2 (x_1-k_1, ..., x_N-k_N) = 1, \tag{41}$$

where $k := (k_1, ..., k_n) \in \mathbb{Z}^N$, $\forall\ x \in \mathbb{R}^N$,
hence

(iii)

$$\sum_{k=-\infty}^{\infty} Z_2 (nx-k) = 1, \tag{42}$$

$\forall\ x \in \mathbb{R}^N$; $n \in \mathbb{N}$,
and

(iv)

$$\int_{\mathbb{R}^N} Z_2 (x)\, dx = 1, \tag{43}$$

that is Z_2 is a multivariate density function.

(v) It is clear that

$$\sum_{\begin{cases} k=-\infty \\ \left\| \frac{k}{n}-x \right\|_{\infty} > \frac{1}{n^{\beta}} \end{cases}}^{\infty} Z_2(nx-k) < \frac{1}{4m\left(n^{1-\beta}-2\right)^{2m}} = c_2(\beta,n), \tag{44}$$

$0<\beta<1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \mathbb{R}^N$, $m \in \mathbb{N}$.

(vi) By Theorem 9 we get that

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_2(nx-k)} < \frac{1}{(\psi_2(1))^N} \cong \left[2\left(\sqrt[2m]{1+4^m}\right)\right]^N := \gamma_2(N), \tag{45}$$

$\forall\, x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $n \in \mathbb{N}$.

Furthermore it holds

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_2(nx-k) \neq 1, \tag{46}$$

for at least some $x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

2.3 About the Gudermannian Activation Function

See also [21,34].

Here we consider $gd(x)$ the Gudermannian function [34], which is a sigmoid function, as a generator function:

$$\sigma(x) = 2\arctan\left(\tanh\left(\frac{x}{2}\right)\right) = \int_0^x \frac{dt}{\cosh t} =: gd(x), \ x \in \mathbb{R}. \tag{47}$$

Let the normalized generator sigmoid function

$$f(x) := \frac{2}{\pi}\sigma(x) = \frac{2}{\pi}\int_0^x \frac{dt}{\cosh t} = \frac{4}{\pi}\int_0^x \frac{1}{e^t+e^{-t}}dt, \ x \in \mathbb{R}. \tag{48}$$

Here

$$f'(x) = \frac{2}{\pi\cosh x} > 0, \ \forall\, x \in \mathbb{R},$$

hence f is strictly increasing on $\mathbb{R}$.

Notice that $\tanh(-x) = -\tanh x$ and $\arctan(-x) = -\arctan x$, $x \in \mathbb{R}$.

So, here the neural network activation function will be:

$$\psi_3(x) = \frac{1}{4}\left[f(x+1) - f(x-1)\right], \ x \in \mathbb{R}. \tag{49}$$

By [21], we get that

$$\psi_3(x) = \psi_3(-x), \ \forall\, x \in \mathbb{R}, \tag{50}$$

i.e. it is even and symmetric with respect to the y-axis. Here we have $f(+\infty) = 1$, $f(-\infty) = -1$ and $f(0) = 0$. Clearly it is

$$f(-x) = -f(x), \ \forall \ x \in \mathbb{R}, \tag{51}$$

an odd function, symmetric with respect to the origin. Since $x + 1 > x - 1$, and $f(x+1) > f(x-1)$, we obtain $\psi_3(x) > 0$, $\forall \ x \in \mathbb{R}$.

By [21], we have that

$$\psi_3(0) = \frac{1}{\pi} gd(1) \cong 0.2757. \tag{52}$$

By [21] ψ_3 is strictly decreasing on $(0, +\infty)$, and strictly increasing on $(-\infty, 0)$, and $\psi_3'(0) = 0$.

Also we have that

$$\lim_{x \to +\infty} \psi_3(x) = \lim_{x \to -\infty} \psi_3(x) = 0, \tag{53}$$

that is the x-axis is the horizontal asymptote for ψ_3.

Conclusion, ψ_3 is a bell shaped symmetric function with maximum $\psi_3(0) \cong 0.551$.

We need

Theorem 10 [21]. *It holds that*

$$\sum_{i=-\infty}^{\infty} \psi_3(x - i) = 1, \ \forall \ x \in \mathbb{R}. \tag{54}$$

Theorem 11 [21]. *We have that*

$$\int_{-\infty}^{\infty} \psi_3(x) \, dx = 1. \tag{55}$$

So $\psi_3(x)$ is a density function.

Theorem 12 [21]. *Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds*

$$\sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}}^{\infty} \psi_3(nx - k) < \frac{2}{\pi e^{(n^{1-\alpha}-2)}} = \frac{2e^2}{\pi e^{n^{1-\alpha}}} =: c_3(\alpha, n). \tag{56}$$

Theorem 13 [21]. *Let $[a, b] \subset \mathbb{R}$ and $n \in \mathbb{N}$, so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_3(nx - k)} < \frac{2\pi}{gd(2)} \cong 4.824 =: \alpha_3, \tag{57}$$

$\forall \ x \in [a, b]$.

We make

Remark 5 [21].

(i) We have that

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi_3(nx-k)\neq 1, \tag{58}$$

for at least some $x\in[a,b]$.

(ii) Let $[a,b]\subset\mathbb{R}$. For large n we always have $\lceil na\rceil\leq\lfloor nb\rfloor$. Also $a\leq\frac{k}{n}\leq b$, iff $\lceil na\rceil\leq k\leq\lfloor nb\rfloor$.

In general it holds

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi_3(nx-k)\leq 1. \tag{59}$$

We introduce (see also [23])

$$Z_3(x_1,...,x_N):=Z_3(x):=\prod_{i=1}^{N}\psi_3(x_i),\ x=(x_1,...,x_N)\in\mathbb{R}^N,\ \ N\in\mathbb{N}. \tag{60}$$

It has the properties:

(i) $Z_3(x)>0,\ \ \forall\ x\in\mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty}Z_3(x-k):=\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}...\sum_{k_N=-\infty}^{\infty}Z_3(x_1-k_1,...,x_N-k_N)=1, \tag{61}$$

where $k:=(k_1,...,k_n)\in\mathbb{Z}^N$, $\forall\ x\in\mathbb{R}^N$,

hence

(iii)

$$\sum_{k=-\infty}^{\infty}Z_3(nx-k)=1, \tag{62}$$

$\forall\ x\in\mathbb{R}^N$; $n\in\mathbb{N}$,

and

(iv)

$$\int_{\mathbb{R}^N}Z_3(x)\,dx=1, \tag{63}$$

that is Z_3 is a multivariate density function.

(v) It is also clear that

$$\sum_{\begin{cases} k=-\infty \\ \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}} \end{cases}}^{\infty} Z_3(nx-k) < \frac{2e^2}{\pi e^{n^{1-\beta}}} = c_3(\beta, n), \tag{64}$$

$0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \mathbb{R}^N$, $m \in \mathbb{N}$.

(vi) By Theorem 13 we get that

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_3(nx-k)} < \left(\frac{2\pi}{gd(2)} \right)^N \cong (4.824)^N =: \gamma_3(N), \tag{65}$$

$\forall \, x \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$, $n \in \mathbb{N}$.

Furthermore it holds

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_3(nx-k) \neq 1, \tag{66}$$

for at least some $x \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$.

2.4 About the Generalized Symmetrical Activation Function

Here we consider the generalized symmetrical sigmoid function [22, 29]

$$f_1(x) = \frac{x}{\left(1 + |x|^{\mu}\right)^{\frac{1}{\mu}}}, \; \mu > 0, \; x \in \mathbb{R}. \tag{67}$$

This has applications in immunology and protection from disease together with probability theory. It is also called a symmetrical protection curve.

The parameter μ is a shape parameter controling how fast the curve approaches the asymptotes for a given slope at the inflection point. When $\mu = 1$ f_1 is the absolute sigmoid function, and when $\mu = 2$, f_1 is the square root sigmoid function. When $\mu = 1.5$ the function approximates the arctangent function, when $\mu = 2.9$ it approximates the logistic function, and when $\mu = 3.4$ it approximates the error function. Parameter μ is estimated in the likelihood maximization [29]. For more see [29].

Next we study the particular generator sigmoid function

$$f_2(x) = \frac{x}{\left(1 + |x|^{\lambda}\right)^{\frac{1}{\lambda}}}, \; \lambda \text{ is an odd number}, \; x \in \mathbb{R}. \tag{68}$$

We have that $f_2(0) = 0$, and

$$f_2(-x) = -f_2(x), \tag{69}$$

so f_2 is symmetric with respect to zero.

When $x \geq 0$, we get that [22]

$$f_2'(x) = \frac{1}{\left(1+x^{\lambda}\right)^{\frac{\lambda+1}{\lambda}}} > 0, \tag{70}$$

that is f_2 is strictly increasing on $[0, +\infty)$ and f_2 is strictly increasing on $(-\infty, 0]$. Hence f_2 is strictly increasing on $\mathbb{R}$.

We also have $f_2(+\infty) = f_2(-\infty) = 1$.

Let us consider the activation function [22]:

$$\psi_4(x) = \frac{1}{4}\left[f_2(x+1) - f_2(x-1)\right] =$$

$$\frac{1}{4}\left[\frac{(x+1)}{\left(1+|x+1|^{\lambda}\right)^{\frac{1}{\lambda}}} - \frac{(x-1)}{\left(1+|x-1|^{\lambda}\right)^{\frac{1}{\lambda}}}\right]. \tag{71}$$

Clearly it holds [22]

$$\psi_4(x) = \psi_4(-x), \ \forall\, x \in \mathbb{R}. \tag{72}$$

and

$$\psi_4(0) = \frac{1}{2\sqrt[\lambda]{2}}, \tag{73}$$

and $\psi_4(x) > 0$, $\forall\, x \in \mathbb{R}$.

Following [22], we have that ψ_4 is strictly decreasing over $[0, +\infty)$, and ψ_4 is strictly increasing on $(-\infty, 0]$, by ψ_4-symmetry with respect to y-axis, and $\psi_4'(0) = 0$.

Clearly it is

$$\lim_{x\to+\infty} \psi_4(x) = \lim_{x\to-\infty} \psi_4(x) = 0, \tag{74}$$

therefore the x-axis is the horizontal asymptote of $\psi_4(x)$.

The value

$$\psi_4(0) = \frac{1}{2\sqrt[\lambda]{2}}, \ \lambda \text{ is an odd number,} \tag{75}$$

is the maximum of ψ_4, which is a bell shaped function.

We need

Theorem 14 [22]**.** *It holds*

$$\sum_{i=-\infty}^{\infty} \psi_4(x-i) = 1, \ \forall\, x \in \mathbb{R}. \tag{76}$$

Theorem 15 [22]**.** *We have that*

$$\int_{-\infty}^{\infty} \psi_4(x)\, dx = 1. \tag{77}$$

So that $\psi_4(x)$ is a density function on $\mathbb{R}$.
We need

Theorem 16 [22]**.** *Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds*

$$\sum_{\begin{cases} j=-\infty \\ : |nx-j| \geq n^{1-\alpha} \end{cases}}^{\infty} \psi_4(nx-j) < \frac{1}{2\lambda \left(n^{1-\alpha}-2\right)^{\lambda}} =: c_4(\alpha, n), \tag{78}$$

where $\lambda \in \mathbb{N}$ is an odd number.

We also need

Theorem 17 [22]**.** *Let $[a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. Then*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_4(|nx-k|)} < 2\sqrt[\lambda]{1+2^{\lambda}} =: \alpha_4, \tag{79}$$

where λ is an odd number, $\forall\, x \in [a,b]$.

We make

Remark 6 [22]. (1) We have that

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_4(nx-k) \neq 1, \text{ for at least some } x \in [a,b]. \tag{80}$$

(2) Let $[a,b] \subset \mathbb{R}$. For large enough n we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$.
In general it holds that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_4(nx-k) \leq 1. \tag{81}$$

We introduce (see also [26])

$$Z_4(x_1, ..., x_N) := Z_4(x) := \prod_{i=1}^{N} \psi_4(x_i), \; x = (x_1, ..., x_N) \in \mathbb{R}^N, \; N \in \mathbb{N}. \tag{82}$$

It has the properties:

(i) $Z_4(x) > 0, \;\; \forall\, x \in \mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty} Z_4(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} Z_4(x_1-k_1,...,x_N-k_N) = 1, \tag{83}$$

where $k := (k_1, ..., k_n) \in \mathbb{Z}^N$, $\forall\, x \in \mathbb{R}^N$,
hence

(iii)

$$\sum_{k=-\infty}^{\infty} Z_4(nx-k) = 1, \tag{84}$$

$\forall\, x \in \mathbb{R}^N$; $n \in \mathbb{N}$,

and

(iv)

$$\int_{\mathbb{R}^N} Z_4(x)\, dx = 1, \tag{85}$$

that is Z_4 is a multivariate density function.

(v) It is clear that

$$\sum_{\substack{k=-\infty \\ \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}}}}^{\infty} Z_4(nx-k) < \frac{1}{2\lambda\left(n^{1-\beta}-2\right)^{\lambda}} = c_4(\beta, n), \tag{86}$$

$0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \mathbb{R}^N$, λ is odd.

(vi) By Theorem 17 we get that

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_4(nx-k)} < \left(2\sqrt[\lambda]{1+2^{\lambda}}\right)^N =: \gamma_4(N), \tag{87}$$

$\forall\, x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$, $n \in \mathbb{N}$, λ is odd.

Furthermore it holds

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_4(nx-k) \neq 1, \tag{88}$$

for at least some $x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$.

Set

$$\lceil na \rceil := (\lceil na_1 \rceil, ..., \lceil na_N \rceil),$$

$$\lfloor nb \rfloor := (\lfloor nb_1 \rfloor, ..., \lfloor nb_N \rfloor),$$

where $a := (a_1, ..., a_N)$, $b := (b_1, ..., b_N)$, $k := (k_1, ..., k_N)$.

Let $f \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, and $n \in \mathbb{N}$ such that $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i=1,...,N$.

We define the multivariate averaged positive linear quasi-interpolation neural network operators ($x := (x_1,...,x_N) \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$); $j=1,2,3,4$:

$$ {}_jA_n(f,x_1,...,x_N) := {}_jA_n(f,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z_j(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_j(nx-k)} = \tag{89} $$

$$ \frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} ... \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n},...,\frac{k_N}{n}\right)\left(\prod_{i=1}^{N}\psi_j(nx_i-k_i)\right)}{\prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor}\psi_j(nx_i-k_i)\right)}. $$

For large enough $n \in \mathbb{N}$ we always obtain $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i=1,...,N$. Also $a_i \leq \frac{k_i}{n} \leq b_i$, iff $\lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor$, $i=1,...,N$.

When $f \in C_B\left(\mathbb{R}^N\right)$ we define ($j=1,2,3,4$)

$$ {}_jB_n(f,x) := {}_jB_n(f,x_1,...,x_N) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z_j(nx-k) := \tag{90} $$

$$ \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} f\left(\frac{k_1}{n},\frac{k_2}{n},...,\frac{k_N}{n}\right)\left(\prod_{i=1}^{N}\psi_j(nx_i-k_i)\right), $$

$n \in \mathbb{N}$, $\forall\ x \in \mathbb{R}^N$, $N \in \mathbb{N}$, the multivariate full quasi-interpolation neural network operators.

Also for $f \in C_B\left(\mathbb{R}^N\right)$ we define the multivariate Kantorovich type neural network operators ($j=1,2,3,4$)

$$ {}_jC_n(f,x) := {}_jC_n(f,x_1,...,x_N) := \sum_{k=-\infty}^{\infty}\left(n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt\right) Z_j(nx-k) := \tag{91} $$

$$ \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty}\left(n^N \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}}\int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}} ... \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1,...,t_N)\,dt_1...dt_N\right) $$

$$ \cdot\left(\prod_{i=1}^{N}\psi_j(nx_i-k_i)\right), $$

$n \in \mathbb{N}$, $\forall\ x \in \mathbb{R}^N$.

Again for $f \in C_B\left(\mathbb{R}^N\right)$, $N \in \mathbb{N}$, we define the multivariate neural network operators of quadrature type ${}_jD_n(f,x)$, $n \in \mathbb{N}$, as follows. Let $\theta = (\theta_1,...,\theta_N) \in \mathbb{N}^N$, $\overline{r} = (r_1,...,r_N) \in \mathbb{Z}_+^N$, $w_{\overline{r}} = w_{r_1,r_2,...r_N} \geq 0$, such that $\sum_{\overline{r}=0}^{\theta} w_{\overline{r}} = \sum_{r_1=0}^{\theta_1}\sum_{r_2=0}^{\theta_2} ... \sum_{r_N=0}^{\theta_N} w_{r_1,r_2,...r_N} = 1$; $k \in \mathbb{Z}^N$ and

$$ \delta_{nk}(f) := \delta_{n,k_1,k_2,...,k_N}(f) := \sum_{\overline{r}=0}^{\theta} w_{\overline{r}} f\left(\frac{k}{n}+\frac{\overline{r}}{n\theta}\right) := $$

$$\sum_{r_1=0}^{\theta_1}\sum_{r_2=0}^{\theta_2}...\sum_{r_N=0}^{\theta_N} w_{r_1,r_2,...r_N} f\left(\frac{k_1}{n}+\frac{r_1}{n\theta_1},\frac{k_2}{n}+\frac{r_2}{n\theta_2},...,\frac{k_N}{n}+\frac{r_N}{n\theta_N}\right), \quad (92)$$

where $\frac{\overline{r}}{\theta} := \left(\frac{r_1}{\theta_1},\frac{r_2}{\theta_2},...,\frac{r_N}{\theta_N}\right)$; $j = 1, 2, 3, 4$.

We put

$$_jD_n(f,x) := {}_jD_n(f,x_1,...,x_N) := \sum_{k=-\infty}^{\infty}\delta_{nk}(f)\,Z_j(nx-k) := \quad (93)$$

$$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}...\sum_{k_N=-\infty}^{\infty}\delta_{n,k_1,k_2,...,k_N}(f)\left(\prod_{i=1}^{N}\psi_j(nx_i-k_i)\right),$$

$\forall\ x \in \mathbb{R}^N$.

For the next we need, for $f \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$ the first multivariate modulus of continuity

$$\omega_1(f,h) := \sup_{\substack{x,y\in\prod_{i=1}^{N}[a_i,b_i]\\ \|x-y\|_\infty\le h}} |f(x)-f(y)|,\ h>0. \quad (94)$$

It holds that

$$\lim_{h\to 0}\omega_1(f,h) = 0. \quad (95)$$

Similarly it is defined for $f \in C_B\left(\mathbb{R}^N\right)$ (continuous and bounded functions on $\mathbb{R}^N$) the $\omega_1(f,h)$, and it has the property (95), given that $f \in C_U\left(\mathbb{R}^N\right)$ (uniformly continuous functions on $\mathbb{R}^N$).

We mention

Theorem 18 *(see [23–26]). Let* $f \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $0<\beta<1$, $x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $N,n\in\mathbb{N}$ *with* $n^{1-\beta}>2$; $j=1,2,3,4$. *Then*

1)

$$|_jA_n(f,x)-f(x)| \le \gamma_j(N)\left[\omega_1\left(f,\frac{1}{n^\beta}\right)+2c_j(\beta,n)\|f\|_\infty\right] =: \lambda_{j1}, \quad (96)$$

and

2)

$$\|_jA_n(f)-f\|_\infty \le \lambda_{j1}. \quad (97)$$

We notice that $\lim_{n\to\infty} {}_jA_n(f) = f$, *pointwise and uniformly.*

In this article we extend Theorem 18 to the fuzzy-random level.

We mention

Theorem 19 *(see [23–26]). Let* $f \in C_B\left(\mathbb{R}^N\right)$, $0<\beta<1$, $x\in\mathbb{R}^N$, $N,n\in\mathbb{N}$ *with* $n^{1-\beta}>2$; $j=1,2,3,4$. *Then*

1)

$$\left|{}_{j}B_{n}\left(f,x\right)-f\left(x\right)\right|\leq\omega_{1}\left(f,\frac{1}{n^{\beta}}\right)+2c_{j}\left(\beta,n\right)\left\|f\right\|_{\infty}=:\lambda_{j2}, \tag{98}$$

2)

$$\left\|{}_{j}B_{n}\left(f\right)-f\right\|_{\infty}\leq\lambda_{j2}. \tag{99}$$

Given that $f\in\left(C_{U}\left(\mathbb{R}^{N}\right)\cap C_{B}\left(\mathbb{R}^{N}\right)\right)$, *we obtain* $\lim_{n\to\infty}{}_{j}B_{n}\left(f\right)=f$, *uniformly.*

We also need

Theorem 20 *(see [23–26]). Let* $f\in C_{B}\left(\mathbb{R}^{N}\right)$, $0<\beta<1$, $x\in\mathbb{R}^{N}$, $N,n\in\mathbb{N}$ *with* $n^{1-\beta}>2$; $j=1,2,3,4$. *Then*

1)

$$\left|{}_{j}C_{n}\left(f,x\right)-f\left(x\right)\right|\leq\omega_{1}\left(f,\frac{1}{n}+\frac{1}{n^{\beta}}\right)+2c_{j}\left(\beta,n\right)\left\|f\right\|_{\infty}=:\lambda_{j3}, \tag{100}$$

2)

$$\left\|{}_{j}C_{n}\left(f\right)-f\right\|_{\infty}\leq\lambda_{j3}. \tag{101}$$

Given that $f\in\left(C_{U}\left(\mathbb{R}^{N}\right)\cap C_{B}\left(\mathbb{R}^{N}\right)\right)$, *we obtain* $\lim_{n\to\infty}{}_{j}C_{n}\left(f\right)=f$, *uniformly.*

We also need

Theorem 21 *(see [23–26]). Let* $f\in C_{B}\left(\mathbb{R}^{N}\right)$, $0<\beta<1$, $x\in\mathbb{R}^{N}$, $N,n\in\mathbb{N}$ *with* $n^{1-\beta}>2$; $j=1,2,3,4$. *Then*

1)

$$\left|{}_{j}D_{n}\left(f,x\right)-f\left(x\right)\right|\leq\omega_{1}\left(f,\frac{1}{n}+\frac{1}{n^{\beta}}\right)+2c_{j}\left(\beta,n\right)\left\|f\right\|_{\infty}=\lambda_{j3}, \tag{102}$$

2)

$$\left\|{}_{j}D_{n}\left(f\right)-f\right\|_{\infty}\leq\lambda_{j3}. \tag{103}$$

Given that $f\in\left(C_{U}\left(\mathbb{R}^{N}\right)\cap C_{B}\left(\mathbb{R}^{N}\right)\right)$, *we obtain* $\lim_{n\to\infty}{}_{j}D_{n}\left(f\right)=f$, *uniformly.*

In this article we extend Theorems 19, 20, 21 to the random level.

We are also motivated by [1–16] and continuing [17]. For general knowledge on neural networks we recommend [31–33].

3 Main Results

I) q-mean Approximation by Fuzzy-Random arctangent, algebraic, Gudermannian and generalized symmetric activation functions based Quasi-Interpolation Neural Network Operators

All terms and assumptions here as in Sects. 1, 2.

Let $f \in C_{\mathcal{FR}}^{U_q}\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $1 \leq q < +\infty$, $n,N \in \mathbb{N}$, $0<\beta<1$, $\overrightarrow{x} \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $(X,\mathcal{B},P)$ probability space, $s \in X$; $j=1,2,3,4$.

We define the following multivariate fuzzy random arctangent, algebraic, Gudermannian and generalized symmetric activation functions based quasi-interpolation linear neural network operators

$$\left({}_jA_n^{\mathcal{FR}}(f)\right)(\overrightarrow{x},s) := \sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor *} f\left(\frac{\overrightarrow{k}}{n},s\right) \odot \frac{Z_j\left(n\overrightarrow{x}-\overrightarrow{k}\right)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor} Z_j\left(n\overrightarrow{x}-\overrightarrow{k}\right)}, \tag{104}$$

(see also (89)).

We present

Theorem 22. *Let $f \in C_{\mathcal{FR}}^{U_q}\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $0<\beta<1$, $\overrightarrow{x} \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $n,N \in \mathbb{N}$, with $n^{1-\beta}>2$, $1 \leq q < +\infty$. Assume that $\int_X \left(D^*\left(f(\cdot,s),\widetilde{o}\right)\right)^q P(ds) < \infty$; $j=1,2,3,4$. Then*

1)

$$\left(\int_X D^q\left(\left({}_jA_n^{\mathcal{FR}}(f)\right)(\overrightarrow{x},s), f(\overrightarrow{x},s)\right)P(ds)\right)^{\frac{1}{q}} \leq \tag{105}$$

$$\gamma_j(N)\left\{\Omega_1\left(f,\frac{1}{n^\beta}\right)_{L^q} + 2c_j(\beta,n)\left(\int_X \left(D^*\left(f(\cdot,s),\widetilde{o}\right)\right)^q P(ds)\right)^{\frac{1}{q}}\right\} =: \lambda_{j1}^{(\mathcal{FR})},$$

2)

$$\left\|\left(\int_X D^q\left(\left({}_jA_n^{\mathcal{FR}}(f)\right)(\overrightarrow{x},s), f(\overrightarrow{x},s)\right)P(ds)\right)^{\frac{1}{q}}\right\|_{\infty,\left(\prod_{i=1}^{N}[a_i,b_i]\right)} \leq \lambda_{j1}^{(\mathcal{FR})}, \tag{106}$$

where $\gamma_j(N)$ as in (25), (45), (65), (87) and $c_j(\beta,n)$ as in (24), (44), (64), (86).

Proof. We notice that

$$D\left(f\left(\frac{\overrightarrow{k}}{n},s\right), f(\overrightarrow{x},s)\right) \leq D\left(f\left(\frac{\overrightarrow{k}}{n},s\right),\widetilde{o}\right) + D\left(f(\overrightarrow{x},s),\widetilde{o}\right) \tag{107}$$

$$\leq 2D^{*}\left(f\left(\cdot,s\right),\widetilde{o}\right).$$

Hence

$$D^{q}\left(f\left(\frac{\overrightarrow{k}}{n},s\right),f\left(\overrightarrow{x},s\right)\right)\leq 2^{q}D^{*q}\left(f\left(\cdot,s\right),\widetilde{o}\right), \tag{108}$$

and

$$\left(\int_{X}D^{q}\left(f\left(\frac{\overrightarrow{k}}{n},s\right),f\left(\overrightarrow{x},s\right)\right)P\left(ds\right)\right)^{\frac{1}{q}}\leq 2\left(\int_{X}\left(D^{*}\left(f\left(\cdot,s\right),\widetilde{o}\right)\right)^{q}P\left(ds\right)\right)^{\frac{1}{q}}. \tag{109}$$

We observe that

$$D\left(\left({}_{j}A_{n}^{\mathcal{FR}}\left(f\right)\right)\left(\overrightarrow{x},s\right),f\left(\overrightarrow{x},s\right)\right)= \tag{110}$$

$$D\left(\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor *}f\left(\frac{\overrightarrow{k}}{n},s\right)\odot\frac{Z_{j}\left(nx-k\right)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}Z_{j}\left(nx-k\right)},f\left(\overrightarrow{x},s\right)\odot 1\right)=$$

$$D\left(\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor *}f\left(\frac{\overrightarrow{k}}{n},s\right)\odot\frac{Z_{j}\left(nx-k\right)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}Z_{j}\left(nx-k\right)},f\left(\overrightarrow{x},s\right)\odot\frac{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}Z_{j}\left(nx-k\right)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}Z_{j}\left(nx-k\right)}\right)= \tag{111}$$

$$D\left(\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor *}f\left(\frac{\overrightarrow{k}}{n},s\right)\odot\frac{Z_{j}\left(nx-k\right)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}Z_{j}\left(nx-k\right)},\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor *}f\left(\overrightarrow{x},s\right)\odot\frac{Z_{j}\left(nx-k\right)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}Z_{j}\left(nx-k\right)}\right)$$

$$\leq\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}\left(\frac{Z_{j}\left(nx-k\right)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}Z_{j}\left(nx-k\right)}\right)D\left(f\left(\frac{\overrightarrow{k}}{n},s\right),f\left(\overrightarrow{x},s\right)\right). \tag{112}$$

So that

$$D\left(\left({}_{j}A_{n}^{\mathcal{FR}}\left(f\right)\right)\left(\overrightarrow{x},s\right),f\left(\overrightarrow{x},s\right)\right)\leq$$

$$\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}\left(\frac{Z_{j}\left(nx-k\right)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}Z_{j}\left(nx-k\right)}\right)D\left(f\left(\frac{\overrightarrow{k}}{n},s\right),f\left(\overrightarrow{x},s\right)\right)= \tag{113}$$

$$\sum_{\substack{\overrightarrow{k}=\lceil na\rceil \\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_{\infty}\leq\frac{1}{n^{\beta}}}}^{\lfloor nb\rfloor}\left(\frac{Z_j\left(nx-k\right)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}Z_j\left(nx-k\right)}\right)D\left(f\left(\frac{\overrightarrow{k}}{n},s\right),f\left(\overrightarrow{x},s\right)\right)+$$

$$\sum_{\substack{\overrightarrow{k}=\lceil na\rceil \\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\lfloor nb\rfloor}\left(\frac{Z_j\left(nx-k\right)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}Z_j\left(nx-k\right)}\right)D\left(f\left(\frac{\overrightarrow{k}}{n},s\right),f\left(\overrightarrow{x},s\right)\right).$$

Hence it holds

$$\left(\int_X D^q\left(\left({}_jA_n^{\mathcal{FR}}\left(f\right)\right)\left(\overrightarrow{x},s\right),f\left(\overrightarrow{x},s\right)\right)P\left(ds\right)\right)^{\frac{1}{q}}\leq \tag{114}$$

$$\sum_{\substack{\overrightarrow{k}=\lceil na\rceil \\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_{\infty}\leq\frac{1}{n^{\beta}}}}^{\lfloor nb\rfloor}\left(\frac{Z_j\left(nx-k\right)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}Z_j\left(nx-k\right)}\right)\left(\int_X D^q\left(f\left(\frac{\overrightarrow{k}}{n},s\right),f\left(\overrightarrow{x},s\right)\right)P\left(ds\right)\right)^{\frac{1}{q}}+$$

$$\sum_{\substack{\overrightarrow{k}=\lceil na\rceil \\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\lfloor nb\rfloor}\left(\frac{Z_j\left(nx-k\right)}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}Z_j\left(nx-k\right)}\right)\left(\int_X D^q\left(f\left(\frac{\overrightarrow{k}}{n},s\right),f\left(\overrightarrow{x},s\right)\right)P\left(ds\right)\right)^{\frac{1}{q}}\leq$$

$$\left(\frac{1}{\sum_{\overrightarrow{k}=\lceil na\rceil}^{\lfloor nb\rfloor}Z_j\left(nx-k\right)}\right)\cdot\left\{\Omega_1^{(\mathcal{F})}\left(f,\frac{1}{n^{\beta}}\right)_{L^q}+\right. \tag{115}$$

$$\left.2\left(\int_X\left(D^*\left(f\left(\cdot,s\right),\widetilde{o}\right)\right)^q P\left(ds\right)\right)^{\frac{1}{q}}\left(\sum_{\substack{\overrightarrow{k}=\lceil na\rceil \\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\lfloor nb\rfloor}Z_j\left(nx-k\right)\right)\right\}$$

(by (24), (25); (44), (45); (64), (65); (86), (87))

$$\leq\gamma_j\left(N\right)\left\{\Omega_1^{(\mathcal{F})}\left(f,\frac{1}{n^{\beta}}\right)_{L^q}+2c_j\left(\beta,n\right)\left(\int_X\left(D^*\left(f\left(\cdot,s\right),\widetilde{o}\right)\right)^q P\left(ds\right)\right)^{\frac{1}{q}}\right\}. \tag{116}$$

We have proved claim.

Conclusion 6. *By Theorem 22 we obtain the pointwise and uniform convergences with rates in the q-mean and D-metric of the operator* ${}_{j}A_{n}^{\mathcal{FR}}$ *to the unit operator for* $f \in C_{\mathcal{FR}}^{U_q}\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $j = 1,2,3,4$.

II) 1-mean Approximation by Stochastic arctangent, algebraic, Gudermannian and generalized symmetric activation functions based full Quasi-Interpolation Neural Network Operators

Let $g \in C_{\mathcal{R}}^{U_1}\left(\mathbb{R}^N\right)$, $0 < \beta < 1$, $\overrightarrow{x} \in \mathbb{R}^N$, $n, N \in \mathbb{N}$, with $\|g\|_{\infty,\mathbb{R}^N,X} < \infty$, $(X, \mathcal{B}, P)$ probability space, $s \in X$.

We define

$$ {}_{j}B_{n}^{(\mathcal{R})}(g)(\overrightarrow{x}, s) := \sum_{\overrightarrow{k}=-\infty}^{\infty} g\left(\frac{\overrightarrow{k}}{n}, s\right) Z_j\left(n\overrightarrow{x} - \overrightarrow{k}\right),\ j = 1,2,3,4, \tag{117} $$

(see also (90)).

We give

Theorem 23. *Let* $g \in C_{\mathcal{R}}^{U_1}\left(\mathbb{R}^N\right)$, $0 < \beta < 1$, $\overrightarrow{x} \in \mathbb{R}^N$, $n, N \in \mathbb{N}$, *with* $n^{1-\beta} > 2$, $\|g\|_{\infty,\mathbb{R}^N,X} < \infty$; $j = 1,2,3,4$. *Then*

1)

$$ \int_X \left| \left({}_{j}B_{n}^{(\mathcal{R})}(g)\right)(\overrightarrow{x}, s) - g(\overrightarrow{x}, s) \right| P(ds) \le \tag{118} $$

$$ \left\{ \Omega_1\left(g, \frac{1}{n^{\beta}}\right)_{L^1} + 2c_j(\beta, n)\|g\|_{\infty,\mathbb{R}^N,X} \right\} =: \mu_{j1}^{(\mathcal{R})}, $$

2)

$$ \left\| \int_X \left| \left({}_{j}B_{n}^{(\mathcal{R})}(g)\right)(\overrightarrow{x}, s) - g(\overrightarrow{x}, s) \right| P(ds) \right\|_{\infty,\mathbb{R}^N} \le \mu_{j1}^{(\mathcal{R})}. \tag{119} $$

Proof. Since $\|g\|_{\infty,\mathbb{R}^N,X} < \infty$, then

$$ \left| g\left(\frac{\overrightarrow{k}}{n}, s\right) - g(\overrightarrow{x}, s) \right| \le 2\|g\|_{\infty,\mathbb{R}^N,X} < \infty. \tag{120} $$

Hence

$$ \int_X \left| g\left(\frac{\overrightarrow{k}}{n}, s\right) - g(\overrightarrow{x}, s) \right| P(ds) \le 2\|g\|_{\infty,\mathbb{R}^N,X} < \infty. \tag{121} $$

We observe that

$$ \left({}_{j}B_{n}^{(\mathcal{R})}(g)\right)(\overrightarrow{x}, s) - g(\overrightarrow{x}, s) = $$

$$ \sum_{\overrightarrow{k}=-\infty}^{\infty} g\left(\frac{\overrightarrow{k}}{n}, s\right) Z_j(nx-k) - g(\overrightarrow{x}, s) \sum_{\overrightarrow{k}=-\infty}^{\infty} Z_j(nx-k) = \tag{122} $$

$$\left(\sum_{\overrightarrow{k}=-\infty}^{\infty} g\left(\frac{\overrightarrow{k}}{n}, s\right) - g\left(\overrightarrow{x}, s\right) \right) Z_j\left(nx-k\right).$$

However it holds

$$\sum_{\overrightarrow{k}=-\infty}^{\infty} \left| g\left(\frac{\overrightarrow{k}}{n}, s\right) - g\left(\overrightarrow{x}, s\right) \right| Z_j\left(nx-k\right) \leq 2\left\|g\right\|_{\infty,\mathbb{R}^N,X} < \infty. \tag{123}$$

Hence

$$\left| \left({}_jB_n^{(\mathcal{R})}\left(g\right)\right)\left(\overrightarrow{x}, s\right) - g\left(\overrightarrow{x}, s\right) \right| \leq$$

$$\sum_{\overrightarrow{k}=-\infty}^{\infty} \left| g\left(\frac{\overrightarrow{k}}{n}, s\right) - g\left(\overrightarrow{x}, s\right) \right| Z_j\left(nx-k\right) = \tag{124}$$

$$\sum_{\substack{\overrightarrow{k}=-\infty \\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_{\infty} \leq \frac{1}{n^\beta}}}^{\infty} \left| g\left(\frac{\overrightarrow{k}}{n}, s\right) - g\left(\overrightarrow{x}, s\right) \right| Z_j\left(nx-k\right) +$$

$$\sum_{\substack{\overrightarrow{k}=-\infty \\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_{\infty} > \frac{1}{n^\beta}}}^{\infty} \left| g\left(\frac{\overrightarrow{k}}{n}, s\right) - g\left(\overrightarrow{x}, s\right) \right| Z_j\left(nx-k\right).$$

Furthermore it holds

$$\left(\int_X \left| \left({}_jB_n^{(\mathcal{R})}\left(g\right)\right)\left(\overrightarrow{x}, s\right) - g\left(\overrightarrow{x}, s\right) \right| P\left(ds\right) \right) \leq$$

$$\sum_{\substack{\overrightarrow{k}=-\infty \\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_{\infty} \leq \frac{1}{n^\beta}}}^{\infty} \left(\int_X \left| g\left(\frac{\overrightarrow{k}}{n}, s\right) - g\left(\overrightarrow{x}, s\right) \right| P\left(ds\right) \right) Z_j\left(nx-k\right) + \tag{125}$$

$$\sum_{\substack{\overrightarrow{k}=-\infty \\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_{\infty} > \frac{1}{n^\beta}}}^{\infty} \left(\int_X \left| g\left(\frac{\overrightarrow{k}}{n}, s\right) - g\left(\overrightarrow{x}, s\right) \right| P\left(ds\right) \right) Z_j\left(nx-k\right) \leq$$

$$\Omega_1\left(g, \frac{1}{n^\beta}\right)_{L^1} + 2\left\|g\right\|_{\infty,\mathbb{R}^N,X} \sum_{\substack{\overrightarrow{k}=-\infty \\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_{\infty} > \frac{1}{n^\beta}}}^{\infty} Z_j\left(nx-k\right) \leq$$

$$\Omega_1\left(g, \frac{1}{n^\beta}\right)_{L^1} + 2c_j\left(\beta, n\right)\left\|g\right\|_{\infty,\mathbb{R}^N,X},$$

proving the claim.

Conclusion 7. *By Theorem 23 we obtain pointwise and uniform convergences with rates in the* 1*-mean of random operators* ${}_jB_n^{(\mathcal{R})}$ *to the unit operator for* $g \in C_{\mathcal{R}}^{U_1}\left(\mathbb{R}^N\right)$, $j = 1, 2, 3, 4$.

III) 1-mean Approximation by Stochastic arctangent, algebraic, Gudermannian and generalized symmetric activation functions based multivariate Kantorovich type neural network operator

Let $g \in C_{\mathcal{R}}^{U_1}\left(\mathbb{R}^N\right)$, $0 < \beta < 1$, $\overrightarrow{x} \in \mathbb{R}^N$, $n, N \in \mathbb{N}$, with $\|g\|_{\infty,\mathbb{R}^N,X} < \infty$, $(X, \mathcal{B}, P)$ probability space, $s \in X$.

We define ($j = 1, 2, 3, 4$):

$$ {}_jC_n^{(\mathcal{R})}(g)(\overrightarrow{x}, s) := \sum_{\overrightarrow{k}=-\infty}^{\infty} \left(n^N \int_{\frac{\overrightarrow{k}}{n}}^{\frac{\overrightarrow{k}+1}{n}} g\left(\overrightarrow{t}, s\right) d\overrightarrow{t} \right) Z_j\left(n\overrightarrow{x} - \overrightarrow{k}\right), \quad (126) $$

(see also (91)).

We present

Theorem 24. *Let* $g \in C_{\mathcal{R}}^{U_1}\left(\mathbb{R}^N\right)$, $0 < \beta < 1$, $\overrightarrow{x} \in \mathbb{R}^N$, $n, N \in \mathbb{N}$, *with* $n^{1-\beta} > 2$; $j = 1, 2, 3, 4$, $\|g\|_{\infty,\mathbb{R}^N,X} < \infty$. *Then*

1)

$$ \int_X \left| \left({}_jC_n^{(\mathcal{R})}(g)\right)(\overrightarrow{x}, s) - g(\overrightarrow{x}, s) \right| P(ds) \leq $$

$$ \left[\Omega_1\left(g, \frac{1}{n} + \frac{1}{n^\beta}\right)_{L^1} + 2c_j(\beta, n)\|g\|_{\infty,\mathbb{R}^N,X} \right] =: \gamma_{j1}^{(\mathcal{R})}, \quad (127) $$

2)

$$ \left\| \int_X \left| \left({}_jC_n^{(\mathcal{R})}(g)\right)(\overrightarrow{x}, s) - g(\overrightarrow{x}, s) \right| P(ds) \right\|_{\infty,\mathbb{R}^N} \leq \gamma_{j1}^{(\mathcal{R})}. \quad (128) $$

Proof. Since $\|g\|_{\infty,\mathbb{R}^N,X} < \infty$, then

$$ \left| n^N \int_{\frac{\overrightarrow{k}}{n}}^{\frac{\overrightarrow{k}+1}{n}} g\left(\overrightarrow{t}, s\right) d\overrightarrow{t} - g(\overrightarrow{x}, s) \right| = \left| n^N \int_{\frac{\overrightarrow{k}}{n}}^{\frac{\overrightarrow{k}+1}{n}} \left(g\left(\overrightarrow{t}, s\right) - g(\overrightarrow{x}, s)\right) d\overrightarrow{t} \right| \leq $$

$$ n^N \int_{\frac{\overrightarrow{k}}{n}}^{\frac{\overrightarrow{k}+1}{n}} \left| g\left(\overrightarrow{t}, s\right) - g(\overrightarrow{x}, s) \right| d\overrightarrow{t} \leq 2\|g\|_{\infty,\mathbb{R}^N,X} < \infty. \quad (129) $$

Hence

$$ \int_X \left| n^N \int_{\frac{\overrightarrow{k}}{n}}^{\frac{\overrightarrow{k}+1}{n}} g\left(\overrightarrow{t}, s\right) d\overrightarrow{t} - g(\overrightarrow{x}, s) \right| P(ds) \leq 2\|g\|_{\infty,\mathbb{R}^N,X} < \infty. \quad (130) $$

We observe that

$$ \left({}_jC_n^{(\mathcal{R})}(g)\right)(\overrightarrow{x}, s) - g(\overrightarrow{x}, s) = $$

$$\sum_{\overrightarrow{k}=-\infty}^{\infty}\left(n^N\int_{\frac{\overrightarrow{k}}{n}}^{\frac{\overrightarrow{k}+1}{n}}g\left(\overrightarrow{t},s\right)d\overrightarrow{t}\right)Z_j\left(n\overrightarrow{x}-\overrightarrow{k}\right)-g\left(\overrightarrow{x},s\right)=$$

$$\sum_{\overrightarrow{k}=-\infty}^{\infty}\left(n^N\int_{\frac{\overrightarrow{k}}{n}}^{\frac{\overrightarrow{k}+1}{n}}g\left(\overrightarrow{t},s\right)d\overrightarrow{t}\right)Z_j\left(n\overrightarrow{x}-\overrightarrow{k}\right)-g\left(\overrightarrow{x},s\right)\sum_{\overrightarrow{k}=-\infty}^{\infty}Z_j\left(n\overrightarrow{x}-\overrightarrow{k}\right)= \tag{131}$$

$$\sum_{\overrightarrow{k}=-\infty}^{\infty}\left[\left(n^N\int_{\frac{\overrightarrow{k}}{n}}^{\frac{\overrightarrow{k}+1}{n}}g\left(\overrightarrow{t},s\right)d\overrightarrow{t}\right)-g\left(\overrightarrow{x},s\right)\right]Z_j\left(n\overrightarrow{x}-\overrightarrow{k}\right)=$$

$$\sum_{\overrightarrow{k}=-\infty}^{\infty}\left[n^N\int_{\frac{\overrightarrow{k}}{n}}^{\frac{\overrightarrow{k}+1}{n}}\left(g\left(\overrightarrow{t},s\right)-g\left(\overrightarrow{x},s\right)\right)d\overrightarrow{t}\right]Z_j\left(n\overrightarrow{x}-\overrightarrow{k}\right).$$

However it holds

$$\sum_{\overrightarrow{k}=-\infty}^{\infty}\left[n^N\int_{\frac{\overrightarrow{k}}{n}}^{\frac{\overrightarrow{k}+1}{n}}\left|g\left(\overrightarrow{t},s\right)-g\left(\overrightarrow{x},s\right)\right|d\overrightarrow{t}\right]Z_j\left(n\overrightarrow{x}-\overrightarrow{k}\right)\leq 2\left\|g\right\|_{\infty,\mathbb{R}^N,X}<\infty. \tag{132}$$

Hence

$$\left|\left({}_jC_n^{(\mathcal{R})}\left(g\right)\right)\left(\overrightarrow{x},s\right)-g\left(\overrightarrow{x},s\right)\right|\leq$$

$$\sum_{\overrightarrow{k}=-\infty}^{\infty}\left[n^N\int_{\frac{\overrightarrow{k}}{n}}^{\frac{\overrightarrow{k}+1}{n}}\left|g\left(\overrightarrow{t},s\right)-g\left(\overrightarrow{x},s\right)\right|d\overrightarrow{t}\right]Z_j\left(n\overrightarrow{x}-\overrightarrow{k}\right)= \tag{133}$$

$$\sum_{\substack{\overrightarrow{k}=-\infty\\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_\infty\leq\frac{1}{n^\beta}}}^{\infty}\left[n^N\int_{\frac{\overrightarrow{k}}{n}}^{\frac{\overrightarrow{k}+1}{n}}\left|g\left(\overrightarrow{t},s\right)-g\left(\overrightarrow{x},s\right)\right|d\overrightarrow{t}\right]Z_j\left(n\overrightarrow{x}-\overrightarrow{k}\right)+ \tag{134}$$

$$\sum_{\substack{\overrightarrow{k}=-\infty\\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_\infty>\frac{1}{n^\beta}}}^{\infty}\left[n^N\int_{\frac{\overrightarrow{k}}{n}}^{\frac{\overrightarrow{k}+1}{n}}\left|g\left(\overrightarrow{t},s\right)-g\left(\overrightarrow{x},s\right)\right|d\overrightarrow{t}\right]Z_j\left(n\overrightarrow{x}-\overrightarrow{k}\right)=$$

$$\sum_{\substack{\overrightarrow{k}=-\infty\\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_\infty\leq\frac{1}{n^\beta}}}^{\infty}\left[n^N\int_0^{\frac{1}{n}}\left|g\left(\overrightarrow{t}+\frac{\overrightarrow{k}}{n},s\right)-g\left(\overrightarrow{x},s\right)\right|d\overrightarrow{t}\right]Z_j\left(n\overrightarrow{x}-\overrightarrow{k}\right)+ \tag{135}$$

$$\sum_{\substack{\overrightarrow{k}=-\infty\\ \left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_\infty>\frac{1}{n^\beta}}}^{\infty}\left[n^N\int_0^{\frac{1}{n}}\left|g\left(\overrightarrow{t}+\frac{\overrightarrow{k}}{n},s\right)-g\left(\overrightarrow{x},s\right)\right|d\overrightarrow{t}\right]Z_j\left(n\overrightarrow{x}-\overrightarrow{k}\right).$$

Furthermore it holds

$$\left(\int_X \left| \left({}_jC_n^{(\mathcal{R})}(g)\right)(\overrightarrow{x}, s) - g(\overrightarrow{x}, s) \right| P(ds)\right) \underset{\text{(by Fubini's theorem)}}{\leq}$$

$$\sum_{\substack{\overrightarrow{k}=-\infty \\ \left\| \frac{\overrightarrow{k}}{n} - \overrightarrow{x} \right\|_\infty \leq \frac{1}{n^\beta}}}^{\infty} \left[n^N \int_0^{\frac{1}{n}} \left(\int_X \left| g\left(\overrightarrow{t} + \frac{\overrightarrow{k}}{n}, s\right) - g(\overrightarrow{x}, s) \right| P(ds) \right) d\overrightarrow{t} \right] Z_j\left(n\overrightarrow{x} - \overrightarrow{k}\right) +$$

$$\sum_{\substack{\overrightarrow{k}=-\infty \\ \left\| \frac{\overrightarrow{k}}{n} - \overrightarrow{x} \right\|_\infty > \frac{1}{n^\beta}}}^{\infty} \left[n^N \int_0^{\frac{1}{n}} \left(\int_X \left| g\left(\overrightarrow{t} + \frac{\overrightarrow{k}}{n}, s\right) - g(\overrightarrow{x}, s) \right| P(ds) \right) d\overrightarrow{t} \right] Z_j\left(n\overrightarrow{x} - \overrightarrow{k}\right) \leq \tag{136}$$

$$\Omega_1\left(g, \frac{1}{n} + \frac{1}{n^\beta}\right)_{L^1} + 2\|g\|_{\infty,\mathbb{R}^N,X} \sum_{\substack{\overrightarrow{k}=-\infty \\ \left\| \frac{\overrightarrow{k}}{n} - \overrightarrow{x} \right\|_\infty > \frac{1}{n^\beta}}}^{\infty} Z_j\left(n\overrightarrow{x} - \overrightarrow{k}\right) \leq$$

$$\Omega_1\left(g, \frac{1}{n} + \frac{1}{n^\beta}\right)_{L^1} + 2c_j(\beta, n)\|g\|_{\infty,\mathbb{R}^N,X}, \tag{137}$$

proving the claim.

Conclusion 8. *By Theorem 24 we obtain pointwise and uniform convergences with rates in the* 1*-mean of random operators* ${}_jC_n^{(\mathcal{R})}$ *to the unit operator for* $g \in C_{\mathcal{R}}^{U_1}(\mathbb{R}^N)$, $j = 1, 2, 3, 4$.

IV) 1-mean Approximation by Stochastic arctangent, algebraic, Gudermannian and generalized symmetric activation functions based multivariate quadrature type neural network operator

Let $g \in C_{\mathcal{R}}^{U_1}(\mathbb{R}^N)$, $0 < \beta < 1$, $\overrightarrow{x} \in \mathbb{R}^N$, $n, N \in \mathbb{N}$, with $\|g\|_{\infty,\mathbb{R}^N,X} < \infty$, $(X, \mathcal{B}, P)$ probability space, $s \in X$, $j = 1, 2, 3, 4$.

We define

$${}_jD_n^{(\mathcal{R})}(g)(\overrightarrow{x}, s) := \sum_{\overrightarrow{k}=-\infty}^{\infty} \left(\delta_{n\overrightarrow{k}}(g)\right)(s) Z_j\left(n\overrightarrow{x} - \overrightarrow{k}\right), \tag{138}$$

where

$$\left(\delta_{n\overrightarrow{k}}(g)\right)(s) := \sum_{\overrightarrow{r}=0}^{\overrightarrow{\theta}} w_{\overrightarrow{r}}\, g\left(\frac{\overrightarrow{k}}{n} + \frac{\overrightarrow{r}}{n\overrightarrow{\theta}}, s\right), \tag{139}$$

(see also (92), (93)).

We finally give

Theorem 25. *Let* $g \in C_{\mathcal{R}}^{U_1}(\mathbb{R}^N)$, $0 < \beta < 1$, $\overrightarrow{x} \in \mathbb{R}^N$, $n, N \in \mathbb{N}$, *with* $n^{1-\beta} > 2$; $j = 1, 2, 3, 4$, $\|g\|_{\infty,\mathbb{R}^N,X} < \infty$. *Then*

1)

$$\int_X \left| \left({}_jD_n^{(\mathcal{R})}(g) \right) (\overrightarrow{x}, s) - g(\overrightarrow{x}, s) \right| P(ds) \leq$$

$$\left\{ \Omega_1 \left(g, \frac{1}{n} + \frac{1}{n^\beta} \right)_{L^1} + 2c_j(\beta, n) \|g\|_{\infty, \mathbb{R}^N, X} \right\} =: \gamma_{j1}^{(\mathcal{R})}, \tag{140}$$

2)

$$\left\| \int_X \left| \left({}_jD_n^{(\mathcal{R})}(g) \right) (\overrightarrow{x}, s) - g(\overrightarrow{x}, s) \right| P(ds) \right\|_{\infty, \mathbb{R}^N} \leq \gamma_{j1}^{(\mathcal{R})}. \tag{141}$$

Proof. Notice that

$$\left| \left(\delta_{n\overrightarrow{k}}(g) \right)(s) - g(\overrightarrow{x}, s) \right| =$$

$$\left| \sum_{\overrightarrow{r}=0}^{\overrightarrow{\theta}} w_{\overrightarrow{r}} \left(g\left(\frac{\overrightarrow{k}}{n} + \frac{\overrightarrow{r}}{n\overrightarrow{\theta}}, s \right) - g(\overrightarrow{x}, s) \right) \right| \leq$$

$$\sum_{\overrightarrow{r}=0}^{\overrightarrow{\theta}} w_{\overrightarrow{r}} \left| g\left(\frac{\overrightarrow{k}}{n} + \frac{\overrightarrow{r}}{n\overrightarrow{\theta}}, s \right) - g(\overrightarrow{x}, s) \right| \leq 2\|g\|_{\infty, \mathbb{R}^N, X} < \infty. \tag{142}$$

Hence

$$\int_X \left| \left(\delta_{n\overrightarrow{k}}(g) \right)(s) - g(\overrightarrow{x}, s) \right| P(ds) \leq 2\|g\|_{\infty, \mathbb{R}^N, X} < \infty. \tag{143}$$

We observe that

$$\left({}_jD_n^{(\mathcal{R})}(g) \right) (\overrightarrow{x}, s) - g(\overrightarrow{x}, s) =$$

$$\sum_{\overrightarrow{k}=-\infty}^{\infty} \left(\delta_{n\overrightarrow{k}}(g) \right)(s) Z_j\left(n\overrightarrow{x} - \overrightarrow{k} \right) - g(\overrightarrow{x}, s) =$$

$$\sum_{\overrightarrow{k}=-\infty}^{\infty} \left(\left(\delta_{n\overrightarrow{k}}(g) \right)(s) - g(\overrightarrow{x}, s) \right) Z_j\left(n\overrightarrow{x} - \overrightarrow{k} \right). \tag{144}$$

Thus

$$\left| {}_jD_n^{(\mathcal{R})}(g)(\overrightarrow{x}, s) - g(\overrightarrow{x}, s) \right| \leq$$

$$\sum_{\overrightarrow{k}=-\infty}^{\infty} \left| \left(\delta_{n\overrightarrow{k}}(g) \right)(s) - g(\overrightarrow{x}, s) \right| Z_j\left(n\overrightarrow{x} - \overrightarrow{k} \right) \leq 2\|g\|_{\infty, \mathbb{R}^N, X} < \infty. \tag{145}$$

Hence it holds

$$\left| \left({}_jD_n^{(\mathcal{R})}(g) \right) (\overrightarrow{x}, s) - g(\overrightarrow{x}, s) \right| \leq$$

$$\sum_{\overrightarrow{k}=-\infty}^{\infty} \left| \left(\delta_{n\overrightarrow{k}}(g) \right)(s) - g(\overrightarrow{x}, s) \right| Z_j\left(n\overrightarrow{x} - \overrightarrow{k} \right) =$$

$$\sum_{\substack{\overrightarrow{k}=-\infty \\ \left\| \frac{\overrightarrow{k}}{n} - \overrightarrow{x} \right\|_{\infty} \leq \frac{1}{n^{\beta}}}}^{\infty} \left| \left(\delta_{n\overrightarrow{k}} (g) \right) (s) - g \left(\overrightarrow{x}, s \right) \right| Z_j \left(n\overrightarrow{x} - \overrightarrow{k} \right) +$$

$$\sum_{\substack{\overrightarrow{k}=-\infty \\ \left\| \frac{\overrightarrow{k}}{n} - \overrightarrow{x} \right\|_{\infty} > \frac{1}{n^{\beta}}}}^{\infty} \left| \left(\delta_{n\overrightarrow{k}} (g) \right) (s) - g \left(\overrightarrow{x}, s \right) \right| Z_j \left(n\overrightarrow{x} - \overrightarrow{k} \right). \tag{146}$$

Furthermore we derive

$$\left(\int_X \left| \left({}_j D_n^{(\mathcal{R})} (g) \right) \left(\overrightarrow{x}, s \right) - g \left(\overrightarrow{x}, s \right) \right| P(ds) \right) \leq$$

$$\sum_{\substack{\overrightarrow{k}=-\infty \\ \left\| \frac{\overrightarrow{k}}{n} - \overrightarrow{x} \right\|_{\infty} \leq \frac{1}{n^{\beta}}}}^{\infty} \sum_{\overrightarrow{r}=0}^{\overrightarrow{\theta}} w_{\overrightarrow{r}} \left(\int_X \left| g \left(\frac{\overrightarrow{k}}{n} + \frac{\overrightarrow{r}}{n\overrightarrow{\theta}}, s \right) - g \left(\overrightarrow{x}, s \right) \right| P(ds) \right) Z_j \left(n\overrightarrow{x} - \overrightarrow{k} \right) \tag{147}$$

$$+ \left(\sum_{\substack{\overrightarrow{k}=-\infty \\ \left\| \frac{\overrightarrow{k}}{n} - \overrightarrow{x} \right\|_{\infty} > \frac{1}{n^{\beta}}}}^{\infty} Z_j \left(n\overrightarrow{x} - \overrightarrow{k} \right) \right) 2 \|g\|_{\infty, \mathbb{R}^N, X} \leq$$

$$\Omega_1 \left(g, \frac{1}{n} + \frac{1}{n^{\beta}} \right)_{L^1} + 2c_j (\beta, n) \|g\|_{\infty, \mathbb{R}^N, X}, \tag{148}$$

proving the claim.

Conclusion 9. *From Theorem 25 we obtain pointwise and uniform convergences with rates in the 1-mean of random operators ${}_j D_n^{(\mathcal{R})}$ to the unit operator for $g \in C_{\mathcal{R}}^{U_1} \left(\mathbb{R}^N \right)$, $j = 1, 2, 3, 4$.*

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