Chapter 12
Recent Trends of History of Mathematics Teacher Education: The Iberic American Tradition

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Abstract This chapter intends to analyze the recent evolution of the Teacher Education programs themselves, the proposals relating to knowledge and practice, the noticing approach, the didactic phenomenology or the ontosemiotic approach, the anthropological perspective, sociocultural and critical perspectives, among others other general theoretical frameworks. In addition, the chapter focalizes the Iberoamerican tradition less analyzed by the English influence but participating as an international perspective. Also, future perspectives are recognized.

Keywords History of mathematics · History of teacher education · Teacher education · Teacher knowledge · Iberoamerican tradition

Introduction

This chapter analyses the evolution of the research about teacher training programs and research proposals developed in the study, mainly focusing on mathematics teacher education from a historical point of view. We do not focus on using history as a resource for teaching but on exploring how mathematics teacher education has evolved in the last decades, resulting from a historical development process.

Two facts mark the emergence of the new status of Teacher Mathematics Education (TME) as a research agenda within Mathematics Education (M.E.) as a scientific discipline: (a) the importance of teacher educators in the field, and (b) the presence of teacher-researchers in international forums. Many studies have been published in the *Journal of Math Teacher Education* and the *European Journal of Teacher Education,* and other scientific sources, including *CIEAEM Conferences*. Such organisation consolidates the role of teachers as teacher-researchers.

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In the last decades, there has been considerable development of the TME. After a first period (1968–2000) in which the research was focused more on students, it started a new period centred on teaching and teacher development.

The year 2000 increased the reflection about the role of Mathematics in schools, also giving opportunities for the preparation for the new role of teachers in society. The ICMI community was ready to receive the message of Ubiratan D'Ambrosio (1986) at ICME-5, which introduced sociocultural dimensions into mathematics education and acknowledged the dignity of various forms of mathematical culture. It started the *Renaissance period* (Furinghetti et al., 2008). In CIEAEM63 (Barcelona), D'Ambrosio delivered one of the keynotes. He encouraged the training of future teachers capable of arousing interest in mathematics as a tool to improve the well-being of the entire society (D'Ambrosio, 2011).

Before that, it was a lack of international communication regarding mathematical education basically because many differences appear concerned the contents of instruction and the epistemology of school mathematics and the methodology of teaching.

In many countries, the training of teachers has become a more professional perspective. The inclusion of an official preparation of teachers in the Universities, leaving the old practice in separated colleges or "escuelas normales", introduced a theoretical and specialized preparation on M.E. (Sierra, 1995). A critical moment was the publication of a collection of a series called "Mathematics Teacher Education" and the definitive inclusion of Teacher Education as an important point in the international agenda. The book "Learning through Teaching Mathematics, edited by Leikin and Zazkis (2010), was also a significant challenge trying to change the regular perspective of observing others to a new view of self-reflection.

Theoretical and Historical Aspects

The changing conditions for the preparation of teachers in a common European framework, and internationally promoting exchanges among prospective teachers and teachers themselves, increase the interest in Teacher Education. The influence was noticeable in the themes and contributions to international meetings. The year 2000 promoted research interest in M.E. from a broader international audience: colleagues from non-industrialized and socialist countries and primary and secondary school teachers. Shifting from a concentration on content and methodological questions in mathematics, new themes were included in the research agenda, such as broader epistemological, psychological, sociological, and technological problems.

The teacher's professional problems create links between scientific knowledge and craft wisdom, reinforcing the collaboration of mathematics education research and practice. Many questions arise from that perspective: What are the strategies in research and practice that support the development and provide essential and appropriate teaching and learning opportunities that ensure access to all levels of institutionalized schooling in elementary, secondary, higher education, and adult education? How to create appropriate social conditions to establish a teaching and learning practice guided by social justice and equity principles?

The insufficient learning opportunities and a lack of transparency of the assessment system mainly concern mathematics as the prominent means of selection in the educational system. Education is in danger of being no longer perceived as a public duty or vital public service. TME should share a common aim: to train future teachers for equity and mathematics for all. TME should support adult education as a critical force for democratization and change in such a framework. It means a new way of seeing Mathematics teacher preparation for all.

The importance is given to training on practice, and from practice is considered. Likewise, collaborative intervention in research on teacher training is also considered. The proposals are analysed according to the need for a more significant link with school practice, the need for programs that face interdisciplinary visions, the tension between globalized training in the pre-K, kindergarten, and primary stages, and specialized postgraduate training in secondary education. The role and analysis of general or specific professional competencies are recognized among the new topics. The new views refer to professional identity, analysis of the holistic nature of specialized knowledge, the incorporation of research results, the essence of practical knowledge, including the history of mathematics in initial training, special needs, etc.

Concerning the curriculum for Teacher Education, the content matter should be extended to subjects too complex for treatment in traditional instruction, and application and problem-solving a much more appropriate simulation of reality is possible. By far less evident is how mathematics education should respond to the change in the notion of reality: the blending of the real and the virtual worlds, the loss of reliable discrimination of reality, and its manipulation. A tremendous problem emerges from the fact that the new technologies open up unprecedented chances and risks in various fields like biotechnology and military development, based on models and simulations beyond theoretical comprehension and beyond the validity of existing empirical knowledge. No attempts have been made so far to furnish reliable intellectual and moral basis equipment to the coming generations that inevitably will have to deal with these challenges (CIEAEM, 2000).

The following sections present different approaches from the international perspective, putting a particular accent on the research done in the Iberic American countries. Therefore, we focus on: (1) cognitive and epistemic aspects, (2) preservice teacher development at different levels, (3) knowledge and practice of mathematics teacher education, introducing lesson study and other frameworks, (4) anthropological and onto-semiotic perspective studies, (5) identity of teachers, attitudes, and beliefs; (6) critical issues and ethnomathematics studies, and (7) the CIEAEM contributions to mathematics teacher training. Finally, new topics and future perspectives are described.

Cognitive and Epistemic Aspects of MTE

In the last decades, several contributions have expanded the classic work done by Shulman (1986, 1987) on teachers' knowledge. This section exemplifies studies related to didactic-epistemological reflections, mainly done at the secondary level. Personal epistemologies refer to individuals' cognition of knowledge and the process of knowing (Pintrich, 2004). In Furinghetti (2000) work, it is argued that the history of mathematics may be an efficient element to provide students with flexibility, open-mindedness, and motivation for mathematics. She presented a historical presentation of 'definition'; it was developed with mathematics students who will become mathematics teachers. Furinghetti's work (2000) claimed the need to consider the cognitive dimension of the knowledge that mathematics teachers will need to teach their students. Later, it is recognized the importance of teacher educators as a learner (Krainer et al., 2014).

Mathematics teachers' knowledge has been a very prominent focus of attention in the last decades. Many of the studies introduced the analysis of mathematics knowledge on different topics. From such a perspective, it is found the work about specialized mathematics knowledge on topics such as divisibility (Almeida et al., 2021), three-dimensional figures (Vanegas & Giménez, 2021); logic (Alfaro et al., 2020), among others.

Ball and her colleagues (2008) expanded Shulman's model exploring more in detail the distinction between content knowledge and pedagogical content knowledge (PCK), adding new concepts (common content knowledge, specialized content knowledge, and horizon content knowledge; knowledge of content and students, knowledge of content and teaching, and knowledge of curriculum).

These proposals have been resonated in the Iberic American context (Contreras et al., 2017; Montes et al., 2013; Pinto Sosa & González Astudillo, 2008). In 2020 the University of Huelva organized the IV Congreso Iberoamericano sobre conocimiento especializado del profesor de matemáticas [IV Ibero-American Congress on specialized knowledge of the mathematics teacher]. Members of the *Red Iberoamericana MTSK* discussed the relationships between the subdomains of PCK (Cabrera-Baquedano & Pezoa-Reyes, 2020).

However, the PCK model (and further developments, such as the ones by Ball et al.) have been criticized for being too static (Venkat & Adler, 2020). Opposite to this type of approach, frameworks focus on the developmental change in teachers' epistemologies (Feucht et al., 2017). Teachers change their beliefs, practices, and pedagogical knowledge while teaching over time or because of being exposed to new teaching evidence (further research studies, additional teacher training programs, etc.).

The team from Huelva in Spain expanded the PCK model toward an MTSK perspective (Carrillo et al., 2013a, b). Drawing on Ball et al. (2008) work, Carrillo and his colleagues expanded the original concept (PCK), including the specialized knowledge that mathematics teachers have (Montes et al., 2013; Flores et al., 2013; Carreño & Climent, 2009; Hill et al., 2008). According to them, mathematics teachers have specialized knowledge emerging from their practice. Teachers need this specialized knowledge to conduct their lessons, which is different from the broader knowledge of mathematics that every mathematics user may have. The MTSK includes six subdomains of knowledge: three regarding the mathematics knowledge (M.K.): knowledge of themes (KoT), knowledge of the mathematics structure (KSM); and three regarding the pedagogical content knowledge: knowledge about the features of the learning mathematics (KFLM), knowledge about mathematics teaching (KMT), and knowledge about the mathematics learning standards (KMLS).

Rowland and his colleagues (2003) coined the *Knowledge Quartet*, focusing on analyzing teacher situations and introducing a model composed of four dimensions: foundation, transformation, connection, and contingency knowledge. This approach has also been used in the Iberic American tradition to provide examples for future teachers of using teachers' professional knowledge to reflect on teachers' practices (Martínez & Arévalo, 2017).

In August 2015, a group of researchers met at the Advanced Study Colloquium (ACS), funded by the EARLI (European Association for Research in Learning and Instruction), and introduced the 3R-EC framework: reflection, reflexivity, and resolved action for epistemic cognition. They discussed that teachers use the review to examine their practices to improve their future praxis. According to Feucht et al. (2017), teachers first reflect on a particular issue of concern. Then they engage in an internal dialogue including diverse structural, cultural and personal factors (as well as personal epistemologies) to finally make teaching decisions that will be enacted in the next opportunity (i.e., in their next lesson). Epistemic cognition refers to "how people acquire, understand, justify, change and use knowledge in formal and informal contexts" (Greene et al., 2016).

This dynamic view of teachers' competence to reflect (and change) their practices have been explored in the Iberic American region. For example, Zamorano (2015) discusses teachers' practices drawing on Rowland's concept of contingency situations. Solar and Deulofeu (2014) explore the contingency, focusing on how teachers justify their statements/claims within the classroom. Similarly, Martínez and Arévalo (2017) use the "knowledge quartet" model to analyze mathematics classes as a strategy for the professional development of primary education teachers.

Preservice Teacher Development at Different Levels

The expression *teacher professional development* arises from the XXI century and tends to encompass the initial preparation with in-service research as a continuity. Therefore, some reflections about language contexts were emphasized for teacher education in the ICMI study held in 2005.

The agenda, in this case, argue on assumptions about classroom practice and preservice teachers' and school students' learning. There is still a discussion about theoretical frameworks and methodologies to discuss potentials and challenges. It continues the reflection about analyzing representations and other mathematical processes.

Recent elaborations introduce the perspective of teachers' competencies resulting from the interaction between personal, situational, and social aspects (Blomeke et al., 2015).

In the case of preparation for secondary level, it is essential to conceptualize and measure knowledge in/for teaching mathematics, core features, basic abilities, and attempts of a normative approach. An unsolved question is what kind of knowledge is relevant for practice, and where and how do teachers learn this knowledge? It is also important how to implement the reflective practice, designing learning environments, changing perceptions, collaborative issues, inquiry learning environments, digital learning (Borba et al., 2016) environments, management issues, Self-efficacy, etc.

Among the different perspectives, many investigations relate to noticing competence in different levels (Llinares et al., 2016; Brown et al., 2020; López, 2021; Tekin-Sitrava et al., 2021; Amador et al., 2021; Dindyal et al., 2021) and study of learning trajectories (Burgués & Giménez, 2007; Ivars et al., 2019) mainly for Primary Teachers; Bernabeu et al., 2021).

Also, to understand opportunities for learning measurement (Callejo et al., 2021) and how prospective early childhood teachers try to instrument a learning trajectory (Moreno et al., 2021) and use the noticing approach as a research-based design for early reasoning (Vanegas et al., 2021).

Knowledge and Practice of Mathematics Teacher Education

Borko and Potari (2020) have pointed out that 'knowledge for teaching' comprises 'subject matter knowledge and "pedagogical matter knowledge." There is an increasing "polarization" between practitioners and researchers in many countries and mathematicians and mathematics educators. Politicians find this an attractive situation and take advantage of it by using the division to minimize academic "interference" in their agenda for education, for example, in a further back-to-basics approach (CIEAEM, 2000).

Sfard (2004) qualified a new period as "the era of the teacher" due to researchers' uncontested focus on teachers. Such attention is also represented in the launching in 1998 of an international journal dedicated to mathematics teachers' education, the Journal of Mathematics Teacher Education. Questions about what teachers need to know and be able to do and how they develop their knowledge, skills, and beliefs have become central to the mathematics education research literature.

In some sense, "mathematics teacher educator" suggests a focus on academics only. This may be true for those countries/regions where MaTED is mainly at universities. But there are countries/regions where MaTED takes place within the instruction system or in teacher education institutes independent of universities.

The recently launched ICMI Study 25 (co-chaired by Potari & Borko) focuses on the idea of mathematics teachers learning through collaboration in schools or larger communities, drawing on an ICME-13 survey team by Robutti et al. (2016). Collaborative groups may be teams, communities, schools, and other educational institutions, professional development courses, local or national networks. This means that mathematics teacher educators can work in formal or informal groupings, in either face-to-face or distance settings. In addition, they can be facilitators such as trainers, coaches, or mentors. Given the variety of ways mathematics teacher educators can work and the different settings they can operate, many papers are needed to identify how to handle these professionals.

Anthropological, Historicist, and Onto-Semiotic Approaches

Recently, new topics have arisen, such as the so-called ecological aspect and analysis of restrictions, which allows us to analyse the so-called study and research paths from the Anthropological Theory of Didactics. Sustainability in training processes, comparative studies, distance training models, and new integrative analyses of the social with the linguistic in analysing processes in initial or in-service teacher training. The role of reflective methodologies such as Lesson Study is now recognized. Also, issues related to social justice, the use of democratic debates, and the analysis of training for citizenship in teacher training. The critical view of Mathematics Education is also analysed regarding teacher training in recent years. It also points to the role of the specific analysis of textbook proposals, homework, analysis of interactions, and other curricular materials.

The origin of the Anthropological theory of didactic (the ATD) is the theory of the didactic transposition held by Chevallard (1991). At that time, Chevallard intended to describe and understand the lessons of mathematics taught by teachers in the classroom as the result of a process of "translation" of the mathematics itself, in the form of mathematics for teaching (which is not mathematics but a reformulation of mathematics in didactical terms). Drawing from this approach, many authors have contributed to the ATD framework (Bosch et al., 2011; Bronner et al., 2010; Chevallard, 2006, 2007; Ruiz-Higueras et al., 2007). Similar formulations also include the so-called "Joint Action Theory in Didactics" (JATD), drawing on Brousseau's (1997) idea of didactic situations. According to the Anthropological perspective, *didactics* includes two main elements: the object of knowledge (O) and the human being x who is studying O. In this sense, for researchers under this approach, the object of study is the system $R(x, O)$, where "R" means the relations between x and O. This minimum system can be further expanded including the person who is teaching O (y), as well as other people involved in the didactical situation (relatives, community members, etc.). The relation $R(x, O)$ expands to $R(x, y, O)$, etc., creating a didactic system in which the general expression is denoted by $S(X, Y, O)$. The authors under this approach are aware of the restrictions/conditions the educational institution imposes on the system $S(X, Y, O)$. In every school, there is a pedagogical approach of the centre, also influenced by the general standards (curriculum). Hence, in every case, the system *S*(*X, Y, O*) may look a bit (or significantly) different. In turn, the object of knowledge (*O*) can also be expanded into different "layers": the domain of the discipline (i.e., geometry), sectors, themes (or topics), and subjects.

The component that makes the system work is called *praxeology*. This component includes the *type of tasks* (*T*); the *technique* t (*tau*), concerning a way of performing the tasks; the *technology* q (*theta*), referring to the way of explaining and justifying the technique t; and the *theory* Q (*big theta*), that explains, justifies, or generates any part of the technology q that may be missing (or embedded). According to the ATD, the object of knowledge (*O*) results in a combination of praxeologies (*T,* t, q, Q).

The JATD introduces another critical aspect in the didactical analysis (to understand how teaching works): the relationship between teacher, student, and knowledge, based on what Ligozat and Schubauer-Leoni (2009) call *epistemic joint act*. According to them, teaching is an "action" in which the teacher teaches an object of knowledge to the student. In a similar vein, other authors (Sensevy, 2011) use the idea of the "Didactic game" as a metaphor to describe the interactions between teacher and student (as two players aiming at achieving the goal of the game, which is learning *O*), as described in Chevallard and Sensevy (2014).

In the Iberic American countries, the Anthropological framework has been used in different contexts, such as integrating theories (Gellert et al., 2013), specific topics (Barbé et al., 2005; Corica & Otero, 2009; Quijano & Corica, 2017). We even found efforts of theoretical dialogue among different approaches, such as ATD and APOS (Bosch et al., 2017).

Radford (2013) proposes the *theory of objectification*. He introduces a more dynamic perspective in the didactical analysis based on a historicist point of view. According to him, "the goal of mathematics education is a dynamic political, societal, historical, and cultural endeavor aiming at the dialectical creation of reflexive and ethical subjects who critically position themselves in historically and culturally constituted and always evolving mathematical discourses and practices" (Radford, 2013, p. 8). Knowledge is constructed through doing, thinking, and reflecting, which is historically and culturally situated. Knowing becomes the actualization of knowledge. Learning is the social practice in which participants internalize that knowledge.

In a different vein, another contribution to the teacher training domain emerging from the Spanish scientific research, primarily echoed along with the Iberic American countries, is the onto-semiotic approach (OSA) (Godino et al., 2007). Initially, this perspective focused on the mathematical object. According to this approach, mathematics is a "socially shared problem activity, a symbolic language and a logically organized conceptual system." (Godino et al., 2007, p. 129) People engage in *mathematical practices* when doing any mathematical activity (namely, solving mathematical problems or situations). When an individual is doing "mathematics," s/he carries out certain shared social practices, and s/he uses particular instruments and tools. Recently, OSA relates very closely to ATD perspectives introducing a prescriptive character of didactics (Godino et al., 2019). Researchers drawing on this approach tend to pay more attention to the systems of operative and discursive practices people use when solving mathematical tasks. These practices may be set up individually or defined/established by the institution (who is the "authority" determining what is right and wrong). Godino et al. (2007) distinguish among different types of institutional meanings: implemented, assessed, intended, and referential. Personal meanings include: global (personal practices that an individual can potentially carry out about a mathematical object), declared (what the individual does in fact), and achieved (personal practices fitting with the institutional meaning expressed by the teacher). Doing mathematics involves using mathematical objects. Godino et al. (2007) distinguish between language (terms, expressions, notations, graphics); situations (problems, extra or intra-mathematical applications, exercises, etc.); concepts (i.e., number, point, straight line, mean, function, etc.); propositions, properties, or attributes; procedures (operations, algorithms, techniques); and arguments (i.e., deductive, inductive, abductive, etc.) Godino et al., 2007, p. 130) Objects are organized into systems of practices. Teachers and students (or participants in the teaching-learning context) use semiotic representations (not just language but also other representations) to discuss the mathematical objects organized within systems of practice. According to authors working from this perspective, mathematical objects are organized in "configurations" that can be epistemic or cognitive. When individuals engage in primary mathematical processes (such as communicating, solving problems, defining, enunciating, elaborate procedures, arguing and/or justifying, etc.), they may use some of the above objects. Drawing on Wittgenstein's (1953) concept of *language game*, Godino (2002) claimed that individuals' interaction when doing mathematics might swing from personal to institutional, ostensive to non-ostensive, extensive to intensive, unitary to systemic and expression to content dimensions. The researcher needs to consider the duality (dialectally) among those binomials to analyze and understand what happens when a subject is doing mathematics.

Recent elaborations of this approach include a focus on didactical problems, practices, processes, and objects, including what is called "didactical suitability criteria (DSC)." (Breda et al., 2018). The DSC include six dimensions: epistemic, cognitive, interactive, mediational, emotional, and ecological. Researchers use them to analyze how fair is a particular mathematical practice (i.e., a lesson, an interaction in small groups, etc.), to design tasks for didactical analysis (Breda, 2021; Díez-Palomar et al., 2020; Giménez et al., 2013), to reflect on the meta-didactic knowledge of teachers (Breda et al., 2017), to push prospective teachers to develop their didactical suitability analysis competence (Giacomone et al., 2018). In addition, some recent studies suggest connections between DSC and other well-known teaching constructs, such as the "lesson study" (Hummes et al., 2019). Other examples of how this approach is being used (and expanded) among Ibero-American countries include assessment (García Marimón et al., 2021), analyzing self-regulation practices (Hidalgo Moncada et al., 2021).

Identity of Teachers, Attitudes, and Beliefs

At the end of the twentieth Century, mathematics emphasized how mathematics teachers could improve situations by introducing "interesting mathematics "and analysing such a situation. The emotional and affective domain became a cornerstone for mathematics educators and researchers, mainly since McLeod (1989) worked on beliefs, attitudes, and emotions. In 1994, he published a paper in the *Journal for Research in Mathematics Education*, providing one of the most important reviews on affect and mathematics learning in the last decades (McLeod, 1994). He tracked research in this domain back to the 1960s when researchers such as Schacter and Singer (1962) or H.A. Simon (1967), among others, started to claim that students may develop a negative attitude towards mathematics and their ability to learn mathematics if they experience many situations in which they are unable to solve a problem, or they do not solve the problem according to their expectations (for instance, finding the solution in a short period -like 2 or 3 minutes-). According to McLeod (1992), when students experience negative episodes several times, they develop negative attitudes towards mathematics. He introduced relevant concepts to clarify the field, namely: beliefs (i.e., a problem can be solved quickly or not at all, only geniuses can be creative in mathematics, mathematics is primarily rule-oriented or concept-oriented, mathematics is helpful, but involves mainly memorizing and following rules, etc. -see Schoenfeld, 1985; Fennema & Peterson, 1985; Dossey et al., 1988-), attitudes (affective responses involving positive or negative feelings of moderate-intensity and reasonable stability -see Leder, 1987; Reyes, 1984-), and emotions (i.e., tension, frustration, happiness, etc. -see Bloom & Broder, 1950; Buxton, 1981-). He also highlighted the importance of confidence, self-concept, self-efficacy, and mathematics anxiety to understand how learners feel when learning mathematics.

Philippou and Christou (1998a, b) paid particular attention to the link between history and mathematics teachers' beliefs and attitudes about mathematics.

Theories of mathematical giftedness entail attitudes to teach mathematics in a school for all to the few: only gifted and "socially useful." To identify the gifted, more selection and individual differentiation in tests is justified, and the chances of collective learning experiences are ignored or neglected. If a social focus on the "gifted" persists, the majority will not be educated appropriately.

Later, authors such as Markus Hannula led the field, creating the TSG on mathematics and affect in CERME and PME (Hannula, 2006, 2019; Hannula et al., 2004).

In the Iberic American context, the pioneer work on mathematics and affect came from Inés María Gómez Chacón. In 2000 she published her dissertation as a book titled *Matemática Emocional. Los afectos en el aprendizaje matemático*. She worked together with McLeod on mathematics and affect. She provided one of the better classifications of beliefs, attitudes and emotions that is still used largely in the field (Hannula et al., 2005). Since then, many studies have addressed this issue in the Iberic American countries (Blanco et al., 2009, Blanco et al., 2010; Ciro & Torres, 2016; Contreras & Moreno, 2019; García-González et al., 2021; García González &

Pascual Martín, 2017; Gil et al., 2006; Ibarra-González & Eccius-Wellmann, 2018; Perdomo Díaz & Fernández Vizcarra, 2018; etc.). This topic has been raised up also in Iberic American conferences, such as RELME (Farfán & Sosa, 2007; Martínez, 2014; Martínez Padrón et al., 2007), or CIBEM (González, 2009; Martínez Padrón, 2009), among others.

Mathematics still is one of those school subjects that provoke strong feelings of anxiety, aversion, and incompetence. Pupils (and teachers) still dislike school mathematics as a compulsory enterprise without significance. How can a subject raise such solid emotions and block both interest and ability to think mathematically? Why is mathematics so meaningless and challenging for most pupils who consider themselves "mentally handicapped" in mathematics and doomed to failure?

Regarding continuous training and research problems, the tension between peer training programs based on specific reflective practice and school team training that emphasize project work proposals. It has gone from the prevalence of generic studies on professional development and analysis of training models, those based on the epistemic, affective, or socio-cognitive component, to studies that combine reflection on beliefs and professional identity with proposals on social and cultural approaches.

Critical Issues and Ethnomathematics Studies

Mathematical abstractions and formalizations applied to social reality create formal systems and hierarchies, models, or ways of argumentation that eventually become quasi-natural social rules. By transforming into technology, application, and continuous use, these formalizations turn into representations of "natural" social order and "natural" patterns of social organization, institutions, and regulations - formatting of the society by mathematics has taken place. Mathematics Teacher education must understand the processes of "mathematization" in society and how to prepare teachers to do it. And it must create a critical judgment about it, transparency of the part of mathematics played in social conditions, and enlightenment about the social use of mathematics. It is a way to improve the metacognitive aspect of teaching and know ways to understand modelling as a deep interrelation between social and human aspects of interpreting and changing the world and learning how to introduce debates, analyse the role of language and metaphors, etc.

Critical perspective recognizes still unsolved problems in the research of Teacher Mathematics Education: Is the perception of excellence or high achievement in mathematics different in different cultures, societies, and communities, perhaps depending on class, gender, and ethnicity? Does it respect social awareness and political responsibility? What are various strategies to counteract conflicts, lack of justice, and equal treatment in teaching and learning mathematics in the classroom and the school or broader society? What are the influences of changing social environments on the attitudes towards mathematics and the performance expectations of teachers and parents?

At ICME VI in 1988, for the first time included the social and political dimensions of mathematics education as a legitimate challenge, a matter of worldwide consciousness and recognition.

The recent analysis of Montecino and Valero (2017) states that by deploying a Foucault-inspired discourse analysis on a series of documents produced by these agencies, we argue that nowadays, the cultural thesis about who the mathematics teacher should be are framed in a double bind of the teacher as a policy product and as a sales agent. Narratives about the mathematics teacher are made possible within a dispositive of control, making mathematics education and mathematics teachers the cornerstone for realizing current market-oriented, competitive, and globalized societies.

The CIEAEM Contributions to Mathematics Teacher Education

CIEAEM was born as a space for bridging research and teaching practice that has contributed significantly to improving the teaching and learning of mathematics. Therefore, its very beginning was already marked by a clear orientation towards teacher training, but with an evident emphasis on creating bridges between practitioners and researchers, which, seventy years later, continues to be a unique space.

Throughout the history of this congress, multiple contributions have been made to the professional development of the mathematics teacher. The contribution of the Iberic American community has also been (and it is) relevant. CIEAEM is a privileged space. It offers a unique perspective on mathematics teacher education and the challenges that the teachers of mathematics face in schools. Almost all CIEAEM conferences have included a working group focused on the professional development of teachers. Over the years, the centres of interest have been changing according to the needs of each moment. However, there have always been common elements that have given continuity to the CIEAEM vision of teacher training. The central recurring concern in all editions is: How to promote connections between school mathematics and academic mathematics in teacher training?

In recent times (from 2014 onwards), the Iberic American community has contributed significantly to establishing research lines in teachers' professional development.

In 2014, Javier Díez-Palomar and Gail FitzSimons (FitzSimons & Díez-Palomar, 2014) coordinated a working group on teachers' education at the annual CIEAEM meeting held in Lyon. On this occasion, the central theme was "mathematics and its teaching about other disciplines." Díez-Palomar et al. (2014) discussed the connections of mathematics with other disciplines as a meeting point for preservice teachers' training programs. Drawing on a case study, they addressed questions such as "how can mathematics interact with other disciplines to support the understanding of a multidimensional problem?", highlighting interdisciplinarity as a way of creating didactic learning units where mathematical objects appear contextualized. These types of proposals encourage the development of the mathematical competence of future teachers.

In 2015, when CIEAEM was held in Aosta, Joaquim Giménez (together with Daniela Ferrarello and Ruhal Floris) coordinated the working group on "teacher education." The main topic of discussion was the obstacles and difficulties that mathematics teachers must face to carry out their work. Attendees discussed questions such as:

- How is it possible to support teachers to develop relevant knowledge and competencies in digital technologies so that they are effective in their mathematics teaching?
- What are the main obstacles to mathematics teacher development?
- How can the social dimension become a resource for teacher education? What are the programs' challenges based on social interaction in communities of practice/inquiry?
- How can the affective dimension become a resource for teacher education?

In 2017 the focus was mathematization. The fourth working group focused on "Mathematization as a didactic principle: looking at teachers of mathematics." The mathematization approach has traditionally been dominated by the contributions of authors influenced by Freudenthal and his Institute in mathematics for pre-and kindergarten, primary and secondary schools. The Iberic American community of the CIEAEM has already made relevant contributions to this research line, such as the piece by Giusti de Souza, Nogueira de Lima, Mendonça Campos and Gerardini (2012).

The most recent contribution of CIEAEM to mathematics teachers' professional development was found in 2019 when a working group on mathematics teacher education was included again (Panero & Mamede, 2020). The main topic for discussion was the connections and complexity in mathematics teacher education on this opportunity. The following research questions were discussed:

- What kind of mathematics training should teachers have to promote learning with understanding?
- How can teacher training contribute to establishing connections between the various areas of Mathematics?
- How can teacher training contribute to establishing connections between Mathematics and other subjects?
- How to promote connections between school mathematics and academic mathematics in teacher training?
- What competencies do we need to include in professional training programs for mathematics teachers to cope with the increasingly complex world challenges?

The research carried out by Iberic American researchers has incorporated all these topics. Different approaches to teacher education have been used. For example, Pereira Gonçalves and Gomes (2020) discuss the use of the MKT model in the case of numbers and operations. In their work, they reflect on the mathematical preparation of future teachers (a concern that is recurrent in other Iberic American countries). Font and his colleagues have used the OSA approach to discuss various aspects related to the training of mathematics teachers. They use the didactic suitability criteria (DSC) to identify cognitive, epistemic, emotional, ecological, interactional, and mediational components of mathematical practices and planning and teaching. Other authors have also discussed some of these components. For instance, complexity frameworks have also been introduced to interpret teacher education issues (Giménez, 2020). Bruna (2015), for example, highlights the impact that mathematics teachers' beliefs have on the willingness of students to solve complex math problems. Dilemmas such as "when to correct students: immediately or let them realize their mistake?" are relevant because they affect how students define their identity as problem solvers. According to Bruna (2015), how math teachers act to support (correct) their students affect the feeling of security and confidence students may have when facing word problems. But not only teachers' beliefs have been the subject of the discussion. So has the aspect of emotions. Hummes et al. (2020) claim that the criteria of didactical suitability can be used to analyze and support teachers' didactical choices, allowing us to consider the emotional dimension in connection with the epistemic and cognitive ones.

Other authors, such as Vale et al. (2015), have contributed to teaching tools, such as math trails, designed to create problems. Their reflections highlight the need to transmit to future mathematics teachers' knowledge about the development of effective tasks in teaching mathematics. Their proposal is situated in the tradition of the problem-posing developed by Silver (1997). Similarly, Lobo da Costa et al. (2017) propose "investigative tasks" as tools for teachers to develop teachers technological pedagogical content knowledge. On the other hand, they also offer problem-based learning as a research-based approach to teaching optimization problems (Lopes Galvao et al., 2017).

Diversity has been a recurring theme that brings together an excellent line of research in the Iberian American scope. Giménez et al. (2012) discuss interculturality and citizenship. They provide evidence suggesting that teacher training must prepare future teachers to make connections between mathematics and everyday life, creating tasks based on mathematical principles, but that also integrates students' personal representations, encouraging them to develop a critical sense to make decisions in our society. More recently, Vanegas et al. (2015) reflected on sociocultural contexts as resources to be incorporated in the training of future mathematics teachers. Another different approach within the domain is proposed by Hitt and Rivera (2017) when they use Bourdieu's theory to suggest sociocultural training.

Another of the questions that we have historically asked ourselves in teacher training is, "what is the best way to teach mathematics? Is there a teaching style that is the "best" way to teach mathematics?" Ferrarello et al. (2015) discusses whether a style based on concrete experience (feeling), reflective observation (watching), abstract conceptualization (thinking), or active experimentation (doing) is better. According to his work, the best proposal is perhaps a combination (in a spiral) of all these styles since they help the mathematics teacher manage the complexity of teaching mathematics better. Other authors, such as Mulat and Berman (2015) draw on the pedagogy of the question (which was initially inspired by Freire), which denotes the profound impact of the Brazilian pedagogue in the mathematics education research that we find in CIEAEM.

The reflection on the best way to teach mathematics leads us to the field of assessment of mathematical knowledge for teaching. Gonçalves and Gomes (2020) adapted a questionnaire and used interviews as research tools. However, they also pointed out methodological aspects that still need to be deepened in mathematics teacher education.

The CIEAEM has also addressed the use of technologies to train preservice teachers. For example, Floris (2015) presents the case of an initial training device to integrate numeric environments in the teaching of mathematics in secondary schools. Drawing on Brousseau's approach (1997), he uses the concept of milieu to integrate a virtual environment in teacher training. In the Iberic American field, this line of research has contributed with studies such as those by Lobo da Costa et al. (2015) and Brito Prado et al. (2015). According to their work, introducing the use of digital technologies as a tool for mathematics teachers requires not only that mathematics teachers know them and know how to use them; they also must be able to reconstruct mathematical concepts in digital environments so that the sense of using these types of resources as tools for training (not as ends in themselves) is not lost.

Interaction (Díez-Palomar et al., 2021) was also raised in CIEAEM as a critical aspect of teacher training. Along with the international scientific community, we find works highlighting the contribution of approaches such as collaborative study groups in the professional development of teachers (Lopes Galvao et al., 2015).

Finally, another relevant contribution of the Iberian American research community has been made in social inclusion. In 2011, Flecha (2011) gave a plenary conference on the teaching of mathematics for social inclusion. He highlighted the transformative role that inclusive mathematics education can have, creating opportunities for those people and groups in a situation of vulnerability. On the other hand, we also find the work of D'Ambrosio (2011), who established an agenda for future teacher training. From his point of view, future mathematics teachers will have to: (a) Promote citizenship and (b) promote creativity.

On the one hand, teachers face the challenge of transmitting past values (which leads to citizenship). They must train students for an uncertain future (which implies creativity). In doing so, D'Ambrosio warns that future teachers must be careful because we neither want to transmit docile citizenship nor promote irresponsible creativity. The key is not justifying the math curriculum we teach simply because it satisfies rigor. D'Ambrosio argues that the mathematics that is needed for the present but above all for the future is that advanced mathematics that promotes the well-being of all people without compromising rigor serves to generate interest motivation. "Education has a responsibility in building up healthy attitudes towards the self, towards society, towards nature." (p.31).

New Themes, New Perspectives

Where is teacher training going? What are the new topics to work on in the coming years? Suppose we refer to the evolution of the contributions made by the Iberic American community in recent years. In that case, we can conclude, at least, that one obvious fact is that teacher training responds to the needs that arise at each historical moment. This trend will continue in the coming years.

In educational research at the international level (and in mathematics education more specifically), there is a growing interest in approaches of a global nature, which attempt to explain the practice of teaching and learning mathematics as a multidimensional fact with multiple Aspects. Teacher training must provide future mathematics teachers with mathematics content (an old and ever-present debate) but also didactic and pedagogical content (this is also already "old"). But in a world where technologies have transformed educational scenarios (mainly because of the covid-19 pandemic, which has normalized the use of online, hybrid teaching, etc.), teacher training will have to incorporate all these new methodologies, tools, etc., and accommodate them to achieve the goals of excellent and quality mathematics teaching.

In this scenario, surely another critical element will be the assessment. An actual, most controversial debate concerns the quality of teaching and learning mathematics and teacher training research. What are the criteria or methods of evaluating quality in teaching and learning mathematics? Quality management is more effective for institutional management and administration in education than for teaching, learning, and research issues. The effects of recent developments on the structure and content of mathematical curricula can be described by many trends which tend to be similar in many countries. Sustainability for professional development is an unsolved problem (Zehetmeier & Krainer, 2011) and has a lot of influence on implementing innovative teaching (Maass et al., 2019). And we need more research on promoting citizenship in mathematics teaching (Vanegas et al., 2013; Maass et al., 2019).

We also need to enlarge the collaborative experiences of teachers and researchers through inquiry design (Jaworski & Potari, 2021) more about the use of technological tools for teacher education and science-mathematics exchanges.

Finally, everything indicates that teaching based on successful educational actions will become widespread (Flecha, 2014; García-Carrión, Padrós Cuxart, et al., 2020b). Future teachers and in-service teachers will increasingly demand to know the scientific evidence that supports the educational actions that they can implement in their classrooms. Lesson design, teaching planning, quality assessment in teaching and learning mathematics, the effectiveness of methods, teaching strategies, etc., all must be supported by scientific evidence with social and educational impact (García-Carrión, López de Aguileta, et al., 2020a). This will be one of the future challenges, and much research will be needed to confirm that a specific practice is (or is not) successful.

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