A Digital Educational Path with an Interdisciplinary Perspective for Pre-service Mathematics Primary Teachers' Professional Development

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Abstract. In this study we present an experimental research which analyses a designed pedagogical device based on the Core Concept (CC) to promote the pre-service primary teachers' (PTs') evolution of cognitive processes and reconstruction of mathematical meanings. The PTs have been involved in an educational path in which they solved an arithmetic-algebraic task, working in small groups in a learning digital environment. The task has been used to provoke a cognitive conflict and social interactions to foster the discussion and make different perspectives emerge. We analyzed the PTs' solutions through Radford's model concerning the levels of generalization. Our analysis, supported by the PTs' reflections, allowed us to highlight the crucial role of the CC, within the pedagogical device, in favoring the PTs' transition from the factual level of generalization, to the contextual one, to the symbolic one. Moreover, previously unforeseen aspects arose: a mutual interplay between the mathematical goal and the understanding of the CC by the PTs arose.

Keywords: Professional development · Core concept · Pre-service mathematics teachers · Relations and functions · Remote learning

1 Introduction

Recently interdisciplinarity and teachers' professional development have been indicated as two of the major challenges for Mathematics Education (Bakker et al. 2021).

Working on multiple dimensions is nowadays necessary to tackle complex problems, but the logic of the disciplines alone, does not guarantee a possible dialogue. Hence the need to grasp the complexity of the processes and distinguish different phases in them.

Moreover, the need of deepening affordances and limitations of remote teacher education has been highlighted and more experimental studies on this approach should be carried out (Perry et al. 2021). In this paper we present an experimental study carried

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out within an interdisciplinary perspective (Choi 2017) and fully conducted in remote learning by three researchers in Mathematics Education and a researcher in Teacher Education. The research involved pre-service primary teachers (PTs) from two different Italian universities who were asked to collaboratively work online to solve an arithmetic-algebraic task with the aim to foster the construction of mathematical meanings through the Core Concept (CC).

In designing the educational path for PTs, we thought that a significant role could be played by the Core Concept (CC), an essential construct of Teacher Education, having metaphorical meaning and allowing easier access to disciplinary ideas. The CCs are essential concepts recurring along a disciplinary curriculum and having an organizing value for the construction of knowledge; they mirror the structure of the discipline, encompassing both epistemological and pedagogical aspects. Some examples of CCs in Mathematics could be “spaces and figures”, “measurement”, “number”, “proof”, “relations and functions” (Robutti 2000).

In our educational path we used the CCs as key elements fostering the PTs’ transition from the student-perspective to the teacher-perspective awareness about their mathematical knowledge as students, should be elements allowing the development of PTs’ professional awareness.

The study pursues two intertwined goals: from an educational point of view, it aims at fostering PTs’ professional development through a pedagogical device (Berten 1999) involving a careful use of communication technology fostering the peer-discussion; from the research point of view, we are interested in investigating to what extent the use of a pedagogical concept like the CC could support the achievement of a key mathematical goal, such as the generalization of the concept of relation (Montone et al. 2021).

To reach these goals, we analyzed the PTs’ solutions of arithmetic-algebraic tasks, using Radford’s level of generalization (Radford 2001, 2010) as a theoretical lens.

Finally, the role of technology is fundamental, as the activities are designed to be held online. It plays an important role in ensuring the random distribution of the tasks to be solved and discussed, the monitoring and sharing of the interactions among PTs from teachers. Furthermore, the digital nature of the learning environment requires a reflection on the role it assumes in shaping the activities about the CC and the evolution of Radford’s level of generalization.

2 Theoretical Framework

Following Radford (2001), we assume a socio-cultural semiotic perspective to identify three levels of generalization when students face an arithmetic-algebraic task: the factual level, in which students solve the task operatively, performing iterated actions on the concrete objects the task is focused on; the contextual level, in which students perceive the emergence of a general structure from the actions performed and recognize it as a mathematical object; the symbolic level, in which students elaborate the mathematical meanings in general terms. The three levels of generalizations manifest through some semiotic means. From the linguistic point of view, the factual level is characterized by the use of personal and deictic pronouns, adverbs of time like “always”, verbs indicating perception or actions, possibly iterated, mainly in the first (singular or plural) person. The contextual level is characterized by a hybrid language, where both abstract and situated elements coexist. The symbolic level is characterized by an impersonal and not ostensive language, which refers to a general mathematical object, possibly by means of specific mathematical formalism. In tune with Radford's model (2001, 2010), we recognize an evolution of the cognitive processes, from the simulation of concrete actions and counting (factual level), to the recognition of structural regularities and their translation in numbers (contextual level), up to not context-bound, general algebraic relations (symbolic level). According to Radford, to favor the transition of students towards higher levels of generalization, two elements are crucial: a good mathematical problem and social interactions.

In this respect, we used a task to provoke a cognitive conflict (Piaget 1975; Laurillard 2012) and social interactions (Vygotsky 1962) to foster the discussion and make different perspectives emerge. But our context, characterized by PTs, required something more (Di Giulio and Defila 2017). Indeed, they need to reconstruct mathematical meanings and progressively assume the teacher's perspective on them. Taking in mind this double aim, we designed a suitable pedagogical device involving a significant open task, the peer discussion and the CC as underlying theoretical element. The CCs, as elements that reflect the structure of the discipline, encompass both historical-epistemological and pedagogical aspects (Olmi 2000; Crosswhite and Anderson 2020). On the one hand, the CCs are key tools for the teachers, supporting the design and action for learning (Berthoz 2012); on the other hand, they favor the students in progressively grasping the epistemology of the discipline. The CCs are characterized by three essential features: verticality, since the CCs cyclically recur along the disciplinary curriculum; horizontality, since they favor the link with other contents of the same subject and with other subjects; generativity, since their metaphorical power supports the teachers' and students' orientation in the discipline and in the exploration of new topics, offering an interpretative model (Rossi and Pentucci 2021).

3 The Research Methodology and the Educational Path

The designed pedagogical device for the educational path is characterized by the two arithmetic-algebraic tasks inspired by Radford (2001, 2010), which the PTs should face:

Task 1

A sequence of symbols begins as follows:

Which symbol do you find in the 17th position? Where is the 5th star?

Generalization question: "Which symbol do you find in the 1987th position?".

Task 2

Silvia uses 3 cans to build a pyramid of 2 floors, as shown by the figure. She uses 6 cans to build a pyramid of 3 floors. How many cans does she use to build a pyramid of 5 floors, with the cans arranged in the same way?

Generalization Question: How many cans does Silvia use to build a pyramid of 100 floors, with the cans arranged in the same way?

The generalization questions aimed to make a cognitive conflict emerge, revealing the inadequacy of the factual approach (involving the drawing and the counting of the concrete objects). The tasks allow different strategies and through the generalization question should force the PTs to link their informal knowledge with the formal one.

Other elements of the pedagogical device are the peer discussion, and the CC.

All the elements of the pedagogical device had a specific function in promoting the PTs' professional development: the task and, in particular, the generalization question aimed to make a cognitive conflict emerge, revealing the inadequacy of the factual approach; the peer discussion aimed at fostering the collective evolution of the different solutions and perspectives; the introduction of the CC aimed to provide students with a trigger to overcome the cognitive conflict (Rossi and Pentucci 2021).

The pedagogical device has been used in an educational path that involved 330 PTs at two different universities, one located in the center of Italy (University of Macerata) and one in the south of Italy (University of Bari). All PTs were attending a five-year course. The students at the University of Macerata attended the course of Teacher Education in their third year, whilst the ones at the University of Bari attended the course of Mathematics Education, in their fourth year. The teachers of these courses are the authors of this paper. The path lasted about one month and was performed using distance learning, due to the Covid-19 pandemic. The participants were divided into groups of four PTs from the same university. All the interactions among students and with teachers happened through digital communication channels, such as Microsoft Teams, Google Meet, Youtube, WhatsApp. The educational path envisaged the following steps:

- (S1) Groups were provided with an arithmetic-algebraic task. They were asked to answer an open-ended online written questionnaire requiring: S1.Q1) How do you solve the task? Explain your reasoning. This step lasted one week.
- (S2) All the PTs were exposed to two 1-h lectures about the CCs and the CCs in Mathematics, during which the general features of the CC (from the Teacher Education perspective and the Mathematics Education perspective) were discussed, and examples of CCs were provided, without any reference to the task the PTs had to solve.
- (S3) The groups, communicating at distance, reworked their first solution, within four days, based on the generalization question. They were asked to answer a written questionnaire including the following questions: S3.Q1) Does your solving process change? In which way? S3.Q2) What connections with other mathematical topics would you highlight if you should present the task in a classroom?
- (S4) Each group discussed at distance the elaborated answers with a group of the other university. This step lasted one week.
- (S5) Each group had five days to answer a written questionnaire to possibly improve the solution and metacognitively reflect on the whole experience. Among other questions,

we asked: S5.Q1) Do you want to modify your previous solution of the task? S5.Q2) How could the CCs influence your design and activities as a teacher?

- (S6) The teachers gave an online lecture to all the PTs, aimed at analyzing their productions and providing joint feedback about them.

According to a design-based approach (Swan 2020), the design of the pedagogical device, its use and the validation of the path were strongly intertwined and evolved during the activities. Also, the self-study methodology (Tidwell et al. 2009) was applied since the researchers were also the practitioners and handled the educational path. To validate the path, a qualitative analysis based on the Radford's model of the three questionnaires (both cognitive and metacognitive) answered by PTs was carried out, according to the criteria of credibility, dependability, transferability, and confirmability (Guba 1981), to ensure trustworthiness. The researchers repeatedly read the PTs's answers and coded them individually, referring to the research goals and the theoretical framework; then they discussed the emerging themes until they agreed on the most relevant ones (Sharma 2013).

3.1 Research Questions

- Can peer interaction supported by a task in which there is a question of generalization lead students in training to reconstruct mathematical concepts?
- How do university students behave when faced with a task with a generalization question? Do they go through the three Radford's levels of generalization?
- Does the CC allow PTs to become aware of the "formal" mathematics underlying an arithmetic-algebraic task?

4 The Project and the Structure of the Educational Path

The project involved 330 PTs from the Italian universities of Macerata and Bari. The PTs of the university of Macerata attended the course of Teacher Education in their third year of study, while the PTs of the university of Bari attended the course of Mathematics Education, in their fourth year. In this project technology plays a major role, as it allows to support the processes involved in the educational path in its three main functions (Albano et al. 2020): sending and viewing; processing and analysis of the data collected during the lessons; provide an interactive environment, where students can interact to work individually or in groups on a task or to explore mathematical/scientific content (Perry et al. 2021). The use of the digital resources in organizing the activities of the educational path allows to customize both the types of tasks and the interaction among the PTs (Montone et al. 2021).

Along the educational path, the PTs were parted in small groups of people of the same university and required to solve an arithmetic-algebraic task working online. Then, they were exposed to two one-hour lectures on the CC in general terms and on the CC in Mathematics and, after having worked together with a group of the other university, they have had the possibility to rework their first solution. In the next phase, the PTs were required to reflect metacognitively on the whole experience. Finally, the teachers of the courses taught an online lecture, in order to analyze the PTs' productions and give joint feedback. The PTs provided their solutions and metacognitive reflections through three online open-ended questionnaires.

Each component of the pedagogical device (Berten 1999) had a specific function within the educational path: 1) the arithmetic-algebraic task, designed according to the Radford's model on the level of generalizations (Radford 2001), should have brought out a cognitive conflict and induced the generalization of the concept of relation; 2) the peer discussion, completely carried out at distance through different communication channels, should have fostered the comparison of different viewpoints and supported either the improvement of communication skills and the evolution of the collective solutions and metacognitive reflections; 3) the CC as underlying theoretical element, should have favored an easier access to the mathematical content by the PTs, supporting the overcoming of the cognitive conflict, and fostered the PTs' transition from the student-perspective to the teacher-perspective.

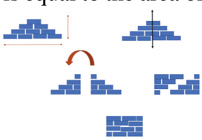
5 Results

The analysis of the texts revealed that at step S1 about 32% of groups solved the task at a contextual level, while the remaining ones solved the task at the factual level. There were no solutions at the symbolic level. In step S2 all students overcame the factual level and about 35% of students reached the symbolic level. At the end of the learning path, 71% of students achieved the symbolic level. In the following tables, we will focus on the works of two groups within steps S1, S2 and S4 of the educational path.

These transcriptions show the PTs' transition through the three levels of generalization (Radford 2001), as expected. The first version of the task induced PTs to find a solution at the factual level, which turned out to be insufficient when the generalization question was added. Hence PTs referred to the CC, which supported them in finding a general and effective formula, thanks to its features of generativity, verticality and horizontality. The CCs acted both epistemologically and pedagogically: they conveyed the idea of generalization, and they are recognized as a fil rouge for the teaching path. In addition, in this answer students put together mathematical and pedagogical knowledge, in an interdisciplinary way, to revise their solution and complete it. PTs gradually modified their solving process: they shifted from a factual operativity, directly acting on the cans by counting, to the recognition of a contextual regular structure connecting numbers (the number of cans on the basis of the pyramid with the number of floors); finally, they identified an algebraic relation referred to mathematical objects and independent from the context. Moreover, in this evolution, we note that PTs switched from the arithmetic semiotic register to the algebraic one, through the geometric one. The different reformulations show how the mathematical meanings and the language evolve from the description of iterated actions, through the calculation of an area, to the general idea to calculate the sum of the first n natural numbers.

As in the previous example, the transition along the levels of generalization emerges, testifying the PTs' evolution of cognitive processes. All the above comments apply also to this case. Besides, here students got to use specific mathematical formalism, then came back to the contextual elements of the task. Moreover, even when they reached the

Table 1. First group's answers to the questionnaires

<table>
<tr><td colspan="2">Task 1. Silvia uses 3 cans to build a pyramid of 2 floors, as shown by the figure. She uses 6 cans to build a pyramid of 3 floors. How many cans does she use to build a pyramid of 5 floors, with the cans arranged in the same way? 
Generalization question (provided within S2): How many cans does Silvia use to build a pyramid of 100 floors, with the cans arranged in the same way?</td></tr>
<tr><th>First group's answers</th><th>Authors' analysis</th></tr>
<tr><td>(S1) We solved the task by drawing a pyramid. Starting from the top, we went down, by adding one can for each floor: at floor 1 I have 1 can, at floor 2, I have 2 cans, at floor 3, I have 3 cans, at floor 4, I have 4 cans, and finally at floor 5, I have 5 cans, which bring us to 15 cans in all</td><td>PTs solved the task operatively, by performing concrete actions and counting (drawing, we went down, adding, at floor 1 I have 1 can…). We recognize a factual level of generalization by the use of verbs, in the first singular or plural person (we solved, I have), indicating iterated actions (I have,…, I have) on the concrete objects (cans) and linked to numeric operations</td></tr>
<tr><td>(S2) Our process changed, becoming more generative and effective. Our first solution works well for small numbers, but not for big numbers. We inevitably needed to change our approach. We chose to refer to the geometry […], we drew a pyramid with 4 floors, we halved it, obtaining two equal ''staircases'', one on the left and one on the right. We rotated the right figure on the other one, thus obtaining a rectangle with basis (n + 1)/2 cans and height n cans. So, the product n(n + 1)/2 is equal to the area of the constructed rectangle
</td><td>PTs reworked their first solution, induced by the generalization question which revealed the inadequacy of the first approach, as they declared. The search for a solution seems to be oriented by the CC "space and figures". In this excerpt the horizontality of the CC emerges, since PTs passed from the arithmetic field to the geometric one. The generativity of the CC also arises, since PTs refer to a metaphorical rectangle formed by cans. We recognize a contextual level of generalization by the use of a hybrid language, where both situated and abstract elements coexist. There are some spatial references (right, left), verbs indicating action (draw, halve), references to material objects (cans) and concrete not material objects (pyramid, figure, rectangle). The cans are not viewed as single objects, but they are ideally grouped into a unique figure (pyramid) on which it is possible to act mathematically (halve, rotate). This process induces a formula. An integration of different perspectives and semiotic registers arises, as the formula and the figure prove</td></tr>
</table>

(*continued*)

Table 1. (*continued*)

(S4) By decomposing the figure, a rectangle having a basis equal to (n + 1)/2 and height equal to n is created. The product n(n + 1)/2 is equal to the area of the rectangle, i.e. the sum of the first n natural numbers	A *symbolic level of generalization* arises in the expression referring to abstract objects (*figure, rectangle*) and in the formula. The verbs are impersonal, there are not pronouns nor spatial references
(S4/Q1) After the additional question about the pyramid of 100 floors, we referred to the CC of "relations and functions", since we would like to find a general formula for all the possible cases. We think that the CC can influence our design for learning since it fosters the synthesis of contents. By the CC we can form a net structuring of the curriculum; moreover, we can give a linear direction to our teaching path, by identifying a *fil rouge*	PTs declared that, facing the additional question aimed at the generalization, they grounded their solution on the CC. They passed from the CC "space and figures" to the CC "relations and functions", highlighting the *horizontality* of CCs. As emerged by their reflections and the reworked solution, they searched for a *general formula for all the cases*. Here the *generativity* of the CC emerges, since it conveys the rich mathematical idea of generalization, hence the epistemology of the discipline. Moreover, PTs showed awareness of the meaning of the CC and its potential influence on the design for learning. They also recognized the *verticality* of the CC, seen as a *fil rouge* for the teaching path

Table 2. Second group's answers to the questionnaires

Task 2. A sequence of symbols begins as follows: 
Which symbol do you find in the 17th position? Where is the 5th star?
Generalization question (provided within S2): Which symbol do you find in the 1987th position?

Second group's answers	**Authors' Analysis**
(S1) In the 17th position we saw the second triangle; from the 15th position, we continued to draw the symbols, according to the same sequence. So, following the multiplication table of 3, in position 6 we found a triangle, in position 9 a star, in position 12 a triangle, in position 15 a star,…, and in position 27 the fifth star	PTs solved the task by drawing the sequence of symbols up to the 17th position. We recognize a *factual level of generalization* using verbs of action and perception, in the first plural person (*we saw, we continued to draw*). These verbs refer to concrete objects (*triangle*, *star*) and are linked to numeric operations (*following the multiplication table of 3*). The last sentence shows the rhythmic repetition of words, suggesting something general, that continues in space and time

(*continued*)

Table 2. (*continued*)

(S2) Our perspective changed when we needed a general rule to find the symbol corresponding to any given position (i.e., 1987). It was necessary to identify and analyse the relations between symbols and numbers. The symbols reoccur according to a cyclic order of 6. This reasoning is based on an arithmetic progression, i.e., the difference between each symbol and the previous one [of the same kind] is a constant number of symbols (5). Since we add 6 to each symbol, we can talk about multiples of 6 (6n)	The additional question revealed the PTs the inadequacy of their first approach. To overcome the impasse, they searched *relations* between the symbols and their positions, i.e., they referred to the CC "relations and functions". The CC turns out to be *generative*, allowing PTs to change their perspective and see the mathematical structure of the problem. The *contextual level of generalization* displayed using a hybrid language, referring to general objects (*symbols, cyclic order*, *arithmetic progression)* but also including references to space, time and operations, as in the expression *the difference between each symbol and the previous one is a constant number of symbols*
(S4) We looked for regularities in the sequence. We started from the star. It is in position 3, then in position 9 by adding 6, then in position 15 by adding 6… In terms of multiples of 6 (6n), the circle will occur in positions 6n + 1 and 6n + 2, the star in position 6n + 3, the triangle in positions 6n + 4, 6n + 5 and 6n + 6. Hence, given a sequence (where symbols reoccur according to cycles of n), in order to find the symbol in position m, you need to identify the nearest multiple of n lower than m, then you take it out of m and see what number (from 1 to n) you obtain	PTs reached a s*ymbolic level of generalization*, as proven by the last sentence. They referred to a general, not context-bound solution, using a not situated language and a specific mathematical formalism. However, to obtain the final general explanation, PTs merge all the levels of generalizations. Indeed, they retraced the whole process, from the iterated action typical of the factual level (*by adding 6*), through the contextual level (*the circle will occur in positions 6n + 1,…*), to the symbolic generalization, where any reference to concrete objects vanishes
(S4.Q1) We experienced the role of the CC as a generative and unifying element. In the elaboration of our solution, the CC suggested how to find a general solution, searching connections with other mathematical topics. The awareness about this aspect will allow us to support our future students in a meaningful construction of knowledge, by using appropriate strategies. The discussion with the peers unfolded two layers: the identification of the conceptual knots of mathematics, and the shift from the empirical solution to the theoretical one	PTs became aware of the *horizontality* of the CC, which induces the link with other mathematical contents. Moreover, the PTs seemed to grasp the *generativity* of the CC in a double sense: epistemologically, it offered them a perspective to find a general solution to the task and, pedagogically, it will be significant for their future teaching action. PTs recognized the effectiveness of the discussion with the peers, both in identifying *conceptual knots* of the subject, and in inducing them towards a general solution

symbolic level in the explanation of the process, they again referred to iterated actions on concrete objects, which is typical of the factual level. We recognize the permeability and the coexistence of the levels of generalizations as new elements. Indeed, students went back and forth through the levels when they solved the task. Finally, on the one hand, the CC turned out to be a key element for the evolution to higher levels of generalizations, allowing the overcoming of the cognitive conflict arisen with the generalization question; on the other hand, the PTs' understanding of the CC evolved thanks to its use in the provided authentic task. As expected, the peer discussion was also a key element of the pedagogical device in the educational path of the PTs, as highlighted in the last row of Table 2.

In conclusion, thanks to different subjects involved in an interdisciplinary key and thanks to the digital nature of the learning environment, an evolution of mathematical levels of generalization has been realized.

6 Conclusion

To investigate the research goals, we collected and analyzed the answers to the open-ended online questionnaires, where the PTs shared their solutions to the arithmetic-algebraic tasks, their reworked solutions and their metacognitive reflections.

The outcomes of the educational path seemed to confirm the effectiveness of the designed pedagogical device based on the CC to promote the PTs' professional development and reconstruction of mathematical meanings. In the first step, the PTs have been required to solve a mathematical task; as expected, they faced it, for the most part, as students. The introduction of the CC, together with the peer discussion, modified their posture, inducing them to review the task also from the perspective of future teachers. Finally, the CC, thanks to its pedagogical trait, also facilitates the access to the involved mathematical aspects, as testified by the transition of PTs' productions towards higher Radford's levels of generalization. The PTs passed from factual solutions, involving drawings and counting in specific cases, to contextual or symbolic solutions, identifying underlying mathematical objects and relations allowing to solve a class of similar problems. Moreover, PTs displayed to have understood the pedagogical scope of this transition. Referring to the CC, PTs seemed to progressively grasp either epistemological aspect of Mathematics, overshadowing the context of the task and focusing on its mathematical structure, and pedagogical aspects, identifying the advantages of the use of the CCs for teaching. Our analysis of the PTs' solutions and metacognitive reflections allowed us to highlight that, as expected, the authentic task and its generalization question made the PTs face a cognitive conflict, and the CC acted as a trigger for its overcoming. A dynamic dialogue arose between the mathematical goal and the understanding of the CC by PTs, which developed in a mutual interplay. The CC revealed its value as a structuring and structured element: it gave structure to the educational path, promoting, within the pedagogical device, the PTs' transition towards higher levels of generalization; conversely, it turned out to be structured by its instrumental use within the mathematical task, through which PTs became aware of the role of CC as an organizing principle for the curriculum and the design for learning, hence as an element of simplexity.

Further, the metacognitive reflections shared by the PTs allowed us to highlight the key role of the peer discussion, fully carried out at distance by different communication technology channels. This confirms the key role of technology in the designing of the educational path. The PTs also declared that the peer discussion fostered the comparison between different viewpoints (students from different universities, attending different courses), and their progressive assumption of the teacher-perspective.

As future work, we would like to deepen the status of CC and its role in the Education of Mathematics Teacher through further theoretical and experimental research.

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