Chapter 9
The Number Line in the Elementary Classroom as a Vehicle for Mathematical Thinking

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Abstract This chapter builds on research related to the number line by exploring it as a vehicle for mathematical understanding in the naturalistic setting of Grade 2 and Grade 3 elementary classrooms. Starting from pupils' embodiment of the number line, and explicitly giving emphasis on the nature of the number line, an instructional sequence was designed and organized around number sequence and recognition, addition and subtraction in the domain from 1 through 1000. Using evidence from pupils' own productions, this chapter points to the role the number line plays in supporting pupils' sense making, the elaboration of informal strategies, leading to the development of more sophisticated ones.

Keywords Number line · Computation strategy · Mathematical understanding · Addition · Subtraction · Mental arithmetic

9.1 Introduction

In recent decades, important research in mathematics education has been devoted in identifying, understanding and fostering pupils' strategies and approaches for performing addition and subtraction (Beishuizen, 2010). In this context, through mathematical models, manipulatives and representations of mathematical ideas and concepts are described and assessed. Representations are significant tools in thinking, reasoning and communicating about mathematical ideas and operations (Kilpatrick et al., 2001). A representation that plays an important role in facilitating pupils to actively construct mathematical meaning, number sense, and understandings of number relationships is the number line (Frykholm, 2010).

The understanding of and ability to use the mathematical number line constitutes an essential facet of pupils' mathematics understanding. Consequently, the number line is an area of interest in cognitive psychology (Booth & Siegler, 2008; Siegler &

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Booth, 2005) and in cognitive neuroscience (Umiltà et al., 2010). There is a large body of literature that discusses the number line and its crucial role in teaching and learning elementary mathematics (Ball, 2003). Despite the widespread use of the number line, doubts about its appropriateness have been raised, with studies reporting difficulties and limitations in its use (Van den Heuvel-Panhuizen, 2008). Hence, there is a need to further our understanding regarding the conditions under which the number line in the elementary classroom is to be a vehicle for mathematical understanding.

This chapter contributes to a growing body of research in mathematics education related with the number line. The aim of this study was to explore the idea of the number line as a vehicle for mathematical understanding in the naturalistic setting of the elementary classroom. To be more descriptive, it sought to examine the use of the structured and empty number line (ENL) as a tool to support and develop Grade 2 and Grade 3 pupils' mathematical thinking specific to addition and subtraction in the number domain 0–1000.

In the first section of this chapter, the theoretical background pertaining to the number line is outlined. The argument is developed and synthesized by focusing on the structured and the ENL in relation to addition and subtraction. In the following parts of this chapter, this general information is made more concrete by concentrating on how this particular study utilised the structured and ENL to foster connections and relations between representations of mathematical ideas concerning quantities, the ways to decompose quantities and how to regroup them, illustrated by examples from the elementary classrooms.

9.2 The Number Line

According to Herbst (1997), a number line is formed by the consecutive translation of a specified segment U, as a unit from zero that can be partitioned in an infinite number of ways. He suggests that the number line is a metaphor of the number system; all kinds of numbers can be represented on the number line. The number line is also considered a geometrical model, involving a continuous interchange between a geometrical and an arithmetic representation (Gagatsis et al., 2003). That is, the numbers presented on the line correspond to vectors and to the set of the discrete points of the line. Simultaneously, points on the line are numbered so that the distance between two points depict the difference between the corresponding numbers. Furthermore, Teppo and Heuvel-Panbuizen (2014) argue that the number line is a figural device, representing particular mathematical abstractions that make it possible to think about and operate with different types of number.

The number line is currently an extensively used model in the teaching of mathematics (Lemonidis, 2016; Reys et al., 2012). The number line is used for estimation (Onslow et al., 2005), measuring time (Moone & Groot, 2005) and length, extending students' knowledge, giving access to possible solution strategies (Thompson, 2010). It allows the representation of numbers and the forming of geometric knowledge for the operation of arithmetic (Herbst, 1997; Kilpatrick et al., 2001). It can be used as a model for teaching percentages (Van de Heuvel-Panhuizen, 2003) and algebra for teaching linear equations (Dickinson & Eade, 2004). It is also employed for the development of the concept of fractions (Sidney et al., 2019).

Within the literature, two major types of number lines can be identified; the structured number line and the ENL (Diezmann et al., 2010; Teppo & van de Heuvel-Panhuizen, 2014). The filled or structured number line is characterized by equidistant points or tick marks, representing whole numbers. Diezmann and Lowrie (2007), elaborate the potential cognitive benefits of the number line for understanding various aspects of mathematics. When pupils experience many variations linked to a mathematical concept and are exposed to the concept through a variety of representations, then abstraction and generalisation, both of which constitute essential aspects of conceptual development is promoted. For instance, the number line can show the continuity of rational numbers. Furthermore, a fraction can be represented on a circular or rectangular area model, an array, and on a number line. The number line can also be considered as a tool for representational transfer, the goal of which is knowing how to use a common representation and deriving the solution procedure on a novel task from this representation (Novick, 1990). For example, the knowledge of sequencing whole numbers on the number line may be transferred when requested to sequence decimal numbers on another number line.

The ENL is a blank line presented without numbers or markers. It is a horizontal line acting as a visual representation for recording and sharing students' thinking strategies during mental computation. Gravemeijer (1999) explicates the didactical and psychological advantages for using the ENL in mathematics education. Initially, he outlines the need for a linear representation of numbers. While models such as blocks reflect situations dealing with quantities, and thus representing the numerosity aspect of number, in situations involving distance or measurement, the number line, which is considered as a linear representation of number, seems more appropriate. Furthermore, he outlines the flexibility the ENL provides in being adapted to fit students' thinking and informal solution strategies (van de Heuvel-Panhuizen, 2008). That is, the ENL reflects students' intuitive mental strategies, from counting-on or counting down to compensation and partitioning strategies. Furthermore, the natural and transparent character of the ENL, may stimulate a mental representation of numbers and number operations, and thus can be exploited for the representation and solution of non-standard context of word problems. The ENL relieves the working memory, as preliminary results can be put down relatively fast (Selter, 1998, p. 6).

The ENL may function as a way of scaffolding by fostering the development of more sophisticated strategies. To be more precise, as students record their thinking strategies, the number line shows which parts of the operation have been completed and which parts remain, the level of thinking as well as any possible errors. The students are cognitively involved in the actions undertaken on the number line. Making students' thinking visible provides opportunities to encourage the development of more efficient and sophisticated strategies. Building on this, it may also stimulate classroom discussion and expression and communication of mental strategies (Bobis, 2007).

In reviewing the research literature that focuses on the strategies adopted by pupils to perform mental arithmetic for numbers up to 1000, a wide range of strategies have been documented and classified. There is a broad consensus in mathematics education that three main strategies for mental addition and subtraction of are identified when using the ENL; (i) splitting where the numbers are divided by multiples of ten and units and processed separately when operations are carried out, (ii) stringing or compensation strategy which refers to keeping the first number intact while splitting the second number into tens and ones, which are then added or subtracted separately from the first number, and (iii) bridging, where the second number is split to facilitate a bridge to the nearest decade and the balance of the number is added or subtracted from the previous set (Beishuizen, 1993, 2010; Beishuizen et al., 1997). Variations of these strategies are also identified in the literature (Hartnett, 2007; Van den Heuvel-Panhuizen, 2008).

A consideration of the aforementioned, points to the crucial role the number line plays in mathematics education. However, whilst generally effective, research findings often raise doubts about the usefulness of the number line as a didactical model. Studies have investigated students' performance on number line tasks of various grade levels either at a single point in time (Skoumpourdi, 2010); over a small period of time (Hartnett, 2007) or through a longitudinal study (Diezman & Lowrie, 2007). These studies point both to instances where the number line functioned as an auxiliary means, as well as instances where students encountered difficulties in utilizing the number line. Common errors on number line items point to difficulties with distance, position, counting or misreading the diagram (Diezmann et al., 2010). Errors in problems originating from the dual nature of the number line may be persistent over time (Pelczer et al., 2011). Thus, neglecting one of the main features of the number line (direction, origin and unit measure) may lead to misconceptions.

Adding to the above, Lemonidis and Gkolfos (2020), argue that some of the difficulties students encounter when using the number line may be interpreted as epistemological obstacles related with; the separation between the numbers and the magnitude or the separation between the numbers and the straight line; the negative numbers and the orientation on the number line in the positive or negative direction; the density of rational numbers and the extra unit intervals needed to place them on the number line and irrational numbers. While they outline that more research is needed in clarifying the nature of these difficulties, they stress that these difficulties should be explicitly addressed by the teaching procedure.

Some of the difficulties students may face in effectively utilizing the number line may be attributed to the level of difficulty of number line problems. Other factors can be associated with the way the number line in presented in the mathematics textbooks and other curriculum resources and the way the teachers use it (Gray & Doritou, 2008; Murphy, 2011). These studies point towards a coherent treatment of the number line throughout the years of mathematics education and presentation of the number line in the school official documentation by focusing on the simultaneous presence of the geometric and the arithmetic conceptualization of number on the number line.

Keeping in mind both the affordances as well as the constraints of the number line, it is argued that the number line should be introduced in early grade instruction. However, it is acknowledged that it is superficial to simply recommend the use of the number line for the students' mathematical development or include it in curriculum materials and other recourses (Van den Heuvel-Panhuizen, 2008). If emphasis is given on the nature of the number line and its use as a representation of sophisticated ideas, then a conceptual way of teaching and learning should be encouraged, contributing to addressing students' difficulties. Various learning strands, didactical models and approaches have been proposed in the literature for the use of the number line in the learning and teaching process of mathematics (Selter, 1998; Thompson, 2010).

9.3 Methodological Considerations

The study was undertaken in a public primary school in Cyprus. The participants of this study were 19 Grade 2 students (11 boys and 8 girls) and 18 Grade 3 students (10 boys and 8 girls). The pupils were of a wide range of abilities and represented a broad spectrum of socioeconomic backgrounds.

An instructional sequence, described in full shortly, was designed to support the use of the number line as a vehicle for making visible mathematical understanding. Informed by a socio-constructivist approach to teaching and learning (Gravemeijer, 2020), pupils' learning processes were viewed from both the individual perspective and the social perspective. To be more elaborative, this instructional sequence gives the pupils a guided opportunity to invent the intended mathematics themselves and build up the targeted mathematics by modelling their own informal mathematical activity (Gravemeijer, 2004, 2020; Van den Heuvel-Panhuizen, 2003). Such a model is the number line, fostering the transition from a model of pupils' informal solution strategies to a model for mathematical reasoning (Freudenthal, 1973).

The instructional sequence was carried out as part of the ordinary mathematics classroom with the teacher as the researcher and author of this chapter and during the regular class time. A variety of data were gathered; audio recordings from the instructional sequence, the field notes of the researcher, and pupils' written work (worksheets, Concept Cartoons, and mathematical journal). The analysis of collected data started simultaneously with the data collection process, in order to identify pupils' thinking, strategies and procedures and their development, as well as further organise the instructional sequence. The ongoing analysis being conducted while the study was in progress led to a focus on several issues and events, which were then placed in a broader theoretical context by conducting a retrospective analysis (Gravemeijer, 2004).

9.4 The Instructional Sequence

The goal of the instructional sequence was to support and develop Grade 2 and Grade 3 pupils' sense making and calculation strategies specific to addition and subtraction in the number domain 0–1000. Considering the previously discussed elaboration, the instructional sequence aimed at creating opportunities for the pupils to build connections or relations between representations of mathematical ideas, to have the freedom to come up with their own notations, find their own ways to decompose quantities and regroup them and express and discuss their ideas. In this process, the number line would be utilized as a means for support, fostering the transition from a model of pupils' informal solution strategies to a model for mathematical reasoning. However, the interpretation of the number line is not self-evident. Thus, the instructional sequence was also focused on providing pupils with the opportunity to appreciate the number line as a rich model that can have different manifestations, by giving them as much initiative as possible, and simultaneously reducing the leading role of the teacher. In doing so, the pupils had the opportunity to understand the nature of the structured and ENL before acting on them and modelling their mental computation strategies.

The instructional sequence started with a story where the classroom's puppet converted its name into a number. The pupils followed the puppet's step. By assigning each letter of the alphabet a number value that is equal to its place in the alphabet, the pupils took each letter in their name, converted it to a number and added up the numbers. In the context of the story, the classroom's puppet invited the pupils to play a board game involving these numbers on a number line. However, what was missing from the game box was the number line. The pupils were to construct a physical number line. Even though they did not invent the tool for themselves, they were involved in the invention process. The pupils were encouraged to construct a definition regarding the structured number line and develop key understandings underpinning the conventions considered to interpret, create, and use structured number lines. After constructing the structured number line, the pupils plotted everyone's name number, engaging in number sequencing and comparing activities. The game cards created opportunities to support pupils' understanding of math processes, by inviting them to act on the structured number line, use their own strategies, and became familiar with different strategies, by explicitly discussing and reflecting on these strategies. The flexibility in the ways of recording results and the flexibility in the jumps pupils make to solve problems was stressed.

The instructional sequence was followed by Concept Cartoons the aim of which was to explicitly explore the dual nature of the number line (Pelczer et al., 2011). The Concept Cartoons involved problems and possible answers which included pupils' misconceptions. Comparing and contrasting ideas facilitates learning and interpretation of knowledge, revealing and eliminating misconceptions (Dabell et al., 2008; Naylor & Keogh, 2013). Similar Concept Cartoons were utilised when the ENL was introduced in the classrooms.

When the pupils reached the stage of becoming familiar with a linear representation of number and understanding and effectively using counting skills, the ENL was introduced in the classroom. In the sequel to the story, the classroom puppet discovered a different number line, and wandered whether this ENL could be utilised as a model of scaffolding and communicating the solution procedures the pupils use. The pupils were encouraged to construct a definition regarding the ENL. Emphasis was put on discriminating between the structured number line and the ENL. By drawing upon the pupils' own informal strategies, Concept Cartoons provided the opportunity to gain insight into pupils' calculation strategies as well as providing pupils with the opportunity to experience and develop a range of calculation strategies and discuss the flexibility of the ENL (Sexton et al., 2009). Speech bubbles proposing various solution procedures encouraged pupils to identify the name of the character that best matched their personal strategy choice for calculating the result, and providing reasons for choosing the specific strategy. The calculation strategies were introduced as possibilities, in an atmosphere of invention.

Three main calculation strategies were explored in both classrooms: splitting, stringing and bridging. When a new strategy for computation was introduced, the discussion was built on examples produced by the pupils. The pupils were involved in situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions. Furthermore, the pupils were encouraged but not forced to use the number line, methods or strategies that were being introduced and discussed in the classroom. The pupils would 'try' the strategy introduced in the classroom, but use the methods they felt comfortable with when solving problems. Thus, the pupils had the freedom to choose the materials and models (hundred chart, arithmetic blocks, dot-fields, money model) that supported their calculation. Through continuous discussion, strategies and procedures were presented, explained and contrasted. Pupils reflected upon the quality of their solution strategies and whether they could make them more manageable and efficient. Even though, no formal labelling was given to the computation strategies that were explored in the classrooms, the pupils were able to create general categories that supported them in communicating with each other and the teacher. At the end, a repertoire of strategies and methods was produced by each classroom in regards both to addition and subtraction.

Throughout the instructional sequence, the pupils also engaged in mathematical diary writing (Burns, 2004). It allowed pupils that are uncomfortable in oral situations to express understanding in a less public form (Yang, 2005). Including pupils in meaningful communication, mathematical diary writing also encouraged pupils to reflect on their own computation strategies (Selter, 1998).

9.5 Results and Discussion

In the following paragraphs, snapshots from the enacted instructional sequence reflect a small proportion of evidence illustrating the way the number line was utilized by Grade 2 and Grade 3 pupils, by concentrating on the development of pupils' computation strategies throughout the year. These snapshots followed the introduction of the ENL in the classroom. It should be emphasized that providing pupils with ongoing opportunities for revisiting, reviewing, engaging and mastering was not a straightforward process.

Initially, the pupils were presented with a context problem introducing the addition of two-digit numbers and three-digit numbers (64 + 30 and 264 + 30 for Grade 2 and 3 pupils accordingly), where the second number is multiple of 10. When the pupils were asked to share their mathematical thinking used for this calculation, they referred to counting-based strategies to calculate, calculating by utilising models.

While no pupil perceived the calculation as difficult, the reliance on counting materials (arithmetic blocks) was evident (seven Grade 2 and seven Grade 3 pupils). Grade 2 pupils' computation strategy was to split 64 into tens and ones and processed separately. These pupils modelled their thinking using the structured number line (four pupils) and the hundred chart (two pupils). When the pupils modelled their thinking on the structured number line, it was observed that three pupils utilised the structured number line for counting. No Grade 2 pupil referred to the ENL, even though it had been introduced in the classroom. The pupils' hesitation in utilising the ENL was related with its recent introduction in the classroom, revealing the need for more opportunities to build up familiarity with the ENL as a supportive model to carry out calculations. Grade 3 pupils referred to both splitting and compensation when working the calculation either mentally (seven pupils) or with the support of the ENL (five pupils).

Pupils were encouraged to record on paper the strategies chosen, Grade 2 pupils made drawings (e.g., iconic representation of arithmetic blocks, money model). This may be explained by the fact that the pupils had not been introduced to written symbolic representations. Three Grade 3 pupils described in writing their underlying pathways that led to how they determined the answer. Four pupils did not know what to do. The need for all pupils to be provided with more opportunities to draw upon number sense, and develop both written and mental computation was evident.

Successively, the instructional sequence was followed with a context problem introducing the addition of two and three-digit numbers without bridging (45 + 32 and 256 + 143 for Grade 2 and 3 pupils accordingly). Various methods of computation were discussed by the pupils. Two Grade 2 pupils chose arithmetic blocks, revealing that pupils would gradually rely less on manipulatives to solve a calculation task. In the same way, three Grade 3 pupils stated that as the numbers are bigger, the arithmetic blocks would assist them in solving correctly the calculation task. Nevertheless, modelling the problem with arithmetic blocks highlighted partitioning and regrouping of numbers. Two Grade 2 pupils and three Grade 3 pupils referred to the standard algorithm. Even though the algorithm had not been introduced in the classroom, it was mentioned previously by the pupils, as knowledge they had acquired either outside school (Grade 2 pupils), or during the previous school year (Grade 3 pupils). It was explicitly discussed, but without expecting pupils to use it as a way of working. The other pupils' strategies were divided between splitting and stringing (using either the ENL or a written computation), with four Grade 2 and

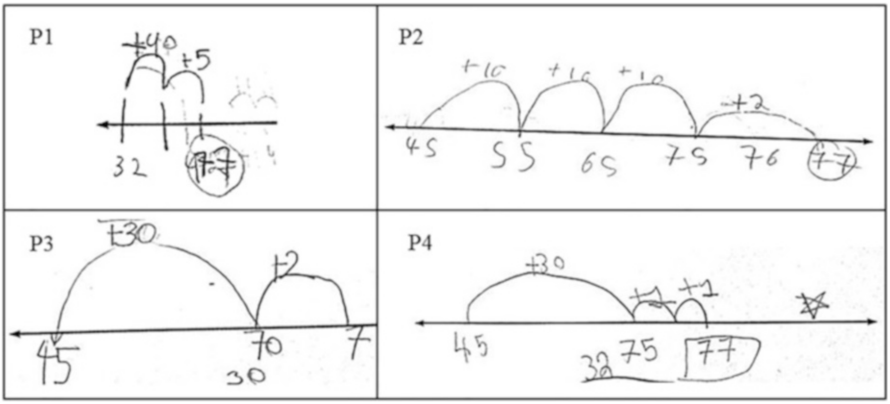

Fig. 9.1 Grade 2 pupils' informal strategies for computing an addition calculation mentally

3 pupils commenting that they did not need a written externalisation because they worked the calculation mentally. These pupils' mental computation was previously supported by the ENL and may suggest the success of the ENL as a mental model.

Figures 9.1 and 9.2 demonstrate Grade 2 and 3 pupils recording their mental computation on the ENL. The written work on the ENL was generated almost simultaneously with the pupils' thinking process revealing the solution procedures and at the same time disclosing the cognitive evidence included in this process. They constitute an indication of how the ENL is adapted by the pupils to fit their thinking and the pupils' increased confidence in their ability to use numbers flexibly.

Grade 2 pupils' strategy was compensation (see Fig. 9.1). The main procedures for implementing compensation, was that of separation from left to right (P2–4) or from right to left (P1). The structure of the calculation sequence involved either calculating in groups of tens and ones (P1 and P3) or a combination of groups of tens or ones and whole tens or ones (P2 and P4).

Accordingly, Fig. 9.2 demonstrates Grade 3 pupils modelling their mathematical thinking on the ENL, with some also providing a written description of their work. The splitting (P2 and P4) and compensation (P1 and P3) strategy used by the pupils led to a correct result. The pupils argued that their recordings on the ENL assisted them in providing an informal written method. This was quite unexpected considering that written work on the ENL has only a secondary function, and that the ENL does not easily lead to an informal written method. It shows that introducing the ENL, may also encourage pupils to invent and develop informal symbolic representations.

The instructional sequence was followed by an addition problem involving bridging (46 + 37 and 248 + 136 for Grade 2 and 3 pupils accordingly). The pupils were again encouraged to generate strategies based on their intuitive understanding of the numbers and actions needed. Initially, it should be explicated that no pupil relied on physical material, suggesting a shift from lower order strategies such as counting to more sophisticated strategies. Concerning the two-digit addition

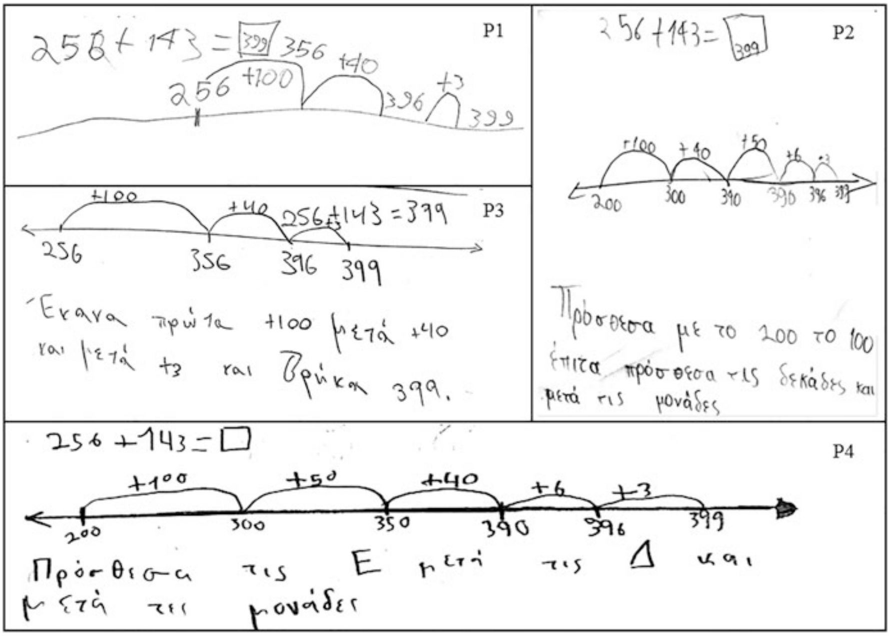

Fig. 9.2 Grade 3 pupils' informal strategies for computing an addition calculation mentally

problem, while not all methods led to a correct result, the classroom's documented work included the ENL as a mental model, language and written symbols.

To be more elaborative, in proposing computation strategies and procedures to calculate the result, six Grade 2 pupils commented that they could solve the calculation mentally by picturing the ENL in their heads. However, they expressed their preference for flexible and mental steps beyond the number line. The discussion led to the informal written computation illustrated in Fig. 9.3. In this occasion, the number line was used with other strategies to solve the problem.

The strategies pupils adopted to perform mental addition was splitting (see Fig. 9.3) and compensation (see Fig. 9.4). Even though this computation entails a calculation demand, no pupil referred to bridging, despite the fact that they engaged in bridging situations with numbers up to 20. While a bridging through ten approach would seem more appropriate, it did not offer a foundation for strategic choice. The pupils' decision about how to calculate was related with their familiarization with the alternative strategies. Closer examination of the pupils' methods indicated that compensation and splitting were combined with counting the remaining units (instead of bridging).

Further discussing the proposed strategies, a pupil commented *'I think there is another way to do it. 46 need 4 to become 50. We can take the 4 from 7 and then add the remaining 3. And then add 30.'* The pupils were encouraged to discuss, get familiar with and reflect on bridging through ten as a strategy for mental

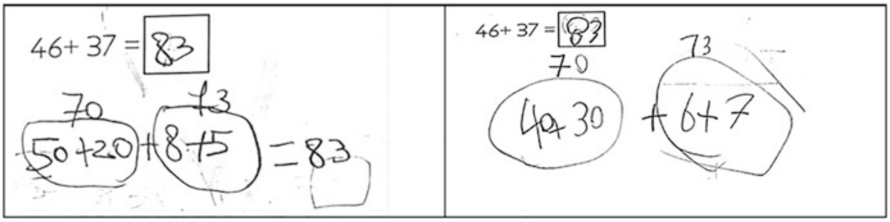

Fig. 9.3 Grade 2 pupils' informal written computation

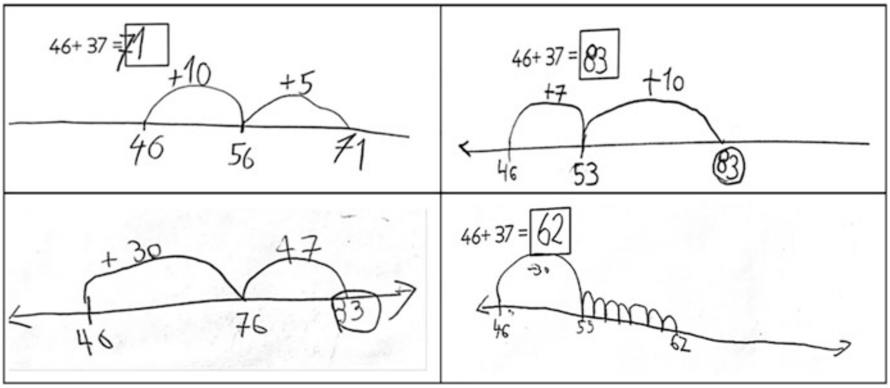

Fig. 9.4 Grade 2 pupils' informal written computation

computation. Figure 9.5 demonstrates the ENL supporting pupils' understanding of bridging. In this occasion, the pupils were also encouraged to provide a written computation that had been proposed in the classroom by their classmate. It is noticed that in the third example, the pupil used splitting stating that *'I understand what to do, but I feel more comfortable with this strategy (splitting)'.*

Concerning Grade 3 pupils approaching the addition problem involving bridging, the following extract from the classroom discussion shows pupils sharing the way they worked to calculate 248 + 136. Pupils 1–3 had modelled their calculation on the ENL.

P1: I started from 136. Then I added 40. 176. 176 + 200 = 376. And then I added 8. 376 plus 8 equals 384.

P2: I did it like P1, but I started from the biggest number. 248. First I added 100, then 30 and then 6. 378 plus 6 is 384.

P3: I am thinking 248 is 200 plus 40 plus 8 and 136 is 100 plus 30 plus 6. I add the hundreds, then the tens with the tens and the ones with the ones. 200 plus 100 is 300. 40 plus 30 is 70 and 8 plus 6 is 14. 300 plus 70 plus 14 is 384.

P4: In my head I worked it differently. 248 + 2 = 250. I took 2 from 6. Then I did 250 plus 130 equals 380. And the 4 that are left . . . 384.

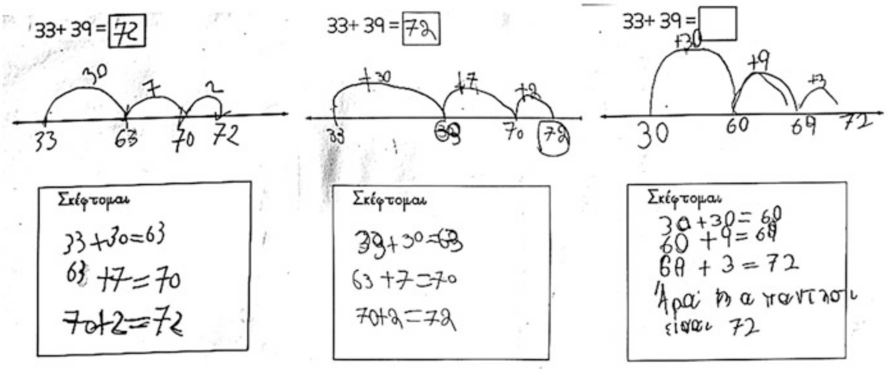

Fig. 9.5 Grade 2 pupils becoming familiar with different calculation strategies

Grade 3 pupils' calculation strategies involved splitting (P3), stringing (P1 and P2) and bridging (P4). The pupils' explanations constitute an indication of the numbers being treated as mathematical entities. This discussion also shows how pupils would engage in a discussion where computation strategies are shared and contrasted. P2 understood that he utilised the same strategy as P1, but commented that instead of starting the process from right to left, his process progressed from left to right. Similarly, P3 and P4 shared different strategies.

As a final example, the following extracts illustrate Grade 2 and 3 pupils sharing, towards the end of the instructional sequence, how they calculated a subtraction mentally.

Grade 2, 84-57

P5: I started from 57. I am thinking 57 + 3 = 60. We need 24 to reach 84. Then I added the 3, 24 + 3 = 27.

P6: It is like the number line is in my head. 84–50 = 34, then I subtracted the 7. 27.

P7: I worked in out in my head. I started from 84 as well, but I subtracted fist the ones, then the tens.

Grade 3, 584-257

P8: I started with the hundreds. 500–200 = 300, 84–57 = 27, 300 + 27 = 327.

P9: I worked differently. 584 minus the hundreds, then minus the tens and at the end I subtracted the ones. 327. It is easier.

P10: I also started from 584, but I subtracted fist the ones, then the tens and then the hundreds.

P11: I worked in out in my head with addition. I started from 257.Pupils 8–10 recorded their mathematical thinking as written computation (vertical equations), commenting that they did not need to use the ENL as a tool to support their mental computation. It is acknowledged that when the calculation involves bigger numbers the need to record the successive stages of the calculation arises. The pupils' mental computation strategies involved splitting (P8), bridging 10 to subtract with varying procedures (P6, P7, P9, P10) and subtraction as addition using bridging (P5 and P11). The pupils show a high degree of flexibility in their explaining. Compared to previous illustrations of pupils' work, it can be argued that the pupils' mathematical thinking became more elegant and sophisticated, involving fewer steps. The above extracts, indicate pupils having a grounded understanding of number and place value and are able to utilize this understanding to successfully complete simple tasks mentally. It can be concluded that the pupils developed computational fluency and deeper sense of number. An additional remark constitutes that fact that the pupils were eventually able to share and discuss their calculation strategies verbally without having to write them down.

It is important to note that Grade 2 and 3 pupils who were able to perform mental computation strategies flexibly, would only use the ENL to document their strategy so as to ensure they reached the correct result, to demonstrate their method to their classmates or as a way to assist them in moving into more appropriate and efficient methods. It should also be noted that there were also instances were pupils from both year groups that struggled to model their mathematical thinking on the ENL. Nevertheless, if students struggle on number line problems, it does not automatically mean that they struggle in understanding the mathematical operation involved. This may be translated into the fact that they do not yet have the specific knowledge and skills necessary for translating the numerical expression into a number line representation (Ernest, 1985). Indeed, these pupils did not struggle in understanding the mathematical operation involved. The errors identified in translating the numerical expression into a number line representation, were related with the fact that they did not understand the flexibility of the ENL, and instead treated it as a structured number line. Despite this, no difficulties were observed when the pupils were asked to explore and discuss an addition or subtraction modelled on the ENL. The pupils were able to understand the conventions used in interpreting the diagram, and thus, reading and identifying the strategy being modelled on the ENL. This is an indication of the number line being a powerful tool in enhancing communication in the classroom (Bobis, 2007).

9.6 Conclusion

In this chapter, I have described an instructional sequence that attempted to provide pupils with the opportunity to appreciate the number line as a rich model that can have different manifestations. Findings from this study show the number line offering pupils the opportunity to develop, express and share their thinking. Evidently, the number line representation functioned as a vehicle for mathematical understanding. It allowed pupils, through the gradual development of both cognitive and metacognitive strategies, to construct and develop their own strategies for accurate and flexible mathematical thinking. Simultaneously, the ENL supported pupils in making sense of numbers and operations, enhancing mathematical reasoning and communication. Associating actions with the number line and communicating mathematical meaning may contribute towards a comprehensive picture of its conceptual structure and complete development of understanding of the number system.

A contribution of this research is to the existing scholarship on whether pupils should be given the opportunity to discover strategies based on their own knowledge and skills (Threlfall, 2000). Additionally, through a classroom discourse encouraging invention, the number line acted as a bridge between mental and written computation. This chapter contributes to the concerns raised by the mathematics education community, regarding the ENL not easily leading to informal written methods. It illustrates that pupils may build on their strategies through the support of the ENL.

For the enactment of such an instructional sequence, the significance of the classroom culture must be stressed (Gravemeijer, 2004). As the pupils' understanding of the number line and development of strategies and procedures is influenced by social and sociomathematical norms negotiated and established in the classroom, the classroom environment encouraged communication, exploration, discussion and reasoning (Kilpatrick et al., 2001; Selter, 1998; Threlfall, 2000). In this respect, the proactive role of the teacher is highlighted in promoting a non-threatening classroom culture that encourages even reluctant or less confident pupils to create, formulate, express and extent their mathematical understanding.

Three paths for further investigation are emerging. Treating the number line in a coherent way by focusing on the simultaneous presence of the geometric and the arithmetic conceptualization of number on the number line is not a straightforward process. The structured and ENL, constitute fundamentally different models that may function in different ways in the learning environment. The number line may foster the transition from a model of pupils' informal solution strategies to a model for higher levels of mathematical understanding. New manifestations of the model also encompass previous manifestations of the model. The shift in applying the model on a progressively higher level is not always successful. The relationship between knowledge of the structured and ENL and their compatibility seems like a valuable avenue for further exploration. This is imperative as learning the number line is considered critical both for current and future success in mathematics (Booth & Siegler, 2006). Secondly, it is acknowledged that the instructional sequence was designed considering natural numbers up to 1000. Thus, the notion of the number line evolving from a unit that could be repeated and partitioned has not been fully explored. The number line is considered a metaphor of the number system (Herbst, 1997). Thus, more research is necessary to further investigate how such a local instruction theory can be realised regarding other number sets.

Finally, the snapshots presented in this chapter, provide an indication of the repertoire of strategies being developed by the pupils. The pupils extended their awareness of possibilities. Regarding strategy selection, differences emerge between the year groups. Grade 2 pupils used mostly the splitting strategy, whereas Grade

3 pupils used most frequently the compensation strategy. It is acknowledged that it is not only important to determine pupils' strategies, but also to identify how these strategies are interrelated, and which might need further instruction. In order to gain a more detailed insight in the development strategies in relation to pupils' mathematical thinking conceptual understanding, future research should further investigate the constituents that influence the flexibility in strategy use.

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