Chapter 1
An Introduction to Mathematical Cognition and Understanding in the Elementary and Middle School Years

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Abstract This volume focuses on the complex and diverse processes and factors affecting the mathematical cognition and understanding of elementary and middle school children, a critical time where they experience a range of developmental, pedagogical, and individual changes that impact their lifelong mathematics education and experience. In this first chapter, we identify the central topics of the individual contributions in this volume, providing a broader framing for the chapters by organising them into two parts (Cognitive Factors, Mathematical Understanding), and highlight broader themes connecting the chapters. We also draw attention to how each chapter provides new theoretical contributions and practical recommendations for teachers, paraprofessionals, parents, and policy makers, with the goal of improving children's success in mathematics.

Keywords Mathematical cognition · Mathematical understanding · Mathematics education · Middle school · Elementary school

1.1 Introduction

Studying the diverse processes and factors contributing to elementary and middle school children's mathematical cognition and understanding requires combining theories and evidence from a broad range of fields. The elementary and middle school years, which are the focus of this volume, roughly correspond to the ages of 6–12 although variability exists between or even within a country's states,

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provinces, territories, or areas. These school years are particularly complex given the vast developmental, pedagogical, and individual changes that occur and combine to affect children's mathematics education and experience. Students make great strides during these years of formal schooling in their ability to attend, plan, and execute cognitive strategies (Xu et al., 2013). They likely experience diverse types of mathematics instruction on complex foundational concepts (e.g., Fazio, 2019; Parr et al., 2019), and they are apt to experience strong emotions and shifting motivations about mathematics as a school subject (Karamarkovich & Rutherford, 2021). As such, researchers, educators, and parents' understanding of the factors that aid, impede, and motivate students to learn mathematics during elementary and middle school is critical and is the focus of this volume.

This volume represents the work of international scholars bringing their perspectives on children's mathematical minds from intersecting areas of inquiry including mathematics education, educational psychology, cognitive development, developmental psychology, mathematical cognition, cognitive neuroscience, and educational neuroscience who all have the goal of ensuring that all children in the elementary and middle school years succeed at mathematics. As it becomes ever more apparent how important early mathematics skills are for many areas of our lives, including academics, career choice, health and well-being, and financial literacy (Every Child a Chance, 2009; OECD, 2019; Parsons & Bynner, 2005; Statistics Canada, 2013), building knowledge about the cognitive factors involved in developing children's mathematical minds and how children understand mathematics is a keystone to ensuring our children's lifelong success.

Our book is designed to be read in a variety of ways. Each individual chapter makes a relevant contribution to how we understand mathematical cognition and understanding in the elementary and middle school years. Chapters are grouped to allow for connections and convergences to be made by readers with approaches that share topics, themes, approaches, and interests, and serve well as chapter focus clusters. These clusters fit within the broader conception of the volume's two-part organization, and so each part is designed to highlight confluences between the chapters. Collectively, this volume's contents provide a series of perspectives that move from pedagogy to research and application, which provides synthesis across the two parts. Further, because the book is uniquely conceived as a set of international and disciplinary perspectives on mathematics and cognition and understanding, a key element of its design is the way in which cumulative knowledge of how mathematics is taught and understood *across and within* various nations and disciplines develops across the entire volume.

This edited collection is an extension of two other edited volumes, *Mathematical Learning and Cognition in Early Childhood: Integrated Interdisciplinary Research* and *Mathematical* and *Teaching and Learning: Perspectives on Mathematical Minds in the Elementary and Middle School Years* (Robinson et al., 2019, 2023). The former installment focused on early childhood and included theories and evidence ranging from mathematics education to neuroscience. The volume covered both the teaching and learning of mathematics in early childhood education and home environments, as well as the cognitive and neurocognitive underpinnings of early numeracy ability. The latter installment, like the current volume, spanned the elementary and middle school years. It included perspectives from international scholars taking psychological and educational approaches to understand the process of mathematics pedagogy in Part 1 of the book and mathematics learning in Part 2 of the book.

The present volume also includes two parts. Part 1 focuses on critical cognitive factors underpinning mathematical thinking during the elementary and middle school years, ranging from domain general cognitive skills like attention and planning to motivation and achievement emotions. The chapters reflect key approaches to understanding more about the processes that impact children's mathematics success including the examination of basic cognitive factors, the role of the brain, and the interaction between cognitive and affective factors.

Part 2 engages with children's understanding of important mathematical concepts, including some persistently tricky topics for children such as word problem solving and fractions. The present volume, like its companion volumes, is aimed at both educators and scholars in the fields of mathematics education and numerical cognition. This is achieved by each chapter including theoretical contributions alongside practical recommendations for teachers, paraprofessionals, parents, and policy makers. Together, both parts present current thinking and research that inform readers on what is known and understood about elementary and middle school students' understanding of mathematics and the cognitive factors related to their success in mathematics. Further, the authors of each chapter have outlined how knowledge gained through advances in theory and research can be translated into practice.

Part 1: Cognitive factors in elementary and middle school mathematics.

Part 1 is organized into three clusters of related chapters to form a strong understanding of three important approaches to investigating cognitive factors in elementary and middle school mathematics. The first approach investigates the key role of basic cognitive skills in mathematics. The second approach examines mathematics from a cognitive neuroscience perspective. Finally, the third approach broadens to include the critical role of affective processes and how they interact with cognitive factors in children's mathematics learning. The first cluster of chapters in Part 1 begins with a focus on specific and basic cognitive factors such as spatial cognition and domain-general cognitive skills such as attention and working memory which have long been identified as critical factors in ensuring children's success in mathematics through the elementary and middle school years (Cragg & Gilmore, 2014; Hawes et al., 2019) and which are often of particular concern for children with or at risk of learning difficulties in mathematics (Yazdani et al., 2021).

In Chap. 2, Hawes, Gilligan-Lee, and Mix examine spatial thinking and its connection to mathematics ability and instruction. The authors argue that mathematics education can be improved via a 'spatializing' of the curriculum. They identify and describe a range of spatial skills and review the literature to show how each of them relates to specific mathematics outcomes. They then engage with how to best introduce spatial instruction to mathematics education, comparing isolated and integrated approaches, and conclude with recommending a mixed approach that transitions from isolated spatial skill training to increasingly embedded lessons that infuse spatial skills into mathematics instruction.

The contribution of domain-general attention skills and their role in the development of mathematics ability for typically developing children and those with neurodevelopmental conditions forms the focus of Chap. 3, by Clark, Perelmiter, and Bertone. They begin by demonstrating the combined contributions of attention, executive functions, working memory, and processing speed to mathematics ability, all framed within cognitive load theory, and argue that this complex contribution is not being addressed by interventions for mathematics remediation. This is supported by a systematic review of task-specific, function-specific, as well as indirect and direct attentional and working memory intervention studies. Their discussion calls for more research assessing the effectiveness of cognitive interventions but concludes that the most promising interventions are ones that align with cognitive load theory and integrate different attentional functions.

Domain-general skills are also of interest to Johnson, Stecker, and Linder in In Chap. 4. They examine the role of working memory in children's arithmetic fact fluency which is a vital mathematical skill. The authors begin by outlining the typical developmental trajectories for fact knowledge acquisition and then go on to detail how working memory deficits are a risk factor for children developing mathematical learning disabilities. Having identified the process of fact fluency acquisition and working memory deficits as a potential limiter on fact fluency development, incremental rehearsal is presented as an instructional intervention to support basic fact knowledge in elementary and middle school students. The chapter concludes with recommendations for practice for teachers, paraprofessionals, and parents on how to effectively implement this approach and identifies how technology could play a role in its success.

In the next cluster of chapters in Part 1, the focus turns to a second important approach to investigating cognitive factors by examining the neuropsychological processes involved in mathematics. Neuroscience can yield critical information not only on the brain processes involved in mathematics (De Smedt & Grabner, 2015) but can also be used to identify the different brain processes used by children who are typically or atypically developing (Skeide et al., 2018).

In Chap. 5, Declercq, Fias, and De Smedt describe how brain imaging research can provide unique insights into arithmetic strategy development. Arithmetic strategies are necessary to developing more advance mathematics understanding and are hallmarked by learners' gradual transition from calculation-based strategies to memory-based fact retrieval. Declercq and colleagues review the few studies providing insights into the brain regions potentially responsible for this transition, specifically looking at the effect of various mathematics interventions on children's brain activity. Taking this approach, the chapter details the potential power of brain imaging research to unravel the fine-grained processes missed by studying behavioural data alone and, thus, builds a fuller understanding of how arithmetic develops.

The influential Planning, Attention, Simultaneous, Successive (PASS) theory is used by Georgiou, Charalambos, and Sergiou in Chap. 6 to understand the role of neuropsychological processes in mathematics. The authors review pertinent literature and then detail data from a clinical case study of six children with mathematics giftedness and six with mathematics disabilities from three cultures. By examining the cognitive profiles of these children, they demonstrate the diversity in planning and simultaneous processing that exists in mathematics processes and go on to discuss how teachers can facilitate students' planning.

In the final cluster of chapters in Part 1, the focus turns to the interaction between cognitive and affective factors in the development of mathematics during the elementary and middle school years. Ensuring that children are engaged with mathematics is often key to success in mathematics (Wang et al., 2021). Research has examined not only how motivated children are but also how motivation and positive emotions can be increased (Hannula, 2006).

Liu, Rutherford, and Karamarkovich describe research on the connection between motivation and cognition in Chap. 7. This is accomplished through a systematic review of works investigating both motivation and cognition. They provide an overview of how three dominant motivational theories (Situated Expectancy-Value Theory (SEVT), Self-Determination Theory, and Achievement Goal theory) are used to understand mathematics outcomes. Using SEVT as a lens, motivational theories are expanded by including cognitive processes to better understand their relative contribution to mathematics achievement. The authors conclude by arguing that future research needs to expand even further, beyond the individual, to consider how environmental-level factors (school, family) affect cognitive and motivational processes.

In Chap. 8, Wen and Dubé delve into how technology, specifically educational mathematics games, can elicit positive emotions critical to mathematics success and further, how these games can be designed to promote mathematics ability. Using control-value theory as a framework, a systematic review and meta-analysis are conducted to determine the effect of five game-based emotional design principles (i.e., Visuals, Music, Mechanics, Narrative, Incentives) on students' achievement emotions and learning outcomes. The results show how design principles contributing to players' control and value appraisals are more likely to generate positive achievement emotions and stronger learning outcomes and the discussion guides teachers and parents on which types of mathematics games to bring into their classrooms and homes.

Part 2: Mathematical understanding in the elementary and middle school years.

Part 2 shifts to elementary and middle school children's understanding of important mathematics concepts. These chapters address central concepts identified in mathematics education and mathematical cognition research and have been divided into two clusters. First, specific concepts have been identified as being core competencies such as the understanding of the number line (Schneider et al., 2009) and the understanding of the operations of addition, subtraction, multiplication, and division (De Corte & Verschaffel, 1981). Second, both educators and psychologists working in the area of mathematical cognition have long been concerned about children's understanding of fractions which constitute a particular hurdle for many children during the elementary and middle school years (Siebert & Gaskin, 2006; Siegler et al., 2013). The last cluster of chapters included in this volume illustrate how researchers are trying to alleviate this concern by attempting to not only understand the obstacles to elementary and middle school children's understanding of fractions but to also address and surmount these obstacles.

In the first cluster of chapters in Part 2, the focus begins on the number line as a basic tool in the development of mathematical understanding and the innovate ways that the number line can be used to promote diverse mathematical knowledge. The focus then turns to how the understanding of arithmetic varies both across the elementary and middle school years but also from student to student.

Pericleous begins in Chap. 9 by detailing how the number line can serve as a vehicle for children's broader mathematical understanding. The chapter describes the naturalistic study of students who were taught how to use number lines as a tool for representing their own mathematical ideas and processes. The instructional approach incorporates puppets, narrative, and games, and encourages students to construct, create and interpret mathematics concepts. The richness of such an approach is captured and presented by the students' own words and illustrations. The chapter culminates in Pericleous contending that number line understanding is influenced by more than just mathematics; it is also a product of the social and sociomathematical norms established in the classroom.

In Chap. 10, Robinson and Buchko argue for and demonstrate the power of longitudinal methods for investigating students' conceptual knowledge of arithmetic. They begin by identifying what is known about children's understanding of equivalence, inversion, and associativity and then highlight a) children's relatively poor understanding for the multiplicative versions of these concepts; and b) that research on these concepts is largely informed by cross-sectional designs and that longitudinal work is needed. To this end, they report on a recent longitudinal study showing an overall increase in knowledge for all three concepts. However, analysis of student profiles shows how the larger trend masks considerable individual variability and the need for direct instruction to improve understanding of these key concepts.

In the final cluster of chapters in Part 2, the focus now hones in on fractions to illustrate their particular importance during the elementary and middle school years. From learning how to share a pie amongst several friends to adding 1/4 to 2/5, fractions are crucial in children's understanding of mathematics (Booth & Newton, 2012). This set of chapters begins by identifying why fractions are such challenging concept for many students and how these challenges can be addressed, continues by showcasing the difficulties children experience moving from concrete to abstract reasoning, and concludes by describing an effective intervention approach to learning fractions for children with mathematics learning difficulties.

In Chap. 11, Gabriel, Van Hoof, Gómez, and Van Dooren outline the most likely barriers preventing students from understanding the notoriously tricky concept of fractions. They begin by defining conceptual and procedural knowledge of fractions and detail how they relate to and reinforce each other. The discussion then turns to how children's natural number processing (NNP) both helps and hinders their understanding of fractions. They identify how children's NNP is responsible for three common misconceptions that children hold about fractions and then conclude by detailing several promising interventions teachers and parents can employ to counter them.

Another tricky area of mathematics, word problems, and specifically word problems involving fractions is the target for Osana and colleagues in Chap. 12. Their investigation focuses on children's word problem solving strategies for equal sharing problems and how they change as a function of the "groundedness" of the to-be-shared object (cf., abstract) and whether the object is measured in units of area or length. Their discussion on the role of problem and object characteristics in world problem solving is illustrated by recent data. The results are complex, as it seems that children may use their experience with concrete objects to reason about abstract ones. The authors conclude that children's word problem solving is informed by children's word knowledge, linguistic competencies, and the specific mental representations triggered by the problems themselves.

Chen, Thannimalai, and Kalyuga in Chap. 13 combine the volume's focus on cognition and understanding by demonstrating how worked examples, informed by cognitive load theory, can improve students' understanding of fraction concepts. The chapter begins by reviewing both cognitive load theory and the worked example effect and by showing how cognitive load theory can be used to explain why worked examples are so effective. They then present new empirical evidence suggesting that worked examples, if properly designed, can be effective for teaching fraction understanding.

Finally, in Chap. 14, Jordan, Dyson, Devlin, and Gesuelli present their creation of an evidence-based fraction intervention for students with mathematics learning difficulties (MLD). Their discussion on how to address children's difficulties with fractions looks at domain specific causes, such as fraction magnitude, equivalence, arithmetic, common errors, and representations, as well as common techniques for supporting students with MLD. The authors explain how they combine literature on the source of fraction errors with literature on effective supports to create their 'fraction sense intervention.' The chapter concludes by detailing the intervention's components and providing evidence of its effectiveness.

As these chapters so well illustrate, scholars from a wide range of perspectives, including across disciplines and countries, are necessary to gain a fuller understanding of mathematical minds in the elementary and middle school years. Translating research and theory into practice is crucial to ensure that students of varying mathematical skills and abilities succeed both in their present and future mathematical endeavours. By targeting the cognitive factors involved in mathematics performance as well as children's understanding of mathematics, this volume aims to inform both researchers and practitioners to help develop elementary and middle school students' mathematical minds.

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