Katherine M. Robinson
Adam K. Dubé
Donna Kotsopoulos *Editors*

Mathematical Cognition and Understanding

Perspectives on Mathematical Minds in the Elementary and Middle School Years

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Contents

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Abbreviations

ADHD	Attention-Deficit/Hyperactivity Disorder
AG	Angular Gyrus
AGT	Achievement Goal Theory
ASD	Autism Spectrum Disorder
CA	Chronological Age
CCSSM	Common Core State Standards in Mathematics
CLT	Cognitive Load Theory
CRA	Concrete-Representational-Abstract
d	Cohen's *d*
EF	Executive Functioning
ENL	Empty Number Line
ERIC	Educational Resources Information Center
EVT	Expectancy-Value Theory
F	Fluent Facts
FC	Functional Connectivity
FRQS	Fonds de recherche du Québec – Santé
FSI	Fraction Sense Intervention
GMV	Grey Matter Volume
GPA	Grade Point Average
HC	Hippocampus
IDR	Integrated Dynamic Representation
IFG	Inferior Frontal Gyrus
IPS	Intraparietal Sulcus
IQ	Intelligence Quotient
LSD	Least Significant Difference
M	Mean
MD	Mathematics Difficulties
MFG	Middle Frontal Gyrus
MID	Mild Intellectual Disability
MLD	Mathematical Learning Difficulties

MTL	Medial Temporal Lobe
MTSS	Multi-Tiered System of Supports
N	Number
NAEP	National Assessment for Educational Progress
NCTM	National Council of Teachers of Mathematics
NDC	Neurodevelopmental Condition
NMAP	National Mathematics Advisory Panel
NNB	Natural Number Bias
NRC	National Research Council
p	Probability
P	Pupil
PASS	Planning, Attention, Simultaneous, and Successive
PFC	Prefrontal Cortex
PNLab	Perceptual Neuroscience Lab for Autism and Development
PPT	Picture Perception Task
PS	Processing Speed
PT	Previous Target Facts
R	Right
RD	Reading Difficulties
RTI	Response to Intervention
SD	Standard Deviation
SDT	Self-Determination Theory
SEM	Standard Error of Mean
SEVT	Situated Expectancy-Value Theory
STEM	Science, Technology, Engineering, and Mathematics
T	Target Facts
t	t-test
TD	Typically Developing
TIMSS	Trends in International Mathematics and Science Study
U.S.	United States
VSTM	Visual Short-Term Memory
WEIRD	Western, Educated, Industrialized, Rich, and Democratic
WISC	Wechsler Intelligence Scale for Children
WM	Working Memory
WNB	Whole Number Bias

Chapter 1
An Introduction to Mathematical Cognition and Understanding in the Elementary and Middle School Years

Adam K. Dubé, Donna Kotsopoulos, and Katherine M. Robinson

Abstract This volume focuses on the complex and diverse processes and factors affecting the mathematical cognition and understanding of elementary and middle school children, a critical time where they experience a range of developmental, pedagogical, and individual changes that impact their lifelong mathematics education and experience. In this first chapter, we identify the central topics of the individual contributions in this volume, providing a broader framing for the chapters by organising them into two parts (Cognitive Factors, Mathematical Understanding), and highlight broader themes connecting the chapters. We also draw attention to how each chapter provides new theoretical contributions and practical recommendations for teachers, paraprofessionals, parents, and policy makers, with the goal of improving children's success in mathematics.

Keywords Mathematical cognition · Mathematical understanding · Mathematics education · Middle school · Elementary school

1.1 Introduction

Studying the diverse processes and factors contributing to elementary and middle school children's mathematical cognition and understanding requires combining theories and evidence from a broad range of fields. The elementary and middle school years, which are the focus of this volume, roughly correspond to the ages of 6–12 although variability exists between or even within a country's states,

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provinces, territories, or areas. These school years are particularly complex given the vast developmental, pedagogical, and individual changes that occur and combine to affect children's mathematics education and experience. Students make great strides during these years of formal schooling in their ability to attend, plan, and execute cognitive strategies (Xu et al., 2013). They likely experience diverse types of mathematics instruction on complex foundational concepts (e.g., Fazio, 2019; Parr et al., 2019), and they are apt to experience strong emotions and shifting motivations about mathematics as a school subject (Karamarkovich & Rutherford, 2021). As such, researchers, educators, and parents' understanding of the factors that aid, impede, and motivate students to learn mathematics during elementary and middle school is critical and is the focus of this volume.

This volume represents the work of international scholars bringing their perspectives on children's mathematical minds from intersecting areas of inquiry including mathematics education, educational psychology, cognitive development, developmental psychology, mathematical cognition, cognitive neuroscience, and educational neuroscience who all have the goal of ensuring that all children in the elementary and middle school years succeed at mathematics. As it becomes ever more apparent how important early mathematics skills are for many areas of our lives, including academics, career choice, health and well-being, and financial literacy (Every Child a Chance, 2009; OECD, 2019; Parsons & Bynner, 2005; Statistics Canada, 2013), building knowledge about the cognitive factors involved in developing children's mathematical minds and how children understand mathematics is a keystone to ensuring our children's lifelong success.

Our book is designed to be read in a variety of ways. Each individual chapter makes a relevant contribution to how we understand mathematical cognition and understanding in the elementary and middle school years. Chapters are grouped to allow for connections and convergences to be made by readers with approaches that share topics, themes, approaches, and interests, and serve well as chapter focus clusters. These clusters fit within the broader conception of the volume's two-part organization, and so each part is designed to highlight confluences between the chapters. Collectively, this volume's contents provide a series of perspectives that move from pedagogy to research and application, which provides synthesis across the two parts. Further, because the book is uniquely conceived as a set of international and disciplinary perspectives on mathematics and cognition and understanding, a key element of its design is the way in which cumulative knowledge of how mathematics is taught and understood *across and within* various nations and disciplines develops across the entire volume.

This edited collection is an extension of two other edited volumes, *Mathematical Learning and Cognition in Early Childhood: Integrated Interdisciplinary Research* and *Mathematical* and *Teaching and Learning: Perspectives on Mathematical Minds in the Elementary and Middle School Years* (Robinson et al., 2019, 2023). The former installment focused on early childhood and included theories and evidence ranging from mathematics education to neuroscience. The volume covered both the teaching and learning of mathematics in early childhood education and home environments, as well as the cognitive and neurocognitive underpinnings of early numeracy ability. The latter installment, like the current volume, spanned the elementary and middle school years. It included perspectives from international scholars taking psychological and educational approaches to understand the process of mathematics pedagogy in Part 1 of the book and mathematics learning in Part 2 of the book.

The present volume also includes two parts. Part 1 focuses on critical cognitive factors underpinning mathematical thinking during the elementary and middle school years, ranging from domain general cognitive skills like attention and planning to motivation and achievement emotions. The chapters reflect key approaches to understanding more about the processes that impact children's mathematics success including the examination of basic cognitive factors, the role of the brain, and the interaction between cognitive and affective factors.

Part 2 engages with children's understanding of important mathematical concepts, including some persistently tricky topics for children such as word problem solving and fractions. The present volume, like its companion volumes, is aimed at both educators and scholars in the fields of mathematics education and numerical cognition. This is achieved by each chapter including theoretical contributions alongside practical recommendations for teachers, paraprofessionals, parents, and policy makers. Together, both parts present current thinking and research that inform readers on what is known and understood about elementary and middle school students' understanding of mathematics and the cognitive factors related to their success in mathematics. Further, the authors of each chapter have outlined how knowledge gained through advances in theory and research can be translated into practice.

Part 1: Cognitive factors in elementary and middle school mathematics.

Part 1 is organized into three clusters of related chapters to form a strong understanding of three important approaches to investigating cognitive factors in elementary and middle school mathematics. The first approach investigates the key role of basic cognitive skills in mathematics. The second approach examines mathematics from a cognitive neuroscience perspective. Finally, the third approach broadens to include the critical role of affective processes and how they interact with cognitive factors in children's mathematics learning. The first cluster of chapters in Part 1 begins with a focus on specific and basic cognitive factors such as spatial cognition and domain-general cognitive skills such as attention and working memory which have long been identified as critical factors in ensuring children's success in mathematics through the elementary and middle school years (Cragg & Gilmore, 2014; Hawes et al., 2019) and which are often of particular concern for children with or at risk of learning difficulties in mathematics (Yazdani et al., 2021).

In Chap. 2, Hawes, Gilligan-Lee, and Mix examine spatial thinking and its connection to mathematics ability and instruction. The authors argue that mathematics education can be improved via a 'spatializing' of the curriculum. They identify and describe a range of spatial skills and review the literature to show how each of them relates to specific mathematics outcomes. They then engage with how to best introduce spatial instruction to mathematics education, comparing isolated and integrated approaches, and conclude with recommending a mixed approach that transitions from isolated spatial skill training to increasingly embedded lessons that infuse spatial skills into mathematics instruction.

The contribution of domain-general attention skills and their role in the development of mathematics ability for typically developing children and those with neurodevelopmental conditions forms the focus of Chap. 3, by Clark, Perelmiter, and Bertone. They begin by demonstrating the combined contributions of attention, executive functions, working memory, and processing speed to mathematics ability, all framed within cognitive load theory, and argue that this complex contribution is not being addressed by interventions for mathematics remediation. This is supported by a systematic review of task-specific, function-specific, as well as indirect and direct attentional and working memory intervention studies. Their discussion calls for more research assessing the effectiveness of cognitive interventions but concludes that the most promising interventions are ones that align with cognitive load theory and integrate different attentional functions.

Domain-general skills are also of interest to Johnson, Stecker, and Linder in In Chap. 4. They examine the role of working memory in children's arithmetic fact fluency which is a vital mathematical skill. The authors begin by outlining the typical developmental trajectories for fact knowledge acquisition and then go on to detail how working memory deficits are a risk factor for children developing mathematical learning disabilities. Having identified the process of fact fluency acquisition and working memory deficits as a potential limiter on fact fluency development, incremental rehearsal is presented as an instructional intervention to support basic fact knowledge in elementary and middle school students. The chapter concludes with recommendations for practice for teachers, paraprofessionals, and parents on how to effectively implement this approach and identifies how technology could play a role in its success.

In the next cluster of chapters in Part 1, the focus turns to a second important approach to investigating cognitive factors by examining the neuropsychological processes involved in mathematics. Neuroscience can yield critical information not only on the brain processes involved in mathematics (De Smedt & Grabner, 2015) but can also be used to identify the different brain processes used by children who are typically or atypically developing (Skeide et al., 2018).

In Chap. 5, Declercq, Fias, and De Smedt describe how brain imaging research can provide unique insights into arithmetic strategy development. Arithmetic strategies are necessary to developing more advance mathematics understanding and are hallmarked by learners' gradual transition from calculation-based strategies to memory-based fact retrieval. Declercq and colleagues review the few studies providing insights into the brain regions potentially responsible for this transition, specifically looking at the effect of various mathematics interventions on children's brain activity. Taking this approach, the chapter details the potential power of brain imaging research to unravel the fine-grained processes missed by studying behavioural data alone and, thus, builds a fuller understanding of how arithmetic develops.

The influential Planning, Attention, Simultaneous, Successive (PASS) theory is used by Georgiou, Charalambos, and Sergiou in Chap. 6 to understand the role of neuropsychological processes in mathematics. The authors review pertinent literature and then detail data from a clinical case study of six children with mathematics giftedness and six with mathematics disabilities from three cultures. By examining the cognitive profiles of these children, they demonstrate the diversity in planning and simultaneous processing that exists in mathematics processes and go on to discuss how teachers can facilitate students' planning.

In the final cluster of chapters in Part 1, the focus turns to the interaction between cognitive and affective factors in the development of mathematics during the elementary and middle school years. Ensuring that children are engaged with mathematics is often key to success in mathematics (Wang et al., 2021). Research has examined not only how motivated children are but also how motivation and positive emotions can be increased (Hannula, 2006).

Liu, Rutherford, and Karamarkovich describe research on the connection between motivation and cognition in Chap. 7. This is accomplished through a systematic review of works investigating both motivation and cognition. They provide an overview of how three dominant motivational theories (Situated Expectancy-Value Theory (SEVT), Self-Determination Theory, and Achievement Goal theory) are used to understand mathematics outcomes. Using SEVT as a lens, motivational theories are expanded by including cognitive processes to better understand their relative contribution to mathematics achievement. The authors conclude by arguing that future research needs to expand even further, beyond the individual, to consider how environmental-level factors (school, family) affect cognitive and motivational processes.

In Chap. 8, Wen and Dubé delve into how technology, specifically educational mathematics games, can elicit positive emotions critical to mathematics success and further, how these games can be designed to promote mathematics ability. Using control-value theory as a framework, a systematic review and meta-analysis are conducted to determine the effect of five game-based emotional design principles (i.e., Visuals, Music, Mechanics, Narrative, Incentives) on students' achievement emotions and learning outcomes. The results show how design principles contributing to players' control and value appraisals are more likely to generate positive achievement emotions and stronger learning outcomes and the discussion guides teachers and parents on which types of mathematics games to bring into their classrooms and homes.

Part 2: Mathematical understanding in the elementary and middle school years.

Part 2 shifts to elementary and middle school children's understanding of important mathematics concepts. These chapters address central concepts identified in mathematics education and mathematical cognition research and have been divided into two clusters. First, specific concepts have been identified as being core competencies such as the understanding of the number line (Schneider et al., 2009) and the understanding of the operations of addition, subtraction, multiplication, and division (De Corte & Verschaffel, 1981). Second, both educators and psychologists working in the area of mathematical cognition have long been concerned about children's understanding of fractions which constitute a particular hurdle for many children during the elementary and middle school years (Siebert & Gaskin, 2006; Siegler et al., 2013). The last cluster of chapters included in this volume illustrate how researchers are trying to alleviate this concern by attempting to not only understand the obstacles to elementary and middle school children's understanding of fractions but to also address and surmount these obstacles.

In the first cluster of chapters in Part 2, the focus begins on the number line as a basic tool in the development of mathematical understanding and the innovate ways that the number line can be used to promote diverse mathematical knowledge. The focus then turns to how the understanding of arithmetic varies both across the elementary and middle school years but also from student to student.

Pericleous begins in Chap. 9 by detailing how the number line can serve as a vehicle for children's broader mathematical understanding. The chapter describes the naturalistic study of students who were taught how to use number lines as a tool for representing their own mathematical ideas and processes. The instructional approach incorporates puppets, narrative, and games, and encourages students to construct, create and interpret mathematics concepts. The richness of such an approach is captured and presented by the students' own words and illustrations. The chapter culminates in Pericleous contending that number line understanding is influenced by more than just mathematics; it is also a product of the social and sociomathematical norms established in the classroom.

In Chap. 10, Robinson and Buchko argue for and demonstrate the power of longitudinal methods for investigating students' conceptual knowledge of arithmetic. They begin by identifying what is known about children's understanding of equivalence, inversion, and associativity and then highlight a) children's relatively poor understanding for the multiplicative versions of these concepts; and b) that research on these concepts is largely informed by cross-sectional designs and that longitudinal work is needed. To this end, they report on a recent longitudinal study showing an overall increase in knowledge for all three concepts. However, analysis of student profiles shows how the larger trend masks considerable individual variability and the need for direct instruction to improve understanding of these key concepts.

In the final cluster of chapters in Part 2, the focus now hones in on fractions to illustrate their particular importance during the elementary and middle school years. From learning how to share a pie amongst several friends to adding 1/4 to 2/5, fractions are crucial in children's understanding of mathematics (Booth & Newton, 2012). This set of chapters begins by identifying why fractions are such challenging concept for many students and how these challenges can be addressed, continues by showcasing the difficulties children experience moving from concrete to abstract reasoning, and concludes by describing an effective intervention approach to learning fractions for children with mathematics learning difficulties.

In Chap. 11, Gabriel, Van Hoof, Gómez, and Van Dooren outline the most likely barriers preventing students from understanding the notoriously tricky concept of fractions. They begin by defining conceptual and procedural knowledge of fractions and detail how they relate to and reinforce each other. The discussion then turns to how children's natural number processing (NNP) both helps and hinders their understanding of fractions. They identify how children's NNP is responsible for three common misconceptions that children hold about fractions and then conclude by detailing several promising interventions teachers and parents can employ to counter them.

Another tricky area of mathematics, word problems, and specifically word problems involving fractions is the target for Osana and colleagues in Chap. 12. Their investigation focuses on children's word problem solving strategies for equal sharing problems and how they change as a function of the "groundedness" of the to-be-shared object (cf., abstract) and whether the object is measured in units of area or length. Their discussion on the role of problem and object characteristics in world problem solving is illustrated by recent data. The results are complex, as it seems that children may use their experience with concrete objects to reason about abstract ones. The authors conclude that children's word problem solving is informed by children's word knowledge, linguistic competencies, and the specific mental representations triggered by the problems themselves.

Chen, Thannimalai, and Kalyuga in Chap. 13 combine the volume's focus on cognition and understanding by demonstrating how worked examples, informed by cognitive load theory, can improve students' understanding of fraction concepts. The chapter begins by reviewing both cognitive load theory and the worked example effect and by showing how cognitive load theory can be used to explain why worked examples are so effective. They then present new empirical evidence suggesting that worked examples, if properly designed, can be effective for teaching fraction understanding.

Finally, in Chap. 14, Jordan, Dyson, Devlin, and Gesuelli present their creation of an evidence-based fraction intervention for students with mathematics learning difficulties (MLD). Their discussion on how to address children's difficulties with fractions looks at domain specific causes, such as fraction magnitude, equivalence, arithmetic, common errors, and representations, as well as common techniques for supporting students with MLD. The authors explain how they combine literature on the source of fraction errors with literature on effective supports to create their 'fraction sense intervention.' The chapter concludes by detailing the intervention's components and providing evidence of its effectiveness.

As these chapters so well illustrate, scholars from a wide range of perspectives, including across disciplines and countries, are necessary to gain a fuller understanding of mathematical minds in the elementary and middle school years. Translating research and theory into practice is crucial to ensure that students of varying mathematical skills and abilities succeed both in their present and future mathematical endeavours. By targeting the cognitive factors involved in mathematics performance as well as children's understanding of mathematics, this volume aims to inform both researchers and practitioners to help develop elementary and middle school students' mathematical minds.

We would like to thank the exemplary scholars who have contributed to this collected work and who continue to grow our knowledge of mathematical cognition and understanding, Springer for their steadfast support of this collective endeavor, and our readers who support these works.

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Part I
Cognitive Factors

Chapter 2
Infusing Spatial Thinking Into Elementary and Middle School Mathematics: What, Why, and How?

Zachary C. K. Hawes, Katie A. Gilligan-Lee, and Kelly S. Mix

Abstract Spatial thinking plays a critical role in the learning, doing, and communication of mathematics. Yet, spatial thinking remains an under-valued, under-recognized, and under-instructed feature of mathematics education. In this chapter, we argue that the teaching and learning of mathematics can be improved through 'spatializing' the curriculum—that is, taking a more explicitly spatial approach to mathematics instruction. The chapter is structured around three main questions, beginning with "*What* is spatial thinking?" We then discuss *why* spatial and mathematical skills are related, including a review of evidence that spatial training/instruction may be causally related to mathematics performance. In the remaining sections of the chapter, we focus on *how* spatial thinking can be improved and leveraged to support mathematics learning in the classroom. In doing so, we provide a simulated spatial training progression appropriate for middle school mathematics. Altogether, we argue that spatial thinking represents untapped potential, a hidden strength in students that can be drawn from and further cultivated to achieve new disciplinary insights, understandings, and appreciation of mathematics.

Keywords Spatial thinking · Mathematics education · Spatial training · Spatial visualization · Intervention

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2.1 Introduction

Spatial thinking is fundamental to learning, doing, inventing, and communicating mathematics. Well known mathematical ideas and proofs, such as the Pythagorean Theorem, Pi, the area of a circle, seven bridges of Königsberg problem, and Pascal's triangle are but a few examples of the entanglement of spatial and mathematical thought (Cain, 2019; Davis, 2015). To riffle through a mathematics text book is to further witness the countless diagrams, graphs, and other visualizations used to communicate mathematical ideas and encourage student understanding. The tools and manipulatives that spill from students' desks and fill the shelves of classrooms offer further insights into the role of spatial thinking in mathematics. For example, the number line is a simple, yet elegant tool used to show how fractions relate to one another – using space as a way of grounding meaning through organizing, structuring, and highlighting numerical relationships (Hamdan & Gunderson, 2017). Many other manipulatives serve a similar purpose such as the abacus, Cuisenaire/relational rods, algebra tiles, and so forth (Mix, 2010). Moving beyond everyday observations and anecdotes, there is over a century of empirical evidence indicating the robust relationship between spatial thinking and mathematics, such that people with stronger spatial skills tend to also have stronger mathematical skills (Atit et al., 2021; Hawes & Ansari, 2020; Lourenco et al., 2018; Mix & Cheng, 2012).

Despite such tight relations, research suggests that spatial thinking rarely plays a central role in mathematics education. In fact, in some places of the world, there are initiatives to remove spatial skill development from mathematics curricula. For example, in 2021, England revised their early years learning standards to reduce the focus on shape and space in favor of increased attention to early number skills. Furthermore, other research suggests that when spatial thinking is included in the curriculum, the instruction tends to focus primarily on verbal tasks (e.g., labeling and sorting shapes), rather than promoting dynamic spatial thinking such as spatial visualization and transformations (Clements, 2004; Moss et al., 2016). There is also the issue of teacher training. Elementary school teachers receive little to no professional development in the area of spatial thinking (Ginsburg et al., 2006). Furthermore, there is little evidence that teacher training programs prepare students to think about the importance of spatial thinking in mathematics education.

Taken together, we are left with an interesting paradox. On the one hand, spatial thinking is fundamental to mathematics. And yet, on the other hand, spatial thinking appears to be a commonly neglected aspect of mathematics education. In this chapter, we will argue that the teaching and learning of mathematics can be improved through 'spatializing' the curriculum—that is, taking a more explicitly spatial approach to mathematics instruction. Although our focus is the spatialization of middle school mathematics (approximately ages 10–14 years), the ideas discussed are relevant across the PreK-12 curriculum. We begin by addressing the question, *"What* is spatial thinking?" We then discuss *why* spatial and mathematical skills are related, including a review of new evidence suggesting that spatial training/instruction may be causally related to mathematics performance. In the remaining sections of the chapter, we focus on *how* spatial thinking can be improved and leveraged to support mathematics learning in the classroom. In addition to providing an overview of evidence-based spatial approaches to improving mathematics, we also offer a simulated spatial training progression appropriate for middle school mathematics.

2.2 What Is Spatial Thinking?

Defining spatial thinking is not easy. Although widely recognized as a critical component of human cognition – distinct from other key facets of intellect, such as verbal and numerical reasoning – spatial thinking represents a vast and varied construct (Hegarty & Waller, 2005). It comprises not just one skill, or subset of skills, but many (Newcombe & Shipley, 2015). We engage in spatial thinking when navigating familiar environments, such as our homes and immediate surrounding area, but also when navigating new and unfamiliar environments, such as exploring a new city. We also engage in spatial thinking when imagining the best way to pack the trunk of a car, using our visualization skills to rotate and fit objects together. Interestingly, the spatial skills in these examples, navigation and spatial visualization respectfully, are dissociable to some extent (Hegarty & Waller, 2004; for a similar distinction made between intrinsic and dynamic spatial skills see: Mix et al., 2018). Being an expert 'trunk packer' does not necessarily translate to being an expert navigator.

That spatial thinking is multidimensional and used differently across different contexts has important implications for education. First, it means that we must be careful not to subscribe to the mantra that "I'm not a spatial thinker," and, in turn, to label students, and indeed ourselves, as either possessing spatial ability or not. Binary interpretations of spatial thinking like this are common but potentially harmful, as they fail to recognize the breadth, complexity, and routine use of this key facet of human reasoning. Although spatial skills tend to be highly related to one another when measured with psychological tests (Mix et al., 2016, 2017), there is an emerging literature that suggests that the use and measurement of spatial thinking in real-world contexts is more variable and context-dependent (Atit et al., 2020). This leads to a second implication. When it comes to the classroom, spatial thinking is likely to present itself in many different ways and to vary in its use and importance across educational contexts.

When learning mathematics, certain spatial skills appear more important than others. For example, there is little research indicating that large-scale spatial thinking skills, such as navigation, is associated with higher levels of mathematics achievement (Lourenco et al., 2018). There is, however, an extensive body of evidence to suggest the importance of small-scale spatial skills, namely, spatial visualization, in the learning and doing of mathematics (Lowrie et al., 2020). Defined as the capacity to recall, create, maintain, manipulate, and transform visual-spatial information, spatial visualization skills have been associated with performance across a wide

Table 2.1 Examples and descriptions of different spatial thinking assessments and their associations with various mathematics outcomes

Spatial skill	Assessment tool	Task description	Associations with math outcomes
Spatial assembly	Test of Spatial Assembly (TOSA) (Verdine et al., 2014)	Participants are asked to copy a model construction using physical bricks/ blocks	Associated with numeracy skills (number, operation and counting skills) at 3 years of age (Bower et al., 2020)
Mental rotation	2D Mental Rotation Test (Neuburger et al., 2011)	Participants are required to identify which images/ objects are rotated versions of one another	Associated with arithmetic and intro to algebra (missing term) problems in children aged 6–8 (Hawes et al., 2015)
Spatial scaling	Spatial Scaling Discrimination Paradigm (Gilligan et al., 2018)	Participants must choose which of 4 choice maps shows a target in the same relative position as a scaled target map	Associated with standardised mathematics performance, number line estimation, and approximate number sense in children aged 6–10 years (Gilligan et al., 2019)
Mental transformation	Children's Mental Transformation Task (Levine et al., 1999)	Participants must choose which of 4 shapes can be created by mentally joining two component pieces together (requiring both mental translation and rotation)	Associated with arithmetic and number-logic problems in kindergarten (children around 6 years of age) and geometry in second grade children (Frick, 2019)
Perspective taking	Level Two Perspective Taking Task (Frick et al., 2014)	Participants are required to imagine what photograph has been taken by a toy photographer who has a different viewpoint (perspective) than themselves.	Associated with geometry, spatial functions, and number-logic problems in children aged 6–8 years
Visuo-spatial working memory	Visual Spatial Working Memory Task (Kaufman & Kaufman, 1983)	Items are presented to participants in a grid, for 5 seconds. Participants are asked to draw an x on a corresponding grid to show where the items were displayed. Difficulty is increased by increasing the cells in the grid and number of items displayed	Associated with overall mathematics performance (a composite of algebra, word problems, number line, fractions, geometry, place value, charts and graphs, and calculation) (Mix et al., 2016)

assortment of mathematics activities, ranging from foundational counting and arithmetic skills all the way up to highly advanced mathematics topics, such as function theory and computational mathematics (Hawes & Ansari, 2020; Lohman, 1996; Mix et al., 2017). Table 2.1 provides examples of different spatial skill assessments, all of which have been found to correlate with mathematics performance. In the next section, we explore why this is the case. We outline reasons why spatial visualization, as well as other spatial skills, are believed to underlie and support mathematics learning and performance.

2.3 Why Are Spatial Skills and Mathematics Related?

There are several reasons why spatial skill might be related to the learning and doing of mathematics. First, there is emerging evidence that spatial and mathematical thinking both make use of the same brain regions. A 2019 meta-analysis found that mental rotation (a prototypical spatial skill), basic number processing (e.g., comparing the magnitude of different number symbols), and arithmetic all involve neural activity in and around the intraparietal sulcus (IPS) (see Fig. 2.1) (Hawes et al., 2019b; see also Hubbard et al., 2009). This suggests that spatial and mathematical thinking may rely on similar brain regions and shared neural mechanisms.

Second, the affordances of spatial visualization may support mathematics problem solving. Spatial visualization may serve as "mental blackboard" on which various mathematical concepts, relations, and operations can be modeled and visualized (Lourenco et al., 2018; Mix, 2019). For example, when confronted with a word problem, learners might create a mental model of the problem, organizing and structuring the problem accordingly. This capacity to mentally visualize and simulate mathematical relations (e.g., bringing sets of objects together or apart in one's mind), is believed to play an especially important role for novel and unfamiliar mathematical problems (Hawes et al., 2019a, b; Mix et al., 2016; Uttal & Cohen, 2012). Even for more practiced mathematics tasks, such as solving memorized arithmetic facts, individuals may rely on one's mental blackboard to record and keep track of the question at hand.

A third explanation for the space-math link relates to the ways in which space is used to represent and communicate mathematical meaning. Consider all the ways space is used to represent and communicate mathematical concepts in the classroom. Mathematical tools and manipulatives, such as number lines, rulers, and protractors, as well as manipulatives, such as relational (Cuisenaire) rods, base-ten blocks,

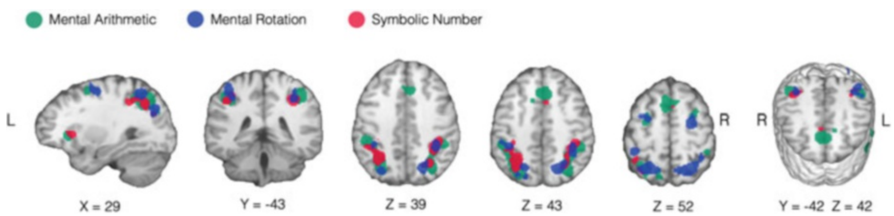

Fig. 2.1 Meta-analysis of fMRI studies examining brain regions associated with mental arithmetic (green), and mental rotation (blue), and basic numerical processing (red)

fraction bars, etc. are ubiquitous, and share in common the mapping of numbers to space to bear mathematical meaning (Mix, 2010). Space is also used to convey mathematical meaning through writing conventions. Place value, for example, is conveyed through the positioning of digits. Diagrams, graphs, and other visual representations that rely on spatial relations are commonly used to illustrate mathematical concepts, as are gestures (Lowrie, 2020; Novack et al., 2014).

From these examples, a fourth, and perhaps the most obvious reason why space and mathematics are related emerges; certain domains of mathematics are inherently spatial. Indeed, the 'study of space' is a core feature of what it means to engage in mathematics. Geometry and measurement are two elementary and middle school mathematics topics where spatial reasoning lies central to disciplinary practice. And yet, as argued elsewhere, even overtly spatial branches of mathematics, such as geometry, are not as 'spatial' as they could be, with instructional efforts failing to recognize the potential use of spatial reasoning as a key approach to effective and efficient mathematical problem solving (Clements, 2004; Moss et al., 2016). This is an important point and one we will examine for much of the remaining chapter. That is, how might we further leverage the link between spatial and mathematical thinking, to give rise to improved spatial skills as well as mathematics understanding and performance?

2.4 Does Spatial Instruction Improve Mathematics Performance?

The idea that spatial instruction (also referred to as spatial training) should lead to improvements in mathematics is not new (e.g., see Bishop, 1980; Smith, 1964). However, only recently have researchers begun to empirically test this hypothesis. According to our own analysis, 95% of the published studies in this area have appeared in or since 2014 (more on this later). Why has it taken researchers so long to test these ideas? One reason might have something to do with the controversies around the malleability of spatial thinking. Until recently, the question of whether spatial thinking can be improved with practice was still an open question (National Research Council, 2006; Sims & Mayer, 2002). There was, and in many cases, there still is, a belief that spatial thinking is a fixed trait – either you are a spatial thinker or not. However, as we have outlined above, spatial thinking is multifaceted and it is a gross oversimplification to categorize individuals' as possessing uniformly 'good' or 'bad' spatial skills. Critically, the question of whether or not spatial skills can be trained and improved with practice has also been put to rest.

There is now substantial evidence that spatial thinking is malleable. A meta-analysis by Uttal et al. (2013) analysed a total of 206 spatial training studies across a 25-year period (1984–2009) and found the average effect size (Hedges's g) for training relative to control was approximately one-half standard deviation (0.47). To put this into context, the authors suggested that an improvement of this magnitude would roughly double the number of U.S. citizens with the spatial skills of a trained engineer. Moreover, the results indicated that spatial thinking can be improved in people of all ages, through a variety of training approaches (e.g., video games, course training, spatial task training), and that the effects were durable (i.e., still present weeks and months later) and generalizable (i.e., training one type of spatial skill was associated with improvements in other aspects of spatial thinking). With the malleability of spatial skills established, the critical follow-up question of whether spatial training might also lead to benefits in mathematics emerged.

To address this question, we conducted a meta-analysis that focused on the effects of spatial training/instruction on mathematics (Hawes et al., 2022). We identified 29 studies that used a controlled pre-post study design ($N = 3765$) with an assessment of mathematics performance both prior to and following participation in a spatial intervention. Furthermore, all studies involved a control group who either completed no intervention (passive control group), or another kind of intervention (active control group; e.g., participating in another form or math instruction). We wanted to know whether people who completed spatial training demonstrated greater improvements in mathematics compared to people assigned to the control conditions. Our results showed that the average effect size (Hedges's g) of spatial training on mathematics relative to the control conditions was 0.28. An effect of this magnitude is comparable to the annual gains that occur in Grades 6–10 on U.S.-based nationally-normed tests of mathematics (see Bloom et al., 2008; Hill et al., 2008). In comparison to younger grades (Kindergarten-Grade 5), our effect of 0.28 is comparable to about 25–50% of the annual gains that occur in mathematics. Against these benchmarks, the effects appear quite large. However, these benchmark estimates are based on U.S. data only, suggesting the need for caution in drawing any direct comparisons. Another way of contextualizing the effect is to compare it to the effects of other cognitive training programs. For example, in comparison to the effects of working memory training on mathematics, our effects are about 2–3 times larger (Melby-Lervåg et al., 2016; Schwaighofer et al., 2015). Overall, the results of this study provided the most conclusive evidence to date that spatial training may indeed lead to gains in mathematics.

Beyond this main effect, there were a few other noteworthy findings from this study. First, relevant to the focus of this book, 23 of the 29 studies (79%) were carried out with children aged 6–14 years; We know less about performance at the extreme ends of the K-12 age range, but our findings provide particularly strong evidence that spatial training may benefit mathematics outcomes in middle childhood and early adolescence. Another age-related finding was that as the age of participants increased from 3–20 years, so too did the effects of spatial training on mathematics performance. This might suggest that advanced mathematics is more reliant on spatial representations than basic mathematics and/or that as children mature, they become more skillful at applying spatial strategies. Moreover, this indicates that spatial skills should not exclusively be prioritised in the early years, but throughout elementary and middle school education. Third, spatial training that involved the use of concrete materials, such as manipulatives and other hands-on learning materials, was associated with significantly larger effects than training offered through computers or worksheets. Although further study is needed, we might tentatively conclude that classroom-based efforts to improve mathematics through spatial instruction may be most potent in hands-on learning opportunities. We return to this idea in the next section.

Lastly, larger improvements in mathematics were observed when the spatial training and mathematics outcomes were closely aligned. That is, larger effects occurred when there was a clear link between the training and the type of mathematics being assessed (e.g., training that targeted improving spatial transformation skills and a geometry assessment involving geometric transformations). This effect was quite pronounced, with effects of 0.65 on 'near transfer' measures compared to an effect size of 0.20 on 'far transfer' measures. While this result might seem fairly obvious, it is actually at odds with the larger correlational literature. For example, a review of this literature suggests that spatial thinking is related to both overtly spatial aspects of mathematics (geometry), but also less obviously spatial aspects of mathematics (basic understandings of number) (Mix & Cheng, 2012; Xie et al., 2019). Thus, based on correlational data, one might predict little difference in whether the spatial training and math outcomes were well aligned. By moving beyond correlational evidence to focus on causal research designs, our meta-analysis suggested that space-math associations may be more dependent on task-specific shared processes and strategies than previously understood. It should also be noted, however, that even when the training and math outcome under question were not clearly linked, there was still a significant effect of spatial training on mathematics. This suggests that spatial thinking might be linked to mathematics through both domain-general as well as domain-specific shared processes, which has important implications for the design and implementation of spatial learning opportunities.

2.5 How to Best Leverage the Space-Mathematics Association?

We now arrive at the crux of this chapter – how to improve mathematics learning through spatial training and instruction. To begin, let us be clear that there is no single 'right' answer to this question, and researchers have approached this question from a variety of perspectives and methodologies. Confusing things further, there are cases in which one approach, such as computerized spatial training, benefits mathematics performance in one study (e.g., Hung et al., 2012), but not another (e.g., Cornu et al., 2019). The same spatial approach does not guarantee the same results across different settings and with different individuals.

With that said, the literature review for our meta-analysis revealed two general approaches that researchers have used successfully to test whether and to what extent spatial training transfers to mathematics. The first approach, referred to hereafter as the 'Isolated Approach to Spatial Training,' involves a general and decontextualized approach to spatial training. The second approach, referred to hereafter as the

'Embedded Approach to Spatial Training,' involves a more math-focused and contextualized approach to spatial training. Importantly, there are some key theoretical differences between these approaches with important implications for practice. In the following sections, we describe each approach in more detail, including an example of each type of training from the literature, discuss the strengths and limitations of each approach, and make connections to the theoretical assumptions underlying each approach.

It is important to note that these two categories are not static. However, our description of each approach represents a generalization of the current literature. Indeed, one of the central purposes of dichotomizing spatial training interventions is to highlight the current benefits and drawbacks of each approach. In doing so, the middle ground is exposed and we can begin to see and appreciate the many unanswered questions that may guide future efforts to improve mathematics learning through spatial instruction.

2.5.1 Isolated Approaches to Spatial Training

Isolated approaches to spatial training are defined here as repeated practice of a specific spatial skill, such as mental rotation, without any attention to the connections to mathematics. The isolated approach usually involves highly controlled experimental studies, often within the confines of a lab, with the intent of manipulating only the key variable of interest (i.e., the spatial skill to be trained). The major strength of this approach is the high internal validity, allowing researchers to make stronger causal inferences about the effect of spatial training on mathematics. For example, in a study by Gilligan et al. (2020), 8-year-olds were trained on either mental rotation or spatial scaling. Training was delivered through instructional videos and involved solving various 'puzzle' tasks. Fig. 2.2 shows an example of each training task.

At no point were children told how these spatial skills might relate to mathematics. Nonetheless, both training tasks shown in Fig. 2.2 led to improvements in spatial thinking, as well as training-related gains in mathematics. Relative to both an active control group and other forms of spatial training, children in the mental rotation group demonstrated gains on an assessment of missing-term problems (e.g., 4 + __ = 7). Children in the spatial scaling group demonstrated gains on an empty number line assessment (e.g., placing 67 on an empty line flanked by 0 and 100 at both ends), relative to the control group and the mental rotation group. These results suggest highly specific effects of spatial training on mathematics. Because the training targeted a single spatial skill, the changes in mathematics performance can reasonably be attributed to the specific skill trained. This in turn, contributes to a more refined theoretical model of the space-mathematics link.

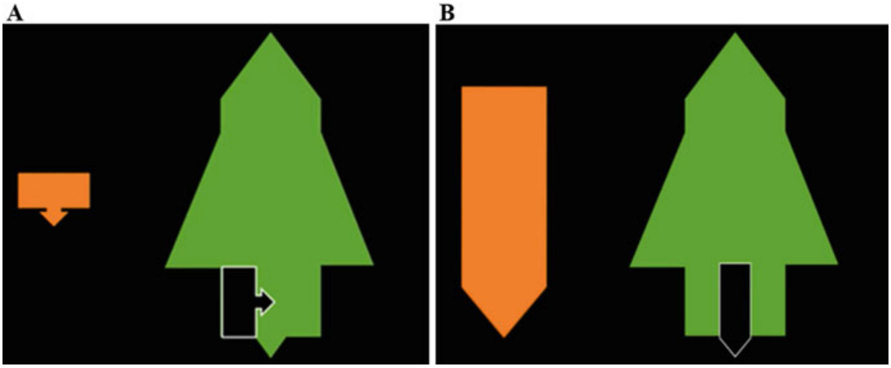

Fig. 2.2 In these activities, participants were asked whether the orange shape could be a rotated or scaled version of the missing piece needed to complete the puzzle shown. Instructional videos demonstrated how the orange shape could be rotated (**a**) or scaled (**b**) to complete the image

2.5.2 Embedded Approaches to Spatial Training

Embedded approaches to spatial training are defined here as spatial training that targets multiple spatial skills and includes more implicit and explicit links to mathematics. Research using the embedded approach usually occurs in the classroom and as part or in replacement of regular mathematics class. Compared to the isolated approach, the classroom teacher assumes a more active role, often facilitating the spatial training. An example of this approach can be seen in the study by Lowrie et al. (2017). Researchers and a group of five sixth grade teachers collaborated to design and implement a 10-week (20 hr) spatial reasoning program in replacement of the standard mathematics curriculum. The intervention program was designed to enhance a range of spatial skills, including spatial visualization, mental rotation, and spatial orientation. Relative to a 'business as usual' control group, those in the spatial group demonstrated improvements in spatial thinking as well as a broad curriculum-based assessment of mathematics (number, geometry, measurement).

A closer inspection of the instructional activities used in the embedded approach reveals its key characteristics. Specific examples included teaching students rotational symmetry, 2D rotation around a point, perspective taking (top, front, side views of 3D objects), nets of solids, and differentiating between rotations and reflections. Critically, although each one of these activities can be considered spatial, these activities are also directly related to the mathematics itself. For example, each of these activities are taught in the Ontario curriculum for students in Grades 5–8 (https://www.dcp.edu.gov.on.ca/en/curriculum/elementary-mathematics). Because of the inherent overlap between spatial skills and mathematics in these activities, we see this type of spatial intervention as one that is embedded in school mathematics. To be clear, the focus and emphasis of this program and other embedded approaches (e.g., see Hawes et al., 2017), is directed at the spatial reasoning inherent in these various mathematical tasks. This approach can also be considered 'spatializing' the mathematics curriculum; taking mathematical content that is spatial in nature or can be approached in spatial ways and using this as a natural training ground for developing students' understanding and skills in mathematics (as well as spatial reasoning). Perhaps the greatest strength of this approach is the clear connection between the content of the spatial training and its applicability to various mathematics curriculum standards. This approach, to date, has capitalized on the natural links between spatial and mathematical thinking, doing so by employing ecologically valid, classroom-based interventions. An interesting question is whether embedded training—which improves both spatial and mathematics skills—would have the same kind of spontaneous transfer to non-spatial mathematics (e.g., addition) than isolated training. If so, this would suggest that even domain-specific training yield general improvements in mathematics. Another open question is whether embedded spatial training that happens to be more domain-general in nature (e.g., training on how to organize and structure/model word problems), transfers widely to domain-specific aspects of mathematics. That is, although embedded training interventions, to date, have focused on spatializing inherently spatial aspects of the curriculum (measurement and geometry), opportunity exists to study how other embedded approaches, such as instruction aimed at improving students' ability to spatially represent mathematics, supports mathematical reasoning.

2.5.3 Strengths, Limitations, and Theoretical Underpinnings

It helps to consider the strengths and limitations of each approach from both a research and practical standpoint. From a research perspective, the closely controlled study designs that are typical of the isolated approach lend themselves more easily to causal inferences. Indeed, randomized controlled study designs, in which a single variable is manipulated, are considered the 'gold standard' when it comes to establishing trustworthy claims about cause-and-effect relations. This research approach also is instrumental in developing solid psychological theories. A major limitation of this approach, at least to date, is that—as the name suggests—it is isolated from both the mathematical content, and the mathematical contexts, such as the classroom, in which mathematical learning takes place. The embedded approach, on the other hand, may lack internal validity but makes up for it in ecological validity. The embedded approach typically occurs within the classroom, with large groups of children, and under the guidance of a professional educator. Moreover, the embedded approach, thus far, has integrated and embedded spatial thinking training within and as part of mathematics instruction. In doing so, the link between spatial and mathematical processes is made more transparent.

This is a critical distinction between approaches and one that has clear implications for practice. While the embedded approach 'spatializes' various mathematics content and makes explicit links between spatial and mathematical thinking, the isolated approach—at least to date—leaves it up to the learner to apply the training to mathematics problems. This difference, in part, may have to do with one's theoretical position on how spatial training transfers to mathematics. For example, a distinction may be drawn here between transfer that occurs due to a shared underlying mechanism vs. transfer that occurs due to strategic recruitment of spatial skills. That spatial and mathematical thinking draw on a common cognitive resource need not involve explicit awareness of how to apply spatial training to mathematics. However, if transfer occurs as a result of intentionally recruiting spatial strategies to solve mathematics problems, then one's awareness of such a connection between training and application clearly matters. We return to this point further below.

Perhaps the most pressing problem to date concerns both approaches. It remains unclear why or how spatial training supports mathematics understanding and performance. Presently, specific theories of change remain scarce, poorly articulated, or more often than not, have to be inferred. For example, if one theorizes that spatial visualization improves mathematics through improving one's ability to organize and model math problems, then one must look to evaluate and elucidate evidence of change at this level. To date, there is little indication that this is happening. Instead, researchers—including ourselves—have established a clear link between 'input' and 'output' without necessarily testing and uncovering the transformations that occur between spatial instruction and mathematical learning. To gain further insights into the space-to-math link requires more detailed analyses of the processes theorized to underly such an association. Practically speaking, this means figuring out the ways in which spatial training changes how one approaches and/or understands the mathematics in question. For example, this research might involve tracking changes in students' problem-solving strategies before and after spatial training. Does spatial visualization training help students in drawing out and diagramming the structure of word problems? Does spatial scaling instruction encourage students to adopt more proportional reasoning strategies? Does spatial composition/decomposition instruction help students see how this sort of thinking can be used to solve and better understand a whole host of measurement problems? The answers to these sorts of questions are critical in better understanding the mechanisms that underpin space-to-math transfer.

2.6 Translating Theory to Practice: Infusing Spatial Training Into Mathematics Teaching

Our analysis of these two approaches exposes a curious gap in the literature. Based on our review, there do not appear to be any studies that fit somewhere in the middle of these two extremes (embedded vs. isolated instruction). Furthermore, many questions remain about the particulars of implementing effective spatial training programs. Practically speaking, what might a classroom-based spatial intervention look like? How might it progress from start to finish? In an attempt to address both of these gaps, we end this chapter by offering a simulated model of a classroom-based

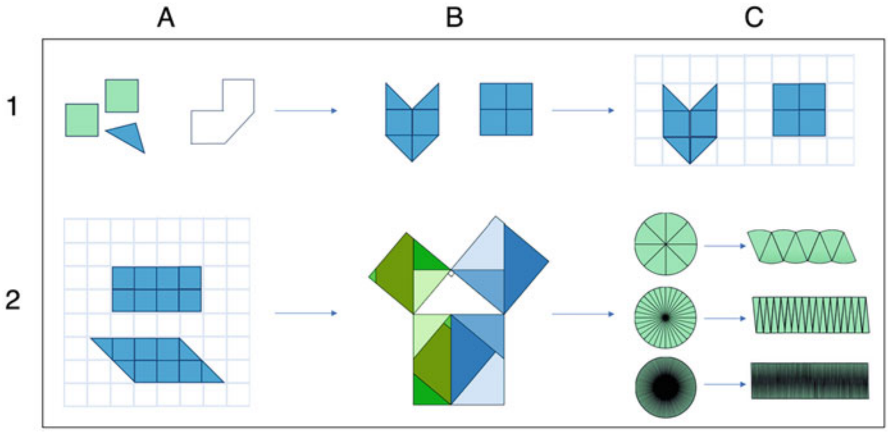

Fig. 2.3 Example of spatial training program in the area of spatial transformations

spatial training program. For this simulation, we target spatial transformation skills, namely composition and decomposition, and present a progression of instruction that starts with an isolated approach to training and continues with increasingly embedded instruction. Fig. 2.3 provides an overview of this progression, illustrating some of the natural connections and affordances that exist between training spatial transformation skills and middle school geometry and measurement.

One of the ways researchers have trained spatial transformation skills is through isolated approaches that involve puzzle-like tasks. Fig. 2.3:1A provides one such example. Here, students are asked to use the three coloured shapes to complete the outlined shape on the right (like a jigsaw). This requires spatial transformation skills, such as mentally rotating the triangular piece into position, spatial translation (imagining sliding/moving the squares into place), and composition (putting all the pieces together to form the whole). Note the similarities of this sort of training to the mental rotation training highlighted earlier (see Fig. 2.1a). Such training is arguably more closely aligned with the isolated approach, in that any explicit connections to symbolic mathematics is absent. This is not to say that this approach is not mathematical. We would argue it is, as evidenced by its presence in various mathematics curricula (e.g., the Ontario mathematics curriculum: https://www.dcp.edu.gov.on.ca/en/curriculum/elementary-mathematics). However, without any explicit connections to formal, symbolic mathematics, this approach remains somewhat isolated. That is, training of this sort (e.g., mental rotation training), is quite general in nature, and for this reason, transfer may be expected to occur across a variety of other domains/skills that also require this general skill (e.g., Science, Technology, Engineering, and Mathematics (STEM) domains more generally). Whether or not the evidence supports such a prediction, remains an open question (for a detailed discussion on this issue see: Stieff & Uttal, 2015; Uttal & Cohen, 2012).

This brings us to the theoretical question of why this relatively isolated approach might transfer to mathematics. As just alluded to, one reason has to do with the underlying relations that exists between spatial skill and mathematics. The argument goes something like this: spatial thinking is fundamental to mathematics. Those with higher spatial skills do better in mathematics. Therefore, improving spatial thinking has the potential to improve mathematics performance. This prediction rests on shared processes between space and math at very general level. Another possibility, and one that has much more to do with specific shared processes, involves strategy use. For some individuals, mere exposure and practice with spatial transformation tasks may be enough to prompt more widespread spatial strategy use during mathematical problem solving via priming existing spatial skills rather than improving them. Such transfer would be expected to occur across mathematical tasks that benefit from spatial transformation strategies. For example, rather than use a purely numeric approach to solve an area measurement problem, some individuals might recognize the usefulness of proving congruence, as one example of many, through spatially transforming the problem. However, for some students, making this leap from isolated training to application—may be too far. Indeed, for some students, especially those with little experience or skills in the area of spatial transformations, isolated spatial training offers an opportunity to build capacity and better master this fundamental spatial skill. Doing so may set up these individuals for later success in solving mathematics problems that benefit from the awareness and use of this skill. That individuals are likely to vary in both their baseline spatial skill as well as capacity to transfer isolated spatial learning to mathematical contexts, suggest the potential usefulness of a more scaffolded approach to spatial instruction.

Indeed, there are a number of ways that spatial transformation instruction can be scaffolded to further reveal its mathematical value. For example, in Fig. 2.3:1B, students might be asked to transform one shape into another. While requiring similar cognitive processes to those used in Fig. 2.3:1A, there is now a need to directly compare areas and (dis)prove congruence (e.g., by simply moving the shapes around, can you prove that they occupy the same area?). Unlike Fig. 2.3:1A, there is also opportunity to integrate spatial and numerical reasoning, recognizing that the square and triangular pieces can be composed into like units to be counted and compared. Surprising to many, students are often quick to approach this sort of question through a pure counting approach, counting the total number of pieces that make-up each shape and reasoning that the shape with the most pieces has the larger area (e.g., see Kamii & Kysh, 2006; Moss et al., 2016). A similar reliance on counting discrete units, rather than reasoning about relative magnitudes, has also been observed for proportional reasoning problems (e.g., see Boyer & Levine, 2015; Boyer et al., 2008). A greater emphasis on spatial approaches to these and similar problems, may help prevent the impulse to immediately resort to rote counting strategies (e.g., see Newcombe et al., 2015).

The introduction of the grid in Figs. 2.3:1C and 2.3:2A provide further opportunity to integrate spatial and numerical strategies and understanding. The grid provides a structural element, allowing students to further see and prove congruence through multiplicative reasoning (i.e., through comparison of arrays). The grid further suggests the need to assign a unit name to the area of the shapes being compared (e.g., square units). In this way, the grid serves as a critical link between more isolated training and training that occurs within and as part of routine mathematics instruction (movement and locations on a grid).

It is at this point—after sufficient practice of proving congruence through composition/decomposition strategies—that students might be introduced to, or reminded of, more standard formulas for solving various area problems. For example, experiencing problems like those presented in Fig. 2.3:1C provides a meaningful space to understand and apply the formula for a rectangle, *Area* = *length* x *width*. Through continued use of spatial transformation strategies and an understanding of how they connect to the formula for the area of a rectangle, students might be encouraged to generalize this knowledge to other shapes (e.g., parallelograms, trapezoids, triangles). For example, in Fig. 2.3:2A, learners can be encouraged to solve congruence through a spatial transformation strategy (decomposing the bottom figure and recomposing it to create a rectangle identical to the one above). This experience lends itself to understanding the formula for a parallelogram, *Area* = *base* x *height*, and its direct and identical relation to the formula for a rectangle, *Area* = *length* x *width.* In these examples, it can be seen how spatial transformation skills, developed and practiced in situ, provide plentiful opportunities to make sense of and ground the mathematical meanings of various measurement phenomena.

The same approach—a combination of spatial and numerical strategies—can be used to structure and give meaning to increasingly complex ideas/proofs, including the Pythagorean Theorem (see inner white triangle in Fig. 2.3:2B). Whether one proves congruence through literally 'squaring' the sides of a right-angle triangle (see Fig. 2.3:2B for squares added on each side of a right-angled triangle) or engaging in various forms of proof through rearrangement (Fig. 2.3:2B offers one such example), spatial transformation skills play a central role. Through engaging in and practicing one's spatial transformation skills in meaningful mathematical contexts, such as this one, students may not only be improving their spatial thinking skills, but also their understanding of the underlying mathematics, and their ability to apply this type of spatial thinking to other mathematical contexts.

One such extension is in proving and understanding the formula for the area of a circle. As shown in Fig. 2.3:2C, one way of making sense of the area of a circle is to relate it back to one's prior knowledge of the area of a rectangle. Through spatially transforming a circle into a rectangle, one can see how the radius and circumference of the circle relate to the height and the width of the rectangle, respectively (for a detailed explanation see the animation: https://youtu.be/YokKp3pwVFc). Similarly, Pi can be better understood through taking the time to physically measure the circumference of a circle (with a string, for example) and comparing (dividing) this distance to the diameter of the circle. These 'spatial' approaches help ground and give meaning to Pi and the formula for the area of a circle. In both cases, powerful mathematical insights can be gleaned through the application of spatial transformations. These examples also highlight some of the many domain-specific relations that exist between spatial thinking and mathematics.

In the progression highlighted here, it has been shown that there are multiple ways in which spatial thinking might be infused into practice in the elementary school classroom. Opportunities to practice spatial thinking exist in isolation and/or embedded into mathematically-focused instruction, highlighting the ways in which spatial thinking can be leveraged and used as an effective tool for problem solving (for a more in-depth discussion see Casey & Fell, 2018). As revealed in the meta-analysis described earlier (Hawes et al., 2022), there may be benefits to both approaches. The training regimens of professional athletes provide a useful analogy for this. There is much to be learned by practicing one's sport in its entirety, but there is also much to be gained by practicing componential skills in relative isolation from the sport. Perhaps, what matters most, is the recognition of the skills and processes most useful to one's domain. When it comes to leveraging space-math connections, we would argue that more needs to be done to recognize this potential. Through developing 'spatial habits of mind,' learners (and indeed instructors) can begin viewing spatial thinking as an integral part of mathematics. Learners might be helped to see how spatial reasoning can be used more widely across mathematics. For example, after sufficient practice of performing spatial transformations in two-dimensional contexts, learners are more prepared to see how spatial transformation skills can be used to solve both one-dimensional/linear measurement and three-dimensional problems. In the end, a spatial approach to mathematics, has the potential to provide students with new entry points into mathematics; new ways of seeing, doing, understanding, and connecting with mathematics.

2.7 Conclusion

In this chapter, we have argued the need to pay more attention to the natural connections and affordances that exist between spatial thinking and mathematics. Arguably, the first step towards this goal is learning to more readily identify spatial thinking, in all its forms, in practice. Although it can be difficult to define, and perhaps even notice at first, spatial thinking is omnipresent in the mathematics classroom. Once aware that such a construct exists, it becomes easier to recognize and label in practice. This is similar to the experience of learning a new word and suddenly hearing that new word on a regular basis. It makes you wonder how you hadn't noticed it in the first place. A similar phenomenon appears to occur with spatial thinking (e.g., see Schwartz, 2017).

However, it is not enough to simply recognize spatial thinking in practice, there is also a need to understand the reasons why spatial thinking is connected to mathematics. In this chapter, we highlighted potential explanations for spatial-mathematical relations. These included shared neural resources, shared domain-general and domain-specific processes, and the active recruitment of spatial strategies during mathematical problem solving. By better understanding why spatial thinking is related to mathematics, teachers are prepared to optimize the space-math link in the classroom.

Moving beyond why, we ventured into the realm of how spatial instruction can be used in the classroom to improve mathematics. The results of our meta-analysis suggest that spatial training has an overall positive effect on mathematics learning. To date, researchers have approached spatial training in two main ways, through an isolated approach vs. an embedded approach. There are likely benefits to both approaches. Consequently, we ended this chapter by walking through a theoretical training program designed to highlight the various connections that exist between spatial transformation skills and middle school geometry and measurement. This simulation aimed to illustrate a progression from an isolated approach to increasingly embedded approaches, highlighting the multiple opportunities to infuse spatial transformation skills into mathematics instruction (specifically geometry and measurement). This is but one of many similar progressions that can be imagined (e.g., the natural connections/overlap between spatial scaling, proportional reasoning, and the broad range of mathematics that involve these skills offers other fertile ground). There is also the open question of whether general, but embedded, approaches to spatial instruction, such as training that involves learning to spatially structure mathematics problems, yields improved reasoning across a wide variety of mathematical tasks where spatial structuring/modeling may assist the learner.

The time is ripe to bring these ideas into the wild; designing, testing, and exploring the varied interventions and forms of instruction that seek to take advantage of the natural connection between space and mathematics. It is also safe to assume that many educators are already taking advantage of the space-mathematics link in their instruction by conveying mathematical concepts through gesture, diagrams, and manipulatives. However, we argue that there remains much untouched potential for spatial reasoning in the classroom, and opportunities exist to further spatialize mathematics curricula, by explicitly leveraging spatial thinking to yield new mathematical skills and understanding.

Teachers and other educational professionals are decidedly the foundational building blocks on which the future success of spatial interventions depend. Indeed, seeking and amplifying the voice of practicing educators as part of the design process is key to developing effective intervention. To close, we are only beginning to understand how to best leverage the space-mathematics link. However, as we have argued throughout, there are many reasons to be optimistic about the future of this undertaking.

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Chapter 3
Understanding the Relationship Between Attention, Executive Functions, and Mathematics: Using a Function-Specific Approach for Understanding and Remediating Mathematics Learning

Emma Clark, Taryn Perelmiter, and Armando Bertone

Abstract Mathematics learning and achievement are integral to academic success and have been shown to predict overall achievement at later grades. Methods for improving mathematics ability are therefore crucial. Domain-general attentional skills are important for the development of mathematical proficiency for typically developing (TD) children and children with neurodevelopmental conditions (NDCs). Despite this, most current methods for remediation are limited to task-specific approaches of targeting and rehearsing specific mathematics skills. Given the evidence supporting the relationship between attentional abilities and mathematics learning and achievement, it is proposed that mathematics difficulties are function-specific and can be remediated as such. In this chapter, the relationship between attentional skills and mathematics learning and achievement in both TD children and those with NDCs will be presented. This literature will serve to contextualize the limited research that has evaluated cognitive training for mathematics remediation; the effectiveness of such cognitive-based interventions will also be evaluated. The chapter will conclude with a discussion regarding practical implications of understanding the role of attention in mathematics learning. This conclusion will aim to inform clinicians and educators about effective identification and remediation of mathematics learning using cognitive-based assessments and interventions. Suggestions for future research and potential cognitive interventions will be discussed.

Keywords Mathematics learning · Attention · Cognitive training · Atypical development · Intervention

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3.1 Introduction

The development and mastery of school-entry mathematical skills are strongly related to later overall academic achievement (Duncan et al., 2007), suggesting early identification and remediation of mathematics difficulties are essential. However, to do so, an accurate understanding of the various factors that contribute to the development of mathematics proficiency in elementary and middle school is required. Commonly referred to as dyscalculia[1], mathematics learning disorders (MLDs) are characterized by a difficulty in acquiring arithmetic skills necessary to solve calculations or problems (Raja & Kumar, 2012). Specific impairments can include individual or combined deficits in number sense, memorization of arithmetic facts, fluent calculation, or mathematics reasoning (American Psychiatric Association, 2013). Specific learning disorders become apparent as children enter elementary school, as their ability to learn foundational skills is impacted (American Psychological Association, 2013). It then follows that proper identification and intervention are integral during the elementary and middle school years.

It is estimated that in the typically developing population (TD), 6–7% of school-aged children present with an MLD (Zentall, 2007). In children with neurodevelopmental conditions (NDCs), the rates of MLDs are almost three times that of TD individuals (Mayes & Calhoun, 2006). For instance, individuals with attention-deficit/hyperactivity disorder (ADHD) experience significant challenges in mathematics learning; in the ADHD population, the rate of MLDs is estimated at 31% (Zentall, 2007). Though there has been recent research into the etiology of MLDs (see Butterworth et al., 2011; Bartelet et al., 2014; Price & Ansari, 2013), there is much less known about numerical cognition and numeracy mastery, compared to our understanding of language acquisition and reading (Kwok & Ansari, 2019; Geary et al., 2011; Gersten et al., 2007; Zentall, 2007). Mathematics difficulties are associated with cognitive deficits that are not explicitly related to numerical processing (Peng et al., 2018). There is substantial empirical literature demonstrating a relationship between various attentional abilities and mathematics proficiency across the lifespan (e.g., Blair & Razza, 2007; Peng et al., 2018). Various types of attentional abilities and related executive functioning skills have been strongly related to mathematics learning across periods of development and clinical populations (Cragg & Gilmore, 2014). For example, inattention, attentional switching, and inhibition have been linked to the development of mathematics skills in early elementary school years (Kindergarten to Grade 3) (Blair & Razza, 2007; Fuchs et al., 2005; Gold et al., 2013). Additionally, studies have demonstrated a functional relationship between working memory and numeracy in 5-year-olds (Kroesbergen et al., 2014) that has been further evidenced in middle childhood-aged samples by neuroimaging research (Dehaene et al., 1999; Kwok & Ansari, 2019; LeFevre et al., 2010; Menon, 2016). The studies mentioned above provide evidence for a relationship between mathematics learning, attention, and executive functions in childhood and thereby indicate that these cognitive skills should be considered in the assessment and remediation of mathematics learning. However, most remediation research centres around mathematics skill-based approaches such as repeated practice, exposure, and feedback (Mong & Mong, 2010; Templeton et al., 2008). While these interventions may be useful, it can be proposed that they are a *task*-specific approach to an arguably *function*-specific problem defined by deficits in attention and related executive functioning skills. Given the importance of mathematics to overall academic achievement (Duncan et al., 2007), and the increased rate of MLD amongst clinical populations, there is a need to understand the cognitive factors associated with mathematics learning to better inform function-specific remediation for both typically and atypically developing individuals (Zentall, 2007).

In this chapter, we will analyze and summarize the key findings of the research that empirically demonstrate the relationship between attention, related executive skills, and mathematics. We will then compare these results to intervention studies and assess whether the mathematics-related components of attention are adequately addressed in mathematics intervention research. Finally, we will assess the results of the interventions qualitatively and argue why they were effective, or not, based on the attentional skill targeted or the outcome measures used. To conclude, we will make recommendations for the future of attention-based interventions for mathematics remediation.

3.2 Attentional Abilities and Mathematics Proficiency

3.2.1 Attention

Broadly defined, attention is the ability to orient and sustain one's focus on stimuli while ignoring distracting information (Tsal et al., 2005). Various aspects of attention, including sustained attention, selective attention, and inattention have been related to mathematical abilities in students from third to eighth grade (e.g., Lindsay et al., 2001; Raghubar et al., 2009; Rueckert & Levy, 1995). For instance, numerosity, a fundamental skill in mathematics, is strongly related to both mathematics achievement and attentional abilities in second-grade children, suggesting a unique contribution of attention to the development of numerosity (Child et al., 2019). Furthermore, the development of attention from 4 to 6 years of age has been shown to contribute to mathematical proficiency above and beyond other vital skills such as visuomotor integration or fine motor coordination (Kim et al., 2018). These two studies provide initial examples of the integral role of attentional abilities in the development of key mathematical skills. While many more will be explored within the scope of this chapter, it is important to identify what is considered attention and related executive functioning skills.

Understanding the relationship between attentional and mathematical ability is complicated because of the often-conflicting operational definitions of "attention" and what are considered as related or distinct executive functioning skills. Sustained and selective attention, working memory, executive functioning, and processing speed are cognitive skills commonly associated with the general term “attention”. Many researchers consider attention, working memory (WM), and executive functioning (EF) to be distinct skills that are related to mathematics ability (Peng et al., 2018; Peterson et al., 2017).

WM can be considered as a mental storage unit that relies on the ability to focus one’s attention and integrate information from several sources, all while inhibiting irrelevant information (Dahlin, 2013). However, these latter processes involved in WM have been described by Miyake et al. (2000) as three interrelated executive functioning skills: shifting, updating, and inhibiting. The successful use of these executive function skills relies on attention being activated.

In addition to shifting, updating, and inhibiting, EF includes processes for self-regulated, goal-directed behaviours including goal identification, planning and initiation, self-regulation, cognitive flexibility, allocation of attention, and using feedback (Anderson, 2002). Attention can be considered a fundamental skill to EF, such that disruption to attention leads to distractibility, impulsivity, forgetfulness, and poor focus (May et al., 2013).

Finally, many studies consider processing speed (PS) as another factor that influences mathematics abilities (Peng et al., 2018). PS has been studied as a stand-alone variable, as well as a consideration within cognitive load theory (Conlin et al., 2005; Geary et al., 2007; Peng et al., 2018;). Cognitive load theory provides one method of conceptualizing the complicated relationship between attention, WM, EF, and PS. This theory proposes that an excess of information leads to inefficient processing of information, as attentional resources are being allocated to multiple pieces of information or processes (Paas et al., 2003). More efficient processing of information frees up cognitive space to perform higher-order functions encompassed by WM and EF (Paas et al., 2003).

Just as there is variability amongst researchers’ beliefs about whether the aforementioned skills are related or distinct, there are two different schools of thought regarding whether executive functions themselves are a set of distinct domain-general skills, or a unilateral skill set. Conceptualized by Miyake et al. (2000), one school of thought proposes that WM, EF, PS, and attention are distinct but related skills whose success depends on a top-down approach. That is, WM is a mental workspace that is integral to mathematical processing (Raghubar et al., 2010). The successful use of that workspace relies on intact executive functions, namely shifting, updating, and inhibiting (Miyake et al., 2000). Lastly, these skills require efficient information processing as well as attentional activation and control. Thus, the role of various related cognitive factors involved in mathematics relies initially on attention. This conceptualization proposed by Miyake is widely cited and accepted in the literature. However, it’s main critique is being an adult-centric framework by those proposing a more unilateral framework of EF in childhood (ex. Doebel, 2020; Garon et al., 2008; Karr et al., 2018). Given our interest in exploring the effects of general attention and the implications it has to math learning, our overview aligns most closely with that of Miyake et al. (2000). However, it is important to highlight the alternative school of thought proposed by Doebel (2020), that is contextualized to a greater extent within a development framework.

Doebel (2020) counters Miyake et al. (2000) by proposing that the development of executive function should be considered as related skills that emerge and coordinate in pursuit of specific goals. There is some research to suggest that the EF system changes as a function of development, with younger children showing less differentiation between skills, suggesting a unitary construct (Bull & Lee, 2014). Karr et al. (2018) suggest that a unidimensional approach is most appropriate for understanding EF in childhood particularly regarding isolating the skill of shifting (Karr et al., 2018). Along these lines, Garon et al. (2008) are also in agreement with a unilateral framework for conceptualizing EF in early childhood and proposes that the development of all EF can be attributed in part to development of the attention system.

The research exploring the relationship between these cognitive skill sets, and mathematics does not always adhere to one framework of EF conceptualization. Thus, for the purposes of isolating and exploring various EFs and the relationships to mathematics skills, we will explore them in separate sections, adhering most closely to Miyake's conceptualization. Following this top-down conceptualization, we will first explore the relationship between mathematics and attention (Sect. 3.2.1), followed by EF, WM, and PS (Sects. 3.2.2, 3.2.3, and 3.2.4, respectively).

3.2.1.1 Attentional Abilities with Mathematics Learning Profiles

Examining the attentional abilities of children with advanced mathematics skills, as well as those with MLDs, provides further insight into the math-attention relationship. Mathematically gifted adolescents (11- to 15-years-old) show significantly faster responses with fewer errors on a selective attention task when compared to typically developing (TD) peers (Rueckert & Levy, 1995). Conversely, compared to TD peers, fifth- to eight-grade students with dyscalculia performed significantly worse on a clinical task of sustained attention (Lindsay et al., 2001). Specifically, students with dyscalculia made more omission errors and demonstrated more inconsistency in reaction times (Lindsay et al., 2001). Dyscalculia has also been associated with elementary school children making more errors on sustained attention tasks even when compared to individuals with ADHD (Kuhn et al., 2016). Inattention has also been related to the severity of mathematics learning difficulties in Grades 3 to 4, with students with MLDs being rated as more inattentive than their low-achieving peers (Raghubar et al., 2009).

3.2.1.2 Mathematical Abilities Within Attentional Profiles

Further evidence for the relationship between attention and mathematics learning comes from assessing the mathematical abilities of individuals with impaired attention. Children in Grades 1–5 diagnosed with ADHD, without a comorbid learning disability, still demonstrate significant mathematics difficulties despite having intact numerical knowledge (Colomer et al., 2013). Elementary school children with

ADHD have been observed to perform significantly lower than TD peers on standardized tests of mathematics (McConaughy et al., 2011). There is potential for a profound impact on future achievement if these children do not receive early intervention. This is evident within ADHD samples, where the proportion of children presenting with moderate to severe difficulties with mathematics increases with age (Colomer et al., 2013). However, for these children with ADHD, the nature of the difficulties shifts with age from counting errors and number dictation (Grades 1 to 2), which rely on automaticity, to more procedural skills such as mental calculation and counting (Grades 3 to 5) (Colomer et al., 2013).

Even within samples of children with impaired attention, the severity of attentional difficulties has an inverse relationship with mathematics performance. Third and fourth-grade students rated as more inattentive have been observed to answer fewer multi-digit computation questions correctly, perform more mathematics fact errors, and commit more procedural errors (Raghubar et al., 2009). Comparably, middle and high school students with lower ADHD-inattentive symptoms demonstrate increased mathematics achievement (Mattison & Blader, 2013). Ross and Randolph (2016) argue that the mathematics deficits exhibited by 7- to 11-year old children with ADHD can be accounted for by their task vigilance, specifically how easily they are distracted and how well they can disengage from distractions and return to tasks. Yet another example of the inverse relationship between attention and mathematics comes from a sample of children with Mild Intellectual Disability (MID). Djuric-Zdravkovic et al. (2011) assessed mathematics and attention in a sample of 12- to 14-year-old students with MID and reported that impaired mathematics learning was accounted for by difficulties with sustained attention, above and beyond what would be explained by general cognitive impairments.

Thus, whether you examine the attentional abilities of individuals with distinct mathematics learning profiles, or the mathematics abilities of those with attention difficulties, the mathematics and attention relationship exists. This is further supported by relationships among various attentional abilities, such as sustained and selective attention, inhibition, and the development of mathematics skills in TD children. The argument in support of the function-specific mathematics difficulties becomes stronger when considering the relationship of cognitive skills that rely on attention activation, such as executive functions.

3.2.2 Executive Functioning

The perceived influence of multiple EFs on mathematical ability is robust, with planning, updating, and inhibition explaining 45% of the variance in childhood numeracy in 5- and 6-year-olds, after controlling for intelligence quotient (IQ) (Kroesbergen et al., 2009). Poorghorban et al. (2018) even observed that high and low mathematics achieving fourth-grade students did not differ in their performance on sustained attention, but rather the difference in mathematics performance was accounted for by EF, specifically shifting. Compared to the *skill*-specific relationship of WM to mathematics skills (e.g., visual WM and magnitude estimation), Peng et al. (2018) conducted a meta-analytic review that identified more general EF deficits that have a global impact on mathematics learning for individuals with mathematics difficulties across the lifespan. Difficulties with inhibition and switching were identified amongst third-grade students with mathematics difficulties by Bull and Scerif (2001). Specifically, individuals with lower mathematical ability demonstrated significant difficulty inhibiting a learned strategy in favour of a new one (Bull & Scerif, 2001). Furthermore, these children were more impaired in their ability to inhibit prepotent responses (Bull & Scerif, 2001). This difficulty was thought to be accounted for by more irrelevant information being held in WM, and a lower WM span (Bull & Scerif, 2001). In addition, domain-general cognitive skills (i.e., inhibition, attention, and WM) were predictive of mathematics skills longitudinally and predicted growth in mathematics in a sample of children aged 3- to 4-years-old (Coolen et al., 2021). This suggests that executive skills serve as a predictor for foundational mathematics skills, before the onset of formal education. Aside from inhibiting and updating, attentional switching is a complex EF skill often impaired in individuals with neurodevelopmental conditions.

May et al. (2013) consider *attentional switching* to be integral to mathematics abilities. In a sample of 7- to 12-year-old children with an Autism Spectrum Disorder (ASD), they found that performance on an attentional switching task explained a significant amount of variance in mathematics achievement (May et al., 2013). A similar effect of attentional switching is observed in non-clinical samples, such that high mathematics achieving fourth-grade students perform significantly better on tasks of attentional shifting than their low mathematics achieving peers (Poorghorban et al., 2018). A meta-analytic review of the literature further supports the role that attentional shifting plays in mathematics ability, with an effect size of .33 (Yeniad et al., 2013). Alternatively, Bull et al. (2008) propose that skills such as attentional shifting, inhibition, goal planning, and monitoring influence learning more generally, as opposed to being specific to math. However, they do report that in early elementary grades, inhibition, planning, and monitoring provide some predictive power in addition to visual short-term memory (Bull et al., 2008). In contrast, the ability to shift attention may be critical in solving more complex mathematics problems in later elementary school (Bull et al., 2008).

A final EF related to mathematics achievement in middle childhood is *planning*: the mental representation of problem-solving to reach a goal (Friedman et al., 2014). Furthermore, better planning abilities are predicted by stronger short-term memory, sustained attention, and inhibition (Friedman et al., 2014). This exemplifies the complex relationship between many different cognitive domains that can be considered under the term of attention.

While general function-specific deficits in EF are related to general mathematics deficits, there are also distinct connections between distinct EFs and task-specific deficits, particularly in problem-solving. This point brings us back to the debate as to whether or not EF can be considered as a domain-specific skill set or a unilateral skillset within childhood. The unilateral framework aligns well with the first argument that general deficits in EF are related to general deficits in mathematics.

However, considering EF as a set of distinct skills leads us to draw more nuanced conclusions between specific EF deficits and impacted mathematics skills. These specific relationships can help support intervention development, discussed later in this chapter. For example, attentional shifting or switching is related not only to general achievement in both typically and atypically developing populations, but also specifically to complex problem-solving skills. Additionally, function-specific deficits in inhibition result in task-specific difficulty with response inhibition and strategy selection during problem-solving. Finally, planning is also explicitly related to problem-solving. The development of one's planning abilities is directly connected to adequate ability in other attentional skills, demonstrating the interrelation of these cognitive processes. Thus, whether in global learning or specific skill development, EF skills are integral to the development of mathematical proficiency. Most importantly, attentional activation and successful use of executive functions support WM, the mental workspace essential for mathematical processing.

3.2.3 Working Memory

General mathematics competence requires the knowledge and flexible application of skills and procedures in different mathematical contexts (Raghubar et al., 2010). Furthermore, math-based problem-solving is dependent on the ability to hold and process information (Raghubar et al., 2010). Thus, WM skills are required to support the fundamentals of many mathematics tasks throughout life (Raghubar et al., 2010). In fact, visual WM has been shown to serve as a significant predictor of mathematics ability in 8- to 16-year-old students, even after controlling for reading and attentional difficulties (Peterson et al., 2017).

An in-depth exploration of the literature suggests the relationship between WM and mathematics ability is complex, with WM's impact on mathematics ability targeting specific mathematics skills only, as opposed to having a global effect on mathematics learning. For instance, Bull et al. (2008) noted that in TD school-aged children, visual short-term memory (VSTM) significantly predicted mathematics achievement, and by the end of third grade, visual WM was a unique predictor of mathematics skills. They suggest that the ability to represent visual-spatial information in WM is integral for non-numerical skills such as estimation and visualization of magnitude (Bull et al., 2008). This is as opposed to other mathematics skills, such as numeracy.

Further evidence for this complicated relationship comes from studies with participants presenting with atypical mathematics or attention abilities, or both. Elementary-school students with dyscalculia perform poorly on tasks of verbal, visual, and numerical WM (Kuhn et al., 2016; Peng et al., 2018). Peng et al. (2018) also noted that WM deficits seemed to be more strongly associated with calculation and comprehensive mathematics difficulties, as opposed to word problem-solving difficulties. Specifically, subtraction skills seem to be particularly impaired in elementary school students with dyscalculia and attention difficulties, suggesting the shared role of inattention and WM in solving subtraction problems (Kuhn et al., 2016). Geary et al. (2007) discovered central executive deficits as a core component that characterized early-elementary school students with mathematics learning disabilities across tasks but noted the phonological loop and visuospatial sketch pad as contributing to more specific deficits; estimation, addition, and counting (Geary et al., 2007). Additionally, deficits in visual WM result in task-specific difficulties with mathematics skills and non-numerical skills such as magnitude estimation for kindergarten and elementary school children (Geary et al., 2007; Raghubar et al., 2010).

Acknowledging the influence of many EFs in mathematics skills, including the important role of visual, but not verbal WM in mathematics processing, Szűcs et al. (2014) proposed an "executive memory function centric" (p. 518) model of mathematical processing in 9-year-old children. This model encompasses selective attention, shifting, updating, monitoring, and inhibition to be central executive memory processes involved during arithmetic tasks (Szűcs et al., 2014). Scűcs's model fits well with the framework of conceptualizing EF in childhood as unilateral. However, this poses challenges when considering remediation, as interventions would have to address each of these central executive memory processes, as opposed to potentially isolating just one.

As we have seen above, each distinct skill relies on attending and successful skill development and use for developing mathematical proficiency. Before exploring intervention, it is important to consider PS, a final cognitive skill related to attention that can impact attention, WM, and EF.

3.2.4 Processing Speed

Processing speed refers to the efficiency with which an individual perceives and processes information and produces a response (Forchelli et al., 2022). One possible explanation for the role of processing speed in mathematical learning is provided by the bottleneck theory. The bottleneck theory states that high-level cognitive skills such as WM and EF are supported by low-level cognitive skills such as processing speed and that deficits in the latter can constrict the information flow necessary for high-level processing during mathematics tasks (Peng et al., 2018). For example, for 7- to 9-year old children, performance on memory-based tasks is most severely impaired in situations where processing activity demands attentional resources (Conlin et al., 2005). This theory is one proposed explanation for the role of *processing speed* impacting all domains of mathematics proficiency. In a sample of 8- to 16-year old students with mathematics difficulties (MD), reading difficulties (RD), and ADHD, processing speed contributed to the relationship between mathematics and attention (Peterson et al., 2017), and has been noted to be a salient cognitive deficit in students with mathematics difficulties (Peng et al., 2018). Furthermore, processing speed deficits were observed across ages and different types of mathematics difficulties, indicating a possible fundamental cognitive correlate to all mathematics difficulties (Peng et al., 2018). The impact of more efficient processing of mathematics problems in the allocation of attentional resources has also been explored in cognitive neuroscience. It has been observed that in seventh-grade students, simple exposure to a "+" sign is related to increased activity in the hippocampus, a brain associated with spatial attention (Mathieu et al., 2018). It is suggested that this elicits a priming effect, resulting in more efficient processing of subsequent information (Mathieu et al., 2018). More efficient processing implies that more cognitive resources can then be allocated to incoming information, mental manipulation, execution of operations, and problem-solving.

Evidentially, function-specific deficits in PS have the potential to impact general mathematics learning and achievement and have task-specific relationships to mathematics processing and fluency. Clearly, many cognitive skills related to attention are involved in mathematics learning and performance. The bottleneck theory is one explanation for the relationship between processing speed to previously mentioned skills (attention, WM, EF). In the next section, Cognitive Load theory will be discussed as another framework that integrates attention-related cognitive skills in a way that explains the dynamic interplay between attention-related skills and how they can be considered harmoniously in mathematics learning.

3.2.5 Cognitive Load Theory

Cognitive load theory proposes that individuals have a limited WM capacity (Paas et al., 2003). The less automatic the processing of a given task is, the more attentional resources one has to allocate to attending to and processing of information during task completion, thus using valuable cognitive space that would otherwise be used for holding, manipulating, or integrating information (Paas et al., 2003). This theory integrates the many functions accounted for by the aforementioned attention-related cognitive skills (e.g., attention, WM, EF, and PS), and can help explain their interrelated contribution to mathematics abilities. For example, 13- to 17-year old students with ADHD demonstrate a deficit in automatized retrieval of mathematics facts when their mathematics fluency is assessed (Zentall, 1990). In turn, their efforts to retrieve these mathematical facts consume already limited attentional resources, further impacting their mathematical performance (Zentall, 1990).

John Sweller is a pioneer of cognitive load theory and has informed the direction of research and application since coining the term. In this section, we will explore some of his early work that influenced instructional design changes that we continue to see today. Then, we will briefly review recent research that supports the role of cognitive load theory in mathematics learning. Tarmizi and Sweller (1988) propose that automaticity, and thus, cognitive load is significantly reduced by schema acquisition. A schema is a cognitive construct that enables someone to recognize a problem as belonging to a specific category and thus requiring specific steps to a solution (Tarmizi & Sweller, 1988). However, the development of these schemas relies on attentional regulation. Furthermore, it is not just an individual's inherent attentional abilities that influence mathematics performance. The way that educators present material to elementary-school students can considerably change the attentional demands of the task, suggesting that instructional design can greatly influence one's achievement (Bobis et al., 1993; Tarmizi & Sweller, 1988). They argue that traditional problem-solving with well-defined goals (e.g., solve for angle A) imposes a high cognitive load because it requires attentional splitting between multiple sources of information, integration of different operations and strategies, and maintenance of a goal state (Tarmizi & Sweller, 1988). The addition of a clear end goal forces the student to maintain this goal state while backward planning necessary steps and calculations (Tarmizi & Sweller, 1988). Thus, compared to open-ended problem solving (e.g., solve for as many angles as possible), what appear to be straightforward problems with clear end goals actually impose a heavy cognitive load (Tarmizi & Sweller, 1988). This then interferes with effective schema acquisition that would facilitate more fluent problem-solving skills (Tarmizi & Sweller, 1988). Additionally, providing students with redundant information to process, or presenting information in a way that physically separates information thus requiring split attention and mental integration of information, also increases extraneous cognitive load (Bobis et al., 1993). While the justification for providing this information is often to improve comprehension, it is argued that the demands on cognitive load eliminate any potential comprehension benefits (Bobis et al., 1993).

Tarmizi and Sweller (1988) discovered that removing goal states, reducing the amount of extraneous information given, and providing a worked-example with numerically ordered steps resulted in significantly more problems being answered when compared to control groups. Therefore, while the intrinsic complexity of a task cannot be reduced, how educators present material can significantly impact cognitive load (Tarmizi & Sweller, 1988; Bobis et al., 1993). In fact, changing instructional design flaws results in faster processing and fewer errors (Bobis et al., 1993).

Cognitive load theory has recently been used to explain more efficient problem-solving skills in second- to fourth-grade students with strong calculation abilities (Watchorn et al., 2014). Specifically, a relationship was observed between attentional flexibility, calculation skill, and the use of superior problem-solving strategies, but only for students with strong calculation skills (Watchorn et al., 2014). The proposed explanation is that better computation skills result in more attentional resources being available for evaluating and selecting more efficient strategies (Watchorn et al., 2014). There are also examples of instructional design being altered to reduce the demands on a child's cognitive load, resulting in improved performance in fourth grade and middle school students (Gillmor et al., 2015; Yung & Paas, 2015).

Given the evidence discussed above, it seems clear that the cognitive domain of attention plays a substantial role in mathematics learning and performance. Whether broken down to relate specific cognitive skills to components of mathematics or integrated with theory to account for its general contribution to math, the relationship is clear. However, this strong relationship is not being addressed in the literature on mathematics remediation. The majority of current remediation methods take a task-specific approach of targeting specific mathematics skills, as opposed to a function-specific approach that addresses underlying cognitive skills, such as those related to attention.

3.3 Intervention for Mathematics Remediation

A systematic approach was used to retrieve articles considered in this chapter. The studies reviewed were identified by searching electronic databases from the field of education and psychology (PsycINFO and Education Resources Information Center (ERIC)) and by conducting a backward search of relevant articles. Searches were limited to peer-reviewed materials. Two separate searches were conducted and are detailed below.

To gather literature on interventions, the following keywords were used to search PsycINFO: Mathematics OR Mathematics Education AND Remedial Education OR Intervention OR School-Based Intervention AND Attention. This search provided 7 results. In ERIC, the keywords Mathematics AND Attention were entered within the field of descriptors (SU) and the keyword Intervention with no specified search field was added. With the limit of <elementary education>, <elementary and secondary education>, <middle schools>, and <primary education>, the search produced 38 papers.

For inclusion in this review, studies had to report on the relationship between sustained attention, selective attention, ADHD symptomology, executive functioning, working memory, processing speed, cognitive load, and mathematics learning or achievement. Correlational and experimental studies, as well as meta-analyses, were retained. Studies assessing populations with atypical mathematics or attentional skills were also included. Intervention literature needed to have mathematics remediation as a noted goal to be considered in this review. The following exclusion criteria were also used: (a) self-monitoring or meta-cognitive literature; (b) studies of teacher knowledge and characteristics; (c) progress-monitoring (response to intervention - RTI) program evaluation literature. After reviewing the titles and abstracts of all papers, 73 papers were retained based on the above criteria. After reviewing these 73 papers in more depth, a total of 34 papers were retained for the review of the relationship between attention and math, and 10 on mathematics remediation.

3.3.1 Task-Specific Intervention and Remediation

Despite overwhelming evidence for the role of attention in mathematical ability, many remediation methods are focused on mathematics instruction, as opposed to addressing the underlying cognitive mechanisms. Mathematics fluency has often been the target of intervention to improve mathematics proficiency. For instance, a program called Great Leaps was used as a supplemental intervention in addition to the daily mathematics curriculum for a small sample of 6 s and third-grade students with various intellectual, learning, and attentional difficulties (Jolivette et al., 2006). The sessions were brief, individualized instruction of single-digit basic operations, and as the students' fluency improved the tasks became more challenging. While three students made gains in their fluency for oral addition, only one student progressed to written fluency and began the oral subtraction component of the program. The other two students showed improvements in addition fluency however, they did not progress further in the program (Jolivette et al., 2006). It was noted that one of the students who did not progress to the next stage had significant attentional impairments that interfered with the one-on-one instruction. The authors noted that "attentional difficulties during any of the sessions would negatively affect student fluency performance" (Jolivette et al., 2006, p. 389).

Other evidence-based interventions, such as integrated dynamic representation (IDR), pose promising results for mathematics remediation for 6- to 9-year old students (González-Castro et al., 2016). While this intervention model appears to be math-knowledge-based, its instructional process addresses many components of attention. The IDR intervention involves a multilevel process carried out on a computer program that presents visual representations of concepts, links, questions, and processes that the participant must work through to reach the final solution of a situational mathematics problem. IDR attempts to remediate mathematics competencies and problem-solving skills using fragmented comprehension, representation, and integration of sets of representations (González-Castro et al., 2016).

However, a closer examination of the levels of representation of IDR reveals how cognitive factors may explain its success. For instance, key concepts (relevant information to solve the problem) are presented in circles, eliminating the demands of selective attention to orient to relevant information while ignoring distractors. There are pictograms that children can select to signify an operation to complete or a relationship between two key concepts. For example, a whole circle around a key concept to suggest adding, whereas a circle with a dotted line reflects subtraction. This reduces the amount of information to be held in WM. Ultimately, the fragmentation and reintegration of information in IDR reduce the demands of many attention skills, thus overall cognitive load. As such, more attentional resources can be allocated to solving the operation or word problem. Though IDR was effective in short-term remediation of mathematics competencies for children 6–9 years of age with MLD, ADHD, and MLD+ADHD, the most significant improvements were noted for children with MLD, followed by the comorbid group, and finally the ADHD group (González-Castro et al., 2016). We argue that the results can be accounted for by the degree to which cognitive strategies were made explicit. Specifically, mathematics knowledge was taught through the use of strategies that alleviated the demands of attentional skills; however, they did not directly intervene or train these skills. While IDR is a step in the right direction in addressing cognitive factors contributing to mathematics learning, the gains are likely to be short-lived without cognitive training.

3.3.2 Function-Specific Intervention and Remediation

As discussed, the research on cognitive interventions for mathematics learning difficulties is limited and often inconsistent. It is important to note that there is extensive literature on cognitive training in general, and while some studies report academic benefits, the goal of their interventions was not mathematics remediation (Peng & Miller, 2016). While they provide evidence for the effectiveness of cognitive training and thus justification for its use in remediating math, this body of literature is not included in this review, as intervention goals were not math-based. Rather, we will explore the small body of research on cognitive interventions with the specific goal of improving mathematics proficiency. We will also examine some studies where it can be argued that cognitive skills were indirectly targeted in attempts to improve mathematics performance. As all intervention studies are explored below, it is important to note that there is a significant gap in cognitive studies that explicitly target sustained attention. Intuitively, one would presume that successful use of processing skills such as EF and WM rely on one's ability to orient and sustain their attention to a task. We then must presume that the results are based on samples of students with relatively intact sustained attention skills. Alternatively, we could assume that results may have been more robust if interventions addressed sustained attention as well. This is certainly an area for future direction and will be discussed again later in this chapter.

Another important consideration before reviewing this research is revisiting the previously discussed debate in the field as to whether EF in childhood should be considered as a set of distinct skills, or a unilateral skillset that works together in pursuit of a goal. As we will see below, the existing literature assessing cognitive interventions for mathematics remediation do not systematically target each component of EF in the way that was previously outlined. In fact, the review did not return any studies that explicitly targeted distinct EF skills. As such, the sections below include studies that claimed to target attention-related skills and WM. This is perhaps a limitation on the existing research, and an area for future exploration that would provide significant knowledge to clinicians and educators working with these students.

3.3.2.1 Indirect Intervention of Attention-Related Skills

Though there are instances of successful cognitive-based interventions specifically for math, it can be argued that the cognitive skills were often addressed within the context of a mathematics task, and not directly. For instance, fourth-grade girls at-risk for ADHD showed decreased off-task behaviours and improved problem-solving when given a mathematics task that had keywords highlighted for them (Kercood et al., 2012). The argument for this intervention was based on optimal stimulation theory and previous evidence that 8-year-old male students with ADHD benefited from having operation signs highlighted (Kercood & Grskovic, 2009).

While the intervention used tools to address cognitive factors such as selective attention, the approach did not facilitate any training or development of attention; instead, it aimed to remove a barrier. In neglecting to train the cognitive skill of selective attention, there are unlikely to be long-term gains to the intervention that will transfer to future academic experiences or translate to real-world applications.

Kang and Zentall (2011) proposed a similar means of intervention, where they increased the stimulation of a task by adding novelty to relevant features through graphic representations, aiming to increase mathematics performance for second- to fourth-grade students with ADHD. Though one could argue against such intervention considering redundancy and split-attention effects (Bobis et al., 1993), the level of intensity was manipulated solely through visual input (e.g., contrast, shadows added to geographical shapes to provide light-source information) therefore not increasing cognitive load. It was noted that second- to fourth-grade students with diagnoses of ADHD and with at-risk inattentive behaviours performed significantly better than controls when questions were presented with high visual intensity (Kang & Zentall, 2011). However, similarly to Kercood et al. (2012), this study addressed cognitive factors related to mathematics performance in elementary school years without explicitly training any specific skill. Therefore, while this supports the effectiveness of cognitive-based interventions for remediating mathematics, these studies did not go further to evaluate more specific cognitive training. Given that children cannot manipulate or change the cognitive demands of real-world math-based tasks, it is logical that we again consider the potential of cognitive interventions to support cognitive load management.

3.3.2.2 Direct Intervention on Attention

As previously discussed, the intervention literature targeting sustained attention is limited. One exception to this is Barnes et al. (2016), who attempted to compare the effectiveness of task-specific mathematics interventions with and without attentional training (vigilance, switching) in very low mathematics performing pre-kindergarten students. They discovered that attention training provided near-transfer effects to improved attention but did not show far-transfer effects to math, as the intervention groups that received mathematics and attention training did not differ from the math-only group (Barnes et al., 2016). However, it is important to note that the improvements in attention that they noted were small (Barnes et al., 2016). The attention intervention was very low in intensity, with students only training once a week for 8 min, which likely accounts for the minimal to null effects of this intervention and is a limitation recognized by the researchers (Barnes et al., 2016).

Paananen et al. (2018) addressed the question of sustained attention in their Maltii intervention for grade one to six students that targets EFs such as inhibition and attentional control. However, their methods for training sustained attention was through a token system that provided external reinforcement and motivation for on-task behaviour. It is known that once external motivators are removed, then intrinsic motivation decreases (Bénabou & Tirole, 2003), calling into question the long-term efficacy of this cognitive intervention. Furthermore, while the results indicated significant improvement for math, the effects were only seen in basic skills such as fluency, and not for more complex operations. One explanation for this is that post-testing was completed directly following the intervention; however, as previously discussed, the transfer effects of cognitive training to academic performance should increase over time. Therefore, it is possible the transfer to more complex skills would be observed at a later post-test. This is particularly true of classroom and testing environments where children are expected to regulate their attention over a prolonged period.

Aligned with Miyake et al.' (2000) conceptualization of EF, one needs to be able to effectively control their attention to adequately use WM and EF skills. The lack of research exploring the academic impacts of training attention is concerning, and has great potential to inform school-based interventions for elementary and middle school students. While we will continue to explore the interventions that target working memory, let us not forget that they may not be fruitful if the child has limited sustained attention capabilities.

3.3.2.3 Direct Intervention on Working Memory

Compared to the indirect studies explored earlier, transferrable gains are more likely when explicit strategies are taught, or when training is specific to cognitive skills. For example, embedding explicit instruction on WM strategies within a third-grade mathematics class (mean age = 8.5 years old) led to a significant increase in WM abilities, on-task behaviour, and therefore increased exposure to learning (Davis et al., 2014). While successful in training WM within the context of mathematics instruction, there was no explicit measure of pre- and post-test mathematics, making conclusions about the academic effectiveness speculative.

Dahlin (2013) answered some outstanding questions in a more individualized and targeted approach to WM training for 9- to 12-year-old children with ADHD and saw significant improvements in mathematics ability immediately and at short-term follow-up for both males and females. For males, the positive effects were still present at the 7-month follow-up. The authors speculate that WM training resulted in increased activity in the prefrontal cortex, and this increased their efficiency of processing and ability to focus (Dahlin, 2013). Though not explicitly stated, the results and interpretation of this intervention can be best understood by cognitive load theory, which integrates the many aspects of attention described earlier. Specifically, it can be speculated that more efficient processing and ability to focus means that more attentional resources can be allocated to learning new information within the classroom context. As there is increased exposure to learning, this will have a cumulative effect over time, such that more advanced concepts can be learned, thus test scores improving at longer-term follow-up.

Interestingly, whether or not WM training is domain-specific (numerically-based) or domain-general does not appear to impact the transfer to mathematics ability (Kroesbergen et al., 2014). For example, two groups of kindergarten children (mean age = 5.78) played games intended to increase WM (one number-based, one domain-general), and both demonstrated significant improvements in WM and numeracy skills compared to controls (Kroesbergen et al., 2014). The lack of difference between the two experimental groups suggests the unique influence of WM, as opposed to numerical WM or exposure to numbers during cognitive training. Despite promising results, this intervention did not take into account individuals with attentional or mathematics difficulties, thus cannot be generalized to these populations. Furthermore, they did not include follow-up assessments to determine the long-term effects of cognitive training. These findings are interesting when compared to Ramani et al. (2017), who targeted the improvement of 5- to 7-year-olds' foundational mathematics skills by utilizing two approaches: a domain-specific training of numerical knowledge and a domain-general training of WM. The results demonstrated that providing children with training on an iPad in both domain-specific knowledge and domain-general skills led to enhanced numerical magnitude knowledge, thus providing evidence for the conceptual framework of mathematics development proposed by Geary and Hoard (2005). These gains in numerical magnitude knowledge are proposed to support their mathematical proficiency through middle childhood. According to this framework, mathematical achievement depends on conceptual understanding and procedural knowledge, which are both supported by an array of cognitive systems, such as attention, inhibition, language, and visuospatial systems.

Finally, more intensive intervention has demonstrated long-term transfer effects to mathematics (Holmes et al., 2009). Children in middle childhood (8- to 11-years-old) who completed 35 min of WM training, for at least 20 days, over 5–7 weeks, showed marked improvements in mathematics performance at a 6-month follow-up (Holmes et al., 2009). Interestingly, these students did not show improvement in mathematics at immediate post-testing. However, it is argued that the effects of cognitive training for supporting learning take time to show significant improvement on standardized tests (Holmes et al., 2009). Thus, to have a more robust understanding of the effects of training-based intervention studies, there is a need for longer-term follow-up testing, as false conclusions could be drawn from only looking at immediate post-test. Such testing should be included in all intervention studies, not just cognitively based ones, to allow for meaningful comparisons of effects.

While the interventions explored throughout Sect. 3.3.2 demonstrate potential for the utility of cognitive interventions to address mathematics learning difficulties, there is notably a substantial gap in addressing one cognitive skill: sustained attention. Intuitively, and in line with Miyake et al.' (2000) conceptualization of EF, higher-order attentional abilities such as WM and other EFs rely on one's ability to orient and sustain their attention to information over a prolonged period, especially in the classroom.

Considering the interventions discussed above, it is evident that certain elements of a cognitive intervention increase the likelihood of success and should be clearly identified to develop successful interventions for clinicians and educators. Specifically, interventions should directly target and train a cognitive skill and the training should be relatively rigorous and intensive. Furthermore, the training should ultimately result in more efficient processing of information and thus overall reduced cognitive load during mathematics tasks. The latter assumption rests on the success of cognitive load theory in not only conceptualizing the relationship of many attentional skills to mathematics learning and performance but also in explaining the success of the few cognitive interventions.

3.4 Discussion

The relationship between attentional abilities and mathematics learning is supported by empirical studies. Despite the nuanced definition of terminology, whether analyzed as domain-general attentional abilities or broken down into parts, there is overwhelming support for the contribution of attentional cognitive skills to mathematics proficiency in children. A large body of empirical work has demonstrated the unique and shared contribution of attentional control, selective attention, sustained attention, WM, EFs, and PS to performance on school-based mathematics tasks. Despite the evidence, the majority of literature and clinical practice still utilize a task-specific approach to remediation. However, with the awareness of mathematics difficulties in ADHD samples, as well as the attentional difficulties seen in mathematics LD profiles and conversely, the superior attentional abilities of individuals gifted in math, it seems the remediation literature is neglecting to address underlying cognitive factors. In other words, there is a trend of using task-specific solutions to a function-specific problem. It can be argued that the risk in this approach is that a task-specific solution is a surface-level, short-term approach that addresses the issue in the here and now. Targeting the function-specific problem through cognitive training is essential because it addresses the underlying mechanisms. This approach has the potential to create long-term, meaningful changes that can impact mathematics abilities, and thus, overall achievement. The question of how to target this function-specific problem becomes complicated by the debate within the literature as to whether EF is a unitary construct, or discrete skills, in childhood. Given the many links from specific EFs to mathematics skills, we argue that initial interventions should specifically target one skill. If these fail to be effective, then considering interventions that address EF as a unitary process should be explored. Most importantly, sustained attention should be the first skill addressed, as the adequate use of WM and EF relies on effortful control and sustained attention.

Despite the significant amount of literature demonstrating a relationship between mathematics and attention, the research exploring cognitive interventions to address this relationship is quite small. There are very few studies that assess the effectiveness of cognitive interventions for remediating math, and within those studies the results are mixed. That said, the interventions with the most immediate and long-term effects seem to align with cognitive load theory, which integrates many different attentional functions (see above, e.g., González-Castro et al., 2016; Kercood & Grskovic, 2009; Kang & Zentall, 2011; Dahlin, 2013). Unfortunately, there is still minimal investigation on the training of sustained attention on mathematics ability, a skill fundamental to academic success. Interventions targeting WM and EF are necessary, and they should follow primary intervention into attentional control skills, as they serve little utility unless a child can sustain their attention to the necessary and relevant information.

3.4.1 Future Research

Future studies should continue to explore and replicate interventions that target cognitive skills to remediate mathematics learning. Based on the review of the current literature, it is integral that the specific attentional skill is well operationalized. Furthermore, researchers should take careful consideration into selecting the means for their cognitive intervention, ensuring that it is adequately addressing the operationalized skill. The interventions should be intensive and should include immediate and follow-up testing to evaluate the impact of cognitive training over time.

Future research should also compare the effectiveness of the same intervention for typically developing students, students with mathematics learning disorders, students with attentional impairments, and students with comorbid mathematics learning and attentional difficulties. Doing so will provide further insight into the profiles of these populations, as well as help in understanding if intervention should be differentiated for different presentations. Finally, this could justify whether mathematics instruction for all students could be improved by addressing attentional components of instruction design or supplementing some cognitive training into the regular curriculum for the general education population.

3.4.2 Implications

While the relationship between cognition and mathematics achievement is not novel to many, it is widely ignored in the intervention literature. For the most part, cognitive training research has been broad, and intended to improve symptomatology for clinical populations (e.g., ADHD). This chapter highlights the lack of intervention training to follow up the empirically evidenced relationship between mathematics and attention. Understanding the unique relationship between attention and mathematics seen in multiple clinical and typically developing populations suggests an incredible potential for a new approach to mathematics remediation that is not being addressed. Furthermore, the implementation of such interventions in schools' clinical practice will require considerable evidence from research. For instance, searching for evidence-based practices for mathematics remediations on What Works Clearinghouse (a public website that reviews and identifies studies to support education) yields 150 results. However once combined with the keyword cognition, there are a total of 4 interventions, none of which are exclusively mathematics remediation programs.

For educators and clinicians, the research is strongly in support of early identification and intervention for students struggling with math, in support of their long-term academic growth and success (Duncan et al., 2007). The literature would suggest that educators should be keenly aware of their students who may be struggling with foundational mathematical skills, and/or those showing difficulties with attention, working memory, and executive functions. Flagging these difficulties in childhood has the potential for efficient identification and thus proper early remediation. For clinicians, students presenting for assessment who are experiencing attentional challenges should strongly be considered for academic testing that explores the acquisition of fundamental mathematics skills. Though the presenting concern may be attentional challenges or difficulties with executive functions, as we have read throughout this chapter, there is a strong likelihood that this child is susceptible to falling behind in mathematics learning. In the same regard, students presenting for assessment to explore mathematics learning challenges should have sufficient testing to explore their attentional profile and related skills (WM/EF) to identify any skills deficits. In investigating these related skill sets through standardized testing; it follows that at-risk students will receive thorough and individualized educational planning to support their immediate and long-term learning.

When considering interventions to support students with mathematics learning challenges, educators are encouraged to reflect on whether their programming is task-specific, and whether this is in the long-term interest of the student's learning. Teachers and school professionals should review the current practices in place for remediation and explore potential for introducing function-specific programming. That said, it is essential that more research is conducted in the field to provide evidence in support of such interventions and programming. Far too often, schools are expected to implement intervention programs with limited research into its effectiveness, and a lack of explanation for the rationale. We strongly encourage that schools have a transparent understanding of their intervention programs, and selected cognitive-intervention programs are based in sound evidence.

3.5 Conclusion

The evidence supporting a relationship between domain-general attentional functions and mathematics learning and performance is clear. There are limited studies that look at using cognitive training as a means for mathematics remediation, despite the notable mathematics difficulties in children with ADHD, as well as attentional deficits in children with MLDs. Research on such interventions has the potential to inform and change current methods for clinical practice and remediation of mathematics difficulties. The value of doing so is pertinent, as effective mathematics intervention has the potential to influence overall academic achievement in later years.

Educators and clinicians are encouraged to use this framework in supporting their students, advocating for thorough assessment, and implementing evidenced-based interventions that address the underlying cognitive issues, as opposed to the observable deficits in learning.

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Chapter 4
Instructional Support for Fact Fluency Among Students with Mathematics Difficulties

Friggita Johnson, Pamela M. Stecker, and Sandra M. Linder

Abstract Fact fluency is a critical component for increasing mathematics proficiency. Effortless and quick retrieval of number combinations reduces cognitive load and allows for more sufficient cognitive resources to execute higher order mathematical problems. This chapter begins by reviewing the developmental trajectories for fact knowledge acquisition among children who are typically developing. Then, we discuss working memory deficit as a common characteristic experienced by children with or at risk of learning disabilities in mathematics in developing fact knowledge. Next, we explain the significance of building fact fluency and describe several effective instructional interventions or strategies to support basic fact knowledge among elementary and middle school students experiencing mathematics difficulties. We focus particularly on incremental rehearsal as a strategy that involves interspersing a high percentage of already known items to unknown targeted items to promote better acquisition and retention of the targeted facts for children with mathematics difficulties who need intensive, individualized intervention.

After reviewing research support for the efficacy of incremental rehearsal, we use a hypothetical vignette to describe an elementary student with mathematics difficulties to illustrate intervention support that can be used for fact retrieval. Steps for teachers, paraprofessionals, and parents to implement incremental rehearsal as well as implications for practice are included.

Keywords Fact fluency · Mathematics difficulties · Working memory deficits · Instructional support · Incremental rehearsal

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4.1 Background

Basic facts are number combinations where each addend in addition, the subtrahend and difference in subtraction, factors in multiplication, the divisor and quotient in division, are single-digit whole numbers (i.e., 0–9); or include numbers 1–9 for divisor and quotient in division (Stein et al., 2018). Fact fluency is the ability to retrieve and recall facts rapidly, accurately, and effortlessly (National Council of Teachers of Mathematics [NCTM], 2000).

Effortless and quick retrieval of number combinations reduces cognitive load and allows for sufficient cognitive resources to execute more advanced mathematical skills (National Mathematics Advisory Panel [NMAP], 2008). Efficient recall of basic facts can be achieved via rote or meaningful memorization. Rote memorization involves repeating facts until remembered and may not involve understanding the pattern and relation among number combinations. However, more meaningful memorization activities include exploration of number combinations and their relations to better promote mathematical thinking and learning and perhaps making automatic recall easier for learners. Most important, meaningful memorization of facts is facilitated by acquiring conceptual and procedural knowledge when using counting and reasoning strategies prior to building fluency (Baroody et al., 2009).

Balancing conceptual understanding (i.e., understanding mathematical concepts and relations) and procedural fluency (i.e., ability to execute procedures efficiently and flexibly and appropriately) instruction is required for developing mathematics proficiency (NCTM, 2000). When students acquire conceptual understanding, it allows them to recognize the relation among number combinations, know more than isolated facts, and learn new facts better by connecting them to known facts (National Research Council [NRC], 2001). Furthermore, the NRC suggests that the mathematical concepts and procedures learned with understanding aid long-term retention because facts forgotten may be more easily retrieved from memory if the individual is able to connect ideas and see an existing pattern between new and known facts. For example, knowing $4 + 4 = 8$ (a doubles fact) may help a student calculate $4 + 5 = 9$ quickly by adding 1 more to 4 to make the sum 9. Consequently, meaningful memorization activities may assist students in moving from mere recall of facts with accuracy to recall with fluency and application of facts when solving nonroutine problems.

4.2 Typical Developmental Trajectories

Knowing the typical developmental trajectories for fact knowledge acquisition has direct implications for how and what instructional supports and interventions are required to promote mathematical proficiency (Allsopp et al., 2007). Typically developing children follow a sequential developmental pattern, although all children do not progress at the same rate (National Association for the Education of Young

Children, 2009). According to Baroody et al. (2009), typically developing children progress through three developmental phases to attain proficiency in basic number combinations. These phases include counting strategies, reasoning strategies, and fluency mastery (for details, see Fig. 4.1). Baroody emphasized that children possess the skills to construct new ideas from already existing knowledge and, thereby, can make meaningful associations for fact recall rather than relying only on rote memorization. Baroody's work also confirms that children typically progress from using concrete objects and verbal counting to using reasoning strategies, both of which appear to be prerequisites or requisites for development of basic fact fluency. However, children with mathematics difficulties may not follow the same developmental sequence due to their cognitive deficits in working memory or ineffective educational experiences (Baroody et al., 2009). Knowing more about the specific weaknesses in mathematics performance may assist educators in providing intervention pertinent to student needs (Gersten et al., 2005).

4.3 Development of Fact Fluency Among Children with Mathematical Difficulties

Children with mathematics difficulties (MD) commonly are those children with either a formal diagnosis of learning disabilities in mathematics or who are at risk of developing mathematics disabilities (Powell et al., 2013). Moreover, children with mathematics disabilities constitute a heterogeneous group with problems in different mathematical areas or have other comorbidities. Dyscalculia is a widely used term for children with mathematics disabilities. However, it primarily refers to children who have difficulty performing early mathematics skills, such as number sense, counting, and calculation, leading to long-term problems with mathematics facts recall (Geary, 2006). Mathematics disabilities may also co-occur with reading disabilities (Hanich et al., 2001), resulting in the slower acquisition of mathematical competence than children with mathematics disabilities only (Gersten et al., 2005). While reading disability is a broad term for students experiencing problems in reading individual words and understanding text, dyslexia is a type of reading disability affecting early reading skills, such as sounding out words, calling out words quickly, and understanding written words that can lead to problems with text comprehension (Hulme & Snowling, 2016). We use the term MD to refer to this larger group of students experiencing difficulties in the area of mathematics. Studies may not distinguish between participants with and without identified mathematics disabilities or comorbidities, or researchers may choose to combine participant findings, in part, because some students at risk may not yet have received a formal diagnosis.

Many children with MD experience working memory deficits that may interfere with their ability to retrieve and recall facts fluently, thereby affecting their performance on higher order mathematical skills (Geary, 2004). According to Baddeley

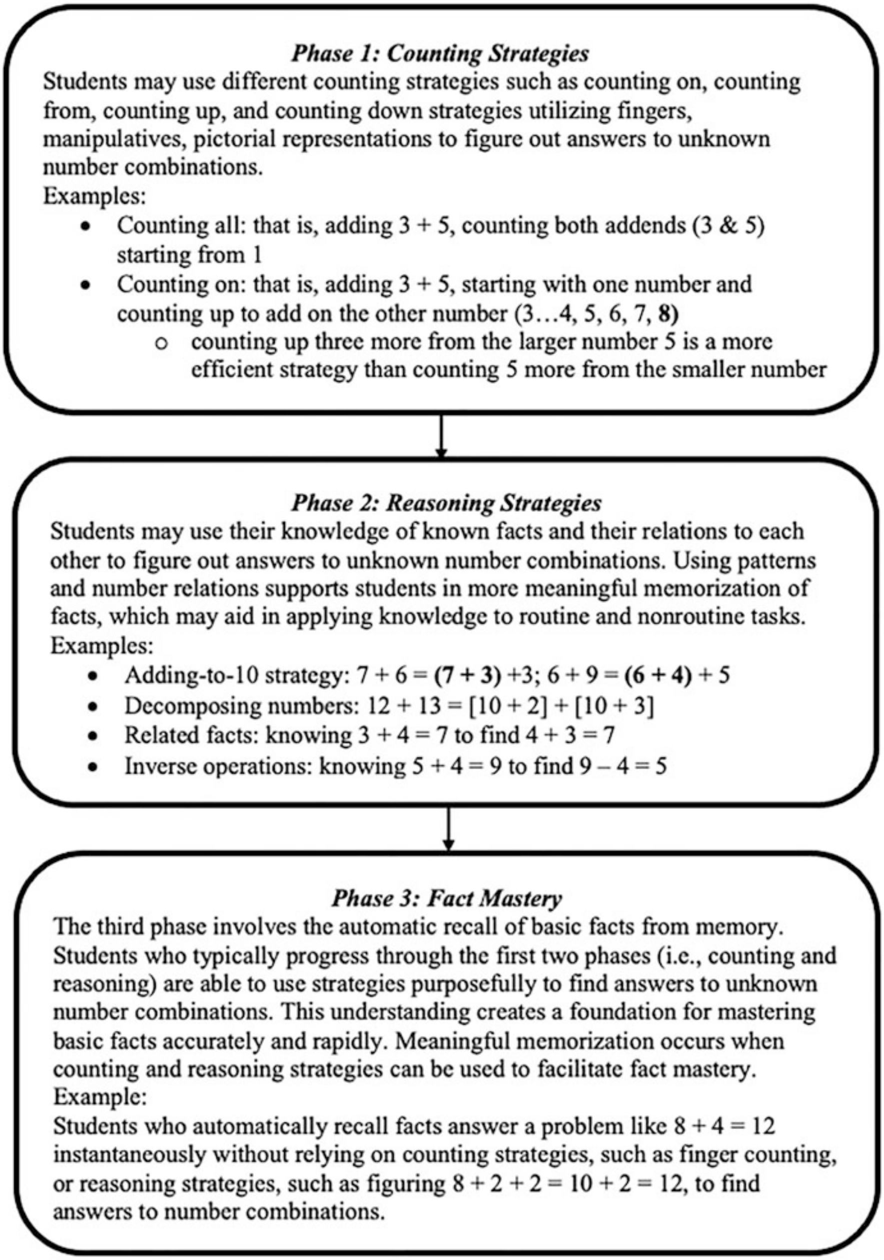

Fig. 4.1 Three phases in typical fact knowledge acquisition. (Source: Adapted from Baroody, 2006 & Baroody et al., 2009)

(1986), working memory is a system for temporarily storing information in the brain and associating it with some information for performing other cognitively demanding tasks. For example, working memory involves keeping a basic fact or formula in mind and using it to solve a multistep word problem. Due to working memory deficits, children with MD may remain at a finger counting stage, rather than moving to reasoning stage to derive answers to number combinations and, consequently, may fail to retrieve basic facts effortlessly from memory (Jordan & Montani, 1997). Hence, when mathematics facts are timed, children with MD perform more poorly compared to typical achievers because they are expected to retrieve facts from memory quickly and accurately (Jordan & Montani, 1997) instead of being able to rely on immature backup strategies, such as finger counting (Geary, 2004). Recognizing that a subgroup of students who are low achieving in mathematics have demonstrated deficits in fact retrieval similar to students with identified mathematics disabilities (Geary et al., 2012a), proficient fact retrieval remains a common concern. Similarly, children with comorbid mathematics and reading disabilities have trouble with basic mathematics fact retrieval when timed (Hanich et al., 2001).

4.4 Importance of Developing Fact Fluency

Improving fact fluency is significant for developing mathematics competency in advanced mathematics skills, such as solving multidigit computation and fractions (Hasselbring et al., 1987; NMAP, 2008). For children with MD due to working memory deficits, it may be cognitively challenging to connect ideas and see relations among number combinations, thus interfering with storage of information as declarative knowledge (i.e., $5 \times 3 = 15$ & $15 \div 3 = 5$) and affecting long-term retention, including recall of facts with automaticity (Hasselbring, 1988). Emphasizing the use of efficient strategies (e.g., commutative properties) to answer number combinations may make recall of basic facts more efficient and consistent in working memory, thus facilitating long-term retention for quick and effortless retrieval (Fuchs et al., 2008; Goldman & Pellegrino, 1987). Consequently, children with MD may need intervention early in building fact fluency. Students lacking fact fluency may expend cognitive resources just to retrieve facts from working memory, which, in turn, makes demands on cognitive resources more daunting when performing complex tasks (Goldman & Pellegrino, 1987). Fact fluency may free up cognitive load for retrieving facts quickly and efficiently and for promoting transfer and generalization of skill to novel situations (NRC, 2001). Additionally, when students are able to retrieve facts quickly and accurately, their frustration and anxiety may be reduced when solving complex mathematics problems (Cates & Rhymer, 2003). Moreover, when students with MD have trouble recalling facts from memory, they rely overtly on immature strategies, such as finger counting or using concrete objects for retrieving facts; when they discover that deriving answers via counting strategies provides a correct solution, they may continue to use these inefficient and time-consuming strategies to retrieve answers (Hasselbring, 1988). Most typically developing children begin retrieving facts from memory automatically by third grade (Fuchs et al., 2008). When children experience challenges recalling facts with automaticity at the beginning of third grade, strategies for delivering fluency intervention to prevent widening of the achievement gap between students with and without MD each year may need to be considered.

4.5 Evidence-Based Instructional Intervention Strategies for Fluency Building

Fuchs et al. (2008) recommended that repetitive practice of basic skills (e.g., arithmetic facts) with feedback is an important mathematics intervention principle for developing automatic recall of basic mathematics facts. For students who struggle, intervening early with strategies that embed brief, frequent, and timed practice may facilitate the shift from concrete to mental representations to promote automaticity of basic mathematics facts (Gersten et al., 2005; Daly et al., 2007). Moreover, fluency activities that incorporate ample opportunities to practice are recommended for basic fact fluency (NMAP, 2008). However, Hasselbring et al. (1987) argued that efficient recall of basic facts often transpires when children develop reasoning strategies and understand the association between number pairs before engaging in practice activities. Because students with learning disabilities often are poor strategic learners, explicit instruction in using specific strategies may be beneficial in overcoming their strategic deficits (Montague, 2008).

Evidence-based rehearsal activities or strategies that incorporate the critical components of an explicit instructional approach support basic fact knowledge acquisition at the student's instructional level. Explicit instruction is an evidence-based, systematic, direct, and concise approach to teach students through demonstration, guided practice, and independent practice (Archer & Hughes, 2011). Some critical components of effective practice strategies include multiple opportunities to respond with immediate corrective feedback (Pool et al., 2012), timed, distributed practice (Fuchs et al., 2019), and reinforcement (Daly et al., 2007). For example, fluency intervention strategies that incorporate these components include Cover, Copy, Compare; Taped Problems; and Detect, Practice, Repair (see Table 4.1 for additional information). The importance of incorporating these elements in intervention is supported by a meta-analytic review conducted by Codding et al. (2011) on basic mathematics fact fluency interventions for students with mathematics difficulties. This review revealed that fluency interventions incorporating components of repeated practice, corrective feedback, and reinforcement demonstrated the largest effect sizes.

Although these fluency interventions have variations in procedures, they use an explicit instructional approach to practice new facts. These practice activities provide students with multiple practice opportunities to master the new facts, immediate error correction, and assessment to evaluate student response to intervention (Joseph et al., 2012b; Poncy et al., 2013).

Table 4.1 Mathematics fact fluency interventions

Intervention	*Cover, copy, compare*	*Taped problems*	*Detect, practice, repair*
Description	An evidence-based, self-monitoring strategy developed by Skinner et al. (1989).	An evidence-based, self-monitoring strategy originally developed by Freeman and McLaughlin (1984).	An evidence-based practice activity for improving fact fluency that focuses on a test-teach-test strategy (Poncy et al., 2006).
Procedure	Student views the number combination and its answer, covers up the problem and the solution, writes the problem and the solution from memory, uncovers the problem and the solution, and compares the student's response with the model. If the response is correct, the student moves to the next problem. If not, the student writes the problem and solution three times.	Student (a) listens to the problem read, (b) writes the answer to the problem on a sheet of paper before the tape recorder answers, (c) listens to the answer provided by the audio tape that employs a time delay, (d) writes a slash on the response if incorrect and writes the correct response as heard on the tape, and (e) continues to listen to the next problem.	Students complete a timed pretest containing a prescribed number of problems. They are given a short interval of time to answer each fact (i.e., 1 minute and 30 seconds across the pretest). The first five unanswered items from the pretest are selected for practice using the cover, copy, compare method with both verbal and written responses. Then, students have the opportunity to repair their performance.
Empirical support	Findings from research studies (Joseph et al., 2012b; Skinner et al., 1997) reveal the efficacy of the intervention in improving fact fluency among struggling learners 7–14 years old.	Data from McCallum et al. (2004) demonstrated an increase in fact fluency among elementary students 8–10 years of age with MD.	Studies (Parkhurst et al., 2010; Poncy et al., 2006, 2013) that investigated the efficacy of the intervention demonstrated improved fact fluency among low-achieving students 8–10 years old in need of remediation in basic facts.

In addition, these fluency interventions can be implemented in any educational setting, such as general education or special education settings, and especially among children who typically are practicing fact fluency. These strategies may be used independently by students who make a decision about fact accuracy based on the answer that is provided as a part of the materials. Elementary-aged students are expected to have memorized addition, subtraction, multiplication, and division facts (Common Core State Standards, 2010). However, some students enter middle school without sufficient fact retrieval, which may interfere with strategies used for problem solving. Fluency interventions may benefit these older middle or high school students with MD, as fluency in basic skills supports solving higher order mathematical problems, such as problems involving fractions, decimals, ratios, algebra, and statistics.

4.6 Incremental Rehearsal Strategy

Although these fluency interventions described may be implemented feasibly in most educational settings among most school-age children, interventions vary based on their level of instructional intensity and implementation time. For example, standard fluency interventions that students may use independently, may not be sufficient for students with MD who experience severe and persistent mathematics difficulties. Students with greater difficulties may benefit from repeated, intensive, one-on-one practice, using evidence-based intervention (Powell & Fuchs, 2015).

Incremental rehearsal (Tucker, 1989) is another evidence-based fluency intervention strategy that incorporates essential components of the explicit instructional approach. The intensity of incremental rehearsal procedures makes it an appropriate strategy to highlight in this chapter focused more on students with intensive needs in mathematics. Studies of incremental rehearsal reveal consistent improvement in mathematics fact fluency among students with (e.g., Burns, 2005) and without disabilities (e.g., Burns et al., 2016) as well as contributions to better retention, efficiency, and generalization effects (e.g., Burns, 2005; Codding et al., 2010; MacQuarrie et al., 2002).

Incremental rehearsal uses a simple flashcard technique and is unique and distinct compared to other practice strategies described because of the interspersal of a high percentage of known facts (i.e., facts already recalled accurately within 0–2 s) with new facts (i.e., facts not able to be recalled accurately in 2 s). Repetition and incremental spacing of the new facts during the practice session provide multiple opportunities for practice and feedback on only a few new facts. It also requires students to rehearse previously known facts to reduce cognitive load during rehearsal. This strategy appears to support the transfer of newly learned facts from short-term to long-term memory for automatic retrieval later (MacQuarrie et al., 2002; Stein et al., 2018). This high percentage of known facts with only a few new facts targeted per practice session also may reduce potential student anxiety and frustration. In contrast, students typically work on only new, unlearned facts in other fluency drill methods (e.g., Cover, Copy, Compare). However, students with MD who have working memory deficits may require a more intensive, individualized intervention. Incremental rehearsal, which uses repeated and spaced practice with only a few new items presented and practiced one at a time, may be selected for more intensive practice to promote automatic retrieval of basic facts.

4.6.1 Research Supporting the Incremental Rehearsal Strategy

Many studies support the effectiveness of the incremental rehearsal strategy for developing automaticity in various academic areas, for example, in reading (letter identification, Bunn et al., 2005; sight words, Kupzyk et al., 2011), writing (spelling, Garcia et al., 2014), and mathematics (basic facts, Burns, 2005). The effectiveness and efficiency of this strategy also have been investigated among children with and without disabilities and English language learners (e.g., DuBois et al., 2014; Haegele & Burns, 2015; Rahn, 2015) across different grades (e.g., preschool, Bunn et al., 2005; elementary, Codding et al., 2010; middle, Zaslofsky et al., 2016), and using different delivery methods (e.g., flashcards, Nist & Joseph, 2008; computer-assisted instruction, Volpe et al., 2011a).

Some studies comparing incremental rehearsal to other flashcard techniques showed mixed findings. However, most studies in reading (Burns & Boice, 2009; Burns & Sterling-Turner, 2010; Joseph et al., 2012a) reported incremental rehearsal as an efficacious method for retention, maintenance, and generalization compared to the traditional drill model that focused solely on new words. Only a few studies (Mule et al., 2015; Volpe et al., 2011b) reported that the traditional drill model was effective.

Compared to the number of studies using incremental rehearsal in reading, investigations for mathematics fluency building are far fewer. However, findings from mathematics studies (i.e., Burns, 2005; Burns et al., 2016, 2019; Codding et al., 2010) suggest that incremental rehearsal is an effective strategy for building mathematics facts fluency for students with MD compared to the more traditional drill. Incremental rehearsal is effective because it integrates features that appear important for facilitating shorter latencies (i.e., the interval between fact presentation and student response), supporting long-term retention, and reducing frustration. These features include (a) providing multiple opportunities to practice on a small set of new facts critical for student accuracy and retention, (b) using a timing feature that supports faster fact retrieval, and (c) practicing with a high percentage of known facts to new facts presented in increments to help students maintain previously learned items while also providing the opportunity for many correct responses within the session, thereby increasing motivation and reducing frustration.

The following hypothetical vignette illustrates when a teacher may select incremental rehearsal as an intervention for a student with intensive needs in mathematics. Following the vignette is a description of the steps of the incremental rehearsal strategy.

Derek is an 11-year-old, sixth-grade student at Beachside Middle School. He has an identified learning disability in mathematics and receives special education services. Derek's special education teacher, Mrs. Valdez, recognizes his need for intensive intervention. She initially was concerned about Derek's lack of progress in solving multidigit addition, subtraction, multiplication, and division calculations. She noted that Derek had acquired understanding of the operations but used finger counting and/or tally marks to derive answers to basic number combinations. Although he was generally accurate, he was very slow in figuring facts and continued to have trouble

(continued)

solving multidigit problems (e.g., 2843–1962 =?). When solving multidigit problems, such as problems with renaming, Mrs. Valdez noticed that Derek reverted to immature strategies, such as finger counting, to recall facts instead of retrieving answers to basic facts from memory. Mrs. Valdez speculated that finger counting was interfering with Derek's ability to solve complex problems accurately and that a more intensive one-on-one intervention to practice target facts may facilitate faster and more accurate recall of basic facts. Mrs. Valdez knew that repeated practice using evidence-based rehearsal strategies can increase accurate and fast recall, leading to better long-term retention. Mrs. Valdez decided to use the incremental rehearsal strategy to help Derek recall facts with automaticity, which, in turn, may lead to improvement with solving multidigit computational problems. With incremental rehearsal, she knew that the design of a high percentage of already fluent facts interspersed with only a few target facts potentially may reduce anxiety, increase accuracy, and improve motivation, especially as compared to other, more typical rehearsal strategies. She thought that obtaining correct answers on his previously learned facts from prior practice sessions may motivate Derek to practice current new facts.

4.6.2 Steps for Implementing Incremental Rehearsal

Implementation of the incremental rehearsal procedure takes approximately 12–20 mins. Knowing the specific facts to present is essential, because the hallmark of intensifying intervention is tailoring instruction to meet the individual needs of the students. Prior to implementing incremental rehearsal, the teacher pretests the student by presenting each number combination from the relevant operation or operations to determine the facts to address. Each fact is printed on a flashcard without the answer, is shuffled with all the cards, and is presented one at a time. A two-second latency between the stimulus and response is allowed. When the student states the correct response within 2 s, the card is placed in the *Fluent* stack. If the student gives an incorrect response or fails to respond within 2 s, then the card is placed in the *Target* stack.

An intentional, systematic approach can be used when selecting new facts for a practice session. For example, the teacher selects a few target facts by basing them on the student's prior knowledge of related facts (e.g., 3 + 2, 2 + 3) and their reversals or using a specific series of facts (e.g., 3 + 2, 4 + 2, 5 + 2). Students with MD may benefit if a small set of target facts (i.e., not more than three or four) are introduced during each practice session to facilitate easy storage of number combinations for efficient and accurate recall later (Hasselbring et al.,1988). When selecting already fluent facts, the total number may vary and may be selected randomly from the *Fluent* stack.

During practice, a high percentage (i.e., 85% to 90%) of fluent facts (i.e., facts recalled accurately within 0–2 seconds) is presented with one target fact (i.e., difficulty recalling facts accurately within 2 seconds) during each session. Spacing the interval of fact presentation by gradually expanding the number of known facts between the presentation of the same target fact helps establish the student's declarative knowledge. Studies that investigated incremental rehearsal with basic mathematics fact fluency used 7, 8, or 9 fluent facts interspersed with 1 target fact with success (Burns, 2005; Burns et al., 2016, 2019). Additionally, a recent dissertation study (Johnson, 2020) investigated the efficacy of incremental rehearsal for improving fact fluency using 6 fluent facts interspersed with one target fact and showed promising results. Thus, teachers may choose to alter the total number of fluent facts (i.e., 6–9 fluent facts) presented with a single target fact. One of the key features of incremental rehearsal is expanded practice using a less challenging ratio of known to new items (i.e., greater than 50% of known facts) to aid student retention (MacQuarrie et al., 2002). Thus, teachers may individualize instruction by varying the number of fluent facts presented depending on the availability of practice time and the functioning level of each student.

Incremental rehearsal follows an explicit instructional approach that includes modeling the target facts one at a time, allowing multiple opportunities to rehearse the target facts, and providing corrective feedback. During modeling, the teacher models each target fact and its correct response. Then, the student is asked to repeat by stating the target fact and its correct response, first with a prompt (e.g., teacher said, "three minus two equals how many?" and the student said, "one"), and then without any prompt (e.g., the student said, "three minus two equals one"). Next, the longest part of the session is corrective practice. The teacher shows fact cards one at a time that have been sequenced according to incremental spacing between the target fact and already fluent facts. Initially, the student is expected to provide a correct response within 2 or 3 seconds (as set by the teacher) for each fact presented. The response time (i.e., the time between the presentation of the fact and the student's response) may vary based on the performance level of the child. That is, the teacher may start with 3 s as the latency allowed during the first session. However, response time may be reduced from 3 s to 2 s in subsequent practice sessions to wean children away from using counting strategies and encouraging them to retrieve answers automatically (Hasselbring, 1988). Eventually, students should be able to respond within 1 s of seeing the flash card to demonstrate mastery of the fact.

The practice session is completed after all the target facts (e.g., three or four target facts) have been practiced while being interspersed with the selected known facts. For example, if three facts are targeted, only one new fact is chosen to be practiced first and is interspersed with the known facts. Keeping to the same order of already fluent facts, the teacher presents the student with a target fact, then one known fact, then the target fact, then two known facts, the target fact again, then three known facts, and so forth until all 6–9 selected known facts have been used. When practice of the first target fact has concluded, another cycle of practice with the second new fact occurs. Last, a cycle of practice occurs with the third new fact. When a student misses a fact or fails to recall accurately within the required seconds, the fact is modeled. The student repeats the fact with the correct answer and then rehearses it a few times without any prompt.

At the end of each practice session, a timed written assessment (e.g., a 2-minute test of all facts in the designated operation) is administered to determine overall fluency and generalization of facts. The total number of facts answered correctly within the specified time is recorded and graphed to determine whether overall fluency increases with incremental rehearsal practice across sessions. As students practice more and more new facts, they should see their progress on the total pool of facts increase. Because the assessment is written, latency is not evaluated for each fact. However, the purpose of the written assessment is to gauge overall fluency and progress over time.

If a student has difficulty responding accurately, the target facts practiced in a session could be rehearsed again in subsequent sessions to improve automaticity. When new target facts are selected, the target facts rehearsed in previous sessions become labeled as fluent facts to be used in subsequent practice sessions. In this way, the initial target facts continue to get practiced in subsequent sessions until enough new facts replace the original fluent facts. Consequently, presentation of the same target facts across several sessions as well as repeated practice on the same facts within a practice session provide multiple, distributed opportunities for practice and feedback. This relatively brief but consistent timed practice 3–4 times a week may bridge the gap between fact accuracy and fact fluency.

Figure 4.2 provides the specific steps and sequence for interspersing three target facts used with six fluent facts for one practice session. The incremental rehearsal procedure outlined in the figure is adapted from Tucker (1989) and Burns (2005) and was used in the Johnson (2020) dissertation study investigating the effectiveness of incremental rehearsal for improving subtraction fact fluency among elementary students with mathematics difficulties. Johnson selected subtraction facts because no other work had addressed subtraction fact fluency. Further, developing fluency in all four mathematics operations, including subtraction facts (Common Core State Standards, 2010) before entering middle school is critical for computing higher-order mathematical problems, such as geometry, fractions, algebra, probability, and statistics.

Johnson modified some features from the Tucker (1989) and Burns (2005) strategies. Although several of these features were explained more fully as a part of the incremental rehearsal procedures detailed above, they are outlined here. First, she used only six fluent facts to make the practice sessions briefer (i.e., rather than using 7–9 fluent facts). Second, she used an intentional, systematic approach for selecting new facts (i.e., rather than using random selection) to support more meaningful memorization. Third, she provided two days of practice on the same set of new facts, rather than changing facts each session. However, she altered the response time allowed during these two practice sessions by starting with 3 s for the first day and reducing the acceptable response time to 2 s on the second day. Last,

T1-F1 **T1-F1-F2** **T1-F1-F2-F3** **T1-F1-F2-F3-F4** **T1-F1-F2-F3-F4-F5** **T1-F1-F2-F3-F4-F5-F6**	1. Present the first target fact, and the student answers aloud. 2. Present the first fluent fact, and the student answers it correctly. 3. Present the first target fact again, and the student answers it correctly. 4. Present the first fluent fact. 5. Present the second fluent fact. 6. Present the target fact again. 7. Present the first, second, and third fluent facts one after the other. 8. Present the first target fact again. 9. Present the first, second, third, and fourth fluent facts. 10. Present the first target fact again. 11. Complete this sequence of presenting the first target fact with all six fluent facts.
T2-PT1 **T2-PT1-F1** **T2-PT1-F1-F2** **T2-PT1-F1-F2-F3** **T2-PT1-F1-F2-F3-F4** **T2-PT1-F1-F2-F3-F4-F5**	12. Replace the first target fact with the second target fact. The first target fact now becomes the first fluent fact; the first fluent fact becomes the second fluent fact, and the second fluent fact becomes the third fluent fact, and so on. Remove the last fluent fact. 13. Model and practice the second target fact as the new first target fact. 14. Complete this sequence of presenting the second target fact with the six fluent facts.
T3-PT2 **T3-PT2-PT1** **T3-PT2-PT1-F1** **T3-PT2-PT1-F1-F2** **T3-PT2-PT1-F1-F2-F3** **T3-PT2-PT1-F1-F2-F3-F4**	15. Replace the second target fact with the third target fact. The second target fact is now the first fluent fact; the first fluent fact (i.e., the old first target fact) becomes the second fluent fact; remove the last fluent fact from the previous cycle. 16. Model and practice the third target fact. 17. Complete this sequence of presenting the third target fact with the six fluent facts. 18. The process is repeated until all the target facts have been practiced in the session. • Any time the student misses a fact or fails to answer within the allotted 3 or 2 seconds, the interventionist models the fact (statement and answer) and asks the student to read the statement and answer correctly. • The total number of flashcards in the deck is always 7 (i.e., six fluent facts and one target fact).

Fig. 4.2 Incremental rehearsal procedure note. T1, T2, and T3 = target facts 1, 2, and 3; F = fluent fact; PT = previous target fact. (Source: Adapted from Tucker (1989) and Burns (2005))

she included the same facts that had been presented previously as new facts as “fluent facts” in subsequent practice sessions. In this way, students were given additional opportunities to practice these newer facts across time (i.e., distributed practice across multiple practice sessions) rather than dropping them after their initial practice. These incremental rehearsal features appeared promising. Even though the study was not completed due to the school’s sudden transition to virtual instruction, positive trends in performance were observed across students.

Mrs. Valdez worked with Derek for 15 minutes daily four times a week using the incremental rehearsal strategy. After a few weeks of this intensive individualized intervention, Derek not only became more fluent on targeted facts, but he started to show improvement in solving multidigit computation problems. It appeared that incremental rehearsal helped Derek recall and retain the targeted facts. Further, due to the interspersing of the high percentage of already fluent facts, Derek had many opportunities to experience correct responses during the practice trials, providing both important review and appearing to motivate Derek to continue practicing using incremental rehearsal. Mrs. Valdez was pleased with Derek's performance and progress over time and was excited about sharing his success using incremental rehearsal with her colleagues.

4.6.3 Implications for Classroom Practice

The incremental rehearsal strategy has several implications for classroom use. Teachers use an explicit instructional approach (Archer & Hughes, 2011) for practice, which is the most commonly recommended approach for teaching any content area effectively for students who need more intensive intervention, including both students identified with or who are at risk for mathematics disabilities. Some of the features of explicit instruction imbedded in incremental rehearsal include modeling the target fact prior to the target fact practice; scaffolding instruction, which appears, in part, through the interspersing of target facts during practice; practicing the same target facts multiple times, distributed across occasions, and providing corrective feedback. All these features likely contribute to improved fact fluency.

Incremental rehearsal intervention is intensive because the new information is selected based on student performance and is practiced repeatedly across time in a systematic way. This distributed practice spaced over time promotes automatic recall of basic facts (Hasselbring, 1988). Specifically, the incremental nature of practice facilitates movement of the new item from short-term to long-term memory (Nist & Joseph, 2008). In addition, the intensive, individualized, and repeated practice used in incremental rehearsal is the hallmark of effective instructional practice used among students with severe and persistent needs.

As suggested by Hasselbring (1988), the small number of facts used in the incremental rehearsal supports easy storage of facts for quick and accurate retrieval later. For students with MD who experience deficits in working memory, this practice strategy may be especially important. Incremental rehearsal implementation is cost-effective because it utilizes simple flashcards with an easy system for recording student responses. Thus, teachers, paraprofessionals, or parents could serve as an interventionist during individualized practice sessions. The interventionist should understand the underlying principles upon which these treatment procedures were developed if fidelity of implementation and similar positive results as those obtained in research are expected in classroom settings.

Although incremental rehearsal implementation is feasible and practical, interventionists implementing this rehearsal strategy may require some initial practice and training, so materials are prepared and sequenced accurately. Additionally, implementation potentially could be time-consuming when used across multiple, individual students. To mitigate this limitation, a technology-based application for implementing incremental rehearsal strategy could be developed. The use of digital mobile devices (e.g., smartphones, iPad®) in delivering mathematics instruction has gained attention (NCTM, 2000; NMAP, 2008), particularly in special education settings (Ok et al., 2019). In addition to portability and use with multiple students simultaneously, benefits of a digital system with incremental rehearsal may include the ability to more easily track student progress, provide immediate feedback, and enhance student motivation (Musti-Rao & Plati, 2015). Though we are not aware of current applications for incremental rehearsal, the potential advantages of using mobile digital devices for reducing teacher time during implementation and management of student data may prompt additional research and development.

4.7 Summary

Intervention support in basic skills, such as basic fact fluency, is imperative because it influences students' mathematical outcomes later (Gersten et al., 2009). Given that approximately 7% of school-aged children have learning disabilities in mathematics and 10% of students are low achieving in mathematics (Geary et al., 2012b), mathematics disability is considered a high-incidence disability, and teachers in general education may have at least several students with MD in each class. Recent data from the National Center for Educational Statistics (2017) indicated no marked improvement in the fourth- and eighth-grade students' mathematics proficiency compared to previous years; thus, instructional support and interventions to foster mathematics fact fluency appear vital. Moreover, research supports the benefit of using individualized, intensive intervention to improve basic mathematics facts fluency for students with MD. Fluency intervention strategies for promoting fact fluency, especially the incremental rehearsal strategy discussed in this chapter, may be beneficial for practitioners who seek resources for developing fact fluency and potentially improving a student's overall achievement in mathematics.

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Chapter 5
The Development of Arithmetic Strategy Use in the Brain

Merel Declercq, Wim Fias, and Bert De Smedt

Abstract Arithmetic development is quintessential for learning more advanced mathematics. A key aspect of arithmetic development is a shift from calculation-based procedural strategies to memory-based fact retrieval. For example, children start to learn 3 × 4 by adding 4 + 4 + 4, which is an example of a procedure. After enough repetitions, this becomes an arithmetic fact. This chapter will review the scarce but growing evidence on how arithmetic strategy development in primary school children is reflected in the brain and how its functional networks change over development. The brain network in children recruited for doing arithmetic includes frontal, parietal, occipital-temporal and medial-temporal areas, with different foci depending on the strategy. We discuss studies that have compared different ages as well as longitudinal research. We review the results of brain imaging research that has examined the effects of educational interventions and experimental manipulations of arithmetic strategy on children's brain activity. Such intervention and experimental studies are critical to unravel the brain mechanisms underlying the successful learning of arithmetic. On a broader note, such studies are able to assess the impact of real-world learning on brain activity patterns and therefore provide an excellent foundation to further the field of educational neuroscience.

Keywords Arithmetic strategies · fMRI · Intervention · Experiments

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5.1 The Development of Arithmetic Strategies

Arithmetic skills are crucial to navigate through the grown-up world as adults are surrounded by numbers. For example, arithmetic skills are needed to manage a budget or to catch a train on time. To master arithmetic, children need substantial training during formal education to develop adequate arithmetic knowledge. As such, arithmetic skill development represents a core subject of the curriculum in primary schools. To support this skill acquisition, it is important to understand the development of these arithmetic skills not only at the level of behavior but also at the level of the brain, which is the key focus of the current chapter. This investigation at multiple levels of analysis has the potential to reveal a more complete theoretical understanding on how children acquire arithmetic knowledge. This knowledge may in turn serve as a ground for the design of effective educational interventions (e.g., Howard-Jones et al., 2016).

Previous research has already taken great steps in understanding how children first learn to solve arithmetic problems (Siegler, 1996; Verschaffel et al., 2007). In general, children's development in arithmetic is characterized by a change in strategy selection and efficiency (Lemaire, 2018; Siegler & Shrager, 1984). At first, children learn to solve basic arithmetic problems with simple counting strategies (e.g., 1, 2 and 3, 4, 5 to solve 2 + 3, or counting-on). Within these counting strategies, children gradually develop more efficient strategies, such as counting on from the larger number (e.g., 3, 4, 5 to solve 2 + 3, or counting-on-larger), either using these counting strategies with or without the aid of finger-based numerical representations (Geary et al., 1992). These counting procedures pave the way to learn more complex procedural strategies also known as decomposition strategies, where advanced number manipulations are used to solve arithmetic problems. Common to these procedural strategies is that children use their understanding of numbers to decompose problems into smaller or more common ones that are easier to answer. Examples of such decomposition strategies include strategies to cross 10 in addition and subtraction (such as $7 + 5 = 7 + 3 + 2$) or derived fact strategies (e.g., $6 \times 4 = 18 + 6$, if the child knows already 6 x 3 and understands that it needs to add the remainder).

While the abovementioned decomposition strategy can be used in all four operations, some of these more advanced procedural strategies are operation-specific. One example is that subtraction problems are sometimes solved using the indirect addition strategy (Torbeyns et al., 2009), where subtraction problems are answered by asking how much needs to be added to the subtrahend to arrive at the minuend (e.g., solving 71–59 through $59 + 12 = 71$). Another example is the occurrence of repeated addition strategies (e.g., $6 \times 4 = 6 + 6$ (=12) + 6 = (18) + 6 = 24) in the context of multiplication.

Because primary school education puts so much emphasis on being fluent in arithmetic (European Education and Culture Executive Agency et al., 2015), specific arithmetic exercises are repeated and rehearsed many times over the course of development. Through this frequent exposure, problem-answer associations are formed and these are labeled as arithmetic facts when they are stored in long-term memory (Siegler, 1996; Siegler & Shrager, 1984). As a result of more frequent fact retrieval, less working memory is consumed, fewer errors are made and less time is needed to arrive at the solution of the arithmetic problem as compared to the more time-consuming procedural strategies (e.g., Bailey et al., 2012). Although this transition from procedure use to fact retrieval occurs for all arithmetic operations, there are differences in the frequency of retrieval between the different operations (Vanbinst et al., 2012), with multiplication being the operation where retrieval is the most common (Campbell & Xue, 2001; Imbo & Vandierendonck, 2008). This transition to arithmetic fact retrieval of single-digit arithmetic problems is a necessary step to be able to execute more advanced and more difficult calculations, because fact retrieval relies less on cognitive systems, such as working memory, and allows more cognitive resources to be recruited for the added difficulty.

The change in children's strategy use is not a sudden transition, but a slow and gradual evolution over time. This is summarized in the overlapping waves theory (Siegler, 1996), which posits that all strategies remain available over development, but that the frequency with which certain strategies are used changes across development (Bailey et al., 2012; Jordan et al., 2003). When children advance to a new strategy, they do so because it improves the efficiency of arithmetic problem solving through increased accuracy and decreased response times (Barrouillet et al., 2008; Barrouillet & Fayol, 1998). The use of a new strategy does not mean that the old strategy is not being employed anymore. Rather, a new strategy allows for more efficient strategy selection for each individual problem. Over time, fact retrieval becomes the dominant strategy for single-digit arithmetic (Siegler, 1996), particularly in multiplication.

It is important to point out that this transition from procedures to retrieval use remains a hotly debated topic in the mathematical cognition literature. An alternative account posits that the repeated exposure to arithmetic problems does not necessarily lead to arithmetic fact retrieval but rather to the use of compacted fast procedures (Barrouillet & Thevenot, 2013; Uittenhove et al., 2016). Proponents of this account claim that the fast response times that are seen when solving arithmetic problems may not exclusively be explained by the emergence of fact retrieval (Baroody, 1983, 1994). Instead, such automatization might also occur because the procedures that are used to solve a problem become fast and automatized, also leading to a decrease of response time and an increase in accuracy. These internally executed fast procedures happen so quickly and subconsciously that the person solving the problems is not aware of them and instead believes they retrieved the answer from memory.

5.2 Arithmetic Strategies in the Developing Brain

The ability to execute different arithmetic strategies relies on a plethora of neurocognitive systems and brain regions that need to work together and that require to be coordinated meticulously. These regions have already been extensively described and summarized in adults (Arsalidou & Taylor, 2011; Menon, 2015). More specifically, brain imaging studies that have used experimentally manipulated transitions in strategy use via an arithmetic drill paradigm simulated the development of strategy use in adults (see Zamarian & Delazer, 2015 for a review). Combining this experimental strategy manipulation with brain imaging techniques, these studies are able to draw causal conclusions on the different neural systems that support different arithmetic strategies. It is critical to emphasize that these findings from adults cannot be merely generalized to children (Ansari, 2010). Previous research has already shown that it is not self-evident that similar neuronal structures will be active for similar tasks in adults and in children, as has been revealed for a range of other cognitive tasks, such as reading (Martin et al., 2015) or working memory (Thomason et al., 2009). It is therefore valuable to focus on the differences in the brain networks that support arithmetic between adults and children, and on the developmental changes that occur in these networks (Menon & Chang, 2021; Peters & De Smedt, 2018). Studies about arithmetic strategies in adults provide a valuable contribution to how arithmetic is carried out in a fully developed individual, but they are unable to explain how that individual has reached that point. Below, studies that focus on arithmetic strategy use in the brain in children are discussed and summarized, with careful attention to the similarities and the differences between children and adults.

Regions that are typically activated during arithmetic (for an anatomical description, see Fig. 5.1) generally include the intraparietal sulcus in the posterior parietal cortex, the dorsolateral and ventrolateral prefrontal cortex, the anterior insula, the ventral temporal-occipital cortex, the supramarginal gyrus and the angular gyrus in the parietal cortex, and the hippocampus and parahippocampal gyrus in the medial temporal lobe (Arsalidou et al., 2018; Menon & Chang, 2021; Peters & De Smedt,

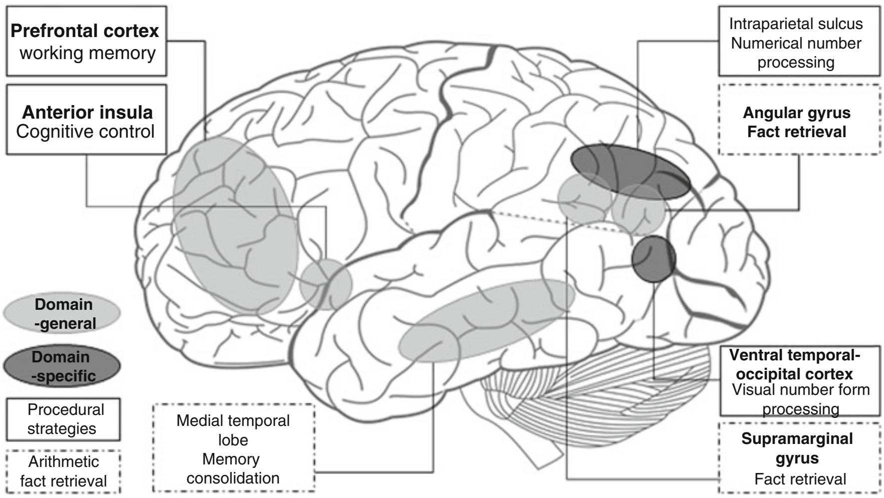

Fig. 5.1 Visual representation of brain regions important during arithmetic

2018). As apparent from these distinct regions in Fig. 5.1, there are many regions involved in the solution of arithmetic problems, with regions that are involved in more domain-specific processes (e.g., number processing), and other regions that support more domain-general processes (e.g., working memory). Activity in these regions is dependent on the specific arithmetic strategies that are being executed, and we organize our discussion of the brain regions that are involved in arithmetic along these strategies, i.e., the use of procedures and the use of fact retrieval.

Figure 5.1 Dark grey regions are more domain-specific, light grey regions are more domain-general ones. Full-line boxes indicate regions that are more active during procedural strategies, dashed-line boxes indicate regions that are more active during arithmetic fact retrieval

5.2.1 Brain Regions that Are Activated During Arithmetic Procedures

Most of the existing body of studies have focused on the intraparietal sulcus when looking at arithmetic. This region is particularly active during the solution of arithmetic procedures and more broadly, the intraparietal sulcus is consistently active in a wider array of number processing tasks (Dehaene et al., 2003; Fias et al., 2013). This activation of the intraparietal sulcus occurs whenever numerical magnitudes are being processed (Fias et al., 2003; Piazza et al., 2007). This processing of numerical magnitudes is an important component of procedural strategies, as procedural strategies manipulate numbers to find the correct solution for the arithmetic problem. For example, when applying a decomposition strategy to solve 15–7, a child needs to know that 7 can be split in 5 and 2, which is a magnitude-based decision, such that the item can be solved by subtracting 15–5 and subsequently – 2. Previous research in children ($M_{\text{age}} = 8.8$ years) has additionally shown that activity in the intraparietal sulcus during the processing of numbers is correlated with their arithmetic scores (Bugden et al., 2012), showing the important role of the intraparietal sulcus in children's arithmetic.

Together with the intraparietal sulcus in the parietal cortex, regions in the prefrontal cortex are often coactivated during arithmetic tasks (Arsalidou et al., 2018). The involvement of the prefrontal cortex in cognitive processes in general is attributed to the need of working memory capacity to solve a particular task (Blankenship et al., 2018; Metcalfe et al., 2013). This happens very frequently during the execution of calculation procedures, during which intermediate results have to be maintained in working memory (e.g., keeping 20 in working memory while solving 17 + 8 through 17 + 3 + 5). Metcalfe et al. (2013) asked 74 children between 7 and 9 years perform a visuo-spatial working memory task outside the scanner, and asked children to solve addition problems (e.g., 3 + 4) during fMRI. Individual differences in visuospatial working memory were associated with increased brain responses during arithmetic problem solving in the prefrontal cortex, more specifically in the left dorsolateral and right ventrolateral prefrontal cortex, showing the link between working memory, prefrontal activity and arithmetic problem solving in children.

Another region that shows increases in brain activity during the execution of arithmetic procedures, is the anterior insula. The anterior insula is a structure that connects the frontal and temporal lobes. It is known to be activated in a wide range of cognitive activities, including cognitive control (i.e., selecting the most efficient strategy for a specific item), and therefore is referred to as a domain-general region (see Uddin et al., 2014 for a meta-analysis). For example, Chang et al. (2018) studied the intrinsic functional connectivity (i.e., the degree of co-activation with other brain regions) of the anterior insula in 8- to 10-year-old children, specifically investigating the shared neural circuits associated with cognitive control in both arithmetic and reading. They found that there was both common network connectivity for the anterior insula for arithmetic and reading, as well as connectivity that was uniquely associated with arithmetic. They found correlations between subtraction performance and the functional connectivity of the anterior insula with the prefrontal cortex (i.e., frontal operculum), supramarginal gyrus and the premotor cortex. Similar analyses for multiplication showed significant correlations between the multiplication performance and functional connectivity of the anterior insula with the medial temporal gyrus, the inferior frontal gyrus, and the supramarginal gyrus (Chang et al., 2018). Due to the general involvement of the insula across many cognitive domains and tasks, one has to be careful with attributing a specific role to the anterior insula in arithmetic. In the meta-analysis of brain areas associated with numbers and calculation in children by Arsalidou et al. (2018), insular activity is contended to be involved in the interaction between cognition, emotion, and interoception (i.e., the sense of the physiological condition of the body, Craig, 2002), and is being linked with intrinsically motivated behaviors while learning. Thus, involvement of the insula in arithmetic tasks may additionally express the intrinsic motivation of children to learn these problems.

Finally, the visual number form processing system, situated in the ventral temporal-occipital cortex, represents another important contributor to the brain network that is active during the execution of arithmetic procedures. At this location, the visual system has specialized areas for the processing of symbolic numerical information (Menon & Chang, 2021; Yeo et al., 2017), such as the different numbers in an arithmetic problem. Visual areas are often reported in studies with children (Peters et al., 2016; Polspoel et al., 2017; Rosenberg-Lee et al., 2015), although their roles are rarely discussed in detail. Some studies suggest that this activity is attributed to differences in visual input between the different experimental conditions, because more difficult items often contain more characters (e.g., 53–21 versus 6–3), which results is more visual input (Polspoel et al., 2017). Prado et al. (2014) found grade-related (second grade to seventh grade) differences of activity in small subtraction problems in the middle occipital gyrus, showing that this region also shows differential activation based on age. The precise role of these visual areas in the development of arithmetic needs to be further clarified.

5.2.2 Brain Regions that Are Activated During Arithmetic Fact Retrieval

Although the brain network that is active during arithmetic procedures in children is fairly consistent across studies and is comparable to what has been observed in adults, this is much less the case for fact retrieval. Several studies in adults, which have used trial-by-trial verbal strategy reports, found greater activation in the left angular gyrus during arithmetic fact retrieval (Grabner et al., 2009; Tschentscher & Hauk, 2014). A similar design was applied in a study with 26 typically developing 9- to 10-year-olds (Polspoel et al., 2017). Greater activity was observed in the angular gyrus for items that were categorized as fact retrieval independent of their operation (i.e., multiplication or subtraction). There are several hypotheses on the exact role of the angular gyrus in arithmetic fact retrieval. First, it has been hypothesized that the left angular gyrus supports the retrieval of arithmetic facts from verbal memory through language-mediated processes (Dehaene et al., 2003). Another suggestion attributes the activation of the angular gyrus in arithmetic fact retrieval to its role in activating a semantic memory network (Ansari, 2008). This theory assumes that arithmetic facts are retrieved from an associative network of facts using a semantic representation of the problem. More recently, it has been suggested that the activity in the angular gyrus reflects an attentional process where the left angular gyrus is needed for the selection of the most efficient strategy to solve a particular problem (Bloechle et al., 2016). Taken together, the precise role of the angular gyrus in arithmetic remains unclear.

It is important to note that many studies on arithmetic in children have not observed increased activity in the angular gyrus during arithmetic fact retrieval (see Peters & De Smedt, 2018 for a systematic review). On the other hand, structures in the medial temporal lobe, such as the hippocampus, show increased brain activity during arithmetic fact retrieval, a result that has not been observed in adults (for an exception, see Bloechle et al., 2016). Greater hippocampal activation has been found in children using different study designs (Cho et al., 2011, 2012; De Smedt et al., 2011). In De Smedt et al. (2011), greater activity in the left hippocampus was found in children between 10 and 12 years, for small problems and addition problems, which are usually solved via arithmetic fact retrieval by children at that age. Similarly, Cho et al. (2011), found in younger children (7 to 9- years) that retrieval strategies elicited more distinct patterns of brain activity in the hippocampus as compared to counting strategies (Cho et al., 2011). Going a step further, they also showed that children who retrieved single-digit arithmetic items more frequently had greater activity in regions of the medial temporal lobe, including the hippocampus and the parahippocampal gyrus (Cho et al., 2012).

By studying children in broad age ranges, one can examine the effect of age on the role of the medial temporal lobe during fact retrieval (Prado et al., 2014; Rivera et al., 2005). Rivera et al. (2005) showed decreased brain activation with age (in 8- to 19-year-olds) during an arithmetic task where participants had to verify whether an addition or subtraction problem was correct or not. These decreases in activation were found in several frontal regions, reflecting the decreasing involvement of working memory and attentional resources to solve these problems, and regions in the left hippocampus and parahippocampal gyrus. Interestingly, Prado et al. (2014) found that multiplication was associated with grade-related increases of activity in the medial temporal lobe in 8- to 13-year-old children in a similar verification task. At first sight, these results are seemingly opposites. However, it has been suggested that the involvement of the hippocampus in fact retrieval might be time-dependent (De Smedt et al., 2011). Against the background of adult data on learning ordered sequences (Van Opstal et al., 2008), De Smedt et al. (2011) suggested that fact retrieval might be a graded phenomenon. During the early stages of retrieval, that is the first phases of consolidation of arithmetic facts into long-term memory during elementary school, the hippocampus might play a prominent role. When the actual retrieval of arithmetic facts becomes more automatic during middle and secondary school and during adolescence, this is process is supported by other brain regions, such as the angular gyrus. This switch in region that supports fact retrieval makes the role of the hippocampus more and more obsolete during development. In other words, there might be different stages of retrieval which are less or more automatized and which are supported by different brain structures. This time-dependent role for the hippocampus and the idea that there are different stages of fact retrieval is in line with a study that combined a longitudinal design in young children with cross-sectional data in adolescents and adults (Qin et al., 2014). This study showed that in 9-year-old children more frequent use of retrieval strategy use was associated with increased hippocampal engagement after one year. The study further showed that although the use of retrieval strategies continued to increase throughout adolescence and adulthood, the reliance on the hippocampus to retrieve arithmetic facts decreased. Altogether, these results support the hypothesis of the time-dependent role of the hippocampus in children and also clearly show that the brain regions that support arithmetic fact retrieval differ in children versus adults.

5.3 The Use of Educational Interventions and Experimental Paradigms to Study the Development of Arithmetic Strategies in the Brain

To further study the development of arithmetic, it is vital to focus on the transition from one strategy to the other, as this is a crucial developmental change in arithmetic. As advocated by Rosenberg-Lee (2018), a powerful way to do this is to couple studies that use an arithmetic intervention or a specific experimental manipulation of an aspect of arithmetic development (e.g., a particular strategy), with neuroimaging. In her discussion, she advocates for study designs that have four key components: (1) an initial behavioral assessment, (2) a pre- and post-neuroimaging assessment, (3) a measure of learning outside the scanning environment to supplement the limited assessment possibilities during functional scanning, such as a strategy assessment or standardized arithmetic tests that are not feasible to do during scanning and (4) an intervention or experimental manipulation of arithmetic (Rosenberg-Lee, 2018). This pre- and post-intervention neuroimaging assessment approach enables the learning of arithmetic to be studied by monitoring changes in the brain that occur as a consequence of learning.

In this chapter, we distinguish between two different approaches that can be used. The first approach that is discussed in this chapter focusses on arithmetic (educational) interventions. These interventions usually take place over a couple of weeks or months, and their content is based on real-life learning and practices. They are also explicitly designed to improve educational outcomes and they reveal how environmental factors (e.g., education) change brain structure and function, thereby providing insights into the plasticity of the brain. Table 5.1 summarizes the main findings of different studies with the same educational intervention design. However, the

Table 5.1 Summary of intervention studies on arithmetic strategy use in the brain in children

Study	Sample (M_{age})	Intervention characteristics	Neurological assessment	Main neurological findings
Supekar et al. (2013)	24 intervention group (8.5)16 control group (9.0)	8 weeks3 times a week40–50 minutes per session1:1 Increasing difficultybasic number propertiescounting Strategiesnumber families	Addition task pre & post intervention	↑ GMV of HC ~ ↑ performance efficiency after intervention↑ FC between HC and PFC ~ ↑ performance efficiency after intervention
Iuculano et al. (2015)	15 TD children (8.54)15 MLD children (8.65)	See Supekar et al. (2013)	Addition task pre & post intervention	After intervention, brain activity MLD = brain activity TD↑ intervention-induced brain plasticity ~ ↑ tutoring-induced performance efficiency
Jolles et al. (2016)	21 intervention group (8.61)21 no-contact control group (9.02)	See Supekar et al. (2013)	Addition and subtraction task pre & post intervention	↑ FC of IPS after intervention ≠ ↑ FC of AG after Interventiononly ↑ FC of IPS ~ ↑ intervention-induced performance
Rosenberg-Lee et al. (2018)	19 intervention group (8.5)15 no-contact control group (8.8)	See Supekar et al. (2013)	Addition task pre & post intervention	↑ activity HC after intervention↑ retrieval ~ ↓ Activation in AG and IFG↑ retrieval ~ ↑ FC of HC with IPS

AG angular gyrus, *FC* functional connectivity, *GMV* grey matter volume, *HC* hippocampus, *IFG* inferior frontal gyrus, *IPS* intraparietal sulcus, *MLD* mathematical learning difficulties, *MRI* magnetic resonance imaging, *PFC* prefrontal cortex, *RS* resting state, *TD* typically developing disadvantage of this more naturalistic approach, is that it includes a variety of manipulations and methods as happens in real classroom learning. This can make it difficult to determine which specific manipulation has causal effects on arithmetic performance.

In contrast to arithmetic interventions, the second approach that are discussed here focus on controlled experiments that manipulate arithmetic strategy use. This allows researchers to draw more specific conclusions regarding causation. The use of a very specific experimental manipulation allows a more focused approach where only one particular aspect of arithmetic, such as strategy use, is manipulated. For example, arithmetic strategy use in adults has already been investigated through a short-term manipulation in which arithmetic items were drilled, forcing the participants to learn new arithmetic problems by heart and to switch from decomposition procedures to arithmetic fact retrieval. Through this more controlled experimental approach, these studies are able to analyze mechanisms behind arithmetic development. These studies are typically also short in time which allows them to eliminate maturational effects on the brain as much as possible. The main findings of studies applying an experimental approach in children are summarized in Table 5.2. However, these experimental manipulations are very different from the educational reality. They also are not designed to improve educational practices but only allow one to draw causal conclusions on the aspect that was manipulated during the experiment. These experimental designs do not reveal how arithmetic should be taught, but they do offer information on the mechanisms through which arithmetic development occurs.

Taken together, the term intervention will be used to indicate studies that have a more naturalistic approach. These studies have provided an education-based intervention that stretches out over a couple of weeks or months, where instruction is focused on broad concepts of arithmetic. These interventions have less power to infer the causal dimensions of a particular arithmetic process, but they are more ecologically valid. The term experimental manipulation or experiment will be used

Table 5.2 Summary of studies that involve experimental manipulations of arithmetic strategy use in children

Study	Sample (M_{age})	Experimental characteristics	Neurological assessment Operation	Main neurological findings
Soltanlou et al. (2018)	20 TD children (11.1)	6 sessions over 2 weeks at home8 simple multiplication items8 complex multiplication items	Multiplication task pre & post experiment	↓ activity R MFG at post-test
Chang et al. (2019)	22 for univariate analyses (9.46) 17 for multivariate analyses (9.42)	5 sessions over 1 week, 1:114 double-digit plus single-digit addition problems	Addition post experiment only	↓ activity fronto-parietal network at post-test↑ activity AG and MLT at post-test

AG angular gyrus, *MFG* middle frontal gyrus, *R* right to indicate studies that used a controlled experimental arithmetic drill paradigm to only manipulate strategy use. As seen in the summary of the included studies in both Table 5.1 and Table 5.2, all studies have small sample sizes, mainly because of the inclusion of neuroimaging techniques. These are often time-consuming, expensive, and they are not straightforward to use in children. It is vital to emphasize that the number of both types of studies on arithmetic in children that use neuroimaging techniques is small, which is important to keep in mind when drawing conclusions.

5.3.1 Arithmetic Educational Interventions

The available studies that combine an arithmetic intervention with pre- and post-intervention neurological assessment all used the same intervention protocol (Iuculano et al., 2015; Jolles et al., 2016; Rosenberg-Lee et al., 2018; Supekar et al., 2013). In this protocol, children underwent an 8-week math tutoring program focused on conceptual instructions with speeded retrieval of additions and subtraction. Full details of the intervention designs can be found in Table 5.1.

All intervention studies reported improvements in arithmetic performance after intervention for the experimental group, either measured in increased accuracy, decreased response times, or in a performance measure that combined accuracy and response times. This improvement is usually interpreted as likely due to an increase in the use of arithmetic fact retrieval. It is important to note that only one of the studies included a specific strategy measure that was able to explicitly confirm a significant increase of fact retrieval (Supekar et al., 2013). It is therefore also plausible that improvements of response times as a consequence of the intervention are due to more efficient procedural strategy use. One possibility is that participants switch from slower procedures such as repeated addition (e.g., $3 \times 7 = 7 + 7 + 7$) to derived facts (e.g., $3 \times 7 = 14 + 7$). Another possibility is that the execution of these procedural strategies becomes faster. This cannot be ruled out by the existing evidence and it emphasizes the need for future studies to carefully document how strategies change throughout the intervention.

All studies included a neurological assessment before and after the intervention. Although the inclusion of a pre-test results in a strong design and the opportunity to directly see changes in brain activation post-intervention, only Rosenberg-Lee et al. (2018) directly contrasted the post-intervention fMRI with the pre-intervention fMRI to see where in the brain the intervention caused changes in brain activity. Rosenberg-Lee et al. (2018) showed increased activation in the left hippocampus after the intervention for the intervention group, while the control group did not show any increased activation. This increase in hippocampal activity is in line with the abovementioned cross-sectional studies in children that have shown increases in hippocampal activity during the acquisition of arithmetic facts (Cho et al., 2011, 2012; De Smedt et al., 2011; Prado et al., 2014; Rivera et al., 2005).

As we have elaborated above, many different brain regions are consistently active when performing arithmetic. These regions are not necessarily neighboring each other and they need to communicate through functional connections or co-activations. For example, Jolles et al. (2016) focused on the functional connections of the intraparietal sulcus and angular gyrus before and after the intervention. Jolles and colleagues reasoned that, although that the intraparietal sulcus and the angular gyrus are two neighboring regions of the parietal cortex, they are assumed to have distinct roles in arithmetic and thus will have different functional connections. Before the intervention, the intraparietal sulcus showed connections with the dorsolateral fronto-parietal network, while the angular gyrus was connected with the default mode network. This latter network consists of medial and lateral parietal, medial prefrontal and medial and lateral temporal cortices, and is characterized by being active during rest when the person in not focused on a specific task (Raichle, 2015). Thus, the functional segregation between the networks of the intraparietal sulcus and the angular gyrus during arithmetic is already present before intervention, which adds to the evidence that the intraparietal sulcus and the angular gyrus have distinct roles in mathematical cognition. This is further corroborated by the fact that the functional connections of the intraparietal sulcus and angular gyrus changed in a different way as a result of the intervention (Jolles et al., 2016). These findings by Jolles et al. (2016) are consistent with the interactive specialization hypothesis of brain function (Johnson, 2011). This theory posits that learning, with arithmetic as an example here, results in the differentiation of functional brain circuits. Interestingly, the only intervention-induced change in connectivity that was common between the network of the intraparietal sulcus and the network of the angular gyrus was the change in the connectivity with the hippocampus (Jolles et al., 2016).

In Rosenberg-Lee et al. (2018) intervention-induced increases in retrieval use were also correlated with increases in connectivity between the intraparietal sulcus and hippocampus only in the intervention-group. Similarly, intrinsic functional connectivity of the hippocampus predicted individual differences in arithmetic skill acquisition after the 8 weeks of a mathematical intervention (Supekar et al., 2013). These results strengthen the idea that the hippocampus, and by extension, the medial temporal lobe, supports learning through the integration of signals from different subdivisions of the posterior parietal cortex of which the intraparietal sulcus and the angular gyrus are both part of. The correlations between intervention-induced increases in either performance or fact retrieval with more hippocampal connectivity with the intraparietal sulcus confirm that the hippocampus plays an important role in the consolidation of arithmetic facts (Jolles et al., 2016; Rosenberg-Lee et al., 2018; Supekar et al., 2013).

Insights into the plasticity of the brain and the circuits associated with learning can also be helpful for the development of remediation programs for children with mathematical learning difficulties. This is particularly relevant in view of the differences in brain activity in these children as compared to typically developing children (Peters & De Smedt, 2018). For example, studies indicate that children with mathematical learning difficulties typically show an overactivity in the abovementioned regions of the arithmetic brain network (Fias et al., 2013; Rosenberg-Lee et al., 2015). Against this background, Iuculano et al. (2015) aimed to compare the effects of an intervention in typically developing children to the effects of the same intervention in children with mathematical learning difficulties. This allowed Iuculano et al. (2015) to test different hypotheses on the effects of interventions in children with mathematical learning difficulties, i.e., the normalization hypothesis, the neural compensation hypothesis, and the neural aberration hypothesis. Following the normalization hypothesis, atypical brain responses in children with mathematical learning difficulties should disappear after an intervention. The neural compensation hypothesis predicts that interventions lead to the compensatory mechanisms or the activation of brain regions outside the arithmetic network. Finally, the neural aberration hypothesis predicts altered brain activity in the same regions after tutoring, even though behavior responses have normalized. Iuculano et al. (2015) reported that the intervention was able to normalize the brain activity of children in the mathematical learning disability group and that no compensatory brain activation was observed. These results support the neural normalization hypothesis, where atypical brain responses in children with mathematical learning difficulties disappear after an intervention.

5.3.2 Experimental Manipulations: Arithmetic Drill Studies

As explained above, experiments that manipulate specific aspects of arithmetic have a different starting point than educational interventions. While interventions studies provide important information about brain plasticity during arithmetic development in general, they are too broad to detect precise mechanisms related to arithmetic development. In the following section, we discuss how a controlled experimental approach has been successfully used in brain imaging studies in adults (see Zamarian & Delazer, 2015, for a review) and has been adapted for children to study the transition from procedural strategies to arithmetic fact retrieval.

In this approach, the experimental manipulation is the repeated presentation of specific arithmetic items that should lead to changes in strategy use. By keeping the manipulation short, researchers avoid potential confounding effects of (natural) maturational brain development on the observed findings. It is important to note that this is not an ecologically valid approach, and consequently it cannot be considered as an approach to improve education. On the other hand, it allows us to understand the mechanisms behind one very particular process, that is the transition from procedures to fact retrieval. The major logic of these short-term experiments, which manipulate the transition from calculation procedures to arithmetic fact retrieval, is to select items that are complex (e.g., multi-digit multiplication items or artificial operations) and are solved using procedural strategies first. Over the course of the study, usually only a few days long, these items are presented several times. Through repeated exposure to these complex arithmetic items, participants start to develop problem-answer associations or arithmetic facts.

In adults, these short-term experiments have already been applied (see Zamarian & Delazer, 2015 for a review). They provide an important scaffold for similar studies that focus on arithmetic strategies in children. In one of the first fMRI studies by

Delazer et al. (2003), healthy adults practiced 18 complex (double-digit x single-digit) multiplication problems daily over the course of a week. At post-test, all participants underwent an fMRI during which both the trained items and untrained items were presented. The untrained arithmetic items were still solved using procedural strategies while the trained items changed strategy and became arithmetic facts. Note that, echoing current debates in the field of mathematical cognition, it could be that these items were solved via compacted procedures as well. Untrained items, which are expected to be solved using procedures, elicited greater activity compared with trained items in a large-scale network, including the intraparietal sulcus, the inferior parietal lobule and the inferior frontal gyrus. This reflects the involvement of number processing and working memory resources that are needed to execute procedures. Trained items, which are expected to be solved using fact retrieval, elicit greater activity compared to untrained items in the angular gyrus, reflecting the retrieval of facts from memory. These findings were replicated in a series of similar subsequent studies with different experimental paradigms, such as comparing rote learning with meaningful learning (Delazer et al., 2005), or using addition rather than multiplication (Ischebeck et al., 2006).

Such short-term experiments in children, although they have been suggested as a promising avenue for research (Rosenberg-Lee, 2018) are very scarce. To the best of our knowledge, only two studies have used this approach (Chang et al., 2019; Soltanlou et al., 2018). Details of these studies can be found in Table 5.2. The increased efficiency and fact retrieval frequency indicate that the short-term experiment was successful in manipulating the arithmetic strategy in children (Chang et al., 2019; Soltanlou et al., 2018).

Chang et al. (2019) identified experiment-related differences between problems in neural activity for children, as found in most studies with adults that are discussed above. The untrained problems showed greater activity in the fronto-parietal network than the trained problems, while the trained problems showed greater activity in the medial temporal lobe and the angular gyrus compared to the untrained problems. The short-term study of Soltanlou et al. (2018) found similar reduced activation in the middle frontal gyrus for the trained items at the post-test, but did not replicate the shift from the intraparietal sulcus to the angular gyrus as in Chang et al. (2019). It is important to note that these studies differed in the brain imaging method they used. Chang et al. (2019) used fMRI to look at brain activity while Soltanlou et al. (2018) used a combination of electroencephalography and functional near-infrared spectroscopy. One problem of these latter two methods is that they have a lower spatial resolution, making it more difficult to detect differences between adjacent brain areas, such as the intraparietal sulcus and angular gyrus. Another problem is that the methods used by Soltanlou et al. (2018) can only record cortical activity while brain activity of deeper sub-cortical structures such as the hippocampus is more difficult to detect. These methodological differences might explain the differences between the two studies, particularly in their findings related to fact retrieval.

5.4 Discussion

Although the number of brain imaging studies on arithmetic has increased over the last decade, there is still a lot of work to be done to further establish the brain activations, patterns, and connections that play a role in arithmetic strategies. The results from previous cross-sectional and longitudinal studies have generally converged to the same findings using different approaches or designs. Similar to adults, children recruit a fronto-parietal network during those items that are assumed to be solved using procedural strategies. This network includes regions that play a role in number magnitude processing and working memory, such as the intraparietal sulcus (Bugden et al., 2012) and the prefrontal cortex (Metcalfe et al., 2013) respectively. Next to these two regions, there also seem to be important contributions of the anterior insula (Arsalidou et al., 2018; Chang et al., 2018) and the ventral temporal-occipital cortex (Menon & Chang, 2021; Prado et al., 2014), although their role in arithmetic strategies have been less investigated and their precise function in arithmetic is not entirely clear.

For arithmetic fact retrieval, children show greater activity in the hippocampus (Cho et al., 2011, 2012; De Smedt et al., 2011; Prado et al., 2014; Qin et al., 2014; Rivera et al., 2005). As the hippocampus is known to play a major role in the consolidation of information from short-term memory to long-term memory (Moscovitch et al., 2016), these studies attributed the role of the hippocampus in arithmetic fact retrieval, and specifically during development, to the consolidation of the association between a problem and its answer. During adulthood, most literature points to the left angular gyrus in fact retrieval, although the exact function of the angular gyrus in arithmetic fact retrieval remains debated. Some studies argue that the function of the angular gyrus during arithmetic fact retrieval is to retrieve verbal information (Dehaene et al., 2003), while others attribute the function of the angular gyrus to the involvement of semantic memory (Ansari, 2008) or to attentional processes (Bloechle et al., 2016). The differences between children and adults during arithmetic fact retrieval have paved the way for the idea of a time-dependent role of the hippocampus (Prado et al., 2014; Qin et al., 2014; Rivera et al., 2005). Once the associations between problems and their solutions become fully consolidated, other parts of the brain, potentially the angular gyrus, take over the retrieval from memory, and the involvement of the hippocampus disappears. So, activity first increases in the hippocampus during the acquisition phase, which might take place over elementary school. Later activity decreases again after consolidation, the process of which might take place over middle and secondary school and adolescents. In all, this indicates that even the basic acquisition of arithmetic facts continues during middle school and adolescence at the level of the brain, although these developments are not always detectable in behavior.

An important topic for further research is the investigation of the time-frame for both the emergence of hippocampal activity, as well as how long it is sustained before the angular gyrus replaces the hippocampus in the role of supporting arithmetic fact retrieval. The current literature points to an increased reliance of the hippocampus starting around 7 or 8 years of age (Rivera et al., 2005; Rosenberg-Lee et al., 2011), with a peak between 9 and 15 years of age (Qin et al., 2014). Intervention studies as well as controlled experiments that use brain imaging techniques are needed to unravel the precise time-course of hippocampal activity which is likely to change during middle childhood.

It is valuable to know how the change from early consolidation to more automatized arithmetic facts takes place at the level of the brain. Many of the abovementioned studies found that the effects of the interventions were not specifically reflected in behavioral measures such as accuracy or reaction times, but that these effects were only observed at the level of the brain (Iuculano et al., 2015; Jolles et al., 2016; Rosenberg-Lee et al., 2018; Supekar et al., 2013). This is also observed in longitudinal designs (Qin et al., 2014). As an example, in a detailed experimental study on fast learners and near transfer, Chang et al. (2019) were able to show that faster learners showed greater overlap in neural representations within the medial temporal lobe, as well as greater segregation of large-scale brain circuits between trained and novel problems. These effects were not observed at the level of behavior. Taken together, this illustrates that brain imaging studies can unravel subtle fine-grained processes that cannot be gleaned from behavioral data alone (De Smedt, 2018). These findings further add to a more complete understanding on how arithmetic develops.

Understanding how these subtle neural processes lead to individual differences in typically developing children are an important gateway to studies on understanding atypical mathematical development. It remains to be determined whether the aberrant activation of the intraparietal sulcus, a critical region important for number processing, is the cause or consequence of mathematical learning difficulties (see Rosenberg-Lee, 2018, for a discussion). Iuculano et al. (2015) showed that an arithmetic educational intervention can normalize brain activation in children with mathematical learning difficulties, after overactivation was detected before the intervention. This finding suggests that the aberrant activation in the intraparietal sulcus in children with mathematical learning difficulties could be a consequence of lowered exposure or expertise in mathematics, although this should be validated in future studies.

Although the sections above show the potential of brain imaging data in educational interventions or short-term experimental studies, some caution is needed. Such studies are not without challenges, both practically and theoretically. Practically, neuroimaging with children is more demanding than neuroimaging with adults. The duration of the neuroimaging sessions are more limited, as the environment is taxing (e.g., confined space, no movement allowed), and children's focus needs to by high for the full duration of the assessment. Another practical consideration to make is that both interventions and experimental studies require a lot of contact moments, often one-on-one. It takes a lot of time and effort to plan and to execute these studies.

Theoretically, one also needs to understand the limitations of these approaches. The studies reviewed in this chapter can help to uncover the mechanisms that underlie the development of arithmetic, yet they cannot determine how arithmetic should be taught (De Smedt, 2018). It is also critical to distinguish between what we have called arithmetic educational interventions and experimental studies in this chapter. As already touched upon, these are two different research designs, with each approach having its own merits and with specific research questions that are more suitable for one or the other approach. For example, the educational interventions are more closely aligned with learning in real-life context than experimental studies. This adds to the ecological validity of the educational interventions but makes it more difficult to draw conclusions on how specific arithmetic processes, such as procedural strategy use, might have changed. The latter can be addressed by experimental studies, yet the ecological validity of such studies is low.

The current chapter has looked at these educational interventions and experimental studies through the lens of mathematical cognition and arithmetic, yet these approaches are relevant for research in other cognitive domains as well, especially in those domains that show processes of automatization upon which the transition from effortful procedures to fluent fact retrieval are dependent. In reading, for example, the acquisition of letter-speech sounds associations is considered a basic requirement for learning to read (Ehri, 2005), as the acquisition of problem-answer associations is necessary for fact retrieval. As in arithmetic, the accuracy of which letter-speech sounds are correctly matched increases throughout development and response times decrease steadily over the course of elementary and middle school reading instruction (Froyen et al., 2009). Short-term, highly controlled experiments can also be applied for other cognitive domains, such as reading, especially if there is a clear role for automatization. In general, these experiments can advance the field of educational neuroscience by providing deeper understanding of how the automatization takes place in the brain.

To combine the goals of both neuroimaging (i.e., understanding how the brain works) and education (i.e., understanding how learning can be improved), one can ask the question 'how do we learn' at multiple levels of analysis (Rosenberg-Lee, 2018). Except for the few studies discussed in this chapter, the potential for this approach remains untouched to a great extent. There is a need for studies with pre/post neuroimaging designs and studies that examine the effect of educational interventions or experimental manipulations on brain activity. These studies should include a sufficient number of participants and robust designs in order to have better powered studies. This allows one to better understand the development of arithmetic fact retrieval with its switch from hippocampus to angular gyrus. There is a need to delineate the time frame, likely in middle school, in which this switch occurs and which individual differences influence this change. Such knowledge will serve as a ground to further understand how the development of arithmetic facts is altered in children with mathematical learning difficulties.

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Chapter 6
The Role of Neuropsychological Processes in Mathematics: Implications for Assessment and Teaching

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Abstract What neuropsychological processes underlie mathematics performance? What cognitive strategies should we teach to boost mathematics performance? Undoubtedly, both questions lie at the heart of mathematics research and are critical for many teachers and practitioners. In this chapter, we aim to provide an answer to both questions by drawing readers' attention to four separate but interrelated neuropsychological processes: Planning, Attention, Simultaneous, and Successive (PASS) processing. In the first section of this chapter, we review the literature on PASS neuropsychological processes and their relation to mathematics performance. In the second section, we present evidence on how information from assessing children on these neuropsychological processes (see Cognitive Assessment System; [Naglieri, J. A., Das, J. P., & Goldstein, S., *Cognitive Assessment System—Second Edition: Brief*, 2014]) can be used to describe children with mathematics difficulties or superior mathematics performance in three countries (Canada, China, and Cyprus) representing three different cultures (Western, East Asian, and European). These profiles clearly show that children have weaknesses (in the case of children with mathematics difficulties) or strengths (in the case of high achievers) in Planning and, to a lesser extent, Simultaneous processing, and this is irrespective of the cultural background of the participants. Finally, in the last section, we discuss the role of planning facilitation that can be used to enhance cognitive planning and mathematics performance.

Keywords Pass theory · Neuropsychological processes · Intelligence · Intervention · Mathematics · Planning

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Several studies have shown that intelligence – operationalized with IQ tasks – is a significant predictor of mathematics performance (e.g., Foster et al., 2015; Geary, 2011; Kyttälä & Lehto, 2008; Manolitsis et al., 2013; Passolunghi et al., 2014). In a meta-analysis, Peng et al. (2019) showed that the average correlation between fluid intelligence and mathematics was .41. Despite this, researchers have expressed some concerns because some tasks in popular psychometric batteries of intelligence (e.g., Vocabulary and Arithmetic in the Wechsler Intelligence Scale for Children [WISC]) have a close resemblance with academic achievement tasks and partly measure what children already "know" (i.e., what they have learned at school or at home) than how children "think", which is the ultimate purpose of assessing intelligence. In addition, a circular argument ensues when mathematics within an IQ assessment is used to predict mathematics.

To bypass these problems, Das et al. (1994) proposed a neurocognitive theory of intelligence called PASS (for Planning, Attention, Simultaneous, and Successive processing) and a way to measure it (Cognitive Assessment System [CAS]; Naglieri & Das, 1997; see also Naglieri et al., 2014, for its second edition). The PASS theory, as operationalized by the CAS, emphasizes that: (a) a test of intelligence should be based on a theory of intelligence; and (b) the test should measure basic neurocognitive processes defined by the intellectual demands of the test, not the content of the questions. Thus, in this chapter we aim to introduce researchers and practitioners to the PASS theory of intelligence and how it relates to middle school mathematics performance.

6.1 The PASS Theory of Intelligence

Alexander R. Luria's (1966, 1973) research on the functional aspects of brain structures formed the basis for the Planning, Attention, Simultaneous, Successive (PASS) theory initially described by Das et al. (1994). Das and colleagues used Luria's work as a blueprint for defining the basic neuropsychological processes that underlie human performance.

According to Luria (1973), human cognitive functions could be conceptualized as three separate but interrelated "functional units" that provide four basic neuropsychological processes. These three functional units have been used by Naglieri and Das (1997) as the basis of Planning (third functional unit), Attention (first functional unit), and Simultaneous and Successive (second functional unit) cognitive processes.[1] The first functional unit provides regulation of cortical arousal and attention, while the second unit codes information using Simultaneous and Successive processes. The third unit is responsible for strategy development, strategy use, self-monitoring, and control of cognitive activities. Although Luria did not have the same

[1] The acronym PASS does not follow the order of the three functional units. Das et al. (1994) chose PASS because it is easy to remember.

neuroimaging techniques that exist today, his conceptualization of how the brain functions is still valid. Studies using fMRI and EEG (Avram et al., 2013; McCrea, 2009; Okuhata et al., 2009) have shown that each area of the brain participates in numerous large- and small-scale functional systems within and across cortical and sub-cortical brain structures.

6.2 Operationalization of PASS Processes

To operationalize the four PASS processes, Naglieri and Das (1997) developed the Das-Naglieri Cognitive Assessment System (D-N CAS). This first edition of D-N CAS had two forms: a Standard Battery with 12 subtests (3 for each PASS process) and a briefer version called Basic Battery with 8 subtests (2 for each PASS process). In 2014, Naglieri et al. published the second edition of CAS and along with it the CAS:2-Brief (4 subtests), the CAS:2 Rating Scale (4 subtests), and a Hispanic version (see Fig. 6.1, for CAS's various versions).

The CAS (both editions) can generate three types of scores: (1) a Full Scale score (expressed as a standard score with a mean of 100 and a standard deviation of 15), which is an index of an individual's overall cognitive functioning, (2) a standard

CAS2 Extended Battery (12 subtests)	CAS2 core Battery (8 subtests)	CAS2: Spanish (12 subtests)	CAS2: Brief (4 subtests)	CAS2: Rating Scale (40 items)
		Planning		
Planned Codes	Planned Codes	Códigos planificados	Planned Codes	Planning items
Planned Connections	Planned Connections	Conexiones planificadas		
Planned Number Matching		Planificación de números pareados		
		Simultaneous		
Matrices	Matrices	Matrices	Simultaneous Matrices	Simultaneous items
Verbal-Spatial Relations	Verbal-Spatial Relations	Relaciónes verbales-espaciales		
Figure Memory		Memoria de figuras		
		Attention		
Expressive Attention	Expressive Attention	Attention expresiva	Expressive Attention	Attention items
Number Detection	Number Detection	Detección de numeros		
Receptive Attention		Atención receptiva		
		Successive		
Word Series	Word Series	Serie de palabras	Successive Digits	Successive items
Sentence Repetition/ Sentence Questions	Sentence Repetition/ Sentence Questions	Repetición de oraciones		
		Preguntas a oraciónes		
Visual Digit Span		Retención visual de digitos		

Fig. 6.1 The various versions of CAS:2 with their sub-tests

score for each PASS process (with a mean of 100 and a standard deviation of 15), which indicates the individual's cognitive functioning on the specific cognitive process (e.g., Planning) and is used for the identification of specific strengths and weaknesses in cognitive processing, and (3) a scaled score (with a mean of 10 and a standard deviation of 3) for each CAS subtest. According to Naglieri et al. (2014), the maximum standard score is 160 and the maximum scaled score 19.

The tasks used to assess each of the PASS processes in CAS:2 are the following: (a) Planning [subtests: Planned Connections, Planned Codes, and Planned Number Matching], (b) Attention [subtests: Expressive Attention, Number Detection, and Receptive Attention], (c) Simultaneous processing [subtests: Matrices, Verbal-Spatial Relations, and Figure Memory], and (d) Successive processing [subtests: Word Series, Sentence Repetition–replaced by Sentence Questions in ages 8–17, and Visual Digit Span].

Test administration and scoring varies among the PASS processes depending on task demands. There are subtests, such as Planned Number Matching, Planned Codes, Expressive Attention, Number Detection, and Receptive Attention, in which scoring begins with recording the time and number correct (accuracy score) for each item. These are combined into ratio scores obtained from a table provided in the Record Form. The ratio scores are then summed across items to obtain subtest raw scores, which are converted to the subtest scaled score. In turn, the scoring for subtests such as Matrices, Verbal-Spatial Relations, Figure Memory, Word Series, and Sentence Repetition/Sentence Questions is the total number correct. Interpretation of CAS-2 for practitioners is detailed in *Essentials of CAS-2 Assessment* (Naglieri & Otero, 2017).

6.3 The Relation of PASS Processes with Mathematics Performance

Each PASS process is theoretically linked to mathematics performance (e.g., Das et al., 1994; Das & Misra, 2015). An obvious one is between Planning and problem solving. In problem solving, individuals must make decisions on how to solve a mathematics problem, monitor their own performance, and revise their initial plan as more information becomes available (Schoenfeld, 1994). All of these are characteristics of cognitive planning. Attention is important for selectively attending to the important components of a problem and for suppressing irrelevant information (e.g., in a word problem asking students to find the difference between the cost of a computer in two different stores, the name of the stores is not that important to attend to while solving the problem). Simultaneous processing is relevant for tasks that consist of different interrelated elements that must be integrated into a whole, as in solving an equation with multiple operations (e.g., $(3 + 5) \times (4 + 4)/2 = ?$)) or in areas of mathematics that involve integration of visual-spatial information (e.g., geometry). Finally, Successive processing is relevant when information has to be processed in a certain order, as in counting.

Three lines of research have generally confirmed these theoretical connections. First, several studies in different languages with typically-developing children have shown significant effects of these skills on mathematics performance (e.g., Best et al., 2011; Cai et al., 2016; Georgiou et al., 2015; Naglieri & Rojahn, 2004; Kroesbergen et al., 2010). Naglieri and Rojahn (2004), for example, found that the average correlations between PASS processes and Broad Math[2] in a sample of 1559 children aged 5 to 17 years range from .45 to .58 (the highest being between simultaneous processing and Broad Math). A recent meta-analysis by Georgiou et al. (2020) has reported similar correlations. More specifically, Georgiou et al. (2020) found the average correlation between PASS processes and mathematics to be .46. However, there were also significant interactions between the PASS processes and the mathematics outcomes. Simultaneous processing produced significantly stronger correlations with mathematics accuracy ($r = .42$) and problem solving ($r = .48$) than mathematics fluency ($r = .18$). In turn, Planning correlated more strongly with mathematics fluency ($r = .42$) than Simultaneous processing ($r = .18$). Finally, Simultaneous processing correlated more strongly with problem solving ($r = .48$) than Attention ($r = .34$).

Second, some studies have examined the role of PASS processes in mathematics disabilities (e.g., Cai et al., 2013; Iglesias-Sarmiento & Deaño, 2011; Iglesias-Sarmiento et al., 2017, 2020; Kroesbergen et al., 2003). Kroesbergen et al. (2003), for example, examined if children with mathematics disabilities (MD) have difficulties in the PASS processes compared to their chronological-age (CA) controls. Their sample consisted of 137 students with MD ($M_{age} = 8.9$ years) recruited from general education classes, 130 children with MD ($M_{age} = 10.5$ years) recruited from special education classes, and 185 children without MD ($M_{age} = 9.8$ years). The results of multivariate analyses showed that the students with MD performed significantly lower than their CA controls on all PASS processes. Students with MD from special education classes also scored significantly lower than their MD peers from general education classes. Similarly, in a study with older students with MD, Cai et al. (2013) found that a group of 55 children with MD recruited from Grades 6, 7, and 8 in Shanghai, China, performed significantly poorer than a control group of 56 typically-developing children on all four PASS processes. However, the differences were more pronounced in Simultaneous processing and Planning.

Finally, researchers have conducted intervention studies and examined the impact of training children in one or more PASS processes on children's mathematics performance (e.g., Iseman & Naglieri, 2011; Naglieri & Gottling, 1997; Naglieri & Johnson, 2000; van Luit & Naglieri, 1999; see also Das & Misra, 2015, for a review). This line of research began with the work of Cormier et al. (1990) and Kar et al. (1992). In both studies, researchers used a dynamic assessment approach that

[2] Broad Math is a cluster score derived from three subtests (Calculations, Math Fluency, and Applied Problems) from the Woodcock Johnson III (WJ-III; Woodcock et al., 2001).

required children (1) to describe carefully and systematically the task at hand, (2) to think-aloud as the problem was solved, (3) to explain why a particular answer is correct or incorrect, and (4) to explain why the non-chosen alternatives were incorrect. Children with poor planning skills benefited the most and even outperformed the group of good planners at post-test on tasks requiring planning and strategy use. This research was extended to mathematics by Naglieri and Gottling (1995, 1997) who found that the use of strategies by learning disabled children ($M_{\text{age}} = 10.10$ years) could be facilitated, rather than directly taught, resulting in improved performance in math calculation. Naglieri and Johnson (2000) further showed that Grade 6 to 8 children (age ranged from 12 to 14 years) with a cognitive weakness in Planning improved considerably over baseline rates both in Planning and in classroom mathematics performance following strategy instruction.

6.4 Clinical Use of CAS

In this section, we aim to further illustrate the role of PASS processes in mathematics (a) by examining with the help of a single case design the cognitive profile of six children with mathematics giftedness as well as six children with mathematics disabilities in three diverse cultures (North American, East Asian, and European) and (b) by examining associations between the verbal responses (i.e., retrospective think-aloud protocols) of the Cypriot children while solving the Planning and Simultaneous processing tasks and their responses when solving selected mathematics tasks.

6.4.1 The Cognitive Profile of Children with Mathematics Giftedness

The children with mathematics giftedness were selected from previous studies (e.g., Dunn et al., 2020; Sergiou et al., 2021; Wang et al., 2018), because they scored in the very superior range (standard score higher than 130) range in Woodcock-Johnson III Broad Math (for the sample from Canada and China).[3] In Cyprus, children with mathematics giftedness were selected if they had scored more than 2 standard deviations above the group mean on two tests of mathematics performance

[3]The children's standard score in WJ-III Broad Math was as follows: George scored 142, Kate scored 134, Jiao scored 148, and Ming-tun scored 136. In Cyprus, the children's z score for the mathematics achievement test was 1.4 (Gabriel) and 1.5 (Andreas), whereas their score in logits for the mathematics reasoning test was 1.8 (Gabriel) and 1.6 (Andreas).

(Mathematics Achievement Test [Kyriakides et al., 2019] and Mathematics Reasoning Test [Sergiou & Charalambous, 2019]).

The two children from Edmonton (Canada), George (a Grade-4 male) and Kate (a Grade-6 female; all names are pseudonyms to protect privacy), were attending an enrichment program in a public elementary school. To attend an enrichment program in Alberta, children must be coded as gifted. Their age was 9.8 and 11.1 years, respectively, and both reported English as their native language. In turn, Jiao (female) and Ming-tun (male) were Grade 6 students from Hangzhou (China). Both were attending an enrichment program in an elementary school and reported Mandarin as their native language. Their mean age was 11.0 and 11.3 years, respectively. Finally, Gabriel and Andreas were both males attending Grade-6 regular classes in pubic elementary schools in Cyprus. Both were Greek speakers and at the time of data collection they were 11.8 and 11.7 years old, respectively.

To examine the cognitive profile of these children, we plotted their PASS scale scores (see Fig. 6.2). The findings were rather consistent across the three sites. First, all children with mathematics giftedness scored in the superior range (>120) in Planning and in the average range (90–110) in Successive processing. Second, the children from Canada and China scored in the superior range in Simultaneous processing and within average (with the exception of Jiao) in Attention. The pattern in Cyprus was a bit different in that both children scored in the high average range in Simultaneous processing, but in the superior range in Attention. This may relate to the fact that one of the mathematics tests used in Cyprus was speeded and this may have inflated the role of Attention that was operationalized with speeded tasks. Taken together, these findings suggest that in upper elementary grades, Planning and Simultaneous processing play a critical role in supporting a child's superior mathematics performance (Dunn et al., 2020). This might be expected given that in

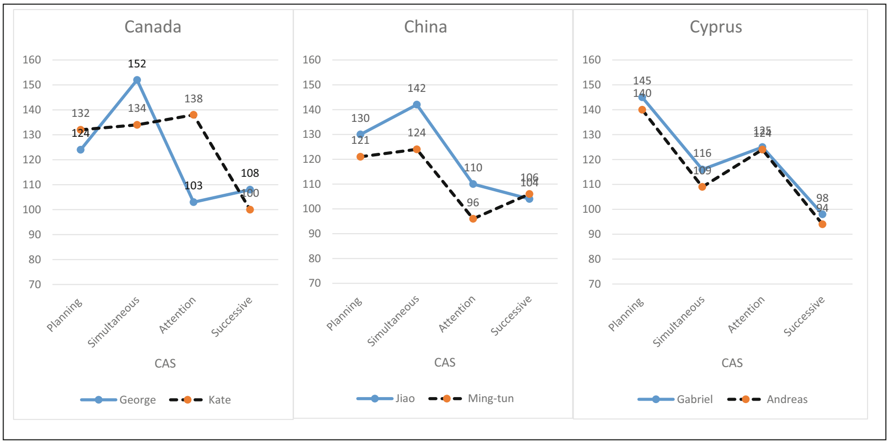

Fig. 6.2 The profiles of the mathematics gifted students. (***Note.*** Score classification: <70: very poor; 70–79: poor; 80–89: below average; 90–109: average; 110–119: above average; 120–129: superior; ≥130: very superior)

these grade levels children engage more frequently in higher level mathematics tasks (e.g., word problems or tasks require mathematical reasoning) than in arithmetic calculations that may relate more to attention (in this case indicating automaticity in fact retrieval) and Successive processing (involved in tasks that require serial processing of information).

6.4.2 *The Cognitive Profile of Children with Mathematics Disabilities*

The children with mathematics disabilities were also selected from previous studies (e.g., Dunn et al., 2020; Sergiou et al., 2021; Wang et al., 2018), because they scored in the below average range (standard score lower than 85) in Woodcock-Johnson III Broad Math (for the sample from Canada and China).[4] In Cyprus, children with mathematics disabilities were selected if they had scored at least 1SD below average in the two aforementioned tests of mathematics.

The two children from Edmonton, Jasper (a Grade 4 male) and Alice (a Grade 6 female) were attending a general education class in the same public school as the two children with mathematics giftedness. Their age was 9.9 and 11.1 years, respectively, and both were native speakers of English. In China, Chiu (female) and Zhan (male) were attending Grade 6 general education classes and were also recruited from the same schools as the children with mathematics giftedness. Both reported Mandarin as their native language. Their mean age was 11.0 and 11.3 years, respectively. The two Cypriot students, Gregory (male) and Anna (female) were both native Greek speakers and both were attending Grade 6. At the time of data collection, they were 11.6 and 11.5 years old, respectively. None of these participants were experiencing any intellectual, sensory, or behavioral difficulties (based on school records) and none were coded for any learning disabilities (we acknowledge though that such diagnosis is rather rare in Cyprus and in China).

Figure 6.3 presents the cognitive profiles of all six students with mathematics disabilities. As can be seen from this figure, all children performed in the low average or below average range (<90) in Planning and Simultaneous processing. Low average performance was also observed in Successive processing (with the exception of Jasper who scored 98). In contrast, in Attention, children scored mostly in the average range (see Gregory for an exception). This suggests that in the case of these children with mathematics disabilities almost all PASS processes are low (see Cai et al., 2013; Iglesias-Sarmiento & Deaño, 2011; Iglesias-Sarmiento et al., 2017; Kroesbergen et al., 2003, for similar findings).

[4] The children's standard score in WJ-III Broad Math was as follows: Jasper scored 83, Alice scored 80, Chiu scored 84 and Zhan scored 78. In Cyprus, the children's z score for the mathematics performance test was −0.61 (Gregory) and − 0.55 (Anna); for the mathematics reasoning test, their score in logits was −0.69 (Gregory) and − 0.60 (Anna).

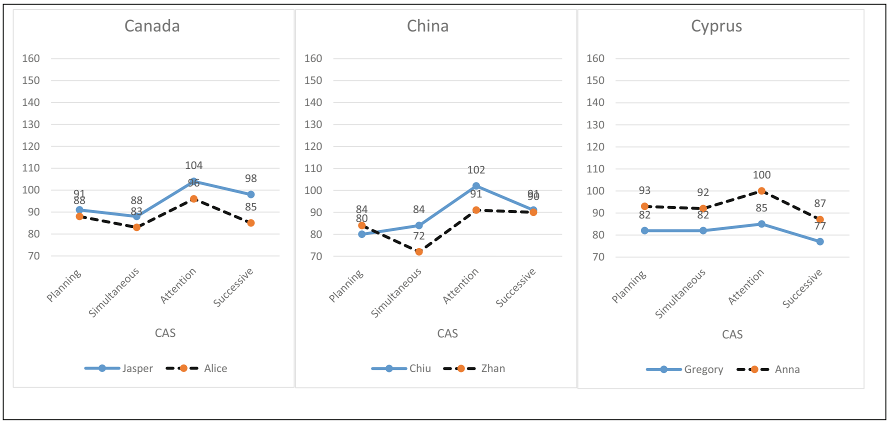

Fig. 6.3 The profiles of the students with mathematics disabilities. (***Note.*** Score classification: <70: very poor; 70–79: poor; 80–89: below average; 90–109: average; 110–119: above average; 120–129: superior; ≥130: very superior)

Because the children from Cyprus were also interviewed while solving the CAS and the mathematics tasks, we were also able to examine whether their performance in the CAS tasks was consistent with their mathematics performance.[5] In the interest of space, below we focus on their performance in the planning and simultaneous processing tasks of CAS and their performance in selected mathematics-reasoning tasks which required pattern noticing and generalizing. We focus on these tasks because they more clearly indicate the association between planning and simultaneous processing on the one hand, and pattern noticing and generalizing, on the other hand.

We first start with the planning task. This task consisted of six similar items. In each item, students were presented with four cards numbered from 1 to 4 (see Items 1 and 3 in Fig. 6.4a). Each card was split into two columns and each column included either an X or an O. Students were asked to complete an 8X4 grid of cards, by noticing the number that appeared at the top of each of these 32 cards and replicating the arrangement of X's and O's in the original cards, accordingly. Because students were given only 60 seconds to complete each item, they needed to identify a pattern that would allow them quickly complete the cards. For example, in Item 1 shown in Fig. 6.4b, students could quickly complete the cards by noticing that same-number

[5]Students were first asked to solve a task or a sub-task silently, and immediately after finishing, articulate their thinking. We preferred a retrospective as opposed to a concurrent think-aloud approach, because of the possibility of reactivity when students articulate their thinking as they work on a task and the excessive cognitive load often involved in this approach, as students try to both think and articulate their thinking (see more in Jaaskelainen, 2010; Someren et al., 1994; van den Haak et al., 2003). As suggested (Ericsson & Simon, 1993), the limitations of the retrospective approach can be minimized when students are asked to articulate their thinking immediately after having worked on a task.

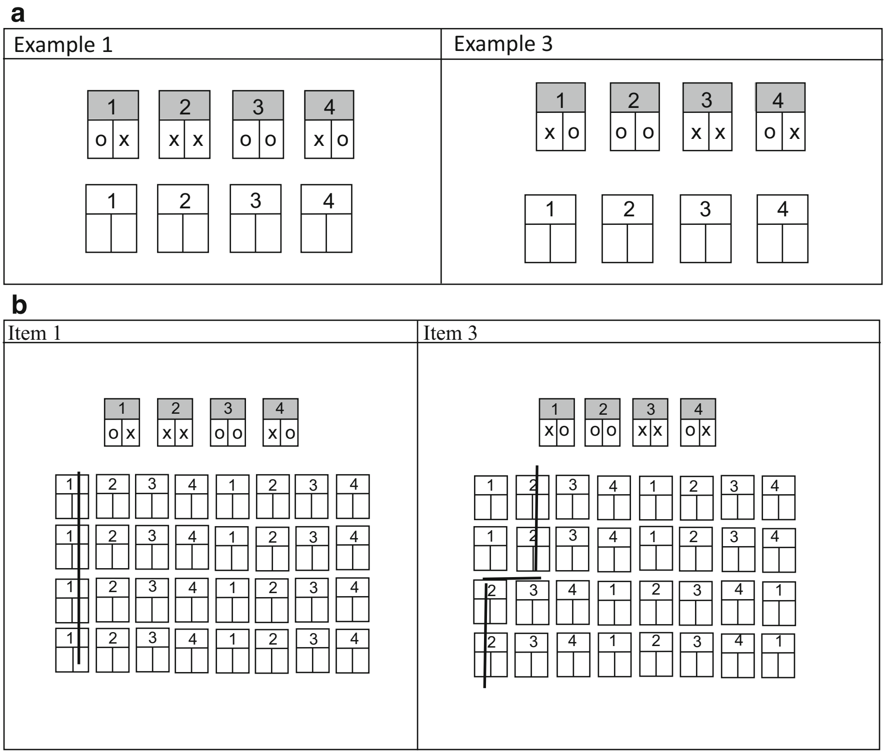

Fig. 6.4 (**a**) Two examples of cards. (**b**) Items 1 and 3 of the planning task

cards were placed vertically; in contrast, in Item 3, the placement of same-number cards followed a "chair" shape.

A cross-examination of the four students' performance in the planning task items suggested that whereas the two gifted students readily identified different patterns to complete the task, something that allowed them to answer the task items much quicker than their counterparts, the two students with disabilities mostly followed only a single pattern at best in solving all items. Specifically, as shown in Fig. 6.5, both Andreas and Gabriel figured out a number of patterns (e.g., moving vertically, moving diagonally) that allowed them to quickly complete the matrices. Even for the more complex matrices (Items 5 and 6), both were able to notice and capitalize upon certain patterns. For example, for Item 5, Gabriel commented that he noticed that the cards with the same number were forming the letter V; although working with a less refined pattern, Andreas commented that he "moved along the diagonal, and then completed what was left." Similarly, for Item 6, both identified two diagonal movements, which helped them complete the item quickly and correctly. It is also informative to consider how these two students reflected on their work on this set of items as a collective. Andreas remarked that in "all items, there were 8 cards with the same number;" to solve each item, he was first trying to figure out a pattern and then

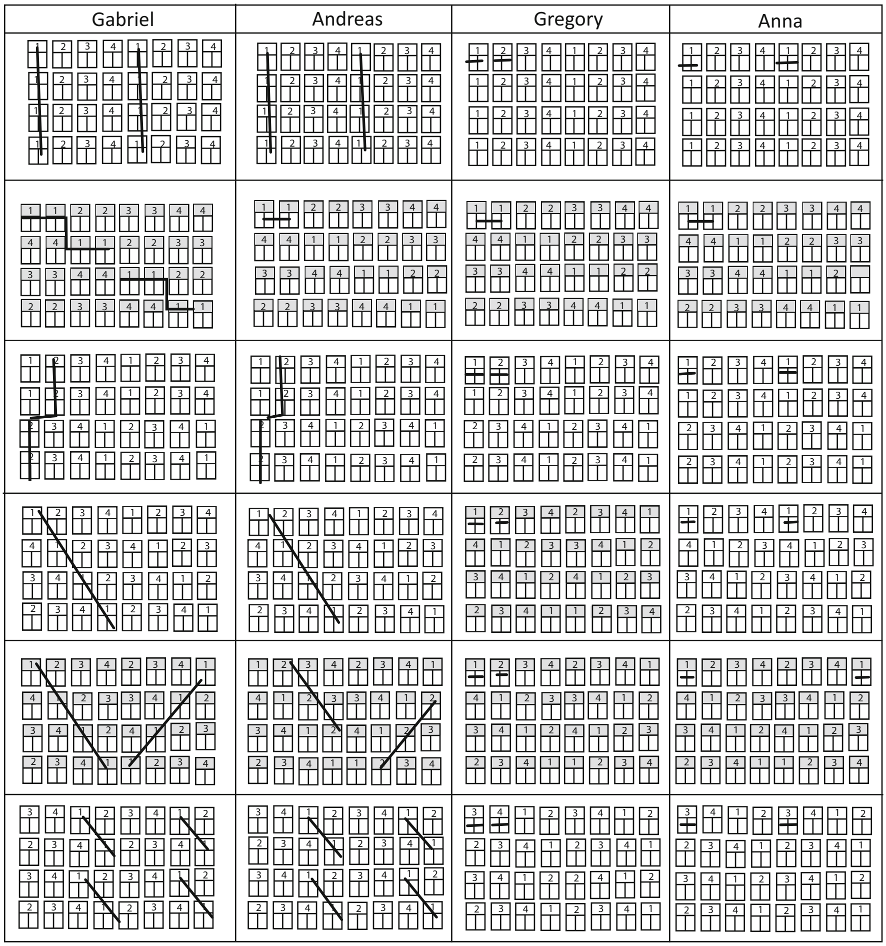

Fig. 6.5 The four Cypriot students' solutions in solving the planning items

complete the item. Along the same lines, Gabriel remarked that he was always trying to first figure out "an easy way in order to move fast;" prompted to clarify what he meant, he explained that key in his work was to first find a pattern.

Compare now these students' performance with the corresponding performance of students with disabilities (last two columns of Fig. 6.5): their work suggests that at best they identified a more restricted pattern, mainly moving only horizontally and considering each row of matrices in isolation. For example, Anna remarked that in solving all the tasks, she was going row by row, simply identifying the cards with the same number. Even less developed was the approach employed by Gregory, who, in solving all tasks simply moved horizontally between adjacent cards.

Students' performance in the simultaneous processing task also suggested notable differences in how these two pairs of students approached the given items. From the

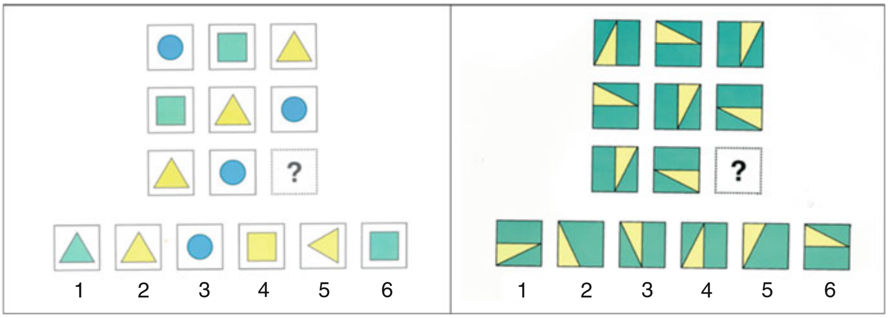

Fig. 6.6 Two items of the simultaneous processing task

items given, here we focus on the last two in which students' performance showed notable differences (see Fig. 6.6). The first item presented students with a 3 × 3 grid including circles, squares, and triangles, asking them to figure out the shape missing at the bottom right corner of the grid. The second item again presented students with a 3 × 3 grid, with triangles in different rotations; students were asked to choose among a set of six triangles with different rotations the one that best matched the missing triangle in the bottom right corner. Both Andreas and Gabriel answered the items correctly, after having identified and capitalized upon correct patterns. For example, in solving the first item, Andreas noted that in total there were 3 triangles, 3 circles, and 2 squares, so the missing shape was a square. Approaching the task differently but still figuring out a correct pattern, Gabriel noticed that every row needed to include one triangle, one circle, and one square. Both students were also able to identify how the triangle in the second item was tilted, thus figuring out the correct answer. Neither Anna nor Gregory was able to answer either of the two items correctly. Gregory was totally baffled as to how he could solve either of the items, saying "I don't know." Anna gave an answer to the first item, but as her clarification suggested, she approached this item rather superficially: "[The missing shape] is a circle, because there is also a circle in the exact opposite corner."

Students' performance and thinking in solving both the planning and the simultaneous tasks was remarkably aligned with their performance and thinking in solving mathematical reasoning tasks that required pattern noticing and generalizing. Here, we consider two such mathematical reasoning tasks (see Fig. 6.7). The first task presented students with a sequence of shapes following the ABC pattern "square-triangle-star." Students were asked to determine the 12th term in this sequence and explain how they worked to figure out the answer. The second task presented students with a sequence of triangular shapes in which each term included 3 N matchsticks (N being the number of the shape in the given sequence); students were asked to first complete a table specifying the number of matchsticks needed for different shapes. The table included the first five consecutive terms in the sequence, followed by the eighth and the tenth terms. To figure out these latter terms, students could not simply add three to the previous term, unless, they first figured out the seventh and ninth terms. The task was also asking students to figure out the number

Item1: A necklace is made of three types of beads, following the pattern shown below.

a)which type of bead will be in the 12^{th} position?

……………………………………………………………………………………

b) Explain your thinking.

……………………………………………………………………………………

……………………………………………………………………………………

Item2: Nikos uses matchsticks to create the following pattern.Look at the patttern and answer the question that follow.

Shape 1 Shape 2 Shape 3

a)How many matchsticks will be needed for the 4^{th} shape;……………………..

b)The following tables shows the shape number and the number of matchsticks needed for that shape.Fill in the table to show how many matchsticks are needed for different shapes.

Shape	Number of matchsticks
1	3
2	
3	
4	
5	
8	
10	

c)How many matchsticks will be needed for the 35^{th} shape?

……………………………………………………………………………………

……………………………………………………………………………………

d)Think about the relation between the number of the shape and the number of matchsticks needed to make this shape.Use words or letters (variables) to write the rule that describes this relation.

……………………………………………………………………………………

……………………………………………………………………………………

e)Nikos claims that he will need 45 matchsticks for shape number 15.Do you agree with him? Explain your thinking.

……………………………………………………………………………………

……………………………………………………………………………………

Fig. 6.7 Two mathematics reasoning tasks of matchsticks needed for the 35th term, which (again) called for identifying a general rule connecting the shape/term with the number of matchsticks, instead of moving consecutively by adding 3 matchsticks each time. Students were also asked to present such a general rule, as well as determine whether 45 matchsticks would be needed for the 15th term.

Consistent with their performance in the CAS planning and simultaneous processing tasks, in approaching both of these tasks, both Gabriel and Andreas used more advanced ways of thinking, largely identifying the patterns and generalizing from them. For example, Andreas explained how he worked on solving the first task as follows: “The star appears in the 3rd place and then in the 6th. Given that 12 is a multiple of 3, if the pattern continues, the star will again appear in the 9th and the 12th place.” Gabriel also noted that 12 is a multiple of 3, thereby there will be a star in the 12th place.” Anna was also able to answer the first task correctly, but unlike her fellow-students, she simply “expanded the pattern” writing down all the terms till she reached the 12th one. Hence, instead of trying to identifying a general rule, this student worked more consecutively, finding the next term each time. Even worse, Gregory could not figure out the term, answering that he could not understand how to work on the task. These differences were even more prevalent in the second task.

Both Andreas and Gabriel successfully answered all the questions of this task, by first figuring out the general rule (pattern)—although they were asked to do so in the fourth question of this task. Prompted to explain how he worked on the task, Andreas explained, “the first term has three [matchsticks], the second six [matchsticks], the third nine [matchsticks]; so, it goes up three each time, so, you multiply [the term/shape] by three.” Prompted to further explain his thinking, he kept repeating that “I simply noticed that each time you multiply the shape with three to figure out the [number of] matchsticks; so, I followed this rule.” Similarly, Gabriel explained that “I was trying to find a relation between the number of the shape and the number of the matchsticks; I noticed that each time you multiply the number of the shape with three; hence, [the sequence] is the multiples of three.” Working more consecutively, as she was working on the CAS planning task, Anna first completed the table as shown in Fig. 6.6; her work suggested that she simply added three each time, a pattern that she continued even when moving from the fifth to the eighth term and from the eighth to the tenth term. Without being able to identify a general pattern, Anna failed to answer the remaining questions. Similarly, Gregory failed to answer all the task questions, mentioning that they were difficult, without, however, providing any additional clarifications.

In sum, the four students’ performance and thinking in the planning and simultaneous processing tasks was consistent with their performance in the mathematics reasoning tasks, suggesting that students’ cognitive processing skills, as captured by the aforementioned CAS tasks, appear to have a critical role in how they approach cognitively challenging tasks in mathematics that require pattern noticing and generalizing.

6.5 Interventions Based on PASS Theory

In this last part of our chapter, we review intervention studies aiming to enhance cognitive planning and subsequently arithmetic calculations and problem solving (e.g., Iseman & Naglieri, 2011; Naglieri & Gottling, 1997; Naglieri & Johnson, 2000). You will notice that even though cognitive planning is an integral component of PASS theory, the strategies used in these interventions are similar to those that target self regulated learning in mathematics problem solving (see e.g., Montague, 2008). Recently, J. P. Das developed a mathematics intervention program called "Math Booster" that also trains successive and simultaneous processing but it targets early grades and, to our knowledge, no intervention studies have tested its effectiveness yet. Clearly, this is an area of future research, particularly in upper elementary grades.

Returning to the existing "planning facilitation" intervention studies, Naglieri and colleagues stated that the goal of their intervention sessions (10–21 sessions, 10 minutes each) was to help children understand the need of setting plans and deploying effective strategies through self-reflection and verbalization of the strategies employed to solve the problem. To help children achieve this general goal, the teachers encouraged them to (a) determine how they completed the different work sheets, (b) verbalize and discuss their ideas, (c) explain which methods worked well and which ones worked poorly, and (d) be self-reflective. Examples of teacher probes were the following: "*Can anyone tell me anything about these problems?*" "*Let's talk about how you did the work*," "*What was the same or different about the problems?*" "*What could you do to make this seem easier?*" "*Why did you do it that way?*" "*How did you solve the problems?*" "*What did it teach you?*" and "*What will you do next time?*" Discussions and further development of ideas followed from the responses of the students. For example, some students shared approaches with classmates including drawing columns in the multiplication problems to keep the answers more organized, and other students discussed strategies such as simplifying fractions before performing addition, subtraction, multiplication, or division. As mentioned earlier, Naglieri and Johnson (2000) found that Grade 6 to 8 children (age ranged from 12 to 14 years) with a cognitive weakness in Planning who received this kind of intervention improved considerably over baseline rates both in Planning and in classroom mathematics performance.

Notably, these practices overlap to a large extent with the meta-cognitive practices included in Montague and Applegate's (1993) cognitive-metacognitive model of mathematical problem solving. More specifically, the three metacognitive strategies in their model (i.e., self-instruction, self-questioning, and self-monitoring) are integral components of Planning (as described by Das et al., 1994). Metacognitive processes focus on self-awareness of cognitive knowledge that is necessary for effective problem solving. Successful problem solvers use self-instruction, self-questioning and self-monitoring to gain access to strategic knowledge, guide execution of strategies, and regulate the use of strategies.

6.6 Conclusion

The role of intelligence in academic performance (particularly mathematics) is undeniable. In this book chapter, we tried to present an alternative way of looking at intelligence (i.e., PASS processes) and how it connects to mathematics performance. The PASS theory – grounded on neuropsychology – has given us the opportunity to profile children with mathematics difficulties or strengths across cultures. The children in our study demonstrated cognitive weaknesses (in the case of children with mathematics disabilities) or strengths (in the case of children with mathematics giftedness) in Planning and, to a lesser extent, in Simultaneous processing. This is not to say that Attention or Successive processing are not important for mathematics. The contribution of all four processes likely depends on the type of mathematics outcome (and the demands of each mathematics task) and the grade level of the children. In upper grades (the focus of this handbook), Planning and Simultaneous appear to play a more important role in mathematics. Future studies could extend this line of work, by examining the association between different types of mathematical tasks—both tasks require reasoning and mathematical thinking, as well as more algorithmic tasks—with the four PASS processes. An examination of such associations in different educational contexts and with students of different ability levels—as attempted in the current chapter—is expected to help better understand how students' cognitive processes contribute to their performance in mathematics.

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Chapter 7
The Interplay Between Motivation and Cognition in Elementary and Middle School Mathematics

Allison S. Liu, Teomara Rutherford, and Sarah M. Karamarkovich

Abstract Getting and keeping children engaged in mathematics is a critical but difficult aspect of mathematics education. This may be especially crucial during middle childhood, when children experience significant changes in cognitive development and declines in mathematics motivation. A growing body of research has investigated motivation's contribution to mathematics learning and achievement; a largely separate literature has researched cognitive and numeracy contributors. Understanding how motivation and cognition jointly contribute to mathematical performance during middle childhood can help to identify the intervention targets that may have the most impact on mathematics achievement. However, little research has investigated the interplay between motivation and cognition in middle childhood mathematics. This chapter reviews existing studies, including the authors' own research, that concurrently investigate motivation and cognition in the context of mathematics, and discusses how these findings can be used to understand mathematics achievement during middle childhood with a focus on grades 3–5. The authors also illustrate how existing motivational theories can be expanded to include cognitive processes, using Situated Expectancy–Value Theory as an example, and how such integration can inform instructional practice. Finally, the chapter recommends educational practices and discusses open questions and limitations within the field that future research can address.

Keywords Achievement goals · Expectancy–value · Mathematics achievement · Mathematics cognition · Self-determination theory

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Understanding how children learn mathematics is a key focus of mathematics education. Many studies have approached this question from a cognitive perspective. For example, the field of numerical cognition has researched domain-specific cognitive factors in depth, such as the cognitive foundations of animal and human number representations, and how symbolic and non-symbolic numerical processing contributes to mathematical skills (De Smedt et al., 2013; Kadosh & Dowker, 2015). Studies have also found a relationship between a variety of domain-general cognitive skills and mathematics, including executive functions (e.g., Geary et al., 2017; Raghubar et al., 2010; Spiegel et al., 2021) and general intelligence or cognitive ability (e.g., Primi et al., 2010; Watkins & Styck, 2017). These cognitive-mathematics relationships have been the basis of many instructional interventions and practices that aim to improve children's mathematical achievement, with mixed success (e.g., Holmes & Gathercole, 2014; Karbach et al., 2015; Ramani et al., 2017).

An equally critical aspect of mathematics education is how to motivate children to *want* to learn mathematics. Evidence shows that higher mathematics motivation is associated with better mathematics learning and later STEM outcomes (e.g., Guo et al., 2015; Jiang et al., 2020; Musu-Gillette et al., 2015; Plante et al., 2013; Shanley et al., 2019). Mathematics motivation may be especially critical during middle childhood (defined here as ages 6–12), a time when children experience significant changes in cognitive development (National Research Council, 1984). During this period, children begin to develop mathematics-specific motivations (Bandura, 1997; Eccles et al., 1993; Marsh & Ayotte, 2003), but mean levels of these motivations also begin to decline (Fredricks & Eccles, 2002; Jacobs et al., 2002; Mullis et al., 2020; Peterson & Hyde, 2017; Wigfield et al., 1997) before stabilizing in high school (Jacobs et al., 2002; Musu-Gillette et al., 2015; Weidinger et al., 2017). These motivational declines are often faster in mathematics than in other subjects (Gottfried et al., 2007; Jacobs et al., 2002). This decline is likely related to transitions that occur during this developmental period. Children enter new schooling contexts with a stronger emphasis on external evaluation (Wigfield et al., 1998), which can influence their social and cognitive growth. Children are also developing their identities, self-concepts, and metacognitive skills (Eccles, 1999; Raffaelli et al., 2005; Stipek & Mac Iver, 1989), allowing them to calibrate their beliefs about their skills more accurately and from more sources (Muenks et al., 2018). Because middle childhood is a tumultuous period for children's motivational beliefs, it may be particularly key to support motivation for mathematics during these ages to keep children interested in pursuing mathematics.

Understanding both mathematics-related cognition and motivation may be essential for supporting students' mathematics learning. First, children with higher motivation show greater gains from domain-general cognitive interventions (e.g., Jaeggi et al., 2011, 2014). Second, motivation is often theorized to be more malleable than cognition (Gutman & Schoon, 2013); thus, interventions that target motivational beliefs about mathematics may be more practical for improving mathematics achievement. However, to fully capitalize on interventions and practices that target motivational factors, it is first necessary to understand how motivation and cognition individually contribute to mathematics performance during middle childhood to determine which targets may have the most impact. Although social-cognitive theories of learning exist (see Panadero, 2017 for a review), most prior research has largely investigated motivational and cognitive factors separately, rather than the interplay between motivation and cognition. Precisely how these two factors jointly contribute and interact to influence mathematics outcomes remains unclear. Answering these questions can inform the design of more effective holistic supports for children's mathematics development, as well as validate and revise existing cognitive and motivational theories.

In this chapter, we conduct a systematic review of existing studies, including our own research, that concurrently investigate cognition and motivation in the context of mathematics learning in middle childhood, with a focus on grades 3–5 (see Appendix A for details about our review methodology). Our goal is to provide a comprehensive overview of the current research, and to recommend instructional practices and future research directions that can support mathematics achievement during middle childhood. We focus on three motivational theories that are commonly used in educational research to understand how instruction relates to academic outcomes: Situated Expectancy–Value Theory, Self-Determination Theory, and Achievement Goal Theory. We further show how existing motivational theories can be expanded to include cognitive processes to better describe mathematics learning, using Situated Expectancy–Value Theory as an example. We chose to use Situated Expectancy–Value Theory because it broadly theorizes how motivational, cognitive, and social processes may relate to academic achievement (compared to Self-Determination Theory and Achievement Goal Theory which primarily focus on motivational processes) and can more easily illustrate the relations between cognition and motivation based on their relative contributions to mathematics achievement. Finally, we discuss open questions and recommendations for instructional practices and future research studies regarding cognitive and motivational processes in mathematics.

7.1 Situated Expectancy–Value Theory

Situated Expectancy–Value Theory (SEVT) posits that people's achievement-related choices and performance are influenced by their expectancies for success on a task and the value they place on the task (Eccles & Wigfield, 2020). *Expectancies* are defined as children's beliefs about how well they will do on upcoming tasks in the immediate or long-term future (e.g., "I think I will do well in my mathematics class this year"). *Values* are conceptualized based on three sub-components: attainment value (how important it is to the individual to do well on the task), intrinsic value (how much enjoyment the individual gains from doing the task), and utility value (how useful the individual finds the task, based on personal or societal standards). For example, a child who places high value on their mathematics performance may consider it important to do well in their mathematics classes, find it enjoyable to engage with mathematics-related activities, or believe that mathematics will help them in a future career. *Academic self-concepts* are another related construct, defined as children's perceptions of their current competence for an academic subject (e.g., "I usually do well in mathematics"); these beliefs are generally thought to be more stable dispositions compared to expectancies or values. The full SEVT model considers the broader social and cognitive contributors that shape children's expectancies and values, including the environments in which individuals are situated, the developmental processes that transform individuals' experiences into self-perceptions, and more proximal social-cognitive aspects that influence decision making. Thus, SEVT provides a theory of motivation that integrates social and cognitive factors, making it a popular framework for studies that investigate motivational and/or cognitive factors in learning.

In line with SEVT's conceptualization, we discuss research that investigates expectancies, values, and/or academic self-concepts in combination with cognitive factors. Because SEVT is a recent re-conceptualization of Wigfield and Eccles' (2000) Expectancy–Value Theory (EVT), many of the studies described in this section utilize EVT as their theoretical framework rather than SEVT; however, the main constructs of interest (i.e., expectancies, values, academic self-concepts) remain the same across EVT and SEVT so that these studies can still be compared. We also include studies that measure mathematics self-efficacy (individuals' beliefs in their ability to succeed on mathematics-related tasks; Bandura, 1997) because it is conceptually similar to mathematics expectancies, and can thus further inform how beliefs about one's abilities influence mathematics learning; however, we acknowledge that mathematics self-efficacy and expectancies may differ in important ways, such as their theoretical roots and specificity (Bong & Skaalvik, 2003; Marsh et al., 2019; Wigfield & Eccles, 2000). We first review research that included expectancies, values, and/or self-concepts with domain-specific cognitive factors, followed by research on domain-general cognitive factors.

7.1.1 Domain-Specific Cognitive Factors and Situated Expectancy–Value Theory

Few studies on expectancy–value include domain-specific cognitive factors. Of those that do, a trend emerges in which domain-specific cognition appears to be important for mathematics learning earlier on in middle childhood (e.g., 4th grade and below), and mathematics expectancy appears to be separately important throughout the period. There is also some early evidence that expectancy's importance may increase as children grow older. One study used a sample of 6435 Turkish 4th grade students from the Trends in International Mathematics and Science Study (TIMSS) 2015 database to investigate how students' confidence in mathematics (comparable to expectancies), interest in learning (comparable to intrinsic value), and performance on early numeracy tasks predicted mathematics scores on the TIMSS, an international mathematics and science assessment given to 4th and 8th grade students across the world (Tomul et al., 2021). Both confidence in mathematics (average $\beta = 0.34$) and early numeracy tasks (average $\beta = 0.20$) significantly predicted mathematics scores across four types of school locations (urban, suburban, medium-sized city, and village). Interest in learning mathematics was also a significant predictor, but only for urban school locations, and the effect size was small ($\beta = 0.06$).

In our own work, we investigated how numerical processing, executive functions (updating, inhibitory control, shifting), visuo-spatial ability, mathematics expectancy, and mathematics value (as two subcomponents of importance-utility and intrinsic value) predicted performance on a standardized state test in a majority-Hispanic sample of 3rd to 5th grade students in the U.S. Further, we looked at whether contributions differed depending on grade level or type of mathematics being tested (Liu et al., 2022). We found that mathematics expectancy (average $\beta = 0.12$) was one of two factors that significantly predicted general mathematics scores for all three grades (the other being updating, average $\beta = 0.09$). We also looked at the contributions of these factors across five types of mathematics content to understand which factors showed the broadest reach across a variety of mathematics subjects. Numerical processing (average $\beta = 0.15$ across five subtests), updating (average $\beta = 0.13$), and shifting (average $\beta = 0.12$) were most predictive for 3rd grade students (predicting scores on four of the five content areas), whereas mathematics expectancy was most predictive for 4th grade students (average $\beta = 0.11$, significantly predicting scores on four of the five content areas) and 5th grade students ($\beta = 0.19$, significantly predicting scores on all five content areas). Similarly to the TIMSS study (Tomul et al., 2021), neither subcomponent of mathematics value were significant predictors of broad-level mathematics performance—in our work, value was only predictive of one subtest for 5th grade. Though we cannot make any causal claims due to the nature of our data, these findings suggest that 3rd to 4th grade students may benefit from instruction that teaches numerical processing skills. In parallel, supplementing this instruction with practices that build students' confidence in their mathematics abilities may support students' mathematics performance during these grades and beyond.

Although educators may feel pressure to emphasize the importance of learning mathematics to their students, mathematics value actually appears to have a very small association with mathematics performance at this age compared to domain-specific cognition or mathematics expectancies, so prioritizing these other factors may have a more immediate impact on mathematics performance. However, it should be noted that these studies are cross-sectional, and thus strong conclusions cannot be made about the development of expectancy and domain-specific cognitive factors, or their contributions to mathematics learning over time.

7.1.2 Domain-General Cognitive Factors and Situated Expectancy–Value Theory

A greater number of studies have investigated expectancy–value together with domain-general cognitive factors, particularly executive functions, general intelligence, and general cognitive ability. Domain-general cognitive factors are regularly found to be stronger predictors of mathematics achievement than expectancies or values, but expectancy often still explains a significant amount of additional variance. As is found with domain-specific cognitive factors, value is a less consistent predictor of mathematics performance compared to expectancy.

Most studies involving executive functions only include working memory capacity (the amount of short-term information that can be held and actively manipulated in one's mind for a task), usually finding that working memory and mathematics expectancy separately and significantly contribute to mathematics performance (e.g., Prast et al., 2018; Weber et al., 2013). For example, in 2nd to 6th grade students from the Netherlands, Prast et al. (2018) found that mathematics achievement positively predicted perceived competence in mathematics (measured as self-efficacy and self-concept combined; $\beta = 0.51$) and task value ($\beta = 0.19$) three months later. Of these factors, only perceived competence significantly predicted mathematics achievement five months later ($\beta = 0.12$) after controlling for prior mathematics achievement and working memory, but perceived competence still explained a much lower amount of variance in mathematics achievement compared to working memory ($\beta = 0.71$). Further, when the sample of students was split by achievement level, this relationship between perceived mathematics competence and mathematics achievement only held for low- and high-achieving students, not average-achieving students, whereas working memory continued to be predictive for all levels of achievement.

In our work described above that included three executive functions (updating, inhibitory control, and shifting), we also found that updating (comparable to others' measures of working memory) and mathematics expectancy significantly predicted broad-level mathematics scores across 3rd, 4th, and 5th grade students, though expectancy had a larger average effect size than updating in our sample (Liu et al., 2022). Thus, similarly to numerical processing, improving students' working memory along with their mathematics confidence may have benefits for students' mathematics performance. The effectiveness of training working memory is still under debate (e.g., Simons et al., 2016; von Bastian et al., 2022), but instructional practices can still provide supports that reduce working memory loads to ensure that students are not limited by their working memory capacities (e.g., providing visual tools for students to offload important information instead of holding it in working memory). The relationship between mathematics expectancies and mathematics performance also provides promise for a more malleable pathway toward improving mathematics performance.

General intelligence (typically measured as one's ability to reason abstractly, using problems that require minimal prior knowledge) also largely contributes to mathematics performance, separate from mathematics expectancy and value. Similar to executive functions, intelligence typically explains a larger amount of unique variance compared to the two motivational factors across various measures of intelligence and mathematics outcomes and across countries (e.g., Chamorro-Premuzic et al., 2010; Orbach et al., 2019; Spinath et al., 2006, 2008). For example, in a sample of 9-year-old children from England and Wales (Spinath et al., 2008), intelligence explained 20% of variance in teachers' assessments of student mathematics achievement ($\beta = 0.38$), and mathematics self-perceptions explained an incremental 7% of variance ($\beta = 0.32$). Meanwhile, intrinsic value only explained an additional 0.1%, and this was likely inflated due to the study's large sample size ($N = 4464$). Other studies have also found that intelligence contributes more to mathematics outcomes (with β estimates ranging from approximately .30 to .50), whereas expectancy contributes a significant but smaller amount (with β estimates ranging from approximately .20 to .40) (e.g., Chamorro-Premuzic et al., 2010; Orbach et al., 2019). The relation between mathematics expectancies and mathematics achievement also appears to hold up to 3 years later (Chamorro-Premuzic et al., 2010) and across all levels of cognitive ability (Tracey et al., 2020), suggesting that the relation of mathematics expectancy is relatively stable and separate from the relation of cognition. Again, mathematics value is rarely a statistically significant predictor of mathematics performance when cognitive factors are controlled, and may even have the opposite directional relationship, such that mathematics achievement is a predictor of subsequent interest rather than vice versa (Jōgi et al., 2015). These findings are consistent with a meta-analysis of 74 studies ($N = 80{,}145$) showing that intelligence is a larger contributor to school achievement than expectancies or values, explaining approximately 66.6% of unique variance as compared to 16.6%, and that school achievement has a stronger association with expectancies than with values (Kriegbaum et al., 2018).

A neuroimaging study may provide insight into a potential mechanism through which mathematics expectancies and values can impact mathematics achievement (Chen et al., 2018). Positive mathematics attitudes (a composite measure of mathematics expectancy, mathematics intrinsic value, and general academic attitudes) were associated with greater activation in the hippocampus and more frequent use of efficient memory-based strategies, controlling for IQ. Hippocampal activation and strategies also partially mediated the relation between positive attitudes and mathematics achievement. As with working memory, intelligence is generally considered less plastic as a skill than motivation (Gutman & Schoon, 2013, though see Blackwell et al., 2014 for evidence of intelligence's malleability), and attempts at improving general intelligence have met with mixed results (Au et al., 2015; Buschkuehl & Jaeggi, 2010). These findings emphasize that mathematics expectancy may be a more effective way to directly support mathematics performance at these ages and may potentially do so by influencing the types of strategies that students utilize during problem solving.

When both working memory and intelligence are included as predictors of mathematics achievement, the contribution of mathematics expectancy is less clear. For example, Weber et al. (2013) found that working memory and intelligence (combined into one "cognitive ability" measure) and self-perceived mathematics ability and intrinsic value (combined into one "motivation" measure) all explained unique amounts of variance of German 4th grade students' mathematics grades (together explaining 71% of variance), though the cognitive ability measure was a stronger predictor ($\beta = 0.59$ compared to 0.49 for motivation). In contrast, Lu et al. (2011) found that only working memory ($\beta = 0.59$) and intelligence ($\beta = 0.37$) significantly predicted 36.4% of variance in Chinese 4th grade students' mathematics exam scores, whereas self-perceived mathematics ability and intrinsic values explained a non- significant 2.2% of variance. The authors suggested that the inconsistencies between their and other studies' findings regarding self-perceived mathematics ability may stem from differences in the cultural specifics in parental attitudes in their Chinese sample compared to other studies' Western samples (i.e., a stronger emphasis on effort as the main source of academic success, as opposed to inherent ability).

As would be theorized by the Expectancy–Value model, parental attitudes should influence students' self-perceptions, and consequently, their relation to mathematics achievement. Thus, although mathematics expectancies may be a pathway to supporting mathematics performance, it is important to remember that students' motivations are inseparably situated within their broader lives in and outside of the school environment, such that effective approaches for supporting their mathematics expectancies may differ depending on students' particular circumstances. For example, a student who enters the classroom believing that their mathematics skills are unchangeable may be less impacted by interventions that encourage them to be confident in their mathematics skills, at least until their beliefs about the malleability of their mathematics abilities are changed first.

Mathematics self-beliefs also appear to contribute directly and indirectly to immediate mathematics outcomes, when working memory is considered. One study of 3rd grade classes in Germany found that mathematics self-concept ($\beta = 0.14$) and working memory ($\beta = 0.32$) directly predicted concurrent mathematics performance, though only working memory predicted mathematics performance six months later ($\beta = 0.25$) (Gunzenhauser & Saalbach, 2020). Mathematics self-concept and working memory were also associated with self-regulation, which contributed to concurrent mathematics performance, providing another indirect pathway through which both self-concept and working memory influenced mathematics performance. However, it should be noted that the Gunzenhauser and Saalbach study only measured affective components of mathematics self-concept with items that may be considered more aligned with intrinsic value rather than self-concept (e.g., "I am interested in math"); as shown above, mathematics value had a generally weak relationship with mathematics performance. Another study of 3rd and 5th grade students also found that mathematics self-concept and working memory fully mediated the relationship between mathematics anxiety and mathematics performance across three mathematics measures (teacher ratings of mathematics achievement, scores on a standardized assessment of mathematics fluency, and scores on a standardized assessment of mathematics problem solving) (Justicia-Galiano et al., 2017). Although self-beliefs are characterized as more stable motivational dispositions (Eccles & Wigfield, 2020), these findings continue to emphasize the potential importance of students' mathematics confidence for their mathematics achievement.

7.1.3 Summary of Situated Expectancy–Value Findings

In summary, mathematics expectancy and self-concepts uniquely contribute to various mathematics outcomes, separately from domain-specific and domain-general cognitive factors during middle childhood. Domain-general cognitive factors consistently have a larger influence on mathematics outcomes than mathematics expectancy or self-concepts, but both motivational constructs still predict an additional and meaningful amount of variance. The contributions of domain-specific cognitive factors may be limited to younger ages within this developmental period, but research that includes domain-specific cognitive factors, expectancies, values, and/or self-concepts is still limited. Because expectancies are thought to be more malleable than most domain-general cognitive factors (Gutman & Schoon, 2013), educators may want to build students' confidence in their ability to succeed at a given mathematics task. Potential instructional practices include providing incremental opportunities for students to feel that they have mastered skills and succeeded at challenging assignments (Usher & Pajares, 2009).

In contrast to mathematics expectancy and self-concepts, mathematics value was rarely a significant contributor to mathematics outcomes when cognitive factors were controlled. Mathematics value is thought to primarily affect mathematics performance indirectly by encouraging students to pursue mathematics activities (e.g., joining after-school mathematics programs or enrolling in optional advanced mathematics classes) (Lauermann et al., 2017; Simpkins et al., 2006). During middle childhood, mathematics opportunities at this age are largely provided by formal classes and assigned classwork, such that children's choices will have less of an influence on mathematics performance. If mathematics value becomes more important when children have fewer required mathematics experiences and greater autonomy over their selection of mathematics opportunities (Updegraff et al., 1996), then mathematics value would not be expected to be a large contributor for this age range until students complete their requisite mathematics courses and have to seek out additional mathematics learning through optional experiences.

7.2 Self-Determination Theory

Self-Determination Theory (SDT) is a social cognitive theory that explains people's choices based on the source of motivation (Adams et al., 2017; Deci & Ryan, 2012). The theory distinguishes between intrinsic and extrinsic motivation, defining *intrinsic motivation* as doing a task because it is interesting or enjoyable to the individual, and *extrinsic motivation* as doing a task because of external pressures or rewards. At a broad level, SDT treats intrinsic motivation as a malleable construct that can be fostered by meeting an individual's need for autonomy (control over whether to act or not), competence (ability to master their environment), and relatedness (sense of social belonging) during a given task. In contrast to intrinsic motivation, extrinsic motivation is generally fostered when individuals feel that they are being forced to act in ways that are out of their control (though there are also more autonomous forms of extrinsic motivation). Few studies have investigated intrinsic and/or extrinsic motivation in combination with domain-specific cognition as contributors to mathematics outcomes; as such, we focus our review on studies involving domain-general cognitive factors.

7.2.1 Domain-General Cognitive Factors and Self-Determination Theory

Studies investigating domain-general cognitive factors with SDT have primarily involved general intelligence or general cognitive ability. Findings suggest that intrinsic motivation may be a less important construct for mathematics achievement during middle childhood. It may still have long-term benefits, as intrinsic motivation—much like mathematics value (Lauermann et al., 2017; Simpkins et al., 2006)—is likely to influence individual choice to participate in certain activities (Lavigne et al., 2007; Ryan et al., 1991).

In one cross-sectional study, Stevens et al. (2006) investigated how general mental ability and intrinsic motivation, as well as mathematics self-efficacy, mathematics interest, sources of self-efficacy, and prior mathematics achievement, predicted performance on standardized mathematics assessments in White and Hispanic youth from ages 8 to 18. Intrinsic motivation did not significantly predict mathematics performance for either White or Hispanic children (though the model was a poor fit for the Hispanic sample's data). Instead, general mental ability ($\beta = 0.21$ for the White sample, $\beta = 0.18$ for the Hispanic sample), emotional feedback ($\beta = .23$ for the White sample), and mathematics self-efficacy ($\beta = .27$ for both samples) predicted mathematics performance. These findings continue to emphasize the importance of domain-general cognitive factors and beliefs in one's mathematics skills for mathematics performance at this age—again, with the latter being the more malleable of the two (Gutman & Schoon, 2013).

Longitudinal studies provide context around intrinsic motivation's missing relationship with mathematics achievement—as discussed above regarding mathematics value, choices about mathematics opportunities may simply be less common in middle childhood. For example, a sample of elementary school children in Quebec were assessed on cognitive and motivational factors in 1st, 2nd, and 4th grade (Garon-Carrier et al., 2016). After controlling for general cognitive abilities, cross-lagged models found that achievement on standardized mathematics tests in 1st grade predicted intrinsic motivation in 1st and 2nd grade, and achievement in 2nd grade similarly predicted intrinsic motivation in 2nd and 4th grade. However, intrinsic motivation did not predict subsequent achievement at any time, suggesting a unidirectional relationship opposite from other studies. The authors proposed that at this age, mathematics learning in school may be primarily driven by external forces (e.g., classroom requirements or teacher-assigned activities), meaning that intrinsic motivation has less opportunity to influence learning behaviors and subsequent mathematics achievement (Garon-Carrier et al., 2016).

In support of this conclusion, a longitudinal study on German students assessed from 5th to 10th grade found that intrinsic and extrinsic motivation related to initial performance on a standardized mathematics assessment at 5th grade, but neither predicted change in mathematics performance. In contrast, intrinsic motivation was a significant predictor of change at 7th grade (Murayama et al., 2013), which may signify that 7th grade children had more opportunities for autonomy regarding their mathematics learning. By 6th grade, another study found that autonomous motivation (in which one voluntarily invests effort into a task because of inherent interest or identified importance) positively predicted initial mathematics achievement ($\beta = 0.09$), as well as a year's mathematics growth ($\beta = 0.12$); intelligence (measured in 3rd grade) also predicted both initial achievement ($\beta = 0.61$) and growth ($\beta = 0.40$). Increases in autonomous motivation were also related to increases in mathematics achievement ($\beta = 0.08$). The opposite was found for controlled motivation (in which one invests effort into a task because of external or internal pressures), which was unrelated to initial levels of mathematics achievement, negatively related to change ($\beta = -0.08$), and increases in controlled motivation were negatively related to change in mathematics ($\beta = -0.07$) (Boncquet et al., 2020). Still, this pattern is not completely straightforward.

Another longitudinal study looking at 1st to 3rd grade students found that intrinsic motivation significantly predicted mathematics outcomes in addition to IQ. However, the mathematics outcome measures included both mathematics report card grades and teacher ratings of mathematics achievement rather than standardized mathematics tests (Gottfried, 1990). It is possible that intrinsic motivation still has a notable influence on *others'* perceptions and ratings of students' mathematics achievement even at these younger ages, even if it has less immediate influence on standardized measures. It should be noted that the amount of variance explained by intrinsic motivation was still relatively small, ranging from 3% to 6% compared to IQ's explanation of 14–33% (Gottfried, 1990). Intrinsic motivation's influence on mathematics achievement may grow stronger as grade increases and choices regarding mathematics opportunities become available (e.g., Gottfried et al., 2013; Taylor et al., 2014).

Group and profile analyses provide further information about potential long-term benefits of intrinsic motivation, as well as ways in which intrinsic and extrinsic motivation may interact to promote mathematics outcomes. Lv et al. (2019) identified five profiles in 4th to 6th grade students based on levels of intrinsic motivation, identified motivation (in which individuals identify with reasons for a behavior or personally find a behavior important), and controlled motivation. After controlling for IQ, the profile characterized by high intrinsic and identified motivation and low controlled motivation showed the highest levels of mathematics achievement compared to the other four profiles. The lowest mathematics achievement was shown by profiles with average to high levels of controlled motivation, regardless of their levels of intrinsic or identified motivation.

Gottfried et al. (2005) also found that, in a longitudinal study of children from 1 to 17 years of age, the children with extremely high levels of intrinsic motivation were more likely to have higher mathematics achievement and cumulative GPAs, be rated more highly by teachers and parents regarding achievement and engagement, take the SAT and score higher on the Quantitative and Verbal sections, score higher on intelligence tests, report higher academic self-concepts, and pursue college degrees directly out of high school. Both intrinsic motivation and IQ also significantly and uniquely contributed to predicting GPA. These findings highlight the potential detriments of extrinsic motivation on immediate mathematics achievement, which may even override benefits gained from high intrinsic motivation. In contrast, although fostering intrinsic motivation during middle childhood may not have immediate mathematics benefits at this age, it may still be worthwhile if it supports positive future outcomes.

7.2.2 Summary of Self-Determination Theory Findings

In sum, studies on Self-Determination Theory show somewhat mixed results on whether intrinsic motivation separately contributes to mathematics outcomes when domain-general cognitive factors are included (and more research is needed on domain-specific cognition). Similar to what was found with mathematics value, intrinsic motivation may be less important during middle childhood because children have few meaningful ways to exercise mathematics-relevant autonomy. Still, there is evidence that intrinsic motivation can have long lasting effects if developed during this period. Thus, it may be constructive to support intrinsic motivation by meeting children's needs of competence and relatedness, or providing small ways for children to act autonomously. For example, educators can present opportunities for students to acknowledge their progress toward important mathematics goals (competence), to work collaboratively on mathematics problems (relatedness), or to choose from a set of pre-planned activities during mathematics classes (autonomy). Supporting these needs may also have the added benefit of protecting against high levels of non-autonomous extrinsic motivation, given consistent evidence that extrinsic motivation has a negative relationship with mathematics achievement even in middle childhood.

7.3 Achievement Goal Theory

Achievement Goal Theory (AGT) focuses on individuals' purposes for engaging in achievement-related behavior. There are several variations of achievement goal models that differ in the number of achievement goals and how the goals are distinguished. The dichotomous model only considers two broad types of achievement goals based on the aim of the behavior: *mastery goals*, in which the aim is to gain competence; and *performance goals*, in which the aim is to demonstrate competence (Ames, 1992; Nicholls et al., 1989). Other models (e.g., the trichotomous and 2 × 2 models; Elliot, 1999, Elliot & Harackiewicz, 1996) further separate mastery and performance based on how an individual tries to attain these goals: mastery- or performance-*approach* goals, in which individuals are motivated to demonstrate they can master a task or perform better compared to others; and mastery- or performance-*avoidance* goals, in which individuals are motivated to avoid showing that they cannot master a task or perform more poorly than others. Each type of achievement goal is thought to have its own pattern of antecedents and consequences. Achievement goals are often conceptualized as being couched in stable self-schemas similarly to academic self-concepts (Tuominen-Soini et al., 2011), but stability may vary depending on the type of goal and task (Fryer & Elliot, 2007; Muis & Edwards, 2009).

Two general hypotheses have been proposed on how achievement goals relate to cognition (Linnenbrink & Pintrich, 2000). The "working memory" hypothesis suggests that achievement goals influence the information held within working memory, as well as the processes involved in manipulating this information. They may also affect the amount of effort that individuals are willing to expend to maintain high levels of performance, which may be reflected in one's executive functions during a task. The "self-regulated learning" hypothesis suggests that achievement goals influence the use of cognitive or self-regulation strategies. Based on these hypotheses, much of the existing work on Achievement Goal Theory and cognition has focused on domain-general factors of executive functions and general intelligence or cognitive ability, as well as cognitive and meta-cognitive strategy use. We review this research below.

7.3.1 Domain-General Cognitive Factors and Achievement Goal Theory

There is some evidence that working memory mediates the relationship between achievement goals and mathematics outcomes. In a sample of children ages 10 and 12, Lee et al. (2013) compared three models on how working memory, mastery and performance goals, and standardized mathematics assessment scores related. They found that a model in which working memory partially mediated mastery and performance goals' relation to standardized mathematics assessment scores provided a better fit to their data compared to either a model in which working memory and achievement goals independently related to mathematics scores or a model in which working memory's relation to mathematics scores were partially mediated by goal orientation. Working memory ($\beta = 0.66$) and mastery goals ($\beta = 0.16$) had positive effects on mathematics scores, whereas performance goals ($\beta = -0.23$) had a negative effect on mathematics scores. Mastery and performance goals also both had indirect effects on mathematics scores via working memory ($\beta = 0.13$ for mastery goals, $\beta = -0.10$ for performance goals). There were significant two-way interactions between performance goals and working memory, in which children with low working memory and high-performance goals had lower mathematics scores, and between mastery goals and performance goals, in which children who reported high performance goals plus low mastery goals also had lower mathematics scores. These findings suggest that fostering students' desires to master tasks is broadly beneficial for mathematics achievement. Further, the types of achievement goals may affect how working memory is used during mathematics learning, and one's achievement goals and working memory capacity may influence one's subsequent mathematics achievement. Understanding students' achievement goals and working memory capacities may serve as a basis for identifying meaningful, qualitative differences in student approaches to mathematics learning, which educators can use to guide their instructional planning.

In contrast to achievement goals' relation with working memory, achievement goals generally make unique contributions to mathematics achievement separate from general intelligence and cognition (e.g., Moyano et al., 2020; Zhang et al., 2016). When intelligence or general cognitive abilities are controlled, mastery goals are found to have a positive association with mathematics achievement (e.g., Moyano et al., 2020; Orbach et al., 2019; Zhang et al., 2016). However, performance goals' influences on mathematics outcomes are more mixed when intelligence is included. Sometimes, performance goals show an association with lower mathematics scores (Kikas et al., 2009) even in students who hold high mastery goals (Zhang et al., 2016). Other times, they show no association with performance at all (e.g., Jõgi et al., 2015; Moyano et al., 2020; Orbach et al., 2019). For example, in an Estonian sample of 2nd and 3rd grade students, researchers found that performance-approach and performance-avoidance goals predicted mathematics interest a year later (positively and negatively, respectively), but were not predictive of mathematics skills in calculation, word problems, or geometrical tasks. Instead, prior mathematics skills had a significant association with children's performance goal orientations.

Performance-approach goals in turn had a particularly strong relation with mathematics interest for low ability students (Jõgi et al., 2015). These findings build on the working memory studies that primarily utilized the dichotomous achievement goal model, showing that performance goals are not uniformly negative for mathematics achievement—a strong desire to demonstrate competence may actually bolster students' interest in mathematics tasks despite lower achievement, or vice versa. This hypothesis fits well within SEVT's broader model, which theorizes that students' goals may influence students' mathematics expectancies and values

(Eccles & Wigfield, 2020). As such, targeting students' achievement goals may be another viable target on the path to better mathematics achievement. In addition to providing information about achievement goals' unique contribution to mathematics achievement, these findings further support the working memory hypothesis by showing that achievement goals can relate to mathematics achievement specifically through working memory processes, not just broad cognitive ability.

7.3.2 Cognitive and Meta-cognitive Strategy Use and Achievement Goal Theory

As with the working memory hypothesis, there is also some evidence for the self-regulated learning hypothesis. Several studies suggest that achievement goals are related to cognitive and meta-cognitive strategy use (e.g., Meyer et al., 1997; Ocak & Yamac, 2013; Seo & Taherbhai, 2009). For example, in 5th grade Korean students, Seo and Taherbhai (2009) found that performance-avoidance goals had a direct negative association ($\beta = -0.11$) with province-ordered mathematics exam scores. Further, mastery goals ($\beta = 0.41$) and performance-approach goals ($\beta = 0.12$) both indirectly associated with mathematics performance through the use of cognitive and meta-cognitive strategies ($\beta = 0.09$). Another study found that children's perceptions of their classroom achievement goal structures influenced their strategy use more than their own goal orientations, as there was no direct relationship found between children's held achievement goals and strategy use (Young, 1997). One type of cognitive strategy that achievement goals promote may be retrieval strategies. Neuroimaging studies have found greater activation in the left inferior frontal gyrus (associated with retrieval strategies use) among low-ability children with more positive mathematics attitudes (a composite measure that included avoidance goals, as well mathematics intrinsic value, utility, and self-efficacy) (Demir-Lira et al., 2020). However, one study showed that although students with approach-oriented achievement goals showed more effort on tasks (controlling for general cognitive ability), this effort did not translate into higher mathematics achievement (Hornstra et al., 2017). In other words, achievement goals may encourage students to work harder, but not necessarily more effectively, and outside support is likely still needed to push students toward appropriate learning strategies. Together, these findings suggest that targeting students' mathematics strategies may be a more direct way of supporting mathematics achievement rather than changing their achievement goals.

7.3.3 Summary of Achievement Goal Theory Findings

In sum, mastery goals appear to have a positive relation with mathematics achievement, whereas performance-avoidance goals generally have a negative relation with mathematics achievement. There is also evidence supporting both the working memory and self-regulatory hypotheses regarding achievement goals' connections to cognition and mathematics achievement: achievement goals may partly influence working memory processes, as well as cognitive and meta-cognitive strategy use, which may then influence subsequent mathematics achievement. Encouraging students' mastery goals is most likely to improve mathematics achievement. Classroom practices such as assigning fewer or more diverse homework assignments, offering students more choices about their learning, and allowing more time for classroom activities may be perceived by students as more mastery-oriented and encourage them to adopt mastery-oriented learning goals (Vedder-Weiss, 2017). This may also influence students' mathematics expectancies and values to provide further benefit for mathematics achievement (Eccles & Wigfield, 2020). Understanding students' achievement goals, or their perceptions of their classroom achievement goals, may also provide insight into the types of strategies and thought processes that students engage in during mathematics learning, which educators can use as a starting point for their instructional plans.

7.4 Combining Cognition and Motivation: An Example Using Situated Expectancy–Value Theory

To demonstrate how existing motivational theories can be expanded to include cognitive processes and aid in understanding mathematics learning, we use Eccles and Wigfield's (2020) SEVT as an example (see Fig. 7.1). Because SEVT encompasses a broad number of influences on general achievement, the theory provides a useful base for mapping motivational, cognitive, and social processes to create a mathematics-specific model, based on current and future research involving cognition and expectancy–value, as well as research on self-concepts, intrinsic and extrinsic motivation, and achievement goals.

Several studies on domain-specific and domain-general cognition offer information that is relevant to the middle and right sections of the SEVT model, which illustrate how expectancies and values directly influence mathematics achievement, as well as the more distal factors that shape individuals' expectancies and values. First, there may be a bi-directional relationship between affective reactions and memories and subjective task values, rather than the unidirectional connection illustrated in the current model, and there may also be a direct connection between affective reactions and memories and expectancies. A study on 3rd to 6th grade students found that expectancy and value interact to influence affective states, controlling for general cognitive ability: children with low self-perceptions of their mathematics ability but high ratings of mathematics' utility and importance predicted higher levels of worry when doing mathematics (Lauermann et al., 2017). Second, the link between task values and goals/self-schemata may also be bi-directional, as subjective values have been shown to influence goal orientations (Seo & Taherbhai, 2009). Within a classroom, this may manifest in changes in

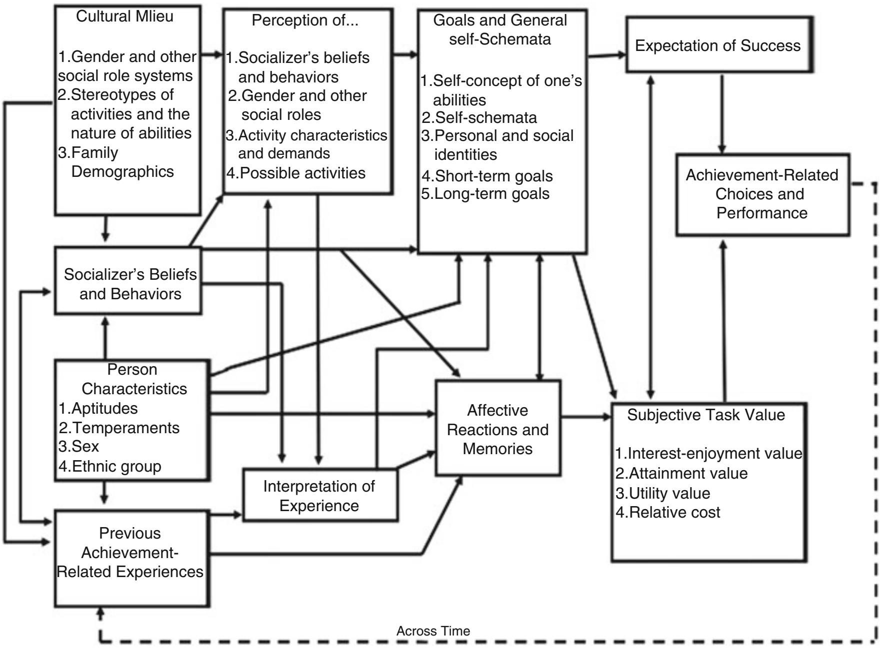

Fig. 7.1 Situated expectancy–value theory model. (Reprinted from Eccles and Wigfield (2020) with permission from Elsevier)

students' affective states as a result of their expectancies, values, and feedback about their mathematics performance. For example, a student with high worries about mathematics, high mathematics value, and low mathematics expectancy may increase their worry and adopt more avoidance goals if they encounter setbacks during mathematics tasks. Meanwhile, a comparable student with high worries about mathematics but low mathematics value and high mathematics expectancy may eventually worry less and adopt more approach goals if their motivations bolster their mathematics confidence in the face of failure. This also has implications for interventions that target such motivations, suggesting that a uniform increase in mathematics expectancies and values may not necessarily work well for all students, depending on other personal and situational variables.

Additional cognitive constructs could also be mapped onto to the SEVT model to describe how expectancies and values influence mathematics achievement. For example, several neuroimaging studies have found that higher expectancies and values promote greater use of efficient retrieval strategies in children, as indicated by greater activation in the hippocampus and left inferior frontal gyrus (e.g., Chen et al., 2018; Demir-Lira et al., 2020; Suarez-Pellicioni et al., 2021). Behavioral studies further show that expectancy and value may relate to slightly different strategies. For example, a study of Greek 5th and 6th grade students found that the use of rehearsal, elaboration, and organizational strategies mediated self-efficacy and task value's relation to teachers' ratings of student mathematics achievement. In contrast, strategies involving planning and monitoring only mediated task value's relation to mathematics achievement (Metallidou & Vlachou, 2007). Research on achievement goals also suggest that the use of these strategies and available working memory mediates the relationship between children's goals and achievement-related outcomes (e.g., Lee et al., 2013; Seo & Taherbhai, 2009). Thus, including these processes as components within the model may help to further detail how self-schemata, expectancies, and values work to influence mathematics outcomes.

Adding these cognitive processes and revisions into an SEVT model created specifically for the mathematics domain can provide a reference for how to support mathematics achievement. For researchers, the model illustrates relationships between both individual and broad level factors that may be worth evaluating (and revising as needed based on one's findings). For instruction, the model could act as a quick guide on targets to influence mathematics achievement, as well as the broader factors that one may need to consider when determining how motivational and cognitive supports will affect mathematics development.

7.5 Open Questions and Recommendations for Instructional Practice and Future Research

The existing research reviewed here shows a consistent association between motivational constructs and mathematics achievement, separate from domain-specific and domain-general cognitive factors. Mathematics expectancies and approach-oriented achievement goals may have the most potential for improving immediate mathematics performance during the elementary and middle school years, in addition to domain-general cognitive factors of working memory and general intelligence. Educators may wish to take advantage of the malleability of these motivational factors to build students' mathematics confidence and self-beliefs. For example, assignments can be scaffolded to provide small successes that gradually build in difficulty. Mastery- and performance-approach achievement goals may also be fostered through practices such as modeling how to learn from mistakes, giving targeted feedback to help students master skills, or laying out clear and varied criteria for students to demonstrate mathematics success. In the longer term, supporting mathematics values and intrinsic motivation may also support future mathematics achievement through students' math-related choices. Although these factors may be less of an immediate priority during the elementary and middle school years, educators can still prepare students for the future by providing space for students to practice autonomy, approval to build feelings of competence, and collaborative assignments that mimic real-world tasks to build relatedness and value.

Although research that involves both motivational and cognitive factors is growing in number, there is still much work to be done to fully understand the interplay between motivation and cognition for mathematics learning in the elementary and middle school years. First, future studies may want to include a larger range of factors to gain a broader understanding of children's mathematics achievement. Within this chapter, we focused on work that included domain-specific cognitive or domain-general cognitive factors, in conjunction with motivational factors. However, few studies have investigated all three types of these factors together, even though our work suggests that all three types jointly contribute to mathematics performance (Liu et al., 2022). There are also other types of factors (as shown by the SEVT model) that influence mathematics achievement that were not included here, such as other motivational constructs or affective factors (e.g., mathematics anxiety). Many of these other factors are likely to have an influence on or be influenced by cognition and motivation. Another important factor when considering cognitive and motivational measures together may be test motivation; that is, students who are more highly motivated in mathematics may also be motivated to try harder on general cognition tests, which may inflate the predictive validity of general intelligence on mathematics achievement (Duckworth et al., 2011). We also focused primarily on individual-level factors in this chapter, but it is important to consider broader environmental-level factors, such as school or family influences, and how these more distal factors influence cognitive and motivational processes. Investigating more varied factors concurrently will provide a better understanding of their unique and overlapping influences on mathematics achievement and inform which factors may be most effective and malleable to interventions for improving mathematics achievement.

Future studies should also consider investigating the relationships between motivation and cognition within a broader variety of samples. Many existing studies involve participant samples that are predominantly White. These "WEIRD" populations—Western, educated, industrialized, rich, and democratic—have been shown to be unrepresentative on measures of many psychological phenomena, and the same is likely true for motivational theories (Henrich et al., 2010). Indeed, there is evidence in a couple of studies reviewed here that models that fit these convenient participants may not fit as well with other samples of participants (e.g., Lu et al., 2011; Stevens et al., 2006). It is an open but important question to understand how motivational processes work for different children and contexts, especially considering the broader social constructs that may impact cognitive and motivational functioning. Models such as SEVT can help researchers to identify potential sample characteristics that may lead to significant differences in motivation, and guide the recruitment of participants. This practice can ensure that interventions and practices based on cognitive and motivational processes can benefit all students. Other useful methods for researchers may be to report sample characteristics in more detail, justify the sample used in one's study, and be explicit about the generalizability of findings based on one's given sample (Rad et al., 2018).

Finally, we encourage researchers working within the motivational and cognitive space to situate their studies within motivational frameworks and models. These models can provide guidance on how cognitive variables and processes may fit with motivational ones, driving research questions and analytical choices (e.g., what pathways, mediations, or interactions to expect and test). Framing analyses within these models can further help to refine existing theories so that models can be revised to better illustrate mathematics learning and inform theory and practice.

Overall, the research space involving the intersection of cognition and motivation is an open and growing area with implications for instructional practices that influence mathematics achievement. Current and future work can elucidate a holistic understanding of mathematics development during the elementary and middle school years, informing future research and effective practices for supporting children's mathematics achievement and future outcomes.

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Appendix A: Review Methodology

Inclusion & Exclusion Criteria The review only included studies that investigated relations between at least one motivational construct, at least one cognitive construct, and at least one mathematics achievement outcome. We required studies to use motivational constructs that fit within the frameworks of Situated Expectancy–Value Theory (SEVT), Self-Determination Theory (SDT), or Achievement Goal Theory (AGT). Cognitive constructs also had to fit within domain-specific cognition (e.g., number sense) or domain-general cognition (e.g., executive functions, general intelligence or cognition). At least part of the study's participant sample had to include 3rd to 5th grade students (8–11 years old). We only included studies written in English.

Literature Identification We conducted three sets of literature searches, based on our three motivation theories of interest. All literature searches began with the keywords "math" and "elementary," combined with additional keywords that were specific to one of our three motivational theories of interest: (1) "expectancy value"; (2) "self determination," "intrinsic motivation," and "extrinsic motivation"; and (3) "achievement goal theory," "performance goal," and "mastery goal." We also included one more keyword about either domain-specific (i.e., "number," "number sense," "domain specific") or domain-cognitive factors (i.e., "cog*," "domain general," "executive function," "intelligence"). For each manuscript, we used the title to determine preliminary relevance. We utilized the Google Scholar database for the search. We reviewed the first ten pages of search results for each of the searches, finding a total of 207 (89 SEVT, 48 SDT, and 70 AGT) potentially relevant articles.

Inclusion Screening We read the abstracts and methods of the 207 studies to decide on their inclusion for the literature review. 175 (74 SEVT, 42 SDT, 59 AGT) studies were excluded for not meeting our inclusion criteria upon closer review (e.g., outside of the intended age range, missing a cognitive and motivational construct in one of the three theories of interest, no mathematics achievement outcome). The remaining articles were then skimmed in full to screen for quality, and additional studies were identified through backward and forward citation searches (8 SEVT, 2 SDT, 3 AGT), for a final total of 32 relevant studies (15 SEVT, 6 SDT, 11 AGT).

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Chapter 8
Design Principles for Digital Mathematical Games that Promote Positive Achievement Emotions and Achievement

Run Wen and Adam K. Dubé

Abstract Digital educational games can be an enjoyable way to improve students' mathematics achievement. However, players may experience other emotions besides enjoyment when learning about mathematics, such as anxiety and boredom. These emotions are also important as they affect learning outcomes via multiple pathways. Loderer et al.' (Emotional foundations of game-based learning. In Plass JL, Mayer RE, Homer BD (eds) Handbook of game-based learning. The MIT Press, 111–152 (2020)) propose five foundational emotional design principles for use in digital game-based learning. However, empirical evidence to support these principles is lacking. This paper conducted a meta-analysis on the effects of emotional design principles. The results showed that among the studies reviewed ($n = 17$), most of games applied multiple principles. In general, emotional design principles had a medium effect size on both achievement emotions ($g = .50$) and learning outcomes ($g = .66$). Principles that influence control and value appraisals had stronger effects ($g = .60/.63$ respectively) compared to those that did not contribute to control and value appraisals. These principles should be adopted for a better emotional experience and learning outcome.

Keywords Achievement emotions · Mathematical games · Digital games · Game design

8.1 Introduction

A great deal of research has been conducted on the effectiveness of digital mathematical games and their ability to improve learning outcomes. In Byun and Joung's (2018) meta-analysis of 17 studies on K-12 mathematics education, digital games are shown to have a positive effect on mathematics learning outcomes ($d = .37$). In a narrative review of 25 studies, Dubé et al. (2019) similarly conclude that digital

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mathematical games can be effective learning tools for engaging students with learning content, increasing performance, and improving attitudes. However, digital games are not equally effective in all domains of learning and some learning outcomes are more studied than others (Ke, 2009). This chapter discusses why digital mathematical games are effective tools for students in the elementary and middle school years by investigating which emotional design features contribute to better mathematics learning experience and outcomes.

8.2 Why Are Digital Mathematical Games Effective?

Dubé et al. (2019) argue that digital educational games are often designed to be "interactive" and "fun", and these characteristics of games may have the potential to overcome the "boring" mindsets and anxious feelings students hold towards mathematics. Students start to develop mathematics anxiety (fear and tension towards mathematics) at an early age, and it grows over time (Aiken, 1970). The level of mathematics anxiety students experience does not differ as a factor of mathematical ability as high performing students also report feeling stressed and anxious towards mathematics (OECD, 2018). Mathematics anxiety matters because it can affect students' performance during assessment as well as their willingness to pursue mathematics as a field of study (Ashcraft & Faust, 1994). In fact, a negative relationship between anxiety and mathematics performance has been reported in different studies spanning all grades (Brassell et al., 1980; Hembree, 1990; Lee, 2009; Ma, 1999; Zakaria & Nordin, 2008).

In contrast to the anxious feelings experienced with mathematics, students report having fun and enjoyment during digital educational games regardless of academic subject (Mekler et al., 2014). Mekler et al.'s (2014) systematic review of 87 digital game studies find that descriptors such as "enjoyable", "enjoying", "fun", and "interest" have appeared in 89% of studies. For mathematics, enjoyment is often reported by students themselves when interacting with mathematical games (Chen et al., 2012). Putwain et al. (2018) further report a reciprocal relation between enjoyment and mathematics achievement among 5th to 6th grade students. Their finding suggests that higher levels of enjoyment experienced during mathematical game learning leads to better learning outcomes (Putwain et al., 2018). Therefore, it seems that the fun part of digital mathematical games not only offset the boring and anxious feelings students experience with mathematics but can also generate positive emotions and lead to better mathematics performance.

8.3 Emotions

Besides enjoyment, there are a range of emotions students may experience during game-based learning and how they affect the learning process is complex. Emotions refer to affective states, either positive or negative (Lazarus, 1993). Ekman et al.

(1987) propose six basic emotions: fear, anger, joy, sadness, disgust, and surprise. However, these general emotions are insufficient to capture the complex roles of emotions in learning (Kapoor et al., 2001; Kort et al., 2001). Since this earlier work, there are a broad range of emotions that researchers consider when studying learning. Craig et al. (2004) includes confusion, frustration, boredom, flow/engagement, interest, and being stuck as affective sates that learners may encounter. Similarly, D'Mello and Graesser (2007) argue that confusion, frustration, and boredom are inevitable emotions during learning. Woolf et al. (2009) adapt Ekman's (1999) basic emotions, while focusing on learning-related subsets of emotions such as pleasure, frustration, boredom, anxiety, novelty, and confidence.

8.3.1 Achievement Emotions

Pekrun and Perry's (2014) three-dimensional taxonomy of achievement emotions is a systematic way to organize and define learning related emotions. They defined learning related emotions as achievement emotions, which are "emotions that are tied to achievement activities (e.g., studying) or achievement outcomes (success and failure)" (Pekrun, 2017, p. 143). There are 17 types of achievement emotions that can be categorized according to three dimensions: object focus, valence, and activation (Pekrun & Perry, 2014). The object dimension distinguishes achievement emotions by activity and outcome (both prospective and retrospective). For example, enjoyment is an activity achievement emotion that occurs during learning, hope is an outcome prospective achievement emotion that occurs before learning, and pride is an outcome retrospective achievement emotion. The valence dimension groups achievement emotions into positive achievement emotions (joy) and negative achievement emotions (sadness). The activation dimension distinguishes the activating achievement emotions (anger) from deactivating achievement emotions (boredom).

Therefore, we adopted Pekrun and Perry's (2014) definition of achievement emotions and refer to learning related emotions as achievement emotions. Achievement emotions can manifest themselves overtly or be internalized. Moreover, they have the potential to impact learning and the pleasure of learning, which is described next.

8.3.2 Importance of Achievement Emotions

Mathematics attitude is closely related to mathematics achievement (Ma & Kishor, 1997; Ma & Xu, 2004). It contains cognitive (beliefs towards mathematics), affective (emotions associate with mathematics), and behavioural (intentional behaviours towards mathematics) components (Wen & Dubé, under review). Achievement emotions are considered as affective components of mathematics attitudes and believed to play an important role in learning as suggested by Pekrun and Perry's (2014) control-value theory. Control-value theory argues that achievement emotions are products of cognitive appraisals of learning events and predictors of students' performance and achievement (Pekrun, 2006; Pekrun et al., 2007).

There are two types of appraisals that play vital roles in arousing achievement emotions: (1) control-related and (2) value-related appraisals. Control-related appraisals refer to the evaluation of one's controllability over achievement activities and achievement outcomes. Value-related appraisals refer to one's value of both achievement activities and achievement outcomes, which can be either intrinsically or extrinsically valued. Achievement emotions are products of control−/value-related appraisals, with high control and value resulting in positive achievement emotions (e.g., enjoyment) while high value but low control leading to negative achievement emotions (e.g., anxiety; Pekrun, 2006).

Achievement emotions are not only the results of control and value appraisals, but also indicators of students' mathematics achievement (Pekrun, 2006). In fact, achievement emotions have an impact on academic performance through motivational factors (Pekrun & Perry, 2014). For example, students with positive achievement emotions (e.g., enjoyment) are more likely to reengagement in the activity and are more likely to have a better performance. Students with negative achievement emotions (e.g., anxiety), on another hand, are more likely to avoid the task and underperform in their learning (Pekrun, 2006). This relationship between achievement emotions and learning achievement proposed by control-value theory is supported by empirical evidence. A study of 3425 5th to 9th grade students find strong evidence for a reciprocal relationship between the two (Pekrun et al., 2017). From a control-value lens, achievement emotions are critical in learning as they reflect student's appraisals and predict their achievement.

So, how do digital mathematical games affect students' achievement emotions, and hence, influence their learning? Given the aforementioned theoretical lens, there must be features/design elements of digital mathematical games that affect students' control and value appraisals, which result in different achievement emotions, and further influence students' mathematics ability. To be specific, game features that contribute to students' control/value appraisals are assumed to lead to positive achievement emotions and better learning outcomes. The next step is to identify what these potential game features could be, and to test if these features lead to better mathematics learning experience and outcomes.

8.4 Emotional Foundations of Digital Game Design

Though achievement emotions have been widely studied in the field of psychology, few researchers have explored achievement emotions in the context of digital mathematical games. Loderer et al.'s (2020) recent study explored the emotional design of digital learning environments resulting in a model that explores the design of emotional foundations for game-based learning. Loderer et al. (2020) argues that emotional support can improve learning for all individuals and identifies five general principles for game design from an emotional design perspective: visual aesthetic design, musical score, game mechanics, narrative, and incentive system.

8.4.1 Visual Aesthetic Design

Loderer et al. (2020) propose that visual aesthetic design elements contain bright colours, round shapes, and the presence of learner-resembled avatars. Bright colours are saturated warm colours (e.g., orange, pink) (Loderer et al., 2020). Round shapes refer to graphic images and user interfaces that adhere to a rounded form (cf., square) (Loderer et al., 2020). Avatars are graphical representations of certain characters, whose faces and expressions evoke learners' emotions (Loderer et al., 2020). Generally, bright colours, round shapes, and avatars are connected with positive achievement emotions (Arroyo et al., 2013; Boyatzis & Varghese, 1994; Kao & Harrell, 2015a, b; Mayer & Estrella, 2014; Plass et al., 2014; Um et al., 2012). Particularly, avatars are better at inducing learners' positive achievement emotions compared to bright colours, which likely occurs because the control over avatars provides players a sense of power, and increased control-related appraisals further arouses positive achievement emotions.

8.4.2 Musical Score

Music has a direct impact on a learner's emotions via tones and rhythms (Loderer et al., 2020). Musical score includes emotional tones, vocal sound, and sound feedback in the game (Loderer et al., 2020). Studies on tones indicate that higher brightness of tones are associated with positive emotions such as happy and joyful (Wu et al., 2013). Human generated sounds are also more likely to evoke positive emptions compared to computer-generated sounds, due to a sense of social presence or connection they provide (Baylor, 2011). Sound feedback occurs when sounds are provided based on learners' performance (e.g., recognize mistakes or celebrate success, Loderer et al., 2020). It is a particularly important complementary source to visual feedback, and both are essential for learning (Fiorella et al., 2012).

8.4.3 Game Mechanics

Game mechanics are "methods invoked by agents for interacting with the game world" (Sicart, 2008). A well-designed game mechanic should (1) match with learning goals, (2) have a clear task, (3) have learner-appropriate difficulty, (4) provide social interaction, and (5) provide scaffolding (Loderer et al., 2020). Loderer et al. (2018) suggest that good game mechanics provide students with a sense of control over the challenges, and thus are more likely to produce positive achievement emotions like enjoyment.

8.4.4 Narrative

Narrative means that a game has a storyline that contextualizes the gameplay situation and provides a sense of belonging to a world (Dickey, 2007). Narrative can be either relevant or irrelevant to learning mechanics. Relative storylines make learning part of the story, with the goal of increasing motivation, engagement, and learning gains (Cordova & Lepper, 1996). Though a meta-analysis found that most games adopting irrelevant storylines had better learning outcomes (Clark et al., 2016). Clark et al. (2016) suggests that games with overly developed, or 'thick' narratives distract players from learning. Therefore, games with relevant narratives to learning should avoid complex storylines as they may make it hard for learners to follow and understand.

8.4.5 Incentive Systems

Incentive systems contain rewards, unlocking mechanisms, and learner choice that keeps them motivated (Loderer et al., 2020). The basic incentives are rewards, which could be in the form of points, scores, stars, or badges. More advanced incentives include unlocking mechanisms, which allow learners to get access to game levels (new levels or new mini games). The opportunity to unlock an unknown game level acts as an intrinsically motivating factor, that captures learners' curiosity (Malone, 1981). Another incentive is learner choice, which gives learners the ability to choose rewards. For example, learners can choose which gifts are earned or change the avatar used in the game. Learner choice provides learners a sense of control over the game (Loderer et al., 2020). Thus, incentive systems link directly to learners' control and value appraisals of the learning activity (McNamara et al., 2010). However, the number of incentives has to be considered carefully as overly frequent rewards can undermine learner's intrinsic value of learning (Abramovich et al., 2013).

8.5 Do Emotional Design Principles Promote Positive Achievement Emotions and Learning Outcomes?

There is limited evidence to support the effectiveness of a particular emotional design principle and whether these five principles facilitate positive achievement emotions during mathematical game-based learning needs to be systematically investigated. Further, control-value theory holds that emotional design principles that contribute to control/value appraisals are more likely to result in positive achievement emotions, but this assumption also needs empirical support. To address this gap, we developed a coding scheme based on Loderer et al.' (2020) five emotional design principles and conducted a meta-analysis of digital mathematical game research to explore (1) which emotional design principles are used in digital mathematical game research and (2) how effective each emotional design principle is at improving achievement emotions and learning outcomes.

8.5.1 Review Process

First, a systematic review was conducted using keywords combinations of mathematics, game, and emotions searched in three databases: PsycINFO, ERIC EBSCO and Scopus (see Table 8.1). The initial search returned 171 articles. After removing duplicates, 144 articles were entered into a two-stage screening process.

Second, selected articles were then coded for the following: study (authors and publication year), grade level, sample size, intervention duration, study design, and emotion measured (see Table 8.2).

Third, a content analysis was conducted to categorize games in the selected studies based on Loderer et al.'s emotional design principles (2020). Table 8.3 shows the coding framework.

Fourth, effect sizes were calculated to compare the effect of the different principles on mathematics achievement emotions and outcomes across studies. This was done due to the advantages effect sizes provide in representing true effects and comparing across studies (Cohen, 1988; Ellis, 2010). Effect size for each study was calculated by using the data provided in the study.

Table 8.1 Keywords for information retrieval

Mathematics	Game	Emotions
Math*	Game*	Emotion*
	Educational game*	Enjoy*
	Digital game*	Hop*
	Serious game*	Anxi*
	Video game*	Boredom*
	Mobile game*	Frustrat*
	Mobile app*	Sad*
	Tablet game*	Anger*
	Tablet app*	Relief*
		Relax*
		Shame*

Note: The asterisk at the end of each keyword indicates truncation searching, allowing researchers to search for a term and its various spellings

Table 8.2 Summary of reviewed studies

Studies	Grade level	Sample size	Durations	Design	Achievement emotions	*g* on Emo	*g* on Achi
Adamo-Villani and Dib (2013)	1–5	13	No data	Qualitative	Enjoyment	NA	NA
Beserra et al. (2019)	2	110	14-week	Mixed	Enjoyment	NA	1.7
Chen et al. (2012)	4	53		Quantitative	Enjoyment	.45	.43
Chiang and Qin (2018)	7	89	4-week	Mixed	Enjoyment	.25	.52
Conati and Gutica (2016)	6–7	15	1–2 h	Mixed	Enjoyment	NA	.64
Gilliam et al. (2017)	9–12	133	5-week	Mixed	Enjoyment Boredom	NA NA	NA NA
Godfrey and Mtebe (2018)	1–3	111	6-week	Mixed	Enjoyment Frustration	NA NA	NA NA
Gresalfi et al. (2018)	3	95	3-day	Mixed	Enjoyment	.69	.43
Hensberry et al. (2015)	4	46	4-day	Mixed	Enjoyment	NA	1.2
Hill et al. (2016)	8–11	322	3-week	Quantitative	Enjoyment	.09	.34
Howard-Jones and Demetriou (2009)	5–6	50	1 session	Mixed	Frustration	NA	NA
Huang et al. (2014)	2	56	8-week	Quantitative	Reduced anxiety	.02	.32
McLaren et al. (2017)	6	153	No data	Quantitative	Enjoyment	.95	.37
Pareto et al. (2012)	3	47	9-week	Mixed	Enjoyment	NA	NA
Plass et al. (2013)	6–8	58	1 session	Quantitative	Enjoyment	1.27	.86
Sedig (2008)	6	58	1 session	Quantitative	Enjoyment	NA	.72
Van Eck (2006)	7–8	123	1 session	Quantitative	Reduced anxiety	.36	NA

8.5.2 *Which Emotional Design Principles Are Used in Mathematical Game Research?*

Among the 17 reviewed studies, many (71%) used more than one emotional design principle, while only one study adopted all five (see Table 8.4). Among the five emotional design principles, visual aesthetic design was the most applied (71%). These studies were more likely to use bright colours and avatars in their games. Three types of avatars controlled by players were found in the reviewed studies, expert avatars (e.g., a scientist agent or a teacher agent), peer avatars (e.g., a student

Table 8.3 Operationalization of Emotional Design Principles

Emotional design principles	Operationalization	Operationalized keywords and instructions
Visual aesthetic design	Basic emotional related visual designs, such as warm colors and round shapes. Avatars or agents that resemble players in the game.	Shape, round, face-liked shapes, oval. Warm, bright, red, yellow, pink, orange. Agents, avatars, peers, experts, virtual selves.
Musical score	Musical score refers to auditory stimulus in the game. Higher musical tempo, vocal sound, and sound feedback promote positive emotions.	Rhythms, tones, higher musical tempo, vocal sound, human voice, volume, pitch, prosody, rate of speech, sound feedback.
Game mechanics	Game mechanics that align with learning goals, task clarity, task demands, scaffolding, and social interaction.	Learning activity matches with learning goals/skills. No extraneous content in the game. The task is simple and clear. The difficulty level matches with players' competencies. Scaffolding is provided such as providing examples, adjusting difficulty, providing hints, offering explanations, repeating contents. Social interactions are allowed between players in the game.
Narrative	Narrative refers to a story in the game. Well-structured narratives have compelling story lines.	Storyline contextualizes the skill to be learned.
Incentive system	Incentives such as rewards and other systems that provide extra control over game progression.	Reward, punishment, progress bars, scores, badges, access to game levels, unlock game levels, trade for gifts.

or boy agent), and animal avatars (e.g., a monkey agent). The second most common principle was game mechanics (65%). These included having a clear task, relevant difficulty, social interactions, and scaffolding. For social interaction, two of the five studies with this principle used collaboration, one study used competition, and the remaining two used both. Incentive systems were found in 53% of the reviewed studies. The most common incentive system was rewards, while unlocking mechanism and learner choice were seldom provided.

Musical score (29%) and narrative (24%) were the least applied principles in the reviewed studies. Studies that applied musical score mostly did so by providing sound feedback while fewer used vocal sounds. Notably, all studies that implemented musical score also adopted visual aesthetic design. This is likely because the combination of sound and visuals are more effective at engaging students than sound alone (Wolfson & Case, 2000). Finally, studies applied narrative all used simple narratives that were relevant to the learning content (cf., complex

Table 8.4 Summary of emotional design principles and achievement emotions in the reviewed studies

Citations	Visual Aesthetic Design			Musical Score			Game Mechanics					Narrative	Incentive System			Emotions
	Round Shape	Bright Colour	Avatars	Tones	Vocal Sound	Sound Feedback	Match with Learning Goals	Clear Task	Relative Difficulties	Social Interaction	Scaffolding	Storyline	Rewards	Unlocking Mechanism	Learner Choice	
Adamo-Villani and Dib (2013)			✓												✓	Enjoyment
Beserra et al. (2019)												✓				Enjoyment
Chen et al. (2012)	✓	✓	✓					✓		✓			✓			Enjoyment
Chiang and Qin (2018)	✓	✓				✓	✓	✓					✓			Enjoyment
Conati and Gutica (2016)			✓						✓		✓	✓				Enjoyment
Gilliam et al. (2017)			✓							✓		✓				Enjoyment Boredom
Godfrey and Mtebe (2018)			✓	✓	✓	✓		x	✓			✓	✓	✓		Enjoyment Frustration
Gresalfi et al. (2018)		✓				✓							✓	✓		Enjoyment
Hensberry et al. (2015)		✓					✓	✓			✓				✓	Enjoyment
Hill et al. (2016)		✓			✓	✓			✓							Enjoyment

Howard-Jones and Demetriou (2009)													x			Frustration
Huang et al. (2014)											✓					Reduced anxiety
McLaren et al. (2017)		✓	✓													Enjoyment
Pareto et al. (2012)										✓	✓					Enjoyment
Plass, et al. (2013)								✓	✓	✓			✓			Enjoyment
Sedig (2008)	✓	✓				✓			✓				✓			Enjoyment
Van Eck (2006)			✓							✓						Reduced Anxiety

Note: A check means the game in the reviewed study has covered the given feature while a cross means the feature was included but was deemed poorly implemented ones that distract). For example, in Beserra et al.'s (2019) work, the story involves an avatar progressing forward by building a bridge with stones, each stone has a number on it, and the avatar needs to choose the correct number to progress. No other distracting elements were presented and there was a central story that was linked to the game mechanic.

8.5.3 How Effective Are Emotional Design Principles at Improving Achievement Emotions and Learning Outcomes?

Both qualitative and quantitative studies were considered when exploring game's effectiveness, as Ke (2009) suggests. Among the reviewed studies, one was qualitative, nine were mixed methods, and seven were quantitative. For the studies that did not have statistical data to calculate effect size, a meta-thematic analysis was conducted based on the themes that students used to describe game-based learning in the interview (Mekler et al., 2014; Talan et al., 2020) (see Table 8.5). Meta-thematic analysis indicated that students enjoyed learning with mathematical games, with "fun" and "interesting" being reported in multiple studies (Adamo-Villani & Dib, 2013; Beserra et al., 2019; Chiang & Qin, 2018; Conati & Gutica, 2016; Gresalfi et al., 2018). Not only did games generate positive achievement emotions, but students believed the game also helped their learning, fostered collaboration, and increased interests in the subject (Chen et al., 2012; Gilliam et al., 2017; Gresalfi et al., 2018).

For quantitative studies, effect sizes were calculated. Below, we provide forest plots that visualize the results as well as report key results (overall average effect size, heterogeneity, and predicted interval range). Forest plots can be interpreted as

Table 8.5 The most frequently used terms regarding games

Emotion-related themes	Examples	N
Fun	"It was fun to play the game every session."	3
Interesting	"Scratch is interesting."	2
Enjoyable/enjoying	"I enjoyed participating in the activities using the game."	1
Awesome	"This is awesome because you feel like you are really in a bakery. . ."	1
Future play intention	"I would want to play this app again"	1
Learning-related terms	Examples	N
Helpful	"I think it's helpful" "I like this game 'cause it helps you learn, for strugglers."	1
Collaboration	". . .it was all about collaboration and helping with the group. . ." "All that collaboration between eight of us, that really helped"	1
Increased interest in learning	"It makes me feel like I want to know more about things"	1

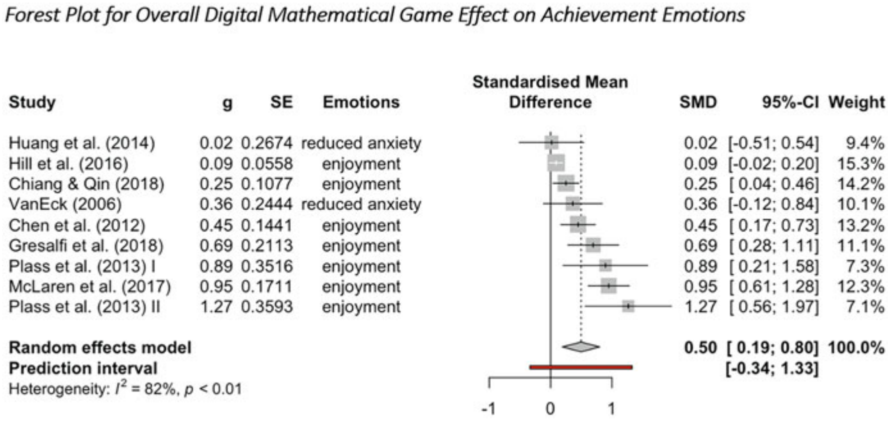

Fig. 8.1 Forest plot for overall digital mathematical game effect on achievement emotions

follows: boxes depict the effects of individual studies while the diamond at the bottom of each plot depicts the average effect for all studies combined. The horizontal lines for each study depict 95% confidence intervals, with narrow ones represent more precision effect sizes. A narrative explanation and interpretation for these results is presented in the subsequent discussion. First, we report the overall effect of mathematical games on achievement emotions and achievement.

Overall, emotional design principles in digital mathematical games had a medium effect on achievement emotions (see Fig. 8.1) ($n = 9$, $g = .50$, 95% CI [.19, .80], $p = .006$). The between-study heterogeneity variance was estimated at $\tau^2 = .11$ (95% CI [.03, .59]), with an I^2 value of 81.6% (95% CI [.66–.90]). The prediction interval ranged from $g = -.34$ to 1.33, indicating that negative intervention effects cannot be ruled out for future studies. Similarly, a medium effect on mathematics achievement was found (see Fig. 8.2) ($n = 12$, $g = .66$, 95% CI [.38, .94], $p < .001$). The between-study heterogeneity variance was estimated at $\tau^2 = .16$ (95% CI [.06, .48]), with an I^2 value of 88% (95% CI [.80–.92]). The prediction interval ranged from $g = -.28$ to 1.6, indicating that negative intervention effects may occur for future studies.

Second, we explore each design principles' effect on achievement emotions and learning outcomes. This was achieved by grouping studies based on their design principles and calculating effect sizes for each grouping.

8.5.3.1 Visual Aesthetic Design

Visual aesthetic design had a small to medium effect on enjoyment and reduced anxiety (see Fig. 8.3) ($n = 6$, $g = .44$, 95% CI [.11, .77], $p = .02$). The between-study heterogeneity variance was estimated at $\tau^2 = .08$ (95% CI [.02, .56]), with an I^2 value of 84% (95%CI [.67–.92]). The prediction interval ranged from $g = -.43$ to 1.31, indicating that negative intervention effects cannot be ruled out for future

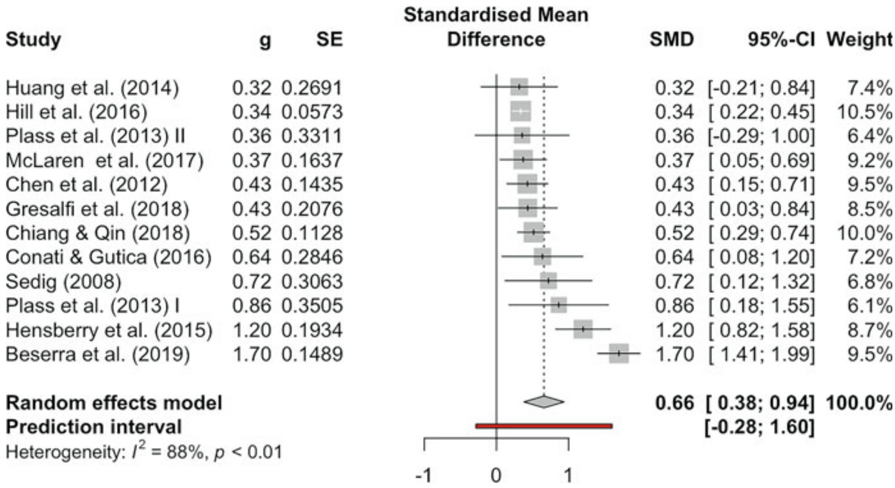

Fig. 8.2 Forest plot for overall digital mathematical game effect on mathematics achievement

studies. Similarly, a slightly higher medium effect on mathematics achievement was found (see Fig. 8.3) ($n = 8$, $g = .54$, 95% CI [.31, .78], $p < .001$). The between-study heterogeneity variance was estimated at $\tau^2 = .05$ (95%CI [0.01, 0.30]), with an I^2 value of 66% (95%CI [.28–.84]). The prediction interval ranged from $g = -.05$ to 1.13, indicating that negative intervention effects may occur for future studies.

8.5.3.2 Musical Score

Musical score had a small but non-significant effect on enjoyment (see Fig. 8.3) ($n = 3$, $g = .29$, 95% CI [−.43, 1.02], $p = .22$). The between-study heterogeneity variance was estimated at $\tau^2 = .06$ (95%CI [.01, 3.85]), with an I^2 value of 77% (95%CI [.25–.93]). The prediction interval ranged from $g = -3.46$ to 4.05, indicating that negative intervention effects cannot be ruled out for future studies. Meanwhile, a significant small to medium effect on mathematics achievement was found (see Fig. 8.3) ($n = 6$, $g = .43$, 95% CI [.28, .59], $p < .001$). The between-study heterogeneity variance was estimated at $\tau^2 = .01$ (95%CI [.00, 0.18]), with an I^2 value of 3% (95%CI [.00–.75]). The prediction interval ranged from $g = .15$ to .72, indicating that negative intervention effects can be ruled out for future studies.

8.5.3.3 Game Mechanics

Game mechanics had a small to medium effect on enjoyment and reduced anxiety (see Fig. 8.3) ($n = 8$, $g = .42$, 95% CI [.11, .73], $p = .02$). The between-study heterogeneity variance was estimated at $\tau^2 = .07$ (95%CI [.01, .66]), with an I^2 value

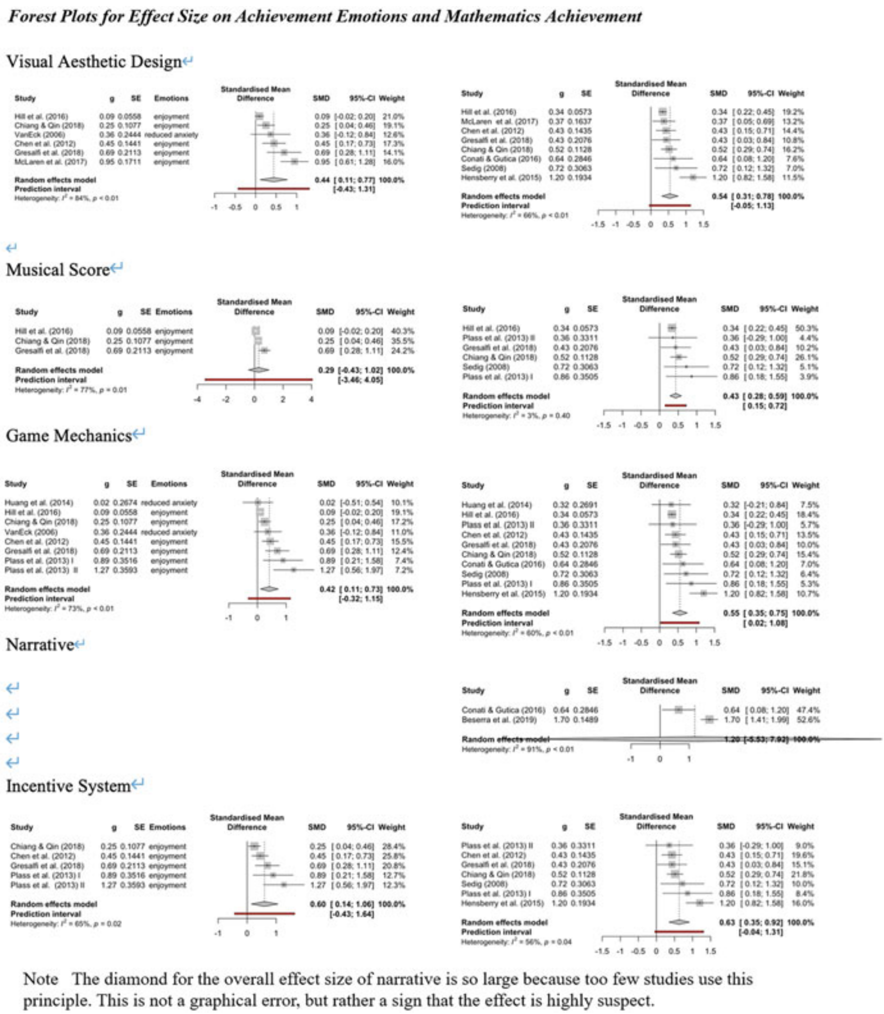

Fig. 8.3 Forest plots for effect size on achievement emotions and mathematics achievement. Note the diamond for the overall effect size of narrative is so large because too few studies use this principle. This is not a graphical error, but rather a sign that the effect is highly suspect

of 73% (95%CI [.45–.87]). The prediction interval ranged from $g = -.32$ to 1.15, indicating that negative intervention effects cannot be ruled out for future studies. Meanwhile, a medium effect on mathematics achievement was found (see Fig. 8.3) ($n = 10$, $g = .55$, 95% CI [.35, .75], $p < .001$). The between-study heterogeneity variance was estimated at $\tau^2 = .04$ (95%CI [.00, .22]), with an I^2 value of 60% (95% CI [.19, .80]). The prediction interval ranged from $g = .02$ to 1.08, indicating that negative intervention effects can be ruled out for future studies.

8.5.3.4 Narrative

There was not enough statistical data to estimate narrative's effectiveness on achievement emotions. A large non-significant effect size on mathematics achievement was found (see Fig. 8.3) ($n = 2$, $g = 1.2$, 95% CI [−5.53, 7.92], $p = .27$). The between-study heterogeneity variance was estimated at $\tau^2 = 0.51$, with an I^2 value of 91% (95%CI [.67, .97]).

8.5.3.5 Incentive System

Incentive system had a medium effect size on enjoyment (see Fig. 8.3) ($n = 5$, $g = .60$, 95% CI [.14, 1.06], $p = .02$). The between-study heterogeneity variance was estimated at $\tau^2 = .08$ (95%CI [.00, 1.21]), with an I^2 value of 65% (95%CI [.09–.87]). The prediction interval ranged from $g = -.43$ to 1.64, indicating that negative intervention effects cannot be ruled out for future studies. Meanwhile, a medium effect size on mathematics achievement was found (see Fig. 8.3) ($n = 7$, $g = .63$, 95% CI [.35, .92], $p = .002$). The between-study heterogeneity variance was estimated at $\tau^2 = .06$ (95%CI [.00, .40]), with an I^2 value of 56% (95%CI [.00, .81]). The prediction interval ranged from $g = -.04$ to 1.31, indicating that negative intervention effects may occur for future studies.

8.6 Summary of Design Principles of Digital Mathematical Games' Impact on Achievement Emotions and Achievement

Most games from reviewed studies applied more than one emotional design principle, indicating that students' emotional experience during digital game-based learning is being considered by mathematics researchers. The most used principles were visual aesthetic design, game mechanics, and incentive system, while musical scores and narrative were the least used. Though multiple emotional design principles were employed to some extent, most games adopted features that are easier to integrate (e.g., bright colours, avatars) while neglecting ones that should be just as important to learning but are difficult to integrate. For example, features such as narratives, unlocking mechanisms, and learner choice were largely neglected in the reviewed studies. The possible reasons could be that writing a simple story that aligns with learning goals is more challenging than presenting the learning content alone (Clark et al., 2016); having levels that unlock requires building a larger game world and system in addition to the core game; and providing learner choice requires developing and designing content that is not absolutely necessary. These design principles should be included in future mathematical game studies as they have the potential to intrinsically motivate learners and give them a sense of control over games (Habgood & Ainsworth, 2011).

Both the qualitative and quantitative studies indicate that digital mathematical games have a positive impact on achievement emotions and mathematics achievement. Incentive systems had the highest effect on both achievement emotions and learning outcomes, followed by visual aesthetic design and game mechanics. Narrative had a large but non-significant effect on achievement. However, due to small sample size, more studies are needed to draw further conclusions. Musical score showed the least effect on achievement emotions and mathematics achievement.

From a control-value theory lens, design principles that contribute to players' control and value appraisals are more likely to generate better achievement emotions and learning outcomes (Pekrun, 2006). Emotional design principles that had stronger effects (incentive system, visual aesthetic design, and game mechanics) likely provide learners with a sense of control over the task (control of avatars, adjusted difficulty level) or add extra value (rewards) to the task (Chen et al., 2012; Gresalfi et al., 2018). Similarly, narrative sets up challenges or goals in the game (Lindley, 2005), which offers extrinsic value to players. In contrast, emotional design principles that showed small effect size (i.e., musical scores) may not impact control appraisals (Beserra et al., 2019; Conati & Gutica, 2016), as sound and music are either pre-set or reflect in-game actions and cannot be predicted or determined by players.

All five emotional design principles had larger effects on learning outcomes than achievement emotions. One possible reason is that fewer studies measured achievement emotions than performance. Therefore, future works not only need to explore the effect of mathematical games on learning, but also need to examine their effect on achievement emotions, and how achievement emotions mediate the effects on achievement (Pekrun, 2006). The results provide guidance on choosing mathematical games for educational purposes. Teachers and parents should pick games that embed more than one forementioned emotional design principle, particularly those that provide a sense of control to players and add value to the activities. For instance, games that have avatars for learners to control or adjust difficulty level to different learners can give learners a sense of control whereas games with rewards add extra value for learners. This increased control and value may potentially lead to better learning experience and outcomes. A good digital educational game not only points out the mistakes made by players (e.g., sound feedback), but also scaffolds them with proper hints and adjusted difficulty levels, while motivating them with appealing visuals, an integrated narrative, and rewarding systems.

Research on achievement emotions and the effects of emotional design principles on game-based learning is lacking. This meta-analysis systematically examines each emotional design principles' impact on both achievement emotions and mathematics achievement. The findings show good evidence that adopting emotional design principles which promote control and value of the achievement activity can lead to positive achievement emotions and better learning outcomes. One limitation of the current work is that games in the reviewed studies covered more than one emotional design principles, thus the reported effect could be the result of an interaction. To better distinguish individual principle's effects, future work needs to adopt a value-added approach that systematically integrates and evaluates one principle at a time (Mayer, 2019). The current results not only provide insights to developers, educators, and parents on how to design and identify digital mathematical games that better facilitate students' learning but also provide a direction for future researchers to investigate the effect of different types of achievement emotions on game-based learning.

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Part II
Mathematical Understanding

Chapter 9
The Number Line in the Elementary Classroom as a Vehicle for Mathematical Thinking

Maria Pericleous

Abstract This chapter builds on research related to the number line by exploring it as a vehicle for mathematical understanding in the naturalistic setting of Grade 2 and Grade 3 elementary classrooms. Starting from pupils' embodiment of the number line, and explicitly giving emphasis on the nature of the number line, an instructional sequence was designed and organized around number sequence and recognition, addition and subtraction in the domain from 1 through 1000. Using evidence from pupils' own productions, this chapter points to the role the number line plays in supporting pupils' sense making, the elaboration of informal strategies, leading to the development of more sophisticated ones.

Keywords Number line · Computation strategy · Mathematical understanding · Addition · Subtraction · Mental arithmetic

9.1 Introduction

In recent decades, important research in mathematics education has been devoted in identifying, understanding and fostering pupils' strategies and approaches for performing addition and subtraction (Beishuizen, 2010). In this context, through mathematical models, manipulatives and representations of mathematical ideas and concepts are described and assessed. Representations are significant tools in thinking, reasoning and communicating about mathematical ideas and operations (Kilpatrick et al., 2001). A representation that plays an important role in facilitating pupils to actively construct mathematical meaning, number sense, and understandings of number relationships is the number line (Frykholm, 2010).

The understanding of and ability to use the mathematical number line constitutes an essential facet of pupils' mathematics understanding. Consequently, the number line is an area of interest in cognitive psychology (Booth & Siegler, 2008; Siegler &

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Booth, 2005) and in cognitive neuroscience (Umiltà et al., 2010). There is a large body of literature that discusses the number line and its crucial role in teaching and learning elementary mathematics (Ball, 2003). Despite the widespread use of the number line, doubts about its appropriateness have been raised, with studies reporting difficulties and limitations in its use (Van den Heuvel-Panhuizen, 2008). Hence, there is a need to further our understanding regarding the conditions under which the number line in the elementary classroom is to be a vehicle for mathematical understanding.

This chapter contributes to a growing body of research in mathematics education related with the number line. The aim of this study was to explore the idea of the number line as a vehicle for mathematical understanding in the naturalistic setting of the elementary classroom. To be more descriptive, it sought to examine the use of the structured and empty number line (ENL) as a tool to support and develop Grade 2 and Grade 3 pupils' mathematical thinking specific to addition and subtraction in the number domain 0–1000.

In the first section of this chapter, the theoretical background pertaining to the number line is outlined. The argument is developed and synthesized by focusing on the structured and the ENL in relation to addition and subtraction. In the following parts of this chapter, this general information is made more concrete by concentrating on how this particular study utilised the structured and ENL to foster connections and relations between representations of mathematical ideas concerning quantities, the ways to decompose quantities and how to regroup them, illustrated by examples from the elementary classrooms.

9.2 The Number Line

According to Herbst (1997), a number line is formed by the consecutive translation of a specified segment U, as a unit from zero that can be partitioned in an infinite number of ways. He suggests that the number line is a metaphor of the number system; all kinds of numbers can be represented on the number line. The number line is also considered a geometrical model, involving a continuous interchange between a geometrical and an arithmetic representation (Gagatsis et al., 2003). That is, the numbers presented on the line correspond to vectors and to the set of the discrete points of the line. Simultaneously, points on the line are numbered so that the distance between two points depict the difference between the corresponding numbers. Furthermore, Teppo and Heuvel-Panbuizen (2014) argue that the number line is a figural device, representing particular mathematical abstractions that make it possible to think about and operate with different types of number.

The number line is currently an extensively used model in the teaching of mathematics (Lemonidis, 2016; Reys et al., 2012). The number line is used for estimation (Onslow et al., 2005), measuring time (Moone & Groot, 2005) and length, extending students' knowledge, giving access to possible solution strategies (Thompson, 2010). It allows the representation of numbers and the forming of geometric knowledge for the operation of arithmetic (Herbst, 1997; Kilpatrick et al., 2001). It can be used as a model for teaching percentages (Van de Heuvel-Panhuizen, 2003) and algebra for teaching linear equations (Dickinson & Eade, 2004). It is also employed for the development of the concept of fractions (Sidney et al., 2019).

Within the literature, two major types of number lines can be identified; the structured number line and the ENL (Diezmann et al., 2010; Teppo & van de Heuvel-Panhuizen, 2014). The filled or structured number line is characterized by equidistant points or tick marks, representing whole numbers. Diezmann and Lowrie (2007), elaborate the potential cognitive benefits of the number line for understanding various aspects of mathematics. When pupils experience many variations linked to a mathematical concept and are exposed to the concept through a variety of representations, then abstraction and generalisation, both of which constitute essential aspects of conceptual development is promoted. For instance, the number line can show the continuity of rational numbers. Furthermore, a fraction can be represented on a circular or rectangular area model, an array, and on a number line. The number line can also be considered as a tool for representational transfer, the goal of which is knowing how to use a common representation and deriving the solution procedure on a novel task from this representation (Novick, 1990). For example, the knowledge of sequencing whole numbers on the number line may be transferred when requested to sequence decimal numbers on another number line.

The ENL is a blank line presented without numbers or markers. It is a horizontal line acting as a visual representation for recording and sharing students' thinking strategies during mental computation. Gravemeijer (1999) explicates the didactical and psychological advantages for using the ENL in mathematics education. Initially, he outlines the need for a linear representation of numbers. While models such as blocks reflect situations dealing with quantities, and thus representing the numerosity aspect of number, in situations involving distance or measurement, the number line, which is considered as a linear representation of number, seems more appropriate. Furthermore, he outlines the flexibility the ENL provides in being adapted to fit students' thinking and informal solution strategies (van de Heuvel-Panhuizen, 2008). That is, the ENL reflects students' intuitive mental strategies, from counting-on or counting down to compensation and partitioning strategies. Furthermore, the natural and transparent character of the ENL, may stimulate a mental representation of numbers and number operations, and thus can be exploited for the representation and solution of non-standard context of word problems. The ENL relieves the working memory, as preliminary results can be put down relatively fast (Selter, 1998, p. 6).

The ENL may function as a way of scaffolding by fostering the development of more sophisticated strategies. To be more precise, as students record their thinking strategies, the number line shows which parts of the operation have been completed and which parts remain, the level of thinking as well as any possible errors. The students are cognitively involved in the actions undertaken on the number line. Making students' thinking visible provides opportunities to encourage the development of more efficient and sophisticated strategies. Building on this, it may also stimulate classroom discussion and expression and communication of mental strategies (Bobis, 2007).

In reviewing the research literature that focuses on the strategies adopted by pupils to perform mental arithmetic for numbers up to 1000, a wide range of strategies have been documented and classified. There is a broad consensus in mathematics education that three main strategies for mental addition and subtraction of are identified when using the ENL; (i) splitting where the numbers are divided by multiples of ten and units and processed separately when operations are carried out, (ii) stringing or compensation strategy which refers to keeping the first number intact while splitting the second number into tens and ones, which are then added or subtracted separately from the first number, and (iii) bridging, where the second number is split to facilitate a bridge to the nearest decade and the balance of the number is added or subtracted from the previous set (Beishuizen, 1993, 2010; Beishuizen et al., 1997). Variations of these strategies are also identified in the literature (Hartnett, 2007; Van den Heuvel-Panhuizen, 2008).

A consideration of the aforementioned, points to the crucial role the number line plays in mathematics education. However, whilst generally effective, research findings often raise doubts about the usefulness of the number line as a didactical model. Studies have investigated students' performance on number line tasks of various grade levels either at a single point in time (Skoumpourdi, 2010); over a small period of time (Hartnett, 2007) or through a longitudinal study (Diezman & Lowrie, 2007). These studies point both to instances where the number line functioned as an auxiliary means, as well as instances where students encountered difficulties in utilizing the number line. Common errors on number line items point to difficulties with distance, position, counting or misreading the diagram (Diezmann et al., 2010). Errors in problems originating from the dual nature of the number line may be persistent over time (Pelczer et al., 2011). Thus, neglecting one of the main features of the number line (direction, origin and unit measure) may lead to misconceptions.

Adding to the above, Lemonidis and Gkolfos (2020), argue that some of the difficulties students encounter when using the number line may be interpreted as epistemological obstacles related with; the separation between the numbers and the magnitude or the separation between the numbers and the straight line; the negative numbers and the orientation on the number line in the positive or negative direction; the density of rational numbers and the extra unit intervals needed to place them on the number line and irrational numbers. While they outline that more research is needed in clarifying the nature of these difficulties, they stress that these difficulties should be explicitly addressed by the teaching procedure.

Some of the difficulties students may face in effectively utilizing the number line may be attributed to the level of difficulty of number line problems. Other factors can be associated with the way the number line in presented in the mathematics textbooks and other curriculum resources and the way the teachers use it (Gray & Doritou, 2008; Murphy, 2011). These studies point towards a coherent treatment of the number line throughout the years of mathematics education and presentation of the number line in the school official documentation by focusing on the simultaneous presence of the geometric and the arithmetic conceptualization of number on the number line.

Keeping in mind both the affordances as well as the constraints of the number line, it is argued that the number line should be introduced in early grade instruction. However, it is acknowledged that it is superficial to simply recommend the use of the number line for the students' mathematical development or include it in curriculum materials and other recourses (Van den Heuvel-Panhuizen, 2008). If emphasis is given on the nature of the number line and its use as a representation of sophisticated ideas, then a conceptual way of teaching and learning should be encouraged, contributing to addressing students' difficulties. Various learning strands, didactical models and approaches have been proposed in the literature for the use of the number line in the learning and teaching process of mathematics (Selter, 1998; Thompson, 2010).

9.3 Methodological Considerations

The study was undertaken in a public primary school in Cyprus. The participants of this study were 19 Grade 2 students (11 boys and 8 girls) and 18 Grade 3 students (10 boys and 8 girls). The pupils were of a wide range of abilities and represented a broad spectrum of socioeconomic backgrounds.

An instructional sequence, described in full shortly, was designed to support the use of the number line as a vehicle for making visible mathematical understanding. Informed by a socio-constructivist approach to teaching and learning (Gravemeijer, 2020), pupils' learning processes were viewed from both the individual perspective and the social perspective. To be more elaborative, this instructional sequence gives the pupils a guided opportunity to invent the intended mathematics themselves and build up the targeted mathematics by modelling their own informal mathematical activity (Gravemeijer, 2004, 2020; Van den Heuvel-Panhuizen, 2003). Such a model is the number line, fostering the transition from a model of pupils' informal solution strategies to a model for mathematical reasoning (Freudenthal, 1973).

The instructional sequence was carried out as part of the ordinary mathematics classroom with the teacher as the researcher and author of this chapter and during the regular class time. A variety of data were gathered; audio recordings from the instructional sequence, the field notes of the researcher, and pupils' written work (worksheets, Concept Cartoons, and mathematical journal). The analysis of collected data started simultaneously with the data collection process, in order to identify pupils' thinking, strategies and procedures and their development, as well as further organise the instructional sequence. The ongoing analysis being conducted while the study was in progress led to a focus on several issues and events, which were then placed in a broader theoretical context by conducting a retrospective analysis (Gravemeijer, 2004).

9.4 The Instructional Sequence

The goal of the instructional sequence was to support and develop Grade 2 and Grade 3 pupils' sense making and calculation strategies specific to addition and subtraction in the number domain 0–1000. Considering the previously discussed elaboration, the instructional sequence aimed at creating opportunities for the pupils to build connections or relations between representations of mathematical ideas, to have the freedom to come up with their own notations, find their own ways to decompose quantities and regroup them and express and discuss their ideas. In this process, the number line would be utilized as a means for support, fostering the transition from a model of pupils' informal solution strategies to a model for mathematical reasoning. However, the interpretation of the number line is not self-evident. Thus, the instructional sequence was also focused on providing pupils with the opportunity to appreciate the number line as a rich model that can have different manifestations, by giving them as much initiative as possible, and simultaneously reducing the leading role of the teacher. In doing so, the pupils had the opportunity to understand the nature of the structured and ENL before acting on them and modelling their mental computation strategies.

The instructional sequence started with a story where the classroom's puppet converted its name into a number. The pupils followed the puppet's step. By assigning each letter of the alphabet a number value that is equal to its place in the alphabet, the pupils took each letter in their name, converted it to a number and added up the numbers. In the context of the story, the classroom's puppet invited the pupils to play a board game involving these numbers on a number line. However, what was missing from the game box was the number line. The pupils were to construct a physical number line. Even though they did not invent the tool for themselves, they were involved in the invention process. The pupils were encouraged to construct a definition regarding the structured number line and develop key understandings underpinning the conventions considered to interpret, create, and use structured number lines. After constructing the structured number line, the pupils plotted everyone's name number, engaging in number sequencing and comparing activities. The game cards created opportunities to support pupils' understanding of math processes, by inviting them to act on the structured number line, use their own strategies, and became familiar with different strategies, by explicitly discussing and reflecting on these strategies. The flexibility in the ways of recording results and the flexibility in the jumps pupils make to solve problems was stressed.

The instructional sequence was followed by Concept Cartoons the aim of which was to explicitly explore the dual nature of the number line (Pelczer et al., 2011). The Concept Cartoons involved problems and possible answers which included pupils' misconceptions. Comparing and contrasting ideas facilitates learning and interpretation of knowledge, revealing and eliminating misconceptions (Dabell et al., 2008; Naylor & Keogh, 2013). Similar Concept Cartoons were utilised when the ENL was introduced in the classrooms.

When the pupils reached the stage of becoming familiar with a linear representation of number and understanding and effectively using counting skills, the ENL was introduced in the classroom. In the sequel to the story, the classroom puppet discovered a different number line, and wandered whether this ENL could be utilised as a model of scaffolding and communicating the solution procedures the pupils use. The pupils were encouraged to construct a definition regarding the ENL. Emphasis was put on discriminating between the structured number line and the ENL. By drawing upon the pupils' own informal strategies, Concept Cartoons provided the opportunity to gain insight into pupils' calculation strategies as well as providing pupils with the opportunity to experience and develop a range of calculation strategies and discuss the flexibility of the ENL (Sexton et al., 2009). Speech bubbles proposing various solution procedures encouraged pupils to identify the name of the character that best matched their personal strategy choice for calculating the result, and providing reasons for choosing the specific strategy. The calculation strategies were introduced as possibilities, in an atmosphere of invention.

Three main calculation strategies were explored in both classrooms: splitting, stringing and bridging. When a new strategy for computation was introduced, the discussion was built on examples produced by the pupils. The pupils were involved in situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions. Furthermore, the pupils were encouraged but not forced to use the number line, methods or strategies that were being introduced and discussed in the classroom. The pupils would 'try' the strategy introduced in the classroom, but use the methods they felt comfortable with when solving problems. Thus, the pupils had the freedom to choose the materials and models (hundred chart, arithmetic blocks, dot-fields, money model) that supported their calculation. Through continuous discussion, strategies and procedures were presented, explained and contrasted. Pupils reflected upon the quality of their solution strategies and whether they could make them more manageable and efficient. Even though, no formal labelling was given to the computation strategies that were explored in the classrooms, the pupils were able to create general categories that supported them in communicating with each other and the teacher. At the end, a repertoire of strategies and methods was produced by each classroom in regards both to addition and subtraction.

Throughout the instructional sequence, the pupils also engaged in mathematical diary writing (Burns, 2004). It allowed pupils that are uncomfortable in oral situations to express understanding in a less public form (Yang, 2005). Including pupils in meaningful communication, mathematical diary writing also encouraged pupils to reflect on their own computation strategies (Selter, 1998).

9.5 Results and Discussion

In the following paragraphs, snapshots from the enacted instructional sequence reflect a small proportion of evidence illustrating the way the number line was utilized by Grade 2 and Grade 3 pupils, by concentrating on the development of pupils' computation strategies throughout the year. These snapshots followed the introduction of the ENL in the classroom. It should be emphasized that providing pupils with ongoing opportunities for revisiting, reviewing, engaging and mastering was not a straightforward process.

Initially, the pupils were presented with a context problem introducing the addition of two-digit numbers and three-digit numbers (64 + 30 and 264 + 30 for Grade 2 and 3 pupils accordingly), where the second number is multiple of 10. When the pupils were asked to share their mathematical thinking used for this calculation, they referred to counting-based strategies to calculate, calculating by utilising models.

While no pupil perceived the calculation as difficult, the reliance on counting materials (arithmetic blocks) was evident (seven Grade 2 and seven Grade 3 pupils). Grade 2 pupils' computation strategy was to split 64 into tens and ones and processed separately. These pupils modelled their thinking using the structured number line (four pupils) and the hundred chart (two pupils). When the pupils modelled their thinking on the structured number line, it was observed that three pupils utilised the structured number line for counting. No Grade 2 pupil referred to the ENL, even though it had been introduced in the classroom. The pupils' hesitation in utilising the ENL was related with its recent introduction in the classroom, revealing the need for more opportunities to build up familiarity with the ENL as a supportive model to carry out calculations. Grade 3 pupils referred to both splitting and compensation when working the calculation either mentally (seven pupils) or with the support of the ENL (five pupils).

Pupils were encouraged to record on paper the strategies chosen, Grade 2 pupils made drawings (e.g., iconic representation of arithmetic blocks, money model). This may be explained by the fact that the pupils had not been introduced to written symbolic representations. Three Grade 3 pupils described in writing their underlying pathways that led to how they determined the answer. Four pupils did not know what to do. The need for all pupils to be provided with more opportunities to draw upon number sense, and develop both written and mental computation was evident.

Successively, the instructional sequence was followed with a context problem introducing the addition of two and three-digit numbers without bridging (45 + 32 and 256 + 143 for Grade 2 and 3 pupils accordingly). Various methods of computation were discussed by the pupils. Two Grade 2 pupils chose arithmetic blocks, revealing that pupils would gradually rely less on manipulatives to solve a calculation task. In the same way, three Grade 3 pupils stated that as the numbers are bigger, the arithmetic blocks would assist them in solving correctly the calculation task. Nevertheless, modelling the problem with arithmetic blocks highlighted partitioning and regrouping of numbers. Two Grade 2 pupils and three Grade 3 pupils referred to the standard algorithm. Even though the algorithm had not been introduced in the classroom, it was mentioned previously by the pupils, as knowledge they had acquired either outside school (Grade 2 pupils), or during the previous school year (Grade 3 pupils). It was explicitly discussed, but without expecting pupils to use it as a way of working. The other pupils' strategies were divided between splitting and stringing (using either the ENL or a written computation), with four Grade 2 and

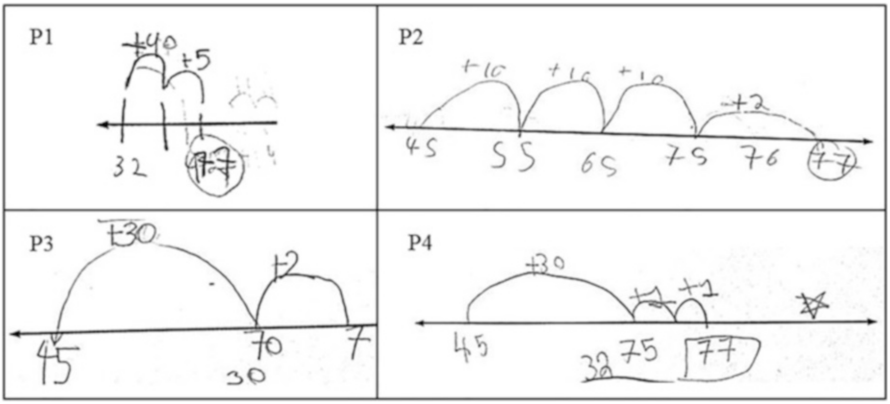

Fig. 9.1 Grade 2 pupils' informal strategies for computing an addition calculation mentally

3 pupils commenting that they did not need a written externalisation because they worked the calculation mentally. These pupils' mental computation was previously supported by the ENL and may suggest the success of the ENL as a mental model.

Figures 9.1 and 9.2 demonstrate Grade 2 and 3 pupils recording their mental computation on the ENL. The written work on the ENL was generated almost simultaneously with the pupils' thinking process revealing the solution procedures and at the same time disclosing the cognitive evidence included in this process. They constitute an indication of how the ENL is adapted by the pupils to fit their thinking and the pupils' increased confidence in their ability to use numbers flexibly.

Grade 2 pupils' strategy was compensation (see Fig. 9.1). The main procedures for implementing compensation, was that of separation from left to right (P2–4) or from right to left (P1). The structure of the calculation sequence involved either calculating in groups of tens and ones (P1 and P3) or a combination of groups of tens or ones and whole tens or ones (P2 and P4).

Accordingly, Fig. 9.2 demonstrates Grade 3 pupils modelling their mathematical thinking on the ENL, with some also providing a written description of their work. The splitting (P2 and P4) and compensation (P1 and P3) strategy used by the pupils led to a correct result. The pupils argued that their recordings on the ENL assisted them in providing an informal written method. This was quite unexpected considering that written work on the ENL has only a secondary function, and that the ENL does not easily lead to an informal written method. It shows that introducing the ENL, may also encourage pupils to invent and develop informal symbolic representations.

The instructional sequence was followed by an addition problem involving bridging (46 + 37 and 248 + 136 for Grade 2 and 3 pupils accordingly). The pupils were again encouraged to generate strategies based on their intuitive understanding of the numbers and actions needed. Initially, it should be explicated that no pupil relied on physical material, suggesting a shift from lower order strategies such as counting to more sophisticated strategies. Concerning the two-digit addition

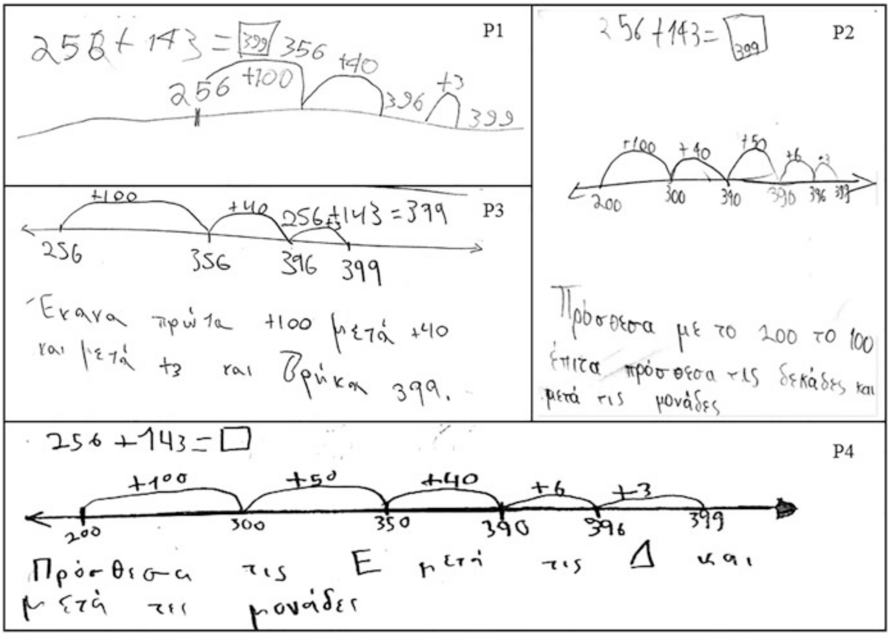

Fig. 9.2 Grade 3 pupils' informal strategies for computing an addition calculation mentally

problem, while not all methods led to a correct result, the classroom's documented work included the ENL as a mental model, language and written symbols.

To be more elaborative, in proposing computation strategies and procedures to calculate the result, six Grade 2 pupils commented that they could solve the calculation mentally by picturing the ENL in their heads. However, they expressed their preference for flexible and mental steps beyond the number line. The discussion led to the informal written computation illustrated in Fig. 9.3. In this occasion, the number line was used with other strategies to solve the problem.

The strategies pupils adopted to perform mental addition was splitting (see Fig. 9.3) and compensation (see Fig. 9.4). Even though this computation entails a calculation demand, no pupil referred to bridging, despite the fact that they engaged in bridging situations with numbers up to 20. While a bridging through ten approach would seem more appropriate, it did not offer a foundation for strategic choice. The pupils' decision about how to calculate was related with their familiarization with the alternative strategies. Closer examination of the pupils' methods indicated that compensation and splitting were combined with counting the remaining units (instead of bridging).

Further discussing the proposed strategies, a pupil commented *'I think there is another way to do it. 46 need 4 to become 50. We can take the 4 from 7 and then add the remaining 3. And then add 30.'* The pupils were encouraged to discuss, get familiar with and reflect on bridging through ten as a strategy for mental

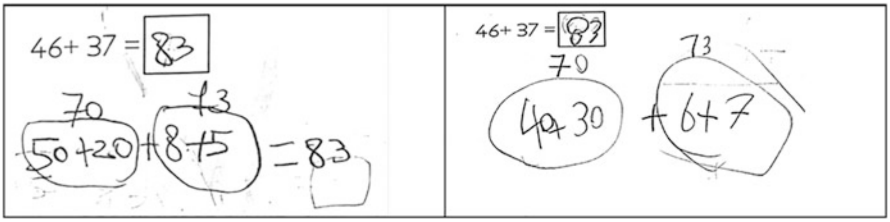

Fig. 9.3 Grade 2 pupils' informal written computation

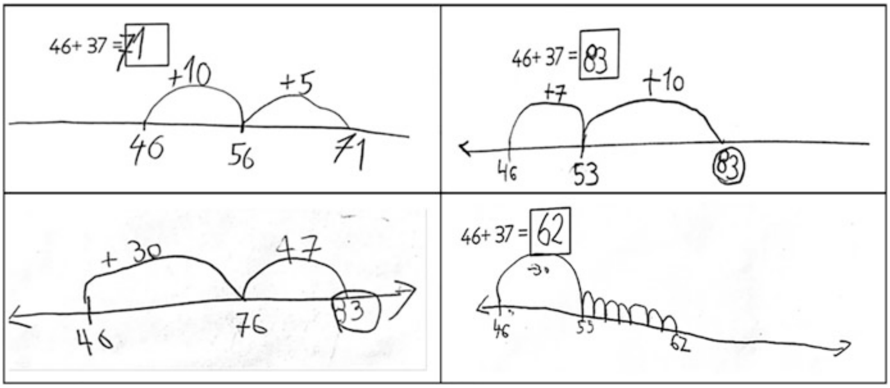

Fig. 9.4 Grade 2 pupils' informal written computation

computation. Figure 9.5 demonstrates the ENL supporting pupils' understanding of bridging. In this occasion, the pupils were also encouraged to provide a written computation that had been proposed in the classroom by their classmate. It is noticed that in the third example, the pupil used splitting stating that *'I understand what to do, but I feel more comfortable with this strategy (splitting)'*.

Concerning Grade 3 pupils approaching the addition problem involving bridging, the following extract from the classroom discussion shows pupils sharing the way they worked to calculate 248 + 136. Pupils 1–3 had modelled their calculation on the ENL.

P1: I started from 136. Then I added 40. 176. 176 + 200 = 376. And then I added 8. 376 plus 8 equals 384.

P2: I did it like P1, but I started from the biggest number. 248. First I added 100, then 30 and then 6. 378 plus 6 is 384.

P3: I am thinking 248 is 200 plus 40 plus 8 and 136 is 100 plus 30 plus 6. I add the hundreds, then the tens with the tens and the ones with the ones. 200 plus 100 is 300. 40 plus 30 is 70 and 8 plus 6 is 14. 300 plus 70 plus 14 is 384.

P4: In my head I worked it differently. 248 + 2 = 250. I took 2 from 6. Then I did 250 plus 130 equals 380. And the 4 that are left . . . 384.

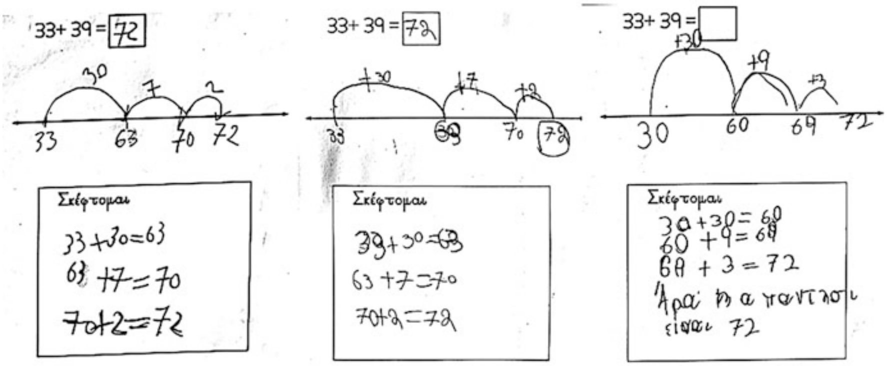

Fig. 9.5 Grade 2 pupils becoming familiar with different calculation strategies

Grade 3 pupils' calculation strategies involved splitting (P3), stringing (P1 and P2) and bridging (P4). The pupils' explanations constitute an indication of the numbers being treated as mathematical entities. This discussion also shows how pupils would engage in a discussion where computation strategies are shared and contrasted. P2 understood that he utilised the same strategy as P1, but commented that instead of starting the process from right to left, his process progressed from left to right. Similarly, P3 and P4 shared different strategies.

As a final example, the following extracts illustrate Grade 2 and 3 pupils sharing, towards the end of the instructional sequence, how they calculated a subtraction mentally.

Grade 2, 84-57

P5: I started from 57. I am thinking 57 + 3 = 60. We need 24 to reach 84. Then I added the 3, 24 + 3 = 27.
P6: It is like the number line is in my head. 84–50 = 34, then I subtracted the 7. 27.
P7: I worked in out in my head. I started from 84 as well, but I subtracted fist the ones, then the tens.

Grade 3, 584-257

P8: I started with the hundreds. 500–200 = 300, 84–57 = 27, 300 + 27 = 327.
P9: I worked differently. 584 minus the hundreds, then minus the tens and at the end I subtracted the ones. 327. It is easier.
P10: I also started from 584, but I subtracted fist the ones, then the tens and then the hundreds.
P11: I worked in out in my head with addition. I started from 257.Pupils 8–10 recorded their mathematical thinking as written computation (vertical equations), commenting that they did not need to use the ENL as a tool to support their mental computation. It is acknowledged that when the calculation involves bigger

numbers the need to record the successive stages of the calculation arises. The pupils' mental computation strategies involved splitting (P8), bridging 10 to subtract with varying procedures (P6, P7, P9, P10) and subtraction as addition using bridging (P5 and P11). The pupils show a high degree of flexibility in their explaining. Compared to previous illustrations of pupils' work, it can be argued that the pupils' mathematical thinking became more elegant and sophisticated, involving fewer steps. The above extracts, indicate pupils having a grounded understanding of number and place value and are able to utilize this understanding to successfully complete simple tasks mentally. It can be concluded that the pupils developed computational fluency and deeper sense of number. An additional remark constitutes that fact that the pupils were eventually able to share and discuss their calculation strategies verbally without having to write them down.

It is important to note that Grade 2 and 3 pupils who were able to perform mental computation strategies flexibly, would only use the ENL to document their strategy so as to ensure they reached the correct result, to demonstrate their method to their classmates or as a way to assist them in moving into more appropriate and efficient methods. It should also be noted that there were also instances were pupils from both year groups that struggled to model their mathematical thinking on the ENL. Nevertheless, if students struggle on number line problems, it does not automatically mean that they struggle in understanding the mathematical operation involved. This may be translated into the fact that they do not yet have the specific knowledge and skills necessary for translating the numerical expression into a number line representation (Ernest, 1985). Indeed, these pupils did not struggle in understanding the mathematical operation involved. The errors identified in translating the numerical expression into a number line representation, were related with the fact that they did not understand the flexibility of the ENL, and instead treated it as a structured number line. Despite this, no difficulties were observed when the pupils were asked to explore and discuss an addition or subtraction modelled on the ENL. The pupils were able to understand the conventions used in interpreting the diagram, and thus, reading and identifying the strategy being modelled on the ENL. This is an indication of the number line being a powerful tool in enhancing communication in the classroom (Bobis, 2007).

9.6 Conclusion

In this chapter, I have described an instructional sequence that attempted to provide pupils with the opportunity to appreciate the number line as a rich model that can have different manifestations. Findings from this study show the number line offering pupils the opportunity to develop, express and share their thinking. Evidently, the number line representation functioned as a vehicle for mathematical understanding. It allowed pupils, through the gradual development of both cognitive and metacognitive strategies, to construct and develop their own strategies for accurate and flexible mathematical thinking. Simultaneously, the ENL supported pupils in making sense of numbers and operations, enhancing mathematical reasoning and communication. Associating actions with the number line and communicating mathematical meaning may contribute towards a comprehensive picture of its conceptual structure and complete development of understanding of the number system.

A contribution of this research is to the existing scholarship on whether pupils should be given the opportunity to discover strategies based on their own knowledge and skills (Threlfall, 2000). Additionally, through a classroom discourse encouraging invention, the number line acted as a bridge between mental and written computation. This chapter contributes to the concerns raised by the mathematics education community, regarding the ENL not easily leading to informal written methods. It illustrates that pupils may build on their strategies through the support of the ENL.

For the enactment of such an instructional sequence, the significance of the classroom culture must be stressed (Gravemeijer, 2004). As the pupils' understanding of the number line and development of strategies and procedures is influenced by social and sociomathematical norms negotiated and established in the classroom, the classroom environment encouraged communication, exploration, discussion and reasoning (Kilpatrick et al., 2001; Selter, 1998; Threlfall, 2000). In this respect, the proactive role of the teacher is highlighted in promoting a non-threatening classroom culture that encourages even reluctant or less confident pupils to create, formulate, express and extent their mathematical understanding.

Three paths for further investigation are emerging. Treating the number line in a coherent way by focusing on the simultaneous presence of the geometric and the arithmetic conceptualization of number on the number line is not a straightforward process. The structured and ENL, constitute fundamentally different models that may function in different ways in the learning environment. The number line may foster the transition from a model of pupils' informal solution strategies to a model for higher levels of mathematical understanding. New manifestations of the model also encompass previous manifestations of the model. The shift in applying the model on a progressively higher level is not always successful. The relationship between knowledge of the structured and ENL and their compatibility seems like a valuable avenue for further exploration. This is imperative as learning the number line is considered critical both for current and future success in mathematics (Booth & Siegler, 2006). Secondly, it is acknowledged that the instructional sequence was designed considering natural numbers up to 1000. Thus, the notion of the number line evolving from a unit that could be repeated and partitioned has not been fully explored. The number line is considered a metaphor of the number system (Herbst, 1997). Thus, more research is necessary to further investigate how such a local instruction theory can be realised regarding other number sets.

Finally, the snapshots presented in this chapter, provide an indication of the repertoire of strategies being developed by the pupils. The pupils extended their awareness of possibilities. Regarding strategy selection, differences emerge between the year groups. Grade 2 pupils used mostly the splitting strategy, whereas Grade

3 pupils used most frequently the compensation strategy. It is acknowledged that it is not only important to determine pupils' strategies, but also to identify how these strategies are interrelated, and which might need further instruction. In order to gain a more detailed insight in the development strategies in relation to pupils' mathematical thinking conceptual understanding, future research should further investigate the constituents that influence the flexibility in strategy use.

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Chapter 10
Longitudinal Approaches to Investigating Arithmetic Concepts Across the Elementary and Middle School Years

Katherine M. Robinson and Denée M. Buchko

Abstract Understanding of arithmetic concepts in the elementary and middle school years is not only essential for developing current mathematical skills and knowledge but also as a foundation for later mathematical skills. Researchers are increasingly interested in arithmetic concepts and valuable knowledge has been gained. However, much of the research has taken a cross-sectional approach and studied and compared different groups of children of different ages or grades. An alternative but less frequently used approach is a longitudinal design in which the same children are studied across several years or grades. In this chapter we discuss the pros and cons of both approaches and focus on some recent longitudinal findings on the development of conceptual knowledge of arithmetic. Research and practical implications for educators and parents are discussed and outlined for promoting the development of children's understanding of arithmetic concepts.

Keywords Arithmetic · Conceptual knowledge · Longitudinal research · Inversion · Associativity · Equivalence

10.1 Introduction

Across the elementary and middle school years, children's understanding of the arithmetic operations of addition, subtraction, multiplication, division, the relations between operations, and the understanding of the equal sign are essential to the development of their current arithmetical knowledge and skills as well as later mathematical knowledge and skills (National Mathematics Advisory Panel, 2008). Researchers have recognized the importance of this understanding and have

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increasingly turned their attention to investigating the development of children's conceptual knowledge of arithmetic. The majority of these research efforts have relied on a cross-sectional research design which compares children of different ages or grades to derive conclusions about development. In this chapter, key findings from these cross-sectional studies are discussed as well as the pros and cons of the cross-sectional design. The pros and cons of a less frequent design, the longitudinal design which follows the same children across development or grades, are then examined and followed by an overview of recent longitudinal findings on the children's development of conceptual understanding of arithmetic. Finally, the implications of these longitudinal findings for researchers, educators, and parents are discussed.

When children start elementary school, one of the key mathematical tasks is to learn basic addition and subtraction (Bisanz & LeFevre, 1990). As they move to middle school years, this extends to simple multiplication and division. Learning basic arithmetic facts is essential to free cognitive resources for when children need to deal with more complex or advanced mathematical tasks introduced in the later elementary and middle school years (Siegler, 1996). Although the emphasis in the elementary school years starts with a focus on learning basic arithmetic facts or factual knowledge, children eventually need to deal with more complex arithmetic problems such as 58×326 or algebra problems such as $5x + 3 = 7x - 1$. At that point, two other forms of arithmetic knowledge will be required: procedural knowledge and conceptual knowledge.

Procedural knowledge refers to the procedures, steps, or problem-solving strategies that are used by children to solve a problem other than the direct retrieval of the answer from memory (Gilmore et al., 2018). Before children are able to encode arithmetic facts into memory, they are taught procedures for answering basic arithmetic questions such as $2 + 3$. Procedures that children might use to solve such a problem include counting on their fingers, using concrete objects to represent 2 and 3 and then counting up the total number of blocks, starting at 2 and then counting up by 3, using previous knowledge that $2 + 2 = 4$ and then adding one more, and so on. Children's procedural knowledge therefore starts developing early.

Over time, children start building their repertoire of problem-solving procedures and start developing conceptual knowledge of arithmetic. Conceptual knowledge of arithmetic is the general understanding of the concepts or principles needed to understand mathematics (Bisanz & LeFevre, 1990) and, more specifically, to understand the concepts and principles needed to implement specific problem-solving procedures (Crooks & Alibali, 2014), thereby demonstrating the strong connections between procedural and conceptual knowledge (Schneider et al., 2011). For example, if a child has conceptual knowledge of commutativity, i.e., that numbers in an addition or multiplication problem can be solved in any order, then that child can use this knowledge when problem solving. On a problem such as $5 + 428$, instead of starting with 5 and adding 428 more, children can use what is known as the min-counting strategy and instead start with 428 and add 5 more which is a faster and less error prone problem-solving procedure and can then lead to better factual knowledge of arithmetic as well (Canobi, 2009).

10.2 The Importance of Conceptual Knowledge of Arithmetic

Conceptual knowledge of arithmetic, unlike factual and procedural knowledge, is somewhat ephemeral and not as easily taught. For example, children can often learn their times tables through repetitive, rote rehearsal (factual knowledge), or be taught how to use a decomposition strategy (e.g., for a problem such as 4 + 5, recall the memorized fact of 4 + 4 and then add 1) (procedural knowledge). In contrast, teaching students why on a problem such as 4 + 5–5 there is no need to add or subtract the 5 s because addition and subtraction are inverse operations can be a conceptually a more difficult task to teach and learn.

Similarly, equivalency is also conceptually difficult for students. On a problem such as 4 + 5 + 6 = 4 +?, children often interpret the equal sign as "do something" and proceed to add 4 + 5 + 6 + the second 4 on the right side of the equation instead of making both sides equal. For those with an understanding of the equals sign, they would instantly recognize the need to make both sides equal. Unfortunately, the conceptual understanding of the equal sign is not obvious to all children and takes time and or explicit instruction to be developed (McNeil et al., 2017).

But why does conceptual knowledge of arithmetic matter so much if children have both factual knowledge and procedural knowledge of arithmetic to rely on in most mathematical situations? As can be seen by the example above of an inversion problem (4 + 5–5 or 4 × 5 ÷ 5), knowing that you do not need to add or subtract the 5 s (or multiple or divide by the 5 s) has the benefit of freeing up cognitive resources. As can be seen by the example above of an equivalence problem, knowing that you need to make both sides equal will be essential when dealing with algebra problems. Indeed, researchers have proposed that the understanding of basic arithmetic concepts is critical for the development of more complex mathematical skills, above and beyond factual and procedural knowledge (Kieran, 1981; Nunes et al., 2008; Rittle-Johnson, 2017). Further, reports have highlighted that children's understanding of both additive concepts (those involving either addition &/or subtraction) but particularly multiplicative concepts (those involving either multiplication &/or division) is concerningly weak and therefore does not prepare children adequately for future mathematical success (National Governors Association Center for Best Practices and Council of Chief School Officers, 2010; National Mathematics Advisory Panel, 2008). This is a concern as basic arithmetic skills are a strong predictor of long-term mental and physical health, employment opportunities, and even the likelihood of incarceration (Every Child a Chance Trust, 2008). The focus of this chapter will be on inversion, associativity, and equivalence as there is an extensive body of research on these three.

10.3 How to Measure Conceptual Knowledge of Arithmetic

Conceptual knowledge is hard to define, hard to teach, and so it is probably not surprising that it is also challenging to measure. This chapter will not go into detail about all these challenges but there are excellent discussions available on the topic (e.g., Prather & Alibali, 2009). For the purpose of this chapter, the focus will be on what can be considered the most common method of assessment of conceptual knowledge which is via problem solving. This is an indirect method for assessing conceptual knowledge as it uses procedural knowledge to gain insights into children's understanding of arithmetic. For example, as previously mentioned, on an inversion problem such as 8 × 32 ÷ 32, multiplication and division are inverse operations so children who apply this understanding during problem solving will often verbally report that "the answer is 8 as the 32s cancel each other out leaving just the 8." Children are inferred to understand inversion when they state that the answer is 8, often have a very fast problem solution times as they did not perform any calculations, and/or verbally report that they did not perform any calculations (Bisanz & LeFevre, 1990; Robinson et al., 2006).

Similarly, it is also possible to assess children's understanding of the associative relation between the operations of addition and subtraction or between multiplication and division which means that the operations within the problem can be solved in any order (Eaves et al., 2021). For example, on an associativity problem such as 8 × 32 ÷ 16, multiplication and division are associative operations so children who apply this understanding during problem solving will often verbally report that they first solved "32 ÷ 16 is 2 and then 2 times 8 makes 16 so the answer is 16." This is a much easier and faster approach to problem solving than using a left-to-right procedure of multiplying 8 × 32 to get 256 and then dividing 256 by 16. Children are inferred as understanding associativity when they state that the answer is 16, often have fast solution times, and verbally report that they started by dealing with the second and third numbers in the problem and then multiplied that number by 8 (Robinson et al., 2006).

Finally, on equivalence problems, children's understanding of the equal sign can be assessed. If children understand that the equal sign is used to indicate that two sides of an equation are equivalent then on the problem used above of 4 × 5 × 6 = 4 ×?, they will understand that the missing number has to result in the same number on both sides of the equation. If children understand this, there are two ways in which children can accomplish this task via problem solving. First, some children will report that they "cancelled out the 4s on each side of the equation and then figured out that 5 times 6 is 30". Second, other children will report they "multiplied 4 × 5 × 6 and got 120 and then figured out what number multiplied by 4 would also equal 120." In both of these problem-solving procedures, children are applying their understanding of equivalence (Hornburg et al., 2018; Robinson et al., 2018).

There are a number of pros and cons for using procedural knowledge to assess conceptual knowledge. The first positive of this assessment method is that it does not require children to verbalize their understanding of inversion, associativity, equivalence, and so forth as this verbalization may be too be challenging, particularly for children (Prather & Alibali, 2009). The second positive is that accuracy, solution times, and/or verbal reports of problem-solving procedure can all be used to infer and support the conclusion that a child has demonstrated conceptual knowledge. This means that if a child reports the correct answer on a problem such as 3 + 578–572, does so quickly, and reports subtracting the 578–572 first and then adding the 6 to the 3, there is corroborating evidence that the child did use a conceptually-based strategy to solve the problem (Robinson et al., 2006). The third positive is that children rarely encounter these three-term inversion, associativity, and equivalence problems in or outside the classroom and therefore these problems are considered to be novel (Robinson, 2017). This is important as it means that children have not had the opportunity to be taught a specific problem-solving procedure (e.g., when you see a problem with two numbers that are both added and subtracted then the answer will be the first number) and must discover and implement the conceptually-based strategy themselves if they have the needed conceptual understanding. This latter point leads to the biggest con of using problem solving to assess conceptual knowledge, which is that even though children might understand inversion, associativity, or equivalence, that does not mean that they will use that knowledge when problem solving. For example, some students may feel that skipping the calculations in an inversion or associativity problem is a form of cheating as they have been taught to always solve problems from left-to-right and to not skip any steps (Robinson & Dubé, 2012). Therefore, studies that use problem solving to infer conceptual knowledge may underestimate conceptual knowledge. Nevertheless, because of the advantages of this approach, many researchers opt to use problem solving to assess conceptual knowledge.

10.4 The Development of Conceptual Knowledge of Arithmetic: Part I

What is currently known about children's understanding of inversion, associativity, and equivalence? This is of particular interest in Grades 4 and beyond when it is possible to assess children's understanding of both additive and multiplicative forms of all three (for a discussion of the understanding of additive concepts in the early elementary years see Robinson, 2019). Several researchers have been studying children's understanding of inversion, associativity, and equivalence and a number of key findings have emerged.

First, we know that children (and adults) have a much stronger understanding of additive versions of inversion, associativity, and equivalence than multiplicative ones (Robinson & Ninowski, 2003; Robinson et al., 2006). For example, Robinson et al. (2018) in a study of Grade 5 to 7 students found that students were more likely to use their conceptual understanding of arithmetic when solving additive versus multiplicative inversion (40 vs. 18%), associativity (20 vs. 8%), and equivalence (56 vs. 44%) problems. This finding has been replicated in other studies of additive and multiplicative versions of inversion and associativity in Grades 6 to 8 as well (Robinson et al., 2006) and suggests that the concepts of inversion, associativity, and equivalence are not equally understood and applied across operation.

Second, we know that understanding one concept does not mean that a child will understand other concepts (Canobi et al., 2003). For example, Robinson et al.'s (2017) study of Grade 4 to 6 students' understanding of additive concepts found that even though 41% of children understood inversion, only 14% and 15% understood associativity and equivalence, respectively. A similar finding with Grade 5 to 7 students also showed more understanding of inversion than associativity and equivalence (Robinson et al., 2018). These findings suggest that conceptual knowledge of arithmetic develops in a piecemeal fashion rather than as a whole.

Third, conflicting findings have been found as to whether children's conceptual understanding of arithmetic increases across grade. A number of studies have examined grade differences on inversion, associativity, and equivalence. For inversion, a meta-analysis by Gilmore and Papadatou-Pastou (2009) suggests a lack of grade-related differences in children's understanding of additive inversion, a finding also replicated by some studies (e.g., Robinson et al., 2017, 2018) but not by others (e.g., Wong et al., 2021). Although there is less research on multiplicative inversion, results are also mixed (e.g., Dubé & Robinson, 2018; Robinson et al., 2018). For associativity, results are also mixed with some studies finding grade differences in additive (Robinson et al., 2017) and multiplicative (Dubé & Robinson, 2018) associativity and others finding no grade differences for both additive (Robinson et al., 2018) and multiplicative associativity (Robinson et al., 2018). For equivalence, similar mixed findings have been found for additive equivalence with some studies finding grade differences (Knuth et al., 2005) and others not (Knuth et al., 2016). In the one study we are aware of that examined grade differences in multiplicative equivalence, no grade differences were found (Robinson et al., 2018). Given the contradictory findings, it is currently impossible to make any strong conclusions about how conceptual knowledge of arithmetic may or may not increase as a function of children's education.

A number of factors may explain this lack of clarity, but one notable commonality exists amongst all of the studies discussed so far and that is that they all used a cross-sectional study design. This means that in all the studies above, not a single study actually followed the same children across several grades to investigate how their conceptual understanding developed. Instead, by using cross-sectional designs, researchers relied on different groups of children, in different grades, to draw conclusions on how conceptual knowledge of arithmetic developed. As discussed in the next section, although there are many good reasons for why most studies use a cross-sectional design, there are limitations that need to be considered.

10.5 Study Designs for Assessing Conceptual Knowledge of Arithmetic

10.5.1 The Cross-Sectional Design

The use of a cross-sectional design not only predominates in the examination of the development of conceptual knowledge of arithmetic but in research on development in general. The cross-sectional design allows researchers to examine age-related developmental changes by collecting data from participants of different age groups, and comparing their average performance/score on measures of interest (Leary, 2017). This design is popular for a number of key reasons: age-related differences can be studied at one point in time, making it both time and cost efficient, and also minimizing loss of data due to withdrawal of participants over time, since all data is captured at one point in time (Spector, 2019).

This design is not without flaw, however. When separated by age, each of the participant groups in a cross-sectional design come from their own cohort experiences. So, while we can compare the average outcomes of one age group to another age group, and efforts should be made to choose appropriate and comparable samples, there still may be differences between individuals that impact the outcome of the study (Leary, 2017). For example, Robinson et al. (2002) found that older adult participants were more likely to use retrieval when solving basic division problems than younger adult participants. This age difference was most likely due to changing mathematics curricula differences in the use of arithmetic drills and practice rather than to developmental differences.

Since cross-sectional studies take, more or less, a snapshot of a phenomenon of interest, they are limited in the ability to identify causal relationships and make definitive conclusions, which can cast doubt on the external validity of the findings. Since the studies are done at a single time point, there is no room to identify developmental changes as they appear, and no way to know if they develop in a continuous and gradual pattern, in stages with periods of stagnation, or in some other way. The cross-sectional research design has its benefits, especially to establish worthy research directions, but another research design better serves to identify developmental patterns.

10.5.2 The Longitudinal Design

The phrase "let's do a longitudinal study" has long been a daunting thought for many researchers. However, the drawbacks of longitudinal research are abated when considering the potential benefits longitudinal research offer to our understanding of the intricacies of human development.

Longitudinal research typically involves the testing, and retesting of one group of participants over a longer period of time, allowing the researcher to re-examine the same individuals at different points of development (Leary, 2017). The results in longitudinal research can be examined both by comparing averages of the data taken at different times, effectively comparing different age groups, as well as by comparing individual scores to their own previous scores (Crone & Elzinga, 2015). This last method allows for the identification and examination of individual differences in development, which can occur in both the how and the when in any developmental phenomenon of interest. Examining individual differences is especially relevant in measures that examine learning, as this is one area of development where individual differences are well documented (Harring & Hancock, 2012).

Longitudinal research also better captures not just the instances but the journey of development as well. The when and the how of a developmental process provide a depth of understanding to the area of interest that the "if" provided by cross-sectional design simply does not cover (Farrington, 1991). For example, children might gradually acquire conceptual knowledge of arithmetic between ages 9 and 12 or acquire it very rapidly during a short window of time at 10 years of age. Supplying learning resources would not be useful if given after the conceptual knowledge has already developed.

So, while cross-sectional research can narrow down an age range in which this occurs, only longitudinal research can provide the depth of information that informs educational decision making. Other patterns of development, like those which develop and then recede, may be mistakenly identified without longitudinal design. Thus, the longitudinal design is an especially good candidate when researchers need more precise information about the development of concepts or phenomena if they are able to watch them unravel over time.

Despite the advantages, collecting data over longer periods of time comes with complications. Much lengthier times for data collection and publication, higher costs for conducting research over a prolonged period of time, and higher participation attrition rates (e.g., due to boredom, moving locations) are all significant disadvantages for researchers.

Another potential limitation of the longitudinal design is that the findings may also be influenced by period effects (Farrington, 1991). For example, if a study on children's understanding of arithmetic concepts was conducted from 2015–2025, a 'period' of time within those years would need to be considered and accounted for as factors such as remote learning as well as the myriad stressors from the covid-19 pandemic could certainly result in changes in the measures of interest. Overall, despite the potential drawbacks, a well-designed longitudinal study can result in rich, useful data in which causal and developmental conclusions are well supported.

10.6 Development of Conceptual Knowledge: Part II

Having argued for the merits of using a longitudinal design to better understand developmental phenomena, we now turn to the issue of what has been learned about the development of conceptual knowledge of arithmetic from longitudinal designs.

Unfortunately, very few such studies exist, presumably due to the concerns raised above. We are aware of only two longitudinal studies investigating our three concepts of interest and both of these focused on additive equivalence problems. First, in their 5-year longitudinal study of children from Grades 2 to 6, Hornburg et al. (2022) found that even by Grade 6 approximately 14% of their participants had only minimal understanding of additive equivalence and almost 18% still had no understanding of equivalence. Second, Alibali et al. (2007) found that understanding of additive equivalence became increasingly sophisticated from Grades 6 to 8 suggesting that this period might be critical for developing a strong understanding of equivalence. The use of conceptually-based strategies to solve additive equivalence problems increased by 25% which is a large improvement over time but is still concerning as the improvement was only from approximately 15% to 40%, indicating that even by Grade 8 many children are still struggling with understanding the meaning of the equal sign. Thus, these two longitudinal studies suggest that when following children across grade, it does appear as though there are grade-related changes, at least for the understanding of equivalence on additive equivalence problems, but that these improvements are slow and gradual. These results together support the earlier point that longitudinal studies can yield important information not only on whether there are grade-related improvements in conceptual understanding but also whether there are individual differences in the development (or not) of conceptual understanding of arithmetic. We next turn to the first, to our knowledge, longitudinal study of both additive and multiplicative inversion, associativity, and equivalence.

10.7 A Longitudinal Study of Additive and Multiplicative Inversion, Associativity, and Equivalence

In a recent study, we examined the use of conceptually-based strategies when solving additive and multiplicative inversion, associativity, and equivalence problems in a group of children we followed from Grades 4 to 6. We started with 49 students (mean age = 9 years and 6 months) in the Fall of Grade 4 and we had them problem solve every six months until the Spring of Grade 6. As anticipated, we lost some participants due to attrition (a few dropped out, moved schools and could not be contacted, or moved out of the province or country) and so our final sample was 42 students. In each of the six sessions, students were asked to solve two sets of problems: additive inversion, associativity, and equivalence problems and multiplicative inversion, associativity, and equivalence.

We first sought to determine how the use of conceptually-based strategies would change from the beginning of the study to the end of the study. When examining conceptual understanding in the Fall of Grade 4 and the Spring of Grade 6, students' conceptual understanding of arithmetic did improve overall, F (1, 41) = 45.31, $p < .001$, $\eta_p^2 = .525$. Interestingly, and contrary to previous findings, conceptual understanding increased globally. Conceptually-based strategy use increased by approximately 20% on each of the additive and multiplicative versions of all three concepts. In contrast, cross-sectional design studies have not found a global increase but instead found that different groups of students in different grades have differing understanding of the three concepts which can vary widely from each other depending on whether additive or multiplicative problems are used. Figure 10.1

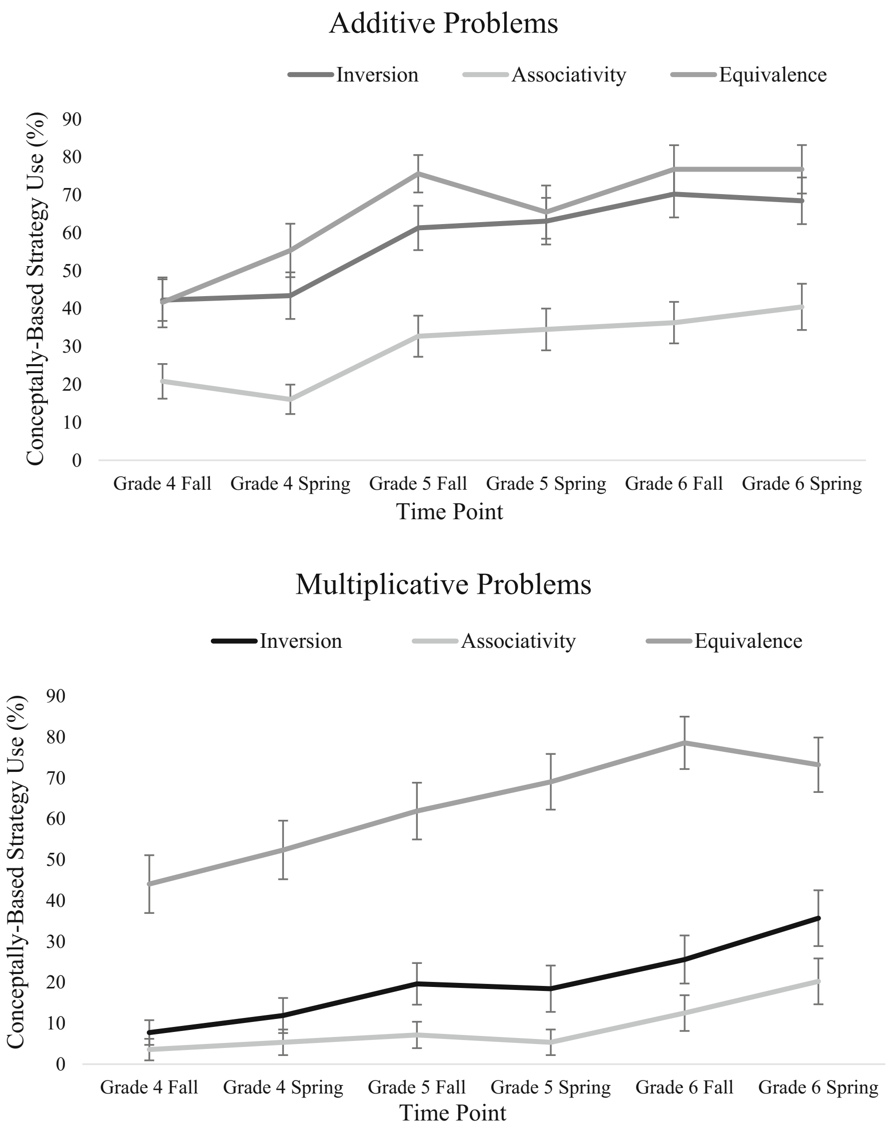

Fig. 10.1 Percentage of conceptually-based strategy use in the fall of Grade 4 and in the Spring of Grade 6 on additive (*top panel*) and multiplicative (*bottom panel*) problem

shows the use of conceptually-based strategies on additive (top panel) and multiplicative (bottom panel) inversion, associativity, and equivalence problems from all six data collection points between the Fall of Grade 4 and the Spring of Grade 6. It is clear from Fig. 10.1 that previous cross-sectional findings were replicated. First, some concepts are easier to grasp than others with equivalence being better understood than associativity. Second, additive versions tend to be better understood than multiplicative versions of the concepts as is the case with inversion and associativity.

Overall, Fig. 10.1 suggests a global increase for all of the students in conceptual understanding across the longitudinal study but the longitudinal design also permitted the examination of the development of conceptual understanding for individuals. Individual differences in the development of conceptual knowledge of arithmetic are, as discussed earlier, best done by following the same children over a period of time. Doing so reveals how strikingly different children are in how they develop their conceptual understanding of arithmetic. In Fig. 10.2 we illustrate these differences by showing how four children in study: Siham, Penelope, Angus, and Kareem (all pseudonyms) developed (or did not develop) their understanding of additive and multiplicative inversion (the two top panels), of additive and multiplicative associativity (the two middle panels), and of additive and multiplicative equivalence (the bottom two panels). These four children were selected as they exemplified the wide individual differences across the entire group of children.

Figure 10.2 demonstrates the difficulties in trying to characterize how children develop their conceptual knowledge of arithmetic as it depends not only on the individual child but also upon the specific concept they are being assessed on and what operations are involved. For example, Penelope's use of conceptually-based strategies on additive inversion problems remained unstable across the study and remained at zero on multiplicative inversion problems across the study. In contrast, Angus and Kareem's use of conceptually-based strategies steadily increased across the longitudinal study regardless of whether the problems were inversion, associativity, or equivalence and their performance on the multiplicative versions of the problems was not as negatively impacted as it was for the other children. However, Angus moved more quickly than Kareem to predominantly using conceptually-based strategies use on additive inversion and associativity problems. Siham used some conceptually-based strategies on additive inversion and associativity problems as the study progressed but never used conceptually-based strategies on any of the multiplicative problems and struggled with both additive and multiplicative equivalence problems. As a whole, these findings highlight the importance for future research to investigate why these marked individual differences exist and the continued need for more longitudinal studies to both determine the predictors of conceptual understanding (Ching & Nunes, 2017) and to identify if interventions could effectively address some of the struggles that individual children are having with conceptual understanding of arithmetic.

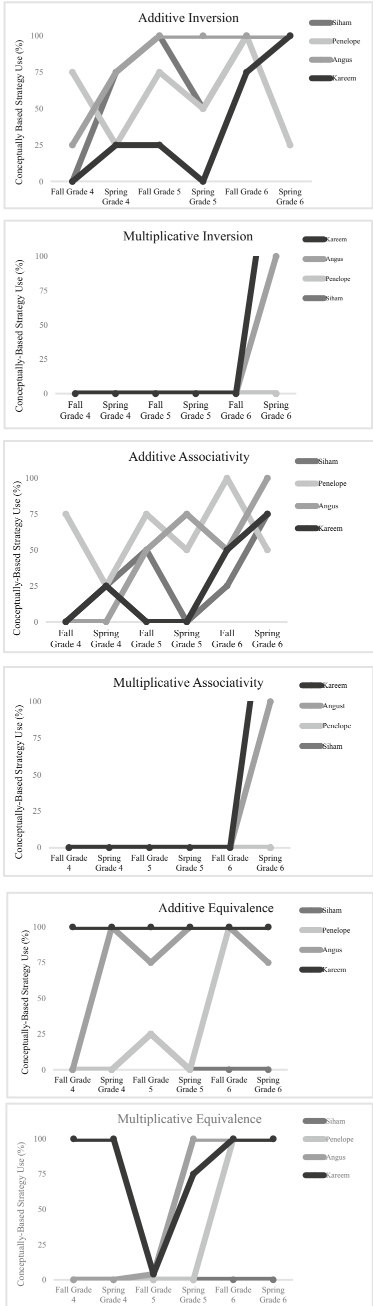

Fig. 10.2 Percentage of conceptually-based strategy use across the longitudinal study for inversion (*top two panels*), associativity (*middle two panels*), and equivalence (*bottom two panels*) problems.

10.8 Future Directions and Practical Implications

Researchers interested in the development of arithmetic concepts have long recognized the need for longitudinal studies. However, that need has yet to be adequately addressed as few longitudinal studies have been conducted, presumably for practical reasons due to their high cost in time and effort. The results of the few longitudinal studies do indicate that the cost may be worthwhile as the longitudinal findings have revealed the intricacy and complexity involved in how children's understanding of arithmetic concepts develop. Because conceptual understanding of arithmetic is considered so critical for later mathematics, and researchers are increasingly interested in both examining individual differences in that understanding as well as the predictors of individual differences, longitudinal designs are essential for moving this area of inquiry forward.

For educators and parents, the findings suggest that children may profit from direct instruction and discussion about the properties of addition, subtraction, multiplication, and division, the relations between the operations, as well as the meaning of the equal sign. Past research has indicated that teachers often underestimate children' understanding of the equal sign (Asquith et al., 2007; Sherman, 2007). The same may be true of parents and teachers when it comes to estimating children's understanding of inversion and associativity as well as estimating how many more difficulties children have with multiplicative versions compared to additive versions of the concepts. Therefore, periodically assessing children's understanding via problem-solving, which was the focus of this chapter, as well as other methods to assess understanding discussed elsewhere (e.g., Crooks & Alibali, 2014; Prather & Alibali, 2009; Wong et al., 2021) may be helpful as a check to see how children's understanding is (or is not) developing, and can provide an opportunity for the necessary direct instruction.

Instruction and practice with varying forms of equivalence, inversion, and associativity problems (both additive and multiplicative) may be helpful in increasing conceptual understanding (e.g., Alibali et al., 2009). One potential approach which continues this chapter's focus on problem-solving to measure conceptual understanding, is to use several different equivalence problems (e.g., 4 + 2 + 3 = 4 +? or 4 + 2 = 3 +? or 4 + 2 =? + 3, Hornburg et al., 2018), inversion problems (e.g., 4 + 22–22 -? or 22 + 4–22, Robinson & Ninowski, 2003), or associativity problems (4 + 25–22 or 25 + 4–22, Eaves et al., 2020). Researchers have primarily focused on increasing children's understanding of equivalence using different instructional methods and interventions (e.g., Fyfe et al., 2015; McNeil et al., 2017) but far less research has focused on inversion and associativity (but see Nunes et al., 2012) and none, to our knowledge, has looked at multiplicative versions of inversion, associativity, or equivalence. One promising way to determine if children have a solid conceptual understanding is to see if they can transfer that understanding to different types of problems assessing the same concept (Gaschler et al., 2013; Godau et al., 2014) or, as found in adults, even different concepts (Eaves et al., 2019).

Overall, the examination of the development of arithmetic concepts in the elementary and middle school years continues to yield exciting new information on the rich individual variability in children's understanding and the call to conduct longitudinal examinations is becoming increasingly more pressing and important. If we want children to have solid factual, procedural, and conceptual knowledge of arithmetic, longitudinal studies of conceptual arithmetic will fill a significant gap in current knowledge and provide the much-needed foundation to help prepare our elementary and middle school students not only for future success in more complex mathematical tasks but, even more importantly, success in life (Every Child a Chance, 2008).

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Chapter 11
Obstacles in the Development of the Understanding of Fractions

Florence Gabriel, Jo Van Hoof, David M. Gómez, and Wim Van Dooren

Abstract Fractions are fundamental in students' mathematical development. However, for many students, they are known to be a major stumbling block. In this chapter, we examine the obstacles elementary school children face when they learn fractions, through the lens of numerical cognition. We start by discussing the discrepancy between children's conceptual and procedural knowledge of fractions, and we review studies showing that the concept of fraction magnitude is particularly difficult to learn. This has wider implications as understanding fraction magnitude has been shown to be a strong predictor of achievement in algebra and overall mathematics achievement in later years. We then discuss the natural number bias (NNB), a well-characterised misconception linked to fraction learning whereby learners are inclined to apply natural number characteristics when reasoning about fractions without considering whether it is appropriate. The NNB is persistent, appearing early in the fraction learning process and lasting through secondary school and beyond. We conclude this chapter by describing interventions aimed at improving fraction learning and we provide suggestions on how to introduce the concept of magnitude more intentionally when teaching fractions.

Keywords Fractions · Natural number bias · Conceptual knowledge · Procedural knowledge

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11.1 Introduction

In the research literature, there is a general agreement that a good understanding of rational numbers is foundational for more advanced mathematics. For example, Siegler et al. (2012) showed that – controlling for natural number knowledge, reading achievement, IQ, working memory, family income and family education – fifth graders' fraction knowledge predicts algebra and overall mathematics scores in high school. Despite their importance, rational numbers are found to be a huge stumbling block in education for a large group of learners (Depaepe et al., 2015; Gabriel et al., 2013; Gómez et al., 2014; McMullen & Van Hoof, 2020; Siegler et al., 2012; Vamvakoussi et al., 2013; Vamvakoussi & Vosniadou, 2010; Van Hoof et al., 2017).

While rational numbers are harder to understand than natural numbers, it seems that within the category of rational numbers, fractions are the most difficult ones (e.g., DeWolf et al., 2015; Iuculano & Butterworth, 2011; Resnick et al., 2019; Wang & Siegler, 2013). Like natural numbers, decimal numbers are written in a place value-based notation system: each digit in its particular place is ten times smaller than the digit preceding it; the structure of the notation fits within a system of ones, tens, hundreds, etc., in the case of natural numbers, and tenths, hundredths, thousandths, etc., in the case of decimals. For fractions, however, this is not the case. They convey a multiplicative relationship between two natural numbers (the ratio between the numerator and the denominator).

Before the insight that rational number understanding was predictive of later mathematical achievement, cognitive (neuro)psychologists had long stressed the importance of a "number sense" for learning mathematics (Schneider et al., 2017). This number sense was assumed to be present from early childhood and it was conceived as the fast, accurate perception of small numerosities along with the ability to compare them and to conduct simple arithmetic operations (e.g., Barth et al., 2003). Using this early conception, the number sense was considered to be natural-number based and was found to predict later general mathematics achievement (e.g., De Smedt et al., 2009, Halberda et al., 2008). However, more recent research indicates that from early childhood onwards, people are also able to quickly and accurately perceive and compare ratios between numerosities (e.g., Denison & Xu, 2014; Matthews et al., 2016). The idea of a number sense has thus been extended to rational numbers (Clarke & Beck, 2021).

So why do learners struggle to understand rational numbers – particularly fractions – more than natural numbers? As argued by Vosniadou, Vamvakoussi, and Skopeliti (2008), in their first years of life, children's intuitions about natural numbers are much more often externalized and systematized in social interaction than any intuitions about rational numbers. Natural numbers are given specific labels and children regularly engage in activities like finger counting. The development of natural number knowledge is further supported and systematized by elementary mathematics instruction, while children's sense of ratios is not necessarily addressed. This means that, long before children are introduced to rational numbers at school, they have already constructed a rich, extended and persistent understanding of numbers grounded almost exclusively in knowledge of natural numbers (Gelman, 2000; Smith et al., 2005, Vamvakoussi & Vosniadou, 2010). Therefore, when rational numbers are then introduced in the curriculum, typically in the middle years of elementary school, several expectations about the features and behaviour of numbers are violated.

We have come to understand that the extension of children's number concept from natural to rational numbers is not a smooth, continuous process. Mathematically speaking, natural numbers are just a subset of the rational numbers, so the same principles apply, but developmentally (and educationally), this connection is often not made (Vosniadou et al., 2008).

This chapter addresses the various implications of having a good understanding of fractions. It brings together research from a range of perspectives, including numerical cognition, cognitive development, and mathematics education. First, fractions come with specific concepts and procedures, and a good conceptual and procedural understanding and a strong link between both types of knowledge is important. While the distinction between and the importance of both knowledge types is widely acknowledged in the literature, we bring together insights on this topic specifically for the topic of fractions, and we identify the questions that require further investigation. Second, recent research has shown that the prior knowledge that learners bring (i.e., experience with natural numbers) may interfere in the development of the understanding of rational numbers, and of fractions in particular. As such, this issue has been studied for many years now. The current chapter synthesizes the different ways in which prior knowledge may interfere (but also facilitate) the learning of fractions (i.e., in terms of understanding the size of fractions, operations with fractions, and their density). Third, the chapter addresses how instruction may be shaped in order to enhance fraction understanding, building on the ideas developed in the preceding sections. We specifically address research-based interventions aimed at enhancing the understanding of fractions as magnitudes (as opposed to the more typical idea of fractions as part of a whole), as well as interventions that address the natural-number based prior knowledge that learners bring to the classroom.

11.2 Conceptual and Procedural Knowledge of Fractions

As in many areas of mathematics, understanding fractions involves learning a constellation of elements. One widely used categorization of mathematical knowledge splits it into (a) knowledge of mathematical objects, definitions, and their relations, and (b) knowledge of mathematical procedures, algorithms, or heuristics that can be used to solve problems. The former knowledge type is named *conceptual knowledge*, sometimes described as "knowing that", whereas the latter type is called *procedural knowledge*, or "knowing how" (Byrnes & Wasik, 1991; Hiebert & Lefevre, 1986; Rittle-Johnson & Schneider, 2015). Many studies in mathematics education and cognitive psychology have investigated these knowledge types in diverse areas of mathematics both with children (e.g., Bempeni & Vamvakoussi, 2015; Byrnes & Wasik, 1991; Gabriel et al., 2013; Hallett et al., 2010, 2012; Hecht & Vagi, 2012; Li, 2014; Özpınar & Arslan, 2021; Rittle-Johnson et al., 2001) and adults (e.g., Engelbrecht et al., 2017; Forrester & Chinnappan, 2010; Van Steenbrugge et al., 2014).

In the domain of fractions, several examples of conceptual knowledge can be inferred from Kieren's (1976) interpretations of rational numbers. These include the understanding that fractions represent numerical relations between quantities based on their ratio, have multiple interpretations such as parts of a whole or points on the number line, are numbers with a given magnitude, and can be presented in infinitely different equivalent forms (e.g., $\frac{1}{2}, \frac{2}{4}, \frac{5}{10}, \frac{1234}{2468}$, etc.). In contrast, procedural knowledge of fractions typically includes tasks for which students know a "series of steps, or actions, done to accomplish a goal" (Rittle-Johnson & Schneider, 2015), such as deciding whether two fractions are equivalent by using cross-multiplication, adding and subtracting fractions, and judging whether a fraction is written in its simplest form. This distinction between these two knowledge types may explain many observations of school children's struggles with fractions. For instance, a child may be proficient in comparing fractions, but fail when asked to identify a fraction between $\frac{1}{3}$ and $\frac{2}{3}$ (Van Hoof et al., 2015a). This pattern of performance suggests that their success in comparing fractions stems from the use of memorized procedures, and not from a conceptual understanding of fractions' numerical magnitude.

A common topic in research has been how conceptual and procedural knowledge relate to one another, and whether the learning of one of them precedes — or should precede in terms of instructional design — the other one (e.g., Byrnes & Wasik, 1991; Castro et al., 2016; Rittle-Johnson et al., 2001). Conceptual and procedural knowledge are deeply related, and many studies have reported that these two types of knowledge are highly correlated (Bailey et al., 2015; Gabriel et al., 2013; Jordan et al., 2013). Moreover, some longitudinal studies have found that both types of knowledge influence one another (Rittle-Johnson & Koedinger, 2009; Rittle-Johnson et al., 2001). These findings have led to a wide agreement that conceptual and procedural knowledge reinforce one another and develop in tandem (Rittle-Johnson et al., 2001; Rittle-Johnson & Schneider, 2015), with researchers calling into question the advantage of teaching approaches that focus extensively on fraction concepts before moving onto procedures (Rittle-Johnson et al., 2015).

Despite all the advances in this topic, a number of relevant issues remain to be addressed by further research. A first issue regards whether and how conceptual and procedural knowledge can be independently and reliably measured. There is broad agreement about the difficulty of classifying specific fraction tasks as conceptual or procedural (Vamvakoussi et al., 2019; Rittle-Johnson & Schneider, 2015). Some fraction tasks are widely believed to be mostly conceptual (e.g., locating a fraction on a number line) and others mostly procedural (e.g., adding fractions with different denominators). Nonetheless, it is likely that different learners approach these tasks with conceptual or procedural strategies (Faulkenberry, 2013). Deciding which one of two fractions is numerically larger is a task believed to reflect children's conceptual knowledge, but it can be successfully solved by either conceptual or procedural means. For instance, a child can judge whether $\frac{7}{6}$ is larger or smaller than $\frac{5}{8}$ by reasoning in at least two different manners. On the one hand, this can be done by recognizing that $\frac{7}{6}$ is larger than one and $\frac{5}{8}$ is smaller than one. This solution taps into conceptual knowledge. On the other hand, the question can also be solved by cross-multiplying the fractions and noting that 7×8 is larger than 5×6. This solution involves procedural knowledge. Therefore, conceptual and procedural knowledge lie not on the task itself but on children's reasoning.

A second, related issue concerns individual differences. Children's conceptual and procedural knowledge may vary independently from one another, which opens the possibility that some children may be very strong in one type while being weak in the other (Hallett et al., 2010, 2012; Hecht & Vagi, 2012; Bempeni & Vamvakoussi, 2015). Hallett et al. (2010) studied 4th- and 5th-grade children, discovering groups of children with distinct patterns of relative strength in conceptual and procedural knowledge (see also Hallett et al., 2012; Hecht & Vagi, 2012). Bempeni and Vamvakoussi (2015) claimed that the imbalance between the degree of development of the two knowledge types can be very extreme, showing data from 9th-grade students who are conceptually very strong but procedurally very weak, and vice versa. Interestingly, these authors provided further evidence that children with strong or poor conceptual knowledge seem to also differ in their approach to mathematics learning. This approach can be *deep*, with an intention to understand, or *superficial*, with an intention to reproduce (Entwistle & McCune, 2004). According to Bempeni and Vamvakoussi (2015), strong conceptual knowledge appeared linked to a deep approach to learning, whereas a pattern of weak conceptual knowledge and strong procedural knowledge was associated with a superficial approach. That is to say, children with strong concepts focused on learning for understanding, whereas children with weak concepts and strong procedures focused on learning for obtaining good grades.

A third issue is the need for a deeper study of mathematical procedures, and of the relation between conceptual and procedural knowledge. While procedures are often defined as sequences of steps to be executed in an automatized manner, authors like Star (2005; see also Star & Stylianides, 2012) argue that procedures can be understood at different depths. A deep understanding of the fraction addition procedure is evident in a student who understands why the procedure starts by converting the two fractions to a common denominator. This understanding allows the student to decide to skip this initial step, for example, when the task calls for an approximate rather than an exact answer. Another case is that of a student who chooses flexibly among different strategies to judge which of two fractions is larger, such as comparing denominators when the fractions share the same numerator (e.g., $\frac{2}{7}$ vs. $\frac{5}{7}$) and attempting to compare against $\frac{1}{2}$ when the fractions share no common component (e.g., $\frac{2}{9}$ vs. $\frac{5}{7}$). From a different perspective, this issue acknowledges that procedures are also mathematical objects, and a learner can therefore acquire conceptual knowledge about them (such as why they work the way they do). This relation between conceptual and procedural knowledge can be used in the design of students' procedural practice, favouring students' noticing of number patterns or solution patterns (an example in the domain of arithmetic has been studied by McNeil et al., 2012).

In summary, a successful learning of fractions requires comprehending concepts and procedures. While some tasks seem naturally conceptual or procedural in essence, it is possible that students solve them in different ways, using their strength in one type of knowledge to hide a weakness in the other type. Therefore, assessing fraction knowledge requires using a diversity of tasks of both types.

11.3 How Natural Number Knowledge Both Facilitates and Hinders Fraction Learning

At the point where fractions are introduced in the classroom, learners have a lot of new and highly complex information to process and understand. The literature distinguishes several reasons explaining why learners struggle so much with the understanding of fractions – and rational numbers more generally (e.g., Moss, 2005; Vamvakoussi, 2015). For example, fractions have several different conceptual meanings that need to be understood. They can be seen as ratios, or as measurements, or as parts of a whole, etc. Moreover, fractions are just one type of representation of rational numbers, and the same rational number has infinitely many different representations: learners need to understand that even though they look very different; $\frac{1}{4}$, 0.25, 2/8, 25% all represent the same numerical magnitude. On top of this, the concept of a 'unit' needs to be reconceptualized (e.g., Moss, 2005; Vamvakoussi, 2015).

There is a large group of learners who struggle with fractions, thinking about them using the properties of natural numbers whether or not it is appropriate (e.g., Christou et al., 2020; Gómez & Dartnell, 2019; McMullen & Van Hoof, 2020; Vamvakoussi et al., 2011; Van Hoof et al., 2015a, b). This phenomenon is called the natural number bias (NNB; see for example Ni & Zhou, 2005), and has been the focus of a vast amount of research over the last couple of decades (for an overview, see Van Hoof et al., 2017). For example, learners who fall into the trap of the NNB tend to think that $\frac{3}{4}$ is smaller than $\frac{7}{12}$, since 3 and 4 are each smaller than 7 and 12, respectively. While there is still a debate on the origin of the NNB (Ni & Zhou, 2005; Van Dooren et al., 2016), a large body of research explains the NNB starting from the conceptual change theory, and more specifically the framework theory approach towards conceptual change (e.g., Vosniadou, 1994; Vosniadou et al., 2008).

As stated by Vosniadou (2013) "research on conceptual change investigates how concepts change with learning and development in different subject matter areas with a focus on explaining students' difficulties in learning the more advanced and counterintuitive concepts in these areas" (p. 1). The gist of the interpretation of the conceptual change theory applied to rational numbers is as follows: Both in children's daily life as in their first years of mathematics education, children encounter natural numbers much more frequently than fractions. Therefore, before fractions are introduced in the classroom, children have already created an initial concept of number, based on their experiences with natural numbers (Vosniadou, 2013). When fractions are then introduced in the classroom, the rules and characteristics of natural numbers are no longer always applicable, leading to misconceptions. Indeed, learners have been found to make systematic mistakes in those fraction tasks where reasoning purely in terms of natural numbers results in an incorrect solution (*incongruent tasks*; e.g., which is larger: $\frac{1}{3}$ or $\frac{1}{4}$? Although 4 is larger than 3, the correct answer is of course $\frac{1}{3}$) while much better performance is found in those fraction tasks where natural number reasoning leads to a correct solution (*congruent tasks*, e.g., which is larger: $\frac{4}{10}$ or $\frac{7}{10}$? 7 is larger than 4, just as $\frac{7}{10}$ is larger than $\frac{4}{10}$) (for an overview, see Van Hoof et al., 2017). For examples of congruent and incongruent items, see Table 11.1.

In this sense, the natural number bias can be seen both as a hindrance (systematic errors in incongruent tasks) as a facilitator (high accuracy levels in congruent tasks) when learners solve fraction tasks. However, it should be noted that previous research showed that learners do not detect a conflict between incorrect intuitive natural number-based reasoning and mathematically correct reasoning (Van Hoof et al., 2013). This indicates that the high accuracy levels on congruent tasks are the result of the misapplication of natural number-based reasoning instead of being the result of good conceptual knowledge of fractions.

The research field distinguishes three main aspects where misconceptions arise in fractions as a consequence of natural number prior knowledge interfering. The first aspect is fractions' numerical magnitude. A common misconception is that the numerical value of a fraction increases when its denominator, numerator or both increase; for example, $\frac{3}{7}$ is judged larger than $\frac{3}{5}$, because 7 is larger than 5 (e.g., Gómez & Dartnell, 2019; Stafylidou & Vosniadou, 2004). The second aspect is their dense structure: unlike natural numbers, which have a discrete nature (i.e., you can always point out the next natural number: after 5 comes 6, after 6 comes 7, etc), fractions have a dense structure (i.e., it is not possible to point out the "next"'rational number because there are always infinitely many numbers between two rational numbers). This difference leads to the misconception that there are no other numbers in between two "pseudo-consecutive" fractions such as $\frac{4}{9}$ and $\frac{5}{9}$ (e.g., McMullen & Van Hoof, 2020; Merenluoto & Lehtinen, 2004; Vamvakoussi et al., 2011). The

Table 11.1 Examples of congruent and incongruent items

Aspect	Congruent	Incongruent
Numerical magnitude	Which is the larger fraction? $\frac{2}{7}$ or $\frac{4}{7}$	Which is the larger fraction? $\frac{2}{7}$ or $\frac{2}{9}$
Structure	Write a number between $\frac{3}{8}$ and $\frac{6}{8}$.	Write a number between $\frac{3}{8}$ and $\frac{4}{8}$.
Effect of operations	Is the outcome of $5 \times \frac{3}{2}$ bigger or smaller than 5?	Is the outcome of $5 \times \frac{2}{5}$ bigger or smaller than 5?

third aspect is the effect of operations. Studies demonstrated learners' misconception that, just like with natural numbers, multiplication with a fraction should always lead to a larger outcome, and vice versa, division with a fraction should always lead to a smaller outcome (Christou et al., 2020; González-Forte et al., 2020). Moreover, the fact that learners struggle to see fractions and decimals as representations of the same numerical magnitude constitutes an additional difficulty that intersects with the three aforementioned aspects of the natural number bias. Previous research showed for example that while some learners already fully understand the dense structure of decimals, they still have a naïve idea of the structure of fractions or vice versa (e.g., Vamvakoussi & Vosniadou, 2010).

The NNB is especially found in the beginning stages of fraction instruction and a clear decline of the strength of the NNB is seen between the beginning years of rational number instruction and adulthood (see for example Siegler & Braithwaite, 2016; Van Hoof et al., 2015a, b, 2018). However, it should be noted that traces of this NNB are still present in secondary school students and even in pre-service teachers (Depaepe et al., 2015) and mathematics experts (Obersteiner et al., 2013). To illustrate this, results from Depaepe et al.'s (2015) study with 158 prospective elementary school teachers and 34 prospective lower mathematics secondary school teachers showed that only a bit more than half of the pre-service teachers could accurately answer that there are infinitely many numbers between 7.2 and 7.4. The most typical wrong answers were: "There is only one (7.3)", "There are 19 (7.21, 7.22, 7.23, . . .), and "There are 20 (7.20–7.40)".

11.4 Educational Interventions and Implications for Teaching

Fractions are complex to learn and to teach. In the early stages of fraction instruction, the focus is often narrow, with teachers and textbooks concentrating on the concept of fractions as parts of a whole. The emphasis on this aspect of fractions is quite common across many different countries and cultures, including Cyprus, Kuwait, USA and the UK among others (Alajmi, 2012; Fuchs et al., 2013; Küchemann, 2017; Pantziara & Philippou, 2012). Consequently, less time is allocated to teaching other conceptual aspects of fractions, leading to students having less opportunity to develop a sound understanding of perhaps the most important concept relating to fractions: magnitude. Emphasising the concept of magnitude in elementary education promotes students' understanding that fractions have different properties to natural numbers, helping counter the NNB.

Although there are few studies that have looked at pedagogical interventions to help learners overcome the NNB (e.g., Gómez & Dartnell, 2015; Van Hoof et al., 2021), they provide a number of implications and suggestions for how to introduce the concept of magnitude more intentionally when teaching fractions.

11.4.1 The Concrete-Representational-Abstract Sequence

The Concrete-Representational-Abstract (CRA) sequence is commonly used in intervention studies aimed at improving children's understanding of fraction magnitude. CRA has been shown to be an effective way to teach mathematics and is particularly suited to teaching fractions (Butler et al., 2003; Chang et al., 2017; Ennis & Losinski, 2019; Gabriel, 2016; Purwadi et al., 2019). Moreover, it is currently included in Singapore's mathematics curriculum on teaching fractions (Chang et al., 2017). The basic approach of the CRA sequence is to present children with different representations of the same problem – from more concrete to more abstract – to improve students' conceptual understanding and learning outcomes in mathematics (Witzel et al., 2003; Purwadi et al., 2019). The first stage of the CRA sequence is the concrete stage where students learn by using manipulatives to model and solve problems. The second stage introduces various pictorial or representational elements. The third and final stage brings in mathematical notation to help children think of fractions as abstract mathematical concepts (i.e., numbers with magnitudes; Morano et al., 2020).

Gabriel et al. (2012) used the CRA sequence to improve elementary students' understanding of fraction magnitude. In this intervention, students were asked to play adaptations of simple card games. For instance, they played a version of the game Memory where cards were laid down in a grid face down and players took turns flipping pairs of cards over. The aim of this game was to identify pairs of fractions representing the same magnitude. If a student identified a matching pair, they had to show it to the other students and make sure they all agreed. In case of disagreement between students, they could use wooden disks cut into different fractions going from halves to twelfths to compare both cards. These wooden disks were used as the concrete manipulative element, cards with multiple pictorial representations of fractions as the representational element and numerical fractions as the abstract element. After ten weeks, the intervention led to a 15 to 20% improvement in students' conceptual understanding of fraction magnitude, and they made fewer NNB-related mistakes when compared to the control group who followed their traditional lessons.

Other studies have investigated using CRA to teach students with learning disabilities. For example, Morano et al. (2020) worked with Grade 5 and 6 students with disabilities. Again, they used a CRA-based intervention to improve students' understanding of fraction magnitude. Fourteen intervention lessons covered the topics of unit fractions, fraction equivalence and the use of equivalent fractions to solve addition and subtraction of fractions with different denominators. There were two concrete lessons (including the use of plastic fraction blocks), two representational lessons for each topic (including number lines and rectangle representations), followed by two abstract lessons covering all three topics (including word problems with no visual representations). Students showed an improved understanding of fraction magnitudes after the intervention and made fewer mistakes linked to the

NNB when performing calculations on fractions. There were limitations to this study (e.g., small sample size and no control group), but it showed promising results.

11.4.2 Playful and Game-Based Interventions

Playful and game-based interventions have also shown great potential to improve students' understanding of fractions. A playful approach to teaching fractions increases children's enjoyment of fractions as well as their motivation and self-efficacy (Riconscente, 2013). And when children are motivated, they are more likely to enjoy learning and see themselves as capable learners. Play also actively encourages students to interact with each other in a way that is not possible with more traditional procedural instruction. This approach gives teachers more opportunities to listen to the children's thoughts, ideas and comments, and so better assess how their understanding develops. Using play also allows teachers to include more open-ended questions in their feedback (e.g., "So, what do you think?" "Is it always the right answer?" "What do you think would happen if. . .?"), thus creating an environment that encourages children to construct their own understanding (Gabriel, 2016).

Games are a powerful tool for increasing student engagement, and they have the added benefit of being easily implemented in digital environments (Gresalfi et al., 2018; Kiili et al., 2018). Digital games present several advantages when teaching mathematics. They can increase students' engagement and attention (Garris et al., 2002). They can also be used to provide immediate feedback (Vogel et al., 2006) and support active learning (Gee, 2003), which will eventually improve learning and achievement (Shin et al., 2012).

In a recent study, Kiili et al. (2018) developed a digital rational number game aimed at improving 4th grade students' conceptual understanding of fractions. The digital game included estimation tasks on number lines where students control a character called Semideus who tries to collect gold coins that were stolen from Zeus by a goblin. The player directs Semideus to the point on a number line where they believe the coins are located based on a fraction given in pictorial, fractional or decimal notation. For every incorrect estimation, Semideus is struck by lightning and the player loses "virtual energy". The game also includes magnitude comparison and ordering tasks where students are asked to arrange stones in ascending order according to the numerical magnitudes depicted on them. The levels are ordered in ascending levels of difficulty and students can track their performance on an analytics page. The game also generates hints to address some common misconceptions about fractions. The intervention group played five 30-minute sessions of the game while the control group attended their regular mathematics lessons. Their results showed that the conceptual understanding of the magnitude of fractions improved more in the intervention than in the control group. While there are limitations linked to the study design and the small sample size, the results are encouraging.

11.4.3 Improving Pre-service Teachers Pedagogical Content Knowledge

As previously discussed, learners' natural number based prior knowledge often interferes with their ability to reason about fractions. This poses serious challenges for teachers, particularly as teachers themselves may be prone to the NNB (Depaepe et al., 2015). Depaepe et al. (2018) developed a lesson series for pre-service elementary teachers aimed at improving not only their content knowledge regarding fractions (based, among others, on the CRA sequence), but also at improving their pedagogical content knowledge (i.e., teachers' knowledge of learners' misconceptions and of possible instructional strategies and representations to address them). As part of the lesson series, the NNB (and its origin in conceptual change theory) was explicitly introduced as a cause for children's difficulties with fractions. The lesson series used a wide range of mathematical representations and strategies to enhance preservice teachers' conceptual understanding. It also included a range of video vignettes showing real classroom situations in which learners exhibited incorrect reasoning. Pre-service teachers were asked to reflect on these videos to find possible ways to react to the classroom situation. The intervention successfully enhanced preservice teachers' content knowledge, while also having a positive impact (albeit limited) on their pedagogical content knowledge.

11.4.4 What Can Parents Do to Help Their Children Learn Fractions?

Parents can play an important role in their children's mathematical development and parental engagement has been shown to have a positive impact on children's achievement in mathematics (Desforges & Abouchaar, 2003; Muir, 2012). Day-to-day activities at home can provide rich contexts for children to learn and apply mathematical concepts (Winter et al., 2004). Parents can explain to their children that fractions are numbers in and of themselves, by emphasising the numerical magnitude of fractions, as some children treat the denominator and numerator as separate entities and never connect them into a unique quantity. One way of doing this is to encourage estimation and to compare fractions to key benchmarks (e.g., $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, etc.) (Clarke et al., 2008). Parents could also use Lego as a concrete hands-on activity that incorporates elements of play, as Lego bricks can be used to create a number line or to compare different fractions for example. Getting children to do simple mathematics when cooking can be another easy way to introduce fractions in daily activities. For instance, children can be encouraged to recognize fraction equivalence (e.g., $\frac{1}{2} = \frac{3}{6}$) by comparing the size of measuring spoons when baking. Another activity that can be useful to introduce fractions as abstract numbers would be to measure and compare quantities when scaling recipes up or down.

11.5 Conclusion

It is important for children to understand fractions as doing so predicts future achievement in mathematics (Booth & Newton, 2012; Siegler et al., 2012). Gains can be made in improving children's conceptual understanding of fraction magnitude and studies have shown this to be a crucial step in reducing errors related to the NNB. The challenge for teachers is to balance the need to teach children the procedural knowledge necessary for using fractions against making sure they are instilled with the conceptual knowledge needed to truly understand how, when and why to use fractions. Teachers also need the relevant pedagogical content knowledge to address the natural-number based prior knowledge that learners bring to the classroom and to use it productively to develop an adequate and deep understanding of fractions.

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Chapter 12
The Role of Groundedness and Attribute on Students' Partitioning of Quantity

Helena P. Osana, Emmanuelle Adrien, Anne Lafay, Katherine Foster, Kiera K. Vaccaro, Vera Wagner, and Rebecca MacCaul

Abstract Equal-sharing word problems involve sharing a quantity among a specified number of groups (e.g., four people share ten brownies equally). The nature of the specific object to be partitioned (e.g., pizzas, ribbons) may differentiate the strategies children use to solve such problems. The present chapter reports an investigation of how children's solutions are impacted by (a) "groundedness," namely whether the problem refers to the to-be-shared object as a concrete, real-world object, and (b) the attribute of the object described in the problem (i.e., length vs. area). Fourth graders ($N = 88$) were randomly assigned to three conditions that differed according to the problems they solved: (1) grounded-area: problems describing objects with area attributes (e.g., brownies), (b) grounded-length: problems describing objects with length attributes (e.g., ropes), and (c) abstract: problems that referred to the object using nonwords (e.g., "porams"). No condition differences in the quality of the students' partitioning strategies were found, but a measure of the students' mental representations of the to-be-shared objects revealed that the participants in the abstract condition imagined the nonwords as objects with physical characteristics, which could account for the comparable strategy performance across problems.

Keywords Equal sharing · Partitioning strategies · Problem solving · Problem features · Mental representations

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12.1 Introduction

In North America and elsewhere, word problems are commonplace in school mathematics at both elementary and secondary levels (Verschaffel et al., 2020). Educators and curriculum designers argue that when students solve word problems, they learn how to reason mathematically, develop conceptual understanding, and transfer their knowledge to novel situations (Hiebert et al., 1996; Liljedahl et al., 2016; Pape, 2003). This chapter focuses on the features of word problems that may influence the types of strategies children use in their solutions. For example, a problem about cookies, objects with which many children have real-world experience, may elicit prior knowledge that could support their strategies. On the other hand, a problem about litchi, a fruit that is relatively less familiar to children in North America, may result in lower performance because children's lack of knowledge and context may make it difficult for them to apply meaningful strategies.

In this chapter, we examine children's solution strategies to equal-sharing word problems. Equal-sharing problems are partitive division problems with fractional remainders (e.g., Four students want to share 13 cookies equally. How many cookies will each student get?). Empson and Levi (2011) argued that equal-sharing word problems are pedagogically effective because students have previous experience with the situations depicted; that is, children have experience sharing objects such as pancakes or brownies among friends and siblings. The authors' claim is that these familiar experiences activate prior knowledge that helps students grasp the problem structure and make sense of fundamental fractions concepts, including the quotient interpretation of fractions (see also Fazio & Siegler, 2011; Steffe & Olive, 2010).

We focus our examination of children's strategies as a function of the type of object that requires partitioning in equal-sharing problems. Specifically, the two factors we tested were (a) the extent to which the to-be-partitioned object is "grounded," or refers to concrete, real-world objects, and (b) the specific attribute of the object, namely whether the object is measured in area or length units. An additional objective of the research was to explore children's mental representations of the objects they are partitioning, which would provide additional insight into the role of groundedness and attribute in their strategies. A deeper understanding of solution strategies in the context of equal sharing has important implications for practitioners, who can modify instructional materials as needed depending on their students' learning needs, world knowledge, and background experiences.

12.2 The Role of Problem Characteristics in Word Problem Solving

Thevenot and Barrouillet (2015) presented three primary approaches to studying children's strategies when solving word problems. The first approach is the examination of problem characteristics, such as the mathematical structure of the problem and ways in which the problem situation is described in the text. The second focus is on individual factors, such as working memory and text comprehension abilities. The third approach seeks to uncover the instructional factors that can enhance children's performance on word problems in mathematics. Results of research emanating from the latter perspective can be used to support the cognitive and instructional theories upon which the instructional interventions are based. Beyond cognitive approaches are those that take an ethnomathematical view, which focuses on the intersection between culture and mathematics (e.g., D'Ambrosio, 2001; Katsap & Silverman, 2016).

We frame our research in the approaches outlined by Thevenot and Barrouillet (2015), with particular attention to the first, namely the effect of problem characteristics on children's problem-solving strategies. Much of the research adopting this perspective focuses on how problem features influence the construction of mental models that guide children's problem-solving efforts (e.g., De Corte & Verschaffel, 1987; García et al., 2006). In one such study, Coquin-Viennot and Moreau (2003) compared the strategies of third and fifth graders on problems such as, "For a prize-giving, the florist prepares for each of the *14* candidates 5 roses and 7 tulips. How many flowers does the florist use in total?" (p. 272) to a problem with the same structure, but reformulated using a so-called "structuring element," namely the word *bouquet*: "For a prize-giving, the florist prepares for each of the *14* candidates a *bouquet* made up of 5 roses and 7 tulips. How many flowers does the florist use in total?" (p. 272). The authors found that the children's strategies were different depending on the presence or absence of the structuring element. Specifically, the first problem was more often solved using the distributive strategy $(14 \times 5) + (14 \times 7)$, and the second using the factorizing strategy, $14 \times (5 + 7)$. The authors concluded that such reformulations can prompt children to construct ad hoc, intermediate representations that are based on real-world knowledge (in this case, of bouquets), which in turn influence the strategies they use in the problem-solving process (see also Thevenot & Oakhill, 2005).

12.3 External Representations in Problem Solving

The finding that certain problem features, such as the semantic structure of a word problem, can prompt children to construct representations based on their real-world knowledge and experience, is related to another body of research that is relevant to the current discussion. Researchers who study the affordances of external representations have argued that intentionally and explicitly "grounding" mathematical tasks in real-world contexts can influence performance (see Belenky & Schalk, 2014, for an overview). Germane to the present chapter, evidence shows that problem solving is enhanced when mathematical tasks, including word problems, make explicit references to concrete objects and events (e.g., De Bock et al., 2011; Goldstone & Sakamoto, 2003; Goldstone & Son, 2005; Koedinger & Nathan, 2004; Kotovsky et al., 1985). We borrow Koedinger et al.'s (2008) definition of "grounding" in our work: Word problems that are grounded incorporate references to concrete and specific real-world objects and everyday events. Scholars have suggested that references to real-world objects in word problems activate their prior knowledge; access to information stored in long-term memory serves to support and verify domain-relevant inferences, which, in turn, assists in generating appropriate solutions (e.g., Belenky & Schalk, 2014).

There is empirical support for the benefits of grounding on students' immediate learning and performance in mathematics. Koedinger and Nathan (2004) found that when high-school students solved simple algebra problems couched in verbal descriptions of real-world situations, their performance was higher than when they worked on analogous problems presented in algebraic symbols. These effects were replicated with undergraduate students (Koedinger et al., 2008). Problems couched in abstract contexts, such as those that are presented in symbolic notation, can present more of a challenge because students have fewer existing knowledge representations to draw on (Fyfe et al., 2014; Weyns et al., 2016).

In a series of studies by Glenberg et al. (2007), third and fourth graders who used realistic figurines (i.e., those that resembled the characters and objects described in the text) solved more word problems correctly, used a larger number of correct strategies, and used less irrelevant numerical information in their solution attempts than those who did not use figurines. Thus, Glenberg and colleagues found support for their "indexing hypothesis," which posits that the realistic figurines serve to ground the symbols (i.e., defined in their theory of embodied cognition as words and phrases in a problem text) to real-world situations.

Other research has shown that grounding problems in real-world contexts can have positive impacts on transfer as well. For instance, McNeil and Fyfe (2012) tested the effects of concreteness fading, an instructional approach where grounded (or "concrete") representations are gradually replaced with more abstract ones. The participants who received the concreteness fading instruction were more successful on learning and transfer tasks than those who had received instruction with abstract representations only. The authors claimed that the students in the fading condition were able to ground their learning of new concepts in real-world contexts with which they had previous experience, which eventually allowed them to make meaningful connections to more abstract representations, thereby promoting transfer.

12.4 Groundedness and Equal-Sharing

Our interest in children's strategies as a function of the type of objects that are partitioned in equal-sharing problems stems from a years-long professional development initiative in a large urban school board in Canada. As part of the professional development, we collected pilot data with fourth-grade children in classrooms and in individual interviews to assess and support their conceptual understanding of fractions (Foster & Osana, 2017; Osana et al., 2017). Our work with the students and their teachers focused on equal sharing as a context in which fractions concepts and their interpretations can be meaningfully explored in the classroom. Our informal observations revealed that some children struggled to accurately represent fractional quantities in their own depictions and to appropriately apply fractions concepts in their partitioning actions. Moreover, we observed that some objects were more difficult to partition than others. In one case, for example, a fourth grader struggled to evenly share the milk in a carton because the child did not know what a milk carton was.

Together, our observations led us to conduct a systematic study the following year (Foster, 2018) to test the degree of groundedness of the to-be-partitioned objects in equal-sharing problems on the quality of children's problem-solving strategies. With 114 fourth-grade students from nine classrooms, we manipulated the objects being shared to create three problem types: (a) grounded, (b) semi-grounded, and (c) abstract. The objects in the grounded problems referred to concrete, real-world objects that we assumed would activate their prior knowledge (e.g., brownies). Objects in the semi-grounded problems involved measures of length in standard units, such as a rope measured in centimetres. While ropes are real-world objects, we predicted that students generally have relatively less everyday experience sharing lengths of rope than sharing food. Finally, the objects being shared in the abstract problems were nonwords and therefore made no reference to actual objects in the real world (e.g., "porams"). We predicted that the degree of grounding would be positively associated with problem-solving performance, which was measured in terms of appropriateness of strategy use.

We found partial support for our prediction. Specifically, equal-sharing problems that involved real-world objects, such as cookies, were associated with more sophisticated strategy use (e.g., Empson et al., 2005) relative to problems that involved partitioning a piece of rope measured in centimetres, an action with which students had relatively less prior experience. What we had not predicted, however, was that the abstract objects did not hinder performance relative to the fully-grounded problems. Our ability to draw definitive conclusions, however, was compromised by the confounding of groundedness and unit type: The grounded objects in the problems involved arbitrary units of measure (e.g., the unit was a brownie), but the semi-grounded objects involved standard units (e.g., the unit was one centimeter). Thus, potential differences in interpreting the unit may have accounted for the observed strategy differences rather than groundedness itself (Mack, 2001).

In this chapter, we describe a third investigation of children's strategies, this time when we removed the unit type confound. This design modification allowed us to better compare grounded problems (those describing objects with area attributes, such as brownies, and those with length attributes, such as rope) to problems with abstract objects. By removing the confound of unit, the improved design also allowed for a more controlled comparison of the effects of object attribute (i.e., area vs. length), which has been shown to influence the strategies children use to solve tasks involving measurement and fraction concepts (Lehrer et al., 1998).

12.5 Role of Object Attribute in Partitioning Strategies

By definition, equal-sharing problems involve the partitioning of objects described in the text. Dividing wholes into parts is a fundamental concept underlying rational number (Empson, 1999; Mack, 2001; Pothier & Sawada, 1983; Streefland, 1997), and partitioning actions can look different depending on the attribute (i.e., length, area, volume) that defines the unit used to quantify the to-be-partitioned object (e.g., Hiebert & Tonnessen, 1978). Therefore, success in solving equal-sharing problems is in part dependent on children's understanding of fundamental measurement concepts, including those related to different types of units and those involving the partitioning of units.

In particular, children should be aware that area units must be used to measure quantities with area attributes and length units for objects with length attributes (Clements & Stephan, 2004; Lehrer, 2003). Furthermore, with respect to partitioning actions specifically, children must know that to measure a surface with an area attribute, for example, one must partition the surface into equal parts (or partitions) and then count them as units to assign a number to the quantity (Curry et al., 2006; Goldenberg & Clements, 2014; Lehrer et al., 1998). Theoretically, therefore, measurement concepts and children's knowledge of different measurement systems are centrally involved in the strategies they use to solve equal-sharing problems.

Evidence suggests that children's application of measurement concepts and their partitioning strategies vary across attributes (e.g., Curry et al., 2006; Hiebert & Tonnessen, 1978). First, Sisman and Aksu (2016) found that children possess numerous misconceptions when measuring objects with area, length, and volume attributes, and that their errors vary depending on the attribute in question. Moreover, children's understanding of measurement principles (e.g., understanding the structure of repeated units, judging the appropriateness of a selected unit) does not develop at the same time or at the same rate for length, area, and volume attributes (Curry et al., 2006). Finally, Nunes et al. (1993) found that children were more successful at measuring lengths with a conventional ruler than with individual pieces of string. In contrast, when measuring quantities with area attributes (i.e., rectangles), the reverse was observed: arbitrary units of individual "bricks" rendered the task more accessible to children than a conventional ruler, which would require a multiplicative strategy, such as applying the length x width procedure. Thus, Nunes et al.'s results suggest that children's measurement strategies are dependent not only on the attribute of the quantity being measured, but also on the attribute of the tools they use (i.e., length vs. area units).

Children's understanding of fraction concepts and operations also appears to be influenced by the type of attribute of the pictorial model they use. Hamdan and Gunderson (2017), for example, found that when second and third graders were trained to represent fractions with a circular representation (i.e., area model), they improved at representing fraction magnitudes with area models. Similarly, children who were trained to represent fractions with a number line (i.e., length model) improved at representing fraction magnitudes on a number line. Interestingly, children who received number line training were able to transfer their knowledge to a fraction magnitude comparison task (using symbols only), but children who received area model training were not. In terms of accuracy and conceptual understanding, Sidney et al. (2019) also found an advantage to using a number line compared to using an area model or no model at all when it comes to solving fraction division problems presented symbolically.

More recently, Osana et al. (2022) asked undergraduates to solve equal-sharing word problems with pictorial models -- that is, the to-be-shared quantities were presented visually with abstract area models (i.e., circles, with the same number of circles as the total quantity described in the problem) or with a number line model, where the number of to-be-shared objects were presented as hash marks on a number line. Performance was superior when the quantities in the problems possessed area attributes (e.g., chocolate bars) than when they possessed length attributes (e.g., ropes), but the difference emerged only when the participants used area models during problem solving. In sum, various features of the quantities involved in fractions and measurement problems, including the attribute of the to-be-partitioned objects, can influence student performance.

12.6 An Investigation of Children's Partitioning Strategies as a Function of Problem Features

Although the nature of children's partitioning strategies in equal-sharing contexts has been examined in previous work (e.g., Charles & Nason, 2000; Empson et al., 2005; Lamon, 1996), questions remain about the affordances of equal-sharing problem features, namely the extent to which the problems refer to real-world objects and the attribute of the to-be-partitioned objects. In the present chapter, we present a description of children's partitioning strategies as a function of the type of objects that require partitioning in equal-sharing word problems. We compared students' strategies on (a) grounded and abstract problems, and (b) problems describing objects with area and length attributes. Eighty-eight fourth-grade students ($M_{age} = 10.13$ years, $SD = 0.35$; 49% female) were recruited from 10 classrooms in four public schools in a large school board in the greater Montreal area in Quebec, Canada.

In a between-groups experiment, participants were randomly assigned to three conditions that were equivalent in terms of age, $F(2, 85) = 2.50$, $p = .09$, and gender distribution, $\chi^2(2, N = 88) = 1.23$, $p = .54$, but differed according to the types of problems they were assigned to solve: (a) grounded-area ($n = 22$), in which participants partitioned rectangular real-world objects with area attributes, such as chocolate bars, (b) grounded-length ($n = 22$), with problems that required partitioning objects with length attributes, such as ropes, and (c) abstract ($n = 44$), in which the problems contained nonwords couched in an equal-sharing context. The nonwords served as abstract objects because they made no reference to concrete, real-world objects with pre-existing physical attributes.

We first predicted an effect of groundedness: Based on previous research highlighting the benefits of grounded contexts for problem solving (e.g., Koedinger & Nathan, 2004; Koedinger et al., 2008), students' performance on both grounded problems (area and length) would be superior to that on abstract problems. We also predicted differences in the quality of the strategies used to solve the grounded-area and grounded-length problems, but because previous research is inconclusive on the role of attribute type in children's partitioning strategies, we made no prediction about the direction of the difference.

We speculate that the students in Foster (2018) may have constructed their own meanings for the abstract objects, perhaps imagining real-world objects that would serve their problem-solving goals (in the pilot study, Foster & Osana, 2017, observed students uttering statements such as, "What's a wog? I am going to pretend a wog is a log"). If so, their mental representations may have attenuated the expected grounding effects. To test this possibility, we included a measure to assess the students' mental representations of the objects in each problem to determine how the students would visualize the abstract to-be-partitioned objects. Data on the participants' internal representations would provide additional insight on the role of groundedness and object attribute in their partitioning strategies. Further, the mental representation measure allowed for a manipulation check that the desired area and length attributes were indeed activated for the grounded-area and grounded-length problems, respectively.

12.7 Documenting Children's Partitioning Strategies

Children's strategies were documented using two tasks: a series of equal-sharing problems and the Picture Perception Task, both of which were administered in individual interviews with a researcher. All interviews were video recorded for subsequent coding and analysis.

12.7.1 Equal-Sharing Problems

The objects, quantities, and number of sharers in all the equal-sharing problems by condition are listed in Table 12.1. The grounded-area problems required partitioning real-world objects that were rectangular in shape, such as chocolate bars. The grounded-length problems involved objects such as ropes. We replaced the standard units of measure in Foster (2018) with arbitrary units: For example, rather than sharing 9 meters of rope, the problem involved sharing 9 ropes. Abstract problems involved partitioning objects that were referenced by nonwords, such as "feeloos" and "figlias." All problems had answers greater than one, with fractional remainders

Table 12.1 Equal-sharing problems in each condition

Condition	Object	Quantities	Sharers
Grounded-area	Brownies	9	4
	Sandwiches	10	8
	Toasts	3	2
	Chocolate bars	15	6
Grounded-length	Ropes	3	2
	Strings	10	8
	Ribbons	9	4
	Straws	15	6
Abstract	Bamoes	9	4
	Porams	3	2
	Figlias	10	8
	Feeloos	15	6

equivalent to either 1/2 or 1/4. Administration of the equal-sharing problems was counterbalanced to control for possible order effects. Students were asked to use paper and pencil to supplement the verbal explanations of their strategies.

We developed a coding rubric based on the development of children's strategies for solving equal-sharing problems as described by Empson et al. (2005) and Empson and Levi (2011). In particular, Empson and colleagues argued that when children begin thinking about equal-sharing situations, they typically do not coordinate the number of sharers in the problem with the number of partitions that are necessary. As such, the coding of the participants' strategies was centred on the developmental hallmark of coordinating the quantity with the number of sharers.

The coding rubric was based on four criteria: (a) partitions had to match the numbers in problem, (b) all objects in the problem had to be used, and (c) shared with all sharers, and (d) objects needed to be distributed equally among the sharers. In line with Empson and Levi (2011), we considered criteria (b) and (c) as essential elements of partitioning (i.e., the coordination of quantity with the number of sharers). Points were awarded according to specific combinations of the four criteria listed above. Table 12.2 presents all the strategies observed in the sample and the corresponding number of awarded points.

Five points were assigned to strategies where all four criteria had been met. Four points were assigned when all criteria had been demonstrated except for the partitions matching the numbers in the problem. For example, a score of 4 was awarded if a student responded that a group of children would share the leftover objects without showing the fractional portion that each child would receive. Students whose strategies contained only the two essential elements of partitioning were given a score of 3. Students who were only able to show one of the two essential elements in their representations were given a score of 2, regardless of how they had fared on the other three criteria. Lastly, a score of 1 was awarded to representations based on the wrong problem structure (e.g., adding the two numbers presented in the problem), and a score of 0 when no strategy was produced to solve the problem (e.g., "I don't

Table 12.2 Strategy coding rubric

	Coding Criterion			
Points	A Partitions match numbers in problem[a]	B Uses all objects	C Shares objects to all sharers	D Distribute objects equally among sharers
5	✓	✓	✓	✓
4		✓	✓	✓
3		✓	✓	
2			✓	✓
2		✓		✓
2	✓		✓	✓
2		✓		
2			✓	
1	Wrong problem structure			
0	Problem not attempted			

Note. The region shaded in grey indicates the strategies that coordinate quantity with the number of sharers

[a] If the number of partitions does not match the number of sharers, this criterion is still met if the strategy is appropriate. For example, in the situation where 6 share 2: A child could partition each whole into thirds and distribute one of the six thirds to each of the six sharers.

know"). Total strategy score was the mean number of points on each item, for a minimum score of 0 and a maximum score of 5 points.

12.7.2 Picture Perception Task

The Picture Perception Task (PPT), designed by the authors, assessed students' mental representations of the to-be-shared object in each problem. The task was composed of five items (see Fig. 12.1 for a sample item). The participant was required to choose between two images – one a closed figure and the other either a curved or straight line – in response to the question, "Please point to the picture that looks most like a [e.g., brownie/rope/feeloo]." All five items of the task were administered after the child solved each equal-sharing problem, and the number of times the participant chose the area model was determined for each problem. The Picture Perception Task score was the mean number of times the participant chose

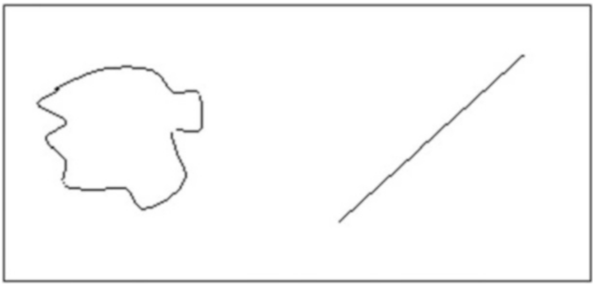

Fig. 12.1 Sample item from the picture perception task

the area model across all five problems, for a minimum score of 0 and a maximum score of 5.

12.8 Results

12.8.1 Mental Representations

A one-way ANOVA was conducted to determine if students' mental representations for the objects in the problems (as assessed by the PPT) were different for grounded-area, grounded-length, and abstract conditions. The data for two participants (one from the grounded-area condition and one from the abstract condition) were excluded because they did not complete the task. Recall that PPT scores reflected the extent to which the area items were selected over the length items. PPT scores were significantly different between the three problem types, $F(2, 83) = 124.56$, $p < .001$, partial eta-squared = .75. Follow-up Fisher LSD comparisons (Levin et al., 1994) revealed that PPT scores for the grounded-area problems ($M = 4.63$, $SD = .56$) were significantly higher than those for the grounded-length problems ($M = 0.82$, $SD = .63$), $t(83) = 15.44$, $p < .001$, $d = 14.89$, and for the abstract problems as well ($M = 3.26$, $SD = .97$), $t(83) = 6.37$, $p < .001$, $d = 5.37$. PPT scores on the abstract problems were significantly higher than on the grounded-length problems, $t(83) = 11.50$, $p < .001$, $d = 9.52$. These results suggest that the desired attributes were activated on the area and length problems and that the students were more likely to mentally represent the nonwords as possessing area attributes than length attributes.

12.8.2 Partitioning Strategies

Means and standard deviations for the strategy scores by condition are presented in Table 12.3.

Table 12.3 Means and standard deviations on strategy scores by condition

Condition	*M*	*SD*
Grounded-area	4.13	1.06
Grounded-length	3.78	1.19
Abstract	3.85	1.17

Orthogonal contrasts were constructed to test the effects of problem features on students' partitioning strategies. The groundedness contrast compared the abstract condition (−2) against the grounded-area (+1) and the grounded-length (+1) conditions. The attribute contrast compared the grounded-area condition (+1) against the grounded-length condition (−1). Results revealed no significant effects for either contrast – that is, students' strategies on grounded problems did not differ significantly from the strategies they used on abstract problems, $t(85) = 0.42$, $p = .68$, and their strategies for solving grounded-area problems were not significantly different from the ones used to solve grounded-length problems, $t(85) = 0.99$, $p = .33$. These data suggest that problems containing references to real-world objects provided no benefits to the quality of the students' partitioning strategies and that strategy quality was comparable regardless of object attribute.

12.9 Discussion

Some scholars have argued that an effective way to introduce fractions to young children is through the introduction of equal-sharing situations (e.g., Charles & Nason, 2000; Empson, 1999). By applying intuitive strategies to partition quantities, children can learn fundamental fractions concepts that are potentially more difficult to learn from traditional instruction emphasizing part-whole interpretations of fractions (e.g., Hamdan & Gunderson, 2017). For example, children can learn that wholes can be divided into parts and that the parts must be the same size. They can also learn that fractional quantities can only be interpreted in the context of the unit (Empson, 1999). Finally, Empson and Levi (2011) argued that through classroom conversations and practice solving equal-sharing problems, children can learn the quotient interpretation of fractions, which is foundational for a flexible conceptual understanding of rational number (Barnett-Clarke et al., 2010).

Our objective was to explore how the features of the to-be-partitioned objects in equal-sharing problems can impact children's partitioning strategies. We first investigated the role of groundedness, operationalized as word problems with references to concrete, real-world objects. The quality of the children's partitioning strategies on grounded problems was compared to their strategies on abstract problems, in which the to-be-partitioned quantity was referenced using nonwords. We also investigated whether the attribute of the objects in the grounded problems (i.e., area, length) made a difference in terms of students' partitioning strategies. A secondary objective was to examine students' mental representations when they attempted to solve equal-sharing problems with abstract quantities. The ways in which the students imagined, or in other ways represented, the abstract objects could assist in the interpretation of the results on the role of groundedness and attribute.

12.9.1 The Role of Object Groundedness

Our hypothesis about the effects of groundedness was not supported. No differences were found in the quality of students' partitioning strategies on the grounded problems and the abstract problems. The data on students' mental representations can shed some light on the reasons for this unexpected result. When students were confronted with objects in equal-sharing problems that made no references to specific real-world objects, they nevertheless represented those objects as possessing specific physical attributes, and in particular, were more likely to imagine objects with area attributes than length attributes. This suggests that even in abstract contexts, students constructed meaning in ways that supported their problem solving, in line with Thevenot and Barrouillet (2015)'s contention that problem features alone cannot account for children's problem-solving performance. Therefore, it is possible that the objects in all three conditions were in fact "grounded," perhaps contributing to the similarities observed in the quality of the students' strategies.

Another possible reason that our results differed from what has been reported in previous research is related to the way we grounded the equal-sharing problems. In Koedinger and Nathan (2004), for example, the effects of grounding were tested by manipulating the number of explicit references to concrete objects and events in mathematical problems. Their grounded problems made several references to concrete objects and situations, such as winning money in a lottery and buying jeans at a discount. Their semi-grounded problems were so-called "word equations," which preserved the mathematical structure of the grounded problems, but removed the situational effects present in the text. The abstract problems were presented using only algebraic symbols and numerals.

In contrast, we tested the effects of groundedness by manipulating only the to-be-partitioned object described in the problem, while maintaining the description of the partitioning situation in the problem text (e.g., "Emma wanted to share 9 brownies evenly among four friends."). The mental representation data demonstrated that the abstract objects appeared to be as grounded to the participants as the real-world objects with area attributes. This may suggest that the reference to the object itself (in this case, the word used to refer to it) may not play as prominent a role in the students' strategies as the partitioning context described in the problem. It is possible that students activated known partitioning schemas, which may have served as the primary grounding mechanism that drove their solution strategies. Thus, students' knowledge of the meaning of the words themselves (e.g., "liquorice"), or their previous experiences with the objects that were referenced, may have had less of an impact on performance than the type of schemas that were activated while problem solving (Čadež & Kolar, 2015).

The types of schemas and representations that are activated when children engage with mathematical word problems, and in particular, the level of abstraction and the ways in which they are constructed, is still not well understood. The research reviewed by Thevenot and Barrouillet (2015), however, provides rather compelling evidence that children construct transient mental models that incorporate their previous experiences and knowledge about the real world, such as partitioning actions in equal-sharing problems. In fact, the authors suggested that the degree of "imageability" of the nouns described in word problems may be related to the extent to which problem solvers bring real-world knowledge to bear. Our data suggest that nonwords were as easily "imagined" as real-world objects with area attributes. As such, the children's mental representations of objects with physical attributes may have facilitated, or at least not interfered with, the partitioning schemas that governed their strategies.

We do not intend to diminish the role of language in the solving of mathematical word problems, however. It has been established that language plays an important role in children's mathematics achievement, particularly on tasks that are language dependent, such as word problems. For example, mathematical vocabulary and receptive vocabulary contribute to word-problem solving accuracy of second and third graders (Xu et al., 2022), and syntactic awareness (i.e., the ability to manage the grammatical structure of language) affects the word-problem solving of 8–9-year-olds (Peake et al., 2015). Indeed, Bale and Barner (2018) posited that if children's quantity judgments of count nouns (e.g., "stones") and mass nouns (e.g., "stone") are based on world knowledge, their performance is at least mediated by the language and syntax in the problem text.

Therefore, despite the need for more research on the relation between problem features and children's sharing strategies, we also suggest that future lines of inquiry focus on the interactions between problem characteristics and individual factors, including children's world knowledge, linguistic competencies, and the types of mental representations that are triggered by the problems themselves. Consistent with this suggestion, Thevenot and Barrouillet (2015) maintained that problem characteristics cannot by themselves account for children's problem solving: Individual factors are responsible for the processes that take place between the presentation of the problem text and the "pure mathematical mental representation" required for computation (Thevenot & Barrouillet, 2015, "Situation Model and Word Problem" section), suggesting an interaction between problem features and cognitive factors.

12.9.2 The Role of Object Attribute

The data also revealed that the students' partitioning strategies did not differ as a function of the attribute of the objects in the problem. That is, the quality of the strategies used was similar whether the object in the problem has a length attribute or an area attribute. This finding was also unexpected, and at first glance, does not align with previous research on children's measurement strategies. A closer look at previous studies on children's measurement, however, revealed that several researchers used tasks that were decontextualized (i.e., not grounded). Like many school-based measurement tasks in American textbooks (Hong et al., 2018), the tasks used by Curry et al. (2006), Sisman and Aksu (2016), and Lehrer et al. (1998), for example, examined measurement tasks that involved finding the length of lines, the area of polygons or circular regions, or the volume of rectangular prisms, none of which were embedded in real-world contexts. In their study on the strategies children use to solve fraction division problems, Sidney et al. (2019) also used decontextualized tasks (e.g., 18 ÷ 2/3) to show differences in performance when using area and length models. Therefore, while more research is warranted to disentangle the sources of children's challenges with different attributes in fractions and measurement contexts, it is difficult to interpret our findings given the methodological differences between our work and what currently exists in the literature.

One of the few studies that assessed children's responses to different attributes in contexts that more closely approximated real-world situations was conducted by Hiebert and Tonnessen (1978), who compared the strategies used by children who shared physical objects evenly among a number of stuffed animals seated around a table. The objects were a stick of liquorice (length attribute), a circular piece of clay to represent a pie (area attribute), and individual pieces of candy (discrete objects). The authors observed differences in the children's strategies when they partitioned length and area. For instance, children easily engaged in repeated halving (i.e., dividing first into halves and then into quarters) when working with area, but could not use the same strategy when dealing with thirds. In contrast, when working with length, the challenges came from a different source: Creating equal partitions become increasingly difficult as the number of parts (i.e., the number of sharers) increased. Hiebert and Tonnessen (1978) only assessed the strategies of nine children, however, reducing the confidence one can place in their conclusions. Thus, we contribute to the literature by showing that the children's measurement strategies may look different in real-world contexts than those used in traditional school-based tasks. It is possible that by grounding partitioning and other measurement actions in schemas that are based on real-world experience, the effect of attribute may be less pronounced, or at least different from what is reported in the literature.

12.9.3 The Role of Unit Type

The type of unit used to measure continuous objects may also play a role in children's work with different attributes in equal sharing contexts. One of our methodological decisions was to remove any reference to standard units of measure in the stimuli used. For example, when partitioning brownies, the unit is arbitrary; that is, the brownie is the unit, regardless of its actual size in standard units of measure. When measuring length, in contrast, the units are contiguous, such as in measuring the length of one rope or the length of a desk. In the equal-sharing problems we administered to the participants, we attempted to make the problems as similar as possible except for the attribute to avoid introducing any confounds. Using arbitrary units to measure one rope, however, would have required additional information about the unit (e.g., Sarah would like to evenly share a length of ribbon that measures 10 paperclips with 4 children), which was not necessary in the case of brownies because each brownie was its own, discrete unit. Alternatively, we could have used standard units to quantify the length of the rope, but this would have introduced the confound of unit type, as was the case in Foster (2018).

To circumvent these methodological issues, we used several pieces of rope in the grounded-length problems, which rendered the objects in all problems – area, length, and abstract – discrete (or "discretized," meaning that the objects were themselves continuous in nature, but the quantities were represented as a collection of individual objects; Rapp et al., 2015). That is, all the objects described in the problems were unattached, countable objects that themselves served as arbitrary units that could be partitioned. It is thus possible that by reducing all objects to discretized units could possibly account for the levelling of strategy performance across all three conditions. More specifically, using individual lengths of rope in the problems may mask the true nature of children's difficulties when partitioning continuous lengths, an issue that does not appear when partitioning discrete objects with area attributes. Indeed, partitioning one object measured in length units is arguably more often encountered in school tasks and in everyday contexts than evenly sharing individual pieces of rope. As such, using individual pieces of rope rendered the grounded-length problems more contrived than the grounded-area problems, but was required for experimental control.

12.10 Practical Implications and Conclusion

Many questions remain regarding the role of problem features and children's mental representations on their partitioning strategies in equal-sharing contexts. As such, it is premature at this stage to prescribe specific instructional strategies for teaching fractions and measurement concepts in the classroom. Nevertheless, our findings imply that teachers should be aware that children are active meaning makers (see Thevenot & Barrouillet, 2015), even when learning about a new number system (i.e., rational number) and grappling with representations of quantities that have no real-world referents. Thus, in addition to considering task characteristics, it is paramount that teachers probe their students' reasoning to gauge how the problems and representations are being interpreted. Gaining a deeper understanding of students' conceptions, particularly as they relate to problems features, such as references to real-world objects, the attributes of those objects, and the types of units used to quantify them, would provide teachers with tools for moving children's learning forward.

In conclusion, the work reported in this chapter, together with prior research in children's problem solving, fractions understanding, and measurement knowledge, allow us to take the first steps in constructing a theoretical picture of the factors that may impact children's solutions to equal-sharing problems. Problem-related features, such as the extent to which the problems are grounded in real-world contexts, the attribute of the objects that are measured and partitioned, and the nature of the quantities described (discrete versus continuous), likely interact to influence children's thinking in complex ways. A coherent theoretical model that identifies the types of equal-sharing problems that optimize the development of students' strategies would be useful for classroom teachers. A deeper understanding of the factors at play can guide educators in the design of appropriate problems and in tailoring their classroom interactions to their students' specific needs and experiences.

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Chapter 13
Designing Worked Examples to Teach Students Fractions

Ouhao Chen, Arunachalam Thannimalai, and Slava Kalyuga

Abstract Cognitive load theory is an instructional theory which aims to generate innovative instructional methods based on the known characteristics of human cognitive architecture. The worked example effect is a well-established phenomenon in cognitive load theory, indicating advantages of explicit instruction over pure problem-solving activities for novice learners. However, it has been mostly investigated with secondary and high school students rather than younger students, such as lower primary school students. This chapter reviews the worked example effect and provides empirical evidence of applying it in teaching fractions to lower primary school students.

Keywords Cognitive load theory · Human cognitive architecture · Worked example effect · Age · Lower primary mathematics · Fractions

13.1 Introduction

Cognitive Load Theory (CLT) is an instructional theory, aiming to generate innovative instructional methods to reduce working memory load in learning (Sweller et al., 2011). The base of CLT is human cognitive architecture, dealing with relations between working memory (i.e., where information is processed and temporarily stored) and long-term memory (i.e., knowledge base) (Sweller & Sweller, 2006), and the five principles of its operation have valuable instructional implications. Within CLT framework, the worked example effect suggests that providing explicit

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instructional guidance to novice learners is superior to engaging them in solving problems, especially at the initial learning stage (Sweller & Cooper, 1985).

There is no agreed definition of a worked example, but atypical worked example normally includes a problem statement and step-by-step solution (Sweller et al., 2011). Novices engaging in solving novel problems must generate possible solutions moves in a rather random way based on their minimal (if any) problem-specific knowledge structures and test the effectiveness of these moves, resulting in a potentially heavy cognitive load. In contrast, studying with worked examples avoids such random search processes, thus reducing the unnecessary cognitive load. However, the effectiveness of worked examples may disappear when dealing with more knowledgeable learners and simpler materials with low levels of interrelations between elements of information, indicating the expertise reversal effect.

Worked examples have been mostly investigated with secondary and high school students, with no clear conclusions for younger children, such as lower primary school students. Therefore, the main goal of this chapter is to provide a review of research related to the worked example effect for younger students. Another goal is to provide some guidance to primary mathematics teachers in how to design worked examples for teaching fractions to this category of learners. The first section of this chapter introduces CLT and human cognitive architecture. The second section provides a review of relevant studies of the worked example effect and its moderators, with a particular focus on the age of participants, followed by the description of two experiments in using worked examples for teaching fractions to lower primary school students. The chapter concludes with implications for teaching primary school mathematics and future research directions.

13.2 Human Cognitive Architecture

Human cognitive architecture, outlining the relations between working memory and long-term memory, serves as the base of CLT. This architecture also informs us how information is processed, stored, and retrieved in human memory systems. There are multiple ways to acquire new knowledge (Sweller & Sweller, 2006). The most efficient way is to borrow the knowledge from other sources and reorganize it for storage, suggesting the *borrowing and reorganizing principle*. If new knowledge is not available for borrowing, humans could randomly generate new knowledge by using and combining elements of prior knowledge (i.e., trial and error technique), suggesting the *randomness as genesis principle*. However, as working memory has very limited capacity (Cowan, 2001; Miller, 1956) and duration time (Peterson & Peterson, 1959), only very limited amount of new knowledge components could be generated and processed in working memory, suggesting the *narrow limits of change principle*. Only new information that was tested as being effective for solving problems could be transferred and stored in long-term memory (the *information store principle*), with ineffective information discarded. Long-term memory unlike working memory has unlimited capacity and could store information for a very long period. By applying the *environmental organizing and linking principle*, the relevant stored information could be retrieved from long-term memory to solve externally presented problems. Worked example effect, one of classic cognitive load effects, was generated based on the human cognitive architecture. The next sections will discuss this effect in detail.

13.3 The Worked Example Effect

In the framework of CLT, the worked example effect indicates that using worked examples to teach novices is superior to engaging these learners in unguided problem-solving activities. The principles of human cognitive architecture provide a general explanation of the advantage of using worked examples for teaching compared to problem solving. Learners directly borrow worked-out steps from others using the *borrowing and reorganising principle*. Engaging in problem solving, particularly for novice learners, may require great mental effort to generate suitable solution moves and test their effectiveness according to the *randomness as genesis principle*. In the latter case, learner working memory capacity is highly likely to be overloaded.

A great number of research studies have provided empirical evidence for the effectiveness of using worked examples in different domains. Sweller and Cooper (1985) firstly investigated the worked example effect in the domain of linear algebra equations in both secondary school and undergraduate mathematics. The worked example followed by solving a similar problem was used to test the worked example effect. In accordance with this paradigm, the participants in the experimental group received a worked example followed by a similar problem to solve, whereas participants in the conventional problems solving group (i.e., problem solving – problem solving paradigm) solved the equivalent number of the same problems without worked-out solutions. Results indicated advantages of using worked examples for teaching: novices (Year 9, Year 11 and undergraduate mathematics students) in the experimental group used less time to solve problems and made less mistakes in the post test, compared to novices in the conventional problems solving group.

In teaching physics, researchers presented secondary school students malfunctioning parallel electrical circuit designs, and students needed to apply their knowledge of electrical circuits to diagnose the faults. Students in the worked example group could study the provided optimal solutions, whereas students in the conventional problems solving group had to answer questions by themselves (Van Gog et al., 2011). In this study, the worked example – problem solving sequence was compared with the problem solving – worked example sequence. The results suggested that using the worked example – problem solving sequence was superior to using problem solving – worked example sequence for students' learning. In other words, learning with instruction before engaging in problem solving was more effective for learning, compared to learning with problem solving before instruction.

Beside well-structured problem domains, such as mathematics and physics, the worked example effect has been also found with ill-structured problems in arts design (Rourke & Sweller, 2009). Participants were taught unique characteristics of five design artists and were randomly assigned to either studying the features of a designer's work and the name of the designer by examples or figuring out the features of a designer's work and the name of the designer on their own. The results were in line with the worked example effect, suggesting that novices who had moderate levels of visual literacy were more successful at recognising the designers' work by learning from examples compared to those who solved problems by themselves.

The worked example effect was found effective not only in individual learning but also in group learning settings. Participants from Year-7 (around 13 years old) in the study by Retnowati et al. (2010) either engaged in problem solving or studied examples individually or in groups. The worked example effect was found in both individual and group learning conditions, thus extending the worked example effect to group learning.

Although using worked examples is effective and superior to engaging in unguided problem solving, a number of factors may moderate the effectiveness of worked examples. The most important of those factors are levels of learner expertise and levels of *element interactivity* of learning materials. The concept of element interactivity is essential and fundamental within cognitive load theory. It indicates the degree of inter-connectiveness of elements in learning materials (Sweller et al., 2011). An element could be a concept or a symbol. Based on the nature of learning materials, they could be high or low in element interactivity. For example, learning mathematics symbols, such as x, y, or sin, is a low element interactivity task, as each symbol (e.g., x) could be learned without referring to another one (e.g., y). Therefore, these symbols could be treated separately and individually, imposing low levels of cognitive load on working memory. However, when solving an equation, such as $5x - 8 = 3$ for x, the task is high in element interactivity. There are at least six elements that must be processed in working memory simultaneously, and they could not be treated separately. In other words, to successfully solve the problem, learners must process these six elements together in working memory, imposing high levels of cognitive load on working memory. However, high element interactivity could become low in element interactivity if the learners' expertise in a task domain is increased.

Another factor is expertise which is closely associated with the amount and structure of knowledge stored in log-term memory (Sweller et al., 2011). The same material which is high in element interactivity for novice learners could be low in element interactivity for more knowledgeable learners. For example, the task of solving the eq. $5x - 8 = 3$ for x, involves six interactive elements for novices, but more knowledgeable learners (experts) could retrieve a relevant schema as an entity from long-term memory (i.e., the combined procedure $(3 + 8)/5$ could be retrieved immediately providing the answer), rendering only effectively one element to be processed in working memory. Therefore, the concept of element interactivity is closely related to levels of learner expertise (Chen et al., 2017).

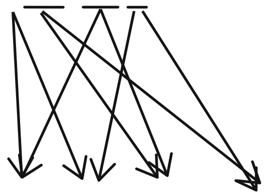
Chen et al. (2015, 2016a, b, 2019, 2020) conducted a series of experimental studies to investigate the relations between expertise and element interactivity, and their influence on the effectiveness of using worked examples. Participants in those studies were from Year-4 (around 10 years old) to Year-8 (around 14 years old). In the early experiments, Chen et al. (2015, 2016a, b) compared the generation effect with the worked example effect with novices and knowledgeable students in the domain of mathematics. The generation effect suggests that encouraging students to generate the solutions by themselves (i.e., engaging in problem solving) is more beneficial for learning than presenting worked-out solutions to them (Slamecka & Graf, 1978), whereas the worked example effect indicates that presenting solutions to students is superior to engaging them in self-generating solutions.

The contradiction between the studies was assumed to be resolved by considering materials with different levels of element interactivity. For example, the tasks that were low in element interactivity involved memorising some mathematical formulas (as they could be memorised separately and individually), whereas the learning tasks that were high in element interactivity required applying those formulas to calculate the areas of compound shapes. When testing these materials with novice learners, the generation effect was found for materials low in element interactivity, while the worked example effect was observed for materials high in element interactivity. However, when the same learning tasks were used with more knowledgeable learners, the generation effect was found for all sets of materials, as all of them turned out to be low in element interactivity for these learners. Therefore, the apparent contradiction between the worked example effect and the generation effect could possibly be resolved by taking into account different levels of element interactivity of learning tasks and levels of learner expertise.

Given that learners' expertise and levels of element interactivity moderate the effectiveness of the worked example effect, Chen et al. (2019, 2020) extended the experiments to mathematical problems with multiple steps and tested different instructional sequences (i.e., worked example – problem solving vs. problem solving – worked example). Chen et al. (2019) compared the performance of worked example and problem-solving groups on the 1st step and 3rd step of opening across bracket problem (Fig. 13.1). A worked example effect was found on the 1st step but no worked example effect on the 3rd step for novices, and no worked example effect was found for more knowledgeable learners. The suggested explanation was that the students had difficulty in successfully accomplishing the 1st step of the solution, as it was high in element interactivity (i.e., when opening across the brackets of $(x + 4)(3x - 2)$, students needed to conduct 4 pairs of multiplications involving 8 interactive elements, such as x and $3x$, 4 and $3x$, simultaneously in working memory), whereas for the 3rd step as the last step of the solution, the element interactivity was lower (i.e., students only needed to do an addition $-2x + 12x$, which is much simpler compared to the 1st step), therefore, worked examples were not superior any more to unguided problem solving.

Similarly, when comparing the two-alternative example-problem instructional sequences with novices for whom the materials were high in element interactivity, the worked example – problem solving sequence was superior to the problem

Task 1: Calculate (x+4)(3x–2)



$(x + 4)\,(3x - 2)$

$= x3x + x(-2) + 4(3x) + 4(-2)$

$= 3x^2 - 2x + 12x - 8$

$= 3x^2 + 10x - 8$

Fig. 13.1 Opening across bracket problem

solving – worked example sequence, but no differences were found between these two sequences for more knowledgeable learners for whom the materials were low in element interactivity.

To summarise the general pattern that has emerged from the above studies, the worked example effect is more likely to be found with materials high in element interactivity, whereas problem solving/generation effect is more likely to be observed with materials low in element interactivity. Worked examples are effective for novices, but their effectiveness disappears or even reverses for more knowledgeable learners, indicating the expertise reversal effect (Kalyuga, 2007).

13.4 The Worked Example Effect and Age Differences

Based on a systematic review conducted by Education Endowment Foundation (UK), the worked example effect has been largely investigated with secondary and above cohort (Perry et al., 2021). Very few studies on the worked example effect have focused on younger learners. Two research studies tested the worked example effect with Year 4 students (the youngest cohort) (Chen et al., 2015; Van Loon-Hillen et al., 2012). Chen et al. (2015) experiments compared the worked example effect with generation effect in geometry, supporting the worked example effect, whereas Van Loon-Hillen et al. (2012) compared lessons using worked examples with lessons taught using RekenRijk method (four teaching lessons, six self-study lessons, one repetition lesson, and one “deepening understanding” lesson) in algebra in a quasi-experiment, with no worked example effect found. Therefore, one of the goals of this chapter was to report new, recently obtained experimental results of studies in the worked example effect with lower primary school students.

13.5 Empirical Evidence of the Effectiveness of Worked Examples with Lower Primary School Students

The schools we collaborated with were interested in improving fraction learning which started from Year 2. Therefore, two experiments were conducted to investigate the effectiveness of worked examples for Year2 (around 8-year-old) students learning fractions in two Singaporean schools. In both studies, participants were novices in the subject matter of interest, and two types of instruction were compared: a set of example-problem pairs against an equivalent set of problem-solving tasks. The worked examples incorporated worked-out solutions of the problems, while the problem-solving tasks had the same problems but did not contain any worked-out solutions.

It was hypothesized that learners who study worked examples would outperform those engaging in problem solving only on post-tests, as using worked example imposes lower level of cognitive load than problem solving. Based on power analysis, to obtain a large effect size ($d = .8$) with power of 0.8 and α error at .05, the sample size should be 42 or higher for both experiments.

Participants in the first experiment were 42 Year 2 students recruited from a registered private primary school in Singapore. The average age of participating children was 8 years old. The learning tasks involved subtraction of two unlike fractions which was a challenging topic for both adults and children (Lortie-Forgues et al., 2015). All participants were novices to this task area. Before the experiment, 22 children were randomly assigned to the worked example group and 20 children were assigned to the problem-solving group.

A sample problem is $3/5-2/10$, with its solution provided in the corresponding worked example (only abstract mathematical expressions were included) $3/5-2/10 = 6/10-2/10 = (6-2)/10 = 4/10$. In the worked example group, two worked example – problem solving pairs were designed based on four chosen task problems. Tasks 1 and 3 were worked examples, while tasks 2 and 4 were just similar problem statements without any worked–out solution steps and final answers, allowing participants to solve on their own after studying the corresponding examples. Thus, task 1 – task 2 were paired together as the first example – problem pair, and tasks 3 and 4 – the second example – problem pair. In the problem–solving group, all four tasks were just problem statements without any worked–out steps and final answers. They were meant for participants to try out solving problems on their own. All the materials in the study were paper based.

After being seated at the assigned place, each child had 16 min to study examples and/or solve problems designed for the learning phase. Immediately following the learning phase, they had 10 min for the post-test. The post-test contained five problems similar to problems used during the learning phase (a sample problem $2/5-3/10$). The internal reliability of the post-test (Cronbach $\alpha = .33$) was rather low due to a wide range of scores (while some students were able to solve all problems correctly, others struggled to solve even one).

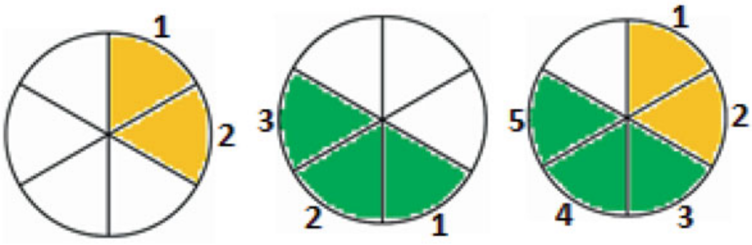
The raw scores obtained in the post-test were converted to the percentage correct scores for the analysis. There was no significant difference found between the worked example (M = 20.91, SD = 31.49) and problem-solving (M = 31.33, SD = 32.45) groups, $t(40) = -1.06$, SED = 9.87, p = .297. Thus, no evidence of the worked example effect was obtained in this experiment. A plausible explanation for this could be that the tasks involving subtraction of two unlike fractions were too difficult for Year 2 students (a 'floor effect' with only around 25% overall success rate in the post-test). Even the students who were studying the worked examples could have found it difficult to decipher the links between worked-out steps shown in the examples to fully understand and make use of these worked examples to benefit their learning.

Thus, no worked example effect was found in this experiment, probably due to the learning materials being too abstract and difficult for lower primary school students. Therefore, in the follow-up experiment, a form of visual support was implemented in worked examples to help children identify the links between worked-out steps, particularly the mapping between abstract fractions and their visual representation, which made the learning materials easier for students.

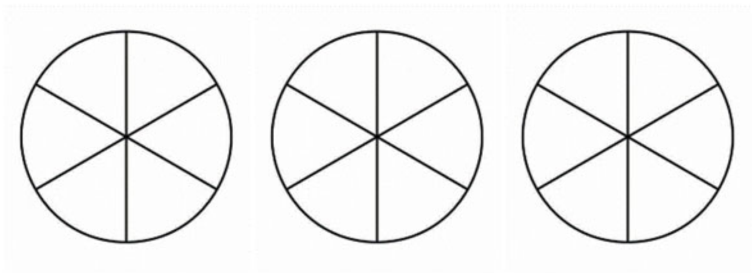
The second experiment involved 48 Year 2 students from a public primary school in Singapore. The mean age of participants was 8. The learning tasks used for this experiment involved addition of two likely fractions. All students were novices to this task area. Before the experiment, all students were randomly assigned to the worked example (N = 24) and problem-solving (N = 24) groups. A sample problem statement was 2/6 + 3/6 (Fig. 13.2). In addition to the numerical form, 2/6 was represented by a circle divided by six equal pieces, with two pieces coloured in yellow, and 3/6 was represented by a circle divided by six equal pieces, with three pieces coloured in green. To obtain the answer for 2/6 + 3/6, it was sufficient to add the two yellow pieces and three green pieces together as five coloured pieces out of six. In this way, using the addition (instead of subtraction) task together with the visual representation was expected to make the material easier for Year 2 students compared to the materials used in the previous experiment.

The sequences of two example problem pairs (for the worked example group) and four problems (for the problem-solving group) were designed similarly to the first experiment. The procedure was also identical to the previous study, with 16 min allocated to the learning phase and 10 min – to the post-test phase. The test paper also contained five problems that were similar to the problems used during the learning phase (Cronbach α = .82). This time, there was a significant difference found between the worked example (M = 93.89, SD = 17.02) and problem-solving (M = 33.06, SD = 35.47) groups, $t(46)$ = 7.58, SED = 8.03, $p < .001$ (1-tailed), d = 1.47, indicating a strong worked example effect.

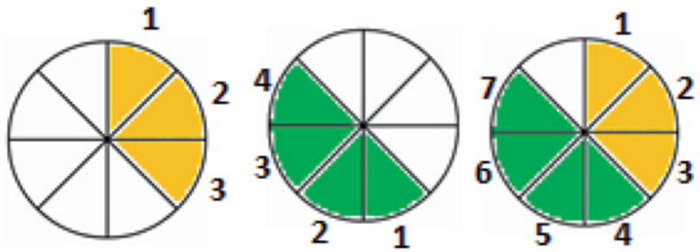
In this experiment, a clear and strong worked example effect was found. Compared to the first experiment, abstract numerical mathematical expressions were represented by concrete geometrical objects, such as pie charts representing fractions, which apparently helped lower primary school students process abstract fractions. When performing likely fractions' addition operations, they could count



[Problem 2] Please use the above method to calculate 36+16



[Problem 3] Calculate 38+48



[Problem 4] Please use the above method to calculate 28+58

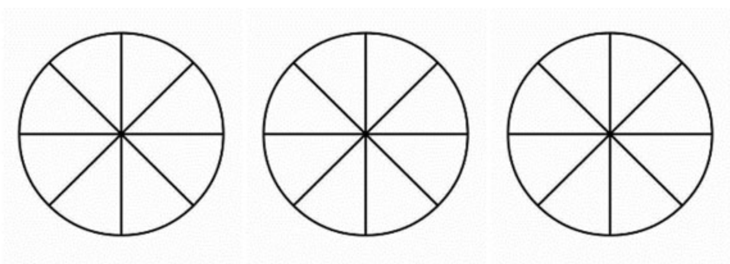


Fig. 13.2 The examples and problems used in Experiment 2

the coloured areas rather than deal with abstract numerical symbols in working memory. Therefore, helping learners visualise abstract mathematical concepts could be beneficial when designing worked examples for lower primary school students.

13.6 Conclusion and Future Research

The worked example effect within the framework of cognitive load theory has been well-documented for over 30 years, indicating that teaching novices with worked examples would be superior to engaging them in problem solving. The explanation is based on the reduced cognitive load when teaching novices with worked examples. However, the worked example effect has been largely investigated with secondary and above students with only a few studies for younger children. Therefore, this chapter reported two experiments testing the worked example effect with Year 2 students in learning fractions.

In the first experiment, the failure in obtaining the worked example effect was likely due to the difficulty of learning materials for both experimental conditions. For Year 2 students, processing abstract mathematical concepts in working memory could impose too much cognitive load, which would result in failed learning. In the second experiment, using pie charts to help children visualise abstract mathematical concepts apparently reduced the heavy information processing burden in learner working memory, leading to more successful learning in the worked example group.

The results supported the worked example effect (Sweller & Cooper, 1985) in teaching fractions to novice primary school students, namely, providing explicit guidance for novice learners in calculating the addition of fractions was superior to engaging them in unguided generation of solution steps, due to reduced cognitive load (Sweller et al., 2011). More importantly, the results indicate that worked examples could be effectively used with much younger children than Year 4 students for whom the worked example effect was found in previous research (Chen et al., 2015). When teaching older students with worked examples, abstract mathematic expressions would be effective, however, the worked examples designed for younger students need to include some visual support, such as diagrams, to make abstract mathematic expressions (i.e., fractions) more concrete.

Using visual representations (i.e., diagrams) in teaching mathematics is in line with the idea that employing more than one representation would promote learning (Ainsworth, 1999). Translating abstract mathematical symbols and relations into supplementary graphs enhances learning (Brenner et al., 1997). Therefore, when designing worked example for teaching primary school students, merely giving numerical solutions might not be effective enough to reduce cognitive load for novice learners. It would be necessary to consider other approaches, such as using multiple representations (Ainsworth, 1999) or adaptive (for learners with different levels of prior knowledge) highlighting of solutions in worked examples (Neubrand et al., 2016). These approaches are consistent with the role of pictorial representations and levels of learner prior knowledge in general principles of cognitive theory of multimedia learning (see Mayer & Fiorella, 2022 for a comprehensive overview).

As a practical implication, worked examples could be an effective tool in teaching mathematics to lower primary school students, if the examples are designed to be understandable to this category of learners (e.g., implementing suitable forms of visual support to represent abstract concepts using concrete objects). Although using worked examples to teach mathematics is a common practice by teachers, some design factors may need to be carefully incorporated to reach its maximum effectiveness.

Future research in the effectiveness of worked examples for teaching lower primary school students' mathematics needs to investigate the effects of implementing various specific forms of visual support (e.g., pie charts) or prompts (e.g., cues or highlights) in worked examples. Such techniques should be an essential aspect of the design of worked examples for lower primary school students. Finally, investigating the effectiveness of worked examples for different categories of lower primary school students should be an important research direction in cognitive load theory, considering the lack of empirical studies in this area.

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Chapter 14
Developing Fraction Sense in Students with Mathematics Learning Difficulties: From Research to Practice

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Abstract Despite considerable investment in research in mathematical cognition and learning over the past decade, students with mathematics learning difficulties are losing ground. Fractions are a particular barrier for many of these learners. Development of evidence-based fraction interventions for students who are still struggling in middle school is essential to help prevent cascading difficulties, particularly when algebra becomes a primary focus. Addressing this need, our research team developed a fraction sense intervention (FSI) for low-performing middle schoolers. To make learning last, the FSI explicitly incorporates general techniques backed by evidence from cognitive science. In this chapter, we address fraction intervention for low achievers in three areas: (a) domain specific concepts, procedures, and representations; (b) general techniques that support learning across domains; and (c) lesson-specific details about how information is presented in the FSI. We describe the iterative development of the FSI and discuss its effectiveness in two contexts: small and larger group settings.

Keywords Fractions · Mathematics learning difficulties · Intervention

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Despite considerable investment in research in mathematical cognition and learning over the past decade (Rittle-Johnson & Jordan, 2016), low-achieving students are losing ground. As an example, the most recent U.S. National Assessment for Educational Progress Report Card in mathematics (NAEP, 2015) reports that lower-performing eighth grade students (i.e., those performing at the 10th and 25th percentiles) scored significantly worse than did a similar cohort of lower-performing students in the previous assessment two years earlier. The loss was greater for students scoring at the 10th percentile. In contrast, mathematics scores of students performing in the higher percentiles (50th, 75th, and 90th percentiles) held steady during this same period, yet only 34% of eighth graders overall met the proficiency criterion. These trends are consequential. Students who show low proficiency in mathematics at the end of secondary school are much less likely to receive a college degree than students with middle to high proficiency (Mamedova et al., 2021).

Educational researcher Tom Loveless (2021) argues that rather than focusing on detailed sets of standards, as has been done in the U.S., more attention should be devoted to developing powerful teaching approaches in crucial components of mathematics proficiency. One content strand of particular importance in the middle grades is fractions. Fractions are typically a student's first introduction to rational number topics, making them foundational to rational number learning more generally; yet fractions are difficult to learn for many students (Jordan et al., 2017). Fraction interventions for students who are still struggling in middle school are especially important because these students experience cascading mathematics difficulties, particularly when algebra becomes a primary focus (Siegler et al., 2012; Siegler & Pyke, 2013). Loveless (2021) observes:

> Fractions are like a gigantic wall that kids hit in fourth, fifth, and sixth grades; some crawl over but many do not. What if the Bill and Melinda Gates Foundation money that went to Common Core, estimated at $300 million, had instead funded dozens of experiments to discover new curricular materials and new ways of teaching fractions, field tested these programs in randomized trials, and then disseminated the findings broadly? (p. 369).

Within multi-tiered system of supports (MTSS) that are common in the U.S., middle-school mathematics teachers are often called on to deliver evidence-based interventions for students with mathematics learning difficulties (MLD), but relatively few such interventions are available, and many teachers do not have specialized training in teaching students who struggle with mathematics. Addressing this need, our research team developed a fraction sense intervention (FSI) for low-performing middle schoolers who have not scaled the fraction "wall", despite conventional instruction and intervention based on national benchmarks. To make learning last, the FSI explicitly incorporates general techniques backed by evidence from cognitive science (Dunlosky et al., 2013).

In the present chapter, we address fraction intervention for low achievers in the context of three instructional categories (Alibali, 2021): (a) domain specific concepts, procedures, and representations; (b) general techniques that support learning across domains; and (c) lesson-specific details about how information is presented in the FSI. We describe the iterative development of the FSI and discuss its effectiveness in two contexts: small and larger group settings.

14.1 Domain Specific Concepts, Procedures, and Representations

14.1.1 Why Are Fractions Hard for So Many Students?

Fractions are complex, and it is not surprising that many students find them hard to learn and teachers find them challenging to teach. Students with fractions difficulties often assume incorrectly that properties of whole numbers or integers apply to all numbers. This phenomenon is referred to as a whole number bias (WNB; Ni & Zhou, 2005; Siegler et al., 2011). Fraction concepts can be challenging to learn when they seem incompatible with the student's existing framework about integers (McMullen et al., 2014; Van Hoof et al., 2015; Vosniadou, 2014). For example, the size of a fraction does not change in ways consistent with the absolute value of its numerator and denominator (Schneider & Siegler, 2010). For example, $\frac{4}{12}$ is less than $\frac{5}{6}$ and $\frac{1}{3}$ is more than $\frac{1}{5}$. Moreover, operational rules with integers do not always apply to fractions (e.g., multiplying a fraction by a proper fraction, which is less than one, always makes the product smaller, while multiplication of integers always makes the product larger). Another source of confusion is that multiple fractions have the same location on the number line $\left(\frac{2}{4}\right.$ falls at the same place as $\frac{1}{2}$ and $\frac{4}{8}$), whereas only one integer falls at a given location.

Many students also fail to grasp that the amount a number represents depends on the measuring unit or the "whole" (Dyson et al., 2018). The whole can be a distance (e.g., miles), an area (e.g., a rectangle), or a group of objects (e.g., pieces of candy). Whole number units can be easily counted but finding fractions of the whole requires equal partitioning skills. For example, if students need to find $\frac{1}{3}$ on the number line, they must be able to partition the distance between 0 and 1 into three *equal* parts – a difficult task for many students with MLD (Rodrigues et al., 2016). If the whole is already partitioned, students may successfully identify the correct fraction when the number of parts matches the denominator (e.g., circle $\frac{2}{5}$ of five objects). However, when asked to indicate $\frac{2}{5}$ of a whole partitioned into 10 equal parts, many students still choose 2 rather than 4.

Another characteristic of fractions that eludes many students is that the number of fractions between any two consecutive integers and any two fractions is infinite – a concept referred to as density (McMullen & Van Hoof, 2020). Although students may recognize there are numbers between zero and one, they often see only familiar unit fractions, such as $\frac{1}{2}$ and $\frac{1}{4}$. These students often misconstrue all fractions as being small – and always less than one (Resnick, et al., 2016). Working with fractions also requires multiplicative reasoning (e.g., 2, 3, 4, and 6 all are factors of 12) to find equivalent fractions. Fluency with multiplication facts helps students find equivalent fractions (e.g., $\frac{3}{4} = \frac{9}{12}$). Slow fact retrieval, however, is a signature characteristic of students who struggle with mathematics (Jordan, et al., 2003), making fraction problem solving especially laborious for these children.

To help students with MLD, the FSI emphasizes three inter-related domain specific areas that are foundational to fraction success: (a) Fraction magnitude, which refers to understanding that fractions are numbers that have a place on the number line and that the fraction's numerator and the denominator work together to determine its size; (b) Fraction equivalence, which refers to the ability to find fractions that are the same size even though they have different numerators and denominators (e.g., $\frac{3}{5} = \frac{6}{10}$) and to order them on the number line; and (c) Fraction arithmetic, which involves conceptual understandings for operating with fractions (e.g., why a common denominator is needed to add fractions but not for fractions multiplication), multiplicative reasoning and fluency, and knowledge of calculation procedures.

14.1.2 Fraction Magnitude and Equivalence

All real numbers have magnitudes that can be assigned specific locations on number lines (Siegler et al., 2011; Siegler & Lortie-Forgues, 2014). The ability to reason about magnitudes of fractions as well as whole numbers is fundamental to working with fractions, predicting both fraction arithmetic proficiency and overall mathematics achievement (Resnick et al., 2016; Siegler et al., 2011). To determine a fraction's magnitude, or size, students must consider the relation between the numerator and the denominator, rather than thinking about each number separately.

The ability to locate fractions on the number line is a reliable indicator of fraction magnitude knowledge. Resnick et al. (2016) investigated the development of students' abilities to estimate the locations of fractions on number lines, ranging from zero to one and zero to two, over multiple longitudinal time points. The data revealed three empirically distinct growth trajectories: One group of students began fourth grade with relatively accurate estimates and became even more accurate over the three-year time period; a second group started with inaccurate estimates but became accurate over the time period, most likely due to instruction; and a third group started with inaccurate estimates and showed little growth over time. Analysis of performance showed that students who fell into the inaccurate, low growth group tended to interpret both proper and improper fractions as having quantities less than one, failing to consider how the numerator and denominator work together to form a magnitude that can be any size. Weak calculation fluency, poor classroom attention, and inaccurate whole number line estimation skill at the start of the study significantly increased the odds of being in the low accuracy and growth group. Consequentially, in sixth grade, about two-thirds of these children failed to meet state mathematics standards versus just 5% of the highly accurate group and 17% of the steep-growth group.

Performance on fractions comparison tasks (e.g., which of two fractions is larger?) also reveals specific information about students' magnitude understanding. Rinne et al. (2017) analyzed student errors and found that many fourth graders show a typical whole number bias – thinking that larger numbers in numerators and denominators produce larger fraction values. Interestingly, however, some children also showed the reverse bias, that fractions with smaller numerators and denominators are larger, suggesting at least a partial understanding that smaller numbers can yield larger magnitudes. Children with this partial understanding were more likely to become accurate over time on fraction comparison tasks than were those who persisted with larger number biases. Recognizing that fractions with integers that are larger in the denominator (more equal parts) can be smaller in magnitude than those with smaller integers (fewer equal parts) in the denominator, as is the case with unit fractions where the numerator is always 1, is a crucial understanding (Empson et al., 2020).

Mathematical equivalence refers to understanding of the equal sign, which is an important foundational concept for algebra learning (Knuth et al., 2016). The equal sign can be viewed as "is the same amount as" or "is the same distance as" to promote the relational interpretation. In the case of fractions, students must be able to think about multiple fractions that are the same value, regardless of their numerators and denominators. Multiplicative reasoning helps students find equivalent fractions (e.g., multiplying the numerator and the denominator by the same number; Hansen et al., 2015; Ye et al., 2016). Equivalent fractions can be simplified to find a fraction that has no common factors other than one (e.g., $\frac{3}{6} = \frac{1}{2}$). Fluency with multiplication facts frees up students' cognitive resources to focus on fraction understandings (Hecht et al., 2003; Seethaler et al., 2011). Overall, students need to be able to recognize equivalent fractions, which is greatly facilitated by fluent multiplicative reasoning.

14.1.3 Fraction Arithmetic

Fraction arithmetic presents added challenges for many students, especially those with MLD who often have weakly grounded knowledge of fraction concepts (Jordan et al. 2017). These arithmetic difficulties persist into secondary school and even adulthood (Calhoon et al., 2007; Kelly et al., 1990; Siegler et al., 2011). For example, Braithwaite et al. (2017) showed that when asked to choose estimated solutions to fraction arithmetic problems, eighth graders, on average, performed near chance level (50%); students' often selected answers that violated basic fraction magnitude principles, providing an estimated sum that was smaller than one or both addends. Given that students struggle with arithmetic estimation, it is not surprising that they go on to have significant difficulties when asked to solve specific fraction arithmetic problems (e.g., Braithwaite et al., 2019; Gabriel et al., 2013; Siegler et al., 2011).

Concepts of operations with whole numbers can be applied to fractions. For example, repeated addition of fractions can be represented as multiplication, and division by a fraction can be modeled as the number of copies of the fraction that fit in to the dividend (Sidney & Alibali, 2015), although those outcomes can be different. Multiplication of whole numbers never leads to an answer smaller than either factor and division by a whole number never leads to an answer larger than the number being divided, but multiplication and division with fractions less than one always produce such outcomes. However, fraction arithmetic requires additional procedural knowledge (Gabriel et al., 2013; Hecht et al., 2007; Lortie-Forgues et al., 2015). For instance, procedures for fraction addition and subtraction vary depending on denominator equality (i.e., common versus uncommon) but remain invariant for whole numbers. Additionally, common denominators are required to operate across fractions for addition and subtraction, but not for multiplication or division. Students must also keep track of multiple components (i.e., numerator and denominator) and perform numerous steps (e.g., finding a common denominator, generating equivalent fractions, and simplifying) which increases cognitive load (Lortie-Forgues et al., 2015).

Many students with MLD show little improvement in their fraction arithmetic skills during the critical intermediate grades (Hansen et al., 2015). Even more concerning, a sizeable portion of students remain highly inaccurate on common denominator addition and subtraction problems in sixth grade (Gesuelli & Jordan, 2021), a basic fraction arithmetic skill that is typically learned by the end of fourth grade (Common Core State Standards in Mathematics (CCSSM); 2010). Siegler and Pyke (2013) found sixth grade students with low mathematics achievement overall possess poorer fraction arithmetic skills than their typically achieving peers; this gap widened by eighth grade with typically achieving students demonstrating gains in their fraction arithmetic skill while low-achieving students' skills remained stagnant.

14.1.4 Common and Persistent Fraction Arithmetic Errors

When operating with fractions, students with MLD make both strategy and execution errors (Bottge et al., 2014; Newton et al., 2014; Schumacher & Malone, 2017). Strategy errors reflect fraction misconceptions, whereas execution errors primarily entail computation errors related to poor number fact skills. Strategy errors include those associated with (a) whole number bias and (b) misapplication of fraction procedures. WNB errors involve the add/subtract across strategy (e.g., $\frac{1}{2} + \frac{3}{8} = \frac{4}{10}$) where the student treats the numerators and denominators of fractions as independent whole numbers. Students may also add the fraction numerators and denominator and then sum together to generate whole numbers (e.g., $\frac{3}{4} + \frac{5}{6} = 3 + 4 + 5 + 6 = 18$) (Bottge et al., 2014; Schumacher & Malone, 2017).

Students also overgeneralize a correct fraction procedure to other arithmetic operations (Braithwaite et al., 2017; Braithwaite et al., 2019; Kelly et al., 1990; Newton et al., 2014; Siegler et al., 2011; Siegler & Pyke, 2013). Examples include applying fraction multiplication procedures to addition and keeping a common denominator for multiplication (e.g., Newton et al., 2014). Students sometimes disregard the fraction arithmetic operation and instead focus on denominator equality when choosing a procedure (Braithwaite et al., 2017; Newton, 2008). Alternatively, they may also overgeneralize the use of the same procedure regardless of the fraction operation and denominator equality (Braithwaite et al., 2019). As a result, students may be highly accurate on certain problems and inaccurate on others. For example, students who consistently operate across the numerators and denominators of fractions were found to be accurate for multiplication problems but inaccurate on addition and subtraction (Braithwaite et al., 2019).

Additional systematic errors also exist for addition and subtraction with uncommon denominators that reflect students' partial understanding of fraction arithmetic (Bottge et al., 2014; Newton, 2008; Schumacher & Malone, 2017). The student may choose the correct common denominator but fail to generate equivalent fractions by adjusting the numerator (e.g., $\frac{1}{2}+\frac{2}{5}=\frac{1}{10}+\frac{2}{10}=\frac{3}{10}$; Bottge et al., 2014; Newton, 2008; Schumacher & Malone, 2017). These transitional errors suggest students are moving away from the overgeneralization of whole number properties and are beginning to solidify their ability to discriminate between certain portions of fraction procedures (Bottge et al., 2014; Schumacher & Malone, 2017).

Students also commit various execution errors where they use the correct fraction procedure but make calculation or equation errors (e.g., performing fraction subtraction instead of addition). Basic computation errors include making a whole number arithmetic error while employing the correct fraction procedure (e.g., $\frac{1}{2}+\frac{2}{5}=\frac{5}{10}+\frac{4}{10}=\frac{8}{10}$; Braithwaite et al., 2017; Newton, 2008; Siegler & Pyke, 2013). Students also make whole number calculation errors when working with mixed numbers or changing a mixed number to an improper fraction (e.g., Bottge et al., 2014; Newton et al., 2014). These errors are observed both independently and in combination with strategy errors. Nevertheless, most students' fraction arithmetic mistakes include strategy rather than calculation errors (e.g., Braithwaite et al., 2017; Hecht, 1998; Newton et al., 2014; Siegler et al., 2013), indicating fundamental fraction misconceptions.

14.1.5 Representations to Build Fraction Knowledge

Research on evidence-based instruction in intervention settings for students with mathematics difficulties suggests that *number line representations* are essential for preparing students for advanced mathematics (Fuchs et al., 2021). Number lines allow students to view the magnitude of fractions more directly than common part-whole "pie" models of fraction representation, which often are over-used for students with MLD. Although students should be exposed to different types of fraction models (e.g., linear, area, and set models), number line representations of fractions help students see that there are fractions between whole numbers and fractions can be less than, equal to, or greater than one (Resnick et al., 2016). Number lines also allow students to see linear relationships between equivalent fractions. For example, they can visualize equivalent values that have different partitions that $\frac{8}{8} = \frac{4}{4} = \frac{2}{2} = 1$ and that the same distance can have an infinite number of numerical representations. Concrete representations of length, such as fraction bars, can be mapped directly onto number line partitions to show, for example, that one half of a fraction is equivalent to two fourth bars and four eighth bars. It is east to see that the whole (from 0 to 1) partitioned into eight equal unit bars has smaller individual parts than when it is divided into 4 equal unit bars, thus making $\frac{1}{8}$ smaller than $\frac{1}{4}$.

14.2 Techniques That Support Learning Across Domains

In addition to considering domain specific fractions content, interventions for students with MLD should incorporate general techniques backed by evidence from cognitive science. These techniques include (a) using integrated models, (b) connecting concrete and abstract representations of concepts, (c) using gestures to promote learning, (d) distributing and interleaving practice, (e) providing retrieval practice with corrective feedback, and (f) presenting side by side comparisons to promote relational thinking.

14.2.1 Using Integrated Models

Attention splitting occurs when instructional materials present visual and verbal/textual information separately or spaced far apart (Sweller et al., 1998; Mayer, 2005). Using an integrated presentation (i.e., presented at the same time and near to each other) reduces attention splitting (Renkl & Scheiter, 2017) and decreases extraneous cognitive load (Ayres & Sweller, 2005). Direct relations between textual and visual information are highlighted by close proximity (Schroeder & Cencki, 2018). Both the visual and auditory channels of working memory are activated to process new information, so that neither channel is overloaded (Clark & Mayer, 2016). Supporting working memory is especially important when designing instruction for struggling learners, as students with MLD often have diminished working memory capacity (Fuchs et al., 2005; Geary, 2004; Swanson, 2011).

14.2.2 Connecting Concrete and Abstract Representations of Concepts

Mathematical concepts can be introduced through representations ranging from concrete (e.g., fraction blocks, pictures of objects) to abstract (e.g., fraction notation on the number line). Although interventions for students with MLD often rely heavily on concrete materials, integrating concrete and abstract representations of mathematical concepts positions students for problem solving with symbols (Pashler et al., 2007). Concreteness fading is a validated instructional practice that links concrete to abstract representation through a multi-step progression, in which concreteness is faded gradually into the abstract representation (Fyfe & Nathan, 2019).

Concreteness fading increases learning and transfer across age groups and mathematics topics (Braithwaite & Goldstone, 2015; Braithwaite & Siegler, 2021; Fyfe et al., 2015). Fyfe et al. (2015), for example, tested the effectiveness of a concreteness fading technique to teach students how to solve mathematical equivalence problems (e.g., 3 + 2 = 4 + __). Students in the concreteness fading condition were first introduced to the equivalence concept in a concrete scenario by making the two sides of a physical balance scale even. Next, a pictorial representation of the scale was presented and labeled with numerical symbols. Finally, students solved the equations with just the symbols but were asked to imagine the scale. Students in the concreteness fading condition correctly solved more transfer problems at posttest than those who were taught with concrete or abstract materials alone, as well as those who were taught using an inverse fading progression that moved from abstract to concrete.

14.2.3 Using Gestures to Promote Learning

When students and teachers discuss mathematics ideas, they often gesture with their hands to direct attention to learning materials, to emphasize their spoken words, or to convey additional information (Alibali et al., 2014). Encouraging students to gesture is a type of embodied cognition that externalizes thought and helps them retain new information (Alibali & Nathan, 2012; Ping & Goldin-Meadow, 2008). Learners produce more spontaneous gestures when task demands are high, in turn leading to better performance (Chu & Kita, 2011). Teachers' gestures also facilitate student learning (Alibali et al., 2014; Ping & Goldin-Meadow, 2008), especially when they simultaneously accompany speech (Congdon et al., 2017). Congdon et al. (2017) found that third graders who learned a problem-solving strategy through teacher instruction that included simultaneous speech and targeted gesture learned more material and showed greater transfer than those who learned through speech alone or speech and gesture presented sequentially.

14.2.4 Distributing and Interleaving Practice

Studying or practicing learned information across two or more sessions that are separated by time leads to better learning and retention than studying for the same amount of time in a single session (Dunlosky et al., 2013). This spacing effect has been reliably demonstrated in lab studies across populations of learners, from elementary to college-aged students (see Carpenter et al., 2012 for review) as well as studies conducted in the classroom (e.g., Powell et al., 2020; Schutte et al., 2015). For example, Schutte et al. (2015) split third graders into three groups to practice math facts fluency. Students completed either (a) four back-to-back administrations in one session; (b) two back-to-back administrations in the morning and two in the afternoon; or (c) four separate administrations throughout the day, spaced two hours apart. Students in the two distributed or spaced practice conditions grew more in their fact fluency across the administrations than those who practiced all at once. Distributed practice benefits learning more complex mathematical concepts (e.g., fraction arithmetic) as well as more simple material (Rohrer & Taylor, 2006, 2007).

Teachers often block mathematics practice problems by common problem-type or content matter (e.g., solving all fraction addition problems followed by all fraction subtraction problems). However, interleaving or mixing problem types is more effective than blocked practice alone for retention of skills over time (Dunlosky et al., 2013). Solving interleaved problems takes increased cognitive effort, as it forces students to think about the type of problem to be solved (Rohrer et al., 2014). An important caveat to the effectiveness of interleaved practice is that it may be better to use blocked practice when first introducing concepts and then employ interleaved concepts and procedures over time (Rau et al., 2010). Despite the effectiveness of interleaving practice, a recent analysis of six middle-school mathematics textbooks frequently used in the United States found that many textbooks rely mainly on blocked practice, and students typically only see an average of one or two interleaved problems per school day (Rohrer et al., 2020).

14.2.5 Providing Retrieval Practice with Corrective Feedback

Retrieval practice can be viewed as an instructional technique, rather than a means of passive assessment (see Rowland & DeLosh, 2014; Yang et al., 2021 for reviews). Students who spend learning time retrieving information (e.g., self-testing) learn and retain more than those who focus the same amount of time on studying information (e.g., highlighting information; Roediger & Karpicke, 2006; Roediger et al., 2011), a phenomenon known as the testing effect. Although there are multiple potential mechanisms behind the testing effect, one prevailing account posits that the effort involved in successfully retrieving information from memory boosts long-term retention by creating multiple retrieval routes (Roediger & Butler, 2011; Rowland & DeLosh, 2014).

Providing corrective feedback during retrieval practice enhances the benefits of the testing effect (Roediger & Butler, 2011). That is, students should provide the correct response so that retrieval practice does not solidify an incorrect response in long-term memory, especially when working with students with learning difficulties. Feedback can also focus on the process or *how* a student solves a task (e.g., using a strategy for determining equivalence), in addition to whether a solution or answer is incorrect (Hattie & Timperley, 2007).

14.2.6 Presenting Side by Side Comparisons to Promote Relational Thinking

The ability to see connections between related concepts (e.g., fraction addition vs. fraction multiplication) improves mathematical proficiency (Richland & Hansen, 2013). However, drawing connections is a large cognitive undertaking that requires instructional scaffolds (Richland et al., 2017). Presenting examples side by side encourages comparison of concepts and strategies. Specifically, comparing more than one example leads learners to create an analogy between the two instances, highlighting the shared relational structure by noting similarities as well as differences (Rittle-Johnson et al., 2020). When using analogy as a teaching tool, it is beneficial to use a familiar concept as the source analog to clarify the relationships of a less familiar target concept (Gray & Holyoak, 2021). For example, Rittle-Johnson (2009) found an interaction between prior knowledge and condition, such that comparing two problem-solving strategies side by side was more effective than studying each strategy separately when students were already familiar with one of the strategies. However, students can also learn from comparing two unfamiliar examples, if they are sufficiently supported (e.g., given more time and less material to cover; Rittle-Johnson et al., 2012). As children are highly susceptible to focusing on irrelevant perceptual features when making comparisons (Richland et al., 2006), it is critical for instructors to link explicitly the two analogs. This can be done through structured questioning, relational language, and gesturing between the examples (Vendetti et al., 2015).

14.3 Development of the FSI

We designed the FSI for middle-school students who struggle with fractions. In the following section, we discuss the efficacy of the FSI across different stages of development and highlight key intervention components.

14.3.1 Description of the FSI

The FSI anchors instruction within the meaningful narrative of a color run race for charity (Bottge et al., 2014). The racecourse incorporates a 0 to 3 number line and helps students see fractions as numbers that have a unique position on the number line, just as whole numbers do. Through a variety of activities, such as placing a drink station at each half mile, a color throw station at each fourth mile, and a marker at every eighth mile, students explore fraction magnitude, fraction equivalency, and ordering of fractions. The story line also provides a context for fraction word problems and helps students make sense of fraction arithmetic apart from procedures.

Consistent and precise mathematical language is used throughout FSI lessons (Fuchs et al., 2021; Karp et al., 2014). For example, using the term "simplifying" fractions rather than "reducing"; using the words "numerator" and "denominator" rather than the "top" and "bottom" when discussing fraction notation; and saying "one fourth" of the whole rather than "one out of four" to emphasize fractions as a number. In order to develop the concept of equivalency, the equal sign is read, "is the same amount as" or "is the same distance as."

When designing the FSI, we sought to *integrate verbal explanation with visual models* and placed both in close proximity on the slides to avoid splitting students' attention. Consistent mathematics vocabulary is paired with visual animations to emphasize the meaning of the word and help students generalize the associated concept across contexts and problems. Multiple visual models are used in the FSI to help students ignore irrelevant properties of the models and abstract the underlying fraction concept shared by all the models (Dienes, 1971). For example, students have a deeper understanding of what it means to find one-fourth by finding one-fourth using a racecourse, a number line, a fraction bar, a ruler, a measuring cup, and a group of marbles. Linear models are favored over part-whole (pie) models to help students think about fractions as numbers (Fuchs et al., 2021).

It is often difficult for teachers to model use of manipulatives and drawings when working in a classroom setting as opposed to small groups or one-on-one. FSI slides make it easy for teachers to model each of the three steps of *concreteness fading* (Fyfe & Nathan, 2019), and even to compare them side by side, as students move from concrete manipulatives, to drawings, and then to fraction symbols. For example, students are given a metal sheet with fraction bar magnets, which show three whole units and the corresponding number of halves, fourths, and eventually eighths (Fig. 14.1). The sheet also has a number line racecourse with a magnetic racer the student can use to show the racer's movement along the racecourse. The fraction bars are sized so that they can be placed on the racecourse to connect the bars to the number line. Students use fraction bars and the racecourse to develop fraction concepts and solve arithmetic word problems.

To address procedural difficulties found with fraction arithmetic in earlier iterations of the FSI (e.g., Barbieri et al., 2020), students now begin modeling the action in an arithmetic or word problem using concrete representations (fraction bars or the racecourse number line with the magnetic racer). Concrete models are faded to model drawings. Students "see" the solution in their model and do not need to rely on procedures. We piloted this approach with a group of students with MLD and produced promising results. Not only were students successful in solving both arithmetic and word problems, but they also intuitively simplified their answers (wrote $\frac{1}{2}$ instead of $\frac{2}{4}$) because the simplified answer was easily seen in the model. Using *concreteness fading* in this way helped to accomplish our goal of developing fraction sense in students rather than misapplying procedures based on whole number bias or other learned misunderstandings. After 24 lessons, students were

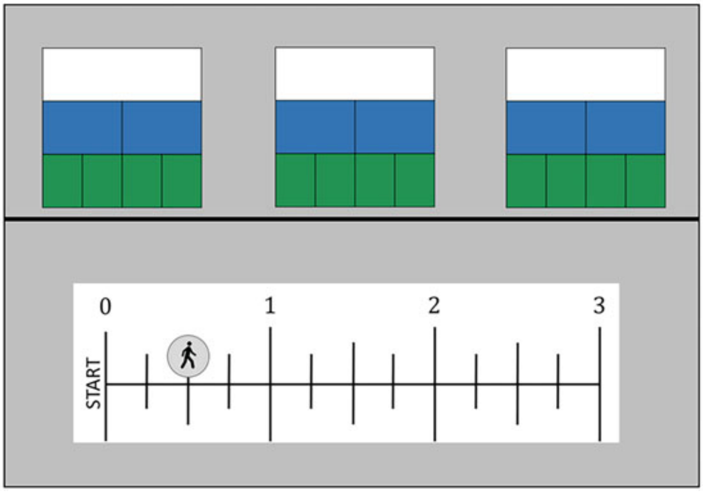

Fig. 14.1 Metal pan with concrete manipulatives: Fraction bar magnets and racecourse number line with racer magnet

beginning to picture the solution in their head and had dropped nonsensical solutions such as $2 + \frac{3}{4} = \frac{5}{6}$. Figure 14.2 shows pretest and posttest solutions for three simple arithmetic problems. At pretest, students misapplied erroneous versions of procedures, but at posttest, students were able to use drawings on paper along with fraction sense to solve these same problems correctly.

As noted previously, students with MLD often confuse operations and their associated procedures. In order to bring these confusions to the students' attention, we present these problems *side by side*. For example, using both the number line racecourse and fraction bars, problems such as $2 + \frac{3}{4}$ and $2 \times \frac{3}{4}$ are presented simultaneously, and the solutions placed side by side for comparison (Fig. 14.3). Attention is drawn to the role of "2″ in each of the problems. In the addition problem, it is a quantity that needs to be "joined" to $\frac{3}{4}$. In the multiplication problem, it is an operator that tells "how many groups" of $\frac{3}{4}$. Several such problems are presented sided by side to give students practice in noticing the difference.

The FSI incorporates *gestures* in three ways: gestures as animations on the slides, teacher gestures, and student gestures. Gestures are used to highlight important information, represent an action, or emphasize a mathematical concept. For example, while moving a racer along the number line racecourse, the animated slide produces curved arrows or "loops" on the racecourse to show that distance is the space from one point to the next and not a single point on the line (an important concept often misunderstood). The teacher mimics this gesture by moving her hand in a looping motion from point to point. Students mimic the gesture as they move their magnetic "racer" along their racecourse or draw loops on their workbook racecourse. Other gestures used repeatedly throughout the intervention are arrows that point to or circles that encircle important information, sweeping vertical bars that show

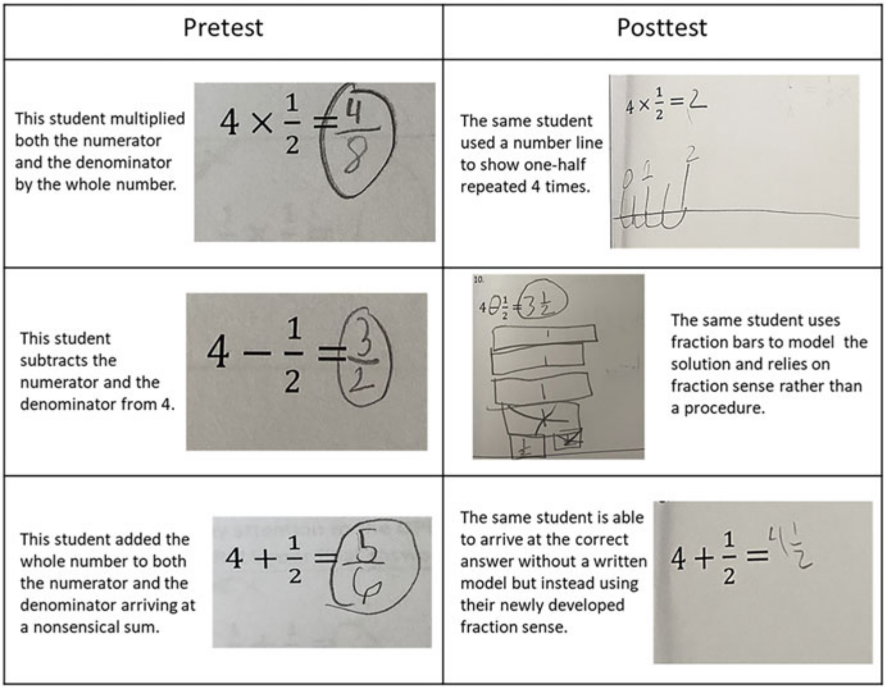

Fig. 14.2 Examples of student solution strategies for pre- and posttest arithmetic

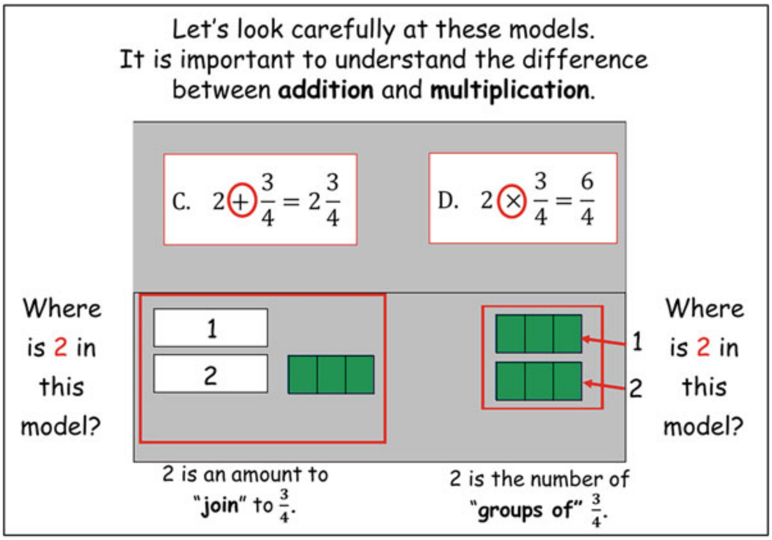

Fig. 14.3 Side by side solutions of similar addition and multiplication problems

equivalence of fractions on the number line, and rectangles that encircle quantities to show sums and products.

The structure of the FSI lessons provides many opportunities for practice. Each lesson begins with *retrieval practice* of multiplication facts to bolster multiplicative reasoning with fractions. The same problems are presented in various formats (i.e., missing the product, missing either of the two factors, product on the right, product on the left). The instructor scores the practice before the next lesson by circling incorrect numbers and writing the number correct. The number correct from the previous lesson has already been recorded on the top of the page allowing students to track their progress. Next, an activity reviews previously learned material to provide *distributed practice*, which activates prior knowledge needed. As students learn new fraction concepts and problem-solving strategies, they are given *process-oriented feedback* (e.g., saying, "Multiplying 3 times three-fourths means making three groups of three-fourths. It is not modeled as the whole number 3 and three-fourths."). Speeded games follow instruction in new concepts and strategies, giving students opportunities to solve problems quickly to develop fluency in equivalency and simple arithmetic operations. Fraction operations are *interleaved* so students must focus their attention and quickly retrieve the correct procedure for the operation (e.g., $2 + \frac{1}{2}\ vs. 2 \times \frac{1}{2}$). Each lesson concludes with a quick written assessment, which uses both distributed and interleaved practice to make learning last and help instructors monitor progress.

14.3.2 *Efficacy of the FSI*

We created and refined the 24-lesson FSI over multiple iterations. The result is a series of lessons that can be implemented by intervention teachers, who may not have training in mathematics difficulties. Table 14.1 summarizes effects (*g*) from a series of randomized studies. All trials were randomized at the student level and used a pre/posttest/delayed posttest design.

In our initial study (Dyson et al., 2018), we examined the effectiveness of the FSI when presented in small groups of children and carried out by trained researchers. Students were randomly assigned to the FSI or a business-as-usual intervention control (which often consisted of computer-based instruction). FSI delivery was supplemental to students' general mathematics instruction. The FSI group demonstrated significantly more learning than the control group from pretest to posttest, with meaningful effect sizes on all fraction measures. The intervention also brought students to a performance level close to average-achieving sixth graders on tasks that required them to estimate the locations of different fractions on the number line. Jayanthi et al. (2021), who tested the efficacy of a similar number-line focused fractions intervention for fifth graders – also carried out in small groups of children – found comparable effects (although their intervention was twice as long and did not examine performance at delayed posttest).

Table 14.1 Summary of Finding from Randomized Studies Assessing the Efficacy of the FSI

Study	Sample	Instructional format	Fraction measures	Effect Size post test	Effect size delayed post test
Dyson et al. (2018)	52 sixth graders screened for participation on a validated fraction screener	Small groups of four; carried out by researcher teachers	Concepts number line arithmetic	.99	.63
				.90	1.02
				.48	.35
Barbieri et al. (2020)	51 sixth graders screened for participation on a validated fraction screener	Small groups of four; carried out by researcher teachers	Concepts number line comparisons arithmetic	1.09	.66
				.85	.60
				.82	.61
				.17	.11
Dyson & Jordan (2019)	81 sixth graders screened for participation on a validated fraction screener	Large groups of 12–15; carried out by researcher teachers	Concepts number line comparisons arithmetic	.68	.58
				.42	.53
				.52	.62
				.13	.11

In a second study (Barbieri et al., 2020), again with small groups of students and researcher instructors, we examined a more impaired population of students than in our initial study (i.e., we used a more stringent screening cut point for participation based on a lower-performing pool of students overall). Nevertheless, both at posttest and delayed posttest, there were medium to large effect sizes for all fraction measures except fraction arithmetic where effect sizes were low and insignificant.

The third study evaluated the effectiveness of the FSI with larger groups of students (Dyson & Jordan, 2019) taught by researcher instructors, still using the more stringent screening cutoff. The format for presenting the lesson visuals was changed from a small tabletop magnetic whiteboard easel to PowerPoint slides with animations. Middle school mathematics interventions in the U.S., unlike those for younger students, are often delivered in classes of 12–16 students. Although effect sizes for this study were smaller due to the increased student/teacher ratio, findings were similar to our second study. Analysis of student arithmetic errors and strategies showed that experimental students in both studies continued to misapply procedures (errors such as $2 \times \frac{3}{4} = \frac{11}{4}$, where the student multiplied 2 times 4 and then added 3 to find the numerator- the procedure for changing the mixed number $2\frac{3}{4}$ to a fraction).

Moving forward, we increased FSI emphasis on fraction arithmetic and refined our animated slides to allow implementation in both smaller and larger groups of students. Another advantage is that slides allow teachers to carry out the FSI remotely for distance education, in the event of school closures or other special circumstances. A tryout of the slide-based FSI with teachers (rather than researcher instructors) produced encouraging results. School intervention teachers remarked how the on-slide scripts and consistent approaches across lessons reduced preparation time. Even teachers with little training in mathematics difficulties carried out the lessons accurately with minimal support from our research team.

14.4 Concluding Remarks

Many students with MLD struggle through successive years of fractions instruction that does not meet their needs. After sixth grade, students' chance of succeeding with more advanced mathematics topics diminishes (Siegler & Pyke, 2013). As a result, students may develop math anxiety or learn to avoid math altogether (Choe et al., 2019). Evidence-based interventions based on linear representations and grounded in techniques from cognitive science help under-achieving students master fundamental yet elusive fraction concepts and procedures, which in turn, will help redirect their math trajectories in school and improve their vocational opportunities. The FSI, which was described in this chapter, has the added benefit of highlighting to teachers' ways in which principles from cognitive science can be incorporated directly into mathematics instruction to improve understanding and retention. With a grant from the Institute of Education Sciences of the U.S. Department of Education, we are currently testing the efficacy of the FSI carried out by classroom teachers (as opposed to researcher instructors) with a large number of classrooms.

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Index

K. M. Robinson et al. (eds.), *Mathematical Cognition and Understanding*,
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F

G

H

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