Monetary Utility Functions and Risk Functionals

Christos Floros, Konstantinos Gkillas, and Christos Kountzakis

Abstract This paper's content is devoted to the study of the monetary utility functions and their use in optimal portfolio choice and optimal risk allocation. In most of the relative papers, the domain of a monetary utility function is a dual space. This approach implies that closed and convex sets are weak-star compact. The main contribution of the present paper is the definition of such a function on any Riesz space, which is not necessarily a dual space, but it formulates a symmetric Riesz dual pair together with its topological dual. This way of definition implies the weak compactness of the sets usually needed for the solution of the above optimization problems.

Keywords Monetary risk measures · Risk functionals · Premium calculation principles · Monetary utility functions · Optimal portfolio choice · Optimal risk allocation

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C. Floros (✉) · K. Gkillas
Department of Accounting and Finance, Hellenic Mediterranean University, Herakleion, Crete, Greece
e-mail: cfloros@hmu.gr; gillask@hmu.gr

C. Kountzakis
Department of Statistics and Actuarial-Financial Mathematics, University of the Aegean, Karlovassi, Samos, Greece
e-mail: chr_koun@aegean.gr

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1 Monetary Utility Functions and Risk Metrics

We obtain the following definition of a monetary utility function:

Definition 1.1 A finite-valued function $U : L^1(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ is called **monetary utility function** if it enjoys the following properties:

(1) $U(X) \geq U(Y)$, if $X(\omega) \geq Y(\omega)$, $\mathbb{P}-$ a.e. (Monotonicity)
(2) $U(t \cdot X + (1-t) \cdot Y) \geq tU(X) + (1-t)U(Y)$, for any $t \in [0, 1]$, where $\cdot$ denotes the usual scalar product (Concavity)
(3) $U(X + m \cdot \mathbf{1}) = U(X) + m$, where $\mathbf{1}(\omega) = 1$, $\mathbb{P}$-a.e. (Cash Invarianve)

A value of some Monetary Utility Function corresponds to an amount of capital, alike in the case of the Principles of Premium Calculation in insurance.

The above definition of a monetary utility function is obtained from Jouini et al. (2007) in $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, where the optimal risk sharing problem is studied. Equilibrium pricing under monetary utility functions is studied in Filipoviĉ and Kupper (2008) in $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ as well. As it is well-known, coherent risk measures are established in Artzner et al. (1999) and convex risk measures in Föllmer and Schied (2002). The main contribution of this paper is that convex and coherent risk measures may be replaced by monetary utility functions and vice versa, under the properties of equivalence defined below. Optimal portfolio selection is the main application of the monetary utility function. Another use of monetary utility functions is that their continuity provides that the optimal risk allocation problem has a non-empty solution. The optimal risk allocation problem is initially studied in Borch (1962). Recent works on the same theme are Kiesel and Rüschendorf (2009), Righi and Moresco (2022). We also provide a way to produce monetary utility functions and corresponding monetary risk measures by Young functions. In general, we notice that a monetary convex risk measure ρ implies the definition of a monetary utility function $u = -\rho$. On the other hand, a monetary utility function u implies the definition of a monetary convex risk measure $\rho = -u$.

2 Risk Functionals and Their Equivalence

Definition 2.1 A **risk measure**, with respect to a nonatomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is some $\rho : L^0 \times \mathcal{F} \to \mathbb{R}$, such that $\rho(X, A) = \rho(X^{-1}(A))$.

Definition 2.2 A **risk functional**, with respect to the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is some $f : \mathbb{P} \times L^0 \times \mathcal{F} \to \mathbb{R}$, such that $f(\mathbb{P}, X, A) = \mathbb{P}(X^{-1}(A))$.

Remark 2.3 A **law-invariant** risk measure is a risk functional. We recall that a risk measure ρ is law invariant if $\mathbb{P}_X = \mathbb{P}_Y$ implies that $\rho(X) = \rho(Y)$, where $\mathbb{P}_Z$ is the distribution probability measure of Z. A monetary risk measure corresponds to the notion of **regulatory capital**. A risk functional which is not a risk measure is value at risk (VaR). Hence, the notion of risk functional is a generalization of the notion of risk measure.

Definition 2.4 Two risk functionals f_i, f_j are called **equivalent**, and we write $f_i \sim f_j$, if for some strictly positive $M_i, M_j \in \mathbb{R}$ we have $M_i f_j \leq f_i \leq M_j f_j$,. A risk functional is called **nontrivial** if it is not equal to the zero function on $\mathcal{F}$.

Proposition 2.5 *The equivalence of risk functionals is actually an equivalence relation in terms of set theory. It is reflexive, symmetric, and transitive.*

Proof If $f_1 \sim f_2$, obviously $f_1 \sim f_1$. If $f_1 \sim f_2$, then $f2 \sim f_1$. Finally, if $f_1 \sim f_2$ and $f_2 \sim f_3$, then $f_1 \sim f_3$. $f_i, i = 1, 2, 3$ are risk functionals according to the above definition. □

We notice that:

Lemma 2.6 *Value at risk and expected shortfall are not equivalent.*

Proof As it is well -known, $ES_a(X) = -\frac{1}{a}\int_0^a VaR_u(X)du$, for any level of significance $a \in (0, 1)$ and any $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. □

Proposition 2.7 *Let $f_i, f_j : \mathcal{F} \to \mathbb{R}$ be two risk functionals which are nontrivial and $f_i \sim f_j$. Moreover, let f_i be coherent. Then, f_j is coherent as well.*

Proof Direct from the properties of coherent risk measures. □

Proposition 2.8 *Let $f_i, f_j : \mathcal{F} \to \mathbb{R}$ be two risk functionals which are nontrivial and $f_i \sim f_j$. Moreover, let f_i be convex. Then, f_j is convex as well.*

Proof Direct from the properties of convex risk measures. □

Another proof of the non-coherence of value at risk is the following one.

Corollary 2.9 *Value at risk is a noncoherent risk functional.*

Proof Direct, from the above proposition and $ES_a(X) = -\frac{1}{a}\int_0^a VaR_u(X)du$, for any level of significance $a \in (0, 1)$ and any $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. □

An example of a premium principle, which does not satisfy the properties of a coherent risk measure, is the **Exponential Principle of Premium Calculation**:

$$P_b(X) := \frac{1}{b}\log \mathbb{E}(e^{bX}), \tag{2.1}$$

for any strictly positive $b \in \mathbb{R}$.

The subset of those $X \in L^0$ in which $\mathbb{E}(e^{bX})$ is not equal to infinity is related to the Orlicz spaces, mentioned below.

Definition 2.10 The parameter b is called **risk aversion coefficient**.

Proposition 2.11 *The monetary utility function $-P_b$ arising from the Exponential Principle of Premium Calculation P_b satisfies the properties of a coherent risk measure, except positive homogeneity.*

Proof First we do prove that $-P_b$ does not satisfy the positive homogeneity: if $t > 0$ is a positive, nonzero real number, then

$$P_b(t \cdot X) = \frac{1}{b} log\mathbb{E}(e^{b(tX)}) = \frac{1}{b} log\mathbb{E}(e^{bt}e^{bX}) = P_b(X) + t,$$

where $\cdot$ denotes the scalar product:

(i) (Translation Invariance): $P_b(X + c\mathbf{1}) = \frac{1}{b}\log\mathbb{E}(e^{b(X+c\mathbf{1})}) = \frac{1}{b}(\log(e^{bc}) + P_b(X) = c + P_b(X)$, for any $c \in \mathbb{R}$. $-P_b$ satisfies the translation invariance property.

(ii) (Monotonicity): If $X \geqslant Y$, $\mathbb{P}$-a.s., then $e^{bX} \geqslant e^{bY}$, $\mathbb{P}$-a.s. This implies $\mathbb{E}(e^{bX}) \geqslant \mathbb{E}(e^{bY})$ and consequently $P_b(X) \geqslant P_b(Y)$.

(iii) (Subadditivity):

$$\frac{1}{b}\log\mathbb{E}(e^{b(X+Y)}) \geqslant \frac{1}{b}\log\mathbb{E}(e^{bX}), \frac{1}{b}\log\mathbb{E}(e^{bY}),$$

hence

$$\frac{1}{b}\log\mathbb{E}(e^{b(X+Y)}) \geqslant \max\left\{\frac{1}{b}\log\mathbb{E}(e^{bX}), \frac{1}{b}\log\mathbb{E}(e^{bY})\right\},$$

namely,

$$P_b(X + Y) \geqslant \max\{P_b(X), P_b(X)\}.$$

Hence, $P_b(X + Y) \geqslant -\min\{-P_b(X), -P_b(X)\}$, and consequently $-U_b(X + Y) \geqslant -\min\{U_b(X), U_b(X)\}$, which implies $\min\{U_b(X), U_b(X)\} \leqslant U_b(X + Y)$. Finally, we get that $U_b(X + Y) \leqslant U_b(X) + U_b(Y)$. □

The last inequality in the above theorem relies on the following:

Lemma 2.12 $P_b(X) \geqslant \mathbb{E}(X)$, *for any* $X \in L^1_+$. *Thus, for any* $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ *taking almost everywhere positive values. For such a* X, $P_b(X) \geq 0$.

Proof It suffices to prove that $\frac{1}{b}\log\mathbb{E}(e^{bX}) \geqslant \mathbb{E}(X)$. From Jensen's inequality, we get that $e^{b\mathbb{E}(X)} \leqslant \mathbb{E}(e^{bX})$. Hence, $b\mathbb{E}(X) \leqslant \log\mathbb{E}(e^{bX})$. □

2.1 The Case of Conditional Value at Risk

As it is well-known expected shortfall ES_a is Conditional Value -at- Risk $CVaR_a$ are equal for any real-valued random variable $X \in L^0(\Omega, \mathcal{F}, \mathbb{P})$, and for any $a \in (0, 1)$. This is true if cumulative distribution function F_X is continuous, except a set $A_a(X) \in \mathcal{B}[0, 1]$, such that $\lambda(A_a(X)) = 0$. $\mathcal{B}[0, 1]]$ denotes the σ-algebra of Borel sets in $[0, 1]$. λ is the Lebesgue measure on $[0, 1]$.

3 Monetary Utility Functions and Equilibrium

Monetary utility functions' impact on investors' decisions may be summarized in terms of "best" portfolio choice for a single investor. That's because the essential problem for any investor is to determine the set of portfolios, which maximizes her monetary utility function U defined on $L^1(\Omega, \mathcal{F}, \mathbb{P})$. $\mathbf{1}$ is the ranodm variable, such that $\mathbf{1}(\omega) = 1$, $\mathbb{P}$ -a.e. Since the order interval $[-e\mathbf{1}, e\mathbf{1}]$ is weakly compact and convex set of $L^1(\Omega, \mathcal{F}, \mathbb{P})$, then $B(p, e, w) = \{X \in L^1_+ | p(X) = w, X \in [-e\mathbf{1}, e\mathbf{1}]\}$ is a weakly compact and convex set. $p \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, such that $p(\omega) > 0$, $\mathbb{P}$ a.e. and $w > 0$ is the cash wealth of the investor.

Then, for any monetary utility function $U : L^1(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$, we obtain the following.

Theorem 3.1 *The problem of maximization of a monetary utility function U over $B(p, e, w)$ has a solution if U is weakly continuous.*

Proof $< L^1(\Omega, \mathcal{F}, \mathbb{P}), L^\infty(\Omega, \mathcal{F}, \mathbb{P}) >$ is a symmetric Riesz pair. Hence $[-e\mathbf{1}, e\mathbf{1}]$ is a weakly compact and convex set. The conclusion arises from the Bauer maximization principle. □

Hence, the Marshallian demand correspondence is well-defined for any investor whose monetary utility function U is convex and weakly continuous. This is a result of special importance if markets are **incomplete**, or else the portfolio payoffs lie in a nontrivial and weakly closed subspace M of $L^1(\Omega, \mathcal{F}, \mathbb{P})$.

4 Optimal Risk Allocations

Monetary utility functions are also related to problems of collective minimization of regulatory capital. We consider a set $\{1, 2, ..., I\}$ consisted of regulators or financial institutions. Risk functionals arise in the problems related to the inf -convolution, which is actually the value functional of the following optimization problem:

$$\inf \left\{ \sum_{i=1}^{I} r_i \rho_i(X_i) \,\middle|\, \sum_{i=1}^{I} X_i = X \in L^p,\ X_i \in L^p \right\}.$$

$r_i > 0$ for any $i = 1, 2, ..., I$ such that $\sum_{i=1}^{I} r_i = 1$, and ρ_i is some risk measure defined on $L^p := L^p(\Omega, \mathcal{F}, \mathbb{P})$ for $p \geq 1$ and $p < \infty$. r_i for any $i = 1, ..., I$ denotes the market power of each $i = 1, ..., I$. Since the optimal risk allocations are related to some class of utility functions, we may consider the class of monetary utility functions. A monetary utility function, which arises from a monetary risk measure $\rho : L^0 \to \mathbb{R}$, is the function $u = -\rho$. On the other hand, a utility function u implies a monetary risk measure $\rho = -u$.

These spaces are in general L^p spaces on a nonatomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $1 \leq p < \infty$. A unified result is the following one:

Proposition 4.1 *The above inf -convolution is well-defined on the symmetric Riesz pair, if ρ_i is weakly continuous, for any $i = 1, ..., I$.*

Proof The conclusion arises from Bauer maximization principle. □

The case of $p = 1$ is of special interest since the probability distributions of the heavy-tailed random variables lie in this one Lebesgue space. We recall that a heavy-tailed random variable is any element $X \in L^0(\Omega, \mathcal{F}, \mathbb{P})$ whose exponential moments $\mathbb{E}(e^{rX}) = +\infty$ for any positive, nonzero real number r. In order to make things more simple, we assume that $X(\omega) \geq 0$, $\mathbb{P}$-a.e.

5 Creating Monetary Utility Functions

As we did notice above, Jensen's inequality implies that for any convex and finite -valued function $C : \mathbb{R} \to \mathbb{R}$:

$$C(\mathbb{E}(X)) \leq \mathbb{E}(C(X)),$$

namely, convex functions imply a way to establish monetary utility functions, whose form is actually an expected utility form. It suffices to assume that $\mathbb{E}(C(X))$ is finite for a subset of L^0. A large class of convex functions is the one of Young functions.

We call *Young function* any convex, even, continuous function Φ satisfying the relations $\Phi(0) = 0$, $\Phi(-x) = \Phi(x) \geq 0$ for any $x \in \mathbb{R}$ and

$$\lim_{x \to \infty} \Phi(x) = \infty\,.$$

The *conjugate function* of Φ is defined by

$$\Psi(y) = \sup_{x \geq 0}\{xy - \Phi(x)\}\,, \qquad \forall\, y \geq 0\,.$$

Definition 5.1 An N-Young function is a Young function Φ defined on $\mathbb{R}$, which satisfies the conditions:

(1)

$$\lim_{x \to 0} \frac{\Phi(x)}{x} = 0\,,$$

(2)

$$\lim_{x \to \infty} \frac{\Phi(x)}{x} = \infty \, ,$$

(3) If $\Phi(x) = 0$, then $x = 0$.

Definition 5.2 We say that a Young function Φ satisfies the Δ_2-property if there exist a constant $k > 0$ and a $x_0 \in \mathbb{R}$ such that holds

$$\Phi(2x) \leq k\Phi(x) \, , \quad \forall \, x \geq x_0 \, .$$

Let us mention some examples of Young functions: $\Phi_0(x) = |x|$ is a Young function. $\Phi_1 = \frac{1}{2}|x|^2$ is a Young function, which satisfies both N and Δ_2 properties. If we would like to specify some Young function which is not of the type of $\Phi_p(x) = \frac{1}{p}|x|^p$, $p > 1$ and satisfies both N properties and Δ_2 properties, then we may mention $\Phi_\ell(x) = (1 + |x|)log(1 + |x|) - |x|$. About the class ∇_2 of Young functions, see (Rao and Ren, 1991, p. 22): a Young function Φ is a ∇_2 -Young function if

$$\Phi(x) \leq \frac{1}{2g}\Phi(x), x \geq x_0 > 0$$

for some $g > 1$. x_0 may be equal to zero. An example of ∇_2 Young function is the conjugate of Φ_ℓ, which is the function $\Psi(x) = e^{|x|} - |x| - 1$.

The book Rao and Ren (1991) is devoted to a complete study on Young functions and Orlicz spaces.

Thus, the monetary utility function implied by some Young function Φ is the following one $\phi : L^0 \to \mathbb{R}$, where $\phi(X) := -\mathbb{E}(\Phi(X))$. Any monetary utility function defined by the way shown above is a **Young monetary utility function**.

In Rao and Ren (1991), the (sub) -set of $X \in L^0(\Omega, \mathcal{F}, \mathbb{P})$ such that $(E)(U(X)) < +\infty$ if $-U$ is a Young function is called *Orlicz Heart* M_U. M_U is in general a convex subset of $L^0(\Omega, \mathcal{F}, \mathbb{P})$. Monetary risk measures defined on Orlicz hearts and Orlicz spaces are initially studied in Cheridito and Li (2009).

6 Analysis Notions and Results Used in the Paper

We add this section in order to make the content of the paper understood in a better manner. The partially ordering implied by some cone K on the vector space E is defined in the following way: $x \geq y \leftrightarrow x - y \in K$. A more detailed study of partially ordered linear spaces and all of the content of this section is obtained from Aliprantis and Border (2006). A non-empty subset K of a vector space E, such that $K + K \subseteq K$, $tK \subset K$ for any $t \in \mathbb{R}_+$ and $K \cap (-K) = \{0\}$ is a cone. Any set of the form $[a, b] = (a + K) \cap (b - K)$, where $a, b \in E$ is an order-interval with respect to the cone K.

The set of upper bounds of $a \in E$, with respect to the cone K, is the set $a + K$. Lower bound of $b \in E$ with respect to the cone K is the set $b - K$. A partially ordered vector space E is a Riesz space (or else a vector lattice) if $\sup\{x, y\} = x \vee y \in E$ and $\inf\{x, y\} = x \wedge y \in E$, where supremum and infimum are the minimum upper bound and the maximum lower bound of $\{x, y\}$, respectively (with respect to the cone K). In such a case, the absolute value of any $x \in E$ is equal to $x \vee (-x) = |x|$ alike in the case of real numbers. The space of all real-valued linear functionals defined on some vector space E is called algebraic dual space of E. A linear functional defined on some partially ordered space E, such that the cone K implying the partially ordering is the cone K, is called order-bounded if it actually maps an order-interval $[a, b]$ to a closed interval of the real numbers. The vector space of all the order-bounded linear functionals of the partially ordered linear space E is called order dual. We denote the order dual of E by E'. An ideal of some Riesz space is any subspace S of E, such that if $|x| \geq |y|$ and $x \in S$, implies that $y \in S$. A dual pair $< E, E^* >$ is called Riesz pair if both E, E^* are Riesz spaces and E^* is an ideal of the order dual E' of E. A dual pair $< E, E^* >$ is called symmetric Riesz pair, if and only if $< E^*, E >$ is a Riesz Pair as well. If $< E, E^* >$ is a symmetric Riesz pair, then the non-empty order intervals of E are weakly compact. The set F of maximizers of some weakly continuous function f is non-empty if the domain of it is some weakly compact set C of E. Moreover, F actually it is an extreme set of C. An extreme set of some convex set is any subset A of it; then every element of $z \in A$, such that $z = tx + (1-t)y \in A$, where $t \in (0, 1)$; then $x, y \in A$. An extreme point is any extremal set consisted of a singleton. This is a proof of Bauer maximization principle refers to the maximization of semicontinuous functions: if C is a compact convex subset C of a locally convex Hausdorff space, then every upper semicontinuous convex function on C has a maximum point that is an extreme point of it. The analog of the above theorem is valid for the minimization of a concave function, which is weakly continuous. The topology under use here is the weak topology over a Riesz pair $< E, E^* >$ as well.

7 Further Research

Further research may be related to the functional form of the efficiency frontiers or the demand functions under different classes of concave functions. This study relies on the equivalence structure for risk functionals as it is defined here.

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