Chapter 6
Equilibrium States of Mean-Field Models and Bogoliubov's Approximation Method

6.1 Topological Framework

Recall that $\mathcal{P}_f$ is the set of all finite subsets of the (cubic) lattice $\Gamma \doteq \mathbb{Z}^d$, for some (space dimension) $d \in \mathbb{N}$. Fix once and for all $N \in \mathbb{N}$ (spin number for quantum spins) or a finite set Ω (the spin set for fermions). These parameters define two different (separable, unital) C^*-algebras, $\mathrm{Spin}(N, \Gamma)$ and $\mathrm{CAR}(\Omega, \Gamma)$, which are always denoted by $\mathcal{U}$, as explained above. In the fermion case, one has additionally to consider the even (CAR) C^*-subalgebra $\mathrm{CAR}(\Omega, \Gamma)^e$, which is denoted by $\mathcal{U}^{\mathrm{e}}$. If one considers the quantum spin case, $\mathcal{U}^{\mathrm{e}}$ is just the original algebra, i.e., $\mathcal{U}^{\mathrm{e}} \doteq \mathcal{U} = \mathrm{Spin}(N, \Gamma)$. Recall that Sect. 5.1 presents the notation in more detail.

For simplicity of notation, as there is no risk of confusion with other objects, the (topological) dual space $\mathcal{U}^{\mathrm{td}}$ of $\mathcal{U}$ is denoted here by $\mathcal{U}^*$, as is usual. The space $\mathcal{U}^*$ is a Banach space when it is endowed with the usual norm for linear functionals on a normed space, that is,

$$\|\rho\|_{\mathrm{op}} \doteq \sup_{A \in \mathcal{U}} \frac{|\rho(A)|}{\|A\|}$$

for all continuous linear functionals $\rho \in \mathcal{U}^*$. However, the norm topology is too strong in practice. The natural topology in the study of infinite systems is given by the $\sigma(\mathcal{U}^*, \mathcal{U})$-topology, usually called the weak* topology of $\mathcal{U}^*$. It is the initial topology of the family of linear mappings $\rho \mapsto \rho(A)$ from $\mathcal{U}^*$ to $\mathbb{C}$ for all algebra elements $A \in \mathcal{U}$. It is, by definition, the coarsest topology on $\mathcal{U}^*$ that makes the mapping $\rho \mapsto \rho(A)$ continuous for every $A \in \mathcal{U}$. See [18, Section 3.8]. The topology of the dual space $\mathcal{U}^*$ is, by default, the weak* topology. In this case, $\mathcal{U}^*$ is a (Hausdorff) locally convex space, and its (topological) dual space is $\mathcal{U}$: Any element of $\mathcal{U}^{**} \equiv \mathcal{U}$ is of the form $\rho \mapsto \rho(A)$ for some algebra element $A \in \mathcal{U}$. See, e.g., [18, Theorem 3.10]. In fact, recall that in Sect. 4.5.1, we define the weak*

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topology for *states* of any separable C^*-algebra (like $\mathcal{U}$) in a more concrete way, via an explicit metric. See Definition 4.80 and Exercise 4.82. It is important to notice that this metric does not reproduce the weak* topology in the whole space $\mathcal{U}^*$, but only in its norm-bounded subsets, like any set of states of $\mathcal{U}$.

The convex subset of invariant states on $\mathcal{U}$ is denoted by $E_1 \subseteq \mathcal{U}^*$. See again Sect. 5.1 for more details. One easily verifies that E_1 is a weak*-closed set. In addition, recall that any continuous linear functional $\rho \in \mathcal{U}^*$ is a state iff $\rho(\mathfrak{1}) = 1$ and $\|\rho\|_{\mathrm{op}} = 1$, $\mathfrak{1} \in \mathcal{U}$ being the unit of $\mathcal{U}$. Hence, from the Banach-Alaoglu theorem [18, Theorem 3.15] and the closedness of E_1, the set E_1 of invariant states is a weak*-compact subset of the unit ball of $\mathcal{U}^*$. See Proposition 4.84 for a direct proof of compactness of the set of states of separable unital algebras, keeping in mind that the (spin or fermion) algebra $\mathcal{U}$ is of this type.

Proposition 7.334 tells us then that the convex weak*-compact space E_1 of invariant states is the weak* closure of the convex hull of the (nonempty) set $\mathcal{E}_1$ of its extreme points:

$$E_1 = \overline{\mathrm{co}\mathcal{E}_1} \ .$$

The set $\mathcal{E}_1 \subseteq E_1 \subseteq \mathcal{U}^*$ also refers in the literature to the extreme boundary of E_1. Here, recall that extreme points of the convex set E_1 are called here *ergodic*, because of the formal analogy to the classical case.

As discussed above, since the (spin or fermion) algebra $\mathcal{U}$ is separable, the weak* topology is metrizable on any weak*-compact subset of $\mathcal{U}^*$. See, e.g., Proposition 4.84 or [18, Theorem 3.16]. In particular, the space E_1 is metrizable, in this case. This is an important property, which strongly simplifies the study of E_1, in particular because it allows for Choquet decompositions of invariant states as barycenters of ergodic ones.

Nonetheless, in spite of the metrizability of the weak* topology in E_1, the space E_1 of all invariant states has still a *fairly complicated* geometrical structure: In 1961, E. T. Poulsen [16] constructed an example of a metrizable simplex with dense set of extreme points. This simplex is now known as *the* Poulsen simplex because it is unique [17, Theorem 2.3], up to an affine homeomorphism. One can show that the set E_1 of invariant states is also a simplex and, in fact, the Poulsen simplex. In particular, the following assertion holds true:

Theorem 6.1 (Density of Ergodic States) *The set $\mathcal{E}_1$ of ergodic states is a weak*-dense subset of the set E_1 of all invariant states.*

Proof Recall that, for all $n \in \mathbb{N}$,

$$\Lambda_n \doteq \{(x_1, \ldots, x_d) \in \Gamma : |x_i| \leq n\} \in \mathcal{P}_f \ .$$

For any invariant state $\rho \in E_1$ and $n \in \mathbb{N}$, let $\tilde{\rho}_n$ be the product state defined by

$$\tilde{\rho}_n = \bigotimes_{x \in \mathbb{Z}^d} \rho|_{\mathcal{U}_{\Lambda_n + (2n+1)x}} \ .$$

It is a periodic state, whose period is $(2n+1, \ldots, 2n+1) \in \mathbb{Z}^d$, and satisfies

$$\tilde{\rho}_n(A) \doteq \rho(A) , \qquad A \in \mathcal{U}_{\Lambda_n} .$$

Here, $\mathcal{U}_\Lambda \doteq \mathrm{Spin}(N, \Lambda)$ (quantum spin case) or $\mathcal{U}_\Lambda \doteq \mathrm{CAR}(\Omega, \Lambda)$ (fermion case) for any $\Lambda \in \mathcal{P}_f$ (see Sect. 5.1). Note that this construction is possible also in the fermion case because any invariant state $\rho \in E_1$ is even, thanks to Theorem 5.3. See Proposition 4.193. Then,

$$\hat{\rho}_n \doteq \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} \tilde{\rho}_n \circ \tau_x \tag{6.1}$$

is a well-defined invariant state, where $\tau_x : \mathcal{U} \to \mathcal{U}$, $x \in \Gamma$, are the translation automorphisms, that is, the unique unital $*$-homomorphisms defined by (5.4), in the quantum spin case, and (5.9), in the fermion case. Fix $A \in \mathcal{U}_{\mathrm{loc}}$, where we recall that $\mathcal{U}_{\mathrm{loc}} \subseteq \mathcal{U}$ is the $*$-algebra of local elements, defined as the (countable) union of $\mathcal{U}_\Lambda$ for all $\Lambda \in \mathcal{P}_f$ (see again Sect. 5.1). Then,

$$\lim_{n \to \infty} \hat{\rho}_n(A) = \rho(A)$$

and so, $\hat{\rho}_n$ converges in the weak* topology to the invariant state $\rho \in E_1$, by density of the $*$-algebra $\mathcal{U}_{\mathrm{loc}} \subseteq \mathcal{U}$ of local elements. Moreover, $\tilde{\rho}_n$ being a product state, there is a constant $C > 0$ (depending on $\Lambda \in \mathcal{P}_f$) such that

$$\tilde{\rho}_n\left(\tau_x(A^*)\tau_y(A)\right) = \tilde{\rho}_n\left(\tau_x(A^*)\right)\tilde{\rho}_n\left(\tau_y(A)\right) ,$$

whenever $|x - y| \geq C$. Then, using the notation $|B|^2 = B^* B$,

$$\begin{aligned}
\tilde{\rho}_n(|A_\ell|^2) &= \frac{1}{|\Lambda_\ell|^2} \sum_{x \in \Lambda_\ell} \sum_{y \in \Lambda_\ell} \tilde{\rho}_n\left(\tau_x(A)\tau_y(A)\right) \\
&= \frac{1}{|\Lambda_\ell|^2} \sum_{x \in \Lambda_\ell} \sum_{y \in \Lambda_\ell : |x-y| \geq C} \tilde{\rho}_n\left(\tau_x(A)\tau_y(A)\right) \\
&\quad + \underbrace{\frac{1}{|\Lambda_\ell|^2} \sum_{x \in \Lambda_\ell} \sum_{y \in \Lambda_\ell : |x-y| < C} \tilde{\rho}_n\left(\tau_x(A)\tau_y(A)\right)}_{\mathcal{O}(\ell^{-d})} .
\end{aligned}$$

Thus,

$$\tilde{\rho}_n(|A_\ell|^2) = \frac{1}{|\Lambda_\ell|^2} \sum_{x,y \in \Lambda_\ell} \tilde{\rho}_n\left(\tau_x(A^*)\right)\tilde{\rho}_n\left(\tau_y(A)\right) + \mathcal{O}(\ell^{-d}) . \tag{6.2}$$

Since $\hat{\rho}_n$ is an invariant state, for any $\Lambda \in \mathcal{P}_f$ and $A \in \mathcal{U}_\Lambda$,

$$\frac{1}{|\Lambda_\ell|}\sum_{x\in\Lambda_\ell}\tilde{\rho}_n\circ\tau_x(A) = \underbrace{\frac{1}{|\Lambda_\ell|}\sum_{x\in\Lambda_\ell}\hat{\rho}_n\circ\tau_x(A)}_{=\hat{\rho}_n(A)} + \mathcal{O}(\ell^{-1}) ,$$

which, combined with (6.2), yields

$$\lim_{\ell\to\infty}\tilde{\rho}_n(|A_\ell|^2) = \left|\hat{\rho}_n\left(A\right)\right|^2 .$$

Using this last equality and Equation (6.1),

$$\begin{aligned}\lim_{\ell\to\infty}\hat{\rho}_n(|A_\ell|^2) &= \lim_{\ell\to\infty}\frac{1}{|\Lambda_n|}\sum_{x\in\Lambda_n}\tilde{\rho}_n\circ\tau_x(|A_\ell|^2)\\ &= \lim_{\ell\to\infty}\frac{1}{|\Lambda_n|}\sum_{x\in\Lambda_n}\tilde{\rho}_n\left(\left|(\tau_x\left(A\right))_\ell\right|^2\right)\\ &= \frac{1}{|\Lambda_n|}\sum_{x\in\Lambda_n}\left|\hat{\rho}_n\circ\tau_x\left(A\right)\right|^2 = \left|\hat{\rho}_n\left(A\right)\right|^2 . \end{aligned} \tag{6.3}$$

By density of the $*$-algebra $\mathcal{U}_{\mathrm{loc}} \subseteq \mathcal{U}$ of local elements, (6.3) holds true for any (spin or fermion) algebra element $A \in \mathcal{U}$, i.e., $\hat{\rho}_n$ is dispersionless at infinity (Definition 5.35). By Theorem 5.37, $\hat{\rho}_n \in \mathcal{E}_1$ is therefore ergodic for each $n \in \mathbb{N}$.

□

In fact, it turns out that also the *full* set of states of the unital C^*-algebra $\mathcal{U}$ associated with an infinitely extended (quantum spin or fermion) system has the property proven above for that set of *invariant* states: $\mathcal{U}$ is a so-called approximately finite-dimensional (AF) C^*-algebra, i.e., it is generated by an increasing family of *finite-dimensional* C^*-subalgebras. In this case, by [22, Lemma 11.2.4], the set $\mathcal{E}$ of extreme points of the set E of all states of $\mathcal{U}$ is weak*-dense in E, i.e.,

$$E = \overline{\mathrm{co}\mathcal{E}} = \overline{\mathcal{E}} . \tag{6.4}$$

For more details, we recommend [21, Section 8]. Note that, astonishingly, (6.4) do not prevent E from having a unique center [24] (i.e., a sort of maximally mixed point).

The property of having a dense extreme boundary should however not be so surprising for mathematicians. The existence of such convex sets is well-known in infinite-dimensional vector spaces. For instance, the unit ball of any infinite-dimensional Hilbert space has a dense extreme boundary in the weak topology. It turns out that this situation is not accidental, but *generic* for weak*-compact convex sets in infinite dimension. See [21, Section 2.3] for more details, which has been extended in [23] for the dual space $\mathcal{X}^*$, endowed with its weak* topology, of any infinite-dimensional, separable topological vector space $\mathcal{X}$.

In the sequel, we will show that such a property is not just a mathematical curiosity, but has important consequences in terms of thermodynamic properties of infinitely extended (quantum spin or fermion) systems.

6.2 Spin and Fermion Mean-Field Models

In Definition 5.5, we introduce spin and fermion interactions on the cubic lattice $\Gamma \doteq \mathbb{Z}^d$ ($d \in \mathbb{N}$). Here, it is convenient to remove from this definition the self-conjugate property of interactions and use the vector space

$$\mathcal{V}^{\mathbb{C}} \doteq \left\{ \Phi + i\Phi' : \Phi, \Phi' \text{ interactions in the sense of Definition 5.5} \right\}$$

of *complex* interactions, where, for all $\Psi, \Psi' \in \mathcal{V}^{\mathbb{C}}$ and $\alpha \in \mathbb{C}$, $\Psi + \Psi' \in \mathcal{V}^{\mathbb{C}}$ and $\alpha\Psi \in \mathcal{V}^{\mathbb{C}}$ are, respectively, defined by

$$(\Psi + \Psi')(\Lambda) \doteq \Psi(\Lambda) + \Psi'(\Lambda) , \qquad (\alpha\Psi)(\Lambda) \doteq \alpha(\Psi(\Lambda)) , \qquad \Lambda \in \mathcal{P}_f .$$

Cf. Eq. (5.11). Invariant (with respect to space translations) complex interactions are defined exactly as in the real case. See Definition 5.5 (iii). In Definition 5.6, we introduce a real Banach space $\mathcal{W}_1$ of invariant (spin or fermion) interactions which is now embedded in a complex Banach space of (complex, invariant, spin, or fermion) interactions:

Definition 6.2 (A Banach Space of Invariant Complex Interactions) The Banach space of (short-range) invariant complex interactions is defined by

$$\mathcal{W}_1^{\mathbb{C}} \doteq \{\Phi \in \mathcal{V}^{\mathbb{C}} : \Phi \text{ is an invariant interaction for which } \|\Phi\| < \infty\} ,$$

where the norm of $\mathcal{W}_1^{\mathbb{C}}$ is defined like in Definition 5.6, that is,

$$\|\Phi\| \doteq \sum_{\Lambda \in \mathcal{P}_f,\, 0 \in \Lambda} \frac{1}{|\Lambda|} \|\Phi(\Lambda)\| \in \mathbb{R}_0^+ \cup \{\infty\} , \qquad \Phi \in \mathcal{V}^{\mathbb{C}} .$$

This space serves to define a much more general Banach space of mean-field models:

Definition 6.3 (A Banach Space of Mean-Field Models) The space of mean-field models is the *real* Banach space $\mathcal{M}_1 \doteq \mathcal{W}_1 \times \ell^2(\mathbb{N}; \mathcal{W}_1^{\mathbb{C}})^2$, where

$$\ell^2(\mathbb{N}; \mathcal{W}_1^{\mathbb{C}}) \doteq \left\{ \Psi \equiv (\Psi_n)_{n\in\mathbb{N}} \subseteq \mathcal{W}_1^{\mathbb{C}} : \|\Psi\|_2^2 \doteq \sum_{n\in\mathbb{N}} \|\Psi_n\|^2 < \infty \right\} ,$$

whose norm is defined by

$$\|\mathfrak{m}\| \doteq \|\Phi\| + \|\Psi_-\|_2 + \|\Psi_+\|_2 , \qquad \mathfrak{m} \doteq (\Phi, \Psi_-, \Psi_+) \in \mathcal{M}_1 .$$

Here, Ψ_- represents the mean-field attraction of the model, while Ψ_+ refers to its mean-field repulsion.

Note that $\mathcal{W}_1 \subseteq \mathcal{M}_1$, using the identification $\Phi \equiv (\Phi, 0, 0)$ for $\Phi \in \mathcal{W}_1$.

Similar to Definition 5.10, local energy observables, or Hamiltonians, are defined for all complex interactions as follows: For all $\Phi \in \mathcal{V}^{\mathbb{C}}$ and $\Lambda \in \mathcal{P}_f$,

$$H_\Lambda^\Phi \doteq \sum_{\Lambda' \in \mathcal{P}_f,\ \Lambda' \subseteq \Lambda} \Phi(\Lambda') \in \mathcal{U}^e .$$

These complex local Hamiltonians are then used to define local Hamiltonians for any mean-field model in $\mathcal{M}_1$:

Definition 6.4 (Local Energy Observables) For any $\mathfrak{m} \doteq (\Phi, \Psi_-, \Psi_+) \in \mathcal{M}_1$ and finite subset $\Lambda \in \mathcal{P}_f$,

$$H_\Lambda^{\mathfrak{m}} \doteq H_\Lambda^\Phi + \frac{1}{|\Lambda|} \sum_{n \in \mathbb{N}} \left(|H_\Lambda^{\Psi_{+,n}}|^2 - |H_\Lambda^{\Psi_{-,n}}|^2 \right) \in \mathrm{Re}\{\mathcal{U}_\Lambda^e\} ,$$

where, as is usual, $|A|^2 \doteq A^* A$. The self-conjugate element $H_\Lambda^{\mathfrak{m}} = (H_\Lambda^{\mathfrak{m}})^*$ is the (local) "Hamiltonian associated with the (finite) region Λ and the mean-field model $\mathfrak{m}$."

Note that the identification $\Phi \equiv (\Phi, 0, 0)$ for $\Phi \in \mathcal{W}_1$ is coherent with Definitions 5.10 and 6.4, since $H_\Lambda^{(\Phi,0,0)} = H_\Lambda^\Phi$ for any $\Lambda \in \mathcal{P}_f$.

By Definition 6.4, the Hamiltonian associated with a mean-field model (Φ, Ψ_-, Ψ_+) has a mean-field attraction term, and a repulsion one, respectively, defined from the components Ψ_- and Ψ_+ of $\mathfrak{m} \doteq (\Phi, \Psi_-, \Psi_+) \in \mathcal{M}_1$. The mean-field model $\mathfrak{m}$ is said to be "purely attractive" iff $\Psi_+ = 0$, while it is "purely repulsive" iff $\Psi_- = 0$. Distinguishing between these two special types of models is important because the effects of mean-field attractions and repulsions on the structure of the corresponding sets of (globally stable) equilibrium states can be very different. For instance, by [1, Theorem 2.25], mean-field attractions have no particular effect on the structure of the set of (generalized) equilibrium states. By contrast, mean-field repulsions have a geometrical effect, by possibly preventing the set of equilibrium states of being a face of the set of all invariant states. See [1, Lemma 9.8].

Exercise 6.5 Show that, for $\Lambda \in \mathcal{P}_f$ and $\mathfrak{m} \in \mathcal{M}_1$,

$$\left\| H_\Lambda^{\mathfrak{m}} \right\| \leq |\Lambda| \, \|\mathfrak{m}\| . \tag{6.5}$$

Example 6.6 Like in Example 5.9, let $\eta \in \mathbb{R}^+$, $\Omega \doteq \{\uparrow, \downarrow\}$, $\mathcal{U} \doteq \mathrm{CAR}(\{\uparrow, \downarrow\}, \Gamma)$, and take the (canonical) Hilbert basis $\{\mathrm{e}_{s,x}\}_{(s,x)\in\{\uparrow,\downarrow\}\times\Gamma}$ of $\ell^2(\{\uparrow, \downarrow\} \times \Gamma)$. The "BCS interaction" $\Psi_{\mathrm{BCS}} \in \mathcal{W}_1^{\mathbb{C}}$ is defined by $\Psi_{\mathrm{BCS}}(\Lambda) \doteq 0$ whenever $|\Lambda| \notin \{1\}$ and $\Psi_{\mathrm{BCS}}(\{x\}) \doteq \eta^{1/2} a\left(\mathrm{e}_{x,\downarrow}\right) a\left(\mathrm{e}_{x,\uparrow}\right)$ for every $x \in \Gamma$. Then, the (reduced) BCS model of superconductivity refers to the purely attractive mean-field model $\mathfrak{n} = (\Phi, (\Psi_{\mathrm{BCS}}, 0, 0, \ldots), 0)$, where $\Phi = \Phi_{\mathrm{Hubb}}$ for $U = 0$ (see Example 5.9). In this case, we get as local Hamiltonians the usual (reduced) BCS Hamiltonians:

$$H_\Lambda^{\mathfrak{n}} \doteq -t \sum_{s\in\{\uparrow,\downarrow\}} \sum_{x,y\in\Lambda,|x-y|=1} a\left(\mathrm{e}_{x,s}\right)^* a\left(\mathrm{e}_{y,s}\right) - \mu \sum_{s\in\{\uparrow,\downarrow\}} \sum_{x\in\Lambda,} a\left(\mathrm{e}_{x,s}\right)^* a\left(\mathrm{e}_{x,s}\right)$$
$$-\frac{\eta}{|\Lambda|} \sum_{x,y\in\Lambda} a\left(\mathrm{e}_{x,\uparrow}\right)^* a\left(\mathrm{e}_{x,\downarrow}\right)^* a\left(\mathrm{e}_{y,\downarrow}\right) a\left(\mathrm{e}_{y,\uparrow}\right) .$$

Here, $\eta \geq 0$ is the "BCS interaction strength." If we take $\Phi = \Phi_{\mathrm{Hubb}}$ for $U \neq 0$, then we obtain the so-called BCS-Hubbard model.

See Sect. 6.6 for more details. Another, more general, example is given in Sect. 6.9.

6.3 Free Energy Density of Mean-Field Models

Recall that the entropy density functional $\mathfrak{s} : E_1 \to \mathbb{R}_0^+$ is the thermodynamic limit of the von Neumann entropy per unit volume:

$$\mathfrak{s}(\rho) \doteq \lim_{\ell\to\infty} \frac{1}{|\Lambda_\ell|} S_\ell(\rho) .$$

See Theorem 5.20, which states that this functional is affine[1] and bounded on the convex weak*-compact space E_1 of all invariant states. We show next its continuity properties with respect to the weak* topology:

Lemma 6.7 (Ergodic Abundance) *The entropy density functional $\mathfrak{s} : E_1 \to \mathbb{R}_0^+$ is affine and weak*-upper semicontinuous. Additionally, for any invariant state $\rho \in E_1$, there is a sequence $(\hat{\rho}_n)_{n\in\mathbb{N}} \subseteq \mathcal{E}_1$ of ergodic states converging to ρ and such that*

$$\mathfrak{s}(\rho) = \lim_{n\to\infty} \mathfrak{s}(\hat{\rho}_n) .$$

[1] Recall that a function h on a convex set K is affine iff $h(\lambda x + (1-\lambda)\, y) = \lambda h(x) + (1-\lambda) h(y)$ for all $x, y \in K$.

Proof Theorem 5.20 tells us that, for any invariant state $\rho \in E_1$,

$$\mathfrak{s}(\rho) = \inf \left\{ \frac{1}{|\Lambda_\ell|} S_\ell(\rho) \ : \ \ell \in \mathbb{N} \right\} .$$

In other words, $\mathfrak{s}$ is given by the infimum of weak*-continuous functionals $S_\ell : E_1 \to \mathbb{R}_0^+$. It is therefore weak*-upper semicontinuous, by Lemma 7.144. Now, it is shown in the proof of Theorem 6.1 that the states

$$\hat{\rho}_n \doteq \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} \tilde{\rho}_n \circ \tau_x$$

for $n \in \mathbb{N}$ are not only invariant but also ergodic and moreover, as $n \to \infty$, they converge to ρ in the weak* topology. Recall that $\tilde{\rho}_n$ is a periodic (product) state, whose period is $(2n+1, \ldots, 2n+1) \in \mathbb{Z}^d$, for which

$$\tilde{\rho}_n (A) \doteq \rho (A)$$

for any $A \in \mathcal{U}_{\Lambda_n}$. If $\mathfrak{s}$ can be defined for invariant states, thanks to Theorem 5.20, then it can also be defined for periodic states by redefining the parameter $N \in \mathbb{N}$ (defining the algebra $\mathrm{Spin}(N, \Gamma)$), in the quantum spin case, or the spin set Ω (defining the algebra $\mathrm{CAR}(\Omega, \Gamma)$), in the fermion case, in order to see any periodic state as an invariant state. In particular, $\mathfrak{s}$ can be defined as an affine functional on periodic states, and in this case, for any fixed $n \in \mathbb{N}$,

$$\mathfrak{s}(\hat{\rho}_n) = \frac{1}{|\Lambda_n|} \sum_{x \in \Lambda_n} \mathfrak{s} \left(\tilde{\rho}_n \circ \tau_x \right) = s(\tilde{\rho}_n) = \frac{1}{|\Lambda_n|} S_n(\rho) .$$

The above sequence $(\hat{\rho}_n)_{n \in \mathbb{N}} \subseteq \mathcal{E}_1$ of ergodic states thus satisfies all the desired properties. For more details, see [1, Lemma 1.29]. ◻

Notice that the convergence of the entropy density along sequences of pure invariant states, referring to the second part of the lemma, has a classical analogue called "ergodic abundance" [11, Section 2.1]. Important applications of this property have been recently found (see [11] and references therein) to the so-called "nonlinear thermodynamic formalism" of classical dynamical systems.

Recall that the energy density observable associated with an invariant interaction $\Phi \in \mathcal{W}_1$ refers to Definition 5.10 (ii). Extended to all complex interactions, it corresponds to

$$e_\Phi \doteq \sum_{\Lambda \in \mathcal{P}_f,\, 0 \in \Lambda} \frac{1}{|\Lambda|} \Phi(\Lambda) \in \mathcal{U}^e \tag{6.6}$$

for any $\Phi \in \mathcal{W}_1^{\mathbb{C}}$. From Proposition 5.11, it defines an energy density functional $\mathfrak{e}_\Phi : E_1 \to \mathbb{R}$ for any interaction $\Phi \in \mathcal{W}_1$. See Definition 5.12. This definition is also extended to all complex interactions: For any invariant state $\rho \in E_1$ and $\Phi \in \mathcal{W}_1^{\mathbb{C}}$,

$$\mathfrak{e}_\Phi(\rho) \doteq \rho(e_\Phi) \ . \tag{6.7}$$

It is clearly an affine functional on the convex weak*-compact space E_1 of all invariant states. Its main basic properties are gathered in the following lemma:

Lemma 6.8 *For any complex interaction $\Phi \in \mathcal{W}_1^{\mathbb{C}}$, the energy density functional $\mathfrak{e}_\Phi : E_1 \to \mathbb{R}$ is affine and weak*-continuous. Moreover, for any $\Phi, \Phi' \in \mathcal{W}_1^{\mathbb{C}}$ and invariant state $\rho \in E_1$,*

$$|\mathfrak{e}_\Phi(\rho) - \mathfrak{e}_{\Phi'}(\rho)| \leq \|\Phi - \Phi'\|.$$

Proof The properties directly follow from Eq. (6.7). Note that the last inequality is already mentioned after Definition 5.12. Its proof results from direct computations using the bound

$$|\mathfrak{e}_\Phi(\rho) - \mathfrak{e}_{\Phi'}(\rho)| = |\mathfrak{e}_{\Phi-\Phi'}(\rho)| \leq \|e_{\Phi-\Phi'}\|$$

and the explicit expression for the algebra element e_Φ, as well as the definition of the norm of interactions given in Definition 6.2. □

In addition to the energy and entropy density functionals, we need the so-called space-averaging functionals, in order to study the thermodynamic properties of mean-field models. This new functionals are defined on the convex weak*-compact space E_1 of all invariant states as follows: Recall that, for any (spin or fermion) algebra element $A \in \mathcal{U}$,

$$A_\ell \doteq \frac{1}{|\Lambda_\ell|} \sum_{x \in \Lambda_\ell} \tau_x(A) \ , \tag{6.8}$$

where the unital $*$-homomorphisms $\tau_x : \mathcal{U} \to \mathcal{U}$, $x \in \Gamma$, are the above-defined translation automorphisms (see Eqs. (5.4), for the quantum spin case, or (5.9), for the fermion case), while, for any natural number $\ell \in \mathbb{N}$, $\Lambda_\ell \in \mathcal{P}_f$ is defined by (5.2).

Then, we use Corollary 5.34 to define a the space-averaging functionals on invariant states:

Definition 6.9 (Space-Averaging Functional) Fix a fixed (spin or fermion) algebra element $A \in \mathcal{U}$. Then, the "space-averaging functional" associated with this algebra element is the mapping Δ_A from the space E_1 of invariant states to $\mathbb{R}$ defined by

$$\rho \mapsto \Delta_A(\rho) \doteq \lim_{\ell \to \infty} \rho\left(A_\ell^* A_\ell\right) \in \left[|\rho(A)|^2, \|A\|^2\right] \ .$$

Observe from Definitions 5.35 and 6.9 and Theorem 5.37 that an invariant state is ergodic iff it is dispersionless at infinity, i.e., $\rho \in \mathcal{E}_1$ iff

$$\Delta_A(\rho) = |\rho(A)|^2 \doteq \delta_A(\rho) , \qquad A \in \mathcal{U} . \tag{6.9}$$

The space-averaging functional is therefore explicitly given on the (dense) set $\mathcal{E}_1$ of ergodic states.

Lemma 6.10 *The space-averaging functional has the following properties:*

(i) *At fixed (spin or fermion) algebra element $A \in \mathcal{U}$, Δ_A is weak*-upper semicontinuous and affine.*
(ii) *At fixed invariant state $\rho \in E_1$ and for all algebra elements $A, B \in \mathcal{U}$,*

$$|\Delta_A(\rho) - \Delta_B(\rho)| \leq (\|A\| + \|B\|)\|A - B\| .$$

Proof Except for the upper semicontinuity property, all the assertions directly follow from the definition. The upper semicontinuity of Δ_A, $A \in \mathcal{U}$, follows by combining Lemma 7.144 with the fact that Δ_A is the infimum over a family of continuous functionals:

$$\Delta_A(\rho) = \inf_{\ell \in \mathbb{N}} \left\{ \rho(|A_\ell|^2) \right\} .$$

This property is proven by using the von Neumann ergodic theorem and the GNS representation of states (Theorem 4.113). See proof of Corollary 5.34 or [1, Section 1.3] for more details. □

Note that the space-averaging functionals cannot be generally weak*-continuous. This is a consequence of the density of the set $\mathcal{E}_1 \subsetneq E_1$ of ergodic states: By Theorem 6.1, if Δ_A is weak*-continuous, then (6.9) holds true for all invariant states, i.e., $\Delta_A = \delta_A$. Therefore, it must exist an algebra element $A \in \mathcal{U}$ such that Δ_A is not weak*-continuous; otherwise, all invariant states would be ergodic, thanks to Theorem 5.37. In fact, we have the following general statement concerning the continuity of Δ_A:

Theorem 6.11 *Fix a (spin or fermion) algebra element $A \in \mathcal{U}$ and let δ_A be the weak*-continuous convex function defined by $\delta_A(\rho) \doteq |\rho(A)|^2$ on the convex weak*-compact convex space E_1 of invariant states. Then, one has:*

(i) *Δ_A is weak*-continuous iff δ_A is a constant function.*
(ii) *Δ_A is weak*-discontinuous on a weak*-dense subset of invariant states, unless δ_A is a constant function.*
(iii) *Δ_A is weak*-continuous on the dense subset $\mathcal{E}_1$ of ergodic states.*
(iv) *For all invariant states, $\rho \in E_1$, $\Delta_A(\rho) = \mu_\rho(\delta_A)$ with μ_ρ being the positive linear functional of Theorem 7.339, on weak*-continuous complex-valued functions on E_1. (See also Theorem 4.68 and related remarks.)*

(v) *We have* $\gamma(\Delta_A) = \delta_A$, *where* $\gamma(\Delta_A)$ *is the so-called* γ*-regularization of* Δ_A *on* E_1*, defined by*

$$\gamma(\Delta_A)(\rho) \doteq \sup\{\rho(B) : B \in \mathrm{Re}\{\mathcal{U}\} \text{ such that } \forall \varpi \in E_1,\ \varpi(B) \leq \Delta_A(\varpi)\} \ .$$

See Definition 7.340 *and Proposition* 7.347.

Proof (i)–(iii) result partially from Theorems 7.339 and 5.37. (iv) follows from Lemma 6.10 (i) combined with Theorems 7.339 and 5.37: By affineness and upper semicontinuity of Δ_A (see Lemma 6.10 (i) and [1, Lemma 10.17]) as well as from Theorems 5.37 and 7.339,

$$\Delta_A(\rho) = \mu_\rho(\Delta_A) = \mu_\rho(\delta_A) \ .$$

It remains to prove (v): By Corollary 7.342, the γ-regularization $\gamma(\Delta_A)$ on E_1 is the largest weak*-lower semicontinuous and convex minorant of Δ_A on E_1. Since

$$\Delta_A(\rho) \doteq \lim_{\ell\to\infty} \rho\left(A_\ell^* A_\ell\right) \in \left[|\rho(A)|^2, \|A\|^2\right]$$

for any invariant state $\rho \in E_1$, the function δ_A is a weak*-continuous convex minorant of Δ_A on E_1. Therefore, for any invariant state, $\rho \in E_1$, $\delta_A \leq \gamma(\Delta_A) \leq \Delta_A$, and it follows that $\gamma(\Delta_A)(\rho) = \delta_A(\rho)$ for any ergodic state $\rho \in \mathcal{E}_1$. By weak*-density of $\mathcal{E}_1 \subseteq E_1$ (Theorem 6.1), (v) follows. □

We are now in a position to define the free energy density associated with mean-field models at fixed (non-zero) temperatures:

Definition 6.12 (Free Energy Density) For any mean-field model $\mathfrak{m} \doteq (\Phi, \Psi_-, \Psi_+) \in \mathcal{M}_1$ and inverse temperature $\beta \in (0, \infty)$, the "free energy density functional" $\mathfrak{f}_{\mathfrak{m},\beta} : E_1 \to \mathbb{R}$ on the space E_1 of all invariant states is defined by

$$\mathfrak{f}_{\mathfrak{m},\beta} \doteq \Delta_{\Psi_+} - \Delta_{\Psi_-} + \mathfrak{e}_\Phi - \beta^{-1}\mathfrak{s} \doteq \Delta_{\Psi_+} - \Delta_{\Psi_-} + \mathfrak{f}_{\Phi,\beta}$$

(see Definitions 5.22 and 6.9, Theorem 5.20, and Eq. (6.7)), where, for any sequence $\Psi \in \ell^2(\mathbb{N}; \mathcal{W}_1^{\mathbb{C}})$ of complex interactions,

$$\Delta_\Psi \doteq \sum_{n\in\mathbb{N}} \Delta_{\mathfrak{e}_{\Psi_n}} \ .$$

The free energy density is clearly the same as the one of Definition 5.22 for any $\Phi \in \mathcal{W}_1 \subseteq \mathcal{M}_1$ and $\beta \in (0, \infty)$. Note that Δ_Ψ is well-defined, because, for any sequence $\Psi \in \ell^2(\mathbb{N}; \mathcal{W}_1^{\mathbb{C}})$ of complex (invariant) interactions,

$$\sum_{n\in\mathbb{N}} \sup_{\rho\in E_1} \left|\Delta_{\mathfrak{e}_{\Psi_n}}(\rho)\right| \leq \sum_{n\in\mathbb{N}} \|\mathfrak{e}_{\Psi_n}\|^2 \leq \sum_{n\in\mathbb{N}} \|\Psi_n\|^2 < \infty \,, \tag{6.10}$$

thanks to Lemma 6.8 and Definition 6.9.

The (previous) free energy density functional $\mathfrak{f}_{\Phi,\beta}$ of Definition 6.12 looks natural, as the energy and entropy per unit volume associated with the invariant interaction Φ in a given invariant state ρ. Nevertheless, the mean-field terms in the new free energy density functional $\mathfrak{f}_{\mathfrak{m},\beta}$, defined above by means of the functionals space-averaging $\Delta_{\Psi_\pm}$, may look more intriguing. To explain the origin of these new terms, we come back to finite-volume systems:

Recall that the Gibbs states of Definition 5.19, i.e., equilibrium states at finite volume, are minimizers of the finite-volume free energy, which leads to the concept of the pressure. See Proposition 3.13 and Definition 3.16. In particular, by considering the local Hamiltonians $H_\Lambda^{\mathfrak{m}}$ of Definition 6.4, given a fixed mean-field model $\mathfrak{m} \in \mathcal{M}_1$ and inverse temperature $\beta \in (0,\infty)$, we can define, for any finite (nonempty) region $\Lambda \in \mathcal{P}_f$, the pressure

$$P_{H_\Lambda^{\mathfrak{m}},\beta} \doteq -\frac{1}{|\Lambda|} \inf \left\{ F_{H_\Lambda^{\mathfrak{m}},\beta}(\rho) : \rho \in E(\mathcal{U}_\Lambda) \right\} \,, \tag{6.11}$$

where the free energy functional $F_{H_\Lambda^{\mathfrak{m}},\beta}$ is the one of Definition 5.19, for $H = H_\Lambda^{\mathfrak{m}}$. Then, by taking, for instance, the sequence (5.2) of cubic boxes in Γ, one may ask about the limit $\ell \to \infty$ of the sequence $(P_{H_{\Lambda_\ell}^{\mathfrak{m}},\beta})_{\ell\in\mathbb{N}}$, as well as the corresponding Gibbs states. Such a limit is known in statistical mechanics as the "thermodynamic limit." Answering such a question naturally yields the free energy density functional of Definition 6.12:

Theorem 6.13 *For any mean-field model* $\mathfrak{m} \in \mathcal{M}_1$ *and inverse temperature* $\beta \in (0,\infty)$,

$$\mathfrak{p}_\beta(\mathfrak{m}) \doteq -\inf \mathfrak{f}_{\mathfrak{m},\beta}(E_1) = \lim_{\ell\to\infty} P_{H_{\Lambda_\ell}^{\mathfrak{m}},\beta} \in \mathbb{R} \,.$$

Idea of Proof Any state $\rho \in E(\mathcal{U})$ on $\mathcal{U}$ can be seen, by restriction, as a state $\rho|_{\mathcal{U}_{\Lambda_\ell}} \in E(\mathcal{U}_{\Lambda_\ell})$ on $\mathcal{U}_{\Lambda_\ell} \subseteq \mathcal{U}$ for any $\ell \in \mathbb{N}$. Using Definition 6.4 and Proposition 3.13, we thus deduce that, for any mean-field model $\mathfrak{m} \doteq (\Phi, \Psi_-, \Psi_+) \in \mathcal{M}_1$, $\beta \in (0,\infty)$ and all states $\rho \in E(\mathcal{U})$,

$$P_{H_{\Lambda_\ell}^{\mathfrak{m}},\beta} \geq \frac{1}{|\Lambda_\ell|^2} \sum_{n\in\mathbb{N}} \rho(|H_{\Lambda_\ell}^{\Psi_{-,n}}|^2) - \frac{1}{|\Lambda_\ell|^2} \sum_{n\in\mathbb{N}} \rho(|H_{\Lambda_\ell}^{\Psi_{+,n}}|^2) - \frac{1}{|\Lambda_\ell|} \rho\left(H_{\Lambda_\ell}^{\Phi}\right)$$
$$+ \frac{1}{\beta|\Lambda_\ell|} S(\rho|_{\mathcal{U}_{\Lambda_\ell}})$$

with equality when $\rho|_{\mathcal{U}_{\Lambda_\ell}}$ is the Gibbs states of Definition 5.19 for $H = H_{\Lambda_\ell}^{\mathfrak{m}}$. When the state ρ is invariant, i.e., $\rho \in E_1$,

$$\lim_{\ell\to\infty} \left\{ \frac{1}{|\Lambda_\ell|} \rho\left(H_{\Lambda_\ell}^{\Phi}\right) - \frac{1}{\beta|\Lambda_\ell|} S(\rho|_{\mathcal{U}_{\Lambda_\ell}}) \right\} = \mathfrak{e}_\Phi(\rho) - \beta^{-1}\mathfrak{s}(\rho) \doteq \mathfrak{f}_{\Phi,\beta}(\rho) \ .$$

See Theorem 5.20 and Definition 5.12. Moreover, for any invariant state $\rho \in E_1$ and any complex (invariant) interaction $\Psi \in \mathcal{W}_1^{\mathbb{C}}$, one checks from direct estimates that

$$\lim_{\ell\to\infty} \left(\frac{1}{|\Lambda_\ell|^2} \rho(|H_{\Lambda_\ell}^{\Psi}|^2) - \rho\left(\left|(e_\Psi)_\ell\right|^2\right) \right) = 0$$

with $(e_\Psi)_\ell \in \mathcal{U}$ being given by

$$(e_\Psi)_\ell \doteq \frac{1}{|\Lambda_\ell|} \sum_{x\in\Lambda_\ell} \tau_x(e_\Psi) \ ,$$

the algebra element $e_\Psi \in \mathcal{U}$ being the energy density observable (6.6). See also Eq. (6.8). By Definition 6.9 of $\Delta_{\mathfrak{e}_\Psi}$, it follows that

$$\lim_{\ell\to\infty} \frac{1}{|\Lambda_\ell|^2} \rho(|H_{\Lambda_\ell}^{\Psi}|^2) = \Delta_{\mathfrak{e}_\Psi}(\rho) \ .$$

Using Corollary 7.314, we deduce that

$$\lim_{\ell\to\infty} P_{H_{\Lambda_\ell}^{\mathfrak{m}},\beta} \geq -\inf \mathfrak{f}_{\mathfrak{m},\beta}(E_1) \doteq -\inf \left\{ \mathfrak{f}_{\mathfrak{m},\beta}(\rho) : \rho \in E_1 \right\} \ .$$

The upper bound is more difficult to derive, in particular for non-zero mean-field attraction $\Psi_- \neq 0$. See [1, Chapter 6] for more details. □

Like in Definition 5.28, we define the pressure as follows:

Definition 6.14 (Pressure Function on $\mathcal{M}_1$) For $\beta \in (0, \infty)$, the function $\mathfrak{p}_\beta : \mathcal{M}_1 \to \mathbb{R}$ defined by

$$\mathfrak{m} \mapsto \mathfrak{p}_\beta(\mathfrak{m}) \doteq -\inf \mathfrak{f}_{\mathfrak{m},\beta}(E_1)$$

is called "pressure function" at temperature $T = \beta^{-1}$.

Similar to Proposition 5.30, for two fixed sequences $\Psi_\pm \in \ell^2(\mathbb{N}; \mathcal{W}_1^{\mathbb{C}})$ of complex (invariant) interactions, the pressure function is a continuous convex real-valued function

$$\Phi \mapsto \mathfrak{p}_\beta(\Phi, \Psi_-, \Psi_+)$$

on the real Banach space $\mathcal{W}_1$ of invariant interactions. In addition, for all Φ, $\Phi' \in \mathcal{W}_1$,

$$|\mathfrak{p}_\beta(\Phi, \Psi_-, \Psi_+) - \mathfrak{p}_\beta(\Phi', \Psi_-, \Psi_+)| \leq \left\|\Phi - \Phi'\right\| .$$

The arguments are the same as those proving Proposition 5.30. We can therefore study tangent functionals to this function, as discussed from Proposition 5.30. However, in the sequel, we perform, instead, a more direct study of the minimizers of the free energy density functional, which are naturally viewed as equilibrium states of the corresponding mean-field model. In fact, this study becomes quite interesting, and highly non-trivial, in the presence of non-zero mean-field terms $\Psi_\pm$.

6.4 Equilibrium States of Mean-Field Models

The free energy density functional on the set E_1 of invariant states is in general **not** weak*-lower semicontinuous: By Lemmata 6.7, 6.8, and 6.10, observe from Definition 6.12 that, for any mean-field model $\mathfrak{m} \doteq (\Phi, \Psi_-, \Psi_+) \in \mathcal{M}_1$ and $\beta \in (0, \infty)$,

$$\mathfrak{f}_{\mathfrak{m},\beta} = \underbrace{\Delta_{\Psi_+}}_{\text{upper semicont.}} + \underbrace{\left(-\Delta_{\Psi_-} + \mathfrak{e}_\Phi - \beta^{-1}\mathfrak{s}\right)}_{\text{lower semicont.}} .$$

The free energy density functional $\mathfrak{f}_{\mathfrak{m},\beta} : E_1 \to \mathbb{R}$, which is an affine functional on the convex weak*-compact space E_1 of invariant states, has thus a **topological** drawback. In particular, it is not clear from the beginning whether there are solutions to the variational problem

$$\inf \mathfrak{f}_{\mathfrak{m},\beta}(E_1) ,$$

or not. The situation is much simpler in the absence of mean-field terms $\Psi_\pm$: When $\Psi_\pm = 0$, the set $M_{\Phi,\beta} \subseteq E_1$ of all minimizers of $\mathfrak{f}_{\Phi,\beta}$, named the globally stable equilibrium states for the interaction $\Phi \in \mathcal{W}_1$ at inverse temperature $\beta \in (0, \infty)$, appearing in Definition 5.22 is always nonempty, $\mathfrak{f}_{\Phi,\beta}$ being lower semicontinuous on a compact set. See Proposition 7.172. The generalization of the notion of globally stable equilibrium states to the mean-field case is done, as is usual, via the (weak*) limits of approximating minimizers:

Definition 6.15 (Equilibrium States) For any mean-field model $\mathfrak{m} \in \mathcal{M}_1$ and $\beta \in (0, \infty)$,

$$\begin{aligned}\Omega_{\mathfrak{m},\beta} \doteq \Big\{\omega \in E_1 : \exists (\rho_n)_{n\in\mathbb{N}} \subseteq E_1 \text{ weak}^* \text{ converging to } \omega \text{ such that } \lim_{n\to\infty} \mathfrak{f}_{\mathfrak{m},\beta}(\rho_n) \\ = \inf \mathfrak{f}_{\mathfrak{m},\beta}(E_1)\Big\} .\end{aligned}$$

The set $\Omega_{\mathfrak{m},\beta}$ is clearly convex, $\mathfrak{f}_{\mathfrak{m},\beta}$ being affine (Lemmata 6.7, 6.8, and 6.10) on a convex set, i.e., E_1. Note also that it is not empty, since any sequence of invariant states has weak*-convergent subsequences, the space E_1 of invariant states being weak*-compact.

Elements of the set $\Omega_{\mathfrak{m},\beta} \subseteq E_1$ of all weak* limits of approximating minimizers of $\mathfrak{f}_{\mathfrak{m},\beta}$ are named again "globally stable equilibrium states" at temperature $T = \beta^{-1}$, associated with the mean-field model $\mathfrak{m}$. The extreme elements of the convex set $\Omega_{\mathfrak{m},\beta}$ are called "pure globally stable equilibrium states." As before, we say that there is a "(first-order) phase transition" for $\mathfrak{m} \in \mathcal{M}_1$ at temperature $T = \beta^{-1}$ if $\Omega_{\mathfrak{m},\beta}$ contains more than one element. Recall that, in contrast with the finite-volume situation, there are possibly many globally stable equilibrium states, even in the absence of mean-field terms. See, for instance, Corollary 5.39. This is reminiscent of the non-uniqueness of irreducible representations of the infinite-dimensional unital C^*-algebra $\mathcal{U}$.

Globally stable equilibrium states in the above sense are directly related with the thermodynamic limit of Gibbs states associated with local Hamiltonians of Definition 6.4. We shortly explain this fact: For any cubic box $\Lambda_\ell \subseteq \Gamma$, $\ell \in \mathbb{N}$, let $\omega_{H^{\mathfrak{m}}_{\Lambda_\ell},\beta} \in E(\mathcal{U}_{\Lambda_\ell})$ be the Gibbs state of Definition 5.19, which is periodically extended (with period $(2\ell + 1)$ in each direction of $\Gamma \doteq \mathbb{Z}^d$), and define

$$\hat{\rho}_{\ell,\mathfrak{m},\beta} \doteq \frac{1}{|\Lambda_\ell|} \sum_{x\in\Lambda_\ell} \omega_{H^{\mathfrak{m}}_{\Lambda_\ell},\beta} \circ \tau_x \in \mathcal{E}_1 \subseteq E_1$$

These invariant states are particular cases of the ones used in the proofs of Theorems 6.1 and Lemma 6.7. They are in particular ergodic. Then, one can prove the following statement:

Theorem 6.16 (Limit of Space-Averaged Gibbs States) *For any mean-field model* $\mathfrak{m} \in \mathcal{M}_1$ *and inverse temperature* $\beta \in (0, \infty)$*, the weak* accumulation points of* $(\hat{\rho}_{\ell,\mathfrak{m},\beta})_{\ell\in\mathbb{N}}$ *belong to* $\Omega_{\mathfrak{m},\beta}$.

Idea of Proof One uses the notion of tangent functionals (Definition 3.18), as explained in Proposition 5.30. See [1, Section 2.6] for more details. □

Observe that Theorem 6.16 does not exactly refer to the limits of Gibbs states. In fact, the set $E(\mathcal{U})$ of all states on $\mathcal{U}$ being weak*-compact, Gibbs states, seen as periodic states on $\mathcal{U}$, have weak*-convergent subsequences, but it is not clear that such limits always belong to the set E_1 of invariant states, as for the sequence $\{\hat{\rho}_{\ell,\mathfrak{m},\beta}\}_{\ell\in\mathbb{N}} \subseteq E_1$. If a weak*-convergent sequence of Gibbs states has an invariant state as limit, then it must belong to $\Omega_{\mathfrak{m},\beta}$. This condition can be ensured by taking periodic boundary conditions, as explained in [1, Chapter 3]. In particular, in this case, the weak*-accumulation points of Gibbs states $\{\omega_{H^{\mathfrak{m}}_{\Lambda_\ell},\beta}\}_{\ell\in\mathbb{N}}$ belong to $\Omega_{\mathfrak{m},\beta}$.

Apart from the fact that E_1 is convex and weak*-compact, recall that it has a weak*−dense set of extreme points, i.e., the set $\mathcal{E}_1$ of ergodic states is dense in E_1. See Theorem 6.1. Moreover, the space-averaging functional of Definition 6.9 takes a simple (explicit) form on this dense set:

$$\Delta_A(\rho) = |\rho(A)|^2 ,$$

for all ergodic states $\rho \in \mathcal{E}_1$, thanks to Theorem 5.37. In particular, the free energy density functional of Definition 6.12 equals the following function on the dense set of ergodic states:

Definition 6.17 (Nonlinear Free Energy Density) For any mean-field model $\mathfrak{m} \doteq (\Phi, \Psi_-, \Psi_+) \in \mathcal{M}_1$ and $\beta \in (0, \infty)$, the "nonlinear[2] free energy density functional" $\mathfrak{g}_{\mathfrak{m},\beta} : E_1 \to \mathbb{R}$ on the space E_1 of all invariant states is defined by

$$\begin{aligned} \mathfrak{g}_{\mathfrak{m},\beta}(\rho) &\doteq \|\mathfrak{e}_{\Psi_+}(\rho)\|_2^2 - \|\mathfrak{e}_{\Psi_-}(\rho)\|_2^2 + \mathfrak{e}_\Phi - \beta^{-1}\mathfrak{s} \\ &= \|\mathfrak{e}_{\Psi_+}(\rho)\|_2^2 - \|\mathfrak{e}_{\Psi_-}(\rho)\|_2^2 + \mathfrak{f}_{\Phi,\beta}(\rho) , \qquad \rho \in E_1 , \end{aligned}$$

(see Definition 5.22, Theorem 5.20, and Eq. (6.7)), where, for any sequence $\Psi \in \ell^2(\mathbb{N}; \mathcal{W}_1^{\mathbb{C}})$ of complex interactions,

$$\mathfrak{e}_\Psi(\rho) \doteq \left(\mathfrak{e}_{\Psi_n}(\rho)\right)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}) .$$

Note that, for any sequence $\Psi \in \ell^2(\mathbb{N}; \mathcal{W}_1^{\mathbb{C}})$ of complex (invariant) interactions,

$$\|\mathfrak{e}_\Psi(\rho)\|_2^2 \leq \sum_{n\in\mathbb{N}} \sup_{\rho\in E_1} |\mathfrak{e}_{\Psi_n}(\rho)|^2 \leq \sum_{n\in\mathbb{N}} \|\Psi_n\|^2 < \infty , \tag{6.12}$$

thanks to Lemma 6.8. The nonlinear free energy density functional $\mathfrak{g}_{\mathfrak{m},\beta}$ is not affine anymore, but has, instead, the following important properties:

Lemma 6.18 *For every mean-field model $\mathfrak{m} \in \mathcal{M}_1$ and inverse temperature $\beta \in (0, \infty)$, $\mathfrak{g}_{\mathfrak{m},\beta}$ is weak*-lower semicontinuous. Additionally, for any invariant state $\rho \in E_1$, there is a sequence $(\hat{\rho}_n)_{n\in\mathbb{N}} \subseteq \mathcal{E}_1$ of ergodic states weak*-converging to ρ, such that*

$$\mathfrak{g}_{\mathfrak{m},\beta}(\rho) = \lim_{n\to\infty} \mathfrak{g}_{\mathfrak{m},\beta}(\hat{\rho}_n) .$$

Proof To prove the lower semicontinuity, combine Lemmata 6.7 and 6.8 together with the weak*-continuity of the functional $\rho \mapsto \mathfrak{e}_\Psi(\rho)$ from E_1 to $\ell^2(\mathbb{N})$, which is deduced from (6.12) and Corollary 7.314. The second part of the lemma directly follows from the corresponding property of the entropy density (see Lemma 6.7) combined with the previously proven continuity of the mapping $\rho \mapsto \mathfrak{e}_\Psi(\rho)$. □

[2] We adopt this terminology, because of the formal analogy to the classical "nonlinear thermodynamic formalism," as, for instance, described in [11].

As already explained above, the nonlinear free energy density functionals equal the usual ones on the dense set of ergodic states:

$$\mathfrak{f}_{\mathfrak{m},\beta}(\rho) = \mathfrak{g}_{\mathfrak{m},\beta}(\rho) \ , \qquad \rho \in \mathcal{E}_1 \ ,$$

thanks to Theorem 5.37. More generally, for (possibly non-ergodic) invariant states, both functionals are related to each other via the following assertion:

Lemma 6.19 *For any mean-field model* $\mathfrak{m} \in \mathcal{M}_1$, $\beta \in (0, \infty)$ *and every invariant states* $\rho \in E_1$, $\mathfrak{f}_{\mathfrak{m},\beta}(\rho) = \mu_\rho\left(\mathfrak{g}_{\mathfrak{m},\beta}\right)$ *with* μ_ρ *being the positive linear functional defined by Theorem 7.339 on the Borel-measurable*[3] *functions on* E_1.

Proof Combine Lemmata 6.7 and 6.8 with Theorems 7.339 and 6.11 (iv). □

The nonlinear free energy density functional is clearly not affine, in contrast with the free energy density functional, but it is, at least, weak*-lower semicontinuous. In fact, being not affine, $\mathfrak{g}_{\mathfrak{m},\beta}$ has a **geometrical** drawback, whereas, being **not** weak*-lower semicontinuous, $\mathfrak{f}_{\mathfrak{m},\beta}$ has a **topological** drawback. Nonetheless, interestingly, both functionals lead to the pressure function of Definition 6.14:

Theorem 6.20 *For any mean-field model* $\mathfrak{m} \in \mathcal{M}_1$ *and inverse temperature* $\beta \in (0, \infty)$,

$$\inf \mathfrak{f}_{\mathfrak{m},\beta}(E_1) = \inf \mathfrak{f}_{\mathfrak{m},\beta}(\mathcal{E}_1) = \inf \mathfrak{g}_{\mathfrak{m},\beta}(\mathcal{E}_1) = \inf \mathfrak{g}_{\mathfrak{m},\beta}(E_1) > -\infty \ ,$$

with $\mathcal{E}_1$ *being the weak*-dense set of ergodic (or extreme) states of* E_1 *(Theorem 6.1).*

Proof We apply the extension of the Bauer maximum principle (Lemma 7.344) to the weak*-compact and convex space $K = E_1$ and the functional

$$\mathfrak{f}_{\mathfrak{m},\beta} = \underbrace{\Delta_{\Psi_+}}_{\text{upper semicont.}} + \underbrace{\left(-\Delta_{\Psi_-} + \mathfrak{e}_\Phi - \beta^{-1}\mathfrak{s}\right)}_{\text{lower semicont.}}$$

for any mean-field model $\mathfrak{m} \doteq (\Phi, \Psi_-, \Psi_+) \in \mathcal{M}_1$ and $\beta \in (0, \infty)$. See Lemmata 6.7, 6.8, and 6.10. In fact, using Lemma 7.344, we conclude that

$$\inf \mathfrak{f}_{\mathfrak{m},\beta}(E_1) = \inf \mathfrak{f}_{\mathfrak{m},\beta}(\mathcal{E}_1) = \inf \mathfrak{g}_{\mathfrak{m},\beta}(\mathcal{E}_1) \ , \tag{6.13}$$

[3] Semicontinuous functions on a metric space, like $\mathfrak{g}_{\mathfrak{m},\beta}$, are special cases of Borel-measurable ones. Moreover, $\mathfrak{g}_{\mathfrak{m},\beta}$ is the supremum of a countable family of continuous functions, because (up to a sign) the entropy density functional has this property. See Lemma 7.144 and related discussions.

keeping in mind that the free energy density functional $\mathfrak{f}_{\mathfrak{m},\beta}$ equals the nonlinear one, $\mathfrak{g}_{\mathfrak{m},\beta}$, on the (dense) set $\mathcal{E}_1$ of ergodic states. Additionally, using Lemma 6.18, we deduce the following facts:

- $\mathfrak{g}_{\mathfrak{m},\beta}$ is weak*-lower semicontinuous, and, thus, there is a minimizer $\omega \in E_1$ for the variational problem

$$\inf \mathfrak{g}_{\mathfrak{m},\beta}(E_1) = \mathfrak{g}_{\mathfrak{m},\beta}(\omega) \ .$$

- There is a sequence $(\hat{\rho}_n)_{n\in\mathbb{N}} \subseteq \mathcal{E}_1$ of ergodic states weak*-converging to ω and such that

$$\mathfrak{g}_{\mathfrak{m},\beta}(\omega) = \lim_{n\to\infty} \mathfrak{g}_{\mathfrak{m},\beta}(\hat{\rho}_n) \ .$$

It follows that

$$\inf \mathfrak{g}_{\mathfrak{m},\beta}(\mathcal{E}_1) = \inf \mathfrak{g}_{\mathfrak{m},\beta}(E_1) \ ,$$

which combined with (6.13), in turn, yields the assertion. □

This theorem opens the door to a new definition of equilibrium states, which can now be defined as minimizers of the weak*-lower semicontinuous nonlinear free energy functional:

Definition 6.21 (Nonlinear Equilibrium States) For any mean-field model $\mathfrak{m} \in \mathcal{M}_1$ and $\beta \in (0,\infty)$,

$$\hat{M}_{\mathfrak{m},\beta} \doteq \left\{\omega \in E_1 : \mathfrak{g}_{\mathfrak{m},\beta}(\omega) = \inf \mathfrak{g}_{\mathfrak{m},\beta}(E_1)\right\} \ .$$

The elements of $\hat{M}_{\mathfrak{m},\beta}$ are called here "nonlinear (globally stable) equilibrium states" of the mean-field model $\mathfrak{m}$ at inverse temperature β.

Clearly, $\hat{M}_{\mathfrak{m},\beta}$ is nonempty, since $\mathfrak{g}_{\mathfrak{m},\beta}$ is weak*-lower semicontinuous (Lemma 6.18). See Proposition 7.172. In general, this set of minimizers differs from the set $\Omega_{\mathfrak{m},\beta}$ of (usual) globally stable equilibrium states of Definition 6.15, i.e., $\hat{M}_{\mathfrak{m},\beta} \neq \Omega_{\mathfrak{m},\beta}$. Recall that

$$\Omega_{\mathfrak{m},\beta} \doteq \Big\{\omega \in E_1 : \exists (\rho_n)_{n\in\mathbb{N}} \subseteq E_1 \text{ weak}^* \text{ converging to } \omega \text{ so that } \lim_{n\to\infty} \mathfrak{f}_{\mathfrak{m},\beta}(\rho_n) = \inf \mathfrak{f}_{\mathfrak{m},\beta}(E_1)\Big\}$$

for any mean-field model $\mathfrak{m} \in \mathcal{M}_1$ and inverse temperature $\beta \in (0,\infty)$. In fact, even if they are generally different sets, there is a strong relation between both notions of equilibrium states: It turns out that $\hat{M}_{\mathfrak{m},\beta} \subseteq \Omega_{\mathfrak{m},\beta}$, i.e., nonlinear equilibrium states are special cases of globally stable equilibrium states of mean-field models. What is more, the nonlinear equilibrium states *generate* the convex set of all equilibrium states, for *all* mean-field models. These properties are precisely stated in the following lemma and Theorem 6.25:

Lemma 6.22 *For any mean-field model* $\mathfrak{m} \in \mathcal{M}_1$ *and inverse temperature* $\beta \in (0, \infty)$*, the following properties hold true:*

(i) $\Omega_{\mathfrak{m},\beta}$ *is a (nonempty) convex weak**-*compact subset of* E_1.
(ii) $\hat{M}_{\mathfrak{m},\beta}$ *is a (nonempty) weak**-*compact subset of* E_1.
(iii) *The weak**-*closed convex hull of* $\hat{M}_{\mathfrak{m},\beta}$ *belong to* $\Omega_{\mathfrak{m},\beta}$*, i.e.,* $\overline{\mathrm{co}(\hat{M}_{\mathfrak{m},\beta})} \subseteq \Omega_{\mathfrak{m},\beta}$.

Proof

(i) The set $\Omega_{\mathfrak{m},\beta}$ is convex, $\mathfrak{f}_{\mathfrak{m},\beta}$ being affine (Lemmata 6.7, 6.8, and 6.10) on the convex set E_1. Since the (spin or fermion) algebra $\mathcal{U}$ is separable, the weak* topology is metrizable on any weak*-compact subset of $\mathcal{U}^*$; see, e.g., Proposition 4.84 or [18, Theorem 3.16]. As E_1 is weak*-compact, one uses the metric generating the weak* topology on E_1 in order to show that $\Omega_{\mathfrak{m},\beta} \subseteq E_1$ is weak*-closed and therefore weak*-compact.
(ii) It is a direct consequence of the weak*-lower semicontinuity of $\mathfrak{g}_{\mathfrak{m},\beta}$ (Lemma 6.18) together with the weak*-compactness of E_1. See Proposition 7.172.
(iii) By Lemma 6.18, for any $\omega \in \hat{M}_{\mathfrak{m},\beta}$, there is a sequence $(\hat{\rho}_n)_{n\in\mathbb{N}} \subseteq \mathcal{E}_1$ weak*-converging to ω such that $\mathfrak{g}_{\mathfrak{m},\beta}(\hat{\rho}_n) = \mathfrak{f}_{\mathfrak{m},\beta}(\hat{\rho}_n)$ converges to $\mathfrak{g}_{\mathfrak{m},\beta}(\omega)$, as $n \to \infty$. Since, by Theorem 6.20,

$$\mathfrak{g}_{\mathfrak{m},\beta}(\omega) = \inf \mathfrak{f}_{\mathfrak{m},\beta}(E_1) \ ,$$

we obtain that $\omega \in \Omega_{\mathfrak{m}}$. As a consequence, the assertion holds true because $\Omega_{\mathfrak{m}}$ is convex and weak*-compact.

□

In fact, we can strengthen Lemma 6.22 (iii) by showing that

$$\overline{\mathrm{co}(\hat{M}_{\mathfrak{m},\beta})} = \Omega_{\mathfrak{m},\beta}$$

for any mean-field model $\mathfrak{m} \in \mathcal{M}_1$ and inverse temperature $\beta \in (0, \infty)$. To prove this equality, we use a relatively recent result of convex analysis [25, Theorem 1.4], which corresponds in our (less general) setting to Theorem 7.345. More precisely, we apply this theorem to the γ-regularization of the free energy density functionals $\mathfrak{g}_{\mathfrak{m},\beta}$ and $\mathfrak{f}_{\mathfrak{m},\beta}$ on the convex weak*-compact space E_1 of invariant states, defined by

$$\gamma\left(\mathfrak{g}_{\mathfrak{m},\beta}\right)(\rho) \doteq \sup\left\{\rho(B) : B \in \mathrm{Re}\{\mathcal{U}\} \text{ so that } \forall \varpi \in E_1, \varpi(B) \leq \mathfrak{g}_{\mathfrak{m},\beta}(\varpi)\right\} \ ,$$

$$\gamma\left(\mathfrak{f}_{\mathfrak{m},\beta}\right)(\rho) \doteq \sup\left\{\rho(B) : B \in \mathrm{Re}\{\mathcal{U}\} \text{ so that } \forall \varpi \in E_1, \varpi(B) \leq \mathfrak{f}_{\mathfrak{m},\beta}(\varpi)\right\} \ ,$$

for any mean-field model $\mathfrak{m} \in \mathcal{M}_1$ and inverse temperature $\beta \in (0,\infty)$. See Definition 7.340 and Proposition 7.347.

Corollary 6.23 *For any mean-field model* $\mathfrak{m} \in \mathcal{M}_1$ *and inverse temperature* $\beta \in (0,\infty)$*,*

$$\begin{aligned}\inf \gamma(\mathfrak{f}_{\mathfrak{m},\beta})(E_1) &= \inf \mathfrak{f}_{\mathfrak{m},\beta}(E_1) = \inf \mathfrak{f}_{\mathfrak{m},\beta}(\mathcal{E}_1) \\ &= \inf \mathfrak{g}_{\mathfrak{m},\beta}(\mathcal{E}_1) = \inf \mathfrak{g}_{\mathfrak{m},\beta}(E_1) = \inf \gamma(\mathfrak{g}_{\mathfrak{m},\beta})(E_1) .\end{aligned}$$

Proof Combine Theorem 6.20 with Theorem 7.345. □

The next question is the following: How are the γ-regularizations $\gamma(\mathfrak{f}_{\mathfrak{m},\beta})$ and $\gamma(\mathfrak{g}_{\mathfrak{m},\beta})$ on the convex weak*-compact space E_1 of invariant states (i.e., the largest weak*-lower semicontinuous and convex minorants of, respectively, $\mathfrak{f}_{\mathfrak{m},\beta}$ and $\mathfrak{g}_{\mathfrak{m},\beta}$ on E_1, by Corollary 7.342) related to each other? A simple and satisfying answer to this question is given by the following lemma:

Lemma 6.24 *For any mean-field model* $\mathfrak{m} \in \mathcal{M}_1$ *and inverse temperature* $\beta \in (0,\infty)$*, we have* $\gamma(\mathfrak{f}_{\mathfrak{m},\beta}) = \gamma(\mathfrak{g}_{\mathfrak{m},\beta})$ *on the space* E_1 *of invariant states.*

Proof
Lower bound: As $\Delta_A(\hat{\rho}) = |\hat{\rho}(A)|^2$ on $\mathcal{E}_1$ (see, e.g., Theorem 6.11 (iv)), for any ergodic state $\hat{\rho} \in \mathcal{E}_1$,

$$\mathfrak{f}_{\mathfrak{m},\beta}(\hat{\rho}) = \mathfrak{f}^{\flat}_{\mathfrak{m},\beta}(\hat{\rho}) = \mathfrak{g}_{\mathfrak{m},\beta}(\hat{\rho}) ,$$

where, for any invariant state $\rho \in E_1$,

$$\mathfrak{f}^{\flat}_{\mathfrak{m},\beta}(\rho) \doteq \underbrace{\|\mathfrak{e}_{\Psi_+}(\rho)\|_2^2}_{\text{convex semicont.}} + \underbrace{\left(-\Delta_{\Psi_-}(\rho) + \mathfrak{e}_{\Phi}(\rho) - \beta^{-1}\mathfrak{s}(\rho)\right)}_{\text{affine lower semicont.}} . \tag{6.14}$$

See Lemmata 6.7, 6.8, and 6.10. Therefore, for any ergodic state $\hat{\rho} \in \mathcal{E}_1$,

$$\mathfrak{f}_{\mathfrak{m},\beta}(\hat{\rho}) = \gamma(\mathfrak{f}_{\mathfrak{m},\beta})(\hat{\rho}) = \mathfrak{g}_{\mathfrak{m},\beta}(\hat{\rho}) . \tag{6.15}$$

By Lemma 6.18, for any invariant state $\rho \in E_1$, there is a sequence $(\hat{\rho}_n)_{n\in\mathbb{N}} \subseteq \mathcal{E}_1$ of ergodic states converging to ρ and such that

$$\lim_{n\to\infty} \mathfrak{g}_{\mathfrak{m},\beta}(\hat{\rho}_n) = \mathfrak{g}_{\mathfrak{m},\beta}(\rho) .$$

By (6.15) and weak*-lower semicontinuity of $\gamma(\mathfrak{f}_{\mathfrak{m},\beta})$, for any invariant state $\rho \in E_1$,

$$\lim_{n\to\infty} \gamma(\mathfrak{f}_{\mathfrak{m},\beta})(\hat{\rho}_n) = \mathfrak{g}_{\mathfrak{m}}(\rho) \geq \gamma(\mathfrak{f}_{\mathfrak{m},\beta})(\rho),$$

implying $\gamma(\mathfrak{f}_{\mathfrak{m},\beta}) \leq \gamma(\mathfrak{g}_{\mathfrak{m},\beta})$.

Upper bound: By Theorem 7.339 and Jensen's inequality (Lemma 7.330; see also [1, Lemma 10.33]),[4] for any invariant state $\rho \in E_1$, there is a (unique) positive linear functional μ_ρ such that

$$h(\rho) \leq \mu_\rho(h)$$

for any convex weak*-lower semicontinuous complex-valued functions h on E_1. By convexity and weak*-lower semicontinuity of $\gamma(\mathfrak{g}_{\mathfrak{m},\beta})$, it follows that

$$\gamma(\mathfrak{g}_{\mathfrak{m},\beta})(\rho) \leq \mu_\rho\left(\gamma(\mathfrak{g}_{\mathfrak{m},\beta})\right) \underset{(=)}{\leq} \mu_\rho\left(\mathfrak{g}_{\mathfrak{m},\beta}\right)$$

which combined with Lemma 6.19 yields $\gamma(\mathfrak{g}_{\mathfrak{m},\beta}) \leq \mathfrak{f}_{\mathfrak{m},\beta}$ and therefore $\gamma(\mathfrak{g}_{\mathfrak{m},\beta}) \leq \gamma(\mathfrak{f}_{\mathfrak{m},\beta})$. □

We are now in a position to prove that the weak*-closed convex hull of the set nonlinear globally stable equilibrium states (see Definition 6.21) is precisely the set of all (usual) globally stable equilibrium states (Definition 6.15):

Theorem 6.25 *For any mean-field model* $\mathfrak{m} \in \mathcal{M}_1$ *and inverse temperature* $\beta \in (0, \infty)$,

$$\Omega_{\mathfrak{m},\beta} = \overline{\mathrm{co}(\hat{M}_{\mathfrak{m},\beta})}\ .$$

Moreover, if $\Psi_- = 0$, *then* $\Omega_{\mathfrak{m},\beta} = \hat{M}_{\mathfrak{m},\beta}$.

Proof Apply Theorem 7.345 to the convex and weak*-compact space $K = E_1$ of invariant states and the real functional $\varphi = \mathfrak{f}_{\mathfrak{m},\beta}$ to show that the set M of minimizers of $\gamma(\mathfrak{f}_{\mathfrak{m},\beta})$ over E_1 is

$$M = \overline{\mathrm{co}\left(\Omega_{\mathfrak{m},\beta}\right)}\ .$$

As $\gamma(\mathfrak{g}_{\mathfrak{m},\beta}) = \gamma(\mathfrak{f}_{\mathfrak{m},\beta})$ (Lemma 6.24), we also deduce from Theorem 7.345 that

$$M = \overline{\mathrm{co}(\hat{M}_{\mathfrak{m},\beta})}\ .$$

[4] For any $\rho \in E$, the positive linear functional μ_ρ is associated with a probability measure on the set $\mathcal{E}_1$ of ergodic states such that

$$\rho = \int_{\mathcal{E}_1} \mathrm{d}\mu_\rho(\hat{\rho})\ \hat{\rho}$$

(in the weak sense). This is reminiscent of the Riesz-Markov theorem. This observation highlights the use of Jensen's inequality, which states that the image of an expectation value of a random variable by a convex function is less than or equal to the expectation value of the image of the random variable by the same function.

By Lemma 6.22, it follows that

$$\Omega_{\mathfrak{m},\beta} = \overline{\mathrm{co}\left(\Omega_{\mathfrak{m},\beta}\right)} = \overline{\mathrm{co}(\hat{M}_{\mathfrak{m},\beta})} \, .$$

Assume now $\Psi_- = 0$. Then, $\mathfrak{g}_{\mathfrak{m},\beta}$ becomes convex. So, $\hat{M}_{\mathfrak{m}}$ is also convex and weak*-compact, because of Lemma 6.22 (ii) and the equality

$$\overline{\mathrm{co}(\hat{M}_{\mathfrak{m},\beta})} = \hat{M}_{\mathfrak{m},\beta} \, . \qquad \square$$

6.5 Approximating Invariant Interactions

In the previous sections, we describe the set of globally stable equilibrium states of mean-field models by means of different variational problems. However, it is a priori not clear how useful these variational formulae are to study phase transitions. To answer to this question, it is convenient to consider the so-called Bogoliubov approximations of mean-field models, which are reminiscent of the "approximating Hamiltonian method" used in the past to compute the pressure associated with particular mean-field models, as explained in [1, Section 2.10]. In [1], we generalize this method in such a way that it can be applied to all elements of the Banach space of mean-field models, as well as to the corresponding equilibrium states. We use the viewpoint of game theory by interpreting the mean-field attractions Ψ_- and repulsions Ψ_+ of any model $\mathfrak{m} \doteq (\Phi, \Psi_-, \Psi_+) \in \mathcal{M}_1$ as attractive and repulsive players, respectively. This leads to a two-person zero-sum game named in [1] the "thermodynamic game," which is defined as follows:

Using the Hilbert space of square-integrable sequences

$$\ell^2(\mathbb{N}) \equiv \ell^2(\mathbb{N};\mathbb{C}) \doteq \left\{ c \equiv (c_n)_{n\in\mathbb{N}} \subseteq \mathbb{C} : \|c\|_2^2 \doteq \sum_{n\in\mathbb{N}} |c_n|^2 < \infty \right\} ,$$

we first define approximating (short-range) invariant interactions associated with mean-field models:

Definition 6.26 (Approximating Interactions) For any mean-field model $\mathfrak{m} \doteq (\Phi, \Psi_-, \Psi_+) \in \mathcal{M}_1$ and sequences $c_-, c^+ \in \ell^2(\mathbb{N})$, we define the corresponding "approximating interaction" to be

$$\Phi_{\mathfrak{m}}(c_-, c_+) \doteq \Phi + 2\sum_{n\in\mathbb{N}} \left(\mathrm{Re}\left\{\overline{c_+}\Psi_{+,n}\right\} - \mathrm{Re}\left\{\overline{c_-}\Psi_{-,n}\right\}\right) \in \mathcal{W}_1 \, .$$

This interaction is a well-defined element of the Banach space $\mathcal{W}_1$ because, for any $\mathfrak{m} \doteq (\Phi, \Psi_-, \Psi_+) \in \mathcal{M}_1$ and $c_-, c_+ \in \ell^2(\mathbb{N})$,

$$\|\Phi_{\mathfrak{m}}(c_-, c_+)\| \leq \|\Phi\| + 2\sum_{n\in\mathbb{N}} \left(\left|c_{+,n}\right| \left\|\Psi_{+,n}\right\| + \left|c_{-,n}\right| \left\|\Psi_{-,n}\right\|\right)$$
$$\leq \|\Phi\| + 2\|c_+\|_2 \|\Psi_+\|_2 + 2\|c_-\|_2 \|\Psi_-\|_2$$
$$\leq \max\left\{1, 2\|c_+\|_2, 2\|c_-\|_2\right\} \|\mathfrak{m}\| < \infty ,$$

thanks to the triangle and Cauchy-Schwarz inequalities. See Definition 6.3.

For each mean-field model $\mathfrak{m} \doteq (\Phi, \Psi_-, \Psi_+) \in \mathcal{M}_1$ and $c_-, c_+ \in \ell^2(\mathbb{N})$, observe from Definition 5.10 that, for any finite subset $\Lambda \in \mathcal{P}_f$,

$$H_\Lambda^{\Phi_{\mathfrak{m}}(c_-,c_+)} \doteq \sum_{\Lambda'\in\mathcal{P}_f,\ \Lambda'\subseteq\Lambda} \Phi(\Lambda') = H_\Lambda^\Phi + 2\sum_{n\in\mathbb{N}} \left(\mathrm{Re}\{\overline{c_+} H_\Lambda^{\Psi_{+,n}}\} - \mathrm{Re}\{\overline{c_-} H_\Lambda^{\Psi_{-,n}}\}\right).$$

Compare this expression with the full Hamiltonian

$$H_\Lambda^{\mathfrak{m}} \doteq H_\Lambda^\Phi + \frac{1}{|\Lambda|}\sum_{n\in\mathbb{N}} \left(|H_\Lambda^{\Psi_{+,n}}|^2 - |H_\Lambda^{\Psi_{-,n}}|^2\right)$$

associated with the mean-field model $\mathfrak{m} \doteq (\Phi, \Psi_-, \Psi_+) \in \mathcal{M}_1$ for $\Lambda \in \mathcal{P}_f$. See Definition 6.4. (Recall that the identification $\Phi \equiv (\Phi, 0, 0)$ for $\Phi \in \mathcal{W}_1$ is coherent with Definitions 5.10 and 6.4.) In particular,

$$|\Lambda|^{-1}(H_\Lambda^{\mathfrak{m}} - H_\Lambda^{\Phi_{\mathfrak{m}}(c_-,c_+)}) + \|c_+\|_2^2 - \|c_-\|_2^2 = \sum_{n\in\mathbb{N}} \Big(|(|\Lambda|^{-1} H_\Lambda^{\Psi_{+,n}} - c_{+,n})|^2$$
$$-|(|\Lambda|^{-1} H_\Lambda^{\Psi_{-,n}} - c_{-,n})|^2\Big) . \tag{6.16}$$

This last expression shall be considered in infinite volume limit $\Lambda \uparrow \Gamma$: If (6.16) would vanish as $\Lambda \uparrow \Gamma$, then one could replace the mean-field model by the simpler model given by the corresponding approximating interaction. Note, however, that this argument is only heuristic, since we compare in the left-hand side of (6.16) a sum over non-commuting elements of the spin or fermion algebra with complex numbers. In fact, the relation between mean-field models and its approximating interactions can be more properly understood via their respective pressures (Definition 6.14). See also Theorem 6.13, which links the pressure function with local Hamiltonians.

Using Definition 6.14 and Theorem 6.20, we first recall the pressures associated with mean-field models $\mathfrak{m} \doteq (\Phi, \Psi_-, \Psi_+) \in \mathcal{M}_1$ and their approximating interactions $\Phi_{\mathfrak{m}}(c_-, c_+)$ for $c_-, c_+ \in \ell^2(\mathbb{N})$, at a given inverse temperature $\beta \in (0, \infty)$:

- Pressure of mean-field models:

$$-\mathrm{p}_{\beta}(\mathfrak{m}) \doteq \inf \mathfrak{f}_{\mathfrak{m},\beta}(E_1) = \inf \mathfrak{f}_{\mathfrak{m},\beta}(\mathcal{E}_1) = \inf \mathfrak{g}_{\mathfrak{m},\beta}(\mathcal{E}_1) = \inf \mathfrak{g}_{\mathfrak{m},\beta}(E_1) \ , \tag{6.17}$$

  where, by Definition 6.17, for any invariant state $\rho \in E_1$,

$$\mathfrak{g}_{\mathfrak{m},\beta}(\rho) \doteq \|\mathfrak{e}_{\Psi_+}(\rho)\|_2^2 - \|\mathfrak{e}_{\Psi_-}(\rho)\|_2^2 + \mathfrak{f}_{\Phi,\beta}(\rho) \, , \tag{6.18}$$

  with $\mathfrak{e}_{\Psi}(\rho) \doteq \left(\mathfrak{e}_{\Psi_n}(\rho)\right)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$ for any $\Psi \in \ell^2(\mathbb{N}; \mathcal{W}_1^{\mathbb{C}})$.
- Pressure of approximating interactions:

$$-\mathrm{p}_{\beta}(\Phi_{\mathfrak{m}}(c_-, c_+)) \doteq \inf \mathfrak{f}_{\Phi_{\mathfrak{m}}(c_-,c_+),\beta}(E_1) = \inf \mathfrak{f}_{\Phi_{\mathfrak{m}}(c_-,c_+),\beta}(\mathcal{E}_1) \ , \tag{6.19}$$

  where, by Definition 6.12, for any invariant state $\rho \in E_1$,

$$\mathfrak{f}_{\Phi_{\mathfrak{m}}(c_-,c_+),\beta}(\rho) = 2\,\mathrm{Re}\left\langle c_+, \mathfrak{e}_{\Psi_+}(\rho)\right\rangle - 2\,\mathrm{Re}\left\langle c_-, \mathfrak{e}_{\Psi_-}(\rho)\right\rangle + \mathfrak{f}_{\Phi,\beta}(\rho) \tag{6.20}$$

  with $\langle \cdot, \cdot \rangle$ being the usual scalar product in the Hilbert space $\ell^2(\mathbb{N})$.

Keeping in mind (6.17) and (6.19), the question we shall answer is whether one can find particular sequences $d_+, d_- \in \ell^2(\mathbb{N})$ such that

$$\mathrm{p}_{\beta}(\mathfrak{m}) = \mathrm{p}_{\beta}(\Phi_{\mathfrak{m}}(d_+, d_-)) \ .$$

In fact, we construct such sequences via the so-called thermodynamic game associated with the given mean-field model. However, before explaining (later, in Sect. 6.7) in detail this game and the related construction of sequences, we make a simple observation, leading us to the appropriate payoff function for the thermodynamic game. In fact, one should compare (6.19) with (6.18) in light of the following equality:

Lemma 6.27 *For any invariant state $\rho \in E_1$ and every sequence $\Psi \in \ell^2(\mathbb{N}; \mathcal{W}_1^{\mathbb{C}})$,*

$$\sup_{c\in\ell^2(\mathbb{N})} \left\{ -\|c\|_2^2 + 2\,\mathrm{Re}\,\langle c, \mathfrak{e}_{\Psi}(\rho)\rangle \right\} = \|\mathfrak{e}_{\Psi}(\rho)\|_2^2$$

with unique maximizer $d(\rho) = \mathfrak{e}_{\Psi}(\rho) \doteq (\mathfrak{e}_{\Psi_n}(\rho))_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$.

Proof Obviously, for any invariant state $\rho \in E_1$, complex number $c \in \mathbb{C}$, and algebra element $A \in \mathcal{U}$,

$$|\rho(A - c)|^2 = |\rho(A)|^2 - 2\,\mathrm{Re}\,\{\rho(A)\bar{c}\} + |c|^2 \geq 0 \, , \tag{6.21}$$

which in turn implies that

$$|\rho(A)|^2 = \sup_{c\in\mathbb{C}} \left\{ -|c|^2 + 2\mathrm{Re}\{\rho(A)\bar{c}\} \right\}$$

with unique maximizer $d = \rho(A)$. This assertion yields the lemma, keeping in mind that $\mathfrak{e}_\Phi(\rho) = \rho(e_\Phi)$ for any invariant state $\rho \in E_1$ and complex interaction $\Phi \in \mathcal{W}_1^{\mathbb{C}}$. Here, $e_\Phi \in \mathcal{U}^e$ is defined by (6.7). □

Keeping in mind Eqs. (6.18) and (6.19) and Lemma 6.27, we define the following approximating free energy density for mean-field models:

Definition 6.28 (Approximating Free Energy Density) For any mean-field model $\mathfrak{m} \in \mathcal{M}_1$ and $\beta \in (0,\infty)$, the corresponding "approximating free energy density" is the function $h_{\mathfrak{m},\beta} : \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N}) \to \mathbb{R}$ defined by

$$h_{\mathfrak{m},\beta}(c_-, c_+) \doteq -\|c_+\|_2^2 + \|c_-\|_2^2 + \inf \mathfrak{f}_{\Phi_{\mathfrak{m}}(c_-,c_+),\beta}(E_1) \ .$$

The thermodynamic game will be the two-person zero-sum game whose payoff function is nothing else than the above-defined approximating free energy density for the given mean-field model. Before explaining this game in Sect. 6.7, as well as its consequences for the theory of equilibrium states of general mean-field models, we first study the special case of purely attractive mean-field models. In fact, considering the special attractive case gives some insight in how to tackle the above-explained problem for general mean-field models.

6.6 Purely Attractive Mean-Field Models and Application to the BCS Theory

6.6.1 Purely Attractive Mean-Field Models

Recall that mean-field models are elements $\mathfrak{m} \doteq (\Phi, \Psi_-, \Psi_+) \in \mathcal{M}_1$, where $\mathcal{M}_1 \doteq \mathcal{W}_1 \times \ell^2(\mathbb{N}; \mathcal{W}_1^{\mathbb{C}})^2$. See Definition 6.3. Recall that, for any such a mean-field model $\mathfrak{m}$, the component Ψ_- represents its mean-field attraction, while Ψ_+ is its mean-field repulsion, and, consequently, a mean-field model (Φ, Ψ_-, Ψ_+) is said to be purely attractive if $\Psi_+ = 0$, while it is purely repulsive if $\Psi_- = 0$. In this section, we are interested in the study of purely *attractive* mean-field models. This is the easiest mean-field case to study. Moreover, (partial) results referring to this particular case are pivotal to analyze the general case, later on. We start by proving a relation between the pressure function (Definition 6.14) of purely attractive mean-field models and the (payoff) function of Definition 6.28:

Proposition 6.29 *For any purely attractive mean-field model* $\mathfrak{m} \doteq (\Phi, \Psi_-, 0) \in \mathcal{M}_1$ *and inverse temperature* $\beta \in (0, \infty)$*,*

$$\mathfrak{p}_\beta(\mathfrak{m}) \doteq -\inf \mathfrak{f}_{\mathfrak{m},\beta}(E_1) = -\inf_{c_- \in \overline{B}_R(0)} h_{\mathfrak{m},\beta}(c_-, 0)$$

with $h_{\mathfrak{m},\beta}(c_-, 0)$ *defined as in Definition 6.28 and* $\overline{B}_R(0) \subseteq \ell^2(\mathbb{N})$ *being a closed ball of sufficiently large radius* $R > 0$*, centered at* 0*.*

Proof Fix $\mathfrak{m} \doteq (\Phi, \Psi_-, 0) \in \mathcal{M}_1$ and $\beta \in (0, \infty)$. By Theorem 6.20 and Definition 6.17,

$$\inf \mathfrak{f}_{\mathfrak{m},\beta}(E_1) = \inf \mathfrak{g}_{\mathfrak{m},\beta}(\mathcal{E}_1) = \inf_{\rho \in \mathcal{E}_1} \left\{ -\| \mathfrak{e}_{\Psi_-}(\rho) \|_2^2 + \mathfrak{f}_{\Phi,\beta}(\rho) \right\} .$$

From Lemma 6.27, Eqs. (6.19)–(6.20) and Definition 6.28, it follows that

$$\begin{aligned} \inf \mathfrak{f}_{\mathfrak{m},\beta}(E_1) &= \inf_{\rho \in \mathcal{E}_1} \inf_{c_- \in \ell^2(\mathbb{N})} \left\{ \|c_-\|_2^2 - 2\,\mathrm{Re}\, \langle c_-, \mathfrak{e}_\Psi(\rho) \rangle + \mathfrak{f}_{\Phi,\beta}(\rho) \right\} \\ &= \inf_{\rho \in \mathcal{E}_1} \inf_{c_- \in \ell^2(\mathbb{N})} \left\{ \|c_-\|_2^2 + \mathfrak{f}_{\Phi_{\mathfrak{m}}(c_-,0),\beta}(\rho) \right\} \\ &= \inf_{c_- \in \ell^2(\mathbb{N})} \left\{ \|c_-\|_2^2 + \inf \mathfrak{f}_{\Phi_{\mathfrak{m}}(c_-,0),\beta}(\mathcal{E}_1) \right\} \\ &= \inf_{c_- \in \ell^2(\mathbb{N})} \left\{ \|c_-\|_2^2 + \inf \mathfrak{f}_{\Phi_{\mathfrak{m}}(c_-,0),\beta}(E_1) \right\} \\ &= \inf_{c_- \in \ell^2(\mathbb{N})} h_{\mathfrak{m},\beta}(c_-, 0) \ . \end{aligned} \tag{6.22}$$

Finally, the existence of a radius $R > 0$ such that

$$\inf_{c_- \in \ell^2(\mathbb{N})} h_{\mathfrak{m}}(c_-, 0) = \inf_{c_- \in \overline{B}_R(0)} h_{\mathfrak{m},\beta}(c_-, 0)$$

directly follows from the fact that, for all sequences $c_- \in \ell^2(\mathbb{N})$,

$$\begin{aligned} |\inf \mathfrak{f}_{\Phi_{\mathfrak{m}}(c_-,0),\beta}(E_1)| &\leq 2 \sup_{\rho \in E_1} \left| \langle c_-, \mathfrak{e}_{\Psi_-}(\rho) \rangle \right| + \left| \inf \mathfrak{f}_{\Phi,\beta}(E_1) \right| \\ &\leq 2\|c\|_2 \|\Psi_-\| + \left| \inf \mathfrak{f}_{\Phi,\beta}(E_1) \right| \ , \end{aligned}$$

by the Cauchy-Schwarz inequality, as well as the bound $|\mathfrak{e}_{\Psi_-}(\rho)| \leq \|\Psi_-\|$ (Lemma 6.8). □

Proposition 6.29 is reminiscent of the so-called Bogoliubov approximation, which formally consists in replacing specific operators appearing in the Hamiltonian of a given physical system with constants that are determined as solutions to some self-consistency equation or to some associated variational problem.

In light of Proposition 6.29, the set of minimizers of the approximating free energy density $h_{\mathfrak{m},\beta}(\cdot, 0)$ should play an important role. As a consequence, we define the set

$$\mathcal{C}_{\mathfrak{m},\beta} \doteq \left\{ d_- \in \ell^2(\mathbb{N}) : h_{\mathfrak{m},\beta}(d_-, 0) = \inf_{c_- \in \ell^2(\mathbb{N})} h_{\mathfrak{m},\beta}(c_-, 0) \right\} \tag{6.23}$$

for any purely attractive mean-field model $\mathfrak{m} \doteq (\Phi, \Psi_-, 0) \in \mathcal{M}_1$ and every inverse temperature $\beta \in (0, \infty)$. The set $\mathcal{C}_{\mathfrak{m},\beta} \subseteq \ell^2(\mathbb{N})$ is nonempty, norm-bounded, and weakly compact when $\Psi_- \neq 0$. See [1, Lemma 8.4]. The next step is to understand the relation between the above set of minimizers of the approximating free energy density and globally stable equilibrium states.

Recall the definition of globally stable equilibrium states: For any mean-field model $\mathfrak{m} \in \mathcal{M}_1$ and $\beta \in (0, \infty)$,

$$\Omega_{\mathfrak{m},\beta} \doteq \Big\{ \omega \in E_1 : \exists (\rho_n)_{n\in\mathbb{N}} \subseteq E_1 \text{ weak}^* \text{ converging to } \omega \text{ so that } \lim_{n\to\infty} \mathfrak{f}_{\mathfrak{m},\beta}(\rho_n) = \inf \mathfrak{f}_{\mathfrak{m},\beta}(E_1) \Big\} .$$

See Definition 6.15. This set is always convex and weak*-compact, by Lemma 6.22 (i). When the model is purely attractive, the set of globally stable equilibrium states is a *face* of E_1. Recall that a face F of a convex set K is defined to be a subset of K with the property that, if $\rho = \lambda_1\rho_1 + \cdots + \lambda_n\rho_n \in F$ with $\rho_1, \ldots, \rho_n \in K$, $\lambda_1, \ldots, \lambda_n \in (0, 1)$ and $\lambda_1 + \cdots + \lambda_n = 1$, then $\rho_1, \ldots, \rho_n \in F$. See Definition 7.333.

Lemma 6.30 *For any model* $\mathfrak{m} \doteq (\Phi, \Psi_-, 0) \in \mathcal{M}_1$ *and* $\beta \in (0, \infty)$,

$$\Omega_{\mathfrak{m},\beta} = \left\{ \omega \in E_1 : \mathfrak{f}_{\mathfrak{m},\beta}(\omega) = \inf \mathfrak{f}_{\mathfrak{m},\beta}(E_1) \right\}$$

with extreme points being all ergodic, i.e., $\mathcal{E}(\Omega_{\mathfrak{m},\beta}) = \Omega_{\mathfrak{m},\beta} \cap \mathcal{E}_1$. *In particular, it is a (nonempty) weak*-closed face of the convex weak*-compact space* E_1 *of invariant states.*

Proof For $\beta \in (0, \infty)$ and any purely attractive mean-field model $\mathfrak{m} \doteq (\Phi, \Psi_-, 0) \in \mathcal{M}_1$, $\mathfrak{f}_{\mathfrak{m},\beta}$ is weak*-lower semicontinuous and affine; see Lemmata 6.7, 6.8, and 6.10 as well as Definition 6.12. The weak*-lower semicontinuity of $\mathfrak{f}_{\mathfrak{m},\beta}$ yields

$$\Omega_{\mathfrak{m},\beta} = \left\{ \omega \in E_1 : \mathfrak{f}_{\mathfrak{m},\beta}(\omega) = \inf \mathfrak{f}_{\mathfrak{m},\beta}(E_1) \right\} ,$$

while its affineness on the convex set E_1 of invariant states implies that the set $\mathcal{E}\left(\Omega_{\mathfrak{m},\beta}\right)$ of extreme points of $\Omega_{\mathfrak{m},\beta}$ belongs to the set $\mathcal{E}_1$ of ergodic states of E_1, i.e.,

$$\mathcal{E}\left(\Omega_{\mathfrak{m},\beta}\right) = \Omega_{\mathfrak{m},\beta} \cap \mathcal{E}_1 .$$

□

Lemma 6.30 of course holds true for all interactions $\Phi \equiv (\Phi, 0, 0) \in \mathcal{M}_1$, in particular for all approximating interactions of Definition 6.26, associated with any (not necessarily purely attractive) mean-field model.

Now, we are in a position to establish a precise relation between the solutions to either variational problems given in Proposition 6.29. This is done through globally stable equilibrium states associated with approximating interactions and leads to self-consistency conditions for these equilibrium states:

Proposition 6.31 (Gap Equations) *For any purely attractive mean-field model $\mathfrak{m} \doteq (\Phi, \Psi_-, 0) \in \mathcal{M}_1$ and inverse temperature $\beta \in (0, \infty)$, the following properties hold true:*

(i) *For all ergodic globally stable equilibrium states $\hat{\omega} \in \Omega_{\mathfrak{m},\beta} \cap \mathcal{E}_1$,*

$$d_- \doteq \mathfrak{e}_{\Psi_-}\left(\hat{\omega}\right) \doteq (\mathfrak{e}_{\Psi_{-,n}}\left(\hat{\omega}\right))_{n\in\mathbb{N}} \in \mathcal{C}_{\mathfrak{m},\beta}$$

and $\hat{\omega} \in \Omega_{\Phi_{\mathfrak{m}}(d_-,0),\beta}$.

(ii) *Conversely, for any fixed $d_- \in \mathcal{C}_{\mathfrak{m},\beta}$,*

$$\Omega_{\Phi_{\mathfrak{m}}(d_-,0),\beta} \cap \mathcal{E}_1 \subseteq \Omega_{\mathfrak{m},\beta} \cap \mathcal{E}_1$$

and every $\omega \in \Omega_{\Phi_{\mathfrak{m}}(d_-,0),\beta}$ satisfies the equality $d_- = \mathfrak{e}_{\Psi_-}(\omega)$.

Proof

(i) Any ergodic equilibrium state $\hat{\omega} \in \Omega_{\mathfrak{m},\beta} \cap \mathcal{E}_1$ is a solution to the right-hand side of (6.22), and the solution $d_- = d_-\left(\hat{\omega}\right)$ of

$$\inf_{c_-\in\ell^2(\mathbb{N})} \left\{\|c_-\|_2^2 + \mathfrak{f}_{\Phi_{\mathfrak{m}}(c_-,0),\beta}\left(\hat{\omega}\right)\right\}$$

satisfies the (Euler-Lagrange) equation $d_-\left(\hat{\omega}\right) = \mathfrak{e}_{\Psi_-}(\omega)$, by Lemma 6.27. The two infima in (6.22) commute with each other and, thus, $d_- = d_-\left(\hat{\omega}\right) \in \mathcal{C}_{\mathfrak{m},\beta}$ and $\hat{\omega} \in \Omega_{\Phi_{\mathfrak{m}}(d_-,0),\beta}$.

(ii) By definition, any sequence $d_- \in \mathcal{C}_{\mathfrak{m},\beta}$ satisfies

$$\|d_-\|_2^2 + \inf_{\rho\in E_1} \mathfrak{f}_{\Phi_{\mathfrak{m}}(d_-,0),\beta}(\rho) = \inf_{c_-\in\ell^2(\mathbb{N})} \left\{\|c_-\|_2^2 + \inf_{\rho\in E_1} \mathfrak{f}_{\Phi_{\mathfrak{m}}(c_-,0),\beta}(\rho)\right\}. \tag{6.24}$$

Since the two infima in the right-hand side of this equality commute with each other as before, any equilibrium state $\omega \in \Omega_{\Phi_{\mathfrak{m}}(d_-,0),\beta}$ satisfies $d_- = \mathfrak{e}_{\Psi_-}(\omega)$ because of Lemma 6.27 and

$$\Omega_{\Phi_{\mathfrak{m}}(d_-,0),\beta} \cap \mathcal{E}_1 \subseteq \Omega_{\mathfrak{m},\beta} \cap \mathcal{E}_1$$

because of Eq. (6.22).

□

Corollary 6.32 *For any purely attractive mean-field model* $\mathfrak{m} \doteq (\Phi, \Psi_-, 0) \in \mathcal{M}_1$ *and inverse temperature* $\beta \in (0, \infty)$,

$$\Omega_{\mathfrak{m},\beta} = \overline{\mathrm{co}\left(\bigcup_{d_- \in \mathcal{C}_{\mathfrak{m},\beta}} \Omega_{\Phi_{\mathfrak{m}}(d_-,0),\beta}\right)} .$$

Proof Combine Proposition 6.31 with Lemma 6.30. □

In the physics literature on superconductors, the self-consistency condition (Euler-Lagrange equation)

$$d_- = \mathfrak{e}_{\Psi_-}(\omega) \ , \qquad d_- \in \mathcal{C}_{\mathfrak{m},\beta}, \ \omega \in \Omega_{\Phi_{\mathfrak{m}}(d_-,0),\beta} \ ,$$

refers to the so-called gap equation. We keep this terminology here, although in a much broader and abstract sense. Proposition 6.31 and Corollary 6.32 demonstrate that, for all ergodic (globally stable) equilibrium states $\hat{\omega} \in \Omega_{\mathfrak{m},\beta} \cap \mathcal{E}_1$, the pair $(\hat{\omega}, \mathfrak{e}_{\Psi_-}(\hat{\omega}))$ solves the gap equation, since $\hat{\omega} \in \Omega_{\Phi_{\mathfrak{m}}(d_-,0),\beta}$. This mathematically justifies the theoretical physics approach using the above self-consistency condition to find the infinite-volume properties of mean-field models. Note that we have shown this property only for purely attractive mean-field models, so far, but we will explain it in the sequel for any general mean-field model.

6.6.2 Application to the BCS Theory on Lattices

The gap equation is pivotal to prove the existence of phase transitions for mean-field models. To illustrate this, as a physically relevant application, we describe the (reduced) BCS model of superconductivity:

(i) General Setup Like in Example 6.6, fix $\Omega \doteq \{\uparrow, \downarrow\}$ and $\mathcal{U} \doteq \mathrm{CAR}(\{\uparrow, \downarrow\}, \Gamma)$. We consider fermions in the cubic box

$$\Lambda_\ell \doteq \{(x_1, \ldots, x_d) \in \Gamma : |x_i| \leq \ell\}$$

for some fixed length $\ell \in \mathbb{N}$. As the BCS model is usually written in Fourier space, we additionally define

$$\Lambda_\ell^* \doteq \frac{2\pi}{(2\ell+1)} \Lambda_\ell \subseteq [-\pi, \pi]^d \ ,$$

the reciprocal lattice of quasi-momenta (referring to periodic boundary conditions). Then, for any spin $s \in \{\uparrow, \downarrow\}$ and (quasi-) momentum $k \in \Lambda_\ell^*$, let

$$\phi_{k,\mathrm{s}}(\mathrm{t}, x) \doteq \frac{1}{|\Lambda_\ell|^{1/2}} \chi_{\Lambda_\ell} \exp(-ik \cdot x)\, \delta_{\mathrm{s,t}}\,, \qquad x \in \Gamma,\ t \in \{\uparrow, \downarrow\}\,,$$

where $\delta_{\mathrm{s,t}}$ is the Kronecker delta, while χ_{Λ_ℓ} is the characteristic function of the cubic box Λ_ℓ.

(ii) The BCS Hamiltonian the Lattice Theoretical foundations of superconductivity go back to the celebrated BCS theory—appeared in the late 1950s (1957)—which explains conventional type I superconductors. The lattice version of this theory is based on the so-called (reduced) BCS Hamiltonian defined, for any $\ell \in \mathbb{N}$, by

$$\mathrm{H}_{\Lambda_\ell}^{BCS} \doteq \underbrace{\sum_{k \in \Lambda_\ell^*,\ \mathrm{s} \in \{\uparrow,\downarrow\}} \hat{\varepsilon}(k)\, \hat{a}_{k,\mathrm{s}}^* \hat{a}_{k,\mathrm{s}}}_{\text{kinetic term}} \underbrace{- \frac{1}{|\Lambda_\ell|} \sum_{k,q \in \Lambda_\ell^*} \eta_{k,q} \hat{a}_{k,\uparrow}^* \hat{a}_{-k,\downarrow}^* \hat{a}_{q,\downarrow} \hat{a}_{-q,\uparrow}}_{\text{attractive interactions}}\,,$$

where $\hat{a}_{k,\mathrm{s}} \doteq a(\phi_{k,\mathrm{s}})$ annihilates a fermion with spin $\mathrm{s} \in \{\uparrow, \downarrow\}$ and (quasi-) momentum $k \in \Lambda_\ell^*$, while $\hat{\varepsilon}$ is the Fourier transform of some real-valued function ε on Γ. In physics, $\{\hat{\varepsilon}(k)\}_{k \in \Lambda_\ell^*}$ is (up to some constant) the spectrum of the discrete Laplacian and

$$\eta_{k,q} = \begin{cases} \eta \geq 0 \text{ for } |k-q| \leq \mathrm{C} \\ 0 \qquad \text{for } |k-q| > \mathrm{C} \end{cases}$$

with constant $\mathrm{C} \in (0, \infty]$. For simplicity, take once and for all $\mathrm{C} = \infty$. In this case, the BCS Hamiltonian can be written in the "x-space" as

$$\mathrm{H}_{\Lambda_\ell}^{BCS} = \sum_{x,y \in \Lambda_\ell,\ \mathrm{s} \in \{\uparrow,\downarrow\}} \varepsilon(x-y)\, a_{x,\mathrm{s}}^* a_{y,\mathrm{s}} - \frac{\eta}{|\Lambda_\ell|} \sum_{x,y \in \Lambda_\ell} a_{x,\uparrow}^* a_{x,\downarrow}^* a_{y,\downarrow} a_{y,\uparrow} \tag{6.25}$$

for any $\ell \in \mathbb{N}$, where $a_{x,\mathrm{s}} \doteq a(\mathrm{e}_{\mathrm{s},x})$ annihilates a fermion with spin $\mathrm{s} \in \{\uparrow, \downarrow\}$ and lattice position $x \in \Gamma$. Here, $\{\mathrm{e}_{\mathrm{s},x}\}_{(\mathrm{s},x) \in \{\uparrow,\downarrow\} \times \Gamma}$ is the (canonical) Hilbert basis of $\ell^2(\{\uparrow, \downarrow\} \times \Gamma)$.

(iii) BCS Mean-Field Model Like in Example 6.6, the "BCS interaction" Ψ_{BCS} is defined by $\Psi_{\mathrm{BCS}}(\Lambda) \doteq 0$ whenever $|\Lambda| \notin \{1\}$ and $\Psi_{\mathrm{BCS}}(\{x\}) \doteq \eta^{1/2} a_{x,\downarrow} a_{x,\uparrow}$ for any $x \in \Gamma$. Then, for the purely attractive mean-field model

$$\mathfrak{n} \doteq (\Phi, (\Psi_{\mathrm{BCS}}, 0, 0, \ldots), 0) \in \mathcal{M}_1\,,$$

where $\Phi \in \mathcal{W}_1$ is some invariant interaction, we observe that

$$H_{\Lambda_\ell}^{\mathfrak{n}} = H_{\Lambda_\ell}^{\Phi} - \frac{\eta}{|\Lambda_\ell|} \sum_{x,y \in \Lambda_\ell} a_{x,\uparrow}^* a_{x,\downarrow}^* a_{y,\downarrow} a_{y,\uparrow}\,,$$

$$H_{\Lambda_\ell}^{\Phi_{\mathfrak{n}}(c_-,0)} = H_{\Lambda_\ell}^{\Phi} - \eta^{1/2} \sum_{x\in\Lambda_\ell} \left(c_{-,1} a_{x,\uparrow}^* a_{x,\downarrow}^* + \overline{c_{-,1}} a_{x,\downarrow} a_{x,\uparrow} \right) ,$$

for any $\ell \in \mathbb{N}$ and $c_- \in \ell^2(\mathbb{N})$. Note that the use of general sequences $c_- \in \ell^2(\mathbb{N})$ is not necessary in this example, since the model has only one non-zero attractive mean-field component, Ψ_{BCS}. One can thus consider constants $c_- \equiv c_{-,1} \in \mathbb{C}$, instead of full sequences $c_- \in \ell^2(\mathbb{N})$. By Corollary 6.32, if one is able to determine the set of states

$$\bigcup_{d_- \in \mathcal{C}_{\mathfrak{n},\beta}} \Omega_{\Phi_{\mathfrak{n}}(d_-,0),\beta} \tag{6.26}$$

then we obtain from it all the equilibrium states of the purely attractive mean-field model $\mathfrak{n}$. Under periodic boundary conditions [1, Chapter 3], we would then know all accumulation points of Gibbs states (in particular all correlation functions) associated with local Hamiltonians $H_{\Lambda_\ell}^{\mathfrak{n}}$, $\ell \in \mathbb{N}$.

(iv) Thermodynamic of the BCS Model Recall that ε is some real-valued function on Γ. We define the parameter $\Phi \in \mathcal{W}_1$ of the mean field model $\mathfrak{n}$ by

$$\begin{aligned}\Phi(\Lambda) \doteq \frac{1}{1+\delta_{x,y}} \Big(&\varepsilon(x-y) \left(a_{x,\uparrow}^* a_{y,\uparrow} + a_{x,\downarrow}^* a_{y,\downarrow} \right) \\ &+\varepsilon(y-x) \left(a_{y,\uparrow}^* a_{x,\uparrow} + a_{y,\downarrow}^* a_{x,\downarrow} \right) \Big)\end{aligned} \tag{6.27}$$

whenever $\Lambda = \{x, y\}$ and $\Phi(\Lambda) = 0$, otherwise. Here, $\delta_{x,y}$ is the Kronecker delta. Observe that, for any $\ell \in \mathbb{N}$,

$$H_{\Lambda_\ell}^{\mathfrak{n}} = \mathrm{H}_{\Lambda_\ell}^{BCS} ,$$

as well as

$$\begin{aligned}H_{\Lambda_\ell}^{\Phi_{\mathfrak{n}}(c_-,0)} = &\sum_{x,y\in\Lambda_\ell,\ \mathrm{s}\in\{\uparrow,\downarrow\}} \varepsilon(x-y)\, a_{x,\mathrm{s}}^* a_{y,\mathrm{s}} \\ &- \eta^{1/2} \sum_{x\in\Lambda_\ell} \left(c_- a_{x,\uparrow}^* a_{x,\downarrow}^* + \overline{c_-} a_{x,\downarrow} a_{x,\uparrow} \right)\end{aligned}$$

for any complex number $c_- \in \mathbb{C}$. This approximating model is quadratic in the annihilation and creation operators. Such Hamiltonians can be exactly diagonalized, which means that the corresponding pressure can be explicitly computed as a function of the parameter c_-. As a consequence, via Theorem 6.13, the approximating free energy density $h_{\mathfrak{n},\beta}(c_-, 0)$ of Definition 6.28, the solutions to the variational problem

$$\inf\left\{h_{\mathfrak{n},\beta}(c_-, 0) : c_- \in \mathbb{C}\right\} = \inf\left\{h_{\mathfrak{n},\beta}(c_-, 0) : |c_-| \leq R\right\}$$

of Proposition 6.29 and the set

$$\bigcup_{d_-\in\mathcal{C}_{\mathfrak{n},\beta}} \Omega_{\Phi_{\mathfrak{n}}(d_-,0),\beta}$$

can be accurately computed by analytic and/or numerical methods. Thus, the full thermodynamic behavior of the (reduced) BCS Hamiltonian $\mathrm{H}_{\Lambda_\ell}^{BCS}$, as $\ell \to \infty$, can be completely determined. In particular, one can show for large temperatures, i.e., $\beta^{-1} \gg 1$, that

$$\mathcal{C}_{\mathfrak{n},\beta} = \{0\} \quad \text{and} \quad \bigcup_{d_-\in\mathcal{C}_{\mathfrak{n},\beta}} \Omega_{\Phi_{\mathfrak{n}}(d_-,0),\beta} = \Omega_{\Phi_{\mathfrak{n}}(0,0),\beta} = \Omega_{\Phi,\beta} = \left\{\omega_\beta\right\} = \Omega_{\mathfrak{n},\beta} ,$$

thanks to Corollary 6.32. Moreover, if $\eta > 0$ is sufficiently large (and fixed for all $\beta > 0$), then there is an inverse temperature β_c such that, for any $\beta > \beta_c$,

$$\mathcal{C}_{\mathfrak{n},\beta} = \left\{\sqrt{\eta r}\exp(i\varphi) : \varphi \in [0, 2\pi)\right\}$$

and

$$\bigcup_{d_-\in\mathcal{C}_{\mathfrak{n},\beta}} \Omega_{\Phi_{\mathfrak{n}}(d_-,0),\beta} = \left\{\omega_{\beta,\varphi} : \varphi \in [0, 2\pi)\right\}$$

for some positive number $r > 0$. As a consequence of the self-consistency condition (gap equation), $\omega_{\beta,\varphi_1} \neq \omega_{\beta,\varphi_2}$ for any $\varphi_1, \varphi_2 \in [0, 2\pi)$ with $\varphi_1 \neq \varphi_2$. This refers to the existence of a superconducting (first-order) phase transition at inverse temperature $\beta_c > 0$, with the breakdown of the gauge invariance. Additionally, the order parameter $r \geq 0$ can be shown to be directly related, at all temperatures, to the Cooper pair condensate density

$$r = \lim_{\ell\to\infty} \frac{1}{|\Lambda_\ell|}\omega_{\mathrm{H}_{\Lambda_\ell}^{BCS},\beta}\left(\mathfrak{c}_0^*\mathfrak{c}_0\right) = \lim_{\ell\to\infty} \frac{1}{|\Lambda_\ell|}\frac{\mathrm{Tr}\left(\mathfrak{c}_0^*\mathfrak{c}_0\exp(-\beta\mathrm{H}_{\Lambda_\ell}^{BCS})\right)}{\mathrm{Tr}\left(\exp(-\beta\mathrm{H}_{\Lambda_\ell}^{BCS})\right)} ,$$

where

$$\mathfrak{c}_0 \doteq \frac{1}{\sqrt{|\Lambda_\ell|}}\sum_{x\in\Lambda_\ell} a_{x,\downarrow}a_{x,\uparrow} = \frac{1}{\sqrt{|\Lambda_\ell|}}\sum_{k\in\Lambda_\ell^*} \hat{a}_{k,\downarrow}\hat{a}_{-k,\uparrow}$$

annihilates one Cooper pair within the condensate, i.e., in the zero mode for electron pairs. The adjoint operator $\mathfrak{c}_0^*$ creates such a pair. Here, $\omega_{\mathrm{H}_{\Lambda_\ell}^{BCS},\beta}$ is the Gibbs state of Definition 5.19 associated with the BCS Hamiltonian $\mathrm{H}_{\Lambda_\ell}^{BCS}$. For more details, we recommend [26].

6.7 Thermodynamic Game

In [1], we generalize the results presented in Sect. 6.6 to all mean-field models of the Banach space $\mathcal{M}_1$. In the current subsection, we explain the main lines of this result. As mentioned above, we use the viewpoint of game theory, via the "thermodynamic game," that we now define precisely. First, recall that $h_{\mathfrak{m},\beta} : \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N}) \to \mathbb{R}$ is the approximating free energy density defined by

$$h_{\mathfrak{m},\beta}(c_-, c_+) \doteq -\|c_+\|_2^2 + \|c_-\|_2^2 + \inf \mathfrak{f}_{\Phi_{\mathfrak{m}}(c_-,c_+),\beta}(E_1)$$

for any mean-field model $\mathfrak{m} \in \mathcal{M}_1$ and inverse temperature $\beta \in (0, \infty)$. See Definitions 6.26 and 6.28. Given $\beta \in (0, \infty)$ and $\mathfrak{m} \doteq (\Phi, \Psi_-, \Psi_+) \in \mathcal{M}_1$, the thermodynamic game associated with the mean-field model $\mathfrak{m}$ is then the two-person zero-sum game whose payoff function is the approximating free energy density $h_{\mathfrak{m},\beta}$:

(i) The two players are denoted by $(-)$ and $(+)$. In fact, we interpret the mean-field attractions Ψ_- and repulsions Ψ_+ of the model $\mathfrak{m} \doteq (\Phi, \Psi_-, \Psi_+)$ as two players that we, respectively, call the attractive and the repulsive player.

(ii) The sets of strategies of the attractive and repulsive player are, respectively, the following subspaces of $\ell^2(\mathbb{N})$:

$$\ell_-^2 \doteq \{c_- \in \ell^2(\mathbb{N}) \ : \ \text{for all } n \in \mathbb{N}, c_{-,n} = 0 \text{ if } \Psi_{-,n} = 0\} ,$$
$$\ell_+^2 \doteq \{c_+ \in \ell^2(\mathbb{N}) \ : \ \text{for all } n \in \mathbb{N}, c_{+,n} = 0 \text{ if } \Psi_{+,n} = 0\} .$$

(iii) The value $h_{\mathfrak{m},\beta}(c_-, c_+) \in \mathbb{R}$ is the loss of the player $(-)$ for the (attractive) strategy $c_- \in \ell_-^2$ and the gain of the second for the (repulsive) strategy $c_+ \in \ell_+^2$:

$(-)$ Without exchange of information, by minimizing

$$h^\sharp_{\mathfrak{m},\beta}(c_-) \doteq \sup_{c_+ \in \ell_+^2} h_{\mathfrak{m},\beta}(c_-, c_+) ,$$

the player $(-)$ obtains her/his least maximum loss

$$\mathrm{F}^\sharp_{\mathfrak{m},\beta} \doteq \inf_{c_- \in \ell_-^2} h^\sharp_{\mathfrak{m},\beta}(c_-) .$$

$(+)$ By maximizing

$$h^\flat_{\mathfrak{m},\beta}(c_+) \doteq \inf_{c_- \in \ell_-^2} h_{\mathfrak{m},\beta}(c_-, c_+) ,$$

the player $(+)$ obtains her/his greatest minimum gain

$$\mathrm{F}_{\mathfrak{m},\beta}^{\flat} \doteq \sup_{c_{+}\in\ell_{+}^{2}} h_{\mathfrak{m},\beta}^{\flat}(c_{+}) \leq \mathrm{F}_{\mathfrak{m},\beta}^{\sharp} .$$

$\mathrm{F}_{\mathfrak{m},\beta}^{\flat}$ and $\mathrm{F}_{\mathfrak{m},\beta}^{\sharp}$ are called the "conservative values" of the thermodynamic game, while $[\mathrm{F}_{\mathfrak{m},\beta}^{\flat}, \mathrm{F}_{\mathfrak{m},\beta}^{\sharp}]$ is its "duality interval." Observe that, in general, $\mathrm{F}_{\mathfrak{m},\beta}^{\flat} < \mathrm{F}_{\mathfrak{m},\beta}^{\sharp}$. That is, the thermodynamic game may not admit a "cooperative equilibrium," which is, by definition, any saddle point of the payoff function $h_{\mathfrak{m},\beta}$. See [1, p. 42].

(iv) The corresponding sets of "conservative strategies" are

$$\begin{aligned} \mathcal{C}_{\mathfrak{m},\beta}^{\flat} &\doteq \left\{ d_{+}\in\ell_{+}^{2} : \mathrm{F}_{\mathfrak{m},\beta}^{\flat} = h_{\mathfrak{m},\beta}^{\flat}(d_{+}) \right\} , \\ \mathcal{C}_{\mathfrak{m},\beta}^{\sharp} &\doteq \left\{ d_{-}\in\ell_{-}^{2} : \mathrm{F}_{\mathfrak{m},\beta}^{\sharp} = h_{\mathfrak{m},\beta}^{\sharp}(d_{-}) \right\} . \end{aligned} \tag{6.28}$$

In the particular case of a purely repulsive mean-field model, i.e., when $\Psi_{-} = 0$, $\mathcal{C}_{\mathfrak{m},\beta}^{\sharp} = \{0\}$, just because $\ell_{-}^{2} = \{0\}$. Similarly, if $\Psi_{+} = 0$, then $\mathcal{C}_{\mathfrak{m},\beta}^{\flat} = \{0\}$. In both cases ($\Psi_{-} = 0$ or $\Psi_{+} = 0$), we have

$$\mathrm{F}_{\mathfrak{m},\beta}^{\flat} = \mathrm{F}_{\mathfrak{m},\beta}^{\sharp} = -\mathfrak{p}_{\beta}(\mathfrak{m}) . \tag{6.29}$$

See Proposition 6.29 for the purely attractive case. For a justification of this equality in the purely repulsive case, see the proof of Theorem 6.34 below. By [1, Lemma 8.4], the sets of conservatives strategies have the following important properties:

Proposition 6.33 *For any mean-field model* $\mathfrak{m} \doteq (\Phi, \Psi_{-}, \Psi_{+}) \in \mathcal{M}_{1}$ *and inverse temperature* $\beta \in (0, \infty)$*, the sets of conservatives strategies have the following properties:*

($\flat$) $\mathcal{C}_{\mathfrak{m},\beta}^{\flat} \subseteq \ell_{+}^{2} \subseteq \ell^{2}(\mathbb{N})$ *has exactly one element* d_{+}*.*
($\sharp$) $\mathcal{C}_{\mathfrak{m},\beta}^{\sharp} \subseteq \ell_{-}^{2} \subseteq \ell^{2}(\mathbb{N})$ *is nonempty and norm-bounded.*

The relevance of the thermodynamic game results from the fact that the conservative values $\mathrm{F}_{\mathfrak{m},\beta}^{\flat}$ and $\mathrm{F}_{\mathfrak{m},\beta}^{\sharp}$ of the game can be written as *variational problems over states*, corresponding in particular to the pressure function (Definition 6.14). This refers to a generalization of Proposition 6.29 to all (not necessarily purely attractive) mean-field models. To state the assertions, we recall two free energy functionals associated with mean-field models $\mathfrak{m} \doteq (\Phi, \Psi_{-}, \Psi_{+}) \in \mathcal{M}_{1}$ at a given inverse temperature $\beta \in (0, \infty)$:

- By Definition 6.12, the usual free energy density functional $\mathfrak{f}_{\mathfrak{m},\beta} : E_{1} \to \mathbb{R}$ is defined by

$$\mathfrak{f}_{\mathfrak{m},\beta} = \Delta_{\Psi_{+}} - \Delta_{\Psi_{-}} + \mathfrak{f}_{\Phi,\beta} .$$

- In the proof of Lemma 6.24, Eq. (6.14), we introduce also a non-conventional free energy density functional $\mathfrak{f}_{\mathfrak{m},\beta}^{\flat} : E_1 \to \mathbb{R}$, defined by

$$\mathfrak{f}_{\mathfrak{m},\beta}^{\flat}(\rho) \doteq \| \mathfrak{e}_{\Psi_+}(\rho) \|_2^2 - \Delta_{\Psi_-}(\rho) + \mathfrak{f}_{\Phi,\beta}(\rho) \ . \tag{6.30}$$

Note that $\mathfrak{f}_{\mathfrak{m},\beta}^{\flat} \leq \mathfrak{f}_{\mathfrak{m},\beta}$, by Theorem 6.11 (v). We are now in a position to give the main statement of this subsection:

Theorem 6.34 *For any mean-field model* $\mathfrak{m} \in \mathcal{M}_1$ *and inverse temperature* $\beta \in (0, \infty)$*, the conservative values equal:*

$$\mathrm{F}_{\mathfrak{m},\beta}^{\flat} \doteq \sup_{c_+ \in \ell_+^2} \inf_{c_- \in \ell_-^2} h_{\mathfrak{m},\beta}(c_-, c_+) = \inf \mathfrak{f}_{\mathfrak{m},\beta}^{\flat}(E_1) \ ,$$

$$\mathrm{F}_{\mathfrak{m},\beta}^{\sharp} \doteq \inf_{c_- \in \ell_-^2} \sup_{c_+ \in \ell_+^2} h_{\mathfrak{m},\beta}(c_-, c_+) = \inf \mathfrak{f}_{\mathfrak{m},\beta}(E_1) \ .$$

Idea of Proof The complete proof of this theorem can be found in [1]. See in particular [1, Theorem 2.36]. This is done in a similar way as in Proposition 6.29. The main issue now is that the infimum and supremum defining $\mathrm{F}_{\mathfrak{m},\beta}^{\flat}$ and $\mathrm{F}_{\mathfrak{m},\beta}^{\sharp}$ do not generally commute with each other. In fact, as already remarked above, one has in general that $\mathrm{F}_{\mathfrak{m},\beta}^{\flat}$ is strictly smaller than $\mathrm{F}_{\mathfrak{m},\beta}^{\sharp}$. To circumvent this problem, we proceed as follows: Note from Proposition 6.29 that

$$\begin{aligned} \mathrm{F}_{\mathfrak{m},\beta}^{\flat} &= \sup_{c_+ \in \ell_+^2} \inf_{c_- \in \ell_-^2} \inf_{\rho \in E_1} \left\{ -\|c_+\|_2^2 + \|c_-\|_2^2 + \mathfrak{f}_{\Phi_{\mathfrak{m}}(c_-, c_+),\beta}(\rho) \right\} \\ &= \sup_{c_+ \in \ell_+^2} \inf_{\rho \in E_1} \left\{ -\|c_+\|_2^2 + \mathfrak{f}_{(\Phi_{\mathfrak{m}}(0,c_+),\Psi_-,0),\beta}(\rho) \right\} \ . \end{aligned}$$

Now, by the von Neumann min-max theorem [1, Theorem 10.50], the new functional

$$(c_+, \rho) \mapsto -\|c_+\|_2^2 + \mathfrak{f}_{(\Phi_{\mathfrak{m}}(0,c_+),\Psi_-,0),\beta}(\rho)$$

on $\ell_+^2 \times E_1$ has a saddle point and the infimum and supremum in the last equality can be interchanged. Doing this, one computes that

$$\mathrm{F}_{\mathfrak{m},\beta}^{\flat} = \inf \mathfrak{f}_{\mathfrak{m},\beta}^{\flat}(E_1) \ . \tag{6.31}$$

Note that by combining this equality with (6.17), one proves the identity (6.29) for the purely repulsive case. To prove the second part of the theorem, i.e., the equality $\mathrm{F}_{\mathfrak{m},\beta}^{\sharp} = \inf \mathfrak{f}_{\mathfrak{m},\beta}(E_1)$, the trick with the saddle point is not necessary anymore, because one can directly use (6.31) instead: In fact, observe that (6.31) yields

$$\sup_{c_+ \in \ell^2_+} h_{\mathfrak{m},\beta}(c_-, c_+) = \|c_-\|_2^2 + \inf_{\rho \in E_1} \{ \| \mathfrak{e}_{\Psi_+}(\rho) \|_2^2 + \mathfrak{f}_{\Phi_{\mathfrak{m}}(c_-,0),\beta}(\rho) \} .$$

Thus, by Lemma 6.27 combined with (6.17),

$$\begin{aligned} \inf_{c_- \in \ell^2_-} \sup_{c_+ \in \ell^2_+} h_{\mathfrak{m},\beta}(c_-, c_+) &= \inf_{\rho \in E_1} \{ \| \mathfrak{e}_{\Psi_+}(\rho) \|_2^2 - \| \mathfrak{e}_{\Psi_-}(\rho) \|_2^2 + \mathfrak{f}_{\Phi,\beta}(\rho) \} \\ &= \inf \mathfrak{f}_{\mathfrak{m},\beta}(E_1) . \end{aligned}$$

Compare this last argument with the proof of Proposition 6.29. □

By Definition 6.14 and Theorem 6.34, note that the pressure of any mean-field model $\mathfrak{m} \in \mathcal{M}_1$ is equal to

$$\begin{aligned} \mathfrak{p}_\beta(\mathfrak{m}) \doteq -\inf \mathfrak{f}_{\mathfrak{m},\beta}(E_1) &= - \inf_{c_- \in \ell^2(\mathbb{N})} \sup_{c_+ \in \ell^2(\mathbb{N})} h_{\mathfrak{m},\beta}(c_-, c_+) \\ &= - \inf_{c_- \in \ell^2_-} \sup_{c_+ \in \ell^2_+} h_{\mathfrak{m},\beta}(c_-, c_+) . \end{aligned}$$

Recall that the infimum and supremum in this expression do not commute in general. A sufficient condition for them to commute is given through Sion's minimax theorem [27] as follows:

Lemma 6.35 *Let $\beta \in (0, \infty)$ and $\mathfrak{m} \doteq (\Phi, \Psi_-, \Psi_+) \in \mathcal{M}_1$ be any mean-field model such that $\Psi_- \neq 0$ and $\Psi_+ \neq 0$. If, for any fixed $c_+ \in \ell^2(\mathbb{N})$, the function $h_{\mathfrak{m},\beta}(\cdot, c_+)$ on $\ell^2(\mathbb{N})$ is quasi-convex, i.e., for all $r \in \mathbb{R}$, the level set*

$$\left\{ c_- \in \ell^2(\mathbb{N}) : h_{\mathfrak{m},\beta}(c_-, c_+) \leq r \right\}$$

is convex, then $\mathrm{F}^\sharp_{\mathfrak{m},\beta} = \mathrm{F}^\flat_{\mathfrak{m},\beta}$.

Proof [28, Lemma 4.2]. □

To conclude, a result like Theorem 6.34 justifies on the level of thermodynamic functions the replacement of specific operators appearing in the Hamiltonian of a given physical system by constants which are determined as solutions to some self-consistency equation or some associated variational problem. This refers to the Bogoliubov approximation, which was used for (purely attractive mean-field) Fermi systems on lattices, already in 1957, to derive the celebrated Bardeen-Cooper-Schrieffer (BCS) theory for conventional type I superconductors [29–31]. The authors were of course inspired by Bogoliubov and his revolutionary paper [32]. A rigorous justification of this theory was given on the level of ground states by Bogoliubov in 1960 [33]. Then a method for analyzing the Bogoliubov approximation in a systematic way—on the level of the pressure—like in Theorem 6.34 with both mean-field repulsions and attractions was introduced by Bogoliubov Jr. in 1966 [34, 35] and by Brankov, Kurbatov, Tonchev, and Zagrebnov during the 1970s and 1980s [36–38]. This method is known in the literature as the *approximating Hamiltonian method* and leads—on the class of Hamiltonians it applies—to a rigorous proof of the exactness of the Bogoliubov approximation on the level of the pressure, provided it is done in an appropriated manner. Note however that the conditions on model imposed by [36–38] are still much more restrictive than those of Theorem 6.34. See discussions in [1, Section 2.10].

6.8 Self-Consistency of Equilibrium States

In Sect. 6.7, we introduce the thermodynamic game, which provides an efficient method to study phase transitions driven by mean-field interactions. It refers to a two-person zero-sum game whose payoff functions is defined as being the approximating free energy density functional of Definition 6.28. By Theorem 6.34, the conservative values of this game are directly related with variational problems over invariant states, naturally associated with any mean-field model. In fact, as we have seen, the largest of both conservative values is nothing else than the conventional pressure.

It turns out that, like in the special case of purely attractive mean-field models (cf. Corollary 6.32), the thermodynamic game also provides a complete characterization of the set of globally stable equilibrium states (Definition 6.15) of mean-field models, as follows:

Recall that the set of globally stable equilibrium states refers to

$$\Omega_{\mathfrak{m},\beta} \doteq \Big\{ \omega \in E_1 : \exists (\rho_n)_{n\in\mathbb{N}} \subseteq E_1 \text{ weak}^* \text{ converging to } \omega \\ \text{so that } \lim_{n\to\infty} \mathfrak{f}_{\mathfrak{m},\beta}(\rho_n) = \inf \mathfrak{f}_{\mathfrak{m},\beta}(E_1) \Big\}$$

for any mean-field model $\mathfrak{m} \in \mathcal{M}_1$ and inverse temperature $\beta \in (0, \infty)$. Having in mind the second variational problem of Theorem 6.34, we also define the set

$$\Omega^\flat_{\mathfrak{m},\beta} \doteq \Big\{ \omega \in E_1 : \mathfrak{f}^\flat_{\mathfrak{m},\beta}(\omega) = \inf \mathfrak{f}^\flat_{\mathfrak{m},\beta}(E_1) \Big\}$$

of non-conventional (globally stable) equilibrium states. Note that $\mathfrak{f}^\flat_{\mathfrak{m}}$ is weak*-lower semicontinuous but only convex (and not affine). In particular, $\Omega^\flat_{\mathfrak{m},\beta}$ is a nonempty weak*-compact convex subset of E_1.

In [1, Lemma 8.3 (♯)], it is proven that, for any $\mathfrak{m} \doteq (\Phi, \Psi_-, \Psi_+) \in \mathcal{M}_1$ with $\Psi_+ \neq 0$, and all functions $c_- \in \ell^2(\mathbb{N})$, the set

$$\left\{ d_+ \in \ell^2_+ : \max_{c_+\in\ell^2_+} h_{\mathfrak{m},\beta}(c_-, c_+) = h_{\mathfrak{m},\beta}(c_-, d_+) \right\} \tag{6.32}$$

has exactly one element, which is denoted by $\mathrm{r}_+(c_-)$. By [1, Lemma 8.8], if $\Psi_+ \neq 0$, then the mapping

$$\mathrm{r}_+ : c_- \mapsto \mathrm{r}_+(c_-) \tag{6.33}$$

defines a continuous functional from ℓ^2_- to ℓ^2_+ itself, i.e., from the set of attractive strategies to the set of repulsive strategies of the mean-field model $\mathfrak{m}$. We call this mapping "the thermodynamic decision rule" of the mean-field model $\mathfrak{m} \in \mathcal{M}_1$. Note that in the particular case of purely attractive mean-field models (i.e., when $\Psi_+ = 0$ and $h_{\mathfrak{m},\beta}$ is thus not depending on c_+), one has $\mathrm{r}_+ = 0$.

For any mean-field model $\mathfrak{m} \doteq (\Phi, \Psi_-, \Psi_+) \in \mathcal{M}_1$, it is convenient to introduce a family of approximating purely attractive mean-field models by

$$\mathfrak{m}(c_+) \doteq (\Phi_{\mathfrak{m}}(0, c^+), \Psi_-, 0) \in \mathcal{M}_1 , \qquad c_+ \in \ell^2(\mathbb{N}) . \tag{6.34}$$

Then, for every pair of strategies $c_- \in \ell^2_-$, $c_+ \in \ell^2_+$, we define the (possibly empty) sets

$$\Omega_{\mathfrak{m},\beta}(c_-, c_+) \doteq \left\{\omega \in \Omega_{\Phi_{\mathfrak{m}}(c_-,c_+),\beta} : \mathfrak{e}_{\Psi_-}(\omega) = c_- \quad \text{and} \quad \mathfrak{e}_{\Psi_+}(\omega) = c_+\right\} \subseteq E_1 \tag{6.35}$$

as well as

$$\Omega_{\mathfrak{m},\beta}(c_+) \doteq \left\{\omega \in \Omega_{\mathfrak{m}(c_+),\beta} : \mathfrak{e}_{\Psi_+}(\omega) = c_+\right\} \subseteq E_1 , \tag{6.36}$$

where, for any fixed invariant state $\rho \in E_1$ and $\Psi \in \ell^2(\mathbb{N}; \mathcal{W}_1^{\mathbb{C}})$,

$$\mathfrak{e}_{\Psi}(\rho) \doteq (\mathfrak{e}_{\Psi_n}(\rho))_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}) \qquad \text{with} \qquad \mathfrak{e}_{\Psi_n}(\rho) \doteq \rho\left(e_{\Psi_n}\right)$$

for all $n \in \mathbb{N}$. By Lemma 6.30, note that $\Omega_{\Phi_{\mathfrak{m}}(c_-,c_+),\beta}$ and $\Omega_{\mathfrak{m}(c_+),\beta}$ are (nonempty) weak*-closed faces of E_1, since $\mathfrak{m}(c_+)$ is a purely attractive mean-field model. Then, we obtain a (static) self-consistency condition for (conventional or non-conventional) globally state equilibrium states, which refers, in a sense, to Euler-Lagrange equations for the variational problem defining the thermodynamic game. More precisely, we have the following statements:

Theorem 6.36 *For any mean-field model* $\mathfrak{m} \doteq (\Phi, \Psi_-, \Psi_+) \in \mathcal{M}_1$ *and fixed inverse temperature* $\beta \in (0, \infty)$*, the following properties hold true:*

(i)

$$\Omega_{\mathfrak{m},\beta} = \overline{\mathrm{co}\left(\bigcup_{d_- \in \mathcal{C}^\sharp_{\mathfrak{m},\beta}} \Omega_{\mathfrak{m},\beta}\left(d_-, \mathrm{r}_+(d_-)\right)\right)} .$$

(ii) *The set $\mathcal{E}(\Omega_{\mathfrak{m},\beta})$ of extreme points of the weak*-compact convex set $\Omega_{\mathfrak{m},\beta}$ is included in the union of the sets*

$$\mathcal{E}\left(\Omega_{\mathfrak{m},\beta}\left(d_-, \mathrm{r}_+(d_-)\right)\right) , \qquad d_- \in \mathcal{C}^\sharp_{\mathfrak{m},\beta} ,$$

of all extreme points of $\Omega_{\mathfrak{m},\beta}(d_-, \mathrm{r}_+(d_-))$, $d_- \in \mathcal{C}^\sharp_{\mathfrak{m},\beta}$, which are nonempty, convex, mutually disjoint, weak-closed subsets of E_1.*

(iii) *When $\Psi_+ \neq 0$,*

$$\mathcal{C}^\flat_{\mathfrak{m},\beta} = \{d_+\} \qquad \mathit{and} \qquad \Omega^\flat_{\mathfrak{m},\beta} = \Omega_{\mathfrak{m},\beta}(d_+) .$$

Proof Assertion (i) results from [1, Theorem 2.21 (i)] and [1, Theorem 2.39 (i)], while (ii) corresponds to [1, Theorem 2.39 (ii)]. As already mentioned, the fact that $\mathcal{C}^\flat_{\mathfrak{m},\beta} = \{d_+\}$ refers to Proposition 6.33. However, the identity $\Omega^\flat_{\mathfrak{m},\beta} = \Omega_{\mathfrak{m},\beta}(d_+)$ was not considered in [1], but its proof is similar to the one of [1, Lemma 9.2]. See [28, Theorem 4.3]. For more details, see also Theorem 7.346 and discussions before and after this theorem, which explain in a general context the strategy of proof used here. □

Theorem 6.36 implies in particular that, for any extreme state $\hat{\omega} \in \mathcal{E}(\Omega_{\mathfrak{m},\beta})$ of $\Omega_{\mathfrak{m},\beta}$, there is a unique $d_- \in \mathcal{C}^\sharp_{\mathfrak{m},\beta}$ such that

$$\mathfrak{e}_{\Psi_-}(\omega) = d_- \quad \text{and} \quad \mathfrak{e}_{\Psi_+}(\omega) = \mathrm{r}_+(d_-) . \tag{6.37}$$

In the physics literature on superconductors, recall that the above equality refers to the so-called gap equations. Conversely, for any $d_- \in \mathcal{C}^\sharp_{\mathfrak{m},\beta}$, there is some generalized equilibrium state ω satisfying the condition above, but ω is not necessarily an extreme point of $\Omega_{\mathfrak{m},\beta}$.

To conclude, note that Theorem 6.36 yields the equality $\Omega_{\mathfrak{m},\beta} = \Omega^\flat_{\mathfrak{m},\beta}$ for any purely repulsive or purely attractive mean-field model $\mathfrak{m} \in \mathcal{M}_1$. However, for mean-field models $\mathfrak{m} \in \mathcal{M}_1$ with both non-trivial attractive and repulsive mean-field interactions, there is no reason for this equality to hold true, in general.

6.9 From Short-Range to Mean-Field Models

Realistic effective interparticle interactions of quantum many-body systems are widely seen as being short-range, not mean-field. However, the rigorous mathematical analysis of phase diagrams of short-range model turns out to be extremely difficult, in general, with many important fundamental questions remaining open still nowadays. By contrast, mean-field models come from different approximations or Ansätze, and are thus less realistic, in a sense, but are technically advantageous, while capturing surprisingly well many real physical phenomena. Indeed, the study of phase diagrams of mean-field models can be performed by self-consistency equations related to the associated thermodynamic game. This is illustrated at the end of Sect. 6.6 for the BCS theory of superconductivity.

Here, we discuss a precise mathematical relation between mean-field and short-range models, by using the long-range limit that is known in the literature as the Kac limit. This is done in [28] in an abstract, model-independent, way. To be more pedagogical, however, we restrict our discussions to a specific example. This gives us, additionally, the opportunity to illustrate results of previous sections, in particular those of Sects. 6.7–6.8, for a specific mean-field model having both positive and attractive mean-field terms.

6.9.1 *The Short-Range Model*

Like in Example 6.6, fix $\Omega \doteq \{\uparrow, \downarrow\}$ and $\mathcal{U} \doteq \mathrm{CAR}(\{\uparrow, \downarrow\}, \Gamma)$. $\{\mathrm{e}_{\mathrm{s},x}\}_{(\mathrm{s},x)\in\Omega\times\Gamma}$ is, as before, the (canonical) Hilbert basis of $\ell^2(\Omega \times \Gamma)$. We use the shorter notation $a_{x,\mathrm{s}} \doteq a\left(\mathrm{e}_{\mathrm{s},x}\right)$ for the "annihilation operator" of a fermion with spin $\mathrm{s} \in \Omega$ and lattice position $x \in \Gamma$. We consider fermions inside the cubic box

$$\Lambda_\ell \doteq \{(x_1, \ldots, x_d) \in \Gamma : |x_i| \leq \ell\}$$

for any $\ell \in \mathbb{N}$. Fix once and for all, in the present subsection, an invariant interaction $\Phi \in \mathcal{W}_1$. For two parameters $\gamma_-, \gamma_+ \in (0, 1)$ and the fixed invariant interaction $\Phi \in \mathcal{W}_1$, we define the local Hamiltonians

$$H_{\Lambda_\ell}(\gamma_-, \gamma_+) \doteq H_{\Lambda_\ell}^{\Phi} + \sum_{x,y\in\Lambda_\ell,\mathrm{s},\mathrm{t}\in\{\uparrow,\downarrow\}} \gamma_+^d \mathrm{v}_+ \left(\gamma_+ (x-y)\right) a_{y,\mathrm{t}}^* a_{y,\mathrm{t}} a_{x,\mathrm{s}}^* a_{x,\mathrm{s}} - \sum_{x,y\in\Lambda_\ell} \gamma_-^d \mathrm{v}_- \left(\gamma_- (x-y)\right) a_{y,\uparrow}^* a_{y,\downarrow}^* a_{x,\downarrow} a_{x,\uparrow}. \tag{6.38}$$

Here, v_+ is a (non-zero) pair potential characterizing interparticle forces, whose range of action is tuned by the parameter $\gamma_+ \in (0, 1)$. The (non-zero) function v_- encodes the hopping strength of Cooper pairs. The corresponding term of the

Hamiltonian thus implements a BCS-type interaction whose range is tuned by the parameter $\gamma_- \in (0,1)$.

As is usual in theoretical physics, $\mathrm{v}_-, \mathrm{v}_+$ are assumed to be fast decaying, reflection-symmetric,[5] and positive definite, i.e., the Fourier transform $\hat{\mathrm{v}}_-, \hat{\mathrm{v}}_+$ of v are positive functions on $\mathbb{R}^d$. This choice for v_+ is reminiscent of a superstability condition, which is essential in the bosonic case [39, Section 2.2 and Appendix G]. For simplicity, we assume that $\mathrm{v}_-, \mathrm{v}_+ \in C_0^{2d}\left(\mathbb{R}^d, \mathbb{R}\right)$ are both compactly supported. Because of some technical issues, we also assume that

$$\hat{\mathrm{v}}_-(\gamma^{-1}k) \leq \hat{\mathrm{v}}_-(k) \ , \qquad k \in \mathbb{R}^d \ , \ \gamma \in (0,1) \ .$$

The definition of the Fourier transform of a function v we used here is

$$\hat{\mathrm{v}}(k) \doteq \int_{\mathbb{R}^d} \mathrm{v}(x)\, \mathrm{e}^{-ik\cdot x} \mathrm{d}^d x \ , \qquad k \in \mathbb{R}^d \ . \tag{6.39}$$

Observe that the sequence of local Hamiltonians $H_{\Lambda_\ell}(\gamma_-,\gamma_+)$, $\ell \in \mathbb{N}$, is the one associated with the invariant interaction:

$$\Phi(\gamma_-,\gamma_+) \doteq \Phi + \Psi_{\mathrm{v}_+,\gamma_+} - \Psi_{\mathrm{v}_-,\gamma_-} \in \mathcal{W}_1 \ ,$$

where the invariant interactions $\Psi_{\mathrm{v}_-,\gamma_-}, \Psi_{\mathrm{v}_+,\gamma_+} \in \mathcal{W}_1$ are defined by

$$\Psi_{\mathrm{v}_-,\gamma_-}(\Lambda) \doteq 0 \doteq \Psi_{\mathrm{v}_+,\gamma_+}(\Lambda)$$

whenever $|\Lambda| > 2$, while, for any $x, y \in \Gamma$,

$$\Psi_{\mathrm{v}_+,\gamma_+}(\{x,y\}) \doteq \left(2-\delta_{x,y}\right) \sum_{\mathrm{s},\mathrm{t}\in\{\uparrow,\downarrow\}} \gamma_+^d \mathrm{v}_+(\gamma_+(x-y))\, a_{y,\mathrm{t}}^* a_{y,\mathrm{t}} a_{x,\mathrm{s}}^* a_{x,\mathrm{s}} \ ,$$

$$\Psi_{\mathrm{v}_-,\gamma_-}(\{x,y\}) \doteq \left(2-\delta_{x,y}\right) \gamma_-^d \mathrm{v}_-(\gamma_-(x-y))\, a_{y,\uparrow}^* a_{y,\downarrow}^* a_{x,\downarrow} a_{x,\uparrow} \ ,$$

$\delta_{x,y}$ being the Kronecker delta. Using these definitions, we have

$$H_{\Lambda_\ell}(\gamma_-,\gamma_+) = H_{\Lambda_\ell}^{\Phi(\gamma_-,\gamma_+)}$$

for all natural numbers $\ell \in \mathbb{N}$ and $\gamma_-, \gamma_+ \in (0,1)$. Therefore, we can apply to $\Phi(\gamma_-,\gamma_+)$ all the above results on the thermodynamic behavior of models of $\mathcal{W}_1 \subseteq \mathcal{M}_1$.

For instance, for all parameters $\gamma_-, \gamma_+ \in (0,1)$, the energy density functional

$$\mathfrak{e}_{\Phi(\gamma_-,\gamma_+)} : E_1 \to \mathbb{R}$$

[5] That is, $\mathrm{v}_\pm(x) = \mathrm{v}_\pm(-x)$. Usually, $\mathrm{v}_\pm(x) = v_\pm(|x|)$ for some function $v_\pm : \mathbb{R}_0^+ \to \mathbb{R}$.

associated with the invariant interaction $\Phi(\gamma_-, \gamma_+) \in \mathcal{W}_1$ is defined by

$$\mathfrak{e}_{\Phi(\gamma_-,\gamma_+)}(\rho) \doteq \lim_{\ell\to\infty} \frac{1}{|\Lambda_\ell|}\rho\left(H_{\Lambda_\ell}(\gamma_-, \gamma_+)\right)$$

for any invariant state $\rho \in E_1$. See Proposition 5.11 and Definition 5.12. It naturally splits into three components:

$$\mathfrak{e}_{\Phi(\gamma_-,\gamma_+)} = \underbrace{\mathfrak{e}_\Phi}_{\text{free term}} + \underbrace{\mathfrak{e}_{\Psi_{\mathrm{v}_+,\gamma_+}}}_{\text{interaction term }+} - \underbrace{\mathfrak{e}_{\Psi_{\mathrm{v}_-,\gamma_-}}}_{\text{interaction term }-} .$$

With this, for any inverse temperature $\beta \in (0, \infty)$ and $\gamma_-, \gamma_+ \in (0, 1)$, the free energy density functional $\mathfrak{f}_{\Phi(\gamma_-,\gamma_+),\beta} : E_1 \to \mathbb{R}$ of Definition 6.12 equals

$$\mathfrak{f}_{\Phi(\gamma_-,\gamma_+),\beta} \doteq \mathfrak{e}_{\Phi(\gamma_-,\gamma_+)} - \beta^{-1}\mathfrak{s} = \mathfrak{e}_{\Psi_{\mathrm{v}_+,\gamma_+}} - \mathfrak{e}_{\Psi_{\mathrm{v}_-,\gamma_-}} + \mathfrak{f}_{\Phi,\beta}, \tag{6.40}$$

where $\mathfrak{s} : E_1 \to \mathbb{R}_0^+$ is the entropy density functional of Theorem 5.20. By Theorem 6.13, the thermodynamic limit of the (grand-canonical) pressure equals

$$P_\beta(\gamma_-, \gamma_+) \doteq \lim_{\ell\to\infty} P_{H_{\Lambda_\ell}(\gamma_-,\gamma_+),\beta} = -\inf \mathfrak{f}_{\Phi(\gamma_-,\gamma_+),\beta}(E_1) < \infty \tag{6.41}$$

for $\beta \in (0, \infty)$ and $\gamma_-, \gamma_+ \in (0, 1)$. See also (6.11). Recall that the globally stable equilibrium states of the short-range model are, by definition, the solutions to this variational problem. They form the set

$$\Omega_{\Phi(\gamma_-,\gamma_+),\beta} \doteq \left\{\omega \in E_1 : \mathfrak{f}_{\Phi(\gamma_-,\gamma_+),\beta}(\omega) = -P_\beta(\gamma_-, \gamma_+)\right\}$$

for any fixed $\beta \in (0, \infty)$ and $\gamma_-, \gamma_+ \in (0, 1)$. By Lemma 6.30, it is a (nonempty) weak*-closed face of the convex weak*-compact space E_1 of invariant states.

6.9.2 The Mean-Field Model

The Kac, or long-range, limits refer here to the limits $\gamma_\pm \to 0^+$ of short-range models that are already in the thermodynamic limit. For small parameters $\gamma_\pm \ll 1$, the short-range model defined in finite volume by (6.38) has an interparticle (+) and BCS (−) interactions with very large range ($\mathcal{O}(\gamma_\pm^{-1})$), but the interaction strength is small as $\gamma_\pm^d$, in such a way that the first Born approximation[6] to the scattering length of the interparticle and BCS potentials remains constant, as is usual. One

[6] That is, $\int_{\mathbb{R}^d} \gamma_\pm^d \mathrm{v}_\pm(\gamma_\pm x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \mathrm{v}_\pm(x)\,\mathrm{d}x \doteq \hat{\mathrm{v}}_\pm(0)$.

therefore expects to have some effective mean-field, or long-range, model in the limits $\gamma_{\pm} \to 0^{+}$.

Given $\Phi \in \mathcal{W}_1$, the effective local Hamiltonians in the limits $\gamma_{\pm} \to 0^{+}$ of short-range models should be

$$H_{\Lambda_\ell}^{\sharp}(\eta_-, \eta_+) \doteq H_{\Lambda_\ell}^{\Phi} + \underbrace{\frac{\eta_+}{|\Lambda_\ell|} \sum_{x,y\in\Lambda_\ell, \mathrm{s},\mathrm{t}\in\{\uparrow,\downarrow\}} a_{y,\mathrm{t}}^{*} a_{y,\mathrm{t}} a_{x,\mathrm{s}}^{*} a_{x,\mathrm{s}}}_{\text{mean-field repulsion } +} \underbrace{- \frac{\eta_-}{|\Lambda_\ell|} \sum_{x,y\in\Lambda_\ell} a_{y,\uparrow}^{*} a_{y,\downarrow}^{*} a_{x,\downarrow} a_{x,\uparrow}}_{\text{mean-field attraction } -} \qquad (6.42)$$

for all natural numbers $\ell \in \mathbb{N}$ and some positive parameters $\eta_-, \eta_+ \in \mathbb{R}_0^+$. Compare this Hamiltonian with (6.38). It refers to the mean-field model

$$\mathfrak{m}(\eta_-, \eta_+) \doteq \left(\Phi, \eta_-^{1/2} \Psi_-, \eta_+^{1/2} \Psi_+\right) \in \mathcal{M}_1$$

where

$$\Psi_- \doteq (\Psi_{\mathrm{BCS}}, 0, \ldots), \Psi_+ \doteq (\Psi_{\mathrm{Int}}, 0, \ldots) \in \ell^2(\mathbb{N}; \mathcal{W}_1^{\mathbb{C}})$$

with $\Psi_{\mathrm{BCS}} \in \mathcal{W}_1^{\mathbb{C}}$ being the "BCS interaction" of Example 6.6 for $\eta = 1$, defined by $\Psi_{\mathrm{BCS}}(\Lambda) \doteq 0$ whenever $|\Lambda| \notin \{1\}$ and

$$\Psi_{\mathrm{BCS}}(\{x\}) \doteq a_{x,\downarrow} a_{x,\uparrow}$$

for all lattice sites $x \in \Gamma$, while $\Psi_{\mathrm{Int}} \in \mathcal{W}_1 \subseteq \mathcal{W}_1^{\mathbb{C}}$ is the invariant interaction defined by $\Psi_{\mathrm{Int}}(\Lambda) \doteq 0$ whenever $|\Lambda| \notin \{1\}$ and

$$\Psi_{\mathrm{Int}}(\{x\}) \doteq a_{x,\uparrow}^{*} a_{x,\uparrow} + a_{x,\downarrow}^{*} a_{x,\downarrow}$$

for all lattice sites $x \in \Gamma$.

We then apply to the mean-field model $\mathfrak{m}(\eta_-, \eta_+)$ the results obtained above for general elements of $\mathcal{M}_1$. For instance, the space-averaging functionals $\Delta_{\Psi_{\pm}} : E_1 \to \mathbb{R}$ associated with the above sequences $\Psi_-, \Psi_+ \in \ell^2(\mathbb{N}; \mathcal{W}_1^{\mathbb{C}})$ are equal to

$$\Delta_{\Psi_{\pm}}(\rho) = \lim_{\ell\to\infty} \frac{1}{|\Lambda_\ell|^2} \sum_{x,y\in\Lambda_\ell} \rho\left(\tau_y\left(A_{\pm}^{*}\right) \tau_x\left(A_{\pm}\right)\right) \in \left[|\rho(A_{\pm})|^2, \|A_{\pm}\|^2\right],$$

for any invariant state $\rho \in E_1$, where

$$A_- \doteq a_{0,\downarrow} a_{0,\uparrow} = e_{\Psi_{\mathrm{BCS}}} \qquad \text{and} \qquad A_+ \doteq a_{0,\uparrow}^{*} a_{0,\uparrow} + a_{0,\downarrow}^{*} a_{0,\downarrow} = e_{\Psi_{\mathrm{Int}}}.$$

See Eqs. (6.6) and (6.8) as well as Definitions 6.9 and 6.12. For any inverse temperature $\beta \in (0,\infty)$ and $\eta_-,\eta_+ \in \mathbb{R}_0^+$, the free energy density functional $\mathfrak{f}_{\mathfrak{m}(\eta_-,\eta_+),\beta} : E_1 \to \mathbb{R}$ of Definition 6.12 equals

$$\mathfrak{f}_{\mathfrak{m}(\eta_-,\eta_+),\beta} \doteq \eta_+\Delta_{\Psi_+} - \eta_-\Delta_{\Psi_-} + \mathfrak{f}_{\Phi,\beta} \,. \tag{6.43}$$

By Theorem 6.13, the thermodynamic limit of the (grand-canonical) pressure equals

$$P_\beta^\sharp(\eta_-,\eta_+) \doteq \lim_{\ell\to\infty} P_{H_{\Lambda_\ell}^\sharp(\gamma_-,\gamma_+),\beta} = -\inf \mathfrak{f}_{\mathfrak{m}(\eta_-,\eta_+),\beta}(E_1) < \infty \tag{6.44}$$

for any $\beta \in (0,\infty)$ and $\eta_-,\eta_+ \in \mathbb{R}_0^+$. As before, the globally stable equilibrium states of the mean-field model are the limits of minimizing sequences for the functional $\mathfrak{f}_{\mathfrak{m}(\eta_-,\eta_+),\beta}$. They form the set

$$\Omega_{\mathfrak{m}(\eta_-,\eta_+),\beta} \doteq \left\{ \begin{array}{c} \omega \in E_1 : \exists (\rho_n)_{n\in\mathbb{N}} \subseteq E_1 \text{ weak}^* \text{ converging to } \omega \text{ so that} \\ \lim_{n\to\infty} \mathfrak{f}_{\mathfrak{m}(\eta_-,\eta_+),\beta}(\rho_n) = -P_\beta^\sharp(\eta_-,\eta_+) \end{array} \right\}$$

for $\beta \in (0,\infty)$ and $\eta_-,\eta_+ \in \mathbb{R}_0^+$. By Lemma 6.22, it is a (nonempty) convex weak*-compact subspace of the space E_1 of invariant states.

6.9.3 Thermodynamic Game and Bogoliubov Approximation

A mathematically rigorous computation of the pressure and equilibrium states of the short-range model to show possible phase transitions is elusive, beyond perturbative arguments, even after decades of mathematical studies. By contrast, such a question can be solved for the mean-field model. This is done by using the thermodynamic game explained in Sect. 6.7.

In this case, the approximating interactions of the mean-field model $\mathfrak{m}(\eta_-,\eta_+)$ equal

$$\Phi_{\mathfrak{m}(\eta_-,\eta_+)}(c_-,c_+) \doteq \Phi + 2\left(\eta_+^{1/2}\,\mathrm{Re}\left\{\overline{c_{+,1}}\right\}\Psi_{\mathrm{Int}} - \eta_-^{1/2}\,\mathrm{Re}\left\{\overline{c_{-,1}}\Psi_{\mathrm{BCS}}\right\}\right) \in \mathcal{W}_1$$

for all sequences $c_-,c_+ \in \ell^2(\mathbb{N})$; see Definition 6.26. Note that the use of full sequences $c_-,c_+ \in \ell^2(\mathbb{N})$ is not necessary here since the model has only one non-zero attractive and repulsive mean-field part. In other words, both sets of attractive and repulsive strategies for the associated thermodynamic game are identified with the set of complex numbers: $c_- \equiv c_{-,1} \in \mathbb{C}$ and $c_+ \equiv c_{+,1} \in \mathbb{C}$. The approximating interaction of the mean-field model leads to the following sequence of local Hamiltonians

$$\tilde{H}_{\Lambda_\ell}(\eta_-, \eta_+, c_-, c_+) \doteq H^{\Phi}_{\Lambda_\ell} + \eta_+^{1/2}\left(\overline{c_+} + c_+\right) \sum_{x \in \Lambda_\ell, s \in \{\uparrow,\downarrow\}} a^*_{x,s} a_{x,s}$$
$$+ \eta_-^{1/2} \sum_{x \in \Lambda_\ell} \left(\overline{c_-} a^*_{x,\uparrow} a^*_{x,\downarrow} + c_- a_{x,\downarrow} a_{x,\uparrow}\right) \tag{6.45}$$

for any two complex numbers $c_-, c_+ \in \mathbb{C}$, natural numbers $\ell \in \mathbb{N}$, and some positive parameters $\eta_-, \eta_+ \in \mathbb{R}_0^+$. Then, by Theorem 6.34 and Eq. (6.44), the conservative values of the thermodynamic game equal

$$\mathrm{F}^\sharp_{\mathfrak{m}(\eta_-,\eta_+),\beta} \doteq \inf_{c_- \in \mathbb{C}} \sup_{c_+ \in \mathbb{C}} \left\{- |c_+|^2 + |c_-|^2 - P_\beta(c_-, c_+, \eta_+, \eta_-)\right\}$$
$$= -P^\sharp_\beta(\eta_-, \eta_+) \tag{6.46}$$

and

$$\mathrm{F}^\flat_{\mathfrak{m}(\eta_-,\eta_+),\beta} \doteq \sup_{c_+ \in \mathbb{C}} \inf_{c_- \in \mathbb{C}} \left\{- |c_+|^2 + |c_-|^2 - P_\beta(c_-, c_+, \eta_+, \eta_-)\right\}$$
$$= -P^\flat_\beta(\eta_-, \eta_+) \ . \tag{6.47}$$

Here, we have the non-conventional pressure defined by

$$P^\flat_\beta(\eta_-, \eta_+) \doteq -\inf \mathfrak{f}^\flat_{\mathfrak{m}(\eta_-,\eta_+),\beta}(E_1) \tag{6.48}$$

where, for any invariant state $\rho \in E_1$,

$$\mathfrak{f}^\flat_{\mathfrak{m}(\eta_-,\eta_+),\beta}(\rho) = \eta_+ \left|\rho(a^*_{0,\uparrow} a_{0,\uparrow} + a^*_{0,\downarrow} a_{0,\downarrow})\right|^2 - \eta_- \Delta_{\Psi_-}(\rho) + \mathfrak{f}_{\Phi,\beta}(\rho)$$
$$\leq \mathfrak{f}_{\mathfrak{m}(\eta_-,\eta_+),\beta}(\rho) \ ,$$

(see (6.30)), while $P_\beta : \mathbb{C}^2 \times \left(\mathbb{R}_0^+\right)^2 \to \mathbb{R}$ is the function defined by

$$P_\beta(c_-, c_+, \eta_+, \eta_-) = \lim_{\ell \to \infty} P_{\tilde{H}_{\Lambda_\ell}(\eta_-,\eta_+,c_-,c_+),\beta}$$
$$= -\inf \mathfrak{f}_{\Phi_{\mathfrak{m}(\eta_-,\eta_+)}(c_-,c_+)}(E_1) < \infty \ , \tag{6.49}$$

thanks to Theorem 6.13. Note that a usual choice for the free interaction $\Phi \in \mathcal{W}_1$ is given by (6.27). In this case, the approximating Hamiltonians (6.45) are quadratic in the annihilation and creation operators. It can be exactly diagonalized, and the variational problems (6.46) and (6.47) can be analytically and numerically studied, in this case. The sets $\Omega_{\mathfrak{m}(\eta_-,\eta_+),\beta}$ and

$$\Omega^{\flat}_{\mathfrak{m}(\eta_-,\eta_+),\beta} \doteq \left\{\omega \in E_1 : \mathfrak{f}^{\flat}_{\mathfrak{m}(\eta_-,\eta_+),\beta}(\omega) = \inf \mathfrak{f}^{\flat}_{\mathfrak{m}(\eta_-,\eta_+),\beta}(E_1) = \mathrm{F}^{\flat}_{\mathfrak{m}(\eta_-,\eta_+),\beta}\right\}$$

of equilibrium states can also be explicitly determined, thanks to Theorem 6.36.

6.9.4 The Kac Limit

We now perform the Kac, or long-range, limits $\gamma_\pm \to 0^+$ of short-range models. First, using a cyclic representation of the C^*-algebra $\mathcal{U}$ induced by any invariant state (Theorem 4.113) as well as the spectral theorem, one can prove [28] that the energy densities associated with the invariant interactions $\Psi_{\mathrm{v}_-,\gamma_-}$ and $\Psi_{\mathrm{v}_+,\gamma_+}$ converge pointwise to

$$\lim_{\gamma_\pm \to 0^+} \mathfrak{e}_{\Psi_{\mathrm{v}_\pm,\gamma_\pm}}(\rho) \doteq \hat{\mathrm{v}}_\pm(0)\Delta_\pm(\rho) \tag{6.50}$$

for any invariant state $\rho \in E_1$, where we have from (6.39) that

$$\hat{\mathrm{v}}_\pm(0) \doteq \int_{\mathbb{R}^d} \mathrm{v}_\pm(x)\,\mathrm{d}^d x \geq 0\,.$$

Recall that $\mathrm{v}_-, \mathrm{v}_+$ are assumed to be positive definite, i.e., the Fourier transforms $\hat{\mathrm{v}}_-, \hat{\mathrm{v}}_+$ of $\mathrm{v}_-, \mathrm{v}_+$, respectively, are positive functions on $\mathbb{R}^d$. Comparing (6.40)–(6.41) and (6.43)–(6.44) in light of (6.50), this suggests that the parameters $\eta_-, \eta_+ \in \mathbb{R}_0^+$ of the mean-field models to be taken in the limits $\gamma_\pm \to 0^+$ are

$$\eta_\pm = \hat{\mathrm{v}}_\pm(0) \in \mathbb{R}_0^+.$$

This is partially confirmed by [28, Theorem 5.15], which in the example presented here refers to the following theorem:

Theorem 6.37 *Let* $\Phi \in \mathcal{W}_1$ *and* $\mathrm{v}_-, \mathrm{v}_+ \in C_0^{2d}\left(\mathbb{R}^d, \mathbb{R}\right)$ *be reflection-symmetric, positive definite functions on* $\mathbb{R}^d$ *with* $\hat{\mathrm{v}}_-(\gamma^{-1}k) \leq \hat{\mathrm{v}}_-(k)$ *for* $k \in \mathbb{R}^d$. *Fix an inverse temperature* $\beta \in (0,\infty)$.

(i) *Convergence of infinite-volume pressures:*

$$\lim_{\gamma_+ \to 0^+} \lim_{\gamma_- \to 0^+} P_\beta(\gamma_-, \gamma_+) = P_\beta^\sharp\left(\hat{\mathrm{v}}_-(0), \hat{\mathrm{v}}_+(0)\right)\,.$$

(ii) *Convergence of equilibrium states: For any* $\gamma_+ \in (0,1)$, *take any weak* accumulation point* ω_{γ_+} *of any net* $(\omega_{\gamma_-,\gamma_+})_{\gamma_- \in (0,1)} \subseteq \Omega_{\Phi(\gamma_-,\gamma_+),\beta}$ *as* $\gamma_- \to 0^+$. *Pick any weak* accumulation point* ω *of the net* $(\omega_{\gamma_+})_{\gamma_+ \in (0,1)}$, *as* $\gamma_+ \to 0^+$. *Then,*

$$\omega_{\gamma_-,\gamma_+} \underset{weak^*,\gamma_-\to 0^+}{\to} \omega_{\gamma_+} \underset{weak^*,\gamma_+\to 0^+}{\to} \omega \in \Omega_{\mathfrak{m}(\hat{\mathrm{v}}_-(0),\hat{\mathrm{v}}_+(0)),\beta} \ .$$

This theorem demonstrates that the mean-field model is generally an idealization of short-range models in the long-range limit. In addition, [28] gives some explicit error estimates, and one can deduce approximated phase diagrams on short-range models for sufficiently small parameters $\gamma_\pm \in (0, 1)$.

Note, however, that Theorem 6.37 uses a special order for the limit of small $\gamma_\pm \in (0, 1)$: First $\gamma_- \to 0^+$ and then $\gamma_+ \to 0^+$. It means that the attractive forces have a much larger range than the one of repulsive forces. One can ask whether this is just a technical artifact. As a matter of fact, it is generally **not** so, and the hierarchy of ranges does have a strong effect on the equilibrium states and pressure of the model:

Proposition 6.38 *Let $\Phi \in \mathcal{W}_1$ and $\mathrm{v}_-, \mathrm{v}_+ \in C_0^{2d}(\mathbb{R}^d, \mathbb{R})$ be reflection-symmetric, positive definite functions on $\mathbb{R}^d$ with $\hat{\mathrm{v}}_-(\gamma^{-1}k) \leq \hat{\mathrm{v}}_-(k)$ for all $k \in \mathbb{R}^d$ and $\gamma \in (0, 1)$. Fix $\beta \in (0, \infty)$. If $(\gamma_{-,n})_{n\in\mathbb{N}}$ and $(\gamma_{+,n})_{n\in\mathbb{N}}$ converges to zero, then*

$$P_\beta^\sharp\left(\hat{\mathrm{v}}_-(0), \hat{\mathrm{v}}_+(0)\right) \leq \liminf_{n\to\infty} P_\beta\left(\gamma_{+,n}, \gamma_{-,n}\right) \leq \limsup_{n\to\infty} P_\beta\left(\gamma_{+,n}, \gamma_{-,n}\right)$$
$$\leq P_\beta^\flat\left(\hat{\mathrm{v}}_-(0), \hat{\mathrm{v}}_+(0)\right) \ .$$

Proof See [28, Proposition 5.14]. □

Recall that the supremum and infimum in (6.46) and (6.47) do not commute, in general. See [1, p. 42]. A sufficient condition for them to commute is given by Lemma 6.35. Thus, we generally have

$$P_\beta^\sharp\left(\hat{\mathrm{v}}_-(0), \hat{\mathrm{v}}_+(0)\right) \neq P_\beta^\flat\left(\hat{\mathrm{v}}_-(0), \hat{\mathrm{v}}_+(0)\right)$$

and Proposition 6.38 suggests that the limits $\gamma_\pm \to 0^+$ of short-range models can lead to a different system from the one described by the *conventional* mean-field model, which is the thermodynamic limit the finite-volume system associated with the local Hamiltonians (6.42). In fact, applying [28, Theorem 5.17] to the model presented above, one can reach the here called "non-conventional mean-field model":

Theorem 6.39 *Let $\Phi \in \mathcal{W}_1$ and $\mathrm{v}_-, \mathrm{v}_+ \in C_0^{2d}(\mathbb{R}^d, \mathbb{R})$ be reflection-symmetric, positive definite functions on $\mathbb{R}^d$ with $\hat{\mathrm{v}}_-(\gamma^{-1}k) \leq \hat{\mathrm{v}}_-(k)$ for $k \in \mathbb{R}^d$. Fix an inverse temperature $\beta \in (0, \infty)$.*

(i) *Convergence of infinite-volume pressures:*

$$\lim_{\gamma_-\to 0^+} \lim_{\gamma_+\to 0^+} P_\beta(\gamma_-, \gamma_+) = P_\beta^\flat\left(\hat{\mathrm{v}}_-(0), \hat{\mathrm{v}}_+(0)\right) \ .$$

(ii) *Convergence of equilibrium states: For any $\gamma_- \in (0, 1)$, take any weak* accumulation point ω_{γ_-} of any net $(\omega_{\gamma_-,\gamma_+})_{\gamma_+\in(0,1)} \subseteq \Omega_{\Phi(\gamma_-,\gamma_+),\beta}$ as $\gamma_+ \to 0^+$. Pick any weak* accumulation point ω of the net $(\omega_{\gamma_-})_{\gamma_-\in(0,1)}$, as $\gamma_- \to 0^+$.*

Then,

$$\omega_{\gamma_-,\gamma_+} \underset{weak^*,\gamma_+\to 0^+}{\to} \omega_{\gamma_-} \underset{weak^*,\gamma_-\to 0^+}{\to} \omega \in \Omega^{\flat}_{\mathfrak{m}(\hat{\mathrm{v}}_-(0),\hat{\mathrm{v}}_+(0)),\beta} \,.$$

As there is no reason to have the equality $\Omega^{\sharp}_{\mathfrak{m}} = \Omega^{\flat}_{\mathfrak{m}}$ for a given arbitrary mean-field model $\mathfrak{m} \in \mathcal{M}_1$, Theorems 6.37 and 6.39 generally describe different physical situations. In fact, one can even prove that the limit of Kac pressures can attain **all** the values of the duality interval

$$\mathrm{I} \doteq \left[P^{\sharp}_{\beta}(\hat{\mathrm{v}}_-(0), \hat{\mathrm{v}}_+(0)),\, P^{\flat}_{\beta}(\hat{\mathrm{v}}_-(0), \hat{\mathrm{v}}_+(0)) \right]$$

of the thermodynamic game associated with the mean-field model $\mathfrak{m}(\hat{\mathrm{v}}_-(0), \hat{\mathrm{v}}_+(0)) \in \mathcal{M}_1$:

Theorem 6.40 *Let* $\Phi \in \mathcal{W}_1$ *and* $\mathrm{v}_-, \mathrm{v}_+ \in C_0^{2d}\left(\mathbb{R}^d, \mathbb{R}\right)$ *be reflection-symmetric, positive definite functions on* $\mathbb{R}^d$ *with* $\hat{\mathrm{v}}_-(\gamma^{-1}k) \leq \hat{\mathrm{v}}_-(k)$ *for* $k \in \mathbb{R}^d$*. Fix an inverse temperature* $\beta \in (0, \infty)$*. For any* $\mathrm{p} \in \mathrm{I}$*, there are two sequences* $(\gamma_{+,n})_{n\in\mathbb{N}}$ *and* $(\gamma_{-,n})_{n\in\mathbb{N}}$ *of real numbers in the interval* $(0, 1)$ *converging to zero, such that*

$$\lim_{n\to\infty} P_\beta\left(\gamma_{-,n}, \gamma_{+,n}\right) = \mathrm{p} \,.$$

Proof See [28, Theorem 5.19]. □

This theorem shows that interplay of the long-range limits $\gamma_\pm \to 0^+$ of short-range models can be highly non-trivial. In fact, as expected, any such long-range (Kac) limit leads to mean-field pressures and equilibrium states. However, in the presence of both repulsive and attractive forces, the limit mean-field model is **not necessarily** what one traditionally guesses. In fact, it strongly depends upon the hierarchy of ranges between attractive and repulsive interparticle forces. We have seen that if the range of repulsive forces is much larger than the range of the attractive ones, then in the Kac limit for these forces, one may get a limit mean-field model that is **unconventional**. See Theorems 6.39 and 6.40.

6.9.5 Historical Observations

The study on long-range limits presented here follows a rather old sequence of works on the Kac limit, basically starting from 1959, with Kac's own work on classical one-dimensional spin systems. The first important result [40] on this subject was provided by Penrose and Lebowitz in 1966, who proved the convergence of the free energy of a classical system toward the one of the van der Waals theory. Shortly after, the results of this seminal paper were extended to quantum systems (Boltzmann, Bose, or Fermi statistics) by Lieb [43]. In 1971, Penrose and Lebowitz went considerably further than [40] with [41]. See also [42] for a review of all these results of classical statistical mechanics. These outcomes form the mainstays of the subsequent results on the Kac limit, and we recommend the book [44] for a more recent review on the subject in classical statistical mechanics, including the so-called Lebowitz-Penrose theorem and a more exhaustive list of references.

Studies on the Kac limit are still performed nowadays in classical statistical mechanics; see, e.g., [45–47]. By contrast, to our knowledge, [28, Theorem 5.19] is the unique recent study on the subject for quantum systems, and the sole important results before [28, Theorem 5.19] are those of [43], which refer to quantum particles in the continuum, but may certainly be extended to lattice systems. The main innovation of [28, Theorem 5.19] is the fact that the convergence in the Kac limit is proven not only for pressure-like quantities (for instance, the thermodynamic limit of the logarithm of canonical or grand-canonical partition functions), as in previous works, but also for equilibrium states, i.e., for **all** correlation functions. These results on states were made possible by the variational approach of [1] for equilibrium states of mean-field models, which we present in a simpler setting in the first part of the current chapter. Additionally, also in contrast with previous results on Kac limits, our method allows for coexistence of both attractive and repulsive long-range forces. This important extension is related to the game theoretical characterization of equilibrium states of mean-field models (cf. thermodynamic game) discussed in Sects. 6.7 and 6.8. This approach thus paves the way for the study of phase transitions,[7] or at least important fingerprints of them like strong correlations at long distances, for models having interactions whose ranges are finite, but very large. It also sheds a new light on mean-field models by connecting them with short-range ones, in a mathematically precise manner. Such studies can be important for future theoretical developments in many-body theory, since long-range interactions are expected to imply effective, classical background fields, in the spirit of the Higgs mechanism of quantum field theory. This is shown in [48–50] for mean-field models.

6.10 The Generalized Hartree-Fock Theory as a Mean-Field Theory

In Sect. 5.7, we introduce the generalized Hartree-Fock theory [77, Definition 3.1], which approximates equilibrium states of fermion systems by means of (general) quasi-free states. Here, we illustrate the affinity of this method with mean-field theories. To this end, we consider an explicit, albeit still very general, fermion system that is similar to the model (6.38) studied in Sect. 6.9 in the context of the Kac limit.

[7] Mean-field repulsions have generally a geometrical effect by possibly breaking the face structure of the set of (generalized) equilibrium states (see [1, Lemma 9.8]). When this appears, we have long-range order for correlations. See [1, Section 2.9].

6.10.1 The Short-Range Model

In the current section, we only consider fermion systems, i.e., $\mathcal{U} \doteq \mathrm{CAR}(\Omega, \Gamma)$. As before, Ω denotes an arbitrary finite subset, which is fixed once and for all, and we use the short notation $a_{x,\mathrm{s}} \doteq a\left(\mathrm{e}_{\mathrm{s},x}\right)$ for the "annihilation operator" of a fermion with spin $\mathrm{s} \in \Omega$ at lattice position $x \in \Gamma$. Again, $\{\mathrm{e}_{\mathrm{s},x}\}_{(\mathrm{s},x)\in\Omega\times\Gamma}$ is the (canonical) Hilbert basis of $\ell^2(\Omega \times \Gamma)$, defined by $\mathrm{e}_{\mathrm{s},x}(\tilde{\mathrm{s}}, \tilde{x}) = 1$ if $(\mathrm{s}, x) = (\tilde{\mathrm{s}}, \tilde{x})$ and $\mathrm{e}_{\mathrm{s},x}(\tilde{\mathrm{s}}, \tilde{x}) = 0$, else. Considering fermions inside the cubic box

$$\Lambda_\ell \doteq \{(x_1, \ldots, x_d) \in \Gamma : |x_i| \leq \ell\}$$

for any $\ell \in \mathbb{N}$, the local Hamiltonians of our prototypical example studied here are equal to

$$H_{\Lambda_\ell} \doteq \sum_{x,y\in\Lambda_\ell,\ \mathrm{s}\in\Omega} \mathrm{h}\,(x-y)\, a_{x,\mathrm{s}}^* a_{y,\mathrm{s}} + \sum_{x,y\in\Lambda_\ell,\ \mathrm{s},\mathrm{t}\in\Omega} \mathrm{v}\,(x-y)\, a_{y,\mathrm{t}}^* a_{y,\mathrm{t}} a_{x,\mathrm{s}}^* a_{x,\mathrm{s}}\ ,$$

where $\mathrm{h} : \Gamma \to \mathbb{R}$ and $\mathrm{v} : \Gamma \to \mathbb{R}$ are two reflection-symmetric[8] functions. The (non-zero) function h encodes the hopping strength of fermions, while v is a (non-zero) pair potential characterizing interparticle forces. In contrast with (6.38), the function v has not necessarily positive values.

Such a family $(H_{\Lambda_\ell})_{\ell\in\mathbb{N}}$ of Hamiltonians is encoded by the (translation) invariant interaction $\Phi_{\mathrm{h},\mathrm{v}} = \Phi_{\mathrm{h}} + \Psi_{\mathrm{v}} \in \mathcal{V} \subseteq \mathcal{V}^{\mathbb{C}}$ (Definition 5.5), where the invariant self-conjugate interactions $\Phi_{\mathrm{h}}, \Psi_{\mathrm{v}} \in \mathcal{V}$ are defined by

$$\Psi_{\mathrm{v}}(\Lambda) \doteq 0 \doteq \Phi_{\mathrm{h}}(\Lambda)$$

whenever $|\Lambda| > 2$, while, for any $x, y \in \Gamma$,

$$\Phi_{\mathrm{h}}\left(\{x,y\}\right) \doteq \left(1 - \frac{1}{2}\delta_{x,y}\right) \sum_{\mathrm{s}\in\Omega} \mathrm{h}\,(x-y) \left(a_{x,\mathrm{s}}^* a_{y,\mathrm{s}} + a_{y,\mathrm{s}}^* a_{x,\mathrm{s}}\right)\ ,$$

$$\Psi_{\mathrm{v}}\left(\{x,y\}\right) \doteq \left(2 - \delta_{x,y}\right) \sum_{\mathrm{s},\mathrm{t}\in\Omega} \mathrm{v}\,(x-y)\, a_{y,\mathrm{t}}^* a_{y,\mathrm{t}} a_{x,\mathrm{s}}^* a_{x,\mathrm{s}}\ ,$$

$\delta_{x,y}$ being the Kronecker delta. In fact, using these definitions, we have

$$H_{\Lambda_\ell} = H_{\Lambda_\ell}^{\Phi_{\mathrm{h},\mathrm{v}}}$$

for all natural numbers $\ell \in \mathbb{N}$ and functions $\mathrm{h} : \Gamma \to \mathbb{R}$ and $\mathrm{v} : \Gamma \to \mathbb{R}$.

[8] That is, $\mathrm{h}\,(x) = \mathrm{h}\,(-x)$ and $\mathrm{v}\,(x) = \mathrm{v}\,(-x)$ for every $x \in \Gamma$.

We additionally impose the two reflection-symmetric functions h and v to be summable, i.e.,

$$\|\mathrm{h}\|_{\Gamma} \doteq \sum_{x\in\Gamma} |\mathrm{h}(x)| < \infty \qquad \text{and} \qquad \|\mathrm{v}\|_{\Gamma} \doteq \sum_{x\in\Gamma} |\mathrm{v}(x)| < \infty \, .$$

This implies that $\Phi_{\mathrm{h,v}} \in \mathcal{W}_1 \subseteq \mathcal{W}_1^{\mathbb{C}}$. See Definitions 5.6 and 6.2. In fact, note that the absolute summability of h and v is a necessary and sufficient condition to have $\Phi_{\mathrm{h,v}} \in \mathcal{W}_1$. It is a very weak condition in view of applications in condensed matter physics. For instance, taking $\mathrm{h}(x) = 0$ when $|x| > 1$ and $\mathrm{v}(x) = 0$ for $x \neq 0$, one obtains the celebrated Hubbard model. Note, moreover, that the summability of h and v is important to ensure the existence of the infinite-volume dynamics, via the celebrated Lieb-Robinson bounds (see, e.g., [94, Sections 4.1–4.2]). In fact, as explained in Paragraph 6.10.3, the existence of an infinite-volume dynamics is used in our arguments in order to link the generalized Hartree-Fock theory to mean-field models, via the KMS theory.

Observe from Proposition 5.11 and Definition 5.12 that the energy density functional

$$\begin{aligned} \mathfrak{e}_{\Phi_{\mathrm{h,v}}} : E_1 &\to \mathbb{R} \\ \rho &\mapsto \mathfrak{e}_{\Phi_{\mathrm{h,v}}}(\rho) \doteq \lim_{\ell\to\infty} \frac{1}{|\Lambda_\ell|} \rho\left(H_{\Lambda_\ell}\right) \end{aligned}$$

associated with the invariant interaction $\Phi_{\mathrm{h,v}} \in \mathcal{W}_1$ naturally splits into two components:

$$\mathfrak{e}_{\Phi_{\mathrm{h,v}}} = \underbrace{\mathfrak{e}_{\Phi_{\mathrm{h}}}}_{\text{free term}} + \underbrace{\mathfrak{e}_{\Psi_{\mathrm{v}}}}_{\text{interaction term}} ,$$

where $\mathfrak{e}_{\Phi_{\mathrm{h}}} : E_1 \to \mathbb{R}$ and $\mathfrak{e}_{\Psi_{\mathrm{v}}} : E_1 \to \mathbb{R}$ are, respectively, equal to

$$\begin{aligned} \mathfrak{e}_{\Phi_{\mathrm{h}}}(\rho) &\doteq \lim_{\ell\to\infty} \frac{1}{|\Lambda_\ell|} \sum_{x,y\in\Lambda_\ell,\ \mathrm{s}\in\Omega} \mathrm{h}(x-y)\, \rho\left(a_{x,\mathrm{s}}^* a_{y,\mathrm{s}}\right) \\ &= \frac{1}{2} \sum_{x\in\Gamma,\ \mathrm{s}\in\Omega} \mathrm{h}(x)\, \rho\left(a_{x,\mathrm{s}}^* a_{0,\mathrm{s}} + a_{0,\mathrm{s}}^* a_{x,\mathrm{s}}\right) \end{aligned} \tag{6.51}$$

and

$$\begin{aligned} \mathfrak{e}_{\Psi_{\mathrm{v}}}(\rho) &\doteq \lim_{\ell\to\infty} \frac{1}{|\Lambda_\ell|} \sum_{x,y\in\Lambda_\ell,\ \mathrm{s,t}\in\Omega} \mathrm{v}(x-y)\, \rho\left(a_{y,\mathrm{t}}^* a_{y,\mathrm{t}} a_{x,\mathrm{s}}^* a_{x,\mathrm{s}}\right) \\ &= \sum_{x\in\Gamma,\ \mathrm{s,t}\in\Omega} \mathrm{v}(x)\, \rho\left(a_{0,\mathrm{t}}^* a_{0,\mathrm{t}} a_{x,\mathrm{s}}^* a_{x,\mathrm{s}}\right) \end{aligned} \tag{6.52}$$

for any invariant state $\rho \in E_1$. With this, for any inverse temperature $\beta \in (0, \infty)$, the free energy density functional $\mathfrak{f}_{\Phi_{\mathrm{h,v}},\beta} : E_1 \to \mathbb{R}$ of Definition 6.12 can be written as

$$\mathfrak{f}_{\Phi_{\mathrm{h,v}},\beta} \doteq \mathfrak{e}_{\Phi_{\mathrm{h,v}}} - \beta^{-1}\mathfrak{s} = \mathfrak{e}_{\Phi_{\mathrm{h}}} + \mathfrak{e}_{\Psi_{\mathrm{v}}} - \beta^{-1}\mathfrak{s} \ ,$$

where $\mathfrak{s} : E_1 \to \mathbb{R}_0^+$ is the entropy density functional of Theorem 5.20. By Theorem 6.13, the thermodynamic limit of the (grand-canonical) pressure equals

$$P_\beta \doteq \lim_{\ell\to\infty} P_{H_{\Lambda_\ell},\beta} = -\inf \mathfrak{f}_{\Phi_{\mathrm{h,v}},\beta}\,(E_1) < \infty \tag{6.53}$$

for any $\beta \in (0, \infty)$ and absolutely summable, reflection-symmetric, functions $\mathrm{h} : \Gamma \to \mathbb{R}$ and $\mathrm{v} : \Gamma \to \mathbb{R}$. See also (6.11). Recall that the globally stable equilibrium states of the short-range model are, by definition, the solutions to this variational problem. They form the set

$$\Omega_{\Phi_{\mathrm{h,v}},\beta} \doteq \left\{\omega \in E_1 : \mathfrak{f}_{\Phi_{\mathrm{h,v}},\beta}\,(\omega) = -P_\beta\right\} \tag{6.54}$$

for any fixed $\beta \in (0, \infty)$ and summable reflection-symmetric functions $\mathrm{h} : \Gamma \to \mathbb{R}$ and $\mathrm{v} : \Gamma \to \mathbb{R}$. By Lemma 6.30, it is a (nonempty) weak*-closed face of the convex weak*-compact space E_1 of invariant states.

6.10.2 Restriction to Quasi-Free States

In Definition 4.217, we introduce the notion of quasi-free states on self-dual CAR C^*-algebras. As explained after Proposition 4.219 with $G = H = \ell^2(\Omega \times \Gamma)$, recall from Corollary 4.207 that

$$\mathcal{U} \doteq \mathrm{CAR}(\Omega, \Gamma) = \mathrm{CAR}(\ell^2(\Omega \times \Gamma))$$

can be naturally identified with $\mathrm{sCAR}(\ell^2(\Omega \times \Gamma)_{\mathrm{sd}})$, where

$$\ell^2(\Omega \times \Gamma)_{\mathrm{sd}} \doteq \ell^2(\Omega \times \Gamma) \oplus_2 \ell^2(\Omega \times \Gamma)^{\mathrm{td}} \ ;$$

see Definition 4.195. We can thus use this identification of C^*-algebras to define from Definition 4.217 quasi-free states on $\mathcal{U}$. They are uniquely defined via Pfaffians and the two-point correlation functions $\rho(a_{x,\mathrm{s}}a_{y,\mathrm{t}})$ and $\rho(a_{x,\mathrm{s}}a_{y,\mathrm{t}}^*)$ for $x, y \in \Gamma$ and $\mathrm{s}, \mathrm{t} \in \Omega$. Such a quasi-free state ρ on $\mathcal{U}$ is called here simple, whenever $\rho(a_{x,\mathrm{s}}a_{y,\mathrm{t}}) = 0$ for all $x, y \in \Gamma$ and $\mathrm{s}, \mathrm{t} \in \Omega$.

Let

$$\mathcal{Q}_1 \doteq \{\rho \in E_1 : \ \rho \text{ is a quasi-free state}\}$$

be the (nonempty[9]) set of quasi-free states on $\mathcal{U}$. Note that a convex combination of quasi-free states is a state that is not necessarily quasi-free. In particular, $\mathcal{Q}_1$ is *not* a convex subset of the convex weak*-compact set E_1, but it is weak*-closed and therefore weak*-compact. This can be straightforwardly deduced from the definition of quasi-free states.

As explained in Sect. 5.7, we follow Bach, Lieb, and Solovej's approach [77] to the Hartree-Fock theory applied to the Hubbard model and, thus, minimize the free energy density functional in the set of all (not necessarily simple) quasi-free states. In other words, instead of (6.53), we study the variational problem

$$\inf \mathfrak{f}_{\Phi_{\mathrm{h,v}},\beta}\left(\mathcal{Q}_1\right)$$

for any inverse temperature $\beta \in (0,\infty)$ and summable reflection-symmetric functions $\mathrm{h}:\Gamma\rightarrow\mathbb{R}$ and $\mathrm{v}:\Gamma\rightarrow\mathbb{R}$. Its solutions form a set denoted by

$$\mathrm{Q}_{\Phi_{\mathrm{h,v}},\beta} \doteq \left\{\omega\in\mathcal{Q}_1 : \mathfrak{f}_{\Phi_{\mathrm{h,v}},\beta}\left(\omega\right) = \inf \mathfrak{f}_{\Phi_{\mathrm{h,v}},\beta}\left(\mathcal{Q}_1\right)\right\} ,$$

which is in general rather different from the set $\Omega_{\Phi_{\mathrm{h,v}},\beta}$ of globally stable equilibrium states defined by (6.54).

To show that $\mathrm{Q}_{\Phi_{\mathrm{h,v}},\beta}$ is not empty, we observe that any invariant state $\rho\in E_1$ uniquely defines a quasi-free state via the corresponding symbol, and we rewrite the variational problem over quasi-free states as follows:

Lemma 6.41 *For any invariant state $\rho\in E_1$, there is a (unique) quasi-free state $q_\rho\in\mathcal{Q}_1$ satisfying*

$$q_\rho\left(a_{x,\mathrm{s}}a_{y,\mathrm{t}}\right) = \rho\left(a_{x,\mathrm{s}}a_{y,\mathrm{t}}\right) \qquad \text{and} \qquad q_\rho\left(a_{x,\mathrm{s}}a_{y,\mathrm{t}}^*\right) = \rho(a_{x,\mathrm{s}}a_{y,\mathrm{t}}^*)$$

for all $x,y\in\Gamma$ and $\mathrm{s},\mathrm{t}\in\Omega$. The mapping $q:\rho\mapsto q_\rho$ from E_1 to $\mathcal{Q}_1$ is weak-continuous and satisfies $q_\rho=\rho$ for any quasi-free state $\rho\in\mathcal{Q}_1$.*

Proof The existence and uniqueness of $q_\rho\in\mathcal{Q}_1$ for any given $\rho\in E_1$ are a consequence of Exercise 4.216 and Proposition 4.219 together with Corollary 4.207, keeping in mind the definition of quasi-free states on CAR algebras just explained above. The weak* continuity of q can be verified by direct computations. See Definitions 4.80 and 4.217. We omit the details. □

Corollary 6.42 *For each inverse temperature $\beta\in(0,\infty)$ and any invariant interaction $\Phi\in\mathcal{W}_1$,*

$$\inf \mathfrak{f}_{\Phi,\beta}\left(\mathcal{Q}_1\right) = \inf \mathfrak{f}_{\Phi,\beta}\circ q\left(E_1\right)$$

and

[9] See, e.g., Proposition 4.219.

$$\left\{\omega \in \mathcal{Q}_1 : \mathfrak{f}_{\Phi,\beta}(\omega) = \inf \mathfrak{f}_{\Phi,\beta}(\mathcal{Q}_1)\right\} = q\left(\left\{\omega \in E_1 : \mathfrak{f}_{\Phi,\beta}(\omega) = \inf \mathfrak{f}_{\Phi,\beta} \circ q\,(E_1)\right\}\right)$$

is a nonempty weak*-compact subset of $\mathcal{Q}_1 \subseteq E_1$.

Proof The assertions are consequences of Lemma 6.41 combined with Lemmata 6.7 and 6.8. Note in particular from these statements that $\mathfrak{f}_{\Phi,\beta} \circ q$ is a weak*-lower semicontinuous functional on E_1, which is a weak*-compact set. As a consequence, the set of minimizers of $\mathfrak{f}_{\Phi,\beta} \circ q$ in E_1 is a nonempty weak*-closed, and thus compact, subset of $\mathcal{Q}_1 \subseteq E_1$. □

Applying this last corollary to the (short-range) model $\Phi_{\mathrm{h,v}} \in \mathcal{W}_1 \subseteq \mathcal{M}_1$ for any summable functions $\mathrm{h} : \Gamma \to \mathbb{R}$ and $\mathrm{v} : \Gamma \to \mathbb{R}$, we conclude in particular that $\mathrm{Q}_{\Phi_{\mathrm{h,v}},\beta}$ is a nonempty weak*-compact subset of $\mathcal{Q}_1 \subseteq E_1$ for any inverse temperature $\beta \in (0,\infty)$. In addition, the energy density functionals, respectively, associated with the kinetic and interparticle interactions have the following properties:

Corollary 6.43 *For any summable reflection-symmetric functions* $\mathrm{h} : \Gamma \to \mathbb{R}$ *and* $\mathrm{v} : \Gamma \to \mathbb{R}$*, we have* $\mathfrak{e}_{\Phi_{\mathrm{h}}} \circ q = \mathfrak{e}_{\Phi_{\mathrm{h}}}$ *and*

$$\mathfrak{e}_{\Psi_{\mathrm{v}}} \circ q\,(\rho) = \mathrm{v}\,(0) \sum_{\mathrm{s}\in\Omega} \rho\left(a_{0,\mathrm{s}}^* a_{0,\mathrm{s}}\right) + \left(\sum_{\mathrm{s}\in\Omega} \rho\left(a_{0,\mathrm{s}}^* a_{0,\mathrm{s}}\right)\right)^2 \sum_{x\in\Gamma} \mathrm{v}\,(x)$$
$$+ \sum_{x\in\Gamma} \mathrm{v}\,(x) \sum_{\mathrm{s,t}\in\Omega} \left(\left|\rho\left(a_{x,\mathrm{s}} a_{0,\mathrm{t}}\right)\right|^2 - \left|\rho\left(a_{0,\mathrm{t}}^* a_{x,\mathrm{s}}\right)\right|^2\right) .$$

Proof To obtain the equality $\mathfrak{e}_{\Phi_{\mathrm{h}}} \circ q = \mathfrak{e}_{\Phi_{\mathrm{h}}}$ it suffices to combine Lemma 6.41 with the explicit expression of the energy density $\mathfrak{e}_{\Phi_{\mathrm{h}}}$ given in Eq. (6.51). Now, recall from (6.52) that

$$\mathfrak{e}_{\Psi_{\mathrm{v}}}(\rho) = \sum_{x\in\Gamma,\ \mathrm{s,t}\in\Omega} \mathrm{v}\,(x)\, \rho\left(a_{0,\mathrm{t}}^* a_{0,\mathrm{t}} a_{x,\mathrm{s}}^* a_{x,\mathrm{s}}\right)$$

for any invariant state $\rho \in E_1$. If $\rho \in \mathcal{Q}_1$ is a quasi-free state, then the 4-point correlation function $\rho\left(a_{0,\mathrm{t}}^* a_{0,\mathrm{t}} a_{x,\mathrm{s}}^* a_{x,\mathrm{s}}\right)$ for $x \in \Gamma$ and $\mathrm{s,t} \in \Omega$ can be written in terms of 2-point correlation functions via the corresponding Pfaffians. More explicitly, one gets from Definitions 4.195 and 4.217 that, for any $\rho \in \mathcal{Q}_1$, $x \in \Gamma$ and $\mathrm{s,t} \in \Omega$,

$$\rho\left(a_{0,\mathrm{t}}^* a_{0,\mathrm{t}} a_{x,\mathrm{s}}^* a_{x,\mathrm{s}}\right) = \rho\left(a\left(\mathrm{e}_{\mathrm{t},0}^*\right) a\left(\mathrm{e}_{\mathrm{t},0}\right) a\left(\mathrm{e}_{\mathrm{s},x}^*\right) a\left(\mathrm{e}_{\mathrm{s},x}\right)\right)$$
$$= \mathrm{Pf}\begin{pmatrix} 0 & \rho(a_{0,\mathrm{t}}^* a_{0,\mathrm{t}}) & \rho(a_{0,\mathrm{t}}^* a_{x,\mathrm{s}}^*) & \rho(a_{0,\mathrm{t}}^* a_{x,\mathrm{s}}) \\ -\rho(a_{0,\mathrm{t}}^* a_{0,\mathrm{t}}) & 0 & \rho(a_{0,\mathrm{t}} a_{x,\mathrm{s}}^*) & \rho(a_{0,\mathrm{t}} a_{x,\mathrm{s}}) \\ -\rho(a_{0,\mathrm{t}}^* a_{x,\mathrm{s}}^*) & -\rho(a_{0,\mathrm{t}} a_{x,\mathrm{s}}^*) & 0 & \rho(a_{x,\mathrm{s}}^* a_{x,\mathrm{s}}) \\ -\rho(a_{0,\mathrm{t}}^* a_{x,\mathrm{s}}) & -\rho(a_{0,\mathrm{t}} a_{x,\mathrm{s}}) & -\rho(a_{x,\mathrm{s}}^* a_{x,\mathrm{s}}) & 0 \end{pmatrix},$$

where $\mathrm{e}_{\mathrm{s},x}^{*} = (0, \langle \mathrm{e}_{\mathrm{s},x}, \cdot\rangle) \in \ell^2(\Omega \times \Gamma)_{\mathrm{sd}}$, the right-hand side of the first equality being written within the self-dual approach. It remains to compute this Pfaffian from its definition. See, e.g., the equation before Definition 4.217. By the CAR (Definition 4.163), note that, for any $x, y \in \Gamma$ and $\mathrm{s}, \mathrm{t} \in \Omega$,

$$a_{x,\mathrm{s}}a_{y,\mathrm{t}} + a_{y,\mathrm{t}}a_{x,\mathrm{s}} = 0 \ , \qquad a_{x,\mathrm{s}}a_{y,\mathrm{t}}^{*} + a_{y,\mathrm{t}}^{*}a_{x,\mathrm{s}} = \delta_{x,y}\delta_{\mathrm{s},\mathrm{t}}\mathbb{1} \ ,$$

keeping in mind that $a_{x,\mathrm{s}} = a\left(\mathrm{e}_{\mathrm{s},x}\right) \in \mathcal{U}$ with $\{\mathrm{e}_{\mathrm{s},x}\}_{(\mathrm{s},x)\in\Omega\times\Gamma}$ being, as is usual, the (canonical) Hilbert basis of $\ell^2(\Omega \times \Gamma)$. By plugging the CAR relations just stated, as well as the self-conjugate (or Hermitian) property of states, into the computation of the above Pfaffian, we arrive at

$$\begin{aligned}
\rho\left(a_{0,\mathrm{t}}^{*}a_{0,\mathrm{t}}a_{x,\mathrm{s}}^{*}a_{x,\mathrm{s}}\right) &= \rho\left(a_{0,\mathrm{t}}^{*}a_{0,\mathrm{t}}\right)\rho\left(a_{x,\mathrm{s}}^{*}a_{x,\mathrm{s}}\right) - \rho\left(a_{0,\mathrm{t}}^{*}a_{x,\mathrm{s}}^{*}\right)\rho\left(a_{0,\mathrm{t}}a_{x,\mathrm{s}}\right) \\
&\quad + \rho\left(a_{0,\mathrm{t}}a_{x,\mathrm{s}}^{*}\right)\rho\left(a_{0,\mathrm{t}}^{*}a_{x,\mathrm{s}}\right) \\
&= \rho\left(a_{0,\mathrm{t}}^{*}a_{0,\mathrm{t}}\right)\rho\left(a_{0,\mathrm{s}}^{*}a_{0,\mathrm{s}}\right) + \rho\left(\left(a_{x,\mathrm{s}}a_{0,\mathrm{t}}\right)^{*}\right)\rho\left(a_{x,\mathrm{s}}a_{0,\mathrm{t}}\right) \\
&\quad - \rho\left(a_{x,\mathrm{s}}^{*}a_{0,\mathrm{t}}\right)\rho\left(a_{0,\mathrm{t}}^{*}a_{x,\mathrm{s}}\right) + \delta_{x,0}\delta_{\mathrm{s},\mathrm{t}}\rho\left(a_{0,\mathrm{s}}^{*}a_{0,\mathrm{s}}\right) \\
&= \rho\left(a_{0,\mathrm{t}}^{*}a_{0,\mathrm{t}}\right)\rho\left(a_{0,\mathrm{s}}^{*}a_{0,\mathrm{s}}\right) + \rho\left(\left(a_{x,\mathrm{s}}a_{0,\mathrm{t}}\right)^{*}\right)\rho\left(a_{x,\mathrm{s}}a_{0,\mathrm{t}}\right) \\
&\quad - \rho\left(\left(a_{0,\mathrm{t}}^{*}a_{x,\mathrm{s}}\right)^{*}\right)\rho\left(a_{0,\mathrm{t}}^{*}a_{x,\mathrm{s}}\right) + \delta_{x,0}\delta_{\mathrm{s},\mathrm{t}}\rho\left(a_{0,\mathrm{s}}^{*}a_{0,\mathrm{s}}\right) \\
&= \rho\left(a_{0,\mathrm{t}}^{*}a_{0,\mathrm{t}}\right)\rho\left(a_{0,\mathrm{s}}^{*}a_{0,\mathrm{s}}\right) + \left|\rho\left(a_{x,\mathrm{s}}a_{0,\mathrm{t}}\right)\right|^2 - \left|\rho\left(a_{0,\mathrm{t}}^{*}a_{x,\mathrm{s}}\right)\right|^2 \\
&\quad + \delta_{x,0}\delta_{\mathrm{s},\mathrm{t}}\rho\left(a_{0,\mathrm{s}}^{*}a_{0,\mathrm{s}}\right)
\end{aligned}$$

for any invariant quasi-free state $\rho \in \mathcal{Q}_1$, lattice position $x \in \Gamma$, and spin $\mathrm{s}, \mathrm{t} \in \Omega$. Using this result together with Lemma 6.41, we deduce the expression in the corollary for the energy density functional $\mathfrak{e}_{\Psi_{\mathrm{v}}} \circ q$. ☐

Since the free energy density functional of the model studied here equals

$$\mathfrak{f}_{\Phi_{\mathrm{h},\mathrm{v}},\beta} = \mathfrak{e}_{\Phi_{\mathrm{h}}} + \mathfrak{e}_{\Psi_{\mathrm{v}}} - \beta^{-1}\mathfrak{s} \ ,$$

for any inverse temperature $\beta \in (0, \infty)$ and summable reflection-symmetric functions $\mathrm{h} : \Gamma \to \mathbb{R}$ and $\mathrm{v} : \Gamma \to \mathbb{R}$, we conclude from Corollaries 6.42 and 6.43 that the variational problem

$$\inf \mathfrak{f}_{\Phi_{\mathrm{h},\mathrm{v}},\beta}\left(\mathcal{Q}_1\right)$$

on the weak*-compact set $\mathcal{Q}_1$ of invariant quasi-free states on $\mathcal{U}$ can be studied by minimizing on the set E_1 of all invariant states the weak*-lower semicontinuous[10]

[10] To prove the lower semicontinuity of $\tilde{g}_{\mathrm{h},f,\beta}$, combine Lemmata 6.7, 6.8, and 6.41, as is already done in the proof of Corollary 6.42.

functional $\mathfrak{f}_{\Phi_{\mathrm{h,v}},\beta} \circ q$, which is *quadratic* in the energy densities and thus similar to the nonlinear free energy density functional $\mathfrak{g}_{\mathfrak{m},\beta} : E_1 \to \mathbb{R}$ of Definition 6.17, for a mean-field model $\mathfrak{m} \in \mathcal{M}_1$. In fact, by Definition 6.17 and Corollary 6.43, there is a mean-field model $\mathfrak{m}_{\mathrm{h,v}} \in \mathcal{M}_1$ such that, for any invariant state $\rho \in E_1$ and every inverse temperature $\beta \in (0, \infty)$,

$$\mathfrak{g}_{\mathfrak{m}_{\mathrm{h,v}},\beta}(\rho) = \tilde{f}_{\mathrm{h,v},\beta}(\rho) + \left(\sum_{\mathrm{s}\in\Omega} \rho\left(a_{0,\mathrm{s}}^{*} a_{0,\mathrm{s}}\right) \right)^{2} \sum_{x\in\Gamma} \mathrm{v}(x) + \sum_{x\in\Gamma} \mathrm{v}(x) \sum_{\mathrm{s,t}\in\Omega} \left(\left|\rho\left(a_{x,\mathrm{s}} a_{0,\mathrm{t}}\right)\right|^{2} - \left|\rho\left(a_{0,\mathrm{t}}^{*} a_{x,\mathrm{s}}\right)\right|^{2} \right)$$

where $\tilde{f}_{\mathrm{h,v},\beta} : E_1 \to \mathbb{R}$ is the weak*-lower semicontinuous and affine functional defined by

$$\tilde{f}_{\mathrm{h,v},\beta}(\rho) \doteq \mathfrak{e}_{\Phi_{\mathrm{h}}}(\rho) + \mathrm{v}(0) \sum_{\mathrm{s}\in\Omega} \rho\left(a_{0,\mathrm{s}}^{*} a_{0,\mathrm{s}}\right) - \beta^{-1} \mathfrak{s}(\rho)$$

for $\rho \in E_1$ and $\beta \in (0, \infty)$. Remark that $\mathfrak{g}_{\mathfrak{m}_{\mathrm{h,v}},\beta} = \mathfrak{f}_{\Phi_{\mathrm{h,v}},\beta}$ on the set $\mathcal{Q}_1$ of invariant quasi-free states but this equality does *not* a priori hold true on the whole set $E_1 \supseteq \mathcal{Q}_1$.

The mean-field model $\mathfrak{m}_{\mathrm{h,v}}$ can be explicitly written by using some bijection from $\mathbb{N}$ to $\Omega \times \Gamma$, but we omit its explicit form to simplify our discussions and focus on the main arguments. Observe only that the mean-field model $\mathfrak{m}_{\mathrm{h,v}} \in \mathcal{M}_1$ refers to a fermion system, whose local Hamiltonians (Definition 6.4) are

$$\begin{aligned} H_{\Lambda}^{\mathfrak{m}_{\mathrm{h,v}}} = & \sum_{x,y\in\Lambda,\ \mathrm{s}\in\Omega} \mathrm{h}(x-y)\, a_{x,\mathrm{s}}^{*} a_{y,\mathrm{s}} + \mathrm{v}(0) \sum_{x\in\Lambda,\ \mathrm{s}\in\Omega} a_{x,\mathrm{s}}^{*} a_{x,\mathrm{s}} \\ & + \frac{1}{|\Lambda|} \sum_{z\in\Gamma} \mathrm{v}(z) \left| \sum_{x\in\Lambda,\ \mathrm{s}\in\Omega} a_{x,\mathrm{s}}^{*} a_{x,\mathrm{s}} \right|^{2} \\ & + \frac{1}{|\Lambda|} \sum_{z\in\Gamma} \mathrm{v}(z) \sum_{\mathrm{s,t}\in\Omega} \left(\left| \sum_{x,x+z\in\Lambda} a_{x+z,\mathrm{s}} a_{x,\mathrm{t}} \right|^{2} - \left| \sum_{x,x+z\in\Lambda} a_{x,\mathrm{t}}^{*} a_{x+z,\mathrm{s}} \right|^{2} \right) \end{aligned}$$

for any finite subset $\Lambda \in \mathcal{P}_f$. This mean-field model is highly non-trivial and even includes BCS-type interactions; see, e.g., Example 6.6. This may give the impression that the original short-range model can imply a superconducting phase transition at low temperatures (for non-positive v), but one shall refrain from making such rapid conclusions, since the generalized Hartree-Fock theory could significantly overestimate the true free energy density.

6.10.3 Thermodynamic Game and Bogoliubov Approximation

The set $\hat{M}_{\mathfrak{m}_{\mathrm{h,v}},\beta}$ of minimizers of the variational problem

$$\inf \mathfrak{g}_{\mathfrak{m}_{\mathrm{h,v}},\beta}(E_1)$$

(see Definition 6.21) can be completely described via Bogoliubov approximations for the associated mean-field models, as explained in Sect. 6.5. This brings us to the thermodynamic game introduced in Sect. 6.7. In fact, similar to Lemma 6.27, the following assertions hold true:

Lemma 6.44 (Bogoliubov Approximation) *Let $\rho \in E_1$ be any invariant state.*

(i) *Given $\gamma \geq 0$, the positive number*

$$r(\rho) = \gamma \sum_{\mathrm{s}\in\Omega} \rho\left(a_{0,\mathrm{s}}^* a_{0,\mathrm{s}}\right)$$

is the unique maximizer of the variational problem

$$\sup_{r\in\mathbb{R}_0^+} \left\{ -r^2 + 2r\gamma \sum_{\mathrm{s}\in\Omega} \rho\left(a_{0,\mathrm{s}}^* a_{0,\mathrm{s}}\right) \right\} = \left(\gamma \sum_{\mathrm{s}\in\Omega} \rho\left(a_{0,\mathrm{s}}^* a_{0,\mathrm{s}}\right) \right)^2 .$$

(ii) *For any function $\xi \in \ell^2(\Omega^2 \times \Gamma, \mathcal{U})$,*

$$\sup_{c\in\ell^2(\Omega^2\times\Gamma)} \left\{ -\|c\|_2^2 + 2\,\mathrm{Re}\,\langle c, \rho(\xi)\rangle \right\} = \|\rho(\xi)\|_2^2$$

with unique a maximizer

$$d(\rho) = \rho(\xi) \doteq (\rho(\xi(\mathrm{s},\mathrm{t},x)))_{(\mathrm{s},\mathrm{t},x)\in\Omega^2\times\Gamma} \in \ell^2(\Omega^2 \times \Gamma) .$$

Proof The proof is the same as the one of Lemma 6.27 and it is therefore omitted. We only remark that the variational problem in (i) can be restricted to positive numbers because $\rho\left(a_{0,\mathrm{s}}^* a_{0,\mathrm{s}}\right) \in \mathbb{R}_0$, a state being by definition a positive functional and $a_{0,\mathrm{s}}^* a_{0,\mathrm{s}} \geq 0$ in $\mathcal{U}$. □

To apply Lemma 6.44, we need to keep track of the sign of the function v at each lattice site. With this aim, as is usual, one splits v into its positive and negative components, $\mathrm{v} = \mathrm{v}_+ - \mathrm{v}_-$, where $\mathrm{v}_+(x) \doteq \sup\{\mathrm{v}(x), 0\}$ and $\mathrm{v}_-(x) \doteq \sup\{-\mathrm{v}(x), 0\}$ for all $x \in \Gamma$. This is similar to what is done for mean-field models for which we must distinguish between the effects of mean-field attractions and repulsions. For the sake of simplicity, we assume from now that $\mathrm{v} = \mathrm{v}_+ \geq 0$. Notice, however, that this special case already yields a non-trivial thermodynamic game, that is, the corresponding mean-field model is neither purely attractive nor repulsive. In fact, the generalization to functions v not having a definite sign is straightforward; it is only a matter of "bookkeeping."

Theorem 6.45 (Hartree-Fock Thermodynamic Game—Repulsive Case) *For any inverse temperature* $\beta \in (0, \infty)$ *and summable reflection-symmetric functions* $\mathrm{h} : \Gamma \to \mathbb{R}$ *and* $\mathrm{v} : \Gamma \to \mathbb{R}_0^+$,

$$\inf \mathfrak{g}_{\mathfrak{m}_{\mathrm{h,v}},\beta}(E_1) = \inf_{c_- \in \ell^2(\Omega^2 \times \mathrm{supp}(\mathrm{v}))} \sup_{r \in \mathbb{R}_0^+} \sup_{c_+ \in \ell^2(\Omega^2 \times \mathrm{supp}(\mathrm{v}))} \left\{ \|c_-\|_2^2 - \|c_+\|_2^2 - r^2 + \inf \mathfrak{f}_{\Phi_{\mathfrak{m}_{\mathrm{h,v}}}(c_-,c_+,r),\beta}(E_1) \right\}$$

where

$$\mathfrak{f}_{\Phi_{\mathfrak{m}_{\mathrm{h,v}}}(c_-,c_+,r),\beta} \doteq \tilde{f}_{\mathrm{h,v},\beta}(\rho) + 2r \|\mathrm{v}\|_{\Gamma}^{1/2} \sum_{\mathrm{s}\in\Omega} \rho\left(a_{0,\mathrm{s}}^* a_{0,\mathrm{s}}\right) + 2 \sum_{x \in \mathrm{supp}(\mathrm{v})} \sqrt{\mathrm{v}(x)} \times \sum_{\mathrm{s,t}\in\Omega} \mathrm{Re}\left\{\overline{c_+(\mathrm{s,t},x)}\rho\left(a_{x,\mathrm{s}} a_{0,\mathrm{t}}\right) - \overline{c_-(\mathrm{s,t},x)}\rho\left(a_{0,\mathrm{t}}^* a_{x,\mathrm{s}}\right)\right\} .$$

Idea of the Proof The proof is a slightly simplified version of the one of Theorem 6.34, which uses among other things Lemma 6.44 together with the von Neumann min-max theorem [1, Theorem 10.50] to be able to exchange the two suprema of the assertion with the infimum over invariant states $\rho \in E_1$. □

Recall that

$$\|\mathrm{v}\|_{\Gamma} \doteq \sum_{x\in\Gamma} |\mathrm{v}(x)| < \infty$$

and $\mathrm{supp}(\mathrm{v}) \subseteq \Gamma$ stands for the support of the function v. Here,

$$\Phi_{\mathfrak{m}_{\mathrm{h,v}}}(c_-, c_+, r) \equiv \Phi_{\mathfrak{m}_{\mathrm{h,v}}}(c_-, (c_+, r)) \in \mathcal{W}_1$$

refers to the approximating interactions associated with the mean-field model $\mathfrak{m}_{\mathrm{h,v}}$. See Definition 6.26.

It is now clear from Theorem 6.45 that the variational problem $\inf \mathfrak{g}_{\mathfrak{m}_{\mathrm{h,v}},\beta}(E_1)$ for non-zero functions v can be seen as the conservative value of a ("Hartree-Fock thermodynamic") game, with players (−) and (+), whose sets of strategies are $\ell^2(\Omega^2 \times \mathrm{supp}(\mathrm{v}))$ for (−) and $\ell^2(\Omega^2 \times \mathrm{supp}(\mathrm{v})) \times \mathbb{R}_0^+$ for (+). The minimizers of $\mathfrak{g}_{\mathfrak{m}_{\mathrm{h,v}},\beta}$ can also be derived from this game:

Note that $\mathfrak{f}_{\Phi_{\mathfrak{m}_{\mathrm{h,v}}}(c_-,c_+,r),\beta}$ is an affine and lower weak* semicontinuous functional on E_1, which is a weak*-compact and convex set. Thus, define its weak*-compact and convex set of minimizers by

$$\Omega_{\Phi_{\mathfrak{m}_{\mathrm{h,v}}}(c_-,c_+,r),\beta} \doteq \left\{\omega \in E_1 : \mathfrak{f}_{\Phi_{\mathfrak{m}_{\mathrm{h,v}}}(c_-,c_+,r),\beta}(\omega) = \inf \mathfrak{f}_{\Phi_{\mathfrak{m}_{\mathrm{h,v}}}(c_-,c_+,r),\beta}(E_1)\right\} .$$

By [1, Lemma 8.3 (♯)], for any $\beta \in (0,\infty)$ and summable reflection-symmetric functions $\mathrm{h} : \Gamma \to \mathbb{R}$ and $\mathrm{v} : \Gamma \to \mathbb{R}_0^+$, $\mathrm{v} \neq 0$, and all $c_- \in \ell^2(\Omega^2 \times \mathrm{supp}(\mathrm{v}))$, there is exactly one element, which is denoted by

$$\mathrm{r}_+(c_-) \doteq (d_+(c_-), \mathbf{r}(c_-)) \in \ell^2(\Omega^2 \times \mathrm{supp}(\mathrm{v})) \times \mathbb{R}_0^+ ,$$

such that

$$\begin{aligned} h^\sharp_{\mathfrak{m}_{\mathrm{h,v}},\beta}(c_-) \doteq \|c_-\|_2^2 + \sup_{r\in\mathbb{R}_0^+} \sup_{c_+\in\ell^2(\Omega^2\times\mathrm{supp}(\mathrm{v}))} &\Big\{- \|c_+\|_2^2 - r^2 \\ &+ \inf \mathfrak{f}_{\Phi_{\mathfrak{m}_{\mathrm{h,v}}}(c_-,c_+,r),\beta}(E_1)\Big\} \\ = \|c_-\|_2^2 - \|d_+(c_-)\|_2^2 - \mathbf{r}(c_-)^2 &+ \inf \mathfrak{f}_{\Phi_{\mathfrak{m}_{\mathrm{h,v}}}(c_-,d_+(c_-),\mathbf{r}(c_-)),\beta}(E_1) . \end{aligned}$$

Using the set

$$\mathcal{C}^\sharp_{\mathfrak{m}_{\mathrm{h,v}},\beta} \doteq \left\{d_- \in \ell^2(\Omega^2 \times \mathrm{supp}(\mathrm{v})) : \inf \mathfrak{g}_{\mathfrak{m}_{\mathrm{h,v}},\beta}(E_1) = h^\sharp_{\mathfrak{m}_{\mathrm{h,v}},\beta}(d_-)\right\} ,$$

one can characterize minimizers of the nonlinear free energy functional $\mathfrak{g}_{\mathfrak{m}_{\mathrm{h,v}},\beta}$ as follows: For all strategies $c_- \in \ell^2(\Omega^2 \times \mathrm{supp}(\mathrm{v}))$ and $(c_+, r) \in \ell^2(\Omega^2 \times \mathrm{supp}(\mathrm{v})) \times \mathbb{R}_0^+$, we define the (possibly empty) set

$$\begin{aligned} \Omega_{\Phi_{\mathfrak{m}_{\mathrm{h,v}}},\beta}(c_-, c_+, r) \doteq \Big\{\omega \in \Omega_{\Phi_{\mathfrak{m}_{\mathrm{h,v}}}(c_-,c_+,r),\beta} \; : \; \|\mathrm{v}\|_\Gamma^{1/2} \sum_{\mathrm{s}\in\Omega} \omega\left(a^*_{0,\mathrm{s}} a_{0,\mathrm{s}}\right) = r , \\ \sqrt{\mathrm{v}(x)}\omega\left(a_{x,\mathrm{s}} a_{0,\mathrm{t}}\right) = c_+(\mathrm{s},\mathrm{t},x) , \; \sqrt{\mathrm{v}(x)}\omega\left(a^*_{0,\mathrm{t}} a_{x,\mathrm{s}}\right) = c_-(\mathrm{s},\mathrm{t},x) , \quad (\mathrm{s},\mathrm{t},x) \in \Omega^2 \\ \times \mathrm{supp}(\mathrm{v})\Big\} \subseteq E_1 . \end{aligned}$$

Note that the above set $\Omega_{\Phi_{\mathfrak{m}_{\mathrm{h,v}}},\beta}(c_-, c_+, r)$ of self-consistent equilibrium states is an instance of (6.35). With these definitions, we have the following assertion:

Theorem 6.46 (Self-Consistency—Repulsive Case) *For any inverse temperature* $\beta \in (0,\infty)$ *and summable reflection-symmetric functions* $\mathrm{h} : \Gamma \to \mathbb{R}$ *and* $\mathrm{v} : \Gamma \to \mathbb{R}_0^+$,

$$\hat{M}_{\mathfrak{m}_{\mathrm{h,v}},\beta} = \bigcup_{d_-\in\mathcal{C}^\sharp_{\mathfrak{m}_{\mathrm{h,v}},\beta}} \Omega_{\Phi_{\mathfrak{m}_{\mathrm{h,v}}},\beta}(d_-, d_+(d_-), \mathbf{r}(d_-))$$

Proof This theorem is proven like Theorem 6.36 (i). For a complete proof, see [1, Theorem 9.4]. □

Similar to Theorem 6.36, this last theorem characterizes minimizers of the nonlinear free energy functional $\mathfrak{g}_{\mathfrak{m}_{\mathrm{h,v}},\beta}$ by (static) *self-consistency conditions*, which refer, in a sense, to Euler-Lagrange equations for the variational problem defining the thermodynamic game.

It turns out that under mild conditions on the functions h and v, the approximating interactions $\Phi_{\mathfrak{m}_{\mathrm{h,v}}}(c_-, c_+, r)$ have exactly one equilibrium state. By Theorem 6.36 (iii), note additionally that $\Omega_{\Phi_{\mathfrak{m}_{\mathrm{h,v}}}(d_-,d_+(d_-),\mathbf{r}(d_-)),\beta}$ is never empty, for any $d_- \in \mathcal{C}^\sharp_{\mathfrak{m}_{\mathrm{h,v}},\beta}$. Thus, under these conditions, the corresponding sets $\Omega_{\Phi_{\mathfrak{m}_{\mathrm{h,v}}},\beta}(c_-, c_+, r)$ of self-consistent equilibrium states have at most one element, and, from the last theorem, we arrive at

$$\hat{M}_{\mathfrak{m}_{\mathrm{h,v}},\beta} = \bigcup_{d_- \in \mathcal{C}^\sharp_{\mathfrak{m}_{\mathrm{h,v}},\beta}} \Omega_{\Phi_{\mathfrak{m}_{\mathrm{h,v}}}(d_-,d_+(d_-),\mathbf{r}(d_-)),\beta} \ .$$

In other words, in this case, the set of minimizers of $\mathfrak{g}_{\mathfrak{m}_{\mathrm{h,v}},\beta}$ is nothing else than the collection of the unique equilibrium states of the corresponding approximating interactions $\Phi_{\mathfrak{m}_{\mathrm{h,v}}}(d_-, d_+(d_-), \mathbf{r}(d_-))$ for $d_- \in \mathcal{C}^\sharp_{\mathfrak{m}_{\mathrm{h,v}},\beta}$. What is more, beyond this nice property of $\mathfrak{g}_{\mathfrak{m}_{\mathrm{h,v}},\beta}$, it turns out that the same condition guaranteeing the uniqueness of the equilibrium state of $\Omega_{\Phi_{\mathfrak{m}_{\mathrm{h,v}}},\beta}(c_-, c_+, r)$ also implies that $\hat{M}_{\mathfrak{m}_{\mathrm{h,v}},\beta} \subseteq \mathcal{Q}_1$, i.e., all the minimizers of $\mathfrak{g}_{\mathfrak{m}_{\mathrm{h,v}},\beta}$ are quasi-free states. In other words, the Hartree-Fock equilibrium states for model considered in this subsection are exactly the minimizers of $\mathfrak{g}_{\mathfrak{m}_{\mathrm{h,v}},\beta}$ in the set of all (i.e., not necessarily quasi-free) invariant states, that is, the nonlinear equilibrium states of an explicit mean-field model $\mathfrak{m}_{\mathrm{h,v}}$.

In order to prove this claim, we now discuss in more detail the relation between the variational problem

$$\inf \mathfrak{g}_{\mathfrak{m}_{\mathrm{h,v}},\beta}(E_1) \ ,$$

along with its set $\hat{M}_{\mathfrak{m}_{\mathrm{h,v}},\beta}$ of minimizers, and the variational problem

$$\inf \mathfrak{g}_{\mathfrak{m}_{\mathrm{h,v}},\beta}(\mathcal{Q}_1) = \inf \mathfrak{f}_{\Phi_{\mathrm{h,v}},\beta}(\mathcal{Q}_1) \ ,$$

along with its set $\mathrm{Q}_{\Phi_{\mathrm{h,v}},\beta}$ of minimizers, given by the generalized Hartree-Fock theory, which is our main concern in Sect. 6.10:

Since $\mathcal{Q}_1 \subseteq E_1$, one has trivially the inequality

$$\inf \mathfrak{g}_{\mathfrak{m}_{\mathrm{h,v}},\beta}(E_1) \leq \inf \mathfrak{g}_{\mathfrak{m}_{\mathrm{h,v}},\beta}(\mathcal{Q}_1) \ .$$

Further, one observes that the approximating (invariant) interaction $\Phi_{\mathfrak{m}_{\mathrm{h,v}}}(c_-, c_+, r)$ associated with the mean-field model $\mathfrak{m}_{\mathrm{h,v}} \in \mathcal{M}_1$ for $c_-, c_+ \in \ell^2(\Omega^2 \times \mathrm{supp}(\mathrm{v}))$ and $r \in \mathbb{R}_0^+$ corresponds to even self-adjoint elements of $\mathcal{U}$ that are *quadratic* in the creation and annihilation elements $a_{x,\mathrm{s}}, a_{x,\mathrm{s}}^*$, for $x \in \Gamma$ and $\mathrm{s} \in \Omega$. For instance, the associated local Hamiltonians (Definition 6.4) (for positive $\mathrm{v} \geq 0$) are equal to

$$\begin{aligned} H_\Lambda^{\Phi_{\mathfrak{m}_{\mathrm{h,v}}}(c_-,c_+,r)} = & \sum_{x,y\in\Lambda,\ \mathrm{s}\in\Omega} \mathrm{h}(x-y)\, a_{x,\mathrm{s}}^* a_{y,\mathrm{s}} + \left(\mathrm{v}(0) + 2r\,\|\mathrm{v}\|_\Gamma^{1/2}\right) \\ & \times \sum_{x\in\Lambda,\ \mathrm{s}\in\Omega} a_{x,\mathrm{s}}^* a_{x,\mathrm{s}} \\ & +2\sum_{z\in\Gamma}\sqrt{\mathrm{v}(z)}\sum_{\mathrm{s,t}\in\Omega}\sum_{x\in\Lambda}\mathrm{Re}\left\{\overline{c_+(\mathrm{s,t},x)}a_{x+z,\mathrm{s}}a_{x,\mathrm{t}}\right. \\ & \left. -\overline{c_-(\mathrm{s,t},x)}a_{x,\mathrm{t}}^* a_{x+z,\mathrm{s}}\right\} \end{aligned}$$

for any finite subset $\Lambda \in \mathcal{P}_f$ and every $c_-, c_+ \in \ell^2(\Omega^2 \times \Gamma)$ and $r \in \mathbb{R}_0^+$. Such kind of quadratic, or bilinear, Hamiltonians can be explicitly diagonalized by a so-called Bogoliubov transformation, as already shown in Berezin's book [95], published in 1966. In other words, the thermodynamic game associated with Theorem 6.45 can be studied from finite-volume systems for which explicit computations can be made.

More generally, if the reflection-symmetric functions h and v are not only summable but also decaying sufficiently fast,[11] as $|x| \to \infty$, such bilinear Hamiltonians are well-known to generate an infinite-volume dynamics which is a strongly continuous group of Bogoliubov $*$-automorphisms, as given by Definition 4.181 and Corollary 4.183. See, for instance, [96, Lemma 2.8]. Araki proves in [69, Theorem 3] the existence of a unique KMS (Kubo-Martin-Schwinger) state associated with such a group of Bogoliubov $*$-automorphisms, which turns out to be a *quasi-free* state. See, for instance, Proposition 3.33 for the KMS condition in finite dimensions. For more details, see, e.g., [55, Sections 5.3–5.4]. If the bilinear model defining the dynamics is invariant, its KMS state is also invariant. If it is gauge-invariant, then the KMS state is also (globally) gauge-invariant and therefore a *simple* quasi-free state.

The relation between KMS states and minimizers of a variational problem derived from the same sufficiently short-range interaction has been studied for lattice fermions by Araki and Moriya [15]: It turns out that all minimizers of a variational problem like

$$\inf \mathfrak{f}_{\Phi_{\mathfrak{m}_{\mathrm{h,v}}}(c_-,c_+,r),\beta}(E_1)$$

for sufficiently decaying reflection-symmetric functions h and v are KMS states. See, e.g., [97, Theorem 3.1]. Since, in this case, the KMS state is unique, invariant, and quasi-free, we conclude that

$$\Omega_{\Phi_{\mathfrak{m}_{\mathrm{h,v}}}(c_-,c_+,r),\beta} = \{\omega\} \subseteq \mathcal{Q}_1 .$$

[11] For instance, they show a sufficiently fast polynomial decay.

Therefore, it follows, in this situation, that

$$\hat{M}_{\mathfrak{m}_{\mathrm{h,v}},\beta} = \bigcup_{d_- \in \mathcal{C}^\sharp_{\mathfrak{m}_{\mathrm{h,v}},\beta}} \Omega_{\Phi_{\mathfrak{m}_{\mathrm{h,v}}}(c_-,d_+(c_-),\mathbf{r}(c_-)),\beta} = \mathrm{Q}_{\Phi_{\mathrm{h,v}},\beta} \subseteq \mathcal{Q}_1 \ ,$$

meaning in particular that

$$\inf \mathfrak{g}_{\mathfrak{m}_{\mathrm{h,v}},\beta}\left(E_1\right) = \inf \mathfrak{g}_{\mathfrak{m}_{\mathrm{h,v}},\beta}\left(\mathcal{Q}_1\right) = \inf \mathfrak{f}_{\Phi_{\mathrm{h},f},\beta}\left(\mathcal{Q}_1\right) \ .$$

This explicitly shows the mean-field character of the general Hartree-Fock theory applied on our prototypical (though very general, quartic) short-range model (Sect. 6.10.1), which includes the celebrated Hubbard model, widely used in Physics, as one simple example.

To conclude, notice that if one would only consider simple (i.e., gauge-invariant) quasi-free states in the Hartree-Fock theory for our prototypical model, everything that is said above still applies, mutatis mutandis, by considering the (simpler) nonlinear energy density functional

$$\mathfrak{g}_{\mathfrak{m}_{\mathrm{h,v}},\beta}\left(\rho\right) = \tilde{f}_{\mathrm{h,v},\beta}\left(\rho\right) + \left(\sum_{\mathrm{s}\in\Omega} \rho\left(a^*_{0,\mathrm{s}} a_{0,\mathrm{s}}\right)\right)^2 \sum_{x\in\Gamma} \mathrm{v}\left(x\right) - \sum_{x\in\Gamma} \mathrm{v}\left(x\right) \sum_{\mathrm{s,t}\in\Omega} \left|\rho\left(a^*_{0,\mathrm{t}} a_{x,\mathrm{s}}\right)\right|^2 \ ,$$

instead of

$$\mathfrak{g}_{\mathfrak{m}_{\mathrm{h,v}},\beta}\left(\rho\right) = \tilde{f}_{\mathrm{h,v},\beta}\left(\rho\right) + \left(\sum_{\mathrm{s}\in\Omega} \rho\left(a^*_{0,\mathrm{s}} a_{0,\mathrm{s}}\right)\right)^2 \sum_{x\in\Gamma} \mathrm{v}\left(x\right) + \sum_{x\in\Gamma} \mathrm{v}\left(x\right) \sum_{\mathrm{s,t}\in\Omega} \left(\left|\rho\left(a_{x,\mathrm{s}} a_{0,\mathrm{t}}\right)\right|^2 - \left|\rho\left(a^*_{0,\mathrm{t}} a_{x,\mathrm{s}}\right)\right|^2\right) \ .$$

In particular, even in this simpler case, for positive $\mathrm{v} \geq 0$, we still have a non-trivial thermodynamic game with two players, $(-)$ and $(+)$, whose sets of strategies are now $\ell^2(\Omega^2 \times \mathrm{supp}(\mathrm{v}))$ (as before) for $(-)$ and $\mathbb{R}^+_0$ (instead of $\ell^2(\Omega^2 \times \mathrm{supp}(\mathrm{v})) \times \mathbb{R}^+_0$) for $(+)$.