

Failure Rate Modeling of Mechanical Components and Systems



Liyang Xie

Abstract For products subjected to many times of load action during service, product life is dominated by load and its capability against load, referred to as strength. This chapter introduces load-strength interference analysis based failure rate modelling method, develops component and system failure rate models, and illustrates the causal relation between failure rate curve shape and load/strength characteristics. For the majority of mechanical components and systems, service load can be described as a random process, material property degrades during load actions, and the dynamic load-strength relationship makes the failure rate change continuously. As failure occurs on load exceeding strength, failure rate models are developed by analyzing the competition behavior between load and strength. By such failure rate models, the effects of load uncertainty, strength uncertainty and strength degradation pattern on failure rate curve shape are demonstrated. Meanwhile, the three stages of the bathtub curve are interpreted in terms of stochastic load-strength competition behavior, the roller coaster type failure rate curve is attributed to the strength diversity of the products in a population.

Keywords Failure rate definition · Life distribution · Strength degradation · Load-strength interference · Bathtub curve

1 Introduction

Failure rate is a frequently used metric for product reliability. By definition, failure rate at time t is the limit of the probability that a product will fail in a time interval $(t, t + \Delta t]$ when Δt approaches to zero, given the product is functioning at time t . Besides direct estimation based on product life data, failure rate function can be derived from the probability density function of product life by the formula $\lambda(t) = f(t)/R(t)$, i.e., failure rate at time t equals to the ratio of the life probability density at time t to the reliability over time t . This formula presents a one to one mapping

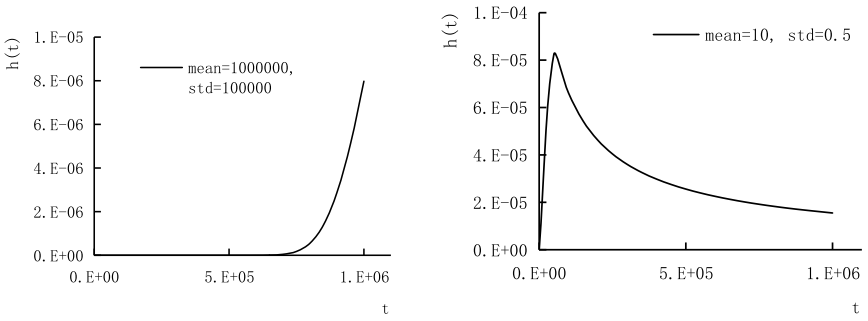
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between failure rate and life distribution. It is easy to know that the exponential life distribution yields constant failure rate, the normal (Gaussian) life distribution yields increasing failure rate, the log-normal life distribution yields unimodal failure rate (first increasing and then decreasing), and the Weibull life distribution may yield increasing, decreasing or constant failure rate depending on shape parameter value (shown in Fig. 1).

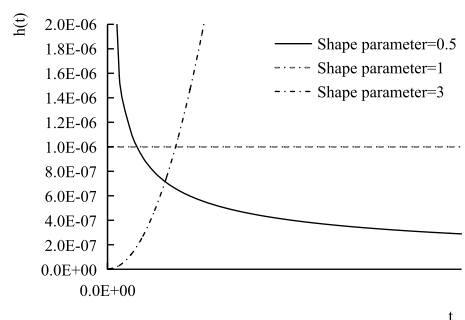
On the other hand, it is traditionally believed that the failure rate curve of bathtub shape is the most typical (shown in Fig. 2). Apparently, none of the commonly used life distributions, such as the exponential distribution, the normal distribution, the log-normal distribution or the Weibull distribution can yield a failure rate curve of bathtub shape, illustrating that either the commonly used probability density functions cannot exactly describe product life distribution, or product failure rate curve does not present bathtub shape.

The three stages in a failure rate curve of bathtub shape as shown in Fig. 2 were conventionally partitioned as infant mortality stage (the decreasing failure rate stage appeared in the early part of the population service life), chance failure stage (the



(a) Failure rate derived from normal life distribution

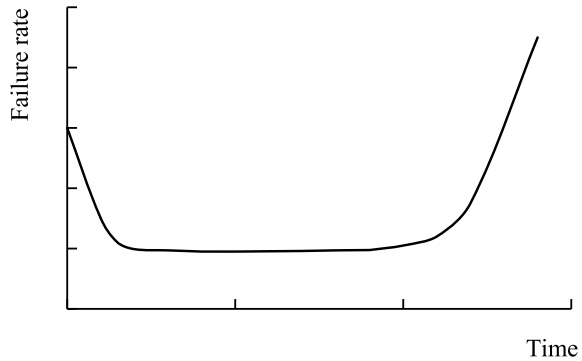
(b) Failure rate derived from log-normal life distribution



(c) Failure rates derived from Weibull life distributions

Fig. 1 Failure rate curves derived from different types of life distributions

Fig. 2 A failure rate curve of bathtub shape



middle part of the failure rate curve, showing a roughly constant failure rate), and wear out stage (the increasing failure rate stage appeared in the last part of the population service life) [1]. They are also called as burn-in period, useful life period and wear-out period, respectively [2]. It is usually explained that the infant mortality stage demonstrates a sub-population dominated by quality-control defects due to poor workmanship, out-of-specification incoming parts and materials, and other substandard manufacturing practices. The other two stages were attributed to stochastic load and product performance deterioration, respectively [1]. In other words, the high failure rate in the initial phase is explained as that there are undiscovered defects in the products. These soon show up when the products are activated. When the product has survived the infant mortality period, the failure rate often stabilizes at a level where it remains for a certain amount of time until it starts to increase as the products begin to wear out [2].

The features of product failure rate have been analyzed from the aspects of reliability function [3], life distribution [4–8] and strength degradation [9, 10]. Some studies on failure rate curve shape thought that mechanical products may not appear to have an infant mortality period or chance failure period [1].

In practice, product failure rates are estimated by means of various methods and models according to life data and/or censored life data. On the other hand, traditional reliability calculation is sometimes carried out based on failure rate function [11]. It means that failure rate should be obtained based on pertinent information different from life distribution or life data. To estimate failure rate directly from product life data needs a large size sample. To derive failure rate equation from product life distribution needs exact life probability density function that is hard to obtain. Therefore, modeling product failure rate in a way different from life data-based approaches is of great significance. Besides, it is helpful to get insight into the meaning of the failure rate curves of different shape.

The complex shape of a bathtub curve implies that failure rate modelling might be difficult. To develop a failure rate model, the basic influence factors must be identified first. Generally, the service time dependent variation of product failure rate depends on load characteristics, product strength, failure mechanism and other operational profile [12–14].

For mechanical components, it is well known that load-strength interference analysis is the most widely applied method to develop reliability model [15]. However, not many studies have been conducted to developed load-strength interference relationship-based failure rate model.

As to the load characteristics and failure mechanisms typical for mechanical components and structures, such as deformation or fracture under static load or fatigue under cyclic load, the times of load action is a more direct parameter to characterize product service life. For instance, taking into account the effect of multiple actions of a random load, a loading number dependent failure probability formula for static strength failure (no strength degradation during load actions) was proposed [16]:

$$P(n) = 1 - \int_0^{\infty} f(x) \left[\int_0^x g(y) dy \right]^n dx \quad (1)$$

where, $P(n)$ stands for the component failure probability after n times of load (stress) action, $f(x)$ stands for the probability density function of component strength, and $g(y)$ stands for the probability density function of the stress subjected to the component.

Obviously, in the situation of one time of load action, Eq. 1 degenerates into the traditional load-strength interference model for failure probability calculation:

$$P = 1 - \int_0^{\infty} f(x) \int_0^x g(y) dy dx \quad (2)$$

In principle, failure rate modeling is much the same as failure probability modeling, both can be achieved through load-strength interference analysis, since both the failure rate and the failure probability is determined by the load distribution, strength distribution, the times of load action, and the strength degradation behavior.

Indeed, life distribution can also be derived by means of load-strength interference analysis [17]. Different load distributions and/or strength distributions, together with their competition relations, yield different life distributions and different failure rate curves [17, 18]. Shown in Fig. 3 are the life distributions and failure rate curves of a mechanical component subjected to random loads, with strength degrading linearly during load actions. All the curves are drawn according to the respective functions formulated based on multiple variates stress-strength interference relationship. The stresses are presumed to follow the Weibull distribution, and strengths are presumed to follow the normal distribution. Figure 3a and b are the life distribution and failure rate curve in the situation of large stress dispersion and small safety margin; Fig. 3c and d are the life distribution and failure rate curve in the situation of small stress dispersion and large safety margin. Both the life distributions and the failure rate curves are considerably different for the two different stress-strength combinations. For the large stress dispersion situation, the life distribution is no longer the conventional unimodal curve, the failure rate curve presents bathtub shape; for the small

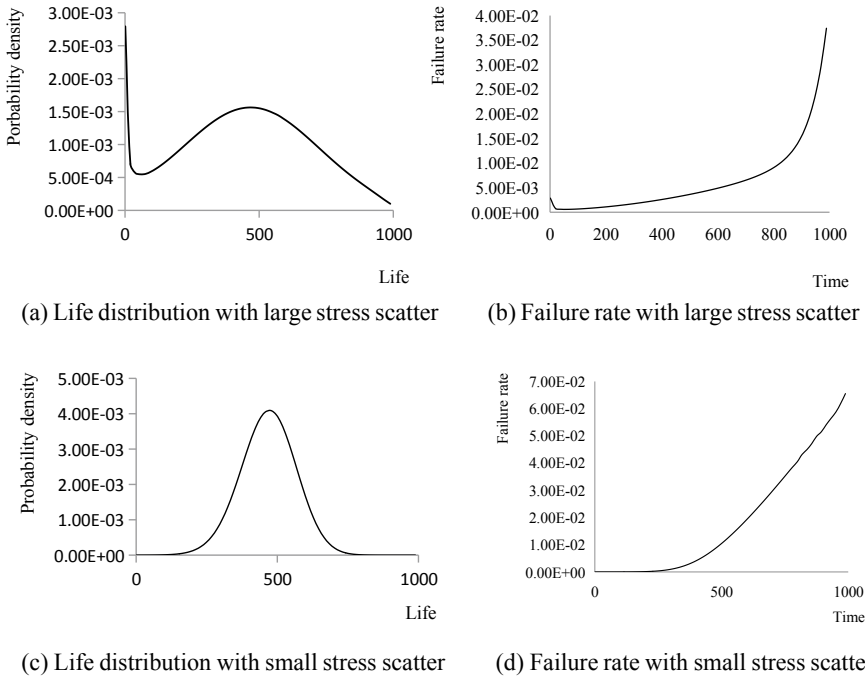


Fig. 3 Life distribution curves and failure rate curves derived from stress-strength distributions

stress dispersion situation, the life distribution presents unimodal curve, the failure rate curve is monotonically increasing.

Product life distributions are usually assumed to be one of the conventional forms such as exponential, normal, log-normal, Weibull, etc. Since none of those might be the true distribution of the product life, the failure rate function derived from life distribution may differ considerably from the true failure rate. That will mislead the understanding to the roles of the influencing factors.

This chapter introduces different ways to formulate failure rate model. First, failure rate functions are established based on stress-strength competition analysis, and the effects of stress distribution and strength distribution on failure rate curve shape are analyzed and demonstrated. Such failure rate models can clearly reveal the mechanism resulting in different shape of failure rate curves. For instance, it is illustrated that the three parts of a bathtub-shaped failure rate curve are not necessarily incurred by different root causes, different influencing factors or different failure mechanisms. Any deteriorate type of failure mechanism such as fatigue under random load history may bring about bathtub-shaped failure rate curve, whereas fatigue under constant amplitude load history leads to monotonically increasing failure rate curve. Besides, failure rate models are formulated based on the definition of failure rate in the condition that the time variable is discrete, and by virtual of random events operation.

2 Load Order Statistics and Stress-Strength Based Failure Rate Modeling

Assume that Y_1, Y_2, \dots, Y_n are independent continuous random variables with probability density function $g(y)$, y_1, y_2, \dots, y_n are n sample values of the random variables, and $y_{(1)} < y_{(2)} < \dots < y_{(n)}$ the sorted sample values from the minimum to the maximum. With the probability density function $g(y)$ and cumulative distribution function $G(y)$ of the random variable Y , the probability density function of the 1st order statistic (the minimum) $Y_{(1)}$ (denoted by $g_1(y)$) and that of the n th order statistic (the maximum) $Y_{(n)}$ (denoted by $g_n(y)$) are, respectively [19]:

$$g_1(y) = n[1 - G(y)]^{n-1}g(y) \quad (3)$$

$$g_n(y) = n[G(y)]^{n-1}g(y) \quad (4)$$

For mechanical equipment and components, most of them will experience many times of random load action during service. In the situation that a product subjects to n times of random load action, the n load values can usually be treated as a set of i.i.d. (independent, identically distributed) random variables. For static strength failure, a product survives n times of load action means that product strength is greater than the maximum load (stress) appeared during the n times of load action. Therefore, the maximum statistic of the n random loads is the most direct parameter for failure rate calculation. Furthermore, it is evident that product failure probability over n times of random load action or failure rate at the n th load action depends on the times of load action as well as the scatter of the random load variable, besides the product strength random variable.

Illustrated in Fig. 4 are the distributions of random load variables and the distributions of the corresponding maximum load order statistics in samples of size 10, 20, 50, and 500, respectively. It clearly shows that, for the random loads with different degrees of uncertainty, the distributions of the maximum load in n times of load action differ from each other considerably.

Incorporating stress order statistic into the conventional stress-strength interference model for multiple times of load action situations, dynamic (load action number dependent) component failure probability model and failure rate model can be developed. The dynamic characteristic of such models is attributed to the ever changing distribution of the maximum load order statistic. That is, the distribution of the maximum load order statistic changes continuously with the increase of the load action number. Component failure probability after n times of load action can be modeled as the following equation which is equivalent to Eq. 1 (see Fig. 5):

$$P(n) = \int_0^{\infty} f(x) \int_x^{\infty} g_n(y) dy dx \quad (5)$$

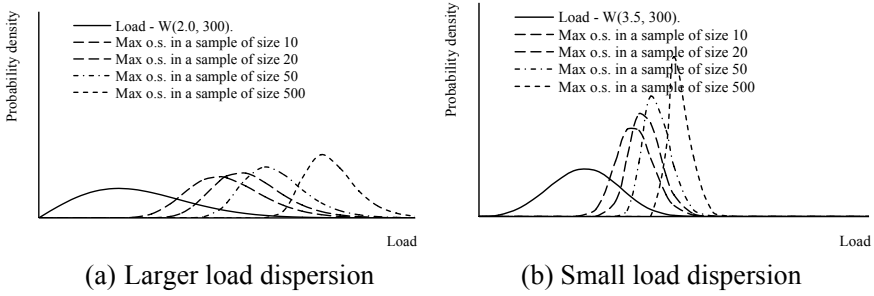


Fig. 4 Distributions of load random variables and their maximum order statistics (o.s.—order statistic)

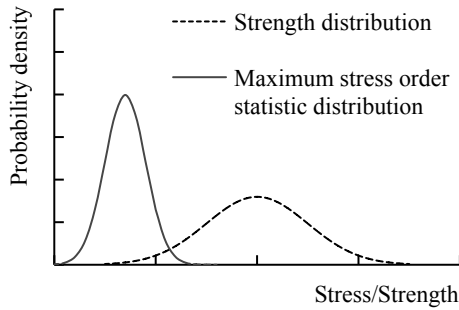


Fig. 5 Stress order statistic-strength interference relationship

In the condition that the times of load action (a discrete variable) is used as the time or life metric (it is usually treated as a continuous variable for failure rate definition), the failure rate of a product at n th load action, denoted by $h(n)$, can be defined as the probability of failure caused by this load action, given that the product has survived all the previous $(n-1)$ times of load action. Therefore, the failure rate $h(n)$ can be derived by means of the relationship between load distribution and the strength distribution of the survived products after $(n-1)$ times of load action, just as product failure probability can be derived by means of the relationship between load distribution and the strength distribution of the product population. To determine failure rate $h(n)$ by means of load-strength interference relationship, it is necessary to know the strength distribution of the survived products after $(n-1)$ times of load action (denoted by $f(x, n)$). Obviously, the strength of a survived product will not be lower than the maximum load in the $(n-1)$ times of load action (assuming no strength degradation during the $(n-1)$ times of load action). Based on the strength distribution of the survived products, failure rate can be expressed as

$$h(n) = \int_0^\infty f(x, n) \int_x^\infty g(y) dy dx \tag{6}$$

where, $f(x, n)$ denotes the strength distribution of the survived products after $(n-1)$ times of load action. Note that the strength of the product population is a random variable distributed in $0 \sim \infty$;, whereas the strength of the products survived $(n-1)$ times of load action is a random variable distributed in $z_{n-1} \sim \infty$; (z_{n-1} is the maximum stress value corresponding to the maximum load in the $(n-1)$ times of load action).

Generally, the strength distribution of the survived products can be obtained by updating the original strength distribution (shown in Fig. 6). As a probability density function, it has to satisfy $\int_0^\infty f(x, n)dx = 1$. It is easy to know that the strength distribution of the products survived $(n-1)$ times of load action is

$$f(x, n) = 0 \quad (0 < x \leq z_{(n-1)})$$

$$f(x, n) = \frac{f(x)}{\int_{z_{(n-1)}}^\infty f(x)dx} \quad (z_{(n-1)} < x < \infty) \tag{7}$$

where, $z_{(n-1)}$ denotes the maximum stress value in $(n-1)$ times of load action.

As load is a random variable, the maximum stress $z_{(n-1)}$ appeared during $(n-1)$ times of load action is a random variable following the maximum order statistic distribution of the stress variable. According to the total probability theorem, a failure rate model can be developed (Ref. Figure 7. For the sake of simplification, $z_{(n-1)}$ is denoted simply by z in the following). That is, failure rate $h(n)$ equals to the statistical average of the probability that failure occurs at the n th load action given survived the previous $(n-1)$ times of load action, weighted by the probability distribution of the maximum stress $z_{(n-1)}$ appeared during the $(n-1)$ times of load action (the probability density function of $z_{(n-1)}$ is denoted by $g_{n-1}(z)$):

$$h(n) = \int_0^\infty g_{n-1}(z) \left\{ \int_z^\infty f(x, n) \left[\int_x^\infty g(y)dy \right] dx \right\} dz \tag{8}$$

or

Fig. 6 Strength distribution of product population and those of the products survived a certain times of load action

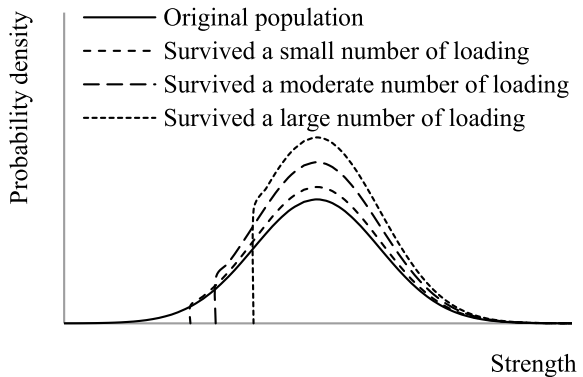
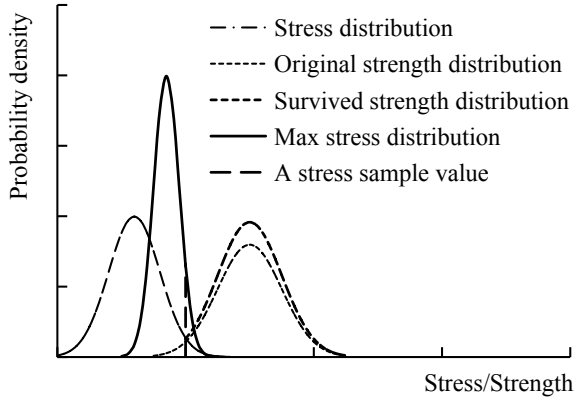


Fig. 7 Variables and their distributions involved in failure rate modeling



$$h(n) = \int_0^\infty g_{n-1}(z) \left\{ \int_z^\infty g(y) \left[\int_z^y f(x, n) dx \right] dy \right\} dz \tag{9}$$

In the situation of deterministic load, i.e., when a constant load is applied n times to a product, Eq. 9 degenerates as

$$h(1) = \int_0^y f(x) dx \tag{10}$$

$$h(n) = \int_y^y f(x, n) dx = 0 \quad (n \geq 2) \tag{11}$$

where, y is the constant stress value produced by the constant load.

These two equations demonstrate that, in the condition that a product subjects to many times of action of the same load, the failure rate at the first time of load action is equal to the failure probability of the product subjected to one time of load action; the failure rate at a load action number equal or greater than two equals to zero. It is easy to understand that a product survived one time of load action will survive forever, since it means that the strength of the product is greater than the stress, and it is assumed that the strength keeps the same during the load actions.

Equations 8 and 9 can also be respectively written as

$$h(n) = \int_0^\infty g_{n-1}(z) \left\{ \int_z^\infty \frac{f(x)}{1 - F(z)} \left[\int_x^\infty g(y) dy \right] dx \right\} dz \tag{12}$$

$$h(n) = \int_0^{\infty} g_{n-1}(z) \left\{ \int_z^{\infty} g(y) \left[\int_z^y \frac{f(x)}{1-F(z)} dx \right] dy \right\} dz \quad (13)$$

and

$$h(n) = \int_0^{\infty} g_{n-1}(z) \left\{ \int_z^{\infty} g(y) \left[\frac{F(y) - F(z)}{1 - F(z)} \right] dy \right\} dz \quad (14)$$

where, $F(\bullet)$ stands for the cumulative distribution function of product strength.

The above failure rate equations are developed based on load-strength interference relationship. The strength distribution can be either component strength or system strength, i.e., a product can be either a component or a system. For a series system (any component failure results in system failure) composed of m components, denoting by $F_i(x)$ the strength distribution of component i , the strength distribution of such a system is

$$F_{seri}(x) = 1 - \prod_{i=1}^m [1 - F_i(x)] \quad (15)$$

for a parallel system (system failure occurs if and only if all components fail) composed of m components, the system strength distribution is

$$F_{para}(x) = \prod_{i=1}^m F_i(x) \quad (16)$$

In the situation that all components in a system simultaneously subject to the same load, system failure rate can be modeled as

$$h(n) = \int_0^{\infty} g_{n-1}(z) \left\{ \int_z^{\infty} g(y) \frac{F_{sys}(y) - F_{sys}(z)}{1 - F_{sys}(z)} dy \right\} dz \quad (17)$$

where, $F_{sys}(x)$ stands for either the series system strength distribution function $F_{seri}(x)$ or the parallel system strength distribution function $F_{para}(x)$.

3 Failure Rate Modeling Based on the Definition with Discrete Time Variable

Product failure rate function $h(n)$ can also be derived through failure rate definition and related events operation. Denote the event that a product fails at the n th load

action by A_n , denote the event that no failure occurs during the preceding $(n-1)$ times of load action by B_{n-1} , the failure rate at n th load action is the probability that event A_n occurs given that event B_{n-1} has occurred, i.e.

$$h(n) = P(A_n|B_{n-1}) \quad (18)$$

According to the conditional probability theorem, the product failure rate (Eq. 18) can be expressed as

$$h(n) = P(A_n|B_{n-1}) = P(A_n B_{n-1})/P(B_{n-1}) \quad (19)$$

Denote the probability that product life N equals to n by $P(N = n)$, denote the product failure probability over n times of load action, i.e., the cumulative probability of product life corresponding to n times of load action by $P(n)$, i.e., $P(n) = P(N \leq n)$.

It is easy to know that

$$P(A_n B_{n-1}) = P(N = n) = P(n) - P(n - 1)$$

and

$$P(B_{n-1}) = 1 - P(n - 1) = R(n - 1)$$

Therefore,

$$\begin{aligned} h(n) &= \frac{P(n) - P(n - 1)}{R(n - 1)} = \frac{(1 - R(n)) - (1 - R(n - 1))}{R(n - 1)} \\ &= \frac{R(n - 1) - R(n)}{R(n - 1)} = 1 - \frac{R(n)}{R(n - 1)} \end{aligned} \quad (20)$$

Equation 20 is equivalent to the failure rate definition in the situation of discrete time variable, where failure rate at the n th time of load action is defined as the probability that the product fails to the n th time of load action given functioning over the $(n-1)$ times of load action, i.e.,

$$\begin{aligned} h(n) &= \frac{P((n - 1) < N \leq n)}{P(N > (n - 1))} = \frac{P(n) - P(n - 1)}{R(n - 1)} \\ &= \frac{R(n - 1) - R(n)}{R(n - 1)} = 1 - \frac{R(n)}{R(n - 1)} \end{aligned} \quad (21)$$

It is equivalent to the conventional form of the failure rate defined in the situation of continuous time variable:

$$h(t) = \lim_{\Delta t \rightarrow 0} \frac{P(t < N \leq (t + \Delta t))}{P(N > t) \cdot \Delta t} = \lim_{\Delta t \rightarrow 0} \frac{P(t + \Delta t) - P(t)}{R(t) \cdot \Delta t} = \frac{f(t)}{R(t)} \quad (22)$$

From Eq. 20,

$$h(n) = 1 - \frac{\int_0^\infty f(x) \left(\int_0^x g_n(y) dy \right) dx}{\int_0^\infty f(x) \left(\int_0^x g_{n-1}(y) dy \right) dx} \tag{23}$$

Or

$$h(n) = 1 - \frac{\int_0^\infty g_n(y) \left(\int_y^\infty f(x) dx \right) dy}{\int_0^\infty g_{n-1}(y) \left(\int_y^\infty f(x) dx \right) dy} \tag{24}$$

On the other hand, event A_n and B_{n-1} occur simultaneously means that failure occurs and only occurs at the n th time of load action during all the n times of loading. The probability that the event A_n (the event that product fails at the n th load action) and the event B_{n-1} (the event that no failure occurs during the preceding $(n-1)$ times of load action) occur simultaneously is

$$P(A_n B_{n-1}) = \int_0^\infty f(x) \left\{ \int_x^\infty g(y) dy \left[\int_0^x g(y) dy \right]^{n-1} \right\} dx \tag{25}$$

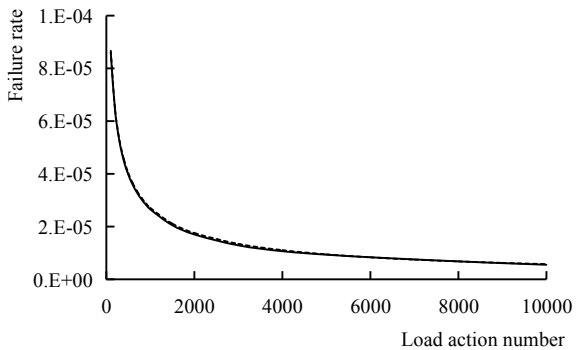
According to Eq. 19,

$$h(n) = \frac{\int_0^\infty f(x) \left\{ \int_x^\infty g(y) dy \cdot \left[\int_0^x g(y) dy \right]^{n-1} \right\} dx}{\int_0^\infty f(x) \left\{ \left[\int_0^x g(y) dy \right]^{n-1} \right\} dx} \tag{26}$$

It is easy to numerically testify that the three types of failure rate equations, i.e. Eqs. 9, 23 and 26, yield perfectly coincident failure rate curves as shown in Fig. 8.

In the situation of deterministic load, i.e., when the same load is applied many times to a product, from Eq. 24

Fig. 8 Failure rate curves yielded by different equations



$$h(1) = 1 - \int_y^\infty f(x)dx = \int_0^y f(x)dx \tag{27}$$

$$h(n) = 0 \quad (n \geq 2) \tag{28}$$

from Eq. 26,

$$h(1) = \int_0^\infty f(x) \left\{ \int_x^\infty g(y)dy \right\} dx = \int_0^\infty g(y) \left\{ \int_0^y f(x)dx \right\} dy = \int_0^y f(x)dx \tag{29}$$

$$h(n) = 0 \quad (n \geq 2) \tag{30}$$

It illustrates that the different types of failure rate equations degenerate into the same failure rate equation in deterministic load condition.

4 Effect of Load/Strength Dispersion on Failure Rate

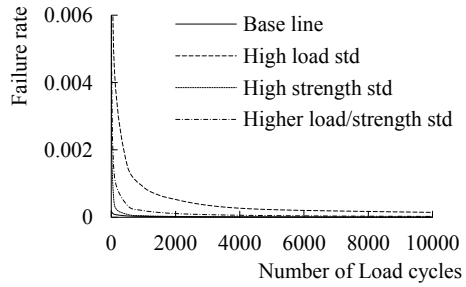
To demonstrate the effects of load uncertainty and strength uncertainty on product failure rate, failure rate curves corresponding to different load-strength combinations are illustrated below. Both the loads and the strengths are assumed to follow the normal distribution. The respective expectations and standard deviations are listed in Table 1. Where, μ_y stands for the mean of stress, σ_y stands for the standard deviation of stress; μ_x stands for the mean of strength, and σ_x stands for the standard deviation of strength. Failure rate curves corresponding to these four load-strength combinations are obtained by means of Eq. 9, shown in Fig. 9 as “base line”, “high load std”, “high strength std” and “higher load/strength std”, respectively. It is demonstrated that the statistical characteristics of load distribution and strength distribution have considerable effect on failure rate curve shape.

The failure rate curves shown in Fig. 9 are decreasing because product strength is assumed no degradation during load actions. The decreasing failure rate is due to the fact that, after a certain times of load action, the survived products are those having higher strength in the population, and the products survived more times of

Table 1 Parameters of four pairs of load-strength distributions

ID	μ_y	σ_y	μ_x	σ_x
Base line	400	40	600	40
High load std	400	80	600	40
High strength std	400	40	600	80
Higher load/strength std	400	60	600	60

Fig. 9 Failure rate curves corresponding to different load-strength combinations



random load action have higher strength than those survived less times of random load action. Therefore, at the next time of load action, the products experienced more times of random load action have lower failure probability.

In the situation of deterministic strength, i.e., all the products have the same strength, the failure rate equation degenerates from Eq. 8 as

$$h(n) = \int_0^\infty g_{n-1}(z) \left\{ \int_z^\infty f(x, n) \left[\int_x^\infty g(y) dy \right] dx \right\} dz = \int_x^\infty g(y) dy \quad (31)$$

from Eq. 23 as

$$\begin{aligned} h(n) &= 1 - \frac{\int_0^\infty f(x) \left(\int_0^x g_n(y) dy \right) dx}{\int_0^\infty f(x) \left(\int_0^x g_{n-1}(y) dy \right) dx} \\ &= 1 - \frac{\int_0^x g_n(y) dy}{\int_0^x g_{n-1}(y) dy} = 1 - \frac{G_n(x)}{G_{n-1}(x)} = 1 - G(x) \\ &= 1 - \int_0^x g(y) dy = \int_x^\infty g(y) dy \end{aligned} \quad (32)$$

from Eq. 26 as

$$\begin{aligned} h(n) &= \frac{\int_0^\infty f(x) \left\{ \int_x^\infty g(y) dy \cdot \left[\int_0^x g(y) dy \right]^{n-1} \right\} dx}{\int_0^\infty f(x) \left\{ \left[\int_0^x g(y) dy \right]^{n-1} \right\} dx} \\ &= \frac{\int_x^\infty g(y) dy \cdot \left[\int_0^x g(y) dy \right]^{n-1}}{\left[\int_0^x g(y) dy \right]^{n-1}} = \int_x^\infty g(y) dy \end{aligned} \quad (33)$$

By these equations, it is illustrated that product failure rate is a constant independent on the times of load action in condition of deterministic strength. Besides, it is proved once more that the three failure rate equations, i.e., Eqs. 9, 24 and 26 are the same.

5 Strength Degradation Effect on Failure Rate

Under cyclic loading, material property will degrade gradually if the stress is high enough, and the strength becomes less and less. Different products or materials have different strength degradation patterns. Some materials show an approximate linear relationship between residual strength and load action number, others show various type of non-linear relationships. First, consider the situation that material strength degrades in power law as described by Eq. 34 and shown in Fig. 10.

$$S(n) = S_0(1 - (\frac{n}{N})^e) \tag{34}$$

where, $S(n)$ stands for the material strength (residual strength) after n times of load action, n stands for the number of load actions, S_0 stands for the original material strength, N stands for the fatigue life under the cyclic stress, e is a constant characterizing strength degradation rate.

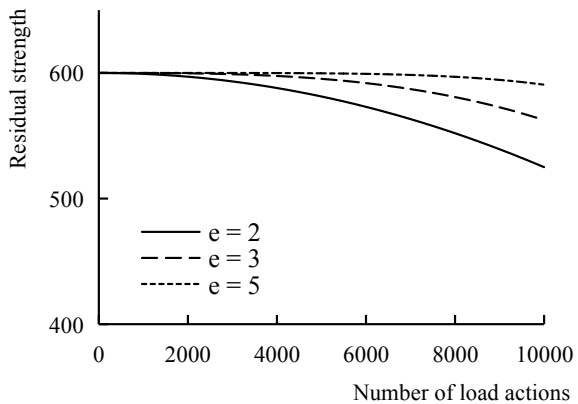
With material strength degradation behavior incorporated, the basic failure rate equation (Eq. 9) becomes

$$h(n) = \int_0^\infty g_{n-1}(z) \left\{ \int_z^\infty g(y) \left[\int_z^y \frac{f_S(x, n)}{\int_z^\infty f_S(x, n) dx} dx \right] dy \right\} dz \tag{35}$$

where, $f_S(x, n)$ stands for the probability density function of the residual strength after $(n-1)$ times of load action.

Shown in Fig. 11 are the strength distribution of the original products and those subjected to a certain times of random load action. Where, “original” is the strength distribution of the new products subjected to no load action, “slightly degraded”, “moderately degraded” and “seriously degraded” are the products after a small, moderate and large numbers of random load actions, respectively. Correspondingly,

Fig. 10 Material strength degradation curves—Power law



shown in Fig. 12 are the strength distribution of the population consist of all the products subjected to no load action, and those of the sub-populations consist of products survived a small, moderate and large numbers of random load actions, respectively.

For the normal-distributed strength $S(n) \sim N(\mu_x(n), \sigma_x(n))$, suppose that the mean value of strength decreases with load action times n as

$$\mu_x(n) = \mu_0(1 - (\frac{n}{N})^\epsilon) \tag{36}$$

where, μ_0 stands for the mean of the original strength.

Under cyclic loading, the dispersion of the residual strength will also change gradually. For the sake of simplicity, especially in the situation of no enough strength degradation data available, strength standard deviation σ_x can be assumed no change, i.e.,

$$\sigma_x(n) = \text{constant} \tag{37}$$

Fig. 11 Distributions of residual strength

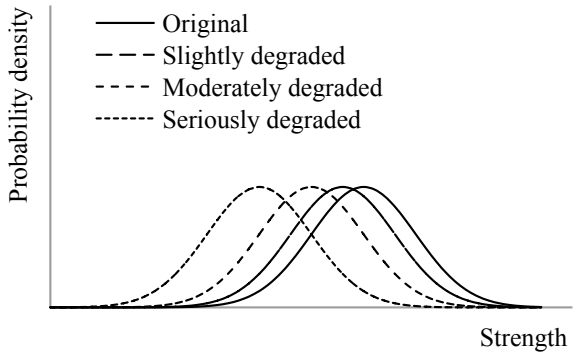
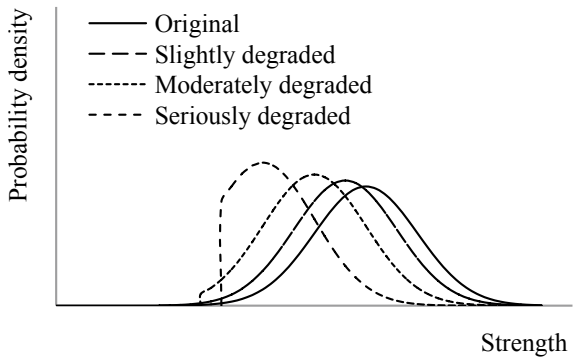
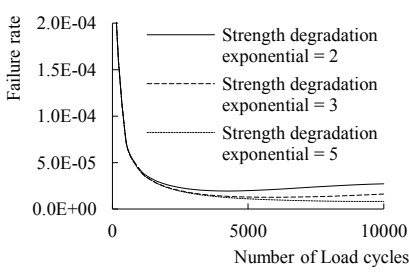
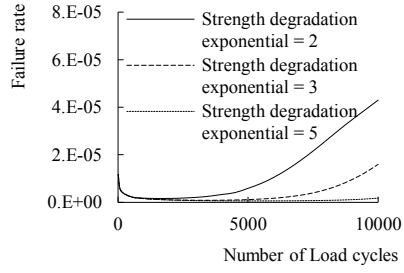


Fig. 12 Distributions of strength of survivals





(a) $\mu_y = 400, \sigma_y = 40; \mu_x = 600, \sigma_x = 60$



(b) $\mu_y = 450, \sigma_y = 20; \mu_x = 600, \sigma_x = 60$

Fig. 13 Failure rate curves in the situation of strength degrading in power law

The typical failure rate curves corresponding to different load-strength combinations and different strength degradation rates are obtained by Eq. 35 and shown in Fig. 13. It demonstrates that different strength degradation rates yield obviously different failure rate curves.

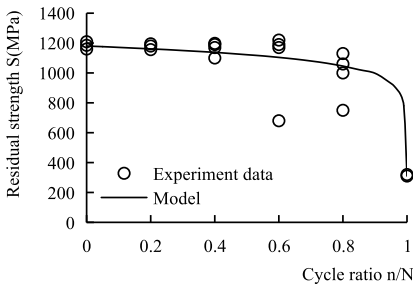
Some materials show logarithmic strength degradation described by the following equation:

$$S(n) = S_0 + (S_0 - \sigma) \ln(1 - n/(N + 1))/\ln(N + 1) \tag{38}$$

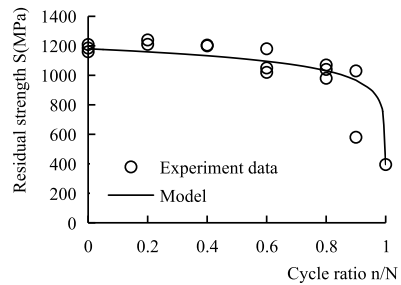
where, $S(n)$ stands for residual strength after n times of load action, n stands for the number of load actions, N stands for the number of load action to material fatigue failure, i.e., the fatigue life under the cyclic stress σ .

Shown in Fig. 14 are test data of specimens made of normalized carbon steel and the logarithmic strength degradation curves, in which the mean of the ultimate tensile strength, i.e., the mean original strength is 1180 MPa.

When the logarithmic strength degradation equation is incorporated into the basic failure rate model, the typical failure rate curve of bathtub shape can be yielded by



(a) Residual strength at stress 310MPa



(b) Residual strength at stress 395MPa

Fig. 14 Strength degradation test data and the logarithmic residual strength curves

Eq. 35 (shown in Fig. 15a). The typical bathtub curve is for a component in the condition that stress follows the normal distribution $N(450, 20^2)$, strength follows the normal distribution $N\left(600 + 300 \times \frac{\ln(1-n/(N+1))}{\ln(N+1)}, 30^2\right)$. Comparing with the failure rate curves obtained in the situation of strength degrading in power law, it demonstrates that strength degradation pattern influences the shape of failure rate curve considerably. Besides, both load distribution and strength distribution influence failure rate curve shape considerably, too. In the extreme situation of deterministic load, failure rate will increase monotonically with the degradation of strength, so it is not surprised to get a monotonically increasing failure rate curve as shown in Fig. 15b. The different curves in Fig. 16 correspond to different load distribution—strength distribution combinations. Shown in Fig. 17a are the failure rate curve of a component in the condition that stress follows the normal distribution $N(450, 20^2)$, strength follows the normal distribution $N\left(600 + 300 \times \frac{\ln(1-n/(N+1))}{\ln(N+1)}, 40^2\right)$, and the failure rate curve of a series system composed of 10 identical components, Fig. 17b are a failure rate curve of a component and that of a parallel system consist of 10 components.

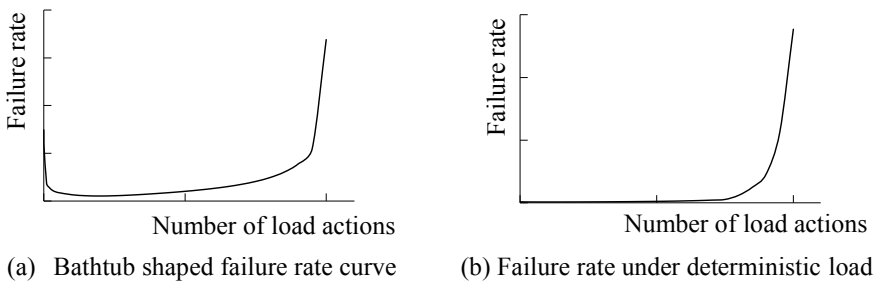
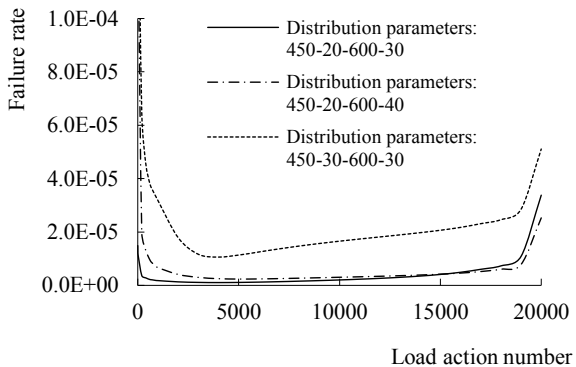


Fig. 15 Failure rate curves yielded by Eq. 35 in the situation of logarithmic strength degradation

Fig. 16 Failure rate curves corresponding to different load/strength distributions (note: parameters shown on graph: stress mean-stress std-strength mean-strength std)



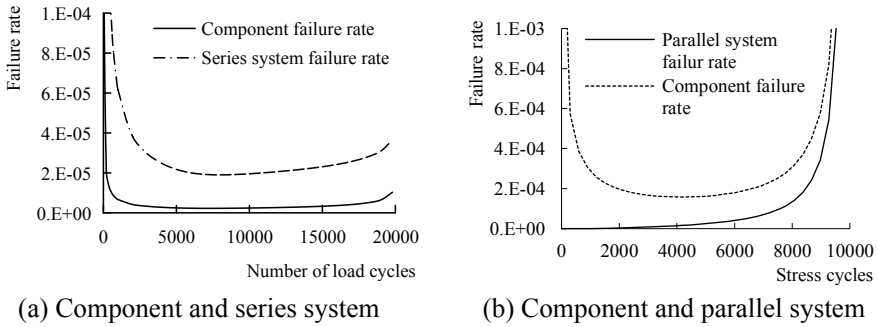


Fig. 17 Failure rate curves of a components and systems consist of 10 components

The above situations demonstrate that products (components or systems) failure rate curves present on different shapes depending on system configuration, the underlying failure mechanism and stress-strength relationship of the components.

6 Mechanism to Yield Failure Rate Curve of Roller Coaster Shape

As mentioned above, the shape of product failure rate curve depends on many factors including product property, load characteristics, failure mechanism, etc. Product failure rate may be as simple as a constant value, a monotonically increasing curve or a monotonically decreasing curve. It may also be as complicated as a bathtub curve or a roller coaster curve as shown in Fig. 18.

The roller-coaster type of failure rate curve shown in Fig. 18a is obtained in the situation that the product population can be divided into two sub-populations. Two potential failure modes exist for the population. A product in the population might fail in either failure mode with a respective probability. Here, one failure mode is related to Weibull distributed stress with shape parameter 2.0 and scale parameter 200,

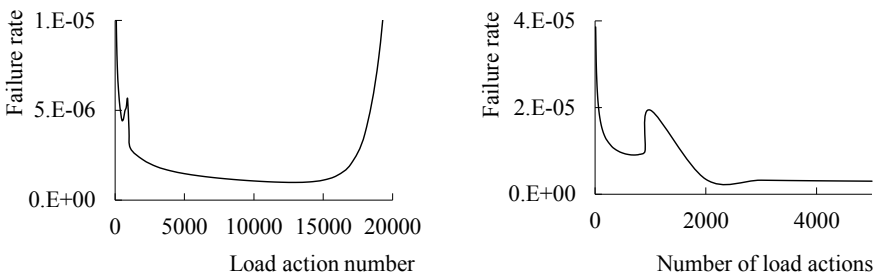


Fig. 18 Failure rate curves of roller coaster shape

normal distributed strength with original mean 600 and standard deviation 60, the mean strength degrades in logarithmic law with a life index 2000; the other is related to normal distributed strength with original mean 600 and standard deviation 120, the strength mean degrades in logarithmic law with a life index 1000. The probability for the first failure mode to take place is 0.7, and that for the second failure mode is 0.3. That is, 70% products in the population belong to one sub-population, the other 30% belong to another sub-population.

Shown in Fig. 18b is the situation of Weibull distributed stress with shape parameter 2.0 and scale parameter 150, strength characteristics are the same as mentioned above despite that the probability for the first failure mode to take place is 0.9, and that for the second failure mode is 0.1.

By the way, for a failure rate curve of bathtub shape, the high failure rate in the infant mortality phase is traditionally attributed to products with material flaw or manufacture defect. The fact is that, if the products can be divided into two groups as perfect and defective, the failure rate curve will present roller coaster shape as shown in Fig. 18. In such a product population, part of the products has manufacture defects manifested as lower strength and higher failure probability in the infant motility stage, others do not have manufacture defect, and therefore, their failures occur mainly in wear out stage.

Furthermore, it is traditionally believed that after the failures of the defective products, the survived products will keep a low and roughly constant failure rate for a long time until they begin to wear out. The failure, i.e., the so-called chance failure in this period is attributed to unexpected factors. Such an explanation is also plausible, since if there are the so called “unexpected factors”, they will exist throughout the product service life, not only in the chance failure stage. In fact, the failure rate model has illustrated that the roughly constant failure rate in the useful life period is the result of stress-stress competition, instead of unexpected factor.

Since product failure or not during service is determined by the dynamic competition relationship between load and strength, the decreasing failure rate in the first stage of product service life is natural for products subjected to multiple times of random load action. If a sub-population, dominated by quality-control defects due to poor workmanship, out-of-specification incoming parts and materials, and other substandard manufacturing practices, exists in the product population, there will be more than one failure mechanisms or failure modes, the corresponding failure rate curve will present roller coaster shape. In other words, if only a part of the products in a population suffers from a certain failure mechanism leading to shorter lifetime, the failure rate curve might present roller coaster shape.

Strength degradation under repeatedly loading results in increasing failure rate. Provide that product property keeps the same during its service life, i.e., no strength degrading during load actions, a product will never fail to a load not greater than those experienced. That is, such a product can only fail when a load higher than all the preceding ones is applied. For a steady random load process, the probability for a higher load to appear is less and less with the increase of the loading history. Therefore, given that strength keeps no change, failure rate will decrease with the increase of product service time as show in Fig. 19a. In the condition that strength

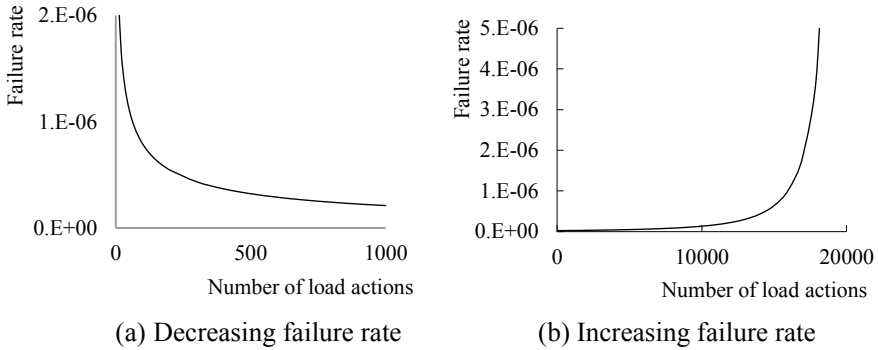


Fig. 19 Failure rate curves in different situations

degrades gradually, the failure rate is increasing for a component with large safety margin or a component subjected to a deterministic load as shown in Fig. 19b.

Although the shapes of failure rate curve are various, failure is essentially the result of the competition between load and strength. In this regard, whether a product fails at a certain load action or not depends on the load and the strength at that moment. Likewise, failure rate at a certain load action number is determined by load distribution and strength distribution including its time dependent degradation behavior. Based on load-strength competition behavior, it is easy to understand that the failure rate curve of bathtub shape comes from the continuously changing competition relationship between the random load and degrading strength, the failure rate curve of roller coaster shape manifests the diversity of product strength in the same population.

7 Conclusion

Failure rate models are developed for components and systems based on dynamic load-strength competition analysis. It is illustrated that whether the failure rate, as a function of the times of load action, takes on bathtub shape or not is mainly determined by stress-strength relationship. The reason for decreasing failure rate is the less and less probability that a higher load appears with the increase of preceding load action numbers, the increasing failure rate is caused by strength degradation. The models, together with the failure rate curves corresponding to the models, highlight the effect of the statistical characteristics of load and strength on the shape of failure rate curve, as well as the role of strength degradation.

It is clearly demonstrated that if product strength doesn't degrade, failure rate curve takes on the feature of the first two stages of a typical three-stage bathtub-shaped curve only, i.e., failure rate decreases continuously with the number of load actions, with lower and lower gradient. When the effect of strength degradation exceeds the effect of the decreasing probability of higher load appearing, the failure

rate begins to increase. In other words, the decreasing failure rate stage is dominated by the statistical risk of load, whereas the increasing failure rate stage is dominated by strength degradation.

For a population containing defective products, the failure rate curve may present roller coaster shape. Generally, a failure rate curve of roller coaster shape will appear in a population where different failure mechanisms or different failure modes exist in different products.

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