Non-local Substitutions for Liouville Equations with Three and Four Independent Variables

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Abstract We obtained the non-local transformations of the Cole—Hopf type, which translate the Liouville equations with three and four independent variables into the Bianchi equations. The solutions with arbitrary functions of these Liouville equations are constructed.

Keywords Liouville equation · Bianchi equation · Non-local transformation

1 On the Group Properties of Bianchi Equations

Consider a homogeneous equation with a dominant partial derivative with variable coefficients (Bianchi equation)

$$u_{xyz} + au_{xy} + bu_{yz} + cu_{xz} + du_x + eu_y + fu_z + gu = 0. \tag{1}$$

In the paper [1] some group properties of this equation have been considered. It is known that the set of equivalence transformations for (1)

$$\overline{x} = \alpha(x), \quad \overline{y} = \beta(y), \quad \overline{z} = \gamma(z), \quad u = \omega(x, y, z)\overline{u}. \tag{2}$$

Two equations of the form (1) are called equivalent in function [2, p 117], if they pass into each other during transformations (2), in which

$$\alpha(x) = x, \quad \beta(y) = y, \quad \gamma(z) = z.$$

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V. Vasilyev (ed.), *Differential Equations, Mathematical Modeling and Computational Algorithms*, Springer Proceedings in Mathematics & Statistics 423,
https://doi.org/10.1007/978-3-031-28505-9_6

In the paper [3] it was shown that two equations of the form (1) are equivalent in function if and only if the Laplace invariants

$$H_1 = a_y + ac - d,\ H_2 = a_x + ab - e,\ H_3 = c_x + bc - f,$$
$$H_4 = b_z + ab - e,\ H_5 = b_y + bc - f,\ H_6 = c_z + ac - d,$$
$$H_7 = a_{xy} + bd + ce + af - 2abc - g, \quad (3)$$
$$H_8 = b_{yz} + bd + ce + af - 2abc - g,$$
$$H_9 = c_{xz} + bd + ce + af - 2abc - g$$

are the same for both equations.

If we look for the operator allowed by the Eq. (1)

$$\alpha\partial_x + \beta\partial_y + \gamma\partial_z + \tau\partial_u,$$

then it turns out that part of the system of defining equations will be

$$\partial_u\alpha = \partial_u\beta = \partial_u\gamma = 0, \quad \partial_u^2\tau = 0.$$

It is known [2, pp. 99–100] that in this case the Lie algebra of the Eq. (1) there is $L = L^r \oplus L^\infty$, where the algebra L^r of dimension r is formed by operators of the form

$$X = \xi^1(x, y, z)\partial_x + \xi^2(x, y, z)\partial_y + \xi^3(x, y, z)\partial_z + \sigma(x, y, z)u\partial_u, \quad (4)$$

and L^∞ is an Abelian subalgebra typical of linear equations with the operator $\omega(x, y, z)\partial_u$, where ω is the solution of the Eq. (1). It is clear that the operator $u\partial_u$ is allowed by any Eq. (1), therefore, this operator can be included in L^∞ and assume that $\sigma(x, y, z)$ is defined in (4) up to a constant summand.

To construct the defining equations we use the third continuation of the operator (4)

$$X_3 = \xi^1\partial_x + \xi^2\partial_y + \xi^3\partial_z + \sigma u\partial_u + \tau^1\partial_{u_1} + \tau^2\partial_{u_2} + \tau^3\partial_{u_3} +$$
$$+\tau^{11}\partial_{u_{11}} + \tau^{12}\partial_{u_{12}} + \tau^{13}\partial_{u_{13}} + \tau^{22}\partial_{u_{22}} + \tau^{23}\partial_{u_{23}} + \tau^{33}\partial_{u_{33}} +$$
$$+\tau^{111}\partial_{u_{111}} + \tau^{112}\partial_{u_{112}} + \tau^{113}\partial_{u_{113}} + \tau^{122}\partial_{u_{122}} + \tau^{123}\partial_{u_{123}} +$$
$$+\tau^{133}\partial_{u_{133}} + \tau^{222}\partial_{u_{222}} + \tau^{223}\partial_{u_{223}} + \tau^{233}\partial_{u_{233}} + \tau^{333}\partial_{u_{333}}.$$

The notation used here is $u_1 = u_x$, $u_2 = u_x$,…, $u_{12} = u_{xy}$,…, $u_{333} = u_{zzz}$. We get

$$\begin{gathered}
\tau^1 = \sigma_x u + (\sigma - \xi^1_x)u_1 - \xi^2_x u_2 - \xi^3_x u_3,\\
\tau^2 = \sigma_y u - \xi^1_y u_1 + (\sigma - \xi^2_y)u_2 - \xi^3_y u_3,\\
\tau^3 = \sigma_z u - \xi^1_z u_1 - \xi^2_z u_2 + (\sigma - \xi^3_z)u_3,\\
\tau^{12} = \sigma_{xy} u + (\sigma_y - \xi^1_{xy})u_1 + (\sigma_x - \xi^2_{xy})u_2 - \xi^3_{xy} u_3 -\\
-\xi^1_y u_{11} + (\sigma - \xi^1_x - \xi^2_y)u_{12} - \xi^3_y u_{13} - \xi^2_x u_{22} - \xi^3_x u_{23},\\
\tau^{13} = \sigma_{xz} u + (\sigma_z - \xi^1_{xz})u_1 - \xi^2_{xz} u_2 + (\sigma_x - \xi^3_{xz})u_3 -\\
-\xi^1_z u_{11} - \xi^2_z u_{12} + (\sigma - \xi^1_x - \xi^3_z)u_{13} - \xi^2_x u_{23} - \xi^3_x u_{33},\\
\tau^{23} = \sigma_{yz} u - \xi^1_{yz} u_1 + (\sigma_z - \xi^2_{yz})u_2 + (\sigma_y - \xi^3_{yz})u_3 -\\
-\xi^1_z u_{12} - \xi^1_y u_{13} - \xi^2_z u_{22} + (\sigma - \xi^1_y - \xi^3_z)u_{23} - \xi^3_y u_{33},\\
\tau^{123} = \sigma_{xyz} u + (\sigma_{yz} - \xi^1_{xyz})u_1 + (\sigma_{xz} - \xi^2_{xyz})u_2 + (\sigma_{xy} - \xi^3_{xyz})u_3 -\\
-\xi^1_{yz} u_{11} + (\sigma_z - \xi^2_{yz} - \xi^1_{xz})u_{12} + (\sigma_y - \xi^1_{xy} - \xi^3_{yz})u_{13} -\\
-\xi^2_{xz} u_{22} + (\sigma_x - \xi^3_{xz} - \xi^2_{xy})u_{23} - \xi^3_{xy} u_{33} -\\
-\xi^1_z u_{112} - \xi^1_y u_{113} - \xi^2_z u_{122} + (\sigma - \xi^1_x - \xi^2_y - \xi^3_z)u_{123} -\\
-\xi^3_y u_{133} - \xi^2_x u_{223} - \xi^3_x u_{233}.
\end{gathered}$$

By applying the operator X_3 to the Eq. (1), we obtain the defining equations

$$\begin{gathered}
\xi^1_y = \xi^1_z = \xi^2_x = \xi^2_z = \xi^3_x = \xi^3_y = 0,\\
\sigma_x + (b\xi^1)_x + b_y\xi^2 + b_z\xi^3 = 0,\\
\sigma_y + c_x\xi^1 + (c\xi^2)_y + c_z\xi^3 = 0,\\
\sigma_z + a_x\xi^1 + a_y\xi^2 + (a\xi^3)_z = 0,\\
\sigma_{xy} + c\sigma_x + b\sigma_y + (f\xi^1)_x + (f\xi^2)_y + f_z\xi^3 = 0,\\
\sigma_{xz} + a\sigma_x + b\sigma_z + (e\xi^1)_x + e_y\xi^2 + (e\xi^3)_z = 0,\\
\sigma_{yz} + a\sigma_y + c\sigma_z + d_x\xi^1 + (d\xi^2)_y + (d\xi^3)_z = 0,\\
\sigma_{xyz} + a\sigma_{xy} + b\sigma_{yz} + c\sigma_{xz} + d\sigma_x + e\sigma_y + f\sigma_z +\\
+(g\xi^1)_x + (g\xi^2)_y + (g\xi^3)_z = 0.
\end{gathered} \tag{5}$$

Defining Eq. (5) can be written using Laplace invariants (3) in the form

$$\begin{gathered}
\xi^1_y = \xi^1_z = \xi^2_x = \xi^2_z = \xi^3_x = \xi^3_y = 0,\\
(\sigma + b\xi^1 + c\xi^2 + a\xi^3)_x = (H_3 - H_5)\xi^2 + (H_2 - H_4)\xi^3,\\
(\sigma + b\xi^1 + c\xi^2 + a\xi^3)_y = (H_5 - H_3)\xi^1 + (H_1 - H_6)\xi^3,\\
(\sigma + b\xi^1 + c\xi^2 + a\xi^3)_z = (H_4 - H_2)\xi^1 + (H_6 - H_1)\xi^2,\\
H_{1x}\xi^1 + (H_1\xi^2)_y + (H_1\xi^3)_z = 0,\\
H_{6x}\xi^1 + (H_6\xi^2)_y + (H_6\xi^3)_z = 0,\\
(H_2\xi^1)_x + H_{2y}\xi^2 + (H_2\xi^3)_z = 0,\\
(H_4\xi^1)_x + H_{4y}\xi^2 + (H_4\xi^3)_z = 0,\\
(H_3\xi^1)_x + (H_3\xi^2)_y + H_{3z}\xi^3 = 0,\\
(H_5\xi^1)_x + (H_5\xi^2)_y + H_{5z}\xi^3 = 0,\\
(H_7\xi^1)_x + (H_7\xi^2)_y + (H_7\xi^3)_z = 0,\\
(H_8\xi^1)_x + (H_8\xi^2)_y + (H_8\xi^3)_z = 0,\\
(H_9\xi^1)_x + (H_9\xi^2)_y + (H_9\xi^3)_z = 0.
\end{gathered} \tag{6}$$

The first row in (6) shows that

$$\xi^i = \xi^i(x_i), \quad i = \overline{1,3}.$$

The second, third and fourth rows from (6) are differential equations for determining the function σ, after ξ^1, ξ^2, ξ^3 have been obtained. The equations starting from the fifth row are responsible for the results of the group classification.

Some consequences can be deduced directly from the defining equations in the form (6). If all $H_i, i = \overline{1,9}$, are identically equal to zero, then the Eq. (1) is equivalent to the equation $u_{xyz} = 0$ and admits an infinite-dimensional Lie algebra of operators of the form

$$\xi^1(x)\partial_x + \xi^2(y)\partial_y + \xi^3(z)\partial_z$$

with arbitrary $\xi^1(x), \xi^2(y), \xi^3(z)$.

Let's introduce the relations into consideration

$$p_{12} = \frac{H_3}{H_5}, \quad p_{13} = \frac{H_2}{H_4}, \quad p_{23} = \frac{H_1}{H_6}, \tag{7}$$

$$\begin{array}{c} q_1 = \frac{(\ln H_1)_{yz}}{H_1}, \quad q_2 = \frac{(\ln H_2)_{xz}}{H_2}, \quad q_3 = \frac{(\ln H_3)_{xy}}{H_3}, \\ q_4 = \frac{(\ln H_4)_{xz}}{H_4}, \quad q_5 = \frac{(\ln H_5)_{xy}}{H_5}, \quad q_6 = \frac{(\ln H_6)_{yz}}{H_6}, \\ q_i = \frac{(\ln H_i)_{xyz}}{H_i}, \quad i = 7,\ 8,\ 9. \end{array} \tag{8}$$

Substitute $H_1 = p_{23}H_6$, $H_6 \neq 0$, in the fifth row (6)

$$p_{23}(H_{6x}\xi^1 + (H_6\xi^2)_y + (H_6\xi^3)_z) + p_{23x}H_6\xi^1 + p_{23y}H_6\xi^2 + p_{23z}H_6\xi^3 = 0.$$

Since the term in parentheses vanishes, it follows

$$\xi^1 p_{23x} + \xi^2 p_{23y} + \xi^3 p_{23z} = 0. \tag{9}$$

The identity (9) means that either $p_{23} = const$ or p_{23} is an invariant of the group G with the operator (4).

If $p_{23} = const$, then from the fifth and sixth rows (6) we get

$$\xi^1(\ln H_6)_x + \xi^2(\ln H_6)_y + \xi^3(\ln H_6)_z + \xi^2_y + \xi^3_z = 0. \tag{10}$$

Differentiating by y, z we get

$$\xi^1\frac{((\ln H_6)_{yz})_x}{(\ln H_6)_{yz}} + \xi^2\frac{((\ln H_6)_{yz})_y}{(\ln H_6)_{yz}} + \xi^3\frac{((\ln H_6)_{yz})_z}{(\ln H_6)_{yz}} + \xi^2_y + \xi^3_z = 0. \tag{11}$$

Subtracting (10) from (11) and then multiplying by $(\ln H_6)_{yz}/H_6$, we get

$$\xi^1 q_{6x} + \xi^2 q_{6y} + \xi^3 q_{6z} = 0.$$

Thus, again either $q_6 = const$ or q_6 is an invariant of the group G with the operator (4).

Then similar identities can be obtained for p_{12}, p_{13}, q_i, $i = \overline{1, 5}$.

Similar identities can be obtained for relations

$$P_1 = \frac{H_7}{H_8}, \quad P_2 = \frac{H_7}{H_9}, \quad P_3 = \frac{H_8}{H_9}.$$

For example, considering the relation P_1, we come to the identity

$$\xi^1 P_{1x} + \xi^2 P_{1y} + \xi^3 P_{1z} = 0.$$

Again, either $P_1 = const$, or P_1 is an invariant of the group G with the operator (4). If $P_1 = const$, then row 12 from (6) gives

$$\xi^1 (\ln H_8)_x + \xi^2 (\ln H_8)_y + \xi^3 (\ln H_8)_z + \xi^1_x + \xi^2_y + \xi^3_z = 0. \tag{12}$$

Differentiating by x, y, z we get

$$\xi^1 \frac{((\ln H_8)_{xyz})_x}{(\ln H_8)_{xyz}} + \xi^2 \frac{((\ln H_8)_{xyz})_y}{(\ln H_8)_{xyz}} + \xi^3 \frac{((\ln H_8)_{xyz})_z}{(\ln H_8)_{xyz}} + \xi^1_x + \xi^2_y + \xi^3_z = 0. \tag{13}$$

Subtracting (12) from (13) and multiplying by $(\ln H_8)_{xyz}/H_8$, we get

$$\xi^1 q_{8x} + \xi^2 q_{8y} + \xi^3 q_{8z} = 0.$$

Thus, either $q_8 = const$ or q_8 is an invariant of the group G with the operator (4).

Based on the above statements, classes of equations of the form (1) admitting Lie algebras of the largest dimensions were listed in the work [1].

In the case when $q_i = const$, $i = \overline{1, 6}$, the invariant H_i is a solution of the Liouville equation (this follows from (8)), the formula of the general solution of which is known [2, p 123]. Similarly, if any of the constructions q_i, $i = \overline{7, 9}$, is constant, then the corresponding invariant H_i is the solution of the equation

$$(\ln H_i)_{xyz} = q_i H_i.$$

In this regard, the task of constructing is of interest exact solutions of the three-dimensional analogue of the Liouville equation

$$u_{xyz} = e^u. \tag{14}$$

We can propose the following method of constructing an exact solution based on the application of Lie groups of point transformations.

The usual algorithm for calculating the group of point transformations allowed by the Eq. (14) leads to the Lie algebra of operators

$$X = \xi(x)\partial_x + \eta(y)\partial_y + \zeta(z)\partial_z - (\xi'(x) + \eta'(y) + \zeta'(z))\partial_u,$$

where $\xi(x)$, $\eta(y)$, $\zeta(z)$ are arbitrary functions.

To determine the invariants of the group allowed by the Eq. (14), we obtain the system

$$\frac{dx}{\xi(x)} = \frac{dy}{\eta(y)} = \frac{dz}{\zeta(z)} = \frac{du}{-\xi'(x) - \eta'(y) - \zeta'(z)}. \tag{15}$$

The first integrals of the system (15) have the form

$$u + \ln|\xi(x)\eta(y)\zeta(z)| = C_1,$$

$$\varphi(x) - \psi(y) = C_2, \quad \varphi(x) - \chi(z) = C_3,$$

$$\varphi'(x) = \frac{1}{\xi(x)}, \quad \psi'(y) = \frac{1}{\eta(y)}, \quad \chi'(z) = \frac{1}{\zeta(z)}.$$

Let's introduce new variables

$$v = u + \ln|\xi(x)\eta(y)\zeta(z)|, \quad t = \varphi(x) - \psi(y), \quad \tau = \varphi(x) - \chi(z).$$

Invariant with respect to the group of point transformations allowed by the Eq. (14) , the solution has the form $v = w(t, \tau)$. As a result, we come to the equation for determining the function w

$$w_{tt\tau} + w_{t\tau\tau} = e^w. \tag{16}$$

The Eq. (16) has a solution

$$w = \ln\frac{-12}{(t+\tau)^3}.$$

Then (here $\xi(x)\eta(y)\zeta(z) > 0$)

$$u = -\ln(\xi(x)\eta(y)\zeta(z)) + \ln\frac{-12}{(2\varphi(x) - \psi(y) - \chi(z))^3} =$$

$$= \ln\frac{-12\frac{1}{\xi(x)}\frac{1}{\eta(y)}\frac{1}{\zeta(z)}}{(2\varphi(x) - \psi(y) - \chi(z))^3}.$$

Denoting $\lambda(x) = 2\varphi(x)$, $\mu(y) = -\psi(y)$, $\nu(z) = -\chi(z)$, we obtain an exact solution of the Eq. (14), depending on three arbitrary functions

$$u = \ln \frac{-6\lambda'(x)\mu'(y)\nu'(z)}{(\lambda(x)+\mu(y)+\nu(z))^3}.$$

In [4, 5] some group properties of the fourth-order Bianchi equation were considered. The homogeneous Bianchi equation of the fourth order is

$$\begin{gathered} u_{x_1x_2x_3x_4} + a_1 u_{x_2x_3x_4} + a_2 u_{x_1x_3x_4} + a_3 u_{x_1x_2x_4} + a_4 u_{x_1x_2x_3} + \\ + a_{12} u_{x_3x_4} + a_{13} u_{x_2x_4} + a_{14} u_{x_2x_3} + a_{23} u_{x_1x_4} + a_{24} u_{x_1x_3} + a_{34} u_{x_1x_2} + \\ + a_{123} u_{x_4} + a_{124} u_{x_3} + a_{134} u_{x_2} + a_{234} u_{x_1} + a_{1234} u = 0. \end{gathered} \tag{17}$$

It is implied here that the coefficients are variable.

The Laplace invariants for this equation have the form

$$\begin{gathered} h_{i,j} = a_{ix_j} + a_i a_j - a_{ij}, \\ h_{i,jk} = a_{ix_jx_k} + a_i a_{jk} + a_j a_{ik} + a_k a_{ij} - 2a_i a_j a_k - a_{ijk}, \\ h_{i,jkl} = a_{ix_jx_kx_l} + a_i a_{jkl} + a_j a_{ikl} + a_k a_{ijl} + a_l a_{ijk} + \\ + a_{ij} a_{kl} + a_{ik} a_{jl} + a_{il} a_{jk} - 2a_i a_j a_{kl} - 2a_i a_k a_{jl} - \\ -2a_i a_l a_{jk} - 2a_j a_k a_{il} - 2a_j a_l a_{ik} - 2a_k a_l a_{ij} + \\ +6a_i a_j a_k a_l - a_{ijkl}, \quad \{i,j,k,l\} = \{1,2,3,4\}, \quad j<k<l. \end{gathered}$$

Here we consider coefficients that differ in the order of the indices to be equal (for example, $a_{123} = a_{231}$). There are a total of 28 Laplace invariants for this equation. Two equations of the form (17) are equivalent in function if and only if they have all the corresponding Laplace invariants equal.

Note that if all Laplace invariants are identically zero, then the Eq. (17) is equivalent to the equation $u_{x_1x_2x_3x_4} = 0$ and admits an infinite-dimensional Lie algebra of operators of the form

$$\xi^1(x_1)\partial_{x_1} + \xi^2(x_2)\partial_{x_2} + \xi^3(x_3)\partial_{x_3} + \xi^4(x_4)\partial_{x_4}$$

with arbitrary $\xi^i(x_i)$.

Similarly to the case of the third-order Bianchi equation, we can introduce into consideration the constructions

$$p_{ij} = \frac{h_{j,i}}{h_{i,j}}, \quad q_{ij} = \frac{(\ln h_{i,j})_{x_ix_j}}{h_{i,j}}, \quad i,\ j = \overline{1,4};$$

$$p^l_{ijk} = \frac{h_{l,l_1l_2}}{h_{i,jk}}, \quad q_{ijk} = \frac{(\ln h_{i,jk})_{x_ix_jx_k}}{h_{i,jk}}, \quad \{l,l_1,l_2\} = \{i,j,k\};$$

$$p^n_{ijkl} = \frac{h_{n,n_1n_2n_3}}{h_{i,jkl}}, \quad q_{ijkl} = \frac{(\ln h_{i,jkl})_{x_1x_2x_3x_4}}{h_{i,jkl}}, \quad \{n,n_1,n_2,n_3\} = \{i,j,k,l\}.$$

These constructions are used in [5] to obtain classes of fourth-order Bianchi equations with certain group properties.

It is easy to notice that for constants q_{ij}, q_{ijk}, q_{ijkl} the Laplace invariants are again solutions of the Liouville equation and its three-dimensional and four-dimensional analogues.

2 Three-Dimensional Analogue of the Liouville Equation

Let us consider an approach to the problem of constructing exact solutions to non-linear equations based on non-local transformations of variables. Equation

$$u_{xyz} = \lambda e^u \tag{18}$$

is a three-dimensional analogue of the Liouville equation

$$u_{xy} = \lambda e^u. \tag{19}$$

Equation (19), in particular, plays a key role in the problem of group classification of second-order hyperbolic equations [2, pp. 116–125]

$$v_{xy} + a(x, y)v_x + b(x, y)v_y + c(x, y)v = 0.$$

The general solution of the Eq. (19) is well known and can be constructed in various ways [2, p. 123], [6, pp. 239–240]. As noted earlier, the Eq. (18) is used in the study of the group properties of the third-order Bianchi Eq. (1).

Here a non-local transformation (such as the Cole—Hopf substitution [7]) is constructed, translating the Eq. (18) into the simplest Bianchi equation

$$v_{xyz} = 0, \tag{20}$$

which has a general solution with three arbitrary functions

$$v = \alpha(x, y) + \beta(x, z) + \gamma(y, z). \tag{21}$$

In this case, an algorithm based on the use of group methods is used [6, pp. 237–241].

Equation (18) admits the Lie algebra of operators

$$X = \xi(x)\partial_x + \eta(y)\partial_y + \zeta(z)\partial_z - (\dot{\xi}(x) + \dot{\eta}(y) + \dot{\zeta}(z))\partial_u,$$

where $\xi(x)$, $\eta(y)$, $\zeta(z)$ are arbitrary functions [1].

On the other hand, the Eq. (20) admits the Lie algebra of operators

$$X_0 = \xi(x)\partial_x + \eta(y)\partial_y + \zeta(z)\partial_z,$$

where $\xi(x), \eta(y), \zeta(z)$ are also arbitrary. In addition, like any linear equation, Eq. (20) admits a stretching operator

$$Y = v\partial_v.$$

In this regard, assume that there is a non-local transformation

$$u = \varphi(v, v_x, v_y, v_z) \tag{22}$$

such that the system of Eqs. (18), (20), (22) admits the Lie algebra of operators

$$X = \xi(x)\partial_x + \eta(y)\partial_y + \zeta(z)\partial_z - (\dot{\xi}(x) + \dot{\eta}(y) + \dot{\zeta}(z))\partial_u,$$
$$Y = v\partial_v.$$

We find the first continuations of operators

$$X_1 = \xi(x)\partial_x + \eta(y)\partial_y + \zeta(z)\partial_z - (\dot{\xi}(x) + \dot{\eta}(y) + \dot{\zeta}(z))\partial_u -$$
$$-(\ddot{\xi}(x) - \dot{\xi}(x)u_x)\partial_{u_x} - (\ddot{\eta}(y) - \dot{\eta}(y)u_y)\partial_{u_y} - (\ddot{\zeta}(z) - \dot{\zeta}(z)u_z)\partial_{u_z} +$$
$$+\dot{\xi}(x)v_x\partial_{v_x} + \dot{\eta}(y)v_y\partial_{v_y} + \dot{\zeta}(z)v_z\partial_{v_z},$$

$$Y_1 = v\partial_v + v_x\partial_{v_x} + v_y\partial_{v_y} + v_z\partial_{v_z}.$$

We get relations

$$Y_1(u - \varphi)|_{u=\varphi} = v\varphi_v + v_x\varphi_{v_x} + v_y\varphi_{v_y} + v_z\varphi_{v_z} = 0, \tag{23}$$

$$X_1(u - \varphi)|_{u=\varphi} = -(\dot{\xi} + \dot{\eta} + \dot{\zeta}) + \dot{\xi}(x)v_x\varphi_{v_x} + \dot{\eta}(y)v_y\varphi_{v_y} + \dot{\zeta}(z)v_z\varphi_{v_z} = 0. \tag{24}$$

Since the function v has the form (21), from (23) and (24) we get the system

$$(\alpha + \beta + \gamma)\varphi_v + (\alpha_x + \beta_x)\varphi_{v_x} + (\alpha_y + \gamma_y)\varphi_{v_y} + (\beta_z + \gamma_z)\varphi_{v_z} = 0,$$
$$-(\dot{\xi} + \dot{\eta} + \dot{\zeta}) + \dot{\xi}(x)(\alpha_x + \beta_x)\varphi_{v_x} + \dot{\eta}(y)(\alpha_y + \gamma_y)\varphi_{v_y} + \dot{\zeta}(z)(\beta_z + \gamma_z)\varphi_{v_z} = 0. \tag{25}$$

The system (25) is satisfied by the relation

$$u = \varphi(v, v_x, v_y, v_z) = \ln\frac{cv_xv_yv_z}{v^3} = \ln c + \ln v_x + \ln v_y + \ln v_z - 3\ln v. \tag{26}$$

Substituting (26) into the Eq. (18) taking into account (21) leads to a formula defining a class of solutions to the Eq. (18) depending on three arbitrary functions

$$u = \ln\left(-\frac{6}{\lambda}\frac{f_1'(x)f_2'(y)f_3'(z)}{(f_1(x) + f_2(y) + f_3(z))^3}\right). \tag{27}$$

Here $f_1(x)$, $f_2(y)$, $f_3(z)$—arbitrary continuously differentiable functions.

3 Fourth-Order Analogue of the Liouville Equation

Now consider the equation

$$u_{xyzt} = \lambda e^u, \tag{28}$$

related to the fourth-order linear Bianchi equation, whose group properties are considered in [4, 5].

Similarly to the case of the Eq. (18), we construct a non-local transformation that translates the Eq. (28) into the equation

$$v_{xyzt} = 0, \tag{29}$$

the general solution of which

$$v = \alpha(x, y, z) + \beta(x, y, t) + \gamma(x, z, t) + \delta(y, z, t). \tag{30}$$

Equation (28) admits a Lie algebra of operators

$$X = \xi(x)\partial_x + \eta(y)\partial_y + \zeta(z)\partial_z + \tau(t)\partial_t - (\dot{\xi}(x) + \dot{\eta}(y) + \dot{\zeta}(z) + \dot{\tau}(t))\partial_u,$$

where $\xi(x)$, $\eta(y)$, $\zeta(z)$, $\tau(t)$ are arbitrary functions [5].

On the other hand, the Eq. (29) admits the Lie algebra of operators

$$X_0 = \xi(x)\partial_x + \eta(y)\partial_y + \zeta(z)\partial_z + \tau(t)\partial_t,$$

as well as the stretching operator

$$Y = v\partial_v.$$

Looking for a non-local transformation

$$u = \varphi(v, v_x, v_y, v_z, v_t) \tag{31}$$

such that the system of Eqs. (28), (29), (31) admits the Lie algebra of operators

$$X = \xi(x)\partial_x + \eta(y)\partial_y + \zeta(z)\partial_z + \tau(t)\partial_t - (\dot{\xi}(x) + \dot{\eta}(y) + \dot{\zeta}(z) + \dot{\tau}(t))\partial_u,$$
$$Y = v\partial_v.$$

We calculate the first continuations of operators

$$X_1 = \xi(x)\partial_x + \eta(y)\partial_y + \zeta(z)\partial_z + \tau(t)\partial_t - (\dot{\xi}(x) + \dot{\eta}(y) + \dot{\zeta}(z) + \dot{\tau}(t))\partial_u -$$
$$-(\ddot{\xi}(x) - \dot{\xi}(x)u_x)\partial_{u_x} - (\ddot{\eta}(y) - \dot{\eta}(y)u_y)\partial_{u_y} - (\ddot{\zeta}(z) - \dot{\zeta}(z)u_z)\partial_{u_z} -$$
$$-(\ddot{\tau}(t) - \dot{\tau}(t)u_t)\partial_{u_t} + \dot{\xi}(x)v_x\partial_{v_x} + \dot{\eta}(y)v_y\partial_{v_y} + \dot{\zeta}(z)v_z\partial_{v_z} + \dot{\tau}(t)v_t\partial_{v_t},$$

$$Y_1 = v\partial_v + v_x\partial_{v_x} + v_y\partial_{v_y} + v_z\partial_{v_z} + v_t\partial_{v_t}$$

and we write down the ratios

$$Y_1(u-\varphi)|_{u=\varphi} = v\varphi_v + v_x\varphi_{v_x} + v_y\varphi_{v_y} + v_z\varphi_{v_z} + v_t\varphi_{v_t} = 0, \tag{32}$$

$$X_1(u-\varphi)|_{u=\varphi} = -(\dot{\xi}+\dot{\eta}+\dot{\zeta}+\dot{\tau}) + \dot{\xi}v_x\varphi_{v_x} + \dot{\eta}v_y\varphi_{v_y} + \dot{\zeta}v_z\varphi_{v_z} + \dot{\tau}v_t\varphi_{v_t} = 0. \tag{33}$$

The function v has the form (30), therefore from (32)–(33) we get the system

$$\begin{gathered}(\alpha+\beta+\gamma+\delta)\varphi_v + (\alpha_x+\beta_x+\gamma_x)\varphi_{v_x} + (\alpha_y+\beta_y+\delta_y)\varphi_{v_y} + \\ +(\alpha_z+\gamma_z+\delta_z)\varphi_{v_z} + (\beta_t+\gamma_t+\delta_t)\varphi_{v_t} = 0, \\ -(\dot{\xi}+\dot{\eta}+\dot{\zeta}+\dot{\delta}) + \dot{\xi}(\alpha_x+\beta_x+\gamma_x)\varphi_{v_x} + \dot{\eta}(\alpha_y+\beta_y+\delta_y)\varphi_{v_y} + \\ +\dot{\zeta}(\alpha_z+\gamma_z+\delta_z)\varphi_{v_z} + \dot{\delta}(\beta_t+\gamma_t+\delta_t)\varphi_{v_t} = 0.\end{gathered} \tag{34}$$

The system (34) is satisfied by the relation

$$u = \ln\frac{cv_x v_y v_z v_t}{v^4}. \tag{35}$$

Substituting (35) into (28) and taking into account (30), we get the solution of the Eq. (28)

$$u = \ln\left(\frac{24}{\lambda}\frac{f_1'(x)f_2'(y)f_3'(z)f_4'(t)}{(f_1(x)+f_2(y)+f_3(z)+f_4(t))^4}\right),$$

where $f_1(x)$, $f_2(y)$, $f_3(z)$, $f_4(t)$—arbitrary continuously differentiable functions.

This paper has been supported by the Kazan Federal University Strategic Academic Leadership Program (Priority—2030).

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