Local Bifurcations of Periodic Traveling Waves in the Generalized Weakly Dissipative Ginzburg-Landau Equation

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Abstract In this paper we consider a periodic boundary value problem for the generalized Ginzburg-Landau. The generalized version of the weakly dissipative Ginzburg-Landau equation differs from the traditional version by replacing the cubic nonlinearity with nonlinearity of arbitrary odd degree. We will show that the periodic boundary value problem has a countable set of solutions that are single-mode and periodic in the evolutionary variable. We will examine the stability question as well as local bifurcations of such solutions when they change stability. In this case, the two-dimensional attracting invariant tori bifurcate emerges when stability is lost from single-mode solutions. These are non-resonant tori that have appeared in the generic situation. The main results are obtained on the basis and development of methods of the theory of dynamical systems with an infinite-dimensional phase space. These include the method of invariant manifolds and normal forms, as well as the principle of self-similarity. This principle allows us to reduce the problem of bifurcations of a countable set of single-mode solutions to the analysis of the corresponding problem.

Keywords Ginzburg-Landau equation · Periodic boundary conditions · Stability · Bifurcations · Normal forms · Invariant tori

1 Introduction

One of the most famous nonlinear evolutionary equations of mathematical physics can be considered as the corresponding partial differential equation

$$u_t = gu - (d + ic)u|u|^2 + (a + ib)u_{xx}, \tag{1}$$

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where $u = u(t, x) = u_1(t, x) + iu_2(t, x), a, b, c, d, g \in R, d > 0, a \geq 0, g > 0$. Note that Eq. (1) is the referred to as the Ginzburg-Landau evolutional complex equation. It appears in several branches of physics as well as chemical kinetics as a mathematical model [1–3]. It is studied together with the periodic boundary conditions [1]

$$u(t, x + 2l) = u(t, x).$$

For chemical kinetics problems, the corresponding boundary conditions of "impenetrability" (homogeneous Neumann boundary conditions) are used

$$u_x(t, 0) = u_x(t, l) = 0.$$

We normalize the variables t, x and the functions $u(t, x)$ as follows:

$$t \to \gamma_1 t,\ x \to \gamma_2 x,\ u \to \gamma_3 u$$

and assume that $l = \pi, d = 1, g = 1$, if these constants are positive. We will study special cases, generalizations and modifications of Eq. (1) and its variations. For instance, if $c = b = 0$, then Eq. (1) is called the variational Ginzburg-Landau equation [1, 4–6]. A variational version of the Ginzburg-Landau equation is found in a section of modern physics as the theory of condensed matter and requires special investigation. Note that if $a = 0$ then we obtain the "weakly dissipative Ginzburg-Landau equation" [7–12]. For this version of the Ginzburg-Landau equation, we also apply the generalized cubic Schrodinger equation [11]. Next observe that if $g = d = a = 0$, then the original version of the Ginzburg-Landau equation is transformed into one of the variations of the nonlinear Schrodinger equation. Analogous to the nonlinear Schrodinger equation, the Ginzburg-Landau equation also occurs in nonlinear optics [8], as well as in some sections of hydrodynamics [2]. In monograph [13] the hypothesis is given that when replacing Eq. (1) with the following

$$u_t = gu - (d + ic)u|u|^4 + (a + ib)u_{xx},$$

according to its authors, a significant change in the dynamics of solutions is possible. In particular, the hard oscillations are possible. In other words the subcritical bifurcations are realized. In this paper, we will consider the generalized weakly dissipative Ginzburg-Landau equation, which includes both variants of the Ginzburg-Landau equation from the introduction.

2 Formulation of the Problem

Our aim is to examine the following boundary value problem

$$u_t = u - (1 + ic)u|u|^{2p} - ibu_{xx}, \tag{2}$$

$$u(t, x + 2\pi) = u(t, x), \tag{3}$$

where $u = u(t, x) = u_1(t, x) + iu_2(t, x)$, $c \in R$, $b > 0$, $p \in N$. Note that if $p = 2$ we obtain one of the versions of the equation in monograph [13]. If $p = 1$, we then obtain the initial version of the weakly dissipative Ginzburg-Landau equation.

Next, if we consider the following initial condition for the boundary value problem (2), (3)

$$u(0, x) = f(x), \tag{4}$$

where $f(x) \in H_2$, then via the results from [14, 15] that the initial-boundary value problem (2), (3), (4) is locally well-solvable. Also recall that the inclusion $f(x) \in H_2$ resembles the following characteristics:

(1) $f(x)$ has period 2π;

(2) $f(x)$ has generalized derivatives up to the inclusive second order derivatives $f(x)$, $f'(x)$, $f''(x) \in L_2(-\pi, \pi)$.

This space H_2 is the phase space of solutions to the initial-boundary value problem (2), (3), (4). The nonlinear boundary value problem (2), (3) has a countably family single-mode solutions in the space variable x and periodic in t

$$u = u_n(t, x) = \eta_n \exp(inx + i\sigma_n t), \tag{5}$$

where $n \in Z$ (Z is the set of integer), $|\eta_n| = 1$, $\sigma_n = bn^2 - c$. Indeed, substitution of the right side of equality (5) into Eq. (2) after elementary simplifications leads to a complex equation for determining η_n, σ_n

$$i\sigma_n = 1 - (1 + ic)|\eta_n|^{2p} + ibn^2.$$

Next notice that along with solution (5), the boundary value problem (2), (3) also has solutions in the corresponding form

$$u_n(t, x, h) = \exp(ih)\exp(inx + i\sigma_n t),\ h \in R.$$

The solutions $u_n(t, x, h)$ form a one-dimensional invariant subspace in the phase space of solutions to the boundary value problem (2), (3). Since h is arbitrary, in further constructions we can assume that $\eta_n = 1$. Replacing an unknown function

$$u(t, x) = \exp(i\omega_n t + inx)v(t, y), \tag{6}$$

where $\omega_n = bn^2$, $y = x + 2bnt$, $n = 0, \pm 1, \pm 2, \ldots$ leads us to the following equation for $v(t, y)$

$$v_t = v - (1 + ic)v|v|^{2p} - ibv_{yy},$$

which should be considered with the corresponding periodic boundary conditions

$$v(t, y + 2\pi) = v(t, y).$$

Notice that substitution (6) transforms the solutions of boundary value problem (2), (3) into solutions of the same boundary value problem. Therefore, the study of the neighborhood of each of the family of solutions (5) can be substituted by a similar problem for one of them: $u_0(t, x) = \exp(i\sigma_0 t)$, where $\sigma_0 = -c$. In physics, the solution $u_0(t, x) = u_0(t)$ is often called a spatially homogeneous cycle (or "thermodynamic" branch, Andronov-Hopf cycle). The remaining solutions of family (5) for $n \neq 0$ are periodic traveling waves and periodically depend on t and x.

3 Stability Analysis of Periodic Traveling Waves

As previously noted, the stability analysis of solutions of $u_n(t, x)$ (stability of one-dimensional manifolds $V_1(u)$) can be reduced by virtue of the principle of self-similarity [9] to the analysis of similar questions for a spatially homogeneous periodic solution of $u_0(t) = \exp(i\sigma_0 t)$, where $\sigma_0 = -c$. In turn, to analyze the stability of the solution $u_0(t)$ by setting

$$u(t, x) = u_0(t)(1 + w(t, x)). \tag{7}$$

For the deviation $w(t, x)$ we obtain the following nonlinear boundary value problem

$$w_t = A(p)w - (1 + ic)F(w, p), \tag{8}$$

$$w(t, x + 2\pi) = w(t, x), \tag{9}$$

where $F(w, p) = F_2(w, p) + F_3(w, p) + F_0(w, p)$ is a two-variable polynomial of degree $2p + 1$. For further constructions, we consider the following terms

$$F_2(w, p) = \frac{1}{2}p\Big((p + 1)w^2 + 2(p + 1)w\overline{w} + (p - 1)\overline{w}^2\Big),$$
$$F_3(w, p) = \frac{1}{6}p\Big((p^2 - 1)w^3 + 3(p^2 + p)w^2\overline{w} + 3(p^2 - 1)w\overline{w}^2 + (p^2 - 3p + 2)\overline{w}^3\Big).$$

$F_0(w, p)$ denotes the terms at zero that have an order of smallness in the variables $w, \overline{w}$ higher than the third. This leads us to $A(p)w = -p(1 + ic)(w + \overline{w}) - ibw_{xx}$.

Next, we will reformulate the linear differential operator A in real form. Instead of the complex-valued function $w = w_1 + iw_2$, we form the vector function $v = colon(w_1, w_2)$. In this case, we rewrite the linear differential operator A as follows

$$Av = \begin{pmatrix} -2p & b\partial^2 \\ -2cp - b\partial^2 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

where we apply the short notation $\partial^2 f = \dfrac{\partial^2 f}{\partial x^2}$. This leads us to the functions in the corresponding form

$$v_k(x) = \exp(ikx) \begin{pmatrix} h_{1k} \\ h_{2k} \end{pmatrix},$$

where $h_{1,k}, h_{2,k}$ are real or complex constants, $k = 0, \pm 1, \pm 2, \ldots$ In this case, the problem of determining the eigenvalues and eigenelements of the linear differential operator A reduces to analyzing the spectrum of the following countable family of matrices

$$A_k = \begin{pmatrix} -2p & -bk^2 \\ bk^2 - 2cp & 0 \end{pmatrix}$$

and to determining of the roots of the family of characteristic equations

$$\lambda^2 + 2p\lambda + q_k = 0,$$

where $k = 0, \pm 1, \pm 2, \ldots, q_k = bk^2(bk^2 - 2cp), p \in Z$. For $k = 0$ we obtain $\lambda_{1,0} = 0, \lambda_{2,0} = -2p < 0$, i.e. for all values of the parameters p and b the linear differential operator A has a zero eigenvalue corresponding to the eigenfunction $v_0(x) = colon(0, 1)$ or $H_0(x) = i$ in the complex record form.

Let now $k \neq 0$. Note that $\lim\limits_{|k| \to \infty} q_k = \infty$. Consequently, the inequalities $q_k > 0$ for all $k \neq 0$ lead to the following inequality

$$Re\lambda_{k,j} \leq -\gamma_0 < 0 \tag{10}$$

for all values of k and $j = 1, 2$. Thus we obtain, $q_k > 0$ for all $k \in Z\backslash\{0\}$, if $b > 2pc$ ($b > 0$ by condition). Otherwise, when $b < 2cp$, the linear differential operator A has at least one of its eigenvalues in the right half-plane of the complex plane (one of the numbers $\lambda_{k,1}$ or $\lambda_{k,2}$ are positive). Finally, for $b = 2pc$ the linear operator A has a triple zero eigenvalue, which in the complex notation corresponds to the following eigenfunctions

$$H_0(x) = i, \; H_1(x) = (-c + i)\cos x, \; H_2(x) = (-c + i)\sin x.$$

The linear differential operator A corresponding to this choice of parameters will be denoted by $A_0 : A_0 w = -p(1+ic)(w+\overline{w}) - 2icpw_{xx}$. Thus, in addition to the zero-equilibrium state, the nonlinear boundary value problem (8), (9) has the following one-parameter family of equilibrium states

$$w_*(t,x) = w_*(h) = \exp(ih) - 1,$$

which is easy to verify indirectly. Next by substituting $w_*(h)$ into formula (7) leads to the following equality

$$u_*(t,x,h) = \exp(i\sigma_0 t)\exp(ih),$$

which is a spatially homogeneous solution of the boundary value problem (2), (3) for all $h \in R$.

Notice that the solutions $w_*(h)$ (family) of equilibrium states form a one-dimensional invariant manifold $M_1(h)$, which exists for all values of the parameters of the boundary value problem (8), (9) and for $h = 0$ we have $w = 0$. Therefore, this one-dimensional invariant manifold is a center manifold in a neighborhood of the zero equilibrium state [16, 17], at least for small $|h|$. This remark and theorems on behavior solutions outside the center manifolds are the base to the assertion.

Theorem 1 *(1) Suppose that $b > 2cp$, then $M_1(h)$ be a local attractor for solutions to the boundary value problem (8), (9). In particular, all equilibria for small $|h|$, including the zero-equilibrium state, are stable but not asymptotically stable. (2) Suppose that $b < 2cp$. Then all the equilibrium states forming $M_1(h)$, including the zero-equilibrium state are unstable (saddle points).*

Remark 1 If $b = 2cp$, then the invariant manifold $M_1(h)$ exists and is formed by a one-parameter family $(w_*(h) = \exp(ih) - 1)$ of equilibrium states. In particular, the equilibrium state $w = 0$ for which the critical case of a threefold zero eigenvalue emerges, belongs to $M_1(h)$. Hence, in this case, an additional analysis of the question of stability of zero state of equilibrium is required. This is due to the fact that the stability theorem with respect to linear approximation cannot be used even in the case of ordinary differential equations.

Corollary 1 *From the previous constructions, when transitioning from the boundary value problem (2), (3) to the auxiliary boundary value problem (8), (9), substitution (7) and from the self-similarity principle, we obtain the following features:*

(1) for $b > 2pc$ all traveling waves $u_n(t,x,h) = \exp(i(nx + \sigma_n t + h))$, where $n = \pm 1, \pm 2, \ldots, h \in R$ and spatially homogeneous solutions $u_0(t,h) = \exp(i(\sigma_0(t+h)))$ are stable;

(2) for $b < 2cp$ they are all unstable;

(3) for $b = 2pc\,(c = b/(2p))$ the critical case stability problem of solutions $u_n(t,x,h)$ are realized.

Further in the next section, the boundary value problems (2), (3) and (8), (9) will be considered in cases where the threefold zero eigenvalue of the operator A is close to critical. This means that the boundary value problems studied below will be considered if

$$b = 2pc(1 + \gamma\varepsilon), \tag{11}$$

where $\varepsilon \in (0, \varepsilon_0)$, $\gamma = \pm 1$ and 0. Appropriate values of γ will be chosen at the final stage of the analysis of the studied boundary value problems.

Before proceeding to the direct analysis of the bifurcation problem, we introduce some notation and also recall one fairly well-known statement from the theory of linear boundary value problems for ordinary differential equations, which we formulate in a form adapted to our case. Consider the differential operator

$$A(\varepsilon)y = A_0 y + \gamma\varepsilon A_1 y,\ \ A_0 y = -p(1 + ic)(y + \overline{y}) - 2pciy'',$$
$$A_1 y = -2pciy'',\ y(x) = y_1(x) + iy_2(x).$$

In this case $y(x)$ is a sufficiently smooth 2π periodic function.

Remark 2 We will consider the following linear nonhomogeneous boundary value problem

$$A_0 y = f(x),\ y(x + 2\pi) = y(x), \tag{12}$$

where the complex-valued function $f(x) \in L_2(-\pi, \pi)$ and has period 2π. The boundary value problem (12) has a solution if $f(x)$ satisfies the following two conditions:

$$(a)\ Re(a_0(c + i)) = 0,\ \text{ where } a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx;$$

$$(b)\ Im a_1 = Im b_1 = 0,\ \text{ where } a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x dx,\ \ b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x dx.$$

The solution of the boundary value problem (12) is unique, for which the following equalities hold:

$$(a)\ Re(y_0(c + i)) = 0,\ \text{ where } y_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} y(x)dx;$$

$$(b)\ Im y_1 = Im z_1 = 0,\ \text{ where } y_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} y(x)\cos x dx,\ \ z_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} y(x)\sin x dx$$

The conditions for solvability when using the complex notation are given. They have a more familiar form. Next, we will consider the corresponding nonhomogeneous boundary value problem

$$A_0 y(x) = f(x),\ y(x + 2\pi) = y(x),$$

where $y(x) = colon(y_1(x), y_2(x))$, $f(x) = colon(f_1(x), f_2(x))$. Now A_0 in real form is expressed as

$$A_0 = \begin{pmatrix} -2p & 2pc\partial^2 \\ -2pc - 2pc\partial^2 & 0 \end{pmatrix}.$$

It has a triple zero eigenvalue, which corresponds to three eigenfunctions

$$H_0(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ H_1(x) = \begin{pmatrix} -c \\ 1 \end{pmatrix} \cos x,\ H_2(x) = \begin{pmatrix} -c \\ 1 \end{pmatrix} \sin x.$$

Conjugate operator

$$A_0^* = \begin{pmatrix} -2p & -2pc - 2pc\partial^2 \\ 2pc\partial^2 & 0 \end{pmatrix}$$

is defined on sufficiently smooth 2π periodic vector functions $z(x) = colon(z_1(x), z_2(x))$. Naturally, it has a triple zero eigenvalue corresponding to the eigenfunctions

$$E_0(x) = \begin{pmatrix} -c \\ 1 \end{pmatrix},\ E_1(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos x,\ E_2(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin x,$$

The solvability conditions arise in the following form

$$< f, E_j >= 0,\ j = 0, 1, 2,$$

where $< f(x), q(x) >$ denotes the scalar product in the corresponding function space

$$< f(x), q(x) >= \int_{-\pi}^{\pi} (f(x), q(x))dx,$$

where $q(x) = colon(q_1(x), q_2(x))$, and the brackets $(*, **)$ inside the integral denote the inner product in R^2 [18, 19].

The statements from Remark 2 are known as solvability conditions.

4 Turing—Prigogine Bifurcation

We will focus on the analysis of the nonlinear boundary value problem (2), (3) for the determined values by equality (11). Next, we will transition to a modified version of the boundary value problem (8), (9) with the following substitution

$$u(t,x) = u_0(t)\exp(i\varphi)(1 + w(t,x)), \tag{13}$$

where, as in substitution (7) $u_0(t) = \exp(i\sigma_0 t)$. As a result, now for $w(t,x)$ we obtain a boundary value problem similar to boundary value problem (8), (9)

$$w_t + i\varphi_t(1 + w) = A(\varepsilon)w - (1 + ic)(F_2(w) + F_3(w) + F_0(w)), \tag{14}$$

$$w(t, x + 2\pi) = w(t,x). \tag{15}$$

In this case, $\varphi = \varphi(t,\varepsilon)$ and $\varphi_t(t,0) = 0$, i.e. $\varphi(t,0) = h$ is an arbitrary real constant.

We indicate an essential feature of the boundary value problem (14), (15). We denote $H_{2,even}$ as the subspace of the function space H_2, containing only even functions $f(x)$. In this case, the specificity of the right side of Eq. (14) is such that this subspace is invariant for solutions of the boundary value problem (14), (15). If $w(0,x) \in H_{2,even}$, then its solution is $w(t,x)$ for all t, when it exists, belongs to $H_{2,even}$. In this case, the periodic boundary conditions (15) can be replaced by the homogeneous Neumann boundary conditions

$$w_x(t,0) = w_x(t,\pi) = 0, \tag{16}$$

assuming that $x \in [0,\pi]$. First we restrict ourselves to the analysis of the auxiliary boundary value problem (14), (16). With this choice of boundary conditions, the linear differential operator A_0 has a double zero eigenvalue, whose corresponding eigenfunctions are

$$H_0(x) = i, \;\; H_1(x) = (-c + i)\cos x.$$

Let us recall some well-known assertions. Denote by $\Omega(r)$ the ball of radius r centered at the zero of the phase space $H_{2,even}$. As is well known (see, for example, [17]), boundary value problem (14), (16) in a neighborhood of the equilibrium $w = 0$ has a smooth two-dimensional invariant manifold $M_2(\varepsilon) \in \Omega(r)$, where r is a sufficiently small positive constant. All solutions of the auxiliary boundary value problem (14), (16) from this neighborhood $\Omega(r)$ approach $M_2(\varepsilon)$ with the exponential rate over time. In this case, solutions to the boundary value problem (14), (16) that belong to $M_2(\varepsilon)$, can be sought in the following form [9, 10, 12]

$$w(t,x,\varepsilon) = \varepsilon^{1/2}Q_1(x,z) + \varepsilon Q_2(x,z) + \varepsilon^{3/2}Q_3(x,z) + \varepsilon Q_4(x,z,\varepsilon). \tag{17}$$

The functions $Q_j(x, z)$, $j = 1, 2, 3$, $Q_4(x, z, \varepsilon)$ reveal the following properties:

(1) they depend on their variables rather smoothly if $|z| < z_0$, $\varepsilon \in (0, \varepsilon_0)$ and, in addition, $Q_4(x, z, 0) = 0$ (z_0, ε_0 – some positive constants);

(2) as a function of x they belong to $W_2^2[0, \pi]$ (the corresponding Sobolev space is denoted by $W_2^2[0, \pi]$) and satisfy the boundary conditions (16).

Further, we assume that the functions $\varphi = \varphi(s, \varepsilon)$, $z = z(s)$ depend on the slow time $s = \varepsilon t$. They satisfy the corresponding system of two ordinary differential equations

$$\varphi_s = \Psi_0(z, \varepsilon),\ z_s = \Psi_1(z, \varepsilon), \tag{18}$$

where the right-hand sides smoothly depend on z, ε, if $|z| < z_0$ and $\varepsilon \in (0, \varepsilon_0)$. The system of differential equations (18) is called the normal form. It can be replaced with a shortened version [20]

$$\varphi_s = \Theta_0(z),\ z_s = \Theta_1(z), \tag{19}$$

where $\Theta_0(z) = \Psi_0(z, 0)$, $\Theta_1(z) = \Psi_1(z, 0)$. Such a variant of system (18), i.e. system (19) is called "truncated normal form". It is this kind of normal form that plays the main role in the analysis of local bifurcations. We substitute the sum (17) into the auxiliary boundary value problem (14), (16) and note that $z_t = z_s\varepsilon$, $\varphi_t = \varphi_s\varepsilon$.

As a result of such a substitution, we obtain a sequence of linear boundary value problems of the terms at equal powers $\varepsilon^{1/2}$. So for $\varepsilon^{1/2}$ we obtain a homogeneous boundary value problem for $Q_1 = Q_1(x, z)$ of the following form

$$A_0 Q_1 = 0,\ \ Q_{1x}(0, z) = Q_{1x}(\pi, z) = 0,$$

as solutions of which the function can be chosen

$$Q_1(x, z) = zH_1(x) = z(-c + i)\cos x.$$

Collecting the terms at $\varepsilon, \varepsilon^{3/2}$, we obtain two nonhomogeneous boundary value problems. Thus, to determine the function $Q_2 = Q_2(x, z)$, we obtain the following boundary value problem

$$A_0 Q_2 = (1 + ic)\Phi_2(x)z^2 + i\Theta_0(z), \tag{20}$$

$$Q_{2x}(0, z) = Q_{2x}(\pi, z) = 0, \tag{21}$$

where $\Phi_2(x)z^2 = F_2(H_1(x, z))$ and we procure

$$\Phi_2(x)z^2 = \frac{p}{2}z^2\big[(p + 1)(c - i)^2 + 2(p + 1)(c^2 + 1) + (p - 1)(c + i)^2\big]\cos^2 x.$$

These computations assumed that $\varphi_t = \varepsilon\varphi_s$ and therefore $\varphi_t = \varepsilon\Theta_0(z) + o(\varepsilon)$. The boundary value problem (20), (21) has a solution from the specified class of functions if $\Theta_0(z) = pc(c^2+1)z^2$. In this case, the corresponding solution (see solvability conditions)

$$Q_2(x,z) = v(x)z^2 = (v_0 + v_2\cos 2x)z^2,$$
$$v_0 = -\frac{1}{4}(1+ic)((2p+3)c^2+1),\ v_2 = \frac{(2p+1)c^2-1}{12} - i\frac{(4p+5)c^2+1}{24c}.$$

We obtain the corresponding linear nonhomogeneous boundary value problem by collecting the terms at $\varepsilon^{3/2}$, at the third step of the implementation of the algorithm:

$$A_0Q_3 = \Theta_1(z)H_1(x) - z\gamma A_1H_1(x) + i\Theta_0(z)zH_1(x) + \\ +(1+ic)(F_3(Q_1) + \Phi_3(Q_1,Q_2))z^3, \tag{22}$$

$$Q_{3x}(0,z) = Q_{3x}(\pi,z) = 0. \tag{23}$$

In the boundary value problem (22), (23) $Q_3 = Q_3(x,z)$,

$$F_3(Q_1) = F_3(H_1(x)) = \frac{p}{6}(1+ic)\Big((p^2-1)(-c+i)^3 + 3p(p+1)(-c+i)^2(-c-i) + \\ +3(p^2-1)(-c+i)(c+i)^2 + (p-1)(p-2)(-c-i)^3\Big)\cos^3 x,$$
$$\Phi_3(Q_1,Q_2) = p\Big((p+1)H_1(x)v(x) + \\ +(p+1)\Big(H_1(x)\overline{v}(x) + \overline{H}_1(x)v(x)\Big) + (p-1)\overline{H}_1(x)\overline{v}(x)\Big).$$

It follows from the solvability conditions for the nonhomogeneous boundary value problem (22), (23) that in this case one should choose

$$\Theta_1(z) = \nu_p z - l_p z^3,$$

where $\nu_p = -2\gamma c^2 p,\ l_p = \frac{p}{6}((4p^2+22p+4)c^4 + (2p-11)c^2 + 1)$.

Thus, the analysis of the boundary value problem (14), (16) has been reduced to the study of a system of ordinary differential equations (the "shortened" or "truncated" normal form). In our case, it is presented in the following form

$$\varphi_s = pc(c^2+1)z^2, \tag{24}$$

$$z_s = \nu_p z - l_p z^3. \tag{25}$$

Lemma 1 *Differential equation (25), in addition to the zero equilibrium state* $S_0(z=0)$, *has nonzero equilibrium states*

$$S_+ : z_+ = \sqrt{\frac{\nu_p}{l_p}}, \ \ S_- : z_- = -\sqrt{\frac{\nu_p}{l_p}},$$

if $\nu_p/l_p > 0$.

For $l_p > 0\ (\nu_p > 0)$, *the equilibrium states* S_+, S_- *are asymptotically stable and they are unstable if* $l_p < 0\ (\nu_p < 0)$. *In turn,* S_0 *is an asymptotically stable equilibrium state if* $\nu_p < 0$ *or* $\nu_p = 0, l_p > 0$.

The proof of Lemma 1 is fairly straight forward. In fact, even in the situation where $\nu_p \neq 0$, one should use the stability theorem in the first (linear) approximation. For $\nu_p = 0$ we get the equation $z_s = -l_p z^3$ and its solution $z(s) \to 0$ for $s \to \infty$, if $l_p > 0$ and $z(s)$ leaves the neighborhood of zero if $l_p < 0$. In our case $l_p > 0$ for any positive integer p and all $c \in R$, due to the discriminant of the square trinomial

$$l_p(\xi) = (4p^2 + 22p + 4)\xi^2 - (2p - 11)\xi + 1$$

is negative.

We choose γ such that $\nu_p > 0$ (for example, $\gamma = -1$). The equilibrium states S_+, S_- of the differential equation (25) correspond to the solutions

$$\varphi_+(s) = (pc(c^2 + 1)z_+^2)s + h_+, \ \ \varphi_-(s) = (pc(c^2 + 1)z_-^2)s + h_-$$

of differential equation (24). Here $h_+, h_- \in R$ and are arbitrary. Transitioning to a more complete system (18) in this case gives us

$$\varphi_\pm(s) = (pc(c^2 + 1)z_\pm^2 + O(\varepsilon))s.$$

It follows from the results of [21, 22] and previous constructions that the assertion is true.

Lemma 2 *There exists a constant* $\varepsilon_p > 0$, *such that for all* $\varepsilon \in (0, \varepsilon_p)$ *there are two sets of functions*

$$\{\varphi_+(t, \varepsilon), w_+(x, \varepsilon); \ \varphi_-(t, \varepsilon), w_-(x, \varepsilon)\},$$

satisfying the nonhomogeneous boundary value problem (14), (16). For such functions, the following asymptotic formulas are valid

$$w_\pm(x, \varepsilon) = \varepsilon^{1/2} z_\pm(-c + i)\cos x + \varepsilon z_\pm^2(v_0 + v_2 \cos 2x) + o(\varepsilon),$$
$$\varphi_\pm(t, \varepsilon) = (pc(c^2 + 1)z_\pm^2 + O(\varepsilon))\varepsilon t.$$

Also observe that these functions satisfy the boundary value problem (14), (15). Moreover, due to the translational invariance for the solutions of the boundary value problem (14), (15), it also has the following pairs of solutions

$$(w_+(x + h_+, \varepsilon), \varphi_+(t, \varepsilon)), \ \ (w_-(x + h_-, \varepsilon), \varphi_-(t, \varepsilon)).$$

Next note that the boundary value problem (14), (16) is invariant under the change

$$x \to \pi - x,\ z_+ \to z_-,$$

then for the boundary value problem (14), (15) there remains only one set

$$(w_+(x + h_+, \varepsilon), \varphi_+(t, \varepsilon)),$$

which includes all the corresponding solutions by choosing an appropriate shift h_+. All these constructions and remarks allow us to formulate the main result, which refers to the boundary value problem (2), (3).

Theorem 2 *There exists $\varepsilon_p > 0$, such that for all $\varepsilon \in (0, \varepsilon_p)$ the nonlinear boundary value problem (2), (3) for $b_p = 2cp(1-\varepsilon)$ $(\gamma = -1)$ has a two-parameter family of the periodic in t solutions $V_0(h_0, h)$*

$$u_0(t, x, \varepsilon) = \exp(i\sigma_0 t + i\varphi_+(t, \varepsilon) + ih_0)(1 + w_+(x + h, \varepsilon)),$$

where $\varphi_+(t, \varepsilon) = \Big(pc(c^2 + 1)z_+^2 + O(\varepsilon)\Big)\varepsilon t,\ z_+ = \sqrt{\nu_p/l_p},$

$$w_+(x + h, \varepsilon) = \varepsilon^{1/2} z_+(-c + i)\cos(x + h) + \varepsilon z_+^2(v_0 + v_2 \cos 2(x + h)) + o(\varepsilon),$$

where $h_0, h \in R$ and are arbitrary, the constants v_0, v_2 were specified earlier. The two-dimensional invariant manifold $V_0(h_0, h)$ is a local attractor.

The validity of the assertion follows from the principle of self-similarity from Eq. (6). The following assertion is corollary from Theorem 2.

Corollary 2 *Boundary value problem (2), (3) has a countable set of two-dimensional attracting invariant manifolds $V_n(h_0, h)$, generated by the following solutions*

$$\begin{aligned} u_n(t, x, \varepsilon) = \exp(i\sigma_n t + inx + i\varphi_+(t, \varepsilon) + ih_0)\times \\ \times\Big(1 + w_+(x + 4npc(1-\varepsilon)t + h, \varepsilon)\Big), \end{aligned} \tag{26}$$

where $n = \pm 1, \pm 2, \ldots, \sigma_n = -c + 2pc(1-\varepsilon)n^2$, and the functions $\varphi_+(t, \varepsilon)$ and $w_+(x, \varepsilon)$ were found earlier in the process of implementing the modified Krylov-Bogolyubov algorithm (see formula (17) and boundary value problems (20), (21) and (22), (23)) and using the principle of self-similarity.

From the asymptotic formulas and the method of constructing solutions $u_0(t, x, \varepsilon), u_n(t, x, \varepsilon)$, it follows that $V_n(h_0, h)$ for all $n \in Z$ are two-dimensional invariant tori. Moreover, the torus $V_0(h_0, h)$ is filled with solutions that are periodic in t, and the solutions that form $V_n(h_0, h)$ as $n \neq 0$ are almost periodic functions of the variable t with a non-resonant set of eigenfrequencies. We emphasize that the solutions that form these two-dimensional tori are stable but cannot be asymptotically stable as in the neighborhood of each of these solutions there is always one more representative of the corresponding family.

5 Conclusions

The aim of this work was to generalize the results of works [9, 10], where particular cases of the boundary value problem (2), (3) for $p = 1, 2$ were considered. In this work we were able to show the following characteristics. Qualitatively, the results for all p are fairly close. In all boundary value problems with different values of p there exists a countable set of traveling waves that are periodic in t. When they lose stability, two-dimensional invariant tori, which are attracting invariant manifolds, bifurcate from each of them. For $n = 0$ the torus $V_0(h_0, h)$ is filled with periodic solutions, and the tori $V_n(h_0, h)$ are non-resonant in the generic situation. Thus, the hypothesis that the replacement of the cubic nonlinearity by the fifth-degree nonlinearity leads to subcritical bifurcations in the vicinity of traveling waves turned out to be not completely consistent. In any case, it is of paramount interest to consider the weakly dissipative versions of Ginzburg-Landau equation.

However, for the basic and generalized versions of the weakly dissipative version of the Ginzburg-Landau equation, the dynamics can be quite complex. The periodic boundary value problem (2), (3), with an appropriate choice of the coefficients of the equation, can have a countable set of local attractors, each of which is a two-dimensional invariant torus. The torus with number n ($n = 0, \pm 1, \ldots$) is formed by solutions (26) whose norm in the phase space (that is, in H_2) tends to infinity if $|n| \to \infty$. At the same time, the norm of all these solutions in the space $L_2(-\pi, \pi)$ is close to $\sqrt{2\pi}$.

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