Invariants of Dynamical Systems with Dissipation on Tangent Bundles of Low-Dimensional Manifolds

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Abstract Tensor invariants (differential forms) for homogeneous dynamical systems on tangent bundles to smooth two-dimensional manifolds are presented in this paper. The connection between the presence of these invariants and the full set of the first integrals necessary for the integration of geodesic, potential and dissipative systems is shown. At the same time, the introduced force fields make the considered systems dissipative with dissipation of different signs and generalize the previously considered ones. We also represent the typical examples from rigid body dynamics.

Keywords Dynamic equations · Nonconservative force field · Integrability · Transcendental tensor invariant

1 Introduction

It is well known [1–3] that a system of differential equations can be completely integrated when it has a sufficient number of not only first integrals (scalar invariants) but also tensor invariants. For example, the order of the considered system can be reduced if there is an invariant form of the phase volume. For conservative systems, this fact is natural. However, for systems having attracting or repelling limit sets, not only some of the first integrals, but also the coefficients of the invariant differential forms involved have to consist of, generally speaking, transcendental (in the sense of complex analysis) functions [4–6].

For example, the problem of a spatial pendulum on a spherical hinge placed in material flow leads to a system on the tangent bundle of the two-dimensional sphere with a special metric on it induced by an additional symmetry group [7]. Dynamical systems describing the motion of such a pendulum have signchanging dissipation, and the complete list of first integrals consists of transcendental functions expressed

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V. Vasilyev (ed.), *Differential Equations, Mathematical Modeling and Computational Algorithms*, Springer Proceedings in Mathematics & Statistics 423,
https://doi.org/10.1007/978-3-031-28505-9_12

in terms of a finite combination of elementary functions. There are also problems concerning the motion of a point over two-dimensional surfaces of revolution, the Lobachevsky plane, etc. The results obtained are especially important in the context of a nonconservative force field present in the system [5, 6].

Below, we present tensor invariants (differential forms) for homogeneous dynamical systems on tangent bundles of smooth two-dimensional manifolds. The relation between the existence of these invariants and the existence of a complete set of first integrals necessary for the integration of geodesic, potential, and dissipative systems is shown. The force fields introduced into the considered systems make them dissipative with dissipation of different signs and generalize previously considered force fields.

2 Example: Plane Pendulum in a Jet Flow

We describe in brief some problem on a physical pendulum on a cylindrical hinge in the flow of the incoming medium. The space of positions of such a pendulum is one-dimensional circle $\mathbf{S}^1\{\theta \bmod 2\pi\}$, and the phase space is the tangent bundle $T\mathbf{S}^1\{\dot{\theta}; \theta \bmod 2\pi\}$, i.e. two-dimensional cylinder.

Under the considered model assumptions, the equation of motion of such a pendulum is written out. statement [8] is proved that the dynamical system describing the behavior of such a pendulum is trajectorically topologically equivalent to the following differential equation on a two-dimensional cylinder (an angle θ is measured 'by the flow'):

$$\ddot{\theta} + h\dot{\theta}\cos\theta + \sin\theta\cos\theta = 0, \;\; h > 0. \tag{1}$$

Equation (1) can be rewritten as a system on a phase cylinder $\mathbf{R}^1\{\omega\} \times \{\alpha \bmod 2\pi\}$ ($\alpha = \theta + \pi$):

$$\dot{\alpha} = -\omega + h\sin\alpha, \;\; \dot{\omega} = \sin\alpha\cos\alpha, \tag{2}$$

the phase portrait of which is shown in [7].

For $h = 0$, the conservative system (2) has a smooth first integral of energy:

$$\frac{\omega^2}{2} + \frac{\sin^2\alpha}{2} = C_0 = \text{const}, \tag{3}$$

at the same time, its phase flow preserves the area on the plane $\mathbf{R}^2\{\alpha, \omega\}$, i.e. the differential 2-form is preserved

$$d\alpha \wedge d\omega. \tag{4}$$

When integrating the system, either the first integral of energy (3) or the fact of phase area conservation (4) can be used.

In the case of $h \neq 0$ is more complicated. Since the system (2) has attractive or repulsive (asymptotic) limit sets, the first integral of the system is a transcendental (in the sense of complex analysis) function, which has the form

$$\Phi_0(\alpha, \omega) = \sin\alpha \exp \Psi_0(t) = C_1 = \text{const}, \ \Psi_0(t) = \int \frac{(t-h)dt}{t^2 - ht + 1}, \ t = \frac{\omega}{\sin\alpha}, \tag{5}$$

in this case, the asymptotic limit sets are found from the system of algebraic equalities $\sin\alpha = 0, \ \omega = 0$ (see also [9]).

Since the system (2) has asymptotic limit sets there is not even an absolutely continuous function that is the density of the measure of the phase plane (cf. with [3, 7, 8]). But it is possible (along with the first integral) to present an invariant differential 2-form with coefficients that are transcendental functions, which has the form

$$T_1(\alpha, \omega) = \exp\{-h\Psi_1(t)\} \, d\alpha \wedge d\omega, \ \Psi_1(t) = \int \frac{dt}{t^2 - ht + 1}, \ t = \frac{\omega}{\sin\alpha}. \tag{6}$$

3 Example of More General System with One Degree of Freedom

We consider the smooth dynamical system on the plane $\mathbf{R}^2\{\alpha, \omega\}$ with one degree of freedom α of the following form:

$$\dot{\alpha} = -\omega + b\delta(\alpha), \ \dot{\omega} = F(\alpha); \tag{7}$$

we can rewrite this system in the form of the equation

$$\ddot{\alpha} - b\tilde{\delta}(\alpha)\dot{\alpha} + F(\alpha) = 0, \ \tilde{\delta}(\alpha) = \frac{d\delta(\alpha)}{d\alpha}. \tag{8}$$

A pair of smooth functions $(F(\alpha), \delta(\alpha))$ defines the force field in the system: the function $F(\alpha)$ describes the conservative component of the field, and the function $\delta(\alpha)$ describes possible scattering or pumping of energy in the system. For $b = 0$, the conservative system (7) has a smooth integral of energy:

$$\frac{\omega^2}{2} + 2\int\limits_{\alpha_0}^{\alpha} F(\xi)d\xi = C_0 = \text{const}, \tag{9}$$

at the same time, its phase flow preserves the area on the plane $\mathbf{R}^2\{\alpha, \omega\}$, i.e. the differential 2-form is preserved

$$d\alpha \wedge d\omega. \tag{10}$$

When integrating the system, either the first integral of energy (9) or the fact of phase area conservation (10) can be used.

The situation is different in the case of $b \neq 0$. Since the system (7) has, generally speaking, attractive or repulsive (asymptotic) limit sets, the first integral of the system is a transcendental (in the sense of complex analysis) [10] function. Let 's give it for the next important case:

$$F(\alpha) = \lambda\delta(\alpha)\tilde{\delta}(\alpha), \ \lambda \in \mathbf{R}. \tag{11}$$

Indeed, the first integral has the form

$$\Phi(\alpha, \omega) = \delta(\alpha) \exp \Psi(t) = C_1 = \text{const}, \ \Psi(t) = \int \frac{(t-b)dt}{t^2 - bt + \lambda}, \ t = \frac{\omega}{\delta(\alpha)}, \tag{12}$$

in this case, the asymptotic limit sets are found from the system of algebraic equalities $\delta(\alpha) = 0, \ \omega = 0$ (see also [9]).

Since asymptotic limit sets appear, there is not even an absolutely continuous function that is the density of the measure of the phase plane (cf. with [7, 8]). But it is possible (along with the first integral) to present an invariant differential 2-form with coefficients that are transcendental functions.

Indeed, the desired 2-form has the form

$$T(\alpha, \omega) = \exp\{-b\Theta(t)\} \, d\alpha \wedge d\omega, \ \Theta(t) = \int \frac{dt}{t^2 - bt + \lambda}, \ t = \frac{\omega}{\delta(\alpha)}. \tag{13}$$

4 Invariants of Systems of Geodesic Equations

Consider a smooth two-dimensional Riemannian manifold $M^2\{\alpha, \beta\}$ with affine connectivity $\Gamma^i_{jk}(\alpha, \beta)$ and study the structure of the equations of geodesic lines on the tangent bundle $TM^2\{\dot{\alpha}, \dot{\beta}; \alpha, \beta\}$ (cf. with [11, 12]). To do this , we will further study a fairly general case of setting kinematic relations in the following form:

$$\dot{\alpha} = z_2 f_2(\alpha), \ \dot{\beta} = z_1 f_1(\alpha), \tag{14}$$

where $f_1(\alpha)$ and $f_2(\alpha)$ are sufficiently smooth functions that are not identically zero. Such coordinates z_1, z_2 in tangent space are introduced when geodesic equations are considered, for example, with three nonzero connectivity coefficients (in particular, on surfaces of rotation, Lobachevsky plane, etc.):

$$\ddot{\alpha} + \Gamma^{\alpha}_{\alpha\alpha}(\alpha, \beta)\dot{\alpha}^2 + \Gamma^{\alpha}_{\beta\beta}(\alpha, \beta)\dot{\beta}^2 = 0, \ \ddot{\beta} + 2\Gamma^{\beta}_{\alpha\beta}(\alpha, \beta)\dot{\alpha}\dot{\beta} = 0, \tag{15}$$

that is, the equalities are met

$$\Gamma^{\alpha}_{\alpha\beta}(\alpha,\beta) \equiv \Gamma^{\beta}_{\alpha\alpha}(\alpha,\beta) \equiv \Gamma^{\beta}_{\beta\beta}(\alpha,\beta) \equiv 0. \tag{16}$$

In the case of (14) the relations on the tangent bundle $TM^2\{z_2, z_1; \alpha, \beta\}$ will take the form

$$\begin{gathered}
\dot{z}_1 = -\frac{f_2^2(\alpha)}{f_1(\alpha)}\Gamma^{\beta}_{\alpha\alpha}(\alpha,\beta)z_2^2 - f_2(\alpha)\left[2\Gamma^{\beta}_{\alpha\beta}(\alpha,\beta) + \frac{d\ln|f_1(\alpha)|}{d\alpha}\right]z_1z_2- \\
-f_1(\alpha)\Gamma^{\beta}_{\beta\beta}(\alpha,\beta)z_1^2, \\
\dot{z}_2 = -f_2(\alpha)\left[\Gamma^{\alpha}_{\alpha\alpha}(\alpha,\beta) + \frac{d\ln|f_2(\alpha)|}{d\alpha}\right]z_2^2 - f_1(\alpha)\cdot 2\Gamma^{\alpha}_{\alpha\beta}(\alpha,\beta)z_1z_2- \\
-\frac{f_1^2(\alpha)}{f_2(\alpha)}\Gamma^{\alpha}_{\beta\beta}(\alpha,\beta)z_1^2,
\end{gathered} \tag{17}$$

and under the conditions (16) will simplify:

$$\begin{gathered}
\dot{z}_1 = -f_2(\alpha)\left[2\Gamma^{\beta}_{\alpha\beta}(\alpha,\beta) + \frac{d\ln|f_1(\alpha)|}{d\alpha}\right]z_1z_2, \\
\dot{z}_2 = -f_2(\alpha)\left[\Gamma^{\alpha}_{\alpha\alpha}(\alpha,\beta) + \frac{d\ln|f_2(\alpha)|}{d\alpha}\right]z_2^2 - \frac{f_1^2(\alpha)}{f_2(\alpha)}\Gamma^{\alpha}_{\beta\beta}(\alpha,\beta)z_1^2,
\end{gathered} \tag{18}$$

and the Eq. (15) geodesics are almost everywhere equivalent to a composite system (14), (18) on the manifold $TM^2\{z_2, z_1; \alpha, \beta\}$ with new coordinates z_1, z_2 on the tangent space.

To fully integrate the system (14), (18) it is necessary to know, generally speaking, three independent tensor invariants: either the first three integrals, or three independent differential forms, or some combination of integrals and forms. At the same time, of course, the first integrals (in particular, for geodesic equations) can be searched for in a more general form than discussed below.

In [6, 8] examples of geodesic systems on a two-dimensional sphere with various metrics are considered, and in [12] examples of geodesic systems on two-dimensional surfaces of rotation and on the Lobachevsky plane are considered too.

Theorem 1 *If the following conditions are satisfied*

$$\begin{gathered}
f_1^2(\alpha)\Gamma^{\alpha}_{\beta\beta}(\alpha,\beta) + f_2^2(\alpha)\left[2\Gamma^{\beta}_{\alpha\beta}(\alpha,\beta) + \frac{d\ln|f_1(\alpha)|}{d\alpha}\right] \equiv 0, \\
\Gamma^{\alpha}_{\alpha\alpha}(\alpha,\beta) + \frac{d\ln|f_2(\alpha)|}{d\alpha} \equiv 0,
\end{gathered} \tag{19}$$

$$\Gamma^{\beta}_{\alpha\beta}(\alpha,\beta) = \Gamma^{\beta}_{\alpha\beta}(\alpha), \tag{20}$$

then the system (14), (18) has a complete set consisting of the first three integrals of the form

$$\Phi_1(z_2, z_1) = z_1^2 + z_2^2 = C_1^2 = const, \tag{21}$$

$$\Phi_2(z_1; \alpha) = z_1\Phi_0(\alpha) = C_2 = const, \;\; \Phi_0(\alpha) = f_1(\alpha)\exp\left\{2\int\limits_{\alpha_0}^{\alpha}\Gamma^{\beta}_{\alpha\beta}(b)db\right\}, \tag{22}$$

$$\Phi_3(\alpha, \beta) = \beta \mp \int\limits_{\alpha_0}^{\alpha} \frac{C_2 f_1(b)}{f_2(b)\sqrt{C_1^2 \Phi_0^2(b) - C_2^2}}\, db = C_3 = const. \tag{23}$$

Moreover, after some reduction of that system, replacing the independent variable

$$\frac{d}{dt} = f_2(\alpha)\frac{d}{d\tau}, \tag{24}$$

and phase one

$$z_1^* = \ln |z_1|, \tag{25}$$

the phase flow of the system (14), (18) preserves the volume on the tangent bundle $TM^2\{z_2, z_1^*; \alpha, \beta\}$, i.e. the corresponding differential form is preserved:

$$dz_2 \wedge dz_1^* \wedge d\alpha \wedge d\beta. \tag{26}$$

The system (19) can be interpreted as the possibility of converting the quadratic form of the metric to a canonical form with the law of conservation of energy (21) (or see below (30)) depending on the problem under consideration. The history and current state of consideration of this more general problem are quite extensive (we note only the works of [12, 13]). Well, the search for both the integral (21) and (22) relies on the presence of additional symmetry groups in the system [5, 6].

5 Invariants of Potential Systems

We modify the system somewhat (14), (18), introducing into it a conservative smooth force field in projections on the axis $\dot{z}_1, \dot{z}_2$, respectively:

$$\tilde{F}(z_2, z_1; \alpha) = \begin{pmatrix} F_1(\beta) f_1(\alpha) \\ F_2(\alpha) f_2(\alpha) \end{pmatrix}. \tag{27}$$

The system under consideration on the tangent bundle $TM^2\{z_2, z_1; \alpha, \beta\}$ will take the form

$$\begin{cases} \dot{\alpha} = z_2 f_2(\alpha), \\ \dot{z}_2 = F_2(\alpha) f_2(\alpha) - f_2(\alpha)\left[\Gamma^{\alpha}_{\alpha\alpha}(\alpha, \beta) + \frac{d \ln |f_2(\alpha)|}{d\alpha}\right] z_2^2 - \\ \quad - \frac{f_1^2(\alpha)}{f_2(\alpha)} \Gamma^{\alpha}_{\beta\beta}(\alpha, \beta) z_1^2, \\ \dot{z}_1 = F_1(\beta) f_1(\alpha) - f_2(\alpha)\left[2\Gamma^{\beta}_{\alpha\beta}(\alpha, \beta) + \frac{d \ln |f_1(\alpha)|}{d\alpha}\right] z_1 z_2, \\ \dot{\beta} = z_1 f_1(\alpha), \end{cases} \tag{28}$$

and it is almost everywhere equivalent to the following system:

$$\begin{aligned} \ddot{\alpha} - F_2(\alpha) f_2(\alpha) + \Gamma^{\alpha}_{\alpha\alpha}(\alpha, \beta)\dot{\alpha}^2 + \Gamma^{\alpha}_{\beta\beta}(\alpha, \beta)\dot{\beta}^2 = 0, \\ \ddot{\beta} - F_1(\beta) f_1(\alpha) + 2\Gamma^{\beta}_{\alpha\beta}(\alpha, \beta)\dot{\alpha}\dot{\beta} = 0, \end{aligned} \tag{29}$$

on the tangent bundle $TM^2\{\dot{\alpha}, \dot{\beta}; \alpha, \beta\}$.

Theorem 2 *If the conditions (19), (20) are satisfied, then the system (28) has a complete set consisting of the first three integrals of the form*

$$\Phi_1(z_2, z_1) = z_1^2 + z_2^2 + V(\alpha, \beta) = C_1 = const, \tag{30}$$

$$V(\alpha, \beta) = V_2(\alpha) + V_1(\beta) = -2\int\limits_{\alpha_0}^{\alpha} F_2(a)da - 2\int\limits_{\beta_0}^{\beta} F_1(b)db, \tag{31}$$

and also with $F_1(\beta) \equiv 0$—by the first integral (22) and

$$\Phi_3(\alpha, \beta) = \beta \mp \int\limits_{\alpha_0}^{\alpha} \frac{C_2 f_1(b)}{f_2(b)\sqrt{\Phi_0^2(b)[C_1 - V(b, \beta_0)] - C_2^2}}\, db = C_3 = const. \tag{32}$$

Moreover, after some reduction of that system, i.e. replacing the independent variable

$$\frac{d}{dt} = f_2(\alpha)\frac{d}{d\tau}, \tag{33}$$

and phase one

$$z_1^* = \ln|z_1|, \tag{34}$$

the phase flow of the system (28) preserves the volume on the tangent bundle $TM^2\{z_2, z_1^*; \alpha, \beta\}$, i.e. the corresponding differential form is preserved:

$$dz_2 \wedge dz_1^* \wedge d\alpha \wedge d\beta. \tag{35}$$

6 Invariants of Systems with Alternating Dissipation

Next, we modify the system somewhat (28) by introducing a smooth force field with dissipation into it. Its presence (generally speaking, alternating signs) characterizes not only the coefficient $b\delta(\alpha)$, $b > 0$, in the first equation of the system (37) (unlike the system (28)), but also the following dependence of the (external) force field in projections on the axis $\dot{z}_1$, $\dot{z}_2$, respectively:

$$\tilde{F}(z_2, z_1; \alpha, \beta) = \begin{pmatrix} F_1(\beta) f_1(\alpha) \\ F_2(\alpha) f_2(\alpha) \end{pmatrix} + \begin{pmatrix} z_1 F_1^1(\alpha) \\ z_2 F_2^1(\alpha) \end{pmatrix}. \tag{36}$$

The system under consideration on the tangent bundle $TM^2\{z_2, z_1; \alpha, \beta\}$ will take the form

$$\begin{cases} \dot{\alpha} = z_2 f_2(\alpha) + b\delta(\alpha), \\ \dot{z}_2 = F_2(\alpha) f_2(\alpha) - f_2(\alpha)\left[\Gamma^{\alpha}_{\alpha\alpha}(\alpha, \beta) + \frac{d \ln |f_2(\alpha)|}{d\alpha}\right] z_2^2 - \\ -\frac{f_1^2(\alpha)}{f_2(\alpha)} \Gamma^{\alpha}_{\beta\beta}(\alpha, \beta) z_1^2 + z_2 F_2^1(\alpha), \\ \dot{z}_1 = F_1(\beta) f_1(\alpha) - f_2(\alpha)\left[2\Gamma^{\beta}_{\alpha\beta}(\alpha, \beta) + \frac{d \ln |f_1(\alpha)|}{d\alpha}\right] z_1 z_2 + z_1 F_1^1(\alpha), \\ \dot{\beta} = z_1 f_1(\alpha), \end{cases} \tag{37}$$

and it is almost everywhere equivalent to the following system:

$$\begin{cases} \ddot{\alpha} - \left\{b\tilde{\delta}(\alpha) + F_2^1(\alpha) + b\delta(\alpha)\left[2\Gamma^{\alpha}_{\alpha\alpha}(\alpha, \beta) + \frac{d \ln |f_2(\alpha)|}{d\alpha}\right]\right\}\dot{\alpha} - \\ -F_2(\alpha) f_2^2(\alpha) + b\delta(\alpha) F_2^1(\alpha) + b^2\delta^2(\alpha)\left[\Gamma^{\alpha}_{\alpha\alpha}(\alpha, \beta) + \frac{d \ln |f_2(\alpha)|}{d\alpha}\right] + \\ +\Gamma^{\alpha}_{\alpha\alpha}(\alpha, \beta)\dot{\alpha}^2 + \Gamma^{\alpha}_{\beta\beta}(\alpha, \beta)\dot{\beta}^2 = 0, \\ \ddot{\beta} - \left\{F_1^1(\alpha) + b\delta(\alpha)\left[2\Gamma^{\beta}_{\alpha\beta}(\alpha, \beta) + \frac{d \ln |f_1(\alpha)|}{d\alpha}\right]\right\}\dot{\beta} - \\ -F_1(\beta) f_1^2(\alpha) + 2\Gamma^{\beta}_{\alpha\beta}(\alpha, \beta)\dot{\alpha}\dot{\beta} = 0, \end{cases} \tag{38}$$

on the tangent bundle $TM^2\{\dot{\alpha}, \dot{\beta}; \alpha, \beta\}$. Here, as above,

$$\tilde{\delta}(\alpha) = \frac{d\delta(\alpha)}{d\alpha}. \tag{39}$$

We will integrate the fourth-order system (37) when performing the properties (19), (20), as well as when $F_1(\beta) \equiv 0$. At the same time, an independent subsystem of the third order is separated:

$$\begin{cases} \dot{\alpha} = z_2 f_2(\alpha) + b\delta(\alpha), \\ \dot{z}_2 = F_2(\alpha) f_2(\alpha) - \frac{f_1^2(\alpha)}{f_2(\alpha)} \Gamma^{\alpha}_{\beta\beta}(\alpha) z_1^2 + z_2 F_2^1(\alpha), \\ \dot{z}_1 = \frac{f_1^2(\alpha)}{f_2(\alpha)} \Gamma^{\alpha}_{\beta\beta}(\alpha) z_1 z_2 + z_1 F_1^1(\alpha), \end{cases} \tag{40}$$

if there is also a fourth equation

$$\dot{\beta} = z_1 f(\alpha). \tag{41}$$

We will also assume that for some $\kappa \in \mathbf{R}$ the equality is satisfied

$$\Gamma^{\alpha}_{\beta\beta}(\alpha)\frac{f_1^2(\alpha)}{f_2^2(\alpha)} = \kappa\frac{d}{d\alpha}\ln|\Delta(\alpha)| = \kappa\frac{\tilde{\Delta}(\alpha)}{\Delta(\alpha)},\ \tilde{\Delta}(\alpha) = \frac{d\Delta(\alpha)}{d\alpha},\ \Delta(\alpha) = \frac{\delta(\alpha)}{f_2(\alpha)}, \tag{42}$$

and for some $\lambda_2^0, \lambda_k^1 \in \mathbf{R}$, $k = 1, 2$, the equalities must be met

$$\begin{aligned} F_2(\alpha) &= \lambda_2^0 \tfrac{d}{d\alpha}\tfrac{\Delta^2(\alpha)}{2} = \lambda_2^0\tilde{\Delta}(\alpha)\Delta(\alpha); \\ F_k^1(\alpha) &= f_2(\alpha)\tfrac{d}{d\alpha}\Delta(\alpha) = \lambda_k^1\tilde{\Delta}(\alpha) f_2(\alpha),\ k = 1, 2. \end{aligned} \tag{43}$$

Condition (42) let's call it 'geometric', and the conditions from the group (43)—'energetic'.

Condition (42) it is called geometric, among other things, because it imposes a condition on the key coefficient of connectivity $\Gamma^{\alpha}_{\beta\beta}$, bringing the corresponding coefficients of the system to a homogeneous form with respect to the function $\Delta(\alpha)$. The conditions of the group (43) are called energetic, among other things, because the forces become, in a sense, 'potential' with respect to the functions of $\Delta^2(\alpha)/2$ and $\Delta(\alpha)$, bringing the corresponding coefficients of the system to a homogeneous form also with respect to the function $\Delta(\alpha)$ (see also [9]).

Theorem 3 *Let the conditions (42) and (43) be satisfied. Then the system (40), (41) has three independent, generally speaking, transcendental [4, 10] first integrals.*

In general, the first integrals are written out cumbersomely (since it is necessary to integrate the Abel equation [14]). In particular, if $\kappa = -1$, $\lambda_1^1 = \lambda_2^1$, the explicit form of the key first integral is:

$$\begin{aligned} &\Theta_1(z_2, z_1; \alpha) = G_1\left(\tfrac{z_2}{\Delta(\alpha)}, \tfrac{z_1}{\Delta(\alpha)}\right) = \\ &= \tfrac{f_2^2(\alpha)(z_2^2+z_1^2)+(b-\lambda_1^1)z_2\delta(\alpha)f_2(\alpha)-\lambda_2^0\delta^2(\alpha)}{z_1\delta(\alpha)f_2(\alpha)} = C_1 = \text{const}. \end{aligned} \tag{44}$$

In this case , the additional first integrals have the following structures:

$$\Theta_2(z_2, z_1; \alpha) = G_2\left(\Delta(\alpha), \frac{z_2}{\Delta(\alpha)}, \frac{z_1}{\Delta(\alpha)}\right) = C_2 = \text{const}, \tag{45}$$

$$\Theta_3(z_2, z_1; \alpha, \beta) = G_3\left(\Delta(\alpha), \beta, \frac{z_2}{\Delta(\alpha)}, \frac{z_1}{\Delta(\alpha)}\right) = C_3 = \text{const}. \tag{46}$$

The expression of functions (44)–(46) through a finite combination of elementary functions also depends on the explicit form of the function $\Delta(\alpha)$. So, for example, with $\kappa = -1$, $\lambda_1^1 = \lambda_2^1$ the additional first integral of the system (40) is found from the differential relation

$$
\begin{gathered}
d\ln|\Delta(\alpha)| = \frac{(b+u_2)du_2}{U_2(C_1,u_2)},\ u_2 = \frac{z_2}{\Delta(\alpha)},\ u_1 = \frac{z_1}{\Delta(\alpha)},\\
U_1(u_2) = u_2^2 + (b-\lambda_1^1)u_2 - \lambda_2^0,\\
U_2(C_1,u_2) = 2U_1(u_2) - \frac{C_1}{2}\left\{C_1 \pm \sqrt{C_1^2 - 4U_1(u_2)}\right\},\ C_1 \neq 0.
\end{gathered}
\tag{47}
$$

The right part of this relation is expressed in terms of a finite combination of elementary functions, and the left—depending on the function $\Delta(\alpha)$.

Theorem 4 *If for systems of the form (40), (41) there are the first integrals of the form (44) to (46), then it also has the following three functionally independent invariant differential forms with transcendental coefficients:*

$$
\begin{gathered}
\rho_1(z_2,z_1;\alpha)dz_2\wedge dz_1\wedge d\alpha,\\
\rho_1(z_2,z_1;\alpha) = \exp\left\{(b+\lambda_1^1)\int\frac{du_2}{U_2(C_1,u_2)}\right\}\cdot\frac{u_2^2+u_1^2-(b-\lambda_1^1)u_2-\lambda_2^0}{u_1},\\
\rho_2(z_2,z_1;\alpha)dz_2\wedge dz_1\wedge d\alpha,\\
\rho_2(z_2,z_1;\alpha) = \Delta(\alpha)\exp\left\{(b+\lambda_1^1)\int\frac{du_2}{U_2(C_1,u_2)}\right\}\cdot\exp\left\{-\int\frac{(b+u_2)du_2}{U_2(C_1,u_2)}\right\},\\
\rho_3(z_2,z_1;\alpha,\beta)dz_2\wedge dz_1\wedge d\alpha\wedge d\beta,\\
\rho_3(z_2,z_1;\alpha,\beta) = \exp\left\{(b+\lambda_1^1)\int\frac{du_2}{U_2(C_1,u_2)}\right\}\cdot G_3\left(\Delta(\alpha),\beta,\frac{z_2}{\Delta(\alpha)},\frac{z_1}{\Delta(\alpha)}\right),
\end{gathered}
\tag{48}
$$

but dependent with the first integrals (44)–(46).

For the complete integrability of the system (40), (41), you can use either the first three integrals, or three independent differential forms, or some combination (only independent elements) of integrals and forms (cf. with [2, 3, 15]).

On the structure of the first integrals for the systems under consideration with dissipation, see also [5, 6, 8]. Note only that for systems with dissipation, the transcendence of functions (in the sense of having essentially singular points) as the first integrals, it is inherited from the presence of attracting or repelling limit sets in the system.

In conclusion, we can refer to numerous applications concerning the integration of systems with dissipation, on the tangent bundle to a two-dimensional sphere, as well as more general systems on the bundle of two-dimensional surfaces of rotation and the Lobachevsky plane [15, 16].

7 Spatial Pendulum in the Flow of the Incoming Medium

Let us briefly describe the problem of a physical pendulum on a spherical hinge in the flow of an incoming medium, started in [8]. The position space of such a pendulum is a two-dimensional sphere $\mathbf{S}^2\{0\le\xi\le\pi,\ \eta \bmod 2\pi\}$, phase space—tangent bundle $T\mathbf{S}^2\{\dot\xi,\dot\eta;\ 0\le\xi\le\pi,\ \eta \bmod 2\pi\}$ to it.

Under the considered model assumptions, the equations of motion of such a pendulum are written out. Further, the statement is proved that the dynamical system describing the behavior of such a pendulum is trajectorically topologically equivalent to the following dynamical system on the tangent bundle of a two-dimensional sphere (the angle ξ is measured "along the flow"):

$$\begin{cases} \ddot{\xi} + b\dot{\xi}\cos\xi + \sin\xi\cos\xi - \dot{\eta}^2\frac{\sin\xi}{\cos\xi} = 0, \\ \ddot{\eta} + b\dot{\eta}\cos\xi + \dot{\xi}\dot{\eta}\frac{1+\cos^2\xi}{\cos\xi\sin\xi} = 0, \ b > 0. \end{cases} \tag{49}$$

The system (49) is almost everywhere equivalent to the system

$$\begin{cases} \dot{\xi} = -w_2 - b\sin\xi, \\ \dot{w}_2 = \sin\xi\cos\xi - w_1^2\frac{\cos\xi}{\sin\xi}, \\ \dot{w}_1 = w_1 w_2\frac{\cos\xi}{\sin\xi}, \end{cases} \tag{50}$$

$$\dot{\eta} = w_1\frac{\cos\xi}{\sin\xi}, \tag{51}$$

on the tangent bundle $T_*\mathbf{S}^2\{(w_2, w_1; \xi, \eta_1) \in \mathbf{R}^4 : \ 0 \leq \xi \leq \pi, \ \eta_1 \bmod 2\pi\}$ of two-dimensional sphere $\mathbf{S}^2\{(\xi, \eta_1) \in \mathbf{R}^2 : \ 0 \leq \xi \leq \pi, \ \eta_1 \bmod 2\pi\}$.

It can be seen that in the fourth-order system (50), (51), due to the cyclicity of the variable η, an independent third-order subsystem (50) is allocated, which can be independently considered on its three-dimensional manifold.

The key first integral of the system (50), (51) has the following form:

$$\Theta_1(w_2, w_1; \xi) = \frac{w_2^2 + w_1^2 + bw_2\sin\xi + \sin^2\xi}{w_1\sin\xi} = C_1 = \text{const}. \tag{52}$$

Remark 1 Consider a system (50) with variable dissipation with zero mean [5, 6, 8] becoming conservative at $b = 0$:

$$\begin{aligned} \dot{\xi} &= -w_2, \\ \dot{w}_2 &= \sin\xi\cos\xi - w_1^2\frac{\cos\alpha}{\sin\alpha}, \\ \dot{w}_1 &= w_1 w_2\frac{\cos\xi}{\sin\xi}. \end{aligned} \tag{53}$$

It has two analytic first integrals of the form

$$w_2^2 + w_1^2 + \sin^2\xi = C_1^* = \text{const}, \tag{54}$$

$$w_1\sin\xi = C_2^* = \text{const}. \tag{55}$$

Obviously, the ratio of two integrals (54), (55) it is also the first integral of the system (53). But with $b \neq 0$ each of the functions

$$w_2^2 + w_1^2 + bw_2 \sin\xi + \sin^2\xi \tag{56}$$

and (55) separately is not the first integral of the system (50). However, the ratio of functions (56), (55) is the first integral of the system (50) for any b.

The additional first integral of the system (50) is expressed in terms of a finite combination of elementary functions and has the following form (due to the bulkiness, we will write out the structural form):

$$\Theta_2(w_2, w_1; \xi) = G\left(\sin\xi, \frac{w_2}{\sin\xi}, \frac{w_1}{\sin\xi}\right) = C_2 = \text{const}. \tag{57}$$

Another (additional) first integral that 'binds' the Eq. (51) can be represented as

$$\Theta_3(w_2, w_1; \xi, \eta) = -\eta \pm \frac{1}{2}\text{arctg}\frac{w_1^2 - w_2^2 - bw_2\sin\xi - \sin^2\xi}{w_1(2w_2 + b\sin\xi)} = C_3 = \text{const}. \tag{58}$$

In the case under consideration, the system of dynamic equations (50), (51) has the first three integrals expressed by the relations (52), (57), (58), which are transcendental functions of phase variables (in the sense of complex analysis) and expressed in terms of a finite combination of elementary functions.

It is also possible to present invariant differential forms for the system of dynamic equations under consideration:

$$\begin{gathered}
\rho_1(w_2, w_1; \xi) dw_2 \wedge dw_1 \wedge d\xi, \\
\rho_1(w_2, w_1; \xi) = \exp\left\{b\int \frac{du_2}{U_2(C_1, u_2)}\right\} \cdot \frac{u_2^2 + u_1^2 + bu_2 + 1}{u_1}, \\
\rho_2(w_2, w_1; \xi) dw_2 \wedge dw_1 \wedge d\xi, \\
\rho_2(w_2, w_1; \xi) = \sin\xi \exp\left\{b\int \frac{du_2}{U_2(C_1, u_2)}\right\} \cdot \exp\left\{-\int \frac{(b+u_2)du_2}{U_2(C_1, u_2)}\right\}, \\
\rho_3(w_2, w_1; \xi, \eta) dw_2 \wedge dw_1 \wedge d\xi \wedge d\eta, \\
\rho_3(w_2, w_1; \xi, \eta) = \exp\left\{b\int \frac{du_2}{U_2(C_1, u_2)}\right\} \cdot \Theta_3(w_2, w_1; \xi, \eta), \\
u_2 = \frac{w_2}{\sin\xi}, \; u_1 = \frac{w_1}{\sin\xi}, \\
U_1(u_2) = u_2^2 + bu_2 + 1, \\
U_2(C_1, u_2) = 2U_1(u_2) - \frac{C_1}{2}\left\{C_1 \pm \sqrt{C_1^2 - 4U_1(u_2)}\right\}, \; C_1 \neq 0.
\end{gathered} \tag{59}$$

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