Chapter 8
Existence, Regularity, and Stability in the α-Norm for Some Neutral Partial Functional Differential Equations in Fading Memory Spaces

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Abstract The aim of this chapter is to study the regularity and the stability in the α-norm for neutral partial functional differential equations in fading memory spaces. We assume that a linear part is densely defined and generates an analytic semigroup. The delayed part is assumed to be Lipschitzian. For illustration, we provide an example for some reaction–diffusion equation involving infinite delay.

Keywords Analytic semigroup · Neutral partial functional differential equations · α-norm · Stability · Fading memory space

8.1 Introduction

Let $(X, |.|)$ be a Banach space, $(\mathscr{L}(X), |.|_{\mathscr{L}})$ be the space of bounded linear operators on X, and α be a constant such that $0 < \alpha < 1$. The aim of this chapter is to study the stability results of the following class of neutral partial functional differential equations in the α-norm in fading memory spaces

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$$\begin{cases} \frac{d}{dt}\mathcal{D}(u_t) = -A\mathcal{D}(u_t) + f(u_t) \quad \text{for} \quad t \geq 0, \\ u_0 = \phi \in \mathcal{B}_\alpha, \end{cases} \tag{8.1}$$

where $f : \mathcal{B}_\alpha \to X$ is a continuous function and $A : D(A) \subseteq X \to X$ is a linear operator such that $(-A)$ generates an analytic semigroup $(T(t))_{t\geq 0}$ on the Banach space X. $D(A)$ is the domain of the operator A. We also denote $R(A)$ the range of the operator A. For $0 < \alpha < 1$, A^α denotes the fractional power of A, and the space X_α will be defined later. The initial function ϕ belongs to a Banach space $\mathcal{B}_\alpha$ of functions mapping $(-\infty, 0]$ into X_α and satisfying some axioms to be introduced later. $\mathcal{D}$ is a bounded linear operator defined on $\mathcal{B}_\alpha$ with values in X as follows:

$$\mathcal{D}(\phi) = \phi(0) - \mathcal{D}_0(\phi) \ \text{ for } \phi \in \mathcal{B}_\alpha, \tag{8.2}$$

where $\mathcal{D}_0$ is also a bounded linear operator defined on $\mathcal{B}_\alpha$ with values in X.

We denote by u_t for $t \in \mathbb{R}^+$ the historic function defined on $(-\infty, 0]$ by

$$u_t(\theta) = u(t + \theta) \quad \text{for all} \quad \theta \leq 0,$$

where u is a function from $\mathbb{R}$ into X_α.

The existence results of neutral partial functional differential equations with delay are an important subject studied by many authors (see [1, 3, 5, 6, 8, 11, 20] and the references therein). One of the qualitative behaviours of solutions of neutral partial functional differential equations with delay developed in many works is the stability (see [2, 4, 7, 9, 10, 15, 21, 22] and the references therein).

One of the most important qualitative results of the functional partial differential equations is the stability, extensively studied by many authors. A mechanical or an electrical device can be constructed to a level of perfect accuracy that is restricted by technical, economic, or environmental constraints. What happens to the expected result if the construction is a little off specifications? Does output remain near design values? How sensitive is the design to variations in fabrication parameters? Stability theory gives some answers to these and similar questions.

Adimy and Ezzinbi in [4] established the stability results in the α-norm for the problem of neutral type of the form

$$\begin{cases} \frac{d}{dt}\mathcal{D}(u_t) = -A\mathcal{D}(u_t) + f(u_t) \quad \text{for} \quad t \geq 0, \\ u_0 = \phi \in C_\alpha, \end{cases}$$

where $f : \mathbb{R} \times C_\alpha \to X$ is a continuous function and $A : D(A) \subseteq X \to X$ is a linear operator;

u_t for $t \in \mathbb{R}$ is the historic function defined on $[-r, 0]$ with $r > 0$ by $u_t(\theta) = u(t+\theta)$ for $\theta \in [-r, 0]$, where u is a continuous function from $\mathbb{R}$ into X_α; $C_\alpha = C([-r, 0]; D(A^\alpha))$ is the space of continuous functions from $[-r, 0]$ into $D(A^\alpha)$ provided with the uniform norm topology, $\mathcal{D}$ is a bounded linear operator from $C = C([-r, 0]; X)$ into X defined by

$$\mathcal{D}(\phi) = \phi(0) - \mathcal{D}_0(\phi) \ \text{ for } \phi \in C,$$

where the operator $\mathcal{D}_0$ is given by

$$\mathcal{D}_0(\phi) = \int_{-r}^{0} d\eta(\theta)\phi(\theta) \ \text{ for } \phi \in C,$$

and $\eta : [-r, 0] \to \mathscr{L}(X)$ is of bounded variation and non-atomic at zero, that is, there exists a continuous nondecreasing function $\delta : [0, r] \to [0, +\infty)$ such that $\delta(0) = 0$ and

$$\left| \int_{-s}^{0} d\eta(\theta)\phi(\theta) \right| \leq \delta(s) \, |\phi|_C \ \text{ for } \phi \in C \quad \text{and } \ s \in [0, r].$$

In our work, we study the stability results of Eq. (8.1) following the results obtained in [2, 4, 7, 9, 10, 21].

To get some stability results in the uniform fading memory spaces, we make use of the spectral theory of linear operators, the fractional power operators, and the linear semigroup theory (see [13, 19]).

The organization of this chapter is as follows: In Sect. 8.2, we introduce some preliminary results on analytic semigroups, fractional powers of operator, and axiomatic phase space adapted to the fractional norm space for infinite delay. In Sect. 8.3, the existence and uniqueness of strict solutions is established. In Sect. 8.4, we are concerned with the smoothness results of the solutions. In Sect. 8.5, we investigate the stability near an equilibrium by using the linearized principle. In the last section, an example is provided to illustrate the applications of the main results of this chapter.

8.2 Analytic Semigroup, Fractional Power of Its Generator, and Partial Functional Differential Equations

Throughout this chapter, we assume the following:

($\mathbf{H}_1$) $(-A)$ is the infinitesimal generator of an analytic semigroup of linear operators $\{T(t)\}_{t\geq 0}$ on a Banach space X. Without loss of generality, we suppose that $0 \in \rho(A)$; otherwise, instead of A, we take $A - \delta I$, where δ is chosen such that $0 \in \rho(A - \delta I)$ and where $\rho(A)$ is the resolvent set of A.

It is well-known that $|T(t)x| \leq Me^{\omega t}|x|$ for all $t \geq 0$, $x \in X$, where $M \geq 1$ and $\omega \in \mathbb{R}$.

For all $0 < \alpha < 1$, we define (see [19]) the operator $A^{-\alpha}$ by

$$A^{-\alpha}x = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} T(t)x dt \text{ for all } x \in X,$$

where $\Gamma(\alpha)$ denotes the well-known gamma function at the point α. The operator $A^{-\alpha}$ is bijective, and the operator A^{α} is defined by

$$A^{\alpha} = (A^{-\alpha})^{-1}.$$

We denote by $D(A^{\alpha})$ the domain of the operator A^{α}. Then, $D(A^{\alpha})$ endowed with the norm $|x|_{\alpha} = |A^{\alpha}x|$ for all $x \in D(A^{\alpha})$ is a Banach space [19]. We denote it by X_{α}. Moreover, we recall the following known results.

Theorem 8.2.1 ([19], p.69–75) *Let* $0 < \alpha < 1$, *and assume that* $(\mathbf{H}_1)$ *holds. Then:*

(a) $T(t) : X \to D(A^{\alpha})$ *for each* $t > 0$ *and* $\alpha \geq 0$.
(b) *For all* $x \in D(A^{\alpha})$, $T(t)A^{\alpha}x = A^{\alpha}T(t)x$.
(c) *For each* $t > 0$, *the linear operator* $A^{\alpha}T(t)$ *is bounded and* $|A^{\alpha}T(t)x| \leq M_{\alpha}t^{-\alpha}e^{\omega t}|x|$, *where* M_{α} *is a positive real constant.*
(d) *For* $0 < \alpha \leq 1$ *and* $x \in D(A^{\alpha})$, $|T(t)x - x| \leq N_{\alpha}t^{\alpha}|A^{\alpha}x|$, *for* $t > 0$, *where* N_{α} *is a positive real constant.*
(e) *For* $0 < \alpha < \beta < 1$, $X_{\beta} \hookrightarrow X_{\alpha}$.

From now on, we use an axiomatic definition of the phase space $\mathcal{B}$ that was first introduced by Hale and Kato in [16]. We assume that $\mathcal{B}$ is the normed space of functions mapping $(-\infty, 0]$ into X and satisfying the following axioms:

(**A**) There exist a positive constant N, a locally bounded continuous function $M(.)$ on $[0, +\infty)$, and a continuous function $K(.)$ on $[0, +\infty)$, such that if $u : (-\infty, a] \to X$ is continuous on $[\xi, a]$ with $u_{\xi} \in \mathcal{B}$ for some $\xi < a$ where $0 < a$, then for all $t \in [\xi, a]$:

  (i) $u_t \in \mathcal{B}$.
  (ii) $t \to u_t$ is continuous on $[\xi, a]$.
  (iii) $N|u(t)| \leq |u_t|_{\mathcal{B}} \leq K(t-\xi) \sup_{\xi \leq s \leq t} |u(s)| + M(t-\xi)|u_{\xi}|_{\mathcal{B}}$.

(**B**) $\mathcal{B}$ is a Banach space.

Lemma 8.2.1 ([7]) *Let* C_{00} *be the space of continuous functions mapping* $(-\infty, 0]$ *into* X *with compact supports and* C_{00}^{a} *be the subspace of functions in* C_{00} *with supports included in* $[-a, 0]$ *endowed with the uniform norm topology. Then* $C_{00}^{a} \hookrightarrow \mathcal{B}$. □

Let

$$\mathcal{B}_\alpha = \left\{\phi \in \mathcal{B} : \ \phi(\theta) \in D(A^\alpha) \text{ for } \theta \leq 0 \text{ and } A^\alpha \phi \in \mathcal{B}\right\}.$$

and provide $\mathcal{B}_\alpha$ with the following norm:

$$|\phi|_{\mathcal{B}_\alpha} = |A^\alpha \phi|_{\mathcal{B}} \ \text{ for } \ \phi \in \mathcal{B}_\alpha.$$

We also assume that

(**H**$_2$) $A^{-\alpha}\phi \in \mathcal{B}$ for all $\phi \in \mathcal{B}$, where the function $A^{-\alpha}\phi$ is defined by

$$(A^{-\alpha}\phi)(\theta) = A^{-\alpha}(\phi(\theta)) \ \text{ for } \theta \leq 0$$

and

(**H**$_3$) $K(0)|\mathcal{D}_0| < 1$.

Lemma 8.2.2 ([7]) *Assume that* (**H**$_1$) *and* (**H**$_2$) *hold. Then,* $\mathcal{B}_\alpha$ *is a Banach space and satisfies the axiom* (**A**). □

For regularity results in the Banach space X, consider the following problem:

$$\begin{cases} \dfrac{d}{dt}\mathcal{D}(u_t) = -A\mathcal{D}(u_t) + f(t) \quad \text{for} \quad t \geq 0, \\ \\ u_0 = \phi. \end{cases} \tag{8.3}$$

Definition 8.2.1 Let $\phi \in \mathcal{B}$. A function $u : (-\infty, a] \to X$ is called a mild solution of Eq. (8.3) associated to ϕ if

$$\begin{cases} \mathcal{D}(u_t) = T(t)\mathcal{D}(u_0) + \displaystyle\int_0^t T(t-s)f(s)ds \ \text{ for } t \in [0, a] \\ \\ u_0 = \phi. \end{cases}$$

Definition 8.2.2 Let $\phi \in \mathcal{B}$. A function $u : (-\infty, a] \to X$ is called a strict solution of Eq. (8.3) associated to ϕ if

$$\begin{cases} t \longmapsto \mathcal{D}(u_t) \quad \text{is continuously differentiable on} \quad [0, a] \\ \\ \mathcal{D}(u_t) \in D(A) \quad \text{for} \quad t \geq 0 \\ \\ u(t) \quad \text{satisfies the system} \quad (8.3) \quad \text{for} \quad t \geq 0. \end{cases}$$

We have the following important result.

Theorem 8.2.2 *Let* $u_0 = \phi$, $\mathcal{D}(\phi) \in D(A)$, *and* $f \in C^1([0, a]; X)$. *The existence of a mild solution* u *of* (8.3) *on* $[0, a]$ *implies the existence of a strict solution of* (8.3) *on* $[0, a]$. □

Proof Let u be a mild solution of (8.3). Then,

$$\mathcal{D}(u_t) = T(t)\mathcal{D}(u_0) + \int_0^t T(t-s)f(s)ds \text{ for } t \in [0, a]. \tag{8.4}$$

Show that $t \mapsto \mathcal{D}(u_t)$ is continuously differentiable. We need to only examine the second term of the right-hand side of (8.4), which will be denoted by $v(t)$. It is well-known that $T(t-s) = -\frac{\partial}{\partial s}(T(t-s))(-A)^{-1}$ since $(-A)$ generates the analytic semigroup $(T(t))_{t\geq 0}$. Hence,

$$\begin{aligned} v(t) &= -\int_0^t \frac{\partial}{\partial s}(T(t-s))(-A)^{-1} f(s)ds \\ &= \Big[-(T(t-s))(-A)^{-1} f(s)\Big]_0^t + \int_0^t T(t-s)(-A)^{-1} f'(s)ds \\ &= -(-A)^{-1} f(t) + T(t)(-A)^{-1} f(0) + \int_0^t T(t-s)(-A)^{-1} f'(s)ds. \end{aligned}$$

Since $\lim_{h\to 0}\Big[\int_0^t \frac{T(t+h-s) - T(t-s)}{h}(-A)^{-1}f'(s)ds + \frac{1}{h}\int_t^{t+h} T(t-s)(-A)^{-1}$ $f'(s)ds\Big] = (-A)^{-1} f'(t) + \int_0^t T(t-s)f'(s)ds$, it is easy to see that

$$\frac{d}{dt}v(t) = T(t)f(0) + \int_0^t T(t-s)f'(s)ds. \tag{8.5}$$

Using Eq. (8.5) and the fact that $f \in C^1([0, a]; X)$ and the semigroup $(T(t))_{t\geq 0}$ is analytic, then $t \mapsto \frac{d}{dt}v(t)$ is continuous. Consequently, $t \mapsto \mathcal{D}(u_t)$ is continuously differentiable on $t \in [0, a]$.

Now, let us show that $\mathcal{D}(u_t) \in D(A)$. Since $T(t)\mathcal{D}(\phi) \in D(A)$, it remains to prove that $v(t) \in D(A)$. We use the relation (8.5) in order to obtain

$$\begin{aligned} \frac{d}{dt}v(t) &= T(t)f(0) + \int_0^t T(t-s)f'(s)ds \\ &= -Av(t) + f(t). \end{aligned}$$

Thus, $Av(t) = -\frac{d}{dt}v(t) + f(t)$ exists and $v(t) \in D(A)$.

To finish, let us prove that u verifies (8.3). Using (8.4), one can write

$$\begin{aligned}\frac{d}{dt}(\mathcal{D}(u_t)) &= T'(t)\mathcal{D}(\phi) + \int_0^t \frac{\partial}{\partial s}(T(t-s))f(s)ds + f(t)\\ &= -AT(t)\mathcal{D}(\phi) - A\int_0^t T(t-s)f(s)ds + f(t)\\ &= -A\left[T(t)\mathcal{D}(\phi) + \int_0^t T(t-s)f(s)ds\right] + f(t)\\ &= -A\mathcal{D}(u_t) + f(t).\end{aligned}$$

□

8.3 Existence and Uniqueness of Strict Solutions

Now, we give the notions of solutions that will be studied in our work.

Definition 8.3.1 Let $\phi \in \mathcal{B}_\alpha$. A function $u : (-\infty, +\infty) \to X_\alpha$ is called a mild solution of Eq. (8.1) associated to ϕ if:

(i) $\mathcal{D}(u_t) = T(t)\mathcal{D}(\phi) + \int_0^t T(t-s)f(u_s)ds$ for $t \geq 0$.
(ii) $u_0 = \phi$.

□

Definition 8.3.2 Let $\phi \in \mathcal{B}_\alpha$. A function $u : (-\infty, +\infty) \to X_\alpha$ is called a strict solution of Eq. (8.1) associated to ϕ if:

(i) $t \longmapsto \mathcal{D}(u_t)$ is continuously differentiable on $[0, +\infty)$.
(ii) $\mathcal{D}(u_t) \in D(A)$ for $t \geq 0$.
(iii) $u(t)$ satisfies the system (8.1) for $t \geq 0$.

□

Often in this chapter, $u_t(., \phi)$ and $u_t(\phi)$ denote the mild solution associated to the initial data ϕ, and we simply denote it by u_t if there is no confusion.

We assume that there exists $k > 0$ such that

$(\mathbf{H}_4)$ $\quad |f(\phi_1) - f(\phi_2)| \leq k|\phi_1 - \phi_2|_{\mathcal{B}_\alpha}$ for all $\phi_1, \phi_2 \in \mathcal{B}_\alpha$.

Theorem 8.3.1 ([14]) *Assume that* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, $(\mathbf{H}_3)$, *and* $(\mathbf{H}_4)$ *hold. Then, for each* $\phi \in \mathcal{B}_\alpha$, *there exists a unique mild solution of Eq.* (8.1) *that is defined for* $t \geq 0$.

Lemma 8.3.1 *Assume that* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, *and* $(\mathbf{H}_3)$ *hold. Let* $\phi \in \mathcal{B}_\alpha$ *and* $h \in C(\mathbb{R}^+; X_\alpha)$ *such that* $\mathcal{D}(\phi) = h(0)$. *Then, there exists a unique continuous function* x *on* $\mathbb{R}^+$ *that solves the following problem:*

$$\begin{cases} \mathcal{D}(x_t) = h(t) \quad for \quad t \geq 0, \\ \\ x(t) = \phi(t) \ \ for \ t \in (-\infty, 0]. \end{cases} \tag{8.6}$$

Moreover, there exist two functions a and b in $L^{\infty}_{loc}(\mathbb{R}^+; \mathbb{R}^+)$ *such that*

$$|x_t|_{\mathcal{B}_\alpha} \leq a(t)|\phi|_{\mathcal{B}_\alpha} + b(t) \sup_{0 \leq s \leq t} |h(s)|_\alpha \quad for \quad t \geq 0. \tag{8.7}$$

Proof We define for $p > 0$ the space

$$W = \{x \in C([0, p]; X_\alpha) : x(0) = \phi(0)\}$$

endowed with the uniform norm topology. For $x \in W$, we define its extension $\tilde{x}$ on $\mathbb{R}^-$ by

$$\tilde{x}(t) = \begin{cases} x(t) \quad \text{for} \quad t \in [0, p] \\ \\ \phi(t) \quad \text{for} \ \ t \in (-\infty, 0]. \end{cases}$$

Using axiom (**A**), one can see that $t \mapsto \tilde{x}_t$ is continuous from $[0, p]$ to $\mathcal{B}_\alpha$. Let us define the function $\mathcal{K}$ on W by

$$(\mathcal{K}(x))(t) = \mathcal{D}_0(\tilde{x}_t) + h(t) \quad \text{for} \ \ t \geq 0.$$

One must show that $\mathcal{K}$ has a unique fixed point on W. Since $h \in C(\mathbb{R}^+; X_\alpha)$, then $h \in C([0, p]; X_\alpha)$. Moreover, $h(0) = \mathcal{D}(\phi) = \phi(0) - \mathcal{D}_0(\phi)$. It follows that

$$\mathcal{K}(W) \subset W.$$

We can also write for $x, y \in W$ with their respective extensions $\tilde{x}$ and $\tilde{y}$ associated to ϕ

$$\begin{aligned} |(\mathcal{K}(x) - \mathcal{K}(y))(t)|_\alpha &\leq |\mathcal{D}_0||\tilde{x}_t - \tilde{y}_t|_{\mathcal{B}_\alpha} \\ &\leq |\mathcal{D}_0| K(t) \sup_{0 \leq s \leq t} |x(s) - y(s)|_\alpha \\ &\leq |\mathcal{D}_0| K(t) |x - y|_W. \end{aligned}$$

Choosing $p > 0$ small enough, one obtains that $\mathcal{K}$ is a strict contraction. Consequently, (8.6) has a unique solution x on $(-\infty, p]$.

It follows for $s \in [0, p]$ that

$$\begin{aligned} |x_s|_{\mathcal{B}_\alpha} &\le K(s) \sup_{0\le\tau\le s} |x(\tau)|_\alpha + M(s)|\phi|_{\mathcal{B}_\alpha} \\ &\le K(s)\Big(|\mathcal{D}_0| \sup_{0\le\tau\le s} |x_\tau|_{\mathcal{B}_\alpha} + \sup_{0\le\tau\le s} |h(\tau)|_\alpha\Big) + M(s)|\phi|_{\mathcal{B}_\alpha} \\ &\le K_p|\mathcal{D}_0| \sup_{0\le\tau\le s} |x_\tau|_{\mathcal{B}_\alpha} + K_p \sup_{0\le\tau\le s} |h(\tau)|_\alpha + M_p|\phi|_{\mathcal{B}_\alpha}, \end{aligned}$$

where $K_p = \sup_{s\in[0,p]} K(s)$ and $M_p = \sup_{s\in[0,p]} M(s)$.

Therefore,

$$\begin{aligned} \sup_{0\le s\le t} |x_s|_{\mathcal{B}_\alpha} &\le \sup_{0\le s\le t} \Big\{K_p|\mathcal{D}_0| \sup_{0\le\tau\le s} |x_\tau|_{\mathcal{B}_\alpha} + K_p \sup_{0\le\tau\le s} |h(\tau)|_\alpha + M_p|\phi|_{\mathcal{B}_\alpha}\Big\} \\ &\le K_p|\mathcal{D}_0| \sup_{0\le s\le t} |x_s|_{\mathcal{B}_\alpha} + K_p \sup_{0\le s\le t} |h(s)|_\alpha + M_p|\phi|_{\mathcal{B}_\alpha}. \end{aligned}$$

Thus, for $p > 0$ small enough and using ($\mathbf{H}_3$), one can write for $t \in [0, p]$,

$$\sup_{0\le s\le t} |x_s|_{\mathcal{B}_\alpha} \le \frac{K_p}{1 - K_p|\mathcal{D}_0|} \sup_{0\le s\le t} |h(s)|_\alpha + \frac{M_p}{1 - K_p|\mathcal{D}_0|}|\phi|_{\mathcal{B}_\alpha}.$$

As a consequence, we have the existence of $a, b \in L^\infty_{loc}([0, p]; \mathbb{R}^+)$ such that

$$|x_t|_{\mathcal{B}_\alpha} \le a(t)|\phi|_{\mathcal{B}_\alpha} + b(t) \sup_{0\le s\le t} |h(s)|_\alpha, \quad \text{for} \quad t \in [0, p].$$

Now, to extend the solution x on $[p, 2p]$, we consider the space

$$W_1 = \{u \in C([p, 2p]; X_\alpha) : u(p) = x(p)\}$$

endowed with the uniform norm topology and the following problem:

$$\tilde{u}(t) = \begin{cases} u(t) & \text{for } \ t \in [p, 2p], \\ \\ x(t) & \text{for } \ t \in (-\infty, p]. \end{cases}$$

We define the function $\mathcal{K}_1$ on W_1 by

$$(\mathcal{K}_1(u))(t) = \mathcal{D}_0(\tilde{u}_t) + h(t), \quad \text{for } \ t \in [p, 2p].$$

Using the same arguments as above, we show that $\mathcal{K}_1$ is a strict contraction on W_1. That leads to the existence of a unique solution u of (8.6) on $(-\infty, 2p]$, and u is the extension of x on $(-\infty, 2p]$.

Also, we have to extend a, b on $[p, 2p]$. Therefore, let $s \in [p, 2p]$. Then, one can write

$$
\begin{aligned}
|x_s|_{\mathcal{B}_\alpha} &\le K(s-p) \sup_{p \le \tau \le s} |x(\tau)|_\alpha + M(s-p) \left|x_p\right|_{\mathcal{B}_\alpha} \\
&\le K_p \sup_{p \le \tau \le s} |x(\tau)|_\alpha + M_p \left|x_p\right|_{\mathcal{B}_\alpha} \\
&\le K_p \sup_{p \le \tau \le s} \left\{ |\mathcal{D}_0| \, |x_\tau|_{\mathcal{B}_\alpha} + |h(\tau)|_\alpha \right\} + M_p \left|x_p\right|_{\mathcal{B}_\alpha} .
\end{aligned}
$$

Therefore, for each $t \in [p, 2p]$ such that $s \le t$, we have

$$
\begin{aligned}
\sup_{p \le s \le t} |x_s|_{\mathcal{B}_\alpha} &\le \sup_{p \le s \le t} \left\{ K_p \sup_{p \le \tau \le s} \left\{ |\mathcal{D}_0| \, |x_\tau|_{\mathcal{B}_\alpha} + |h(\tau)|_\alpha \right\} + M_p \left|x_p\right|_{\mathcal{B}_\alpha} \right\} \\
&\le K_p \, |\mathcal{D}_0| \sup_{p \le s \le t} |x_\tau|_{\mathcal{B}_\alpha} + K_p \sup_{p \le s \le t} |h(\tau)|_\alpha + M_p \left|x_p\right|_{\mathcal{B}_\alpha} .
\end{aligned}
$$

Thus, for $t \in [p, 2p]$,

$$
|x_t|_{\mathcal{B}_\alpha} \le \frac{K_p}{1 - K_p \, |\mathcal{D}_0|} \sup_{p \le s \le t} |h(s)|_\alpha + \frac{M_p}{1 - K_p \, |\mathcal{D}_0|} \left|x_p\right|_{\mathcal{B}_\alpha} .
$$

Since $p \in [0, p]$, one can write

$$
|x_p|_{\mathcal{B}_\alpha} \le a(p) |\phi|_{\mathcal{B}_\alpha} + b(p) \sup_{0 \le s \le p} |h(s)|_\alpha .
$$

Consequently,

$$
\begin{aligned}
|x_t|_{\mathcal{B}_\alpha} &\le \frac{K_p}{1 - K_p \, |\mathcal{D}_0|} \sup_{p \le s \le t} |h(s)|_\alpha + \frac{M_p}{1 - K_p \, |\mathcal{D}_0|} \left|x_p\right|_{\mathcal{B}_\alpha} \\
&\le \frac{K_p}{1 - K_p \, |\mathcal{D}_0|} \sup_{p \le s \le t} |h(s)|_\alpha + \frac{M_p a(p)}{1 - K_p \, |\mathcal{D}_0|} |\phi|_{\mathcal{B}_\alpha} \\
&\quad + \frac{M_p b(p)}{1 - K_p \, |\mathcal{D}_0|} \sup_{0 \le s \le p} |h(s)|_\alpha \\
&\le \frac{M_p a(p)}{1 - K_p \, |\mathcal{D}_0|} |\phi|_{\mathcal{B}_\alpha} + \max\left\{ \frac{K_p}{1 - K_p \, |\mathcal{D}_0|}, \frac{M_p b(p)}{1 - K_p \, |\mathcal{D}_0|} \right\} \sup_{0 \le s \le p} |h(s)|_\alpha \\
&\quad + \max\left\{ \frac{K_p}{1 - K_p \, |\mathcal{D}_0|}, \frac{M_p b(p)}{1 - K_p \, |\mathcal{D}_0|} \right\} \sup_{p \le s \le t} |h(s)|_\alpha \\
&\le \frac{M_p a(p)}{1 - K_p \, |\mathcal{D}_0|} |\phi|_{\mathcal{B}_\alpha} + 2 \max\left\{ \frac{K_p}{1 - K_p \, |\mathcal{D}_0|}, \frac{M_p b(p)}{1 - K_p \, |\mathcal{D}_0|} \right\} \sup_{0 \le s \le t} |h(s)|_\alpha .
\end{aligned}
$$

Thus, for all $t \in [p, 2p]$,

$$|x_t|_{\mathcal{B}_\alpha} \leq a_1(t)\,|\phi|_{\mathcal{B}_\alpha} + b_1(t) \sup_{0\leq s\leq t} |h(s)|_\alpha\,,$$

where a_1 can be seen as the extension of a on $[0, 2p]$ and b_1 the extension of b on $[0, 2p]$. It is exactly to say there exist $a, b \in L^\infty_{loc}([0, 2p]; \mathbb{R}^+)$ such that

$$|x_t|_{\mathcal{B}_\alpha} \leq a(t)|\phi|_{\mathcal{B}_\alpha} + b(t) \sup_{0\leq s\leq t} |h(s)|_\alpha, \quad \text{for} \quad t \in [0, 2p].$$

Inductively, one can show the existence of an extension u of x on $[np, (n+1)p]$ and the extension a_{np} of a, b_{np} of b on $[np, (n+1)p]$. Finally, the solution x is unique and continuous defined on $\mathbb{R}^+$. Also, the functions $a \in L^\infty_{loc}(\mathbb{R}^+; \mathbb{R}^+)$ and $b \in L^\infty_{loc}(\mathbb{R}^+; \mathbb{R}^+)$ are well-defined. □

We have the following result.

Theorem 8.3.2 ([14]) *Assume that* $(\mathbf{H}_1)$*,* $(\mathbf{H}_2)$*,* $(\mathbf{H}_3)$*, and* $(\mathbf{H}_4)$ *hold. Let u and v be two mild solutions of Eq.* (8.1) *on* $\mathbb{R}$*, respectively, associated to the initial data* ϕ *and* ψ*. Then, for any* $a > 0$*, there exists* $l(a) > 0$ *such that*

$$|u_t(\phi) - v_t(\psi)|_{\mathcal{B}_\alpha} \leq l(a)|\phi - \psi|_{\mathcal{B}_\alpha} \text{ for } t \in [0, a]. \tag{8.8}$$

For the regularity of the mild solution, we suppose that $\mathcal{B}$ satisfies the following axiom:

$(\mathbf{B}_1)$ If $(\phi_n)_{n\geq 0}$ is a Cauchy sequence in $\mathcal{B}$ and converges compactly to ϕ in $(-\infty, 0]$, then $\phi \in \mathcal{B}$ and $|\phi_n - \phi|_{\mathcal{B}} \to 0$ as $n \to +\infty$.

Now, we can claim the existence and uniqueness of strict solution for Eq. (8.1).

Theorem 8.3.3 *Assume that* $(\mathbf{H}_1)$*,* $(\mathbf{H}_2)$*,* $(\mathbf{H}_3)$*, and* $(\mathbf{H}_4)$ *hold. Furthermore, assume that* $\mathcal{B}$ *satisfies axiom:* $(\mathbf{B}_1)$ $f : \mathcal{B}_\alpha \to X$ *is continuously differentiable with* f' *locally Lipschitz continuous. Let* $\phi \in \mathcal{B}_\alpha$ *be such that*

$$\phi' \in \mathcal{B}_\alpha, \;\; \mathcal{D}(\phi) \in D(A) \;\; \textit{and} \;\; \mathcal{D}(\phi') = -A\mathcal{D}(\phi) + f(\phi).$$

Then, the mild solution u of the problem (8.1) *is a strict solution of the problem* (8.1). □

Proof Let $p > 0$ and u be the mild solution of the problem (8.1) associated to ϕ. We consider the following problem:

$$\begin{cases} \mathcal{D}(w_t) = T(t)\mathcal{D}(\phi') + \displaystyle\int_0^t T(t-s)f'(u_s)w_s ds, \quad \text{for } t \in [0, p] \\ \\ w_0 = \phi' \end{cases} \tag{8.9}$$

and $z \in C((-\infty, p]; X_\alpha)$ defined by

$$z(t) = \begin{cases} \phi(0) + \int_0^t w(s)ds, & \text{for } t \in [0, p] \\ \\ \phi(t) & \text{for } t \leq 0. \end{cases} \tag{8.10}$$

Then (8.9) has a unique mild and continuous solution w on $(-\infty, p]$. Also, one can recall the following lemma that plays an important role in the proof of this current theorem.

Lemma 8.3.2 ([7]) *The function z defined above verifies*

$$z_t = \phi + \int_0^t w_s ds, \qquad \textit{for} \quad t \in [0, p]. \tag{8.11}$$

Note that our objective is to show that $u = z$ on $[0, p]$. Using (8.9), we get

$$\int_0^t \mathcal{D}(w_s)ds = \int_0^t T(t-s)\mathcal{D}(\phi')ds + \int_0^t \int_0^s T(s-\tau)f'(u_\tau)w_\tau d\tau ds. \tag{8.12}$$

For $t \in [0, p]$, we have

$$\frac{d}{dt}\int_0^t T(t-s)f(z_s)ds = T(t)f(\phi) + \int_0^t T(t-s)f'(z_s)w_s ds. \tag{8.13}$$

Consequently,

$$\int_0^t T(s)f(\phi)ds = \int_0^t T(t-s)f(z_s)ds - \int_0^t \int_0^s T(s-\tau)f'(z_\tau)w_\tau d\tau ds. \tag{8.14}$$

Using Eq. (8.11), it follows that

$$\begin{aligned} \mathcal{D}(z_t) &= \mathcal{D}(\phi) + \int_0^t T(t-s)\Big(-A\mathcal{D}(\phi) + f(\phi)\Big)ds \\ &\quad + \int_0^t \int_0^s T(s-\tau)f'(u_\tau)w_\tau d\tau ds \\ &= T(t)\mathcal{D}(\phi) + \int_0^t T(s)f(\phi)ds + \int_0^t \int_0^s T(s-\tau)f'(u_\tau)w_\tau d\tau ds. \end{aligned}$$

Using Eq. (8.14), we have

$$\begin{aligned} \mathcal{D}(z_t) &= T(t)\mathcal{D}(\phi) + \int_0^t T(t-s)f(z_s)ds \\ &\quad + \int_0^t \int_0^s T(s-\tau)\Big(f'(u_\tau) - f'(z_\tau)\Big)w_\tau d\tau ds. \end{aligned} \tag{8.15}$$

Therefore,

$$\mathcal{D}(u_t - z_t) = \int_0^t T(t-s)\Big(f(u_s) - f(z_s)\Big)ds - \int_0^t \int_0^s T(s-\tau)\Big(f'(u_\tau) - f'(z_\tau)\Big)w_\tau d\tau ds. \tag{8.16}$$

By Fubini's theorem, we get that

$$\mathcal{D}(u_t - z_t) = \int_0^t T(t-s)\Big(f(u_s) - f(z_s)\Big)ds - \int_0^t \Big(\int_0^{t-s} T(\tau)d\tau\Big)\Big(f'(u_s) - f'(z_s)\Big)w_s ds. \tag{8.17}$$

Then, we put for $t \in [0, p]$,

$$h(t) = \int_0^t T(t-s)\Big(f(u_s) - f(z_s)\Big)ds - \int_0^t \Big(\int_0^{t-s} T(\tau)d\tau\Big)\Big(f'(u_s) - f'(z_s)\Big)w_s ds,$$

to obtain for some positive constants k and C_1,

$$\begin{aligned}
|h(t)|_\alpha &= \Big|\int_0^t T(t-s)\Big(f(u_s) - f(z_s)\Big)ds \\
&\quad - \int_0^t \Big(\int_0^{t-s} T(\tau)d\tau\Big)\Big(f'(u_s) - f'(z_s)\Big)w_s ds\Big|_\alpha \\
&\le \int_0^t \Big|T(t-s)\Big(f(u_s) - f(z_s)\Big)\Big|_\alpha ds \\
&\quad + \int_0^t \int_0^{t-s} \Big|T(\tau)(f'(u_s) - f'(z_s))w_s\Big|_\alpha d\tau ds \\
&\le kM_\alpha \int_0^t \frac{e^{\omega(t-s)}}{(t-s)^\alpha}|u_s - z_s|_{\mathcal{B}_\alpha} ds \\
&\quad + C_1 M_\alpha \int_0^t \Big(\int_0^{t-s} \frac{e^{\omega\tau}}{\tau^\alpha}d\tau\Big)|u_s - z_s|_{\mathcal{B}_\alpha} ds.
\end{aligned}$$

One can write for $\omega > 0$

$$\begin{aligned}
\int_0^{t-s} \frac{e^{\omega\tau}}{\tau^\alpha}d\tau &\le e^{\omega(t-s)} \int_0^{t-s} \frac{1}{\tau^\alpha}d\tau \\
&\le e^{\omega(t-s)}\Big[\frac{1}{1-\alpha}\frac{1}{\tau^{\alpha-1}}\Big]_0^{t-s}
\end{aligned}$$

$$\le e^{\omega(t-s)} \frac{t-s}{1-\alpha} \frac{1}{(t-s)^\alpha}$$
$$\le e^{\omega(t-s)} \frac{p}{1-\alpha} \frac{1}{(t-s)^\alpha}.$$

Therefore,

$$|h(t)|_\alpha \le kM_\alpha \int_0^t \frac{e^{\omega(t-s)}}{(t-s)^\alpha} |u_s - z_s|_{\mathcal{B}_\alpha} ds + \frac{C_1 p M_\alpha}{1-\alpha} \int_0^t \frac{e^{\omega(t-s)}}{(t-s)^\alpha} |u_s - z_s|_{\mathcal{B}_\alpha} ds.$$

Moreover, since for all $\theta \in (-\infty, 0]$, $u(\theta) = z(\theta)$, then one has for all $s \in [0, t]$,

$$|u_s - z_s|_{\mathcal{B}_\alpha} \le \max_{0 \le \tau \le t} \left| u(\tau) - z(\tau) \right|_\alpha.$$

Thus,

$$|h(t)|_\alpha \le \left(kM_\alpha + \frac{C_1 p M_\alpha}{1-\alpha} \right) \left(\int_0^p \frac{e^{\omega\tau}}{\tau^\alpha} d\tau \right) \max_{0 \le \tau \le p} \left| u(\tau) - z(\tau) \right|_\alpha.$$

Using Lemma 8.3.1, one obtains

$$|u_t - z_t|_{\mathcal{B}_\alpha} \le \left(kM_\alpha + \frac{C_1 p M_\alpha}{1-\alpha} \right) \left(\int_0^p \frac{e^{\omega\tau}}{\tau^\alpha} d\tau \right) \max_{0 \le \tau \le p} \left| u(\tau) - z(\tau) \right|_\alpha.$$

One can choose $p > 0$ small enough such that

$$\left(kM_\alpha + \frac{C_1 p M_\alpha}{1-\alpha} \right) \left(\int_0^p \frac{e^{\omega\tau}}{\tau^\alpha} d\tau \right) < 1.$$

It follows that $u = z$ in $(-\infty, p]$ and that leads to u continuously differentiable on $[0, p]$ with respect to the α-norm. In order to extend the solution to $[p, 2p]$, we consider the following problems:

$$\begin{cases} \mathcal{D}(w_t) = T(t-p)\mathcal{D}(u'_p) + \displaystyle\int_p^t T(t-s) f'(u_s) w_s ds \quad \text{for } t \in [p, 2p] \\ w_p = u'_p, \end{cases}$$

and $\tilde{z} \in C((-\infty, 2p]; X_\alpha)$ defined by

$$\tilde{z}(t) = \begin{cases} u_p(0) + \displaystyle\int_p^t w(s) ds \quad \text{for } t \in [p, 2p] \\ z(t) \quad \text{for } t \le p. \end{cases}$$

Using the same technique, one obtains that $u = \tilde{z}$ on $(-\infty, 2p]$. Proceeding inductively, solution u is uniquely extended to $[np, (n+1)p]$ for all $n \in \mathbb{N}^*$ with respect to the α-norm. Since $X_\alpha \hookrightarrow X$, one obtains that $u \in C^1([0, +\infty); X)$. Finally, using Theorem 8.2.2 , u is the strict solution defined on $\mathbb{R}$. □

8.4 Smoothness Results of the Operator Solution

Let $K : D(K) \subseteq Y \to Y$ be a closed linear operator with dense domain $D(K)$ in a Banach space Y. We denote by $\sigma(K)$ the spectrum of K.

Definition 8.4.1 The essential spectrum $\sigma_{ess}(K)$ of K is the set of all $\lambda \in \mathbb{C}$ such that at least one of the following relations holds:

(i) The range $Im(\lambda I - K)$ is not closed.
(ii) The generalized eigenspace $M_\lambda(K) = \bigcup_{n\geq 0} ker(\lambda I - K)^n$ of λ is infinite-dimensional.
(iii) λ is a limit of $\sigma(K)$, that is, $\lambda \in \overline{\sigma(K) - \{\lambda\}}$.

□

The essential radius denoted by $r_{ess}(K)$ is given by

$$r_{ess}(K) = \sup\{|\lambda| : \lambda \in \sigma_{ess}(K)\}.$$

Definition 8.4.2 The spectral bound $s(A)$ of the linear operator A is defined as

$$s(A) = \sup\{Re\lambda : \lambda \in \sigma(A)\}.$$

Definition 8.4.3 The type of the linear operator $(T(t))_{t\geq 0}$ is defined by

$$\omega_0(T) = \inf\left\{\omega \in \mathbb{R} : \sup_{t\geq 0}\left\{e^{-\omega t}|T(t)| < \infty\right\}\right\}.$$

In the sequel, we recall the χ measure of noncompactness, which will be used in the next to analyse the spectral properties of semigroup solution. The χ measure of noncompactness for a bounded set H of a Banach space Y with the norm $|.|_Y$ is defined by

$$\chi(H) = \inf\{\epsilon > 0 : \ H \text{ has a finite cover of diameter } < \epsilon\}.$$

The following results are some basic properties of the χ measure of noncompactness.

Lemma 8.4.1 ([17]) *Let A_1 and A_2 be bounded sets of a Banach space Y. Then:*

(i) $\chi(A_1) \leq dia(A_1)$, *where* $dia(A_1) = \sup_{x,y\in A_1} |x-y|$.
(ii) $\chi(A_1) = 0$ *if and only if* A_1 *is relatively compact in* Y.
(iii) $\chi(A_1 \bigcup A_2) = max\,\{\chi(A_1), \chi(A_2)\}$.
(iv) $\chi(\lambda A_1) = |\lambda|\chi(A_1)$, $\lambda \in \mathbb{R}$, *where* $\lambda A_1 = \{\lambda x :\ x \in A_1\}$.
(v) $\chi(A_1 + A_2) \leq \chi(A_1) + \chi(A_2)$, *where* $A_1 + A_2 = \{x + y :\ x \in A_1,\ y \in A_2\}$.
(vi) $\chi(A_1) \leq \chi(A_2)$ *if* $A_1 \subseteq A_2$.

Definition 8.4.4 The essential norm of a bounded linear operator K on Y is defined by

$$|K|_{ess} = \inf\{M \geq 0 :\ \chi(K(B)) \leq M\chi(B) \text{ for any bounded set B in } Y\}.$$

Let $V = (V(t))_{t\geq 0}$ be a c_0-semigroup on a Banach space Y.

Definition 8.4.5 The essential growth $\omega_{ess}(V)$ of $(V(t))_{t\geq 0}$ is defined by

$$\omega_{ess}(V) = \inf\left\{\omega \in \mathbb{R} :\ \sup_{t\geq 0} e^{-\omega t}|V(t)|_{ess} < \infty\right\}.$$

Theorem 8.4.1 ([7]) *The essential growth bound of $(V(t))_{t\geq 0}$ is given by*

$$\omega_{ess}(V) = \lim_{t\to+\infty} \frac{1}{t}\log|V(t)|_{ess} = \inf_{t>0} \frac{1}{t}\log|V(t)|_{ess}. \tag{8.18}$$

Moreover,

$$r_{ess}(V(t)) = exp(t\omega_{ess}(V)),\ \ for\ t \geq 0. \tag{8.19}$$

Assume now that:

($\mathbf{H}_5$) The semigroup $(T(t))_{t\geq 0}$ is compact for $t > 0$.

Theorem 8.4.2 *Assume that* ($\mathbf{H}_1$), ($\mathbf{H}_2$), ($\mathbf{H}_3$), ($\mathbf{H}_4$), *and* ($\mathbf{H}_5$) *hold. Then, the solution $u(., \phi)$ of Eq. (8.1) is decomposed as follows:*

$$u_t(., \phi) = \mathcal{U}(t)\phi + \mathcal{W}(t)\phi,\ \ for\ \ t \geq 0,$$

where $\mathcal{W}(t)$ is a compact operator on $\mathcal{B}_\alpha$, for each $t > 0$, and $\mathcal{U}(t)$ is the semigroup solution of the following equation:

$$\begin{cases} \dfrac{d}{dt}\mathcal{D}(x_t) = -A\mathcal{D}(x_t) \quad for\ \ t \geq 0, \\ \\ x_0 = \phi \in \mathcal{B}_\alpha. \end{cases} \tag{8.20}$$

Proof Let $\mathscr{U}(t)$ be defined by

$$(\mathscr{U}(t)\phi)(\theta) = \begin{cases} \phi(t+\theta) & \text{for} \quad t+\theta \le 0 \\ \\ v(t+\theta) & \text{for} \quad t+\theta \ge 0, \end{cases} \tag{8.21}$$

where v is a unique solution of the problem

$$\begin{cases} \mathcal{D}(v_t) = T(t)\mathcal{D}(\phi) & \text{for} \quad t \ge 0 \\ \\ v(t) = \phi(t) & \text{for} \quad t \le 0. \end{cases} \tag{8.22}$$

We can write $\mathscr{W}(t)\phi = w_t(.,\phi) = u_t(.,\phi) - \mathscr{U}(t)\phi = u_t(.,\phi) - v_t(.,\phi)$. Then,

$$\mathcal{D}(\mathscr{W}(t)\phi) = \mathcal{D}(u_t(.,\phi)) - \mathcal{D}(v_t(.,\phi)) = \int_0^t T(t-s)f(u_s)ds.$$

Consequently,

$$\begin{cases} \mathcal{D}(w_t) = h(t,\phi) = \displaystyle\int_0^t T(t-s)f(u_s)ds & \text{for} \quad t \ge 0, \\ \\ w_0 = 0 & \text{for} \quad t \le 0. \end{cases} \tag{8.23}$$

Let $\{\phi_k\}_{k\ge 0}$ be a bounded sequence in $\mathcal{B}_\alpha$. We will show that the family $\{h(.,\phi_k) : k \ge 0\}$ is equicontinuous and bounded on $C([0,\sigma]; X_\alpha)$, for any $\sigma > 0$ fixed. For all $0 < \alpha < \beta < 1$, there exists a positive constant C such that

$$\begin{aligned} |A^\beta h(t,\phi_k)| &= |A^\beta \int_0^t T(t-s)f(u_s(.,\phi_k))ds| \\ &\le \int_0^t |A^\beta T(t-s)f(u_s(.,\phi_k))|ds \\ &\le M_\beta C \int_0^t \frac{e^{\omega s}}{s^\beta} ds, \end{aligned}$$

for every $k \ge 0$.

Using the compactness of the operator $A^{-\beta} : X \to X_\alpha$, we get that the set $\{h(t,\phi_k) : k \ge 0\}$ is relatively compact in X_α for each $t \ge 0$. Now, let us prove the equicontinuity of the family $\{h(.,\phi_k) : k \ge 0\}$ in the α-norm. For this purpose, let $t > t_0 \ge 0$. Then,

$$
\begin{aligned}
A^\alpha h(t,\phi_k) - A^\alpha h(t_0,\phi_k) &= \int_0^t A^\alpha T(t-s) f(u_s)ds - \int_0^{t_0} A^\alpha T(t_0-s) f(u_s)ds \\
&= \int_0^{t_0} A^\alpha [T(t-s) - T(t_0-s)] f(u_s)ds \\
&\quad + \int_{t_0}^t A^\alpha T(t-s) f(u_s)ds \\
&= [T(t-t_0) - I] \int_0^{t_0} A^\alpha T(t_0-s) f(u_s)ds \\
&\quad + \int_{t_0}^t A^\alpha T(t-s) f(u_s)ds.
\end{aligned}
$$

We obtain that

$$
\left| \int_{t_0}^t A^\alpha T(t-s) f(u_s)ds \right| \le M_\alpha k \int_{t_0}^t \frac{e^{\omega s}}{s^\alpha} ds \to 0 \quad \text{as} \quad t \to t_0 \text{ uniformly in } \phi_k.
$$

Moreover, since $\{A^\alpha \int_0^{t_0} T(t_0-s) f(u_s(.,\phi_k))ds : \ k \ge 0\}$ is relatively compact in X, then there is a compact set Γ in X such that

$$
\int_0^{t_0} A^\alpha T(t_0-s) f(u_s(.,\phi_k))ds \subset \Gamma \quad \text{for all} \quad \phi_k.
$$

It is well-known by the Banach–Steinhaus theorem that

$$
\lim_{t\to t_0} \sup_{x\in\Gamma} |(T(t-t_0) - I)x| = 0.
$$

Thus,

$$
\lim_{t\to t_0} |h(t,\phi_k) - h(t_0,\phi_k)|_\alpha = 0 \quad \text{uniformly in} \quad \phi_k.
$$

Using the same argument, we also obtain for $t_0 > t$,

$$
\lim_{t\to t_0} |h(t,\phi_k) - h(t_0,\phi_k)|_\alpha = 0 \quad \text{uniformly in} \quad \phi_k.
$$

Therefore, the family $\{h(.,\phi_k) : \ k \ge 0\}$ is relatively compact on $C([0,\sigma]; X_\alpha)$ for each $\sigma > 0$. Then, there exists a subsequence $\{\phi_k : \ k \ge 0\}$ such that $h(t,\phi_k)$ converges as $k \to +\infty$ uniformly on $[0,\sigma]$ to some function $h(t)$ with respect to the α-norm. Let w_t^k be the solution of problem (8.23) with the initial data $\phi = \phi_k$. Then,

$$
\mathcal{D}(w_t^j - w_t^k) = h(t,\phi_j) - h(t,\phi_k).
$$

Using Lemma 8.3.1, we obtain

$$|w_t^j - w_t^k|_{\mathcal{B}_\alpha} \leq b(t) \sup_{0 \leq s \leq t} |h(t, \phi_j) - h(t, \phi_k)|_\alpha,$$

which implies that $\{w_t^k\}_{k\geq 0} = \{w_t(., \phi_k)\}_{k\geq 0}$ is a Cauchy sequence in $\mathcal{B}_\alpha$. Therefore, $\mathscr{W}(t)$ is compact in $\mathcal{B}_\alpha$. □

Definition 8.4.6 $\mathcal{D}$ is said to be stable if the zero solution of the difference system

$$\begin{cases} \mathcal{D}(x_t) = 0 \quad \text{for} \quad t \geq 0, \\ x_0(t) = \phi(t) \quad \text{for} \quad t \leq 0 \end{cases}$$

is exponentially stable. □

Now, we give the definitions of fading memory spaces that will be used later on. For $\phi \in \mathcal{B}$, $t \geq 0$ and $\theta \leq 0$, we define the following:

$$[S(t)\phi](\theta) = \begin{cases} \phi(0) \quad \text{if} \quad t + \theta \geq 0, \\ \phi(t+\theta) \quad \text{if} \quad t + \theta < 0. \end{cases} \tag{8.24}$$

Then, $\{S(t)\}_{t\geq 0}$ is a strongly continuous semigroup on $\mathcal{B}$. We set

$$S_0(t) = S(t)/\mathcal{B}_0, \text{ where } \mathcal{B}_0 = \{\phi \in \mathcal{B} : \phi(0) = 0\}.$$

Definition 8.4.7 [7] We say that $\mathcal{B}$ is a uniform fading memory space if the following conditions hold:

(i) If a uniformly bounded sequence $(\phi_n)_{n\in\mathbb{N}}$ in C_{00} converges to a function ϕ compactly on $(-\infty, 0]$, then ϕ is in $\mathcal{B}$ and $|\phi_n - \phi|_{\mathcal{B}} \to 0$ as $n \to +\infty$.
(ii) $|S_0(t)|_{\mathcal{B}} \to 0$ as $t \to +\infty$. □

Lemma 8.4.2 ([7]) *If $\mathcal{B}$ is a uniform fading memory space, then K and M can be chosen such that K is bounded on $\mathbb{R}^+$ and $M(t) \to 0$ as $t \to +\infty$.* □

Lemma 8.4.3 *If $\mathcal{B}$ is a uniform fading memory space, then $\mathcal{B}_\alpha$ is a uniform fading memory space.* □

Proof Let $(\phi_n)_{n\in\mathbb{N}}$ in C_{00} be a uniformly bounded sequence that converges to a function ϕ compactly on $(-\infty, 0]$. Then ϕ is in $\mathcal{B}$ and $|\phi_n - \phi|_{\mathcal{B}} \to 0$ as $n \to +\infty$ since $\mathcal{B}$ is a uniform fading memory space. Using $(\mathbf{H}_2)$, one can write $A^{-\alpha}\phi \in \mathcal{B}$ since $\phi \in \mathcal{B}$. $A^{-\alpha}\phi \in \mathcal{B}$ leads to the existence of $A^{-\alpha}\phi(\theta)$. We know that $R(A^{-\alpha}) = D(A^\alpha)$. For this reason, $\left|A^{-\alpha}\phi(\theta)\right|_\alpha$ is well-defined. The fact that $A^{-\alpha}$ is bounded linear operator implies $|\phi(\theta)|_\alpha$ exists. Therefore, $\phi(\theta) \in D(A^\alpha)$ for all $\theta \leq 0$. Also,

$$\left|A^{-\alpha}A^\alpha\phi\right|_{\mathcal{B}} = |\phi|_{\mathcal{B}} < \infty.$$

Using again the boundedness of $A^{-\alpha}$, one obtains the existence of $|A^{\alpha}\phi|_{\mathcal{B}}$. Thus, $A^{\alpha}\phi \in \mathcal{B}$. Hence, we establish that $\phi \in \mathcal{B}_{\alpha}$. Moreover,

$$|\phi_n - \phi|_{\mathcal{B}} = \left|A^{-\alpha}A^{\alpha}(\phi_n - \phi)\right|_{\mathcal{B}} \to 0 \quad \text{as} \quad n \to +\infty.$$

Since $A^{-\alpha}$ is a bounded linear operator, one obtains

$$\left|A^{\alpha}(\phi_n - \phi)\right|_{\mathcal{B}} = |\phi_n - \phi|_{\mathcal{B}_{\alpha}} \to 0 \quad \text{as} \quad n \to +\infty.$$

Consequently, the condition (i) of Definition 8.4.7 is satisfied.

Now, we have to show that the condition (ii) of Definition 8.4.7 is verified. In order to do this, we use the fact that $A^{-\alpha}$ is a bounded linear operator and $\mathcal{B}$ is a uniform fading memory space to write

$$|S_0(t)|_{\mathcal{B}} = \left|A^{-\alpha}A^{\alpha}S_0(t)\right|_{\mathcal{B}} \to 0 \quad \text{as} \quad t \to +\infty$$

and

$$|S_0(t)|_{\mathcal{B}_{\alpha}} \to 0 \quad \text{as} \quad t \to +\infty.$$

Hence, the condition (ii) is satisfied. Finally, $\mathcal{B}_{\alpha}$ is a uniform fading memory space. □

Now, we have to prove that $\mathcal{U}(t)$ is exponentially stable. It is known that $\mathcal{U}(t)$ in Theorem 8.4.2 is defined by

$$(\mathcal{U}(t)\phi)(\theta) = \begin{cases} \phi(t+\theta) & \text{for} \quad t+\theta \le 0 \\ v(t+\theta) & \text{for} \quad t+\theta \ge 0, \end{cases}$$

where v is a unique solution for the same initial data ϕ of the following problem:

$$\begin{cases} \mathcal{D}(v_t) = T(t)\mathcal{D}(\phi) & \text{for} \quad t \ge 0 \\ v(t) = \phi(t) & \text{for} \quad t \le 0. \end{cases}$$

Using the superposition principle of solutions of linear systems, we have

$$v(t) = x(t) + y(t) \ \text{ for } t \in \mathbb{R},$$

where

$$\begin{cases} \mathcal{D}(x_t) = 0 & \text{for} \quad t \ge 0, \\ x(t) = \phi(t) & \text{for} \quad t \le 0 \end{cases} \tag{8.25}$$

and

$$\begin{cases} \mathcal{D}(y_t) = T(t)\mathcal{D}(\phi) \quad \text{for} \quad t \geq 0, \\ \\ y(t) = 0 \quad \text{for} \quad t \leq 0. \end{cases} \tag{8.26}$$

Now, let $K_\infty = \sup_{s \geq 0} K(s)$. We have the following result.

Theorem 8.4.3 *Assume that* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, *and* $(\mathbf{H}_3)$ *hold. Moreover, suppose that* $\mathcal{B}_\alpha$ *is a uniform fading memory space,* $\mathcal{D}$ *is stable, the semigroup* $\{T(t)\}_{t\geq 0}$ *is exponentially stable, and* $K_\infty|\mathcal{D}_0| < 1$. *Then, the semigroup solution* $\{\mathcal{U}(t)\}_{t\geq 0}$ *defined in Theorem 8.4.2 is exponentially stable.* □

Proof Since y verifies problem (8.26) and $\mathcal{B}_\alpha$ is a uniform fading memory space, then, using Axiom (**A**)-(iii), one can write for $t \geq s \geq \epsilon > 0$

$$\begin{aligned} |y_s|_{\mathcal{B}_\alpha} &\leq K(\epsilon) \sup_{s-\epsilon\leq\tau\leq s} |y(\tau)|_\alpha + M(\epsilon)|y_{s-\epsilon}|_{\mathcal{B}_\alpha} \\ &\leq K(\epsilon)|\mathcal{D}_0| \sup_{s-\epsilon\leq\tau\leq s} |y_\tau|_{\mathcal{B}_\alpha} + K(\epsilon) \sup_{s-\epsilon\leq\tau\leq s} |T(\tau)\mathcal{D}(\phi)|_\alpha + M(\epsilon)|y_{s-\epsilon}|_{\mathcal{B}_\alpha} \\ &\leq K(\epsilon)|\mathcal{D}_0| \sup_{s-\epsilon\leq\tau\leq s} |y_\tau|_{\mathcal{B}_\alpha} + K(\epsilon) \sup_{s-\epsilon\leq\tau\leq s} |T(\tau)\mathcal{D}(\phi)|_\alpha \\ &\quad + M(\epsilon) \sup_{s-\epsilon\leq\tau\leq s} |y_\tau|_{\mathcal{B}_\alpha}. \end{aligned}$$

Therefore, taking $\epsilon > 0$ such that $s - \epsilon \geq t - 2\epsilon \geq 0$, then

$$\begin{aligned} |y_s|_{\mathcal{B}_\alpha} \leq K(\epsilon)|\mathcal{D}_0| \sup_{t-2\epsilon\leq\tau\leq s} |y_\tau|_{\mathcal{B}_\alpha} + K(\epsilon) \sup_{t-2\epsilon\leq\tau\leq s} |T(\tau)\mathcal{D}(\phi)|_\alpha \\ + M(\epsilon) \sup_{t-2\epsilon\leq\tau\leq s} |y_\tau|_{\mathcal{B}_\alpha}. \end{aligned}$$

Now, one can write

$$\begin{aligned} \sup_{t-2\epsilon\leq s\leq t} |y_s|_{\mathcal{B}_\alpha} &\leq \sup_{t-2\epsilon\leq s\leq t} \Big\{ K_\infty|\mathcal{D}_0| \sup_{t-2\epsilon\leq\tau\leq s} |y_\tau|_{\mathcal{B}_\alpha} \\ &\quad + K_\infty \sup_{t-2\epsilon\leq\tau\leq s} |T(\tau)\mathcal{D}(\phi)|_\alpha + M(\epsilon) \sup_{t-2\epsilon\leq\tau\leq s} |y_\tau|_{\mathcal{B}_\alpha} \Big\} \\ &\leq K_\infty|\mathcal{D}_0| \sup_{t-2\epsilon\leq s\leq t} |y_s|_{\mathcal{B}_\alpha} + K_\infty \sup_{t-2\epsilon\leq s\leq t} |T(s)\mathcal{D}(\phi)|_\alpha \\ &\quad + M(\epsilon) \sup_{t-2\epsilon\leq s\leq t} |y_s|_{\mathcal{B}_\alpha}. \end{aligned}$$

Since $M(\epsilon) \to 0$ as $\epsilon \to +\infty$, then we choose ϵ big enough such that $0 < 1 - K_\infty|\mathcal{D}_0| - M(\epsilon)$. We obtain that

$$|y_t|_{\mathcal{B}_\alpha} \leq \frac{K_\infty}{\left(1 - K_\infty|\mathcal{D}_0| - M(\epsilon)\right)} \sup_{t-2\epsilon \leq s \leq t} |T(s)\mathcal{D}(\phi)|_\alpha.$$

Since $\{T(t)\}_{t\geq 0}$ is exponentially stable, then there exist positive constants α' and β' such that $|y_t|_{\mathcal{B}_\alpha} \leq \beta' e^{-\alpha' t}$ for all $t \geq 0$.

Since $\mathcal{D}$ is stable, then $x_t(\phi) \to 0$ as $t \to +\infty$. On the other hand, we have

$$\mathcal{U}(t)\phi = x_t(\phi) + y_t(\phi).$$

Then, it follows that $\mathcal{U}(t) \to 0$ as $t \to 0$ and $\{\mathcal{U}(t)\}_{t\geq 0}$ is exponentially stable. □

In the sequel, we give the following.

Theorem 8.4.4 *Assume that there exists $r > 0$ such that the elements $\phi \in \mathcal{B}_\alpha$ are continuous from $[-r, 0]$ to X_α. If $\mathcal{D}(\phi) = \phi(0) - q\phi(-r)$ for all $\phi \in \mathcal{B}_\alpha$ with $0 < q < 1$ and $\mathcal{B}_\alpha$ a uniform fading memory space, then $\mathcal{D}$ is stable.* □

Proof Since $\mathcal{D}(x_t) = 0$ and $x_0 = \phi$, then for all $t \in [0, r]$, we have $x(t) = qx(t - r)$. Therefore,

$$|x(t)|_\alpha \leq q|\phi(t - r)|_\alpha.$$

Also, for all $t \in [r, 2r]$,

$$|x(t, \phi)|_\alpha \leq q^2|\phi(t - 2r)|_\alpha.$$

Inductively, for all $t \in [(n - 1)r, nr]$, we have

$$|x(t, \phi)|_\alpha \leq q^n|\phi(t - nr)|_\alpha;$$

since $t \in [(n - 1)r, nr]$, then $t - nr \in [-r, 0]$. Furthermore, $\mathcal{B}_\alpha$ is assumed to be the space of functions from $(-\infty, 0]$ to X_α that are continuous on $[-r, 0]$. Thus, for all $t \in [(n - 1)r, nr]$,

$$|x(t, \phi)|_\alpha \leq q^n \sup_{-r\leq s\leq 0} |\phi(s)|_\alpha,$$

for all $\phi \in \mathcal{B}_\alpha$.

Thus, there exist $\alpha = -\frac{\ln(q)}{r} > 0$ and $C > 0$ such that

$$\begin{aligned}|x(t, \phi)|_\alpha &\leq q^n \sup_{-r\leq s\leq 0} |\phi(s)|_\alpha \\ &\leq Ce^{-\alpha t}.\end{aligned}$$

Hence, for all $\phi \in \mathcal{B}_\alpha$,

$$\lim_{t \to +\infty} x(t, \phi) = 0.$$

Now, let $\phi \in \mathcal{B}_\alpha$ such that $|\phi|_{\mathcal{B}_\alpha} \leq 1$.

Using again Axiom (**A**)-(iii) and the fact that $\mathcal{B}_\alpha$ is a uniform fading memory space, we have for $t \geq s \geq \epsilon > 0$

$$\begin{aligned} |x_s(., \phi)|_{\mathcal{B}_\alpha} &\leq K(\epsilon) \sup_{s-\epsilon \leq \tau \leq s} |x(\tau, \phi)|_\alpha + M(\epsilon)|x_{s-\epsilon}(., \phi)|_{\mathcal{B}_\alpha} \\ &\leq K_\infty \sup_{s-\epsilon \leq \tau \leq s} |x(\tau, \phi)|_\alpha + M(\epsilon)|x_{s-\epsilon}(., \phi)|_{\mathcal{B}_\alpha}. \end{aligned}$$

Choosing $\epsilon > 0$ such that $s - \epsilon \geq t - 2\epsilon \geq 0$, we have

$$\begin{aligned} |x_s(., \phi)|_{\mathcal{B}_\alpha} &\leq K_\infty \sup_{s-\epsilon \leq \tau \leq s} |x(\tau, \phi)|_\alpha + M(\epsilon)|x_{s-\epsilon}(., \phi)|_{\mathcal{B}_\alpha} \\ &\leq K_\infty \sup_{t-2\epsilon \leq \tau \leq s} |x(\tau, \phi)|_\alpha + M(\epsilon)|x_{s-\epsilon}(., \phi)|_{\mathcal{B}_\alpha} \\ &\leq \sup_{t-2\epsilon \leq s \leq t} \left\{ K_\infty \sup_{t-2\epsilon \leq \tau \leq s} |x(\tau, \phi)|_\alpha + M(\epsilon)|x_{s-\epsilon}(., \phi)|_{\mathcal{B}_\alpha} \right\} \\ &\leq K_\infty \sup_{t-2\epsilon \leq s \leq t} |x(s, \phi)|_\alpha + M(\epsilon) \sup_{t-2\epsilon \leq s \leq t} |x_s(., \phi)|_{\mathcal{B}_\alpha}. \end{aligned}$$

Thus,

$$\sup_{t-2\epsilon \leq s \leq t} |x_s(., \phi)|_{\mathcal{B}_\alpha} \leq K_\infty \sup_{t-2\epsilon \leq s \leq t} |x(s, \phi)|_\alpha + M(\epsilon) \sup_{t-2\epsilon \leq s \leq t} |x_s(., \phi)|_{\mathcal{B}_\alpha}.$$

Since $M(\epsilon) \to 0$ as $\epsilon \to +\infty$, then we can choose ϵ big enough such that $0 < 1 - M(\epsilon)$. Therefore,

$$|x_t(., \phi)|_{\mathcal{B}_\alpha} \leq \frac{K_\infty}{(1 - M(\epsilon))} \sup_{t-2\epsilon \leq s \leq t} |x(s, \phi)|_\alpha, \quad \text{for all } \phi \in \mathcal{B}_\alpha \text{ with } |\phi|_{\mathcal{B}_\alpha} \leq 1.$$

Thus, $x_t(., \phi) \to 0$ as $t \to +\infty$ whenever $\phi \in \mathcal{B}_\alpha$ and $|\phi|_{\mathcal{B}_\alpha} \leq 1$. Hence, $\mathcal{D}$ is stable. □

Example 8.4.1 Let γ be a real number, $1 \leq p < +\infty$, and $r > 0$. We define the space $C_r \times L^p_\gamma$ that consists of measurable functions $\varphi : (-\infty, 0] \to X$ that are continuous on $[-r, 0]$ such that $e^{\gamma\theta}|\varphi(\theta)|^p$ is measurable on $(-\infty, -r]$. Let us provide the space $C_r \times L^p_\gamma$ with the following norm:

$$|\varphi|_{\mathcal{B}} = \sup_{-r \leq \theta \leq 0} |\varphi(\theta)| + \int_{-\infty}^{-r} e^{\gamma\theta}|\varphi(\theta)|^p d\theta.$$

$(C_r \times L^p_\gamma, |.|_{\mathcal{B}})$ is a normed linear space satisfying Axioms (**A**) and (**B**).

Corollary 8.4.1 *Suppose that assumptions* ($\mathbf{H}_1$), ($\mathbf{H}_2$), *and* ($\mathbf{H}_3$) *hold, and there exists a positive constant r such that all $\phi \in \mathcal{B}_\alpha$ imply that ϕ is continuous on $[-r, 0]$ with values in X_α. Moreover, suppose that $\mathcal{B}_\alpha$ is a uniform fading memory space, $\mathcal{D}(\phi) = \phi(0) - q\phi(-r)$ for all $\phi \in \mathcal{B}_\alpha$, the semigroup $\{T(t)\}_{t\geq 0}$ is exponentially stable, and $K_\infty|\mathcal{D}_0| < 1$. Then, the semigroup solution $\{\mathcal{U}(t)\}_{t\geq 0}$ defined in Theorem 8.4.2 is exponentially stable.* □

8.5 Linearized Stability of Solutions

Coming back to the operator $U(t)$ for $t \geq 0$ defined on $\mathcal{B}_\alpha$ by

$$U(t)(\phi) = u_t(., \phi),$$

where $u_t(., \phi)$ is the unique mild solution of the problem (8.1) for the initial condition $\phi \in \mathcal{B}_\alpha$, it is proved that the following result holds.

Proposition 8.5.1 ([4]) *The family $(U(t))_{t\geq 0}$ is a nonlinear strongly continuous semigroup on $\mathcal{B}_\alpha$, that is:*

(i) $U(0) = I$.
(ii) $U(t + s) = U(t)U(s)$, *for* $t, s \geq 0$.
(iii) *For all* $\phi \in \mathcal{B}_\alpha$, $U(t)(\phi)$ *is a continuous function of* $t \geq 0$ *with values in* $\mathcal{B}_\alpha$.
(iv) *For* $t \geq 0$, $U(t)$ *is continuous from* $\mathcal{B}_\alpha$ *to* $\mathcal{B}_\alpha$.
(v) $(U(t))_{t\geq 0}$ *satisfies the following translation property, for* $t \geq 0$ *and* $\theta \leq 0$:

$$(U(t))(\theta) = \begin{cases} (U(t+\theta)(\phi))(0), & \text{if } t+\theta \geq 0, \\ \phi(t+\theta) & \text{if } t+\theta \leq 0. \end{cases} \tag{8.27}$$

It is now interesting to investigate the stability results of the equilibriums of the problem (8.1). Recalling that equilibrium means a constant solution u^* of the problem (8.1). To preserve the generality, we can suppose that $u^* = 0$.

Now, let us assume that:

($\mathbf{H}_6$) $f : \mathcal{B}_\alpha \to X$ is differentiable at zero.

It is well-known that the linearized problem associated to problem (8.1) is given by

$$\begin{cases} \dfrac{d}{dt}\mathcal{D}(y_t) = -A\mathcal{D}(y_t) + L(y_t), & \text{for } t \geq 0, \\ y_0 = \phi \in \mathcal{B}_\alpha, \end{cases} \tag{8.28}$$

with $L = f'(0)$.

Let $(V(t))_{t\geq 0}$ be the semigroup solution on $\mathcal{B}_\alpha$ associated to the problem (8.28).

Theorem 8.5.1 *Assume that* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, $(\mathbf{H}_3)$, $(\mathbf{H}_4)$, *and* $(\mathbf{H}_6)$ *hold. Then, for every* $t > 0$, *the derivative of* $U(t)$ *is* $V(t)$. □

Proof Let $t \geq 0$ be fixed and $\phi \in \mathcal{B}_\alpha$. One has

$$\mathcal{D}\Big[U(t)\phi - V(t)\phi\Big] = \int_0^t T(t-s)\Big[f(U(s)(\phi)) - L(V(s)(\phi))\Big]ds.$$

Let us set

$$w_t = U(t)(\phi) - V(t)(\phi)$$

and

$$h(t) = \int_0^t T(t-s)\Big[f(U(s)(\phi)) - L(V(s)(\phi))\Big]ds.$$

Then, we can write

$$h(t) = \int_0^t T(t-s)\Big[f(U(s)(\phi)) - f(V(s)(\phi))\Big]ds + \int_0^t T(t-s)\Big[f(V(s)(\phi)) - L(V(s)(\phi))\Big]ds.$$

Using Lemma 8.3.1, we obtain

$$|w_t|_{\mathcal{B}_\alpha} \leq b(t) \sup_{0\leq s\leq t} |h(s)|_\alpha, \quad \text{for} \quad t \in [0, T].$$

Moreover,

$$|h(t)|_\alpha \leq kM_\alpha \int_0^t \frac{e^{\omega(t-s)}}{(t-s)^\alpha} |w_s|_{\mathcal{B}_\alpha} ds + M_\alpha \int_0^t \frac{e^{\omega(t-s)}}{(t-s)^\alpha} \Big| f(V(s)(\phi)) - L(V(s)(\phi)) \Big| ds.$$

Using the fact that f is differentiable at zero with differential L at zero, we can state that for all $\epsilon > 0$, there exists $\eta > 0$ such that

$$M_\alpha \int_0^t \frac{e^{\omega(t-s)}}{(t-s)^\alpha} \Big| f(V(s)(\phi)) - L(V(s)(\phi)) \Big| ds \leq \epsilon |\phi|_{\mathcal{B}_\alpha} \quad \forall \phi \in \mathcal{B}_\alpha \text{ with } |\phi|_{\mathcal{B}_\alpha} < \eta.$$

Note that $w_0 = 0$, so we can write

$$|w_s|_{\mathcal{B}_\alpha} \le \sup_{0\le\tau\le t} |w(\tau)|_\alpha \le |w_t|_{\mathcal{B}_\alpha}, \quad \text{for } s \in [0, t].$$

Therefore, for $t \in [0, T]$,

$$\begin{aligned}\sup_{0\le s\le t} |h(s)|_\alpha &\le \epsilon|\phi|_{\mathcal{B}_\alpha} + kM_\alpha\Big(\int_0^t \frac{e^{\omega s}}{s^\alpha} ds\Big)|w_t|_{\mathcal{B}_\alpha}\\ &\le \epsilon|\phi|_{\mathcal{B}_\alpha} + kM_\alpha\Big(\int_0^T \frac{e^{\omega s}}{s^\alpha} ds\Big)|w_t|_{\mathcal{B}_\alpha}.\end{aligned}$$

We can choose $T > 0$ small enough such that $kM_\alpha b(T)\Big(\int_0^T \frac{e^{\omega s}}{s^\alpha} ds\Big) < 1$.

Consequently, for all $|\phi|_{\mathcal{B}_\alpha} < \eta$,

$$|w_t|_{\mathcal{B}_\alpha} \le \frac{b(T)}{1 - kM_\alpha b(T)\Big(\int_0^T \frac{e^{\omega s}}{s^\alpha} ds\Big)} \epsilon|\phi|_{\mathcal{B}_\alpha}.$$

Thus, $U(t)$ is differentiable at zero for all $t \in [0, T]$ with $d_\phi U(t)(0) = V(t)$. Proceeding by steps, one can prove that $d_\phi U(t)(0) = V(t)$, for all $t > 0$. □

Theorem 8.5.2 *Assume that* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, $(\mathbf{H}_3)$, $(\mathbf{H}_4)$, *and* $(\mathbf{H}_6)$ *hold. If the zero equilibrium of* $(V(t))_{t\ge0}$ *is exponentially stable, then the zero equilibrium of* $(U(t))_{t\ge0}$ *is locally exponentially stable, which means that there exist* $\eta > 0$, $\beta > 0$, *and* $C \ge 1$ *such that for* $t \ge 0$,

$$|U(t)(\phi)|_{\mathcal{B}_\alpha} \le Ce^{-\beta t}|\phi|_{\mathcal{B}_\alpha} \quad \textit{for all } \phi \in \mathcal{B}_\alpha \textit{ with } |\phi|_{\mathcal{B}_\alpha} \le \eta.$$

Moreover, if $\mathcal{B}_\alpha$ *can be decomposed as* $\mathcal{B}_\alpha = \mathcal{B}_\alpha^1 \oplus \mathcal{B}_\alpha^2$, *where* $\mathcal{B}_\alpha^i$ *are* V*-invariant subspaces of* $\mathcal{B}_\alpha$ *and* $\mathcal{B}_\alpha^1$ *a finite-dimensional with*

$$\omega_0 = \lim_{h\to+\infty} \frac{1}{h} \log \Big|V(h)/\mathcal{B}_\alpha^2\Big|_\alpha$$

and

$$\inf\{|\lambda| : \quad \lambda \in \sigma(V(t)/\mathcal{B}_\alpha^1)\} > e^{\omega_0 t},$$

then the zero equilibrium of $(U(t))_{t\ge0}$ *is not stable, in the sense that there exist* $\epsilon > 0$, *a sequence* $(\phi_n)_{n\in\mathbb{N}}$ *converging to* 0, *and a sequence* $(t_n)_{n\in\mathbb{N}}$ *of positive real numbers such that* $|U(t_n)\phi_n|_\alpha > \epsilon$. □

The proof of this theorem is based on the Theorem 8.5.1 and the following theorem.

Theorem 8.5.3 ([12]) *Let $(W(t))_{t\geq 0}$ be a nonlinear strongly continuous semigroup on the subset Ω of a Banach space $(X; ||.||)$. Assume that $x_0 \in \Omega$ is an equilibrium of $(W(t))_{t\geq 0}$ such that $W(t)$ is differentiable at x_0 for each $t \geq 0$ with $Z(t)$ the derivative of $W(t)$ at x_0. Then, $(Z(t)_{t\geq 0}$ is a strongly nonlinear continuous semigroup of bounded linear operators on X, and if the zero equilibrium of $(Z(t))_{t\geq 0}$ is exponentially stable, then the equilibrium x_0 of $(W(t))_{t\geq 0}$ is locally exponentially stable. Moreover, if X can be decomposed as $X = X_1 \oplus X_2$, where X_i are Z-invariant subspaces of X, X_1 a finite-dimensional with*

$$\omega = \lim_{h\to+\infty} \frac{1}{h} \log \left| |Z(h)/X_2| \right|$$

and

$$\inf\{|\lambda| : \quad \lambda \in \sigma(Z(t)/X_1)\} > e^{\omega t},$$

then the zero equilibrium of $(W(t))_{t\geq 0}$ is not stable, in the sense that there exist $\epsilon > 0$, a sequence $(\phi_n)_{n\in\mathbb{N}}$ converging to 0, and a sequence $(t_n)_{n\in\mathbb{N}}$ of positive real numbers such that

$$|W(t_n)\phi_n|_\alpha > \epsilon.$$

Lemma 8.5.1 ([19], Corollary 1.2, page 43) *Let Θ be a continuous and right differentiable function on $[a, b)$. If the right derivative function $d^+\Theta$ is continuous on $[a, b)$, then Θ is continuously differentiable on $[a, b)$.* □

Now, we make some sufficient conditions on $\mathcal{B}$ in order to determine $(A_V, D(A_V))$, the generator of the semigroup $(V(t))_{t\geq 0}$. So, we assume the following axiom:

(**C**): Let $(\phi_n)_{n\geq 0}$ be a sequence in $\mathcal{B}$ such that $\phi_n \to 0$ as $n \to +\infty$ in $\mathcal{B}$; then, $\phi_n(\theta) \to 0$ as $n \to +\infty$ for all $\theta \leq 0$.

We can state the following result.

Theorem 8.5.4 *Assume that* ($\mathbf{H}_1$), ($\mathbf{H}_2$), ($\mathbf{H}_3$), ($\mathbf{H}_4$), *and* ($\mathbf{H}_6$) *hold. Moreover, suppose that $\mathcal{B}$ satisfies axioms* (**A**), (**B**), *and* (**C**). *If $\mathcal{B}$ is a subspace of the space of continuous functions from $(-\infty, 0]$ into X, then $(A_V, D(A_V))$ is given by*

$$\begin{cases} D(A_V) = \Big\{\phi \in \mathcal{B}_\alpha : \ \phi' \in \mathcal{B}_\alpha, \ \mathcal{D}(\phi) \in D(A) \quad \textit{and} \\ \quad \mathcal{D}(\phi') = -A\mathcal{D}(\phi) + L(\phi)\Big\}, \\ \\ A_V\phi = \phi' \quad \textit{for} \ \ \phi \in D(A_V). \end{cases}$$

Proof Let B be the infinitesimal generator of the semigroup $(V(t))_{t\geq 0}$ on $\mathcal{B}_\alpha$ and $\phi \in D(B)$. Then, one can write

$$\begin{cases} \lim_{t\to 0^+} \dfrac{1}{t}(V(t)\phi - \phi) = \psi \text{ exists in } \mathcal{B}_\alpha, \\ \\ B\phi = \psi. \end{cases}$$

Using axiom (**C**), one obtains

$$\lim_{t\to 0^+} \frac{1}{t}(\phi(t+\theta) - \phi(\theta)) = \psi(\theta) \quad \text{ for } \theta \in (-\infty, 0).$$

It follows that the right derivative $d^+\phi$ exists on $(-\infty, 0)$ and is equal to ψ. The fact that each function in $\mathcal{B}_\alpha$ is continuous on $(-\infty, 0]$ leads to $d^+\phi$ continuous on $(-\infty, 0)$.

Using Lemma 8.5.1, we deduce that the function ϕ is continuously differentiable and $\phi' = \psi$ on $(-\infty, 0)$. Moreover,

$$\lim_{\theta\to 0} d^+\phi(\theta) = \psi(0),$$

which implies that the function ϕ is continuously differentiable from $(-\infty, 0]$ to X_α and $\phi' = \psi$ on $(-\infty, 0]$.

We have

$$\frac{1}{t}(T(t)\mathcal{D}(\phi) - \Leftarrow\phi)) = \frac{1}{t}\mathcal{D}(V(t)\phi - \phi) - \frac{1}{t}\int_0^t T(t-s)L(V(s)\phi)ds.$$

It is well-known that

$$\lim_{t\to 0} \frac{1}{t}\int_0^t T(t-s)L(V(s)\phi)ds = L(\phi)$$

in X-norm and

$$\lim_{t\to 0} \frac{1}{t}\mathcal{D}(V(t)\phi - \phi) = \mathcal{D}(\phi')$$

in α-norm. The fact that $X_\alpha \hookrightarrow X$ implies

$$\lim_{t\to 0} \frac{1}{t}\mathcal{D}(V(t)\phi - \phi) = \mathcal{D}(\phi')$$

in X-norm. Consequently,

$$\mathcal{D}(\phi) \in D(A) \text{ and } \lim_{t\to 0} \frac{1}{t}(T(t)\mathcal{D}(\phi) - \Leftarrow\phi)) = A\mathcal{D}(\phi)$$

in X-norm. It follows that

$$\begin{cases} D(B) \subseteq \Big\{\phi \in \mathcal{B}_\alpha : \ \phi' \in \mathcal{B}_\alpha, \ \mathcal{D}(\phi) \in D(A) \text{ and } \mathcal{D}(\phi') = -A\mathcal{D}(\phi) + L(\phi)\Big\}, \\ B(\phi) = \phi'. \end{cases}$$

Conversely, let $\phi \in \mathcal{B}_\alpha$ be such that

$$\phi' \in \mathcal{B}_\alpha, \ \mathcal{D}(\phi) \in D(A) \quad \text{and} \ \mathcal{D}(\phi') = -A\mathcal{D}(\phi) + L(\phi).$$

Since $t \to T(t)\phi$ is continuously differentiable from $\mathbb{R}^+$ to X_α, then $\phi \in D(B)$.

□

Now, let us study the spectral of the linear equation. We assume that $\mathcal{B}_\alpha$ satisfies the following axiom:

(**D**) There exists a constant $\nu \in \mathbb{R}$ such that for every $x \in X$ and $\lambda \in \mathbb{C}$ with $\Re(\lambda) > \nu$, one has

$$\epsilon_\lambda \otimes x \in \mathcal{B}_\alpha \quad \text{and} \quad \sup_{|x|\leq 1} |\epsilon_\lambda \otimes x| < \infty,$$

where $(\epsilon_\lambda \otimes x)(\theta) = e^{\lambda\theta}x$ for $\theta \leq 0$.

For $\lambda \in \mathbb{C}$ such that $\Re(\lambda) > \nu$, we define the linear operator $\Delta(\lambda)$ by

$$\begin{cases} D(\Delta(\lambda)) = \Big\{x \in X_\alpha : \ \mathcal{D}(e^{\lambda \cdot}x) \in D(A) \quad \text{and} \ A\mathcal{D}(e^{\lambda \cdot}x) - L(e^{\lambda \cdot}x) \in X_\alpha\Big\}, \\ \Delta(\lambda) = \lambda\mathcal{D}(e^{\lambda \cdot}I) + A\mathcal{D}(e^{\lambda \cdot}I) - L(e^{\lambda \cdot}I). \end{cases}$$

Let $(A_V, D(A_V))$ be the infinitesimal generator of the semigroup $(V(t))_{t\geq 0}$ and $\sigma_p(A_V)$ be the point spectrum of A_V.

Theorem 8.5.5 *Assume that* (**H**$_1$)*,* (**H**$_2$)*,* (**H**$_3$)*,* (**H**$_4$)*, and* (**H**$_6$) *hold. Assume furthermore that the axioms* (**A**)*,* (**B**)*,* (**C**)*, and* (**D**) *are satisfied. Let $\lambda \in \mathbb{C}$ with $\Re(\lambda) > \nu$. If $\mathcal{B}_\alpha$ is a uniform fading memory space and $\mathcal{D}$ is stable, then the following are equivalent:*

(i) $\lambda \in \sigma_p(A_V)$.
(ii) $ker\,\Delta(\lambda) \neq \{0\}$.

□

Proof Let $\lambda \in \sigma_p(A_V)$ with $\Re(\lambda) > \nu$. Then, there exists $\phi \in D(A_V)$, $\phi \neq 0$, with $A_V\phi = \lambda\phi$. That leads to

$$\lim_{t\to 0} \frac{1}{t}(V(t)\phi - \phi) = \lambda\phi$$

and

$$\lim_{t\to 0}\frac{1}{t}\mathcal{D}(V(t)\phi-\phi)(0)=\lambda\mathcal{D}(\phi(0)).$$

Since for all $t>0$,

$$\frac{1}{t}(T(t)\mathcal{D}(\phi(0))-\Leftarrow\phi(0)))=\frac{1}{t}\mathcal{D}(V(t)\phi-\phi)(0)-\frac{1}{t}\int_0^t T(t-s)L(V(s)\phi)ds,$$

then letting t goes to 0, and one obtains

$$\mathcal{D}(\phi(0))\in D(A)\quad\text{and}\quad -A\mathcal{D}(\phi(0))=\lambda\mathcal{D}(\phi(0))-l(\phi). \tag{8.29}$$

Moreover, using the spectral mapping (Theorem 2.4 in [18]), we have

$$e^{\lambda t}\in\sigma_p(V(t))\quad\text{and}\quad V(t)\phi=e^{\lambda t}\phi\quad\text{for all}\quad t>0.$$

Letting $t>0$ and $\theta\le 0$ such that $t+\theta\ge 0$, the translation property of the semigroup solution leads to

$$(V(t)\phi)(\theta)=(V(t+\theta)\phi)(0)=e^{\lambda t}\phi(\theta)=e^{\lambda(t+\theta)}\phi(0).$$

Thus, $\phi(\theta)=e^{\lambda\theta}\phi(0)$ for $\theta\ge 0$. Since $\phi\neq 0$, using (8.29), it follows that $\mathcal{D}(\phi(0))\in ker\Delta(\lambda)$.

Conversely, if ϕ verifies all conditions of Theorem 8.3.3, then $A_V\phi=\phi'$. Taking $x\in D(A)$ such that $x\neq 0$ and $\Delta(\lambda)x=0$, then the function $\epsilon_\lambda\otimes x$ satisfies all conditions of Theorem 8.3.3, and we deduce that

$$A_V(\epsilon_\lambda\otimes x)=\lambda(\epsilon_\lambda\otimes x).$$

□

Now, let

$$\nu_0=\inf\{\nu\in\mathbb{R}:\quad\text{such that}\quad\textbf{(D)}\quad\text{is satisfied}\}.$$

Lemma 8.5.2 ([18]) *If $\mathcal{B}$ is a uniform fading memory space, then $\nu_0<0$.* □

Definition 8.5.1 $\lambda\in\mathbb{C}$ is a characteristic value of Eq. (8.28) if

$$\Re(\lambda)>\nu_0\quad\text{and}\quad ker\Delta(\lambda)\neq\{0\}.$$

Let

$$s'(A_V)=\sup\{\Re(\lambda):\quad\lambda\in\sigma(A_V)-\sigma_{ess}(A_V)\}.$$

It is well-known that $\sigma(A_V) - \sigma_{ess}(A_V)$ contains a finite number of eigenvalues of A_V. Consequently, the stability of $(V(t))_{t\geq 0}$ is completely determined by $s'(A_V)$.

Theorem 8.5.6 *Assume that* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, $(\mathbf{H}_3)$, $(\mathbf{H}_4)$, $(\mathbf{H}_5)$, *and* $(\mathbf{H}_6)$ *hold. Furthermore, assume that the axioms* $(\mathbf{A})$, $(\mathbf{B})$, $(\mathbf{C})$, *and* $(\mathbf{D})$ *are satisfied. If* $\mathcal{B}$ *is a uniform fading memory space and* $\mathcal{D}$ *is stable, then the following holds:*

(i) *If* $s'(A_V) < 0$, *then* $(V(t))_{t\geq 0}$ *is exponentially stable.*
(ii) *If* $s'(A_V) = 0$, *then there exists* $\phi \in \mathcal{B}_\alpha$ *such that* $|V(t)\phi|_{\mathcal{B}_\alpha} = |\phi|_{\mathcal{B}_\alpha}$.
(iii) *If* $s'(A_V) > 0$, *then there exists* $\phi \in \mathcal{B}_\alpha$ *such that* $\lim_{t\to+\infty} |V(t)\phi|_{\mathcal{B}_\alpha} = +\infty$.

□

We deduce the following stability result in the nonlinear case, from Theorem 8.5.2.

Theorem 8.5.7 *Assume that* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, $(\mathbf{H}_3)$, $(\mathbf{H}_4)$, $(\mathbf{H}_5)$, *and* $(\mathbf{H}_6)$ *hold. Furthermore, assume that the axioms* $(\mathbf{A})$, $(\mathbf{B})$, $(\mathbf{C})$, *and* $(\mathbf{D})$ *are satisfied. If* $\mathcal{B}$ *is a uniform fading memory space and* $\mathcal{D}$ *is stable, then the following holds:*

(i) *If* $s'(A_V) < 0$, *then the zero equilibrium of* $(U(t))_{t\geq 0}$ *is locally exponentially stable.*
(ii) *If* $s'(A_V) > 0$, *then the zero equilibrium of* $(U(t))_{t\geq 0}$ *is unstable.*

□

8.6 Application

To apply the theoretical results of this chapter, we consider the following nonlinear system with infinite delay:

$$\begin{cases} \dfrac{\partial}{\partial t}\Big[v(t,\xi) - qv(t-r,\xi)\Big] = \dfrac{\partial^2}{\partial \xi^2}\Big[v(t,\xi) - qv(t-r,\xi)\Big] \\ +b\frac{\partial}{\partial \xi}\Big[v(t,\xi) - qv(t-r,\xi)\Big] \\ +c\displaystyle\int_{-\infty}^{0} g(\theta, v(t+\theta,\xi))d\theta \quad \text{for } t \geq 0 \text{ and } \xi \in [0,\pi] \\ \\ v(t,0) - qv(t-r,0) = v(t,\pi) - qv(t-r,\pi) = 0 \text{ for } t \geq 0 \\ \\ v(\theta,\xi) = \psi(\theta,\xi) \text{ for } \theta \in (-\infty,0] \text{ and } \xi \in [0,\pi], \end{cases} \tag{8.30}$$

where $g : (-\infty, 0] \times \mathbb{R} \to \mathbb{R}$ is a function and $c \in \mathbb{R}_+^*$, $b \in \mathbb{R}$. q is a positive constant such that $|q| < 1$. $H : \mathbb{R}^2 \to \mathbb{R}$ is a Lipschitz continuous with $H(0,0) = 0$. The initial data ψ will be precised in the next.

In order to write system (8.30) in an abstract form, we introduce the space $X = L^2((0,\pi);\mathbb{R})$. Let A be the operator defined on X by

$$\begin{cases} D(A) = H^2((0,\pi);\mathbb{R}) \cap H_0^1((0,\pi);\mathbb{R}), \\ Ay = -y'' \text{ for } y \in D(A). \end{cases}$$

Then, $(-A)$ generates an analytic semigroup $(T(t))_{t\geq 0}$ on X. Moreover, $T(t)$ is compact on X for every $t > 0$. The spectrum $\sigma(-A)$ is equal to the point spectrum $P\sigma(-A)$ and is given by $\sigma(-A) = \{-n^2 : n \geq 1\}$, and the associated eigenfunctions $(\phi_n)_{n\geq 1}$ are given by $\phi_n = \sqrt{\frac{2}{\pi}}\sin(nx)$ for $x \in [0,\pi]$; the associated analytic semigroup is explicitly given by

$$T(t)y = \sum_{n=1}^{\infty} e^{-n^2 t}\,(y,\phi_n)\,\phi_n \quad \text{for } t \geq 0 \text{ and } y \in X,$$

where $(.,.)$ is an inner product on X.

Lemma 8.6.1 ([21]) *If $\alpha = \frac{1}{2}$, then*

$$Ay = \sum_{n=1}^{+\infty} n^2 (y,\phi_n)\phi_n \ \textit{for}\ y \in D(A),$$

$$A^{\frac{1}{2}}y = \sum_{n=1}^{+\infty} n(y,\phi_n)\phi_n \ \textit{for}\ y \in X,$$

$$A^{\frac{1}{2}}T(t)y = \sum_{n=1}^{+\infty} n e^{-n^2 t}(y,\phi_n)\phi_n \ \textit{for}\ y \in X,$$

$$A^{-\frac{1}{2}}y = \sum_{n=1}^{+\infty} \left(\frac{1}{n}\right)(y,\phi_n)\phi_n \ \textit{for}\ y \in X,$$

and

$$A^{-\frac{1}{2}}T(t)y = \sum_{n=1}^{+\infty} \left(\frac{1}{n}\right)e^{-n^2 t}(y,\phi_n)\phi_n \ \textit{for}\ y \in X.$$

There exists $M \geq 1$ (see [21]) such that for $t \geq 0$, $|T(t)| \leq Me^{\omega t}$ for some $-1 < \omega < 0$.

Then, the semigroup $\{T(t)\}_{t\geq 0}$ is exponentially stable.

Note also that (see [21]) there exists $M_{\frac{1}{2}} \geq 0$ such that

$$|A^{\frac{1}{2}}T(t)| \leq M_{\frac{1}{2}} t^{-\frac{1}{2}} e^{\omega t} \text{ for each } t > 0.$$

Therefore, hypotheses $(\mathbf{H}_1)$ and $(\mathbf{H}_5)$ are satisfied.

Lemma 8.6.2 ([7]) *If $m \in D(A^{\frac{1}{2}})$, then m is absolutely continuous, $\frac{\partial}{\partial x}m \in X$. Moreover, there exist positive constants N_0 and M_0 such that*

$$N_0|A^{\frac{1}{2}}m|_X \leq |\frac{\partial}{\partial x}m|_X \leq M_0|A^{\frac{1}{2}}m|_X.$$

Let $\gamma > 0$. We consider the following phase space

$$\mathcal{B} = C_\gamma = \left\{\phi \in C((-\infty, 0]; X) : \lim_{\theta \to -\infty} e^{\gamma\theta}|\phi(\theta)| \text{ exists in } X\right\}$$

provided with the following norm:

$$|\phi|_{C_\gamma} = \sup_{\theta \leq 0} e^{\gamma\theta}|\phi(\theta)|_X \text{ for } \phi \in C_\gamma.$$

According to [7], $\mathcal{B}$ satisfies Axioms $(\mathbf{A})$, $(\mathbf{B})$ and is a uniform fading memory space. Moreover, it is well-known that $K(t) = 1$ for every $t \in \mathbb{R}^+$ and $M(t) = e^{-\gamma t}$ for $t \in \mathbb{R}^+$. Therefore, the norm in $\mathcal{B}_{\frac{1}{2}}$ is given (see [7]) by

$$|\phi|_{\mathcal{B}_{\frac{1}{2}}} = \sup_{\theta \leq 0} e^{\gamma\theta}|A^{\frac{1}{2}}\phi(\theta)|_X.$$

One can write (see, [21], p.144)

$$\int_0^\pi \Big(\phi(\theta)(\xi)\Big)^2 d\xi \leq |A^{\frac{1}{2}}\phi(\theta)|_X^2 = \int_0^\pi \Big(\frac{\partial}{\partial\xi}\phi(\theta)(\xi)\Big)^2 d\xi. \tag{8.31}$$

Next, we assume the following.

$(\mathbf{H}_7)$ For $\theta \leq 0$ and $\zeta_1, \zeta_2 \in \mathbb{R}$, $|g(\theta, \zeta_1) - g(\theta, \zeta_2)| \leq s(\theta)|\zeta_1 - \zeta_2|$, $g(\theta, 0) = 0$, $\frac{\partial}{\partial\zeta}g(\theta, 0) \neq 0$, where s is some nonnegative function that verifies

$$\int_{-\infty}^0 e^{-2\gamma\theta}s(\theta) < \infty.$$

Let f_1, f_2, and f be defined on $\mathcal{B}_{\frac{1}{2}}$ by

$$f_1(\phi)(\xi) = c\int_{-\infty}^{0} g(\theta, \phi(\theta)(\xi))d\theta \text{ for } \xi \in [0, \pi],$$

$$f_2(\phi)(\xi) = b\frac{\partial}{\partial \xi}\Big[\phi(0)(\xi) - q\phi(-r)(\xi)\Big] \text{ for } \xi \in [0, \pi],$$

and

$$f(\phi)(\xi) = f_1(\phi)(\xi) + f_2(\phi)(\xi) \text{ for } \xi \in [0, \pi].$$

Proposition 8.6.1 *For each* $\phi \in \mathcal{B}_{\frac{1}{2}}$, $f(\phi) \in L^2((0, \pi); \mathbb{R})$, *and* f *is continuous on* $\mathcal{B}_{\frac{1}{2}}$. □

Proof Let $\phi \in \mathcal{B}_{\frac{1}{2}}$. Since for all $\xi \in [0, \pi]$ and for all $\theta \in (-\infty, 0]$, we have

$$\begin{aligned} |g(\theta, \xi)| &\le s(\theta)|\xi| + |g(\theta, 0)| \\ &= s(\theta)|\xi|, \end{aligned}$$

then for all $\xi \in [0, \pi]$,

$$|f_1(\phi)(\xi)| \le c\int_{-\infty}^{0} |s(\theta)|\,|\phi(\theta)(\xi)|d\theta.$$

Let us set

$$B(\xi) = \int_{-\infty}^{0} |s(\theta)||\phi(\theta)(\xi)|d\theta \text{ for } \xi \in [0, \pi].$$

Using Hölder inequality, one can write

$$\begin{aligned} B(\xi) &= \int_{-\infty}^{0} e^{-2\gamma\theta}|s(\theta)||\phi(\theta)(\xi)|e^{2\gamma\theta}d\theta \\ &\le \left(\int_{-\infty}^{0} |e^{-2\gamma\theta}s(\theta)|^2 d\theta\right)^{\frac{1}{2}} \left(\int_{-\infty}^{0} |\phi(\theta)(\xi)e^{2\gamma\theta}|^2 d\theta\right)^{\frac{1}{2}}. \end{aligned}$$

Then, using the above inequality and the inequality (8.31),

$$\begin{aligned} \int_0^{\pi} |B(\xi)|^2 d\xi &\le \int_0^{\pi} \left(\left(\int_{-\infty}^{0} |e^{-2\gamma\theta}s(\theta)|^2 d\theta\right)\left(\int_{-\infty}^{0} |\phi(\theta)(\xi)e^{2\gamma\theta}|^2 d\theta\right)\right) d\xi \\ &= \int_0^{\pi} \left(|e^{-2\gamma\cdot}s|^2_{L^2(\mathbb{R}^-)} \int_{-\infty}^{0} |\phi(\theta)(\xi)e^{2\gamma\theta}|^2 d\theta\right) d\xi \end{aligned}$$

$$
\begin{aligned}
&\le |e^{-2\gamma\cdot}s|^2_{L^2(\mathbb{R}^-)}\left(\int_{-\infty}^0 e^{2\gamma\theta}\left(e^{2\gamma\theta}\int_0^\pi |\phi(\theta)(\xi)|^2 d\xi\right)d\theta\right)\\
&\le |e^{-2\gamma\cdot}s|^2_{L^2(\mathbb{R}^-)}\left(\int_{-\infty}^0 e^{2\gamma\theta}\left(e^{2\gamma\theta}\int_0^\pi |\frac{\partial}{\partial\xi}\phi(\theta)(\xi)|^2 d\xi\right)d\theta\right)\\
&= |e^{-2\gamma\cdot}s|^2_{L^2(\mathbb{R}^-)}\left(\int_{-\infty}^0 e^{2\gamma\theta}\left(e^{2\gamma\theta}|\frac{\partial}{\partial\xi}\phi(\theta)|^2_{L^2([0,\pi];\mathbb{R})}\right)d\theta\right)\\
&\le |e^{-2\gamma\cdot}s|^2_{L^2(\mathbb{R}^-)}\left(\int_{-\infty}^0 e^{2\gamma\theta}\left(\sup_{\theta\le 0} e^{2\gamma\theta}|\frac{\partial}{\partial\xi}\phi(\theta)|^2_{L^2([0,\pi];\mathbb{R})}\right)d\theta\right)\\
&\le |e^{-2\gamma\cdot}s|^2_{L^2(\mathbb{R}^-)}\left(\int_{-\infty}^0 e^{2\gamma\theta}\left(\sup_{\theta\le 0} e^{2\gamma\theta}|A^{\frac{1}{2}}\phi(\theta)|^2_{L^2([0,\pi];\mathbb{R})}\right)d\theta\right)\\
&\le |e^{-2\gamma\cdot}s|^2_{L^2(\mathbb{R}^-)}\left(\int_{-\infty}^0 e^{2\gamma\theta}|\phi|^2_{\mathcal{B}_{\frac{1}{2}}}d\theta\right)\\
&\le |e^{-2\gamma\cdot}s|^2_{L^2(\mathbb{R}^-)}|\phi|^2_{\mathcal{B}_{\frac{1}{2}}}\int_{-\infty}^0 e^{2\gamma\theta}d\theta\\
&< \infty.
\end{aligned}
$$

Also, we refer to Minkowski inequality to obtain

$$
\begin{aligned}
\int_0^\pi |f_2(\phi)(\xi)|^2 d\xi &= \int_0^\pi \left|\frac{\partial}{\partial\xi}\Big[\phi(0)(\xi) - q\phi(-r)(\xi)\Big]\right|^2 d\xi\\
&\le \int_0^\pi \left|\frac{\partial}{\partial\xi}\phi(0)(\xi)\right|^2 d\xi + \int_0^\pi \left|q\frac{\partial}{\partial\xi}\phi(-r)(\xi)\right|^2 d\xi\\
&\quad +2\Big(\int_0^\pi \left|\frac{\partial}{\partial\xi}\phi(0)(\xi)\right|^2 d\xi\Big)^{\frac{1}{2}}\Big(\int_0^\pi \left|q\frac{\partial}{\partial\xi}\phi(-r)(\xi)\right|^2 d\xi\Big)^{\frac{1}{2}}\\
&\le \left|A^{\frac{1}{2}}\phi(0)\right|^2_{L^2([0,\pi];\mathbb{R})} + q^2\left|A^{\frac{1}{2}}\phi(-r)\right|^2_{L^2([0,\pi];\mathbb{R})}\\
&\quad +2q\left|A^{\frac{1}{2}}\phi(0)\right|_{L^2([0,\pi];\mathbb{R})}\left|A^{\frac{1}{2}}\phi(-r)\right|_{L^2([0,\pi];\mathbb{R})}\\
&\le \sup_{\theta\le 0} e^{2\gamma\theta}\left|A^{\frac{1}{2}}\phi(\theta)\right|^2_{L^2([0,\pi];\mathbb{R})}\\
&\quad +q^2 e^{2\gamma r}\sup_{\theta\le 0} e^{2\gamma\theta}\left|A^{\frac{1}{2}}\phi(\theta)\right|^2_{L^2([0,\pi];\mathbb{R})}\\
&\quad +2q\sup_{\theta\le 0} e^{2\gamma\theta}\left|A^{\frac{1}{2}}\phi(\theta)\right|_{L^2([0,\pi];\mathbb{R})} e^{2\gamma r}\\
&\quad \times\sup_{\theta\le 0} e^{2\gamma\theta}\left|A^{\frac{1}{2}}\phi(\theta)\right|_{L^2([0,\pi];\mathbb{R})}
\end{aligned}
$$

$$\leq \sup_{\theta\leq 0} e^{2\gamma\theta}\left|A^{\frac{1}{2}}\phi(\theta)\right|^2_{L^2([0,\pi];\mathbb{R})}$$
$$+q^2e^{2\gamma r}\sup_{\theta\leq 0} e^{2\gamma\theta}\left|A^{\frac{1}{2}}\phi(\theta)\right|^2_{L^2([0,\pi];\mathbb{R})}$$
$$+2qe^{2\gamma r}\sup_{\theta\leq 0} e^{2\gamma\theta}\left|A^{\frac{1}{2}}\phi(\theta)\right|^2_{L^2([0,\pi];\mathbb{R})}$$
$$< \infty.$$

We conclude that $f(\phi) = (f_1 + f_2)(\phi) \in L^2([0,\pi];\mathbb{R})$ for all $\phi \in \mathcal{B}_{\frac{1}{2}}$.

Let us show that f is continuous. For this purpose, let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{B}_{\frac{1}{2}}$ and $\phi \in \mathcal{B}_{\frac{1}{2}}$ such that $\phi_n \to \phi$ in $\mathcal{B}_{\frac{1}{2}}$ as $n \to +\infty$. Then

$$\Big(f_1(\phi_n) - f_1(\phi)\Big)(\xi) = c\int_{-\infty}^0 g(\theta,\phi_n(\theta)(\xi))d\theta - c\int_{-\infty}^0 g(\theta,\phi(\theta)(\xi))d\theta$$
$$= c\int_{-\infty}^0 \Big[g(\theta,\phi_n(\theta)(\xi)) - g(\theta,\phi(\theta)(\xi))\Big]d\theta,$$

and we obtain that

$$|(f_1(\phi_n) - f_1(\phi))(\xi)| \leq c\int_{-\infty}^0 |s(\theta)|\,|\phi_n(\theta)(\xi) - \phi(\theta)(\xi))|d\theta.$$

Let us set for all $\xi \in [0,\pi]$,

$$J_n(\xi) = c\int_{-\infty}^0 |s(\theta)|\Big|\phi_n(\theta)(\xi) - \phi(\theta)(\xi)\Big|d\theta.$$

Then

$$|J_n(\xi)| \leq c\int_{-\infty}^0 e^{-2\gamma\theta}|s(\theta)|\Big|\phi_n(\theta)(\xi) - \phi(\theta)(\xi)\Big|e^{2\gamma\theta}d\theta$$
$$\leq c\left(\int_{-\infty}^0 \left|e^{-2\gamma\theta}s(\theta)\right|^2 d\theta\right)^{\frac{1}{2}}\left(\int_{-\infty}^0 \left|\Big(\phi_n(\theta)(\xi) - \phi(\theta)(\xi)\Big)e^{2\gamma\theta}\right|^2 d\theta\right)^{\frac{1}{2}},$$

which leads to

$$\int_0^\pi |J_n(\xi)|^2 d\xi \leq |c|^2|e^{-2\gamma\cdot}s|^2_{L^2(\mathbb{R}^-)}\int_{-\infty}^0 \left(e^{2\gamma\theta}e^{2\gamma\theta}\int_0^\pi \Big|\phi_n(\theta)(\xi)\right.$$
$$\left. - \phi(\theta)(\xi)\Big|^2 d\xi\right)d\theta$$

$$
\begin{aligned}
&\leq |c|^2 |e^{-2\gamma\cdot}s|^2_{L^2(\mathbb{R}^-)} \int_{-\infty}^0 \left(e^{2\gamma\theta} e^{2\gamma\theta} \int_0^\pi \left| \frac{\partial}{\partial\xi}\phi_n(\theta)(\xi) \right.\right. \\
&\quad \left.\left. - \frac{\partial}{\partial\xi}\phi(\theta)(\xi) \right|^2 d\xi \right) d\theta \\
&\leq |c|^2 |e^{-2\gamma\cdot}s|^2_{L^2(\mathbb{R}^-)} \int_{-\infty}^0 e^{2\gamma\theta} \left(\sup_{\theta\leq 0} e^{2\gamma\theta} \int_0^\pi \left| \frac{\partial}{\partial\xi}\phi_n(\theta)(\xi) \right.\right. \\
&\quad \left.\left. - \frac{\partial}{\partial\xi}\phi(\theta)(\xi) \right|^2 d\xi \right) d\theta \\
&\leq |c|^2 |e^{-2\gamma\cdot}s|^2_{L^2(\mathbb{R}^-)} \int_{-\infty}^0 e^{2\gamma\theta} \Big(\sup_{\theta\leq 0} e^{2\gamma\theta} \Big| A^{\frac{1}{2}}(\phi_n(\theta) \\
&\quad -\phi(\theta)) \Big|^2_{L^2([0,\pi];\mathbb{R})} \Big) d\theta \\
&\leq |c|^2 \Big| e^{-2\gamma\cdot}s \Big|^2_{L^2(\mathbb{R}^-)} \Big| \phi_n - \phi \Big|^2_{\mathcal{B}_{\frac{1}{2}}} \int_{-\infty}^0 e^{2\gamma\theta} d\theta.
\end{aligned}
$$

Since $\phi_n \to \phi$ in $\mathcal{B}_{\frac{1}{2}}$, then $\int_0^\pi |J_n(\xi)|^2 d\xi \to 0$ as $n \to +\infty$. Therefore, f_1 is continuous. Moreover,

$$
\begin{aligned}
\int_0^\pi \Big| f_2\Big(\phi_n(\xi) - \phi(\xi)\Big)\Big|^2 d\xi &= \int_0^\pi \Big| \frac{\partial}{\partial\xi}\Big[\Big(\phi_n(0) - \phi(0)\Big)(\xi) \\
&\quad -q\Big(\phi_n(-r) - \phi(-r)\Big)(\xi)\Big]\Big|^2 d\xi \\
&\leq \int_0^\pi \Big| \frac{\partial}{\partial\xi}\Big(\phi_n(0)(\xi) - \phi(0)(\xi)\Big)\Big|^2 d\xi \\
&\quad + \int_0^\pi \Big| q\frac{\partial}{\partial\xi}\Big(\phi_n(-r)(\xi) - \phi(-r)(\xi)\Big)\Big|^2 d\xi \\
&\quad +2\Big(\int_0^\pi \Big| \frac{\partial}{\partial\xi}\Big(\phi_n(0)(\xi) - \phi(0)(\xi)\Big)\Big|^2 d\xi\Big)^{\frac{1}{2}} \\
&\quad \times\Big(\int_0^\pi \Big| q\frac{\partial}{\partial\xi}\Big(\phi_n(-r)(\xi) - \phi(-r)(\xi)\Big)\Big|^2 d\xi\Big)^{\frac{1}{2}} \\
&\leq \Big| A^{\frac{1}{2}}\Big(\phi_n(0) - \phi(0)\Big)\Big|^2_{L^2([0,\pi];\mathbb{R})} \\
&\quad +q^2 \Big| A^{\frac{1}{2}}\Big(\phi_n(-r) - \phi(-r)\Big)\Big|^2_{L^2([0,\pi];\mathbb{R})}
\end{aligned}
$$

$$
\begin{aligned}
&+2q\left|A^{\frac{1}{2}}\Big(\phi_n(0)-\phi(0)\Big)\right|_{L^2([0,\pi];\mathbb{R})}\left|A^{\frac{1}{2}}\right.\\
&\left.\times\Big(\phi_n(-r)-\phi(-r)\Big)\right|_{L^2([0,\pi];\mathbb{R})}\\
\leq\ &\sup_{\theta\leq 0} e^{2\gamma\theta}\left|A^{\frac{1}{2}}\Big(\phi_n(\theta)-\phi(\theta)\Big)\right|^2_{L^2([0,\pi];\mathbb{R})}\\
&+q^2e^{2\gamma r}\sup_{\theta\leq 0} e^{2\gamma\theta}\left|A^{\frac{1}{2}}\Big(\phi_n(\theta)-\phi(\theta)\Big)\right|^2_{L^2([0,\pi];\mathbb{R})}\\
&+2q\sup_{\theta\leq 0} e^{2\gamma\theta}\left|A^{\frac{1}{2}}\Big(\phi_n(\theta)-\phi(\theta)\Big)\right|_{L^2([0,\pi];\mathbb{R})}\\
&\times e^{2\gamma r}\sup_{\theta\leq 0} e^{2\gamma\theta}\left|A^{\frac{1}{2}}\Big(\phi_n(\theta)-\phi(\theta)\Big)\right|_{L^2([0,\pi];\mathbb{R})}\\
\leq\ &\left|\phi_n-\phi\right|^2_{\mathcal{B}_{\frac{1}{2}}}+q^2e^{2\gamma r}\left|\phi_n-\phi\right|^2_{\mathcal{B}_{\frac{1}{2}}}\\
&+2qe^{2\gamma r}\left|\phi_n-\phi\right|^2_{\mathcal{B}_{\frac{1}{2}}}.
\end{aligned}
$$

Using the fact that $\phi_n \to \phi$ in $\mathcal{B}_{\frac{1}{2}}$ as $n \to +\infty$, we obtain that $\int_0^\pi \left|f_2\Big(\phi_n(\xi) - \phi(\xi)\Big)\right|^2 d\xi \to 0$ when $n \to +\infty$.

Hence, $f(\phi_n) \to f(\phi)$ in $L^2([0,\pi];\mathbb{R})$ as $n \to +\infty$ and the proof is complete. □

Let

$$
\begin{cases}
u(t)(x) = v(t,x) & \text{for } t \geq 0 \text{ and } x \in [0,\pi],\\
u_0(\theta)(x) = \psi(\theta,x) & \text{for } \theta \in (-\infty,0] \text{ and } x \in [0,\pi].
\end{cases}
$$

We need the following result to prove that $(\mathbf{H}_3)$ is satisfied.

Proposition 8.6.2 *Assume that* $(\mathbf{H}_7)$ *holds. Then,* f *is Lipschitzian.*

Proof We have to show that f_1 and f_2 are Lipschitz functions. So, let ϕ and ψ be in $\mathcal{B}_{\frac{1}{2}}$. Then, for $\xi \in [0,\pi]$, one has

$$
(f_1(\phi) - f_1(\psi))(\xi) = c\int_{-\infty}^0 \Big[g(\theta,\phi(\theta)(\xi)) - g(\theta,\psi(\theta)(\xi))\Big]d\theta \text{ for } \xi \in [0,\pi].
$$

Note that using Hölder inequality, one can write

$$
\begin{aligned}
|(f_1(\phi) - f_1(\psi))(\xi)| &\le |c| \int_{-\infty}^{0} \Big| g(\theta, \phi(\theta)(\xi)) - g(\theta, \psi(\theta)(\xi)) \Big| d\theta \\
&\le c \int_{-\infty}^{0} |s(\theta)| \Big| \phi(\theta)(\xi) - \psi(\theta)(\xi) \Big| d\theta \\
&= c \int_{-\infty}^{0} e^{-2\gamma\theta} |s(\theta)|\, e^{2\gamma\theta} \Big| \phi(\theta)(\xi) - \psi(\theta)(\xi) \Big| d\theta \\
&\le c \left(\int_{-\infty}^{0} |e^{-2\gamma\theta} s(\theta)|^2 d\theta \right)^{\frac{1}{2}} \left(\int_{-\infty}^{0} e^{4\gamma\theta} \Big| \phi(\theta)(\xi) \right. \\
&\qquad \left. - \psi(\theta)(\xi) \Big|^2 d\theta \right)^{\frac{1}{2}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|f_1(\phi)(\xi) - f_1(\psi)(\xi)|^2 \le |c|^2 \Big(\int_{-\infty}^{0} |e^{-2\gamma\theta} s(\theta)|^2 d\theta \Big) \Big(\int_{-\infty}^{0} e^{4\gamma\theta} \Big| \phi(\theta)(\xi) \\
- \psi(\theta)(\xi) \Big|^2 d\theta \Big),
\end{aligned}
$$

for which we deduce that

$$
\begin{aligned}
&\int_0^{\pi} |f_1(\phi)(\xi) - f_1(\psi)(\xi)|^2 d\xi \\
&\quad \le |c|^2 \Big(\int_{-\infty}^{0} |e^{-2\gamma\theta} s(\theta)|^2 d\theta \Big) \times \int_{-\infty}^{0} e^{4\gamma\theta} \left(\int_0^{\pi} |\phi(\theta)(\xi) - \psi(\theta)(\xi)|^2 \, d\xi \right) d\theta \\
&\quad \le |c|^2 \Big(\int_{-\infty}^{0} |e^{-2\gamma\theta} s(\theta)|^2 d\theta \Big) \\
&\qquad \times \int_{-\infty}^{0} e^{2\gamma\theta} \left(\sup_{\theta \le 0} e^{2\gamma\theta} \int_0^{\pi} \left| \frac{\partial}{\partial \xi} \phi(\theta)(\xi) - \frac{\partial}{\partial \xi} \psi(\theta)(\xi) \right|^2 d\xi \right) d\theta \\
&\quad \le |c|^2 \Big(\int_{-\infty}^{0} |e^{-2\gamma\theta} s(\theta)|^2 d\theta \Big) \\
&\qquad \times \int_{-\infty}^{0} e^{2\gamma\theta} \left(\sup_{\theta \le 0} e^{\gamma\theta} \sqrt{\int_0^{\pi} \left| \frac{\partial}{\partial \xi} \phi(\theta)(\xi) - \frac{\partial}{\partial \xi} \psi(\theta)(\xi) \right|^2 d\xi} \right)^2 d\theta \\
&\quad \le |c|^2 \Big(\int_{-\infty}^{0} |e^{-2\gamma\theta} s(\theta)|^2 d\theta \Big)
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_{-\infty}^{0} e^{2\gamma\theta} \left(\sup_{\theta \le 0} e^{\gamma\theta} |A^{\frac{1}{2}}(\phi(\theta) - \psi(\theta))|_{L^2([0,\pi];\mathbb{R})} \right)^2 d\theta \\
&\le |c|^2 \Big(\int_{-\infty}^{0} |e^{-2\gamma\theta} s(\theta)|^2 d\theta \Big) \int_{-\infty}^{0} e^{2\gamma\theta} \, |\phi - \psi|^2_{\mathcal{B}_{\frac{1}{2}}} \, d\theta \\
&\le \frac{|c|^2}{2\gamma} \Big| e^{-2\gamma \cdot} s \Big|^2_{L^2(\mathbb{R}^-)} \Big| \phi - \psi \Big|^2_{\mathcal{B}_{\frac{1}{2}}} .
\end{aligned}
$$

Finally, we obtain that

$$
|f_1(\phi) - f_2(\psi)|_{L^2([0,\pi];\mathbb{R})} \le k' \, |\phi - \psi|_{\mathcal{B}_{\frac{1}{2}}} \quad \text{for } \phi, \psi \in \mathcal{B}_{\frac{1}{2}},
$$

where

$$
k' = \frac{|c|}{\sqrt{2\gamma}} \left(\int_{-\infty}^{0} |e^{-2\gamma\theta} s(\theta)|^2 d\theta \right)^{\frac{1}{2}} .
$$

Moreover,

$$
\begin{aligned}
\Big| f_1(\phi) - f_2(\psi) \Big|^2_{L^2([0,\pi];\mathbb{R})} &\le \Big| \phi - \psi \Big|^2_{\mathcal{B}_{\frac{1}{2}}} + q^2 e^{2\gamma r} \Big| \phi - \psi \Big|^2_{\mathcal{B}_{\frac{1}{2}}} + 2q e^{2\gamma r} \Big| \phi - \psi \Big|^2_{\mathcal{B}_{\frac{1}{2}}} \\
&\le k'' \Big| \phi - \psi \Big|^2_{\mathcal{B}_{\frac{1}{2}}} .
\end{aligned}
$$

Therefore, f is Lipschitzian and ($\mathbf{H}_4$) is satisfied. □

Let us define the operators $\mathcal{D}$ and $\mathcal{D}_0$ on $\mathcal{B}_{\frac{1}{2}}$ by

$$
(\mathcal{D}(\phi)(\xi)) = \phi(0)(\xi) - q\phi(-r)(\xi) \quad \text{for all} \quad \xi \in [0,\pi]
$$

and

$$
(\mathcal{D}_0(\phi))(\xi) = q\phi(-r)(\xi) \quad \text{for all} \quad \xi \in [0,\pi].
$$

Then, $\mathcal{D}(\phi) = \phi(0) - \mathcal{D}_0(\phi)$.

Proposition 8.6.3 $\mathcal{D} \in \mathcal{L}(\mathcal{B}_{\frac{1}{2}}; X)$. □

Proof Let $\phi \in \mathcal{B}_{\frac{1}{2}}$. Then, $\mathcal{D}_0(\phi)(\xi) = q\phi(-r)(\xi)$ for all $\xi \in [0,\pi]$. We can write

$$
\int_0^{\pi} |\mathcal{D}_0(\phi)(\xi)|^2 d\xi = \int_0^{\pi} q^2 |\phi(-r)(\xi)|^2 d\xi
$$

$$
\begin{aligned}
&= q^2 e^{2\gamma r} e^{-2\gamma r} \int_0^{\pi} |\phi(-r)(\xi)|^2 \\
&= q^2 e^{2\gamma r} e^{-2\gamma r} \int_0^{\pi} |\frac{\partial}{\partial \xi}\phi(-r)(\xi)|^2 \\
&\leq q^2 e^{2\gamma r} \sup_{\theta \leq 0} e^{2\gamma\theta} |A^{\frac{1}{2}}\phi(\theta)|^2_{L^2([0,\pi];\mathbb{R})} \\
&= q^2 e^{2\gamma r} |\phi|^2_{\mathcal{B}_{\frac{1}{2}}}.
\end{aligned}
$$

Hence, $\mathcal{D}_0 \in \mathcal{L}(\mathcal{B}_{\frac{1}{2}}; X)$. It is obvious that $\phi(0) \in \mathcal{L}(\mathcal{B}_{\frac{1}{2}}; X)$. Therefore, we can conclude that $\mathcal{D} \in \mathcal{L}(\mathcal{B}_{\frac{1}{2}}; X)$ and the proof is complete. □

Since $0 < q < 1$, then $\mathcal{D}$ is stable and $|\mathcal{D}_0| < 1$. Thus, hypothesis $(\mathbf{H}_3)$ is satisfied.

Now, let φ be defined by $\varphi(\theta)(\xi) = \psi(\theta, \xi)$ for all $\theta \in (-\infty, 0]$ and $\xi \in [0, \pi]$. We make the following additional assumption.

$(\mathbf{H}_8)$ $\quad \varphi(\theta) \in D(A^{\frac{1}{2}})$ for all $\theta \leq 0$, with

$$
\sup_{\theta \leq 0} e^{\gamma\theta} \sqrt{\int_0^{\pi} \left(\frac{\partial}{\partial \xi}\psi(\theta, \xi)\right)^2 d\xi} < \infty
$$

and

$$
\lim_{\theta \to \theta_0} \int_0^{\pi} \left(\frac{\partial}{\partial \xi}\psi(\theta, \xi) - \frac{\partial}{\partial \xi}\psi(\theta_0, \xi)\right)^2 d\xi = 0 \text{ for all } \theta_0 \leq 0.
$$

Remark that $(\mathbf{H}_8)$ implies $\varphi \in \mathcal{B}_{\frac{1}{2}}$. Then, Eq. (8.30) can be written as follows:

$$
\begin{cases}
\dfrac{d}{dt}\mathcal{D}(u_t) = -A\mathcal{D}(u_t) + f(u_t) \text{ for } t \geq 0, \\
\\
u_0 = \varphi.
\end{cases}
\tag{8.32}
$$

Consequently, we obtain the existence and uniqueness of a mild solution of problem (8.32). Furthermore, it is clear that f_1 and f_2 are continuously differentiable and their differential functions are given for $\phi, \psi \in \mathcal{B}_{\frac{1}{2}}$ and $\xi \in [0, \pi]$ by

$$
f_1'(\phi)(\psi)(\xi) = c \int_{-\infty}^{0} \frac{\partial}{\partial \zeta} g(\theta, \phi(\theta)(\xi))\psi(\theta)(\xi) d\theta
$$

and

$$
f_2'(\phi)(\psi)(\xi) = b \frac{\partial}{\partial \xi}\Big[\psi(0)(\xi) - q\psi(-r)(\xi)\Big] \text{ for } \xi \in [0, \pi].
$$

Let $v_0 = \psi \in \mathcal{B}_{\frac{1}{2}}$ such that:

(a) $v_0(0,.) - qv_0(-r,.) \in H^2(0,\pi) \cap H_0^1(0,\pi)$ and $\frac{\partial v_0}{\partial \theta} \in \mathcal{B}_{\frac{1}{2}}$.

(b) $\dfrac{\partial v_0(0,\xi)}{\partial \theta} - q\dfrac{\partial v_0(-r,\xi)}{\partial \theta} = \dfrac{\partial^2}{\partial \xi^2}\Big[v_0(0,\xi) - qv_0(-r,\xi)\Big] + b\dfrac{\partial}{\partial \xi}\Big[v_0(0,\xi) - qv_0(-r,\xi)\Big] + c\displaystyle\int_{-\infty}^0 g(\theta, v_0(\theta,\xi))d\theta$ for and $\xi \in [0,\pi]$.

We deduce that

$$\psi \in \mathcal{B}_{\frac{1}{2}}, \psi' \in \mathcal{B}_{\frac{1}{2}}, \quad \mathcal{D}(\psi) \in D(A) \quad \text{, and} \quad \mathcal{D}(\psi') = -A\mathcal{D}(\psi) + f(\psi).$$

Then, problem (8.32) has a unique strict solution for every $\phi \in \mathcal{B}_{\frac{1}{2}}$.

Now, we can see that $f = f_1 + f_2$ is continuously differentiable, and zero is a solution of (8.30), ,i.e., $f(0) = 0$. The differential of f in 0 is given for $\phi, \psi \in \mathcal{B}_{\frac{1}{2}}$ and $\xi \in [0,\pi]$ by

$$L(\psi)(\xi) = f'(0)(\psi)(\xi) = c\int_{-\infty}^0 \frac{\partial}{\partial \zeta} g(\theta,0)\psi(\theta)(\xi)d\theta + b\frac{\partial}{\partial \xi}\Big[\psi(0)(\xi) - q\psi(-r)(\xi)\Big].$$

Consequently, the linearized equation of (8.30) can be written as follows:

$$\begin{cases} \dfrac{\partial}{\partial t}\Big[v(t,\xi) - qv(t-r,\xi)\Big] = \dfrac{\partial^2}{\partial \xi^2}\Big[v(t,\xi) - qv(t-r,\xi)\Big] \\ +b\frac{\partial}{\partial \xi}\Big[v(t,\xi) - qv(t-r,\xi)\Big] + c\displaystyle\int_{-\infty}^0 p(\theta)v(t+\theta,\xi)d\theta \quad \text{for } t \geq 0 \text{ and } \xi \in [0,\pi] \\ v(t,0) - qv(t-r,0) = v(t,\pi) - v(t-r,\pi) = 0 \text{ for } t \geq 0 \\ v(\theta,\xi) = \psi(\theta,\xi) \text{ for } \theta \in (-\infty,0] \text{ and } \xi \in [0,\pi], \end{cases} \tag{8.33}$$

where $p = \frac{\partial}{\partial \zeta} g(.,0) : (-\infty,0] \to \mathbb{R}$ is a continuous and measurable function.

We state the main result of the stability of the solutions.

Theorem 8.6.1 *Assume that* $(\mathbf{H}_7)$ *and* $(\mathbf{H}_8)$ *hold. Furthermore, suppose that*

$$0 < c\int_{-\infty}^0 |p(\theta)|d\theta < \left(1 + \frac{b^2}{4}\right)(1-q). \tag{8.34}$$

Then, the semigroup solution of (8.33) *is exponentially stable.* □

The proof of Theorem 8.6.1 makes use of this following lemma.

Lemma 8.6.3 ([4]) *The spectrum* $\sigma(\tilde{A})$ *of the operator* $\tilde{A} = \dfrac{\partial^2}{\partial \xi^2} + b\dfrac{\partial}{\partial \xi}$ *is equal to the point spectrum* $P\sigma(\tilde{A}) = \{-n^2 - \dfrac{b^2}{4} : \ n \in \mathbb{N}^*\}$. □

Proof of Theorem 8.6.1 The exponential stability of (8.33) is obtained when $s'(\tilde{A}) < 0$, which is true only if

$$\sup\Big\{\Re(\lambda) : \ \lambda \in \sigma(\tilde{A}) - \sigma_{ess}(\tilde{A}) \quad \text{and} \quad \Re(\lambda) > -\gamma\Big\} < 0.$$

Moreover, the characteristic equation is given by

$$\begin{cases} \Re(\lambda) > -\gamma, \quad f \in D(A), \quad f \neq 0 \\ \lambda(1 - qe^{-\lambda r})f - (1 - qe^{-\lambda r})(f'' + bf') - c\Big(\int_{-\infty}^{0} p(\theta)e^{\lambda\theta}d\theta\Big)f = 0, \end{cases} \tag{8.35}$$

which leads to

$$\lambda - \frac{c}{1 - qe^{-\lambda r}} \int_{-\infty}^{0} p(\theta)e^{\lambda\theta}d\theta \in \sigma_p\left(\frac{\partial^2}{\partial \xi^2} + b\frac{\partial}{\partial \xi}\right).$$

Since

$$\sigma_p\left(\frac{\partial^2}{\partial \xi^2} + b\frac{\partial}{\partial \xi}\right) = P\sigma(\tilde{A}) = \{-n^2 - \frac{b^2}{4} : \ n \in \mathbb{N}^*\},$$

then the characteristic equation (8.35) becomes

$$\begin{cases} \Re(\lambda_n) > -\gamma, \\ \lambda_n = \dfrac{c}{1 - qe^{-\lambda_n r}} \displaystyle\int_{-\infty}^{0} p(\theta)e^{\lambda_n\theta}d\theta - n^2 - \dfrac{b^2}{4} \quad \text{for some} \quad n \in \mathbb{N}^*. \end{cases} \tag{8.36}$$

Let $k_n = n + \dfrac{b^2}{4}$. Then, using (8.36), we obtain that

$$(\lambda_n + k_n)(1 - qe^{-\lambda_n r}) = c\int_{-\infty}^{0} p(\theta)e^{\lambda_n\theta}d\theta.$$

Therefore,

$$|\lambda_n + k_n||1 - qe^{-\lambda_n r}| = |c\int_{-\infty}^{0} p(\theta)e^{\lambda_n \theta}d\theta|$$
$$\leq c\int_{-\infty}^{0} |p(\theta)|e^{\Re(\lambda_n \theta)}d\theta.$$

We have also

$$|\lambda_n + k_n| \geq \sqrt{(\Re(\lambda_n) + k_n)^2}$$

and

$$\left|1 - qe^{-\lambda_n r}\right| \geq \left||1| - |qe^{-\lambda_n r}|\right| = |1 - qe^{-\Re(\lambda_n r)}|.$$

It follows that

$$\sqrt{(\Re(\lambda_n) + k_n)^2}\left|1 - qe^{-\Re(\lambda_n r)}\right| \leq c\int_{-\infty}^{0} |p(\theta)|e^{\Re(\lambda_n \theta)}d\theta.$$

Now, assume that $\Re(\lambda_n) \geq 0$. Then,

$$\left|1 - qe^{-\Re(\lambda_n r)}\right| \geq (1 - q).$$

Consequently,

$$(1 - q)\Big[\Re(\lambda_n) + k_n\Big] \leq c\int_{-\infty}^{0} |p(\theta)|d\theta.$$

Finally, since $(1 - q)\Re(\lambda_n)$, we obtain

$$(1 - q)k_n \leq c\int_{-\infty}^{0} |p(\theta)|d\theta.$$

Taking $n = 1$, we obtain a contraction with condition (8.34). That leads to $\Re(\lambda) < 0$. □

References

1. M. Adimy, K. Ezzinbi, A class of linear partial neutral functional differential equations with nondense domains. J. Differ. Equ., **147**, 285–332 (1998)
2. M. Adimy, K. Ezzinbi, Existence and linearized stability for partial neutral functional differential equations with nondense domains. Differer. Equ. Dyn. Syst. **7**, 371–417 (1999)

3. M. Adimy, K. Ezzinbi, Strict solutions of nonlinear hyperbolic neutral differential equations. Appl. Math. Lett. **12**, 107–112 (1999)
4. M. Adimy, K. Ezzinbi, Existence and stability in the α-norm for some partial functional differential equations of neutral type. Ann. Mate. Pura ed Appl. **185**(3), 437–460 (2006)
5. M. Adimy, H. Bouzahir, K. Ezzinbi, Local existence for a class of partial neutral functional differential equations with infinite delay. Differ. Equ. Dyn. Syst. **12**(3–4), 353–370 (2004)
6. C. Avramescu, A fixed point theorem for multivalued mappings. Electron. J. Qual. Theory Differ. Equ. **2004**(17), 1–10 (2004)
7. R. Benkhalti, K. Ezzinbi, Existence and stability in the α-norm for some partial functional differential equations with infinite delay. Differ. Integral Equ. **19**(5), 545–572 (2006)
8. H. Bouzahir, On neutral functional differential equations with infinite delay. Fixed Point Theory **6**(1), 11–24 (2005)
9. T. Burton, *Stability and Periodic Solutions of Ordinary Differential Equations and Functional Differential Equations* (Academic Press, San Diego, 1985), pp. 197–308
10. M.A. Cruz, J.K. Hale, Stability of functional differential equations of neutral type. J. Differ. Equ. **7**, 334–355 (1970)
11. R. Datko, Linear autonomous neutral differential equations in a Banach space. J. Differ. Equ. **25**, 258–274 (1977)
12. W. Desch, W. Schappacher, Linearized stability for nonlinear semigroup, in *Differential Equations in Banach Spaces*, ed. by A. Favini, E. Obrecht. Lecture Notes, vol. 1223 (Springer, Berlin, 1986)
13. K. Engel, R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*. Graduate Texts in Mathematics, vol. 194 (Springer, Berlin, 2001)
14. K. Ezzinbi, B.A. Kyelem, S. Ouaro, Periodicity in α-Norm for partial functional differential equations in fading memory spaces. Nonlinear Anal. Theory Methods Appl. **97**, 30–54 (2014)
15. J.K. Hale, *Asymptotic Behavior of Dissipative Systems* (American Mathematical Society, Providence, 1988)
16. J. Hale, J. Kato, Phase space for retarded equations with infinite delay. Funkcial. Ekvac. **21**, 11–41 (1978)
17. V. Lakshmikantham, S. Leela, *Differential and Integral Inequalities*, vol. 1 (Academic, New York, 1969)
18. T. Naito, J.S. Shin, S. Murakami, The generator of the solution semigroup for the general linear functional differential equation. Bull. Univ. Electro-Commun. **11**(1), 29–36 (1998)
19. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations* (Springer, New-York, 1993)
20. C.C. Travis, G.F. Webb, Partial differential equations with deviating arguments in the time variable. J. Math. Anal. Appl. **56**, 397–409 (1976)
21. C.C. Travis, G.F. Webb, Existence, stability and compactness in α-norm for partial functional differential equations. Trans. Am. Math. Soc. **240**, 129–143 (1978)
22. J. Wu, H. Xia, Self-sustained oscillations in a ring array of coupled lossless transmission lines. J. Differ. Equ. **124**, 247–278 (1996)