Chapter 1
Advancing Mathematical Modelling and Applications Educational Research and Practice

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Abstract This volume provides a snapshot of the current state-of-the-art in theory, research, and practice in the area of mathematical modelling in education. Recognising at the outset the important development of the subfield of mathematical modelling in mathematics education in the last decades, this chapter presents the main themes of a set of contributions concerning educational research and practice on the teaching and learning of mathematical modelling. The various chapters reflect the work carried out at ICME-14 held in Shanghai in July 2021, within the scope of Topic Study Group 22 and Survey Team 4, whose mission was to systematise the current state-of-the-art on the teaching and learning of mathematical modelling considering interdisciplinary aspects. In the collection are systematic literature reviews that offer an overview of mathematical modelling in mathematics education, and empirical studies adopting different theoretical perspectives and research aims addressing key issues that fall within the learning of mathematical modelling at the school level and the tertiary level, teacher education in modelling and teaching methods. This set of studies is an indicator of the consistency of ongoing research in mathematical modelling and applications. Diversity and complementarity are evident. Opportunities for international cooperation are apparent and key for the advancement of the subfield of mathematical modelling in mathematics education.

Keywords Assessment · Interdisciplinary · Mathematics · Mathematical modelling · Modelling competencies · Teacher education · Technology · STEM

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1.1 Introduction

The research on teaching and learning of mathematical applications and modelling is an expanding subfield of mathematics education research. It has been an important theme for teachers and researchers especially during the last 50 years, and its profile has been growing worldwide during the last decade. This prominence is evident, for example, in the International Congress on Mathematical Education (ICME) regular topic study groups and lectures on applications and modelling, and in the series of conferences of the International Community on the Teaching of Mathematical Modelling and Applications (ICTMA) since 1983, as well as in the publications arising from both. Other well-known international forums, such as the Congresses of the European Society for Research in Mathematics Education (CERME), have hosted continuous research and debate on the topic of applications and modelling in mathematics education (Carreira et al., 2019). This increasing interest is a consequence of several factors. There is public demand for the usefulness of mathematics outside the discipline (Brez & Allen, 2016; Vos, 2020), and there has been an increasing number of research projects (e.g. Achmetli et al., 2019; Greefrath 2020; Vincent Geiger et al., 2018) and empirical studies (Schukajlow et al., 2018) which focus on specific aspects of applications and modelling in mathematics teaching and learning at all levels of education, from the early years (e.g. Suh et al., 2021) to tertiary (e.g. Durandt et al., 2022). The case for curricular changes in mathematics around the world explicitly targeting the modelling process (e.g. Greefrath & Vorhölter, 2016; Lo et al., 2022; OECD, 2018) as well as the concomitant challenge of assessing modelling in standardised tests or rubrics (e.g. Greefrath & Frenken, 2021; Kohen & Gharra-Badran, 2022) have increased the visibility of modelling and also extended the research field. In addition, there is an increasing number of conceptual and theoretical works (e.g. Barquero et al., 2019; Blomhøj & Niss, 2021) which act as starting points for new lines of research inquiry and development.

Many recent qualitative and quantitative research studies on mathematical modelling in school and higher education have focused on students (e.g. Baioa & Carreira, 2021; Carreira et al., 2020; Durandt et al., 2022) and their modelling processes (e.g. Czocher et al., 2022; Stillman & Brown, 2021); however, teachers clearly play an important role in implementing and fostering students' modelling in classrooms (see, e.g., Cetinkaya et al., 2016; Greefrath et al., 2022; Wendt et al., 2022). Furthermore, classroom settings also play an important role (Schukajlow & Blum, 2018). Setting the focus on teacher practice in proposing and implementing intervention activities, there has been research on the design of single modelling lessons (Beckschulte, 2020) as well as whole modelling learning environments (Orey & Rosa 2018) at different school levels.

At ICME-14 held in Shanghai in July 2021, Topic Study Group 22 (Greefrath and Carreira in press) considered the importance of exploring relations between mathematics and the real world that occur in educational environments. The value of examining the discussion in research and development on mathematical applications and modelling issues at school and university, including mathematics teacher education and the interplay between research and development of modelling learning environments, were also seen as important and timely. In addition, ICME-14 Survey Team 4 reviewed the current state-of-the-art on the teaching and learning of mathematical modelling considering interdisciplinary aspects (Stillman et al. in press). In particular, the importance of a well-understood relation between mathematics and the real world was in their focus.

Following the Congress, a number of authors were invited to contribute to a volume in order to produce a snapshot of the current state-of-the-art in theory, research, and practice in the area of mathematical modelling in education. After a rigorous review process, the remaining chapters from this selection have been collated into this edited collection.

1.2 Overviewing Mathematical Modelling in Education

The promotion of mathematical modelling competencies is recognised worldwide as an important goal of mathematics teaching. Researchers have taken different approaches to this integration and are still in the process of developing empirical evidence on the impact of these approaches on the integration of modelling in school practice (Kaiser, 2017). Stillman (2019) has seen as an important impetus theoretical approaches to research on mathematical modelling, among others, the study of modelling frameworks, modelling competence, and metacognition. The current research landscape shows a variety of case study approaches and cognitively oriented studies. Somewhat less frequently, one finds studies that use quantitative research methods or focus on affect-related topics (Schukajlow et al., 2018). There have been a number of meta-studies on mathematical modelling, including on specific topics such as modalities of assessment of modelling (Frejd, 2013), the role of technology in mathematical modelling (Molina-Toro et al., 2019), and modelling competencies (Cevikbas et al., 2022; Hidayat et al., 2022). Overall, this shows the need for further theoretical work on mathematical modelling competencies. Currently, however, a wealth of developed empirical approaches and their implementation at different educational levels can already be identified (Cevikbas et al., 2022). A review chapter on the analysis of current literature is part of this book.

ICME-14 survey team 4 was asked to review the literature from the last ICME in 2016 to the current state-of-the-art on the teaching and learning of *mathematical modelling* considering *interdisciplinary aspects*. In particular, the importance of a well-understood *relation between mathematics and the real world* was in the focus brief. The aim was to establish an in-depth review of the most important developments and contributions and current tendencies and trends to July 2021, including new perspectives and emerging challenges. Based on a systematic, qualitative, analytical review of literature (Newman & Gough, 2020) from this time period, **Stillman, Ikeda, Schukajlow, Araújo**, and **Ärlebäck** identified four major threads of contributions relating to schooling. These threads were: the continued importance of a well-understood relation between mathematics and the real world supporting interdisciplinary work in mathematics education; the contribution to knowledge about modelling of interdisciplinary research and teaching teams; issues and challenges in the relationships among mathematical modelling, mathematics, the real world and interdisciplinarity; and, mathematical modelling providing critical high-leverage to ensure mathematical depth in STEM integration. Each thread is exemplified with purposively selected examples from their research synthesis. The authors suggest that it is timely to reconsider the rather loosely used notion in both research and curriculum documents that mathematical modelling is a common-sense enabler of interdisciplinarity as well as how mathematical modelling educational and STEM research communities interact. One emergent question thus concerns the clarification of the connection between modelling and integrated STEM, in school and university curricula. This discussion may be traced back to the seminal work of Blum and Niss (1991), where various possibilities of linking mathematics to other subjects were raised and differentiated, namely the mathematics curriculum integrated approach and the interdisciplinary integrated approach. More work on modelling from a curriculum point of view is apparently needed.

Preciado Babb, Peña Acuña, Ortiz Rocha, and **Solares Rojas** analysed a selection of recent mathematical modelling literature identifying local and global tendencies and the diversity of approaches to mathematical modelling. The systematic literature review comprised more than 500 documents from articles published in a selection of journals and selected books from ICTMA (2010–2020), the ICMI-14 study volume (Blum et al., 2007) and the volume from ICME-13 TSG with a focus on mathematical applications and modelling (Stillman & Brown, 2019). Babb et al. identified six countries with significant numbers of publications and present relative percentage distributions corresponding to mathematical modelling perspective, educational content, and unit of analysis (school level, job, or profession). The review showed that authors from Germany, United States, and Australia contributed half of the publications and that the educational content played a different role depending on the corresponding modelling perspective in specific countries.

1.3 Learning Mathematical Modelling at School

In recent decades, the potential of integrating mathematical modelling into mathematics education has been widely investigated (Kaiser, 2017). There are various approaches to possible meaningful measures to support learning mathematical modelling. For example, knowledge about the modelling cycle (Galbraith & Clatworthy, 1990), the provision of heuristic solution examples (Zöttl et al., 2010), or the use of a strategic solution plan (Beckschulte, 2020) can support and promote modelling processes. Furthermore, the creation of mathematical drawings (Rellensmann et al., 2017) and the use of technological tools and resources (Galbraith et al., 2003; Greefrath et al., 2018) can also contribute to an increase in modelling competence. In this section, we summarise a number of contributions that approach mathematical modelling in school in different ways. These chapters present results of studies concerning several questions ranging from elementary to secondary school: the value of whole-class discussions and reflections on solutions to modelling problems; the use of technological environments in modelling tasks and its relationship with self-regulated learning; the student's modelling activity within interdisciplinary tasks; changes in students' beliefs about mathematics and problem-solving after experience with modelling challenges; and the interplay between modelling competence and the mathematics competence of elementary school students.

A key distinction among traditions in modelling research, according to **Brady, Jung McLean, Dominguez, and Glancy,** is whether modelling is primarily viewed as a curricular *topic* to be learnt or as a favourable context for supporting and studying mathematical thinking. For modelling-as-context traditions, modelling tasks can be designed to illuminate student thinking; to position groups of students as inventive creators of mathematics; or to spur them on to engage in forms of mathematising that are valued in the discipline of mathematics. In this chapter, Brady et al. argue that whole-class presentations of solutions to modelling tasks can be particularly rich settings to research such topics. Their focus is on how presentation sessions offer opportunities to engage in reflective discourse, in which the class can convert modelling actions that various student teams have engaged in into objects of collective discussion. To illustrate this point, Brady et al. analyse three episodes of reflective discourse from a mathematical modelling summer camp for students aged 10–13. For each episode, they describe the specific mathematical value of reflective discourse as it emerged in the context.

Modelling processes can be supported, enriched, and made more authentic using ICT which can be combined in a computer-based learning environment. The pre-structured form of computer-based learning environments has the potential to promote and stimulate self-regulated learning. However, as **Frenken** points out, from a theoretical perspective, we can anticipate that modelling within such environments can also pose difficulties for student modellers. In an exploratory study, two Year 9 classes worked independently within a computer-based learning environment for two weeks during distance learning. In that learning environment, students started with tutorials, including videos and applets, to learn about the tools and solve exercises in GeoGebra. Afterwards, they were invited to solve modelling tasks using GeoGebra applets provided in the computer-based learning environment. The results on the relationship between performance in modelling tasks and self-regulated learning were obtained with cluster analysis. The interpretation of the clusters showed that mathematical modelling-specific performance in an independent learning environment is strongly related to self-regulatory skills. More precisely, students who achieved high success in the modelling tasks, made frequent use of the help and the possibilities offered by the environment. On the contrary, those who spent very little time in the computer-based learning environment and did not use help were the ones who had less success in the modelling tasks. In subsequent studies, it is suggested that data collected from navigation and performance on the computer-based learning environment be complemented with video recordings and student interviews.

It has been known for some time that students' beliefs about the nature of mathematics greatly influence their interests and attitudes towards school mathematics (Schukajlow et al., 2017). Common beliefs such as mathematics problems always having a unique and exact answer can become obstacles to student learning. Research has found that mathematical modelling experiences could help students see the relevance of mathematics in the real world and their lives, but more attention is needed as to whether they affect other beliefs. **Guiñez and González** focus on exploring high-school students' views about mathematics when they work independently on solving real-world mathematical modelling problems during the selection process of the teams that represented Chile at the International Mathematical Modelling Challenge. The findings suggest that exposure to these modelling tasks has the potential to modify participants' beliefs, for instance, with regard to the existence of many solutions and correct procedures for mathematical problem-solving.

Moutet shows that the extended Mathematical Working Space (MWS) framework makes it possible to analyse school tasks by considering the relationships between the cognitive plane of students, and the epistemological planes of mathematics and of physics according to stages of the Blum and Leiß modelling cycle (2007). He analysed a problem-solving activity incorporating a multidisciplinary approach involving physics and mathematics. The problem-solving activity investigates the possibilities of using a solenoid formed by winding copper wire covered with an insulating film to produce an intense magnetic field as might be required for a medical imaging device, for example. A group of 12th grade volunteer students in France completed the online activity. In line with the empirical results of other studies, this investigation led to the conclusion that the epistemological planes of physics and mathematics are mobilised in different ways, depending on the stage of the modelling cycle that is being carried out by students.

Wang, Xie, and Liu explored 298 grade four Chinese students' competence in mathematical modelling and its relationship to their mathematics competence. Descriptive analysis, t-tests, and correlation coefficients were used and reported to describe the mathematical modelling competence and sub-competencies of the students and to analyse the relationship between mathematics competence and these sub-competencies. The results indicated that the students hardly engaged mathematical modelling and that of the sub-competencies of mathematical modelling, the competence of mathematical working was the best in comparison. In addition, a strong positive correlation between mathematical modelling competence and mathematics competence was found in the data collected.

1.4 Mathematical Modelling at University

Engineering, science, and technology applications in undergraduate courses are in many cases difficult for students to understand. Even those studying STEM subjects are often unable to connect the mathematical world with the real world (Crouch & Haines, 2004). It is furthermore clear that there are strong institutional constraints to the widespread diffusion of mathematics as a modelling activity, especially in universities as compared to schools (Barquero et al., 2013). One approach at university is to make basic content such as linear algebra more accessible with the help of mathematical modelling and Realistic Mathematics Education (Stewart et al., 2019). This is one of the issues dealt with in this section, together with the need for validating assessment instruments on students' modelling competencies, tensions arising from interdisciplinary projects where mathematics and other scientific fields intervene, and engaging students in projects where citizenship is emphasised, namely implying the statistical analysis of messy data.

Ramirez-Montes, Carreira, and **Henriques** report on a study of two classes of undergraduate students participating in a linear algebra course at a university in Costa Rica. They attempted different versions of a modelling task using digital coordinate geometry technological tools. The modelling problem involved the manipulation of an image of Big Ben that is transformed and incorporates the concept of linear transformation. This qualitative study focused on students' modelling processes and the influence of technology use on their linear transformation models. Students' use of technology was rather rudimentary, being used mainly for constructing a mathematical model of the real situation. Students' modelling also exhibited difficulties in interpreting the real situation as a case of geometrical transformations, in using linear transformation properties, and in validating results of their models. There was no evidence that the technology enabled these students to relate algebraic and geometrical meanings of a linear transformation. Further research is suggested to better understand how technology may help students in interpreting linear algebra models in terms of geometrical representations and in solving real-world problems involving linear transformations.

As part of a larger project focused on exploring development of mathematical modelling competencies among post-secondary STEM majors enrolled in advanced mathematics, **Czocher, Kularajan, Roan,** and **Sigley** developed a pair of parallel multiple-choice modelling competencies assessments. The chapter provides a technical report of item development, scale calibration, and validation of the assessment. Multiple statistical approaches used included classical test theory, item response theory, and principal component analysis. These documented item behaviours, scale properties, and dimensionality of the developing multiple-choice assessment of mathematical modelling competencies designed for post-secondary STEM majors. Czocher et al. share analyses and inferences, making recommendations for the field in pursuing such assessments. The authors were able to ensure the validity of the two multiple-choice instruments to assess collective gains in students' modelling competencies resulting from pedagogical interventions. This represents a step towards obtaining a valid and reliable instrument to generate the empirical basis to evaluate pedagogical interventions, possibly at all levels of education.

Rogovchenko used Activity Theory to analyse the work of biology undergraduates at a Norwegian university with biologically meaningful mathematical modelling tasks. Tensions related to collaboration in an interdisciplinary team, students' engagement, understanding of a modelling task, comprehension of its mathematical content and solution manifest multiple primary and secondary contradictions in the activity system. Rogovchenko identifies these contradictions and discusses possibilities for their resolution through expansive learning. While the contradictions that could be identified remained unresolved, the motivation of the project team and the positive feedback from the student group opened promising perspectives for expansive learning. For example, the students' statements about assumptions in modelling were very promising despite the difficulties involved.

Citizen Science (Crain et al., 2014) provides the means for students to engage in collecting and analysing data important to their local environments. **McLean, Brady, Jung, Dominguez**, and **Glancy** describe how undergraduate students in the United States participated in a model-eliciting activity to make sense of large, complex, and messy datasets gathered in connection with a citizen science project. Focusing on the data moves that students performed to manipulate the data into a manageable form they showed how student groups oriented towards the data as capturing a phenomenon in the records. They assert that filtering is a key component of the modelling process, especially when citizen science often involves using large datasets to solve a real-world problem. McLean et al. argue that model-eliciting activities offer entry points to appreciate the complexity of citizen science as a practice and the value of the scientific questions that citizen science projects are engaging.

1.5 Teacher Education in Mathematical Modelling

Teachers are faced with a variety of demands, especially in mathematical modelling (Berget, 2022; Vince Geiger et al., 2022; Wendt et al., 2022). International research in teacher education could provide the coherence needed to develop a knowledge base for effective pedagogical interventions in teaching mathematics through applications and modelling (Doerr, 2007). In teacher education research, there is a focus on the development of professional competencies, building on different dimensions of knowledge (Kaiser et al., 2020). This is composed of different areas of knowledge such as mathematical content knowledge, pedagogical content knowledge, and pedagogical-psychological knowledge. Furthermore, professional competence includes affective-value-oriented aspects in addition to the cognitively oriented knowledge dimensions mentioned. In a comprehensive model by Blömeke et al. (2015) of teacher knowledge, in which the analytical and holistic approaches to conceptualising and measuring competence are combined to represent competence as a process, the so-called situation-specific skills (perceiving, interpreting, and internalising) serve as a link to performance (Sherin et al., 2011). In this section, we summarise five contributions that approach teacher education in mathematical modelling in different ways.

In classrooms, students can use metacognitive strategies to overcome obstacles and to ensure smooth working when independently solving complex problems, such as mathematical modelling problems. Thus, it is important for teachers to know about metacognition and to be able to perceive and interpret students' use, or lack of use, of metacognitive modelling strategies. **Alwast and Vorhölter** analysed the development of 52 pre-service teachers' knowledge and noticing competencies for teaching mathematical modelling with respect to students' use of metacognitive strategies as well as the relationship between these. While the pre-service teachers' knowledge regarding metacognition significantly improved during a university modelling seminar, pre-service teachers' noticing competencies barely changed. There was, however, a correlation between them.

Ekol reports a small study aimed at understanding the contribution of assessment for learning in a pre-service secondary teacher mathematical modelling course at a university in South Africa. A matched-pairs design was adopted to analyse assessment data collected during and at the end of the course. Descriptive and inferential data analysis detected no statistically significant increase in the mean score from the formative phase to the final assessment at the end of the course. The study contributes to research on different assessment approaches in pre-service mathematics education courses that include mathematical modelling and understanding their practical contributions to the learning gains at the end of the courses.

Saeki, Kaneko, Kawakami, and **Ikeda** focus on Japanese in-service teachers with less experience in teaching mathematical modelling. They describe and analyse the novices' activities to design modelling tasks based on mathematised tasks. Results of analysis of the in-service teachers' activities and artefacts revealed that Lesson Study enabled novice modelling teachers to understand the characteristics of each criterion of the modelling task through activities that transform familiar textbook mathematised tasks into modelling tasks; and to develop and implement modelling lessons incorporating examples from students' realities. This arose during discussions between teachers from different backgrounds and researchers.

Siller, Greefrath, Wess, and **Klock** focus on the professionalisation of pre-service teachers through reflective practice when they participated in a 12-session university seminar about mathematical modelling over one semester. They consider the pre-service teachers' self-efficacy beliefs as an important aspect of professional competence for teaching mathematical modelling. A quasi-experimental study with a pre-post design was used to examine the extent to which self-efficacy of mathematics pre-service teachers for mathematical modelling can be increased through a variety of different teaching–learning laboratories (i.e. a focus on task self-design or a focus on diagnostic and intervention competencies). Clearer effects were seen when the pre-service teachers themselves created modelling tasks for use with grade nine students. The study contributes to research on the possibilities for teaching–learning laboratories in teacher education that focus on the acquisition of competencies by pre-service teachers.

Project-based instruction focuses on real-world tasks as a vehicle for learning, which **Park** proposes as a platform to drive the teaching of mathematical modelling. She reports a case study of the task design and implementation for a mathematical modelling 5-lesson sequence in project-based instruction of a team of two pre-service teachers. Data sources were open-ended questionnaires, pre-service teachers' lesson plans, and video-recorded classroom observations. Example lessons designed by two pre-service teachers are examined. How the lessons were designed and implemented is discussed with a view to informing future research on pre-service teachers' preparation for problem-based instruction with mathematical modelling.

1.6 Teaching Mathematical Modelling at School

There are increasing demands on the teaching of mathematical modelling. For example, prospective teachers should gain experience with stochastic models in their preparation programmes; such a change would mean a move away from the current dominance of deterministic models in mathematical teacher preparation (Doerr, 2007). However, teaching can also take into account the diversity of cultural forms of mathematics (Rosa & Orey, 2013). Teaching methods can also be of central importance for the teaching of modelling, which can have an effect not only on the students' achievement but also on their affect (Schukajlow et al., 2012).

Ärlebäck and Kawakami present arguments and examples highlighting the similarities and differences between statistics, statistical modelling, and mathematical modelling. They elaborate on the potentially productive connections for the development of, and research on, the teaching and learning of statistics, statistical modelling, and mathematical modelling. They outline their development of an ongoing research agenda that pursues a framework for conceptualising connections between statistics and statistical modelling and mathematical modelling. In addition, they suggest how to extend this emerging framework to provide a richer, more nuanced, and useful picture of the relationships between statistics, statistical modelling, and mathematical modelling.

Ethnomodelling is an alternative methodological approach suited to diverse sociocultural realities and proposes the rediscovery of mathematical knowledge systems developed, accumulated, adopted, and adapted in other cultural contexts. **Orey and Rosa** focus on the glocal (dialogic) approach of ethnomodelling and how the interaction between local (emic) and global (etic) approaches can promote understanding of cultural dynamism through the elaboration of ethnomodels. It is important to discuss epistemological stances regarding how cultural aspects are integrated into the ethnomodelling perspective and how this integration enables showing the relevance of different issues with respect to local (emic), global (etic), or glocal (dialogic) approaches. This confirms that the central content of ethnomodelling may represent a significant contribution to mathematical modelling educational research and its pedagogical action.

An important goal of mathematics education is to develop and to examine methods for teaching modelling problems. **Schukajlow and Blum** identify guided instruction and a constructivist view of teaching as two general principles of teaching methods for modelling problems. They exemplify these principles by teaching methods developed in the DISUM (Blum, 2015) and MultiMa projects (e.g. Achmetli et al., 2019). These teaching methods vary in the degree of guidance given by teachers or learning materials and in the degree of self-regulation experienced by students. The effects of these teaching methods have been evaluated in prior studies. Schukajlow and Blum report under which conditions and pre-requisites for students these teaching methods worked. They also raise some challenges for future research.

1.7 Discussion

Activities to promote mathematical modelling in research and practice already have an international history of more than 50 years. In recent years, international visibility has increased significantly, not least through a series of relevant thematic issues in important research journals (Carreira & Blum, 2021a, 2021b; Kaiser & Schukajlow, 2022; Schukajlow et al., 2018, 2021). In addition to new theoretical contributions, there are several empirical studies on mathematical modelling in all school levels up to university. Both students and teachers, as well as the context and the design of teaching, are considered. This spectrum was also reflected by Topic Study Group 22 and Survey Team 4 at ICME-14 in Shanghai.

The activities are also evident in many current review articles and meta-studies on the teaching and learning of mathematics. In addition to the many journal articles, the ICTMA books are of particular importance for these studies (see from Berry et al., 1984 to Leung et al., 2021). These overviews also show that not all countries and continents are equally represented. The chapter from the ICME survey team 4 brings new insights onto the table regarding mathematical modelling research. Specifically, the often loosely used term of interdisciplinarity. Certainly, education and mathematical modelling education need this research as a starting point to explore interdisciplinary issues. The call by the survey team for more cooperation between the mathematical modelling and STEM research communities is an opportunity for the advancement of the STEM agenda, raised by many in the STEM literature (e.g. Bajuri et al., 2018; English, 2021; Hallström & Schönborn, 2019), to use the inherent interdisciplinary nature of modelling to support pedagogical innovations (Ekici & Alagoz, 2021). This is also an opportunity for further international collaboration within and between these research communities.

Various aspects of mathematical modelling in schools are currently being discussed. In addition to mathematical modelling as a context of teaching, the expansion of possibilities through the use of digital learning environments and the influence on learners' attitudes are also being considered. The qualitative and quantitative analysis of students' modelling processes through appropriate frameworks and the measurement of modelling competencies are important research directions. The holistic and atomistic views of modelling competencies are also relevant here (Kaiser & Brand, 2015).

Furthermore, there are a number of new approaches to mathematical modelling at university, such as modelling in linear algebra, and others better known, such as modelling with differential equations, continue to be of interest. Many studies involve students from STEM subjects. In some cases, approaches to investigating modelling skills are similar to those used at school, both through appropriate test instruments that are being increasingly developed and through more in-depth case studies. Modelling in the context of interdisciplinary projects and problems is also gaining prominence in tertiary education as also happens at school.

A particularly active field of research at present is teacher education in mathematical modelling. On the basis of competence models, different areas of professional competence for teaching mathematical modelling are considered. These include knowledge aspects as well as noticing and affective aspects such as self-efficacy. Knowledge of diverse models and content is important for teaching mathematical modelling. In addition to these areas, there are also a number of research activities in the field of teaching methods for mathematical modelling.

Overall, this book showcases current research activities in four areas of educational research, as summarised in Fig. 1.1, as well as more global survey and review research into the focusses of mathematical modelling educational research itself. The chapters in this book thus contribute to several lines of investigation in advancing and consolidating research on mathematical applications to the real-world and mathematical modelling in mathematics education. Many individual research activities in the different areas complement each other. Further international cooperation is, therefore, important to advance this subfield of mathematics education and at the same time can promote the teaching and learning of mathematical modelling worldwide.

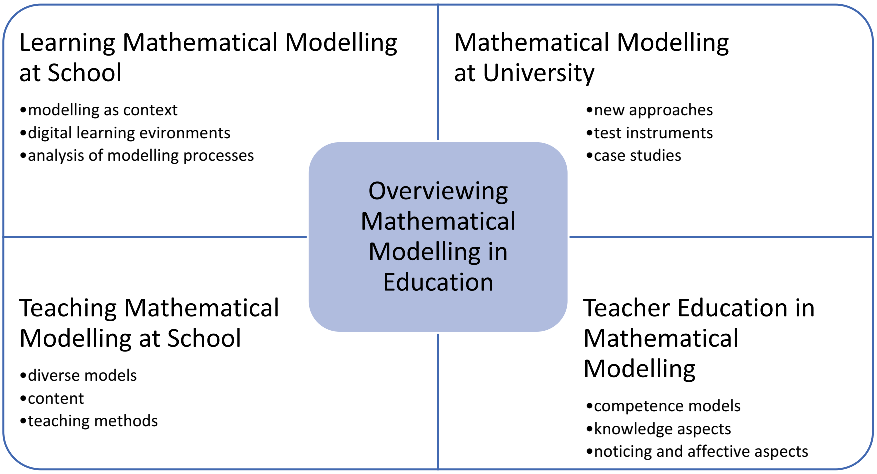

Fig. 1.1 Current research activities on teaching and learning mathematical modelling

1.8 Final Reflections and Outlook for Future Work

The current research activities show a wide range of methods and content. There is theoretical and both qualitative and quantitative empirical work. Individuals from all school levels and also from university are studied. Both students and teachers as well as the context and design of teaching are taken into account.

The contributions collected in this volume offer some advances on different lines of research. About the students' modelling processes at different educational levels, we may see that different modalities and perspectives, namely relating mathematics to other subjects (physics, biology, ecology, engineering, citizen science, culture, and society), are focused on the potential for developing students' modelling competencies and improving the learning and understanding of mathematical topics. The integration of digital resources and tools continues to be an important issue, so it is necessary to reconsider questions such as how to best help students master the power of technology to increase their ability to solve modelling problems. Teacher education and the central role of the teacher as a stimulator and supporter of the student's modelling activity are also on the agenda. Significant issues that the studies brought up include the quest for effective teacher education models that attend to different aspects, be it the specialised knowledge for teaching modelling or practical issues such as monitoring, noticing and task designing, in addition to the affective component of the challenging work of solving modelling problems.

International research is occurring in many countries. At present, some of the activities, in particular some lines of research and perspectives, are still concentrated on the work of researchers from certain countries. For some topics, stronger links could be made between the different research areas, such as university mathematics and school mathematics and the transition between the two. International cooperation is the key to success here. Conferences such as ICME and ICTMA play an important role as they bring together researchers with different research aims, approaches, and research methods and thus promote scientific exchange on the current areas of research on mathematical modelling and therefore on the practice of mathematical modelling across educational levels.

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