A Note on Central Limit Theorems for Trimmed Subordinated Subordinators

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1 Introduction

Ipsen et al [3] and Mason [7] have proved under general conditions that a trimmed subordinator satisfies a *self-standardized* central limit theorem [CLT]. One of their basic tools was a classic representation for subordinators (e.g., Rosiński [9]). Ipsen et al [3] used conditional characteristic function methods to prove their CLT, whereas Mason [7] applied a powerful normal approximation result for standardized infinitely divisible random variables by Zaitsev [12]. In this note, we shall examine self-standardized CLTs for trimmed subordinated subordinators. It turns out that there are two ways to trim a subordinated subordinator. One way leads to CLTs for the usual trimmed subordinator treated in [3] and [7], and a second way to a closely related *subordinated trimmed subordinator* and CLTs for it.

We begin by describing our setup and establishing some basic notation. Let $V = (V(t), t \geq 0)$ and $X = (X(t), t \geq 0)$ be independent 0 drift subordinators with Lévy measures Λ_V and Λ_X on $\mathbb{R}^+ = (0, \infty)$, respectively, with *tail function* $\overline{\Lambda}_V(x) = \Lambda_V((x, \infty))$, respectively, $\overline{\Lambda}_X(x) = \Lambda_X((x, \infty))$, defined for $x > 0$, satisfying

$$\overline{\Lambda}_V(0+) = \overline{\Lambda}_X(0+) = \infty. \tag{1}$$

For $u > 0$, let $\varphi_V(u) = \sup\{x : \overline{\Lambda}_V(x) > u\}$, where $\sup \varnothing := 0$. In the same way, define φ_X.

Remark 1 Observe that we always have

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$$\varphi_V(u) \to 0, \text{ as } u \to \infty.$$

Moreover, whenever $\overline{\Lambda}_V(0+) = \infty$, we have

$$\varphi_V(u) > 0 \text{ for all } u > 0.$$

For details, see Remark 1 of Mason [7]. The same statement holds for φ_X.

Recall that the Lévy measure Λ_V of a subordinator V satisfies

$$\int_0^1 x\Lambda_V(\mathrm{d}x) < \infty, \text{ equivalently, for all } y > 0, \int_y^\infty \varphi_V(x)\,\mathrm{d}x < \infty.$$

The subordinator V has Laplace transform defined for $t \geq 0$ by

$$E\exp\left(-\theta V(t)\right) = \exp\left(-t\Phi_V(\theta)\right), \ \theta \geq 0,$$

where

$$\Phi_V(\theta) = \int_0^\infty \left(1 - \exp\left(-\theta v\right)\right)\Lambda_V(\mathrm{d}v),$$

which can be written after a change of variable to

$$= \int_0^\infty \left(1 - \exp\left(-\theta\varphi_V(u)\right)\right)\mathrm{d}u.$$

In the same way, we define the Laplace transform of X.

Consider the subordinated subordinator process

$$W = \left(W(t) = V(X(t)), t \geq 0\right). \tag{2}$$

Applying Theorem 30.1 and Theorem 30.4 of Sato [11], we get that the process W is a 0 drift subordinator W with Lévy measure Λ_W defined for Borel subsets B of $(0, \infty)$ by

$$\Lambda_W(B) = \int_0^\infty P\{V(y) \in B\}\Lambda_X(\mathrm{d}y), \tag{3}$$

with Lévy tail function

$$\overline{\Lambda}_W(x) = \Lambda_W((x, \infty)), \ \text{for } x > 0.$$

Remark 2 Notice that (1) implies

$$\overline{\Lambda}_W(0+) = \infty.$$

To see this, we have by (3) that

$$\overline{\Lambda}_W(0+) = \lim_{n\to\infty} \int_0^\infty P\left\{V(y) \in \left(\frac{1}{n}, \infty\right)\right\} \Lambda_X(\mathrm{d}y).$$

Now $\overline{\Lambda}_V(0+) = \infty$ implies that for all $y > 0$, $P\{V(y) \in (0,\infty)\} = 1$. Hence by the monotone convergence theorem,

$$\lim_{n\to\infty} \int_0^\infty P\left\{V(y) \in \left(\frac{1}{n}, \infty\right)\right\} \Lambda_X(\mathrm{d}y) = \overline{\Lambda}_X(0+) = \infty.$$

For later use, we note that W has Laplace transform defined for $t \geq 0$ by

$$E\exp(-\theta W(t)) = \exp(-t\Phi_W(\theta)),\ \theta \geq 0,$$

where

$$\Phi_W(\theta) = \int_0^\infty \left(1 - e^{-\theta x}\right) \Lambda_W(\mathrm{d}x)$$

$$= \int_0^\infty \int_0^\infty \left(1 - e^{-\theta x}\right) P(V(y) \in \mathrm{d}x)\, \Lambda_X(\mathrm{d}y)$$

$$= \int_0^\infty \left(1 - e^{y\Phi_V(\theta)}\right) \Lambda_X(\mathrm{d}y).$$

Definition 30.2 of Sato [11] calls the transformation of V into W given by $W(t) = V(X(t))$ subordination by the subordinator X, which is sometimes called the directing process.

2 Two Methods of Trimming W

In order to talk about trimming W, we must first discuss the ordered jump sequences of V, X, and W. For any $t > 0$, denote the ordered jump sequence $m_V^{(1)}(t) \geq m_V^{(2)}(t) \geq \cdots$ of V on the interval $[0, t]$. Let $\omega_1, \omega_2, \ldots$ be i.i.d. exponential random variables with parameter 1, and for each $n \geq 1$, let $\Gamma_n = \omega_1 + \ldots + \omega_n$. It is well-known that for each $t > 0$,

$$\left(m_V^{(r)}(t)\right)_{r\geq 1} \stackrel{\mathrm{D}}{=} \left(\varphi_V\left(\frac{\Gamma_r}{t}\right)\right)_{r\geq 1}, \tag{4}$$

and hence for each $t > 0$,

$$V(t)=\sum_{r=1}^{\infty}m_V^{(r)}(t)\stackrel{\mathrm{D}}{=}\sum_{r=1}^{\infty}\varphi_V\left(\frac{\Gamma_r}{t}\right)=:\widetilde{V}(t). \tag{5}$$

See, for instance, equation (1.3) in IMR [3] and the references therein. It can also be inferred from a general representation for subordinators due to Rosiński [9].

In the same way, we define for each $t>0$, $\left(m_X^{(r)}(t)\right)_{r\geq 1}$ and $\left(m_W^{(r)}(t)\right)_{r\geq 1}$, and we see that the analogs of the distributional identity (4) hold with $m_V^{(r)}$ and φ_V replaced by $m_X^{(r)}$ and φ_X , respectively, $m_W^{(r)}$ and φ_W. Recalling (2), observe that for all $t>0$,

$$W(t)=\sum_{0<s\leq t}\Delta W(s)=V(X(t))=\sum_{0<s\leq X(t)}\Delta V(s). \tag{6}$$

From (6) and the version of (4) with $m_V^{(r)}$ and φ_V replaced by $m_W^{(r)}$ and φ_W, we have for each $t>0$

$$W(t)=\sum_{r=1}^{\infty}m_W^{(r)}(t)\stackrel{\mathrm{D}}{=}\sum_{r=1}^{\infty}\varphi_W\left(\frac{\Gamma_r}{t}\right)=:\widetilde{W}(t).$$

Let V, X and $(\Gamma_r)_{r\geq 1}$ be independent. In particular, V is independent of

$$\left\{\left(m_X^{(r)}(y)\right)_{r\geq 1}, y>0\right\} \text{ and } (\Gamma_r)_{r\geq 1}.$$

Next consider for each $t>0$

$$\left(m_V^{(r)}(X(t))\right)_{r\geq 1}.$$

Note that conditioned on $X(t)=y$

$$\left(m_V^{(r)}(X(t))\right)_{r\geq 1}\stackrel{\mathrm{D}}{=}\left(m_V^{(r)}(y)\right)_{r\geq 1}.$$

Therefore, using (4), we get for each $t>0$

$$\left(m_V^{(r)}(X(t))\right)_{r\geq 1}\stackrel{\mathrm{D}}{=}\left(\varphi_V\left(\frac{\Gamma_r}{X(t)}\right)\right)_{r\geq 1},$$

and thus by (5),

$$V(X(t))=\sum_{r=1}^{\infty}m_V^{(r)}(X(t))\stackrel{\mathrm{D}}{=}\sum_{r=1}^{\infty}\varphi_V\left(\frac{\Gamma_r}{X(t)}\right)=:\widetilde{V}(X(t)).$$

Here are two methods of trimming $W(t) = V(X(t))$.

Method I For each $t > 0$, trim $W(t) = V(X(t))$ based on the ordered jumps of V on the interval $(0, X(t)]$. In this case, for each $t > 0$ and $k \geq 1$, define the kth trimmed version of $V(X(t))$

$$V^{(k)}(X(t)) := V(X(t)) - \sum_{r=1}^{k} m_V^{(r)}(X(t)),$$

which we will call the *subordinated trimmed subordinator process.* We note that

$$V^{(k)}(X(t)) \stackrel{\mathrm{D}}{=} \widetilde{V}(X(t)) - \sum_{r=1}^{k} \varphi_V\left(\frac{\Gamma_r}{X(t)}\right) =: \widetilde{V}^{(k)}(X(t)).$$

Method II For each $t > 0$, trim $W(t)$ based on the ordered jumps of W on the interval $(0, t]$. In this case, for each $t > 0$ and $k \geq 1$, define the kth trimmed version of $W(t)$

$$W^{(k)}(t) := W(t) - \sum_{r=1}^{k} m_W^{(r)}(t)$$

$$\stackrel{\mathrm{D}}{=} \widetilde{W}(t) - \sum_{r=1}^{k} \varphi_W\left(\frac{\Gamma_r}{t}\right) =: \widetilde{W}^{(k)}(t).$$

Remark 3 Notice that in method I trimming for each $t > 0$, we treat $V(X(t))$ as the subordinator V randomly evaluated at $X(t)$, whereas in method II trimming we consider $W = V(X)$ as the *subordinator*, which results when the subordinator V is randomly time changed by the subordinator X.

Remark 4 Though for each $t > 0$, $V(X(t)) = W(t)$, typically we cannot conclude that for each $t > 0$ and $k \geq 1$

$$V^{(k)}(X(t)) \stackrel{\mathrm{D}}{=} W^{(k)}(t).$$

This is because it is not necessarily true that

$$\left(m_V^{(r)}(X(t))\right)_{r\geq 1} \stackrel{\mathrm{D}}{=} \left(m_W^{(r)}(t)\right)_{r\geq 1}.$$

See the example in Appendix 1.

3 Self-Standardized CLTs for W

3.1 Self-Standardized CLTs for Method I Trimming

Set $V^{(0)}(t) := V(t)$, and for any integer $k \geq 1$, consider the trimmed subordinator

$$V^{(k)}(t) := V(t) - m_V^{(1)}(t) - \cdots - m_V^{(k)}(t),$$

which on account of (4) says for any integer $k \geq 0$ and $t > 0$

$$V^{(k)}(t) \stackrel{D}{=} \sum_{i=k+1}^{\infty} \varphi_V\left(\frac{\Gamma_i}{t}\right) =: \widetilde{V}^{(k)}(t). \tag{7}$$

Let T be a strictly positive random variable independent of

$$\left\{\left(m_V^{(r)}(t)\right)_{r\geq 1}, t > 0\right\} \text{ and } (\Gamma_r)_{r\geq 1}. \tag{8}$$

Clearly, by (4), (7), and (8), we have for any integer $k \geq 0$

$$V^{(k)}(T) \stackrel{D}{=} \widetilde{V}^{(k)}(T).$$

Set for any $y > 0$

$$\mu_V(y) := \int_y^{\infty} \varphi_V(x)\,\mathrm{d}x \text{ and } \sigma_V^2(y) := \int_y^{\infty} \varphi_V^2(x)\,\mathrm{d}x.$$

We see by Remark 1 that (1) implies that

$$\sigma_V^2(y) > 0 \text{ for all } y > 0.$$

Throughout these notes, Z denotes a standard normal random variable. We shall need the following formal extension of Theorem 1 of Mason [7]. Its proof is nearly exactly the same as the proof of the Mason [7] version, and just replace the sequence of positive constants $\{t_n\}_{n\geq 1}$ in the proof of Theorem 1 of Mason [7] by $\{T_n\}_{n\geq 1}$. The proof of Theorem 1 of Mason [7] is based on a special case of Theorem 1.2 of Zaitsev [12], which we state in the digression below. Here is our self-standardized CLT for method I trimmed subordinated subordinators.

Theorem 1 *Assume that* $\overline{\Lambda}_V(0+) = \infty$. *For any sequence of positive integers* $\{k_n\}_{n\geq 1}$ *and sequence of strictly positive random variables* $\{T_n\}_{n\geq 1}$ *independent of* $(\Gamma_k)_{k\geq 1}$ *satisfying*

$$\frac{\sqrt{T_n}\sigma_V\left(\Gamma_{k_n}/T_n\right)}{\varphi_V\left(\Gamma_{k_n}/T_n\right)} \stackrel{P}{\to} \infty, \text{ as } n \to \infty,$$

we have uniformly in x, as n → ∞,

$$\left| P\left\{ \frac{\widetilde{V}^{(k_n)}(T_n) - T_n\mu_V\left(\Gamma_{k_n}/T_n\right)}{\sqrt{T_n}\sigma_V\left(\Gamma_{k_n}/T_n\right)} \le x|\Gamma_{k_n}, T_n \right\} - P\{Z \le x\} \right| \stackrel{P}{\to} 0,$$

which implies as n → ∞

$$\frac{\widetilde{V}^{(k_n)}(T_n) - T_n\mu_V\left(\Gamma_{k_n}/T_n\right)}{\sqrt{T_n}\sigma_V\left(\Gamma_{k_n}/T_n\right)} \stackrel{D}{\to} Z. \tag{9}$$

The remainder of this subsection will be devoted to examining a couple of special cases of the following example of Theorem 1.

Example For each $0 < \alpha < 1$, let $V_\alpha = (V_\alpha(t), t \ge 0)$ be an α-stable process with Laplace transform defined for $\theta > 0$ by

$$E\exp(-\theta V_\alpha(t)) = \exp\left(-t\int_0^\infty (1-\exp(-\theta x))\,\alpha\Gamma(1-\alpha)\,x^{-1-\alpha}\mathrm{d}x\right)$$

$$= \exp\left(-t\int_0^\infty \left(1-\exp(-\theta c_\alpha u^{-1/\alpha})\right)\mathrm{d}u\right) = \exp\left(-t\theta^\alpha\right), \tag{10}$$

where

$$c_\alpha = 1/\Gamma^{1/\alpha}(1-\alpha).$$

(See Example 24.12 of Sato [11].) Note that for V_α,

$$\varphi V_\alpha(x) =: \varphi_\alpha(x) = c_\alpha x^{-1/\alpha}\mathbf{1}_{\{x>0\}}.$$

We record that for each $t > 0$

$$V_\alpha(t) \stackrel{D}{=} \widetilde{V}_\alpha(t) := c_\alpha \sum_{i=1}^\infty \left(\frac{\Gamma_i}{t}\right)^{-1/\alpha}. \tag{11}$$

For any $t > 0$, denote the ordered jump sequence $m_\alpha^{(1)}(t) \ge m_\alpha^{(2)}(t) \ge \dots$ of V_α on the interval $[0, t]$. Consider the kth trimmed version of $V_\alpha(t)$ defined for each integer $k \ge 1$

$$V_\alpha^{(k)}(t) = V_\alpha(t) - m_\alpha^{(1)}(t) - \cdots - m_\alpha^{(k)}(t), \tag{12}$$

which for each $t > 0$

$$\stackrel{\mathrm{D}}{=} \widetilde{V}_{\alpha}^{(k)}(t) := c_{\alpha} \sum_{i=1}^{\infty} \left(\frac{\Gamma_{k+i}}{t} \right)^{-1/\alpha}. \tag{13}$$

In this example, for ease of notation, write for each $0 < \alpha < 1$ and $y > 0$, $\mu_{V_\alpha}(y) = \mu_\alpha(y)$ and $\sigma_{V_\alpha}^2(y) = \sigma_\alpha^2(y)$. With this notation, we get that

$$\mu_\alpha(y) = \int_y^\infty c_\alpha v^{-1/\alpha} \mathrm{d}v = \frac{c_\alpha \alpha}{1-\alpha} y^{1-1/\alpha}$$

and

$$\sigma_\alpha^2(y) = \int_y^\infty c_\alpha^2 v^{-2/\alpha} \mathrm{d}v = \frac{c_\alpha^2 \alpha}{2-\alpha} y^{1-2/\alpha}.$$

From (13), we have that for any $k \geq 1$ and $T > 0$

$$\frac{\widetilde{V}_{\alpha}^{(k)}(T) - T\mu_\alpha\left(\frac{\Gamma_k}{T}\right)}{T^{1/2}\sigma_\alpha\left(\frac{\Gamma_k}{T}\right)} = \frac{\sum_{i=1}^{\infty}(\Gamma_{k+i})^{-1/\alpha} - \frac{\alpha}{1-\alpha}\Gamma_k^{1-1/\alpha}}{\sqrt{\frac{\alpha}{2-\alpha}}\Gamma_k^{1/2-1/\alpha}}. \tag{14}$$

Notice that

$$\frac{\sqrt{T}\sigma_\alpha\left(\frac{\Gamma_k}{T}\right)}{\varphi_\alpha(\frac{\Gamma_k}{T})} = (\Gamma_k)^{1/2}\sqrt{\frac{\alpha}{2-\alpha}}. \tag{15}$$

Clearly by (15) for any sequence of positive integers $\{k_n\}_{n\geq 1}$ converging to infinity and sequence of strictly positive random variables $\{T_n\}_{n\geq 1}$ independent of $(\Gamma_k)_{k\geq 1}$,

$$\frac{\sqrt{T_n}\sigma_\alpha\left(\Gamma_{k_n}/T_n\right)}{\varphi_\alpha\left(\Gamma_{k_n}/T_n\right)} = \left(\Gamma_{k_n}\right)^{1/2}\sqrt{\frac{\alpha}{2-\alpha}} \stackrel{\mathrm{P}}{\to} \infty, \text{ as } n \to \infty.$$

Hence, by rewriting (9) in the above notation, we have by Theorem 1 that as $n \to \infty$

$$\frac{\widetilde{V}_{\alpha}^{(k_n)}(T_n) - T_n\mu_\alpha\left(\frac{\Gamma_{k_n}}{T_n}\right)}{T_n^{1/2}\sigma_\alpha\left(\frac{\Gamma_{k_n}}{T_n}\right)} \stackrel{\mathrm{D}}{\to} Z. \tag{16}$$

Digression To make the presentation of our Example more self-contained, we shall show in this digression how a special case of Theorem 1.2 of Zaitsev [12] can be used to give a direct proof of (16).

It is pointed out in Mason [7] that Theorem 1.2 of Zaitsev [12] implies the following normal approximation. Let Y be an infinitely divisible mean 0 and variance 1 random variable with Lévy measure Λ and Z be a standard normal random variable. Assume that the support of Λ is contained in a closed interval $[-\tau, \tau]$ with $\tau > 0$; then for universal positive constants C_1 and C_2 for any $\lambda > 0$ all $x \in \mathbb{R}$

$$P\{Z \le x - \lambda\} - C_1 \exp\left(-\frac{\lambda}{C_2 \tau}\right) \le P\{Y \le x\}$$

$$\le P\{Z \le x + \lambda\} + C_1 \exp\left(-\frac{\lambda}{C_2 \tau}\right). \tag{17}$$

We shall show how to derive (16) from (17). Note that

$$\frac{\sum_{i=1}^{\infty} (\Gamma_{k+i})^{-1/\alpha} - \frac{\alpha}{1-\alpha}\Gamma_k^{1-1/\alpha}}{\sqrt{\frac{\alpha}{2-\alpha}}\Gamma_k^{1/2-1/\alpha}} \stackrel{D}{=} \frac{\sum_{i=1}^{\infty} \left(1 + \frac{\Gamma_i'}{\Gamma_k}\right)^{-1/\alpha} - \frac{\alpha}{1-\alpha}\Gamma_k}{\sqrt{\frac{\alpha}{2-\alpha}}\Gamma_k^{1/2}}, \tag{18}$$

where $\left(\Gamma_i'\right)_{i \ge 1} \stackrel{D}{=} (\Gamma_i)_{i \ge 1}$ and is independent of $(\Gamma_i)_{i \ge 1}$. Let $Y_\alpha = (Y_\alpha(y), y \ge 0)$ be the subordinator with Laplace transform defined for each $y > 0$ and $\theta \ge 0$, by

$$E \exp(-\theta Y_\alpha(y)) = \exp\left(-y \int_0^1 (1 - \exp(-\theta x))\, \alpha x^{-\alpha-1} \mathrm{d}x\right)$$

$$=: \exp\left(-y \int_0^1 (1 - \exp(-\theta x))\, \Lambda_\alpha(\mathrm{d}x)\right). \tag{19}$$

Observe that the Lévy measure Λ_α of Y_α has Lévy tail function on $(0, \infty)$

$$\overline{\Lambda}_\alpha(x) = \left(x^{-\alpha} - 1\right) \mathbf{1}_{\{0 < x \le 1\}}$$

with φ function

$$\varphi_{Y_\alpha}(u) = (1 + u)^{-1/\alpha} \mathbf{1}_{\{u > 0\}}.$$

Thus from (5), for each $y > 0$,

$$Y_\alpha(y) \stackrel{D}{=} \sum_{i=1}^{\infty} \left(1 + \frac{\Gamma_i'}{y}\right)^{-1/\alpha}.$$

Also, we find by differentiating the Laplace transform of $Y_\alpha(y)$ that for each $y > 0$

$$EY_\alpha(y) = \frac{\alpha y}{1-\alpha} =: \beta_\alpha y \text{ and } VarY_\alpha(y) = \frac{\alpha y}{2-\alpha} =: \gamma_\alpha^2 y, \tag{20}$$

and hence,

$$Z_\alpha(y) := \frac{Y_\alpha(y) - \beta_\alpha y}{\gamma_\alpha \sqrt{y}}$$

is a mean 0 and variance 1 infinitely divisible random variable whose Lévy measure has support contained in the closed interval $[-\tau(y), \tau(y)]$, where

$$\tau(y) = 1/\left(\gamma_\alpha \sqrt{y}\right). \tag{21}$$

Thus by (17) for universal positive constants C_1 and C_2 for any $\lambda > 0$ all $x \in \mathbb{R}$ and $\lambda > 0$,

$$P\{Z \le x - \lambda\} - C_1 \exp\left(-\frac{\lambda}{C_2 \tau(y)}\right) \le P\{Z_\alpha(y) \le x\}$$

$$\le P\{Z \le x + \lambda\} + C_1 \exp\left(-\frac{\lambda}{C_2 \tau(y)}\right). \tag{22}$$

Clearly, since $\left(\Gamma_i'\right)_{i\ge1} \stackrel{\mathrm{D}}{=} (\Gamma_i)_{i\ge1}$ and $\left(\Gamma_i'\right)_{i\ge1}$ is independent of $\left(\Gamma_{k_n}\right)_{n\ge1}$, we conclude by (22) and (21) that

$$P\{Z \le x - \lambda\} - C_1 \exp\left(-\frac{\lambda\gamma_\alpha\sqrt{\Gamma_{k_n}}}{C_2}\right) \le P\left\{Z_\alpha\left(\Gamma_{k_n}\right) \le x | \Gamma_{k_n}\right\}$$

$$\le P\{Z \le x + \lambda\} + C_1 \exp\left(-\frac{\lambda\gamma_\alpha\sqrt{\Gamma_{k_n}}}{C_2}\right). \tag{23}$$

Now by the arbitrary choice of $\lambda > 0$, we get from (23) that uniformly in x, as $k_n \to \infty$,

$$\left| P\left\{\frac{Y_\alpha\left(\Gamma_{k_n}\right) - \beta_\alpha \Gamma_{k_n}}{\gamma_\alpha\sqrt{\Gamma_{k_n}}} \le x | \Gamma_{k_n}\right\} - P\{Z \le x\}\right| \stackrel{\mathrm{P}}{\to} 0.$$

This implies as $n \to \infty$

$$\frac{Y_\alpha\left(\Gamma_{k_n}\right) - \beta_\alpha \Gamma_{k_n}}{\gamma_\alpha\sqrt{\Gamma_{k_n}}} \stackrel{\mathrm{D}}{\to} Z. \tag{24}$$

Since the identity (14) holds for any $k \ge 1$ and $T > 0$, (16) follows from (18) and (24). Of course, there are other ways to establish (24). For instance, (24) can be shown to be a consequence of Anscombe's Theorem for Lévy processes. For details, see Appendix 2.

Remark 5 For any $0 < \alpha < 1$ and $k \geq 1$, the random variable $Y_\alpha(\Gamma_k)$ has Laplace transform

$$E \exp\left(-\theta Y_\alpha(\Gamma_k)\right) = \left(1 + \int_0^1 (1 - \exp(-\theta x))\, \Lambda_\alpha(\mathrm{d}x)\right)^{-k}, \theta \geq 0.$$

It turns out that for any $t > 0$

$$Y_\alpha(\Gamma_k) \stackrel{\mathrm{D}}{=} V_\alpha^{(k)}(t) / m_\alpha^{(k)}(t),$$

where $V_\alpha^{(k)}(t)$ and $m_\alpha^{(k)}(t)$ are as in (12). See Theorem 1.1 (i) of Kevei and Mason [6]. Also refer to page 1979 of Ipsen et al [4].

Next we give two special cases of our example, which we shall return to in the next subsection when we discuss self-standardized CLTs for method II trimming.

Special Case 1: Subordination of Two Independent Stable Subordinators
For $0 < \alpha_1, \alpha_2 < 1$, let V_{α_1}, respectively V_{α_2}, be an α_1-stable process, respectively an α_2-stable process, with a Laplace transform of the form (10). Assume that V_{α_1} and V_{α_2} are independent. Set for $t \geq 0$

$$W(t) = V_{\alpha_1}\left(V_{\alpha_2}(t)\right)$$

and

$$W = (W(t), t \geq 0).$$

One finds that for each $t \geq 0$

$$W(t) = V_{\alpha_1}\left(V_{\alpha_2}(t)\right) = \sum_{0<s\leq V_{\alpha_2}(t)} \Delta V_{\alpha_1}(s).$$

Moreover, W is a stationary independent increment process, and for each $t \geq 0$ and $\theta \geq 0$,

$$\begin{aligned} E \exp(-\theta W(t)) &= E \exp\left(-V_{\alpha_2}(t)\, \theta^{\alpha_1}\right) \\ &= \exp\left(-t\theta^{\alpha_1\alpha_2}\right). \end{aligned} \tag{25}$$

This says that W is the $\alpha_1\alpha_2$-stable subordinator $V_{\alpha_1\alpha_2}$ with Laplace transform (25). (See Example 30.5 on page 202 of Sato [11].) Thus for each $t \geq 0$ and $\theta \geq 0$,

$$E \exp(-\theta W(t)) = E \exp\left(-\theta V_{\alpha_1\alpha_2}(t)\right). \tag{26}$$

Therefore, with $c(\alpha_1\alpha_2) = \frac{1}{\Gamma^{1/(\alpha_1\alpha_2)}(1-\alpha_1\alpha_2)}$, we get

$$c(\alpha_1\alpha_2)\sum_{i=1}^{\infty}\left(\frac{\Gamma_i}{t}\right)^{-1/(\alpha_1\alpha_2)} =: \widetilde{V}_{\alpha_1\alpha_2}(t),$$

which by (11), (25), and (26) for each fixed $t > 0$ is

$$\stackrel{\mathrm{D}}{=} V_{\alpha_1}\left(V_{\alpha_2}(t)\right).$$

Here we get that for any sequence of positive integers $\{k_n\}_{n\geq 1}$ converging to infinity and sequence of positive constants $\{s_n\}_{n\geq 1}$, by setting $T_n = V_{\alpha_2}(s_n)$, for $n \geq 1$, we have by (16) that as $n \to \infty$

$$\frac{\widetilde{V}_{\alpha_1}^{(k_n)}\left(V_{\alpha_2}(s_n)\right) - V_{\alpha_2}(s_n)\,\mu_{\alpha_1}\left(\frac{\Gamma_{k_n}}{V_{\alpha_2}(s_n)}\right)}{\sqrt{V_{\alpha_2}(s_n)}\sigma_{\alpha_1}\left(\frac{\Gamma_{k_n}}{V_{\alpha_2}(s_n)}\right)} \stackrel{\mathrm{D}}{\to} Z.$$

Special Case 2: Mittag-Leffler Process

For each $0 < \alpha < 1$, let V_α be the α-stable process with Laplace transform (10). Now independent of V, let $X = (X(s), s \geq 0)$ be the standard Gamma process, i.e., X is a zero drift subordinator with density for each $s > 0$

$$f_{X(s)}(x) = \frac{1}{\Gamma(s)}x^{s-1}e^{-x}, \text{ for } x > 0,$$

mean and variance

$$EX(s) = s \text{ and } VarX(s) = s,$$

and Laplace transform for $\theta \geq 0$

$$E\exp(-\theta X(s)) = (1+\theta)^{-s},$$

which after a little computation is

$$= \exp\left[-s\int_0^{\infty}(1-\exp(-\theta x))\,x^{-1}e^{-x}\mathrm{d}x\right].$$

Notice that X has Lévy density

$$\rho(x) = x^{-1}e^{-x}, \text{ for } x > 0.$$

(See Applebaum [1] pages 54–55.)

Consider the subordinated process

$$W = (W(s) := V_\alpha(X(s)), s \geq 0).$$

Applying Theorem 30.1 and Theorem 30.4 of Sato [11], we see that W is a drift 0 subordinator with Laplace transform

$$E\exp(-\theta W(s)) = E\exp(-V(X(s)))$$

$$= E\exp\left(-X(s)\,\theta^\alpha\right) = \left(1+\theta^\alpha\right)^{-s}$$

$$= \exp\left[-s\int_0^\infty \left(1-\exp\left(-\theta^\alpha y\right)\right) y^{-1}e^{-y}\mathrm{d}y\right],\ \theta \geq 0.$$

It has Lévy measure Λ_W defined for Borel subsets B of $(0, \infty)$, by

$$\Lambda_W(B) = \int_0^\infty P\{V_\alpha(y) \in B\}\, y^{-1}e^{-y}\mathrm{d}y.$$

In particular, it has Lévy tail function

$$\overline{\Lambda}_W(x) = \int_0^\infty P\{V(y) \in (x,\infty)\}\, y^{-1}e^{-y}\mathrm{d}y, \text{ for } x > 0.$$

For later use, we note that

$$\int_0^\infty \left(1-e^{-\theta x}\right)\Lambda_W(\mathrm{d}x) = \int_0^\infty\int_0^\infty \left(1-e^{-\theta x}\right) P_{V_\alpha(y)}(\mathrm{d}x)\, ay^{-1}e^{-by}\mathrm{d}y$$

$$= \int_0^\infty \left(1-e^{y\theta^\alpha}\right) y^{-1}e^{-y}\mathrm{d}y.$$

Such a process W is called the Mittag-Leffler process. See, e.g., Pillai [8].

By Theorem 4.3 of Pillai [8] for each $s > 0$, the exact distribution function $F_{\alpha,s}(x)$ of $W(s)$ is for $x \geq 0$

$$F_{\alpha,s}(x) = \sum_{r=0}^\infty (-1)^r \frac{\Gamma(s+r)\, x^{\alpha(s+r)}}{\Gamma(s)\, r!\,\Gamma(1+\alpha(s+r))},$$

which says that for each $s > 0$ and $x \geq 0$

$$P\{W(s) \leq x\} = P\{V_\alpha(X(s)) \leq x\}$$

$$= P\left\{\widetilde{V}_\alpha(X(s)) \leq x\right\} = F_{\alpha,s}(x).$$

In this special case, for any sequence of positive integers $\{k_n\}_{n\geq 1}$ converging to infinity and sequence of positive constants $\{s_n\}_{n\geq 1}$, by setting $T_n = X(s_n)$, for $n \geq 1$, we get by (16) that as $n \to \infty$

$$\frac{\widetilde{V}_{\alpha}^{(k_n)}\left(X\left(s_n\right)\right) - X\left(s_n\right)\mu_{\alpha}\left(\Gamma_{k_n}/X\left(s_n\right)\right)}{\sqrt{X\left(s_n\right)}\sigma_{\alpha}\left(\Gamma_{k_n}/X\left(s_n\right)\right)} \stackrel{\mathrm{D}}{\to} Z.$$

3.2 *Self-Standardized CLTs for Method II Trimming*

Let W be a subordinator of the form (2). Set for any $y > 0$

$$\mu_W(y) := \int_y^{\infty} \varphi_W(x)\,\mathrm{d}x \text{ and } \sigma_W^2(y) := \int_y^{\infty} \varphi_W^2(x)\,\mathrm{d}x.$$

We see by Remarks 1 and 2 that (1) implies that

$$\sigma_W^2(y) > 0 \text{ for all } y > 0.$$

For easy reference for the reader, we state here a version of Theorem 1 of Mason [7] stated in terms of a self-standardized CLT for the method II trimmed subordinated subordinator W.

Theorem 2 *Assume that* $\overline{\Lambda}_W(0+) = \infty$. *For any sequence of positive integers* $\{k_n\}_{n\geq 1}$ *and sequence of positive constants* $\{t_n\}_{n\geq 1}$ *satisfying*

$$\frac{\sqrt{t_n}\sigma_W\left(\Gamma_{k_n}/t_n\right)}{\varphi_W\left(\Gamma_{k_n}/t_n\right)} \stackrel{\mathrm{P}}{\to} \infty, \text{ as } n \to \infty,$$

we have uniformly in x, as $n \to \infty$,

$$\left| P\left\{ \frac{\widetilde{W}^{(k_n)}\left(t_n\right) - t_n\mu_W\left(\Gamma_{k_n}/t_n\right)}{\sqrt{t_n}\sigma_W\left(\Gamma_{k_n}/t_n\right)} \leq x|\Gamma_{k_n} \right\} - P\left\{Z \leq x\right\} \right| \stackrel{\mathrm{P}}{\to} 0,$$

which implies as $n \to \infty$

$$\frac{\widetilde{W}^{(k_n)}\left(t_n\right) - t_n\mu_W\left(\Gamma_{k_n}/t_n\right)}{\sqrt{t_n}\sigma_W\left(\Gamma_{k_n}/t_n\right)} \stackrel{\mathrm{D}}{\to} Z.$$

Remark 6 Theorem 1 of Mason [7] contains the added assumption that $k_n \to \infty$, as $n \to \infty$. An examination of its proof shows that this assumption is unnecessary. Also we note in passing that Theorem 1 implies Theorem 2.

For the convenience of the reader, we state the following results. Corollary 1 is from Mason [7]. The proof of Corollary 2 follows after some obvious changes of notation that of Corollary 1.

Corollary 1 *Assume that $W(t)$, $t \geq 0$, is a subordinator with drift* 0, *whose Lévy tail function $\overline{\Lambda}_W$ is regularly varying at zero with index $-\alpha$, where $0 < \alpha < 1$. For any sequence of positive integers $\{k_n\}_{n\geq 1}$ converging to infinity and sequence of positive constants $\{t_n\}_{n\geq 1}$ satisfying $k_n/t_n \to \infty$, we have, as $n \to \infty$,*

$$\frac{\widetilde{W}^{(k_n)}(t_n) - t_n \mu_W(k_n/t_n)}{\sqrt{t_n}\sigma_W(k_n/t_n)} \xrightarrow{\mathrm{D}} \sqrt{\frac{2}{\alpha}} Z. \tag{27}$$

Corollary 2 *Assume that $W(t)$, $t \geq 0$, is a subordinator with drift* 0, *whose Lévy tail function $\overline{\Lambda}_W$ is regularly varying at infinity with index $-\alpha$, where $0 < \alpha < 1$. For any sequence of positive integers $\{k_n\}_{n\geq 1}$ converging to infinity and sequence of positive constants $\{t_n\}_{n\geq 1}$ satisfying $k_n/t_n \to 0$, as $n \to \infty$, we have* (27).

The subordinated subordinator introduced in Special Case 1 above satisfies the conditions of Corollary 1, and the subordinated subordinator in Special Case 2 above fulfills the conditions of Corollary 2. Consider the two cases.

Special Case 1 To see this, notice that in Special Case 1, by (25) necessarily W has Lévy tail function on $(0, \infty)$

$$\overline{\Lambda}_W(y) = \Gamma(1 - \alpha_1\alpha_2)\, y^{-\alpha_1\alpha_2}\mathbf{1}_{\{y>0\}},$$

for $0 < \alpha_1, \alpha_2 < 1$, which is regularly varying at zero with index $-\alpha$, where $0 < \alpha = \alpha_1\alpha_2 < 1$. In this case, from Corollary 1, we get (27) as long as $k_n \to \infty$ and $k_n/t_n \to \infty$, as $n \to \infty$.

Special Case 2 In Special Case 2, observe that $W = V_\alpha(X)$, with $0 < \alpha < 1$, where $V_\alpha = (V_\alpha(t), t \geq 0)$ is an α-stable process with Laplace transform (10), $X = (X(s), s \geq 0)$ is a standard Gamma process, and V_α and X are independent. The process $r^{-1/\alpha}W(r)$ has Laplace transform $(1 + \theta^\alpha/r)^{-r}$, for $\theta \geq 0$, which converges to $\exp(-\theta^\alpha)$ as $r \to \infty$. This implies that for all $t > 0$

$$r^{-1/\alpha}W(rt) \xrightarrow{\mathrm{D}} V_\alpha(t), \text{ as } r \to \infty.$$

By part (ii) of Theorem 15.14 of Kallenberg [5] and (10) for all $x > 0$

$$r\overline{\Lambda}_W\left(r^{1/\alpha}x\right) \to \Gamma(1-\alpha)\, x^{-\alpha}, \text{ as } r \to \infty.$$

This implies that W has a Lévy tail function $\overline{\Lambda}_W(y)$ on $(0, \infty)$, which is regularly varying at infinity with index $-\alpha$, $0 < \alpha < 1$. In this case, by Corollary 2, we can conclude (27) as long as $k_n \to \infty$ and $k_n/t_n \to 0$, as $n \to \infty$.

4 Appendix 1

Recall the notation of Special Case 1. Let V_{α_1}, V_{α_2}, and $(\Gamma_k)_{k\geq 1}$ be independent and $W = V_{\alpha_1}\left(V_{\alpha_2}\right)$. For any $t > 0$, let $m^{(1)}_{V_{\alpha_1}}(V_{\alpha_2}(t)) \geq m^{(2)}_{V_{\alpha_1}}(V_{\alpha_2}(t)) \geq \cdots$ denote the ordered jumps of V_{α_1} on the interval $\left[0, V_{\alpha_2}(t)\right]$. They satisfy

$$\left(m^{(k)}_{\alpha_1}(V_{\alpha_2}(t))\right)_{k\geq 1} \stackrel{\mathrm{D}}{=} \left(c(\alpha_1)\left(\frac{\Gamma_k}{V_{\alpha_2}(t)}\right)^{-1/\alpha_1}\right)_{k\geq 1}.$$

Let $m^{(1)}_W(t) \geq m^{(2)}_W(t) \geq \cdots$ denote the ordered jumps of W on the interval $[0, t]$. In this case, for each $t > 0$

$$\left(m^{(k)}_W(t)\right)_{k\geq 1} \stackrel{\mathrm{D}}{=} \left(c(\alpha_1\alpha_2)\left(\frac{\Gamma_k}{t}\right)^{-1/(\alpha_1\alpha_2)}\right)_{k\geq 1}.$$

Observe that for all $t > 0$

$$W(t) = \sum_{0<s\leq t} \Delta W(s) = \sum_{0<s\leq V_{\alpha_2}(t)} \Delta V_{\alpha_1}(s) = \sum_{k=1}^{\infty} m^{(k)}_{\alpha_1}(V_{\alpha_2}(t)). \tag{28}$$

Note that though (28) holds, $\left(m^{(k)}_{\alpha_1}(V_{\alpha_2}(t))\right)_{k\geq 1}$ is not equal in distribution to $\left(m^{(k)}_W(t)\right)_{k\geq 1}$. To see this, notice that

$$\left(\frac{m^{(k)}_{\alpha_1}(V_{\alpha_2}(t))}{m^{(1)}_{\alpha_1}(V_{\alpha_2}(t))}\right)_{k\geq 1} \stackrel{\mathrm{D}}{=} \left(\left(\frac{\Gamma_k}{\Gamma_1}\right)^{-1/\alpha_1}\right)_{k\geq 1}, \tag{29}$$

whereas

$$\left(\frac{m^{(k)}_W(t)}{m^{(1)}_W(t)}\right)_{k\geq 1} \stackrel{\mathrm{D}}{=} \left(\left(\frac{\Gamma_k}{\Gamma_1}\right)^{-1/(\alpha_1\alpha_2)}\right)_{k\geq 1}. \tag{30}$$

Obviously, the sequences (29) and (30) are not equal in distribution and thus

$$\left(m^{(k)}_{\alpha_1}(V_{\alpha_2}(t))\right)_{k\geq 1} \stackrel{\mathrm{D}}{\neq} \left(m^{(k)}_W(t)\right)_{k\geq 1}.$$

5 Appendix 2

A straightforward modification of the proof of Theorem 1 of Rényi [10] gives the following Anscombe's theorem for Lévy processes.

Theorem A *Let* $X = (X(t), t \geq 0)$ *be a mean zero Lévy process with* $EX^2(t) = t$ *for* $t \geq 0$*, and let* $\eta = (\eta(t), t > 0)$ *be a random process such that* $\eta(t) > 0$ *for all* $t > 0$ *and for some* $c > 0$*,* $\eta(t)/t \xrightarrow{P} c$*, as* $t \to \infty$*, then*

$$X(\eta(t))/\sqrt{\eta(t)} \xrightarrow{D} Z.$$

A version of Anscombe's theorem is given in Gut [2]. See his Theorem 3.1. In our notation, his Theorem 3.1 requires that $\{\eta(t), t \geq 0\}$ *be a family of stopping times.*

Example A Let $Y_\alpha = (Y_\alpha(y), y \geq 0)$ be the Lévy process with Laplace transform (19) and mean and variance functions (20). We see that

$$X := \left(X(y) = \frac{Y_\alpha(y) - \beta_\alpha y}{\gamma_\alpha},\ y \geq 0 \right)$$

defines a mean zero Lévy process with $EX^2(y) = y$ for $y \geq 0$. Now let $\eta = (\eta(t),\ t \geq 0)$ be a standard Gamma process independent of X. Notice that $\eta(t)/t \xrightarrow{P} 1$, as $t \to \infty$. Applying Theorem A, we get as $t \to \infty$,

$$X(\eta(t))/\sqrt{\eta(t)} \xrightarrow{D} Z.$$

In particular, since for each integer $k \geq 1$, $\eta(k) \stackrel{D}{=} \Gamma_k$, this implies that (24) holds for any sequence of positive integers $(k_n)_{n \geq 1}$ converging to infinity as $n \to \infty$, i.e.,

$$\frac{Y_\alpha\left(\Gamma_{k_n}\right) - \beta_\alpha \Gamma_{k_n}}{\gamma_\alpha \sqrt{\Gamma_{k_n}}} \xrightarrow{D} Z.$$

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