Chapter 6
The Age of Analysis and the French Revolution

Everything our minds can comprehend is interrelated.

Leonhard Euler

The Age of Analysis

The King of the Amateurs

Considering the art of the Renaissance, it is not difficult to arrive at the conclusion that painting as the representative of the spatial arts is intimately tied to geometry, just as in Ancient Greece, the Pythagoreans recognized that algebra or arithmetic is closely related to music, the representative of the temporal arts. It is interesting to note in this light that the great masters of modern music in Europe did not appear until late in the seventeenth century, with the appearance of such figures as Antonio Vivaldi (1678–1741), Johann Sebastian Bach (1685–1750) in Germany, and George Frideric Handel (1685–1759), who was also born in Germany, but spent most of his life in England. They arrived on the scene much later than the master painters and sculptors of the Renaissance. Perhaps this is related to the fact that prior to the invention of calculus, geometry occupied the undisputed place of prominence in mathematics, with Euclidean geometry as its core (Fig. 6.1).

From antiquity, most mathematicians in Europe referred to themselves as geometers; this is exemplified by the most famous epigrams associated with ancient mathematics, Euclid's remark that "there is no royal road to geometry," and the inscription "let no one ignorant of geometry enter here" at the entrance to Plato's Academy. Much later, Pascal refers to geometers in the broad sense in his melancholic aphorism from the *Pensées*, "all geometers would be intuitive, if only they had clear sight, ... and all intuitive minds would become geometers, if only they could direct their sight to the unfamiliar principles of geometry."

The establishment of the Cartesian coordinate system provided a bridge linking the study of geometry to the use of algebraic methods, and the impression of algebra as a subordinate discipline in mathematics also changed. All the same, the primary

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Fig. 6.1 Pierre de Fermat

focus of research in algebra at the time revolved still around problems relation to solving equations, and it would have to wait until the nineteenth century for truly revolutionary changes in algebra to appear, as for that matter in geometry. Rather the first branch of mathematics to experience true breakthroughs was number theory, the most ancient topic in mathematics, concerned with the properties of the natural numbers or the integers and their interrelationships, a topic which could be described as frequently stepping out from the garden of algebra. This was due mainly to the private interest and efforts of an unassuming amateur named Pierre de Fermat (1607–1665), a civilian official in the small town of Toulouse in the south of France (Fig. 6.2).

At his provincial remove far from the capital city Paris, Fermat spent his days occupied in judicial affairs and devoted the evenings and the holidays almost exclusively to his passion for mathematics and its study. Partially, this was on account of opposition in France at the time to private social activity among its councilors, in light of the fact that friends and acquaintances might someday find themselves before the court. This forced isolation from the upper echelons of Toulouse society to which he might otherwise naturally have belonged enabled Fermat to focus on his research hobby; he spent nearly all his nights engrossed in mathematics, and he was drawn especially to problems in number theory. He proposed a wealth of propositions and conjectures, many of which have kept mathematicians busy through the centuries since.

There are not so many complete conclusions associated with Fermat for which he himself provided proofs; among these, the most famous are as follows: every

Fig. 6.2 Andrew Wiles, who proved Fermat's last theorem

odd prime number can be expressed as a difference of two square numbers in one and only one way, and every odd prime number of the form $4n + 1$ denotes the hypotenuse of a right triangle with integer sides in exactly one way when not raised to any power, in two ways when squared, in three ways when cubes, and so on, as, for example,

$$5^2 = 3^2 + 4^2,$$
$$25^2 = 15^2 + 20^2 = 7^2 + 24^2,$$
$$125^2 = 75^2 + 100^2 = 35^2 + 120^2 = 44^2 + 117^2.$$

More often, Fermat would present his results either in his correspondences or by way of a mathematical challenge simply with a statement of the conclusion, without any proof. These include the following: the area of a right triangle with integer side lengths can never be a square number, and every natural number can be written as a sum of four or fewer square numbers. There is a famous generalization of this latter conclusion known as Waring's problem. Research on Waring's problem attracted international attention to the autodidact Chinese mathematician Hua Luogeng (1910–1985), who was partially paralyzed by typhoid fever in his youth and made contributions to mathematics in the fields of analytic number theory, algebra, the theory of functions of several complex variables, numerical analysis, and others.

The two propositions just mentioned were only proved later by the French mathematician Joseph-Louis Lagrange; the Swiss mathematician Leonhard Euler also devoted considerable energy to the resolution of various questions left unresolved by Fermat, and it is for this reason that we have delayed a discussion of Fermat until the beginning of this chapter, since both Lagrange and Euler were mathematicians of the eighteenth century. In fact, throughout his long career, Euler carried out deep and meticulous research into almost problem posed by Fermat. As one example, Fermat famously conjectured that for every nonnegative inter n, the number

$$F_n = 2^{2^n} + 1$$

is prime; such prime numbers are called Fermat primes. Fermat himself verified this conjecture for the cases $0 \leq n \leq 4$. But Euler discovered that F_5 is not prime and even identified a prime factor 641 of F_5. Since that time, no new Fermat primes have been found.

For another example, Fermat had proposed in 1740 in a correspondence with a friend that following divisibility result: if p is a prime number, and a is any integer relatively prime to p (i.e., the greatest common factor of a and p is 1, which means simply that a is not a multiple of p when p is prime), then $a^{p-1} - 1$ is divisible by p. Nearly a century later, Euler not only proved this proposition but also generalized it considerably to the case where p in the proposition is replaced by any positive integer. For this generalization, he introduced what has since come to be called the Euler totient function $\phi(n)$, which counts the number of positive integers not exceeding n that are relatively prime to n. So $\phi(1) = \phi(2) = 1$, $\phi(3) = \phi(4) = \phi(6) = 2$ (because, e.g., 1 and 5 are the only two positive integers not exceeding 6 that are relatively prime to it), $\phi(5) = 4$, and so on. Euler's generalization states that for any two relatively prime positive integers n and a, $a^{\phi(n)} - 1$ is divisible by n.

The special case and its generalization just discussed are known as Fermat's little theorem and Euler's theorem, respectively. It is somewhat astonishing that Euler's theorem has sprouted important applications in modern society several centuries later: it plays an important role in the RSA public key cryptosystem developed in 1977 and widely used today for secure data transmission. But in contrast with Fermat's little theorem, Euler could make no dents in the conjecture and eventual theorem that came to be known as Fermat's last theorem, first written down by Fermat in 1637. Fermat's last theorem states that there are no solutions x, y, z in positive integers for the equation

$$x^n + y^n = z^n$$

whenever $n \geq 3$. Of course when $n = 2$, there are infinitely many solutions; these are precisely the Pythagorean triples, which can be easily and completely characterized. Fermat himself proved that there are no solutions when $n = 4$, and Euler resolved the case $n = 3$ (which is more difficult than the case $n = 4$). But a fully general proof remained completely out of reach.

For more than three centuries after it was first written down, this conjecture continued to attract innumerable bright and intelligent mathematicians to make their own contributions to it, until finally it was proved toward the end of the twentieth century by the British mathematician Andrew Wiles (1953-), working at Princeton in the United States. This news made the front page of *The New York Times*, alongside a portrait of Fermat. In fact, what Wiles, with assistance from his student

Richard Taylor (1962-), proved is actually something known at the time as the Taniyama-Shimura conjecture, proposed in 1957 by two Japanese mathematicians and now referred to as the modularity theorem. More precisely, Wiles and Taylor proved a special case of the conjecture as applied to semistable elliptic curves, which was sufficient to prove Fermat's last theorem as a corollary. The modularity theorem elucidates the relationship between elliptic curves and modular forms; the former are geometric objects with profound arithmetic properties, and the latter are highly periodic functions derived from the field of analysis.

In addition to the two Japanese mathematicians just mentioned, many mathematicians have made important contributions to general mathematics along the road toward a proof of Fermat's last theorem. Particularly worthy of mention is the German mathematician Ernst Kummer (1810–1893), who introduced the theory of ideal numbers and thereby established the discipline of algebraic number theory, a development that is probably more important than Fermat's last theorem itself. His extended family also included the composer Felix Mendelssohn and the mathematician Peter Gustav Lejeune Dirichlet.

Finally, there is a famous story concerning the origins of Fermat's last theorem: Fermat wrote his conjecture in the margins to his Latin copy of the book *Arithmetica* by the Ancient Greek mathematician Diophantus. Following it, the mischievous recluse scribbled an additional remark: "I have discovered a truly marvelous proof of this, which this margin is too narrow to contain."

Fermat also carried out important research outside the scope of number theory. In optics, there is Fermat's principle, which states that the path taken by a ray of light between two points is that which can be travelled in the least amount of time, whether a straight line or bent due to refraction. A corollary is that light travels in straight lines through a vacuum. Returning to mathematics, Fermat also discovered the basic principles of analytic geometry independent of Descartes, and his methods for finding the maxima and minima of curves established him as a founder of differential calculus. And in his correspondence with Pascal, the two mathematicians inaugurated probability as mathematical subject. They were interesting in particular in a gambling problem: suppose A and B are two gamblers with a comparable level of skills playing a game in which A needs to earn at least two points in a round to win, while B needs at least three; what is the probability of victory for each?

Fermat analyzed the situation in a table as follows, using the lowercase letters a and b to indicate a point earned by A or B, respectively, and taking into account that every game is completed in at most four rounds:

aaaa	aaab	abba	bbab
baaa	baba	abab	babb
abaa	bbaa	aabb	abbb
aaba	baab	bbba	bbbb

From this, the solution can be read off directly: the probability of victory for A is $\frac{11}{16}$, and for B, it is $\frac{5}{16}$.

It is necessary here also to include some discussion of statistics, which appeared later than probability and consists mainly of the collection of data, the use of probability theory for the construction of models, quantitative analysis, consolidation of results, and ultimately inference and prediction; all of this makes statistics an invaluable tool for research and decision-making, in areas as diverse as physics and the social sciences, the humanities, and business and government. Major applications in particular are in insurance, epidemiology, census making, and public polling. In the modern zoology of disciplines, statistics has separated from mathematics and established itself like computer science as an independent field of research derived from mathematical origins.

We mentioned in the first chapter that statistics had its earliest development in the work of Aristotle, but this did not yet include its maturation as an independent discipline. Modern statistics, like the theory of probability, grew from not altogether reputable origins, the latter from the study of gambling and the former from the analysis of death. In 1666, the Great Fire of London swept through the city destroying such notable buildings as St. Paul's Cathedral and possibly helping to bring an end to the plague years. One of its victims was a local haberdasher named John Graunt (1620–1674), who was bankrupted by the devastation, who had made a study of 130 years worth of death records in London. He used survival rates at ages 6 and 76 to extrapolate the proportion of the population that had lived to other ages and determine their life expectancies. A similar study was carried out in 1693 by the British astronomer Edmond Halley (1656–1742), who conducted a statistical survey of the mortality rate in the German city of Breslau (now known as Wrocław and part of Poland).

We close this section with some further remarks on Fermat's last theorem, which has been likened to a goose that lays golden eggs. When Wiles announced that he had conquered this problem, the mathematical community was at once overjoyed but also concerned that there would be no more such problems that would stimulate so fruitfully the development of number theory. But within a few years, the *abc* conjecture emerged as an important candidate for its replacement. The *abc* conjecture is an inequality relating the two fundamental integer operations of addition and multiplication. We introduce first a bit of notation: if n is a natural number, define its radical $\mathrm{rad}(n)$ as the product of its distinct prime factors. For example, $\mathrm{rad}(12) = 6$, since the distinct prime factors of 12 are 2 and 3. The *abc* conjecture, which was proposed in 1985 independently by the French mathematician Joseph Oesterlé (1954-) and the British mathematician David Masser (1948-), states, in its weaker form, that if a, b, and c are relatively prime integers such that $a + b = c$, then

$$c \leq (\mathrm{rad}(abc))^2 .$$

The resolution of the *abc* conjecture or its weaker version could lead to the solution of a number of important and outstanding problems in number theory. It is also easy to derive directly some well-known theorems and conjectures as corollaries to the *abc* conjecture, including four results that earned Fields Medals for their proofs, one of these being Fermat's last theorem. Taking this as an example, suppose $n \geq 3$ and $x^n + y^n = z^n$. Then with $a = x^n$, $b = y^n$, and $c = z^n$, the weak form of the *abc* conjecture states that

$$z^n \leq (\mathrm{rad}(x^n y^n z^n))^2 < (xyz)^2 < z^6.$$

This limits the possible values of n to $n = 3$, 4, or 5, and these cases can be handled by purely elementary methods.

The Further Development of Calculus

For the Western European powers at the center of the recent scientific developments, the transition from the seventeenth to the eighteenth century was a relatively smooth period, but the northern regions experience some turbulence and change during this time. In the year 1700, Tsar Peter the Great of Russia adopted the Julian calendar, with January 1st as the first day of the new year, and at the same time began the undertaking of various reforms of a military nature. That summer, only a week after the conclusion of a 30-year truce agreement with Turkey, Russia, with Poland and Denmark as allies, launched the Great Northern War against Sweden. Denmark withdrew from the effort not long afterward, however, when King Charles XII of Sweden, who was fond incidentally of mathematics, painting, and architecture, led his troops to Copenhagen. In Germany at this time, the Royal Prussian Academy of Sciences was established in Berlin, with Leibniz as its first president.

The rapid development of calculus shortly after its invention was facilitated precisely by the peace and prosperity of this era. Its applications also spread wide and quickly, resulting in many new branches of mathematics, collected together under the umbrella term *analysis* as an ensemble of distinct concepts and methods. The eighteenth century became known in mathematics as the era of analysis, an important period of transition from ancient to modern mathematics. Intriguingly, just as analysis presented a synthesis of geometry and algebra, there also appeared a new synthesis in the arts between spatial art and temporal art. The characteristic form of synthetic art is theater, and eventually film, which comprises both a spatial component alike to the visual arts such as painting and sculpture and a temporal component, for which the classical analogues are poetry and music. It was after the Renaissance that European theater began its rapid development (Fig. 6.3).

In France, the golden age of drama was the seventeenth century, in which time the great dramaturges Pierre Corneille (1606–1684), Molière (born Jean-Baptiste Poquelin, 1622–1673), and Jean Racine (1639–1699) all lived and worked. Much as English Elizabethan drama, and most notably Shakespeare, was heavily influenced by the Italian Renaissance (see, e.g., *The Merchant of Venice*, *Romeo and Juliet*, *The Tempest*, and so on, all of which were set in the Apennines), modern French drama

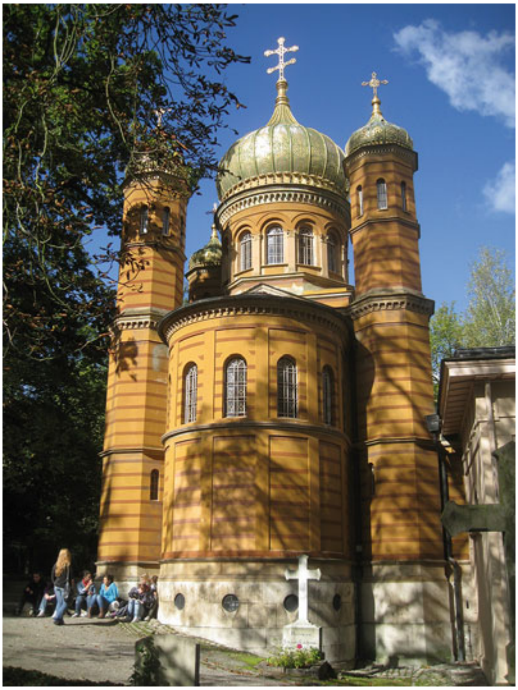


Fig. 6.3 The Russian Orthodox Chapel of Weimar, situated next to the Weimarer Fürstengruft, which houses the coffins of Goethe and Schiller; photograph by the author

drew upon Spanish dramas; for example, the protagonist of *Le Cid* by Corneille was a Spanish national hero. In Germany, drama sprang to life in the eighteenth century, with the emergence of such figures as Gotthold Lessing (1729–1781), Johann Wolfgang von Goethe (1749–1832), and Friedrich Schiller (1759–1805).

Returning to the development of calculus, the mathematicians of the eighteenth century were presented with a full plate of new problems and even disciplines left over in germinal form in the original works of Newton and Leibniz. But before these developments could be seen through to their completion, it was necessary to carry out the perfection and expansion of calculus itself, and the first task at hand was to achieve a full understanding of elementary functions. An example of the issues involved is the logarithmic function, which had originated as a description of the termwise relationship between the geometric and arithmetic series and was later recognized as the integral of the rational function $\frac{1}{1+x}$; at the same time, this function also serves as the inverse of the exponential functional, a particularly simple characterization.

In the period after Newton, the main results in British mathematics were in the study of power series expansions. A particularly important result is due to Brook Taylor (1685–1731), known today as the Taylor series:

$$f(x+h) = f(x) + hf^{(1)}(x) + \frac{h^2}{2!} f^{(2)}(x) + \cdots ,$$

which makes it possible to expand any function as a power series and quickly proved to be a powerful tool for the development of calculus, to the extent that the French mathematician Lagrange even later referred to it as the basic principle of differential calculus.

On the other hand, Taylor's proof of this result was by no means rigorous, and he did not even take into account the question of the convergence or divergence of this series. These shortcomings in his work can perhaps be overlooked in light of his additional talents as a painter, which inspired him to compile a comprehensive treatment of perspective in his 1715 essay *Linear Perspective*, in which he introduced for the first time the term *vanishing point*, and provided the first full explanation of the geometry of multipoint perspective. An important special case of the Taylor series is the Maclaurin series, corresponding to the evaluation at $x = 0$, familiar today to any high school student.

Colin Maclaurin (1698–1746) was 13 years younger than Taylor and arrived at his result later, but it is his name that has been attached to it ever since. Partially, this is because Taylor was not well known during his lifetime, but Maclaurin was also a savvy academician, an early promoter of Newton's method of fluxions, who was admitted as a member of the Royal Society at the age of 21. After the deaths of these two mathematicians, British mathematics suffered a long period of decline. One cause of this was a conservative and nationalistic mentality among British mathematicians of the period inspired by the priority dispute over the invention of calculus. They were loath to acknowledge let alone overcome the weaknesses of the fluxion formulation associated with Newton during a time when their continental counterparts were taking full advantage of the symbolic and conceptual clarity of the calculus as developed by Leibniz to achieve fast and fruitful results.

Consider Switzerland, for example. This small, landlocked country in Central Europe was home to several of the most important mathematicians of the eighteenth century. These included Johann Bernoulli, the first to provide a formal definition for the concept of a function and who also introduced various integration techniques such as substitution of variables and integration by parts, and then his student at the University of Basel, Leonhard Euler (1707–1783), arguably the greatest mathematician of the century, who carried out meticulous research into every corner touched by calculus (Fig. 6.4).

Euler proceeded from the loose notion of a function as consisting of an analytical expression of a certain form involving a variable and constants; this was enough to encompass polynomials, power series, exponential and logarithmic functions, trigonometric functions, and even multivariate functions. Euler also separated the algebraic operations involved in the definition of a function into two categories: rational operations, involving only the four basic arithmetic operations, and irrational operations, involving, for example, square roots (Fig. 6.5).

Euler gave the definition of some important functions in terms of limits, for example, the logarithmic function, which he defined for $x > 0$ by

$$\log x = \lim_{n\to\infty} n\left(x^{1/n} - 1\right),$$

Fig. 6.4 The tomb of Euler; photograph by the author, in St. Petersburg

Fig. 6.5 Leonhard Euler

and along with this the exponential function defined as

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n .$$

The symbol e is generally regarded now as a tribute to Euler, although he does not seem to have introduced it for this constant with any special meaning in mind.

The Euler family included many generations of craftsmen, originally based on the shore of Lake Constance at the border between Switzerland and Germany. Toward the end of the seventeenth century, they had made their way down along the Rhine River to Basel, where Euler was born in 1707. He graduated from the University of

Fig. 6.6 Euler's formula, relating five of the most important mathematical constants

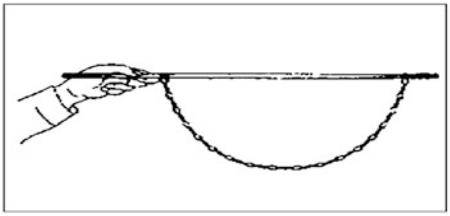


Fig. 6.7 Bernoulli's catenary

Basel at the age of 15 and earned a master's degree a year later. He also came to the attention of the Bernoulli family early on, and they became important mentors and friends. In 1727, he entered the Paris Academy prize competition for the first time; he would go on to win this prize a total of 12 times (Fig. 6.6).

When he was 20, Euler moved to Russia after having failed to earn a physics professorship at his alma mater. He obtained a position with the Imperial Russian Academy of Sciences in Saint Petersburg and succeeded Daniel Bernoulli as a professor of mathematics in 1733. Although he never returned to his home country, Euler retained his Swiss nationality his entire life. He spent 25 years in Berlin at the Prussian Academy of Sciences. The remainder of his life took place in Saint Petersburg. Euler was a remarkably prolific mathematician and father to 13 children, only 5 of whom survived to adulthood. He made seminal contributions to number theory, analysis, geometry, topology, graph theory, and mechanics (Fig. 6.7).

Euler further introduced distinctions between explicit and implicit functions, single-valued and multiple-valued functions, and algebraic and transcendental functions and provided a definition for continuous functions equivalent to the

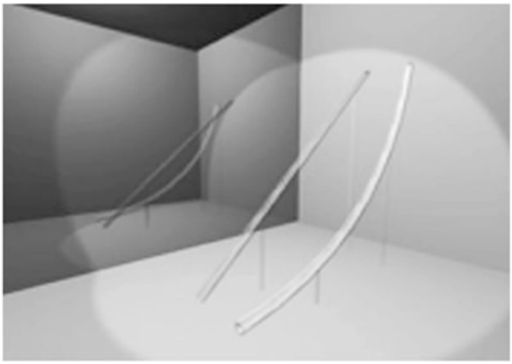


Fig. 6.8 The path of fastest descent between two points is not a straight line

modern notion of an analytic function. He considered the power series expansions of various functions and made the assertion that any function can be expanded in a power series, which is not strictly correct from a modern perspective. His work touched deeply upon physics, astronomy, architecture, and navigation. Euler was a remarkably productive mathematician, who also found time to raise a large family of many children. He famously remarked: "everything our minds can comprehend is interrelated."

The Influence of Calculus

At the same time that the calculus was undergoing internally a continuous development, rigorization, and refinement, and the concept of functions was to become more and more deep, the scope of calculus was also expanding widely and rapidly in its application to other fields, leading to the formation of some new branches of mathematics. One of the most notable developments was that mathematics and mechanics grew more closely related to one another than they had ever been. Most of the Western mathematicians of the period also carried out work in mechanics,[1] much as in ancient times in the east most mathematicians were also astronomers. These emerging disciplines included among them the study of ordinary and partial differential equations, the calculus of variations, differential geometry, and the theory of algebraic equations. Moreover, the influence of the calculus extended beyond simply mathematics and the natural sciences and penetrated even into the humanities and the social sciences (Fig. 6.8).

The theory of ordinary differential equations sprang up directly concomitantly with the growth of calculus. Starting at the end of the seventeenth century, practical problems related to cycloid motion, the theory of elasticity, and celestial mechanics

[1] Later, in the twentieth century, many colleges and universities in China established departments of mathematics and mechanics.

Fig. 6.9 Mathematician and Enlightenment thinker Jean le Rond d'Alembert

produced a series of equations involving differentials, laying down a challenge at the feet of the mathematicians. The most famous of these was the catenary problem, which asks for the equation of the curve formed by an idealized flexible but inelastic cable hanging between two fixed points in a uniform gravitational field. The problem was first posed explicitly as a challenge by Jacob Bernoulli, the brother of Johann Bernoulli, and given its name by Leibniz. Johann Bernoulli derived the equation

$$y = c \cosh \frac{x}{c}$$

for the catenary curve, where c is a constant determined by the weight per unit rope length and cosh is the hyperbolic cosine function.

Subsequently, the theory of ordinary differential equations developed from first-order equations, to higher-order equations with constant coefficients, and then on to higher-order equations with variable coefficients. Finally, this topic was perfected by the two great mathematicians Euler and Lagrange. Euler also established the important distinction between the particular and general solutions of an ordinary differential equation (Fig. 6.9).

Partial differential equations appeared later and were first studied in 1747 by the French mathematician and polymath of the Age of Enlightenment Jean le Rond d'Alembert (1717–1783), who published a paper on the mechanics of string vibrations containing within it the concept of the partial derivative. D'Alembert had been abandoned by his parents as an infant and was later adopted by the wife of a glazier. His name was taken from the patron saint of the church on the steps of which he was found, in keeping with the custom of the time. His knowledge of mathematics was almost entirely self-taught. Later, Euler provided a particular solution involving the sine and cosine functions under the assumption that the initial condition is sinusoidal. Motivated by applications to musical aesthetics and instrument design, Euler and Lagrange both also studied the vibration of tympanic membranes and the wave equation generated by the propagation of sound.

Another important contribution to the development of partial differential equations came from the French mathematician Pierre-Simon Laplace (1749–1827), who introduced the so-called Laplace equation:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

Here, V refers to a potential function, and for this reason, this equation is sometimes also called the potential equation. Potential theory provided a solution to a problem much clamored about in mechanics: the determination of the gravitational force between two objects. If the mass of the objects is negligible in comparison with the distance between them, then the partial derivative of V is the gravitational component between them, determined by Newton's formula for universal gravitation.

In contrast, the genesis of the calculus of variations was more dramatic, and its eventual applications were extremely broad, including both the study of soap bubbles and the theory of relativity, geodesics and minimal surfaces, and isoperimetric problems (the determination of maximal areas enclosed by a curve of a fixed perimeter). The original problem for which this discipline was invented however was a simple one: the identification of the line of fastest descent. This problem is as follows: given two points that do not lie in the same plane horizontally or along the same line vertically, determine the curve between them along which a particle travels in the least time subject only to the action of gravity. After Johann Bernoulli publicly posed this problem in 1696, it attracted the great mathematical minds from around Europe, including Newton, Leibniz, and Johann's brother Jacob Bernoulli. At its core, the problem boiled down to the identification of a pole of a certain special function. Among the various correct solutions that appeared, Newton submitted a solution anonymously, but Johann Bernoulli quickly discerned the identity of its author, famously remarking that he could be recognized "as the lion from its claw."

Through the joint effort of many mathematicians in the establishment of the above various offshoots of calculus, the broad mathematical discipline of *analysis* was born. This became one of the three major areas of modern mathematics, alongside algebra and geometry, and its fruits in this time were the most numerous of the three. Even today, greater weight is placed on mathematical analysis than algebra or geometry as the foundation of mathematical education at the undergraduate level. Calculus also exerted a profound influence on the study of algebra and geometry, starting with the birth of differential geometry. But in the eighteenth century, this was limited to a discussion of geometrical properties in the region near a point or local differential geometry as we would say today; we discuss this in some detail below.

Due to the rise of calculus and its connection with the other natural sciences, this period aroused the enthusiasm of remarkable and careful thinkers and inspired in them confidence in the power of rational thought and the application of mathematical methods to physics and even the normative sciences. There was much faith that this success could be extended to the totality of knowledge. Descartes, for example, believed early on that all problems can be reduced to mathematical problems, all mathematical problems can be reduced to algebraic problems, and all algebraic problems can be reduced to basic equation solving. It could be said that he regarded mathematical reasoning as the only reliable method of thought and sought to reconstruct all knowledge atop these sure foundations.

Leibniz went further even than the ambitious goals outlined by Descartes in his attempt to create a framework for universal logical calculation and universal conceptual language that would render the solution of all human problems trivial. Mathematics was not only the starting point for his program but also its beating heart. Among his other proposals, he suggested that the human mind can be factored into basic and distinct parts, just as the number 24 can be written as a product of its prime factors 2 and 3. Although neither Leibniz nor his successors could ever see this program through to completion, the development of mathematical logic in the second half of the nineteenth century and the twentieth century was based on his idea of a purely formal language, and for this reason, he has sometimes been celebrated as the father of modern logic.

The birth of calculus and turn toward faith in mathematics had an even more direct and obvious influence on religion, which at that time played a central role in both spiritual and secular life. Although Newton attributed to God the power to create the universe, he limited his role in daily life, and Leibniz further depreciated his influence. Although Leibniz too acknowledged his role in creation, he believed that God was constrained to proceed according to established mathematical order. The increased emphasis on reason in this period also contributed to a decrease in devotion to religion, although was not necessarily an outcome intended by mathematicians and scientists of the time. Just as Plato described God as a geometer, Newton believed him to be a capable physicist and mathematician (Fig. 6.10).

In the eighteenth century, the further development of calculus introduced further changes to the spiritual and intellectual landscape. The pioneer and spiritual leader of the French Enlightenment François-Marie Arouet (1694–1778), better known as Voltaire, was a stalwart advocate of Newtonian mathematics and physics and simultaneously a leading proponent of the emerging philosophy of Deism, a theological system in which reason and nature were equated to one another that quickly gained popularity among the intellectuals of the period. In the United States, its adherents included Thomas Jefferson and Benjamin Franklin, the former of whom did much to encourage the instruction of advanced mathematics. In fact, none of the first seven presidents of the United States, including its first president George Washington, identified themselves as Christian. Among the disciples of Deism, nature was God, and Newton's *Principia* is its bible. With philosophy and theology as its accomplices, calculus has exerted a remarkably broad-reaching influence on just about every sphere of human activity, including economics, law, literature, and aesthetics (Fig. 6.11).

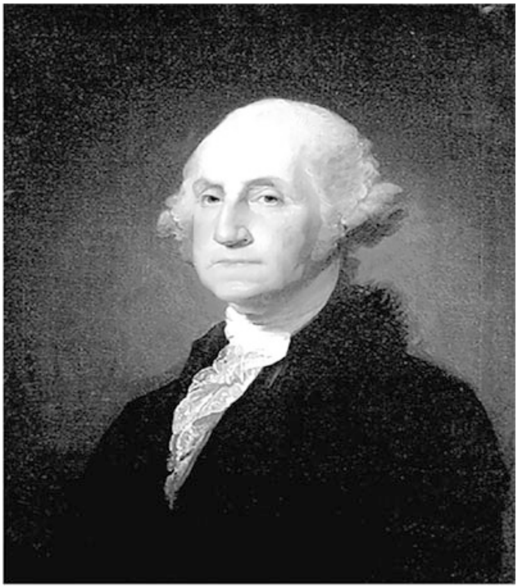


Fig. 6.10 George Washington, during his time as public land surveyor

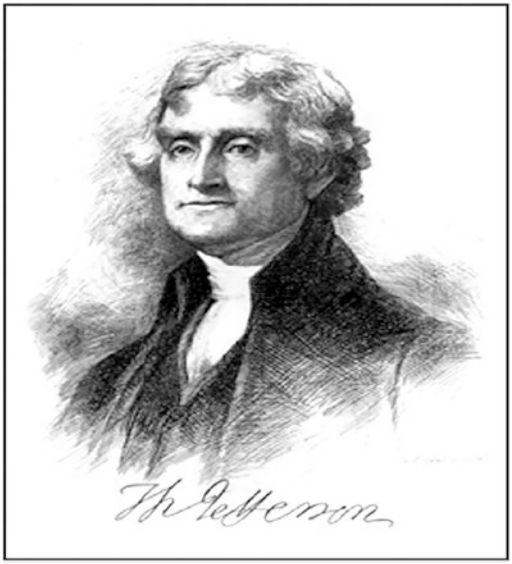


Fig. 6.11 Thomas Jefferson, coauthor of the *Declaration of Independence*

The Bernoulli Family

We have mentioned several times already in the preceding sections the outstanding contributions of the Bernoulli brothers Johann and Jacob, and their Swiss compatriot Euler, in the development and application of the calculus. We discuss now in detail this most famous mathematical family in history, a family seemingly destined for calculus. The Bernoulli family was originally based in Antwerp in Belgium, at that time part of the Spanish Netherlands, where they were practicing Huguenots, a protestant sect which suffered persecution by the Catholic church, like the Calvinists and the Puritans. As a result, Jakob Bernoulli fled his hometown in 1583, taking refuge first in Frankfurt in Germany and eventually settling in Basel in Switzerland, where he married into a prominent local family and established a career as a well-connected merchant of medicinal herbs.

More than a century later, the first of many mathematicians in this family was born; this was Jacob Bernoulli (1654–1705), who mastered the new discipline of calculus as formulated by Leibniz through diligent self-study and later served as a professor of mathematics at the University of Basel. Initially intended for a life as a man of the cloth, Jacob initially studied theology and entered the ministry, but he became obsessed with mathematics over the objections of his father and eventually rejected his church appointments. In 1690, he was the first to introduce the term *integral* into the mathematical lexicon, and in the following year, he studied catenary curves and applied the fruits of his research to bridge design. His other important research areas included the theory of permutations and combinations, the law of large numbers in probability, the Bernoulli numbers derived from the sums of integer powers, and the calculus of variations, discussed already above (Fig. 6.12).

Fig. 6.12 Map of the ancient city of Carthage; photograph by the author, in Tunisia

The basic idea of the calculus of variations makes some interesting and beautiful appearances in ancient literature: according to Greek legend, the founder Queen Dido of Carthage cleverly offered a high price to purchase for the establishment of her new city all the land that could be enclosed in a single leather hide. She then proceeded to cut the hide into a very thin continuous strip, long enough to enclose all the territory she needed for the city. In another version of this story, Dido fled her home city of Tyre upon discovering that her cruel brother Pygmalion had orchestrated the murder of her husband; she made her way to the coast of Africa and purchased a plot of land for the establishment in Carthage, the territory demarcated in such a way as to match the size of a ditch dug in a single day. A very moving and tragic love story involving Dido and Aeneas, the legendary founder of Rome, as well as Dido's sister Anna, appears in the *Aeneid* by the Roman poet Virgil (70 BCE–19 BCE) and in the *Heroides* by Ovid (43 BCE–14 CE).

Returning to Jacob Bernoulli, the Bernoulli numbers B_n named in his honor play an invaluable role in number theory. These numbers can be defined recursively as

$$B_0 = 1,\ B_1 = \frac{1}{2},\ B_n = \sum_{k=0}^{n} \binom{n}{k} B_k\ (n \geq 2),$$

where the numbers $\binom{n}{k}$ are the usual binomial coefficients. From this, it is obvious that every B_n is a rational number, and these numbers exhibit some remarkable properties. For example, it is easy to prove that $B_n = 0$ whenever $n \geq 3$ is an odd number; and for odd prime numbers p, the special case of Fermat's last theorem with exponent given by p can be directly resolved by way of the number B_{p-3}. The Bernoulli polynomials, which also play an important role in number theory, as well as in the theory of functions, are also defined in terms of the Bernoulli numbers. Upon his death, Jacob Bernoulli requested that his gravestone be engraved with a logarithmic spiral and the motto *Eadem mutata resurgo* (*Although changed, I rise again the same*), but instead it was engraved with an Archimedean spiral.

The mathematical contributions of his younger brother Johann Bernoulli (1667–1748) were no less significant; some of them have been discussed already above. Johann first studied medicine and earned a doctorate in Basel for a thesis on muscle contraction. Later, like Jacob over the objections of his father, he studied mathematics with his brother and went on to become a professor of mathematics at the University of Groningen in the Netherlands. He returned to Basel only many years later, shortly after his brother had succumbed to tuberculosis.

The best-known mathematical discovery associated with Johann Bernoulli is his method for determining the limit of a fraction of functions as both numerator and denominator tend to zero, a familiar favorite of calculus students. This rule states that if two functions $f(x)$ and $g(x)$ both admit derivatives $f'(x)$ and $g'(x)$ in the neighborhood of a certain point a with

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0,$$

Fig. 6.13 Daniel Bernoulli: the second generation of Bernoulli family mathematicians

and $g'(x) \neq 0$ for all $x \neq a$ near a, then if $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, we can calculate

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

This rule was included in the first systematic textbook on infinitesimal calculus, written by a French former student of Johann Bernoulli's named Guillaume de l'Hôpital, and it has ever since been referred to as l'Hôpital's rule. Johann also used calculus to tackle the problem of fastest descent, known as the brachistochrone problem, and to determine the lengths and enclosed areas of related tautochrone curves (Fig. 6.13).

The brothers Johann and Jacob Bernoulli were academic colleagues and both were friends to Leibniz, but between them, there developed frequent academic rivalry. Johann was known for a quick temper and jealous disposition. In spite of this, he seems to have been an impressive teacher: not only did he nurture such impressive students as l'Hôpital, but he also trained his three sons for lives as mathematicians, although he encouraged both the eldest Niklaus and the second eldest Daniel to pursue careers other than mathematics; the former studied law and the latter medicine, but they both eventually took posts as professors of mathematics at the newly founded Saint Petersburg Academy. It was these two who introduced their close friend Euler to Russia, where he spent the better part of his life. The youngest of the three, Johann II, succeeded his father as professor of mathematics at the University of Basel, after an earlier stint as a professor of rhetoric. The legacy did not end with this generation: the two sons Johann and Jakob of Johann II also found their way to mathematics, after some detours (Fig. 6.14).

In general, the second and third generations of Bernoulli mathematicians did not achieve the same heights as the first, with the notable exception of Daniel Bernoulli (1700–1782), who at several points throughout his life could be said to rival his

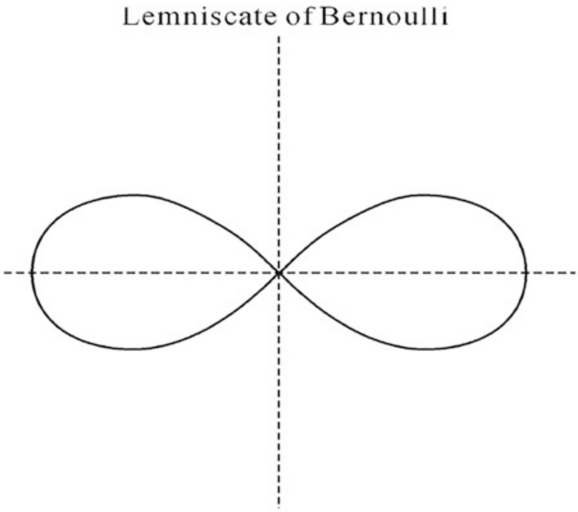


Fig. 6.14 The lemniscate of Bernoulli

friend and contemporary, the great Euler, with whom he shared several of the ten prizes he was awarded by the French Academy of Science. Upon his return to Basel from Saint Petersburg, Daniel successively served as a professor of medicine, metaphysics, and natural philosophy while continuing to make contributions in a number of different fields within mathematics, including calculus, differential equations, and the theory of probability.

The most famous result due to Daniel Bernoulli is Bernoulli's principle, a result in fluid dynamics that has direct applications in modern aircraft design. This theorem states that the total energy of a moving fluid (gas or liquid) remains constant; this includes its kinetic energy and dynamic pressure, potential energy due to gravity and static pressure, and internal energy. As an example, a fluid flowing horizontally experiences no change in its gravitational potential energy, and from this, it follows that its static pressure decreases with an increase in the speed of its flow. This principle provides the theoretical basis for many problems in engineering, notably in the design of aircraft wings: since the airflow along the curved upper surface of the wing is faster than along its lower surface, Bernoulli's principle implies that the pressure along the lower surface is greater than along the upper surface, thereby generating lift.

In the 1990s, the Bernoulli Society for Mathematical Statistics and Probability in the Netherlands introduced the *Bernoulli Journal* in commemoration of the mathematical contributions of the Bernoulli family. This is now the second mathematical journal we have encountered to derive its name from a significant mathematician or mathematical family, after the *Fibonacci Quarterly*.

The French Revolution

Napoleon Bonaparte

In the year 1769, the situation of the two French mathematicians Laplace and Lagrange was as follows: Lagrange was 31 years old and serving as the director of mathematics at the Prussian Academy of Sciences in Berlin; Laplace, 11 years his junior, was employed as a professor of mathematics at the École Militaire. At this time, their future student and friend Napoleon Bonaparte was born in Ajaccio, capital of the Mediterranean island of Corsica. Only a year earlier, this island had belonged to the Republic of Genoa in the Apennine Peninsula. If its transfer to France had been delayed for even a few more years, Napoleon might have found himself as an adult fighting instead for the territorial defense and expansion of Italy, or a part of the underground resistance against France, as indeed his father had been. In fact, his paternal ancestors the Bonapartes descended from a family of minor nobles in Tuscany, whose capital city Florence had been the central city of the Italian Renaissance (Fig. 6.15).

The Corsican resistance against France quickly collapsed, however, and Napoleon's father was obliged to submit to French rule and serve in his capacity as an attorney for the new regime, eventually becoming the representative of Corsica to the court of Louis XVI. All this paved the way for young Napoleon, at the age of 9, to move to the French mainland and enroll briefly in a religious school

Fig. 6.15 Napoleon, the amateur geometer

Fig. 6.16 *La Marseillaise*, the national anthem of France, was born during the French Revolution

before obtaining a scholarship to a military academy at Brienne-le-Château, and eventually, many transfers later, to graduate from the École Militaire in Paris. It was while Napoleon was at the École Militaire that his father died, compelling him to complete his course in a single year. He exhibited some talent for mathematics during his studies and was examined for graduation at the age of 16 by none other than Laplace (Fig. 6.16).

After he graduated from the military academy, Napoleon became a second lieutenant in an artillery regiment. During this time, he carried out an extensive reading of military treatises. Not long afterward, he returned for 2 years to Corsica, and it seems that he sustained a strong affection for his homeland, to which he returned again several times in later years, and which had things gone differently he might still have helped to achieve its independence. But as the French Revolution, Napoleon was more and more attracted to Paris; he was a loyal student of Voltaire and Rousseau and fervently believed that political change was necessary for France. All the same, when the revolution actually arrived and the Parisians stormed the Bastille, a fortress that served as a symbol of the tyranny of the king, on July 14, 1789 (later the national holiday of France), Napoleon was in the provinces.

The French Revolution not only brought an end to the ancient regime in France but also marked a change in the entire political climate of Europe. Although historians disagree about the precise causes of this revolution, there are five generally agreed-upon explanations: (1) France had the largest population of any European nation at the time and could no longer adequately support it; (2) the rise of a wealthy bourgeoisie class and its exclusion from political power, a divide more extreme than that which existed in other nations of the period; (3) a deepening understanding among the peasants of the situation and with it their inability to tolerate a feudal system that subjected them to fraud and exploitation; (4) the appearance of radical philosophers promoting political and social change and the widespread circulation of their works; and (5) the depletion of the national treasury due to French participation in the American Revolution.

There is no doubt however that Napoleon's march into Paris required many blessings from the goddess of fate. In January of 1793, King Louis XVI was executed by guillotine for high treason. This was a period of intense crisis within France and even throughout Europe, as the revolutionaries had earlier declared war against the counterrevolutionary forces in various European countries. The following winter, Napoleon led the artillery of the republican forces in the port city of Toulon to defeat the royalist British navy assembled at the behest of the Baron d'Imbert and force them to evacuate. This battle brought him fame and recognition and earned him a promotion to brigadier general. Another year passed, in which time the royalists carried out their White Terror campaign and an attempt to seize power in Paris. Napoleon crushed their efforts; by the age of 26, this young Corsican soldier was widely recognized as the savior and hero of the French Revolution.

Also in the year 1795, the old University of Paris and the Academy of Sciences in Paris were abolished by the National Assembly in the name of egalitarianism, replaced by the École Polytechnique and the University of France, the latter of which absorbed the Paris Academy of Sciences as one of its three branches, as well as a new normal academy in Paris founded the year earlier and later rebranded in 1808 as the École Normale Supérieure. Although these two schools were originally conceived as training schools for engineers and teachers, respectively, both placed a high importance on mathematics, perhaps not unrelated to the fact that the Marquis de Condorcet, whose was involved in the establishment of the new state education, was himself a mathematician. He brought in the most respected mathematicians in France at the time: Lagrange, Laplace, Legendre, and Gaspard Monge, who later also served as the first Director of the École Polytechnique.

It was a few more years however before Napoleon became the First Consul of the Republic; during this time, he led military campaigns both northward and southward, leaving behind him his footprints in Italy, Malta, and Egypt, where he commanded more victories than losses. It was after he returned to France that he could truly be said to have consolidated beneath him the military and political power, not unlike the return of Caesar from Egypt to Rome. On the last Christmas of the eighteenth century, France established a new national constitution in which Napoleon was designated the first consul for a period of 10 years, with basically unlimited powers, including the appointment of ministers, generals, civil officers, magistrates, and legislators. Afterward, Napoleon included his mathematician friends among the high-ranking officials.

Although his rise to power was facilitated by the French Revolution, Napoleon himself was a man of tremendous ambition, and his belief in the sovereignty of the people and the virtue of free legislative debate quickly proved illusory.

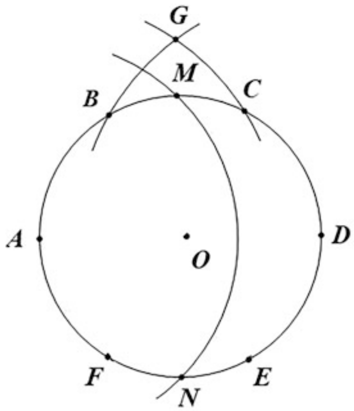


Fig. 6.17 Answer to a question raised by Napoleon

Rather Napoleon took up the mantle of the philosopher king of pure reason and intellect, with mathematics and jurisprudence as his advisors. The war effort was still unresolved however, and his campaign of territorial expansion had just begun. In his capacity as First Consul, Napoleon considered the management of the army as in need of most careful attention in order to consolidate power and achieve his imperial ends. As a result, the École Polytechnique was militarized and charged with the training of artillery officers and engineers, and its professors were encouraged to turn their attention to mechanics, the development of artillery shells and other weaponry, and to maintain close contact with the consulate (Fig. 6.17).

The mathematical talent nourished in his early years and his continued contact with eminent mathematicians encouraged Napoleon to propose a question in geometry: using only a compass but no straightedge, how to divide a circle into four equal parts? This problem was solved by the Italian mathematician Lorenzo Mascheroni (1750–1800), who had been trapped in Paris by the war. Mascheroni also wrote a book entitled *Geometria del Compasso*, dedicated to Napoleon, in which he proved that any geometrical construction than can be accomplished by compass and straightedge can also be accomplished by compass alone, that is, that the straightedge of classical Euclidean geometry is superfluous. It was discovered by later generations that this result had in fact already been proven in an obscure book by the Danish mathematician Jørgen Mohr (1640–1697) (Fig. 6.18).

The specific method for the division of the circle into four parts is as follows: let A be any point on the given circle O, and with one bisector at A, set up a total of six bisectors at A, B, C, D, E, and F, dividing the circle into six equal parts as shown in the figure. Construct two circles with centers A and D and radius AC or BD, intersecting in the point G. Construct another circle with A as its center and with radius OG, meeting the circle O in points M and N. Then the points A, M, D, and N divide the circle into four equal arcs. Indeed, according to the Pythagorean theorem, $AG^2 = AC^2 = (2r)^2 - r^2 = 3r^2$, and therefore, $AM^2 = OG^2 = AG^2 - r^2 = 2r^2$, $AM = \sqrt{2}r$, so AO and MO are perpendicular.

Fig. 6.18 Marquis de Condorcet, a revolutionary and a mathematician

The Lofty Pyramid

We turn now to Joseph-Louis Lagrange (1736–1813), considered alongside Euler as one of the two greatest mathematicians of the eighteenth century. As for which of the two was in fact the greater, this has been a topic of much debate and not immune to the preference in mathematical interests of the supporters of one or the other. Lagrange was born in Turin, a famous city in northwestern Italy, known today as the home of Fiat and the Juventus Football Club. Its close proximity to France had meant that Turin was for a time occupied by France, during the sixteenth century, and by the time that Lagrange was born, it was the capital of the Kingdom of Sardinia. Its status did not afterward change until the nineteenth century, when Turin was at the political and ideological center of the struggle for Italian unification, to the extent that it was even briefly the capital of the newly independent Kingdom of Italy.

Lagrange was of mixed French and Italian ancestry. His great-grandfather had been a captain in the French cavalry, who settled in Turin and married into a prominent local family after having served under the king of Sardinia, a Mediterranean island that is today a part of Italy. His father briefly had charge of the king's military chest and served as Treasurer of the Office of Public Works and Fortifications in Turin, but all the same he failed to effectively manage his family property, and Lagrange, who was the firstborn of 11 children, received only a small inheritance. Later he regarded this as the luckiest thing that could have happened to him, reasoning that a large fortune might have cut him off from his fate as a mathematician.

In his school years, Lagrange was drawn first to classical literature and not much inspired by his encounters with the geometric works of Euclid and Archimedes. Later he stumbled by accident across a popular work written by Edmond Halley (1656–1742), a friend of Newton responsible for the discovery of Halley's Comet, in which the topic of calculus was introduced and exalted. Lagrange became fascinated with this new subject and quickly mastered through self-study the full body of knowledge in analysis of his era. At the age of either 19 or 16 (accounts vary), Lagrange was appointed Sostituto del Maestro di Matematica (assistant professor of mathematics) at the Royal Military Academy of the Theory and Practice of Artillery and embarked upon one of the most glorious careers in the history of mathematics. By the age of 25, Lagrange was already regarded as one of the greatest mathematicians in the world (Fig. 6.19).

Unlike any earlier mathematician, Lagrange was an analyst right from the start of the career, further evidence that analysis had already become the most popular branch of mathematics in that period. This preference achieved its full realization in his *Mécanique analytique* (*Analytical Mechanics*), which Lagrange first conceived at the age of 19, although its publication in Paris did not appear until he was already

Fig. 6.19 Lagrange, a descendant of France and Italy

52, by which time he had largely lost interest in mathematics. In the preface to this work, Lagrange writes:

> No diagrams will be found in this work. The methods that I explain require neither geometrical, nor mechanical, constructions or reasoning, but only algebraical operations in accordance with regular and uniform procedure.

All the same, his framework for mechanics is novel in its appeal to the geometry of four dimensions: three coordinates representing spatial position and a fourth coordinate representing time. According to this conception, the mechanics of a moving point is determined entirely by its geometrical description.

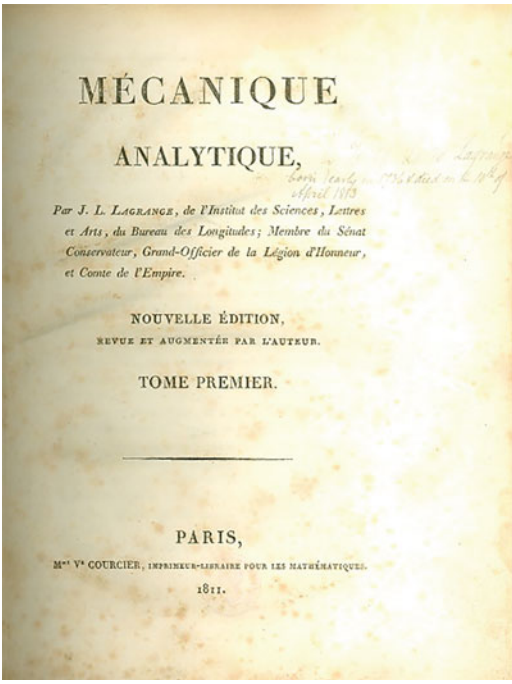
Lagrange also introduced the notations $f'(x)$, $f^{(2)}(x)$, $f^{(3)}(x)$, etc., for the derivatives of a function $f(x)$ with which we are familiar today, and discovered an early version of the mean value theorem, sometimes referred to as Lagrange's mean value theorem. This theorem states that if a function $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists at least one point ζ in the interval $a < \zeta < b$ satisfying

$$f'(\zeta) = \frac{f(b) - f(a)}{b - a}.$$

In addition, Lagrange developed approximation methods for determining the real roots of polynomial equations using continued fractions and investigated the question of the representation of arbitrary functions by power series.

In his *Analytical Mechanics*, which the Irish mathematician William Rowan Hamilton referred to in the nineteenth century as "a scientific poem," Lagrange reduced the general equations of solid and fluid dynamics to a single principle from which he derived the general equations of dynamical systems, including what have since come to be called the Lagrange equations. This work also includes some of his best known results concerning differential equations, partial differential equations, and the calculus of variations. Its importance to general mechanics is as great as the importance of Newton's law of universal gravitation for celestial mechanics. Not to say, however, that Lagrange paid no heed to the celestial bodies; in fact, he solved the problem of the moon's libration, that is, why is it that the moon presents the same face to the earth at almost all times. His analytical approach to problem-solving in mechanics marked a departure from the classical Greek tradition, and even from the study of mechanics by Newton and his immediate successors, which still made use of geometry and figures (Fig. 6.20).

From the start of his career, Lagrange received generous praise and support from his potential rival Euler, almost 30 years his senior, and the intellectual selflessness of their relationship has become one of the pivotal stories in the history of mathematics. Like Euler, Lagrange applied himself primarily to analysis and its applications but also indulged his curiosity with countless investigations into number theoretic questions: we have seen already that he resolved two important conjectures left over by Fermat. There is also Lagrange's theorem in modular arithmetic, which states that if some prime number p does not divide every


MÉCANIQUE

ANALYTIQUE,

Par J. L. Lagrange, de l'Institut des Sciences, Lettres et Arts, du Bureau des Longitudes; Membre du Sénat Conservateur, Grand-Officier de la Légion d'Honneur, et Comte de l'Empire.

NOUVELLE ÉDITION,
REVUE ET AUGMENTÉE PAR L'AUTEUR.

TOME PREMIER.

PARIS,
Mme Ve COURCIER, Imprimeur-Libraire pour les Mathématiques.
1811.

Fig. 6.20 First volume of *Mécanique analytique*, by Lagrange (1811)

coefficient of a polynomial $f(x)$ of degree n, then the congruence $f(x) \equiv 0 \pmod{p}$ has at most n distinct solutions modulo p. But the most famous theorem associated with the name Lagrange is Lagrange's theorem in group theory, which states that the order of any subgroup of a finite group G is a factor of the order of G.

In light of his achievements, Lagrange obtained funding from the king of Sardinia to travel to Paris and London, but he fell ill during his time in Paris and returned early to Turin upon his recovery. Not long afterward, he travelled again, this time to Berlin at the invitation of King Frederick of Prussia, and he remained there for 11 years until the death of the king, at which time France did not miss a second opportunity to invite him to Paris at the behest of King Louis XVI. This was in the year 1787, and Lagrange had already turned his attention mainly to the humanities, medicine, and botany. He became close with the king and his queen Marie Antoinette, who looked after him with care and did her best to soothe his bouts of depression (Fig. 6.21).

Two years later, the French Revolution reached its climax in Paris, and this seems to have penetrated through the intellectual lethargy into which Lagrange had sunk and inspired him to become active in mathematics. He wrote several academic works and textbooks and declined an invitation to return to Berlin, surviving through the reign of terror by virtue of his silence and discretion; his friend the chemist Antoine Lavoisier (1743–1794) was not so lucky and died under the guillotine. When the École Normale Supérieure was established, Lavoisier was appointed as a professor, and later, he also became the first professor at the École Polytechnique, teaching mathematicians to young military engineers in the service of Napoleon; among

Fig. 6.21 *The Death of Marat*, by Jacques-Louis David (1793)

them was the future mathematician Augustin-Louis Cauchy. Napoleon, who turned his attention to internal affairs in the intermission between two campaigns, paid frequent visits to Lagrange to discuss mathematics and philosophy and honored him by making him a Senator and a Count of the Empire. This towering emperor who had invaded Egypt described Lagrange as "the lofty pyramid of the mathematical sciences."

The French Newton

In his later years, Lagrange referred to Newton with a measure of envy, remarking that although he was no doubt a particularly gifted man, also he was the luckiest of scientists, since there history admits but one opportunity to explain the universe. In this sense, Pierre-Simon Laplace (1749–1827) could be said to have been less fortunate than Lagrange; he too came too late to achieve the monumental revolutions of Newton, and his career spread evenly across the eighteenth and nineteenth centuries, the former dominated by the shadow of Euler and Lagrange and the latter by that of Gauss; he is automatically disqualified therefore for such titles as the greatest mathematician of this or that century. All the same, his was a brilliant life marked by a tremendous intellect, diligence, and his association with his student Napoleon (Fig. 6.22).

Laplace was born to farmer parents in Beaumont-en-Auge in Calvados, Lower Normandy, not far from the English Channel, site of the Allied invasion of Western Europe during World War II. He exhibited considerable talent as a student at the

Fig. 6.22 Pierre-Simon Laplace, the "French Newton"

Fig. 6.23 Laplace metro station, Paris; photograph by the author

village school, including special eloquence in theological debates, attracting the attention of his wealthy neighbors and securing for him a place as a day student at a local military school. It may have been on account of his prodigious memory rather than his mathematical ability, but in any case, he secured a letter of recommendation from an influential figure to travel for the first time to Paris at the age of 18 to advance his fortune (Fig. 6.23).

This letter nearly proved his undoing. It was addressed to d'Alembert, a famous mathematician and co-editor of the *Encyclopédie*, who did not as a rule pay much attention to letters of introduction submitted to him and turned the young man away. Returning dejected to his residences, Laplace wrote overnight a new introduction containing a treatment of the principles of mechanics and possibly the solution of a problem posed to him in passing by d'Alembert, and it was this letter rather that caught the attention of d'Alembert, who wrote back after having read it and invited Laplace for an immediate audience, remarking that he should not have brought any letter of recommendation in the first place as he had introduced himself so much more capably. Just a few days later, at the recommendation of d'Alembert, Laplace was awarded a teaching position at the École Militaire, where he came into contact with his future student Napoleon.

Laplace devoted less energy to pure mathematics and achieved in it fewer results than Lagrange, preferring instead to carry out his research toward applications in astronomy. Among the results associated with him, there is the Laplace expansion for the calculation of the determinant of a matrix; in its most general form, this states that the determinant can be obtained as an expansion along any arbitrarily selected k rows or columns of the matrix by taking a weighted sum of the products of determinants of various submatrices and their complements determined by the particular choice of k rows or columns. There is also the Laplace transform for differential equations; this transform replaces a suitable function $F(t)$ with another type of function $f(p)$ via the improper integral

$$f(p) = \int_0^\infty e^{-pt} F(t) dt.$$

But the work for which Laplace is best known is of course his treatise *Celestial Mechanics* (*Traité de mécanique céleste*) in five volumes which earned for him this nickname as the French Newton. Starting from the age of 24, Laplace had carried out research into the application of the Newtonian theory of gravity to the solar system as a whole and sought to answer why it is that the orbit of Saturn is expanding while that of Jupiter is shrinking. He proved that the mutual action of two planets could only ever produce small changes to their eccentricities and inclinations. He showed also that the acceleration of the moon is related to the eccentricity of the orbit of the earth, providing a theoretical solution to the last anomaly in the dynamical observations of the solar system. His name is inseparably linked with the nebular theory for the formation and evolution of planetary systems throughout the universe, and another testimony to his achievements is the Laplace equation for potential energy that we have introduced already in our discussion of the influence of calculus.

The respective and comparative work of Laplace and Lagrange, two giants in the history of science, is a topic much discussed among their successors. The nineteenth-century French mathematician Siméon Denis Poisson observed a profound difference between the thought and working methods of the two in everything they did, from pure mathematics to studying the libration of the moon.

"Lagrange," Poisson observed, "often appeared to see in the questions he treated only mathematics, of which the questions were the occasion – hence the high value he put upon elegance and generality. Laplace saw in mathematics principally a tool, which he modified ingeniously to fit every special problem as it arose."

There were also stark differences between them in terms of their personal attributes. Joseph Fourier remarked of Lagrange:

> By his whole life he proved, in the moderation of his desires, his immovable attachment to the general interests of humanity, by the noble simplicity of his manners and the elevation of his character, and finally by the accuracy and depth of his scientific works.

In contrast, Laplace developed a reputation among mathematicians as a political actor and a snob. The historian of mathematics E.T. Bell summarizes his character by his greed for titles, casual political flexibility, and an intense desire to gain the respect of the public at the center of its ever-shifting attention.

But Laplace was not without his candid and sincere side. His dying words were reported by Fourier to have been, "What we know is not much. What we do not know is immense." Napoleon also found much fault with his administrative work as Minister of the Interior on account of his fastidiousness and tendency to look for the subtlest nuances in all things; he even quipped that Laplace brought the spirit of the infinitesimal with him into his efforts as an administrator. All the same, Napoleon heaped upon him many honors, making him a senator, a Count of the Empire, appointing him to the Bureau of Longitudes, and awarding him the Legion of Honour. In spite of this, Laplace signed the decree to banish Napoleon and continued to flourish under the Bourbon Restoration, during which time he was awarded the further title of Marquis de Laplace and a seat in the Chamber of Peers; he served also during this time as chairmen of the committee for the reorganization of the École Polytechnique.

In the eighteenth century and early nineteenth century, French mathematicians spoke of the *three Ls* of mathematics: Lagrange, Laplace, and Legendre. Legendre spent his entire life in Paris; he became a professor at the École Militaire at the age of 23. Later in 1795, he became a professor at the École Normale Supérieure. His outstanding work on elliptic integration provided fundamental analytic tools for mathematical physics, and along with Gauss, he introduced the least squares method, proposed the prime number theorem as a conjecture, and proved the law of quadratic reciprocity in number theory. There is also the Legendre symbol in number theory, which appears in every introductory course in that topic. His book *Éléments de géométrie* replaced Euclid's *Elements* as the basic geometry textbook in European and American universities.

The Emperor's Friend

There is a widely circulated legend about Laplace that he had presented Napoleon with a copy of the *Celestial Mechanics* after the latter had become emperor and received from him the pointed question how is it that he had written a work on the system of the world without any mention of its author. Laplace replied, “I had no need of that hypothesis.” This sentence recalls to mind the response of Euclid to Ptolemy I, “There is no royal road to geometry.” In fact, Laplace may have had in mind with this omission a contrast the Newtonian tradition, since Newton did in fact make reference to God in his works, and Laplace considered his celestial mechanics to accommodate a wider scope than the solar system as conceived by Newton.

Both Laplace and Lagrange enjoyed a relationship with Napoleon that fit the classical conception of the relation between scientist and enlightened monarch; in particular, the distinction between monarch and subject was well delineated. Not so with Monge. Gaspard Monge (1746–1818) was 3 years older than Laplace and somewhat less talented in mathematics, but his personal experiences and open personality led him to establish a close friendship with the young Napoleon. As a result, during the Bourbon Restoration, Monge was not showered in glory as Laplace was but rather he became a wanted man and went into hiding, widely regarded as a close confidante of the Corsican emperor, as indeed he was: Napoleon had said of him that Monge loved him as a man loves a mistress (Fig. 6.24).

Monge was born in Beaune, a small town in Côte-d’Or in central France, belonging to the Burgundy region, famous for its wine, and located to the southwest of Dijon. Today, it is a stopping point along the high-speed rail line between Monte Carlo and Paris. His father was a hawker and knife sharpener who placed great emphasis on the education of his son, and as a result, his son took naturally to a leadership position in everything from sports to crafts. When he was 14 years old, Monge designed a fire truck without reference to any preexisting diagrams, relying only on his own perseverance and dexterity, and presented his construction with geometrical precision. Two years later, he drew up a detailed map of his hometown

Fig. 6.24 Gaspard Monge, among the few who dared to contradict Napoleon

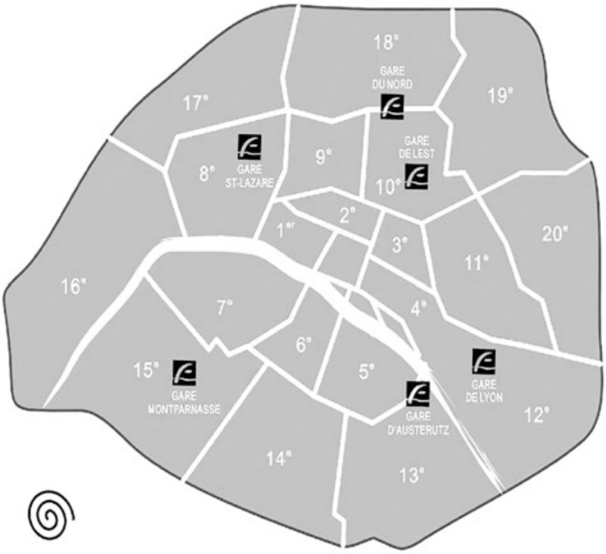


Fig. 6.25 Zoning map of Paris, exhibiting an Archimedean spiral

on a large scale and was recommended for a teaching position in physics at the Collège de la Trinité at Lyon (Fig. 6.25).

On one occasion, Monge encountered in the course of his return home from Lyon an officer of engineers who had seen his map and recommended him for a position at the École Royale du Génie at Mézières, capital of the Champagne-Ardenne region in northern France. This city is only 14 kilometers from the Belgian border, and nearby Charleville was the birthplace of the great poet Rimbaud more than a century later. Monge worked during this time as draftsman, responsible for measurements and drawing, and took advantage of this experience to create a new form of geometry, now known as descriptive geometry, which involves the representation of three-dimensional objects in the two-dimensional plane according to specific rules. He also worked as a teacher, and one of his students Lazare Carnot (1753–1823) later enjoyed a fruitful career as a geometer and participated in the French Revolution.

In 1768, when he was 22 years old, Monge began to teach mathematics at the École Royale du Génie at Mézières, and a few years later, he was appointed there as a professor of mathematics and physics. He left the city only in 1783 when he travelled to Paris and took a post as an examiner of naval candidates. Before moving to Paris, he married a young widow renowned for her beauty and devotion. He was surrounded in Paris by powerful figures and became enmeshed in the petty struggles of the city elite; inevitably, he was drawn into the French Revolution when it broke out. He was compelled to serve for a time as the Minister of the Marine after the formation of an executive council by the new Legislative Assembly. When the École Polytechnique was established in Paris in 1795, Monge was heavily involved in its founding and served there afterward as a professor of descriptive geometry. The

Fig. 6.26 The tombs at the Panthéon in Paris, where Lagrange, Monge, Carnot, and Condorcet are buried; photograph by the author

birth of this school and the École Normale Supérieure in the same year marked the beginning of a glorious period in the history of French mathematics (Fig. 6.26).

The next year, Monge received a letter from the young Corsican who had already ascended quite some way along his rise to power. This letter recalled first the cordial welcome that Napoleon had received as an unknown artillery officer from Monge in his capacity as an examiner for the navy, remarking with gratitude that he had now already risen to the rank of general of the army and was on an expedition to Italy. As an expression of his appreciation, Napoleon appointed Monge as a commissioner to select various paintings, sculptures, and other works of art for return to Paris. Fortunately, Monge succumbed to his conscience after carrying out his task to a suitable degree of completion counselled moderation rather than strip Italy completely of its masterpieces. Subsequently, Monge and Napoleon began a long and close friendship; it has been remarked that after Napoleon had become emperor, Monge was along among his friends who dared to contradict him or speak plain truths in his hearing.

Napoleon however was not at this time entirely occupied by domestic affairs, and in 1798, he led an expedition to Egypt, and Monge accompanied him as a member of the Legion of Culture, alongside Fourier, inventor of the well-known Fourier series expansion of functions. Along the voyage to the Mediterranean, Napoleon seems to have summoned Monge and others to his flagship each morning for a discussion on the same major topic, for example, the age of the earth, the possibility of its destruction by fire or flood, the existence of any other habitable planets, and so on. Upon their arrival to Cairo, Monge helped to establish the Institut d'Égypte, after the model of the Institut de France.

Finally, we turn to the contributions of Gaspard Monge to mathematics. In addition to the creation of descriptive geometry, Monge is also remembered as the father of differential geometry, a form of geometry that makes use of the tools of calculus to study curves, surfaces, and their various extensions and applications. Monge greatly advanced the theory of curves and surfaces in space, a topic characterized by its close connection with differential equations; various properties of curves and surfaces can be represented in terms of differential equations, which is also why this branch of mathematics is called differential geometry. As one example, Monge obtained the general representation of a class of surfaces known as developable surfaces and showed that with the exception of cylindrical surfaces perpendicular to the xy plane, such surfaces always satisfy the partial differential equation

$$\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - - \left(\frac{\partial^2 z}{\partial x \partial y} \right)^2 = 0.$$

Monge served for a time as director of the École Polytechnique and continued to occasionally deliver lectures to its students. During one such lecture, he discovered an ingenious theorem in geometry concerning the properties of the tetrahedron. Recall that a tetrahedron is a solid with four faces and six edges, each of which meets every other edge in a point except for one, called its opposite edge. Monge's tetrahedron theorem states that the six planes passing through the midpoints of each edge and perpendicular to its opposite edge all meet in a point, now known as the Monge point. We close this section with the remark any readers who have the opportunity to visit Paris can find in that city a Rue Monge, a Place Monge, and a Café Monge.

While Monge was serving at the École Polytechnique, there was a student there named Jean-Victor Poncelet (1788–1867), who went on to become the proper founder of modern projective geometry and serve as the director of his alma mater. Poncelet was born in Metz in eastern France, an illegitimate child, later legitimated. At the age of 24, he participated in an expedition to Moscow under Napoleon as an engineer lieutenant. He was captured and turned his attention to mathematical problems during his time in a prison camp along the Volga River, using charcoal intended for heating to scribble on the walls. During this time, he wrote his most influential work, the *Traité des propriétés projectives des figures*, which presented his central projection treatment of conic sections, now the starting point for projective geometry of three dimensions. This paved the way for his career as a mathematician. Starting a year after his death, the French Academy of Sciences began to offer the Poncelet Prize for mechanics, applied mathematics, and the advancement of science.

Conclusion

The development of mathematics has proven throughout its history to need occasional nutrition from external sources; among these, physics has been consistently the most fruitful, and of course, it is also physics that has benefited most from the contributions of mathematics. Problems in physics have given much impetus to mathematics, especially in analysis, which has been closely linked with mechanics right from the birth of the calculus, and perhaps since the late nineteenth century also in geometry. This was the source of Lagrange's great masterpiece, the *Mécanique analytique*. But Lagrange himself perhaps loved number theory most of all the mathematical disciplines, and he was very proud of his proof that every positive integer can be represented as a sum of four or fewer squares. Another impetus for mathematics came from the demands for military and technological innovation ignited by the French Revolution. Since that time, the revolving door linking the development of mathematics and its applications has never closed.

It is necessary to observe that during the time after Newton and Leibniz had completed their work but before the appearance of Lagrange, the greatest mathematical minds in Europe were all concentrated in the small mountain country of Switzerland, at that time with a still relatively underdeveloped economy, culture, and scientific atmosphere: these of course were Leonhard Euler and the various members of the Bernoulli family, all of them from the same small city of Basel. The first-generation Johann and Jacob of the Bernoullis served as teachers to Euler in their capacities as professors at the University of Basel. After he graduated from this university, Euler spent most of his life in two distant and exotic cities, Berlin and Saint Petersburg. After his death, Euler appeared on the 10 Swiss francs banknote; alongside Newton on the 1 Pound Sterling banknote in Britain and Niels Henrik Abel on the 500 Norwegian Kroner banknote, he was one of three major mathematicians to appear on European currency still in circulation today. We mention also that Euler made his European debut in a prize competition organized by the Paris Academy of Sciences, which prize he won 12 times.

The École Polytechnique played an important historical role as the start of a new type of university; it also provided reliable employment for mathematicians, especially applied mathematicians. Lagrange and Monge were the first mathematical luminaries to appear among the professorship of this institution, and young students competed fiercely for admission with the goal of entry into service as an officer or engineer. The most significant member of this next generation was Augustin-Louis Cauchy, responsible for deep and humanistic achievements, although in later years some shallowness of mind or conceit caused him to ignore his younger colleagues, in particular Abel. The tradition of this institution later spread across the globe, with notable examples after its prototype being the Massachusetts Institute of Technology and California Institute of Technology in the United States, Tsinghua University in China, and the Indian Institute of Technology, with seven independent campuses located in different cities across the country.

Prior to Cauchy, there were two other great mathematicians to pass through the École Polytechnique. These were Joseph Fourier (1768–1830) and Siméon Denis Poisson (1781–1840). The greatest work by Fourier was his *The Analytic Theory of Heat*, which later James Clerk Maxwell is said to have called a great mathematical poem. In this book, he proved the important result that any function (today this claim requires some further qualification) can be expanded as a series of sine functions in multiples of the variable. Such expansions, known as Fourier series expansions, are important in the theory of boundary-constrained partial differential equations and also contributed an extension of the scope of the concept of a function. Poisson was the son of a former soldier and district president and became the first person to study integration along paths in the complex plane. His name comes up frequently in university mathematics: there is the Poisson integral, Poisson's equation in potential theory, Poisson's ratio in the theory of elasticity, the Poisson distribution and Poisson law in probability, the Poisson bracket in the theory of differential equations, and so on (Fig. 6.27).

Fourier famously remarked: "The deep study of nature is the most fruitful source of mathematical discoveries." There are also various interesting rumors surrounding Fourier and Poisson. Among them, it is said that during his time as Prefect in Egypt, he adopted the habit of wearing thick layers of clothes in the hot desert as part of his research into thermodynamics, and this aggravated his heart condition; when he died in Paris at the age of 63, he was alleged to have been as hot as if he had just been boiled. Poisson was looked after by a caretaker in his childhood. One day his father came for a visit and discovered the caretaker in absentia and his son hanging

Fig. 6.27 French mathematician Joseph Fourier

Fig. 6.28 Tomb of Fourier, at Père Lachaise Cemetery, Paris

in a cloth bag from a stud in the wall. The caretaker later offered the explanation that this was to prevent the child from catching an illness from the floor. Perhaps there is some connection between this event and his later years devoted to the study of pendulums (Fig. 6.28).

It is not unreasonable to say that the number of great mathematicians to emerge in the eighteenth century was greater than in any previous period, including even the seventeenth century, which was not lacking for geniuses. On the other hand, no singular giant of Renaissance proportions appeared among them, and the increasingly pragmatic turn of the times led to a separation between mathematics and philosophy. For this reason, the eighteenth century has sometimes been referred to as the century of invention. As a point of fact, there was not a single mathematician-philosopher of note during this time, and both Euler and Lagrange came to feel in their later years that the supply of mathematical ideas was beginning to run out. They could not have anticipated that this would merely mark a new turning point in the story of the development of mathematics (Fig. 6.29).

On the other hand, the astonishing new achievements in mathematics and through its applications and with it the elevated light in which mathematics came to be regarded shook longstanding systems of philosophical and religious thought. For intellectuals of the period, a devout piety and religious was increasingly impossible, and philosophers took this as an opportunity to inquire more deeply into the foundations of truth and its discovery. The German philosopher Immanuel Kant (1724–1804) wrote extensively on the subject, and he took as an example the Euclidean axiom that a straight line is the shortest distance between two points

Fig. 6.29 The philosopher Immanuel Kant; like Fermat, he remained in his small hometown (Königsberg) his entire life

to argue that truth cannot be obtained from experience alone but rather requires comprehensive rational judgment.

Another example from Kant was his introduction of the antinomies; these comprise inherent contradictions between two propositions each established in accordance with generally recognized principles. Antinomy is a fundamental concept in his philosophy, in particular in his *Critique of Pure Reason*, in which Kant presented with proofs four sets of antinomies in the form of thesis and antithesis. Among them, two are mathematical in nature and take the form of mathematical paradoxes:

> **Third Antinomy**
> Thesis: Causality as determined by the laws of nature is not sufficient to derive one and all of the appearances of the world; there must also be another form of causality in the form of spontaneity.
> Antithesis: There is no such thing as spontaneity, and everything in the world takes place solely according to the laws of nature.
> **Fourth Antinomy**
> Thesis: There exists in the world either as part of it or as its cause some being that is absolutely necessary.
> Antithesis: There exists no absolutely necessary being in the world, nor does one exist outside of it as its cause.

The main pillar of the philosophical system established by Kant, at least with respect to mathematical truths then, is that mathematical truths contain both Euclidean geometry and paradox.