Chapter 5
From the Renaissance to the Birth of Calculus

I would wish that the painter could be as learned as possible in the liberal arts, but first and foremost I would wish that he know geometry.

Leon Battista Alberti

Here is buried Isaac Newton, Knight, who by a strength of mind almost divine, and mathematical principles peculiarly his own, explored the course and figures of the planets, the paths of comets, the tides of the sea...

Inscription at Newton's monument

The Renaissance in Europe

Medieval Europe

During the period when the ancient civilizations of China, India, and Arabia in the east were making new contributions in mathematics, Europe was in the midst of its long Dark Age, a term first used by the Italian poet and scholar Petrarch (1304–1374), often considered the father of the Renaissance. The start of this period is marked by the collapse of Roman civilization in the fifth century, but there is no universal agreement as to its end, which could be considered to belong to the fourteenth, fifteenth, or even sixteenth century, with the start of the European Renaissance. The Dark Ages, which lasted for a thousand years, was later called the Middle Ages by Italian humanists in order to highlight their own works and ideals and to mark out the echoes of classical Greece and Rome formed by their era in contrast with the intervening centuries (Fig. 5.1).

Prior to the Middle Ages, the European territories outside of Greece and Rome had not done much to leave behind any deep marks on the history of human civilization, and since later there was no sign of intellectual revival in Greece, such terms as the Dark Age and Middle Ages alike, with the exception of the epidemic of the Black Plague, were mainly technical terms of academic humanism, limited in scope to Italy. In fact, even along the Apennines, the situation of mathematics

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Fig. 5.1 Likeness of Pope Sylvester II on a French stamp

during these times was not so bleak. Pope Sylvester II (ca. 945–1003) in particular admired and endorsed mathematics, and his election to the papacy was not unrelated to his mathematical facility, establishing him as something of a legend in the history of mathematics.

This pope, originally known as Gerbert of Aurillac, was born in central France and spent 3 years in Spain in his youth, where he studied the quadrivium at a monastery north of Barcelona, where the level of mathematics was high as a result of the legacy of Muslim Spain. When later he visited Rome, where he met the pope and the emperor, who were impressed by his mathematical knowledge, the latter hired him as a tutor for the young prince. With the further support of the subsequent emperor, Gerbert was elected eventually to the papacy and took the name Sylvester II. He is also said to have constructed an abacus and an armillary sphere, reintroducing them to Europe, and to have invented the first mechanical clock. In his mathematical work *De geometria*, he solved an open problem of the period: given the hypotenuse and area of a right triangle, determine the length of its remaining two sides (Fig. 5.2).

The period of Pope Sylvester II corresponds more or less with the era of translation in the history of science, when the classic works of Greek mathematics and science began to reappear in Western Europe, having been for centuries preserved primarily in the Islamic world long after they had disappeared from Alexandria and other centers of Greek academic activity. Whereas the translation of these works from Greek into Arabic had taken place mainly in the House of

Fig. 5.2 Toledo, Spanish capital city after the fall of the Roman empire; photograph by the author

Wisdom in Baghdad, the route from Arabic to Latin was more varied, taking place in the ancient Spanish city Toledo (which flooded with European scholars after the Christian defeat of the Muslims) and Sicily (which had been under Muslim rule for a period) and involving also diplomats in Baghdad and Constantinople.

The works translated into Latin included not only Euclid's *Elements*, Ptolemy's *Almagest*, *Measurement of a Circle* by Archimedes, *Conics* by Apollonius of Perga, and other Greek classics but also the more recent gems from the Islamic world, such as al-Khwarizmi's *Algebra*. All this took place mostly in the twelfth century, as the center of economic power in this part of the world shifted gradually from the eastern Mediterranean to the west. The primary mover of this change came from developments in agriculture, when the cultivation of pulses provided for the first time in history a guaranteed source of protein, leading to a population explosion that became one of the factors contributing to the disintegration of the old feudal structure.

By the thirteenth century, an endless proliferation of different social organizations emerged in Italy, including various guilds, associations, civic councils, churches, and so on, all of them desperate for some measure of autonomy. The idea of representation in the determination of important laws developed and spread

Fig. 5.3 Portrait of Fibonacci as a court mathematician

until finally a political assembly was formed whose members had authority to make binding decisions on behalf of all citizens participating in their election. In art, the classic models of Gothic architecture and sculpture sprang into being, and in terms of cultural and intellectual life, the methodology of scholastic philosophy took place of prominence. The representative figure of this trend was St. Thomas Aquinas (ca. 1225–1274), a Christian philosopher who was born in Sicily and took tremendous inspiration from the works of Aristotle (the French philosopher Jacques Maritain mentioned in the previous chapter was an important modern Thomist). For the first time, longstanding conservative beliefs came up against scientific rationalism (Fig. 5.3).

Fibonacci's Rabbits

In this relatively open and humanistic political atmosphere, mathematics did not lag far behind. The most outstanding mathematician of the European Middle Ages Fibonacci (ca. 1170–ca. 1250) was born during this time, a bit later than Bhaskara II in India and a bit earlier than Li Ye in China. Fibonacci, known in his time as Leonardo Bonacci or Leonardo of Pisa, was born in Pisa. His father was a merchant and customs official who brought his young son with him to Bugia (now Algeria), where Fibonacci was exposed to Islamic mathematics and learned to use Hindu-Arabic numerals. He subsequently visited Egypt, Syria, Byzantium, and Sicily, acquainting himself with the calculation of both the East and the Middle East. Not

Fig. 5.4 Graphical representation of the Fibonacci numbers

long after he returned to Pisa, he wrote and published his masterpiece the *Liber Abaci* (*The Book of Calculation*). The title of this book suggests some connection with the abacus, but this is misleading: actually, it is in reference to sand table calculations without the use of an abacus. The original 1202 manuscript is not known to exist; rather, the work survives in a copy from 1227, dedicated to one of the scientific advisors of the Holy Roman Emperor Frederick II (1194–1250).

The first section of the *Liber Abaci* introduces the basic arithmetic of numbers, with calculations in sexagesimal; a noteworthy innovation is the introduction also of the horizontal bar demarcating the numerator and denominator of a fraction, which notation is still in use today. The second section consists of word problems related to commerce, including the *Hundred Fowls Problem* from China. This problem, first posed by Zhang Qiujian, seems to have spread to the Arabic world. The third section contains miscellaneous problems and mathematical oddities, including a problem concerning rabbits that has proved significant. This problem asks how many rabbits can be bred in 1 year, starting from a single pair, under the stipulations that each pair of rabbits begins to breed at the age of 2 months and can produce thereafter a new pair of rabbits each month (Fig. 5.4).

Subsequent generations have referred to the sequence of numbers determined by the rabbit problem as the *Fibonacci sequence*:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \dots.$$

This sequence can also be described by the recurrence relation

$$\begin{cases} F_1 = F_2 = 1 \\ F_n = F_{n-1} + F_{n-2} \text{ (for } n \geq 3) \end{cases},$$

one of the first recurrence relations to appear in mathematics. There is also a remarkable explicit expression for the terms of this sequence involving the irrational number $\sqrt{5}$, that is:

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

The Fibonacci sequence has many interesting properties and important applications. For example, as n grows larger and larger approaching infinity ($n \longrightarrow \infty$),

$$\frac{F_{n+1}}{F_n} \longrightarrow \frac{1+\sqrt{5}}{2} \approx 1.618.$$

This number is related to the golden ratio identified as a ratio of line segments by Pythagoras in the early years of the history of mathematics. In addition to tendrils stretching into many areas of mathematics, the Fibonacci sequence has also turned up in applied problems related to the reproduction of bees, the petals of certain flowers, and aesthetics.

Around the year 1220, Fibonacci was summoned by Frederick II, who was visiting Pisa at the time. His scientific advisors posed to Fibonacci a series of mathematical problems, which Fibonacci answered one by one. One of these problems was to find the roots of the cubic equation $x^3+2x^2+10x = 20$. Fibonacci used an approximation method to give the answer in sexagesimal, accurate up to nine digits after the decimal point. Afterward, Fibonacci maintained a long correspondence with the emperor and his court, where mathematics was held in high regard (according to some accounts, he was rather hired by the emperor to serve at the palace and became the first court mathematician in European history). Frederick II seems to have had almost limitless energy and served simultaneously as the King of Sicily, the King of Germany, and later the Holy Roman Emperor and the King of Jerusalem.

Fibonacci devoted his second substantial work, *The Book of Squares* (*Liber Quadratorum*), to Frederick II. In this book, Fibonacci presents the profound proposition that $x^2 + y^2$ and $x^2 - y^2$ cannot both be perfect squares simultaneously. This book is perhaps the first monograph ever devoted to a specific class of problems in number theory and established Fibonacci as the significant number theorist between the times of Diophantus and Fermat. Considering the legacy of Fibonacci, he not only played a pioneering role in the revival of European mathematics but also served as an important bridge in the transfer of mathematics from east to west. Gerolamo Cardano, the finest Italian mathematician of the sixteenth century, remarked: “We can conclude that all the knowledge we have of mathematics outside of Greece is due to the appearance of Fibonacci.”

Judging from the surviving likenesses of Fibonacci, he had a charm similar to that of his compatriot the painter Raphael, who lived three centuries later, and he seems to have regarded himself as a kind of wanderer. The name Leonardo of Pisa by which he is also known places him in the company of Leonardo da Vinci, the painter of the *Mona Lisa*. In the year 1963, a group of American mathematicians inspired by the rabbit problem established the Fibonacci Association and began to publish *Fibonacci Quarterly* in the United States, dedicated to mathematical research papers related to the Fibonacci sequence. Since 1984, the Fibonacci Association has also hosted biannually an *International Conference on Fibonacci Numbers and Their Applications* around the world. The development of such a rich universe of research from a simple model of rabbit reproduction is another miraculous legend in the history of mathematics.

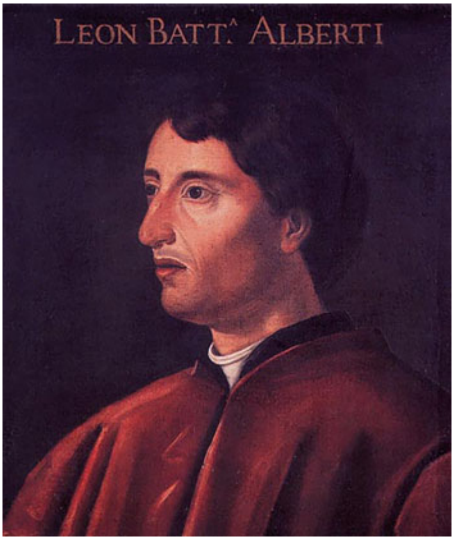


Fig. 5.5 Alberti the humanist

Alberti's Perspective Method

After the collapse of the old feudal structure in Europe, there followed an astonishing sequence of events that taken together signified the birth of a new era governed by a totally new mental outlook: the strengthening of the Italian city-states; the rise of the monarchies in Spain, France, and England; the development of secular education; the discovery of new maritime routes and the New World; the radical proposal of a heliocentric solar system by Nicolaus Copernicus;[1] the invention and application of movable type printing; and so on. This new era recalled and took inspiration from the scholarship, wisdom, and values of the classical world; for this reason, it was called the Renaissance.

The Italian thinkers of the Renaissance period embraced a humanistic ideal with man at the center of the universe and capable of unlimited development. It followed naturally for many such thinkers that is incumbent on humankind to pursue the total acquisition of knowledge and the development of abilities, that is, the refinement of skill and capability not only in every intellectual field but also in physical training, social activity, literature, and art. Such polymathic individuals are often referred to today as Renaissance men in honor of this ideal. The archetypal Renaissance man was Leon Battista Alberti (1404–1472), a humanist, artist, writer, mathematician, and thinker, who also excelled in horsemanship and martial arts (Fig. 5.5).

[1] Copernicus was studying at the University of Kraków, a medium-sized city in Poland that was home to the two Nobel Laureates Czesław Miłosz and Wisława Szymborska at the same time in the early twenty-first century, at the time when Columbus reached the New World.

Alberti was born in Genoa, the illegitimate son of a wealthy Florentine banker, who taught him mathematics in his youth. He took to writing early on, composing Latin comedies, and later obtained a doctorate in law, took holy orders, and served the papal court. Alberti used his knowledge of geometry to determine for the first time in history precise laws for the representation of a three-dimensional scene on a flat wooden block or the surface of a wall. This had an immediate effect on Italian painting and relief-making and facilitated the production of an accurate, rich, and geometrically correct perspective style. Alberti wrote, "a man can do anything if he but wills it," and "I would wish that the painter could be as learned as possible in the liberal arts, but first and foremost I would wish that he know geometry.[2]"

Prior to Alberti, Florence had already produced the great architect Filippo Brunelleschi (1377–1446), responsible for the dome over the most famous cathedral to this day in this art capital. According to one saying, he loved mathematics since childhood and took up painting only in order to engage with geometry; of course, this was not enough to achieve mastery in mathematics, and he turned later to engineering and architecture, but nonetheless he was the first person to study perspective, and Alberti's own interest in perspective came about because of his connection to such predecessors. The basic principles of perspective that Alberti introduced can be described as follows:

> Place upright a glass screen between the eyes of the observer and the scene, and imagine rays of light emitted from one eye to every point of the scene, creating a cross-section as they pass through the screen. This cross-section should present to the eye the same impression as the scene itself, so that the problem of realistic painting is precisely the production of this cross-section of glass on the canvas. Alberti noticed that if two such glass screens are placed between the eyes and the scene at different locations, the results will be different, and similarly if the eyes look through the same screen from two different positions, the cross-section in glass again will be in each case different.

In every case, Alberti asked what is the mathematical relationship between any two parallel scenes; this question is the starting point of projective geometry (Fig. 5.6).

Alberti also discovered that in a realistic representation of the scene, parallel lines (except those parallel to the glass screen or to the plane of the image) must intersect in a certain point. This point is called the vanishing point, and its discovery was a turning point in the history of painting. In the past, it was rare to paint so accurately, but subsequent generations of painters mostly followed this principle, although of course the vanishing point itself need not appear in the painting. The origin or rather cause of the existence of the vanishing point is as follows: any two parallel lines in the real scene form two intersecting planes containing the observation point, and the point where the line of intersection of these two planes meets the glass screen forms the vanishing point. It is precisely because of his work with perspective and the vanishing point that Alberti became the most important art theorist of the Renaissance (Fig.,5.7).

[2] *On Painting*, Tr. Rocco Sinisgalli

Fig. 5.6 Dome of the Florence Cathedral, designed by Brunelleschi

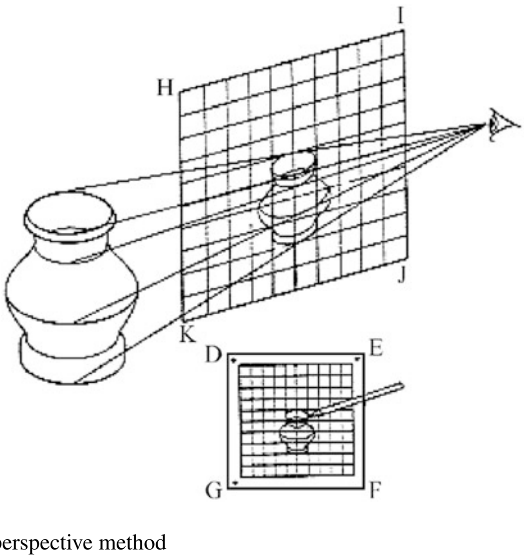


Fig. 5.7 Alberti's perspective method

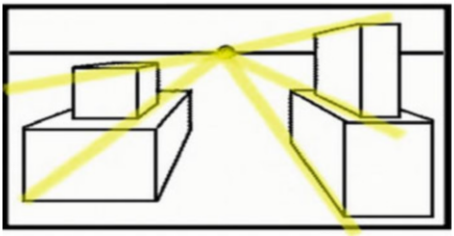


Fig. 5.8 Alberti's vanishing point

Throughout his entire body of work, Alberti always maintained the outlook of civic humanism that flourished in Florence at the time. For example, he wrote the first Italian grammar, arguing that Italian was as regular as Latin and equally suited to literary composition; he also wrote a pioneering treatise in cryptography, in which appear the first polyalphabetic cipher and the first table of letter frequencies. His final work, written several years before his death, was a dialogue entitled *De iciarchia* (*On Ruling the Household*), in which he praises human accomplishment and public service as virtues, fully in line with the spirit of humanism in pursuit of public welfare. According to the biographer Giorgio Vasari (1511–1574), Alberti died quietly and contented (Fig. 5.8).

Da Vinci and Dürer

When Alberti was approaching 50 years of age, the most glorious figure of the Renaissance period was born in a village called Vinci on the outskirts of Florence: Leonardo da Vinci (1452–1519). He was born out of wedlock to a peasant woman who later married a craftsman and a successful Florentine notary and landlord, who also married shortly afterward. His first of several wives was unable to bear children, however, and Leonardo's father took custody of him early on and provided for his elementary education in reading, writing, and arithmetic. He became a studio boy in his adolescence and took up painting as an apprentice and after the age of 30 turned his attention to advanced geometry and arithmetic. His two famous works *The Last Supper* and the *Mona Lisa* were painting in his middle age and old age, respectively (Fig. 5.9).

The artistic achievements of Leonardo da Vinci are common knowledge and need no further introduction here; his name even inspired a suspense novel in the twenty-first century that became a worldwide bestseller. He believed deeply that the foundation of painting was the accurate reproduction of the original impression, which can be achieved only through rigorous adherence to the mathematics of perspective, which he referred to as the steering wheel and guiding principle of painting. It was probably in response to this attitude that the twentieth-century French avant-garde painter Marcel Duchamp made his *L.H.O.O.Q.*, consisting of a cheap postcard reproduction of the *Mona Lisa* on top of which the artist drew a

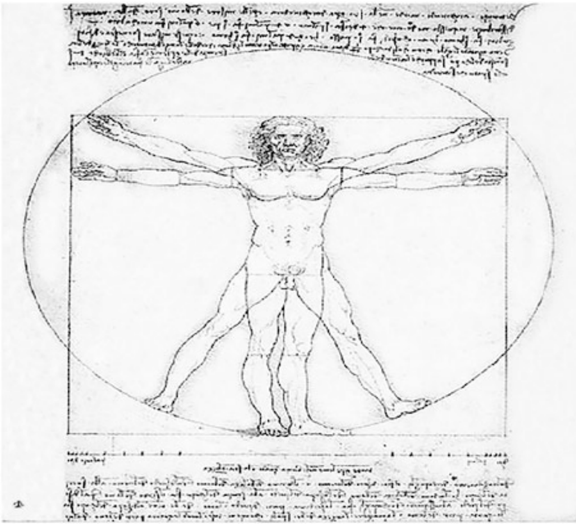


Fig. 5.9 Leonardo da Vinci's famous Vitruvian Man, drawn around 1487

Fig. 5.10 Statue of Leonardo da Vinci in Amboise, France

moustache and beard in pencil. In geometry, Leonardo's main achievement was the determination of the position of the center of gravity of the tetrahedron, given as a quarter of the distance along the line to the opposite vertex from the center of gravity of the base triangle. On the other hand, he made an error in his similar determination of the center of gravity of the isosceles trapezoid, providing two methods, only one of which was correct (Fig. 5.10).

Leonardo also achieved outstanding results beyond the scope of art and mathematics. His observations of the celestial bodies led him to secretly record in his notebooks that "il sole no si muove" or "the sun does not move" earlier than Copernicus, in contradiction with the doctrine of the Bible that God created the sun and the moon and made them to travel around the earth. The flight of birds inspired him to investigate air resistance and sketch the first designs for a flying

Fig. 5.11 *Self Portrait*, Albrecht Dürer (1498)

machine. Some dynamicists believe today that if Leonardo had access to a light fuel source at the time, he could have made it to the heavens. He also personally dissected more than 30 corpses in his research in the mysteries of human anatomy and life. All of these various researches were abandoned midway, but contributed to the development of his observational powers and accuracy in painting (Fig. 5.11).

Also in the fifteenth century, another versatile artist and Renaissance figure appeared in Nuremburg in Bavaria, Germany, in Northern Europe. This was Albrecht Dürer (1471–1528), born 1 year before the death of Alberti, whose humanistic ideals lent to his art its characteristic air of knowledge and rationality. Dürer spent about 20 years of his life travelling and living in Holland, Switzerland, Italy, and other places. He also maintained some connection to his fellow religious reformer Martin Luther (1483–1546), a few years his junior, and the various figures surrounding him. He produced creative work in a very broad range of fields, including oil painting, printmaking, woodcutting, illustration, and so on, and it is obvious from his work that Dürer was well versed in the perspective method introduced by Alberti (Fig. 5.12).

Among all Renaissance artists, Dürer is generally considered to be the one with the greatest knowledge of mathematics. His *Four Books on Measurement* (or *Instructions for Measuring with Compass and Ruler*) deals mainly with geometry and touches also on linear perspective. Among its innovations is a treatment of the projections on to the plane of curves in space and the introduction of the epicycloid, the curve traced by the trajectory of a fixed point on the circumference of a rolling circle. Even more impressive, Dürer considered the orthogonal projections of curves or figures onto two or three mutually perpendicular planes, a very advanced topic that was not developed further until the eighteenth century, when the French

Fig. 5.12 *Melencolia I*, Albrecht Dürer (1514)

mathematician Gaspard Monge created the field of descriptive geometry, earning a place for his own name in the history of mathematics.

In his large 1514 engraving *Melencolia I*, Dürer depicts in the foreground a winged young woman sitting in contemplative manner with her head resting in her left hand. In the background, there is a fourth-order magic square:

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

.

In this magic square, every row, column, and diagonal sums to 34, as do the terms of the five second-order submatrices in the four corners and the center, and even four of the third-order submatrices and the unique fourth-order submatrix, as well as other such arrays.

Comparing this to the example cited in the works of Yang Hui of the Southern Song dynasty in China, the only difference is in the order of the rows. The presence of this magic square undoubtedly contributes to the enigmatic melancholia of the engraving; it is interesting also to note that the middle two numbers of the final row serve to date the picture: 1514. This year saw the death of his mother, and he may have made this engraving as an expression of his grief. The magic square contained within it, however, is not altogether perfect. There is one inscribed at the entrance to Parshvanatha temple in Khajuraho, India, in the twelfth century which is more satisfactory in some respects although less so in others. In particular, the terms of all nine of its second-order submatrices also sum to 34.

Speaking broadly with respect to painting, it is generally the case that colors are more expressive of emotion, while line is more expressive of reason. In line with the German reputation for rational thought, German painters have proved excellent in their use of line. Certainly this holds for Dürer. His precise line drawings directly reveal the subtlety of his observation and the complexity of his conception, and the combination of his cerebral approach with his ardent ideals produces a unique effect. In addition to the visual arts and mathematics, Dürer worked in art theory and scientific writing, including works on draftsmanship, human proportions, and architectural engineering, featuring his own illustrations.

The Invention of Calculus

The Awakening of New Mathematics

Although the artists of the Renaissance offered novel insights into mathematics, the revival of mathematics, and indeed the rise of modern mathematics, did not take place until the sixteenth century. The first new advances in mathematics began with algebra: for example, trigonometry had been separated from astronomy, the study of perspective gave rise to projective geometry, and the invention of logarithms facilitated easier computation, but the main breakthrough was in the solution of cubic and quartic equations and the development of symbolic algebra. After the *Algebra* of al-Khwarizmi was translated into Latin, it was widely circulated and used as a textbook throughout Europe. At this time, people considered the solution of cubic and quartic equations to be a problem as difficult as the three unsolved geometric problems of Ancient Greece. But at the turn of the century, two Italian mathematicians were born who managed to settle this issue completely: Tartaglia and Cardano (Fig. 5.13).

Tartaglia (1499–1557), whose name at birth was Niccolò Fontana, was born in Brescia, not far from Milan, to a dispatch rider father who was murdered several years later. In a further misfortune, Tartaglia's jaw and palate were sliced with a saber by an invading French soldier in 1512, leaving him with a speech impediment that earned him the nickname by which he is remembered today (*tartaglia* means *stammerer*). As an adult, Tartaglia made his living teaching mathematics; he made a name for himself with his claim that he could solve any cubic equation lacking either the linear or quadratic term, that is, equations of the form $x^3+mx^2 = n$ or $x^3+mx =$

Fig. 5.13 Gerolamo Cardano, prominent physician, lawyer, and politician

n with $m, n > 0$. A professor at the University of Bologna doubted this claim and sent a student to challenge Tartaglia, which challenge Tartaglia readily met, since his opponent could handle only those cubic equations lacking the quadratic term.

In 1539, a mathematics enthusiast and medical practitioner in Milan named Gerolamo Cardano (1501–1576) invited Tartaglia to stay at his home as a guest for 3 days to discuss mathematics. After sharing together a full and satisfying meal, Cardano cajoled Tartaglia into revealing his solution to the cubic equation with the promise never to publish it. Tartaglia presented his solution encoded in a poem of 25 lines. Several years later, Cardano encountered the same solution in an unpublished work and determined that his promise was no longer binding. Tartaglia was shocked to see his solution published by Cardano in his book *Ars Magna* to quite some fanfare, and a bitter enmity developed between the two mathematicians.

Tartaglia's solution, which we present here in modernized exposition, was as follows. In light of the identity

$$a^3 - b^3 = (a-b)^3 + 3ab(a-b),$$

choose appropriate a and b such that

$$\begin{cases} 3ab = m, \\ a^3 - b^3 = n \end{cases}.$$

Then $a - b$ is a solution of the equation $x^3 + mx = n$, and it is not difficult to further solve for a and b to obtain them as

$$\sqrt[3]{\pm\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}}.$$

This solution is what is known as Cardano's equation, although Cardano was careful to give credit for it to Tartaglia. Cardano also considered the case $m < 0$ and gave in this case also the complete solution. As for cubic equations lacking a linear term, they can always be transformed into equations of this type by a change of variables.

Even more impressively, the *Ars Magna* also gave a general solution for the quartic equation, which was also not due to Cardano, but rather to his former servant and eventual student Lodovico Ferrari (1522–1565). Ferrari had begun his career at the age of 15 as a house servant to Cardano, known at that time mainly as a doctor. Cardano quickly recognized his intelligence and began to teach him mathematics. And indeed, Ferrari quickly discovered a way to convert quartic equations into cubic ones and became as a result the first mathematician to successfully solve the quartic equation. He also represented Cardano in a second mathematical challenge against Tartaglia, this time in Milan; on this occasion, Tartaglia did not emerge victorious (Fig. 5.14).

After making a name for himself while still in his teens, Ferrari quickly obtained a prestigious teaching post in Rome, from which he retired at the age of 42 to move back to his hometown and serve as a professor of mathematics at the University of Bologna. Unfortunately, he died not long after at the young age of 43 of arsenic poisoning, given to him according to legend by a widowed and greedy sister. The question of polynomials of degree five and higher was not resolved until the Norwegian mathematician Niels Henrik Abel proved their insolvability by radicals in the nineteenth century; from this, it is evident that the achievements and stories associated with these Italian mathematicians circulated among their mathematical colleagues and successors for a long stretch of time.

From the discussion above, we can conclude that although Tartaglia and Ferrari were more adept at discovering clever solutions to specific problems, Cardano played a more important and unifying role in this story. In this respect, he was a kind of Euclidean character for this period in the history of mathematics. Another such character emerged in France in the sixteenth century: François Viète (1540–1603), more commonly known by the Latinized form of his name, Franciscus Vieta. Vieta is credited with the creation of the first symbolic algebra, with which he was able to make substantial contributions to the theory of equations. Middle school mathematics textbooks today include a special case of Vieta's formula, which relates the two roots x_1 and x_2 of the quadratic polynomial $ax^2 + bx + c$ to its coefficients:

$$x_1 + x_2 = -\frac{b}{a}, \text{ and } x_1 x_2 = \frac{c}{a}.$$

Fig. 5.14 François Viète, lawyer and politician

Vieta was a lawyer and politician by profession. During the wars between France and Spain, Vieta used his mathematical talent to uncover the key to a Spanish cipher. It was during a period of political frustration that he devoted himself to mathematical research and he developed the ideas of using letters to represent algebraic terms while reading the writings of Diophantus. Although most of the symbols he himself used have since been replaced, Vieta is still recognized as the father of symbolic algebra. In particular, Vieta used consonants to represent known constants, and vowels to represent unknown terms, with the symbol $\sim$ indicating a negative quantity. Considering more generally the standard notation in modern mathematics texts, the addition and subtraction signs $+$ and $-$ and notation for powers were introduced in the fifteenth century, the equals and greater-than signs $=$ and $>$ were introduced in the sixteenth century, and the less-than sign $<$, the radical $\sqrt{\ }$, the multiplication and division signs $\times$ and $\div$, the use of the bottom of the alphabet (a, b, c, ...) for knowns and the top of the alphabet (..., x, y, z) for unknowns, and exponential notation were all introduced in the seventeenth century (Fig. 5.15).

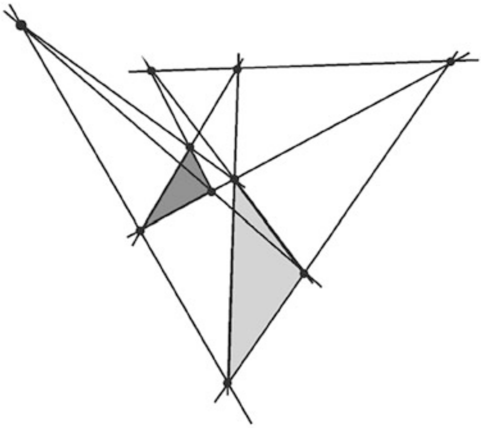


Fig. 5.15 Desargues's theorem

Analytic Geometry

After the advent of the seventeenth century, various mathematical theories and branch began to spring up like bamboo shoots after a rain. It is impossible for us to analyze here these developments in comprehensive detail, and inevitably, we will have to leave out even some of the more important mathematicians. We press on all the same and turn first to the French mathematician Girard Desargues (1591–1661). It was Desargues who answered the outstanding questions about perspective left over from Alberti and established the discipline of projective geometry. Indeed, Desargues is considered as the founder of this branch of mathematics. Desargues was originally a soldier and later earned his living as an engineer and architect. He was a regular participant in the mathematical salons organized by the priest and polymath Marin Mersenne, where he won the respect of such young mathematicians as Descartes, Pascal, and others (Fig. 5.16).

One of the fundamental contributions that Desargues made to projective geometry is the concept of the point at infinity, unifying the classes of parallel lines and intersecting lines in the plane by allowing parallel lines to intersect in the point at infinity; this point of view would later prove very fruitful for the development of non-Euclidean geometry. It follows that in projective geometry, every pair of lines lying in the same plane eventually intersect, which is the starting point on which the theory is built. An additional innovation is that Desargues concerned himself only with the interrelationships between geometric figures without any reference to measurement, also a novel and forward-looking idea in geometry. Finally, there is also Desargues' theorem, which states that if the lines formed by three corresponding pairs of vertices of two triangles all intersect, respectively, then the three sets of intersection points so obtained are each individually colinear. From the point of view of the painter, this theorem can be stated as follows: if two triangles can be seen in perspective from a single external point (which turns out to be just at two different sections of the cone), the points of intersection of the extended corresponding edges are colinear, provided none of the edge pairs are parallel.

Fig. 5.16 Fashion show with designs based on Desargues's theorem

More generally, geometric research in the seventeenth century broke out along two main strands. The path taken by Desargues can be described as a continuation of the tradition of synthetic geometry, but under conditions of a broader generality. The second path proved ultimately to be the more brilliant and influential; this was to introduce the use of algebraic tools to the study of geometry, specifically, the discipline of analytic geometry established by Descartes.

At its essence, the contrast between modern mathematics and ancient mathematics is that modern mathematics is concerned with variables, whereas ancient mathematics was concerned with constants. The development of capitalist production following upon the Renaissance created new demands on science and technology: the widespread use of machinery, for example, necessitated the study of mechanical motion; the development of a maritime industry driven by trade created a demand for more accurate and convenient methods for the determination of the positions of ships, leading people to study the laws of motion governing the celestial bodies; and the improvement of weapons technology stimulated research into problems of ballistics. All of these various topics and questions indicate that the study of movement and change had become the central research topic in the natural sciences and mathematics (Fig. 5.17).

The first milestone in the mathematics of variables was the invention of analytic geometry. The basic idea of analytic geometry is to introduce coordinates to the

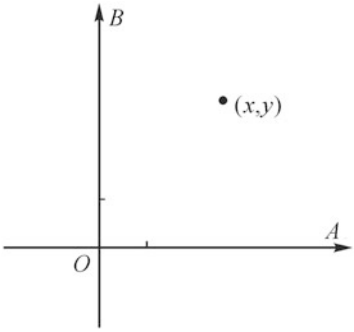


Fig. 5.17 Cartesian coordinates

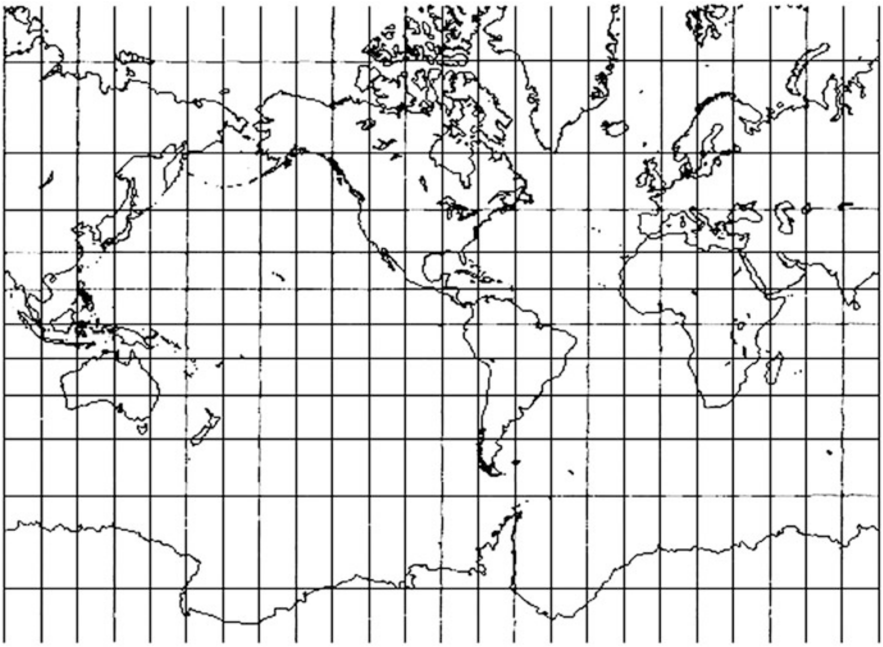


Fig. 5.18 Map of the world by Mercator

plane; for this reason, analytic geometry is known also by the name coordinate geometry. The coordinates are determined by a coordinate system as follows: fix any two intersection straight lines A and B in the plane and designate their point of intersection O as the *origin* of the system. The two lines A and B are referred to as the coordinate axes, and the coordinate system is established by fixing unit coordinates along the two axes. In this way, every ordered pair (x, y) of real numbers corresponds to a unique point in the coordinate plane, and vice versa (Fig. 5.18).

With the tools of analytic geometry in hand, it becomes possible to associate the solution set of any algebraic equation of the form

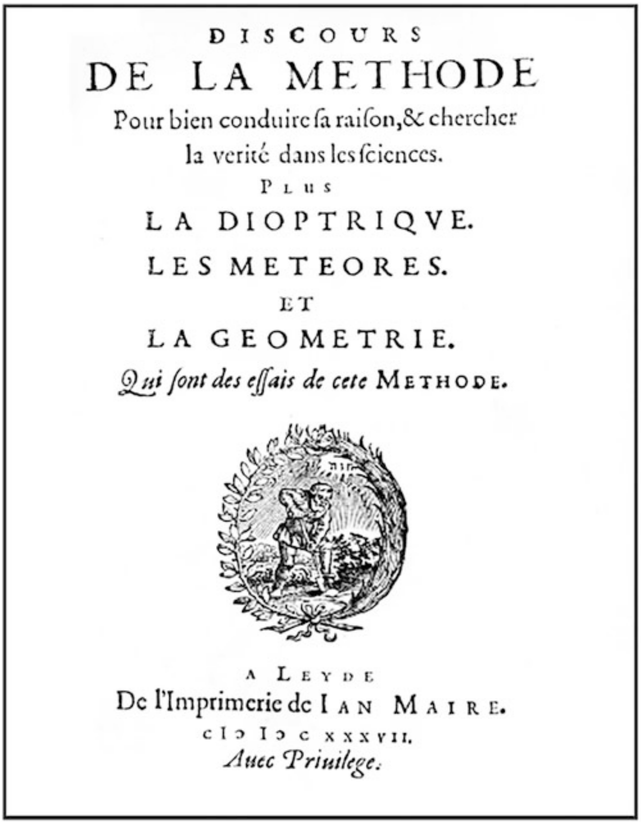
$$f(x, y) = 0$$

with a curve in the plane. This amounts to a reduction of problems in geometry to algebraic problems, so that new geometric results can be obtained through the study of problems in algebra. In the other direction, this association produces a natural geometric interpretation of algebraic problems.

There were several precursors to this innovation throughout history: the fourteenth-century French mathematician Nicole Oresme (ca. 1320–1382) borrowed from geography the terms *longitude* and *latitude* to describe his geometric figures (he was also the inventor of the + symbol in mathematics), and in the sixteenth century, the Flemish geographer Gerardus Mercator (1512–1594) used orthogonal longitudinal and latitudinal lines to draw the first atlas in history. He was also the first to use the term *atlas*. He was deeply versed in the mathematics and physics of his time and applied his knowledge freely in his work; he was in addition an excellent engraver and calligrapher. But in any case, neither of these two forerunners took the further step of establishing a direct association between numbers and geometric figures. Rather the credit for the invention of analytic geometric belongs properly to two later French mathematics, Descartes and Fermat.

It is necessary to point out that Descartes and Fermat alike both took as their starting point the general consideration of oblique coordinate systems, with the system of rectangular coordinates with axes A and B perpendicular to one another, say as horizontal and vertical, considered only as a kind of special case. They also both discussed the further possibility of a coordinate system in three dimensions. It has since become customary to refer to the coordinate system as Cartesian coordinates, or to the plane equipped with a coordinate system as the Cartesian plane, although this should not be taken to mean that Descartes achieved earlier or more brilliant results in this domain than Fermat. The main difference between the two is that Descartes considered his invention to mark a sharp break from Greek tradition and in particular emphasized the power of algebraic methods, while Fermat regarded his work as a straightforward restoration of the mathematics of Apollonius. But Fermat was also decidedly more explicit in his emphasis on the use of equations to define trajectories and curves. He gives directly the modern forms for the equations of many curves, including straight lines, circles, ellipses, parabolas, and hyperbolas (Fig. 5.19).

Although Descartes and Fermat arrived at analytic geometry by different routes and for different purposes, nevertheless, they became caught up in a priority dispute. Descartes had published his results in analytic geometry, in 1637, under the title *Geometry* (*La Géométrie*), an appendix to his *Discourse on Method* (*Discours de la méthode*), a broad philosophical treatise. Fermat never published his work, but he had discovered the basic principles of coordinate geometry as early as 1629. This was published only after his death in 1665, along with many other of his mathematical discoveries. Perhaps because they were both French, this dispute



DISCOURS
DE LA METHODE
Pour bien conduire ſa raiſon, & chercher
la verité dans les ſciences.
PLUS
LA DIOPTRIQVE.
LES METEORES.
ET
LA GEOMETRIE.
Qui ſont des eſſais de cete METHODE.

A LEYDE
De l'Imprimerie de IAN MAIRE.
CIƆ IƆ C XXXVII.
Auec Priuilege.

Fig. 5.19 Title page of *Discourse on Method* by Descartes

never bubbled over into troubling proportions, but each had their supporters: Pascal took up with Fermat, and Desargues with Descartes (Fig. 5.20).

This was not the only coordinate system invented in this period. In 1671, 2 years after the publication of Fermat's work in coordinate geometry, Isaac Newton in Britain invented his own system of coordinates, known today as polar coordinates. In modern terminology, polar coordinates are determined by a fixed point O in the plane and a half-line A extended in any direction from O. Then any point B in the plane is determined by the distance r between the points O and B, and the angle θ formed by the intersection of lines OA and OB. The elements of the ordered pair (r, θ) are called the polar coordinates of the point B. As everybody has learned in middle school, some geometric figures lend themselves to simpler expression in polar coordinates than in Cartesian coordinates, for example, Archimedean spirals, catenary curves, cardioids, three- and four-leafed rose curves, and so on.

Fig. 5.20 Descartes

The Pioneers of Calculus

The invention of analytic geometry not only enabled the application of algebraic methods to geometric problems but also introduced variables into mathematics, paving the way for the creation of calculus; but an even more crucial development was the establishment of the concept of a function.

In the year 1642, 5 years after the publication of Descartes' *Geometry*, Isaac Newton (1642–1727) was born in a hamlet in the country of Lincolnshire in England. It was also the same year in which Galileo died. Newton was a posthumous child, born 3 months after the death of his father, and did exhibit in childhood the signs of a prodigy. He did however develop a love for extracurricular reading and in middle school picked up the habit of compiling notebooks, which he referred to as his *waste book*. This habit, which some generations later was also practiced by Gauss, would prove very important, and later he brought his notebooks with him to Cambridge University, where he used them for notes on mathematics and mechanics, including his work on calculus and the theory of gravity.

At around the age of 22, Newton began to include in his notes a record of his work on calculus, in which he always used the word *fluent* to denote a relationship between variables. It was the German mathematician Leibniz who first used the word *function* to designate to any quantity that changes according to the change in position of a point on a curve. The familiar notation $f(x)$ to represent a function in the variable x was only introduced in 1734, by the Swiss mathematician Leonhard

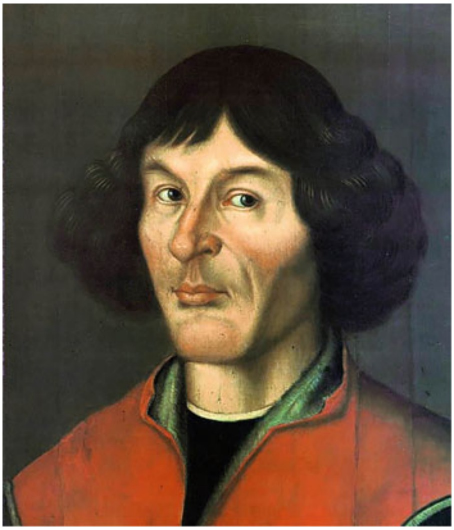


Fig. 5.21 Polish astronomer Copernicus

Euler, when functions had long already been the centerpiece of the conceptual machinery of calculus (Fig. 5.21).

In fact, the basic ideas of calculus, and in particular integral calculus, can be traced back to ancient times. As we have discussed already, the calculation of areas and volumes has been a topic of interest to mathematicians since ancient times, and there appear many examples of the use of infinitesimal arguments to compute the areas, volumes, or arc lengths of various special figures in the mathematical writings of Ancient Greece, China, and India. These include the work of Archimedes in Greece and Zu Chongzhi and his son in China on the calculation of the volume of a sphere. The example of Zeno's paradox also introduces the idea of the infinite division of an ordinary constant. As for differential calculus, Archimedes and Apollonius discussed, respectively, the tangent lines to spirals and conic sections, although only individually or statically. But calculus in its modern form was introduced mainly in order to solve the scientific problems of the seventeenth century (Fig. 5.22).

The first half of the seventeenth century in Europe saw successive major advances in the fields of astronomy and mechanics. First, a Dutch lensmaker invented the telescope in 1608, and when the Italian scientist Galileo Galilei (1546–1642) heard of its invention, he quickly built a powerful telescope of his own and used it to discover many hitherto unknown secrets of the solar system. In particular, his observations confirmed the validity of the heliocentric model of the solar system first proposed in modern times in the fifteenth century by Polish astronomer Nicolaus Copernicus (1473–1543). This remarkable achievement however brought upon Galileo a series of disasters, including interrogation and persecution by the church and leading eventually to blindness and despondency at the end of his life. Simultaneously, the German astronomer Johannes Kepler (1571–1630), 7 years

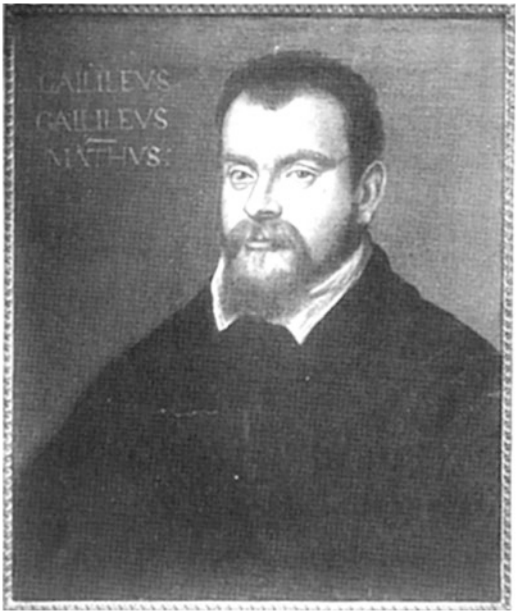


Fig. 5.22 Italian physicist Galileo

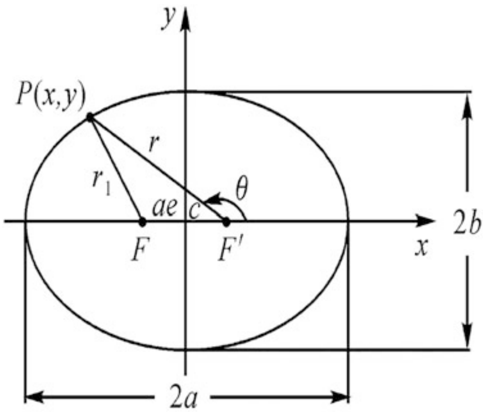


Fig. 5.23 Kepler determined that the orbits of the planets are elliptical

Galileo's junior, was in the process of obtaining a more precise mathematical argument for the heliocentric model on the basis of data collected by his predecessor and employer, the Danish astronomer Tycho Brahe (1546–1601) (Fig. 5.23).

Copernicus and Tycho Brahe however both believed that the orbits of the planets were circular (and Galileo too said nothing against this theory). It was Kepler who first stated as his first law of planetary motion, which states that:

> the orbit of every planet is an ellipse, with the sun at one of its two foci.

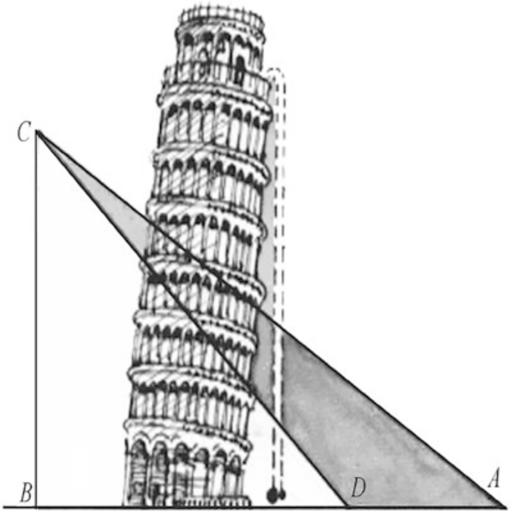


Fig. 5.24 The Leaning Tower of Pisa and associated with it a famous experiment on freely falling bodies

His second and third laws of planetary motion demonstrate even more thoroughly his mathematical ability, which was probably greater even than that of Galileo. The second states that:

> the line segment joining a planet and the sun sweeps out an equal area in equal intervals of time,

and the third that:

> the square of the orbital period of a planet is proportional to the cube of the length of the semi-major axis of its orbit.

This is not to say however that the achievements of Galileo lagged behind those of Kepler. In the first half of his life, that is, in the late sixteenth century, Galileo introduced the law of free fall $s = \frac{1}{2}gt^2$ and the law of inertia or Galilean relativity; he was also a great pioneer in the use of experimental methods in science (Fig. 5.24).

Neither Stuttgart, not far from which Kepler was born, nor Prague, where he later lived, was at the center of European civilization at the time, and his work did not receive as much attention as it deserved. On the other hand, he also avoided the religious persecutions suffered by Galileo. Still, his life was not altogether a happy one: he was a premature and sickly baby and the child of an unhappy marriage, and he himself later suffered through two disastrous marriages and a series of family troubles. He was comforted in his difficulties by his belief in the mathematical harmony of the heavens as revealed to him in mathematics and astronomy, a doctrine showing the distance influences of Pythagoras and Plato, and it was this conviction that set him in pursuit of the laws of planetary motion. In another story from his life, Kepler apparently was at one time deeply dissatisfied with the rough calculation for the volume of a wine barrel employed by the merchant who sold it to him and took it upon himself to discover a method for determining

Fig. 5.25 Statue of Tycho Brahe and Johannes Kepler in Prague

precisely the volume contained in a surface of revolution, a generalization of the spherical volume formula discovered by Archimedes (Fig. 5.25).

Kepler discovered the first two of his three laws of planetary motion in the year 1609, but it took him another 10 years to produce the third. The main obstacle was the complexity of the date left behind by Tycho Brahe and the computational challenges posed by it, in particular the continuous need to multiply together very large numbers. In 1614, the Scottish landowner and mathematician John Napier (1550–1617) invented the logarithm, which simplified the calculations involved in multiplication and division to addition and subtraction. But the practical use of logarithms only became possible 2 years later, when the British mathematician Henry Briggs (1561–1630) paid a visit to Napier in Scotland and encouraged him to reformulate his logarithms in base ten and to compile the first comprehensive logarithm tables. News of this innovation reached Kepler and played a critical role in the development of his third law of planetary motion.

The method that Kepler employed was precisely the method of infinitesimal elements of integral calculus; in modern terminology, this is to take the sum of infinitely many infinitely small elements to determine the area contained in a curve or the volume contained in a surface of revolution. A contemporary Italian mathematician Bonaventura Cavalieri (1598–1647), who was a disciple of Galileo but more committed to pure mathematics, devoted his life to the study of indivisible elements, another precursor to infinitesimal calculus, according to which lines, surfaces, and solids, respectively, are considered as composed of infinitely many surfaces, lines, and planes. Using this method, Cavalieri was able to calculate the definite integral of the power function x^n subject to the constraint that n is a positive integer. The British mathematician John Wallis (1616–1703) considered the more general power function $x^{p/q}$, but only managed to resolve the case $p = 1$. John Wallis was the most direct predecessor of Isaac Newton in terms at least of chronology.

In another direction, tracing backward the roots of differential calculus, we cite also the works of three different predecessors: Descartes, Fermat, and Isaac Barrow (1630–1677), who was Newton's teacher. Descartes and Barrow had endeavored to calculate the tangent line to a generic curve in the plane, using, respectively, an algebraic method known as the circle method or the method of normals and a geometric method making use of the so-called differential triangle. Fermat meanwhile used the nascent methods of differential calculus to determine the extreme values of a function, except for a difference in sign. He realized in fact that it was also possible to obtain tangent lines with this method but mentioned it only in passing in a letter to Mersenne, accompanied by the remark that he would discuss it on another occasion. All things considered, Fermat came closest of the various mathematicians discussed above to success, but it remained to Newton and Leibniz to complete the work.

Newton and Leibniz

As we have seen in the previous section, the seventeenth century brought with it a host of new scientific problems that were closely related to the development of calculus. For example, tangent lines to a curve can be used to determine not only the direction of motion of a moving body at a given moment but also the angle of refraction formed between a ray of light entering a lens and the normal line of the lens; the extreme values of a function can be used to determine the launch angle of a projectile such that it achieves its maximum range and also to determine the closest and furthest distances between a planet and the sun. There was also the basic problem of dynamics: given the distance travelled by a moving body as a position of time, to calculate its velocity and acceleration at any moment. It was above all this uncomplicated problem and its inverse that prompted Newton to the creation of the calculus (Fig. 5.26).

Newton established his formulation of calculus using what he called the method of fluxions, which concept started brewing in his thoughts during his time as a student at Cambridge and burst forth in full maturity during 2 years he spent in his hometown in Lincolnshire during the years of the plague. According to his own accounting, Newton invented the fluxion calculus (differential calculus) in

Fig. 5.26 Newton's apple tree; photograph by the author, Cambridge

November of 1665 and the inverse fluxion calculus (integral calculus) in May of the following year. It follows that Newton, in contrast with all his colleagues who were working toward the calculus earlier, considered and resolved the two topics of differential calculus and integral calculus as inverse operations, as did his contemporary competitor Leibniz. It is interesting to note that Newton indicates in his *Waste Book* that although he had studied under Isaac Barrow at Cambridge, he had been more deeply influenced by the work of John Wallis, who taught at Oxford, and that of Descartes; rather, it was Leibniz who absorbed the teachings of Barrow, during his time in Paris. Barrow himself also proved a the fundamental theorem of calculus in geometric formulation several years later, in his 1670 treatise the *Lectiones Geometricae.*

In the year 1669, upon his return to Cambridge, Newton distributed to his colleagues a mathematical work entitled *De analysi per aequationes numero terminorum infinitas*[3] (*On Analysis by Equations with an Infinite Number of Terms*), having previously made public some similar considerations from a kinematic

[3] At that time, Latin was the universal language of academia, and Newton composed his major works in Latin.

perspective. In this paper, Newton considered a curve y such that the area beneath it is given by the equation

$$z = ax^n$$

where n is an integer or rational number. An infinitesimal increment in x is written as o, and the area enclosed by the x-axis, the y-axis, the curve, and the ordinate at $x + o$ is represented by $z + oy$, where oy is the incremental area:

$$z + oy = a(x + o)^n.$$

Making use of his own generalization of the binomial theorem, the right-hand side of this equation is written as an infinite series; subtracting it from the previous equation, dividing each side of the equation by o, and omitting any terms in which the factor o still occurs give

$$y = nax^{n-1}.$$

In the language of modern mathematics, the rate of change of the area under the curve at any point x is the value of y at x. Conversely, if the curve $y = nax^{n-1}$ is given, then the area underneath it is given by $z = ax^n$. This is the basic prototype of the differential and integral calculus. Two years later, Newton presented a fuller account in a book entitled *Method of Fluxions*; in his terminology, a variable was called a *fluent*, and its rate of change the *fluxion*, from which he derived the name for his method (Fig. 5.27).

In the same period, Newton put his calculus of fluxions and inverse fluxions to work in the calculation of tangent lines, curvatures, inflection points, arc lengths, the force of gravity, centers of mass, and so on. But like Fermat, he was reluctant to publish his results: the first of the two treatises discussed in the previous paragraph was published only inn 1711, after much urging by his colleagues, and the *Method of Fluxions* only in 1736, after his death. In his landmark work *Philosophiæ Naturalis Principia Mathematica*, which was published earlier in 1687, Newton cloaked his calculus in geometric costume, and its significance was not fully recognized straight away. This book nevertheless quickly earned its reputation as the greatest scientific work of modern times, on the basis of the establishment of the law of universal gravitation and a strict mathematical derivation of Kepler's three laws of planetary motion; this was easily enough to ensure its immortality (Fig. 5.28).

In contrast with Newton, Gottfried Wilhelm Leibniz (1646–1716) published his results in calculus earlier, in 1684 and 1686, although he arrived at their invention later; this sparked a protracted and bitter debate over priority. Leibniz also proceeded from a geometric rather than kinematic point of view. Specifically, he took his first inspiration from a paper by Pascal on circles that he read in 1673: consider as in the figure a characteristic small triangle with hypotenuse parallel to the tangent line

Fig. 5.27 Statue of Isaac Newton at Trinity College Chapel; photograph by the author, Cambridge

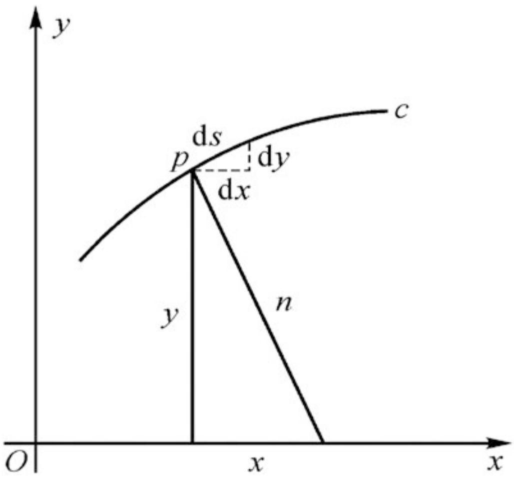


Fig. 5.28 Leibniz formulation of the principles of differential calculus

at any point P on a curve C; then from the proportional relationship between the sidelengths of similar triangles,

$$\frac{ds}{n} = \frac{dx}{y},$$

where n represents the normal line to the curve C at P. Taking the sum,

$$\int yds = \int ndx.$$

This result however was expressed rather vaguely, in words rather than in mathematical notation, and it was 4 years later that Leibniz explicitly stated the fundamental theorem of calculus in a manuscript.

On the other hand, as early as 1666, Leibniz had considered in a published paper entitled *De Arte Combinatoria* (*On the Combinatorial Art*) the first-order and second-order differences of the square sequence

$0, 1, 4, 9, 16, 25, 36, \ldots,$

which are

$1, 3, 5, 7, 9, 11, \ldots$

and

$2, 2, 2, 2, 2, 2, \ldots$

respectively. He noticed that the original sequence is obtained by taking successive sums of the first terms in the sequence of first-order differences, indicating the inverse relationship between summation and difference. It was this that led him to the relationship in calculus between differentiation and integration. In the notation of the Cartesian coordinate system, he wrote the ordinates of an infinite sequence of points on a curve as y and the corresponding abscissas as x. If the ordinates are given in terms of x and the sequence of differences between any consecutive values of y is considered, Leibniz was thrilled to discover that the derivative is simply a kind of difference and the integral a sum.

From this observation, although his progress was not altogether smooth, Leibniz gradually arrived from the notion of the discrete difference to consider the increments of any arbitrary function. In 1675, he introduced the important symbol $\int$ to represent the integral, and in the following year, he obtained the derivative and integral formulas for the power function. As for the fundamental theorem of calculus, it can be stated in modern terminology as follows: in order to find the area under a curve whose ordinate is y, it is only necessary to find a curve with ordinate z such that the slope $\frac{dz}{dx}$ of its tangent line is given by the rule $\frac{dz}{dx} = y$. If the interval

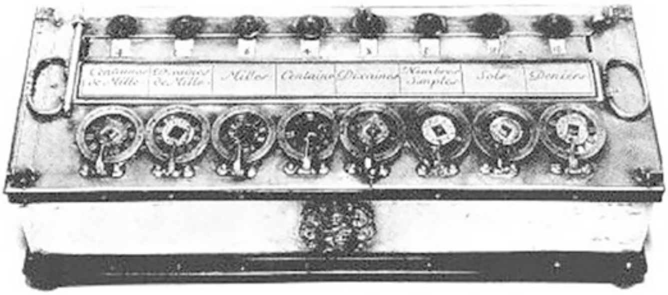


Fig. 5.29 Pascal's calculator

under consideration is $[a, b]$, subtract the area on the interval $[0, a]$ from the area on the interval $[0, b]$ to obtain

$$\int_a^b y dx = z(b) - z(a).$$

This is also known as the Newton-Leibniz formula.

It is interesting and unusual that Leibniz developed his enthusiasm for mathematics initially for reasons of political ambition. At that time, Germany was in a divided state of separate feudal rule, not unlike the situation in the Spring and Autumn and Warring States periods in China more than two millennia prior. During one summer, Leibniz met the former chief minister of the Elector of Mainz.[4] Although he had at that time been dismissed from his post, the erudite former chief minister retained his connection to the Elector of Mainz, to whom he recommended the learned and entertaining young Leibniz for a position as an assistant.

France had become a major power center in Europe by that time, the peak of the rule of the Sun King Louis XIV, and was prone to attack its neighbors to the north at any time. As an assistant to the legal advisor of the Elector, Leibniz proposed a brilliant strategy to distract the French king with the prospect of conquering Egypt. Leibniz was sent at the age of 26 as a diplomatic to Paris where he spent 4 years. Although Descartes, Pascal, and Fermat had already passed away by that time, Leibniz came into contact during his time in Paris with the Dutch mathematician Christiaan Huygens (1629–1695), the inventor of the pendulum clock and the wave theory of light (Fig. 5.29).

Leibniz soon realized the limitations of his mathematical education in technologically regressive Germany, and he applied himself with humility and diligence to his studies under the careful guidance of Huygens. Due to his persistence and talent, and the still incompletely developed mathematical foundations of that era, Leibniz had already made major mathematical discoveries by the time he left Paris,

[4] Historically, the Archbishops of Mainz were the most important of the electors of the Holy Roman Emperor; it was also in Mainz that Gutenberg invented his movable type printing press.

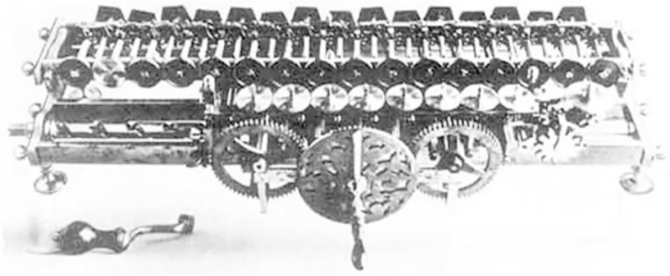


Fig. 5.30 Leibniz's calculating machine, capable of multiplication

although his original plans for intrigue involving France and Egypt had already been shelved. During this time, he first introduced the binary system and subsequently made improvements to Pascal's mechanical calculator, inventing the first calculating machine capable of multiplication, division, and the operation of squaring a number. Of course, his most important contribution was his work with infinitesimals, that is, the invention of the calculus (Fig. 5.30).

This was indeed an epochal contribution to the history of science, and it was precisely because of this innovation that mathematics came to play an outsize role in the natural sciences and social life. It also created a space for thousands of careers in mathematics in subsequent generations, not unlike the role played by the invention of the computer in the twentieth century. In addition, Leibniz also created the elegant theory of determinants and extended the binomial theorem to any number of variables with a beautifully symmetric formulation. Perhaps the most aesthetic of his results for the layman is the infinite series expression for π that he discovered during a visit to London in 1673 at the age of 27:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

In fact, this formula had been independently discovered and forgotten at least twice earlier: by the Scottish mathematician and astronomer James Gregory (1638–1675) and in South India in the fourteenth century by Madhava of Sangamagrama. Madhava derived it from the power series expansion of the arctangent function and made use of his work to calculate π to 13 decimal places. In 1424, 1 year before the death of Madhava, the Persian mathematician Jamshid al-Kashi had used an ancient technique to approximate π to 17 decimal places. Madhava and his followers also obtained power series expansions for the sine and cosine functions, as well as the Taylor series expansions of various functions. Their work was recorded by an astronomer of the Kerala school in the 1530 in a treatise entitled *Yuktibhasa*, meaning *Rationale* in the Malayalam language. This treatise consists of seven chapters, the last of which includes the results just mentioned. It was first published as a book in modern India in 1948, and a critical edition was published with English

Fig. 5.31 Newton's rival, Leibniz

Fig. 5.32 Leibniz's grave; photograph by the author, in Hanover

alongside the original by Springer in 2008. For these reasons, the formula above is sometimes referred to as the Madhava-Gregory-Leibniz formula (Fig. 5.31).

Not long after Leibniz returned from Paris, his patron passed away; his repeated applications to the French Academy of Science as a foreign honorary member were rejected, and he was forced to earn his living as a tutor. In October of 1676, at the age of 30, Leibniz accepted the invitation from the Duke of Brunswick to travel north to Hanover to serve as a legal advisor and librarian. Leibniz continued however to devote himself to mathematics, philosophy, and science with remarkable results and became an honored guest in many of the royal houses of Europe (Fig. 5.32).

I would like to close this section with a discussion of mathematical inheritance, in a sense broader than that of the relationship between mentor and student; rather, I mean something closer to intellectual telepathy or synchronicity. Just as later Euler carefully learned from that mathematical legacy of Fermat, Leibniz developed a particular affinity for the work of Pascal. His original inspiration for the invention of the calculus came from his familiarity with the characteristic triangle invented by Pascal, and his mechanical calculator was also an improvement upon one of the latter's inventions. Pascal is also famous for his work with the binomial coefficients, known familiarly as Pascal's triangle, and Leibniz extended its scope to expansions in any number of variables. In philosophy and the humanities more generally, Leibniz also walked a path paved by the footsteps of Pascal, and the two were alike even in never having married.

Conclusion

Starting from the twelfth century, the Europeans learned from China by way of the Arabian Peninsula the art of papermaking from hemp and cotton as a replacement for parchment and papyrus, and in the middle of the fifteenth century, Johannes Gutenberg (ca. 1400–1468) invented his movable type printing press. In short order, a large number of works on mathematics and astronomy appeared in print. As we have discussed in the previous chapter, the scholarly works of Ancient Greece were translated into Latin by way of the Arabic translations in which they had survived and in this way reappeared in Europe. In 1482, the first Latin edition of Euclid's *Elements* was published in Venice. During this time, the compass and gunpowder were also introduced to Europe from China, the former facilitating voyages across the seas and the latter changing the nature of warfare and the structure and design of military fortifications. In particular, the study of ballistics became important.

As Greek texts began to proliferate across Europe, certain concepts associated in the popular imagination with Ancient Greece also experienced a revival, especially in Italy, including an emphasis on the exploration of nature, admiration for and dependence upon reason, enjoyment of the material world, the pursuit of physical and intellectual perfection, desire and freedom of expression, and so forth. Artists were the first to embody these principles through their love of nature and commitment to the Greek doctrine that mathematics is the essence of nature. They learned their mathematics through practice, in particular geometry, and this led to the rise of such Renaissance figures as Alberti and Leonardo da Vinci. Alberti also contributed directly to the birth of projective geometry as a branch of mathematics through his interest in perspective.

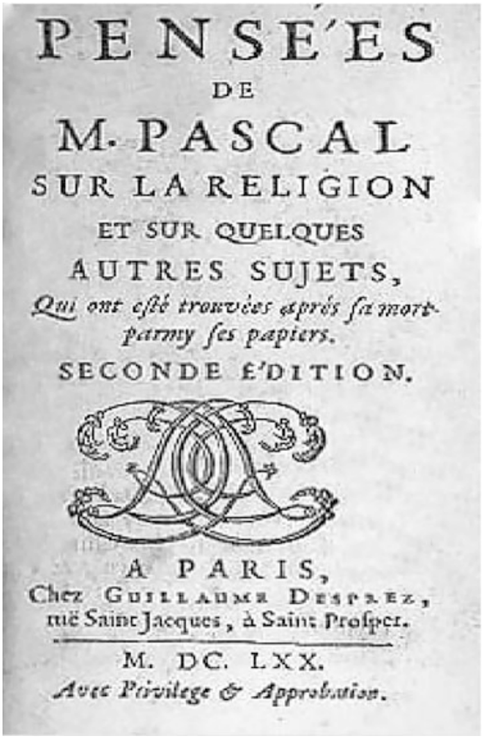
The natural sciences were also increasingly dominated throughout this period by deductive reasoning, which led them to become more mathematical in nature and to an increase in the importance of mathematical terminology, methods, and results. The integration of mathematics and the sciences also fostered an acceleration in their development. From Galileo through to Descartes, the prominent thinkers of the age all believed the world is composed of matter in motion and that the purpose of science is to reveal the mathematical laws governing the motion of moving bodies. The finest examples of this movement are the law of universal gravitation and the three laws of motion, all of them due to Newton.

Fig. 5.33 French mathematician Blaise Pascal

The most important invention in mathematics since the appearance of the Euclidean geometry was the calculus, and as such, it emerged from the background of a rich social tapestry. Most directly, the calculus was designed expressly to handle the major scientific problems of the seventeenth century, in physics, astronomy, optics, and military science alike. But also it met the needs for internal development in pure mathematics, posed by such problems as the determination of the tangent lines to given curves. And its path was paved by the advent of analytic geometry, which introduced the notion of a variable into mathematics and allowed for the quantitative representations of change and motion (Fig. 5.33).

The history of great mathematics is also the history of great mathematicians, and the seventeenth century in particular has been dubbed the century of genius by the British philosopher Alfred North Whitehead. It is no exaggeration to say that the seventeenth century played a crucial role in the developmental history of human civilization, and this was in no small part due to the expansion in the scope and depth of mathematics exemplified by the birth of analytic geometry and calculus. It was also during this time that philosophy and mathematics reentwined in the works of such great thinkers as Descartes, Pascal, and Leibniz, after a long period of separation since the decline of Ancient Greece. All in all, a glorious chapter in the book of history (Fig. 5.34).

I have not yet discussed the upbringings of the two French mathematicians René Descartes (1596–1650) and Blaise Pascal (1623–1662). They were both born in the provinces (as was Fermat), lost their mothers in childhood, and were known to be frail as children; moreover, both of them had fathers who provided for them a good education, and they both came to an interest in mathematics spontaneously. At the age of 12, Pascal discovered on his own without any relevant training the theorem in geometry that the three interior angles of a triangle sum to two right angles; it


PENSÉES
DE
M. PASCAL
SUR LA RELIGION
ET SUR QUELQUES
AUTRES SUJETS,
Qui ont esté trouvées aprés sa mort parmy ses papiers.
SECONDE EDITION.

A PARIS,
Chez GUILLAUME DESPREZ,
ruë Saint Jacques, à Saint Prosper.
M. DC. LXX.
Avec Privilege & Approbation.

Fig. 5.34 *Pensées* by Pascal

was only later that his father, who was himself an amateur mathematician, began to give Pascal lessons in Euclidean geometry. Descartes for his part developed an interest in mathematics when he saw the solutions to mathematical problems written on the barracks blackboard of the military camp during his time as a soldier in the Netherlands.

Despite their considerable achievements in the discovery of new results in mathematics and science, Descartes and Pascal both resisted the honors associated with their work and preferred to direct their scientific interests toward the spiritual world. Descartes composed the important philosophical texts *Discourse on Method*, *The World*, *Meditations on First Philosophy*, and *Principles of Philosophy*. Pascal left behind his *Lettres provinciales* and *Pensées*. Of the two of them, Descartes was the more committed to abstract metaphysics, even perhaps to the point of indulgence; this was probably in response to the trial and conviction of Galileo, punished as one might suppose for having grounded his doctrine too much in reality. This made for remarkable philosophy, but somewhat less successful science. Pascal, who led a lonely life of deep but terrifying piety, composed aphoristic works of intense feeling and spirituality, marking a fascinating chapter in the history of French literature, indeed of world literature.

In philosophy, Descartes is regarded as the liberator of philosophical thinking from the shackles of the scholastic tradition, and later generations have referred to him as the father of modern philosophy. He is famous for his philosophy of dualism, promoting a stark division between the mind and the body; this has been often encapsulated by reference to his famous dictum *cogito ergo sum*, or *I think, therefore I am*, one of the most powerful and well-known propositions in the history of philosophy. This was in contrast to the philosophy of the Greeks, including Pythagoras, who tended to believe that all the phenomena of the world were composed of a single substance. Pascal was a thinker more grounded in human reality: he understood early and all too well the limitations of human faculties, our frailties and faults. His work in mathematics contributed an awe bordering on terror for the concepts of the infinitely large and infinitely small, and his mathematical discoveries as a result were also confined to a limited space.

It was worth also discussing here a bit the relationship of Pascal's triangle to mathematical induction. We have seen already that various interesting properties of the triangle of binomial coefficients were known to Chinese, Indian, and Persian mathematicians many centuries before the life of Pascal. But it was Pascal who first made use of mathematical induction to give rigorous proofs that (e.g.) the sum of the k-th and $k+1$-th elements in the n-th row is equal to the $k+1$-th element of the $n+1$-th row. In fact, this is perhaps the first explicit and clear formulation and use of mathematical induction in the history of mathematics, although its prototype can be traced back to the proof in Euclid's *Elements* that there exist infinitely many prime numbers. Since that time, mathematical induction has become a basic tool in the arsenal of mathematics, used to prove all manner of propositions about infinite sets of numbers, and in particular the positive integers. It provides an effective mean to prove infinite results from finite hypotheses. The name *mathematical induction* was coined in the nineteenth century by the British mathematician and philosopher Augustus de Morgan.

Descartes and Pascal are both giants in this history of human thought, in both the sciences and the humanities. It is probably in no small part due to their influence that mathematics became such an integral part of the traditional intellectual culture of the French people, perhaps indeed its most excellent aspect. French mathematics has prospered and proliferated since the seventeenth century, with great masters emerging one after another. In typical French fashion, their mathematical geniuses have accepted the honors accrued to them without ever having viewed mathematics as a mere stepping stone. Since the establishment of the Fields Medal in 1936, 11 French mathematicians have been awarded this highest honor, second only in number to the 13 Fields Medalists of the United States.

It was also because of his encounter with French mathematics and the intellectual atmosphere of France that Leibniz turned to mathematics during his stay in Paris and eventually developed as a thinker to such an extent that Bertrand Russell later said of him that "Leibniz was one of the supreme intellects of all time." In addition to having invented the calculus simultaneously to Newton, Leibniz propounded an influential philosophy which he referred to as *monadology*. The central tenet of this philosophy was that the universe is composed of infinitely many *windowless monads*, each resembling the soul to varying degrees. These monads are the ultimate, inextensible, spiritual essence at the foundation of all things. In particular, this implies that humans differ from animals and indeed living things from inanimate objects only as a matter of degree, and indeed, Leibniz pointed out in support of this that many of our thoughts and behaviors occur only at the trigger of subconscious impulses, which doctrine brings closer together than had ever been previously suggested the spheres of human and animal behavior. He derived also from this philosophy his belief that all things are interconnected and in particular that any singular entity is inseparable from its connection with every other entity in existence.