Chapter 4
India and Arabia

One might write a history of India coming down to four hundred years ago and hardly mention the sea.

H.G. Wells

In the Rubáiyát we read that the history of the universe is a spectacle that God conceives, stages, and watches

Jorge Luis Borges

From the Indus River to the Ganges

The Indo-European Past

About 4000 years ago, at the time when the Egyptians, Babylonians, and Chinese were developing their river valley civilizations after their separate fashions, a nomadic people speaking a language that has since been classified as Indo-European made a long journey from Central Asia across the Gangdisi Mountains in the Transhimalaya system and settled in what is now northern India. These people later referred to themselves as Aryans, a word derived from Sanskrit and meaning *noble* or *landowner*. Some of them also travelled westward and became the ancestors of the Iranian people and some European peoples. It was believed in the middle nineteenth century and early twentieth century that the most purely Aryan people are found in Germany and the Nordic countries, a fallacy that saw widespread use by Hitler and his followers in the 1930s and 1940s to justify their theory of the noble race (Fig. 4.1).

Prior to the arrival of the Aryans, there were already indigenous people living in India, known as the Dravidian peoples. The history of these peoples can be traced back at least a further thousand years, when they are believed to have spread across the Indus River basins from western Pakistan. In modern times, about a quarter of the population of India still speaks languages belonging to the Dravidian language family, and four such languages, including Tamil and Telugu, are among the many official languages of India. Unfortunately the hieroglyphs used by the early

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Fig. 4.1 Bathing pilgrimage festival at Allahabad

Dravidians are as difficult to decipher as the oracle bone inscriptions of ancient China, and very little is known today about the Indus valley civilization, including the state of their mathematical knowledge.

After they had gained a foothold in northwestern India, the Aryan people continued to advance eastward, crossing the Indo-Gangetic Plain and eventually reaching the region known today as the state of Bihar, with a population of more than 100 million and twice the population density of Japan. They conquered the Dravidian people and made the northern regions into the cultural core of India. The major religions of India, including the predecessor Vedicism of Hinduism, Jainism, Buddhism, and much later Sikhism, were all born here, and the Aryan influence gradually spread throughout India. Sometime in the first millennium after their arrival, written and spoken Sanskrit were developed, as well as the Vedic religion, the oldest documented religion in India. It could be said that the Vedic religion and the Sanskrit language are the roots of the culture of ancient India.

The historical Vedic religion was a kind of polytheism with a heavy emphasis on ritual, in particular the ritual worship of various male deities associated with the sky and other natural phenomena, very different from the later tradition of Hinduism. The Vedic rituals involved animal sacrifice and the consumption of a sacred drink called soma, derived from a plant the identity of which is now unknown. This drink, which was extracted from the stems, pressed through wool, and mixed with water or milk, was cherished by adherents of the Vedic religion for its effects as a stimulant and even a hallucinogen. As for the purposes of the sacrifices, they sprang from the belief that the gods would repay them with material offerings such as plentiful livestock, good fortune, health, longevity, and male descendants. However the abundance of cumbersome rituals and exacting precepts may have contributed to the gradual decline of Vedicism.

The Vedic religion takes its name from its holy texts, the *Vedas*, transmitted and written down over a period of about a thousand years lasting from the fifteenth century to the fifth century BCE. The original meaning of the word *veda* (वेदः) in the Vedic Sanskrit in which they were written is *knowledge* or *light*. These books present sacrificial speeches in prose and poetry, as well as commentary, exegesis, and various philosophical ruminations. These texts also introduce the division of Indian society into four castes: Brahmins (priests), Kshatriyas (warriors), Vaishyas (skilled traders, merchants), and Shudras (unskilled workers or slaves), a division which persisted through later Hinduism (Fig. 4.2).

The exegetic materials appear primarily in the *Brahmanas*, the *Aranyakas*, and the *Upanishads*. The *Brahmanas* contain explanations of ritual procedure, the *Aranyakas* discuss the meaning of ritual sacrifice and the worship of heaven, and the *Upanishads* discuss the relation between individual souls and cosmic reality, the obliteration of ignorance, and concomitant the liberation from attachments to

Fig. 4.2 Cover page of a Chinese edition of the *Upanishads*

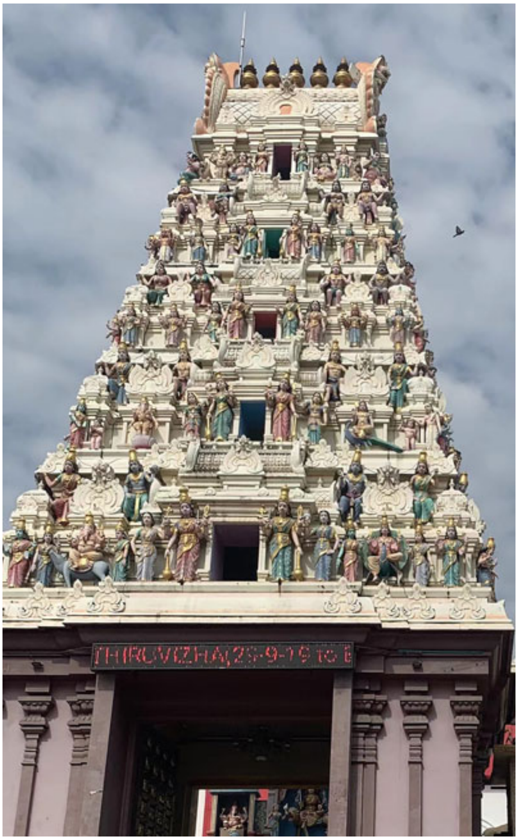

Fig. 4.3 A typical Hindu temple, with the form of an isosceles trapezoid; photograph by the author, in Malaysia

the material world, secular temptations, and physical ego. These texts comprise the sacred *heard text* (*shruti*) tradition; there is also a tradition of *remembered text* (*smriti*), with the famous *Bhagavad Gita* as the most representative example. Among the doctrines of this book, it is remembered for the teaching that tranquility is achieved through yogic practice (Fig. 4.3).

The *Vedas* were originally transmitted orally and later recorded on palm leaves or bark. Although many of the original texts have been lost, it is fortunate from the perspective of the history of mathematics that the *Shulba Sutras* are among those that have survived. The word *shulba* means *string*, *cord*, or *rope*, and the *Shulba Sutras* discuss the design and measurement of altars and temples. This is the earliest mathematical literature in the history of India, excluding a few fragments of mathematical notation appearing on coins and inscriptions. The mathematical content of the *Shulba Sutras* includes some geometric figures and algebraic calculations related to altar design, including an application of the Pythagorean theorem and a general treatment of the diagonal of a rectangular figure, some discussion of similar plane figures, and some basic instructions for the construction of geometric diagrams.

The most essential features of the *Shulba Sutras* are the method of measurement by drawn ropes and basic area calculations.

The Shulba Sutras and Buddhism

The *Shulba Sutras* were written sometime between the eighth century BCE and second century CE, no later than the two classical Indian epics *Mahābhārata* and *Rāmāyaṇa*. Four major *Shulba Sutras* have survived, each named individually after its author or the school of thought represented by its author. These books contain detailed instructions for the construction of fire altars, including in particular the appropriate shapes and sizes. The three most commonly used shapes are square, circle, and semicircle, but an important regulation is that the altar should have a fixed area regardless of its shape. For these reason, ancient Indian mathematicians had to have learned or already known how to construct a circle with the same area as a given square or twice the area of a given square. Another common shape is the isosceles trapezoid, and even more exotic geometric figures appear, all constrained to a given area. This raises a host of new and interesting problems in plane geometry.

The design of the altar according to a prescribed shape requires some basic geometric knowledge and results, for example, the Pythagorean theorem. The Indian mathematicians give a unique statement of this theorem:

> The areas produced separately by the lengths of the breadths of a rectangle taken together equal the area produced by its diagonal.

This is very obviously different in character from the solar height presentation in the *Zhoubi Suanjing*. More generally, the Indian mathematics of this period is a scattered assemblage of approximate laws for the calculation of areas and volumes, expressed in words rather than in notation. A few of these laws included deductive proofs, while most of them were purely empirical.

As an example, the construction of a circular area with twice the area of a given square (say, for a semicircular altar) requires an approximation of π. The *Shulba Sutras* record the following approximate value, which we present in an equivalent formulation with unit fractions:

$$\pi \approx 4\left(1 - \frac{1}{8} + \frac{1}{8 \cdot 29} - \frac{1}{8 \cdot 29 \cdot 6} - \frac{1}{8 \cdot 29 \cdot 6 \cdot 8}\right) \approx 3.0883.$$

The approximations $\pi \approx 3.004$ and $\pi \approx 4 \times \left(\frac{8}{9}\right)^2 \approx 3.16049$ seem also to have been in use. Similarly, the construction of a square altar with area 2 requires a value for $\sqrt{2}$. The *Shulba Sutras* give

$$\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34} \approx 1.414215686,$$

Fig. 4.4 Statue of a reclining Buddha in the Ajanta Caves

accurate to the fifth decimal place. It is interesting that both of these approximations can be recognized as unit fractions, suggesting perhaps that such fractions also held significance for Indian mathematicians as they did for Egyptian mathematicians, whether by coincidence or as the product of cultural exchange (Fig. 4.4).

The founding promulgator of Jainism in India, Mahavira, was born in about 599 BCE Bihar, not far from where the founder Gautama Buddha of Buddhism was born some 36 years later. The parallels between the lives of these two pivotal figures are manifold. Both grew up in an environment of prosperity and material abundance, which later they renounced along with all family and property at about the age of 30, when both left their homes to lead perambulatory lives in pursuit of truth. One minor difference is Gautama Buddha is said to have left behind a wife and infant son in course of his renunciation and Mahavira is said to have had a daughter instead. In any case, Buddhism and Jainism arose in India at almost exactly the same time and probably due to a common cause: opposition to the bureaucratic fuss of Vedicism and the elitism of the Brahman caste in the caste system.

The original Sanskrit word *jain* derives from the base *ji* (जि) , meaning *victor* or *to conquer*. Jainism is a religion without any central creative deity; rather both time and the universe are held to be eternal and endless, with all things divided into the categories of souls and nonsouls. The foundational texts of Jainism cover a wide range of intellectual territory. In addition to expounding and clarifying the basic doctrine, they also include important contributions to literature, drama, art, architecture, and other topics, as well as some basic principles and conclusions in mathematics and astronomy. Many Jain texts were written between the fifth century BCE and second century CE in the vernacular Prakrit language; the word *prakrit* means *natural*, distinguished it from the classical *sanskrit* language, meaning *refined*. These texts include some mathematical results, including approximations for the circumference of a circle $C = \sqrt{10}d$, for an arc length $s = \sqrt{a^2 + 6h^2}$, and other approximation formulas of a similar flavor (Fig. 4.5).

In contrast, Buddhist doctrine holds that all things are impermanent and transient, subject to constant change both within and without the human experience. Such things as the area of an altar are impossible to state with fixed specificity. Buddhism

Fig. 4.5 The pattern on the altar

accepted all people without regard for caste and does not recognize any essential differences between different people. Buddhism is more of a philosophical concept than a precise religious doctrine than either Jainism or Hinduism, especially as it is practiced in India. The concept of time in Buddhism is also very unique and somewhat of a mathematical character. For example, perhaps because India has three seasons (rainy season, hot season, and dry season), the Buddhist scriptures also divide the days and the nights each into three parts: early day, midday, and late day and similarly early night, midnight, and late night. As for the years, 100 years is one lifespan, 500 years is an evolution, 1000 years is a turn, and 12,000 years is an epoch.

More interesting is the subdivision of time. The shortest period of time is the single moment (kṣāṇa), the time between 2 thoughts, and 60 of these make up the time of a finger snap. From this we get also the saying "the sixty-three moments of youth is a snap of a finger." However, the true duration of a moment cannot be known except to the Buddha. It is as in the following poem:

> We see the fullness of the moon and understand that time is always moving
> We understand the birth and death of our minds, we know the shortness of time.

At the same time that Buddhism and Jainism emerged in India in the sixth century BCE, the concepts of the reincarnation of the soul, karma, and the transcendence of reincarnation through meditation became widespread amongst the Vedics. This became what is now known as the Hindu religion.

Since that time, this reform religion involving almost every aspect of life has gradually come to dominate almost the entire Indian continent, and its philosophy and moral code have even spread into the beliefs, customs, and social religious systems of many ethnic groups in South Asia, including the Nepalese and Sri Lankans. This is in contrast with Buddhism, which developed a broad influence in Southeast Asia, but in India comprised mainly a philosophical system, and with Jainism, the sphere of influence of which remained restricted primarily to some western and northern Indian states. Around this time, mathematics broke free of its connection with religion and developed independently instead as a powerful astronomical tool (Fig. 4.6).

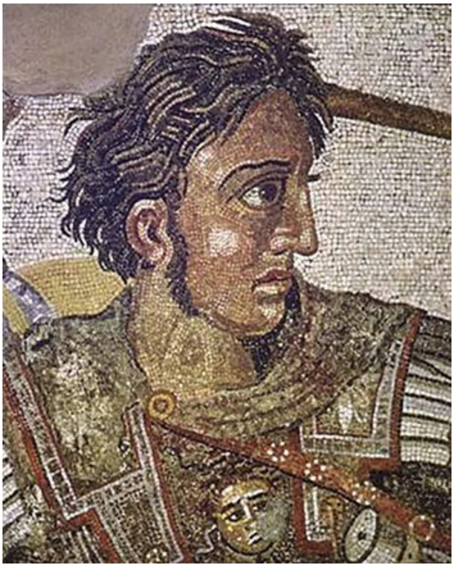

Fig. 4.6 Alexander the Great, whose expeditions bridged the East and the West

The Number Zero and Hindu Numerals

By the middle of the fifth century BCE, the people of the Magadha region, encompassing Bihar, had expanded the boundaries of their territory to include the entire Indo-Gangetic Plain. This laid the foundations for a period of prosperity under the Maurya Empire (322 BCE–185 BCE), which reach its peak under the rule of the emperor Ashoka the Great, often remembered as the greatest monarch in the history of India. He devoted his life and his rule to the promotion and dissemination of Buddhism and served a role in that religion similar to that of Paul the apostle in Christianity, facilitating its eventual spread around the world. His grandfather Chandragupta Maurya had been the founder of the Maurya Empire, who expelled the forces left behind by Alexander the Great during his campaign in India and around the same time or slightly later conquered and unified northern India. This established the first empire in Indian history. This campaign by Alexander was in itself a remarkable expedition, forming a bridge so to speak between Greece in the west and India in the east. They reached the southern shores of the Caspian Sea and continued to march eastward and laid the foundations for the modern cities of Kandahar and Herāt in Afghanistan, before heading north into Samarkand in Central Asia. He did not occupy this territory but rather used it to send his troops southward again through the crevices of the Hindu Kush mountain range and enter India via the Khyber Pass east of Kabul. His original intent was to continue further and further to the east, crossing the desert to the Ganges river basin. Years of fighting had exhausted his soldiers, however, and in 325 BCE Alexander withdrew with his troops from the Indus valley and returned to Persia. He left behind an army and a governor in Punjab, later driven away by Chandragupta Maurya.

Despite its failure, this short campaign forged indelible traces, including the facilitation of future trade and exchange between India and Greece. It is believed that by Roman times Alexandrian merchants had established many settlements throughout South India and even built a temple to Augustus in Muziris. These settlements were usually guarded by two teams of Roman troops, and at this time, the Roman emperor also sent envoys to South India. Moreover, this encounter with Greek civilization certainly had an influence in India on the development of mathematics and the other sciences. A fifth-century Indian astronomer observed that although the Greeks were impure from the perspective of Indian religious belief, nevertheless they ought to be revered for their training in the sciences, in which direction they surpass all other peoples.

In the summer of 1881, in the small village of Bakhshali (near Peshawan in modern Pakistan, but a part of India for most of its history), a local peasant unearthed a birch bark manuscript containing written text. This is the Bakhshali document, which records a rich compilation of Jain mathematics, including treatments of fractions, squares, sequences, proportions, problems involving revenue, expenditure, and the calculation of profits, series summations, algebraic equations, and so forth. This manuscript makes use of a sign for negative numbers, identical visually to the modern addition sign +, but placed to the right of a numeral rather than to its left. And even more consequentially, this manuscript includes numerals written in complete decimal notation, with a solid dot to represent the number zero.

This solid dot evolved over time into a circle and finally into the modern symbol 0. The earliest verified evidence of the modern notation appears in an engraving at a temple in Gwalior date to the year 876. The city of Gwalior is in northern India, in its most densely populated state of Madhya Pradesh, adjacent to Bihar in the Ganges river basin. This engraving is on a stele in a garden and supposed to indicate the number of wreaths or corollas to be supplied daily to local temples. Although the two occurrences of the numeral 0 are small, they are clear and unmistakable (Fig. 4.7).

The use of a circle to indicate zero and more broadly the introduction of the concept of zero as an independent number are a great contribution to mathematics from India. The use of zero as a placeholder in decimal notation had predecessors in early Babylonian cuneiform and in Chinese arithmetic prior to the Song and Yuan dynasties, where it was indicated by an empty space, and with a specific symbol in later Babylonian texts and in Mayan mathematics (which made use of a shell or eye symbol), but it was not regarded as in India as an independent number with a role to play in calculations. Perhaps there is some connection here with religious culture: in Indian religions the idea that the universe was born from nothing was widespread. The Indian mathematicians were also comfortable with the distinction between positive numbers to represent property and negative numbers to represent debts.

The numerals engraved at the Gwalior temple are also closer in form to the so-called Arabic numerals used throughout the world than the numbers used in the Arabic world. After the eighth century CE, Indian digits and the zero numeral were spread successively first to the Arabic-speaking world and later to Europe. After these notations were accepted and formalized in Europe, having been introduced in the influential book *Liber Abaci* by the thirteenth-century Italian mathematician Fibonacci, they have played an invaluable role in the progress of modern science. Since that time, the history of mathematics in India is also the history of some of the leading mathematicians in the world.

From North India to South India

Aryabhata

In the year 476 CE, on the south bank of the Ganges not far from the historical city Pataliputra, known today as the capital city Patna of Bihar, Aryabhata, the earliest Indian mathematician known by name, was born. This city obtained its modern name when the Afghans invaded and rebuilt it in the sixteenth century. Gautama Buddha also taught here in his later years, and this city was the capital of the two most powerful empires in Indian history: the Maurya Empire and the Gupta Empire

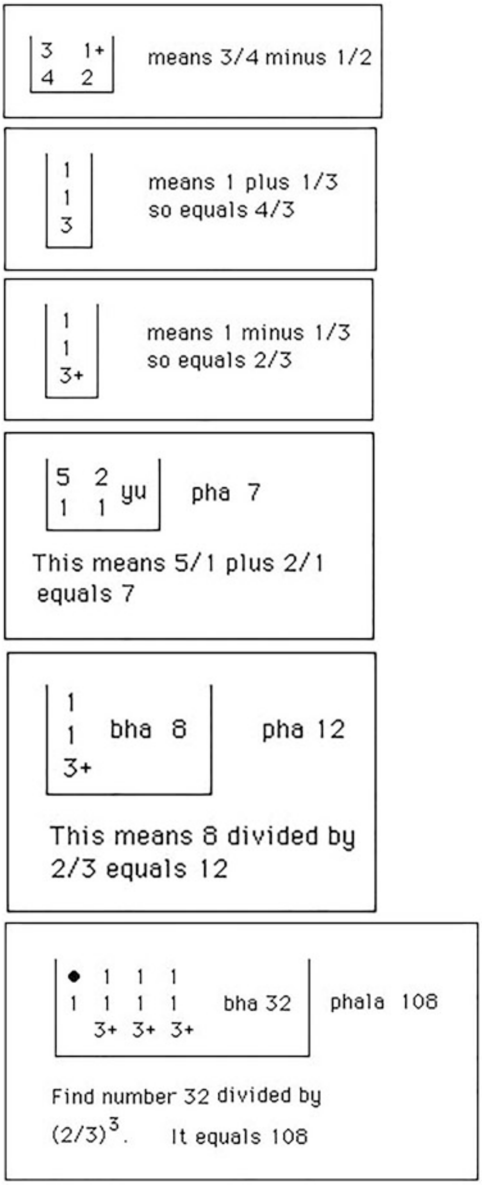

Fig. 4.7 Arithmetic problems from the Bakhshali document

Fig. 4.8 Statue of Aryabhata in Pune

(ca. 320–540), the first empire to unify India during the Middle Ages. The territory of the Gupta Empire included most of the northern, central, and western parts of modern India. It was during this period that decimal notation, Hindu art, and the great Sanskrit epic *Abhijnanashakuntalam* or *The Sign of Shakuntala* and its author the poet Kālidāsa (ca. fifth century) were all born. The Eastern Jin dynasty Buddhist monk Faxian (法显), who travelled by foot from China to India, also came to this city to study during his travels (Fig. 4.8).

By the time of Aryabhata's birth, the capital of the Gupta Empire had moved further west, and the city of Pataliputra had begun to decline from the height of its glory. Nevertheless it was still a significant academic center: indeed, the influential Tang dynasty Buddhist monk Xuanzang (玄奘) visited in around the year 631. As with later Indian mathematicians, Aryabhata studied mathematics mainly for its uses in astronomy and astrology. His two representative works are the *Aryabhatiya* and a book of computations the *Arya-siddhanta*, which has since been lost. The *Aryhabhata* is primarily an astronomical compendium, but it contains significant mathematical content, including arithmetic, a discussion of the measurement of time, spheres, and so on. Around the year 800, this book was translated into Latin and made its way into Europe. Its influence in Indian intellectual history has been huge, especially in South India, and it has been the subject of many commentaries and annotations by subsequent mathematicians.

Aryabhata gives expressions for the sums of squares and cubes of the first n consecutive positive integers:

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$
$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

He was also the first person in India to obtain the valued $\pi \approx 3.1416$, but it is not known by what method he arrived at this approximation, which can be compared with ancient approximations in China. There is some speculation that he calculated the perimeter of an inscribed regular polygon with 384 sides. In trigonometry, Aryabhata is remembered for his sine tables. Earlier such tables had been compiled by Ptolemy in Greece, but Ptolemy used different units of length for circular arcs and the straight radius, an unwieldy system. Aryabhata assumed instead a common measure of length for straight and curved lines and completed his table on this basis from 0° to 90° in 3°45 increments. Aryabhata used the word *ardha-jya* meaning half-chord for *sine*. This was later simplified to *jya* (the bowstring of a hunter), which became *jaib*, meaning *pocket* or *fold*, in Arabic and eventually by way of Arabic *sinus* (meaning *cove* or *bay*) in Latin transliteration. This is the origin of English word *sine*.

In arithmetical work, Aryabhata frequently makes use of trial solutions and working backward from the given conditions. As an example, he states a certain problem as:

> Oh beautiful girl with smiling eyes, can you tell me please, what number multiplied by 3 and then adding again $\frac{3}{4}$ of this product, dividing by 7, subtracting $\frac{1}{3}$ of the result and multiplying it with itself, then subtracting from this 52, taking the square root and adding to it 8, and finally dividing by 10, gives 2?

According to the method of working backward, the solution is found by starting from the solution 2 and inverting the operations as follows:

$$((2 \times 10) - 8)^2 + 52 = 196,$$
$$\sqrt{196} = 14,$$
$$14 \times \frac{3}{2} \times 7 \times \frac{4/7}{3} = 28,$$

which is the required number. We see in this example also that Indian mathematics were prone to express their mathematical work in the language of poetry.

Aryabhata's most significant contribution is the solution of the first-order indefinite equation

$$ax + by = c.$$

He uses a method he calls *kuṭṭaka*, meaning *pulverization*. For example, assume $a > b > 0$ and $c = \gcd(a, b)$ is the greatest common divisor of a, b. Then we find integers $q_1, r_1, \ldots, q_{n+1}, r_n$ such that

$$\begin{aligned} a &= bq_1 + r_1 (0 \le r_1 < b), \\ b &= r_1 q_2 + r_2 (0 \le r_2 < r_1), \\ &\vdots \\ r_{n-2} &= r_{n-1} q_n + r_n (0 \le r_n < r_{n-1}), \\ r_{n-1} &= r_n q_{n+1}. \end{aligned}$$

By iteration, the number $c = \gcd(a, b) = r_n$ can be expressed as a linear combination of a and b, providing integer solutions x and y to the indeterminate equation given above.

In fact, this method was also used later by Qin Jiushao in his *Da Yan Shu* and the positive and negative evolution method, with a rudimentary version appearing earlier in the *Nine Chapters on the Mathematical Art*. In western countries, this method is known as the extended Euclidean algorithm, but the version given by the Greek is incomplete; even the last master of number theory among the Greeks, Diophantus, considered only positive integer solutions to such equations. This restriction was lifted by Aryabhata and his followers.

Aryabhata also made important contributions to astronomy. He used mathematical methods to calculate the circumference of the earth, the movement of the ascending and descending nodes of the ecliptic and lunar path, the latest points and points of slowest motion of certain stars, and even proposed methods for the accurate prediction of solar and lunar eclipses. He also promoted the notion of the rotation of the earth, but this idea was not recognized and carried forward by subsequent generations. In the year 1975, India launched its first man-made satellite, named *Aryabhata*, in honor of his achievements and significance in the history of Indian science.

Brahmagupta

After Aryabhata, it was more than a century for the appearance of another great mathematician: Brahmagupta (ca. 598–ca. 668). It is interesting to note that in fact there do not seem to have been any important mathematicians anywhere in the world during this century. Brahmagupta may have had ancestors from the region of the Sindh province in modern Pakistan, home to its capital and largest city Karachi, but Brahmagupta was probably born in Bhillamāla in Gurjaradesa and spent the better part of his life in Ujjain, a city in the southwestern part of the Indian state of Madhya Pradesh. Alongside its neighboring state of Bihar, these two states formed

Fig. 4.9 Brahmagupta, deep in calculation

the political, cultural, and scientific hub of ancient India, a role similar to that of the Guanzhong Basin and the Central Plains in China (Fig. 4.9).

Although Ujjain was never the capital of a unified empire in India (and indeed India had been divided since the end of the Gupta Empire), it is regarded as one of seven holy cities in India. The Tropic of Cancer passes through the northern suburbs of the city, as does the first meridian set down by Indian geographers. After the decline of Pataliputra, Ujjain was the second city to occupy the center of ancient Indian mathematics and astronomy. It was also the birthplace of the poet and dramaturge Kālidāsa, often considered the greatest writer in the history of India. Since these two cities are separated by nearly a thousand kilometers, with Ujjain closer to Mumbai than Pataliputra, this represented a shift in the center of Indian intellectual activity toward the southwest. It is also believed that Ashoka spent time in Ujjain as a governor before his succession to the throne. Brahmagupta served as the head of the astronomical observatory in Ujjain, one of the oldest and most prestigious observatories in the world prior to the invention of the telescope.

Brahmagupta left behind two important astronomical works: the *Brāhmasphuṭasi ddhānta* or *Correctly Established Doctrine of Brahma*, composed in the year 628, and the *Khaṇḍakhādyaka* (meaning *a morsel*), written around 665 and published after the death of its author. The latter work includes a new sine table, calculated using a different method than that of Aryabhata: the quadratic interpolation method. The *Brāhmasphuṭasiddhānta* contains more extensive mathematical content. This book was written entirely in verse, divided into 24 chapters, with 2 of them devoted entirely to mathematics. These are *Lectures on Arithmetic* and *Lectures on Indefinite Equations*. The former discusses triangles, quadrilaterals, the quadratic formula, and the arithmetical properties and operations related to zero and negative numbers. The latter introduces the study of first-order and second-order indefinite equations. The remaining chapters discuss astronomical research but call upon a rich foundation of mathematical knowledge.

For example, Brahmagupta gives the following rules for calculations involving zero:

> A negative number minus zero is negative, a positive number minus zero is positive; zero minus zero is zero ... the product of zero with a positive number or with a negative number or with zero is zero ... zero divided by zero is zero, and a positive or a negative number divided by zero is a ratio with zero as the divisor.

This is the earliest record of the problem of division by zero in the Indian literature, although the conclusion is different from the modern understanding of the situation. The idea of operating with zero as with any other number survived in the works of later Indian mathematicians.

Brahmagupta also elucidated the concept and notation for negative numbers and provided rules for their treatment in computations:

> The sum of a positive number and a negative is the difference of their absolute values or zero if they are equal; the product of a positive number and a negative number is negative, and the product of a two positive numbers or two negative numbers is positive.

This is the first consideration of its kind in the history of the world.

The most important mathematical achievement due to Brahmagupta is his solution of the indefinite equation

$$nx^2 + 1 = y^2$$

where n is a nonsquare integer. Brahmagupta was the first mathematician to consider such equations; they were later misattributed by the eighteenth-century Swiss mathematician Leonhard Euler to a British mathematician from the previous century named John Pell, and they are known today collectively as Pell's equation. In fact, the first mathematician to discuss this equation in Europe was Fermat, and it was Lagrange who resolved it. Brahmagupta found solutions in the special case $n = 92$ (and other special cases) using a method he called *samasa* and which is known today as the *method of compositions* or an application of *Brahmagupta's identity*. His approach was clever and powerful and deserves its special place in the history of mathematics.

Brahmagupta also gave a general formula for a root of a quadratic equation in one unknown, although he neglected to account for the second root of such equations. He provided a formula for the area of a quadrilateral with sidelengths a, b, c, d

$$s = \sqrt{(p-a)(p-b)(p-c)(p-d)},$$

where $p = \frac{1}{2}(a + b + c + d)$. This result is only valid for a quadrilateral inscribed in a circle, as Brahmagupta himself pointed out, but it is still a very impressive result. Finally, he also gave a beautiful proof of the Pythagorean theorem, using the proportional relationship between the sidelengths of two adjacent triangles.

Mahāvīra

Brahmagupta was a sophisticated mathematician, but unfortunately very little information about his thought, life, and work is known to us. He wrote that just as the stars are eclipsed by the rays of the sun, so too are the works of learned scholars eclipsed by those who can put forward problems in algebra and even more so by those who can solve them. Presumably the academic atmosphere during his lifetime was very refined; there was also a movement in the study of history known as the Ujjain school. But for four centuries after the death of Brahmagupta, no further mathematicians of note appeared in Ujjain, perhaps due to political turmoil and dynastic change. Rather during this time, two mathematicians of genius emerged in the relatively remote state of Karnataka in southwestern India: Mahāvīra and Bhāskara.

Although the area within the borders of India is only about 3 million square kilometers, and the distance between its eastern and western borders is greater than between its northern and southern borders, nevertheless the distinct character of South India has long played a deep role in the self-conception of the Indian people. The towering Deccan Plateau of South India and the two mountains at its northern edge, along with the Narmada river, form a natural defense against incursion by the northern imperial powers (the name of the Deccan Plateau derives from a Sanskrit word meaning *south*). And indeed, many attempts at conquest by the north were met with fierce resistance by the south. As a result, the dietary habits of the Indo-Aryan peoples never spread to the south, the armies of Alexander never set foot there, the various invasions of the Mongols and Muslim armies all fell short, and even in later times the influence of France and Britain was more muted in southern India (Fig. 4.10).

Very little is known about the history of South India prior to the time of Ashoka, but it is clear that even in spite of factionalism, its culture was no less advanced or deep than that of the north with respect to religion, philosophy, moral customs, artistic expression, and material development. Several large states or dynasties competed for dominance throughout history, but none of them succeeded in unifying the entire southern territories of India. Every such dynasty maintained a developed maritime trade with Southeast Asia and established variously a capital city as a cultural center, characterized by extensive temple architecture.

Among these several dynasties, one was the Rashtrakuta dynasty, who ruled the Deccan Plateau and a strip of nearby land from about the year 755 to 975. The originators of this dynasty may have been Dravidian farmers, and their empire

Fig. 4.10 Birthplaces of Indian mathematicians

became so great that a Muslim traveller and commentator of the time referred to its ruler as one of the four great emperors of the world, the other three being the Caliph, the emperor of Byzantium, and the emperor of China. The Trinidad and Tobago-born British writer V.S. Naipaul (1932–2018) observed that there remains a relic of the capital of the kingdom of Vijayanagar some 200 miles from Bangalore. This city was among the most resplendent in the world by the fourteenth century.

The Jain mathematician Mahāvīra, who shared a name with one of the founding figures of that religion, was born during the peak of the Rashtrakuta dynasty, probably in the city of Mysore. Mysore was the second largest city in the state of Karnataka (the word Karnataka probably originates from a word meaning *highlands*), located on the southwestern coast of India, between the two famous cities Bangalore and Kolkata. Bangalore is the capital city of Karnataka, now widely regarded as the Silicon Valley of India and home to the National Institute of Mathematics. Kolkata is a famous port city, the city where the Chinese explorer Zheng He died and to which the Portuguese explorer Vasco da Gama arrived much later via the first voyage to India by sea from Europe, around the Cape of Good Hope.

Not much is known about the life of Mahāvīra: as an adult, he lived in the court of the Rashtrakuta dynasty, where he served as court mathematician. Around the year 850, he wrote a book entitled *Gaṇitasārasañgraha* or *Compendium on the Gist of Mathematics*, later widely used as a textbook throughout South India. This book was translated into English in 1912 and published in Madras. This is the first book in the history of India with the form of a modern mathematical textbook, and it contains already glimpse of the topics and structure of current mathematical textbooks. It is especially remarkable that the *Gaṇitasārasañgrah* really is a treatise on pure mathematics, with hardly any mention of astronomical matters. The book is divided into nine chapters, and its most valuable research results include further discussion of zero, quadratic equations, calculations with interest rates, the properties of integers, and topics in what is now known as combinatorics.

Mahāvīra observed that multiplication of a number by zero yields zero and subtracting zero from any number does not reduce its value. He further noticed that division by a number is equivalent to multiplication by its reciprocal and even put forward that division by zero yields an infinitely large value. On the other hand, he asserted that negative numbers have no square roots, contrary to the modern theory of complex numbers. It is interesting to note that in the same way as the Chinese mathematicians who were fascinated by magic squares, Mahāvīra devoted considerable energy to the study of a numerical game he called *garland numbers*. These are pairs of numbers whose product is a palindromic number

$$14287143 \times 7 = 100010001,$$

$$12345679 \times 9 = 111111111,$$

$$27994681 \times 441 = 12345654321.$$

This terminology of garland number survives in Chinese poetry to indicate palindromic numbers. There is also a variation on palindromic numbers known as Scherezade numbers, after the storyteller in *One Thousand and One Nights*. Palindromic numbers also frequently occur as powers, for example, $121 = 11^2$, $343 = 7^3$, and $14641 = 11^4$, but nobody has ever found a palindromic number occurring as a fifth power.

Earlier classic of Jain mathematics had already introduced some simple problems in permutations and combinations. In the course of summarizing these topics, Mahāvīra introduced for the first time the formula for binomial coefficients with which we are familiar today

$$\binom{n}{r} = \frac{n \times (n-1) \times \cdots \times (n-r+1)}{r \times (r-1) \times \cdots \times 1},$$

where $1 \leq r \leq n$. This was two centuries before the birth of Jia Xian in China.

Mahāvīra improved upon the *kuṭṭaka* method for indeterminate equations of Aryabhata and also carried out extensive research into Egyptian fractions. He determined that the number 1 can be expressed as a sum of any fixed number *n* unit fractions and found ways to express arbitrary fractions as sums of unit fractions subject to various constraints. He also studied in detail the solutions to certain higher-order indeterminate equations and construction problems in plane geometry and gave approximation formulas for the perimeter of an ellipse and area of a circular arc, in agreement with results that appear in the Chinese classic *Nine Chapters on the Mathematical Art*.

Bhāskara II

We turn finally to the greatest mathematician and astronomer of ancient and medieval India: Bhāskara II, the second of two mathematicians named Bhāskara in the history of India. The first of them lived in the seventh century, and Bhāskara II in the twelfth century. He was born in 1114, probably in Bidar in the western part of the Deccan Plateau in South India. This city is located today along the railway connecting Mumbai and Hyderabad (host city of the 2010 International Congress of Mathematicians) and like the hometown of Mahāvīra a part of the Karnataka state. His father was on orthodox Hindu Deshastha Brahmin who authored a popular book on astrology. As an adult, Bhāskara worked at the astronomical observatory at Ujjain and later became its director, a worthy successor to Brahmagupta.

By the twelfth century, Indian mathematics had already accumulated a considerable corpus of research results, which Bhāskara assimilated and built upon to achieve deeper results than any of his predecessors. His literary talents were also formidable, and a poetic flavor permeates his works. His two most significant works are entitled *Līlāvatī* (named after his daughter) and *Bījagaṇita* (*Algebra*), two volumes of a larger work entitled *Siddhānta-Śiromani* (*Crown of Treatises*) (Fig. 4.11).

The *Bījagaṇita* includes discussion of positive and negative numbers, linear equations, low-order indeterminate equations in integer coefficients, and so on. There appear also two beautiful proofs of the Pythagorean theorem, one of which is very similar to the method of Zhao Shuang, the other of which was only rediscovered in the seventeenth century, by the British mathematician John Wallis. With reference

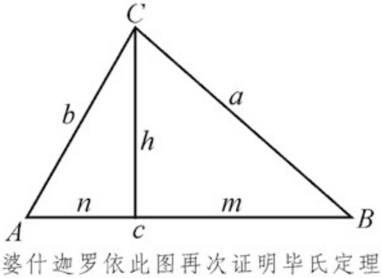

Fig. 4.11 Diagram of the proof of the Pythagorean theorem by Bhaskara

to the figure, we can obtain from the properties of similar triangles that

$$\frac{c}{a} = \frac{a}{m},$$
$$\frac{c}{b} = \frac{b}{n},$$

from which it follows that $cm = a^2$, $cn = b^2$; adding these two identities,

$$a^2 + b^2 = c(m + n) = c^2.$$

Bhāskara also discusses formally a crude version of the concept of mathematical infinity. He writes:

> If we divide a number by zero the result is a quotient with divisor zero, for example three divided by zero is the quotient $3/0$. Such a quotient is called infinity. Just as during the period of cosmic dissolution beings merge into the creator and during the period of creation beings emerge out of him, but the creator himself remains unaffected, likewise nothing happens to the number infinity when any other number is added to it or subtracted from it; it remains unchanged.

The content of *Līlāvatī*, which opens with a salutary Hindu prayer, is more extensive. There is also an interesting legend surrounding the authorship of this book. According to the legend, Bhāskara had concluded through his studies into the horoscope of his daughter Līlāvatī after whom the book was named that she would remain unmarried and childless throughout her life, unless her wedding were held at a precise time on a certain auspicious day. He placed a cup with a small hole in a vessel of water such that the cup would sink at precisely the correct moment for the wedding and warned his daughter not to approach or disturb it. But her curiosity led her to investigate the device, and in the process, a pearl from her bridal headdress dropped into the cup, upsetting it and causing the specified moment to pass by unnoticed. After the marriage, the unfortunate Līlāvatī lost her husband, and Bhāskara promised his devastated daughter to teach her arithmetic and write a book in her honor that would preserve her name to history.[1]

Bhāskara's main mathematical contributions include the use of abbreviated words and symbols to represent unknown quantities and operations; a thorough mastery of trigonometric identities, including the now familiar sum and difference rules; and a more comprehensive discussion of negative quantities, which he referred to as *losses* or *deficits* and indicated by way of a small dot over the numeral. He observed that the square of a positive or a negative number is always positive and that positive numbers have two square roots, one positive and one negative. He also held that negative numbers are not square numbers and have no square roots.

[1] Since the 2010 meeting of the International Congress of Mathematicians in Hyderabad, this institution now offers once every 4 years a *Leelavati Award* for work in the dissemination and popularization of mathematics.

More significantly, Bhāskara and other Indian mathematicians made free use of irrational numbers and did not distinguish them from rational numbers in arithmetic computations. This is in contrast with the Greeks, who had a worked out theory of incommensurable qualities but did not admit irrational numbers into the pantheon of numbers proper.

As heir to the mathematical tradition of Brahmagupta, Bhāskara studied carefully and understood deeply the complete works of his predecessor and made improvements to some of his results, in particular with respect to Pells' equation $nx^2 + 1 = y^2$. He was also productive and successful as an astronomer, which discipline he always pursued from the perspective of a mathematician, introducing innovations in spherical trigonometry, cosmography, astronomical instruments, and so on. Perhaps most astonishing is his use of a technique of instantaneities essentially identical with what would centuries later be the differential calculus to study the motion of the planets. It is said that some generations after his lifetime, a stone stele was discovered in Pataliputra with records of the donation of a sum of money on August 9th, 1207, by local dignitaries to a local educational institution to sponsor the study of his writings. At that time he had already passed away more than 20 years earlier, around the year 1185 (Fig. 4.12).

Remarkably, another very advanced mathematician and astronomer appeared in Kerala at the southern tip of India more than two centuries later. This was Mādhava of Sangamagrāma (ca. 1340–ca. 1425). He was the leading representative of the Kerala school of mathematicians, and the results attributed to him include power series expansions for trigonometric functions and their derivations. We will say something about this work in connection with Leibniz in the final section of Chapter

Fig. 4.12 Srinivasa Ramanujan, Indian mathematician of genius

Fig. 4.13 Indian mathematician Kanakanahalli Ramachandra; photograph by the author, in Bangalore

5. Indeed it is necessary to point out that at the time that Mādhava was born, two pioneers of the European Renaissance, the poet Dante Alighieri and Giotto, the father of European painting, had already died.

We close this section with a discussion of a few mathematicians born in India in the nineteenth and early twentieth centuries, at which time India was under the control of the British Empire as a colony. In addition to the British writers William Thackeray, Eric Blair (better known by his pen name George Orwell), and Rudyard Kipling, two British mathematicians were also born in India during this period. The first of these was the mathematical logician Augustus De Morgan, born in Madurai in the state of Tamil Nadu in South India in 1806, and about a century later, the algebraic topologist J.H.C. Whitehead was born in Chennai, at that time known as Madras, in the same state. De Morgan was responsible for a significant revolution in mathematical logic, freeing it from the restrictive laws passed down by Aristotle, and is recognized today as one of the founders of modern mathematical logic. Whitehead made significant contributions to the development of homotopy theory, one of the fundamental tools of algebraic topology, and also gave the first precise definition of differentiable manifolds (Fig. 4.13).

It was also in this same state of Tamil Nadu that one of the great geniuses of Indian mathematics was born: Srinivasa Ramanujan (1887–1920). Ramanujan was an almost entirely self-taught prodigy who made remarkable contributions to number theory, especially in the theory of integer partitions, but also in the fields of elliptic functions, hypergeometric functions, and divergent series. Alongside the polymath and poet Rabindranath Tagore (1861–1941), these two figures are the pride of modern Indian intellectual history. The inspiration provided by Ramanujan in particular spurred India to achieve great progress in mathematics and the natural sciences in the second half of the twentieth century. The Indian mathematician Kanakanahalli Ramachandra (1933–2011) formed an Indian school of number theory, establishing him as perhaps the true successor of Ramanujan. In physics, the University of Madras has produced two Nobel Laureates: CV Raman (1888–1970) and Subrahmanyan Chandrasekhar (1910–1995). The latter of these two was 10 years old in the year that Ramanujan died.

Sacred Land

The Arabian Empire

We turn now from India to the Arabian Peninsula, where the prosperity of the Arabian Empire contributes one of the most exciting episodes in the history of humanity. The starting point of this narrative is of course the legendary life of the prophet Muhammad. Muhammad ibn Abdullah was born around the year 570 in Mecca in the southwestern Arabian Peninsula. In contrast with the founding figures of Jainism and Buddhism, who were raised in opulence, Muhammad was orphaned in childhood, which, although his grandfather was an important tribal leader, cut him off from his inheritance. At that time Mecca was also a remote place, far removed from every commercial, cultural, and artistic centers, and Muhammad grew up under difficult conditions. When he was 25 years old, Muhammad married the businesswoman widow of a successful businessman, and his economic situation improved considerably, but it was not until the age of 40 that the events that have cemented his place in history took place (Fig. 4.14).

Muhammad experienced a revelation that there is one almighty god Allah with dominion over the world, who had chosen Muhammad as prophet to preach his faith. This was the birth of the religion of Islam. The word *islām* means *submission* in Arabic, and the believers in this religion are called Muslim, meaning *those who submit*. Islam teaches that the end of the world will see the resurrection of the dead and the judgment of all people according to their actions. Muslims have an

Fig. 4.14 Mosque geometry

obligation to relieve poverty and the suffering of others, the accumulation of wealth or mistreatment of the poor are considered social ills that will receive the harsh judgment of later generations. It is also emphasized in Islam that all believers are brothers and should live together in a close group. One saying has it that Allah is closer to his followers than the blood vessels in their necks.

In the year 622, Muhammad had gathered around him about 70 followers, and persistent persecution caused them to migrate from Mecca to Medina, about 200 kilometers to the north. This marked a turning point in the story of Islam and led to a rapid accumulation of new followers. The Bedouin living at that time on the Arabian Peninsula were nomadic Arabic-speaking tribes who were known for their bravery and martial talents, but they had long been divided and had never united in rivalry against the tribes living on the arable land further north along the peninsula. Muhammad brought them together through religious and institutional maneuvers such as intermarriage and began an unprecedented period of conquest, including a campaign that he himself led that pushed as far as the edge of Syria (Fig. 4.15).

In the 10 years following Muhammad's death in the year 632, the army he had assembled, led by two heirs to the caliphate, both of whom were father-in-law to Muhammad, defeated the Sassanid army of Persia; occupied Mesopotamia, Syria, and Palestine; and seized Egypt by way of Byzantium, landing the final blow to the Alexandrian civilization. Around the year 650, a holy text known as the Quran

Fig. 4.15 Cover page of the Quran

was published, based on the words of Muhammad and his followers. This book is considered by Muslims to be a revelation directly from God, written in the language of Allah, and forms the basic text of the Islam religion, one of the four valid sources of law in Usuli jurisprudence, alongside the teachings and normative examples of Muhammad known as the *sunnah* and composed of oral accounts called *hadith*, the consensus of scholars (referred to as *ijmā* ʿ), and the guidance of the rational faculty of the soul (*ʻaql*).

The Muslim campaign continued its rapid expansion: in the year 711, they swept North Africa and proceeded toward the Atlantic Ocean; they crossed the Strait of Gibraltar to the north and occupied Spain. During this period, China was still experiencing the peace and prosperity of the Tang Dynasty, the poet Li Bai was still in his youth, and his younger friend and fellow poet Du Fu was still in his mother's womb. From the perspective of mathematical history, it had been half a century since the death of Brahmagupta, and there were not at this time any significant mathematical figures alive in either the east or the west. It was not inconceivable that the Muslim army would conquer the entirety of Christian Europe. But in the year 732, the Muslim invaders had reached the Aquitane region in what is now central France, where they were defeated in the Battle of Tours.

The Muslim army had expanded their territory to a vast area from India in the east to the Atlantic Ocean to the west, encompassing areas north of the Caspian Sea stretching into Central Asia. This was the Umayyad Caliphate, probably the largest empire in human history to this date, committed throughout its expansion to the dissemination of Islam. But internal power struggles caused the caliphate to split into two powers, one with its capital at Córdoba in Spain and the other with its capital in Damascus, in Syria. The latter became the Abbasid Caliphate, founded by descendants of Muhammad's uncle Abbas ibn Abdul-Muttalib, and its center of power switched gradually eastward to Baghdad in Iraq. Here the Abbasids built a city unparalleled in the world, a center of science, culture, philosophy, and invention during the period later known as the Golden Age of Islam. The Abbasid Caliphate became the longest and most renowned dynastic power in the history of Islam.

The House of Wisdom in Baghdad

Baghdad is located along the Tigris River, at the point where the distance between the Tigris and the Euphrates is smallest, and surrounded by a flat alluvial plain. The word *baghdad* is most likely a Middle Persian compound meaning *a gift from God*, and this city quickly began to prosper after it was established as the capital of the Abbasid Caliphate by its second caliph, Al-Mansur in 762. Palaces and various buildings sprang up from the ground between the circular city walls, and the city reached the peak of its economic and academic prosperity by the late eighth century and the first half of the ninth century, under the governance of Al-Mahdi and his successors Harun al-Rashid and Al-Ma'mun. At this time, Baghdad was one of the richest cities in the world (another was Chang'an in China) (Fig. 4.16).

Fig. 4.16 1237 illustration of the House of Wisdom

The ninth century in world history begins with the appearance on the world stage of two emperors with a dominant role in international affairs: Charlemagne, King of the Franks, and Harun al-Rashid. Charlemagne was the grandson of Charles Martel, who had defeated the Umayyad invasion of Aquitaine at the Battle of Tours, and Charlemagne was crowned Emperor of the Romans by Pope Leo III on Christmas Day in the year 800. Harun al-Rashid exerted an even greater influence on the course of history, and these two leaders in the east and the west, respectively, established a diplomatic alliance and even a personal friendship, characterized by frequent exchanges of valuable gifts. Charlemagne hope that Harun could aid him in his efforts against the Byzantine efforts, and Harun in turn hoped for Charlemagne's assistance in his struggle against the surviving Umayyad dynasty in Spain.

Both legend and historical evidence confirm that the period under Harun al-Rashid was the most glorious time in the history of Baghdad. In less than half a century, the city had grown from its humble beginnings as a deserted village to a cosmopolitan center of extraordinary wealth. Only Constantinople in Byzantium could compete with it for splendor at the time. Harun was a typical Muslim monarch, whose magnetic generosity attracted poets, musicians, singers, dancers, and trainers for hounds and cock-fighting, in a word all manner of remarkable people, to the capital. He was immortalized as an extravagant, even profligate, caliph in *The Thousand and One Nights*.

About the year 771, 9 years after the establishment of Baghdad, an Indian traveller brought with him two scientific papers to the capital. One of these was an astronomical treatise, which Al-Mansur had translated into Arabic. This

Fig. 4.17 Arabic translation of Greek writings

translator became the first astronomer of the Islamic world. Previously, the nomadic inhabitants of the Arabian Peninsula had sustained a deep interest in the position of the stars, but they had not conducted any systematic scientific research into this phenomenon. But the rise of Islam provided a need for careful astronomical calculations due to the requirement to pray facing the direction of Mecca five times daily. This obligation to prayer (*Salah*) is one of the five pillars of Islam; the others are the declaration of faith (*Shahada*), almsgiving (*Zakat*), fasting (*Sawm*), and pilgrimage (*Hajj*) (Fig. 4.17).

The second paper was a mathematical essay by Brahmagupta. The Lebanese-American historian Philip K. Hitti has observed that the numerals known to Europeans as Arabic numerals, and known in Arabic as Indian numerals, were first introduced to the Muslim world by this paper. Apart from this however, the cultural exports of India were very few, and eventually the Greeks came to have the most substantial influence on Arabic thought of this period, especially after the conquest of Syria and Egypt. They began to actively seek out Greek works, including Euclid's *Elements*, Ptolemy's *Almagest*, and the dialogues of Plato, all of which were translated one after another into Arabic.

Here we point out that also Chinese papermaking technology had not long earlier been introduced to the Arabic world at this time, four centuries before its introduction to Europe by way of the Middle East and North Africa (bypassing the Mediterranean, like Hindu-Arabic numerals). Indeed, there was a paper mill in the city of Baghdad already. The Chinese had kept the art of papermaking a deeply held secret for many centuries after improvements were made to this craft in the second century by Cai Lun of the Eastern Han dynasty. But in 751, a Tang dynasty army was defeated by Muslim forces in the Battle of Talas in Kazakhstan, and a group of paper workers were taken prisoner to Samarqand, where they were compelled to divulge their knowledge.

The Greek influence reached its peak after Harun al-Rashid was succeeded by his son al-Ma'mun, who became obsessed with rational inquiry. He is said to have been visited by Aristotle in a dream, who assured him that there was no substantial conflict between reason and then teachings of Islam. In 830, al-Ma'mun ordered the construction of the *Bayt al-Ḥikmah* or *The House of Wisdom* in Baghdad. This was a joint institution with integrated functions as a library, science academy, and translation center. This was undoubtedly the most important academic institution to appear since the establishment of the Library of Alexandria in the third century BCE, and it quickly became the center of the academic world, a place of extensive research activity in philosophy, medicine, zoology, botany, astronomy, mathematics, mechanical technology, architecture, Islamic theology, Arabic grammar, and more.

The Algebra of al-Khwarizmi

In the latter half of the long and effective era of translations under the Abbasid caliphate, Baghdad became home to an age of scientific originality. The most important figure in this story is the mathematician and astronomer Muḥammad ibn Mūsā al-Khwārizmī (780–850). He was born more than a century after the death of Brahmagupta and before the birth of Mahāvīra. Relatively little is known about his life, although it is generally believed on the basis of his name that he was born in the Khwarazm region of Greater Iran, where the Amu Darya river flows into the Aral Sea, not far from the city of Khiva in modern Uzbekistan. Another theory has it that al-Khwarizmi was born on the outskirts of Baghdad but descended from people of the Khwarazm region (Fig. 4.18).

One of his epithets also implies that al-Khwarizmi had among his ancestors adherents of the ancient Zoroastrian religion. Zoroastrianism, also known as

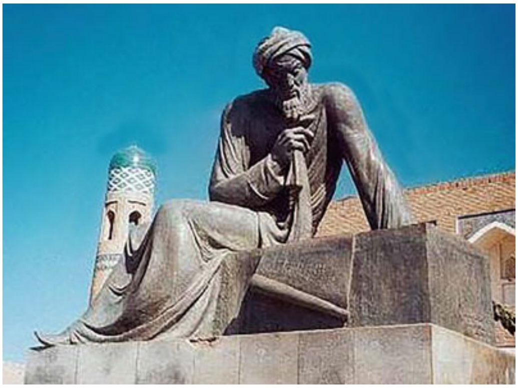

Fig. 4.18 Statue of al-Khwarizmi in Samarkand

Mazdayasna, is one of the oldest continuously practiced religions in the world, with a history stretching back even at that time more than two and a half millennia. Its tenets include a form of elementalism with special prominence given to the role of fire, a theology of cosmic dualism, and its opposition to abstinence, celibacy, fasting, and other such forms of ascetic self-denial. The founding prophet Zoroaster (or Zarathustra) of Zoroastrianism probably lived about the same time as the early Jainist preacher Mahavira, perhaps about 30 years earlier, somewhere in the northern parts of what is today Iran. Since his death, the religion he founded became at several times the state religion of the Persian Empire.

For these reasons, it can be inferred that al-Khwarizmi had Persian ancestors; even if this is not the case, for example, if he was rather of Central Asian descent, his spiritual life was essentially linked to Persia, a nation with a long and deep cultural tradition, and he was proficient in Arabic. Al-Khwarizmi spent the years of his early education in his hometown, before travelling to the ancient city of Merv in Central Asia to continue his studies. He also visited Afghanistan, India, and other places in pursuit of learning. He quickly became a well-known and respected scholar, and al-Ma'mun, at that time governor of Khorāsān in the eastern Iranian Plateau, summoned him back to Merv. Later, after al-Ma'mun became caliph of the Abbasid Caliphate, he hired al-Khwarizmi to work in the capital Baghdad, where al-Khwarizmi eventually became one of the lead researchers at the House of Wisdom. After the death of al-Ma'mun, his successor continued to employ al-Khwarizmi, who remained in Baghdad until his own death. This was a time of tremendous political stability, economic development, and cultural and scientific prosperity for the Islamic empire (Fig. 4.19).

Al-Khwarizmi left behind two influential mathematical treatises: *Algebra* and *On the Calculation with Hindu Numerals*. The full title of the book referred to colloquially as *Algebra* was *The Compendious Book on Calculation by Completion*

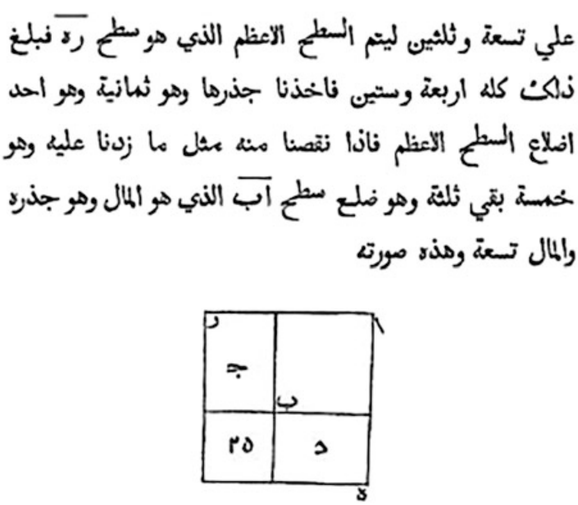
علي تسعة وثلثين ليتم السطح الاعظم الذي هو سطح ره فبلغ
ذلك كله اربعة وستين فاخذنا جذرها وهو ثمانية وهو احد
اضلاع السطح الاعظم فاذا نقصنا منه مثل ما زدنا عليه وهو
خمسة بقي ثلثة وهو ضلع سطح اب الذي هو المال وهو جذره
والمال تسعة وهذه صورته

Fig. 4.19 Manuscript of the *Algebra* of al-Khwarizmi *and Balancing*, and the modern word *algebra* derives from the term *al-jabr* contained within it, meaning *restoration* and designating the process of adding a number to both sides of an equation to cancel or consolidate terms. This book was translated into Latin in the twelfth century during a period that saw a burst of translation activity and exerted an inestimable influence on the subsequent development of European sciences. It would not be inaccurate to say that algebra was an Arabic invention, as perhaps geometry was an Egyptian invention; perhaps algebra was even something like the grammar through which the people of the Arabian Peninsula understood the workings of numbers.

Al-Khwarizmi completed *Algebra* in about the year 820. The problems he discusses are not more difficult than those in Diophantus or Brahmagupta, but since he concerns himself with general rather than particular solutions, his perspective is much closer to that of modern elementary algebra than anything that appears in the ancient Greek or Indian mathematical literature and constitutes a remarkable achievement in the history of mathematics. The book discusses the algebraic treatment of linear equations and provides a general algebraic solution to the quadratic equation. More importantly, it also introduces generalized algebraic tools, such as shifting terms from one side of an equation to the other and merging like terms, paving the way for algebra to develop as the science of equation solving. It is hardly surprising that al-Khwarizmi's book became a standard textbook in Europe for several centuries, an unusual situation for a scientific work from the east.

Whereas Brahmagupta provided only a single solution for quadratic equations in one variable, al-Khwarizmi gives both. He was perhaps the first mathematician in world history to observe explicitly that quadratic equations have two roots. On the other hand, although he was aware that such equations can have negative roots, he did not admit either negative roots or zero roots as solutions to quadratic equations. He also pointed out that if the discriminant of a quadratic equation is negative, then there are no (real) roots (here we introduce modern terminology to describe this observation). After having provided solutions to various typical equations, al-Khwarizmi provides also geometric proofs for his results, a practice which shows the obvious fingerprints of Euclid and the Greeks. Therefore it is reasonable to say that unlike other mathematicians of the Arabian Peninsula, al-Khwarizmi was influenced by the two civilizations of Greece and India, which of course also reflects their relative geographic positions.

On the Calculation with Hindu Numerals was another immensely important book in the history of mathematics, since it provides a systematic introduction to Hindu numerals and decimal notation, both of which had previously been described to scientists in Baghdad by Indian visitors, but which had not yet attracted widespread attention. Like *Algebra*, this book was translated into Latin and widely disseminated in the twelfth century. An early Latin edition is housed today in the University of Cambridge library. Subsequently, the Indian system of numeral notation gradually came to replace the alphabetic system employed by the Greeks, and the system of Roman numerals, until eventually it became the universal number system across the globe. On account of the history of its adoption, such numerals are generally known today as Arabic numerals or Hindu-Arabic numerals. It is also worth mentioning that the original title of the Latin translations of *On the Calculation with Hindu Numerals* was *Algoritmi de numero Indorum*, where *Algoritmi* was the Romanization of the name al-Khwarazmi. It is from this that the modern term *algorithm* in mathematics and computer science derives.

Al-Khwarizmi also made contributions to geometry, in particular to the measurement of area. He classified triangles and quadrilaterals and gave, respectively, the formulas for the calculation of their areas. He also gave an approximate formula for the area of a circle

$$A = \left(1 - \frac{1}{7} - \frac{1}{2} \times \frac{1}{7}\right) d^2,$$

where d is the diameter of the given circle, corresponding to the value $\pi \approx 3.14$. We see also here that the Arabic mathematics like the Indians adopted the Egyptian preference for unit fractions. Al-Khwarizmi also gave an area formula for a circular segment, considering separately the cases where the segment is larger or smaller than the semicircle, respectively.

In astronomy, al-Khwarizmi compiled trigonometric and astronomical tables to calculate the positions of the stars and lunar and solar eclipses and wrote numerous books on astrolabes, the sine quadrant, sundials, and the calendar. In astronomy in particular, al-Khwarizmi's work inspired a remarkable successor: al-Battani (ca. 858–929), who was born in Syria and discovered for the first time that the radius between the earth and the sun varies throughout the year and determined that the apogee of the sun (the point when it is furthest from the earth) produces an annular solar eclipse. Al-Battani replaced the geometric method in astronomy with trigonometry, introduced the use of the sine function in calculations, and corrected some of errors in the works of Ptolemy, including improved calculations for the orbits of the sun and certain planets. His astronomical works were also translated into Latin in the twelfth century and became the best known works of their kind in medieval Europe.

The intellectual achievements of al-Khwarizmi were not limited to mathematics and astronomy. He also wrote the first works of academic history in Arabic and promoted thereby the development of history as a research subject. He also participated in an important project of the period, the creation of a world map, which was in high demand for its military and commercial utility (the people of the Arabian Peninsula have proved to be especially savvy businesspeople). This led to the composition of a book entitled *Kitâb Sûrat al-Ard*, or *Book of the Image of the Earth*, the first geographical monograph in medieval Muslim history, which describes the important settlements, mountains, rivers, lakes, seas, and islands of the known world at that time, accompanied by four maps.

The Scholars of Persia

Omar Khayyam

Although the mathematics and science of the Arabian Peninsula in the medieval period exhibited mainly the influence of Greece and India, the main influence on the culture of this civilization was undoubtedly Persia, in a relationship similar to that of Macedonia under the sway of the Greek civilization, the fruit of which was the great polymath Aristotle. The people of the Arabian Peninsula were known not only for their decisiveness and bravery but also for their excellent organizational and management skills, and an attitude of tolerance and generosity, but in rational philosophy they lagged behind the Persians. In the end, there were two main indigenous features of this civilization: the rise of Islam, which became the state religion, and the Arabic language, which was preserved as the official national language. In other respects, the Persian influence was everywhere: in Baghdad, Persian titles, Persian wine, Persian romance, Persian songs, and more gradually became more and more fashionable (Fig. 4.20).

Fig. 4.20 Mausoleum of Omar Khayyam, Nishapur

According to legends, it was al-Mansur himself who first came to revere and emulate the Persians, and his subjects naturally followed his lead. He also initiated a Persian revival under his government, including a heavy dependence on Persian diplomats. Al-Mansur appointed the first Persian vizier Yahya ibn Khalid, and the two became close to the point that their wives nursed each of the other's children, who were born around the same time, and Jafar bin Yayha, son of Yayha ibn Khalid, served as a tutor to Harun al-Rashid. Eventually however there was a falling out between these two families, according to story, because Jafar bin Yayha violated the terms of what was intended to be a purely formal marriage with Harun's beloved sister Abbāsa, and the two of them had conceived a child. In any case, the Barmakid family to which Yayha belonged fell out of favor, Jafar was beheaded, and many of the remaining family members were imprisoned.

Following the death of al-Ma'mun, the Abbasid Caliphate went into decline, and many smaller autonomous dynastic powers begin to spring up at the periphery of Baghdad. The political situation became increasingly turbulent, competing religious factions proliferated, the empire gradually fractured, and what central power remained fell to the military over the course of multiple troop riots and finally a slave uprising known as the Zanj rebellion. All this provided an opening for the Turks and the Persians to direct again the point of their blades at the heart of Baghdad.

In spite of all this, noteworthy intellectual activity continued in Baghdad. In the tenth century, the Persian mathematician al-Karaji (953–1029) carried out research into the binomial coefficients (later than the Indians, but earlier than Jia Xian) and the algebra of exponents, and made contributions to the theories of linear equations and mathematical induction. In 1065, the first official institution of higher learning in the Muslim world, the Al-Nizamiyya, was established in Baghdad, but it failed to attract the brightest young talents of the period, such as Omar Khayyam, perhaps the greatest mind of the medieval Islamic world.

Omar Khayyam (1048–1131) was born in 1048 in Nishapur, an ancient city in the Khorasan region of northeastern Iran. The name *khayyam* means *tent-maker*, which suggests that his father or more distant forebears were engaged in the practice of tent-making. Perhaps for this reason, his childhood was spent roaming freely with his father, first within the confines of his hometown, later in the province of Bukhara, home to a famous library, and the small city of Balkh in northern Afghanistan, where he studied, before finally he moved to Samarkand, the oldest city in Central Asia, where he gained political favor. Here he began to compose his substantial mathematical works, under the patronage of the governor and chief judge of the city (Fig. 4.21).

Already in Euclid's *Elements*, there appear geometric solutions to quadratic equations of the form $x^2 + ax = b^2$, where one of the solutions is given by

$$\sqrt{\left(\frac{a}{2}\right)^2 + b^2} - \frac{a}{2}.$$

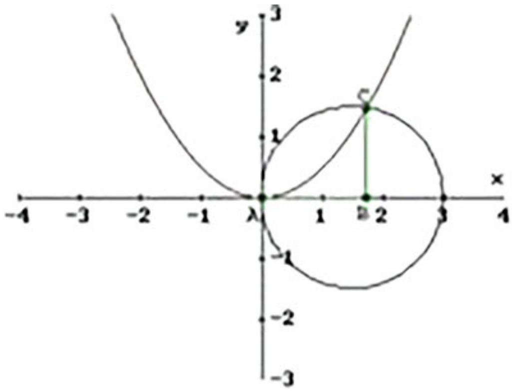

Fig. 4.21 Graph used by Omar Khayyam to find solutions to cubic equations

This can be proven by using right triangles and the Pythagorean theorem: construct a right triangle with sides of length $\frac{a}{2}$ and b about the right angle. Then after removing a length $\frac{a}{2}$ from the hypotenuse, what remains will be the solution as given. But the solution of cubic equations is obviously more complicated. Omar Khayyam considers 14 types of cubic equations in his research and determined their roots by considering the intersections of two conic sections.

Consider as an example the cubic equation $x^3 + ax = b$. This can be rewritten as $x^3 + c^2x = c^2h$, which equation Omar Khayyam considered as determined by the abscissa x of the intersection C of the parabola $x^2 = cy$ and the semicircle $y^2 = x(h-x)$ (see the figure), since the variable y can be eliminated from the latter two equations to recover the original cubic equation. As a result, Omar Khayyam obtained solutions to cubic equations by way of conic sections and paved the way for the study of polynomial equations of higher degree. He presented his results in an important treatise *Maqāla fi l-jabr wa l-muqābala* (*On proofs for problems concerning Algebra*). He almost contributed a historically important attempt at a proof of Euclid's fifth postulate, the parallel postulate.

In the eleventh century, a Turko-Persian empire called the Seljuk Empire swept into power and gained control over a vast area, stretching from western Anatolia and the Levant in the west to Hindu Kush in the east and from Central Asia in the north to the Persian Gulf in the south, also under the banner of Islam. Omar Khayyam was commissioned by the Sultan Malik-Shah I of the Seljuk Empire to travel to the Isfahan to preside over the establishment of a new observatory and the reform of the Persian calendar. This became his main source of livelihood, while mathematical research remained an important recreational pursuit. In his calendar work, he proposed the addition of 8 leap days every 33 years to the basic 365 days of the normal year. This reduces the difference from the actual year to an error of 19.37 seconds, or 1 day every 4460 years, more accurate even than the Gregorian calendar in use around the world today. A change of leadership however had the unfortunate result that his reforms were not implemented (Fig. 4.22).

Omar Khayyam spent most of his life in Isfahan. It can be said that he experienced in his lifetime each of the three dominant strands of his period: Islam, the Suljuk

Fig. 4.22 Poem of four lines by Omar Khayyam, with illustration

court, and Persian culture. But all this turmoil and an eccentric personality made a solitary and somewhat unsettled life, and he recorded his occasionally untimely thoughts in the form of Persian poems of four lines, a style popular in Khorasan at the time. Probably he could not have imagined that some eight centuries later in 1859 an Englishman named Edward FitzGerald would translate and publish a selection of these poems under the title *Rubáiyát of Omar Khayyám* with the result that Omar Khayyam became known the world over for his poetry, while his mathematical achievements drifted under its shadow. In one of these poems (No. 57 in the numbering of the *Rubáiyát*), he laments the failure of his calendar reform:

> Ah, but my Computations, People say,
> Reduced the Year to better reckoning?–Nay
> 'Twas only striking from the Calendar
> Unborn To-morrow, and dead Yesterday.[2]

The ancient Iranian people referred to themselves as *Aryans* and probably belonged to the same group of Indo-European speaking Central Asian nomads who travelled toward Europe sometime between the years 2000 and 1000 BCE. Indeed, the modern Persian name of Iran means *the land of the Aryans*, and likely these were the same people who earlier had migrated into India, where some remained and intermarried with the aboriginal Dravidian peoples. The name Persia itself derives from *Parsa*, the name of the people from whom Cyrus the Great of the Achaemenid

[2] Tr. Edward FitzGerald

dynasty emerged. These people gave their name to the region where they settled as *Fars* (or *Pars*), today a province of Iran containing the central city of Shiraz, known as the city of poets and flowers.

This was the birthplace of modern Persia. Cyrus the Great was born in the sixth century BCE and started as a local leader in his native region, before eventually managing to defeat the Babylonians and many other powers and establishing a great empire between India and the Mediterranean. After the death of Cyrus, one of his sons, and the son Darius of one of his ministers, continued to expand the territories of the empire to encompass Egypt, which is when legend has it that Pythagoras was taken captive in Babylon. The cuneiform inscriptions at Mount Behistun in western Iran that we have already encountered relate how Darius came to the throne. It seems also that after the decline of Plato's Academy in Greece, many Greek scholars travelled to Persia and contributed to the development of its civilization.

Nasir al-Din al-Tusi

About 70 years after the death of Omar Khayyam, during which time both the Italian mathematician Fibonacci and the Chinese mathematician Li Ye were born, the great Persian polymath Nasir al-Din al-Tusi (1201–1274) was born in the city of Tus, also in Khorasan. At this time Tus was the intellectual center of the Arabian Peninsula, and Harun al-Rashid also died there. Nasir-al-Din Tusi's father was a scholar of jurisprudence, who encouraged his son to take seriously his studies and saw himself to his elementary education before his early death, while an uncle in the same city taught Nasir-al-Din al-Tusi logic and philosophy. He also took up algebra and geometry during this time. In his later youth, Nasir-al-Din al-Tusi moved to Nishapur, home of Omar Khayyam, to study medicine and mathematics with disciples of the legendary Persian philosopher and scientist Ibn Sina (ca. 980–1037), more famous in western countries by his Latin name Avicenna. Gradually Nasir-al-Din al-Tusi made a name for himself as a thinker (Fig. 4.23).

At this time, the armies of Genghis Khan were sweeping westward, and the vestiges of the Islamic empire were crumbling. In the absence of any stable academic environment, Nasir-al-Din al-Tusi was invited to move from stronghold to stronghold at the behest of the Nizari Ismaili state, and it was in this context that he composed several important works of mathematics and philosophy. Finally in 1256 the grandson Hulagu Khan of Genghis Khan (the brother of Mongke Khan and Kublai Khan) conquered northern Persia and occupied the stronghold Maymun-Diz where Nasir-al-Din al-Tusi had settled. He was captured, but managed to earn the respect of Hulagu Khan, who appointed him as scientific advisor to the Mongols. Two years later, Nasir-al-Din al-Tusi served under Hulagu Khan in a brutal and bloody expedition against Iraq that signaled the final end of the Abbasid Caliphate.

After the death of Mongke Khan, Kublai Khan succeeded to the throne, and Hulagu Khan was made king of the Ilkhanate territory and charged with the subjugation of Persia and any remaining Muslim states in southwestern Asia, which

Fig. 4.23 Iranian commemorative stamp in honor of Nasir al-Din al-Tusi

he accomplished at the head of a massive Mongol army, establishing a capital in the city of Tabriz in northwestern Iran adjacent to Azerbaijan. With the approval and funding of Hulagu Khan, Nasir-al-Din al-Tusi had established the Maragheh observatory in this region, and he set about recruiting talented scholars, writing and carrying out research, and commissioning the production of advanced instruments of observation, so that the Maragheh observatory became an important center of academic activity. Nasir-al-Din al-Tusi has been twice featured on commemorative stamps in Azerbaijan on account of his importance to the region.

In 1274, Nasir-al-Din al-Tusi paid a visit to Baghdad, where he succumbed to an illness and was buried in the suburbs. Hulagu Khan had already died at this time, having conquered an area including all of Persia and of course Baghdad in particular. By the time his grandson came into power, the Ilkhanate extended from the Amu Darya river in the east to the Mediterranean Sea in the west, from the Caucasus in the north to the Indian Ocean in the south (Fig. 4.24).

Nasir-al-Din al-Tusi was a diligent writer throughout his entire life, leaving behind a trove of treatises and letters in Arabic, as well as a small number of philosophical texts in Persian. He was also reputed to be familiar with the Greek language, and Turkish makes an appearance in some of his writings. He touched upon every aspect of the Islamic intellectual world of the time, in particular mathematics, astronomy, logic, philosophy, ethics, and theology. These works are not only classics of Islamic scholarship but also influenced deeply the awakening of

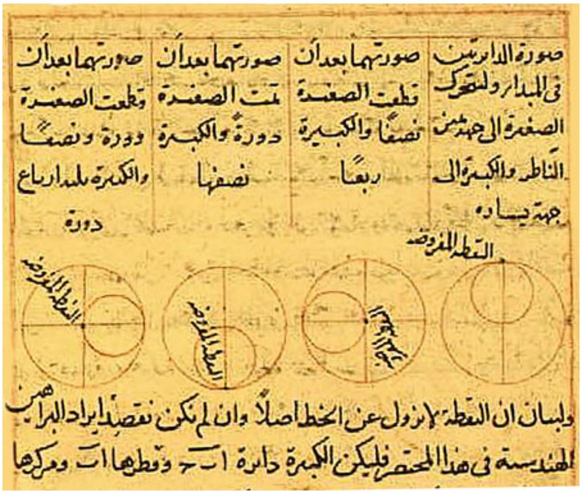

Fig. 4.24 A manuscript of Nasir al-Din al-Tusi

European science. Some of his astronomical instruments may also have made their way to China where they served as reference for the astronomers there.

Three of his books in particular are noteworthy mathematical treatises. In a collection of arithmetical writings entitled *Jami' al-hisab bi-' l-Takht wa- 'l-turab*, Nasir-al-Din al-Tusi presents familiar results from Omar Khayyam and extends the research of numbers more deeply in the direction of irrational numbers and other fields. The numerals are Hindu throughout and include some discussion of Pascal's triangle and methods for finding fourth roots and other higher roots of numbers. This seems to be the earliest extant work on this topic. It is also fascinating that Nasir-al-Din al-Tusi discovered the important number theoretic result that the sum of squares of two odd numbers cannot be square, which is usually proven by way of the theory of congruences (Fig. 4.25).

A more substantial work is contained across the *Al-Risala al-shafiya 'an al-shak fi al-khutut mutawaziyya* (*Treatise healing the doubt about the parallel lines*) and the *Tahrir al-Usul al-Handasiya li-Uqlidis* (*Exposition of Euclid's Elements*), comprising two revisions and annotations of the *Elements* and containing a more detailed consideration of the parallel postulate. Nasir-al-Din al-Tusi argued that the parallel postulate ought not to be a postulate but rather something than can be proved on the basis of the four other fundamental postulates of Euclidean geometry. He followed in this regard the methodology of Omar Khayyam: if $ABCD$ is a quadrilateral, with segments DA and CB equal in length and perpendicular to the side AB, then the angles at C and D are equal. If these two angles are acute, Nasir-al-Din al-Tusi showed that it follows that the sum of the interior angles of a triangle is less than two right angles, which is the basic starting point of Lobachevskian geometry, although Nasir-al-Din al-Tusi did not pursue this line of thought any further.

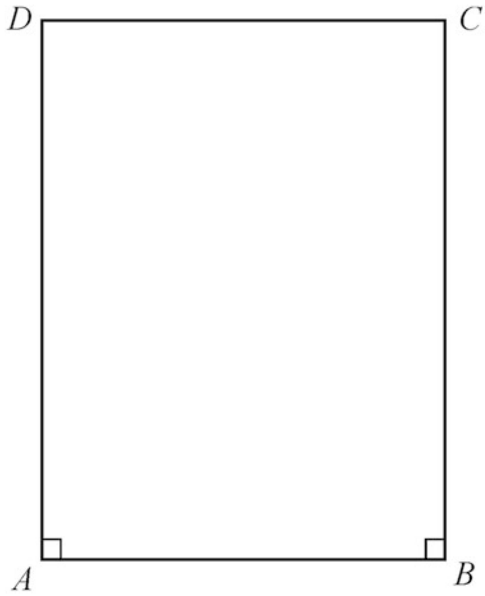

Fig. 4.25 Quadrilateral used by Nasir al-Din al-Tusi in connection with the parallel postulate

The most important mathematical work of Nasir-al-Din al-Tusi is his *Treatise on the Quadrilateral*, the first mathematical monograph devoted solely to trigonometry in the history of mathematics – prior to this, trigonometric results appeared only in astronomical texts, as a computational tool, and it was only after the work of Nasir-al-Din al-Tusi that trigonometry developed as an independent branch of pure mathematics. This book contains also the first statement of the sine law for plane triangles: if A, B, C are three angles in a triangle, and a, b, c the lengths of the sides opposite to them, respectively, then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Nasir-al-Din al-Tusi made equally outstanding contributions to astronomy, which we will not go into in more detail here. His two sons seem to have also worked at the Maragheh observatory, as well as a Chinese astronomer whose name and origins cannot be identified. The *History of Yuan* (元史) lists *seven western instruments* of Arabic design, some of which are very similar to those of Nasir-al-Din al-Tusi. Much later in the eighteenth century, several observatories built by Indians in Delhi and other places imitated the structure and appearance of the observatory established by Nasir-al-Din al-Tusi at Maragegh.

Jamshīd al-Kāshī

It is a historical fact concerning the broad appeal of Islam that although the territory captured by the various Muslim powers may be lost in time, the people inhabiting almost always will have long converted to Islam. Iran, or Persia, is a typical example. After the Arabic conquest in 640 and the advent of Islam after a series of debilitating wars with the Byzantine Empire, the rule over this territory changed hands again and again, as it was occupied by various powers. But the influence of Islam has remained a constant and is reflected through to today in the national emblem and national flag of Iran. The former contains four crescents, a sword in the shape of a water lily, and a stylized script, symbolizing, respectively, Islam, power, and the Quran. The latter is green, white, and red, with that Takbir ("*Allah is great*") written 11 times each in the Kufic script along the edges of the green and red bands.

We turn now to the last great mathematician and astronomer of the ancient Arabic world and indeed of the ancient east: Jamshīd al-Kāshī, whose death in 1429 marks the end of an era. There is no record of the year of his birth, but probably it was in around 1380. Rather the earliest record of his existence dates from June 2nd, 1406, when he observed a lunar eclipse from his hometown Kashan, in the eastern foothills of the central mountain range of Iran, today along the railway line between the old capital Isfahan and the modern capital Tehran. Although al-Kāshī seems to have come from a humble background, nevertheless like Omar Khayyam and Nasir-al-Din al-Tusi before him, his talents were early recognized and appreciated by the political elite.

At the end of the fourteenth century, a descendent of Genghis Khan named Timur (or Timūr Gurkānī), who had been crippled for life during a failed raid in his youth, established the Timurid Empire, with its capital at Samarkand. Timur belonged to the Turco-Mongol tradition and a believer in Islam, and he carried out a fierce and unstoppable campaign across lands stretching from Russia to India and the Mediterranean in pursuit of the restoration of the Mongol empire. He did not return to Samarkand until he had secured tribute from the Sultan of Egypt and the Byzantine emperor. Although Timur himself was illiterate, he enjoyed the company of learned scholars with whom he could play chess and discuss various questions of history, Islamic theology, and applied science (Fig. 4.26).

In 1405, as he was preparing to embark upon a new campaign against China, where the Yuan dynasty had already come to an end, Timur died of an illness. His grandson Ulugh Beg had no appetite for martial affairs, but rather developed an obsession with the sciences and astronomy in particular; he himself discovered through his own observations of calculation errors in the works of Ptolemy. He also wrote poetry, studied history and the Quran, and became a powerful patron and protector of art and science. He established early on an institute for science and theology in Samarkand and made plans to build an observatory as well. Samarkand quickly became the most important academic center in the eastern world at the time (Fig. 4.27).

The academic career of al-Kāshī was closely linked with Ulugh Beg. Although he was trained as a doctor, his passion was for mathematics and astronomy, and he

Fig. 4.26 Uzbekistani commemorative stamp in honor of Ulugh Beg

Fig. 4.27 Ancient city gates of Samarkand

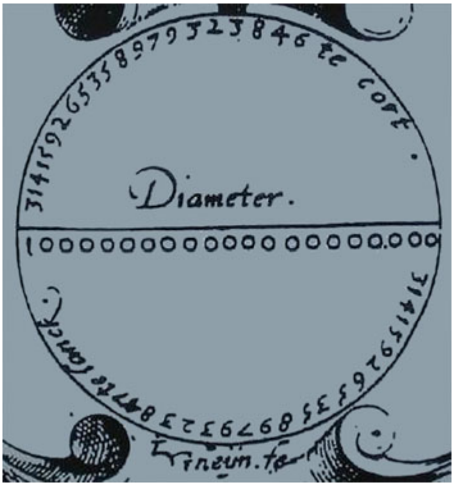

Fig. 4.28 Illustration of the value of π obtained by Al-Kāshī

found stable patronage at Samarkand in the court of Ulugh Beg after a long period of poverty and hesitation. He participated in the construction of the observatory and the installation of its various instruments and became its first director after its completion. In his astronomical works, such as *Sullam al-sama'* (*Ladder to the Sky*), al-Kāshī discussed the distance and size of the stars and other celestial bodies and introduced the armillary sphere and other astronomical instruments, some of which were his own inventions. He also participated like almost every other ancient scholar in calendar reform.

In a letter to his father, al-Kāshī praised in the highest terms the knowledge, organizational skills, and mathematical talents of Ulugh Beg and emphasized especially the spirit of academic freedom in the court at that time, which he considered a necessary prerequisite for scientific progress. Ulugh Beg seems to have been deeply sympathetic to the scientists working under him and was very willing to tolerate in al-Kāshī his lack of refined court etiquette and somewhat unconventional habits. In the preface of a calendar book bearing his name as its title, Ulugh Beg praised al-Kāshī as an outstanding scientist and one of the greatest scholars in the world, well versed in the ancient scientific sources and capable of solving the most difficult problems (Fig. 4.28).

Al-Kāshī contributed two landmark mathematical achievements: the first is his approximation of π, and the second is his approximation of $\sin 1^\circ$. Throughout ancient times, research into the calculation of π reflected to a certain extent the mathematical development of a civilization, just as today the calculation of large prime numbers stands as a benchmark for the computer power available to a corporation or even a country. In 1424, Al-Kāshī set a new record for accuracy in the calculation of π, 962 years after Zu Chongzhi had established the previous record of accuracy up to the seventh decimal place. Al-Kāshī obtained

$$\pi \approx 3.1415926535897932 5,$$

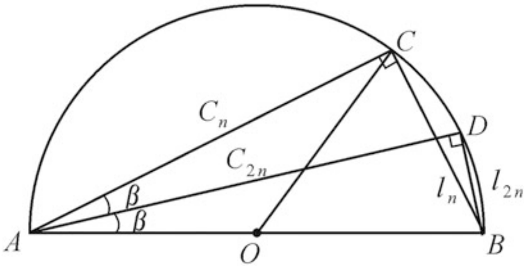

Fig. 4.29 Calculation of π by Al-Kāshī

accurate to 17 decimal places, by determining the perimeter of a polygon with 3×2^{28} sides. This record held until 1596, when Dutch mathematician Ludolph von Ceulen used inscribed and circumscribed polygons of 60×2^{33} sides to obtain an accuracy of 20 decimal places (Fig. 4.29).

We present here the method used by Al-Kāshī in his calculation. As shown in the figure, suppose $AB = d = 2r$ is the diameter of the circle and l_n (respectively, l_{2n}) is the length of one side of the regular polygon with n (respectively, $2n$) sides inscribed in it. Then the other two sides about the right angle have a recurrence relationship given by

$$c_{2n} = d \cos \beta = d\sqrt{\frac{1 + \cos 2\beta}{2}} = \sqrt{r(2r + c_n)}.$$

Then by the Pythagorean theorem,

$$l_n = \sqrt{(2r)^2 - c_n^2}.$$

A similar calculation gives the length of one side of the circumscribed regular polygon, and taking the arithmetic average of the two provides an approximation for the circumference of the circle from which the value of π can be obtained. In comparison with Liu Hui's method of circle division, Al-Kāshī is able to double the sides of the relevant polygon by calculating a single root, using the half-angle formula for cosines.

Conclusion

Bhāskara II died in Ujjain in about 1185, and afterward scientific activity in India went into gradual decline and mathematical progress more or less ceased entirely. In 1206, the long-lasting Delhi Sultanate was established, and India came into the Muslim sphere. About a century later, parts of the south became independent, and there began a protracted struggle for power. In contrast with India, mathematics in Persia both rose later and declined later. But shortly after the assassination of Ulugh

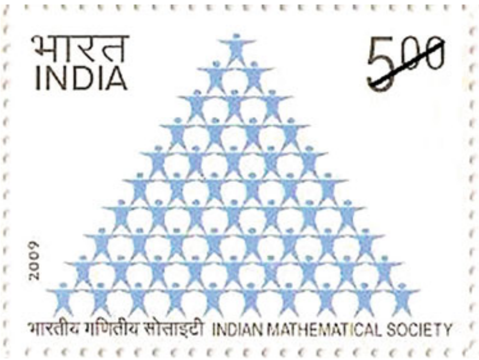

Fig. 4.30 Commemorative stamp of the Indian Mathematical Society

Beg, allegedly orchestrated by his own son, the Safavid dynasty, which was still essentially martial and internally constrained in nature, took power, and the glorious age of mathematics in Persia and indeed the entire Arabic world came to an end. But at precisely this time, the European Renaissance lit a new fire starting in the Apennine Mountains (Fig. 4.30).

As in Egypt, the mathematical minds of India were almost all clerical figures or otherwise belonged to a higher caste. This is in contrast with Greece, with the doors of mathematics in spirit at least were open to all. Another point of contrast is the Indian mathematicians, with the exception of Mahāvīra, were almost all astronomers by profession, whereas the Greeks viewed mathematics from the start as an independent discipline, worthy of study in its right (mathematics for the sake of mathematics). Third, the Indians expressed their mathematical thought in poetic language; the works could be mysterious or mystical, although of course they also introduced the zero numeral, and the results were mainly empirical, without derivations and proofs. The Greeks preferred a logical and even austere presentation and required proofs for every result. The astronomical bent of the Indian mathematicians produced however some splendid results. For example, Indian astronomers recognized that when the moon is half-full, the positions of the sun, the moon, and the earth form the vertices of a right triangle and were able to use this fact and their knowledge of the sine function to conclude that the distance between the moon and the earth is one-fortieth the distance between the sun and the earth.

The Persians were more accomplished in geometry than the Indians, though not so much as the Greeks, with a natural peak given by the geometric solution of cubic equations by Omar Khayyam. Like the Indian mathematics, the Arabic mathematicians for the most part considered themselves to be astronomers, and this emphasis on astronomy facilitated substantial contributions in trigonometry. The four mathematicians mentioned previously all carried out excellent work in astronomy, and the names of many stars even today use Latinate forms of Arabic words, for example, Aldebaran in Taurus, Vega in Lyra, Betelgeuse in Orion, Megrez in the Big Dipper asterism of Ursa Major, and Algol in Perseus. Arabic mathematicians also made substantial contributions to algebra, and many of the questions that appear later in Fibonacci's *Liber Abaci* are taken from al-Khwarizmi's *Algebra*.

The emphasis on astronomy under Islamic rule stemmed in part from the requirement to pray five times a day while facing toward Mecca, a challenging problem of coordination across a vast empire. To this end, they spared no expense in the construction of observatories and the recruitment of talented mathematical minds to staff them. The main work of these scholars included improvements to astronomical data, the construction of observatory sites, and the development of the science of optics. The Arabic interest in mathematics can be said to have arisen primarily through the practical demands of astronomy, astrology, and optics. But they were also excellent businesspeople and so had need to calculate distributions, inheritance, dividends, and so on. This led to the emphasis on algebra and especially calculation.

From the perspective of the history of mathematics, the Arabic mathematicians also served as a conduit through which the mathematical writings of India and ancient Greece were transmitted to Europe during a period of intense interest in translation. Translations of various mathematical works, including Euclid's *Elements*, survived in good condition in the House of Wisdom in Baghdad, long after the original texts had been lost or burnt. Later, this book was translated into Latin by European scholars, primarily in the western part of the Islamic empire centered in Toledo, the former capital of Spain. But as in the Indian and Chinese civilizations of the Middle Ages, Arabic mathematicians emphasized mostly practical results and did not inaugurate any new theoretical peaks or sustained development.

We now compare a bit the different philosophical attitudes of the Greeks and the thinkers to the east. The twentieth-century French philosopher Jacques Maritain (1882–1974) argued that the Indian philosophers viewed wisdom as a form of liberation, salvation, or divine wisdom and concomitantly their metaphysics never took the form of pure speculation in the practical sciences associated with the Greeks, who regarded wisdom rather as the target of rational human inquiry. This begins in the lower realm of earthly things, the tangible reality of change and movement, and the diversity of existence. It is somewhat paradoxical to note that the divine perspective of Indian philosophy contributed to simple and practical mathematical requirements, while for the earthly considerations of the Greeks and eventually all of Europe, mathematics developed its independent existence through the perfection of logical deduction.

It is worth a mention in closing that since the start of the twenty-first century, two mathematicians of Indian descent and two mathematicians of Persian or Iranian descent have earned Fields Medals. These are Manjul Bhargava in 2014 and Akshay Venkatesh in 2018 and Caucher Birkar in 2018 and Maryam Mirzakhani in 2014; Mirzakhani remains the first and only female recipient of the Fields Medal. Other developing countries have contributed several Fields Medalists in this century: Chi-Shen Tao, better known as Terence Tao, of Chinese descent in 2006, Ngô Bảo Châu of Vietnamese descent in 2010, and Artur Avila from Brazil in 2014.